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LONG-TIME BEHAVIOR OF RANDOM HAMILTONIAN SYSTEMS

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by

Hai Le

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The dissertation of Hai Le was reviewed and approved by the following:

Alexei Novikov
Professor of Mathematics
Dissertation Adviser
Chair of Committee
Director of Graduate Studies

Iouri Soukhov
Professor of Mathematics

Anna Mazzucato
Professor of Mathematics and Distinguished Senior Scholar

Guodong Pang
Professor of Computational and Applied Mathematics
Rice University, Houston, TX

Abstract

The long-time behavior of dynamical systems driven by fractional Brownian motions is an interesting and challenging problem. The main difficulty of analysis is that fractional Brownian motion is neither a Markov process nor a semimartingale. Therefore, the machinery of classical stochastic calculus is not suitable to analyze such systems. This dissertation considers Hamiltonian systems driven by algebraically decorrelating noises, which are approximations to fractional Brownian motion noises. The main result of the thesis is that in certain conditions, the long-time behavior of the Hamiltonian does not depend on the memory-property of the driven noises: the limiting process is a diffusion. In other words, the Hamiltonian dynamics are strong enough to overcome long-range correlations of the noise and make the system converge. The goal is to establish a universal limit independent of the nature of the noise.

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Chapter 1

Introduction

1.1 The model and research problem

Consider a stochastically forced nonlinear oscillator with 1 degree of freedom:

$$\ddot{x}_t + f(x_t) = \varepsilon v(t), \quad x_0 \in \mathbb{R}, \quad \dot{x}_0 = y_0 \in \mathbb{R}\,. \tag{1.1}$$

Here $f\colon \mathbb{R} \to \mathbb{R}$ is a given smooth function, and v is a stochastic process representing the noise. Our interest is to study the asymptotic long-time behavior of x when the noise v has an algebraically decaying covariance function. In this case, the rescaled noise converges to fractional Brownian motion (fBm), a process that has a memory. We aim to study how the nonlinear dynamics affects the limiting behavior, and study whether or not the rescaled oscillator also converges to a process with memory. More precisely, we first introduce the Hamiltonian

$$H(x,y) \stackrel{\text{def}}{=} \frac{1}{2}y^2 + \int_0^x f(s)\,ds\,,$$

and recast (1.1) as a stochastically perturbed Hamiltonian system

$$\dot{x}_t = y_t = \partial_y H(x_t, y_t)\,, \tag{1.2a}$$
$$\dot{y}_t = \ddot{x}_t = -f(x_t) + \varepsilon v(t) = -\partial_x H(x_t, y_t) + \varepsilon v(t)\,. \tag{1.2b}$$

Setting $X_t \stackrel{\text{def}}{=} (x_t, y_t)^T$, we rewrite (1.2a)–(1.2b) compactly as

$$\dot{X}_t = \nabla^{\perp} H(X_t) + \varepsilon v(t) e_2, \quad X_0 = (x_0, y_0) \in \mathbb{R}^2\,. \tag{1.3}$$

where $\nabla^{\perp} \stackrel{\text{def}}{=} (\partial_y, -\partial_x)^T$, and $e_2 = (0,1)^T$.

To study the long-time behavior, we consider the time rescaled process $X^{\varepsilon}(t) \stackrel{\text{def}}{=} X(t/\varepsilon^2)$, and observe

$$\dot{X}^{\varepsilon}_t = \frac{1}{\varepsilon^2}\nabla^{\perp} H(X^{\varepsilon}_t) + \frac{1}{\varepsilon} v\left(\frac{t}{\varepsilon^2}\right) e_2\,, \quad X^{\varepsilon}_0 = X_0 = (x_0, y_0) \in \mathbb{R}^2. \tag{1.4}$$

The main question we want to answer is whether the processes $H(X^{\varepsilon}_{\cdot})$ converge to a limit and whether the limit is a diffusion when v is a noise with memory. Our expectation is that no matter how the noise v behaves there is the universal behavior that the Hamiltonian process always converge to a diffusion because of the oscillatory nature of Hamiltonian dynamics. This type of research problem falls under the theory of *singular pertubations in stochastic processes*, where we replace the ideal process (such as *white* *noise*) by a more realistic but mathematically less tractable "perturbation" (in our case, a *colored noise* v). The goal is to study real-life phenomena more accurately and obtain a simpler description but still retaining essential features of the original randomly-behaving dynamics. This goes back to the works of Khasminski [Kha80], Papanicolaou, Stroock and Varadhan [PSV76].

When the noise v is white, the behavior of X^ε as $\varepsilon \to 0$ is completely described by the averaging principle of Freidlin and Wentzell [FW93, FW94, FW98]. To briefly explain this, we note that in the absence of noise (i.e. when $v \equiv 0$), the process X^ε travels very fast on level sets of the Hamiltonian. In the presence of noise, the process X^ε will also diffuse slowly across these level sets. To capture the limiting behavior we factor out the fast motion by projecting to the *Reeb graph* of the Hamiltonian, which has the effect of identifying all closed trajectories of the Hamiltonian system to points. Now, as $\varepsilon \to 0$, this projection converges to a diffusion driven by a standard Brownian motion. One may also study the behavior of X on time scales shorter than $1/\varepsilon^2$. In some scenarios, a robust and stable limiting behavior is also observed on these time scales and has been studied by several authors [You88, YJ91, Bak11, HV14a, HV14b, HKPG16, TV16, HIK$^+$18].

1.2 My contribution

Our interest is to study the case when the driving noise in (1.4) is colored in time and has algebraically decaying correlations. It is well known that this noise can be renormalized to converge to a fractional Brownian motion (fBm), with Hurst parameter that is determined from the decay rate of the correlation function. When the Hurst parameter is not half (which happens when the correlation function decays like $1/t^\gamma$ with $\gamma \neq 1$), the renormalized limit of the noise has memory and is a non-Markovian process. We aim to study whether the memory effect is also present in the $\varepsilon \to 0$ limit of the nonlinear oscillator (1.4), where the noise is combined with the Hamiltonian dynamics.

In this thesis, we are only able to analyze the scenario where the Hamiltonian is smooth with exactly one non-degenerate critical point. In this case, it is convenient to think about the dynamics of X^ε in terms of action-angle coordinates. The angular coordinate of X^ε changes very fast, and has no meaningful limit as $\varepsilon \to 0$. The action coordinate of X^ε, on the other hand, changes slowly as a result of the interaction between the noise and the averaged angular coordinate and has a non-trivial limit $\varepsilon \to 0$. To study this we use the Hamiltonian itself as a proxy for the action coordinate and state our results in terms of convergence of the process $H(X^\varepsilon)$. (To relate this to the Freidlin–Wentzell framework, we note that when H has exactly one non-degenerate critical point, the Reeb graph has exactly one vertex and one edge, and $H(X^\varepsilon)$ is precisely the projection of X^ε onto the Reeb graph.)

In this thesis, we will prove two results. They show that even though the driving noise has memory, the time correlations are destroyed by the oscillatory dynamics of (1.4), and $H(X^\varepsilon)$ converges to a diffusion driven by standard Brownian motion (a process without memory). Roughly speaking our main results are as follows:

- If the Hamiltonian H is quadratic, and the noise v is any stationary Gaussian process with an algebraically decaying covariance function, then $H(X^\varepsilon)$ converges to a diffusion process driven by a Brownian motion. In this particular case, the limiting diffusion is a rescaling of the square of the 2-dimensional Bessel process.
- If the Hamiltonian is not quadratic (but still smooth with exactly one non-degenerate critical point), then we can show $H(X^\varepsilon)$ converges to a diffusion driven by Brownian motions, provided the noise v is chosen suitably. (The noise v is still a stationary Gaussian process with an algebraically decaying covariance function. However, it will be expressed as a superposition of Ornstein–Uhlenbeck processes as we make use of certain exponential mixing estimates in the proof.)

It is well known [Taq75, Taq77, Mar05] that stationary Gaussian noise with an algebraically decaying covariance function can be renormalized to converge to an fBm. Thus, in the above results, the renormalized noise converges to an fBm (a process with memory), whereas the processes $H(X^\varepsilon)$ themselves converge to a memory-less diffusion. Another striking point is the fact that when the noise has long-range correlations the noise increments are *diverging* even though the process $H(X^\varepsilon)$ converges. So not only do the oscillatory dynamics of (1.4) destroy the memory effect but they also coerce $H(X^\varepsilon)$ into converging on a time scale where the driving noise diverges.

Our strategy of the proof is influenced by several previous works. The works of Freidlin and Wentzell [FW93, FW94, FW98] provide a complete description of the limiting process that we should expect our processes to converge to. In particular, in the case of quadratic Hamiltonian, their results match with our results in deriving the limiting diffusion. The construction of the noise that we use is inspired by the noises described in the works of Fannjiang and Komorowski [FK00b, FK00a]. Their noise is also used in analyzing a related problem of advected passive scalar [KNR14]. In that case, the limiting behavior is also diffusive. The difference in our noise is because of technicalities that arise in our method of proving weak convergence, which comes from the work of Kushner [Kus84] in the name of the *perturbed test function (PTF) method*. This method is very useful in analyzing systems that are perturbed by non-Markovian processes. We apply this method after transforming the original dynamics (1.4) into a fast-slow system, where the slow variable is mean-zero in the fast variable. We remark that our fast-slow system is similar to those considered by Hairer and Li [HL20, HL21]. In [HL20], the authors proved an averaging principle for a coupled fast-slow system driven by fBm, with Hurst index $\mathcal{H} > \frac{1}{2}$, and they obtained convergence in probability to the naively averaged system that is still driven by fBm. This result however does not apply to our problem because the Hamiltonian dynamics in our case give rise to a mean-zero property that will make the stochastically driven term in the limiting system vanish. In [HL21], they then consider the case when there is mean-zero property but only when the fast variable does not depend on the slow variable. Our present problem does have that the slow variable feeds into the dynamics of the fast variable.

1.3 Thesis outline

The thesis is organized as follows

- In Chapter 2, we will consider the case of quadratic Hamiltonian $H(X) = |X|^2/2$ and show that the rescaled Hamiltonian $H(X_t^\varepsilon)$ will converge to a rescaling of the square of the 2-dimensional Bessel process. The proof relies on analyzing the convergence of second moments of Gaussian processes.
- In Chapter 3, we will consider the case of Hamiltonians with one non-degenerate critical point with specific contruction of the noise. The proof relies on the concept of pseudogenerator and the method of perturbed test function developed by Kushner [Kus84]. Moreover, the proof also makes use of Markov-like properties of our noises.

A large part of this thesis was presented in our paper [GILN21].

Chapter 2

Quadratic Hamiltonian

2.1 Convergence of the Hamiltonian to diffusion

We first introduce the concept of *slowly varying (at infinity)* functions.

Definition 2.1. *A function $L : (0, \infty) \to (0, \infty)$ is called slowly varying (at infinity) if for all $a > 0$,*

$$\lim_{x \to \infty} \frac{L(ax)}{L(x)} = 1.$$

Our main result when the Hamiltonian is quadratic is as follows:

Proposition 2.2. *Let H be the quadratic Hamiltonian*

$$H(x) \stackrel{\text{def}}{=} \frac{|x|^2}{2}, \qquad x = (x_1, x_2) \in \mathbb{R}^2, \tag{2.1}$$

and suppose the noise v is a stationary Gaussian process whose covariance function

$$R(t) \stackrel{\text{def}}{=} \boldsymbol{E} v(t) v(0) = \boldsymbol{E} v(t+s) v(s) \tag{2.2}$$

is of the form

$$R(t) = \frac{L(t)}{t^\gamma} \tag{2.3}$$

for some $\gamma \in (0, 2)$ and a function L that is slowly varying at infinity. If $\gamma \in (0, 1]$, we further assume that L has the slow increase property

$$\lim_{t \to \infty} \frac{L'(t)}{L(t)/t} = 0. \tag{2.4}$$

Then the family of processes $H(X^\varepsilon)_{\varepsilon > 0}$ converges in distribution to $H(W^D)$, where W^D is a 2D Brownian motion with $W_0 = X_0$ and covariance matrix D given by

$$D_{11} = D_{22} \stackrel{\text{def}}{=} \lim_{x \to \infty} \int_0^x R(z) \cos(z)\, dz \qquad \text{and} \qquad D_{12} = D_{21} = 0. \tag{2.5}$$

Remark 2.3. *For $\gamma \in (0, 1]$ we note that (2.4) implies that $R' < 0$ near infinity. Since $\cos(z)$ oscillates periodically, the limit in (2.5) exists and is finite. For $\gamma \in (1, 2)$, we note $R \in L^1(\mathbb{R})$ and so the limit in (2.5) also exists and is finite.*

Remark 2.4. *It is well known [Taq75, Taq77, Mar05] that the renormalized noise converges to a fractional Brownian motion. The proof is given in the Section 2.3 below.*

The main idea behind the proof is that when H is quadratic, the deterministic dynamics rotates with constant angular speed in all trajectories. Performing a spatial rotation will now reduce the problem to studying convergence of Gaussian processes, which can be resolved by computing covariances.

Proof of Proposition 2.2. When H is given by (2.1), we use Duhamel's formula to write the solution of (1.4) as

$$X_t^\varepsilon = \begin{pmatrix} \cos(\frac{t}{\varepsilon^2}) & \sin(\frac{t}{\varepsilon^2}) \\ -\sin(\frac{t}{\varepsilon^2}) & \cos(\frac{t}{\varepsilon^2}) \end{pmatrix} X_0 + \frac{1}{\varepsilon}\int_0^t v\left(\frac{\tau}{\varepsilon^2}\right)\begin{pmatrix} \sin(\frac{t-\tau}{\varepsilon^2}) \\ \cos(\frac{t-\tau}{\varepsilon^2}) \end{pmatrix} d\tau.$$

Let $M(t)$ be the rotation matrix

$$M(t) \stackrel{\text{def}}{=} \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix},$$

and define the rotated process Y^ε by $Y_t^\varepsilon = M(-t/\varepsilon^2)X_t^\varepsilon$. Clearly $H(X^\varepsilon) = H(Y^\varepsilon)$, and so convergence of the processes $H(X^\varepsilon)$ reduces to convergence of the processes $H(Y^\varepsilon)$.

We claim that the processes Y^ε converge to a Brownian motion as $\varepsilon \to 0$. To see this, observe

$$Y_t^\varepsilon = X_0 + \frac{1}{\varepsilon}\int_0^t v\left(\frac{\tau}{\varepsilon^2}\right)\begin{pmatrix} -\sin\left(\frac{\tau}{\varepsilon^2}\right) \\ \cos\left(\frac{\tau}{\varepsilon^2}\right) \end{pmatrix} d\tau\,.$$

We will now show that the second term above converges to a Brownian motion. For this, define

$$w_1^\varepsilon(t) \stackrel{\text{def}}{=} \frac{1}{\varepsilon}\int_0^t v\left(\frac{\tau}{\varepsilon^2}\right)\sin\left(\frac{\tau}{\varepsilon^2}\right) d\tau\,, \tag{2.6}$$

$$w_2^\varepsilon(t) \stackrel{\text{def}}{=} \frac{1}{\varepsilon}\int_0^t v\left(\frac{\tau}{\varepsilon^2}\right)\cos\left(\frac{\tau}{\varepsilon^2}\right) d\tau\,. \tag{2.7}$$

Note, the noise v (when normalized by $\sigma(\varepsilon)$) converges to fBm. However, the expressions above use a normalization factor of ε instead of $\sigma(\varepsilon)$, and have an oscillatory factor. For this reason we claim $(w_1^\varepsilon, w_2^\varepsilon)$ converges to a non-fractional Brownian motion. To prove convergence of w_1^ε, w_2^ε, we first state two lemmas:

Lemma 2.5. *For any $T > 0$, there exists a constant $C > 0$ such that for every $s, t \in [0, T]$, $i \in \{1, 2\}$, we have*

$$\boldsymbol{E}(w_i^\varepsilon(t) - w_i^\varepsilon(s))^2 \leq \begin{cases} C|t-s|^{1-\gamma} & \gamma \in (0,1)\,, \\ C|t-s|^{2-\gamma} & \gamma \in [1,2)\,. \end{cases} \tag{2.8}$$

Lemma 2.6. *For every $s, t \geq 0$, and $i \in$ we have*

$$\lim_{\varepsilon \to 0} \boldsymbol{E}(w_i^\varepsilon(t) - w_i^\varepsilon(s))^2 = D_{11}|t-s|\,, \tag{2.9}$$

$$\lim_{\varepsilon \to 0} \boldsymbol{E}(w_1^\varepsilon(t) - w_1^\varepsilon(s))(w_2^\varepsilon(t) - w_2^\varepsilon(s)) = 0\,. \tag{2.10}$$

The proofs of both Lemmas 2.5 and 2.6 are lengthy, but direct computations. Thus, for clarity of presentation, we postpone the proofs to the Section 2.2 below. Once Lemmas 2.5 and 2.6 are established, the proof of Proposition 2.2 follows quickly. Indeed, Lemma 2.5 implies that the family $(Y^\varepsilon)_{\varepsilon>0}$ is tight on $\mathcal{C}([0,T],\mathbb{R}^2)$

To see this, it suffices to show that the processes w_i^ε are tight. Without loss of generality suppose $T=1$, and suppose $\gamma\in(0,1)$ (the case when $\gamma\in[1,2)$ is similar). Choose an integer M such that $M_\gamma = M(1-\gamma)>1$. Since w_i^ε is Gaussian process, we know

$$\boldsymbol{E}(w_i^\varepsilon(t)-w_i^\varepsilon(s))^{2M} = C_M[\boldsymbol{E}(w_i^\varepsilon(t)-w_i^\varepsilon(s))^2]^M$$

for some constant C_M. We will allow C_M to change from line to line as long as it only depends on M, and remains independent of ε. The above implies

$$\boldsymbol{E}(w_i^\varepsilon(t)-w_i^\varepsilon(s))^{2M} \leq C_M|t-s|^{M_\gamma}\,. \tag{2.11}$$

Since $M_\gamma>1$ by choice, Kolmogorov's criterion (see for instance [EK86] Proposition 10.3) implies that the family w_i^ε is tight. This implies that the processes Y^ε converge in distribution along a subsequence to a continuous process Y.

Since each process Y^ε is a Gaussian process, the limiting process Y must also be a Gaussian process. Now Lemma 2.6 implies Y is a Brownian motion in $\mathbb{R}^2$ with $Y_0 = X_0$ and covariance matrix D. Since the law of the limiting process is uniquely determined, the family Y^ε itself must converge in distribution, without having to select a subsequence. Finally, since $H(X_t^\varepsilon)=H(Y_t^\varepsilon)$, we obtain convergence of $(H(X^\varepsilon))_{\varepsilon>0}$ as claimed. □

2.2 Convergence of second moments

In this section, we prove Lemmas 2.5 and 2.6 that ensure the convergence of second moments of the processes w_i^ε $(i=1,2)$.

