The Pennsylvania State University
The Graduate School

SPHERICALLY SYMMETRIC LOOP QUANTUM GRAVITY: CONNECTIONS TO TWO-DIMENSIONAL MODELS AND APPLICATIONS TO GRAVITATIONAL COLLAPSE

A Dissertation in
Physics
by
Juan Daniel Reyes

Submitted in Partial Fulfillment
of the Requirements
for the Degree of

Doctor of Philosophy

December 2009

The dissertation of Juan Daniel Reyes was reviewed and approved* by the following:

Martin Bojowald
Associate Professor of Physics
Dissertation Advisor, Chair of Committee

Abhay Ashtekar
Eberly Professor of Physics

Radu Roiban
Associate Professor of Physics

Ping Xu
Professor of Mathematics

Richard Robinett
Professor of Physics
Director of Graduate Studies

*Signatures are on file in the Graduate School.

Abstract

We review the spherically symmetric sector of General Relativity and its midisuperspace quantization using Loop Quantum Gravity techniques. We exhibit anomaly-free deformations of the classical first class constraint algebra. These consistent deformations incorporate corrections presumably arising from a loop quantization and accord with the intuition suggesting that not just dynamics but also the very concept of spacetime manifolds changes in quantum gravity.

Our deformations serve as the basis for a phenomenological approach to investigate geometrical and physical effects of possible corrections to classical equations. In the first part of this work we couple the symmetry reduced classical action to Yang–Mills theory in two dimensions and discuss its relation to dilaton gravity and the more general class of Poisson sigma models. We show that quantum corrections for inverse triad components give a consistent deformation without anomalies. The relation to Poisson sigma models provides a covariant action principle of the quantum corrected theory with effective couplings. We also use our results to provide loop quantizations of spherically symmetric models in arbitrary D space-time dimensions.

In the second part, we turn to Lemaître–Tolman–Bondi models of spherical dust collapse and study implications of inverse triad quantum corrections, particularly for potential singularity resolution. We consider the whole class of LTB models, including nonmarginal ones, and as opposed to the previous strategy in the literature where LTB conditions are implemented first and anomaly-freedom is used to derive consistent equations of motion, we apply our procedure to derive anomaly-free models which first implements anomaly-freedom in spherical symmetry and then the LTB conditions. While the two methods give slightly different equations of motion, which may be expected given the ubiquitous sprawl of quantization ambiguities, conclusions are the same in both cases: Bouncing solutions for effective geometries, as a mechanism for singularity resolution, seem to appear

less easily in inhomogeneous situations as compared to quantizations of homogeneous models, and even the existence of homogeneous solutions as special cases in inhomogeneous models may be precluded by quantum effects.

Table of Contents

List of Figures

Acknowledgments

The work in this thesis was supported in part by the National Science Foundation grants PHY0653127 and PHY0748336 and the Consejo Nacional de Ciencia y Tecnología (CONACyT). I am deeply grateful for their generous support.

I thank Prof. Ping Xu for introducing me to the beauty of Mathematical Physics and Geometry. I am specially thankful to my advisor Prof. Martin Bojowald, for the opportunity to work in the exciting field of Quantum Gravity and for his guidance, lessons and inspiration throughout the years. I am indebted to both of them for their patience and help in difficult times.

I also thank all professors and visitors of the Institute for Gravitation and the Cosmos and the Physics Department from whom I have learned my basic GR and field theory. Particularly Prof. Abhay Ashtekar for his leadership and stimulating lectures, Bernd Bruegmann, John Collins, Alejandro Corichi, Radu Roiban, Hanno Sahlmann, Parampreet Singh and Pablo Laguna, for his initial help and guidance. For invaluable discussions I thank Miguel Campiglia, Mikhail Kagan, Victor Taveras, Rakesh Tibrewala and Nicolas Yunes. Megan Meinecke and Randi Neshteruk are also acknowledged for their always friendly help and professionalism.

Last but not least I thank my parents and family for their unconditional support and love and my friends in graduate school who shared unforgettable moments with me.

A Mis Padres
Gloria y Daniel

Chapter 1

Introduction

Undoubtedly, at the fundamental level, theoretical physics is in the need of reconciling General Relativity and quantum theory. Despite valiant efforts like Loop Quantum Gravity and String Theory, a final theory is still not in sight. Yet, many physical and conceptual questions seek an answer.

With the lack of a complete theory of quantum gravity, a general strategy to tackle these questions has been to apply symmetry reduction to obtain simplified models that make the problem of quantization and subsequent analysis more tractable. Nevertheless, it is critical to realize that symmetric models in a quantum theory play a different role than its classical counterparts. While the latter are exact solutions of a full classical theory, the mini or midisuperspace quantizations can only be approximations of the complete quantum theory since they are obtained by freezing many degrees of freedom which violates their uncertainty relations. These reduced models provide insights into the full theory but their predictions should be taken with caution and results must be compared amongst less symmetric models to ensure their robustness.

A related approach to examine conceptual questions or particular aspects of the sought for quantum theory, is to use toy models specifically designed to bypass technical complications of more realistic models or full quantum gravity. One simplifying assumption is the dimensional reduction of spacetime. In the last two decades, lower dimensional gravity models have been used as a theoretical laboratory to test ideas on quantization of gravity and further have sharpen our conceptual ideas on how to approach complicated problems at the interface between black holes and quantum mechanics.

In particular, the formulation of gravity in two spacetime dimensions allows to isolate precisely black holes of General Relativity. Upon definition of appropiate two - dimensional gravitational actions, the space of their clasical solutions does not only contain 2d black holes, but it only consists of such spacetimes. Many of the considerations performed for their 4d analogs, like the prediction of Hawking radiation as well as the analogy to thermodynamics, may be transferred also into the two-dimensional setting. The study of such models for example, has recently given important clues on the problem of information loss [2].

The study of lower dimensional theories also has more direct physical relevance. Generically, symmetry reduced or compactifications of higher dimensional theories result in lower dimensional theories with additional 'matter' fields. The dimension of the effective theory and the number of additional fields depends obviously on the original theory and the dimension of the symmetry group or number of compactified dimensions. In particular, by inserting the spherically symmetric ansatz for the metric in the four - dimensional Einstein-Hilbert action and integrating angular variables, the spherically symmetric sector of General Relativity may be seen as a two - dimensional gravity theory with a scalar or dilaton field. Indeed, by symmetric criticality for compact symmetry groups, variation of the effective two dimensional action gives Einstein's classical equations with spherical symmetry imposed. In this thesis we will focus on this sector of Einstein's theory and explore some of the possible implications of its loop quantization.

Loop Quantum Gravity has emerged as a widely studied candidate for a quantum theory of gravity [3, 4, 5]. The particular representation which uses holonomies of a connection and fluxes of triad variables instead of a metric has allowed to overcome many of the difficulties associated with the geometrodynamical approach and has given a mathematically rigorous and essentially unique kinematical framework [6]. This background independent framework has extensively and successfully been applied to homogeneous situations in cosmology [7, 8] and the Schwarzschild interior [9] where, as a consequence of the discreteness of spatial geometry in this representation, singularities are removed [10, 11].

The application of loop techniques to the simplest inhomogeneous situation of spherical symmetry has also provided similar results concerning singularity avoidance at the level of quantum equations [12]. An advantage of a loop treatment is precisely that a direct connection with the full theory can be made and therefore the effects of characteristic loop quantum corrections can be studied. Also, the essential uniqueness of the general theory at the kinematical level can be used to justify some of the choices made in the quantization of the model, and going the other direction, consistency in the reduced setting may help resolve some of the quantization ambiguities in the dynamical sector of the full theory.

The spherically symmetric setting allows exploration of field theoretic aspects and open issues in the the full theory, like the Dirac consistency or anomaly problem, that trivialize for the homogeneous models in the minisuperspace quantization of Loop Quantum Cosmology. Unfortunately, even in this simplified case, the analysis of the dynamical difference equations obtained from a loop quantization has proved difficult to deal with so that further simplifications or methods are required.

One such method is the framework of effective theories [13, 14]. Indeed, these have proved very useful in LQC where effective equations for isotropic models have given an intuitive picture of singularity resolution in terms of a 'bounce' produced by repulsive forces at short scales [15, 16]. We intend to extend some of these techniques to the spherically symmetric context.

Rigorous derivation of effective equations from the full loop quantum theory or its spherically symmetric sector is currently inaccessible because constructions of appropriate (semiclassical) states are not available. Instead, we can extract information by imposing consistency conditions, particularly anomaly-freedom of the quantum corrected constraint algebra, to derive effective equations incorporating some of the effects expected from a loop quantization. This serves at least four purposes: (i) to obtain physically relevant information arising from specific quantum corrections (phenomenology), (ii) to check for robustness of the predictions of homogeneous models in LQC, (iii) to check for consistency of the full model of Loop Quantum Gravity (anomaly problem and semiclassical limit), and (iv) to obtain insights into the full theory (for example to resolve quantization ambiguities).

In the canonical approach underlying Loop Quantum Gravity, corrections to classical equations of motion arise from three sources: corrections of inverse components of the densitized triad which is used instead of the spatial metric [17, 18], higher order corrections of connection components (due to the use of holonomies

[19]), and genuine quantum effects due to the back-reaction of higher moments of a state on its expectation values. The latter is generic for any interacting quantum system, where it leads, e.g., to effective potentials. The first two sources of corrections are characteristic of Loop Quantum Gravity and directly related to the underlying discreteness of its spatial geometry. Instead of incorporating all these effects at once however, on a first step for this work, we will focus on inverse triad corrections and study their effect on singularity behavior.

Consistent anomaly-free effective systems with quantum corrections from inverse triads have been shown to exist for perturbative inhomogeneities [20] and nonperturbative (but gauge fixed) inhomogeneous systems [21], and here we will show consistent deformations for the first class constraint algebra of spherically symmetric gravity-matter systems including 2d Yang-Mills fields, dust, and scalar fields.

In the first part of this thesis we will exploit the fact that spherically symmetric gravity, or more generally two-dimensional dilaton gravity [22, 23], can be formulated as a Poisson sigma model (PSM). Poisson sigma models are a general class of topological field theories in two dimensions which algebraically have a very rigid structure: a consistent deformation (such as an anomaly-free quantization) of a PSM can only be another PSM [24]. There is thus a clear way of interpreting any consistent way of introducing quantum corrections. Applying this to loop quantum gravity will then provide a covariant interpretation of corrections in the form of a corrected Poisson structure of the target manifold of the sigma model, or an effective dilaton potential. This is a nontrivial result since it is not always straightforward to derive action principles corresponding to quantum corrected Hamiltonian formulations. Furthermore, it is not obvious how to recover general covariance from the underlying discreteness of the quantum theory and in fact, the general structure of our deformed constraint algebras reflects how the underlying symmetries of the theory are changed compared to the classical ones. We also note that PSM's play a role as models of string theory with connections to Noncommutative Geometry [25, 26], such that an analysis along the lines followed in this work may shed light on the relation between Loop Quantum Gravity and String Theory or Noncomutative Geometry.

In the second part of this thesis we will look at possible implications of inverse triad corrections to gravitational collapse, focusing on the issue of singularity resolution. The concrete model we propose to study here is the Lemaitre-Tolman-Bondi (LTB) family of metrics [27, 28, 29]. This is an exact set of solutions to Einstein's equations describing a spherically symmetric collapse system sourced by inhomogeneous pressureless dust, and parameterized by the dust's initial energy density and velocity profile.

Gravitational collapse has been one of the most fruitful subjects in gravitational physics. In particular in the search for a theory of quantum gravity, black holes play a prominent role to understand conceptual issues such as singularity resolution and the information loss problem and, on a technical level, to test consistency of candidate theories.

Singularities are inevitable in Einstein's general theory of relativity, as shown by the singularity theorems of Hawking and Penrose [30, 31]. They signal the break down and incompleteness of the classical framework and therefore, candidates for quantum gravity or any fundamental theory, are expected to provide mechanisms to resolve these singularities.

The fact that quantum gravitational effects modify the nature of singularities resulting from gravitational collapse is best exemplified by the semiclassical treatment giving Hawking radiation [32]; once quantum effects are taken into account, black holes are shown to radiate with a thermal spectrum, giving the laws of black hole mechanics a physical interpretation as thermodynamical laws. Any quantum theory of gravity should then give a statistical interpretation of black hole entropy and elucidate the final stages of gravitational collapse and evaporation where the semiclassical treatment breaks down. These questions lie at the core of fundamental physics.

The LTB model allows the study of many of these features, it encompasses cosmological FRW models and the Schwarzschild solution, and it is simple enough to be amenable for technical treatment. Indeed, the classical and semiclassical model has been extensively studied in the literature in connection with the cosmic censorship hypothesis. Depending on the initial conditions of the matter distribution and energy, the model gives rise to different types of singularities, mild so called shell crossing singularities, or strong shell focusing ones may form which can be covered behind an event horizon or naked and behave qualitatively differently in the presence of quantum fields. The existence of non-space-like singularities in these models thus opens up a new arena for effects in quantum gravity.

The midi-superspace quantization of the LTB model has had partial but important successes in the last decades, particularly in the canonical approach. The program of canonical quantization in geometrodynamical variables was started in [33, 34]. The Wheeler-DeWitt quantization of the LTB model has given exact solutions to the wave equation and has provided concrete results concerning the entropy count and corrections to Hawking radiation [35].

Unfortunately, a complete loop quantization of the LTB system is currently out of grasp. Whereas the LTB conditions take a simple form at the kinematical level [21], their implementation and analysis at the dynamical quantum level is a difficult problem. Again to circumvent this obstacle and obtain meaningful information from the quantum system, we will use anomaly-free effective equations.

The analysis performed here and in [21] provides nontrivial results regarding the behavior of classical singularities and shows that, even though spherically symmetric loop quantum gravity is singularity-free at the fundamental level, none of the corrections studied resolves the singularities by effective geometries (at least in a naive way).

The numerical results of [21] suggest that although gravitational collapse is slowed down by loop corrections, implying indeed a repulsive effect at short distances, more subtle resolution mechanisms other that a bounce may be at work. Furthermore, as mentioned above and discussed later in this work, the structure of the quantum corrected constraint algebra implies that, even at an effective level, the interpretation of a pseudo-Riemmannian manifold structure for the quantum dynamics is a delicate issue.

In summary, this work exhibits anomaly-free consistent deformations of the constraint algebra of classical General Relativity which incorporate possible corrections from a loop quantization. It extends the program in Loop Quantum Gravity to analyze the issue of singularity resolution in non-perturbative inhomogeneous situations, particularly black holes, and provides partial consistency and robustness checks for the results obtained so far for homogeneous models. But before delving into the details of our results we step back to review in chapter 2, the quantization of spherically symmetric gravity by loop techniques, we discuss its relation to PSM models in chapter 3 and the results for the LTB model in chapter 4. Appendices A and B give respectively calculations of the deformed constraint algebra and canonical transformations between loop and PSM variables and appendix C presents further results for scalar fields.

Chapter 2

Spherically Symmetric Loop Quantum Gravity

In this chapter we briefly review the formulation of General Relativity in terms of Ashtekar variables and the midisuperspace loop quantization of its spherically symmetric sector, initiated in references [36, 37, 38, 39]. This also serves to fix the notation used throughout this work and motivates our choice of representation as opposed to other approaches.

As mentioned before, even this symmetry reduced sector of the theory is difficult to tackle and the program of quantization is currently incomplete. So we propose a strategy, via effective equations, to gain insights into the reduced and the full theory, and to extract physically relevant information coming from quantum corrections.

2.1 Full Theory

Loop quantum gravity is the most successful attempt for a nonperturbative background independent quantization of gravity. Its starting point is the canonical formulation of General Relativity (GR) obtained from the $3+1$ splitting of Einstein's theory [40].

In the Geometrodynamical approach [41], the canonical variables used are the metric $q_{ab}(\bar{x})$ on the three dimensional spatial manifold Σ with coordinates $\bar{x}$, and its conjugate momenta related to the extrinsic curvature $K_{ab}(\bar{x})$ of Σ. In contrast, in a loop quantization the phase space of GR is extended and the symplectic coordinate chart used consists of the $\mathfrak{su}(2)$-valued Ashtekar-Barbero connection $A_a^i(\bar{x})$ and the density one triad vector fields $E_i^a(\bar{x})$.

The densitized triad $E_i^a(\bar{x})$ encodes all the information of the intrinsic geometry of Σ. The spatial 3d metric q_{ab} is constructed from it via $q\,q^{ab} = E_i^a E_i^b$, where $q :=$ det q_{ab}. Equivalently, $E_i^a = e e_i^a$, where $e := det\, e_a^i$, and e_i^a is the inverse of a cotriad e_a^i, a set of three one-forms defining a frame at each point in Σ so that $q_{ab} = \delta_{ij} e_a^i e_b^j$.

Since the vector representation of $SO(3)$ is isomorphic to the Adjoint representation of $SU(2)$, we may also think of E_i^a as a $\mathfrak{su}(2)$-valued vector field. That is $E^a = E_i^a \tau^i$, for a basis $\{\tau^i\}_{i=1,2,3}$ of the Lie algebra of $SU(2)$. Following convention, we fix $\tau^i = \tau_i = -\frac{i}{2}\sigma_i$, where σ_i's are the Pauli matrices.

The Ashtekar–Barbero connection $A_a = A_a^i \tau_i$ [42, 43] is related to the extrinsic curvature $K_a^i := (\det E)^{-\frac{1}{2}} K_{ab} E^{bi}$ and the spin connection Γ_a^i compatible with the triad by the formula

$$A_a^i = \Gamma_a^i + \gamma K_a^i\,. \tag{2.1}$$

The real constant $\gamma > 0$, called the Barbero-Immirzi parameter [43, 44], does not play a role classically because it can be changed by canonical transformations. It will become important after quantization where transformations which could change γ are not represented unitarily.

Diffeomorphism invariance of the theory translates into hamiltonian evolution completely determined by constraints. Time evolution in GR is pure gauge. The canonical action of the theory reads[1]

$$S_H = \int \mathrm{d}t \left[\frac{1}{8\pi G\gamma} \int_\Sigma \mathrm{d}^3x\, E_i^a \dot{A}_a^i - \int_\Sigma \mathrm{d}^3x\, (\lambda^i \tilde{\mathcal{G}}_i + N^a \tilde{\mathcal{D}}_a + N\tilde{\mathcal{H}}) \right] \tag{2.2}$$

with G the gravitiational constant in four dimensions, and where N and N^a are the lapse function and shift vector respectively.

[1]Even though the Ashtekar connection is not the pullback to Σ of a spacetime connection [], this action does follow from the splitting and Legendre transform of the fully covariant Holst action [].

We recognize the symplectic structure given by the Poisson brackets

$$\{A_a^i(\bar{x}), E_j^b(\bar{y})\} = 8\pi G\gamma \delta_a^b \delta_j^i \delta(\bar{x}, \bar{y}) \tag{2.3}$$

and the seven constraints per point of the theory[2] which are also infinitesimal generators of the gauge symmetries: The three components of the Gauss constraint, generator of $SU(2)$-transformations

$$G_{\rm grav}[\lambda^i] = \frac{1}{8\pi G\gamma} \int d^3x\, \lambda^i (\partial_a E_i^a + \epsilon_{ij}{}^k A_a^j E_k^a) \tag{2.4}$$

The three components of the diffeomorphism constraint, generator of spatial diffeomorphims

$$D_{\rm grav}[N^a] = \frac{1}{8\pi G\gamma} \int d^3x\, N^a \left[(\partial_a A_b^j - \partial_b A_a^j) E_j^b - A_a^j \partial_b E_j^b \right] \tag{2.5}$$

And the Hamiltonian constraint, generator of coordinate time evolution

$$H_{\rm grav}[N] = \frac{1}{16\pi G} \int d^3x\, \frac{E_i^a E_j^b}{\sqrt{\det E}} \left[\epsilon^{ij}{}_k F_{ab}^k - 2(1+\gamma^2) K_{[a}^i K_{b]}^j \right] \tag{2.6}$$

where $F_{ab}^k = 2\partial_{[a} A_{b]}^k + \epsilon_{ij}{}^k A_a^i A_b^j$ is the curvature of the Ashtekar connection.

2.2 Spherical Symmetry Reduction

The spatial Riemannian manifold (Σ, q_{ab}) is said to be spherically symmetric, if the rotation group $SO(3)$ (or its universal cover $SU(2)$) acts effectively on Σ and the symmetry orbits are two-dimensional spheres. That is, we assume that for all points $\bar{x}$ except for possible symmetry axes or centers, the isotropy subgroup is $SO(2)$ (or $U(1)$) and $\Sigma \cong B \times (SO(3)/SO(2)) \cong B \times S^2$, where $B \cong \Sigma/SO(3)$ is a one dimensional radial manifold. Furthermore, the Killing vectors of the symmetry group are also Killing vectors of the metric q_{ab}, so that the metric is invariant under

[2] The Hamiltonian and diffeomorphism (vector) constraints contain additional terms proportional to the Gauss constraint. The expressions written here are the ones used for quantization and for the spherical symmetry reduction.

the action of the rotation group[3].

We will use angular coordinates (ϑ, φ) adapted to the invariance of the metric on the symmetry orbits. A coordinate on the radial manifold B will be called x but will not be fixed. As is well known, in these coordinates the spatial metric q_{ab} is determined by two functions $L(x)$ and $R(x)$ and takes the form

$$dq^2 = L^2 dx^2 + R^2 d\Omega^2 = L^2 dx^2 + R^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2)$$

this choice of coordinates (partial gauge) requires that $N^x(t, x)$ be the only non-vanishing component of the shift vector so that the full spacetime metric is

$$ds^2 = -N(t, x)^2 dt^2 + L^2(t, x)(dx + N^x(t, x)dt)^2 + R^2(t, x)d\Omega^2$$

In spherical symmetry, the connection $A = A_a dx^a$ and triad $E = E^a \partial_a$ take a special form just as the metric does. As in any gauge theory, the requirement that the fields be invariant under a group of spacetime transformations leads to a reduction of the number of free variables and a reduction of the gauge freedom to those changes of gauge that preserve the invariance condition.

The gauge group $\mathcal{G}$ of $SU(2)$-valued functions g on Σ acts on the phase space of pairs (A, E) of connection and triads by

$$(A \cdot g, E \cdot g) = (g^{-1} A g + g^{-1} dg, g^{-1} E g)$$

and all gauge related pairs are physically equivalent. Hence, unlike the metric, the connection and triad need not be exactly invariant under the rotation group acting on space, but it is enough for them to be invariant only up to a gauge transformation.

The problem of systematically finding the general form of all invariant connections on Σ can be formulated in terms of fiber bundles as the determination of all inequivalent principal $SU(2)$-bundles P with an action of the rotation group (as automorphisms of P) projecting to the symmetry action on Σ, and a determination of all invariant connections on such bundles [45, 46].

[3]More precisely $\sigma^*_{h(s)} q_{ab} = q_{ab}$, where $\sigma^*_{h(s)} q_{ab}$ is the pullback of the metric by diffeomorphisms $\sigma_{h(s)} : \Sigma \to \Sigma$ generated by one parameter subgroups $h : \mathbb{R} \to SO(3)$

There are different spherical symmetry types of SU(2)-principal fiber bundles[4], but only one of them provides non-degenerate triads in its associated vector bundle [37]. Using spatial coordinates (x, ϑ, φ) the reduced densitized triad in this non-degenerate sector takes the form:

$$E = E^x(x)\tau_3 \sin\vartheta \frac{\partial}{\partial x} + (E^1(x)\tau_1 + E^2(x)\tau_2)\sin\vartheta \frac{\partial}{\partial \vartheta} + (E^1(x)\tau_2 - E^2(x)\tau_1)\frac{\partial}{\partial \varphi} \quad (2.7)$$

which contains only three functions E^x, E^1 and E^2 (the superscripts 1 and 2 are not spacetime indices here but labels for two different functions).

The form of connections A preserving the corresponding symmetry up to gauge is

$$A = A_x(x)\tau_3 dx + (a_1(x)\tau_1 + a_2(x)\tau_2)d\vartheta + (a_1(x)\tau_2 - a_2(x)\tau_1)\sin\vartheta d\varphi + \tau_3 \cos\vartheta d\varphi \quad (2.8)$$

The SU(2)-gauge freedom of the original variables leaves a residual U(1)-gauge in the reduced theory. E^x is an invariant quantity under this Abelian gauge group, while A_x may be seen as a $U(1)$-connection on the radial manifold[5]. It is convenient to further introduce $U(1)$-invariant fields

$$A_\varphi := \sqrt{a_1^2 + a_2^2} \qquad \text{and} \qquad E^\varphi := \sqrt{(E^1)^2 + (E^2)^2}\,. \quad (2.9)$$

The remaining freedom of the four functions E^1, E^2 and a_1, a_2 not contained in E^φ and A_φ is pure gauge and can be parameterized by gauge angles $\eta(x)$ and $\tilde{\beta}(x)$, defined such that

$$\tau_1 \cos\eta + \tau_2 \sin\eta := (E^1\tau_2 - E^2\tau_1)/E^\varphi \quad , \quad \tau_1 \cos\tilde{\beta} + \tau_2 \sin\tilde{\beta} := (a_1\tau_2 - a_2\tau_1)/A_\varphi\,.$$

Each of these can be changed by the same amount with a gauge transformation, so only the difference $\tilde{\alpha} := \eta - \tilde{\beta}$ is gauge invariant.

[4]They are classified (constructed) by homomorphisms $\lambda : U(1) \to SU(2)$ from the isotropy subgroup to the gauge group, and $Z(\lambda(U(1)))$-subbundles with gauge group the centralizer of the image of the corresponding homomorphism in the gauge group.

[5]This splitting of the Ashtekar connection as reduced connection plus scalar fields on B is critical for an embedding of the reduced quantum theory into full LQG.

In terms of these quantities the spatial metric reads

$$dq^2 = E^{\varphi\,2}|E^x|^{-1}dx^2 + |E^x|d\Omega^2$$

The directions in the $\tau_1\tau_2$-plane of the angular components ϑ and φ of the densitized triad

$$\bar{\Lambda} := \tau_1 \sin\eta - \tau_2 \cos\eta \qquad \text{and} \qquad \Lambda := \tau_1 \cos\eta + \tau_2 \sin\eta$$

also determine the corresponding directions for the cotriad

$$e^i_a \tau_i dx^a = \mathrm{sgn}(E^x)E^\varphi |E^x|^{-\frac{1}{2}}\tau_3 dx + |E^x|^{\frac{1}{2}}\bar{\Lambda}d\vartheta + |E^x|^{\frac{1}{2}}\Lambda \sin\vartheta d\varphi$$

and therefore for extrinsic curvature $K = K^i_a\tau_i dx^a$, and spin connection $\Gamma = \Gamma^i_a \tau_i dx^a$

$$\begin{aligned} K &= K_x\tau_3 dx + K_\varphi \bar{\Lambda} d\vartheta + K_\varphi \Lambda \sin\vartheta d\varphi \\ \Gamma &= \Gamma_x \tau_3 dx + \Gamma_\varphi \Lambda d\vartheta + \Gamma_\varphi \bar{\Lambda} \sin\vartheta d\varphi + \tau_3 \cos\vartheta d\varphi \end{aligned}$$

$K_x(x)$ and $K_\varphi(x)$ are the $U(1)$-invariant radial and angular parts of the extrinsic curvature. As in the full theory the inhomogeneous radial component of the spin connection is not invariant and is in fact pure gauge $\Gamma_x = \eta'$, but the homogeneous angular part is gauge invariant

$$\Gamma_\varphi = -\frac{E^{x\,\prime}}{2E^\varphi}\,. \tag{2.10}$$

Relation (2.1) translates into

$$A_\varphi \cos\tilde{\alpha} = \gamma K_\varphi \quad , \quad A_x + \eta' = \gamma K_x \quad \text{and} \quad A^2_\varphi = \Gamma^2_\varphi + \gamma^2 K^2_\varphi \tag{2.11}$$

Substituting the symmetric form of the triad and connection variables (2.7) and (2.8) into the action for General Relativity (2.2), and integrating angular variables, gives the canonical form of the reduced theory. The symplectic structure of the original variables (2.3) makes the functions E^x, E^1 and E^2 canonically conjugate to A_x, a_1 and a_2, respectively. However, E^φ is not canonically conjugate to A_φ due to the non-linear transformation from the original field components. It turns out that E^φ is instead canonically conjugate to K_φ [39]. The most suitable polarization for a loop quatization of spherically reduced gravity turns out to be given by the canonical pairs:

$$\{A_x(x), \frac{1}{2\gamma}E^x(y)\} = \{K_\varphi(x), E^\varphi(y)\} = \{\eta(x), \frac{1}{2\gamma}P^\eta(y)\} = G\delta(x,y)\,. \qquad (2.12)$$

with

$$P^\eta(x) := 2A_\varphi E^\varphi \sin\tilde{\alpha} = 4\mathrm{tr}\left((E^1\tau_1 + E^2\tau_2)(A_2\tau_1 - A_1\tau_2)\right)$$

canonically conjugate to the gauge angle η.

The choice of K_φ as opposed to A_φ for configuration variable to construct 'holonomies' has two motivations. First the asymptotic fall off behavior for the extrinsic curvature $K_\varphi \to 0$ as $x \to \infty$, unlike the one for A_φ, gives the correct semiclassical limit from 'holonomies' in the Hamiltonian constraint. And secondly, the use of densitized triad variable E^φ as conjugate momentum to construct flux variables allows for a considerable simplification of the volume operator eigenstates and the Hamiltonian in the quantum theory[6].

The reduced 2d action is then

$$S = \int dt \Big[\frac{1}{2G\gamma}\int dx\,(E^x\dot{A}_x + 2\gamma E^\varphi\dot{K}_\varphi + P^\eta\dot{\eta}) - \int dx\,(\lambda\tilde{\mathcal{G}}_{\mathrm{grav}} + N^x\tilde{\mathcal{D}}_{\mathrm{grav}} + N\tilde{\mathcal{H}}_{\mathrm{grav}})\Big] \qquad (2.13)$$

where the only nonzero τ_3-component of (2.4) gives the Gauss constraint

$$G_{\mathrm{grav}}[\lambda] = \frac{1}{2G\gamma}\int dx\,\lambda(E^{x\,\prime} + P^\eta) \qquad (2.14)$$

generating U(1)-gauge transformations.The x-component in (2.5) gives the diffeomorphism constraint

$$D_{\mathrm{grav}}[N^x] = \frac{1}{2G}\int dx\,N^x\left[2E^\varphi K_\varphi' - \frac{1}{\gamma}A_x E^{x\,\prime} + \frac{1}{\gamma}\eta' P^\eta\right]$$

[6]One may argue that these considerations permit the construction of a quantum theory which is in some respects closer to full LQG [39], but it is also true that the embedding of reduced states in the full theory does not follow exactly as in [38], and requires a more careful analysis

$$= \frac{1}{2G} \int dx\, N^x \left[2E^\varphi K_\varphi' - K_x E^{x\,\prime} + \frac{1}{\gamma} \eta' (E^{x\,\prime} + P^\eta) \right] \tag{2.15}$$

generating diffeomorphisms on the one-dimensional radial manifold. And the Hamiltonian constraint (2.6) reads

$$H_{\rm grav}[N] = -\frac{1}{2G} \int dx\, N |E^x|^{-\frac{1}{2}} (K_\varphi^2 E^\varphi + 2K_\varphi K_x E^x + (1 - \Gamma_\varphi^2) E^\varphi + 2\Gamma_\varphi' E^x) \tag{2.16}$$

and generates dynamical evolution.

2.3 Loop quantization

To quantize spherically symmetric gravity, we have to define a well-defined algebra of quantities which seperate points on the classical phase space of the fields $(A_x, K_\varphi, \eta; E^x, E^\varphi, P^\eta)$. In particular, some of the fields must be integrated ("smeared") in suitable ways so as to provide an algebra under taking Poisson brackets free of the delta functions which appear for the fields in (2.12). The resulting algebra is then well-defined and can be represented on a Hilbert space to provide the basic representation of the quantum theory. In gravitational systems, the smearing must be done with care because there is no background metric to define the integrations (in addition to the physical metric given by E^x and E^φ which are to be quantized).

A loop quantization is based on holonomies

$$h_e[A_x] = \exp\left(\tfrac{1}{2} i \int_e A_x dx \right) \quad , \quad h_v[K_\varphi] = \exp(i\gamma K_\varphi(v)) \quad , \quad h_v[\eta] = \exp(i\eta(v)) \tag{2.17}$$

as smeared versions of the configuration variables. Here, we have used arbitrary curves e and points v in the radial line as labels. Varying e and v allows one to recover smooth local fields unambiguously. The appearance of integrations without auxiliary structures is dictated by the tensorial nature of the variables: The U(1)-connection A_x can naturally be integrated to define parallel transport, while the remaining components are scalars which we simply exponentiate without integrations. In this framework, exponentiations are not strictly necessary, but we use them in order to take into account the origin of these objects from non-Abelian holonomies in the full setting. Using exponentials instead of linear expressions in connection or extrinsic curvature components will not spoil the linear nature of the underlying basic algebra; see Eq. (2.19) below.

Similarly, we define flux variables

$$F_v[E^x] = E^x(v) \quad , \quad F_e[E^\varphi] = \int_e E^\varphi dx \quad , \quad F_e[P^\eta] = \int_e P^\eta dx \tag{2.18}$$

for the momenta. Also here, the integrations are naturally dictated by transformation properties of the fields, E^x being scalar while E^φ and P^η are densities of weight one. Without introducing a background metric or integration measure, we have thus managed to integrate all fields such that a well-defined algebra results:

$$\begin{aligned} \{h_e[A_x], F_v[E^x]\} &= i\gamma G \delta_{v\in e} h_e[A_x] \\ \{h_v[K_\varphi], F_e[E^\varphi]\} &= i\gamma G \delta_{v\in e} h_v[K_x] \\ \{h_v[\eta], F_e[P^\eta]\} &= 2i\gamma G \delta_{v\in e} h_v[\eta] \end{aligned} \tag{2.19}$$

where $\delta_{v\in e}$ is one if $v \in e$ and zero otherwise.

An irreducible representation of this algebra can easily be constructed. In the connection representation (which is customarily used in the full theory), an orthonormal basis of states is given by

$$T_{g,k,\mu}[A_x, K_\varphi, \eta] = \prod_{e\in g} \exp\left(\tfrac{1}{2} i k_e \int_e A_x dx\right) \prod_{v\in g} \exp(i\mu_v \gamma K_\varphi(v)) \exp(i k_v \eta(v)) \tag{2.20}$$

with integer labels k_e, k_v and positive real labels μ_v on edges e and vertices v, respectively, forming a finite graph g in the 1-dimensional radial line. The labels determine the connection dependence by irreducible representations of the groups spanned by the holonomies. (These groups are U(1) for A_x- and η-holonomies and the Bohr compactification $\bar{\mathbb{R}}_{\rm Bohr}$ of the real line for K_φ-holonomies; see [38] for details.)

Holonomies then simply act as multiplication operators. Specifically:

$$\hat{h}_e[A_x] T_{g,k,\mu} = T_{g,k+\delta_e,\mu} \quad , \quad \hat{h}_v[K_\varphi] T_{g,k,\mu} = T_{g,k,\mu+\delta_v} \quad , \quad \hat{h}_v[\eta] T_{g,k,\mu} = T_{g,k+\delta_v,\mu} \tag{2.21}$$

where δ_e is a function on the set of all edges which is one for the edge e and zero otherwise, and analogously for the function δ_v on the set of vertices. The state $T_{g,0,0}[A_x, K_\varphi, \eta] = 1$ is cyclic for this representation.

The densitized triad components, which are momenta conjugate to the connection components, act as derivative operators:

$$\hat{F}_v[E^x]T_{g,k,\mu} = \gamma\ell_{\mathrm{P}}^2 \frac{k_{e^+(v)} + k_{e^-(v)}}{2} T_{g,k,\mu} \tag{2.22}$$

$$\hat{F}_e[E^\varphi]T_{g,k,\mu} = \gamma\ell_{\mathrm{P}}^2 \sum_{v\in e} \mu_v T_{g,k,\mu} \tag{2.23}$$

$$\hat{F}_e[P^\eta]T_{g,k,\mu} = 2\gamma\ell_{\mathrm{P}}^2 \sum_{v\in e} k_v T_{g,k,\mu} \tag{2.24}$$

where $\ell_{\mathrm{P}}^2 = G\hbar$ is the Planck length squared and $e^\pm(v)$ denote the edges neighboring a point v, distinguished from each other using a given orientation of the radial line. (We have $k_{e^+(v)} = k_{e^-(v)}$ if v is not a vertex of the graph.) All flux operators have discrete spectra: eigenstates as seen in (2.22) and (2.23) are normalizable. But only $\hat{E}^x$ has a discrete set of eigenvalues, while $\hat{E}^\varphi$-eigenvalues fill the real line. (Their eigenstates are elements of the non-separable Hilbert space of square integrable functions on the Bohr compactification of the real line.)

These basic operators can be used for composite operators as well, providing well-defined but rather complicated constraint operators. The Gauss constraint is linear in triad components and can directly be quantized in terms of the basic flux operators and implies

$$k_v = -\tfrac{1}{2}(k_{e^+(v)} - k_{e^-(v)})\,. \tag{2.25}$$

A basis for gauge invariant states in the kernel of the Gauss constraint is thus

$$\bar{T}_{g,k,\mu}[A_x + \eta', K_\varphi] = \prod_{e\in g} \exp\left(\tfrac{1}{2} i k_e \int_e (A_x + \eta')dx\right) \prod_{v\in g} \exp(i\mu_v \gamma K_\varphi(v)) \tag{2.26}$$

where the labels k_v are eliminated by imposing (2.25). Accordingly, states solving the Gauss constraint only depend on $A_x + \eta'$, not on A_x and η' separately. The diffeomorphism constraint can directly be represented in its finite version, where its action simply moves labelled graphs in the radial manifold by a spatial diffeomorphism Φ: $\Phi T_{g,k,\mu} = T_{\Phi(g,k,\mu)}$ where $\Phi(g, k, \mu) = (\Phi(g), k', \mu')$ is the graph $\Phi(g)$

with labels $k'_{\Phi(e)}=k_e$ and $\mu'_{\Phi(v)}=\mu_v$.