Proof of Lemma 2.5. From (2.6) we note

$$w_1^\varepsilon(t) = \varepsilon\int_0^{t/\varepsilon^2}\sin(\tau)v(\tau)\,d\tau\,, \tag{2.12}$$

and hence

$$\begin{aligned}\boldsymbol{E}(w_1^\varepsilon(t)-w_1^\varepsilon(s))^2 &= \varepsilon^2\int_{s/\varepsilon^2}^{t/\varepsilon^2}\int_{s/\varepsilon^2}^{t/\varepsilon^2}\sin(\theta)\sin(\tau)R(\theta-\tau)\,d\theta\,d\tau\\ &= \frac{\varepsilon^2}{2}\int_{s/\varepsilon^2}^{t/\varepsilon^2}\int_{s/\varepsilon^2}^{t/\varepsilon^2}\left[\cos(\theta-\tau)-\cos(\theta+\tau)\right]R(\theta-\tau)\,d\theta\,d\tau\\ &= \frac{1}{2}(I_1^\varepsilon(s,t)-I_2^\varepsilon(s,t))\,, \end{aligned} \tag{2.13}$$

where

$$I_1^\varepsilon(s,t) \stackrel{\text{def}}{=} \varepsilon^2 \int_{s/\varepsilon^2}^{t/\varepsilon^2} \int_{s/\varepsilon^2}^{t/\varepsilon^2} \cos(\theta-\tau)R(\theta-\tau)\,d\theta\,d\tau\,,$$

$$\text{and} \qquad I_2^\varepsilon(s,t) \stackrel{\text{def}}{=} \varepsilon^2 \int_{s/\varepsilon^2}^{t/\varepsilon^2} \int_{s/\varepsilon^2}^{t/\varepsilon^2} \cos(\theta+\tau)R(\theta-\tau)\,d\theta\,d\tau\,.$$

We first analyze $I_1^\varepsilon(s,t)$. Making a change of variables $z=\theta-\tau$ and integrating by parts with respect to variable τ, we have

$$\begin{aligned} I_1^\varepsilon(s,t) &= \varepsilon^2 \int_{s/\varepsilon^2}^{t/\varepsilon^2} \int_{s/\varepsilon^2-\tau}^{t/\varepsilon^2-\tau} \cos(z)R(z)\,dz\,d\tau \\ &= \varepsilon^2 \left[\tau \int_{s/\varepsilon^2-\tau}^{t/\varepsilon^2-\tau} \cos(z)R(z)\,dz\right]_{\tau=s/\varepsilon^2}^{\tau=t/\varepsilon^2} \\ &\quad - \varepsilon^2 \int_{s/\varepsilon^2}^{t/\varepsilon^2} \tau \left(\cos\left(\frac{s}{\varepsilon^2}-\tau\right) R\left(\frac{s}{\varepsilon^2}-\tau\right) - \cos\left(\frac{t}{\varepsilon^2}-\tau\right) R\left(\frac{t}{\varepsilon^2}-\tau\right)\right) d\tau \end{aligned}$$

The first term on the right reduces to

$$\varepsilon^2 \left[\tau \int_{s/\varepsilon^2-\tau}^{t/\varepsilon^2-\tau} \cos(z)R(z)\,dz\right]_{\tau=s/\varepsilon^2}^{\tau=t/\varepsilon^2} = (t-s)\int_0^{(t-s)/\varepsilon^2} \cos(z)R(z)\,dz\,.$$

For the second term on the right, we note

$$\begin{aligned} &\varepsilon^2 \int_{s/\varepsilon^2}^{t/\varepsilon^2} \tau\cos\left(\frac{s}{\varepsilon^2}-\tau\right) R\left(\frac{s}{\varepsilon^2}-\tau\right) d\tau \\ &\qquad\qquad = \varepsilon^2 \int_0^{(t-s)/\varepsilon^2} \cos(z)R(z)zdz + s\int_0^{(t-s)/\varepsilon^2} \cos(z)R(z)dz\,, \end{aligned}$$

and

$$\begin{aligned} &\varepsilon^2 \int_{s/\varepsilon^2}^{t/\varepsilon^2} \cos\left(\frac{t}{\varepsilon^2}-\tau\right) R\left(\frac{t}{\varepsilon^2}-\tau\right) \tau d\tau \\ &\qquad\qquad = -\varepsilon^2 \int_0^{(t-s)/\varepsilon^2} \cos(z)R(z)zdz + t\int_0^{(t-s)/\varepsilon^2} \cos(z)R(z)dz\,. \end{aligned}$$

Therefore, we have

$$\begin{aligned} I_1^\varepsilon(s,t) &= 2(t-s)\int_0^{(t-s)/\varepsilon^2} \cos(z)R(z)\,dz - 2\varepsilon^2 \int_0^{(t-s)/\varepsilon^2} z\cos(z)R(z)\,dz \\ &= I_{11}^\varepsilon - 2I_{12}^\varepsilon\,. \end{aligned} \tag{2.14}$$

Clearly

$$|I_{11}^{\varepsilon}| \le 2(t-s) \sup_{x\in\mathbb{R}} \int_0^x \cos(z)R(z)\, dz\,. \tag{2.15}$$

Note that the supremum on the right-hand side is finite for all $\gamma \in (0, 2)$, as by Remark 2.3 we know $\lim_{x\to\infty} \int_0^x \cos(z)R(z)\, dz$ exists and is finite. To estimate I_{12}, we divide the analysis into cases depending on the value of γ.

- *For* $\gamma \in (0, 1)$: Integrating by parts we note

$$\begin{aligned} I_{12}^{\varepsilon} &= \varepsilon^2 \left[\sin(z)zR(z)\right]_{z=0}^{z=(t-s)/\varepsilon^2} - \varepsilon^2 \int_0^{(t-s)/\varepsilon^2} \sin(z)(R'(z)z + R(z))dz \\ &= \varepsilon^{2\gamma}(t-s)^{1-\gamma} L\Big(\frac{t-s}{\varepsilon^2}\Big) - \varepsilon^2 \int_0^{(t-s)/\varepsilon^2} \sin(z)(R'(z)z + R(z))\, dz\,. \end{aligned} \tag{2.16}$$

  For the first term on the right we note that

$$\varepsilon^{2\gamma}(t-s)^{1-\gamma} L\Big(\frac{t-s}{\varepsilon^2}\Big) \le 2\varepsilon^{2\gamma} L\Big(\frac{1}{\varepsilon^2}\Big)(t-s)^{1-\gamma}\,,$$

  for all sufficiently small ε.

  For the second term, we note that (2.4) implies

$$|R'(z)z| = |L'(z)z^{1-\gamma} - \gamma L(z)z^{-\gamma}| \le 2|R(z)|\,,$$

  for all sufficiently large z. Thus

$$\begin{aligned} \varepsilon^2 \int_0^{(t-s)/\varepsilon^2} \sin(z)(R'(z)z + R(z))\, dz &\le 3\varepsilon^2 \int_0^{(t-s)/\varepsilon^2} |R(z)|\, dz \\ &\le 4\varepsilon^{2\gamma} L\left(\frac{1}{\varepsilon^2}\right)(t-s)^{1-\gamma}\,, \end{aligned} \tag{2.17}$$

  where the last inequality followed from Karamata's theorem [BIKS18, Th 6.2.1], and the fact that L is slowly varying. Combining these, we see

$$|I_{12}^{\varepsilon}| \le 10\varepsilon^{2\gamma} L\left(\frac{1}{\varepsilon^2}\right)(t-s)^{1-\gamma}\,, \tag{2.18}$$

  for all sufficiently small ε.

- *For* $\gamma = 1$: We follow the proof in the case $\gamma < 1$, with a few minor changes. For the first term on the right of, we note that $L(z) \le 2R(0)z$ for all sufficiently small z. Since L is slowly varying at infinity we must certainly have $L(z) \le z$ for all sufficiently large z. Hence we can find a finite constant C such that $L(z) \le Cz$ for all $z \ge 0$. As a result, we have

$$\varepsilon^2 L\left(\frac{t-s}{\varepsilon^2}\right) \le C(t-s)\,.$$

For the remaining terms, note that the function $g(x) \stackrel{\text{def}}{=} \int_0^x R(z)\,dz$ is slowly varying (see for instance [BGT89] Proposition 1.5.9a, p. 26). Thus using (2.17) we see that

$$\varepsilon^2 \int_0^{(t-s)/\varepsilon^2} \sin(z)(R'(z)z + R(z))\,dz \leq 3\varepsilon^2 \int_0^{(t-s)/\varepsilon^2} |R(z)|\,dz \tag{2.19}$$

$$= 3\varepsilon^2 g\left(\frac{t-s}{\varepsilon^2}\right). \tag{2.20}$$

Since $g(0) = 0$, $g'(0) = R(0) < \infty$, and g is slowly varying at infinity, we must have $g(z) \leq Cz$ for all $z \geq 0$ and a finite constant C. (As before, we allow the constant C to change from line to line, provided it does not depend on ε, t and s.) Consequently,

$$|I_{21}^\varepsilon| \leq C(t-s),$$

as desired.

- *For* $\gamma \in (1,2)$: Directly integrating I_{12}^ε and using Karamata's theorem we see

$$|I_{12}^\varepsilon| \leq 4\varepsilon^{2(\gamma-1)} L\left(\frac{1}{\varepsilon^2}\right)(t-s)^{2-\gamma}.$$

We now turn our attention to the term I_2^ε. Substituting $z = \theta - \tau$, we note

$$I_2^\varepsilon = \varepsilon^2 \int_{s/\varepsilon^2}^{t/\varepsilon^2} \int_{s/\varepsilon^2-\tau}^{t/\varepsilon^2-\tau} \cos(z+2\tau)R(z)\,dz\,d\tau$$

$$= \varepsilon^2 \int_{s/\varepsilon^2}^{t/\varepsilon^2} \int_{s/\varepsilon^2-\tau}^{t/\varepsilon^2-\tau} [\cos(z)\cos(2\tau) - \sin(z)\sin(2\tau)]R(z)\,dz\,d\tau \tag{2.21}$$

$$= I_{21}^\varepsilon - I_{22}^\varepsilon, \tag{2.22}$$

where

$$I_{21}^\varepsilon \stackrel{\text{def}}{=} \varepsilon^2 \int_{s/\varepsilon^2}^{t/\varepsilon^2} \int_{s/\varepsilon^2-\tau}^{t/\varepsilon^2-\tau} \cos(z)\cos(2\tau)R(z)\,dz\,d\tau, \tag{2.23}$$

$$\text{and} \qquad I_{22}^\varepsilon \stackrel{\text{def}}{=} \varepsilon^2 \int_{s/\varepsilon^2}^{t/\varepsilon^2} \int_{s/\varepsilon^2-\tau}^{t/\varepsilon^2-\tau} \sin(z)\sin(2\tau)R(z)\,dz\,d\tau.$$

We again divide the analysis into cases.

- *For* $\gamma \in (0,1)$: Integrating by parts with respect to τ, we observe

$$I_{21}^{\varepsilon} = \frac{\varepsilon^2}{2} \sin(2\tau) \int_{s/\varepsilon^2-\tau}^{t/\varepsilon^2-\tau} \cos(z) R(z)\, dz \Big|_{\tau=s/\varepsilon^2}^{\tau=t/\varepsilon^2} \tag{2.24}$$

$$- \varepsilon^2 \int_{s/\varepsilon^2}^{t/\varepsilon^2} \Big(\cos\left(\frac{s}{\varepsilon^2} - \tau\right) R\left(\frac{s}{\varepsilon^2} - \tau\right) \tag{2.25}$$

$$- \cos\left(\frac{t}{\varepsilon^2} - \tau\right) R\left(\frac{t}{\varepsilon^2} - \tau\right)\Big) \sin(2\tau)\, d\tau\,.$$

and hence by Karamata's theorem,

$$|I_{21}^{\varepsilon}| \leq C\varepsilon^2 \int_0^{(t-s)/\varepsilon^2} |R(z)|\, dz \leq C\varepsilon^{2\gamma} L\left(\frac{1}{\varepsilon^2}\right) (t-s)^{1-\gamma}\,, \tag{2.26}$$

for some finite constant C.

- *For* $\gamma = 1$: As in the previous case, we know from (2.26) that

$$|I_{21}^{\varepsilon}| \leq C\varepsilon^2 \int_0^{(t-s)/\varepsilon^2} |R(z)|\, dz\,.$$

Using the same argument as that used to bound (2.20) we obtain

$$|I_{21}^{\varepsilon}| \leq C(t-s)$$

as desired.

- *For* $\gamma \in (1,2)$: As before, set $g(x) \stackrel{\text{def}}{=} \int_0^x |R(z)|\, dz$, and note

$$|I_{21}^{\varepsilon}| \leq \varepsilon^2 \int_{s/\varepsilon^2}^{t/\varepsilon^2} g\left(\frac{t}{\varepsilon^2} - \tau\right) - g\left(\frac{s}{\varepsilon^2} - \tau\right) d\tau = 2\varepsilon^2 \int_0^{(t-s)/\varepsilon^2} g(z)\, dz\,.$$

Let $G(x) = \int_0^x g(z)\, dz$ and note that G is regularly varying with index $2-\gamma$. Thus

$$|I_{21}^{\varepsilon}| \leq 2\varepsilon^2 G\left(\frac{t-s}{\varepsilon^2}\right) = 3\varepsilon^{2\gamma-2} \tilde{L}\left(\frac{1}{\varepsilon^2}\right)(t-s)^{2-\gamma}$$

The estimates for I_{22}^{ε} are identical to those for I_{21}^{ε}. Combining the above estimates we obtain (2.8) for $i=1$. The proof when $i=2$ is identical, and this finishes the proof of Lemma 2.5. □

Proof of Lemma 2.6. We first prove equation (2.9) for $i=1$. To do this, we follow the same computation as in the proof of Lemma 2.5 up to (2.14). Now note

$$\lim_{\varepsilon \to 0} I_{11}^{\varepsilon} = 2 \int_0^{\infty} \cos(z) R(z)\, dz\,,$$

where the above integral converges absolutely for $\gamma \in (1,2)$, and conditionally for $\gamma \in (0,1]$ (see Remark 2.3). When $\gamma \neq 1$, the proof of Lemma 2.5 already shows that I^ε_{12}, I^ε_{21} and I^ε_{22} all vanish as $\varepsilon \to 0$. When $\gamma = 1$, the proof of Lemma 2.5 shows

$$|I^\varepsilon_{12}| + |I^\varepsilon_{21}| \leq C\varepsilon^2 \int_0^{(t-s)/\varepsilon^2} |R(z)|\,dz = C\varepsilon^2 g\left(\frac{t-s}{\varepsilon^2}\right),$$

where $g(x) = \int_0^x |R(z)|\,dz$. Since g is slowly varying this vanishes as $\varepsilon \to 0$. This proves (2.9) as claimed.

To prove (2.10), we note

$$\begin{aligned}
\boldsymbol{E}(w_1^\varepsilon(t) - w_1^\varepsilon(s))(w_2^\varepsilon(t) - w_2^\varepsilon(s)) &= \varepsilon^2 \int_{s/\varepsilon^2}^{t/\varepsilon^2} \int_{s/\varepsilon^2}^{t/\varepsilon^2} R(\theta - \tau)\sin(\theta)\cos(\tau)\,d\theta\,d\tau \\
&= \frac{\varepsilon^2}{2} \int_{s/\varepsilon^2}^{t/\varepsilon^2} \int_{s/\varepsilon^2}^{t/\varepsilon^2} R(\theta-\tau)[\sin(\theta+\tau) + \sin(\theta-\tau)]\,d\theta\,d\tau\,, \\
&= \tilde{I}^\varepsilon_1 + \tilde{I}^\varepsilon_2\,,
\end{aligned}$$

where

$$\begin{aligned}
\tilde{I}^\varepsilon_1 &\stackrel{\text{def}}{=} \varepsilon^2 \int_{s/\varepsilon^2}^{t/\varepsilon^2} \int_{s/\varepsilon^2}^{t/\varepsilon^2} \sin(\theta-\tau)R(\theta-\tau)\,d\theta\,d\tau, \\
\tilde{I}^\varepsilon_2 &\stackrel{\text{def}}{=} \varepsilon^2 \int_{s/\varepsilon^2}^{t/\varepsilon^2} \int_{s/\varepsilon^2}^{t/\varepsilon^2} \sin(\theta+\tau)R(\theta-\tau)\,d\theta\,d\tau.
\end{aligned}$$

We claim $\tilde{I}^\varepsilon_1 = 0$. To see this, make the changes of variables

$$\theta' \stackrel{\text{def}}{=} \frac{(t+s)}{\varepsilon^2} - \theta\,, \quad \tau' \stackrel{\text{def}}{=} \frac{(t+s)}{\varepsilon^2} - \tau\,, \tag{2.27}$$

and rewrite $\tilde{I}^\varepsilon_1$ as

$$\begin{aligned}
\tilde{I}^\varepsilon_1 &= \varepsilon^2 \int_{t/\varepsilon^2}^{s/\varepsilon^2} \int_{t/\varepsilon^2}^{s/\varepsilon^2} \sin(\tau'-\theta')R(\tau'-\theta')\,d\theta'\,d\tau' \\
&= -\varepsilon^2 \int_{s/\varepsilon^2}^{t/\varepsilon^2} \int_{s/\varepsilon^2}^{t/\varepsilon^2} \sin(\theta'-\tau')R(\theta'-\tau')\,d\theta'\,d\tau' \\
&= -\tilde{I}^\varepsilon_1\,.
\end{aligned}$$

This implies $\tilde{I}^\varepsilon_1 = 0$ as claimed.

We now turn our attention to $\tilde{I}_2^\varepsilon$. Making the change of variable $z = \theta - \tau$ we note

$$\begin{aligned}\tilde{I}_2^\varepsilon &= \varepsilon^2 \int_{s/\varepsilon^2}^{t/\varepsilon^2} \int_{s/\varepsilon^2-\tau}^{t/\varepsilon^2-\tau} \sin(z+2\tau) R(z)\, dz\, d\tau \\ &= \varepsilon^2 \int_{s/\varepsilon^2}^{t/\varepsilon^2} \int_{s/\varepsilon^2-\tau}^{t/\varepsilon^2-\tau} [\sin(z)\cos(2\tau) + \cos(z)\sin(2\tau)] R(z)\, dz\, d\tau\,.\end{aligned}$$

This is similar to the expression for I_2^ε (equation (2.21)) and following the proof of Lemma 2.5 we see $\tilde{I}_2^\varepsilon \to 0$ as $\varepsilon \to 0$. Since $\tilde{I}_1^\varepsilon$ and $\tilde{I}_2^\varepsilon$ both vanish as $\varepsilon \to 0$ we obtain (2.10) as claimed. □

2.3 Convergence of the noise to fBm

We have the following result showing the convergence of the normalized version of the noise to the fractional Brownian motion.