Operators quantizing the Hamiltonian constraint have been constructed, but their constraint equations for physical states are more difficult to solve. Several steps are involved in this particular case: First, we have to quantize the inverse of E^x as it appears in the classical constraint. There is no direct operator inverse because $\hat{E}^x$ has a discrete spectrum containing zero. Nevertheless, well-defined operators with the correct classical limit exist, which are based on quantizing, e.g., the right hand side of the classical identity

$$4\pi\gamma G\frac{\mathrm{sgn}(E^x)E^\varphi}{\sqrt{|E^x|}}=\{A_x,V\}=2ih_e[A_x]\{h_e[A_x]^{-1},V\} \qquad (2.27)$$

instead of the left hand side [17]. This can be done using a quantization of the spatial volume $V=4\pi\int dxE^\varphi\sqrt{|E^x|}$ in terms of fluxes and turning the Poisson bracket into a commutator divided by $i\hbar$. Secondly, connection and extrinsic curvature components in the classical constraint are turned into holonomies and then quantized directly using the basic operators. Finally, one can quantize the spin connection terms making use of discretizations of the spatial derivatives. In this process, a well-defined operator

$$\begin{aligned}\hat{H}[N]=\frac{i}{2\pi G\gamma^3\delta^2\ell_{\mathrm{P}}^2}\sum_{v,\sigma=\pm1}\sigma N(v)\mathrm{tr}\Bigg(&\Big(h_\vartheta h_\varphi h_\vartheta^{-1}h_\varphi^{-1}-h_\varphi h_\vartheta h_\varphi^{-1}h_\vartheta^{-1}\\ &+2\gamma^2\delta^2(1-\hat{\Gamma}_\varphi^2)\tau_3\Big)h_{x,\sigma}[h_{x,\sigma}^{-1},\hat{V}]\\ &+\Big(h_{x,\sigma}h_\vartheta(v+e^\sigma(v))h_{x,\sigma}^{-1}h_\vartheta(v)^{-1}-h_\vartheta(v)h_{x,\sigma}h_\vartheta(v+e^\sigma(v))^{-1}h_{x,\sigma}^{-1}\\ &+2\gamma^2\delta\int_{e^\sigma(v)}\hat{\Gamma}'_\varphi\Lambda(v)\Big)h_\varphi[h_\varphi^{-1},\hat{V}]\\ &+\Big(h_\varphi(v)h_{x,\sigma}h_\varphi(v+e^\sigma(v))^{-1}h_{x,\sigma}^{-1}-h_{x,\sigma}h_\varphi(v+e^\sigma(v))h_{x,\sigma}^{-1}h_\varphi(v)^{-1}\\ &+2\gamma^2\delta\int_{e^\sigma(v)}\hat{\Gamma}'_\varphi\bar{\Lambda}(v)\Big)h_\vartheta[h_\vartheta^{-1},\hat{V}]\Bigg)\end{aligned} \qquad (2.28)$$

with the correct classical limit results. Here, δ is a parameter appearing in the exponent of angular holonomies $h_\vartheta=\exp(\delta K_\varphi\bar{\Lambda})$ and $h_\varphi=\exp(\delta K_\varphi\Lambda)$. We use the SU(2)-form of holonomies for compactness and for an easier comparison with the full theory; as matrix elements we have our basic holonomies $h_v[K_\varphi]$ from h_ϑ

and h_φ, as well as $h_{e^\sigma}(v)[A_x]$ from $h_{x,\sigma}(v) := \exp(\int_{e^\sigma(v)} A_x\tau_3 dx)$. Moreover, matrix elements of $\Lambda(v) := \tau_1\cos\eta(v) + \tau_2\sin\eta(v)$ and $\bar\Lambda(v) := \tau_1\sin\eta(v) - \tau_2\cos\eta(v)$ act by multiplication with holonomies $h_v[\eta]$. The parameter δ in angular holonomies may be a function of E^x instead of a constant; this represents the possibility of lattice refinements [47, 21] in the symmetry orbits which a full Hamiltonian constraint operator in general implies since it creates new edges in a graph by acting with the corresponding holonomies. In the reduced model, there are no edges in the symmetry orbits, and thus no direct way exists to implement the creation of such new edges. Instead, an E^x-dependence of δ can be used to make the edge parameter δ depend on the area of the orbit. A value δ shrinking with the area would imply that more edges are created because the coordinate length of each new edge used is smaller at a larger areal size.

The constraint equation $\hat H[N]\psi = 0$ for all N can be formulated as a set of coupled difference equations for states labeled by the triad quantum numbers k_e and μ_v, which have the form

$$\begin{aligned}
&\hat C_{\mathrm{R}+}(k_-,k_+-2)^\dagger\psi(\ldots,k_-,k_+-2,\ldots) + \hat C_{\mathrm{R}-}(k_-,k_++2)^\dagger\psi(\ldots,k_-,k_++2,\ldots)\\
&+\hat C_{\mathrm{L}+}(k_--2,k_+)^\dagger\psi(\ldots,k_--2,k_+,\ldots) + \hat C_{\mathrm{L}-}(k_-+2,k_+)^\dagger\psi(\ldots,k_-+2,k_+,\ldots)\\
&\qquad\qquad +\hat C_0(k_-,k_+)^\dagger\psi(\ldots,k_-,k_+,\ldots) = 0\,, \qquad (2.29)
\end{aligned}$$

one for each vertex. Only the edge labels k_e are written explicitly in this difference expression, but states also depend on vertex labels μ_v on which the coefficient operators $\hat C_I$ act; see [39] for details. We will only require the central coefficient

$$\begin{aligned}
\hat C_0|\vec\mu,\vec k\rangle \;=\; & \frac{\ell_{\mathrm{P}}}{2\sqrt{2}G\gamma^{3/2}\delta^2}\Big(|\mu|\left(\sqrt{|k_++k_-+1|}-\sqrt{|k_++k_--1|}\right)\\
&\times\big(|\mu_-,k_-,\mu+2\delta,k_+,\mu_+\rangle + |\mu_-,k_-,\mu-2\delta,k_+,\mu_+\rangle\\
&\quad -2(1+2\gamma^2\delta^2(1-\Gamma_\varphi^2(\vec\mu,\vec k)))|\mu_-,k_-,\mu,k_+,\mu_+\rangle\big)\\
&-4\gamma^2\delta^2\mathrm{sgn}_{\delta/2}(\mu)\sqrt{|k_++k_-|}\Gamma'_\varphi(\vec\mu,\vec k)|\mu_-,k_-,\mu,k_+,\mu_+\rangle\Big)\\
&+\hat H_{\mathrm{matter},v}|\mu_-,k_-,\mu,k_+,\mu_+\rangle \qquad (2.30)
\end{aligned}$$

below. For these basic difference equations, one can show that they are free of gravitational singularities in the sense of quantum hyperbolicity [48]: the recurrence scheme they define provides an evolution of the wave function which does not stop where a classical singularity would form [12].

2.4 Effective Equations

The difference equations (2.29) obtained by imposing the Hamiltonian constraint operator are linear but quite involved; moreover, they are formulated for a wave function which is not easy to interpret in a generally covariant system. Instead of solving the equations directly, one can make use of an effective analysis which provides approximate equations of a simpler form for expectation values and possibly higher moments of the state.

These effective equations are obtained as a Hamiltonian system whose Hamiltonian function $H^Q := \langle\hat{H}\rangle$ is the expectation value of the Hamiltonian operator $\hat{H}$ in a general state [13], and which are constrained by expectation values of products of constraint operators [49]. This framework is based on a geometrical formulation of quantum mechanics [50], where the imaginary part of the inner product of the quantum theory gives a symplectic structure for the Hilbert space, and operators define functions on it by expectation values. The Poisson structure is given by

$$\{\langle\hat{F}\rangle, \langle\hat{G}\rangle\} := \left\langle \frac{1}{i\hbar}[\hat{F},\hat{G}] \right\rangle \tag{2.31}$$

for expectation values of self-adjoint operators $\hat{F}$ and $\hat{G}$.

Expectation values of the basic variables such as A_x and E^x would correspond to the classical values in constraint expressions, but there is an additional step involved in deriving effective equations: the expectation values of general operators depend on infinitely many quantum variables such as the spread of states or deformations of the wave function, say for example from a Gaussian, which do not have classical analogs. In purely mechanical systems these additional quantum variables can be suitably parameterized in the form[7]

$$G_q^{a,n} = \langle(\hat{q}-\langle\hat{q}\rangle)^{n-a}(\hat{p}-\langle\hat{p}\rangle)^a\rangle_{\mathrm{Weyl}}\,, \qquad 1 < n \in \mathbb{N},\ 0 \le a \le n$$

[7]Weyl here denotes totally symmetric ordering

for any canonical pair (q, p) in the classical theory. These variables, along with $\langle \hat{q} \rangle$ and $\langle \hat{p} \rangle$, coordinatize the space of quantum states (rays in the Hilbert space), which may be seen as an (infinite dimensional) fiber bundle with base manifold the classical phase space. Thus every classical degree of freedom does not only give rise to expectation values $\langle \hat{q} \rangle$ and $\langle \hat{p} \rangle$ but to infinitely many additional quantum variables. All of these variables are dynamical, and are in general coupled to each other. Generically then $\langle \hat{H}(\hat{q}, \hat{p}) \rangle \neq H(\langle \hat{q} \rangle, \langle \hat{p} \rangle)$, and the quantum Hamiltonian H^Q contains (infinitely) many more terms depending on these $G_q^{a,n}$. Effective equations are obtained by truncating this infinite set of variables to finitely many fields, resulting in equations of motion of the classical form corrected by quantum terms. In our case we will keep the same number of fields as in the classical theory and these correction terms will arise from the choice of representation by the use of holonomies and the particular quantization of inverse triads.

The most common way to perform a truncation of the infinite number of quantum variables to a finite set is by using a certain class of semiclassical states to compute expectation values of the Hamiltonian operator. The regime under consideration determines what a suitable set of semiclassical states is. In particular, spread and higher moments of a semiclassical state are usually constructed or assumed to be subdominant compared to expectation values. The construction of such states can be quite involved, even for the spherically symmetric case, but without explicitly constructing semiclassical states satisfying such conditions, one can make semiclassicality assumptions for those parameters to be negligible. This is what we will do in this thesis as a first step and as a shortcut to deriving effective equations from the quantum theory in order to gain intuition on how characteristic features of the quantized Hamiltonian, such as deviations of the quantized $1/E^x$ from the classical behavior, enter the effective Hamiltonian and give rise to potential physical effects (e.g. for black hole collapse in chapter 4).

2.4.1 Anomaly-free constraint algebra

In a Hamiltonian formulation of a gauge theory, like General Relativity, the constraints serve three purposes: they restrict the fields and their initial values to those which make the constraints vanish, they generate gauge transformations which in

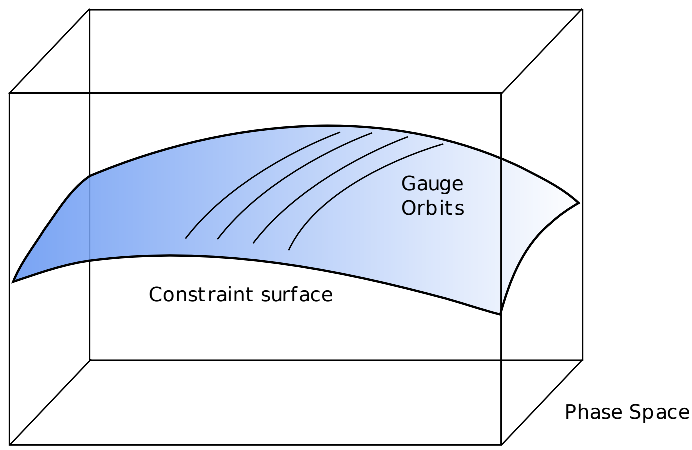


Figure 2.1. First class constraints restrict dynamics to the submanifold of phase space where the constraints vanish. Their Hamiltonian flow is tangential to this 'constraint surface' and determines the 'gauge orbits'.

the case of GR coincide (on shell) with coordinate transformations or equivalently spacetime diffeomorphisms and, they provide equations of motion for the fields in any coordinate time parameter. All this is necessary to ensure the system described is covariant, even though distinguishing momentum variables invariably removes manifest spacetime covariance in a Hamiltonian description.

For this to be consistent, it is critical that constraints be preserved under the time evolution they generate. This is automatically guaranteed if the constraints generate a closed Poisson algebra, or in Dirac's terminology, if they constitute a first class set. Constraints then define, via the Poisson structure, hamiltonian vector fields on field space which are tangent to the submanifold defined by the vanishing of the constraints, and their flow may then be interpreted as gauge; and evolution in the case of GR (Fig.2.1).

Classically, this is certainly realized as a reflection of the general covariance of the underlying theory. However, if quantum aspects are implemented, one must ensure that this consistency condition is preserved: the quantization must be anomaly free. In particular effective formulations must satisfy this requirement if equations are to be consistent and preserve the gauge symmetries. Anomaly-freedom is thus a key requirement not only for the consistency of an underlying fundamental theory but also for the possibility of applications. Quantum corrections to constraints cannot appear in arbitrary forms, but must be restricted so that the deformed

Poisson algebra closes.

We will thus use the requirement of anomaly-freedom to reduce some of the ambiguities or arbitrariness in the form of quantum correction functions we will introduce. As we will see, while this is certainly not enough to pinpoint unique solutions or recipes for the quantization of the Hamiltonian, it does impose non-trivial restrictions and further shows that anomaly-free effective formulations are possible in Loop Quantum Gravity

That anomaly-free deformations of GR with equations incorporating corrections from LQG are realizable is not obvious a priori. Quantum Geometry corrections result from an underlying spatial discreteness, which may cast doubt on whether they can leave the theory covariant. There are arguments at the level of the full theory [17] stating that quantum operators may be anomaly-free, but it remains unknown how to descend from this statement to anomaly-free effective space-time geometries.

We will not compute explicitly expectation values of the Hamiltonian constraint operator. Instead we will introduce appropriate functions of the canonical variables to account for different possibilities of quantum corrections. The general modified effective Hamiltonian constraint we consider here is:

$$H^Q_{\rm grav}[N] = -\frac{1}{2G}\int \mathrm{d}x\, N\big(\alpha\,|E^x|^{-\frac{1}{2}}E^\varphi f_1 + 2s\bar{\alpha}\,|E^x|^{\frac{1}{2}}f_2 + \alpha\,|E^x|^{-\frac{1}{2}}E^\varphi \\ - \alpha_\Gamma\,|E^x|^{-\frac{1}{2}}E^\varphi\Gamma_\varphi^2 + 2s\bar{\alpha}_\Gamma\,|E^x|^{\frac{1}{2}}\Gamma'_\varphi\big)\,. \qquad (2.32)$$

The functionals $\alpha[E^x, E^\varphi]$ and $\bar{\alpha}[E^x, E^\varphi]$ account for possible corrections from the quantization of inverse triads coming from the factors $|E^x|^{-\frac{1}{2}}E^\varphi$ and $|E^x|^{\frac{1}{2}}$. As functions, they are assumed to depend only on triad variables E^x and E^φ but are otherwise unrestricted. Similarly $\alpha_\Gamma[E^x, E^\varphi]$ and $\bar{\alpha}_\Gamma[E^x, E^\varphi]$ depend only on the triad variables and incorporate corrections from inverse triads and the quantization of the spin connection. Their classical limit $\alpha = \bar{\alpha} = \alpha_\Gamma = \bar{\alpha}_\Gamma = 1$ will later be imposed to obtain specific closed algebras.

Classically $f_1 = K_\varphi^2$ and $f_2 = K_\varphi(A_x + \eta')$, but to consider corrections from the use of holonomies instead of the connection, we use general functionals $f_1[K_\varphi, E^x, E^\varphi]$ and $f_2[A_x + \eta', K_\varphi, E^x, E^\varphi]$ depending on extrinsic curvature components K_φ and

$\gamma K_\varphi = A_x + \eta'$, as well as triad variables.

As a first approximation, all the correction functions here (except for f_2 which depends on η') are restricted to depend on phase space variables but not on their spatial derivatives.[8] Of course, while our effective Hamiltonian (2.32) is motivated by the constructions and form of the quantum operator (2.28), by no means it exhausts all possibilities for the quantum Hamiltonian $H^Q = \langle \hat{H} \rangle$ (not even with the assumption of the same number of degrees of freedom for the effective system as the classical one; subdominant higher moments for states, and no spatial derivatives).

Since diffeomorphism invariance is implemented by finite transformations in the quantum theory rather than by constructing the infinitesimal quantum generator. The diffeomorphism constraint remains unchanged.

We take the Poisson bracket derived from (2.31) to coincide with the classical one (2.12). There are important subtleties arising from the choice of the holonomy-flux algebra (2.19) as more fundamental for the quantum theory but we will not consider these issues here. The Poisson algebra of quantum corrected constraints is then

$$\{H^Q_{\rm grav}[M], H^Q_{\rm grav}[N]\} =$$

$$= \frac{1}{2G}\int {\rm d}x\,(MN' - NM')\Bigg[\,\bar{\alpha}\bar{\alpha}_\Gamma \frac{|E^x|}{E^{\varphi\,2}}\left(2\gamma\left(\frac{\partial f_2}{\partial A_x}\right)' E^\varphi - \frac{\partial f_2}{\partial K_\varphi}E^{x\,\prime}\right)$$

$$+ s\left(\bar{\alpha}\alpha_\Gamma\gamma\frac{\partial f_2}{\partial A_x} - \alpha\bar{\alpha}_\Gamma\frac{1}{2}\frac{\partial f_1}{\partial K_\varphi}\right)\frac{E^{x\,\prime}}{E^\varphi} + 2\left(\bar{\alpha}_\Gamma\bar{\alpha}' - \bar{\alpha}\bar{\alpha}_\Gamma'\right)\gamma\frac{\partial f_2}{\partial A_x}\frac{|E^x|}{E^\varphi}\;\Bigg] \tag{2.33}$$

[8]The usual expectation that quantum gravity gives rise to low energy effective (covariant) actions with higher curvature terms, translates in the canonical framework to higher powers and higher spatial derivatives of extrinsic curvature and triad. Higher time derivatives are more difficult to see in a canonical treatment and usually correspond to the presence of the additional quantum variables alluded before. Since these two types of correction terms should combine to give higher curvature terms in a covariant formulation and since we are assuming independent quantum variables are suppressed, it is consistent to ignore higher spatial terms.

$$
\begin{aligned}
\{H^Q_{\rm grav}[N], D_{\rm grav}[N^x]\} = &-H^Q_{\rm grav}[N^x N'] \\
&-\frac{1}{2G}\int {\rm d}x\, N N^{x\,\prime} E^\varphi \Bigg[\frac{\partial \alpha}{\partial E^\varphi}|E^x|^{-\frac{1}{2}}E^\varphi f_1 + 2s\frac{\partial\bar\alpha}{\partial E^\varphi}|E^x|^{\frac{1}{2}} f_2 + \frac{\partial \alpha}{\partial E^\varphi}|E^x|^{-\frac{1}{2}}E^\varphi \\
&\qquad - \frac{\partial \alpha_\Gamma}{\partial E^\varphi}|E^x|^{-\frac{1}{2}}E^\varphi \Gamma_\varphi^2 + 2s\frac{\partial\bar\alpha_\Gamma}{\partial E^\varphi}|E^x|^{\frac{1}{2}}\Gamma'_\varphi\Bigg] \\
&-\frac{1}{2G}\int {\rm d}x\, N N^{x\,\prime} E^\varphi \Bigg[\alpha|E^x|^{-\frac{1}{2}}E^\varphi\frac{\partial f_1}{\partial E^\varphi} + 2s\bar\alpha|E^x|^{\frac{1}{2}}\frac{\partial f_2}{\partial E^\varphi} \\
&\qquad + 2s\bar\alpha\frac{|E^x|^{\frac{1}{2}}}{E^\varphi}\left(\frac{\partial f_2}{\partial A_x}A_x + \frac{\partial f_2}{\partial \eta'}\eta' - f_2\right)\Bigg]
\end{aligned} \tag{2.34}
$$

As shown in appendix A, the algebra can be made first class for both nontrivial holonomy and inverse triad corrections. Since specific holonomy corrections are difficult to discern in the spherically symmetric case and the constraint algebra incorporating their effect is harder to control, we will focus solely on inverse triad corrections.

All quantum corrections (including genuine quantum effects due to the backreaction of higher moments of a state on its expectation values), occur at the same time and must be combined in a complete treatment. While one type of correction might be dominant in certain regimes, this would not be known a priori but would have to be shown by a detailed analysis. While holonomy corrections have been show to play a more significant role in homogeneous models [51, 52], inhomogeneous constructions in perturbative LQG suggest inverse triad corrections are dominant in that context [53]. Nevertheless, it is legitimate at first to separate the different corrections, analyze individual effects and later combine results.

For inverse triad corrections only, $f_1 = K_\varphi^2$ and $f_2 = K_\varphi(A_x + \eta')$, and the Poisson bracket (2.33) gives explicitly

$$
\begin{aligned}
\{H^Q_{\rm grav}[M], H^Q_{\rm grav}[N]\} = &D_{\rm grav}[\bar\alpha\bar\alpha_\Gamma|E^x|(E^\varphi)^{-2}(MN' - NM')] \\
&- G_{\rm grav}[\bar\alpha\bar\alpha_\Gamma|E^x|(E^\varphi)^{-2}(NM' - MN')\eta'] \\
&+ \frac{1}{2G}\int {\rm d}x\,(MN' - NM')(\bar\alpha\alpha_\Gamma - \alpha\bar\alpha_\Gamma)\frac{sK_\varphi(E^x)'}{E^\varphi}
\end{aligned}
$$

$$+\frac{1}{2G}\int \mathrm{d}x\,(MN'-NM')(\bar{\alpha}'\bar{\alpha}_\Gamma-\bar{\alpha}\bar{\alpha}_\Gamma')\frac{2K_\varphi|E^x|}{E^\varphi} \quad (2.35)$$

For a first class algebra the last two terms, which are not related to constraints, must vanish, restricting the correction functions with conditions (A.11):

$$\bar{\alpha}\alpha_\Gamma-\alpha\bar{\alpha}_\Gamma-2E^x\left(\bar{\alpha}_\Gamma\frac{\partial\bar{\alpha}}{\partial E^x}-\bar{\alpha}\frac{\partial\bar{\alpha}_\Gamma}{\partial E^x}\right)=0 \quad (2.36)$$

There are additional conditions which are also imposed by the Poisson bracket (2.34), which reads:

$$\begin{aligned}\{H^Q_{\rm grav}[N],&D_{\rm grav}[N^x]\}=-H^Q_{\rm grav}[N^xN']\\&-\frac{1}{2G}\int\mathrm{d}x\,N(N^x)'E^\varphi\Big(\frac{\partial\alpha}{\partial E^\varphi}|E^x|^{-\frac{1}{2}}K_\varphi^2E^\varphi+2s\frac{\partial\bar{\alpha}}{\partial E^\varphi}K_\varphi K_x|E^x|^{\frac{1}{2}}\\&+\frac{\partial\alpha}{\partial E^\varphi}|E^x|^{-\frac{1}{2}}E^\varphi-\frac{\partial\alpha}{\partial E^\varphi}|E^x|^{-\frac{1}{2}}\Gamma_\varphi^2E^\varphi+2s\frac{\partial\bar{\alpha}}{\partial E^\varphi}\Gamma_\varphi'|E^x|^{\frac{1}{2}}\Big)\end{aligned} \quad (2.37)$$

and gives conditions (A.11):

$$\alpha^{-1}\frac{\partial\alpha}{\partial E^\varphi}=\alpha_\Gamma^{-1}\frac{\partial\alpha_\Gamma}{\partial E^\varphi}=\bar{\alpha}_\Gamma^{-1}\frac{\partial\bar{\alpha}_\Gamma}{\partial E^\varphi} \quad (2.38)$$

There are infinitely many solutions to these equations and we will show explicit examples in subsequent chapters, where we will further couple gravity to different matter sources and use these results to construct anomaly-free systems which may be used for applications.

2.4.1.1 Deformations of General Relativity and spacetime

For correction functions satisfying (2.36) and (2.38), the quantum corrected constraint algebra becomes modulo the Gauss constraint

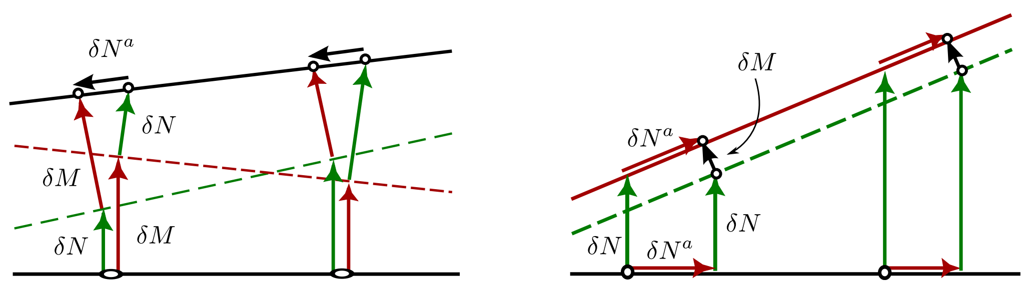
$$\{H^Q_{\rm grav}[M],H^Q_{\rm grav}[N]\}=D_{\rm grav}[\bar{\alpha}\bar{\alpha}_\Gamma|E^x|(E^\varphi)^{-2}(MN'-NM')] \quad (2.39)$$

$$\{H^Q_{\rm grav}[N],D_{\rm grav}[N^x]\}=-H^Q_{\rm grav}[N^xN'-\alpha^{-1}\frac{\partial\alpha}{\partial E^\varphi}E^\varphi NN^{x\,\prime}] \quad (2.40)$$

which certainly coincides in the classical limit with the so called 'algebra of hypersurface deformations' of General Relativity and generally spacetime covariant systems:

$$\begin{aligned}
\{H[M], H[N]\} &= D[q^{ab}(M\partial_b N - N\partial_b M)] \\
\{H[N], D[N^a]\} &= -\,H[N^a\partial_a N] \\
\{D[N^a], D[M^b]\} &= D[[N^a, M^b]]
\end{aligned} \tag{2.41}$$

This algebra is a fundamental object, encoding not only the gauge symmetries of Einstein's theory but the structure of spacetime. The Poisson bracket relations (2.41) express the fact that dynamics takes place on spacelike hypersurfaces embedded in a pseudo-Riemannian spacetime [1, 54]. In a Hamiltonian formulation the dynamics of a general field are obtained by prescribing the field on a spacelike hypersurface and then deforming this hypersurface through spacetime. The deformations of hypersurfaces in a pseudo-Riemannian spacetime observe a simple geometrical pattern[9] (Fig.2.2), and any dynamics taking place on such a spacetime must reflect the structure of this pattern.



(a) The commutator of the generators of 'pure deformations' or 'translations': $[\mathcal{H}_{\delta M}, \mathcal{H}_{\delta N}] = \mathcal{D}_{\delta N^a}$. The 'stretching' $\delta N^a = q^{ab}(\delta M \partial_b \delta N - \delta N \partial_b \delta M)$ is needed to compensate reverse order of the two 'translations' δM and δN.

(b) The commutator of a 'pure deformation' and a 'stretching' or spatial diffeomorphism: $[\mathcal{H}_{\delta N}, \mathcal{D}_{\delta N^a}] = -\mathcal{H}_{\delta M}$. A 'translation' $\delta M = -\delta N^a \partial_a \delta N$ compensates for reversing the order of a 'translation' δN and a 'stretching' δN^a.

Figure 2.2. Hypersurface deformation algebra [1]: The closing relations ensure that consecutive deformations of hypersurface embeddings result in the same final embedding.

The closure of the deformed constraint algebra ensures that the effective quantum corrected theory has the same number of gauge symmetries as the classical one, but due to the explicit appearance of correction functions on the right hand

[9] The bracket $\{D[N^a], D[M^b]\} = D[[N^a, M^b]]$ is a representation of the Lie algebra of infinitesimal spatial diffeomorphims.

side of (2.39) and (2.40), this algebra is not a representation of the algebra of hypersurface deformations[10]. Not only the dynamics (reflected by the modified Hamiltonian constraint operator (2.32)) but the structure or symmetries of the spacetime manifold are changed by these quantum corrections. Gauge symmetries will, in general, no longer coincide with coordinate transformations in our models.

It is difficult to determine what the general structure of these deformed spacetimes is. For particular corrections and models with matter, one may interpret these corrections as effective couplings only (as we do in chapter 3). For inverse triad corrections one may try to interpret these modifications by effective geometries (as we attempt in chapter 4), but this cannot give a fully satisfactory answer since, for example, corrections to the constraint algebra may depend on connection variables too as in (A.14).

2.4.2 Inverse triad corrections

Inverse triad corrections arise from every Hamiltonian operator quantized by loop techniques, where inverse components of the densitized triad appear. Such corrections are directly related to spatial discreteness of quantum geometry since, as seen from equations (2.22) and (2.23), densitized triads as basic variables are quantized to flux operators with discrete spectra containing zero . Since such operators do not have densely defined inverses, no direct inverse operator is available. Instead as sketched before, well-defined quantizations exist based on techniques introduced in [17, 56], implying corrections to the classical inverse which we parameterize by general correction functions.

To gain insight into what the specific form of quantum corrections may look like in our effective Hamiltonian, we may look at expectation values of operators like $\widehat{\frac{1}{\sqrt{E^x}}}$ as quantized in the Hamiltonian constraint.

For example, using equation (2.27), we may quantize

$$\int_e \frac{\widehat{\mathrm{sgn}(E^x)}E^\varphi}{\sqrt{|E^x|}} = -\frac{i}{2\pi\gamma\ell_{\mathrm{P}}^2}\mathrm{tr}(\tau_3 h_x[h_x^{-1},\hat{V}])$$

[10]Indeed, in vacuum, General relativity is the unique canonical representation of the hypersurface deformation algebra if the metric and its conjugate momenta are the sole canonical variables of the theory [55]

using the identity $\exp(c\tau_i) = \cos(c/2)\mathbb{I} + 2\tau_i \sin(c/2)$ we have

$$\begin{aligned} h_x[h_x^{-1}, \hat{V}] =& \hat{V} - \cos(c/2)\hat{V}\cos(c/2) - \sin(c/2)\hat{V}\sin(c/2) \\ &+ 2\tau_3[\cos(c/2)\hat{V}\sin(c/2) - \sin(c/2)\hat{V}\cos(c/2) \end{aligned}$$

for $c = \int_e A_x\, dx$, so

$$\int_e \frac{\widehat{\mathrm{sgn}(E^x)}E^\varphi}{\sqrt{|E^x|}} = -\frac{i}{2\pi\gamma\ell_{\mathrm{P}}^2}(\sin(c/2)\hat{V}\cos(c/2) - \cos(c/2)\hat{V}\sin(c/2))$$

From

$$(\sin(c/2))T_{g,k,\mu} = \frac{e^{ic/2} - e^{-ic/2}}{2i} T_{g,k,\mu} = \frac{1}{2i}(T_{g,k+\delta_e,\mu} - T_{g,k-\delta_e,\mu})$$

and a similar expression for the $\cos(c/2)$ operator, and the volume operator eigenvalues

$$\hat{V}\, T_{g,k,\mu} = 4\pi\gamma^{3/2}\ell_{\mathrm{P}}^3 \left(\sum_{v\in g} |\mu_v| \sqrt{\frac{1}{2}|k_{e^+(v)} + k_{e^-(v)}|} \right) T_{g,k,\mu}$$

we have eigenvalues

$$\begin{aligned} \left(\int \frac{\widehat{\mathrm{sgn}(E^x)}E^\varphi}{\sqrt{|E^x|}} \right) T_{g,k,\mu} =& 2\gamma^{1/2}\ell_{\mathrm{P}} \Bigg[\sum_{v\in g} |\mu_v| \big(\sqrt{|k_{e^+(v)} + k_{e^-(v)} + 1|} \\ & - \sqrt{|k_{e^+(v)} + k_{e^-(v)} - 1|} \big) \Bigg] T_{g,k,\mu} \end{aligned}$$

From this we can read the eigenvalues of $\widehat{\frac{1}{\sqrt{E^x}}}$ and parameterize their deviation from the classical expression by a correction function α.

We may also heuristically take into account the additional quantum degrees of freedom in the symmetry orbits coming from the full theory, as it is done in the framework of 'lattice refinements' [47, 57]. The discrete spectra of geometrical operators shows that the spatial geometry is made up from basic constituents determined by the number of vertices and spin labels of a spin network state. These 'elementary plaquettes' can form a macroscopic geometry in many different ways and dynamically one expects them to change in size as well as in number. A possible form for the correction function incorporating these effects is [21]:

$$\alpha(\Delta) = 2\frac{\sqrt{|\Delta + \gamma\ell_{\mathrm{P}}^2/2|} - \sqrt{|\Delta - \gamma\ell_{\mathrm{P}}^2/2|}}{\gamma\ell_{\mathrm{P}}^2}\sqrt{|\Delta|} \qquad (2.42)$$

where Δ is the size of an elementary plaquette in a discrete state underlying a classical geometry. For corrections in inverse powers of E^x, which is proportional to the area of a spherical orbit, the relevant operators give rise to a dependence on plaquette sizes on orbits. For a nearly spherical distribution of $\mathcal{N}(E^x)$ such plaquettes making up the whole orbit, we thus have $\Delta = E^x/\mathcal{N}$. Since we refer only to the orbit size, corrections thus naturally depend on E^x only but not on E^φ. (That this is required will later be shown independently when we use anomaly freedom to rule out that α could depend on E^φ.) Such a correction function then multiplies any classical appearance of $(E^x)^{-1}$ in a Hamiltonian operator. In particular, classical divergences of inverse factors of E^x are cut off as one can see from the plot in Fig. 4.1. Correspondingly, the dynamics given by such a Hamiltonian will change from quantum corrections. Classically, i.e. for $\ell_{\mathrm{P}} \to 0$, we have $\alpha(E^x) = 1$, and this limit is approached for large E^x. Also this behavior of the correction function is illustrated in Fig. 4.1.

For a large number $\mathcal{N}$ of discrete blocks, the scale Δ is reduced compared to E^x and corrections from α can be significant even for large E^x. Since typically $\mathcal{N}$, in relation to the underlying state, is not a constant but would depend on the size E^x, different kinds of behaviors can arise. This phenomenon of lattice refinement is important to capture the full dynamics of quantum gravity and its elementary degrees of freedom [47, 57]. It is also crucial for realizing the correct scaling behavior of correction terms under changes of coordinates [58]. In this work, we will mainly be looking at general implications in local equations of motion, where the value or behavior of $\mathcal{N}$ is not important. More detailed investigations could at some point provide restrictions on the possible form of $\mathcal{N}(E^x)$, and thus give insights in the required behavior of discrete quantum gravity states.

2.5 Related Approaches

Before closing this chapter we briefly mention here other canonical approaches to the quantization of spherically symmetric spacetimes. The classical theory was first studied in [59] and amended by Unruh in [60]. Kuchař [61] presented a reduced phase space quantization of the extended Schwarzschild solution, where the mass and time at infinity are the conjugate variables. Kastrup and Thiemann also gave a reduced phase quantization starting from (dual) Ashtekar variables in [62] and a Dirac quantization using BRST methods (but not loop methods) in [36].

More recently, coupling the spherically symmetric system to dust and extending the methods of [61], a Dirac quantization for the LTB solutions was constructed in [33, 34]. In this quantization the mass, radius, dust proper time, and their conjugate momenta are chosen as a canonical coordinate chart and the diffeomorphism constraint is used to eliminate one of the momenta leading to a simplified Wheeler-DeWitt equation.

Hussain and Winkler [63, 64] gave a quantization of gravity coupled to a scalar field using loop inspired techniques. This 'polymer' quantization uses Painlevé-Gullstrand coordinates and exponentials of the (conjugate momenta) geometrodynamical variables as a starting point.

The gravity-scalar field system has also been recently studied using loop techniques by Gambini, Pullin and Rastgoo [65]. Gambini and Pullin have also constructed in [66] a loop quantization of the extended Schwarzschild spacetime using partial gauge fixing and discretization as in [67].

Chapter 3

Dilaton Gravity Poisson Sigma Models and Loop Quantum Gravity

In this chapter we will explore the connection of spherically symmetric models in loop variables with general two-dimensional dilaton gravity and Poisson sigma models[1]. We first review in Section 3.1 the general class of two dimensional models considered here and derive an explicit canonical transformation to relate them in Sec. 3.2. After discussing applications in Sec. 3.4, we exhibit consistent deformations for one type of inverse triad corrections in the loop formulation and use our canonical transformation to translate this back to the PSM framework in Sec. 3.5, where we relate quantum corrections to changes in the underlying Poisson structure of the PSM formulation. The analysis will be done in the presence of a Yang–Mills source which can, as detailed in Sec. 3.3, be added as an extension of the original PSM.

3.1 Formulations of gravity in two dimensions

In two dimensions the analog of the Einstein–Hilbert action in vacuum is trivial, but the presence of extra fields gives rise to interesting models. Such fields may, for instance, arise after dimensional reduction of a field theory in four or higher dimensions. In the case of 2d gravity this leads to the presence of the dilaton field.

[1]This chapter is based on the work by the author in [68]

Most of the studied gravity theories in two dimensions can be described by the so-called Generalized Dilaton Gravity action

$$S[\mathbf{g},\Phi] = \frac{1}{2}\int_M d^2x\sqrt{-\mathbf{g}}\left(\mathbf{F}(\Phi)R(\mathbf{g}) + \mathbf{U}(\Phi)\mathbf{g}^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi + \mathbf{V}(\Phi)\right) \tag{3.1}$$

which is the most general diffeomorphism invariant action giving second order differential equations for a metric $\mathbf{g}$ and a scalar dilaton field Φ on a two-dimensional manifold M; for a comprehensive review see [23]. Here, $\mathbf{F}$, $\mathbf{U}$ and $\mathbf{V}$ are three functions parameterizing different models (which should be sufficiently well behaved, and $\mathbf{F}$ being invertible).

3.1.1 Poisson Sigma Models

Poisson sigma models are a more general and unifying structure encompassing and generalising 2d gravity theories as well as 2d Yang-Mills theories [69, 70, 71, 22]. These are topological two-dimensional field theories which encode all the content of a particular model in a single Poisson tensor $\mathcal{P}$ defined on an n-dimensional target manifold N (in local coordinates $\mathbf{X}^i$, $\mathcal{P} = \frac{1}{2}\mathcal{P}^{ij}(\mathbf{X})\frac{\partial}{\partial \mathbf{X}^i}\wedge\frac{\partial}{\partial \mathbf{X}^j}$).