Proposition 2.7. *Let v be a stationary Gaussian process with covariance function R given by (2.3) for some $\gamma \in (0, 2)$ and a slowly varying function L. If $\gamma \in (1, 2)$, we additionally suppose*

$$\int_0^\infty R(t)\, dt = 0\,. \tag{2.28}$$

Let

$$\sigma(\varepsilon) = \begin{cases} L(\varepsilon^{-2})^{1/2} \varepsilon^\gamma & \gamma \neq 1\,, \\ L(\varepsilon^{-2})^{1/2} \varepsilon |\ln \varepsilon|^{1/2} & \gamma = 1\,, \end{cases} \quad \text{and} \quad u^\varepsilon(t) \stackrel{\text{def}}{=} \frac{1}{\sigma(\varepsilon)} \int_0^t v\left(\frac{s}{\varepsilon^2}\right) ds\,. \tag{2.29}$$

Then, as $\varepsilon \to 0$, the family of processes $(u^\varepsilon)_{\varepsilon>0}$ converges in distribution to $\sigma_{\mathcal{H}} B^{\mathcal{H}}$, where $B^{\mathcal{H}}$ is a standard fractional Brownian motion with Hurst index $\mathcal{H} = 1 - \gamma/2$, and

$$\sigma_{\mathcal{H}}^2 \stackrel{\text{def}}{=} \begin{cases} \dfrac{1}{\mathcal{H}|2\mathcal{H}-1|} & \mathcal{H} \neq \dfrac{1}{2}\,, \\ 1 & \mathcal{H} = \dfrac{1}{2}\,. \end{cases} \tag{2.30}$$

Proof of Proposition 2.7 when $\gamma \in (0, 1)$. Using (2.2) and (2.29) we note

$$\boldsymbol{E}(u^\varepsilon(t) - u^\varepsilon(s))^2 = \frac{2}{\sigma(\varepsilon)^2} \int_{r_1=s}^t \int_{r_2=s}^{r_1} R\left(\frac{r_1 - r_2}{\varepsilon^2}\right) dr_2\, dr_1\,. \tag{2.31}$$

By (2.3) and the uniform convergence theorem, we know

$$\frac{1}{\sigma(\varepsilon)^2} \left| R\left(\frac{r_1 - r_2}{\varepsilon^2}\right) \right| = \frac{1}{L(\varepsilon^{-2})|r_1 - r_2|^\gamma} L\left(\frac{r_1 - r_2}{\varepsilon^2}\right) \leq \frac{2}{(r_1 - r_2)^\gamma}\,, \tag{2.32}$$

for all $r_1, r_2 \in [s,t]$ and all sufficiently small ε. Hence

$$\boldsymbol{E}(u^\varepsilon(t) - u^\varepsilon(s))^2 \leq 4 \int_{r_1=s}^{t} \int_{r_2=s}^{r_1} \frac{dr_2\, dr_1}{(r_1 - r_2)^\gamma} = \frac{4(t-s)^{2-\gamma}}{(1-\gamma)(2-\gamma)}\,. \tag{2.33}$$

Moreover (2.3), (2.32) and the dominated convergence theorem imply

$$\lim_{\varepsilon \to 0} \boldsymbol{E}(u^\varepsilon(t) - u^\varepsilon(s))^2 = 2 \int_{r_1=s}^{t} \int_{r_2=s}^{r_1} \frac{dr_2\, dr_1}{(r_1 - r_2)^\gamma} = \frac{(t-s)^{2\mathcal{H}}}{(2\mathcal{H}-1)\mathcal{H}}\,.$$

Since u^ε is Gaussian this implies that the finite-dimensional distributions of u^ε converge to that of $\sigma_{\mathcal{H}} B^{\mathcal{H}}$. For convergence in law, note (2.33) implies

$$\begin{aligned} \boldsymbol{E}[(u^\varepsilon(s) - u^\varepsilon(r))^2 (u^\varepsilon(t) - u^\varepsilon(s))^2] &\leq \boldsymbol{E}[(u^\varepsilon(s) - u^\varepsilon(r))^4]^{1/2} \boldsymbol{E}[(u^\varepsilon(t) - u^\varepsilon(s))^4]^{1/2} \\ &\leq 3\boldsymbol{E}[(u^\varepsilon(s) - u^\varepsilon(r))^2] \boldsymbol{E}[(u^\varepsilon(t) - u^\varepsilon(s))^2] \\ &\leq C|t-r|^{4\mathcal{H}}\,. \end{aligned}$$

Since $4\mathcal{H} > 1$, Theorems 13.4 and 13.5 in [Bil99] imply that u^ε converges to $\sigma_{\mathcal{H}} B^{\mathcal{H}}$ in law. □

Proof of Proposition 2.7 when $\gamma \in (1,2)$. Using (2.2), (2.29) and a change of variable we note

$$\begin{aligned} \boldsymbol{E}(u^\varepsilon(t) - u^\varepsilon(s))^2 &= \frac{2}{\sigma(\varepsilon)^2} \int_{r_1=s}^{t} \int_{r_2=s}^{r_1} R\left(\frac{r_1 - r_2}{\varepsilon^2}\right) dr_2\, dr_1 \\ &= \frac{2}{\sigma(\varepsilon)^2} \int_{r_1=s}^{t} \int_{z=0}^{r_1 - s} R\left(\frac{r_1 - r_2}{\varepsilon^2}\right) dz\, dr_1 \\ &= \frac{-2}{\sigma(\varepsilon)^2} \int_{r_1=s}^{t} \int_{z=r_1-s}^{\infty} R\left(\frac{z}{\varepsilon^2}\right) dz\, dr_1\,, \end{aligned} \tag{2.34}$$

where the last equality followed from (2.28). Since L is slowly varying, (2.3) implies

$$\frac{1}{\sigma(\varepsilon)^2} R\left(\frac{z}{\varepsilon^2}\right) \leq R(0) \wedge \left(\frac{C}{z^{\gamma'}}\right)\,, \tag{2.35}$$

where $\gamma' = (1+\gamma)/2$ and C is a finite constant that is independent of ε. We will subsequently allow C to change from line to line, as long as it doesn't depend on ε.

Note (2.34) and (2.35) immediately imply

$$\boldsymbol{E}(u^\varepsilon(t) - u^\varepsilon(s))^2 \leq 2C \int_{r_1=s}^{t} \int_{z=r_1-s}^{\infty} \frac{dz\, dr_1}{z^{\gamma'}} = C(t-s)^{2-\gamma'}\,. \tag{2.36}$$

Moreover, by (2.3), (2.35) and the dominated convergence theorem, we see

$$\lim_{\varepsilon \to 0} \boldsymbol{E}(u^\varepsilon(t) - u^\varepsilon(s))^2 = 2 \int_{r_1=s}^{t} \int_{z=r_1-s}^{\infty} \frac{dz\, dr_2}{z^\gamma} = \frac{(t-s)^{2\mathcal{H}}}{(1-2\mathcal{H})\mathcal{H}}\,.$$

Now the remainder of the proof is identical to the case when $\gamma \in (0,1)$. □

Proof of Proposition 2.7 when $\gamma = 1$. In this case we write

$$\boldsymbol{E}(u^\varepsilon(t) - u^\varepsilon(s))^2 = \frac{2}{\sigma(\varepsilon)^2} \int_{r=s}^{t} \int_{z=0}^{r-s} R\left(\frac{z}{\varepsilon^2}\right) dz\, dr \tag{2.37}$$

$$= \frac{2}{L(\varepsilon^{-2})|\ln \varepsilon|} \int_{r=s}^{t} \int_{z=0}^{1} R(z)\, dz\, dr + \frac{2}{\sigma(\varepsilon)^2} \int_{r=s}^{t} \int_{z=\varepsilon^2}^{r-s} R\left(\frac{z}{\varepsilon^2}\right) dz\, dr \tag{2.38}$$

The first term on the right vanishes as $\varepsilon \to 0$. Using the uniform convergence theorem [BGT89, Th 1.2.1] on the second term we see

$$\lim_{\varepsilon \to 0} \boldsymbol{E}(u^\varepsilon(t) - u^\varepsilon(s))^2 = \lim_{\varepsilon \to 0} \frac{2}{|\ln \varepsilon|} \int_{r=s}^{t} \int_{z=\varepsilon^2}^{r-s} \frac{1}{z}\, dz\, dr = t - s\,,$$

and the convergence is uniform when s, t belong to any bounded interval. Hence by the same argument as in the previous two cases we see that (u^ε) converges in law to $\sigma_{\mathcal{H}} B^{\mathcal{H}}$ as claimed. □

Chapter 3

Hamiltonians with one non-degenerate critical point

3.1 Algebraically decorrelating noises

In this section, we will construct the noise v that will be used in the upcoming theorems. Unlike in the case of quadratic Hamiltonian, we need the noise to have certain exponential estimates for the proofs to work. Specifically, we require the covariance function R to be of the form

$$R(t) \stackrel{\text{def}}{=} \int_S r(p) e^{-\mu |p|^{2\beta} |t|} \, dp \,, \tag{3.1}$$

where $S = (-r_s, r_s)$ is a symmetric, bounded, open interval, $\mu, \beta > 0$ are constants, and $r \colon S \setminus \{0\} \to [0, \infty)$ is defined by

$$r(p) \stackrel{\text{def}}{=} \frac{\lambda(p)}{|p|^{2\alpha}} \,.$$

Here $\lambda \colon S \to \mathbb{R}$ is a smooth bounded even function such that $\lambda(0) \neq 0$ and

$$\int_S r(p) \, dp > 0 \,,$$

and

$$\alpha < \frac{1}{2} \,. \tag{3.2}$$

We will now construct a stationary Gaussian process v with covariance function R. The chosen form of R allows us to construct v by superimposing Ornstein–Uhlenbeck processes as follows. Let ξ be 2D white noise and define a Gaussian random measure $\mathcal{B}$ by

$$\mathcal{B}(du, dp) = \sqrt{2\mu r(p)} \, |p|^{\beta} \mathbf{1}_S(p) \, \xi(du, dp) \,,$$

Clearly, the covariance of $\mathcal{B}$ is given by

$$\boldsymbol{E} \mathcal{B}(du, dp) \, \mathcal{B}(du', dp') = 2\mu \, r(p) \, |p|^{2\beta} \delta(u - u') \delta(p - p') \, du \, du' \, dp \, dp' \,.$$

Now define the measure-valued Gaussian random process V by

$$V(t, dp) = \int_{-\infty}^{t} e^{-\mu |p|^{2\beta} (t-u)} \, \mathcal{B}(du, dp) \,,$$

and finally define the noise v by

$$v(t) = \int_S V(t, dp)\,.$$

We now establish a few properties of the process v that will be useful later.

Lemma 3.1. *The process v defined above is a stationary Gaussian process with covariance function R.*

Proof. Clearly v is a Gaussian process. To see that v is stationary with covariance function R we choose any $s \leq t$ and compute

$$\begin{aligned}\boldsymbol{E} v(s)v(t) &= \int_{-\infty}^{s} \int_S \exp\left(-\mu\left(|p|^{2\beta}(s-u) + |p|^{2\beta}(t-u)\right)\right) \cdot 2\mu|p|^{2\beta} r(p)\, du\, dp \\ &= \int_S r(p) e^{-\mu|p|^{2\beta}(t-s)}\, dp = R(t-s)\,.\end{aligned}$$

□

Lemma 3.2. *If R is given by* (3.1)*, then*

$$\lim_{t\to\infty} t^{\gamma} R(t) = c_0$$

where c_0 and γ are defined by

$$\gamma \stackrel{\text{def}}{=} \frac{1-2\alpha}{2\beta} \qquad \text{and} \qquad c_0 \stackrel{\text{def}}{=} \lambda(0) \int_{p\in\mathbb{R}} \frac{e^{-\mu|p|^{2\beta}}}{|p|^{2\alpha}}\, dp\,.$$

Proof. By a change of variable, note that

$$t^{\gamma} R(t) = t^{\gamma} \int_S \frac{\lambda(p)}{|p|^{2\alpha}} e^{-\mu|p|^{2\beta} t}\, dp = \int_{(t^{1/(2\beta)}S)} \frac{1}{|p'|^{2\alpha}} \lambda\left(\frac{p'}{t^{1/(2\beta)}}\right) e^{-\mu|p'|^{2\beta}}\, dp'\,,$$

which converges to c_0 as $t \to \infty$. □

Let $\{\mathcal{G}_t\}$ be the augmented filtration generated by V. That is,

$$\mathcal{G}_t \stackrel{\text{def}}{=} \sigma\left(\mathcal{N} \cup \{\sigma(V(s,\cdot)) : 0 \leq s \leq t\}\right)\,, \tag{3.3}$$

where $\mathcal{N}$ is the set of all null sets known at time infinity. Our next lemma computes conditional expectations of $V(\cdot, dp)$ with respect to the filtration $\mathcal{G}$.

Lemma 3.3. *We have for any $t, h \geq 0$*

$$\boldsymbol{E}\big[V(t+h, dp)|\mathcal{G}_t\big] = e^{-\mu|p|^{2\beta}h}\, V(t, dp) \tag{3.4}$$

and

$$\boldsymbol{E}\Big[V(t+h,dp)V(t+h,dq)\Big|\mathcal{G}_t\Big] - \boldsymbol{E}\big[V(t+h,dp)|\mathcal{G}_t\big]\,\boldsymbol{E}\big[V(t+h,dq)|\mathcal{G}_t\big] \\ = (1-e^{-2\mu|p|^{2\beta}h})r(p)\delta(p-q)\,dp\,dq\,. \tag{3.5}$$

Our last result concerns the boundedness of our noise process.

Lemma 3.4. *Let $T>0$, $M>0$,*

$$D_{k,M} \stackrel{\text{def}}{=} [0,T]\times L^\infty([0,T],W_{k,M})$$

with

$$W_{k,M} \stackrel{\text{def}}{=} \big\{\varphi\in W^{1,k}(S): \quad \|\varphi\|_{W^{1,k}}\le M\big\}$$

and where $W^{1,k}(S)$ stands for the Sobolev space with $k\in(1,\infty]$. We have

$$\boldsymbol{E}\Big[\sup_{(t,\varphi)\in D_{k,M}}\Big|V\Big(\frac{t}{\varepsilon^2},\varphi(t,\cdot)\Big)\Big|\Big] \le C + \frac{C(\varepsilon)}{\varepsilon}\,, \tag{3.6}$$

and for any $n\in\mathbb{N}^*$

$$\sup_{\varepsilon}\sup_{t\in[0,T]}\boldsymbol{E}\Big[\sup_{\varphi\in W_{k,M}}\Big|V\Big(\frac{t}{\varepsilon^2},\varphi\Big)\Big|^n\Big] \le C_n\,, \tag{3.7}$$

where C, C_n and $C(\varepsilon)$ are three positive constants where the latter satisfies

$$\lim_{\varepsilon\to 0} C(\varepsilon) = 0.$$

Here the notation $V(t/\varepsilon^2,\varphi)$ denotes integral of a function $\varphi(p)$ with respect to the measure $V(t/\varepsilon^2,dp)$. That is,

$$V\Big(\frac{t}{\varepsilon^2},\varphi\Big) \stackrel{\text{def}}{=} \int_S \varphi(p)\,V\Big(\frac{t}{\varepsilon^2},dp\Big)\,,$$

and we will drop the S in the above notation for simplicity unless specified otherwise. The proofs of these lemmas will be postponed to Sections 3.7.1 and 3.7.2.

3.2 Action-angle variables

One canonical way to analyze integrable Hamiltonian systems is to use a set of *action-angle coordinates.* These coordinates separate the slow and fast motion, and preserves the Hamiltonian structure. The Liouville-Arnold theorem [Arn89, SP18] asserts that there exists a symplectic canonical transformation $\varphi\colon X=(x,y)\mapsto(I,\theta)\in\mathbb{R}\times\mathbb{T}$, where the action variable I and the angle variable θ satisfy

$$K(I) = H(x,y) \qquad \text{and} \qquad \{I,\theta\} = 1\,, \tag{3.8}$$

where K is a given one variable smooth enough increasing function such that $K(0) = 0$, $\{\cdot, \cdot\}$ stands for the standard Poisson bracket and is defined by

$$\{g, h\} = \partial_x g \partial_y h - \partial_y g \partial_x h \,,$$

for our Hamiltonian system with one degree of freedom. The relation on the right-hand side of (3.8) can be also stated as

$$\nabla I \cdot \nabla^{\perp} \theta = 1 \,.$$

Note that in the action-angle coordinate the Hamiltonian is a function of the action coordinate alone.

The existence of such transformation φ is guaranteed by the Liouville-Arnold theorem, and can be constructed through a generating function [Arn89, Section 50 pp. 281], and implicitly defined by the relations

$$y = \partial_x \mathcal{S}(I, x) \,, \qquad \theta = \partial_I \mathcal{S}(I, x) \,, \qquad \text{and} \qquad H(x, \partial_x \mathcal{S}(I, x)) = K(I) \,. \tag{3.9}$$

Such construction is not unique. For convenience in the forthcoming analysis we will choose $\mathcal{S}(I, 0) = 0$ so that

$$\mathcal{S}(I, x) = \int_0^x \partial_{x'} \mathcal{S}(I, x') \, dx' \,,$$
$$\partial_I \mathcal{S}(I, x) = \int_0^x \partial_{Ix'} \mathcal{S}(I, x') \, dx' \,,$$

and hence

$$\partial_I \mathcal{S}(I, 0) = 0 \,.$$

The meaning of the second relation is that the zero angle corresponds to either the positive part of the y-axis or the negative part. Consequently, we have for any $I > 0$

$$\varphi_1^{-1}(I, \theta = 0) = 0 \qquad \text{and} \qquad \varphi_2^{-1}(I, \theta = 0) \neq 0 \,, \tag{3.10}$$

and in particular,

$$\partial_I \varphi_1^{-1}(I, \theta = 0) = 0 \,. \tag{3.11}$$

Let us remark that (3.10) follows from the fact that K cancels only for $I = 0$, and then for any $I > 0$, the component x and y cannot cancel at the same time.

Now let us write $\varphi = (\varphi_1, \varphi_2)$ and define

$$I_t \stackrel{\text{def}}{=} I(X_t) = \varphi_1(X_t) \,, \qquad \text{and} \qquad \theta_t \stackrel{\text{def}}{=} \theta(X_t) = \varphi_2(X_t) \,.$$

In the absence of random fluctuations (i.e. when $v = 0$), the Hamiltonian system $\dot{X}_t = \nabla^{\perp} H(X_t)$ becomes

$$\dot{I}_t = 0 \,, \qquad \text{and} \qquad \dot{\theta}_t = \omega(I_t) \,, \qquad \text{with} \qquad \omega(I) = K'(I) \,.$$

In the presence of random fluctuations, the Hamiltonian system (1.3) becomes

$$\dot{I}_t = \varepsilon v(t) a(I_t, \theta_t)\,, \qquad \text{and} \qquad \dot{\theta}_t = \omega(I_t) + \varepsilon v(t) b(I_t, \theta_t)\,, \tag{3.12}$$

where

$$a = e_2 \cdot \nabla \varphi_1 \circ \varphi^{-1}\,, \quad \text{and} \quad b = e_2 \cdot \nabla \varphi_2 \circ \varphi^{-1}\,.$$

For illustration purposes, we now explicitly compute the action-angle variables when the Hamiltonian is quadratic. From (3.9), we have

$$x = \sqrt{\frac{I}{\pi}} \cos\left(2\pi\theta - \frac{\pi}{2}\right) \qquad \text{and} \qquad y = \sqrt{\frac{I}{\pi}} \sin\left(2\pi\theta - \frac{\pi}{2}\right)$$

where θ gives the angle of $X = (x, y)^T$ from the negative part of the vertical axis. One can then see that the action variable is a multiple of the Hamiltonian,

$$K(I) = \frac{I}{2\pi} \qquad \text{and} \qquad \omega(I) = \frac{1}{2\pi},$$

while the angle variable corresponds to the angle on a trajectory, which is a circle in the case of $H(X) = |X|^2/2$, with period 1. In this case, the functions a and b are as follows

$$a(I, \theta) = 2\sqrt{\pi I} \sin\left(2\pi\theta - \frac{\pi}{2}\right) \quad \text{and} \quad b(I, \theta) = \frac{1}{2\sqrt{\pi I}} \cos\left(2\pi\theta - \frac{\pi}{2}\right)\,. \tag{3.13}$$

3.3 Main theorem

To study the long-time behavior of the system (3.12), we let

$$I_t^\varepsilon = I_{t/\varepsilon^2} \qquad \text{and} \qquad \theta_t^\varepsilon = \theta_{t/\varepsilon^2}\,.$$

Now the system (3.12) becomes

$$\dot{I}_t^\varepsilon = \frac{1}{\varepsilon} v\Big(\frac{t}{\varepsilon^2}\Big) a(I_t^\varepsilon, \theta_t^\varepsilon)\,, \qquad \dot{\theta}_t^\varepsilon = \frac{\omega(I_t^\varepsilon)}{\varepsilon^2} + \frac{1}{\varepsilon} v\Big(\frac{t}{\varepsilon^2}\Big) b(I_t^\varepsilon, \theta_t^\varepsilon)\,, \tag{3.14}$$

with $I_0^\varepsilon = I_0$ and $\theta_0^\varepsilon = \theta_0$. Using the approach in [Gar97] to separate fast and slow motions, we study this system by splitting the angle variable θ_t^ε into two parts

$$\theta_t^\varepsilon = \psi_t^\varepsilon + \tau_t^\varepsilon\,.$$

The evolution of τ_t^ε (fast motion) is obtained by averaging (3.14) over the angular coordinate, and ψ_t^ε (slow motion) is the remainder. That is we require

$$\dot{\tau}_t^\varepsilon = \frac{\omega(I_t^\varepsilon)}{\varepsilon^2} + \frac{1}{\varepsilon} v\Big(\frac{t}{\varepsilon^2}\Big) \langle b(I_t^\varepsilon, \cdot) \rangle \tag{3.15}$$

with initial condition $\tau_0^\varepsilon = 0$, and where we have the notation

$$\langle g \rangle \stackrel{\text{def}}{=} \int_{\theta=0}^{1} g(\theta)\, d\theta\,.$$

Considering only the slow motion variables $(I_t^\varepsilon, \psi_t^\varepsilon)$, the system (3.14) becomes

$$\begin{aligned} \dot{I}_t^\varepsilon &= \frac{1}{\varepsilon} v\Big(\frac{t}{\varepsilon^2}\Big) A(I_t^\varepsilon, \psi_t^\varepsilon, \tau_t^\varepsilon) \\ \dot{\psi}_t^\varepsilon &= \frac{1}{\varepsilon} v\Big(\frac{t}{\varepsilon^2}\Big) B(I_t^\varepsilon, \psi_t^\varepsilon, \tau_t^\varepsilon) \end{aligned} \tag{3.16}$$

with

$$A(I,\psi,\tau) \stackrel{\text{def}}{=} a(I,\psi+\tau) \qquad \text{and} \qquad B(I,\psi,\tau) \stackrel{\text{def}}{=} b(I,\psi+\tau) - \langle b(I,\cdot)\rangle\,.$$

The above equations are coupled with the initial conditions $I_0^\varepsilon = I_0$, and $\psi_0^\varepsilon = \theta_0$. Note that A and B are both 1-periodic and mean-zero with respect to τ. This latter property is mandatory to deal with the long-range correlation case as illustrated in Proposition 2.2. In fact, if the system possesses a component with zero frequency, a noise with long-range correlations will charge this component and cause the system to blow up as $\varepsilon \to 0$.