For a given Poisson tensor, the corresponding PSM action has the form

$$S_{\text{PSM}} = \int_M \mathbf{A}_i \wedge d\mathbf{X}^i + \frac{1}{2}\mathcal{P}^{ij}\mathbf{A}_i\wedge\mathbf{A}_j \tag{3.2}$$

or, written explicitly with coordinates x^μ on M,

$$S_{\text{PSM}} = \int_M dx^\mu \wedge dx^\nu \left[\mathbf{A}_{i\mu}(x)\frac{\partial \mathbf{X}^i}{\partial x^\nu}(x) + \frac{1}{2}\mathcal{P}^{ij}(\mathbf{X}(x))\mathbf{A}_{i\mu}(x)\mathbf{A}_{j\nu}(x)\right].$$

The dynamical fields are $\mathbf{X}^i(x^\mu)$, which parameterize a map $\mathcal{X}\colon M \to N$ from the two-dimensional spacetime manifold M to the target Poisson manifold N, as well as $\mathbf{A}_i = \mathbf{A}_{i\mu}dx^\mu$, a one-form on M taking values in $\mathcal{X}^*(T^*N)$.

More abstractly, the pair $(\mathbf{X}^i, \mathbf{A}_i)$ defines a vector bundle morphism $TM \to T^*N$ with base map $\mathcal{X}$, so that the action may be viewed as a functional of vector bundle morphisms. The equations of motion of the PSM may be shown to require these morphisms to preserve the standard Lie algebroid structures[2] on TM and

[2] A Lie algrebroid is essentially a fiber bundle with a Lie bracket defined on its sections as well

T^*N: solutions to PSMs are Lie algebroid morphisms, and gauge symmetries are related to homotopies of morphisms [72].

The key step in establishing 2d gravity as a PSM is the reformulation of (3.1) in first order form by using Einstein-Cartan variables: dyads and the spin connection instead of the metric. First, using the field redefinition $\phi = \mathbf{F}(\Phi)$ and replacing $\mathbf{U}$ by $\mathbf{U}/\mathbf{F}'^{\,2}$, the coefficient $\mathbf{F}$ for the curvature term may always be assumed to be the identity function. The kinetic term can be eliminated by means of a conformal transformation $g := \Omega^2(\phi)\mathbf{g}$, with

$$\Omega(\phi) = \exp\left(\int^{\phi} \frac{\mathbf{U}(z)}{2} dz + \text{const}\right)$$

and $V = -\mathbf{V}(\phi)/\Omega^2(\phi)$. The action

$$S = \frac{1}{2}\int_M d^2x\sqrt{-g}(\phi R - V(\phi))$$

can then be expressed in first order form using dyads $e^a_\mu dx^\mu$ and connection one-forms $\omega_\mu{}^a{}_b dx^\mu$:

$$S = -\int_M \phi d\omega + \frac{1}{2}V(\phi)\varepsilon + X_a De^a\,. \tag{3.3}$$

Here, we have used the two-dimensional identity $R\varepsilon = -2d\omega$ where ε is the two-dimensional volume form and ω is defined by $\omega^a_b = \omega\varepsilon^a_b$, with ε^a_b being the single generator of the Lorentz gauge algebra $\mathfrak{so}(1,1)$. Lagrange multipliers X^a are introduced to enforce the condition of torsion freedom; see App. B.2 for theories with torsion.

3.1.1.1 Equations of motion

Variation of (3.3) with respect to ϕ, ω, X_a and e^a respectively gives the equations of motion

$$d\omega + \frac{1}{2}V'(\phi)\varepsilon = 0 \tag{3.4}$$

$$d\phi + X_a\varepsilon^a{}_b e^b = 0 \tag{3.5}$$

as an anchor map from the bundle to the tangent bundle over the same base manifold. A Lie algebra is a Lie algebroid whose base manifold is a single point.

$$De^a = de^a + \varepsilon^a{}_b \omega \wedge e^b = 0 \tag{3.6}$$

$$\frac{1}{2} V(\phi) \varepsilon_{ab} e^b + dX_a + \varepsilon_{ab} X^b \omega = 0\,. \tag{3.7}$$

It is convenient to introduce a light cone basis $e^{\pm}$ for dyads, so that $g_{\mu\nu} = 2e^+_{(\mu} e^-_{\nu)}$ and raising and lowering indices is accomplished by replacing a lower $+$ $(-)$ by an upper $-$ $(+)$ and vice versa. Solving the condition (3.6) for torsion freedom with coordinates (t, x) on M and a light cone basis, we get the spin connection in terms of the dyad:

$$\omega_x = \frac{(e^-_x e^+_x)^{\cdot} - (e^-_t e^+_x + e^+_t e^-_x)' + e^-_t e^+_x{}' + e^+_t e^-_x{}'}{e^+_t e^-_x - e^+_x e^-_t} \tag{3.8}$$

$$\omega_t = \frac{\dot{e}^-_x e^+_t + \dot{e}^+_x e^-_t - (e^-_t e^+_t)'}{e^+_t e^-_x - e^+_x e^-_t} \tag{3.9}$$

From Eq. (3.5) we obtain

$$X^{\mp} = \frac{\phi' e^{\mp}_t - \dot{\phi} e^{\mp}_x}{e^+_t e^-_x - e^+_x e^-_t}\,. \tag{3.10}$$

These equations will be useful below.

3.1.1.2 Spherically Symmetric Gravity as a Poisson sigma model

To relate dilaton gravity and PSMs to spherically reduced gravity in four dimensions, we start from the Einstein-Hilbert action

$$S_{\text{EH}} = \frac{1}{16\pi G} \int_{M \times S^{D-2}} d^D x \sqrt{-{}^D g}\; {}^D R$$

in D dimensions and insert the ansatz

$$ds^2 = \mathbf{g}_{\mu\nu}(x^\mu) dx^\mu dx^\nu + \Phi^2(x^\mu) d\Omega^2_{S^{D-2}} \tag{3.11}$$

for a spherically symmetric metric with $\mathbf{g}_{\mu\nu}$ a metric of signature $(-, +)$ on the two-dimensional spacetime M, and $d\Omega^2_{S^{D-2}}$ the area element of the $(D-2)$-sphere S^{D-2} (for the 2-sphere $d\Omega^2 = d\vartheta^2 + \sin^2\vartheta d\varphi^2$). After integration of the angular variables (see e.g. App. B of [73] or App. C of [74]), the reduced 2d dilaton action is

$$S = \frac{\mathcal{O}_{D-2}}{16\pi G}\int_M d^2x\sqrt{-\mathbf{g}}\Big[\Phi^{D-2}\mathbf{R}(\mathbf{g}) + (D-2)(D-3)\Phi^{D-4}\mathbf{g}^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi + (D-2)(D-3)\Phi^{D-4}\Big]\,. \tag{3.12}$$

with $\mathcal{O}_{D-2}$ the volume of S^{D-2}.

Defining

$$\phi := \Phi^{D-2} \quad , \quad g_{\mu\nu} := \phi^{\frac{D-3}{D-2}}\mathbf{g}_{\mu\nu} \quad , \quad V(\phi) := -(D-2)(D-3)\phi^{-1/D-2} \tag{3.13}$$

and introducing Lagrange multipliers X_a as before to implement torsion-freedom, this action is seen to be of the form (3.3):

$$S = -\frac{\mathcal{O}_{D-2}}{8\pi G}\int_M X_a de^a + X_a\varepsilon^a{}_b\omega\wedge e^b + \phi d\omega + \frac{1}{2}V(\phi)\varepsilon\,.$$

Further, integrating by parts and discarding boundary terms we have

$$S = -\frac{\mathcal{O}_{D-2}}{8\pi G}\int_M e^a\wedge dX_a + \omega\wedge d\phi + X_a\varepsilon^a{}_b\omega\wedge e^b + \frac{1}{2}V(\phi)\varepsilon\,.$$

If we collect the zero- and one-forms appearing in the last equation into the multiplets

$$(\mathbf{X}^i) := (X^a,\phi) = (X^-, X^+, \phi)$$
$$(\mathbf{A}_i) := (e_a,\omega) = (e^+, e^-, \omega)$$

and use the Poisson bivector

$$\mathcal{P}^{ij} = \begin{pmatrix} 0 & -V/2 & -X^- \\ V/2 & 0 & X^+ \\ X^- & -X^+ & 0 \end{pmatrix},$$

the action finally takes the form of a Poisson sigma model:

$$S = -\frac{\mathcal{O}_{D-2}}{8\pi G}\int_M \mathbf{A}_i\wedge d\mathbf{X}^i + \frac{1}{2}\mathcal{P}^{ij}\mathbf{A}_i\wedge\mathbf{A}_j \tag{3.14}$$

with a three-dimensional target space, and $i \in \{-,+,3\}$.

From here on we specialize to four dimensions for which

$$\phi := \Phi^2 \quad , \quad g_{\mu\nu} := \sqrt{\phi}\mathbf{g}_{\mu\nu} \quad , \quad V(\phi) := -\frac{2}{\sqrt{\phi}} \tag{3.15}$$

3.1.1.3 Canonical Form

The action for a Poisson sigma model is already in first order form. Using coordinates (t,x) on M, (3.14), specialized to four dimensions, reads:

$$S = \frac{1}{2G}\int dt \int dx[A_i\dot{\mathbf{X}}^i - \Lambda_i(\mathbf{X}^{i\,\prime} + \mathcal{P}^{ij}A_j)] \tag{3.16}$$

with

$$A_i := \mathbf{A}_{xi} \quad , \quad \Lambda_i := \mathbf{A}_{ti} \quad , \quad \dot{\mathbf{X}} := \partial_t\mathbf{X} \quad , \quad \mathbf{X}' := \partial_x\mathbf{X}\,. \tag{3.17}$$

The canonically conjugate variables are thus $\mathbf{X}^i$ and A_i with

$$\{\mathbf{X}^i(x), A_i(y)\} = 2G\delta^i_j\delta(x-y)\,,$$

subject to the total constraint

$$\int dx\, \Lambda_i\tilde{C}^i \approx 0$$

with Lagrange multipliers Λ_i and

$$\tilde{C}^i := \frac{1}{2G}(\mathbf{X}^{i\,\prime} + \mathcal{P}^{ij}A_j)\,.$$

These constraints form the first class algebra

$$\{C[\Lambda_i], C[K_j]\} = -\frac{1}{2G}C[\Lambda_iK_l\partial_k\mathcal{P}^{il}]\,. \tag{3.18}$$

For a linear Poisson tensor, this is an algebra with structure constants, equivalent to the Gauss constraint of a gauge theory with structure constants $\partial_k\mathcal{P}^{ij}$. For non-linear Poisson tensors, on the other hand, the system has structure functions. The Lie algebroid formulation of Poisson sigma models, alluded to above, provides an interesting perspective on systems with structure functions whose constraints generate the symmetries of a Lie algebroid rather than a local Lie algebra. Similar interpretations exist for a large class of algebroid Yang-Mills theories [75] or Dirac sigma models [76].

The action (3.3) is space-time diffeomorphism and SO(1,1)-gauge invariant. In particular, $\tilde{C}^3$ is the canonical generator of local gauge transformations, and the spatial diffeomorphism constraint is the combination

$$\tilde{D} := A_i \tilde{C}^i = A_i \mathbf{X}^{i\,\prime}\,.$$

For the relation to variables underlying the loop formulation, it will be convenient to introduce the SO(1,1) invariant quantities

$$X^2 := X^- X^+ \quad , \quad e^2 := e_x^- e_x^+ \tag{3.19}$$

as well as gauge angles Υ and β defined by

$$X^\pm = X \exp(\pm\beta) \quad , \quad e_x^\pm = e \exp(\pm\Upsilon)\,.$$

The angles are well-defined as long as $X \neq 0$ and $e \neq 0$, which is the case except at horizons. In what follows, we analyze the local constraint algebra such that global problems of this transformation of variables do not play a role. (As expected for an Abelian gauge transformation, $C^3[\Lambda]$ then generates $\Upsilon \to \Upsilon - \Lambda$, $\beta \to \beta - \Lambda$, $\omega_x \to \omega_x + \Lambda'$. In the regions where our change of variables is valid one can even Abelianize the full constraint system; see [22].)

The symplectic structure Ω in the new variables becomes

$$\begin{aligned}
\Omega &= \frac{1}{2G}\int dx\,\delta A_i \wedge \delta \mathbf{X}^i \\
&= \frac{1}{2G}\int dx\,\delta(e\exp\Upsilon)\wedge\delta(X\exp-\beta) + \delta(e\exp-\Upsilon)\wedge\delta(X\exp\beta) + \delta\omega_x\wedge\delta\phi \\
&= \frac{1}{2G}\int dx\,\delta e\wedge\delta(2X\cosh(\Upsilon-\beta)) + \delta\Upsilon\wedge\delta(2Xe\sinh(\Upsilon-\beta)) + \delta\omega_x\wedge\delta\phi \\
&= \frac{1}{2G}\int dx\,\delta e\wedge\delta Q^e + \delta\Upsilon\wedge\delta Q^\Upsilon + \delta\omega_x\wedge\delta\phi
\end{aligned}$$

with

$$Q^e := 2X\cosh(\Upsilon - \beta) \quad , \quad Q^\Upsilon := 2Xe\sinh(\Upsilon - \beta)\,, \tag{3.20}$$

which provides the new canonically conjugate pairs

$$\{Q^e(x), e(y)\} = \{Q^\Upsilon(x), \Upsilon(y)\} = \{\phi(x), \omega_x(y)\} = 2G\delta(x,y)\,. \tag{3.21}$$

We can invert this transformation to find

$$X^\mp = \frac{eQ^e \pm Q^\Upsilon}{2e}\exp(\mp\Upsilon) \tag{3.22}$$

and insert it into the PSM constraints:

$$\tilde{C}^\mp = \frac{1}{2G}\left[\left(\frac{eQ^e \pm Q^\Upsilon}{2e}\right)' \mp \left(\frac{eQ^e \pm Q^\Upsilon}{2e}\right)(\omega_x + \Upsilon') \mp \frac{1}{2}V(\phi)e\right]\exp(\mp\Upsilon) \tag{3.23}$$

$$\tilde{C}^3 = \frac{1}{2G}(\phi' + Q^\Upsilon)\,. \tag{3.24}$$

By the same combination as before, the diffeomorphism constraint is

$$\tilde{D} = \frac{1}{2G}(eQ^{e\,\prime} - Q^\Upsilon\Upsilon' + \omega_x\phi')\,. \tag{3.25}$$

3.2 Relating the two models

The actions (3.16) and (2.13) represent equivalent canonical formulations of spherically reduced general relativity, so there must exist a canonical transformation between the PSM and Ashtekar variables. To find such a transformation we first compare the form of the reduced 2-dimensional metric in terms of these two sets of variables. This relates the dilaton field ϕ and the gauge invariant part e of the dyad directly to the densitized triad components E^x and E^φ. Using this and imposing the canonical relations (3.21) gives a system of differential equations for the remaining PSM variables. We then use equations of motion (3.8) to fix some of the ambiguities and check for consistency.

3.2.1 Comparison of metrics

The general canonical line element $ds^2 = -N^2dt^2 + q_{ab}(dx^a + N^a dt)(dx^b + N^b dt)$ adapted to spherical symmetry with coordinates $(t, x, \vartheta, \varphi)$, lapse function $N(t,x)$ and shift vector $N^x(t,x)$ is

$$ds^2 = -N(t,x)^2dt^2 + L^2(t,x)(dx + N^x(t,x)dt)^2 + R^2(t,x)d\Omega^2 \tag{3.26}$$

where L^2 and R^2 are components of the spatial metric $dq^2 = L^2(t,x)dx^2 + R^2(t,x)d\Omega^2$. In terms of the densitized triad, we have

$$L^2 = \frac{E^{\varphi\,2}}{|E^x|} \quad , \quad R^2 = |E^x|\,.$$

Comparing this form of the metric with (3.11) directly gives

$$q_{\varphi\varphi} = \Phi^2 \quad , \quad \mathbf{g}_{\mu\nu} = \begin{pmatrix} -N^2 + L^2(N^x)^2 & L^2N^x \\ L^2N^x & L^2 \end{pmatrix}\,. \tag{3.27}$$

This relates the densitized spherically symmetric triad variables to the dyads of the conformally transformed metric $g_{\mu\nu} = \sqrt{\phi}\mathbf{g}_{\mu\nu}$ and the dilaton field ϕ:

$$\begin{aligned} \phi = \Phi^2 &= |E^x| \\ g = 2e^+e^- &= \sqrt{\phi}\mathbf{g} \\ \begin{pmatrix} 2e_t^+e_t^- & e_t^+e_x^- + e_x^+e_t^- \\ e_t^+e_x^- + e_x^+e_t^- & 2e_x^+e_x^- \end{pmatrix} &= |E^x|^{\frac{1}{2}} \begin{pmatrix} -N^2 + \frac{E^{\varphi\,2}}{|E^x|}N^{x\,2} & \frac{E^{\varphi\,2}}{|E^x|}N^x \\ \frac{E^{\varphi\,2}}{|E^x|}N^x & \frac{E^{\varphi\,2}}{|E^x|} \end{pmatrix}\,. \end{aligned}$$

From this we obtain

$$e^2 = e_x^+e_x^- = \frac{E^{\varphi\,2}}{2|E^x|^{\frac{1}{2}}}$$

and

$$\begin{aligned} e_x^+ &= p\frac{E^\varphi}{\sqrt{2}|E^x|^{\frac{1}{4}}}\exp\Upsilon\,, \quad e_x^- = p\frac{E^\varphi}{\sqrt{2}|E^x|^{\frac{1}{4}}}\exp(-\Upsilon) \\ e_t^+ &= p\frac{N^xE^\varphi \pm N|E^x|^{\frac{1}{2}}}{\sqrt{2}|E^x|^{\frac{1}{4}}}\exp\Upsilon\,, \quad e_t^- = p\frac{1}{\sqrt{2}|E^x|^{\frac{1}{4}}}\left(\frac{-N^2|E^x| + N^{x\,2}E^{\varphi\,2}}{N^xE^\varphi \pm N|E^x|^{\frac{1}{2}}}\right)\exp(-\Upsilon) \end{aligned} \tag{3.28}$$

with $p=\pm 1$ distinguishing different solutions.

Using equations (3.8) and (3.10) and the equations of motion

$$\dot{E}^x = 2sNK_\varphi|E^x|^{\frac{1}{2}} + N^x E^{x\,\prime}$$
$$\dot{E}^\varphi = N(K_\varphi E^\varphi + K_x E^x)|E^x|^{-\frac{1}{2}} + (N^x E^\varphi)' \tag{3.29}$$

for the spherically symmetric loop variables where s is the sign of E^x, we get the dependence of the spin connection ω_x and Lagrange multipliers $X^\pm$ in terms of $(E^x, E^\varphi, K_x, K_\varphi)$:

$$\omega_x = \pm s K_x \pm \frac{E^\varphi}{2|E^x|}K_\varphi - \Upsilon' \tag{3.30}$$

and

$$X^- = p\sqrt{2}|E^x|^{\frac{1}{4}}\left(-s\frac{E^{x\,\prime}}{2E^\varphi} \mp K_\varphi\right)\exp(-\Upsilon) \tag{3.31}$$
$$X^+ = p\sqrt{2}|E^x|^{\frac{1}{4}}\left(s\frac{E^{x\,\prime}}{2E^\varphi} \mp K_\varphi\right)\exp\Upsilon\,. \tag{3.32}$$

3.2.2 Canonical transformation

We now look for a canonical transformation between the two sets of variables

$$(Q^e, Q^\Upsilon, \phi; e, \Upsilon, \omega_x) \quad \leftrightarrows \quad (E^x, E^\varphi, P^\eta; A_x, K_\varphi, \eta)\,.$$

The Poisson bracket relations (3.21) give a system of partial differential equations for the functional dependence of Q^e, Q^Υ, Υ, and ω_x on the spherically symmetric loop variables. (There are 15 nontrivial relations that must be simultaneously satisfied to ensure consistency: $\{Q^e, e\} = 2G, \{Q^e,\phi\} = \{Q^\Upsilon, e\} = \{Q^\Upsilon,\phi\} = 0, \{\phi,\omega_x\} = 2G, \{e,\omega_x\} = \{\phi,\Upsilon\} = \{e,\Upsilon\} = 0, \{Q^\Upsilon,\Upsilon\} = 2G, \{Q^e,\Upsilon\} = \{Q^e, Q^\Upsilon\} = \{Q^e,\omega_x\} = \{Q^\Upsilon,\omega_x\} = \{\Upsilon,\omega_x\} = 0$. The remaining $\{\phi, e\} = 0$ is automatically satisfied given the functional dependence of ϕ and e on E^x and E^φ.)

These equations are solved in App. B.1, providing the canonical transformation

$$Q^e = p2\sqrt{2}|E^x|^{\frac{1}{4}}K_\varphi + h[|E^x|^{-\frac{1}{4}}E^\varphi] \quad , \quad e = p\frac{E^\varphi}{\sqrt{2}|E^x|^{\frac{1}{4}}}$$
$$\phi = |E^x| \quad , \quad \omega_x = -sK_x - \frac{E^\varphi}{2|E^x|}K_\varphi + \frac{1}{k}\eta'$$

$$Q^{\Upsilon} = \frac{k}{\gamma}P^{\eta} + \left(\frac{k - s\gamma}{\gamma}\right) E^{x\,\prime} \quad , \quad \Upsilon = -\frac{1}{k}\eta \tag{3.33}$$

with inverse transformation

$$\begin{aligned} E^x &= s\phi \quad &,& \quad E^{\varphi} = p\sqrt{2}\phi^{\frac{1}{4}}e \\ K_{\varphi} &= p\frac{(Q^e - h)}{2\sqrt{2}\phi^{\frac{1}{4}}} \quad &,& \quad K_x = -s(\omega_x + \Upsilon' + \frac{e}{4\phi}(Q^e - h)) \\ \eta &= -k\Upsilon \quad &,& \quad P^{\eta} = \frac{\gamma}{k}Q^{\Upsilon} + \left(\frac{\gamma - sk}{k}\right)\phi'\,. \end{aligned} \tag{3.34}$$

Here again, $s = \mathrm{sign}(E^x)$, k is an arbitrary constant, and h an arbitrary function of one variable.

3.2.3 Constraints

We take $h = 0$ to provide a specific canonical transformation. As mentioned in App. B.1, with this solution the C^3 constraint (3.24) reproduces the Gauss constraint (2.14):[3]

$$C^3[\lambda] = \frac{1}{2G}\int dx\,\lambda(\phi' + Q^{\Upsilon}) = \frac{k}{2G\gamma}\int dx\,\lambda(E^{x\,\prime} + P^{\eta}) = k\,G_{\mathrm{grav}}[\lambda]\,. \tag{3.35}$$

The diffeomorphism constraint (3.25) reads:

$$\begin{aligned} D[N^x] &= \frac{1}{2G}\int dx\,N^x(eQ^{e\,\prime} - Q^{\Upsilon}\Upsilon' + \omega_x\phi') \\ &= \frac{1}{2G}\int dx\,N^x(2E^{\varphi}K_{\varphi}' - K_xE^{x\,\prime} + \gamma^{-1}\eta'(P^{\eta} + E^{x\,\prime})) = D_{\mathrm{grav}}[N^x]\,. \end{aligned} \tag{3.36}$$

Using (2.10), the remaining independent linear combination becomes

$$C^+[\frac{\sqrt{2}}{2}N\phi^{1/4}\exp(-\Upsilon)] - C^-[\frac{\sqrt{2}}{2}N\phi^{1/4}\exp(\Upsilon)] =$$

[3]There is a local agreement of the infinitesimal Gauss symmetries generated by the constraints. Globally, however, the formulations differ, one having a compact group U(1), the other the noncompact SO(1,1). In fact, different time gauges have been used to reduce space-time metrics to objects in canonical form, which turns out to imply different topological properties of the gauge orbits.

$$= \frac{\sqrt{2}}{4G}\int dx\, N\phi^{1/4}[Q^e(\omega_x+\Upsilon') - \left(\frac{Q^\Upsilon}{e}\right)' + Ve]$$
$$= \frac{p}{2G}\int dx\, N\Bigg[-|E^x|^{-\frac{1}{2}}K_\varphi^2E^\varphi - 2s|E^x|^{\frac{1}{2}}K_xK_\varphi + \frac{E^\varphi V}{2} + \frac{|E^x|^{-\frac{1}{2}}E^{x\,\prime 2}}{4E^\varphi} - \frac{s|E^x|^{\frac{1}{2}}E^{x\,\prime}E^{\varphi\,\prime}}{E^{\varphi\,2}} + \frac{s|E^x|^{\frac{1}{2}}E^{x\,\prime\prime}}{E^\varphi}\Bigg]$$
$$- \frac{pk}{2G\gamma}\int dx\, N|E^x|^{1/4}\left[\frac{|E^x|^{\frac{1}{4}}}{E^\varphi}(E^{x\,\prime}+P^\eta)\right]' \qquad (3.37)$$

and reproduces the Hamiltonian constraint (2.16) with $V(\phi) = -\frac{2}{\sqrt{\phi}} = -2|E^x|^{-\frac{1}{2}}$ (up to the Gauss constraint):

$$C^+[\frac{\sqrt{2}}{2}N\phi^{1/4}\exp(-\Upsilon)] - C^-[\frac{\sqrt{2}}{2}N\phi^{1/4}\exp(\Upsilon)] =$$
$$= pH_{\rm grav}[N] - \frac{pk}{2G\gamma}\int dx\, N|E^x|^{1/4}\left[\frac{|E^x|^{\frac{1}{4}}}{E^\varphi}(E^{x\,\prime}+P^\eta)\right]'.$$

To summarize,

$$\tilde{C}^3 = k\tilde{\mathcal{G}}_{\rm grav} \qquad (3.38)$$
$$e\,\exp(\Upsilon)\tilde{C}^- + e\,\exp(-\Upsilon)\tilde{C}^+ + \omega_x\tilde{C}^3 = \tilde{\mathcal{D}}_{\rm grav} \qquad (3.39)$$
$$-\exp(\Upsilon)\tilde{C}^- + \exp(-\Upsilon)\tilde{C}^+ + \left(\frac{\tilde{C}^3}{e}\right)' = \sqrt{2}p\phi^{-\frac{1}{4}}\tilde{\mathcal{H}}_{\rm grav} \qquad (3.40)$$

verifying once more that (3.16) and (2.13) represent equivalent constrained systems.

3.3 Inclusion of Yang-Mills fields

Before discussing our main topic, the role of quantum corrections, we extend the formalism to include 2-dimensional Yang–Mills fields [22]. This will provide a non-trivial model in the presence of quantum corrections. The general 2-dimensional Yang–Mills action with an arbitrary coupling ζ, allowed to depend on the dilaton field ϕ, reads

$$S_{\rm YM} = -\frac{1}{4}\int_M \zeta\,{\rm tr}(\mathcal{F}\wedge *\mathcal{F})$$

where $\mathcal{F}^I = d\mathcal{A}^I + \frac{1}{2}c^I{}_{JK}\mathcal{A}^J \wedge \mathcal{A}^K$ is the usual curvature 2-form of the connection $\mathcal{A}^I T_I$, with T_I a basis for the Lie algebra $\mathfrak{g}$ with structure constants $c^I{}_{JK}$ of the chosen n-dimensional internal gauge group. (It should be stressed that from a physical point of view this is a toy model; spherically reduced Yang–Mills theory contains extra fields and does not coincide with the purely 2d model.)

In first order form this action is

$$S_{\rm YM} = -\int_M \mathcal{E}^I \mathcal{F}_I + 2\zeta(\phi)\mathcal{E}^I\mathcal{E}_I\varepsilon\,. \tag{3.41}$$

(Assuming $\mathrm{tr}(T_I T_J) = \frac{1}{2}\delta_{IJ}$, this equivalence is seen most easily by inserting the field equation $\mathcal{E}^I = \frac{1}{4\zeta} * \mathcal{F}^I$ into the original Yang-Mills action.) Then $S_{\rm grav} + S_{\rm YM}$ reads

$$S_{\rm gravYM} = -\frac{1}{2G}\int_M X^a De_a + \phi d\omega + 2G\mathcal{E}^I\mathcal{F}_I + \left(\frac{1}{2}V(\phi) + 4G\zeta(\phi)\mathcal{E}^I\mathcal{E}_I\right)\varepsilon \tag{3.42}$$

where indices a run over $+$ and $-$, and $I = 1, \ldots, n$. The coupling of Yang–Mills theory to gravity thus changes the dilaton potential in a way which depends on the value of the dilaton through the coupling function ζ. Moreover, the target manifold of the PSM has a higher dimension due to the degrees of freedom of the Yang–Mills field: After integrating by parts and dropping the corresponding surface terms, and with the identifications

$$(\mathbf{X}^i) := (X^a, \phi, \mathcal{E}^I) \quad , \quad (\mathbf{A}_i) := (e_a, \omega, \mathcal{A}_I)\,,$$

the previous action turns out to be of Poisson sigma form (3.2) on an $(n+3)$-dimensional target space N with Poisson brackets

$$\begin{aligned} \{X^+, X^-\} &= V/2 + 4G\zeta\mathcal{E}^I\mathcal{E}_I \quad , \quad && \{X^\pm, \phi\} = \pm X^\pm \\ \{\mathcal{E}^I, \mathcal{E}^J\} &= c^{IJ}{}_K\mathcal{E}^K \quad , \quad && \{X^\pm, \mathcal{E}^I\} = \{\phi, \mathcal{E}^I\} = 0\,. \end{aligned}$$

The Poisson bivector can thus be decomposed as

$$(\mathcal{P}^{ij}) = \begin{bmatrix} \mathbf{P} & 0 \\ 0 & \mathbf{Q} \end{bmatrix}$$

with the 3×3 matrix

$$\mathbf{P} = \begin{pmatrix} 0 & -V/2 - 4G\zeta\mathcal{E}^I\mathcal{E}_I & -X^- \\ V/2 + 4G\zeta\mathcal{E}^I\mathcal{E}_I & 0 & X^+ \\ X^- & -X^+ & 0 \end{pmatrix} \tag{3.43}$$

and the $n \times n$ matrix

$$\mathbf{Q} = (Q^{IJ}) = (c^{IJ}{}_K \mathcal{E}^K)\,.$$

The canonical formulation of Sec. 3.1.1.3 proceeds almost unchanged: The symplectic structure (2.3) is extended by the Yang-Mills pairs

$$\{\mathcal{E}^I(x), \mathcal{A}_I(y)\} = \delta(x,y)\,.$$

The constraints $C^{\mp}$ (3.23) (and (B.12)) receive additional terms $\mp 2\zeta\mathcal{E}^I\mathcal{E}_I e \exp(\mp\Upsilon)$:

$$\tilde{C}^{\mp} = \frac{1}{2G}\left[\left(\frac{eQ^e \pm Q^{\Upsilon}}{2e}\right)' \mp \left(\frac{eQ^e \pm Q^{\Upsilon}}{2e}\right)(\omega_x + \Upsilon') \mp \frac{1}{2}V(\phi)e \mp 4G\zeta\mathcal{E}^I\mathcal{E}_I e\right]\exp(\mp\Upsilon) \tag{3.44}$$

and we have the usual n-component Gauss constraints

$$\tilde{C}^{3+I} = \mathcal{E}^{I\,\prime} + c^{IJ}_K \mathcal{A}_J \mathcal{E}^K$$

for the Yang–Mills part of the theory.

3.4 Loop quantization of general 2-dimensional dilaton gravity

We can now obtain the first application of our canonical relation between dilaton gravity and spherically symmetric gravity in Ashtekar variables: a loop quantization of general 2-dimensional dilaton gravity models. So far in this context, loop quantizations have only been performed for one specific dilaton potential $V(\phi) \propto 1/\sqrt{\phi}$, corresponding to spherically symmetric gravity in four dimensions [38]. Looking at the Hamiltonian constraint in the form (3.37) shows that the potential appears only at one place, which is in fact a rather simple term in the

Hamiltonian (resulting from the term E^φ in the parenthesis in (2.16), which is the only term independent of K_φ or Γ_φ). This observation allows us immediately to extend the existing loop quantization of spherically symmetric gravity to an arbitrary potential $V(\phi)$, provided only that the expression $V(\phi)$ can be turned into a well-defined operator. Since this is merely a function of the triad component E^x, such quantizations easily exist.

3.4.1 Applications

Ashtekar variables only exist in three and four dimensions, such that an immediate loop quantization is possible only in those cases or in models which are obtained from them by symmetry reduction. With the formulation of general dilaton gravity models in spherically symmetric Ashtekar variables, however, we are now in a position to extend the loop quantization to arbitrary dilaton potentials: The Gauss and diffeomorphism constraint remain unaffected; in the Hamiltonian constraint, we simply insert the appropriate operator quantizing the given dilaton potential $V(\phi)$ in the spherically symmetric constraint operator; this will only change the coefficient $\hat{C}_0$, (2.30), in a straightforward way: Instead of $1-\Gamma_\varphi^2$ we then have $-V(E^x)\sqrt{|E^x|}-\Gamma_\varphi^2$, where E^x in a triad representation will simply be replaced by $\frac{1}{2}\gamma\ell_{\mathrm{P}}^2(k_+ + k_-)$ after quantization. All the other coefficients in the difference equation remain unchanged, and so does the conclusion about the absence of singularities. Thus, all dilaton gravity models are singularity-free in this loop quantization.

This allows several specific applications. First, we can choose a linear dilaton potential $V(\phi) \propto \phi$, which provides a loop quantization of BF-theory. As one can see, the quantization does not simplify considerably in this case because most terms of the Hamiltonian constraint remain unchanged compared to spherically symmetric gravity. This is quite unexpected given that BF-theory can be quantized rather easily in different formulations. However, transformations between Ashtekar-type variables and variables which allow simple quantizations are non-trivial. Their quantizations can thus differ considerably. The BF-case of PSMs, quantized in Ashtekar variables as obtained here, can provide an interesting model for a quantization of a simple classical theory, quantized using techniques and basic objects as they apply to full gravity. A direct loop quantization of 2d BF-theory is discussed in [77].

Secondly, we can provide loop quantizations of spherically symmetric models in arbitrary D space-time dimensions. Here, we insert quantizations of the corresponding potentials $V(\phi) \propto \phi^{d(D)}$ which $d(D) = -1/(D-2)$ from (3.13). Even though loop quantizations of general D-dimensional theories with $D > 4$ have not been performed, at least non-rotating black holes can be studied by these models and compared with results from alternative quantizations in higher dimensions.

3.5 Inverse triad quantization as a consistent deformation

We are now ready to introduce inverse triad corrections to the classical models considered so far. As sketched before one can quantize inverse densitized triad components in a well-defined way [56], providing densely defined quantum constraint operators. For small values of the triad components near the classical divergence at zero, however, expectation values of the inverse triad operators in coherent states differ from the classical expression of the inverse. This deviation is captured by introducing a quantum correction function in terms of the constraint where inverse triad components appear via properties of triad operators in the loop representation.

Specifically, we have an inverse E^x multiplying all terms in the Hamiltonian constraint (2.16) in Ashtekar variables. We choose here $\alpha = \bar{\alpha} = \alpha_\Gamma = \bar{\alpha}_\Gamma$ for our effective Hamiltonian (2.32), so that

$$H^Q_{\rm grav}[N] = \int dx\, N\alpha\tilde{\mathcal{H}}_{\rm grav} \tag{3.45}$$

where $H_{\rm grav}[N] = \int dx\, N\tilde{\mathcal{H}}_{\rm grav}$ is the classical Hamiltonian (2.16). As before, the Gauss and diffeomorphism constraints remain unaltered since they do not contain inverse triad components.

Assuming again that α depends only on the densitized triad components E^x and E^φ (but not on their spatial derivatives and not on connection components), this specific correction satisfies consistency conditions (2.36) and (2.38) trivially, and the full constraint algebra is

$$\{G_{\rm grav}[\lambda_1], G_{\rm grav}[\lambda_2]\} = 0$$

$$\{G_{\rm grav}[\lambda], D_{\rm grav}[N^x]\} = -G_{\rm grav}[\mathcal{L}_{N^x}\lambda] = -G_{\rm grav}[N^x\lambda']$$
$$\{G_{\rm grav}[\lambda], H^Q_{\rm grav}[N]\} = 0$$
$$\{D_{\rm grav}[N^x], D_{\rm grav}[M^x]\} = D_{\rm grav}[[\bar{N}, \bar{M}]] = D_{\rm grav}[N^x M^{x\,\prime} - M^x N^{x\,\prime}]$$
$$\{H^Q_{\rm grav}[N], D_{\rm grav}[N^x]\} = -H^Q_{\rm grav}[N^x N' - \frac{1}{\alpha}\frac{\partial\alpha}{\partial E^\varphi}E^\varphi N N^{x\,\prime}]$$

$$\begin{aligned}\{H^Q_{\rm grav}[N], H^Q_{\rm grav}[M]\} =& D_{\rm grav}[\alpha^2\frac{|E^x|}{E^{\varphi\,2}}(NM' - MN')] \\ &- G_{\rm grav}[\alpha^2\frac{|E^x|}{E^{\varphi\,2}}(NM' - MN')\eta'] \qquad (3.46)\end{aligned}$$

Thus, any correction of this form provides a first-class algebra, even though coefficients in the algebra are corrected compared to the classical case. We thus have a consistent deformation of the classical theory where the number of all gauge symmetries is preserved (even though the algebra does change). Note that $H^Q_{\rm grav}$ transforms as a scalar only if α is independent of E^φ since E^φ is the only quantity of density weight one. However, the vacuum algebra is first class even if α does depend on E^φ. So far, this result is not surprising because the correction function α simply multiplies the total Hamiltonian constraint and could thus be absorbed in the lapse function. (This by itself could change observable properties, as also discussed in [78], because it would still be the classical lapse function which enters the space-time metric (3.26) while the lapse function entering the Hamiltonian would be corrected by α. However, as far as consistency of the deformation is concerned, the vacuum case is rather trivial.)

The situation changes if we add matter terms by coupling a two-dimensional Yang-Mills system as in Sec. 3.3,

$$S_{\rm YM} = \int dt \int dx\, \mathcal{A}_{xI}\dot{\mathcal{E}}^I - \mathcal{A}_{tI}\tilde{\mathcal{G}}_{\rm YM} - N\tilde{\mathcal{H}}_{\rm YM}$$

with Gauss constraint

$$G_{\rm YM}[\Lambda_I] = \int dx\, \Lambda_I(\mathcal{E}^{I\,\prime} + c^I_{JK}\mathcal{A}_J\mathcal{E}_K)$$

and Hamiltonian

$$H_{\rm YM}[N] = \int dx\, N\zeta E^\varphi \mathcal{E}^I\mathcal{E}_I\,.$$

There is no inverse triad component in this Hamiltonian, which remains uncorrected under the effects studied here. Thus, α can no longer be absorbed in N for the total constraint. The quantum corrected gravity-Yang-Mills Hamiltonian is

$$H^Q[N] = H^Q_{\rm grav}[N] + H_{\rm YM}[N]$$

resulting in the full constraint algebra

$$\{G_{\rm YM}[\Lambda_I], G_{\rm YM}[V_J]\} = -G_{\rm YM}[c^{IJK}\Lambda_I V_J]$$

$$\{G_{\rm YM}[\Lambda_I], G_{\rm grav}[\lambda]\} = \{G_{\rm YM}[\Lambda_I], D_{\rm grav}[N^x]\} = 0$$
$$\{H^Q[N], G_{\rm grav}[\lambda]\} = \{H^Q[N], G_{\rm YM}[\Lambda_I]\} = 0$$

$$\begin{aligned}\{H^Q[N], D_{\rm grav}[N^x]\} =& - H^Q_{\rm grav}[N^x N' - \frac{1}{\alpha}\frac{\partial\alpha}{\partial E^\varphi}E^\varphi N N^{x\,\prime}] \\ & - H_{\rm YM}[N^x N'] - G_{\rm YM}[2NN^x\zeta E^\varphi E_I]\end{aligned}$$

$$\begin{aligned}\{H^Q[N], H^Q[M]\} =& D_{\rm grav}[\alpha^2\frac{|E^x|}{E^{\varphi\,2}}(NM' - MN')] \\ & - G_{\rm grav}[\alpha^2\frac{|E^x|}{E^{\varphi\,2}}(NM' - MN')\eta'] \end{aligned} \qquad (3.47)$$

In contrast to the vaccum case, there is now a non-trivial condition for the correction function: Only when α does not depend on E^φ can all the terms in $\{H^Q[N], D_{\rm grav}[N^x]\}$ be combined to constraints. The dependence on E^x, on the other hand, is unrestricted. Thus, quantum corrections due to the loop quantization can provide non-trivial consistent deformations.

On the other hand, it was proved in [24] that a consistent deformation of the

PSM in the sense of [79] must always be a PSM with the same dimension. Since the corrected constraint algebras (3.46) and (3.47) remain first class, the number of gauge symmetries does indeed stay fixed. It must be possible to formulate the quantum corrected system as a PSM. However, the result that any consistent deformation of a PSM must again be a PSM, as it follows from a BRST analysis, is obtained for equivalence classes of theories up to field redefinitions. This does not provide a constructive procedure to determine a corresponding PSM formulation for a given consistent deformation, and thus further input is required.

Rewriting the PSM constraints in terms of the standard gravitational constraints by inverting (3.38), (3.39) and (3.40):

$$\tilde{C}^{\mp} = \left[\frac{1}{2e}\tilde{\mathcal{D}}_{\text{grav}} \mp \frac{p}{\sqrt{2}}\phi^{-\frac{1}{4}}\tilde{\mathcal{H}} - \frac{\omega_x}{2e}\tilde{C}^3 \pm \left(\frac{\tilde{C}^3}{2e}\right)'\right]\exp(\mp\Upsilon) \tag{3.48}$$

(with $\tilde{\mathcal{H}}$ either $\tilde{\mathcal{H}}_{\text{grav}}$ or $\tilde{\mathcal{H}}_{\text{grav}} + \tilde{\mathcal{H}}_{\text{YM}}$) and susbstituting $\tilde{\mathcal{H}}_{\text{grav}}$ for the quantum corrected Hamiltonian $\alpha\tilde{\mathcal{H}}_{\text{grav}}$, we obtain a deformation of the PSM. It must be possible to cast the anomaly free algebras (3.46) and (3.47) as a PSM of some form. Finding this form will provide an action formulation for the quantum corrected system, and thus a covariant interpretation of the quantum correction function.

Inserting the correction function α directly in (3.48) gives explicitly, in terms of PSM variables,

$$\tilde{C}^{\mp}_{\text{deformed}} = \tilde{C}^{\mp} \mp (\alpha[\phi]-1)\frac{1}{2G}\left[\frac{1}{2}V(\phi)e + \frac{Q^e}{2}(\omega_x + \Upsilon') + \left(\frac{\phi'}{2e}\right)'\right]\exp(\mp\Upsilon) \tag{3.49}$$

with $\tilde{C}^{\mp}$ as in (3.23) or (3.44). This is not yet of a form suitable for a PSM interpretation due to the extra terms involving e.g. derivatives of ϕ which cannot simply be put in the dilaton potential. (The potential must be a function on the target space, which cannot accomodate space-time derivatives.)

Instead, we can use the requirement of the PSM form to find the corresponding formulation. In the previous equation, we have simply taken the same combinations of loop variable constraints as in the classical case. But if the constraints are corrected, we may well have to use different combinations of the constraints, with corrected coefficients, to bring them in a PSM form. We thus change the coefficients in front of the gravity constraints on the right hand side of (3.48) so as to exactly cancel the unwanted terms depending on ϕ-derivatives. For this there is a unique way up to a total factor: the coefficient of $\tilde{\mathcal{H}} = \alpha\tilde{\mathcal{H}}_{\rm grav}+\tilde{\mathcal{H}}_{\rm YM}$ in the combination $\tilde{C}^{\mp}$ of constraints must carry an extra factor of $1/\alpha(\phi)$. In this way, the ϕ-derivatives cancel in the combination of constraints as they do classically. The system is then described by a Poisson sigma model with constraints

$$\tilde{C}_Q^{\mp} = \left[\frac{1}{2e}\tilde{\mathcal{D}}_{\rm grav} \mp \frac{p}{\sqrt{2}}\phi^{-\frac{1}{4}}\left(\tilde{\mathcal{H}}_{\rm grav} + \frac{1}{\alpha[\phi]}\tilde{\mathcal{H}}_{\rm YM}\right) - \frac{\omega_x}{2e}\tilde{C}^3 \pm \left(\frac{\tilde{C}^3}{2e}\right)'\right]\exp(\mp\Upsilon)\,.$$

Here, the correction function appears only in one place multiplying the Yang–Mills Hamiltonian. The correction is thus non-trivial and changes the coupling of Yang–Mills to gravity: We now have the effective potential

$$\frac{1}{2}V(\phi) + 4G\frac{\zeta(\phi)}{\alpha(\phi)}\mathcal{E}^I\mathcal{E}_I\,. \tag{3.50}$$

In these models, the arbitrariness of ζ (in a ϕ-dependent way) is thus enough to account for our consistent deformations: the deformed Poisson sigma model for (3.43) is of the same type with ζ replaced by ζ/α. This is in accordance with our condition for a consistent deformation derived from Eq. (3.47), namely that α only depends on E^x which is identified with the dilaton ϕ. Any other dependence could not be combined with the Yang–Mills coupling function $\zeta(\phi)$.

3.6 Conclusions

We have studied the canonical relation between 2-dimensional dilaton gravity, Poisson sigma models and spherically symmetric gravity in Ashtekar variables. This is of interest because Ashtekar variables allow a background independent quantization of the full theory, while other quantization methods have been applied to dilaton gravity in two dimensions, such as a rigorous path integral quantization. Moreover, Poisson sigma models allow an interpretation of their structure functions as defining Lie algebroid symmetries, generalizing the Lie algebra symmetries of systems with structure constants. Given the explicit canonical transformation to Ashtekar variables we have derived, one may ask whether an analogous reformulation as a Lie algebroid sigma model could exist in four dimensions. If this would be the case, the structure function issue of general relativity could be substantially simplified. Unfortunately, the canonical transformation uses several special features realized only in two dimensions. For instance, as shown by Eq. (3.37) we need to eliminate a second derivative of E^x which appears in the Hamiltonian constraint in Ashtekar variables but not in the PSM constraints. This can be done in two dimensions by means of a spatial derivative of the Gauss constraint. In four dimensions, on the other hand, the Gauss constraint contains the total divergence of the triad, which cannot provide all terms needed to remove all second triad derivatives from the full Hamiltonian constraint.

As a side result, we have used some of our derivations to extend the loop quantization to spherically symmetric systems in arbitrary D space-time dimensions. This extends the proofs of singularity-freedom of spherically symmetric loop quantizations to spherically symmetric systems in arbitrary dimensions. A word of caution is in place here since the canonical transformations we have derived are local and global aspects have not been studied here.

While dilaton gravity in two dimensions has been quantized covariantly by path integral methods, loop quantum gravity is a canonical quantization. In this context, the consistency issue of the resulting quantum constrained system is probably the most important one in loop quantum gravity, whose analysis will tell whether the diverse effects studied in simple models can be viable and covariant in general. What our analysis of consistency in two dimensions shows is that there is indeed room for non-trivial effects due to the quantization. Quantum corrections of the canonical quantization are then related to a covariant action, where effective couplings to the Yang–Mills ingredients arose. We will give further examples of consistent deformations in the next chapter and appendix C.

For further corrections existing in a loop quantization, consistency has not yet been demonstrated. Among those we highlight the general phenomenon of quantum back-reaction here, which implies that moments of a state such as fluctuations and correlations influence the dynamical behavior of expectation values. If this is included, new quantum degrees of freedom arise in an effective theory, as explained in chapter 2). In our context, a consistent deformation of this type will provide a higher-dimensional target space of the Poisson sigma model. Since the number of fields changes, the rigidity proofs for consistent deformations of PSMs no longer apply. Such effective theories could even generate new algebroid sigma models beyond PSMs, e.g. of the forms introduced in [75, 76].

Chapter 4

Lemaître–Tolman–Bondi Models

Quantum gravity changes the structure and dynamics of space-time on small distance scales, which should have implications for the final stages of matter collapse[1]. An interesting class of models to shed light on this issue is given by Lemaître–Tolman–Bondi (LTB) space-times. Classically, these models describe inhomogeneous collapsing dust balls, and contain Friedmann–Robertson–Walker solutions as special cases. They thus provide an interesting extension of models beyond homogeneity, an extension which is particularly important to understand in the case of quantum gravity.

Marginal models, which are a subclass of general LTB models, have been analyzed using effective equations in Ref. [21]. This has resulted in consistent deformations which implement some types of quantum corrections without spoiling general covariance, and made possible an initial analysis of implications regarding effective pictures of collapse singularities. It turned out that there is no clear generic avoidance of either space-like or null singularities by an obvious mechanism, in contrast to several homogeneous models of loop quantum cosmology [8] where phenomenological mechanisms such as bounces could be found easily. While this outcome is not entirely unexpected given the types of corrections analyzed in the marginal case, it does show that further analysis is required. Marginal models, after all, provide spatially flat Friedmann–Robertson–Walker models in the homogeneous limiting case which give rise to phenomenological singularity avoidance in their loop quantization (including a positive matter potential) only with

[1]This chapter is based on the work by the author in [80]

holonomy corrections [81, 10], which were not fully included in [21] due to technical complications. It is thus natural to extend the constructions to non-marginal models which would provide a homogeneous model of positive (as well as negative) spatial curvature as limit. In that case, loop quantum cosmology can give rise to phenomenological singularity resolution even in the presence of inverse triad corrections alone [82], which in inhomogeneous situations are easier to control than holonomy corrections. If the behavior seen in homogeneous models should be generic and apply also to inhomogeneous situations, loop quantized non-marginal LTB models must give rise to singularity resolution more easily than marginal ones.

To find anomaly-free versions of non-marginal LTB models including inverse triad corrections from loop quantum gravity, we follow two derivations. First, we review and extend the methods of [21] where constraints already incorporating the LTB reduction of metric components are made anomaly-free by consistency conditions between correction functions. Secondly, we use our general anomaly-free system of spherically symmetric constraints (2.32), on which we then apply the LTB reduction in a second step. As we will show, the two steps of LTB reduction and deriving consistency conditions almost commute: in the end, we obtain consistent equations of motion of similar structure, although they do differ by some terms. This outcome considerably supports the constructions of [21].

Using these consistent equations, gravitational collapse can be analyzed. We are specifically interested here in the possibility of a turn-around of the collapse, or a bounce, in the corrected equations, which are suggested to exist by models where homogeneous interiors have been matched, Oppenheimer–Snyder-style, to spherically symmetric exteriors [83]. Also here, as in the marginal case but in contrast to homogeneous models, we do not find a clear indication for singularity resolution, although several extra terms do seem to make a bounce more likely. As in the marginal case, this part of the result is not conclusive since not all corrections have been included and no complete analysis has been performed. Our results thus do not mean that there is no bounce in these inhomogeneous models. But they do show that an outright treatment of inhomogeneous models is different from matching homogeneous results. In fact, we also confirm the observation of [21] that quantum corrections of the type studied here prevent the existence of an exact homogeneous limit. "Effective" homogeneous geometries thus have to be taken with care, but consistent relationships with inhomogeneous ones do provide insights in their structure.

4.1 Classical LTB solutions

Lemaître–Tolman–Bondi models [27, 28, 29] are solutions of the spherically symmetric Einstein's field equations, $G_{\alpha\beta} = 8\pi G T_{\alpha\beta}$, with vanishing cosmological constant and with stress-energy tensor describing inhomogeneous, pressureless dust given by $T_{\alpha\beta} = \epsilon U_\alpha U_\beta$.

In co-moving coordinates $(t, x, \vartheta, \varphi)$, where $x \geq 0$ labels the spherical shells of dust and t is the proper time along the world lines of dust particles given by $r = const.$ (so that $U^\alpha = \delta^\alpha_t$), the space-time metric reads

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \frac{R'^2}{1+\kappa(x)}\mathrm{d}x^2 + R^2\mathrm{d}\Omega^2 \,. \tag{4.1}$$

Again $\mathrm{d}\Omega^2 = \mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2$. The function $R(t, x)$ can depend on both time and the radial coordinate, but not on the angular coordinates to leave the metric spherically symmetric, it is the physical or areal radius of the dust shells. Its behaviour is dictated by Einstein's equations which give the relations[2]

$$\epsilon = \frac{F'}{R^2 R'}\,, \qquad \dot{R} = \pm\sqrt{\kappa + \frac{F}{R}} \tag{4.2}$$

The functions $\kappa(x) > -1$, and $F(x) > 0$, are arbitrary functions of the radial coordinate x only and parameterize different solutions. The proper area $4\pi R^2$ of the dust shells goes to zero when $R(t, x) = 0$ so for gravitational collapse we only consider the negative sign in equation (4.2). It is easy to see that positively curved Friedmann–Robertson–Walker models with scale factor $a(t)$ are obtained for $\kappa(x) = -x^2$ and $R(t, x) = a(t)x$. Although we will not need it here, equation

[2]we will give a canonical derivation of these equations in section 4.3.3.

(4.2) can be integrated and gives the general solution

$$t - t_0(x) = -\frac{R^{\frac{3}{2}} Q(-\kappa R/F)}{\sqrt{F}} \tag{4.3}$$

where the positive and bounded real function $Q(y)$ is defined by

$$Q(y) = \begin{cases} -\frac{\operatorname{arcsinh}\sqrt{-y}}{(-y)^{\frac{3}{2}}} - \frac{\sqrt{1-y}}{y}, & \text{for } -\infty \leq y < 0; \\ \frac{2}{3}, & \text{if } y = 0; \\ \frac{\operatorname{arcsinh}\sqrt{y}}{y^{\frac{3}{2}}} - \frac{\sqrt{1-y}}{y}, & \text{for } 0 < y \leq 1. \end{cases}$$

The integration 'constant' $t_0(x)$ arises from the freedom to rescale the radial coordinate or shell index x in (4.1). This is usually fixed by demanding $R(0, x) = x$, and we will do so here. This gives $t_0(x) = x^{3/2} Q(-\kappa/F)/\sqrt{F}$.

Depending on whether κ, called the energy function, vanishes or not the spacetime is classified as marginal or nonmarginal. For the marginal case $\kappa = 0$ we can write the explicit solution

$$R(t, x) = \left(x^{\frac{3}{2}} - \frac{3}{2}\sqrt{F(x)}\, t \right)^{\frac{2}{3}}$$

4.1.1 Singularities and the Misner-Sharp mass

To gain physical insight for the role of the so called mass function $F(x)$ we may notice that it represents the mass (weighted by the factor $\sqrt{1+\kappa}$) contained within the matter shell labeled by x and that with our scaling, it can be expressed in terms of the energy density at $t = 0$:

$$F(x) = \int \epsilon(0, x) x^2 dx$$

but most importantly, we recognize $\frac{1}{2}F(x)$ as the Misner-Sharp mass [84] of the spherically symmetric LTB spacetime.

We recall here that, while there is no generic notion of local effective energy in General Relativity, in spherical symmetry, the Misner-Sharp mass or more appropriately the Misner-Sharp energy gives a definition of energy which has many desirable physical properties [85].

The Misner-Sharp mass m for a spherically symmetric metric $g_{\alpha\beta}$ is a geometrical invariant defined by

$$m := \frac{R}{2}(1 - g^{\alpha\beta}(dR)_\alpha(dR)_\beta)$$

or

$$m = \frac{R}{2}(1 - \nabla_\mu R \nabla^\mu R)$$

where R is again the areal radius and $\mu = (0, 1)$ corresponds to the $t - r$ manifold. For a metric of the form

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + L^2(t,x)\mathrm{d}x^2 + R^2(t,x)\mathrm{d}\Omega^2 \tag{4.4}$$

the Misner-Sharp mass then becomes

$$m = \frac{R}{2}\left(1 + \dot{R}^2 - \frac{R'^2}{L^2}\right) . \tag{4.5}$$

For the LTB metric $R'^2/L^2 = 1 + \kappa$, and therefore, after using Einstein's equations (4.2), the Misner-Sharp mass is precisely

$$m = \frac{F}{2}$$

In the Newtonian limit, the Misner-Sharp mass yields the Newtonian mass to leading order and the Newtonian kinetic and potential energy in the next order. The Misner-Sharp mass is the corresponding charge of the conserved Kodama current[]. In asymptotically flat spacetimes it reduces to the Bondi-Sachs and Arnowitt-Deser-Misner energies at null and spatial infinity. Furthermore, m identifies spherical trapped surfaces and horizons ($2m > R$ and $2m = R$), and provides information on the causal nature of central singularities.

Singularities are the points of spacetime where the normal differentiability and manifold structures break down. In other words points where geodesics terminate and where the energy density ϵ or scalar curvature quantities constructed from the Riemann tensor $\mathcal{R}^\mu_{\nu\sigma\rho}$ diverge. Here we should describe the singularities that may appear in spherically symmetric collapse. A shell-crossing singularity is characterized by $R' = 0$ and $R > 0$. A shell-focusing singularity is characterized by $R = 0$. If the singularity occurs at $x = 0$ it is called a central singularity since the point $x = 0$ is (assumed to be) a center, that is a boundary of the two dimensional spacetime obtained by taking the quotient by the symmetry orbits.

From the first equation in (4.2) and the expressions for the scalar curvature $\mathcal{R} = 8\pi\epsilon$ or $\mathcal{R}^{\mu\nu}\mathcal{R}_{\mu\nu} = 64\pi^2\epsilon^2$, the divergence of the energy density directly implies a scalar curvature singularity.

Shell-crossing singularities are argued to be avoidable by more realistic matter sources and furthermore, it is possible to extend solutions through them at least in a distributional sense [86, 87, 88]. Hence in what follows we will only consider shell-focusing singularities. Equation (4.3) shows that $t = t_0(x)$ is the dust proper time at which the shell labeled by x reaches the physical singularity $R = 0$.

As mentioned above, the Misner-Sharp mass is closely related to the nature of singularities. For example [85], if $\frac{1}{2}F$ is positive and bounded below in a neighborhood of a central singularity then the singularity will be trapped and (if the singularity is differentiable) spatial. Simarly if $\frac{1}{2}F$ is negative and bounded above the singularity will be unbounded and temporal. Hence, by adjusting the form of the mass function $F(x)$ parameterizing LTB solutions, we may obtain models exhibiting space-like as well as time-like or even null singularities which may be hidden inside a black hole or naked.

4.2 Hamiltonian LTB reduction

We now proceed to review and extend the methods of [21] in a Hamiltonian formulation using Loop variables.

We start with the spherically symmetric Gauss (2.14), Diffeomorphism (2.15) and Hamiltonian (2.16) constraints of General relativity. Solving the Gauss constraint removes the canonical pair (η, P^η) and implies that invariant objects can depend on the x-component of the Ashtekar connection only through the extrinsic curvature component $A_x + \eta' = \gamma K_x$. After this step we can work with the canonical pairs

$$\{K_x(x), E^x(y)\} = \{K_\varphi(x), 2E^\varphi(y)\} = 2G\delta(x,y)\,.$$

The relation to the usual spherically symmetric geometrodynamical variables

$$\{R(x), P_R(y)\} = \{L(x), P_L(y)\} = G\delta(x,y) \tag{4.6}$$

as used for example in [33, 34], can be obtained directly by comparing the spatial metric

$$\mathrm{d}q^2 = L^2\mathrm{d}x^2 + R^2\mathrm{d}\Omega^2 = \frac{(E^\varphi)^2}{|E^x|}\mathrm{d}x^2 + |E^x|\mathrm{d}\Omega^2 \tag{4.7}$$

in each set of variables and making use of the equations of motion (3.29) in the definition of canonical momenta P_L and P_R. The canonical point transformation is

$$\begin{aligned} L = E^\varphi|E^x|^{-\frac{1}{2}} \quad &, \quad R = |E^x|^{\frac{1}{2}}\,, \\ P_L = -K_\varphi|E^x|^{\frac{1}{2}} \quad &, \quad P_R = -sK_x|E^x|^{\frac{1}{2}} - K_\varphi E^\varphi|E^x|^{-\frac{1}{2}} \end{aligned} \tag{4.8}$$

where again $s = \mathrm{sgn}(E^x)$.

Specializing the general spherically symmetric metric

$$\mathrm{d}s^2 = -N(t,x)^2\mathrm{d}t^2 + L^2(t,x)(\mathrm{d}x + N^x(t,x)\mathrm{d}t)^2 + R^2(t,x)\mathrm{d}\Omega^2$$

to the LTB form (4.1) requires a vanishing shift function $N^x = 0$ and lapse $N = 1$ for comoving coordinates of the dust, and on using the first equation in (4.8) gives the LTB condition in terms of triads

$$2\sqrt{1+\kappa(x)}E^\varphi = (E^x)' \tag{4.9}$$

If we solve the diffeomorphism constraint (2.15) identically, which requires $2E^\varphi K_\varphi' - K_x(E^x)' = 0$, the LTB condition (4.9) for triad variables gives rise to a condition for the extrinsic curvature components

$$K_\varphi' = \sqrt{1+\kappa(x)}K_x \tag{4.10}$$

The triad condition (4.9) implies a particular form for the spin connection component and its derivative

$$\Gamma_\varphi = -\frac{(E^x)'}{2E^\varphi} = -\sqrt{1+\kappa(x)}, \qquad \Gamma'_\varphi = -\frac{\kappa'(x)}{2\sqrt{1+\kappa(x)}} \tag{4.11}$$

which, after substituting in the Hamiltonian constraint (2.16), gives

$$H_{\rm grav}[N] = -\frac{1}{2G}\int \mathrm{d}x N|E^x|^{-1/2}\left[K_\varphi^2 E^\varphi + 2K_\varphi K_x E^x - \kappa(x)E^\varphi - \frac{\kappa'(x)E^x}{\sqrt{1+\kappa(x)}}\right]. \tag{4.12}$$

For a consistent LTB formulation, the two LTB conditions must be preserved by evolution generated by this Hamiltonian constraint with lapse $N = 1$. This is indeed the case as can be seen by explicitly computing Poisson brackets between the Hamiltonian constraint and each of the LTB conditions. For the Poisson bracket of the two smeared LTB conditions we get

$$\left\{\int \mathrm{d}x\nu(x)\left(\sqrt{1+\kappa(x)}K_x - K'_\varphi\right), \int \mathrm{d}y\mu(y)\left(2\sqrt{1+\kappa(x)}E^\varphi - (E^x)'\right)\right\}$$
$$= 2G\int \mathrm{d}z\sqrt{1+\kappa(z)}(\mu\nu)'. \tag{4.13}$$

For appropriate smearing functions $\mu(x)$ and $\nu(x)$ this is zero for the marginal case but for $k \neq 0$ it is non-vanishing. Although we will not follow this route here, we note that this will have an impact on implementing the LTB conditions at the state level, as done in [21] for the marginal case. Another complication for such a construction is the explicit κ-dependence of the LTB conditions, which makes their integrated version used as conditions on holonomies more complicated.

Equations of motion in this canonical formulation are derived using $\dot{E}^x = \{E^x, H_{\rm grav}[1]\}$, with a similar equation for E^φ. With these we can first eliminate K_x and K_φ, and finally E^φ using the LTB condition to obtain an equation entirely in terms of E^x. After replacing E^x by R^2 we obtain

$$H_{\rm grav} = \frac{-2R\dot{R}\dot{R}' - \dot{R}^2R' + \kappa' R + \kappa R'}{2G\sqrt{1+\kappa(x)}}, \tag{4.14}$$

which has to be equated to the matter part of the Hamiltonian for dust given by

$H_{\rm dust} = -\frac{1}{2G}F'/\sqrt{1+\kappa(x)}$. (A general canonical derivation of the gravity-dust system will be given in Sec. 4.3.2.) Thus

$$2R\dot{R}\dot{R}' + \dot{R}^2R' - \kappa'R - \kappa R' = F' \tag{4.15}$$

is the equation of motion, in agreement with the spatial derivative of Einstein's equation (4.2), $R\dot{R}^2 = \kappa(x)R + F(x)$.

4.2.1 Inverse triad corrections from loop quantum gravity

We can now repeat the canonical analysis using a Hamiltonian constraint containing correction functions as they are suggested by constraint operators in loop quantum gravity. Consistency then requires conditions for the possible terms, which show how quantum corrections can be realized in an anomaly-free way. We discuss here only inverse triad corrections which are easier to implement, and which already provide insights into one of the main classes of quantum geometry corrections.

As already stressed, the exact form of these corrections and their appearance in the Hamiltonian is far from unique. To study the phenomenology they imply and to test robustness of predictions, we consider two versions here. Without loss of generality we choose $E^x > 0$ in this section.

4.2.1.1 First version

We turn to the case where only those terms in the Hamiltonian with explicit $1/\sqrt{|E^x|}$ dependence are corrected by a factor $\alpha(E^x)$. Starting as in the marginal case [21], we will first assume that the classical expression for the spin connection can be used, but show that this is inconsistent. The Hamiltonian is

$$\begin{aligned} H^I_{\rm grav}[N] = -\frac{1}{2G}\int {\rm d}x\, N\Bigg[&\alpha(E^x)\frac{K_\varphi^2E^\varphi}{\sqrt{E^x}} + 2K_\varphi K_x\sqrt{E^x} \\ &- \alpha(E^x)\frac{\kappa(x)E^\varphi}{\sqrt{E^x}} - \frac{\kappa'(x)\sqrt{E^x}}{\sqrt{1+\kappa(x)}}\Bigg]. \end{aligned} \tag{4.16}$$

As in the marginal case, it turns out that the correction in the Hamiltonian can lead to consistent LTB-type solutions only if we also change the LTB conditions by a correction function $f(E^x)$:

$$(E^x)' = 2\sqrt{1+\kappa(x)}f(E^x)E^\varphi \quad \text{and} \quad K'_\varphi = \sqrt{1+\kappa(x)}f(E^x)K_x\,. \tag{4.17}$$

These relations still solve the classical diffeomorphism constraint identically, which does not receive corrections in loop quantum gravity. The main consistency requirement is that Poisson brackets of the LTB conditions with the Hamiltonian constraint vanish.

Each of the Poisson brackets gives a differential equation for $f(E^x)$: For the LTB condition corresponding to the triad variables we obtain

$$2E^x \frac{\mathrm{d}f}{\mathrm{d}E^x} = f(1-\alpha)\,. \tag{4.18}$$

The solution to this equation depends of course on the particular form of α. For example, for the particular form (2.42) derived in Chapter 2, an explicit solution for $\mathcal{N}=1$ can be given by

$$f(E^x) = \begin{cases} \dfrac{c_1\sqrt{E^x}e^{-\alpha/2}}{\left(\sqrt{E^x}+\sqrt{E^x-\gamma\ell_{\mathrm{P}}^2/2}\right)^{1/2}\left(\sqrt{E^x}+\sqrt{E^x+\gamma\ell_{\mathrm{P}}^2/2}\right)^{1/2}}\,, & \text{for } E^x > \gamma\ell_{\mathrm{P}}^2/2; \\ \dfrac{c_2\sqrt{E^x}\exp\left(-\frac{1}{2}\alpha+\frac{1}{2}\arctan\left(\sqrt{E^x/(\gamma\ell_{\mathrm{P}}^2/2-E^x)}\right)\right)}{\left(\sqrt{E^x}+\sqrt{E^x+\gamma\ell_{\mathrm{P}}^2/2}\right)^{1/2}} & \text{for } E^x < \gamma\ell_{\mathrm{P}}^2/2. \end{cases}$$

Here $c_1 = 2\sqrt{e}$ and $c_2 = 2^{5/4}e^{1/2-\pi/4}\gamma^{-1/4}\ell_{\mathrm{P}}^{-1/2}$ are constants of integration fixed, respectively, by the condition $\lim_{E^x\to\infty} f(E^x) \to 1$ and by requiring that $f(E^x)$ be continuous at $E^x = \gamma\ell_{\mathrm{P}}^2/2$. The graph of this solution is shown in Fig. 4.1.

This is the same equation as found for the marginal case in [21]. It is clear that we obtain the same equation because κ does not appear in the terms containing K_x and K_φ in the Hamiltonian and thus, the value of κ has no affect in the evaluation of the Poisson bracket. However, the differential equation for $f(E^x)$ obtained by demanding that the corrected LTB condition for extrinsic curvature components is also preserved in time gives the equation

$$-2\sqrt{1+\kappa}K_\varphi K_x\sqrt{E^x}\frac{\mathrm{d}f}{\mathrm{d}E^x} + \frac{\sqrt{1+\kappa}K_\varphi K_x f}{\sqrt{E^x}} - \frac{\alpha\sqrt{1+\kappa}K_\varphi K_x f}{\sqrt{E^x}} - \frac{\kappa' f}{2\sqrt{E^x}} + \frac{\kappa'\alpha}{2\sqrt{E^x}} = 0 \tag{4.19}$$

which, due to the κ'-terms, is different from and in fact inconsistent with equation

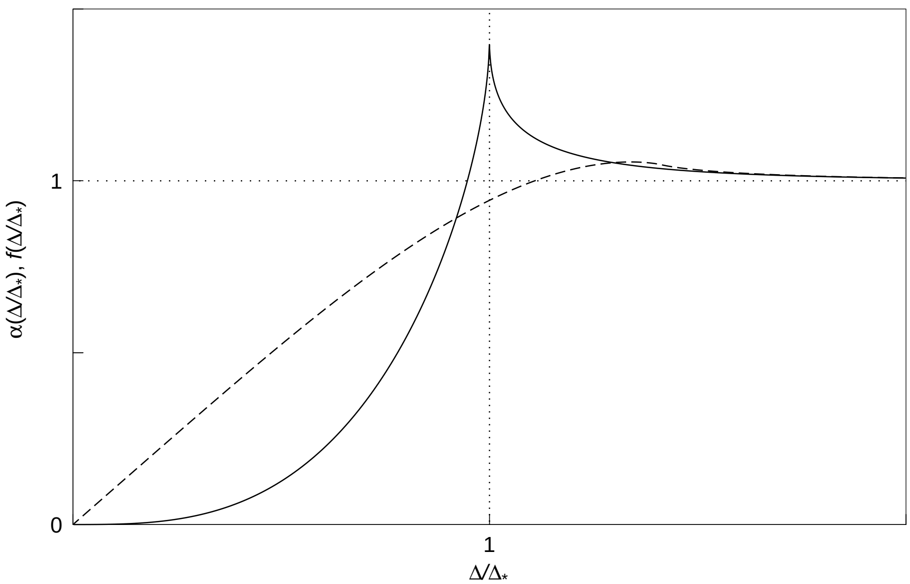


Figure 4.1. The correction functions $\alpha(\Delta)$ (solid line) and $f(\Delta)$ (dashed line) where Δ is taken relative to $\Delta_* := \sqrt{\gamma/2}\ell_{\mathrm{P}}$.

(4.18) obtained from the LTB condition for triads.

4.2.1.2 Second version

We now repeat the above procedure for a Hamiltonian corrected by $\alpha(E^x)$ in all terms, as it is suggested from the full theory. This Hamiltonian reads

$$H^{II}_{\mathrm{grav}}[N] = -\frac{1}{2G}\int \mathrm{d}x\, N\frac{\alpha(E^x)}{\sqrt{E^x}}\left(K_\varphi^2 E^\varphi + 2K_\varphi K_x E^x - \kappa(x)E^\varphi - \frac{\kappa'(x)E^x}{\sqrt{1+\kappa(x)}}\right). \tag{4.20}$$

Again, the LTB-like conditions in the form

$$(E^x)' = 2\sqrt{1+\kappa(x)}g(E^x)E^\varphi \quad \text{and} \quad K_\varphi' = \sqrt{1+\kappa(x)}g(E^x)K_x \tag{4.21}$$

solve the diffeomorphism constraint identically. However, for the same reasons as noted above the conditions cannot be consistent: since the terms containing K_x and K_φ in the Hamiltonian do not involve κ, the Poisson bracket for the triads gives the same result as in the marginal case where the differential equation was

$$\alpha \frac{\mathrm{d}g}{\mathrm{d}E^x} = g \frac{\mathrm{d}\alpha}{\mathrm{d}E^x}\,, \tag{4.22}$$

with the solution $g(E^x) = \alpha(E^x)$. (This solution is unique with the boundary condition imposing that $g = 1$ for large arguments.)

For the Poisson bracket involving the condition on extrinsic curvature, on the other hand, the terms in the Hamiltonian involving κ are important and we get a different result:

$$-2\sqrt{1+\kappa(x)}\alpha K_\varphi K_x \sqrt{E^x}\frac{\mathrm{d}g}{\mathrm{d}E^x} + 2\sqrt{1+\kappa(x)}K_\varphi K_x\sqrt{E^x}\frac{\mathrm{d}\alpha}{\mathrm{d}E^x}g - \frac{\kappa'\alpha g}{2\sqrt{E^x}}$$
$$-\kappa'\sqrt{E^x}\frac{\mathrm{d}\alpha}{\mathrm{d}E^x}g + \frac{\kappa'\alpha}{2\sqrt{E^x}} = 0 \tag{4.23}$$

As in the previous case the presence of κ' terms spoils the consistency: Using $\alpha = g$ the first two terms cancel while the rest would require $\mathrm{d}\alpha/\mathrm{d}E^x = (\alpha - 1)/2E^x$ with a solution $\alpha = 1 + c\sqrt{E^x}$ violating the classical limit at large arguments.

4.2.2 Inclusion of corrections in the spin connection

A direct extension of the results from marginal to non-marginal models is thus impossible. Here we have an example for the information gained by a phenomenological treatment: LTB-type solutions require additional corrections to compensate inconsistencies seen so far. Such corrections may be more difficult to derive from a full Hamiltonian, but they follow directly from a phenomenological treatment. A successful consistent implementation thus provides feedback on the full theory: additional corrections required for consistency must eventually follow from the full theory just like the primary correction α followed from inverse triad operators.

In particular, to resolve inconsistencies, we thus have to include further corrections in terms not affected yet, the chief candidate being the spin connection terms in the Hamiltonian constraint. They vanish in the marginal case, such that results from there do not provide much directions for more general models. Moreover, such terms are in fact more difficult to derive from a full Hamiltonian so that not much is known about their form. We will now look for corrections in the spin connection terms which are such that they combine with those already used to provide a consistent formulation.

4.2.2.1 Implementation

Classically we started with the expression $\Gamma_\varphi = -(E^x)'/2E^\varphi = -\sqrt{1+\kappa(x)}$. However, with quantum corrections to the Hamiltonian the LTB conditions are also corrected. For example for the second version the modified LTB condition $(E^x)' = 2\sqrt{1+\kappa}g(E^x)E^\varphi$, implies that now the spin connection is

$$\Gamma_\varphi = -\frac{(E^x)'}{2E^\varphi} = -\sqrt{1+\kappa(x)}g(E^x)\,. \tag{4.24}$$

We include the additional factor of $g(E^x)$ in the Hamiltonian by replacing any occurrence of the classical spin connection $\Gamma_\varphi^{\rm class}$ from (4.11) with $g(E^x)\Gamma_\varphi^{\rm class}$.

Furthermore, the derivative of the spin connection then is

$$\Gamma'_\varphi = -\frac{\kappa' g(E^x)}{2\sqrt{1+\kappa}} - \sqrt{1+\kappa}(E^x)'\frac{{\rm d}g}{{\rm d}E^x}\,. \tag{4.25}$$

This introduces an explicit $(E^x)'$ in the LTB-reduced Hamiltonian, which would imply $\{H[N], H[M]\} \neq 0$ even though the diffeomorphism constraint has been solved identically. The system would thus be anomalous. We are finally led to incorporate another correction function multiplying Γ'_φ in a function $h(E^x)$ so that in the Hamiltonian Γ'_φ is to be replaced with $h(E^x)(\Gamma_\varphi^{\rm class})'$. The form of $h(E^x)$ will be determined by the requirement of consistency. With all the possible corrections, the new Hamiltonian in the second version is

$$\begin{aligned} H^{II}_{\rm grav}[N] &= -\frac{1}{2G}\int {\rm d}x\, N\Big(\frac{\alpha K_\varphi^2 E^\varphi}{\sqrt{E^x}} + 2\alpha K_\varphi K_x\sqrt{E^x} + \frac{\alpha E^\varphi}{\sqrt{E^x}} \\ &\quad -\frac{\alpha g^2 E^\varphi}{\sqrt{E^x}} - \frac{\kappa\alpha g^2 E^\varphi}{\sqrt{E^x}} - \frac{\kappa'\alpha h\sqrt{E^x}}{\sqrt{1+\kappa}}\Big)\,. \end{aligned} \tag{4.26}$$

We now demand that this Hamiltonian Poisson commutes with the LTB conditions in (4.21). Here we note that the terms containing the spin connection (and its derivative) in the Hamiltonian do not contain K_x or K_φ and therefore in evaluations of the Poisson bracket with the first LTB condition there will be no changes.