It is straightforward to see this mean-zero property in τ for B, but for A we use that

$$\operatorname{Jac}\varphi^{-1}(I,\theta) = [\operatorname{Jac}\varphi(\varphi^{-1}(I,\theta))]^{-1}$$

and that $\det \operatorname{Jac}\varphi(X) = 1$ since φ is a symplectic transformation, to obtain the following relations

$$\begin{aligned} \partial_I \varphi_1^{-1}(I,\theta) &= \partial_y \varphi_2(X), \quad \partial_I \varphi_2^{-1}(I,\theta) = -\partial_x \varphi_2(X)\,, \\ \partial_\theta \varphi_1^{-1}(I,\theta) &= -\partial_y \varphi_1(X), \quad \partial_\theta \varphi_2^{-1}(I,\theta) = \partial_x \varphi_1(X)\,. \end{aligned} \tag{3.17}$$

Therefore, we have

$$a(I,\theta) = -\partial_\theta \varphi_1^{-1}(I,\theta)$$

which is clearly mean-zero with respect to θ since we have the derivative of a periodic function. Note also that from these relations the mean-zero property in θ for b is not clear. This is the reason why we introduce the compensation $\langle b \rangle$ in the definition of B.

Our main result obtains the limiting behavior of the system (3.16) as $\varepsilon \to 0$ under the following assumptions.

- The function K is smooth, and

$$\inf_{I\geq 0} K'(I) = \inf_{I \geq 0} \omega(I) > \omega_0 > 0 \tag{3.18}$$

  for some strictly positive number ω_0.

- There exist $r > 0$, and positive constants $c_{1,r}$, $c_{2,r} > 0$ such that for any $I \in (0,r)$

$$c_{1,r} I \leq K(I) \leq c_{2,r} I \qquad \text{and} \qquad |\omega'(I)| \leq \frac{c_{2,r}}{I}\,. \tag{3.19}$$

Note that these conditions imply that $K(0) = 0$ and that K is an increasing function in I. These assumptions are not too restrictive since any Hamiltonian satisfies (3.18) and (3.19) near non-degenerate critical points. The following lemma is a consequence of the above assumptions, which concerns bounds on the function a.

Lemma 3.5. *Under* (3.18) *and* (3.19) *there exist* $r > 0$ *and a constant* $C_r > 0$ *such that for any* $I \in (0, r)$ *we have*

$$\sup_{\theta \in \mathbb{T}} \Big(|a(I,\theta)| + |\partial_\theta a(I,\theta)| \Big) \le C_r \sqrt{I}\,,$$

and

$$\sup_{\theta \in \mathbb{T}} |\partial_I a(I,\theta)| \le \frac{C_r}{\sqrt{I}}\,.$$

The lemma mainly describes the behavior of $a, \partial_I a$ and $\partial_\theta a$ around $I = 0$. These bounds will allow us to prove the main result that characterizes the limiting process of the sequence of processes $(I^\varepsilon, \psi^\varepsilon)_{\varepsilon>0}$. The proof of this lemma is postponed to Section 3.7.3 below. We state the main result about the convergence of action-angle processes as follows.

Theorem 3.6. *Assume* (3.18) *and* (3.19) *hold, the family* $(I^\varepsilon, \psi^\varepsilon)_{\varepsilon>0}$ *(defined in* (3.15)–(3.16)*) converges in distribution in* $\mathcal{C}([0,\infty), \mathbb{R}^2)$ *to a process* $(I_t, \psi_t)_{t \ge 0}$*, where* $(I_t)_{t\ge 0}$ *is the unique weak solution of the SDE*

$$\begin{aligned} dI_t = &\int_{\tau=0}^{1} a(I_t,\tau)\, dW_t(I_t,\tau)\, d\tau \\ &+ \Big[\int_{u=0}^{\infty} R(u) \int_{\tau=0}^{1} \partial_I \big(a(I, \tau + \omega(I)u) \big)_{|I=I_t} a(I_t,\tau) \\ &\qquad + a(I_t, \tau + \omega(I_t)u) \partial_I a(I_t,\tau)\, d\tau\, du \Big]\, dt\,, \end{aligned} \tag{3.20}$$

with initial condition $I_{t=0} = I_0$*. Here,* W *is a real valued Brownian field with covariance function*

$$\begin{aligned} \boldsymbol{E}[W_t(y,\phi_1) W_s(y,\phi_2)] = t \wedge s &\int_{u=0}^{\infty} R(u) \\ &\times \int_{\tau=0}^{1} \phi_1(\tau + \omega(y)u)\phi_2(\tau) + \phi_1(\tau)\phi_2(\tau + \omega(y)u)\, d\tau\, du\,, \end{aligned} \tag{3.21}$$

for any $\phi_1, \phi_2 \in L^2_0(\mathbb{T})$ *where*

$$L^2_0(\mathbb{T}) \stackrel{\text{def}}{=} \big\{ \phi \in L^2(\mathbb{T}) : \langle \phi \rangle = 0 \big\}\,.$$

Explicitly, we can write

$$W_t(y,\tau) = \frac{1}{\sqrt{2}} \sum_{n \in \mathbb{Z}^*} e^{2i\pi n\tau} R_n^{1/2}(y) (B^1_{t,n} - iB^2_{t,n})$$

where $B^1_{t,n}$ *and* $B^2_{t,n}$ *are two independent real valued cylindrical Brownian motions satisfying* $B^1_{t,-n} = B^1_{t,n}$, $B^2_{t,-n} = -B^2_{t,n}$, *and* $R_n = 2\int_0^\infty R(u)\cos(2\pi n\omega(I)u)\,du$.

Also, we have

$$\begin{aligned} d\psi_t = &\int_{\tau=0}^1 b(I_t,\tau)\,dW_t(I_t,\tau)\,d\tau \\ &+ \Big[\int_{u=0}^\infty R(u)\int_{\tau=0}^1 \partial_I\big(b(I,\tau+\omega(I)u)\big)_{|I=I_t} a(I_t,\tau) \\ &\quad + b(I_t,\tau+\omega(I_t)u)\partial_I a(I_t,\tau)\,d\tau\,du\Big]dt\,, \end{aligned} \tag{3.22}$$

with $\psi_{t=0} = \theta_0$.

Note that the action variable I does not depend on the slow angular motion ψ. However, the distribution of the slow angular motion ψ is completely determined by the motion of the action variable I and the Brownian field.

We also note that the functions $\partial_I a$ and $\partial_I b$ appearing in (3.20) and (3.22) may be singular at $I = 0$. Nevertheless, since the minimum of H is non-degenerate, the function $a\partial_I a$ has no singularity at $I = 0$ and there is no singularity on the right-hand side of equation (3.20). The term $a\partial_I b$ appearing in (3.22) may still have an $O(I^{-1/2})$ singularity at $I = 0$. However, since the point $I = 0$ is inaccessible, the right-hand side of (3.22) is well defined.

We now use Theorem 3.6 to prove the following theorem that shows the convergence of the processes $H(X^\varepsilon)$.

Theorem 3.7. *Let* $H\colon \mathbb{R}^2 \to \mathbb{R}$ *be a smooth Hamiltonian with exactly one non-degenerate minimum at* $(0,0)$. *Then*

$$H(X^\varepsilon) \to \mathcal{X}\,, \tag{3.23}$$

where $\mathcal{X}$ *is a diffusion driven by standard Brownian motion.*

Proof of Theorem 3.7. Using action angle coordinates, we convert the system (1.4) to (3.16). Since the Hamiltonian H has exactly one non-degenerate critical point, the assumptions (3.18)–(3.19) are satisfied. Now by Theorem 3.6 we see that the family of processes $(I^\varepsilon,\psi^\varepsilon)_{\varepsilon>0}$ converges in distribution to the pair (I,ψ) which solves (3.20), (3.22). Since $H(X^\varepsilon) = K(I^\varepsilon)$, we now obtain Theorem 3.7 by applying the Itô formula to $\mathcal{X} = K(I)$. □

For the quadratic Hamiltonian we can explicitly derive the stochastic differential equations for $(I_t)_{t\geq 0}$ and $(\psi_t)_{t\geq 0}$. Using equation (3.13) we obtain

$$dI_t = 2\sqrt{mI_t}dB^1_t + 2m\,dt \qquad \text{and} \qquad d\psi_t = \frac{m}{2\pi\sqrt{I_t}}\,dB^2_t \tag{3.24}$$

with

$$m = \pi\int_0^\infty R(u)\cos(u)\,du\,,$$

and where B^1 and B^2 are two independent standard Brownian motions. One can easily remark that $(I_t/m)_{t\geq 0}$ is a 2-dimensional squared Bessel process, so that $(I_t/2\pi)_{t\geq 0}$

has the same law as $(|W_t|^2/2)_{t\geq 0}$, for W being a 2-dimensional Brownian motion with covariance matrix given by (2.5). Then, we recover the result of Proposition 2.2.

3.4 Truncated processes

In this section, we describe the strategy used to prove Theorem 3.6. In addition to the difficulties concerning the behavior of a around $I = 0$ that we mentioned in Theorem 3.6, we have no *apriori* estimates for the process $(I^\varepsilon_t, \psi^\varepsilon_t)_{t\geq 0}$ that are uniform in ε, in probability. This will cause a problem in showing the tightness property, and when trying to identify the law of the subsequential limits. To bypass this issue, we follow the strategy developed in [SV06, Chapter 11] by introducing a *truncated process* that does not suffer from the above problems. To relate the truncated process with the original one, we introduce a family of stopping times. As we will see, these times go to ∞ when we remove the truncation and then finally obtain the weak convergence of the original processes.

Let $M > 0$ and consider the process $(I^{\varepsilon,M}_t, \psi^{\varepsilon,M}_t)_{t\geq 0}$ which is the unique solution to

$$\begin{aligned} \dot I^{\varepsilon,M}_t &= \frac{1}{\varepsilon} v\Big(\frac{t}{\varepsilon^2}\Big)\phi_M(I^{\varepsilon,M}_t, \psi^{\varepsilon,M}_t) a(I^{\varepsilon,M}_t, \psi^{\varepsilon,M}_t + \tau^{\varepsilon,M}_t) \\ \dot \psi^{\varepsilon,M}_t &= \frac{1}{\varepsilon} v\Big(\frac{t}{\varepsilon^2}\Big)\phi_M(I^{\varepsilon,M}_t, \psi^{\varepsilon,M}_t) (b(I^{\varepsilon,M}_t, \psi^{\varepsilon,M}_t + \tau^{\varepsilon,M}_t) - \langle b(I^{\varepsilon,M}_t, \cdot)\rangle) \end{aligned} \tag{3.25}$$

coupled with the initial conditions $I^{\varepsilon,M}_0 = I_0$, and $\psi^{\varepsilon,M}_0 = \theta_0$, with

$$\dot\tau^{\varepsilon,M}_t = \frac{\omega(I^{\varepsilon,M}_t)}{\varepsilon^2} + \frac{1}{\varepsilon} v\Big(\frac{t}{\varepsilon^2}\Big)\phi_M(I^{\varepsilon,M}_t, \psi^{\varepsilon,M}_t)\langle b(I^{\varepsilon,M}_t, \cdot)\rangle\,.$$

Here ϕ_M is a smooth function on $\mathbb{R}^2$ such that

$$0 \leq \phi_M \leq 1, \quad \phi_M(I,\psi) = 1 \quad \text{if} \quad 1/M \leq I \leq M \quad \text{and} \quad |\psi| \leq M\,,$$

and

$$\phi_M(I,\psi) = 0 \quad \text{if} \quad I > 2M \quad \text{or} \quad I < 1/(2M) \quad \text{or} \quad |\psi| > 2M\,.$$

Thanks to the truncation provided by the cutoff function ϕ_M, for both action and angle variables, the process $(I^{\varepsilon,M}_t, \psi^{\varepsilon,M}_t)_{t\geq 0}$ does not suffer from the regularity problem for the action variable around 0 as mentioned above. Also, its convergence in distribution in $\mathcal{C}([0,\infty), \mathbb{R}^2)$ can be proved using the perturbed-test function method and martingale properties [Kus84, PSV76] thanks to the boundedness of the process. All these points are developed more precisely in Sections 3.5.1 and 3.5.2.

To simplify the notation, we now omit the superscript M and denote

$$Y^\varepsilon_t \stackrel{\text{def}}{=} \begin{pmatrix} I^{\varepsilon,M}_t \\ \psi^{\varepsilon,M}_t \end{pmatrix}\,.$$

Now the system (3.25) can be rewritten as

$$\dot{Y}_t^\varepsilon = \frac{1}{\varepsilon} v\left(\frac{t}{\varepsilon^2}\right) F(Y_t^\varepsilon, \tau_t^\varepsilon) \tag{3.26}$$

with

$$\dot{\tau}_t^\varepsilon = \frac{\omega(Y_t^\varepsilon)}{\varepsilon^2} + \frac{1}{\varepsilon} v\left(\frac{t}{\varepsilon^2}\right) G(Y_t^\varepsilon)\,. \tag{3.27}$$

Here

$$\begin{aligned} F(Y,\tau) &\stackrel{\text{def}}{=} \begin{pmatrix} \phi_M(I,\psi)a(I,\psi+\tau) \\ \phi_M(I,\psi)(b(I,\psi+\tau) - \langle b(I,\cdot)\rangle) \end{pmatrix}, \\ G(Y) &\stackrel{\text{def}}{=} \phi_M(I,\psi)\langle b(I,\cdot)\rangle\,, \end{aligned} \tag{3.28}$$

and a, b are defined in equation (3.12). Note that F is a smooth bounded function (with bounded derivatives) 1-periodic and mean-zero with respect to τ (the truncation only affects ψ, not τ), and $\omega(Y_t^\varepsilon) = \omega(I_t^{\varepsilon,M})$.

To relate the truncated and original processes we introduce first some notations. In the remaining of the thesis, the space of all possible outcomes is $\mathcal{C}([0,\infty),\mathbb{R}^2)$, and we denote the corresponding canonical filtration by $\mathcal{M}_t = \sigma(y_s, 0 \le s \le t)$. The laws of the truncated and original processes on $\mathcal{C}([0,\infty),\mathbb{R}^2)$ will be denoted respectively by $\boldsymbol{P}^{\varepsilon,M}$ and $\boldsymbol{P}^\varepsilon$. Now, we consider the stopping times

$$\eta_{1,M}(y) \stackrel{\text{def}}{=} \inf\{t \ge 0 : \|y_t\| \ge M\} \quad \text{and} \quad \eta_{2,M}(y) \stackrel{\text{def}}{=} \inf\{t \ge 0 : |y_t^1| \le 1/M\}\,,$$

for any $y = (y^1, y^2) \in \mathcal{C}([0,\infty),\mathbb{R}^2)$ and

$$\eta_M \stackrel{\text{def}}{=} \eta_{1,M} \wedge \eta_{2,M}\,. \tag{3.29}$$

It is not hard to see that $\eta_{1,M}$ and $\eta_{2,M}$ are lower semi-continuous so that η_M is also lower semi-continuous, that is their lower level sets are closed subsets of $\mathcal{C}([0,\infty),\mathbb{R}^2)$. The latter property will be used to obtain the convergence of the original process from the truncated one.

From the definition of the above stopping times and the cutoff function ϕ_M it is clear that

$$\boldsymbol{P}^{\varepsilon,M} = \boldsymbol{P}^\varepsilon \qquad \text{on} \quad \mathcal{M}_{\eta_M}\,. \tag{3.30}$$

This is the relation that links the truncated and original processes and that will be also used below in Section 3.6.4 to show the convergence of the original process.

For now, we have the following convergence results for the truncated process. We return the index M for a moment to emphasize the truncation.

Proposition 3.8. *The family* $(Y^\varepsilon)_{\varepsilon>0} \stackrel{\text{def}}{=} (I^{\varepsilon,M}, \psi^{\varepsilon,M})_{\varepsilon>0}$ *converges in distribution in* $\mathcal{C}([0,\infty),\mathbb{R}^2)$ *to the unique solution* $(Y_t^M)_{t\ge0} = (I_t^M, \psi_t^M)_{t\ge0}$ *to the martingale problem with generator defined by*

$$\mathcal{L}h(y) \stackrel{\text{def}}{=} \int_{\tau=0}^{1} \int_{u=0}^{\infty} R(u)\tilde{F}(0,y,\tau)\cdot\nabla(\tilde{F}(u,y,\tau)\cdot\nabla h(y))\,du\,d\tau\,, \tag{3.31}$$

where

$$\tilde{F}(u,y,\tau) \stackrel{\text{def}}{=} F(y,\tau+\omega(y)u)\,,$$

with F defined in (3.28) *and starting point* (I_0,ψ_0).

From this characterization, we can deduce the SDEs satisfied by the limiting process $Y_t^M = (I_t^M, \psi_t^M)$ up to the stopping time η_M.

Proposition 3.9. *Denoting $\tilde{I}_t^M \stackrel{\text{def}}{=} I_{t\wedge\eta_M}^M$, we have*

$$\begin{aligned} d\tilde{I}_t^M = &\int_{\tau=0}^{1} a(\tilde{I}_t^M,\tau)\,dW_{t\wedge\eta_M}(\tilde{I}_t^M,\tau)\,d\tau \\ &+ \int_{u=0}^{\infty} R(u)\Big(\int_{\tau=0}^{1} \partial_I\big(a(I,\tau+\omega(I)u)\big)_{|I=\tilde{I}_t^M} a(\tilde{I}_t^M,\tau) \\ &\qquad + a(\tilde{I}_t^M,\tau+\omega(\tilde{I}_t^M)u)\partial_I a(\tilde{I}_t^M,\tau)\Big)\,d\tau\,du\,dt, \end{aligned} \tag{3.32}$$

with $\tilde{I}_{t=0}^M = I_0$, and where W is defined in (3.21). *Also, we have*

$$\begin{aligned} d\psi_{t\wedge\eta_M}^M = &\int_{\tau=0}^{1} b(\tilde{I}_t^M,\tau)\,dW_{t\wedge\eta_M}(\tilde{I}_t^M,\tau)\,d\tau \\ &+ \int_{u=0}^{\infty} R(u)\Big(\int_{\tau=0}^{1} \partial_I\big(b(I,\tau+\omega(I)u)\big)_{|I=\tilde{I}_t^M} a(\tilde{I}_t^M,\tau) \\ &\qquad + b(\tilde{I}_t^M,\tau+\omega(\tilde{I}_t^M)u)\partial_I a(\tilde{I}_t^M,\tau)\Big)\,d\tau\,du\,dt, \end{aligned} \tag{3.33}$$

with $\psi_{t=0}^M = \theta_0$.