This leads to the same differential equation (4.22) for $g(E^x)$ as obtained earlier, implying $g(E^x) = \alpha(E^x)$. Evaluating the Poisson bracket of the Hamiltonian with the second condition and using the solution for $g(E^x)$ gives a differential equation for $h(E^x)$:

$$\frac{\mathrm{d}(\alpha h\sqrt{E^x})}{\mathrm{d}E^x} = \frac{\alpha^2}{2\sqrt{E^x}}\,. \tag{4.27}$$

Here, we can thus have a consistent formulation for the non-marginal case for a suitable h by correcting the spin connection terms.

We proceed in a similar manner for the first version of the inverse triad corrections. From the LTB condition for triads in (4.17) we find that the spin connection would be replaced with $f(E^x)\Gamma_\varphi^{\mathrm{class}}$, and the derivative of the spin connection receives a correction function $l(E^x)$ in the form $l(E^x)(\Gamma_\varphi^{\mathrm{class}})'$. With these changes the Hamiltonian is

$$\begin{aligned} H^I_{\mathrm{grav}}[N] \;&=\; -\frac{1}{2G}\int \mathrm{d}x\, N\Big(\frac{\alpha K_\varphi^2 E^\varphi}{\sqrt{E^x}} + 2K_\varphi K_x\sqrt{E^x} + \frac{\alpha E^\varphi}{\sqrt{E^x}} \\ &\quad -\frac{\alpha f^2 E^\varphi}{\sqrt{E^x}} - \frac{\kappa\alpha f^2 E^\varphi}{\sqrt{E^x}} - \frac{\kappa' l\sqrt{E^x}}{\sqrt{1+\kappa}}\Big)\,. \end{aligned} \tag{4.28}$$

Evaluating the Poisson bracket of this with the LTB conditions and equating them to zero implies that the equation for $f(E^x)$ is unchanged compared to (4.18). The other Poisson bracket then gives a differential equation for $l(E^x)$:

$$\frac{\mathrm{d}(l\sqrt{E^x})}{\mathrm{d}E^x} = \frac{\alpha f}{2\sqrt{E^x}}\,. \tag{4.29}$$

Differential equations for the correction functions h and l are difficult to solve in general for given α and f. For a near center analysis done later we will need the lowest term in h and l in a power series expansion in x. Integrating (4.27) and (4.29) keeping only the lowest order term in α and f we find the solution

$$h(x\approx 0) = \left(\frac{2}{\gamma\ell_{\mathrm{P}}^2}\right)^{\frac{3}{2}}\frac{R^3}{7} \quad , \quad l(x\approx 0) = \frac{8e^{\frac{1}{2}-\frac{\pi}{4}}}{5(\gamma\ell_{\mathrm{P}}^2)^2}R^4 \tag{4.30}$$

valid near the center.

4.2.2.2 Ambiguities

The E^x-dependence of α follows from the consideration of inverse triad operators in the full quantum theory. Although it is not determined uniquely in this way (see [89, 90] for a discussion), the general shape of this correction function is known well. No such arguments exist for some other correction functions such as h, whose form is thus less clear. More ambiguities are thus expected to arise for it.

A basic condition on multiplicative corrections is that they be scalar to preserve the transformation properties of corrected expressions under changing coordinates. Among the basic triad variables, E^x is the only one free of a density weight and thus can appear in correction functions in an unrestricted way. The other component E^φ, on the other hand, is a density of weight one and would have to appear in combination with other densities to result in a scalar. If only triad components are considered for the dependence, the only other density would be $(E^x)'$. Scalars made from these densities, such as $E^\varphi/(E^x)'$ are however unsuitable for corrections since they are not always finite.

In the present situation, we use the function κ for a non-marginal LTB model, which means that we have another density, κ', at our disposal. Scalars of the form $(E^x)'/\kappa'$ or E^φ/κ' are well-defined for most functions κ of interest, and can thus arise in corrections. This enlargement of the space of acceptable variables means that additional ambiguities can arise. In the next section we will see how several of these ambiguities can be fixed by an analysis of the constraint algebra. The equations of motion will remain structurally similar, so that we proceed for now with an analysis of the equations resulting from the treatment done so far.

4.2.3 Equations of motion

Given the consistency conditions between correction functions we can derive consistent equations of motion even without having explicit solutions for the differential equations (4.27) and (4.29). Once consistent constraints are available, the derivation follows the classical lines which we briefly illustrate first: The first order equation (in time) has already been worked out in (4.15), so that we can go on to the evolution equation. From $\dot{E}^x = \{E^x, H_{\rm grav}^{\rm class}\}$ we have

$$K_\varphi = \frac{\dot{E}^x}{2\sqrt{E^x}}\,. \tag{4.31}$$

Similarly, using $\dot{K}_\varphi = \{K_\varphi, H_{\rm grav}^{\rm class}\}$ we obtain

$$\dot{K}_\varphi = \frac{1}{2}\left(\frac{\kappa}{\sqrt{E^x}} - \frac{K_\varphi^2}{\sqrt{E^x}}\right)\,. \tag{4.32}$$

Eliminating K_φ from the above two equations we obtain

$$\ddot{E}^x = \kappa + \frac{(\dot{E}^x)^2}{4E^x} \tag{4.33}$$

which, using $E^x = R^2$, can be written as

$$2R\ddot{R} + \dot{R}^2 = \kappa\,. \tag{4.34}$$

Eqs. (4.15) and (4.34) are automatically consistent, which can be seen explicitly by subtracting a time derivative of (4.15) from a space derivative of (4.34).

The same procedure is then applied to constrained systems including consistent correction terms. To get the first order equation in version two we use (4.26) in the equations of motion $\dot{E}^a = \{E^a, H_{\rm grav}^{II}\}$ to solve for the extrinsic curvature components

$$K_\varphi = \frac{\dot{E}^x}{2\alpha\sqrt{E^x}} \qquad \text{and} \qquad K_x = \frac{\dot{E}^\varphi}{\alpha\sqrt{E^x}} - \frac{K_\varphi E^\varphi}{E^x}\,. \tag{4.35}$$

Using these along with the LTB condition $E^\varphi = (E^x)'/2\sqrt{1+\kappa g}$ in (4.26) we rewrite the Hamiltonian in terms of E^x only:

$$\begin{aligned} H_{\rm grav}^{II} \;=\; & -\frac{1}{2G}\Big(-\frac{(\dot{E}^x)^2(E^x)'}{8\sqrt{1+\kappa}\alpha^2(E^x)^{3/2}} + \frac{\dot{E}^x(\dot{E}^x)'}{2\sqrt{1+\kappa}\alpha^2\sqrt{E^x}} - \frac{(\dot{E}^x)^2(E^x)'}{2\sqrt{1+\kappa}\alpha^3\sqrt{E^x}}\frac{{\rm d}\alpha}{{\rm d}E^x} \\ & +(1-\alpha^2-\kappa\alpha^2)\frac{(E^x)'}{2\sqrt{1+\kappa}\sqrt{E^x}} - \frac{\kappa'\alpha h\sqrt{E^x}}{\sqrt{1+\kappa}}\Big)\,, \end{aligned} \tag{4.36}$$

where we have already used the condition $g = \alpha$. When equated to the dust

Hamiltonian after using $E^x = R^2$ a first order equation in time ensues:

$$\frac{\dot{R}^2 R'}{\alpha^2} + \frac{2R\dot{R}\dot{R}'}{\alpha^2} - \frac{2R\dot{R}^2 R'}{\alpha^3}\frac{\mathrm{d}\alpha}{\mathrm{d}R} + (1-\alpha^2-\kappa\alpha^2)R' - \kappa'\alpha h R = F' \,. \tag{4.37}$$

To obtain the evolution equation we use

$$\dot{K}_\varphi = \{K_\varphi, H^{II}_{\mathrm{grav}}\} = -\frac{1}{2}\frac{\alpha - \alpha g^2 - \kappa\alpha g^2 + \alpha K_\varphi^2}{\sqrt{E^x}} \tag{4.38}$$

together with $K_\varphi = \dot{E}^x/2\alpha\sqrt{E^x}$ from (4.35), such that

$$\ddot{E}^x - \frac{(\dot{E}^x)^2}{4E^x} - \frac{(\dot{E}^x)^2}{\alpha}\frac{\mathrm{d}\alpha}{\mathrm{d}E^x} = -\alpha^2(1-g^2-\kappa g^2)\,. \tag{4.39}$$

With $g = \alpha$ and $E^x = R^2$ this becomes

$$2R\ddot{R} + \left(1 - 2\frac{\mathrm{d}\log\alpha}{\mathrm{d}\log R}\right)\dot{R}^2 = -\alpha^2(1-\alpha^2-\kappa\alpha^2)\,. \tag{4.40}$$

It is easy to see that this equation has the correct classical limit and (using (4.27)) is consistent with the first order equation.

Proceeding in a similar manner for the first version of the inverse triad correction we find that the first order equation is

$$\alpha\dot{R}^2 R' + 2\dot{R}\dot{R}'R + \alpha(1-f^2-\kappa f^2)R' - \kappa' f l R = fF' \tag{4.41}$$

and the evolution equation

$$2R\ddot{R} + \alpha\dot{R}^2 = -\alpha(1-f^2-\kappa f^2)\,. \tag{4.42}$$

Using (4.29) along with (4.22) one can verify explicitly that the first order and the second order equations are consistent with each other.

4.2.4 Effective density

To interpret effects from correction terms it is often useful to formulate them in terms of effective densities rather than new terms in equations of motion.

We may define an effective density in terms of the Misner-Sharp mass m (4.5) by

$$\epsilon_{\text{eff}} = \frac{m'}{4\pi G R^2 R'} \tag{4.43}$$

and we have seen that for the classical collapse it is

$$\epsilon_{\text{eff}}^{\text{class}} = \frac{F'}{8\pi G R^2 R'} \,. \tag{4.44}$$

This is in agreement with the 00 component of Einstein's equation, $G_{00} = 8\pi G\epsilon(t,x)$ with $\epsilon(t,x)$ the dust density. Classically the effective density, defined in terms of the Misner-Sharp mass, is the same as the dust density.

We now proceed in the same way to find the effective density for the first version of corrected equations. With the corrected LTB condition $E^\varphi = (E^x)'/2\sqrt{1+\kappa} f(E^x)$, or equivalently $L = R'/\sqrt{1+\kappa} f(R)$, the Misner-Sharp mass (4.5) is now

$$m^I = \frac{R}{2}(1 + \dot{R}^2 - (1+\kappa) f^2) \,. \tag{4.45}$$

The corresponding effective density as implied by (4.43) is

$$\epsilon_{\text{eff}}^I = \frac{1}{8\pi G R^2} \left(\frac{fF'}{R'} + (\alpha - 1)(3f^2 + 3\kappa f^2 - \dot{R}^2 - 1) + \frac{\kappa' f l R}{R'} - \frac{\kappa' f^2 R}{R'} \right) \tag{4.46}$$

where we have made use of (4.18) after substituting for E^x in terms of R, and of (4.41). We note that this equation has the correct classical limit.

Similarly for the second version the Misner-Sharp mass is

$$m^{II} = \frac{R}{2}(1 + \dot{R}^2 - (1+\kappa) g^2) \,. \tag{4.47}$$

and using the relation $\alpha' = R' \mathrm{d}\alpha/\mathrm{d}R$ (the prime denotes derivative with respect to x) along with (4.37) we find that the effective density is

$$\begin{aligned} \epsilon_{\text{eff}}^{II} =& \frac{1}{8\pi G} \left(\frac{\alpha^2 F'}{R^2 R'} + \frac{1-\alpha^2}{R^2}(1 - \alpha^2 - \kappa\alpha^2) \right. \\ & \left. + \frac{2}{\alpha R}(\dot{R}^2 - \alpha^2 - \kappa\alpha^2)\frac{\mathrm{d}\alpha}{\mathrm{d}R} + \frac{\kappa'\alpha^3 h}{RR'} - \frac{\kappa'\alpha^2}{RR'} \right) \end{aligned} \tag{4.48}$$

As in the marginal case, these effective densities imply that the near center expansion for the mass function F can have different behavior compared to the classical case, as discussed in Sec. 4.5. The matter contribution to the effective density, as given by the first terms of (4.46) and (4.48), is the same as in the marginal case.

4.2.5 Quantum correction to the energy function κ?

Physically one would expect that the energy function κ, which is related to the velocity of the dust cloud, should also receive corrections after including quantum effects. To derive those, we have to find an independent definition of κ referring only to the constraints or evolution equations derived from them. One possibility, in the classical case, is to use (4.34) whose right hand side only contains the energy function. Once brought into an analogous form, a corrected evolution equation can directly be used to read off a corrected energy function. Specifically for version one, where the evolution equation is given by (4.42), the effective energy function is

$$\kappa_{\rm eff}^{I} = \alpha f^2 \kappa - \alpha(1 - f^2) \tag{4.49}$$

while for version two, where the evolution equation is given by (4.40), the effective energy function becomes

$$\kappa_{\rm eff}^{II} = \alpha^4 \kappa - \alpha^2(1 - \alpha^2)\,. \tag{4.50}$$

This correction in effect would imply that the near center expansion for κ can be different for the quantum corrected equations as we will see when we come to the near center analysis below.

4.3 Spherically symmetric constraints

We have now several versions of consistent sets of equations of motion for non-marginal LTB models including inverse triad corrections as expected from loop quantum gravity. To make these equations consistent, we had to introduce several correction functions in different terms of the Hamiltonian constraint, which were then related to each other by consistency conditions following from the requirement that the LTB conditions be preserved. Since there is some freedom in choosing the places and forms of corrections in the constraint as well as the LTB conditions, one may question how reliable such an analysis is regarding the structure of resulting equations of motion or implications for gravitational collapse.

Before analyzing corrected equations of motion further, we now present an independent derivation which starts with the consistent set of corrected spherically symmetric constraints from chapter 2, and then implements the LTB reduction. As we will see, the structure of the resulting equations is nearly unchanged, while much less assumptions about different corrections are required. With these two procedures we thus demonstrate the robustness of consistently including corrections at a phenomenological level. Note that this would not have been possible had we chosen to fix the gauge generated by the Hamiltonian constraint in any way instead of dealing with the anomaly-issue head-on.

4.3.1 Gravitational variables and constraints

Quantum corrections due to inverse powers of the densitized triad are introduced in the classical Hamiltonian constraint (2.16) as in (2.32):

$$H^Q_{\rm grav}[N] = -\frac{1}{2G}\int \mathrm{d}x\, N\big(\alpha\,|E^x|^{-\frac{1}{2}}K_\varphi^2 E^\varphi + 2s\bar{\alpha}\,K_\varphi K_x|E^x|^{\frac{1}{2}} + \alpha\,|E^x|^{-\frac{1}{2}}E^\varphi - \alpha_\Gamma\,|E^x|^{-\frac{1}{2}}\Gamma_\varphi^2 E^\varphi + 2s\bar{\alpha}_\Gamma\,\Gamma'_\varphi|E^x|^{\frac{1}{2}}\big)\,. \tag{4.51}$$

with general functions $\alpha(E^x,E^\varphi)$ and $\bar{\alpha}(E^x,E^\varphi)$. To account for possible corrections from the quantization of the spin connection, as suggested by the previous analysis, we have generally different functions $\alpha_\Gamma(E^x,E^\varphi)$ and $\bar{\alpha}_\Gamma(E^x,E^\varphi)$ in those terms. The only restriction so far is that we have the same α in the first and third term of the Hamiltonian constraint due to their common origin from the inverse $|E^x|^{-1/2}$. The two main cases of interest here are $\bar{\alpha}=1$ or $\bar{\alpha}=\alpha$, corresponding to two versions of inverse triad corrections.

We now proceed to make the corrected constraints anomaly-free before implementing LTB conditions. To ensure anomaly-freedom, we must determine conditions under which the system of constraints, including its corrections in the Hamiltonian constraint, remains first class. The Poisson bracket $\{H^Q_{\rm grav}[M], H^Q_{\rm grav}[N]\}$ (2.35) is written here again:

$$\begin{aligned}\{H^Q_{\rm grav}[M], H^Q_{\rm grav}[N]\} =& D_{\rm grav}[\bar{\alpha}\bar{\alpha}_\Gamma|E^x|(E^\varphi)^{-2}(MN'-NM')] \\ &- G_{\rm grav}[\bar{\alpha}\bar{\alpha}_\Gamma|E^x|(E^\varphi)^{-2}(NM'-MN')\eta'] \\ &+ \frac{1}{2G}\int {\rm d}x\,(MN'-NM')(\bar{\alpha}\alpha_\Gamma-\alpha\bar{\alpha}_\Gamma)\frac{sK_\varphi(E^x)'}{E^\varphi} \\ &+ \frac{1}{2G}\int {\rm d}x\,(MN'-NM')(\bar{\alpha}'\bar{\alpha}_\Gamma-\bar{\alpha}\bar{\alpha}'_\Gamma)\frac{2K_\varphi|E^x|}{E^\varphi}\,.\end{aligned} \tag{4.52}$$

For a first class algebra the last two terms, which are not related to constraints, must vanish, providing condition (2.36) on the correction functions. One possibility is that they vanish independently. The vanishing of the last term implies $\bar{\alpha}_\Gamma \propto \bar{\alpha}$, upon which the third term gives $\alpha_\Gamma \propto \alpha$. To recover the classical limit we then have:

$$\alpha_\Gamma = \alpha \quad , \quad \bar{\alpha}_\Gamma = \bar{\alpha}\,. \tag{4.53}$$

In this case, anomaly freedom is realized with corrections to the spin connection terms to be only due to the inverse power of the densitized triad factors they contain. This may look contradictory to what we derived earlier, where additional correction functions such as h were needed. However, the previous case (where LTB conditions were used instead of the diffeomorphism constraint) implicitly makes h dependent on $(E^x)'$ as well: Comparing the correction terms we have

$$-\frac{\kappa'}{\sqrt{1+\kappa}}\alpha h = 2\bar{\alpha}_\Gamma\Gamma'_\varphi = -2\bar{\alpha}_\Gamma\left(\frac{1}{2}\frac{\kappa'}{\sqrt{1+\kappa}}g[E^x] + \sqrt{1+\kappa}\frac{{\rm d}g[E^x]}{{\rm d}E^x}(E^x)'\right)$$

and we can write, using $g=\alpha$:

$$h = \bar{\alpha}_\Gamma + 2\frac{1+\kappa}{\kappa'}\frac{{\rm d}\log\alpha}{{\rm d}E^x}(E^x)'\,.$$

Thus, to match the current equations the correction function h used earlier must depend on $(E^x)'$, which has a density weight. (Similar considerations apply to the correction function l.) As the expression demonstrates, this is made possible since in our earlier procedure we had the function κ at our disposal in addition to the triad components. Its derivative κ' provides an extra density, which can be combined with $(E^x)'$ to provide a scalar correction function. In the current setting, by contrast, we have not yet introduced any such function by LTB conditions, and so a possible dependence on $(E^x)'$ is more restricted. Imposing anomaly freedom in the unreduced constraint algebra first is clearly less ambiguous, while the final results will be very close. This again demonstrates the robustness.

To continue with the analysis of anomaly-freedom, the other bracket (2.37):

$$\begin{aligned}\{H^Q_{\rm grav}[N], D_{\rm grav}[N^x]\} &= -H^Q_{\rm grav}[N^x N'] \\ &- \frac{1}{2G}\int {\rm d}x\, N(N^x)'E^\varphi\Big(\frac{\partial\alpha}{\partial E^\varphi}|E^x|^{-\frac{1}{2}}K_\varphi^2 E^\varphi + 2s\frac{\partial\bar\alpha}{\partial E^\varphi}K_\varphi K_x|E^x|^{\frac{1}{2}} \\ &+ \frac{\partial\alpha}{\partial E^\varphi}|E^x|^{-\frac{1}{2}}E^\varphi - \frac{\partial\alpha}{\partial E^\varphi}|E^x|^{-\frac{1}{2}}\Gamma_\varphi^2 E^\varphi + 2s\frac{\partial\bar\alpha}{\partial E^\varphi}\Gamma'_\varphi|E^x|^{\frac{1}{2}}\Big)\,. \end{aligned} \tag{4.54}$$

gives conditions (2.38). In the case $\bar\alpha = \alpha$,

$$\{H^Q_{\rm grav}[N], D_{\rm grav}[N^x]\} = -H^Q_{\rm grav}[N^x N' - (\partial\log\alpha/\partial E^\varphi)E^\varphi N(N^x)']\,.$$

The corrected Hamiltonian $H^Q_{\rm grav}$ transforms as a scalar only if α is independent of E^φ since E^φ is the only basic quantity of density weight one. However, the vacuum algebra is first class even if α depends on E^φ. In contrast, when $\bar\alpha = 1$ (or more generally $\bar\alpha \neq \alpha$), α must be independent of E^φ. (As discussed in chapter 3, the case $\alpha = \bar\alpha$ in vacuum is special because any such correction could be absorbed in the lapse function, making the algebra formally first class.)

In summary, for corrections α (and $\bar\alpha$) independent of E^φ we have

$$\{H^Q_{\rm grav}[N], D_{\rm grav}[N^x]\} = -H^Q_{\rm grav}[N^x N']$$

and

$$\begin{aligned}\{H^Q_{\rm grav}[M], H^Q_{\rm grav}[N]\} &= D_{\rm grav}[\bar\alpha^2|E^x|(E^\varphi)^{-2}(MN' - NM')] \\ &\quad -G_{\rm grav}[\bar\alpha^2|E^x|(E^\varphi)^{-2}(MN' - NM')\eta']\end{aligned} \tag{4.55}$$

To proceed, we will include matter in the form of dust as it is assumed in LTB models.

4.3.2 Dust

For a full consistency analysis based on the constraint algebra we have to use a dynamical formulation of the dust matter source, rather than a phenomenological implementation via the dust profile $F(x)$. It is convenient to use a canonical formulation for dust with stress-energy tensor $T_{\alpha\beta} = \epsilon\, U_\alpha U_\beta$ as developed in [91]. The dust four-velocity is given by the Pfaff form $U_\alpha = -\tau,_\alpha + W_k Z^k,_\alpha$, where as canonical coordinates the dust proper time τ and comoving dust coordinates Z^k with $k = 1, 2, 3$ appear. Their respective conjugate momenta will be called P and P_k. Matter contributions to the diffeomorphism and Hamiltonian constraint read

$$\begin{aligned} D_{\rm dust}[N^a] &= \int \mathrm{d}^3x\, N^a \tilde{D}_a = \int d^3x\, N^a (P\tau,_a + P_k Z^k,_a) \\ H_{\rm dust}[N] &= \int \mathrm{d}^3x\, N \sqrt{P^2 + q^{ab}\tilde{D}_a\tilde{D}_b}\,. \end{aligned} \tag{4.56}$$

Imposing spherical symmetry and using adapted coordinates $\Phi := Z^1$, $Z^2 = \vartheta$, $Z^3 = \varphi$ the constraints become

$$\begin{aligned} D_{\rm dust}[N^x] =& 4\pi \int \mathrm{d}x\, N^x\, (P_\tau \tau' + P_\Phi \Phi') \\ H_{\rm dust}[N] =& 4\pi \int \mathrm{d}x\, N \sqrt{P_\tau^2 + \frac{|E^x|}{(E^\varphi)^2}(P_\tau\tau' + P_\Phi\Phi')^2} \end{aligned} \tag{4.57}$$

with the remaining canonical pairs

$$\{\tau, P_\tau\} = \{\Phi, P_\Phi\} = \frac{1}{4\pi}$$

whose momenta P_τ and P_Φ are defined by the relations $P = P_\tau \sin\vartheta$ (in terms of the P of the full 3-dimensional theory) and $P_\Phi = -P_\tau W_1$.

For non-rotating dust, as must be the case with spherical symmetry, the constraints $P_k = 0$ can be imposed by requiring that the dust motion be described with respect to the frame orthogonal foliation, so that the state does not depend on the frame variables Z^k. As a result P_Φ is usually taken to be zero. However, we will not choose to do so until we try to solve the equations of motion.

From the form of the Hamiltonian (4.56) in the full theory and the relation $q^{ab} =$

$(\det E^c_k)^{-1}E^a_iE^b_i$, we can expect quantum corrections $\beta[E^x,E^\varphi]$ from a quantization of inverse triads inside the square root:

$$H^Q_{\rm dust}[N]=4\pi\int {\rm d}x\, N\sqrt{P_\tau^2+\beta\frac{|E^x|}{(E^\varphi)^2}(P_\tau\tau'+P_\Phi\Phi')^2}\,.$$

Also here, the form of β will be restricted by the requirement of anomaly freedom.

Adding the individual contributions, the diffeomorphism and corrected Hamiltonian constraint for the gravity-dust system are $D[N^x]=D_{\rm grav}[N^x]+D_{\rm dust}[N^x]$ and $H^Q[N]=H^Q_{\rm grav}[N]+H^Q_{\rm dust}[N]$. Now, the Poisson bracket for the matter part of the Hamiltonian with the diffeomorphism constraint is

$$\{H^Q_{\rm dust}[N],D[N^x]\}=-H^Q_{\rm dust}[N^xN']+\int {\rm d}x\, NN^{x\,\prime}\frac{\partial\beta}{\partial E^\varphi}\frac{|E^x|}{2E^\varphi}\frac{\tilde{D}_x^2}{\sqrt{P_\tau^2+\beta|E^x|(E^\varphi)^{-2}\tilde{D}_x^2}}$$

with $\tilde{D}_x:=P_\tau\tau'+P_\Phi\Phi'$. The closure of $\{H^Q[N],D[N^x]\}$ consistently imposes the condition that α and β be independent of E^φ, upon which

$$\{H^Q[N],D[N^x]\}=-H^Q[N^xN']\,.$$

Finally,

$$\begin{aligned}\{H^Q[N],H^Q[M]\}=&\{H^Q_{\rm grav}[M],H^Q_{\rm grav}[N]\}+\{H^Q_{\rm dust}[M],H^Q_{\rm dust}[N]\}\\ =&D_{\rm grav}[\bar{\alpha}^2|E^x|(E^\varphi)^{-2}(MN'-NM')]\\ &-G_{\rm grav}[\bar{\alpha}^2|E^x|(E^\varphi)^{-2}(MN'-NM')\eta']\\ &+\ D_{\rm dust}[\beta|E^x|(E^\varphi)^{-2}(MN'-NM')]\end{aligned}$$

gives the relation

$$\beta=\bar{\alpha}^2\,. \tag{4.58}$$

Note that the presence of matter makes this consistent deformation of the classical constraint algebra non-trivial: corrections can no longer be absorbed in the lapse function. We also point out that a deformation of the constraint algebra is required to implement the corrections consistently. This seems to be an interesting difference to a reduced phase space quantization which is possible in this class of models based on a deparameterization [92].

4.3.3 LTB-like solutions

Using the transformation equations (4.8), the quantum corrected Hamiltonian in ADM variables (4.6) reads

$$H^Q[N] = \frac{1}{G}\int \mathrm{d}x\, N\Bigg[-\bar{\alpha}\frac{P_L P_R}{R} + (2\bar{\alpha}-\alpha)\frac{LP_L^2}{2R^2} - \alpha\frac{L}{2} - (2\bar{\alpha}-\alpha)\frac{R'^{\,2}}{2L} + \bar{\alpha}\left(\frac{RR'}{L}\right)' + 4\pi G P_\tau\left(1+\beta\frac{\tau'^{\,2}}{L^2}\right)^{\frac{1}{2}}\Bigg]$$

and the diffeomorphism constraint is

$$D[N^x] = \frac{1}{G}\int \mathrm{d}x\, N^x(R'P_R - LP_L' + 4\pi G P_\tau \tau')$$

where we have already used $P_\Phi = 0$. Paralleling our treatment of LTB-reduced constraints our further analysis will be split into two different cases of correction functions. We choose to work here in ADM variables instead of triad variables, but of course identical results follow using the latter.

4.3.3.1 Case $\bar{\alpha} = \alpha$

The equations of motion $\dot{R} = \{R, H^Q[N] + D[N^x]\}$, $\dot{L} = \{L, H^Q[N] + D[N^x]\}$, $\dot{P}_R = \{P_R, H^Q[N] + D[N^x]\}$, $\dot{P}_L = \{P_L, H^Q[N] + D[N^x]\}$, $\dot{\tau} = \{\tau, H^Q[N] + D[N^x]\}$ and $\dot{P}_\tau = \{P_\tau, H^Q[N] + D[N^x]\}$ are respectively,

$$P_L = \frac{R}{\alpha N}\left(-\dot{R} + N^x R'\right) \tag{4.59}$$

$$P_R = \frac{1}{\alpha N}\left(-L\dot{R} - \dot{L}R + (N^x RL)'\right) \tag{4.60}$$

$$\begin{aligned}\dot{P}_R = {}& -N\alpha\left(\frac{P_L P_R}{R^2} - \frac{LP_L^2}{R^3}\right) - N\frac{\mathrm{d}\alpha}{\mathrm{d}R}\left(-\frac{P_L P_R}{R} + \frac{LP_L^2}{2R^2} - \frac{L}{2}\right)\\ & - \left(N\alpha\frac{R'}{L}\right)' + N'\alpha\frac{R'}{L} - \left(N'\alpha\frac{R}{L}\right)' + N'\frac{\mathrm{d}\alpha}{\mathrm{d}R}\frac{RR'}{L}\\ & + \left(\frac{\mathrm{d}^2\alpha}{\mathrm{d}R^2}R + \frac{\mathrm{d}\alpha}{\mathrm{d}R}\right)\frac{NR'^{\,2}}{L} - \left(2N\frac{\mathrm{d}\alpha}{\mathrm{d}R}\frac{RR'}{L}\right)' + (N^x P_R)'\end{aligned}$$

$$- 2\pi G N P_\tau \left(1+\beta\frac{\tau'^{\,2}}{L^2}\right)^{-\frac{1}{2}} \frac{\tau'^{\,2}}{L^2}\frac{\mathrm{d}\beta}{\mathrm{d}R} \tag{4.61}$$

$$\dot{P}_L = N\alpha\left(\frac{1}{2}-\frac{P_L^2}{2R^2}-\frac{R'^{\,2}}{2L^2}\right) - (N\alpha)'\frac{RR'}{L^2} + N^x P_L'$$

$$+ 4\pi G N\beta P_\tau \frac{\tau'^{\,2}}{L^3}\left(1+\beta\frac{\tau'^{\,2}}{L^2}\right)^{-\frac{1}{2}} \tag{4.62}$$

$$\dot{\tau} = N\left(1+\beta\frac{\tau'^{\,2}}{L^2}\right)^{\frac{1}{2}} + N^x\tau' \tag{4.63}$$

$$\dot{P}_\tau = \left[N\beta\frac{P_\tau\tau'}{L^2}\left(1+\beta\frac{\tau'^{\,2}}{L^2}\right)^{-\frac{1}{2}} + N^x P_\tau\right]' . \tag{4.64}$$

To try to find an LTB-like solution, following [29] and the recent [92], we choose an embedding by coordinates such that $\tau = t$, or equivalently, from (4.63), $N = 1$. Setting $N^x = 0$, equation (4.64) becomes $\dot{P}_\tau = 0$ so that $P_\tau(x)$ is a function of the spatial coordinate only.

Substituting this and (4.59) with (4.60) in the diffeomorphism constraint

$$\frac{\delta D}{\delta N^x} = R'P_R - LP_L' + 4\pi G P_\tau \tau' = 0$$

gives the equation

$$\left(\frac{R'}{\alpha L}\right)^{\boldsymbol{\cdot}} = 0$$

or $R'/\alpha L = \mathcal{E}$, with $\mathcal{E}(x)$ an arbitrary function of the spatial coordinate only. To make contact with the classical LTB solution we may define κ by

$$\mathcal{E} = \sqrt{1+\kappa} \tag{4.65}$$

and obtain the corrected LTB condition

$$L = \frac{R'}{\alpha\mathcal{E}} .$$

Note that this is exactly the form of corrections used earlier in (4.21), which we now have derived from the corrected constrained system without extra assumptions about the possible form of LTB solutions.

With all this, the Hamiltonian constraint

$$\alpha\left[-\frac{P_L P_R}{R}+\frac{L P_L^2}{2R^2}-\frac{L}{2}-\frac{R'^{\,2}}{2L}+\left(\frac{RR'}{L}\right)'\right]+4\pi G P_\tau=0$$

becomes

$$\left[\frac{R\dot{R}^2}{\alpha^2}+R(1-\alpha^2\mathcal{E}^2)\right]'=8\pi G\mathcal{E}P_\tau\,. \tag{4.66}$$

Again, we may define F by the equation $8\pi G\mathcal{E}P_\tau = F'$, so finally the integrated Hamiltonian constraint reads

$$R\dot{R}^2=\alpha^2 F+R\alpha^2(\alpha^2\mathcal{E}^2-1)+c(t) \tag{4.67}$$

and the second order, evolution equation from (4.62) is

$$2R\ddot{R}+\dot{R}^2=2(\dot{R}^2+\alpha^4\mathcal{E}^2)\frac{\mathrm{d}\log\alpha}{\mathrm{d}\log R}-\alpha^2(1-\alpha^2\mathcal{E}^2)\,. \tag{4.68}$$

This is precisely the time derivative of (4.67), provided $c(t) = c =$ const, and shows consistency. In the limit $\alpha = 1$ we recover the classical LTB condition and evolution equation. In this limit the integration constant can be absorbed in F so we may set $c = 0$ here.

Even though the corrected LTB condition coincides with the one derived earlier, the first order and evolution equations are slightly different from the corresponding ones (4.37) and (4.40) derived before:

$$\left(\frac{R\dot{R}^2}{\alpha^2}\right)'+R'(1-\alpha^2\mathcal{E}^2)-2\mathcal{E}\mathcal{E}'\alpha h R=F' \tag{4.69}$$

which unlike (4.66) is not a spatial derivative, and the second order equation

$$2R\ddot{R}+\dot{R}^2=2\dot{R}^2\frac{\mathrm{d}\log\alpha}{\mathrm{d}\log R}-\alpha^2(1-\alpha^2\mathcal{E}^2)\,. \tag{4.70}$$

While several terms in the two sets of the equations agree, the spatial derivative of (4.67) differs from (4.69) by a term $2\mathcal{E}\mathcal{E}'\alpha R(h-\alpha)+2\mathcal{E}^2R\alpha\alpha'$, and (4.68) from

(4.70) by $2\alpha^4\mathcal{E}^2\mathrm{d}\log\alpha/\mathrm{d}\log R$. It is interesting to note that the more restrictive new method of this section provides a Hamiltonian constraint which can be spatially integrated.

4.3.3.2 Case $\bar{\alpha} = 1$

Similarly to the previous case, choosing $N = 1$ and $N^x = 0$ gives the equations of motion

$$P_L = -\, R\dot{R} \tag{4.71}$$

$$P_R = -\, (RL)^{\boldsymbol{\cdot}} - (1-\alpha)L\dot{R} \tag{4.72}$$

$$\dot{P}_L = \frac{\alpha}{2} + (2-\alpha)\left(-\frac{P_L^2}{2R^2} - \frac{R'\,2}{2L^2}\right) \tag{4.73}$$

$$\begin{aligned}\dot{P}_R = &\left(\frac{LP_L^2}{R^3} - \frac{P_L P_R}{R^2}\right) - \left(\frac{R'}{L}\right)' + \frac{\mathrm{d}\alpha}{\mathrm{d}R}\left(\frac{LP_L^2}{2R^2} + \frac{L}{2} - \frac{R'^{\,2}}{2L}\right) \\ &+ (1-\alpha)\frac{LP_L^2}{R^3} - \left((1-\alpha)\frac{R'}{L}\right)' .\end{aligned} \tag{4.74}$$

Substituting the first two equations in the diffeomorphism constraint gives

$$\left(\frac{R'}{L}\right)^{\boldsymbol{\cdot}} = (1-\alpha)\frac{R'\dot{R}}{RL} .$$

Inserting the ansatz (4.17)

$$L = \frac{R'}{\mathcal{E}f}$$

gives the same equation (4.18) for the function $f(E^x)$ as in the previous treatment:

$$R\frac{\mathrm{d}f}{\mathrm{d}R} = f(1-\alpha)\,. \tag{4.75}$$

Substituting equations (4.71) and (4.72) into the Hamiltonian constraint gives

$$(2-\alpha)R'\dot{R}^2 + 2R\dot{R}\dot{R}' - 2RR'\dot{R}\frac{\dot{f}}{f} + \alpha R' - \alpha R'\mathcal{E}^2 f^2 - R(\mathcal{E}^2 f^2)' = 8\pi G\mathcal{E} f P_\tau$$

or using (4.75) and defining $F' = 8\pi G\mathcal{E}P_\tau$

$$\alpha R'\dot{R}^2 + 2R\dot{R}\dot{R}' + \alpha R'(1-\mathcal{E}^2f^2) - R(\mathcal{E}^2f^2)' = fF' \tag{4.76}$$

which reproduces the classical equation in the limit $\alpha = f = 1$.

The second order equation from (4.73) is

$$2R\ddot{R} + \alpha\dot{R}^2 = -\alpha(1+\mathcal{E}^2f^2) + 2\mathcal{E}^2f^2 \,. \tag{4.77}$$

Also here, compared to (4.41) and (4.42) as obtained earlier:

$$\alpha R'\dot{R}^2 + 2R\dot{R}\dot{R}' + \alpha R'(1-\mathcal{E}^2f^2) - R(\mathcal{E}^2)'fl = fF'$$

$$2R\ddot{R} + \alpha\dot{R}^2 = -\alpha(1-\mathcal{E}^2f^2)\,,$$

the structure of the resulting equations remains similar up to a few extra terms.

As in sections 4.2.4 and 4.2.5 we may interpret the effects of correction terms using effective densities and energy functions.

4.4 Possibility of singularity resolution through bounces

We can now use the different sets of consistent equations to analyze properties of gravitational collapse, such as the formation of singularities. Classically we have the first order equation

$$\dot{R}^2 = \kappa + \frac{F}{R}\,. \tag{4.78}$$

Compared with the marginal case where $\kappa = 0$, there exists the possibility that $\dot{R} = 0$ even for positive mass functions F. However, to conclude whether there is a bounce or not we also need to look at the evolution equation

$$2R\ddot{R} + \dot{R}^2 = \kappa \tag{4.79}$$

and see whether we can have $\ddot{R} > 0$ in addition to $\dot{R} = 0$. From the first equation we see that $\dot{R} = 0$ implies $\kappa + F/R = 0$ and with both F and R positive we get

$\kappa < 0$; bounces would be possible only for negative κ. On the other hand, for $\dot{R} = 0$ the second order equation implies $2R\ddot{R} = \kappa$. Since $R > 0$ and $\kappa < 0$ we conclude that $\ddot{R} < 0$ and thus there is no bounce classically.

We would like to proceed in a similar manner for the quantum corrected case also. However, there the first order equation in time contains terms with spatial derivative as well (which cannot be integrated out in all cases). Therefore the analysis for the possibility of a bounce cannot necessarily be done as easily as for the classical case, a feature clearly related to the fact that we are dealing with inhomogeneous models. Furthermore, because of the inhomogeneous nature of the problem a bounce also makes the analysis difficult by the fact that after the bounce there will be the possibility of shell crossing (unless shells with larger values of x bounce at larger values of R).

We note that (4.37) and (4.41) imply, as for the classical case, that $\dot{R} = 0$ is possible in both versions of inverse triad corrections. Whether this corresponds to a bounce is what we want to analyze. We start by putting $\dot{R} = 0$ in (4.37) which gives

$$(1 - \alpha^2 - \kappa\alpha^2)R' - \kappa'\alpha h R = F' \,. \tag{4.80}$$

Similarly, the evolution equation (4.40) becomes

$$2R\ddot{R} = -\alpha^2(1 - \alpha^2 - \kappa\alpha^2) \,. \tag{4.81}$$

Using (4.80) on the right of the above equation we have

$$\ddot{R} = -\alpha^2 \frac{F' + \kappa'\alpha h R}{2RR'} \,. \tag{4.82}$$

For a bounce, $\ddot{R} > 0$ which implies that $-\alpha^2(F' + \kappa'\alpha h R)/2RR'$ should be greater than zero. We need to check whether this condition can be satisfied in the non-marginal case where, as mentioned before, there are two possibilities $\kappa > 0$ and $-1 < \kappa < 0$. In what follows we will assume that $R' > 0$, which locally around a potential bounce is a valid assumption even though a collapsing shell with $x = x_1$, say, would start expanding after it experiences a bounce: when $\dot{R}(t, x_1) = 0$ the radius of this particular shell is not changing with time. The assumption then essentially implies that a shell with $x = x_2 > x_1$, which may still be contracting, does not immediately catch up with the x_1 shell. With this assumption and because R and α are positive, the condition that (4.82) be positive becomes

$$\kappa' < -F'/\alpha h R \tag{4.83}$$

as a condition on κ' which needs to be satisfied for a bounce, implying in particular that for a bounce κ' has to be negative. Whether (4.83) can be satisfied generically is not clear and must be determined from a numerical analysis of the equations.

For the first version of the inverse triad correction, $\dot{R}=0$ in (4.41) gives

$$\alpha(1-f^2-\kappa f^2)R' - \kappa' f l R = fF' \,. \tag{4.84}$$

The evolution equation (4.42) becomes

$$2R\ddot{R} = -\alpha(1-f^2-\kappa f^2) \tag{4.85}$$

and using (4.84) on the right we get

$$\ddot{R} = -f\frac{F'+\kappa' l R}{2RR'} \,. \tag{4.86}$$

This should be greater than zero for a bounce. Again because of the presence of spatial derivatives in the above expressions it is difficult to say whether there is a bounce in general and whether or not the singularity can be avoided.

With (4.67) we have a corrected equation which can be spatially integrated, allowing an analysis similar to the classical one. The condition $\dot{R}=0$ at a bounce implies

$$R = \frac{F}{1-\alpha^2\mathcal{E}^2} = \frac{F}{1-\alpha^2-\kappa\alpha^2}$$

which is positive provided $\mathcal{E} < 1/\alpha$. The second derivative

$$2R\ddot{R}|_{\dot{R}=0} = 2\alpha^4\frac{\mathrm{d}\log\alpha}{\mathrm{d}\log R}\mathcal{E}^2 - \alpha^2(1-\alpha^2\mathcal{E}^2)$$

can be positive under this condition only if the derivative $\mathrm{d}\log\alpha/\mathrm{d}\log R$ is sufficiently positive. This is not the case in semiclassical regimes (for geometries to the right of the peak in Fig. 4.1), where bounces are thus prohibited. The correction function is increasing to the left of the peak, which is a regime with strong quantum geometry corrections. Since we have not included all quantum corrections, a conclusion of a bounce in this regime would be unreliable. The only semiclassical option for a bounce is to have a geometry above the peak of inverse triad corrections, but have decreasing patch sizes Δ which appear as the argument of α. Thus, the number $\mathcal{N}$ of patches would have to increase sufficiently rapidly. In this regime, we have $\alpha > 1$ and thus $\mathcal{E} < 1$ by our condition for $R > 0$. Such a bounce would thus be possible only for $\kappa < 0$. (As the argument shows, without lattice refinement a bounce from inverse triad corrections would at best be possible only in the strong quantum regime.)

While bounces seem possible in the present situation, they cannot be considered generic. They require a regime where the patch number is increasing sufficiently rapidly in such a way that the patch size decreases. Since in the discrete quantum geometry of loop quantum gravity the patch size has a positive lower bound, the patch number of an orbit of fixed size cannot increase arbitrarily. Tuning then seems required to have the right behavior just when a shell is about to bounce.

4.4.1 Near center analysis regarding bounces

On the basis of a general analysis it seems difficult to conclude whether quantum corrections generically resolve the singularity in non-marginal LTB models through bounces. The simplest possibilty seems to be one where the central shell is prevented from becoming singular because of a bounce. For almost complete collapse, we should expect the relevant regime to be one of small R. In this case the subsequent study of the outer shells will become difficult due to possible shell crossings, but presumably these outer shells will not become singular either. We therefore now proceed to a near center analysis.

As in the marginal case of [21], we use techniques similar to those in [93] and assume that near the center of the dust cloud we can expand $R(t,x)$ as

$$R(t,x) = R_1(t)x + R_2(t)x^2 + \cdots \tag{4.87}$$

For the classical collapse the mass function can be expanded as

$$F(x) = F_3 x^3 + F_4 x^4 + \cdots \tag{4.88}$$

Substituting the expansion for $R(t,x)$ and for $F(x)$ in the classical first order equation $\dot{R}^2 R = \kappa R + F$, we find that the lowest order term on the left (as also the second term on the right) of the above equation goes as x^3. This suggests that the series expansion for the energy function $\kappa(x)$ should be

$$\kappa(x) = \kappa_2 x^2 + \kappa_3 x^3 + \cdots \tag{4.89}$$

It also implies that at $x = 0$, the center of the cloud, $\kappa(x) = 0$ and therefore for the case where $\kappa(x) > 0$ outside $x = 0$, κ_2 should be greater than zero. On the other hand, for $-1 < \kappa(x) < 0$, κ_2 should be less than zero. However, if we consider our effective κ as in (4.49) then we can have lower order terms in κ. Since the lowest order term in α is of order x^3 and that in f is of order x we can have the lowest order term in κ behave as x^{-3}. This would then imply that at the center κ blows up whereas classically for negative κ we have the condition $-1 < \kappa < 0$.

With this caveat we now consider (4.84) to lowest order after using various series expansions:

$$c_1 R_1^3 x^3 \left(1 - c_2^2 R_1^2 x^2 - \frac{c_2^2 \kappa_{-3} R_1^2}{x}\right) R_1 + 3 c_2 c_3 \kappa_{-3} R_1^6 x^2 = 2 c_2 F_2 R_1 x^2 \,. \tag{4.90}$$

Here c_1, c_2, c_3 and κ_{-3} are the coefficients of the lowest order terms in the expansion of α, f, l and κ, respectively. Here we assume the orbital vertex number $\mathcal{N}$ to be nearly constant around the center. (This gives rise to the strongest effect from inverse triad corrections and allows direct comparisons with the matching results from [83].) The lowest order term in F is F_2 instead of F_3 because of the effective density correction. For $x \approx 0$ the first two terms in the parenthesis on the left can be ignored compared to the third term and thus the above equation implies that $\dot{R}_1 = 0$ for

$$\kappa_{-3} = \frac{2 c_2 F_2}{(3 c_2 c_3 - c_1 c_2^2) R_1^5} \,. \tag{4.91}$$

It turns out that $3c_2c_3 - c_1c_2^2 < 0$ implying that $\dot{R}_1$ can be zero only for $\kappa_{-3} < 0$

as in the classical case. If we now look at (4.85) near the center we find that the condition for a bounce is

$$\kappa_{-3} > \frac{2F_2}{3c_3R_1^5} \tag{4.92}$$

which means that, as in the classical case, κ should be positive implying that we do not have a bounce for the central shell.

For version two the expression for the effective density (4.48) and the effective energy function (4.50), respectively, imply that to lowest order the mass function can behave as x^{-3} and the energy function as x^{-10}. To look at the possibility of a bounce at the center of the cloud consider (4.80) to lowest order:

$$R_1 - c_1^2R_1^7x^6 - \frac{\kappa_{-10}c_1^2R_1^7}{x^4} + \frac{10\kappa_{-10}c_1c_4R_1^7}{x^4} = -\frac{3F_{-3}}{x^4}\,. \tag{4.93}$$

Here c_4, F_{-3} and κ_{-10} are the coefficients in the expansion of h, F and κ respectively. Ignoring the first two terms for $x \approx 0$, the above equation implies

$$\kappa_{-10} = -\frac{3F_{-3}}{(10c_1c_4 - c_1^2)R_1^7} \tag{4.94}$$

as the condition for $\dot{R}_1 = 0$. We note that the denominator here is positive implying that $\kappa_{-10} > 0$ if $F_{-3} < 0$ (negative mass function near the center) and $\kappa_{-10} < 0$ if $F_{-3} > 0$ (positive but decreasing mass function near the center). None of these behaviors could occur classically; either negative total energy ($F < 0$) or a negative density ($F' < 0$) would be required. To see if the above condition implies a bounce we use (4.81) to lowest order and find

$$\kappa_{-10} > -\frac{3F_{-3}}{10c_1c_4R_1^7} \tag{4.95}$$

as the condition for getting a bounce. This means that for $F_{-3} < 0$, κ_{-10} has to be positive (in agreement with the condition for $\dot{R}_1 = 0$ found above) implying that a bounce for the central shell is possible if the above inequality is satisfied. For $F_{-3} > 0$, κ_{-10} has to be greater than a negative number and thus if it is positive then a bounce again seems possible.

4.5 Collapse behavior near the center

Using (4.87), (4.88) and (4.89) we see that to lowest order in x (which is x^3) the classical equation $\dot{R}^2 R = \kappa R + F$ implies

$$\mathrm{d}t = \pm \frac{\mathrm{d}R_1}{\sqrt{\kappa_2 + \frac{F_3}{R_1}}} . \tag{4.96}$$

This has the solution (choosing the minus sign which corresponds to collapse)

$$t = -\frac{R_1\sqrt{\kappa_2 + \frac{F_3}{R_1}}}{\kappa_2} + \frac{F_3}{2\kappa_2^{3/2}} \log(F_3 + 2\kappa_2 R_1 + 2\sqrt{\kappa_2}\sqrt{\kappa_2 + \frac{F_3}{R_1}} R_1) \tag{4.97}$$

for $\kappa_2 > 0$ and

$$t = \frac{R_1\sqrt{\frac{F_3}{R_1} - |\kappa_2|}}{\kappa_2} + \frac{F_3}{2|\kappa_2|^{3/2}} \arctan\left[\frac{\sqrt{\frac{F_3}{R_1} - |\kappa_2|}(2|\kappa_2|R_1 - F_3)}{2\sqrt{|\kappa_2|}(\kappa_2 R_1 - F_3)}\right] \tag{4.98}$$

for $\kappa_2 < 0$.

We now proceed with a similar analysis for the first version of the inverse triad correction. Near the center the various quantities (α, f, l) behave as

$$\alpha = \left(\frac{2}{\gamma l_{\mathrm{P}}^2}\right)^{3/2} R_1^3 x^3 \quad , \quad f = \sqrt{\frac{8e^{1-\pi/2}}{\gamma l_{\mathrm{P}}^2}} R_1 x \quad , \quad l = \frac{1}{5}\left(\frac{2}{\gamma l_{\mathrm{P}}^2}\right)^{3/2} \sqrt{\frac{8e^{1-\pi/2}}{\gamma l_{\mathrm{P}}^2}} R_1^4 x^4 \tag{4.99}$$

where the way the near center behavior for l has been determined is described around Eq. (4.30). In what follows we will denote the constants appearing in the expansion of (α, f, l) by (c_1, c_2, c_3) respectively. Thus substituting the series expansions in (4.41) we get

$$c_1(1 - c_2^2 R_1^2 x^2 - \kappa_2 c_2^2 R_1^2 x^4) R_1^4 x^3 + c_1 \dot{R}_1^2 R_1^4 x^5 + 2\dot{R}_1^2 R_1 x^2 - 2\kappa_2 c_2 c_3 R_1^6 x^7 = 3c_2 F_3 R_1 x^3 \tag{4.100}$$

We now consider three different possibilities.

Case I: No correction to the expansion of F and κ

If we work with (4.100) directly then we find that the lowest order term on the left hand side is $2\dot{R}_1^2 R_1 x^2$ and the lowest order term on the right hand side goes as x^3 implying $\dot{R}_1 = 0$.

Case II: Modification to the expansion of F and no modification to κ

However because of the presence of an extra factor of x on the RHS we can start the expansion of F with the leading term behaving as x^2. In this case the lowest order term on both the LHS and RHS are of order x^2 and we get

$$2\dot{R}_1^2 R_1 x^2 = 2c_2 F_2 R_1 x^2 \tag{4.101}$$

which has the solution (for collapsing dust cloud)

$$R_1(t) = 1 - \sqrt{c_2 F_2}(t - t_0) \tag{4.102}$$

where we choose the initial condition $R_1(t = t_0) = 1$. We see that the central singularity $R_1 = 0$ forms in a finite time $t_s = (1 + \sqrt{c_2 F_2} t_0)/\sqrt{c_2 F_2}$.

Case III: Modifications to the series expansion of F and κ

There is a third option which is suggested by the possibility of a corrected energy function as discussed earlier. If we consider this correction then the lowest order term in the expansion of κ goes as κ_{-3}/x^3. With this included the matching of lowest order terms in (4.100) gives

$$-\kappa_{-3} c_1 c_2^2 R_1^5 + 2\dot{R}_1^2 + 3\kappa_{-3} c_2 c_3 R_1^5 = 2c_2 F_2 \tag{4.103}$$

and thus

$$\mathrm{d}t = -\frac{\sqrt{2}\mathrm{d}R_1}{\sqrt{2c_2 F_2 + (\kappa_{-3} c_1 c_2^2 - 3\kappa_{-3} c_2 c_3) R_1^5}} \tag{4.104}$$

with the solution

$$t = -\frac{\sqrt{2} R_1 \sqrt{1 + \frac{(\kappa_{-3} c_1 c_2^2 - 3\kappa_{-3} c_2 c_3) R_1^5}{2c_2 F_2}}\, {}_2F_1\left(\frac{1}{5}, \frac{1}{2}, \frac{6}{5}, -\frac{(\kappa_{-3} c_1 c_2^2 - 3\kappa_{-3} c_2 c_3) R_1^5}{2c_2 F_2}\right)}{\sqrt{2c_2 F_2 + (\kappa_{-3} c_1 c_2^2 - 3\kappa_{-3} c_2 c_3) R_1^5}} + c_0 \tag{4.105}$$

where ${}_2F_1(a,b;c;x)$ is the hypergeometric function and where c_0 is constant of integration.

4.6 Homogeneous limit and Minkowski space

In the classical case, the first order equation is $\dot{R}^2 R = \kappa(x) R + F(x)$ with corresponding expression for the mass function $F' = 8\pi G R^2 R' \epsilon$. This allows isotropic space-times as special solutions. We use the ansatz $R(t,x) = a(t)x$ with the condition that at time $t = 0$, $a(0) = a_0$ and assume that the density $\epsilon = \epsilon_0$ is a constant and that the energy function goes as $\kappa = -kx^2$ where k is a constant. When used in the first order equation, this gives

$$\dot{a}^2 a = -ka + \frac{8\pi G\epsilon_0}{3} \tag{4.106}$$

which is the Friedmann equation.

We would now like to see if we can get a Friedmann-like solution within the LTB class with inverse triad corrections included. This would indicate whether there can be an effective geometry of the classical homogeneous form, although the notion of homogeneity itself might change on a quantum space-time. Using the ansatz for R and the assumed form for the mass function and the energy function we find that for the first version of the inverse triad correction (4.41) we get

$$\frac{\alpha(1-f^2)}{f}a + k\alpha f^2 a x^2 + \frac{\alpha \dot{a}^2 a x^2}{f} + \frac{2a\dot{a}^2 x^2}{f} + 2klax^2 = 8\pi G\epsilon_0 x^2\,. \tag{4.107}$$

Although the resulting expression is not as simple as in the marginal case, we can see that Friedmann like solutions are not possible since x^2 does not cancel from the first term while a is allowed to depend only on t.

The second version of the inverse triad correction (4.37) gives

$$(1-\alpha^2+k\alpha^2 x^2)a + \frac{3\dot{a}^2 a x^2}{\alpha^2} - \frac{2\dot{a}^2 a^2}{\alpha^3}\frac{\mathrm{d}\alpha}{\mathrm{d}(ax)} + 2k\alpha h a x^2 = 8\pi G\epsilon_0 x^2\,. \tag{4.108}$$

Again the Friedmann solution is prohibited. Since the first term, which spoils the homogeneous limit, is the same for an analysis based on (4.67), there is no homogeneous limit in that case, either. One can also see from the first term that no other x-dependent κ, which might implement quantum corrections to the notion of homogeneity, can resolve the non-existence of homogeneous effective geometries subject to our equations.

The homogeneous limit, as a special case, would also include Minkowski space as the vacuum solution. Classically the first order equation for $F = 0$ and $\kappa = 0$, implies that $R \equiv R(x)$ and we recover the Minkowski spacetime. However from (4.41) we see that after choosing the mass function and the energy function equal to zero the equation becomes

$$\alpha(1 - f^2)R' + \alpha\dot{R}^2 R' + 2\dot{R}\dot{R}'R = 0 \tag{4.109}$$

which, due to the presence of the first term, implies that R will be dependent on both (t, x). Even though the equation has the correct classical limit, it is not straight-forward to see how the time dependence of R should disappear in the Minkowski limit.

For the second version (4.37) gives

$$(1 - \alpha^2)R' + \frac{\dot{R}^2 R'}{\alpha^2} + \frac{2R\dot{R}\dot{R}'}{\alpha^2} - \frac{2R\dot{R}^2 R'}{\alpha^3}\frac{\mathrm{d}\alpha}{\mathrm{d}R} = 0 \tag{4.110}$$

which again implies that even though in the classical limit we do have a Minkowski solution, there is still time dependence in R in corrected solutions.

Strong corrections are suggested at small values of the argument of α, which, given that R determines that value, may seem unacceptable because the center in Minkowski space is not physically distinguished. However, the radius R and thus the center is directly relevant for the size of corrections only if there is no lattice refinement in which case the only parameter which α depends on is R. The primary argument of α is, however, the size Δ of discrete patches rather than R^2 of whole spherical orbits. With a non-trivial refinement scheme, $\alpha(R^2/\mathcal{N})$ will also depend on the number of vertices per orbit, which for a good semiclassical state must provide a more uniform distribution of quantum corrections not distinguishing a center: $\mathcal{N}$ must be small when R^2 is small. If discrete patch sizes on all orbits are nearly similar, quantum corrections are uniform and do not distinguish a center. A detailed discussion would go beyond the scope of this paper, but one can already see the crucial role played by lattice refinement for the correct semiclassical limit.

4.7 Conclusions

We have extended the treatment of [21] to nonmarginal models, where additional corrections from spin connection terms arise. With these additional terms the original derivation appears more arbitrary, which led us to provide an independent derivation of equations corrected by the treatment of inverse triads in loop quantum gravity. In this derivation, anomaly-freedom is implemented first and LTB conditions are imposed afterwards to select a special class of solutions. The structure of the resulting equations is very similar in both derivations, showing the robustness. In details, however, the resulting equations do differ which is always possible due to quantization ambiguities. The effects analyzed in this work do not appear to depend sensitively on the method, but further analysis may well provide restrictions on acceptable equations, and thus on quantization ambiguities.

Our analysis in this chapter has been done for inverse triad corrections, while holonomy corrections, which to some degree were treated in [21], are technically more involved. Already for inverse triad corrections, the extension provided here is an interesting step in the analysis of inhomogeneous collapse and singularities. Comparing with homogeneous models and matching results of [83] would suggest easy resolutions of singularities by bounces. Marginal models are not entirely conclusive in this regard since their homogeneous analog is that of a spatially flat Friedmann–Robertson–Walker model which under inverse triad corrections gives rise to bounces only with a negative matter potential [94]. In the collapse analysis, however, we have used positivity conditions for the mass function which indicate that bounces in marginal LTB models with inverse triad corrections should not be expected. For nonmarginal models, on the other hand, homogeneous special solutions with positive spatial curvature exist, which do show bounces with inverse triad corrections and positive matter terms [82]. One would thus expect nonmarginal models to result in bounces much more easily than marginal ones do.

This, however, is not the case: we mostly confirm the results found in marginal models where (i) bounces are not obvious and (ii) a homogeneous limit of quantum corrected solutions may not even exist. As for the first property, bounces seem somewhat easier to achieve than in marginal models, but in contrast to the expectation turn out to be hard to realize generically. Moreover, a complete analysis would have to involve an investigation of shell-crossing singularities which can be involved even classically. (See [21] for more discussions on this in marginal corrected models.) As for property (ii) about the homogeneous limit, one can evade ruling out a homogeneous limit at the dynamical level only if one assumes a quantum notion of symmetry which would imply effective isotropic space-time metrics different from classical Friedmann–Robertson–Walker models. This may well be expected, as indeed the deformed constraint algebra (4.55) shows that there is a corrected quantum space-time structure. It would be interesting to see how this influences space-time symmetries.

Finally, several issues discussed here involved the role of lattice refinement for the semiclassical limit. As treatable models between homogeneous ones and the full theory, LTB models turn out to be quite instructive. This should also be expected for an implementation at the state (rather than phenomenological) level which was started in [21] for marginal models but which we have not attempted here for nonmarginal models.

Appendix A

Poisson Algebra of Modified Constraints

Here we present an explicit and detailed calculation of the Poisson Algebra of quantum corrected constraints. The Gauss and diffeomorphism constraint remain unaltered:

$$G_{\rm grav}[\lambda] = \frac{1}{2G\gamma}\int {\rm d}x\, \lambda((E^x)' + P^\eta) \tag{A.1}$$

$$D_{\rm grav}[N^x] = \frac{1}{2G}\int {\rm d}x\, N^x \left(2E^\varphi K_\varphi' - \frac{1}{\gamma}A_x(E^x)' + \frac{1}{\gamma}\eta' P^\eta\right) \tag{A.2}$$

To account for different possibilities of quantum corrections we consider the general modified Hamiltonian constraint:

$$\begin{aligned} H^Q_{\rm grav}[N] = -\frac{1}{2G}\int {\rm d}x\, N\big(&\alpha\,|E^x|^{-\frac{1}{2}}E^\varphi f_1 + 2s\bar\alpha\,|E^x|^{\frac{1}{2}}f_2 + \alpha\,|E^x|^{-\frac{1}{2}}E^\varphi \\ &- \alpha_\Gamma\,|E^x|^{-\frac{1}{2}}E^\varphi\Gamma_\varphi^2 + 2s\bar\alpha_\Gamma\,|E^x|^{\frac{1}{2}}\Gamma_\varphi'\big)\,. \end{aligned} \tag{A.3}$$

The functionals $\alpha[E^x, E^\varphi]$ and $\bar\alpha[E^x, E^\varphi]$ account for possible corrections from the quantization of inverse triads coming from the factors $|E^x|^{-\frac{1}{2}}E^\varphi$ and $|E^x|^{\frac{1}{2}}$. As functions, they are assumed to depend only on triad variables E^x and E^φ but are otherwise unrestricted. Similarly $\alpha_\Gamma[E^x, E^\varphi]$ and $\bar\alpha_\Gamma[E^x, E^\varphi]$ depend only on the triad variables and incorporate corrections from inverse triads and the quantization of the spin connection.

To consider corrections from the use of holonomies instead of the connection, we use general functionals $f_1[K_\varphi, E^x, E^\varphi]$ and $f_2[A_x+\eta', K_\varphi, E^x, E^\varphi]$ depending on extrinsic curvature components K_φ and $\gamma K_\varphi = A_x + \eta'$.

All the correction functions here (except for f_2 which depends on η') are restricted to depend on phase space variables but NOT on their spatial derivatives.

The Poisson bracket of functions f and g on the phase space of spherically symmetric gravity in Ashtekar variables is

$$\{f,g\} = 2G\int \mathrm{d}x\bigg(\gamma\frac{\delta f}{\delta A_x}\frac{\delta g}{\delta E^x} + \frac{1}{2}\frac{\delta f}{\delta K_\varphi}\frac{\delta g}{\delta E^\varphi} + \gamma\frac{\delta f}{\delta \eta}\frac{\delta g}{\delta P^\eta} - \gamma\frac{\delta f}{\delta E^x}\frac{\delta g}{\delta A_x} - \frac{1}{2}\frac{\delta f}{\delta E^\varphi}\frac{\delta g}{\delta K_\varphi} - \gamma\frac{\delta f}{\delta P^\eta}\frac{\delta g}{\delta \eta}\bigg)\,. \tag{A.4}$$

We recall here the 'definition' of functional derivative

$$\frac{\delta F[N,q]}{\delta q(x)} := \frac{\delta(N\tilde{\mathcal{F}})}{\delta q}\bigg|_{q=q(x)} \tag{A.5}$$

of a functional

$$F[N,q] := \int \mathrm{d}x N\tilde{\mathcal{F}}(q(x), q'(x), q''(x), ...)$$

where $\frac{\delta(N\tilde{\mathcal{F}})}{\delta q}$ is the 'variational derivative' of $N\tilde{\mathcal{F}}$.

$$\frac{\delta(N\tilde{\mathcal{F}})}{\delta q} := N\frac{\partial\tilde{\mathcal{F}}}{\partial q} - \left(N\frac{\partial\tilde{\mathcal{F}}}{\partial q'}\right)' + \left(N\frac{\partial\tilde{\mathcal{F}}}{\partial q''}\right)'' - \dots \tag{A.6}$$

For an explicit calculation of the Poisson brackets of the constraints it is convenient to split the Hamiltonian constraint (A.3) into three terms:

$$H_0[N] := -\frac{1}{2G}\int \mathrm{d}x\, N(\alpha\,|E^x|^{-\frac{1}{2}}E^\varphi f_1 + \alpha\,|E^x|^{-\frac{1}{2}}E^\varphi)$$
$$H_A[N] := -\frac{1}{2G}\int \mathrm{d}x\, N(2s\bar{\alpha}\,|E^x|^{\frac{1}{2}}f_2)$$
$$H_\Gamma[N] := H_\Gamma^1[N] + H_\Gamma^2[N] + H_\Gamma^3[N]$$

with

$$H^1_\Gamma[N] := \frac{1}{2G}\int \mathrm{d}x\, N\,\alpha_\Gamma \frac{|E^x|^{-\frac{1}{2}}(E^{x\,\prime})^2}{4E^\varphi}$$
$$H^2_\Gamma[N] := \frac{1}{2G}\int \mathrm{d}x\, N\, s\bar{\alpha}_\Gamma \frac{|E^x|^{\frac{1}{2}}E^{x\,\prime\prime}}{E^\varphi}$$
$$H^3_\Gamma[N] := -\frac{1}{2G}\int \mathrm{d}x\, N\, s\bar{\alpha}_\Gamma \frac{|E^x|^{\frac{1}{2}}E^{x\,\prime}E^{\varphi\,\prime}}{E^{\varphi\,2}}$$

A.1 $\{H^Q_{\rm grav}[M], H^Q_{\rm grav}[N]\}$

First we compute $\{H^Q_{\rm grav}[M], H^Q_{\rm grav}[N]\}$. Calculations simplify using basic properties of the brackets. From linearity and anti-symmetry:

$$\begin{aligned}\{H^Q_{\rm grav}[M], H^Q_{\rm grav}[N]\} &= \{\,(H_0+H_A+H_\Gamma)[M]\,,\,(H_0+H_A+H_\Gamma)[N]\,\}\\ &= \{H_0[M],H_0[N]\}+\{H_A[M],H_A[N]\}+\{H_\Gamma[M],H_\Gamma[N]\}\\ &\quad+\{H_0[M],H_A[N]\}-\{H_0[N],H_A[M]\}+\{H_0[M],H_\Gamma[N]\}\\ &\quad-\{H_0[N],H_\Gamma[M]\}+\{H_A[M],H_\Gamma[N]\}-\{H_A[N],H_\Gamma[M]\}\end{aligned}$$

The first three brackets on the right hand side clearly vanish: since H_0 does not contain spatial derivatives of the basic variables all the terms in $\{H_0[M], H_0[N]\}$ are proportional to MN and cancel (the first three terms in (A.4) cancel with the last three). Because of the dependence on η', the only terms that would not cancel in $\{H_A[M], H_A[N]\}$ are $\frac{\delta H_A[M]}{\delta\eta}\frac{\delta H_A[N]}{\delta P^\eta}$ and the corresponding one with M and N exchanged, but these terms are zero since $H_A[N]$ does not depend of P^η. Similarly, $\{H_\Gamma[M], H_\Gamma[N]\}$ is zero because $H_\Gamma[N]$ only contains momentum variables.

Expressions of the form $\{f[M], g[N]\} - \{f[N], g[M]\}$ will not vanish if f (or g) contains spatial derivatives of the phase space variables and g (f) contains the corresponding conjugate variables. The pair $\{H_0[M], H_A[N]\} - \{H_0[N], H_A[M]\}$ cancels but the other two pairs do not:

$$\{H^Q_{\rm grav}[M], H^Q_{\rm grav}[N]\} = \{H_0[M], H_\Gamma[N]\} + \{H_A[M], H_\Gamma[N]\} - (M \leftrightarrow N)$$

To illustrate the procedure we compute the first pair explicitly:

$$\{H_0[M], H_\Gamma[N]\} - (M \leftrightarrow N) = \{H_0[M], H_\Gamma^3[N]\} - (M \leftrightarrow N)$$

and

$$\{H_0[M], H_\Gamma^3[N]\} = 2G \int \mathrm{d}x \, \frac{1}{2} \frac{\delta H_0[M]}{\delta K_\varphi(x)} \frac{\delta H_\Gamma^3[N]}{\delta E^\varphi(x)}$$

$$2G \frac{\delta H_0[M]}{\delta K_\varphi(x)} = M(x)\alpha |E^x(x)|^{-\frac{1}{2}} E^\varphi(x) \frac{\partial f_1}{\partial K_\varphi}\bigg|_{K_\varphi = K_\varphi(x)}$$
$$\frac{\delta H_\Gamma^3[N]}{\delta E^\varphi(x)} = \frac{1}{2G} \int \mathrm{d}x_2 \, N s \left[\frac{\delta}{\delta E^\varphi} \left(\bar{\alpha}_\Gamma \frac{|E^x|^{\frac{1}{2}} E^{x\,\prime}}{E^{\varphi\,2}} \right) E^{\varphi\,\prime} + \bar{\alpha}_\Gamma \frac{|E^x|^{\frac{1}{2}} E^{x\,\prime}}{E^{\varphi\,2}} \frac{\delta E^{\varphi\,\prime}}{\delta E^\varphi} \right]$$

the first derivative term in $\frac{\delta H_\Gamma^3[N]}{\delta E^\varphi(x)}$ gives terms proportional to N (to MN when inserted in the Poisson bracket) which will cancel with terms from the other bracket with $M \leftrightarrow N$. The second derivative term gives, after 'integration by parts'

$$N s \bar{\alpha}_\Gamma \frac{|E^x|^{\frac{1}{2}} E^{x\,\prime}}{E^{\varphi\,2}} \delta(x, x_2) \bigg|_{x_2 \in \partial B} - \left(N s \bar{\alpha}_\Gamma \frac{|E^x|^{\frac{1}{2}} E^{x\,\prime}}{E^{\varphi\,2}} \right)'$$

so the only term that will not cancel is $N'(x) s \bar{\alpha}_\Gamma \frac{|E^x(x)|^{\frac{1}{2}} E^{x\,\prime}(x)}{E^\varphi(x)^{\,2}}$ and there are no boundary terms left.

In summary, we may short cut using formula (A.6) noticing that the only term that contributes is $-N' \frac{\partial \tilde{\mathcal{F}}}{\partial q'}$ (and the terms $-(N' \frac{\partial \tilde{\mathcal{F}}}{\partial q''})' - N' \frac{\partial \tilde{\mathcal{F}}}{\partial q''}$ if there are second derivatives).