As we can see, this result is similar to the one of Theorem 3.6, with coefficients a and b themselves, but involving the cutoff M through the stopping time η_M. The proofs of Proposition 3.8 and 3.9 are presented in the next section by introducing the concept of pseudo-generator.

3.5 Convergence of truncated processes

3.5.1 Pseudogenerators

In this section, we follow [Kus84, Chapter 3]. Even though the pair of processes $(Y_t^\varepsilon, \tau_t^\varepsilon)_{t\geq 0}$ is Markov for each $\varepsilon > 0$, the process $(Y_t^\varepsilon)_{t\geq 0}$ by itself is *not* Markov. Since analyzing the pair $(Y_t^\varepsilon, \tau_t^\varepsilon)_{t\geq 0}$ is difficult, we study the process $(Y_t^\varepsilon)_{t\geq 0}$ on its own by using the *pseudo-generator*. This concept allows us to apply martingale techniques to the non-Markovian process $(Y_t^\varepsilon)_{t\geq 0}$. Let

$$\mathcal{G}_t^\varepsilon \stackrel{\text{def}}{=} \mathcal{G}_{t/\varepsilon^2}\,,$$

where $\mathcal{G}_t$ is defined by (3.3), and $\mathcal{S}^\varepsilon$ be the set of all measurable functions $h(t)$, adapted to the filtration $(\mathcal{G}_t^\varepsilon)$, for which $\sup_{t\leq T} \boldsymbol{E}[|h(t)|] < +\infty$, and where $T > 0$ is fixed. The p-lim and the pseudo-generator are defined as follows. Let h and h^δ in $\mathcal{S}^\varepsilon$ for all $\delta > 0$. We say that $h = \text{p-lim}_\delta\, h^\delta$ if

$$\sup_{t,\delta} \boldsymbol{E}|h^\delta(t)| < +\infty \quad \text{and} \quad \lim_{\delta\to 0} \boldsymbol{E}|h^\delta(t) - h(t)| = 0 \quad \forall t \geq 0\,.$$

We say that $h \in \mathcal{D}(\mathcal{A}^\varepsilon)$ the domain of $\mathcal{A}^\varepsilon$ and $\mathcal{A}^\varepsilon h = g$ if h and g are in $\mathcal{S}^\varepsilon$ and

$$\underset{\delta\to 0}{\text{p-lim}}\Big(\frac{\boldsymbol{E}_t^\varepsilon h(t+\delta) - h(t)}{\delta} - g(t)\Big) = 0\,,$$

where $\boldsymbol{E}_t^\varepsilon$ is the conditional expectation given $\mathcal{G}_t^\varepsilon$. The difference with respect to the usual infinitesimal generator is that we have *average* limit (via p-lim) rather than pointwise limit. This greatly helps us average out the noise in the limit $\varepsilon \to 0$. A useful result [Kus84, Theorem 1, p. 39] about the pseudo-generator $\mathcal{A}^\varepsilon$ that will be central in our proof is the following.

Proposition 3.10 (Theorem 1, p. 30 in [Kus84])**.** *Let $h \in \mathcal{D}(\mathcal{A}^\varepsilon)$. Then*

$$M_h^\varepsilon(t) = h(t) - h(0) - \int_0^t \mathcal{A}^\varepsilon h(u)\,du$$

is a $(\mathcal{G}_t^\varepsilon)$-martingale.

In other words, we introduce an operator $\mathcal{A}^\varepsilon$ on functions of Y^ε having similar properties to those of an infinitesimal operator of a Markov process; more specifically, it satisfies a martingale problem.

3.5.2 Perturbed test function method

The proof of Proposition 3.8 relies on the perturbed test function method to show the convergence of the truncated processes (Y^ε) defined in 3.26. The ideas are as follows. To characterize the limiting process through a martingale problem with generator $\mathcal{A}$, we can then try to compare $\mathcal{A}^\varepsilon h$ with $\mathcal{A}h$ for some test function h. Unfortunately, $\mathcal{A}^\varepsilon h$ may have singular terms in ε, and we cannot proceed that way directly. Instead, try and extract the effective statistical behavior of the system under consideration from $\mathcal{A}^\varepsilon$ by introducing h^ε an appropriate perturbation of h. In other words, if we can show $\lim_{\varepsilon\to 0} \boldsymbol{E}|h^\varepsilon(t) - h(t)| = 0$ and $\lim_{\varepsilon\to 0} \boldsymbol{E}|A^\varepsilon h^\varepsilon(t) - \mathcal{A}h(Y_t^\varepsilon)| = 0$ for each t, then we can use Proposition 3.10 to conclude that the limiting process $(Y_t)_{t\geq 0}$ satisfies the martingale problem with generator $\mathcal{A}$. We begin by showing tightness of $(Y^\varepsilon)_{\varepsilon>0}$, which implies imply weak subsequential convergence of stochastic processes.

Before going into the proofs, we remark that $(Y^\varepsilon)_{\varepsilon>0}$ is defined via a truncated process in equation (3.26) and there is an omitted index M in the functions F and G to indicate the truncation. This allows us to bound several expressions in the proofs in terms of M.

3.5.3 Tightness

In this section, we prove the tightness of the family $(Y^\varepsilon)_{\varepsilon>0}$, seen as a family of continuous-time processes. Then, according to [Bil99, Theorem 13.4] it suffices to prove the tightness of $(Y^\varepsilon)_{\varepsilon>0}$ in $\mathcal{D}([0,T],\mathbb{R}^2)$ (which is the set of càdlàg functions with values in $\mathbb{R}^2$ equipped with the Skorokhod topology).

Proposition 3.11. *The family $(Y^\varepsilon)_{\varepsilon>0}$ is tight in $\mathcal{D}([0,T],\mathbb{R}^2)$.*

The proof of Proposition 3.11 consists of applying [Kus84, Theorem 4, p. 48]. In what follows, let h be a bounded smooth function on $\mathbb{R}^2$ with bounded derivatives, and set $h_0^\varepsilon(t) = h(Y_t^\varepsilon)$. Our goal is to construct perturbations to h_0^ε as described above. The pseudo-generator at h_0^ε is then given by

$$\mathcal{A}^\varepsilon h_0^\varepsilon(t) = \frac{1}{\varepsilon}\int V\Big(\frac{t}{\varepsilon^2}, dp\Big) F\big(Y_t^\varepsilon, \tau_t^\varepsilon\big)\cdot\nabla h(Y_t^\varepsilon) \tag{3.34}$$

which is simply differentiating $h_0^\varepsilon(t)$ with respect to t (the derivative $\dot{Y}_t^\varepsilon$ is stated in equation (3.26)). The goal now is to modify the test function h_0^ε using a small perturbation h_1^ε so that the pseudo-generator $\mathcal{A}^\varepsilon(h_0^\varepsilon + h_1^\varepsilon)$ does not blow up in ε anymore. The first perturbation of h_0^ε is defined as follows.

$$\begin{aligned} h_1^\varepsilon(t) &= \frac{1}{\varepsilon}\int_t^\infty du \int \boldsymbol{E}_t^\varepsilon\Big[V\Big(\frac{u}{\varepsilon^2}, dp\Big) F\Big(Y_t^\varepsilon, \tau_t^\varepsilon + \frac{u-t}{\varepsilon^2}\,\omega(Y_t^\varepsilon)\Big)\Big]\cdot\nabla h(Y_t^\varepsilon) \\ &= \varepsilon\int_0^\infty du \int \boldsymbol{E}_t^\varepsilon\Big[V\Big(u + \frac{t}{\varepsilon^2}, dp\Big) F\Big(Y_t^\varepsilon, \tau_t^\varepsilon + u\,\omega(Y_t^\varepsilon)\Big)\Big]\cdot\nabla h(Y_t^\varepsilon) \\ &= \varepsilon\int_0^\infty du \int \sum_{n\in\mathbb{Z}^*} \boldsymbol{E}_t^\varepsilon\Big[V\Big(u + \frac{t}{\varepsilon^2}, dp\Big)\Big] e^{2i\pi n(\tau_t^\varepsilon + u\,\omega(Y_t^\varepsilon))} F_n(Y_t^\varepsilon)\cdot\nabla h(Y_t^\varepsilon), \end{aligned} \tag{3.35}$$

where

$$F_n(y) \overset{\text{def}}{=} \int_0^1 F(y,\tau)e^{-2i\pi n\tau}\,d\tau\,.$$

According to Lemma 3.3, we have

$$\boldsymbol{E}_t^\varepsilon V\Big(u + \frac{t}{\varepsilon^2}, dp\Big) = e^{-\mu|p|^{2\beta}u} V\Big(\frac{t}{\varepsilon^2}, dp\Big),$$

so that

$$\begin{aligned} h_1^\varepsilon(t) &= \varepsilon\sum_{n\in\mathbb{Z}^*} e^{2i\pi n\tau_t^\varepsilon}\int V\Big(\frac{t}{\varepsilon^2}, dp\Big) F_n(Y_t^\varepsilon)\cdot\nabla h(Y_t^\varepsilon)\int_0^\infty e^{-\mu|p|^{2\beta}u} e^{2i\pi n\omega(Y_t^\varepsilon)u}\,du \\ &= \varepsilon\sum_{n\in\mathbb{Z}^*} e^{2i\pi n\tau_t^\varepsilon}\int \frac{V(t/\varepsilon^2, dp)}{\mu|p|^{2\beta} - 2i\pi n\,\omega(Y_t^\varepsilon)} F_n(Y_t^\varepsilon)\cdot\nabla h(Y_t^\varepsilon) \end{aligned}$$

To apply [Kus84, Theorem 4 p. 48] and then prove Proposition 3.11, we only have to prove the two following lemmas, which can be thought of as a version of *Arzela–Ascoli* theorem.

Lemma 3.12. *For any $T > 0$, and $\eta > 0$*

$$\lim_{\varepsilon\to 0} \boldsymbol{P}\Big(\sup_{t\in[0,T]} |h_1^\varepsilon(t)| > \eta \Big) = 0 \qquad \text{and} \qquad \lim_{\varepsilon\to 0} \sup_{t\in[0,T]} \boldsymbol{E}|h_1^\varepsilon(t)| = 0\,.$$

Lemma 3.13. *For any $T > 0$, $\big\{\mathcal{A}^\varepsilon\big(h_0^\varepsilon + h_1^\varepsilon\big)(t), \varepsilon \in (0,1), 0 \le t \le T\big\}$ is uniformly integrable.*

Before going through the proofs of these two lemmas, we make one remark that the test function h_1^ε, above, is well defined since we sum over $n \neq 0$. This is because F has mean zero respect to the τ, and explains why this mean-zero condition is crucial to the analysis. For $n = 0$, the above function h_1^ε would not be defined in the case of long-range correlations.

Proof of Lemma 3.12. We rewrite h_1^ε as

$$h_1^\varepsilon(t) = \varepsilon V\Big(\frac{t}{\varepsilon^2}, \varphi_{t,\varepsilon}\Big),$$

with

$$\varphi_{t,\varepsilon}(p) \stackrel{\text{def}}{=} \sum_{n\in\mathbb{Z}^*} e^{2i\pi n\tau_t^\varepsilon} \frac{1}{\mu|p|^{2\beta} - 2i\pi n\,\omega(Y_t^\varepsilon)} F_n(Y_t^\varepsilon)\cdot\nabla h(Y_t^\varepsilon)\,. \tag{3.36}$$

By the Cauchy-Schwarz inequality

$$\begin{aligned}
|\varphi_{t,\varepsilon}(p)| &\le C_{h,\omega_0} \sum_{n\in\mathbb{Z}^*} \frac{1}{|n|} \|F_n(Y_t^\varepsilon)\| \\
&\le C_{h,\omega_0} \Big(\sum_{n\in\mathbb{Z}^*} \frac{1}{|n|^2}\Big)^{1/2} \Big(\sum_{n\in\mathbb{Z}^*} \|F_n(Y_t^\varepsilon)\|^2\Big)^{1/2} \\
&\le C_{h,\omega_0} \Big(\sum_{n\in\mathbb{Z}^*} \frac{1}{|n|^2}\Big)^{1/2} \Big(\int_0^1 \|F(Y_t^\varepsilon,\tau)\|^2\,d\tau\Big)^{1/2} \\
&\le C_{h,\omega_0} \sup_{|y|\le 2M} \Big(\int_0^1 \|F(y,\tau)\|^2\,d\tau\Big)^{1/2}.
\end{aligned}$$

If $\beta \ge 1/2$, we have the Lipschitz bound

$$\begin{aligned}
|\varphi_{t,\varepsilon}(p) - \varphi_{t,\varepsilon}(q)| &\le (|p|^{2\beta} - |q|^{2\beta}) C_{h,\omega_0,F} \sum_{n\in\mathbb{Z}^*} \frac{1}{|n|^2} \\
&\le \big||p| - |q|\big| \sup_{r\in S} |r|^{2\beta-1} C_{h,\omega_0,F} \sum_{n\in\mathbb{Z}^*} \frac{1}{|n|^2} \le L|p-q|
\end{aligned}$$

with

$$L = \sup_{r\in S} |r|^{2\beta-1} C_{h,\omega_0,F} \sum_{n\in\mathbb{Z}^*} \frac{1}{|n|^2} < \infty\,.$$

Using inequality (3.6) of Lemma 3.4 with $k=\infty$, along with the Chebychev's inequality, we have

$$\boldsymbol{P}\Big(\sup_{t\in[0,T]}|h_1^\varepsilon(t)|>\eta\Big)\le\frac{1}{\eta}\boldsymbol{E}\sup_{t\in[0,T]}|h_1^\varepsilon(t)|\le C'_{h,\omega_0,F}\,C(\varepsilon)\,,$$

with $\lim_\varepsilon C(\varepsilon)=0$.

If $\beta<1/2$, the function $\varphi_{t,\varepsilon}$ belongs to $W^{1,k}(S)$ with $k\in(1,1/(1-2\beta))$ since

$$\int_S|\partial_p\varphi_{t,\varepsilon}(p)|^k\,dp\le C''_{h,\omega_0,F}\int_S|p|^{k(2\beta-1)}\,dr<\infty\,,$$

so that using inequality (3.6) of Lemma 3.4 but with $k\in(1,1/(1-2\beta))$ leads again to

$$\boldsymbol{P}\Big(\sup_{t\in[0,T]}|h_1^\varepsilon(t)|>\eta\Big)\le C'_{h,\omega_0,F}\,C(\varepsilon)\,.$$

To prove the second limit in the lemma, one can just remark that

$$\boldsymbol{E}|h_1^\varepsilon(t)|\le\boldsymbol{E}\sup_{t\in[0,T]}|h_1^\varepsilon(t)|$$

for any $t\in[0,T]$, and then concludes the proof of Lemma 3.12. □

Proof of Lemma 3.13. For proving uniform integrability, it is sufficient to show that

$$\sup_{\varepsilon,t}\boldsymbol{E}|\mathcal{A}^\varepsilon(h_0^\varepsilon+h_1^\varepsilon)(t)|^2\le C\,.$$

Following the definition of pseudo-generator, we have that

$$\begin{aligned}\mathcal{A}^\varepsilon(h_1^\varepsilon)(t)&=-\mathcal{A}^\varepsilon(h_0^\varepsilon)(t)\\&+\frac{1}{\varepsilon}\int_t^\infty du\int\boldsymbol{E}_t^\varepsilon\Big[V\Big(\frac{u}{\varepsilon^2},dp\Big)\Big]\frac{d}{ds}\Big[F\Big(Y_s^\varepsilon,\tau_s^\varepsilon+\frac{u-s}{\varepsilon^2}\,\omega(Y_s^\varepsilon)\Big)\cdot\nabla h(Y_s^\varepsilon)\Big]\Big|_{s=t}\end{aligned}$$

which is essentially differentiating $h_1^\varepsilon(t)$ with respect to t (we however remark that we do not differentiate the t in $\boldsymbol{E}_t^\varepsilon$). After some lengthy but straightforward computations (similarly to those used to obtain (3.35)), we obtain

$$\begin{aligned}
\mathcal{A}^\varepsilon(h_1^\varepsilon)(t) = &-\mathcal{A}^\varepsilon(h_0^\varepsilon)(t) + \sum_{n,m\in\mathbb{Z}^*} e^{2i\pi(n+m)\tau_t^\varepsilon} \iint \frac{V(t/\varepsilon^2,dp)V(t/\varepsilon^2,dp')}{\mu|p|^{2\beta}-2i\pi n\omega(Y_t^\varepsilon)} \\
&\cdot\Big(F_n(Y_t^\varepsilon)^T\nabla^2 h(Y_t^\varepsilon)F_m(Y_t^\varepsilon) + \nabla h(Y_t^\varepsilon)^T \mathrm{Jac}F_n(Y_t^\varepsilon)F_m(Y_t^\varepsilon) \\
&\quad - \frac{2i\pi n}{\mu|p|^{2\beta}-2i\pi n\omega(Y_t^\varepsilon)} F_n(Y_t^\varepsilon)\cdot\nabla h(Y_t^\varepsilon)F_m(Y_t^\varepsilon)\cdot\nabla\omega(Y_t^\varepsilon)\Big) \\
&+ \sum_{n\in\mathbb{Z}^*} 2i\pi n\, e^{2i\pi n\tau_t^\varepsilon} \iint \frac{V(t/\varepsilon^2,dp)V(t/\varepsilon^2,dp')}{\mu|p|^{2\beta}-2i\pi n\omega(Y_t^\varepsilon)} \\
&\cdot F_n(Y_t^\varepsilon)\cdot\nabla h(Y_t^\varepsilon)G(Y_t^\varepsilon) \\
= &-\mathcal{A}^\varepsilon(h_0^\varepsilon)(t) + V\Big(\frac{t}{\varepsilon^2},\varphi^1_{t,\varepsilon}\Big)V\Big(\frac{t}{\varepsilon^2},\mathbf{1}_S\Big) + V\Big(\frac{t}{\varepsilon^2},\varphi^2_{t,\varepsilon}\Big)V\Big(\frac{t}{\varepsilon^2},\mathbf{1}_S\Big) \\
\overset{\text{def}}{=} &-\mathcal{A}^\varepsilon(h_0^\varepsilon)(t) + \mathcal{P}_1^\varepsilon(t) + \mathcal{P}_2^\varepsilon(t)
\end{aligned} \tag{3.37}$$

where $\varphi^j_{t,\varepsilon}$, $j=1,2$ are defined similarly as in (3.36). As in the proof of Lemma 3.12, it is not hard to see that the $\varphi^j_{t,\varepsilon}$, $j=1,2$ are Lispschitz in p on S, and that

$$\begin{aligned}
|\varphi^1_{t,\varepsilon}(p)| &\leq C_{h,\omega_0,\nabla\omega} \sum_{m\in\mathbb{Z}^*} |F_m(Y_t^\varepsilon)| \sum_{n\in\mathbb{Z}^*} \frac{1}{|n|}\Big((1+|n|)|F_n(Y_t^\varepsilon)| + \|\mathrm{Jac}F_n(Y_t^\varepsilon)\|\Big) \\
&\leq \tilde{C}_{h,\omega_0,\nabla\omega} \sup_{|y|\leq 2M} \int_0^1 \Big(|F(y,\tau)|^2 + |\partial_\tau F(y,\tau)|^2 + \|\mathrm{Jac}F(y,\tau)\|^2\Big)\, d\tau\,.
\end{aligned}$$

The same lines give also

$$|\varphi^2_{t,\varepsilon}(p)| \leq C_{h,\omega_0} \sup_{|y|\leq 2M} \big|G(y)\big|\Big(\int_0^1 |\partial_\tau F(y,\tau)|^2\, d\tau\Big)^{1/2}.$$

Then, we can use inequality (3.7) of Lemma 3.4, and obtain

$$\sup_{\varepsilon,t} \boldsymbol{E}[|\mathcal{P}_j^\varepsilon(t)|^2] \leq C, \quad j=1,2\ ,$$

which concludes the proof of Lemma 3.13, and then Proposition 3.11 as well. □

3.5.4 Identification of the limit

In this section, we identify all the limit points of $(Y^\varepsilon)_{\varepsilon>0}$ via a well-posed martingale problem as stated in the following proposition. By abuse of notations, we still denote by $(Y^\varepsilon)_{\varepsilon>0}$ a converging subsequence and by $(Y_t)_{t\geq 0}$ a limit point.

Proposition 3.14. *All the limit points $(Y_t)_{t\geq 0}$ of $(Y^\varepsilon)_{\varepsilon>0}$ are solutions of a well-posed martingale problem with generator*

$$\mathcal{L}h(y) = \int_0^1 d\tau \int_0^\infty du\, R(u)\tilde{F}(0,y,\tau)\cdot\nabla_y(\tilde{F}(u,y,\tau)\cdot\nabla_y h(y)) \tag{3.38}$$

with

$$\tilde{F}(u,y,\tau) = F(y,\tau+\omega(y)u)\,.$$

To prove this proposition we use the notion of pseudo-generator introduced in Section 3.5.1 and the perturbed-test-function technique that we have already used in Section 3.5.3 for the proof of tightness. Thanks to the pseudo-generator we can characterize the subsequential limits of $(Y^\varepsilon)_{\varepsilon>0}$ as solutions of a well-posed martingale problem.

The outline of the proof is as follows. Recall in Section 3.5.3 we saw that $\mathcal{A}^\varepsilon(h_0^\varepsilon)$ has a singular $1/\varepsilon$ term, and modified the test function h_0^ε with a small perturbation h_1^ε to remove this singular term. As a result, $\mathcal{A}^\varepsilon(h_0^\varepsilon+h_1^\varepsilon)$ is not singular any more. However, it is not yet the generator of a martingale problem as $\mathcal{A}^\varepsilon(h_0^\varepsilon+h_1^\varepsilon)$ still has oscillatory terms in the form of $e^{2i\pi n\tau_t^\varepsilon}$. We will show that these terms essentially vanish as $\varepsilon\to 0$ by introducing a small perturbation $h_2^\varepsilon(t)$ to cancel these oscillations, and then construct another perturbation $h_3^\varepsilon(t)$ to cancel oscillations that result from computing $\mathcal{A}^\varepsilon(h_2^\varepsilon)(t)$. After we have constructed the perturbed test function

$$h^\varepsilon(t) = h_0^\varepsilon(t)+h_1^\varepsilon(t)+h_2^\varepsilon(t)+h_3^\varepsilon(t)\,,$$

we show that the pseudo-generator

$$\mathcal{A}^\varepsilon(h_0^\varepsilon+h_1^\varepsilon+h_2^\varepsilon+h_3^\varepsilon)(t)$$

has the desired form of a generator related to a martingale problem, plus a negligible term. Combining with tightness, we can show that the limiting process $(Y_t)_{t\geq 0}$ satisfies a martingale problem.

We now carry out this approach. For convenience, the proofs of needed auxiliary lemmas will be deferred to later sections.

Proof of Proposition 3.14. We split $\mathcal{A}^\varepsilon(h_0^\varepsilon+h_1^\varepsilon)$ into three parts

$$\mathcal{A}^\varepsilon(h_0^\varepsilon+h_1^\varepsilon)(t) = \mathcal{A}_1^\varepsilon(t)+\mathcal{A}_2^\varepsilon(t)+\mathcal{P}_2^\varepsilon(t)\,,$$

where $\mathcal{P}_2^\varepsilon$ is defined in (3.37), and $\mathcal{A}_1^\varepsilon(t)$, $\mathcal{A}_2^\varepsilon(t)$ are given by

$$\begin{aligned}\mathcal{A}_1^\varepsilon(t) &= \sum_{\substack{n,m\in\mathbb{Z}^*\\ n+m\neq 0}} e^{2i\pi(n+m)\tau_t^\varepsilon}\iint V\Big(\frac{t}{\varepsilon^2},dp\Big)V\Big(\frac{t}{\varepsilon^2},dp'\Big)W_{n,m}(p,Y_t^\varepsilon)\,,\\ \mathcal{A}_2^\varepsilon(t) &= \sum_{n\in\mathbb{Z}^*}\iint V\Big(\frac{t}{\varepsilon^2},dp\Big)V\Big(\frac{t}{\varepsilon^2},dp'\Big)W_n(p,Y_t^\varepsilon)\,. \end{aligned} \tag{3.39}$$

Here

$$W_{n,m}(p,Y_t^\varepsilon) \stackrel{\text{def}}{=} \frac{1}{\mu|p|^{2\beta}-2i\pi n\omega(Y_t^\varepsilon)}\Big(F_n(Y_t^\varepsilon)^T\nabla^2 h(Y_t^\varepsilon)F_m(Y_t^\varepsilon) + \nabla h(Y_t^\varepsilon)^T \mathrm{Jac}F_n(Y_t^\varepsilon)F_m(Y_t^\varepsilon)\Big) - \frac{2i\pi n}{(\mu|p|^{2\beta}-2i\pi n\omega(Y_t^\varepsilon))^2}F_n(Y_t^\varepsilon)\cdot\nabla h(Y_t^\varepsilon)F_m(Y_t^\varepsilon)\cdot\nabla\omega(Y_t^\varepsilon)\,, \quad (3.40)$$

and

$$W_n(p,Y_t^\varepsilon) = W_{n,-n}(p,Y_t^\varepsilon)\,.$$

The term $\mathcal{A}_2^\varepsilon$ no longer has oscillatory components as $\mathcal{A}_2^\varepsilon$ only picks up the terms with $n+m=0$. We now show that this term is related to the generator of a martingale problem.

Lemma 3.15. *For all $t \geq 0$, we have*

$$\lim_{\varepsilon\to 0}\boldsymbol{E}\Big|\int_0^t (\mathcal{A}_2^\varepsilon(u) - \mathcal{L}h(Y_u^\varepsilon))\,du\Big| = 0\,.$$

with

$$\mathcal{L}h(y) = \sum_{n\in\mathbb{Z}^*}\int dp\, r(p)W_n(p,y)\,,$$

where W_n is defined through (3.40).

We present the proof of Lemma 3.15 in Section 3.5.6. To show the well-posedness of the limiting martingale problem, we need to rewrite $\mathcal{L}$ as a second-order differential operator, which eventually gives us the form in (3.38).

For the oscillatory term $\mathcal{A}_1^\varepsilon$, we introduce two more small perturbations h_2^ε and h_3^ε to prove that the oscillations will vanish as $\varepsilon \to 0$. Note that $\mathcal{P}_2^\varepsilon$ also has oscillatory terms. These can be treated in a manner similar to the oscillatory terms in $\mathcal{A}_1^\varepsilon$, so that these terms vanish in the limit $\varepsilon \to 0$. Since the treatment of this will be similar to the treatment of $\mathcal{A}_1^\varepsilon$, we omit the details.

Similarly to the construction of h_1^ε in equation (3.35), we introduce the second perturbation

$$h_2^\varepsilon(t) = \int_t^\infty du \sum_{\substack{n,m\in\mathbb{Z}^*\\ n+m\neq 0}} e^{2i\pi(n+m)(\tau_t^\varepsilon+(u-t)\omega(Y_t^\varepsilon)/\varepsilon^2)} \cdot \iint \Big(\boldsymbol{E}_t^\varepsilon\Big[V\Big(\frac{u}{\varepsilon^2},dp\Big)V\Big(\frac{u}{\varepsilon^2},dp'\Big)\Big] - \boldsymbol{E}[V(0,dp)V(0,dp')]\Big)W_{n,m}(p,Y_t^\varepsilon)\,.$$

Making the change of variable $u \to t + \varepsilon^2 u$, we have

$$h_2^\varepsilon(t) = \varepsilon^2 \int_0^\infty du \sum_{\substack{n,m\in\mathbb{Z}^* \\ n+m\neq 0}} e^{2i\pi(n+m)(\tau_t^\varepsilon + u\omega(Y_t^\varepsilon))} W_{n,m}(p, Y_t^\varepsilon)$$

$$\cdot \iint \Big(\boldsymbol{E}_t^\varepsilon\Big[V\Big(u + \frac{t}{\varepsilon^2}, dp\Big) V\Big(u + \frac{t}{\varepsilon^2}, dp'\Big)\Big] - \boldsymbol{E}[V(0,dp)V(0,dp')]\Big) .$$

According to formula (3.5), one has

$$\boldsymbol{E}_t^\varepsilon\Big[V\Big(u + \frac{t}{\varepsilon^2}, dp\Big) V\Big(u + \frac{t}{\varepsilon^2}, dp'\Big)\Big] - \boldsymbol{E}[V(0,dp)V(0,dp')]$$
$$= e^{-\mu(|p|^{2\beta} + |p'|^{2\beta})u} \Big(V\Big(\frac{t}{\varepsilon^2}, dp\Big) V\Big(\frac{t}{\varepsilon^2}, dp'\Big) - r(p)\delta(p-p')\, dp\, dp' \Big) ,$$

so that

$$h_2^\varepsilon(t) = \varepsilon^2 \sum_{\substack{n,m\in\mathbb{Z}^* \\ n+m\neq 0}} e^{2i\pi(n+m)\tau_t^\varepsilon}$$
$$\times \iint \frac{V(t/\varepsilon^2, dp)V(t/\varepsilon^2, dp') - r(p)\delta(p-p')\, dp\, dp'}{\mu(|p|^{2\beta} + |p'|^{2\beta}) - 2i\pi(n+m)\omega(Y_t^\varepsilon)} W_{n,m}(p, Y_t^\varepsilon) .$$

We claim that h_2^ε is uniformly small.

Lemma 3.16. *We have*

$$\lim_{\varepsilon\to 0} \sup_{t\in[0,T]} \boldsymbol{E}|h_2^\varepsilon(t)| = 0 .$$

As before, following the definition of pseudo-generator, we have

$$\mathcal{A}^\varepsilon(h_2^\varepsilon)(t) = -\mathcal{A}_1^\varepsilon(t) + \mathcal{D}_\varepsilon(t) + \mathcal{R}_\varepsilon^1(t) ,$$

with

$$\mathcal{D}_\varepsilon(t) \stackrel{\text{def}}{=} \sum_{\substack{n,m\in\mathbb{Z}^* \\ n+m\neq 0}} e^{2i\pi(n+m)\tau_t^\varepsilon} \int dp\, r(p) W_{n,m}(p, Y_t^\varepsilon) ,$$

and $\mathcal{R}_\varepsilon^1(t)$ to be the remainder. Even though oscillatory terms are present in $\mathcal{R}_\varepsilon^1$, we make the following claim.

Lemma 3.17. *We have*

$$\lim_{\varepsilon\to 0} \sup_{t\in[0,T]} \boldsymbol{E}|\mathcal{R}_\varepsilon^1(t)| = 0 .$$

The proof of this lemma is presented in Section 3.5.5, below. To treat the oscillations in $\mathcal{D}_\varepsilon(t)$ we introduce the third perturbation

$$\begin{aligned} h_3^\varepsilon(t) &= \int_t^\infty du \sum_{\substack{n,m\in\mathbb{Z}^* \\ n+m\neq 0}} e^{2i\pi(n+m)(\tau_t^\varepsilon+(u-t)\omega(Y_t^\varepsilon)/\varepsilon^2)} e^{-\varepsilon(u-t)} \int dp\, r(p) W_{n,m}(p, Y_t^\varepsilon) \\ &= \varepsilon^2 \sum_{\substack{n,m\in\mathbb{Z}^* \\ n+m\neq 0}} \frac{e^{2i\pi(n+m)\tau_t^\varepsilon}}{\varepsilon^3 - 2i\pi(n+m)\omega(Y_t^\varepsilon)} \int dp\, r(p) W_{n,m}(p, Y_t^\varepsilon) \end{aligned}$$

We will see that

$$\mathcal{A}^\varepsilon(h_3^\varepsilon)(t) = -\mathcal{D}_\varepsilon(t) + \mathcal{R}_\varepsilon^2(t)$$

with explicit formula for $\mathcal{R}_\varepsilon^2(t)$ given later. We claim h_3^ε and R_ε^2 both vanish as $\varepsilon \to 0$.

Lemma 3.18. *We have*

$$\lim_{\varepsilon\to 0} \sup_{t\in[0,T]} \boldsymbol{E}|h_3^\varepsilon(t)| = 0 \qquad \textit{and} \qquad \lim_{\varepsilon\to 0} \sup_{t\in[0,T]} \boldsymbol{E}|\mathcal{R}_\varepsilon^2(t)| = 0\,.$$

We will prove Lemma 3.18 in Section 3.5.5, below. As a result, we now have a perturbed test function

$$h(t) = h_0^\varepsilon(t) + h_1^\varepsilon(t) + h_2^\varepsilon(t) + h_3^\varepsilon(t)\,,$$

with $h_0^\varepsilon(t) = h(Y_t^\varepsilon)$, for which

$$\mathcal{A}^\varepsilon(h_0^\varepsilon + h_1^\varepsilon + h_2^\varepsilon + h_3^\varepsilon)(t) = \mathcal{A}_2^\varepsilon(t) + o(\varepsilon)\,. \tag{3.41}$$

Using Theorem 3.10 we see that for all $N \geq 1$, any bounded continuous function Ψ, and every sequence $0 < s_1 < \cdots < s_N \leq s < t$ we must have

$$\boldsymbol{E}\Big[\Psi\big(Y_{s_1}^\varepsilon, \ldots, Y_{s_N}^\varepsilon\big)\Big(M_h^\varepsilon(t) - M_h^\varepsilon(s)\Big)\Big] = 0\,. \tag{3.42}$$

Together with Lemmas 3.12, 3.15, 3.16, 3.17, 3.18, and estimate (3.41), we have

$$\lim_{\varepsilon\to 0} \boldsymbol{E}\Big[\Psi\big(Y_{s_1}^\varepsilon, \ldots, Y_{s_N}^\varepsilon\big)\Big(h(Y_t^\varepsilon) - h(Y_s^\varepsilon) - \int_s^t \mathcal{L}h(u)\,du\Big)\Big] = 0\,, \tag{3.43}$$

where $\mathcal{L}$ is defined in (3.38). Combining with $Y_{t=0}^\varepsilon = Y_0$ for every $\varepsilon > 0$, the equation (3.43) implies that any subsequential limit $(Y_t)_{t\geq 0}$ of $(Y_\varepsilon)_{\varepsilon>0}$ is a solution of the martingale problem for the generator $\mathcal{L}$ with initial condition $Y_{t=0} = Y_0$. We claim that the solution to this martingale problem is unique. Given this, along with the tightness proven in Proposition 3.11, Prokhorov's theorem implies the weak convergence of the sequence of processes $(Y^\varepsilon)_{\varepsilon>0}$ to the process $(Y_t)_{t\geq 0}$ as desired. This completes the proof. □

3.5.5 Estimates for the correction terms

In this section, we prove all the auxiliary lemmas in the previous section that involves the perturbation terms. We first estimate the term h_2^ε in the Lemma 3.16. By using Lemma 3.4, the proof is similar that of Lemma 3.13.

Proof of Lemma 3.16. We split the integrand in h_2^ε into two parts as follows

$$\begin{aligned} h_2^\varepsilon(t) = \varepsilon^2 \sum_{\substack{n,m\in\mathbb{Z}^* \\ n+m\neq 0}} & e^{2i\pi(n+m)\tau_t^\varepsilon} \\ & \cdot \iint \frac{V(t/\varepsilon^2, dp)V(t/\varepsilon^2, dp') - r(p)\delta(p-p')\, dp\, dp'}{\mu(|p|^{2\beta} + |p'|^{2\beta}) - 2i\pi(n+m)\omega(Y_t^\varepsilon)} W_{n,m}(p, Y_t^\varepsilon) \\ \stackrel{\text{def}}{=} h_{21}^\varepsilon(t) - h_{22}^\varepsilon(t) \end{aligned}$$

where h_{21}^ε contains the term $V(t/\varepsilon^2, dp)V(t/\varepsilon^2, dp')$ and h_{22}^ε contains the term $r(p)\delta(p-p')$.

For h_{22}^ε, it is straightforward to see that

$$\begin{aligned} |h_{22}^\varepsilon(t)| \le \varepsilon^2 C_{h,\omega_0,\nabla\omega} & \int dp\, r(p) \\ & \cdot \sum_{\substack{n,m\in\mathbb{Z}^* \\ n+m\neq 0}} \frac{1}{|n(n+m)|} |F_m(Y_t^\varepsilon)| \Big((1+|n|)|F_n(Y_t^\varepsilon)| + \|\mathrm{Jac} F_n(Y_t^\varepsilon)\| \Big) \\ \le \varepsilon^2 C_{h,\omega_0,\nabla\omega} & \int dp\, r(p) \\ & \cdot \sum_{n\in\mathbb{Z}^*} \frac{1}{|n|^2} \sup_{|y|\le 2M} \int_0^1 \Big(|F(y,\tau)|^2 + |\partial_\tau F(y,\tau)|^2 + \|\mathrm{Jac} F(y,\tau)\|^2 \Big) d\tau\,. \end{aligned} \tag{3.44}$$

We rewrite

$$h_{21}^\varepsilon(t) = \varepsilon^2 V\Big(\frac{t}{\varepsilon^2}, \varphi_{1,t,\varepsilon}\Big),$$

with

$$\varphi_{1,t,\varepsilon}(p) = V\Big(\frac{t}{\varepsilon^2}, \varphi_{2,t,p,\varepsilon}\Big)$$

and

$$\begin{aligned} \varphi_{2,t,p,\varepsilon}(p') = \sum_{\substack{n,m\in\mathbb{Z}^* \\ n+m\neq 0}} & e^{2i\pi(n+m)\tau_t^\varepsilon} W_{n,m}(p, Y_t^\varepsilon) \\ & \cdot \frac{1}{\mu(|p|^{2\beta} + |p'|^{2\beta}) - 2i\pi(n+m)\omega(Y_t^\varepsilon)}\,. \end{aligned}$$

For convenience, we pause to recall the meaning of the above notation. The function $\varphi_{1,t,\varepsilon}$ of the variable p is obtained by applying the measure $V(t/\varepsilon^2, dp')$ to the function

$\varphi_{2,t,p,\varepsilon}(p')$, where $\varphi_{2,t,p,\varepsilon}$ (a function of the variable p'), is defined as above. Finally, $h_{21}^{\varepsilon}(t)$ is obtained by applying the measure $V(t/\varepsilon^2, dp)$ to the function $\varphi_{1,t,\varepsilon}(p)$.

For h_{22}^{ε}, one has

$$|\varphi_{2,t,p,\varepsilon}(p')| \leq C\,,$$

uniformly in p and p'. Following the same lines as for the first perturbation of the test function h_1^{ε} but using inequality (3.7) of Lemma 3.4 (with $k = \infty$ if $\beta \geq 1/2$ and $k \in (1, 1/(1-2\beta))$ if $\beta < 1/2$), we have for some $C' > 0$

$$\sup_{p\in S} |\varphi_{1,t,\varepsilon}(p)| \leq \sup_{\varphi\in W_{k,C'}} \left| V\Big(\frac{t}{\varepsilon^2}, \varphi\Big)\right|.$$

In fact, the derivative of $\varphi_{2,t,p,\varepsilon}$ in p' can be uniformly bounded in p in the corresponding $L^k(S)$ space.