Applying these results

$$\{H_0[M], H_\Gamma[N]\} - (M \leftrightarrow N) = -\frac{1}{2G} \int \mathrm{d}x (MN' - NM') \alpha \bar{\alpha}_\Gamma \frac{s}{2} \frac{\partial f_1}{\partial K_\varphi} \frac{E^{x\,\prime}}{E^\varphi}$$

Similarly

$$\begin{aligned} \{H_0[M], H_\Gamma^1[N]\} - (M \leftrightarrow N) =& 2G \int \mathrm{d}x \, \gamma \frac{\delta H_A[M]}{\delta A_x(x)} \frac{\delta H_\Gamma^1[N]}{\delta E^x(x)} - (M \leftrightarrow N) \\ =& \frac{1}{2G} \int \mathrm{d}x (MN' - NM') s \bar{\alpha} \alpha_\Gamma \gamma \frac{\partial f_2}{\partial A_x} \frac{E^{x\,\prime}}{E^\varphi} \end{aligned}$$

$$\{H_0[M], H_\Gamma^2[N]\} - (M \leftrightarrow N) = 2G \int \mathrm{d}x\, \gamma \frac{\delta H_A[M]}{\delta A_x(x)} \frac{\delta H_\Gamma^2[N]}{\delta E^x(x)} - (M \leftrightarrow N)$$
$$= \frac{1}{2G} \int \mathrm{d}x\, (MN' - NM') \times$$
$$\times \left[2 \left(\bar{\alpha}\gamma \frac{\partial f_2}{\partial A_x} |E^x|^{\frac{1}{2}} \right)' \bar{\alpha}_\Gamma \frac{|E^x|^{\frac{1}{2}}}{E^\varphi} - 2\bar{\alpha}\gamma \frac{\partial f_2}{A_x} |E^x|^{\frac{1}{2}} \left(\bar{\alpha}_\Gamma \frac{|E^x|^{\frac{1}{2}}}{E^\varphi} \right)' \right]$$

$$\{H_0[M], H_\Gamma^3[N]\} - (M \leftrightarrow N) =$$
$$= 2G \int \mathrm{d}x\, \left(\gamma \frac{\delta H_A[M]}{\delta A_x(x)} \frac{\delta H_\Gamma^3[N]}{\delta E^x(x)} + \frac{1}{2} \frac{\delta H_A[M]}{\delta K_\varphi(x)} \frac{\delta H_\Gamma^3[N]}{\delta E^\varphi(x)} \right) - (M \leftrightarrow N)$$
$$= -\frac{1}{2G} \int \mathrm{d}x (MN' - NM') \bar{\alpha}\bar{\alpha}_\Gamma \left[2\gamma \frac{\partial f_2}{\partial A_x} E^{\varphi\,\prime} + \frac{\partial f_2}{\partial K_\varphi} E^{x\,\prime} \right] \frac{|E^x|}{E^{\varphi\,2}}$$

Putting everything together, we finally get

$$\{H^Q_{\mathrm{grav}}[M], H^Q_{\mathrm{grav}}[N]\} =$$
$$= \frac{1}{2G} \int \mathrm{d}x\, (MN' - NM') \Bigg[\bar{\alpha}\bar{\alpha}_\Gamma \frac{|E^x|}{E^{\varphi\,2}} \left(2\gamma \left(\frac{\partial f_2}{\partial A_x} \right)' E^\varphi - \frac{\partial f_2}{\partial K_\varphi} E^{x\,\prime} \right)$$
$$+ s \left(\bar{\alpha}\alpha_\Gamma \gamma \frac{\partial f_2}{\partial A_x} - \alpha\bar{\alpha}_\Gamma \frac{1}{2} \frac{\partial f_1}{\partial K_\varphi} \right) \frac{E^{x\,\prime}}{E^\varphi} + 2\left(\bar{\alpha}_\Gamma \bar{\alpha}' - \bar{\alpha}\bar{\alpha}_\Gamma' \right) \gamma \frac{\partial f_2}{\partial A_x} \frac{|E^x|}{E^\varphi} \Bigg] \qquad \text{(A.7)}$$

A.2 $\{H^Q_{\mathrm{grav}}[N], D_{\mathrm{grav}}[N^x]\}$

For general canonical variables in one dimension $\{S_i, T^i\} = \sigma(i)\delta_i^j$ with S_i a scalar and T^i of density one, and $\sigma(i) = \pm 1$, the canonical generator of diffeomorphisms has the form

$$D[N^x] = \int \mathrm{d}x N^x \sigma(i) T^i S_i'$$

Given a functional $F[N] := \int \mathrm{d}x\, N\tilde{\mathcal{F}}(S_i(x), T^i(x), S_i'(x), T^i{}'(x))$, using (A.6), it is easy to compute

$$\begin{aligned}\{F[N], D[N]\} =& \int \mathrm{d}x\, \sigma(i) \left(\frac{\delta F[N]}{\delta S_i(x)} \frac{\delta D[N^x]}{\delta T^i(x)} - \frac{\delta F[N]}{\delta T^i(x)} \frac{\delta D[N^x]}{\delta S_i(x)} \right) \\ =& \int \mathrm{d}x\, N^x \left(\frac{\delta F[N]}{\delta S_i(x)} S_i{}' + \frac{\delta F[N]}{\delta T^i(x)} T^i{}' \right) + \int \mathrm{d}x\, N^x{}' \frac{\delta F[N]}{\delta T^i(x)} T^i \\ =& - F[N^x N'] + \int \mathrm{d}x\, N N^x{}' \left(-\tilde{\mathcal{F}} + \frac{\partial \tilde{\mathcal{F}}}{\partial T^i} T^i + \frac{\partial \tilde{\mathcal{F}}}{\partial S_i'} S_i{}' + 2 \frac{\partial \tilde{\mathcal{F}}}{\partial T^i{}'} T^i{}' \right) \\ &+ \int \mathrm{d}x\, N N^x{}'' \frac{\partial \tilde{\mathcal{F}}}{\partial T^i{}'} T^i \end{aligned} \tag{A.8}$$

We may use this formula to find the action of $D_{\mathrm{grav}}[N^x]$ on each term of the modified Hamiltonian, except for $H_\Gamma^2[N]$ that contains second derivatives, and for which we can derive a similar formula or explicitly compute:

$$\begin{aligned}\{H_0[N], D[N^x]\} =& - H_0[N^x N'] \\ &- \frac{1}{2G} \int \mathrm{d}x\, N N^x{}' |E^x|^{-\frac{1}{2}} E^{\varphi\, 2} \left[\alpha \frac{\partial f_1}{E^\varphi} + \frac{\partial \alpha}{\partial E^\varphi} (f_1 + 1) \right]\end{aligned}$$

$$\begin{aligned}\{H_A[N], D[N^x]\} =& - H_A[N^x N'] \\ &- \frac{1}{2G} \int \mathrm{d}x\, 2s N N^x |{}' E^x|^{\frac{1}{2}} \times \\ &\times \left[\frac{\partial \bar{\alpha}}{\partial E^\varphi} E^\varphi f_2 + \bar{\alpha} \left(\frac{\partial f_2}{\partial E^\varphi} E^\varphi + \frac{\partial f_2}{\partial A_x} A_x + \frac{\partial f_2}{\partial \eta'} \eta' - f_2 \right) \right]\end{aligned}$$

$$\{H_\Gamma^1[N], D[N^x]\} = - H_\Gamma^1[N^x N'] - \frac{1}{2G} \int \mathrm{d}x\, N N^x{}' \frac{E^\varphi}{\alpha_\Gamma} \frac{\partial \alpha_\Gamma}{\partial E^\varphi} \tilde{\mathcal{H}}_\Gamma^1$$

$$\begin{aligned}\{H_\Gamma^2[N], D[N^x]\} =& - H_\Gamma^2[N^x N'] - \frac{1}{2G} \int \mathrm{d}x\, N N^x{}' \frac{E^\varphi}{\bar{\alpha}_\Gamma} \frac{\partial \bar{\alpha}_\Gamma}{\partial E^\varphi} \tilde{\mathcal{H}}_\Gamma^2 \\ &+ \frac{s}{2G} \int \mathrm{d}\, N N^x{}'' \bar{\alpha}_\Gamma \frac{|E^x|^{\frac{1}{2}} E^x{}'}{E^\varphi}\end{aligned}$$

$$\{H^3_\Gamma[N], D[N^x]\} = - H^3_\Gamma[N^x N'] - \frac{1}{2G}\int \mathrm{d}x\, N N^{x\,\prime} \frac{E^\varphi}{\bar{\alpha}_\Gamma}\frac{\partial \bar{\alpha}_\Gamma}{\partial E^\varphi}\tilde{\mathcal{H}}^3_\Gamma$$
$$- \frac{s}{2G}\int \mathrm{d}\, N N^{x\,\prime\prime}\bar{\alpha}_\Gamma \frac{|E^x|^{\frac{1}{2}}E^{x\,\prime}}{E^\varphi}$$

Collecting all terms:

$$\begin{aligned}
\{H^Q_{\rm grav}[N], D_{\rm grav}[N^x]\} &= -H^Q_{\rm grav}[N^x N'] \\
&-\frac{1}{2G}\int \mathrm{d}x\, N N^{x\,\prime} E^\varphi \Bigg[\frac{\partial \alpha}{\partial E^\varphi}|E^x|^{-\frac{1}{2}}E^\varphi f_1 + 2s\frac{\partial\bar{\alpha}}{\partial E^\varphi}|E^x|^{\frac{1}{2}} f_2 + \frac{\partial \alpha}{\partial E^\varphi}|E^x|^{-\frac{1}{2}}E^\varphi \\
&\qquad - \frac{\partial \alpha_\Gamma}{\partial E^\varphi}|E^x|^{-\frac{1}{2}}E^\varphi \Gamma_\varphi^2 + 2s\frac{\partial\bar{\alpha}_\Gamma}{\partial E^\varphi}|E^x|^{\frac{1}{2}}\Gamma_\varphi'\Bigg] \\
&-\frac{1}{2G}\int \mathrm{d}x\, N N^{x\,\prime} E^\varphi \Bigg[\alpha|E^x|^{-\frac{1}{2}}E^\varphi\frac{\partial f_1}{\partial E^\varphi} + 2s\bar{\alpha}|E^x|^{\frac{1}{2}}\frac{\partial f_2}{\partial E^\varphi} \\
&\qquad + 2s\bar{\alpha}\frac{|E^x|^{\frac{1}{2}}}{E^\varphi}\left(\frac{\partial f_2}{\partial A_x}A_x + \frac{\partial f_2}{\partial \eta'}\eta' - f_2\right)\Bigg]
\end{aligned} \tag{A.9}$$

A.3 Holonomy effects

We first analyse the effects of 'holonomies' of K_φ. we set

$$f_1 = F_1^2 \qquad \text{and} \qquad f_2 = F_2(A_x + \eta')/\gamma$$

where $F_2[K_\varphi, E^x, E^\varphi]$ only depends on extrinsic curvature K_φ and possibly on triad variables, and we keep the linear dependence of f_2 on $A_x + \eta'$. Classically $F_1 = F_2 = K_\varphi$.

Next we derive conditions on the correction functions for an anomaly free algebra. We re-write the bracket between Hamiltonians (A.7) which already gives the term proportional to the diffeomorphism constraint:

$$\{H^Q_{\rm grav}[M], H^Q_{\rm grav}[N]\} =$$

$$= \frac{1}{2G}\int \mathrm{d}x\,(MN' - NM')\Bigg[\,\bar{\alpha}\bar{\alpha}_\Gamma\frac{|E^x|}{E^{\varphi\,2}}\frac{\partial F_2}{\partial K_\varphi}\left(2E^\varphi K_\varphi' - \frac{(A_x+\eta')}{\gamma}E^{x\,\prime}\right)$$

$$+s\left(\bar{\alpha}\alpha_\Gamma F_2 - \alpha\bar{\alpha}_\Gamma F_1 \frac{\partial F_1}{\partial K_\varphi}\right)\frac{E^x{}'}{E^\varphi} + 2\left(\bar{\alpha}_\Gamma\bar{\alpha}' - \bar{\alpha}\bar{\alpha}_\Gamma'\right) F_2 \frac{|E^x|}{E^\varphi}$$

$$+ 2\bar{\alpha}\bar{\alpha}_\Gamma \frac{|E^x|}{E^\varphi}\left(\frac{\partial F_2}{\partial E^x}E^x{}' + \frac{\partial F_2}{\partial E^\varphi}E^\varphi{}'\right)\Bigg] \tag{A.10}$$

The last three summands cannot combine to give a term proportional to the Hamiltonian density or diffeomorphism constraint. They thus must vanish like in the classical case, giving the conditions

$$\left[s\bar{\alpha}\alpha_\Gamma - 2|E^x|\left(\bar{\alpha}_\Gamma\frac{\partial\bar{\alpha}}{\partial E^x} - \bar{\alpha}\frac{\partial\bar{\alpha}_\Gamma}{\partial E^x}\right)\right]F_2 + 2\bar{\alpha}\bar{\alpha}_\Gamma|E^x|\frac{\partial F_2}{\partial E^x} = s\alpha\bar{\alpha}_\Gamma F_1\frac{\partial F_1}{\partial K_\varphi} \tag{A.11}$$

and

$$\left(\bar{\alpha}_\Gamma\frac{\partial\bar{\alpha}}{\partial E^\varphi} - \bar{\alpha}\frac{\partial\bar{\alpha}_\Gamma}{\partial E^\varphi}\right)F_2 = \bar{\alpha}\bar{\alpha}_\Gamma\frac{\partial F_2}{\partial E^\varphi} \tag{A.12}$$

which is also obtained from (A.9):

$$\begin{aligned}\{H^Q_{\rm grav}[N], D_{\rm grav}[N^x]\} &= -H^Q_{\rm grav}[N^x N'] \\ &\quad -\frac{1}{2G}\int {\rm d}x\, N N^x{}' E^\varphi\Bigg[\left(\frac{\partial\alpha}{\partial E^\varphi}F_1 + 2\alpha\frac{\partial F_1}{\partial E^\varphi}\right)|E^x|^{-\frac{1}{2}}E^\varphi F_1 \\ &\qquad + 2s\frac{\partial(\bar{\alpha}F_2)}{\partial E^\varphi}|E^x|^{\frac{1}{2}}(A_x + \eta') + \frac{\partial\alpha}{\partial E^\varphi}|E^x|^{-\frac{1}{2}}E^\varphi \\ &\qquad - \frac{\partial\alpha_\Gamma}{\partial E^\varphi}|E^x|^{-\frac{1}{2}}E^\varphi\Gamma_\varphi^2 + 2s\frac{\partial\bar{\alpha}_\Gamma}{\partial E^\varphi}|E^x|^{\frac{1}{2}}\Gamma_\varphi'\Bigg]\end{aligned}$$

and imposes the additional conditions

$$\frac{\partial F_1}{\partial E^\varphi} = 0, \qquad \alpha^{-1}\frac{\partial\alpha}{\partial E^\varphi} = \alpha_\Gamma^{-1}\frac{\partial\alpha_\Gamma}{\partial E^\varphi} = \bar{\alpha}_\Gamma^{-1}\frac{\partial\bar{\alpha}_\Gamma}{\partial E^\varphi} \tag{A.13}$$

To find simple non-trivial solutions, we can set each of the offending terms in (A.10) to zero. The vanishing of the last term is the condition for F_2 to be independent of the triad, and the other two combined imply

$$\bar{\alpha}_\Gamma = \bar{\alpha} \qquad \text{and} \qquad F_2 = \left(\frac{\alpha}{\alpha_\Gamma}\right) F_1 \frac{\partial F_1}{\partial K_\varphi}.$$

If F_1 is also independent of E^x then $\alpha_\Gamma = \alpha$. Taking for example:

$$F_1 = \frac{\sin \delta\gamma K_\varphi}{\delta\gamma}$$

requires

$$F_2 = \frac{\sin \delta\gamma K_\varphi \cos \delta\gamma K_\varphi}{\delta\gamma} = \frac{\sin 2\delta\gamma K_\varphi}{2\delta\gamma}.$$

and gives an anomaly-free 'holonomized' version of the constraint algebra which is different from the classical one:

$$\begin{aligned}
\{H^Q_{\text{grav}}[M], H^Q_{\text{grav}}[N]\} =& D_{\text{grav}}[|E^x|(E^\varphi)^{-2}(MN' - NM') \cos(2\delta\gamma K_\varphi)] \\
&- G_{\text{grav}}[|E^x|(E^\varphi)^{-2}(NM' - MN')\eta' \cos(2\delta\gamma K_\varphi)] \\
\{H^Q_{\text{grav}}[N], D_{\text{grav}}[N^x]\} =& -H^Q_{\text{grav}}[N^x N']
\end{aligned} \tag{A.14}$$

Considering corrections due to holonomies of A_x is more difficult because the structure of the algebra makes it harder to reproduce the diffeomorphism constraint term in the $\{H[M], H[N]\}$ bracket and, since A_x is a density, the $\{H[N], D[N^x]\}$ bracket is problematic too.

Appendix B

Canonical Transformations between PSM's and Loop Variables

B.1 Canonical transformation

Here, we explicitly compute the canonical relation between Poisson sigma models and Ashtekar variables. The equation $\{Q^e, e\} = 2G$ gives

$$-\gamma\frac{psE^\varphi}{4|E^x|^{\frac{5}{4}}}\frac{\delta Q^e}{\delta A_x} + \frac{p}{2|E^x|^{\frac{1}{4}}}\frac{\delta Q^e}{\delta K_\varphi} = \sqrt{2}$$

and $\{Q^e, \phi\} = 0$ reads $\delta Q^e/\delta A_x = 0$. Together they imply:

$$Q^e = p2\sqrt{2}\||E^x|^{\frac{1}{4}}K_\varphi + \tilde{Q}^e[E^x, E^\varphi, P^\eta, \eta] \tag{B.1}$$

for arbitrary function $\tilde{Q}^e[E^x, E^\varphi, P^\eta, \eta]$.

Equations $\{Q^\Upsilon, e\} = 0$ and $\{Q^\Upsilon, \phi\} = 0$ give, respectively,

$$-\gamma\frac{sE^\varphi}{4|E^x|^{\frac{5}{4}}}\frac{\delta Q^\Upsilon}{\delta A_x} + \frac{1}{2|E^x|^{\frac{1}{4}}}\frac{\delta Q^\Upsilon}{\delta K_\varphi} = 0 \quad , \quad \frac{\delta Q^\Upsilon}{\delta A_x} = 0$$

which taken together imply

$$Q^\Upsilon = Q^\Upsilon[E^x, E^\varphi, P^\eta, \eta]\,. \tag{B.2}$$

The equation $\{\phi, \omega_x\} = 2G$ just implies

$$-\gamma\frac{s}{2}\frac{\delta\omega_x}{\delta A_\varphi} = \frac{1}{2}$$

and $\{e, \omega_x\} = 0$ is equivalent to

$$\frac{\delta\omega_x}{\delta K_\varphi} = \gamma\frac{sE^\varphi}{2|E^x|}\frac{\delta\omega_x}{\delta A_x}\,.$$

These two equations give the dependence of ω_x on K_x and K_φ:

$$\omega_x = -\frac{sA_x}{\gamma} - \frac{E^\varphi}{2|E^x|}K_\varphi + \tilde{\omega}_x[E^x, E^\varphi, P^\eta, \eta] \qquad \text{(B.3)}$$

which fixes the sign ambiguity in (3.30) and gives

$$\tilde{\omega}_x = -\frac{s\eta'}{\gamma} - \Upsilon'\,. \qquad \text{(B.4)}$$

Equations $\{\phi, \Upsilon\} = 0$ and $\{e, \Upsilon\} = 0$ are, respectively,

$$\frac{\delta\Upsilon}{\delta A_x} = 0 \quad , \quad \frac{\delta\Upsilon}{\delta K_\varphi} = \gamma\frac{sE^\varphi}{2|E^x|}\frac{\delta\Upsilon}{\delta A_x}\,.$$

Thus,

$$\Upsilon = \Upsilon[E^x, E^\varphi, P^\eta, \eta] \qquad \text{(B.5)}$$

The remaining six equations $\{Q^\Upsilon, \Upsilon\} = 2G$ and $\{Q^e, \Upsilon\} = \{Q^e, Q^\Upsilon\} = \{Q^e, \omega_x\} = \{Q^\Upsilon, \omega_x\} = \{\Upsilon, \omega_x\} = 0$ are

$$\frac{\delta Q^\Upsilon}{\delta\eta}\frac{\delta\Upsilon}{\delta P^\eta} - \frac{\delta Q^\Upsilon}{\delta P^\eta}\frac{\delta\Upsilon}{\delta\eta} = \frac{1}{\gamma} \qquad \text{(B.6)}$$

$$\frac{\delta\tilde{Q}^e}{\delta\eta}\frac{\delta\Upsilon}{\delta P^\eta} - \frac{\delta\tilde{Q}^e}{\delta P^\eta}\frac{\delta\Upsilon}{\delta\eta} = -p\frac{\sqrt{2}|E^x|^{\frac{1}{4}}}{\gamma}\frac{\delta\Upsilon}{\delta E^\varphi} \qquad \text{(B.7)}$$

$$\frac{\delta\tilde{Q}^e}{\delta\eta}\frac{\delta Q^\Upsilon}{\delta P^\eta} - \frac{\delta\tilde{Q}^e}{\delta P^\eta}\frac{\delta Q^\Upsilon}{\delta\eta} - = -p\frac{\sqrt{2}|E^x|^{\frac{1}{4}}}{\gamma}\frac{\delta Q^\Upsilon}{\delta E^\varphi} \qquad \text{(B.8)}$$

$$\frac{\delta\tilde{Q}^e}{\delta\eta}\frac{\delta\tilde{\omega}_x}{\delta P^\eta} - \frac{\delta\tilde{Q}^e}{\delta P^\eta}\frac{\delta\tilde{\omega}_x}{\delta\eta} = -\frac{1}{2\gamma}\left(2s\frac{\delta\tilde{Q}^e}{\delta E^x} + \frac{E^\varphi}{2|E^x|}\frac{\delta\tilde{Q}^e}{\delta E^\varphi} + p2\sqrt{2}|E^x|^{\frac{1}{4}}\frac{\delta\tilde{\omega}_x}{\delta E^\varphi}\right) \qquad \text{(B.9)}$$

$$\frac{\delta Q^{\Upsilon}}{\delta \eta}\frac{\delta \tilde{\omega}_x}{\delta P^{\eta}} - \frac{\delta Q^{\Upsilon}}{\delta P^{\eta}}\frac{\delta \tilde{\omega}_x}{\delta \eta} = -\frac{1}{2\gamma}\left(2s\frac{\delta Q^{\Upsilon}}{\delta E^x} + \frac{E^{\varphi}}{2|E^x|}\frac{\delta Q^{\Upsilon}}{\delta E^{\varphi}}\right) \tag{B.10}$$

$$\frac{\delta \Upsilon}{\delta \eta}\frac{\delta \tilde{\omega}_x}{\delta P^{\eta}} - \frac{\delta \Upsilon}{\delta P^{\eta}}\frac{\delta \tilde{\omega}_x}{\delta \eta} = -\frac{1}{2\gamma}\left(2s\frac{\delta \Upsilon}{\delta E^x} + \frac{E^{\varphi}}{2|E^x|}\frac{\delta \Upsilon}{\delta E^{\varphi}}\right) . \tag{B.11}$$

To try to find a solution to this system of equations we assume $\tilde{Q}^e$ to be independent of P^η and η. Then (B.7) and (B.8) imply that Υ and Q^Υ are independent of E^φ, and we are left with equation (B.6) and

$$\begin{aligned} 2s\frac{\delta \tilde{Q}^e}{\delta E^x} + \frac{E^{\varphi}}{2|E^x|}\frac{\delta \tilde{Q}^e}{\delta E^{\varphi}} + p2\sqrt{2}|E^x|^{\frac{1}{4}}\frac{\delta \tilde{\omega}_x}{\delta E^{\varphi}} &= 0 \\ \frac{\delta Q^{\Upsilon}}{\delta \eta}\frac{\delta \tilde{\omega}_x}{\delta P^{\eta}} - \frac{\delta Q^{\Upsilon}}{\delta P^{\eta}}\frac{\delta \tilde{\omega}_x}{\delta \eta} &= -\frac{s}{\gamma}\frac{\delta Q^{\Upsilon}}{\delta E^x} \\ \frac{\delta \Upsilon}{\delta \eta}\frac{\delta \tilde{\omega}_x}{\delta P^{\eta}} - \frac{\delta \Upsilon}{\delta P^{\eta}}\frac{\delta \tilde{\omega}_x}{\delta \eta} &= -\frac{s}{\gamma}\frac{\delta \Upsilon}{\delta E^x} . \end{aligned}$$

Using (B.4) which implies $\frac{\delta \tilde{\omega}_x}{\delta E^{\varphi}} = 0$, the equation for $\tilde{Q}^e$ is generically solved for $\tilde{Q}^e[E^x, E^\varphi] = h(E^\varphi/|E^x{}^{\frac{1}{4}})$ with h a function of one variable. If we further require that C^3 reproduce the Gauss constraint (2.14), $C^3 = kG_{\rm grav}$ for an arbitrary constant k, we find the solution giving rise to the canonical transformation (3.33).

B.2 Dilaton gravity with torsion

For completeness, we summarize here the constructions necessary in the presence of torsion. This will not change the main results of the paper.

B.2.1 Generalized Dilaton Gravity and comparison of metrics

Using the definitions [23]

$$\phi := \Phi^2 \quad , \quad g_{\mu\nu} := \mathbf{g}_{\mu\nu} \quad , \quad \mathbf{U}(\phi) := -\frac{1}{2\phi} \quad , \quad \mathbf{V}(\phi) := 1$$

instead of (3.15), the spherically symmetric reduced 2d action (3.12) is reexpressed as the generalized 2d Dilaton action

$$S_{\text{dilaton}} = \frac{1}{2G}\int d^2x\sqrt{-g}\left(\frac{1}{2}\phi R + W(-(\nabla\phi)^2,\phi)\right) .$$

This in turn is equivalent to a general 2d gravity action with torsion:

$$S = -\frac{1}{2G}\int_M \phi d\omega - W(X^a X_a,\phi)\varepsilon + X_a De^a$$

with

$$W(X^a X_a,\phi) := \mathbf{U}(\phi)\frac{X^a X_a}{2} + \mathbf{V}(\phi)$$

and $R = 2 * d\bar{\omega}$ the curvature for the torsion free part of the spin connection $\omega = \bar{\omega} + e_a\frac{\partial W}{\partial X_a}$. There is no conformal transformation, so the metric g represents the physical metric in this approach.

The Poisson sigma model in this case is determined by the more general Poisson bivector

$$\mathcal{P}^{ij} = \begin{pmatrix} 0 & W & -X^- \\ -W & 0 & X^+ \\ X^- & -X^+ & 0 \end{pmatrix}$$

so that in the constraints (3.23), $V/2$ is replaced by $-W$:

$$\tilde{C}^{\mp} = \frac{1}{2G}\left[\left(\frac{eQ^e \pm Q^\Upsilon}{2e}\right)' \mp \left(\frac{eQ^e \pm Q^\Upsilon}{2e}\right)(\omega_x + \Upsilon') \pm W(2X^+X^-,\phi)e\right]\exp(\mp\Upsilon) \tag{B.12}$$

and from (3.22)

$$2X^+X^- = 2\left(\frac{eQ^e - Q^\Upsilon}{2e}\right)\left(\frac{eQ^e + Q^\Upsilon}{2e}\right) .$$

Comparison of the metrics now yields different values for ϕ and the dyads (3.28) which get an extra factor of $|E^x|^{-\frac{1}{4}}$:

$$\phi = |E^x|$$

$$e_x^+ = p\frac{E^\varphi}{\sqrt{2}|E^x|^{\frac{1}{2}}}\exp\Upsilon\,, \quad e_x^- = p\frac{E^\varphi}{\sqrt{2}|E^x|^{\frac{1}{2}}}\exp(-\Upsilon)$$

$$e_t^+ = p\frac{N^x E^\varphi \pm N|E^x|^{\frac{1}{2}}}{\sqrt{2}|E^x|^{\frac{1}{2}}}\exp\Upsilon\,, \quad e_t^- = p\frac{1}{\sqrt{2}|E^x|^{\frac{1}{2}}}\left(\frac{-N^2|E^x| + N^{x\,2}E^{\varphi\,2}}{N^x E^\varphi \pm N|E^x|^{\frac{1}{2}}}\right)\exp(-\Upsilon)$$

The dependence of the Lagrange multipliers $X^\pm$ in terms of $(E^x, E^\varphi, K_x, K_\varphi)$ (as in equations (3.31,3.32)) consequently gets an extra factor of $|E^x|^{\frac{1}{4}}$:

$$X^- = p\sqrt{2}|E^x|^{\frac{1}{2}}\left(-s\frac{E^{x\,\prime}}{2E^\varphi} \mp K_\varphi\right)\exp(-\Upsilon) \tag{B.13}$$

$$X^+ = p\sqrt{2}|E^x|^{\frac{1}{2}}\left(s\frac{E^{x\,\prime}}{2E^\varphi} \mp K_\varphi\right)\exp(\Upsilon) \tag{B.14}$$

The torsion free part of the spin connection ω_x is $\bar{\omega}_x = \pm K_x - \Upsilon'$, and the torsion dependent part $e_{xa}\frac{\partial W}{\partial X_a} = \mathbf{U}(\phi)e_{xa}X^a = \pm 2\mathbf{U}(E^x)E^\varphi K_\varphi$, so

$$\omega_x = \pm sK_x \mp 2\mathbf{U}(E^x)E^\varphi K_\varphi - \Upsilon' \tag{B.15}$$

B.2.2 Canonical transformation

The Poisson bracket relations

$$\{Q^e(x), e(y)\} = \{Q^\Upsilon(x), \Upsilon(y)\} = \{\phi(x), \omega_x(y)\} = 2G\delta(x,y)$$

give the following functional dependence of the set $(Q^e, Q^\Upsilon, \phi; e, \Upsilon, \omega_x)$ in terms of $(E^x, E^\varphi, P^\eta; K_x, K_\varphi, \eta)$:

$$\phi = |E^x|\,, \quad e = \frac{E^\varphi}{\sqrt{2}|E^x|^{\frac{1}{2}}}$$

$$\omega_x = -s\frac{A_x}{\gamma} - \frac{E^\varphi}{|E^x|}K_\varphi + \tilde{\omega}_x[E^x, E^\varphi, P^\eta, \eta]\,, \quad Q^e = p\sqrt{2}|E^x|^{\frac{1}{2}}K_\varphi + \tilde{Q}^e[E^x, E^\varphi, P^\eta, \eta]$$

$$Q^\Upsilon = Q^\Upsilon[E^x, E^\varphi, P^\eta, \eta]\,, \quad \Upsilon = \Upsilon[E^x, E^\varphi, P^\eta, \eta] \tag{B.16}$$

and the following differential equations analogous to (B.6), (B.7), (B.8), (B.9), (B.10), (B.11):

$$\frac{\delta Q^\Upsilon}{\delta\eta}\frac{\delta\Upsilon}{\delta P^\eta} - \frac{\delta Q^\Upsilon}{\delta P^\eta}\frac{\delta\Upsilon}{\delta\eta} = \frac{1}{\gamma} \tag{B.17}$$

$$\frac{\delta\tilde{Q}^e}{\delta\eta}\frac{\delta\Upsilon}{\delta P^\eta}-\frac{\delta\tilde{Q}^e}{\delta P^\eta}\frac{\delta\Upsilon}{\delta\eta}=-p\frac{\sqrt{2}|E^x|^{\frac{1}{2}}}{\gamma}\frac{\delta\Upsilon}{\delta E^\varphi} \tag{B.18}$$

$$\frac{\delta\tilde{Q}^e}{\delta\eta}\frac{\delta Q^\Upsilon}{\delta P^\eta}-\frac{\delta\tilde{Q}^e}{\delta P^\eta}\frac{\delta Q^\Upsilon}{\delta\eta}=-p\frac{\sqrt{2}|E^x|^{\frac{1}{2}}}{\gamma}\frac{\delta Q^\Upsilon}{\delta E^\varphi} \tag{B.19}$$

$$\frac{\delta\tilde{Q}^e}{\delta\eta}\frac{\delta\tilde{\omega}_x}{\delta P^\eta}-\frac{\delta\tilde{Q}^e}{\delta P^\eta}\frac{\delta\tilde{\omega}_x}{\delta\eta}=-\frac{1}{2\gamma}\left(2s\frac{\delta\tilde{Q}^e}{\delta E^x}+\frac{E^\varphi}{|E^x|}\frac{\delta\tilde{Q}^e}{\delta E^\varphi}+p2\sqrt{2}|E^x|^{\frac{1}{2}}\frac{\delta\tilde{\omega}_x}{\delta E^\varphi}\right) \tag{B.20}$$

$$\frac{\delta Q^\Upsilon}{\delta\eta}\frac{\delta\tilde{\omega}_x}{\delta P^\eta}-\frac{\delta Q^\Upsilon}{\delta P^\eta}\frac{\delta\tilde{\omega}_x}{\delta\eta}=-\frac{1}{2\gamma}\left(2s\frac{\delta Q^\Upsilon}{\delta E^x}+\frac{E^\varphi}{|E^x|}\frac{\delta Q^\Upsilon}{\delta E^\varphi}\right) \tag{B.21}$$

$$\frac{\delta\Upsilon}{\delta\eta}\frac{\delta\tilde{\omega}_x}{\delta P^\eta}-\frac{\delta\Upsilon}{\delta P^\eta}\frac{\delta\tilde{\omega}_x}{\delta\eta}=-\frac{1}{2\gamma}\left(2s\frac{\delta\Upsilon}{\delta E^x}+\frac{E^\varphi}{|E^x|}\frac{\delta\Upsilon}{\delta E^\varphi}\right) \tag{B.22}$$

Already, from the form of ω_x in (B.16) and in (B.15) we see that a canonical transformation is only consistent for $\mathbf{U}(\phi)=-\frac{1}{2\phi}$. So we only have freedom to change the functional form of $\mathbf{V}(\phi)$ if we try to generalize to other models (just as in the conformal approach).

Attempting to solve these differential equations by assuming $\tilde{Q}^e=0$ and requiring C^3 to reproduce the Gauss constraint as before gives the transformation

$$\begin{aligned} Q^e=p2\sqrt{2}|E^x|^{\frac{1}{2}}K_\varphi+h[|E^x|^{-\frac{1}{2}}E^\varphi] \quad &, \quad e=p\frac{E^\varphi}{\sqrt{2}|E^x|^{\frac{1}{2}}} \\ \phi=|E^x| \quad &, \quad \omega_x=-sK_x-\frac{E^\varphi}{|E^x|}K_\varphi+\frac{1}{k}\eta' \\ Q^\Upsilon=\frac{k}{\gamma}P^\eta+\left(\frac{k-s\gamma}{\gamma}\right)E^{x\,\prime} \quad &, \quad \Upsilon=-\frac{1}{k}\eta \end{aligned} \tag{B.23}$$

and its inverse

$$\begin{aligned} E^x=s\phi \quad &, \quad E^\varphi=p\sqrt{2}\phi^{\frac{1}{2}}e \\ K_x=-s(\omega_x+\Upsilon'+\frac{e}{2\phi}(Q^e-h)) \quad &, \quad K_\varphi=p\frac{(Q^e-h)}{2\sqrt{2}\phi^{\frac{1}{2}}} \\ \eta=-k\Upsilon \quad &, \quad P^\eta=\frac{\gamma}{k}Q^\Upsilon+\left(\frac{\gamma-sk}{k}\right)\phi' \end{aligned} \tag{B.24}$$

where again, $s=\text{sign}(E^x)$, k is an arbitrary constant, and h an arbitrary function of one variable.

B.2.3 Constraints

Again, for $h=0$ we have

$$C^3[\lambda] = kG_{\rm grav}[\lambda] \quad , \quad D[N^x] = D_{\rm grav}[N^x]$$

and the remaining linear combination is

$$\begin{aligned}
C^+[N\exp(-\Upsilon)] - C^-[N\exp(\Upsilon)] &= \frac{1}{2G}\int dx\, N\left(-Q^e(\omega_x+\Upsilon') - \left(\frac{Q^\Upsilon}{e}\right)' - 2We\right)\\
&= \frac{p\sqrt{2}}{2G}\int dx\, N\Big[-|E^x|^{-\frac{1}{2}}K_\varphi^2E^\varphi - 2s|E^x|^{\frac{1}{2}}K_xK_\varphi - |E^x|^{-\frac{1}{2}}E^\varphi\mathbf{V}\\
&\qquad + \frac{|E^x|^{-\frac{1}{2}}E^{x\,\prime 2}}{4E^\varphi} - \frac{s|E^x|^{\frac{1}{2}}E^{x\,\prime}E^{\varphi\,\prime}}{E^{\varphi\,2}} + \frac{s|E^x|^{\frac{1}{2}}E^{x\,\prime\prime}}{E^\varphi}\Big]\\
&\quad + \frac{k}{2G\gamma}\int dx\, N\Big[\frac{sE^{x\,\prime}}{2|E^x|^{\frac{1}{2}}E^\varphi}(E^{x\,\prime}+P^\eta) - \frac{k}{4\gamma|E^x|^{\frac{1}{2}}E^\varphi}(E^{x\,\prime}+P^\eta)^2\\
&\qquad - \left(\frac{|E^x|^{\frac{1}{2}}}{E^\varphi}(E^{x\,\prime}+P^\eta)\right)'\Big]\,.
\end{aligned}$$

For $\mathbf{V}(\phi)=1$, this gives

$$\begin{aligned}
&C^+[N\exp(-\Upsilon)] - C^-[N\exp(\Upsilon)] = p\sqrt{2}H_{\rm grav}[N]\\
&\quad + \frac{k}{2G\gamma}\int dx\, N\Big[\frac{sE^{x\,\prime}}{2|E^x|^{\frac{1}{2}}E^\varphi}(E^{x\,\prime}+P^\eta) - \frac{k}{4\gamma|E^x|^{\frac{1}{2}}E^\varphi}(E^{x\,\prime}+P^\eta)^2\\
&\qquad - \left(\frac{|E^x|^{\frac{1}{2}}}{E^\varphi}(E^{x\,\prime}+P^\eta)\right)'\Big]
\end{aligned}$$

In summary,

$$\tilde{C}^3 = k\tilde{\mathcal{G}}_{\rm grav} \tag{B.25}$$

$$e\,\exp(\Upsilon)\tilde{C}^- + e\,\exp(-\Upsilon)\tilde{C}^+ + \omega_x\tilde{C}^3 = \tilde{\mathcal{D}}_{\rm grav} \tag{B.26}$$

$$-\exp(\Upsilon)\tilde{C}^- + \exp(-\Upsilon)\tilde{C}^+ - \frac{\phi' - G\tilde{C}^3}{2e\phi}\tilde{C}^3 + \left(\frac{\tilde{C}^3}{e}\right)' = p\sqrt{2}\tilde{\mathcal{H}}_{\rm grav}\,. \tag{B.27}$$

Appendix C

Scalar fields

In this last appendix we present another anomaly-free system by (minimally) coupling gravity to a spherically symmetric scalar field. The equations derived should be the starting point for more explorations of implications of loop quantum corrections to gravitational collapse. This system also illustrates how simple quantum corrections may give rise to complicated couplings in equations of motion and highlights the importance of consistency of evolution equations for an analysis of quantum corrections in gravitational collapse.

C.1 Coupling to Scalar Field

Starting with the Einstein-Hilbert action $S_{\rm grav} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}R$, we couple a scalar field χ with general potential $U(\chi)$. The matter part of the action we start with is

$$S_{\rm matter} = -\frac{1}{2}\int d^4x\sqrt{-g}\,[g^{\mu\nu}\partial_\mu\chi\partial_\nu\chi + U(\chi)]$$

or after the usual 3+1 decomposition of the 4-metric $g^{\alpha\beta} = q^{\alpha\beta} - n^\alpha n^\beta$, with normal vector $n^\alpha = \frac{1}{N}((\frac{\partial}{\partial t})^\alpha - N^\alpha)$ in terms of lapse N and shift vector N^α, and $\sqrt{-g} = N\sqrt{q}$

$$S_{\rm matter} = \int dt\int d^3x\left[\Pi\dot{\chi} - N^a\Pi\partial_a\chi - N\left(\frac{\Pi^2}{2\sqrt{q}} + \frac{1}{2}\sqrt{q}q^{ab}\partial_a\chi\partial_b\chi + \frac{1}{2}\sqrt{q}U\right)\right]$$

where $\Pi = \frac{\sqrt{q}}{N}(\dot{\chi} - N^a\partial_a\chi)$ is the canonically conjugate momentum of χ. From this form of the action we immediately identify the matter part of the diffeomorphism constraint and the kinetic, gradient and potential terms of the matter Hamiltonian.

Imposing spherical symmetry and defining p_χ by the relation $\Pi = p_\chi \sin\vartheta$ gives, after integration of the angular variables, the symplectic structure for a spherically symmetric scalar field minimally coupled to gravity:

$$\{\chi(x), p_\chi(y)\} = \frac{1}{4\pi}\delta(x,y)$$

or, explicitly, the Poisson bracket of functions f and g

$$\{f,g\} = \frac{1}{4\pi}\int dx \left(\frac{\delta f}{\delta\chi}\frac{\delta g}{\delta p_\chi} - \frac{\delta f}{\delta p_\chi}\frac{\delta g}{\delta\chi}\right)$$

The diffeomorphism constraint reads:

$$D_{\text{matter}}[N^x] = 4\pi\int dx\, N^x p_\chi \chi'$$

and, using the metric $dq^2 = \frac{E^{\varphi\,2}}{|E^x|}dx^2 + |E^x|d\Omega^2$ and $\sqrt{q} = |E^x|^{\frac{1}{2}}E^\varphi \sin\vartheta$, the Hamiltonian constraint is

$$H_{\text{matter}}[N] = \int dx\, N(\tilde{\mathcal{H}}_\pi + \tilde{\mathcal{H}}_\nabla + \tilde{\mathcal{H}}_U)$$

where the kinetic, gradient and potential terms are respectively:

$$\tilde{\mathcal{H}}_\pi = 4\pi\,\frac{p_\chi^2}{2|E^x|^{\frac{1}{2}}E^\varphi} \tag{C.1}$$

$$\tilde{\mathcal{H}}_\nabla = 4\pi\,\frac{|E^x|^{\frac{3}{2}}\chi'^{\,2}}{2E^\varphi} \tag{C.2}$$

$$\tilde{\mathcal{H}}_U = 4\pi\,|E^x|^{\frac{1}{2}}E^\varphi\frac{U[\chi]}{2} \tag{C.3}$$

General quantum correction functions $\nu[E^x, E^\varphi]$ and $\sigma[E^x, E^\varphi]$ are introduced into the matter part of the Hamiltonian constraint to account for the quantization of inverse triad operators as:

$$H^Q_{\rm matter}[N] = \int dx\, N(\nu\tilde{\mathcal{H}}_\pi + \sigma\tilde{\mathcal{H}}_\nabla + \tilde{\mathcal{H}}_U)$$

Again we consider an inverse triad correction of the form

$$H^Q_{\rm grav}[N] = \int dx\, N\alpha\tilde{\mathcal{H}}_{\rm grav}. \tag{C.4}$$

for function $\alpha[E^x, E^\varphi]$ (the general case (4.51) gives similar results).

Notice the potential term is not expected to acquire primary quantum corrections because there is no inverse of the triad in this term. Thus, even if ν and σ would equal α, the correction does not simply amount to a rescaling of the lapse function and the closure of the constraint algebra becomes a nontrivial requirement that restricts the form of the correction functions.