Now, let us look at the derivative of $\varphi_{1,t,\varepsilon}(p)$ in p. One can write

$$\partial_p \varphi_{1,t,\varepsilon}(p) = |p|^{2\beta-1} V\Big(\frac{t}{\varepsilon^2}, \varphi'_{2,t,p,\varepsilon}\Big) + V\Big(\frac{t}{\varepsilon^2}, \varphi''_{2,t,p,\varepsilon}\Big),$$

where $\varphi'_{2,t,p,\varepsilon}$ and $\varphi''_{2,t,p,\varepsilon}$ belong to some $W_{k,C''}$, and where C'' does not depend on p. Then,

$$|\partial_p \varphi_{1,t,\varepsilon}(p)| \leq (|p|^{2\beta-1} + 1) \sup_{\varphi\in W_{k,C''}} \left| V\Big(\frac{t}{\varepsilon^2}, \varphi\Big)\right|.$$

As a result, considering $\tilde{C} = \max(C', C'')$, and

$$\tilde{h}_{21}^{\varepsilon}(t) \overset{\text{def}}{=} \frac{h_{21}^{\varepsilon}(t)}{\sup_{\varphi\in W_{k,\tilde{C}}} |V(t/\varepsilon^2, \varphi)|} = \varepsilon^2 V\Big(\frac{t}{\varepsilon^2}, \tilde{\varphi}_{1,t,\varepsilon}\Big)\,,$$

with

$$\tilde{\varphi}_{1,t,\varepsilon} = \frac{\varphi_{1,t,\varepsilon}}{\sup_{\varphi\in W_{k,\tilde{C}}} |V(t/\varepsilon^2, \varphi)|},$$

we have

$$\|\tilde{\varphi}_{1,t,\varepsilon}\|_{W^{1,k}(S)} \leq 1.$$

Therefore,

$$|\tilde{h}_{21}^{\varepsilon}(t)| \leq \varepsilon^2 \sup_{\varphi\in W_{k,\tilde{C}}} \left| V\Big(\frac{t}{\varepsilon^2}, \varphi\Big)\right|,$$

and hence

$$\boldsymbol{E}|h_{21}^{\varepsilon}(t)| \leq \varepsilon^2 \boldsymbol{E} \sup_{\varphi\in W_{k,\max(1,\tilde{C})}} \left| V\Big(\frac{t}{\varepsilon^2}, \varphi\Big)\right|^2 = \varepsilon^2 \boldsymbol{E} \sup_{\varphi\in W_{k,\max(1,\tilde{C})}} |V(0,\varphi)|^2.$$

Using Lemma 3.4, this concludes the proof. □

Next, we estimate the term $\mathcal{R}_{\varepsilon}^1$ in the Lemma 3.17.

Proof of Lemma 3.17. By following the computations in (3.37), we obtain the following explicit formula for $\mathcal{R}_1^\varepsilon(t)$:

$$\begin{aligned}\mathcal{R}_1^\varepsilon(t) = \varepsilon v\Big(\frac{t}{\varepsilon^2}\Big) \sum_{\substack{n,m\in\mathbb{Z}^*\\ n+m\neq 0}} \iint & \frac{2i\pi(n+m)e^{2i\pi(n+m)\tau_t^\varepsilon}}{\mu(|p|^{2\beta}+|p'|^{2\beta}) - 2i\pi(n+m)\omega(Y_t^\varepsilon)} \\ & \cdot\Big(G(Y_t^\varepsilon) - \frac{\nabla\omega(Y_t^\varepsilon)\cdot F(Y_t^\varepsilon,\tau_t^\varepsilon)}{\mu(|p|^{2\beta}+|p'|^{2\beta}) - 2i\pi(n+m)\omega(Y_t^\varepsilon)}\Big) \\ & \cdot\Big(V\Big(\frac{t}{\varepsilon^2},dp\Big)V\Big(\frac{t}{\varepsilon^2},dp'\Big) - r(p)\delta(p-p')\,dp\,dp'\Big)W_{n,m}(p,Y_t^\varepsilon) \\ + \varepsilon v\Big(\frac{t}{\varepsilon^2}\Big) \sum_{\substack{n,m\in\mathbb{Z}^*\\ n+m\neq 0}} \iint & \frac{e^{2i\pi(n+m)\tau_t^\varepsilon}}{\mu(|p|^{2\beta}+|p'|^{2\beta}) - 2i\pi(n+m)\omega(Y_t^\varepsilon)} \\ & \cdot\Big(V\Big(\frac{t}{\varepsilon^2},dp\Big)V\Big(\frac{t}{\varepsilon^2},dp'\Big) - r(p)\delta(p-p')\,dp\,dp'\Big) \\ & \cdot\nabla_y W_{n,m}(p,y)_{|y=Y_t^\varepsilon}\cdot F(Y_t^\varepsilon,\tau_t^\varepsilon)\,.\end{aligned}$$

From here, we can see that $\mathcal{R}_1^\varepsilon(t)$ can be treated in the same manner as we treated $h_2^\varepsilon(t)$. The completes the proof. □

Next, we estimate h_3^ε and $\mathcal{R}_\varepsilon^2$ in the Lemma 3.18.

Proof of Lemma 3.18. The bounds for $h_3^\varepsilon(t)$ can be obtained in the same manner as the bounds for h_{22}^ε in equation (3.44). As a result, we obtain

$$\lim_{\varepsilon\to 0}\sup_{t\in[0,T]} \boldsymbol{E}|h_3^\varepsilon(t)| = 0\,.$$

Treating $R_2^\varepsilon(t)$ in the same manner as $R_1^\varepsilon(t)$ we obtain

$$\begin{aligned}\mathcal{R}_2^\varepsilon(t) = \varepsilon v\Big(\frac{t}{\varepsilon^2}\Big) \sum_{\substack{n,m\in\mathbb{Z}^*\\ n+m\neq 0}} \int dp\, & \frac{2i\pi(n+m)r(p)e^{2i\pi(n+m)\tau_t^\varepsilon}}{\varepsilon^3 - 2i\pi(n+m)\omega(Y_t^\varepsilon)} \\ & \cdot\Big(G(Y_t^\varepsilon) - \frac{\nabla\omega(Y_t^\varepsilon)\cdot F(Y_t^\varepsilon,\tau_t^\varepsilon)}{\varepsilon^3 - 2i\pi(n+m)\omega(Y_t^\varepsilon)}\Big)W_{n,m}(p,Y_t^\varepsilon) \\ + \varepsilon^3 \sum_{\substack{n,m\in\mathbb{Z}^*\\ n+m\neq 0}} \int dp\, & \frac{r(p)e^{2i\pi(n+m)\tau_t^\varepsilon}}{\varepsilon^3 - 2i\pi(n+m)\omega(Y_t^\varepsilon)} W_{n,m}(p,Y_t^\varepsilon) \\ + \varepsilon v\Big(\frac{t}{\varepsilon^2}\Big) \sum_{\substack{n,m\in\mathbb{Z}^*\\ n+m\neq 0}} \int dp\, & \frac{r(p)e^{2i\pi(n+m)\tau_t^\varepsilon}}{\varepsilon^3 - 2i\pi(n+m)\omega(Y_t^\varepsilon)} \nabla_y W_{n,m}(p,y)_{|y=Y_t^\varepsilon}\cdot F(Y_t^\varepsilon,\tau_t^\varepsilon)\,.\end{aligned}$$

From here, we can see that each of the terms in $\mathcal{R}_2^\varepsilon(t)$ can be treated in the same manner as $h_2^\varepsilon(t)$. This concludes the proof. □

3.5.6 Convergence of pseudo-generator to infinitesimal generator

We devote this section to proving Lemma 3.15. The proof analyzes how the product $V(t/\varepsilon^2, dp)V(t/\varepsilon^2, dp')$ in $\mathcal{A}_2^\varepsilon$ (defined through Formula (3.39)) averages in the limit $\varepsilon \to 0$. This is essentially similar to the analysis of the covariance in the proof of Proposition 2.2 in the case when the Hamiltonian was quadratic.

Let $\eta > 0$ and write

$$\begin{aligned}\int_0^t (\mathcal{A}_2^\varepsilon(u) - \mathcal{L}^M h(Y_u^\varepsilon))\, du &= \Big(\int_0^{[t/(\varepsilon\eta)]\varepsilon\eta} + \int_{[t/(\varepsilon\eta)]\varepsilon\eta}^t\Big)(\mathcal{A}_2^\varepsilon(u) - \mathcal{L}h(Y_u^\varepsilon))\, du \\ &= \sum_{q=0}^{[t/(\varepsilon\eta)]-1} \int_{q\varepsilon\eta}^{(q+1)\varepsilon\eta} (\mathcal{A}_2^\varepsilon(u) - \mathcal{L}h(Y_u^\varepsilon))\, du \\ &\quad + \int_{[t/(\varepsilon\eta)]\varepsilon\eta}^t (\mathcal{A}_2^\varepsilon(u) - \mathcal{L}h(Y_u^\varepsilon))\, du \\ &\stackrel{\text{def}}{=} R_1^\varepsilon(t) + R_2^\varepsilon(t)\,.\end{aligned}$$

Following the computations of Lemma 3.16, we have

$$\boldsymbol{E}|R_2^\varepsilon(t)| \leq \varepsilon\tilde{C}\,.$$

so that we only have to take care of $R_1^\varepsilon(t)$. For this term we consider the decomposition

$$\begin{aligned}R_1^\varepsilon(t) &= \sum_{q=0}^{[t/(\varepsilon\eta)]-1} \int_{q\varepsilon\eta}^{(q+1)\varepsilon\eta} (\mathcal{A}_2^\varepsilon(u) - \mathcal{L}h(Y_u^\varepsilon))\, du \\ &= \sum_{q=0}^{[t/(\varepsilon\eta)]-1} \int_{q\varepsilon\eta}^{(q+1)\varepsilon\eta} du \\ &\qquad\qquad \cdot \sum_{n\in\mathbb{Z}^*} \iint \Big(V\Big(\frac{u}{\varepsilon^2}, dp\Big)V\Big(\frac{u}{\varepsilon^2}, dp'\Big) - r(p)\delta(p-p')\, dp\, dp'\Big) \\ &\qquad \cdot W_n(p, Y_{q\varepsilon\eta}^\varepsilon) \\ &\quad + \sum_{q=0}^{[t/(\varepsilon\eta)]-1} \int_{q\varepsilon\eta}^{(q+1)\varepsilon\eta} du \\ &\qquad\qquad \cdot \sum_{n\in\mathbb{Z}^*} \iint \Big(V\Big(\frac{u}{\varepsilon^2}, dp\Big)V\Big(\frac{u}{\varepsilon^2}, dp'\Big) - r(p)\delta(p-p')\, dp\, dp'\Big) \\ &\qquad \cdot \Big(W_n(p, Y_u^\varepsilon) - W_n(p, Y_{q\varepsilon\eta}^\varepsilon)\Big) \\ &\stackrel{\text{def}}{=} R_{11}^\varepsilon(t) + R_{12}^\varepsilon(t)\,.\end{aligned}$$

For $R^{\varepsilon}_{12}(t)$, we have

$$\boldsymbol{E}|R^{\varepsilon}_{12}(t)| \le \frac{C_{h,\omega_0,\nabla\omega,F}}{\varepsilon} \sum_{q=0}^{[t/(\varepsilon\eta)]-1} \int_{q\varepsilon\eta}^{(q+1)\varepsilon\eta} du \int_{q\varepsilon\eta}^{u} du' \int dp\, r(p) \le \eta \tilde{C}\,,$$

following the computations in the proof of Lemma 3.16. It only remains to treat $R^{\varepsilon}_{11}(t)$, and this term requires more care. We have

$$\boldsymbol{E}|R^{\varepsilon}_{11}(t)| \le C_{h,\omega_0,\nabla\omega,F^M} \sum_{q=0}^{[t/(\varepsilon\eta)]-1} \sqrt{I^{\varepsilon}_{q,\eta}}\,,$$

with

$$I^{\varepsilon}_{q,\eta} = \boldsymbol{E}\Big[\Big|\int_{q\varepsilon\eta}^{(q+1)\varepsilon\eta} du \iint \Big[V\Big(\frac{u}{\varepsilon^2}, dp\Big)V\Big(\frac{u}{\varepsilon^2}, dp'\Big) - r(p)\delta(p-p')\, dp\, dp'\Big] F_{q,\eta}(u,p)\Big|^2\Big]$$

where

$$F_{q,\eta}(u,p) = \sum_{n\in\mathbb{Z}^*} \Big(W_n(p, Y^{\varepsilon}_u) - W_n(p, Y^{\varepsilon}_{q\varepsilon\eta})\Big)\,.$$

Using Gaussian property of V, we have

$$\begin{aligned} I^{\varepsilon}_{q,\eta} \overset{\text{def}}{=} & \int_{q\varepsilon\eta}^{(q+1)\varepsilon\eta} du_1 \int_{q\varepsilon\eta}^{(q+1)\varepsilon\eta} du_2 \int F_{q,\eta}(u_1,p_1)F_{q,\eta}(u_2,p_2) \\ & \cdot \Big(\boldsymbol{E}\Big[V\Big(\frac{u_1}{\varepsilon^2}, dp_1\Big)V\Big(\frac{u_2}{\varepsilon^2}, dp_2\Big)\Big]\boldsymbol{E}\Big[V\Big(\frac{u_1}{\varepsilon^2}, dp_1'\Big)V\Big(\frac{u_2}{\varepsilon^2}, dp_2'\Big)\Big] \\ & \quad + \boldsymbol{E}\Big[V\Big(\frac{u_1}{\varepsilon^2}, dp_1\Big)V\Big(\frac{u_2}{\varepsilon^2}, dp_2'\Big)\Big]\boldsymbol{E}\Big[V\Big(\frac{u_1}{\varepsilon^2}, dp_1'\Big)V\Big(\frac{u_2}{\varepsilon^2}, dp_2\Big)\Big]\Big), \end{aligned}$$

and by the symmetry between u_1 and u_2, we can write

$$\begin{aligned} I^{\varepsilon}_{q,\eta} \le C_{h,\omega_0,\nabla\omega,F} & \iint dp\, dp'\, r(p)r(p') \\ & \cdot \int_{q\varepsilon\eta}^{(q+1)\varepsilon\eta} du_1 \int_{q\varepsilon\eta}^{u_1} du_2\, e^{-\mu(|p|^{2\beta}+|p'|^{2\beta})(u_1-u_2)/\varepsilon^2}. \end{aligned}$$

Now, we split the integration domain of p into the region where $|p| \le \nu$, and the region where $|p| > \nu$. This leads to the decomposition

$$I^{\varepsilon}_{q,\eta} \le J^{\varepsilon}_{1,q,\nu} + J^{\varepsilon}_{2,q,\nu}.$$

For the second term (when $|p| > \nu$), we have

$$\int_{q\varepsilon\eta}^{(q+1)\varepsilon\eta} du_1 \int_{q\varepsilon\eta}^{u_1} du_2\, e^{-\mu(|p|^{2\beta}+|p'|^{2\beta})(u_1-u_2)/\varepsilon^2} \le \frac{\varepsilon^2}{\mu|p|^{2\beta}} \int_0^{\varepsilon\eta} (1 - e^{-\mu|p|^{2\beta}u_1/\varepsilon^2})\, du$$
$$\le C_{\eta,\nu}\varepsilon^3\,,$$

so that

$$\sum_{q=0}^{[t/(\varepsilon\eta)]-1} \sqrt{J^{\varepsilon}_{2,q,\eta}} \le \varepsilon^{1/2} C_{\eta,\nu,T}\,.$$

For the first term (when $|p| \le \nu$), we have

$$\sum_{q=0}^{[t/(\varepsilon\eta)]-1} \sqrt{J^{\varepsilon}_{1,q,\nu}} \le C_{T,\eta}\Big(\int_{\{|p|\le\nu\}\times S} dp\, dp'\, r(p) r(p')\Big)^{1/2}$$
$$\le C_{T,\eta}\Big(\int_{\{|p|\le 1/\nu\}} \frac{dp}{|p|^{2\alpha}}\Big)^{1/2}.$$

Since $\alpha < 1/2$, the dominated convergence theorem implies that this converges to 0. This concludes the proof of Lemma 3.15.

3.5.7 Well-posedness of the martingale problem

We devote this section to showing that the martingale problem with generator $\mathcal{L}$ (see equation (3.38)) is well-posed. First, we rewrite $\mathcal{L}h(y)$ as

$$\mathcal{L}h(y) = \frac{1}{2}\sum_{j,l=1}^{2} \mathfrak{a}_{jl}(y)\partial^2_{x_l x_j} h(y) + \sum_{j=1}^{2} \partial_{x_j} h(y)\mathfrak{b}_j(y),$$

where

$$\mathfrak{a}_{jl}(y) \stackrel{\text{def}}{=} \sum_{n\in\mathbb{Z}^*} R_n(y) F_{j,n}(y) F_{-l,n}(y) = \sum_{n\in\mathbb{Z}^*} R_n(y) F_{j,n}(y) \overline{F_{l,n}(y)} \tag{3.45}$$

with (equivalently as in formula (3.53))

$$\begin{aligned} R_n(y) &\stackrel{\text{def}}{=} \int dp\, r(p)\Big(\frac{1}{\mu|p|^{2\beta} - 2i\pi n\omega(y)} + \frac{1}{\mu|p|^{2\beta} + 2i\pi n\omega(y)}\Big) \\ &= \int dp\, r(p) \frac{2\mu|p|^{2\beta}}{\mu|p|^{4\beta} + 4\pi^2 n^2 \omega^2(y)} \\ &= 2\int_0^\infty du\, \boldsymbol{E}[v(u)v(0)] \cos(2\pi n\omega(y)u)\,, \end{aligned} \tag{3.46}$$

and

$$\mathfrak{b}_j(y) \stackrel{\text{def}}{=} \sum_{n\in\mathbb{Z}^*} \int dp \, \frac{r(p)}{\mu|p|^{2\beta} - 2i\pi n\omega(y)}$$
$$\cdot \Big(\sum_{l=1}^{2} \Big(\partial_{x_l} F_{j,n}(y) F_{l,-n}(y) - \frac{2i\pi n}{\mu|p|^{2\beta} - 2i\pi n\omega(y)} F_{j,n}(y) F_{l,-n}(y) \partial_{x_l}\omega(y) \Big) \Big). \quad (3.47)$$

The uniqueness of the martingale problem is guaranteed by [KS91, Corollary 4.9 p. 317, Proposition 3.20 p. 309, and Theorem 2.5 p. 287] together with [Fri06, Theorem 1.2 p. 129], where the latter provides the locally Lipschitz behavior of a nonnegative square root. It is not hard to see that $\mathfrak{b}$ is locally Lipschitz with respect to y, as well as $\mathfrak{a}(y)$ is a symmetric nonnegative matrix, which is $\mathcal{C}^2(\mathbb{R}^2)$ with respect to y. The smoothness property is direct from (3.45) and (3.46). Regarding the nonnegativity, we have for any $x \in \mathbb{R}^2$

$$x^T \mathfrak{a}(y) x = \big\langle \sigma^*(y)(x), \sigma^*(y)(x) \big\rangle_{\mathcal{T}} \geq 0 \quad (3.48)$$

with the Hilbert space

$$\mathcal{T} = \Big\{ u : \mathbb{Z}^* \to \mathbb{C} : \overline{u_{-n}} = u_n \quad \text{and} \quad \sum_{n\in\mathbb{Z}^*} |u_n|^2 < \infty \Big\},$$

the adjoint σ^*_M is an operator from $\mathbb{R}^2$ to $\mathcal{T}$ defined by

$$\sigma^*_M(y)(n,x) \stackrel{\text{def}}{=} R_n^{1/2}(y) F_{1,n}(y) x_1 + R_n^{1/2}(y) F_{2,n}(y) x_2 \,,$$

and σ, from $\mathcal{T}$ to $\mathbb{R}^2$, is defined by

$$\sigma(y)(u,j) \stackrel{\text{def}}{=} \sum_{n\in\mathbb{Z}^*} R_n^{1/2}(y) F_{j,n}(y) u_{-n} \qquad j = 1,2 \,. \quad (3.49)$$

Hence, we have an explicit self-adjoint Hilbert-Schmidt square root for $\mathfrak{a}$ that we use in Section 3.5.8 to derive a SDE for the process $(Y_t)_{t\geq 0}$. It is also clear that σ is locally Lipschitz with respect to y, so that we can conclude the uniqueness of the martingale problem.