C.1.1 Constraint Algebra

If the correction functions α, ν, σ depend only on the densitized triad components E^x and E^φ (but not on their spatial derivatives), then the total Hamiltonian $\mathbf{H^Q}[N] = H^Q_{\rm grav}[N] + H_{\rm matter}[N]$ and diffeomorphism constraint $\mathbf{D}[N^x] = D_{\rm grav}[N^x] + D_{\rm matter}[N^x]$ satisfy the algebra:

$$\begin{aligned}\{\mathbf{H^Q}[N], \mathbf{D}[N^x]\} =& -\mathbf{H^Q}[N^x M'] \\ & - H^Q_{\rm grav}[\frac{1}{\alpha}\frac{\partial\alpha}{\partial E^\varphi}E^\varphi N N^{x\,\prime}] \\ & - H^Q_\pi[\frac{1}{\nu}\frac{\partial\nu}{\partial E^\varphi}E^\varphi N N^{x\,\prime}] - H^Q_\nabla[\frac{1}{\sigma}\frac{\partial\sigma}{\partial E^\varphi}E^\varphi N N^{x\,\prime}]\end{aligned}$$

$$\begin{aligned}\{\mathbf{H^Q}[N], \mathbf{H^Q}[M]\} =& D_{\rm grav}[\alpha^2\frac{|E^x|}{E^{\varphi\,2}}(NM' - MN')] + D_{\rm matter}[\nu\sigma\frac{|E^x|}{E^{\varphi\,2}}(NM' - MN')] \\ & - G_{\rm grav}[\alpha^2\frac{|E^x|}{E^{\varphi\,2}}(NM' - MN')\eta']\end{aligned}$$

The requirement of anomaly freedom then restricts the correction functions α,

ν and σ to be independent of E^φ, and further imposes the condition

$$\alpha^2 = \nu\sigma \tag{C.5}$$

C.1.2 Equations of Motion

The canonical equations of motion with the quantum corrected Hamiltonian are

$$\dot{E}^x = s2N\alpha K_\varphi |E^x|^{\frac{1}{2}} + N^x E^{x\,\prime} \tag{C.6}$$

$$\dot{E}^\varphi = N\alpha(sK_x|E^x|^{\frac{1}{2}} + K_\varphi E^\varphi |E^x|^{-\frac{1}{2}}) + (N^x E^\varphi)' \tag{C.7}$$

$$\dot{\chi} = \frac{N\nu}{|E^x|^{\frac{1}{2}} E^\varphi} p_\chi + N^x \chi' \tag{C.8}$$

$$\dot{p}_\chi = \left(\frac{N\sigma |E^x|^{\frac{3}{2}} \chi'}{E^\varphi} \right)' - \frac{1}{2} N |E^x|^{\frac{1}{2}} E^\varphi \frac{\partial U}{\partial \chi} + (N^x p_\chi)' \tag{C.9}$$

$$\begin{aligned} \dot{K}_\varphi =& \frac{N\alpha}{2} |E^x|^{-\frac{1}{2}} \left[-K_\varphi^2 + \frac{E^{x\,\prime\,2}}{4E^{\varphi\,2}} \right] + \frac{N\alpha}{4} V \\ &+ N^x K_\varphi' + s(N\alpha)' \frac{|E^x|^{\frac{1}{2}} E^{x\,\prime}}{2E^{\varphi\,2}} \\ &- 2\pi G N \left[\nu \frac{p_\chi^2}{|E^x|^{\frac{1}{2}} E^{\varphi\,2}} + \sigma \frac{|E^x|^{\frac{3}{2}} \chi'^{\,2}}{E^{\varphi\,2}} - |E^x|^{\frac{1}{2}} U[\chi] \right] \end{aligned} \tag{C.10}$$

$$\begin{aligned} \dot{K}_x =& - N\alpha |E^x|^{-\frac{1}{2}} K_x K_\varphi + sN\alpha \frac{|E^x|^{-\frac{3}{2}} E^\varphi}{2} \left(K_\varphi^2 - \frac{E^{x\,\prime\,2}}{4E^{\varphi\,2}} \right) + N\alpha \frac{E^\varphi}{2} \frac{dV}{dE^x} \\ &+ N\alpha |E^x|^{-\frac{1}{2}} \left(\frac{E^{x\,\prime\prime}}{2E^\varphi} - \frac{E^{x\,\prime} E^{\varphi\,\prime}}{2E^{\varphi\,2}} \right) \\ &+ (N\alpha)' \left(\frac{|E^x|^{-\frac{1}{2}} E^{x\,\prime}}{2E^\varphi} - s \frac{|E^x|^{\frac{1}{2}} E^{\varphi\,\prime}}{E^{\varphi\,2}} \right) \\ &+ s(N\alpha)'' \frac{|E^x|^{\frac{1}{2}}}{E^\varphi} + (N^x K_x)' \\ &+ s2\pi G N \left(-\nu \frac{p_\chi^2}{|E^x|^{\frac{3}{2}} E^\varphi} + \sigma \frac{3|E^x|^{\frac{1}{2}} \chi'^{\,2}}{E^\varphi} + \frac{E^\varphi U[\chi]}{|E^x|^{\frac{1}{2}}} \right) \end{aligned}$$

$$+ 2GN\left(\frac{\partial\alpha}{\partial E^x}\tilde{\mathcal{H}}_g + \frac{\partial\nu}{\partial E^x}\tilde{\mathcal{H}}_\pi + \frac{\partial\sigma}{\partial E^x}\tilde{\mathcal{H}}_\nabla\right) \tag{C.11}$$

C.2 Gravitational collapse in double null coordinates

As discussed in chapter 3, a spherically symmetric scalar field coupled to gravity in arbitrary dimensions may be cast as a two dimensional theory. The use of double null coordinates in the 2d manifold allows for a unified analysis of gravitational collapse in four or higher dimensions [95], giving simple evolution equations amenable for a numerical treatment.

The D-dimensional action for gravity coupled to a massless scalar field is

$$S = \frac{1}{16\pi G}\int d^Dx\,\sqrt{-^Dg}\;[^DR - \Lambda\,] - \frac{1}{2}\int d^Dx\,\sqrt{-^Dg}\;{}^Dg^{\alpha\beta}\partial_\alpha\chi\partial_\beta\chi$$

with G now the gravitational constant in D dimensions and Λ a cosmological constant. Specializing to the spherically symmetric case gives the equivalent two-dimensional dilaton gravity action

$$S = \frac{1}{4G}\int d^2x\sqrt{-g}[\phi R - V(\phi)] - \frac{1}{2}\int d^2x\sqrt{-g}H(\phi)g^{\mu\nu}\partial_\mu\chi\partial_\nu\chi$$

with

$$V(\phi) = \frac{\mathcal{O}_{D-2}}{4\pi}\left(\Lambda\phi^{\frac{1}{D-2}} - (D-2)(D-3)\phi^{-\frac{1}{D-2}}\right)$$
$$H(\phi) = \mathcal{O}_{D-2}\phi$$

and $\mathcal{O}_{D-2}$ the volume of the $(D-2)$-sphere. For $D = 4$ then $V = \Lambda\sqrt{\phi} - \frac{2}{\sqrt{\phi}}$ and $H = 4\pi\,\phi)$

In double null coordinates u, v the 2d metric may be parameterized as

$$ds^2 = -2\bar{g}(u,v)\phi^\nabla(u,v)dudv \tag{C.12}$$

Variation of the metric gives in these coordinates the equations:

$$\mathring{\phi}^\nabla = \frac{\bar{g}\phi^\nabla}{2}V \tag{C.13}$$

$$-\overset{\circ\circ}{\phi} + \left(\frac{\overset{\circ}{g}}{g} + \frac{\overset{\circ}{\phi}{}^{\triangledown}}{\phi^{\triangledown}}\right)\overset{\circ}{\phi} - 2GH\overset{\circ}{\chi}{}^2 = 0 \tag{C.14}$$

$$-\phi^{\triangledown\triangledown} + \left(\frac{g^{\triangledown}}{g} + \frac{\phi^{\triangledown\triangledown}}{\phi^{\triangledown}}\right)\phi^{\triangledown} - 2GH\chi^{\triangledown\,2} = 0 \tag{C.15}$$

and the corresponding equations for the variation of the dilaton ϕ, and scalar field χ read respectively

$$2\left(\frac{\overset{\circ}{g}}{g} + \frac{\overset{\circ}{\phi}{}^{\triangledown}}{\phi^{\triangledown}}\right)^{\triangledown} - \bar{g}\phi^{\triangledown}\frac{dV}{d\phi} + 4G\frac{dH}{d\phi}\overset{\circ}{\chi}\chi^{\triangledown} = 0 \tag{C.16}$$

$$(H\chi^{\triangledown})^{\circ} + (H\overset{\circ}{\chi})^{\triangledown} = 0 \tag{C.17}$$

where the circle denotes differentiation with respect to u and the triangle differentiation with respect to v.

Equations (C.13), (C.15) (which is simply $\bar{g}^{\triangledown}\phi^{\triangledown} = 2G\bar{g}H\chi^{\triangledown\,2}$), and (C.17) contain all information of the dynamics. For example, the uu equation (C.14) can be obtained by combining the u-derivative of the vv eq. (C.15) and the v-derivative of the uv eq. (C.13).

Gravitational collapse in this model has been carefully studied numerically, both classically [96, 95] and incorporating quantum effects [97]. Equations of motion with inverse triad like corrections from a related polymer quantization of spherically symmetric gravity have been studied numerically in [97], but as already noted there, consistency of simple approximations in equations of motion must be carefully tested. As our preliminary calculations exemplify here, simple looking corrections in the Hamiltonian may not necessarily lead to similar corrections in the equations of motion this Hamiltonian generates.

C.2.1 Quantum corrections in null coordinates

To investigate the modifications to the classical equations (C.13 - C.17) we define coordinates (t, x) from the null coordinates (u, v) by the relations

$$t = \frac{1}{2}(u + v), \qquad f(x) = \frac{1}{2}(v - u)$$

for an arbitrary increasing function f.

In these coordinates the metric (C.12) is

$$ds^2 = 2\bar{g}\phi^{\nabla}\left[-dt^2 + \left(\frac{df}{dx}\right)^2 dx^2\right]$$

and can be directly compared with

$$ds^2 = |E^x|^{\frac{1}{2}}\left[\left(-N^2 + \frac{E^{\varphi\,2}}{|E^x|}N^{x\,2}\right)dt^2 + 2\frac{E^{\varphi\,2}}{|E^x|}N^{x\,2}dt\,dx + \frac{E^{\varphi\,2}}{|E^x|}dx^2\right]$$

fixing the gauge $N^x = 0$, $N = (\frac{df}{dx})^{-1}|E^x|^{-\frac{1}{2}}E^{\varphi}$, and $E^{\varphi} = \frac{df}{dx}\sqrt{2\bar{g}}\phi^{\frac{1}{4}}(\phi^{\nabla})^{\frac{1}{2}}$. The angular part of the full metric gives $|E^x| = \phi$.

Substituting these expressions into the constraints and equations of motion (C.9,C.10,C.11) with $U = 0$ for a massless scalar field, and using $\frac{\partial}{\partial t} = \frac{\partial}{\partial u} + \frac{\partial}{\partial v}$ and $\frac{\partial}{\partial x} = \frac{df}{dx}(-\frac{\partial}{\partial u} + \frac{\partial}{\partial v})$ gives the quantum corrected equations in null coordinates.

The diffeomorphism constraint gives the equation

$$\begin{aligned}
-\overset{\circ\circ}{\phi} + \left(\frac{\mathring{g}}{g} + \frac{\mathring{\phi}^{\nabla}}{\phi^{\nabla}}\right)\mathring{\phi} - \left(\frac{\alpha}{\nu}\right)8\pi G\phi\mathring{\chi}^2 \\
+\phi^{\nabla\nabla} - \left(\frac{g^{\nabla}}{g} + \frac{\phi^{\nabla\nabla}}{\phi^{\nabla}}\right)\phi^{\nabla} + \left(\frac{\alpha}{\nu}\right)8\pi G\phi\chi^{\nabla\,2} \\
+\frac{(\mathring{\alpha} - \alpha^{\nabla})}{\alpha}(\mathring{\phi} + \phi^{\nabla}) = 0
\end{aligned} \tag{C.18}$$

The (quantum) Hamiltonian constraint gives

$$\begin{aligned}
-2\mathring{\phi}^{\nabla} + \bar{g}\phi^{\nabla}V \\
+\overset{\circ\circ}{\phi} - \frac{(\alpha^2+1)}{2\alpha^2}\left(\frac{\mathring{g}}{g} + \frac{\mathring{\phi}^{\nabla}}{\phi^{\nabla}}\right)\mathring{\phi} + \left(\frac{1}{\alpha\nu} + \frac{\sigma}{\alpha}\right)4\pi G\phi\mathring{\chi}^2 \\
+\phi^{\nabla\nabla} - \frac{(\alpha^2+1)}{2\alpha^2}\left(\frac{g^{\nabla}}{g} + \frac{\phi^{\nabla\nabla}}{\phi^{\nabla}}\right)\phi^{\nabla} + \left(\frac{1}{\alpha\nu} + \frac{\sigma}{\alpha}\right)4\pi G\phi\chi^{\nabla\,2} \\
+\frac{(\alpha^2-1)}{2\alpha^2}\left[\left(\frac{\mathring{g}}{g} + \frac{\mathring{\phi}^{\nabla}}{\phi^{\nabla}}\right)\phi^{\nabla} + \left(\frac{g^{\nabla}}{g} + \frac{\phi^{\nabla\nabla}}{\phi^{\nabla}}\right)\mathring{\phi}\right] + \left(\frac{1}{\alpha\nu} - \frac{\sigma}{\alpha}\right)8\pi G\phi\mathring{\chi}\chi^{\nabla} = 0
\end{aligned} \tag{C.19}$$

Equation (C.9) reads

$$\left[\left(\frac{1}{\nu}+\sigma\right)\phi\chi^{\triangledown}\right]^{\circ}+\left[\left(\frac{1}{\nu}+\sigma\right)\phi\mathring{\chi}\right]^{\triangledown}+\left[\left(\frac{1}{\nu}-\sigma\right)\phi\mathring{\chi}\right]^{\circ}+\left[\left(\frac{1}{\nu}-\sigma\right)\phi\chi^{\triangledown}\right]^{\triangledown}=0 \tag{C.20}$$

Equation (C.10) is

$$\begin{aligned}
&\frac{2}{\alpha^2}\mathring{\phi}^{\triangledown}-\bar{g}\phi^{\triangledown}V\\
&+\frac{\mathring{\mathring{\phi}}}{\alpha^2}-\frac{(\alpha^2+1)}{2\alpha^2}\left(\frac{\mathring{g}}{g}+\frac{\mathring{\phi}^{\triangledown}}{\phi^{\triangledown}}\right)\mathring{\phi}+\left(\frac{1}{\alpha\nu}+\frac{\sigma}{\alpha}\right)4\pi G\phi\mathring{\chi}^2\\
&+\frac{\phi^{\triangledown\triangledown}}{\alpha^2}-\frac{(\alpha^2+1)}{2\alpha^2}\left(\frac{g^{\triangledown}}{g}+\frac{\phi^{\triangledown\triangledown}}{\phi^{\triangledown}}\right)\phi^{\triangledown}+\left(\frac{1}{\alpha\nu}+\frac{\sigma}{\alpha}\right)4\pi G\phi\chi^{\triangledown\,2}\\
&+\frac{(\alpha^2-1)}{2\alpha^2}\left[\left(\frac{\mathring{g}}{g}+\frac{\mathring{\phi}^{\triangledown}}{\phi^{\triangledown}}\right)\phi^{\triangledown}+\left(\frac{g^{\triangledown}}{g}+\frac{\phi^{\triangledown\triangledown}}{\phi^{\triangledown}}\right)\mathring{\phi}\right]+\left(\frac{1}{\alpha\nu}+\frac{\sigma}{\alpha}\right)8\pi G\phi\mathring{\chi}\chi^{\triangledown}\\
&-\frac{(\mathring{\alpha}-\alpha^{\triangledown})}{\alpha}(\mathring{\phi}-\phi^{\triangledown})-\frac{(\mathring{\alpha}+\alpha^{\triangledown})}{\alpha^3}(\mathring{\phi}+\phi^{\triangledown})=0
\end{aligned} \tag{C.21}$$

Classically equation (C.20) gives the wave equation (C.17). The diffeomorphism constraint (C.18) is the difference $(C.14)-(C.15)$. The Hamiltonian constraint (C.19) is $-(C.13)-(C.14)-(C.15)$, while (C.21) gives $(C.13)-(C.14)-(C.15)$. So we obtain (C.13) by subtracting (C.19) from (C.21) and from this and the diffeomorphism constraint we get (C.14) and (C.15).

Fianally equation (C.11) gives classically $-(C.16)-\frac{1}{2\phi}((C.14)+(C.15))$.

For completeness, equation (C.11) is

$$\begin{aligned}
&-\frac{(\alpha^2+1)}{\alpha}\left(\frac{\mathring{g}}{g}+\frac{\mathring{\phi}^{\triangledown}}{\phi^{\triangledown}}\right)^{\triangledown}+\bar{g}\phi^{\triangledown}\frac{dV}{d\phi}-\left(3\sigma+\frac{1}{\nu}\right)4\pi G\mathring{\chi}\chi^{\triangledown}\\
&+\frac{(\alpha^2+1)}{4\alpha\phi}\left[\mathring{\mathring{\phi}}-\left(\frac{\mathring{g}}{g}+\frac{\mathring{\phi}^{\triangledown}}{\phi^{\triangledown}}\right)\mathring{\phi}\right]+\left(3\sigma-\frac{1}{\nu}\right)2\pi G\mathring{\chi}^2\\
&+\frac{(\alpha^2+1)}{4\alpha\phi}\left[\phi^{\triangledown\triangledown}-\left(\frac{g^{\triangledown}}{g}+\frac{\phi^{\triangledown\triangledown}}{\phi^{\triangledown}}\right)\phi^{\triangledown}\right]+\left(3\sigma-\frac{1}{\nu}\right)2\pi G\chi^{\triangledown\,2}\\
&+\frac{(\alpha^2-1)}{4\alpha\phi}\left[-2\mathring{\phi}^{\triangledown}+\left(\frac{\mathring{g}}{g}+\frac{\mathring{\phi}^{\triangledown}}{\phi^{\triangledown}}\right)\phi^{\triangledown}+\left(\frac{g^{\triangledown}}{g}+\frac{\phi^{\triangledown\triangledown}}{\phi^{\triangledown}}\right)\mathring{\phi}\right]
\end{aligned}$$

$$
\begin{aligned}
&+\frac{(\alpha^2-1)}{\alpha}\left[\left(\frac{\mathring{g}}{g}+\frac{\mathring{\phi}^{\nabla}}{\phi^{\nabla}}\right)^{\circ}+\left(\frac{g^{\nabla}}{g}+\frac{\phi^{\nabla\nabla}}{\phi^{\nabla}}\right)^{\nabla}\right] \\
&+\frac{(\alpha^2+1)}{\alpha^2}\left[\left(\frac{\mathring{g}}{g}+\frac{\mathring{\phi}^{\nabla}}{\phi^{\nabla}}\right)\mathring{\alpha}+\left(\frac{g^{\nabla}}{g}+\frac{\phi^{\nabla\nabla}}{\phi^{\nabla}}\right)\alpha^{\nabla}-\frac{\mathring{\alpha}\mathring{\phi}+\alpha^{\nabla}\phi^{\nabla}}{4\phi}\right] \\
&+\frac{(1-\alpha^2)}{\alpha^2}\left[\left(\frac{\mathring{g}}{g}+\frac{\mathring{\phi}^{\nabla}}{\phi^{\nabla}}\right)\alpha^{\nabla}+\left(\frac{g^{\nabla}}{g}+\frac{\phi^{\nabla\nabla}}{\phi^{\nabla}}\right)\mathring{\alpha}-\frac{\alpha^{\nabla}\mathring{\phi}+\mathring{\alpha}\phi^{\nabla}}{4\phi}\right] \\
&+\mathring{\mathring{\alpha}}-\alpha^{\nabla\nabla}-2\mathring{\alpha}^{\nabla}+\frac{d\alpha}{d\phi}\Bigg\{\quad \mathring{\mathring{\phi}}+\phi^{\nabla\nabla}-2\mathring{\phi}^{\nabla}+\bar{g}\phi^{\nabla}V \\
&\qquad -\frac{(\alpha^2+1)}{2\alpha^2}\left[\left(\frac{\mathring{g}}{g}+\frac{\mathring{\phi}^{\nabla}}{\phi^{\nabla}}\right)\mathring{\phi}+\left(\frac{g^{\nabla}}{g}+\frac{\phi^{\nabla\nabla}}{\phi^{\nabla}}\right)\phi^{\nabla}\right] \\
&\qquad +\frac{(\alpha^2-1)}{2\alpha^2}\left[\left(\frac{\mathring{g}}{g}+\frac{\mathring{\phi}^{\nabla}}{\phi^{\nabla}}\right)\phi^{\nabla}+\left(\frac{g^{\nabla}}{g}+\frac{\phi^{\nabla\nabla}}{\phi^{\nabla}}\right)\mathring{\phi}\right]\Bigg\} \\
&+\left(\frac{1}{\nu^2}\frac{d\nu}{d\phi}+\frac{d\sigma}{d\phi}\right)4\pi G\phi(\mathring{\chi}^2+\chi^{\nabla\,2})+\left(\frac{1}{\nu^2}\frac{d\nu}{d\phi}-\frac{d\sigma}{d\phi}\right)8\pi G\mathring{\chi}\chi^{\nabla}=0 \qquad \text{(C.22)}
\end{aligned}
$$

Bibliography

[1] Claudio Teitelboim. How commutators of constraints reflect the space-time structure. *Ann. Phys.*, 79:542–557, 1973.

[2] Abhay Ashtekar, Victor Taveras, and Madhavan Varadarajan. Information is Not Lost in the Evaporation of 2- dimensional Black Holes. *Phys. Rev. Lett.*, 100:211302, 2008.

[3] C. Rovelli. *Quantum Gravity.* Cambridge University Press, Cambridge, UK, 2004.

[4] A. Ashtekar and J. Lewandowski. Background independent quantum gravity: A status report. *Class. Quantum Grav.*, 21:R53–R152, 2004.

[5] T. Thiemann. *Introduction to Modern Canonical Quantum General Relativity.* Cambridge University Press, Cambridge, UK, 2007.

[6] J. Lewandowski, A. Okołów, H. Sahlmann, and T. Thiemann. Uniqueness of diffeomorphism invariant states on holonomy-flux algebras. *Commun. Math. Phys.*, 267:703–733, 2006.

[7] A. Ashtekar, M. Bojowald, and J. Lewandowski. Mathematical structure of loop quantum cosmology. *Adv. Theor. Math. Phys.*, 7:233–268, 2003.

[8] M. Bojowald. Loop quantum cosmology. *Living Rev. Relativity*, 11:4, 2008. `http://www.livingreviews.org/lrr-2008-4`.

[9] A. Ashtekar and M. Bojowald. Quantum geometry and the schwarzschild singularity. *Class. Quantum Grav.*, 23:391–411, 2006.

[10] A. Ashtekar, T. Pawlowski, and P. Singh. Quantum nature of the big bang: An analytical and numerical investigation. *Phys. Rev. D*, 73:124038, 2006.

[11] A. Ashtekar, T. Pawlowski, and P. Singh. Quantum nature of the big bang: Improved dynamics. *Phys. Rev. D*, 74:084003, 2006.

[12] M. Bojowald. Non-singular black holes and degrees of freedom in quantum gravity. *Phys. Rev. Lett.*, 95:061301, 2005.

[13] M. Bojowald and A. Skirzewski. Effective equations of motion for quantum systems. *Rev. Math. Phys.*, 18:713–745, 2006.

[14] M. Bojowald and A. Skirzewski. Quantum gravity and higher curvature actions. *Int. J. Geom. Meth. Mod. Phys.*, 4:25–52, 2007. Proceedings of "Current Mathematical Topics in Gravitation and Cosmology" (42nd Karpacz Winter School of Theoretical Physics), Ed. Borowiec, A. and Francaviglia, M.

[15] J. Willis. *On the Low-Energy Ramifications and a Mathematical Extension of Loop Quantum Gravity*. PhD thesis, The Pennsylvania State University, 2004.

[16] Victor Taveras. Corrections to the friedmann equations from loop quantum gravity for a universe with a free scalar field. *Physical Review D (Particles, Fields, Gravitation, and Cosmology)*, 78(6):064072, 2008.

[17] T. Thiemann. quantum spin dynamics (QSD). *Class. Quantum Grav.*, 15:839–873, 1998.

[18] M. Bojowald. Inverse scale factor in isotropic quantum geometry. *Phys. Rev. D*, 64:084018, 2001.

[19] C. Rovelli and L. Smolin. Loop space representation of quantum general relativity. *Nucl. Phys. B*, 331:80–152, 1990.

[20] M. Bojowald, G. Hossain, M. Kagan, and S. Shankaranarayanan. Anomaly freedom in perturbative loop quantum gravity. *Phys. Rev. D*, 78:063547, 2008.

[21] M. Bojowald, T. Harada, and R. Tibrewala. Lemaitre-tolman-bondi collapse from the perspective of loop quantum gravity. *Phys. Rev. D*, 78:064057, 2008.

[22] T. Strobl. *Gravity in Two Spacetime Dimensions*. PhD thesis, 2000.

[23] D. Grumiller, W. Kummer, and D. V. Vassilevich. Dilaton gravity in two dimensions. *Phys. Rept.*, 369:327–430, 2002.

[24] K. I. Izawa. On nonlinear gauge theory from a deformation theory perspective. *Prog. Theor. Phys.*, 103:225, 2000.

[25] V. Schomerus. D-branes and deformation quantization. *JHEP*, 06:030, 1999.

[26] N. Seiberg and E. Witten. String theory and noncommutative geometry. *JHEP*, 09:032, 1999.

[27] G. Lemaitre. *Ann. Soc. Sci. Bruxelles I A*, 53:51, 1933.

[28] R. C. Tolman. Effect of inhomogeneity on cosmological models. *Proc. Nat. Acad. Sci.*, 20:169–176, 1934.

[29] H. Bondi. Spherically symmetrical models in general relativity. *Mon. Not. R. Astron. Soc.*, 107:410–425, 1947.

[30] S. W. Hawking and R. Penrose. The singularities of gravitational collapse and cosmology. *Proc. Roy. Soc. Lond. A*, 314:529–548, 1970.

[31] S. W. Hawking and G. F. R. Ellis. *The Large Scale Structure of Space-Time.* Cambridge University Press, 1973.

[32] S. W. Hawking. Particle Creation by Black Holes. *Commun. Math. Phys.*, 43:199–220, 1975.

[33] C. Vaz, L. Witten, and T. P. Singh. Toward a midisuperspace quantization of lemaitre-tolman-bondi collapse models. *Phys. Rev. D*, 63:104020, 2001.

[34] C. Kiefer, J. Müller-Hill, and C. Vaz. Classical and quantum ltb model for the non-marginal case. *Phys. Rev. D*, 73:044025, 2006.

[35] C. Kiefer, J. Müller-Hill, T. P. Singh, and C. Vaz. Hawking radiation from the quantum lemaitre-tolman-bondi model. *Phys. Rev. D*, 75:124010, 2007.

[36] T. Thiemann and H. A. Kastrup. Canonical quantization of spherically symmetric gravity in ashtekar's self-dual representation. *Nucl. Phys. B*, 399:211–258, 1993.

[37] M. Bojowald and H. A. Kastrup. Symmetry reduction for quantized diffeomorphism invariant theories of connections. *Class. Quantum Grav.*, 17:3009–3043, 2000.

[38] M. Bojowald. Spherically symmetric quantum geometry: States and basic operators. *Class. Quantum Grav.*, 21:3733–3753, 2004.

[39] M. Bojowald and R. Swiderski. Spherically symmetric quantum geometry: Hamiltonian constraint. *Class. Quantum Grav.*, 23:2129–2154, 2006.

[40] R. Arnowitt, S. Deser, and C. W. Misner. *The Dynamics of General Relativity.* Wiley, New York, 1962.

[41] Bryce S. DeWitt. Quantum theory of gravity. i. the canonical theory. *Phys. Rev.*, 160(5):1113–1148, Aug 1967.

[42] A. Ashtekar. New hamiltonian formulation of general relativity. *Phys. Rev. D*, 36(6):1587–1602, 1987.

[43] J. F. Barbero G. Real ashtekar variables for lorentzian signature space-times. *Phys. Rev. D*, 51(10):5507–5510, 1995.

[44] G. Immirzi. Real and complex connections for canonical gravity. *Class. Quantum Grav.*, 14:L177–L181, 1997.

[45] J. Harnad, S. Shnider, and L. Vinet. Group actions on principal bundles and invariance conditions for gauge fields. *J. Math. Phys.*, 21(12):2719–2724, 1980.

[46] O. Brodbeck. On symmetric gauge fields for arbitrary gauge and symmetry groups. *Helv. Phys. Acta*, 69:321–324, 1996.

[47] M. Bojowald. Loop quantum cosmology and inhomogeneities. *Gen. Rel. Grav.*, 38:1771–1795, 2006.

[48] M. Bojowald. Singularities and quantum gravity. In *AIP Conf. Proc.*, volume 910, pages 294–333, 2007. Proceedings of the XIIth Brazilian School on Cosmology and Gravitation.

[49] M. Bojowald, B. Sandhöfer, A. Skirzewski, and A. Tsobanjan. Effective constraints for quantum systems. *Rev. Math. Phys.*, 21:111–154, 2009.

[50] A. Ashtekar and T. A. Schilling. *Geometrical Formulation of Quantum Mechanics*, pages 23–65. Springer, New York, 1999.

[51] K. Vandersloot. On the hamiltonian constraint of loop quantum cosmology. *Phys. Rev. D*, 71:103506, 2005.

[52] Abhay Ashtekar, Alejandro Corichi, and Parampreet Singh. Robustness of key features of loop quantum cosmology. *Physical Review D (Particles, Fields, Gravitation, and Cosmology)*, 77(2):024046, 2008.

[53] M. Bojowald, H. Hernández, M. Kagan, and A. Skirzewski. Effective constraints of loop quantum gravity. *Phys. Rev. D*, 75:064022, 2007.

[54] S. A. Hojman, K. Kuchar, and C. Teitelboim. Geometrodynamics Regained. *Annals Phys.*, 96:88–135, 1976.

[55] Karel Kuchar. Geometrodynamics regained: A lagrangian approach. *Journal of Mathematical Physics*, 15(6):708–715, 1974.

[56] T. Thiemann. QSD V: Quantum gravity as the natural regulator of matter quantum field theories. *Class. Quantum Grav.*, 15:1281–1314, 1998.

[57] M. Bojowald, D. Cartin, and G. Khanna. Lattice refining loop quantum cosmology, anisotropic models and stability. *Phys. Rev. D*, 76:064018, 2007.

[58] Martin Bojowald. Consistent loop quantum cosmology. *Class. Quantum Grav.*, 26:075020, 2009.

[59] B. K. Berger, D. M. Chitre, V. E. Moncrief, and Y. Nutku. Hamiltonian formulation of spherically symmetric gravitational fields. *Phys. Rev. D*, 5(10):2467–2470, May 1972.

[60] W. G. Unruh. Notes on black-hole evaporation. *Phys. Rev. D*, 14(4):870–892, Aug 1976.

[61] K. V. Kuchař. Geometrodynamics of schwarzschild black holes. *Phys. Rev. D*, 50(6):3961–3981, 1994.

[62] H. A. Kastrup and T. Thiemann. Spherically symmetric gravity as a completely integrable system. *Nucl. Phys. B*, 425:665–686, 1994.

[63] V. Husain and O. Winkler. Quantum resolution of black hole singularities. *Class. Quantum Grav.*, 22:L127–L133, 2005.

[64] V. Husain and O. Winkler. Quantum black holes from null expansion operators. *Class. Quantum Grav.*, 22:L135–L141, 2005.

[65] Rodolfo Gambini, Jorge Pullin, and Saeed Rastgoo. Quantum scalar field in quantum gravity: the vacuum in the spherically symmetric case. *Class. Quant. Grav.*, 26:215011, 2009.

[66] Rodolfo Gambini and Jorge Pullin. Black holes in loop quantum gravity: The complete space-time. *Physical Review Letters*, 101(16):161301, 2008.

[67] M. Campiglia, R. Gambini, and J. Pullin. Loop quantization of spherically symmetric midi-superspaces. *Class. Quantum Grav.*, 24:3649, 2007.

[68] M. Bojowald and J. D. Reyes. Dilaton gravity, poisson sigma models and loop quantum gravity. *Class. Quantum Grav.*, 26:035018, 2009.

[69] N. Ikeda. Two-dimensional gravity and nonlinear gauge theory. *Ann. Phys.*, 235:435–464, 1994.

[70] P. Schaller and T. Strobl. Poisson structure induced (topological) field theories. *Mod. Phys. Lett. A*, 9:3129–3136, 1994.

[71] N. Ikeda and K. I. Izawa. General form of dilaton gravity and nonlinear gauge theory. *Prog. Theor. Phys.*, 90:237, 1993.

[72] M. Bojowald, A. Kotov, and T. Strobl. Lie algebroid morphisms, poisson sigma models, and off-shell closed gauge symmetries. *J. Geom. Phys.*, 54:400–426, 2005.

[73] M. Bojowald, H. A. Kastrup, F. Schramm, and T. Strobl. Group theoretical quantization of a phase space $s^1 \times r^+$ and the mass spectrum of schwarzschild black holes in d space-time dimensions. *Phys. Rev. D*, 62:044026, 2000.

[74] D. Grumiller. *Quantum Dilaton Gravity in Two Dimensions with Matter*. PhD thesis, TU Vienna.

[75] T. Strobl. Algebroid yang-mills theories. *Phys. Rev. Lett.*, 93:211601, 2004.

[76] A. Kotov, P. Schaller, and T. Strobl. Dirac sigma models. *Commun. Math. Phys.*, 260:455–480, 2005.

[77] Clisthenis P. Constantinidis, Olivier Piguet, and Alejandro Perez. Quantization of the jackiw-teitelboim model. *Physical Review D (Particles, Fields, Gravitation, and Cosmology)*, 79(8):084007, 2009.

[78] M. Bojowald and G. Hossain. Quantum gravity corrections to gravitational wave dispersion. *Phys. Rev. D*, 77:023508, 2008.

[79] G. Barnich and M. Henneaux. Consistent couplings between fields with a gauge freedom and deformations of the master equation. *Phys. Lett. B*, 311:123, 1993.

[80] M. Bojowald, J. D. Reyes, and R. Tibrewala. Nonmarginal lemaitre-tolman-bondi-like models with inverse triad corrections from loop quantum gravity. *Phys. Rev. D*, 80(8):084002, October 2009.

[81] G. Date and G. M. Hossain. Genericity of big bounce in isotropic loop quantum cosmology. *Phys. Rev. Lett.*, 94:011302, 2005.

[82] P. Singh and A. Toporensky. Big crunch avoidance in k = 1 semi-classical loop quantum cosmology. *Phys. Rev. D*, 69:104008, 2004.

[83] M. Bojowald, R. Goswami, R. Maartens, and P. Singh. A black hole mass threshold from non-singular quantum gravitational collapse. *Phys. Rev. Lett.*, 95:091302, 2005.

[84] Charles W. Misner and David H. Sharp. Relativistic equations for adiabatic, spherically symmetric gravitational collapse. *Phys. Rev.*, 136(2B):B571–B576, Oct 1964.

[85] Sean A. Hayward. Gravitational energy in spherical symmetry. *Phys. Rev. D*, 53(4):1938–1949, Feb 1996.

[86] G. Rein, A. D. Rendall, and J. Schaeffer. A regularity theorem for solutions of the spherically symmetric vlasov-einstein system. *Commun. Math. Phys.*, 168:467–478, 1995.

[87] C. J. S. Clarke. *The Analysis of Space-Time Singularities*. Cambridge University Press, Cambridge, UK, 1993.

[88] B. C. Nolan. Dynamical extensions for shell-crossing singularities. *Class. Quantum Grav.*, 20:575–585, 2003.

[89] M. Bojowald. Quantization ambiguities in isotropic quantum geometry. *Class. Quantum Grav.*, 19:5113–5130, 2002.

[90] M. Bojowald, J. E. Lidsey, D. J. Mulryne, P. Singh, and R. Tavakol. Inflationary cosmology and quantization ambiguities in semi-classical loop quantum gravity. *Phys. Rev. D*, 70:043530, 2004.

[91] J. D. Brown and K. V. Kuchař. Dust as a standard of space and time in canonical quantum gravity. *Phys. Rev. D*, 51:5600–5629, 1995.

[92] K. Giesel, J. Tambornino, and T. Thiemann. Ltb spacetimes in terms of dirac observables. 2009.

[93] T. Tanaka and T. P. Singh. Analytic derivation of the map of null rays passing near a naked singularity. *Phys. Rev. D*, 63:124021, 2001.

[94] M. Bojowald, R. Maartens, and P. Singh. Loop quantum gravity and the cyclic universe. *Phys. Rev. D*, 70:083517, 2004.

[95] M. Birukou, V. Husain, G. Kunstatter, E. Vaz, and M. Olivier. Spherically symmetric scalar field collapse in any dimension. *Phys. Rev. D*, 65(10):104036, May 2002.

[96] David Garfinkle. Choptuik scaling in null coordinates. *Phys. Rev. D*, 51(10):5558–5561, May 1995.

[97] V. Husain. Critical behaviour in quantum gravitational collapse. *Adv. Sci. Lett.*, 2(214), 2009.

Vita

Juan Daniel Reyes

Born in Toluca, México on April 18, 1975. He received his B. S. in Electrical Engineering with Honors from the Tecnológico de Monterrey in 1997 and a M.A. in Mathematics from the Pennsylvania State University in 2003. From 2000 to 2001, he worked for the Large Millimeter Telescope collaboration at the University of Massachusetts and the Instituto Nacional de Astrofísica Óptica y Electrónica. He has worked in the research group of Martin Bojowald at the Pennsylvania State University since 2006. His publications are:

1. M. Bojowald and J. D. Reyes, Dilaton Gravity, Poisson Sigma Models and Loop Quantum Gravity, *Class. Quantum Grav.* 26 (2009) 035018, [arXiv:0810.5119]

2. M. Bojowald, J. D. Reyes, and R. Tibrewala, Nonmarginal Lemaitre-Tolman-Bondi-like models with inverse triad corrections from loop quantum gravity, *Phys. Rev. D*, 80(8):084002, [arXiv:0906.4767]