Let us finish by rewriting the martingale problem in a more convenient way and let us start with $\mathfrak{b}$. Using the fact that

$$\frac{-2i\pi n \partial_{x_l}\omega(y)}{(\mu|p|^{2\beta} - 2i\pi n\omega(y))^2} = \partial_{y_l} \int_0^\infty e^{-(\mu|p|^{2\beta} - 2i\pi n\omega(y))u} \, du$$
$$= 2i\pi n \partial_{x_l}\omega(y) \int_0^\infty du \, u \, e^{-(\mu|p|^{2\beta} - 2i\pi n\omega(y))u} \,,$$

we sum Fourier series in (3.47) to obtain

$$\mathfrak{b}(y) = \int_0^1 d\tau \int_0^\infty du \, R(u) \mathrm{Jac}_y[\tilde{F}(u,y,\tau)] \tilde{F}(0,y,\tau) \,,$$

where

$$\tilde{F}(u, y, \tau) = F(y, \tau + \omega(y)u) .$$

For $\mathfrak{a}$ we have the following formulation

$$\begin{aligned}\mathfrak{a}_{jl}(y) &= \int_0^1 d\tau \int_0^\infty du\, R(u)\Big(\tilde{F}_j(u, y, \tau)\tilde{F}_l(0, y, \tau) + \tilde{F}_j(0, y, \tau)\tilde{F}_l(u, y, \tau)\Big)\\ &= 2\int_0^1 d\tau \int_0^\infty du\, R(u)\tilde{F}_j(u, y, \tau)\tilde{F}_l(0, y, \tau), \quad j, l \in \{1, 2\} .\end{aligned}$$

Finally, to conclude, from definition of $\mathfrak{a}$ involving the integration of the periodic variable over a whole period, we obtain the formulation (3.38).

3.5.8 The limiting SDE

We devote this section to proving the Proposition 3.9. The proof of this proposition follows the treatment in [KS91, Proposition 4.6 p. 315] and consists of deriving the SDE corresponding to the martingale problem with generator (3.31). To prove this correspondence, we consider first the function $h_j(y) = y_j$ for any $j \in \{1, 2\}$, so that

$$M_j^0(t) = Y_{j,t} - Y_{j,0} - \int_0^t \mathfrak{b}_j(Y_s)\, ds \tag{3.50}$$

is a martingale, where $\mathfrak{b}$ is defined as $\mathfrak{b}^M$ in (3.47). Proceeding the same way choosing $h_{jl}(y) = y_j y_l$ one can show that for any $j, l \in \{1, 2\}$

$$M_j^0(t)M_l^0(t) - \int_0^t \mathfrak{a}_{jl}(Y_s)\, ds$$

is a martingale. (Recall, $\mathfrak{a}$ is defined by (3.45).) Therefore M^0 is a martingale with quadratic variation given by

$$t \mapsto \int_0^t \mathfrak{a}_{jl}(Y_s)\, ds .$$

Using (3.48), we note $\mathfrak{a} = \sigma\sigma^*$, where σ is defined by (3.49). We can now apply a standard martingale representation theorem involving Hilbert spaces (see [DPZ96, Theorem 8.2] for instance). This guarantees the existence of a complex valued cylindrical Brownian motion B ($B = (B^1 - iB^2)/\sqrt{2}$, with B^1 and B^2 being two independent real valued cylindrical Brownian motions) on the Hilbert space $\mathcal{T}$, possibly defined on an extension of the probability space under consideration, and such that with probability one

$$M^0(t) = \int_0^t \sigma(Y_s)\, dB_s$$

for all $t \geq 0$. As a result, from (3.50) we obtain the following SDE for $(Y_t)_{t\geq 0}$,

$$dY_t = \sigma(Y_t)dB_t + \mathfrak{b}(Y_t)dt .$$

Note that, because of symmetries, and that for the cylindrical Brownian motion on $\mathcal{T}$ we necessarily have $B^1_{-n} = B^1_n$ and $B^2_{-n} = -B^2_n$, this defines a real valued equation.

Regarding σ, one can write

$$\begin{aligned}\sigma_j(Y_t)dB_t &= \frac{1}{\sqrt{2}} \sum_{n\in\mathbb{Z}^*} \tilde{F}_{j,n}(Y_t) R_n^{1/2}(Y_t)(dB^1_{t,n} - idB^2_{t,n}) \\ &= \int_0^1 d\tau \tilde{F}_j(Y_t,\tau)\, dW_t(Y_t,\tau)\end{aligned} \tag{3.51}$$

where

$$dW_t(y,\tau) = \frac{1}{\sqrt{2}} \sum_{n\in\mathbb{Z}^*} e^{2i\pi n\tau} R_n^{1/2}(y)(dB^1_{t,n} - idB^2_{t,n})$$

is a real valued Brownian field with covariance function

$$\begin{aligned}\boldsymbol{E} W_t(y,\tau)W_s(y,\tau') &= t\wedge s \sum_{n\in\mathbb{Z}^*} e^{2i\pi n(\tau-\tau')} R_n(y) \\ &= t\wedge s \sum_{n\in\mathbb{Z}^*} e^{2i\pi n(\tau-\tau')} \int dp\, r(p) \\ &\quad \cdot \int_0^\infty du \left[e^{-(\mu|p|^{2\beta}+2i\pi n\omega(y))u} + e^{-(\mu|p'|^{2\beta}-2i\pi n\omega(y))u} \right] \\ &= t\wedge s \int_0^\infty du\, R(u) \\ &\quad \cdot \sum_{n\in\mathbb{Z}^*} \left[e^{2i\pi n(\tau-\tau'+\omega(y)u)} + e^{2i\pi n(\tau-\tau'-\omega(y)u)} \right].\end{aligned}$$

so that for any test functions ϕ_1, ϕ_2 in the Hilbert space

$$L^2_0(\mathbb{T}) = \left\{ \phi \in L^2(\mathbb{T}) : \int_0^1 \phi(\tau)d\tau = 0 \right\},$$

the covariance function reads

$$\begin{aligned}\boldsymbol{E} W_t(y,\phi_1)W_s(y,\phi_2) &= t\wedge s \int_0^\infty du\, R(u) \\ &\quad \cdot \int_0^1 d\tau \Big[\phi_1(\tau+\omega(y)u)\phi_2(\tau) + \phi_1(\tau)\phi_2(\tau+\omega(y)u) \Big].\end{aligned}$$

To determine the SDE satisfied by $(I^M_t)_{t\geq 0}$ we just have to consider the first coordinate of $(Y_t)_{t\geq 0}$. Then, we have

$$dI^M_t = \int_0^1 d\tau\, a^M(I^M_t,\tau)\, dW_t(I^M_t,\tau) + \int_0^1 d\tau \int_0^\infty du\, R(u)\tilde{A}(u, I^M_t,\tau)\, dt$$

with

$$\tilde{A}(u,I,\tau) = \partial_I\big(a^M(I,\tau+\omega(I)u)\big)a^M(I,\tau)$$
$$+ \partial_\tau a^M(I,\tau+\omega(I)u)\big(b^M(I,\tau) - \langle b^M(I)\rangle\big)\,.$$

Using the fact that a^M is mean-zero in τ, and integrating by parts, we have

$$\int_0^1 d\tau\, \partial_\tau a^M(I,\tau+\omega(I)u)\big(b^M(I,\tau) - \langle b^M(I)\rangle\big)$$
$$= \int_0^1 d\tau \partial_\tau a^M(I,\tau+\omega(I)u)b^M(I,\tau) - \int_0^1 d\tau a^M(I,\tau+\omega(I)u)\partial_\tau b^M(I,\tau)\,.$$

Now, we restrict our study for $t \leq \eta_M$ so that, according to (3.17), we have for $I \in (1/M, M)$

$$\partial_\tau b^M(I,\tau) = \partial_\tau b(I,\tau) = \partial_\tau \partial_I \varphi_1^{-1}(I,\tau) = \partial_I \partial_\tau \varphi_1^{-1}(I,\tau) = -\partial_I a(I,\tau)\,,$$

so that

$$\int_0^1 d\tau \tilde{A}(u,I,\tau) = \int_0^1 d\tau\, \partial_I\big(a(I,\tau+\omega(I)u)\big)a(I,\tau) + a(I,\tau+\omega(I)u)\partial_I a(I,\tau).$$

We then obtain the desired result for the stopped process $\tilde{I}_t^M = I^M_{t\wedge\eta_M}$. Now for the process $\psi^M_{t\wedge\eta_M}$, we look at the second coordinate of $(Y_t)_{t\geq 0}$, and we have

$$d\psi_t^M = \int_0^1 d\tau\, b^M(I_t^M,\tau)\, dW_t(I_t^M,\tau) + \int_0^1 d\tau \int_0^\infty du\, R(u)\tilde{B}(u,I_t^M,\tau)\, dt$$

with

$$\tilde{B}(u,I,\tau) = \partial_I\big(b^M(I,\tau+\omega(I)u) - \langle b^M(I)\rangle\big)a^M(I,\tau)$$
$$+ \partial_\tau(b^M(I,\tau+\omega(I)u) - \langle\rangle b^M(I)\rangle)\big(b^M(I,\tau) - \langle\rangle b^M(I)\rangle\big)\,.$$

Using again that a^M and $\partial_\tau b^M$ are mean-zero in τ, together with an integration by parts, we obtain

$$\int_0^1 d\tau\, \tilde{B}(u,I,\tau) = \int_0^1 d\tau\, \partial_I\big(b^M(I,\tau+\omega(I)u)\big)a^M(I,\tau)$$
$$- \int_0^1 d\tau\, b^M(I,\tau+\omega(I)u)\partial_\tau b^M(I,\tau)\,.$$

Since for $I \in (1/M, M)$ we know

$$\partial_\tau b^M(I,\tau) = -\partial_I a(I,\tau)\,,$$

the proof of Proposition 3.9 is complete.

3.6 Proof of the main theorem

The purpose of this section is to justify the truncation procedure of the original processes $(I^\varepsilon, \psi^\varepsilon)$ to complete the proof of Theorem 3.6. The boundedness of the truncated processes is crucial in estimating several quantities in the proof. Therefore, we study properties of the limiting equation (3.20). The main issue concerns the behavior of a at $I = 0$, but we will see how to define a unique global solution for this equation that does not reach 0 with probability one. In the case of a quadratic Hamiltonian, we have seen in (3.24) that $(I_t)_{t\geq 0}$ is related to a 2-dimensional squared Bessel process (see (3.24)) which is known to reach 0 with probability zero [KS91, Proposition 3.22 p. 161].

3.6.1 Existence of a solution

First, we remark that the function a is not Lipschitz in I as it typically has a square-root singularity near 0. So it is not immediately apparent that equation (3.20) has solutions. To construct solutions to equation (3.20), define

$$a_n(I) \stackrel{\text{def}}{=} \int_{\tau=0}^{1} a(I,\tau)e^{-2i\pi n\tau}\, d\tau \tag{3.52}$$

$$R_n(I) \stackrel{\text{def}}{=} 2\int_{u=0}^{\infty} R(u)\cos(2\pi n\omega(I)u)\, du\,. \tag{3.53}$$

and let

$$\begin{aligned}
\mathbf{a}(I) &\stackrel{\text{def}}{=} \sum_{n\in\mathbb{Z}^*} |a_n(I)|^2 R_n(I) \\
&= 2\int_{u=0}^{\infty} R(u)\int_{\tau=0}^{1} a(I,\tau)a(I,\tau+\omega(I)u)\, d\tau\, du\,, \qquad (3.54)\\
\mathbf{b}(I) &\stackrel{\text{def}}{=} \int_{u=0}^{\infty} R(u)\int_{\tau=0}^{1} \Big[\partial_I(a(I,\tau+\omega(I)u))a(I,\tau) \\
&\qquad\qquad + \partial_I a(I,\tau)a(I,\tau+\omega(I)u)\Big]\, d\tau\, du\,. \qquad (3.55)
\end{aligned}$$

Note that equality in (3.54) is justified by using the Fourier series expansion for $a(I,\tau)$, and using the fact that $a(I,\tau)$ is mean-zero in τ. That is,

$$a(I,\tau) = \sum_{n\in\mathbb{Z}^*} e^{2i\pi n\tau} a_n(I)\,.$$

Definition 3.19. *A weak solution in the interval $(0,\infty)$ to equation (3.20) is a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \boldsymbol{P})$ satisfying the usual conditions and a continuous pair of processes (I_t, W_t) such that the followings hold.*

(i). The process I takes values in $[0,\infty]$ with $I_0 \in (0,\infty)$,

(ii). The process W is a standard $\mathcal{F}_t$-adapted Brownian motion on the Hilbert space $L^2_0(\mathbb{T})$ with covariance function given by (3.21).

(iii). For any $t, M > 0$ we have

$$\int_{s=0}^{t\wedge\zeta_M} (\mathbf{a}(I_s) + |\mathbf{b}(I_s)|)\, ds < \infty\,, \tag{3.56}$$

$$I_{t\wedge\zeta_M} = I_0 + \int_{s=0}^{t\wedge\zeta_M}\int_{\tau=0} a(I_s,\tau)\, dW_s(I_s,\tau) + \int_{s=0}^{t\wedge\zeta_M} \mathbf{b}(I_s)\, ds \quad \forall t \geq 0\,, \tag{3.57}$$

almost surely. Here $\zeta_M \stackrel{\text{def}}{=} \zeta_{1,M} \wedge \zeta_{2,M}$ where $\zeta_{1,M}$ and $\zeta_{2,M}$ are defined by

$$\zeta_{1,M} \stackrel{\text{def}}{=} \inf\{t \geq 0 |\, |I_t| \geq M\}$$
$$\zeta_{2,M} \stackrel{\text{def}}{=} \inf\{t \geq 0 |\, |I_t| \leq 1/M\}\,.$$

For notational convenience we denote

$$\zeta_\infty \stackrel{\text{def}}{=} \lim_{M\to\infty} \zeta_M = \inf\{t \geq 0 |\, I_t \notin (0,\infty)\} \tag{3.58}$$

to be the first exit time of I from $(0,\infty)$.

We will now work with the canonical probability space $\mathcal{C}([0,\infty),\mathbb{R})$ and canonical filtration

$$\mathcal{M}_t = \sigma(h_s,\, 0 \leq s \leq t)\,.$$

To construct a solution to (3.20), we note first that when restricted to the interval $[1/M, M]$, the function a is Lipschitz. Thus we know (3.20) has a strong solution when truncated to this interval (see Proposition 3.8, below). If we denote this truncated solution by (I^M, ζ_M) we obtain an increasing sequence of stopping times ζ_M, an increasing family of σ-algebras $\mathcal{M}_{\zeta_M}$, and a sequence of probability measures $\boldsymbol{P}_I^M$ such that

$$\boldsymbol{P}_I^{M+1} = \boldsymbol{P}_I^M \qquad \text{on} \quad \mathcal{M}_{\zeta_M} \qquad \forall M > 0\,.$$

As a result, we can define $\boldsymbol{P}^0$ on $\bigcup_M \mathcal{M}_{\zeta_M}$ by

$$\boldsymbol{P}_I^0(O) = \boldsymbol{P}_I^M(O) \qquad \text{for} \quad O \in \mathcal{M}_{\zeta_M}\,,$$

and extend $\boldsymbol{P}_I^0$ to a probability measure on the σ-algebra $\mathcal{M}_{\zeta_\infty}$ such that for any $M \in \mathbb{N}$ we have

$$\boldsymbol{P}_I^0 = \boldsymbol{P}_I^M \qquad \text{on} \quad \mathcal{M}_{\zeta_M}\,. \tag{3.59}$$

Note that

$$\mathcal{M}_{\zeta_\infty} = \sigma(\bigcup_M \mathcal{M}_{\zeta_M})\,, \tag{3.60}$$

which can be easily verified from the fact that

$$\mathcal{M}_{\zeta_\infty} = \sigma(h_{t\wedge\zeta_\infty}, t \geq 0)\,,$$

(see for instance [SV06, Lemma 1.3.3 p. 33]). As a result $\boldsymbol{P}_I^0$ provides a solution to (3.20), that we can denote by $I = (I_t)_{t\geq 0}$, in the sense of Definition 3.19. Note that the process $(I_t)_{t\geq 0}$ is nonnegative with probability one thanks to the Portmanteau theorem [Bil99, Theorem 2.1 p. 16].

In the next two sections, we will first show (using the bounds in Proposition 3.5) that $\boldsymbol{P}_I^0(\zeta_\infty = \infty) = 1$. That is, the process I does not reach 0 or blow up in finite time. This will show that $\boldsymbol{P}_I^0$ is the unique extension on $\mathcal{M} \stackrel{\text{def}}{=} \sigma(\bigcup_{t\geq 0} \mathcal{M}_t)$ satisfying (3.59). Additionally, we will show that $I = (I_t)_{t\geq 0}$ is the unique global solution to (3.20).

3.6.2 Non-explosion and inaccessibility to 0

Our aim in this section is to show that the process I cannot explode in finite time nor reach 0.

Proposition 3.20. *Consider a weak solution to* (3.20) *(as in definition 3.19), and let* ζ_∞ *be as in* (3.58). *Then*

$$\boldsymbol{P}_I^0(\zeta_\infty = \infty) = 1\,.$$

This result together with (3.59) implies that for any $t \geq 0$

$$\lim_{M\to\infty} \boldsymbol{P}_I^M(\zeta_M \leq t) = \lim_{M\to\infty} \boldsymbol{P}_I^0(\zeta_M \leq t) = 0\,.$$

Therefore, [SV06, Theorem 1.3.5 p. 34] guarantees that $\boldsymbol{P}_I^0$ is the unique extension satisfying (3.59).

Before proving Proposition 3.20 we need to introduce the *scale function*

$$p(x) \stackrel{\text{def}}{=} \int_1^x \exp\left(-2\int_1^\xi \frac{\mathbf{b}(\nu)}{\mathbf{a}(\nu)}\,d\nu\right) d\xi, \quad x \in (0,\infty)$$

where $\mathbf{a}$ and $\mathbf{b}$ are defined by (3.54) and (3.55) respectively. Observe

$$p''(x) = -2\frac{\mathbf{b}(x)}{\mathbf{a}(x)}p'(x). \tag{3.61}$$

Standard results express the exit probability of I in terms of the scale function, and we prove them in our context for completeness.

Lemma 3.21. *For any* $0 < x_- < x_+ < \infty$ *we have*

$$\boldsymbol{P}(I_{\zeta_{x_-,x_+}} = x_-) = \frac{p(x_+) - p(I_0)}{p(x_+) - p(x_-)} \quad \textit{and} \quad \boldsymbol{P}(I_{\zeta_{x_-,x_+}} = x_+) = \frac{p(I_0) - p(x_-)}{p(x_+) - p(x_-)}\,,$$

where $\zeta_{x_-,x_+} = \zeta_{1,x_+} \wedge \zeta_{2,x_-}$.

Proof of Lemma 3.21. The proof consists of applying the Itô formula for $p(I_t)$ with $t \leq \zeta_{x_-,x_+} \leq \zeta_\infty$. To do this we rewrite the stochastic integral of (3.20) in the Fourier domain, that is

$$\int_{s=0}^t \int_{\tau=0}^1 a(I_s,\tau)\,dW_s(I_s,\tau)\,d\tau = \int_{s=0}^t \sum_{n\in\mathbb{Z}^*} a_n(I_s) R_n^{1/2}(I_s) \frac{1}{\sqrt{2}}(dB_{s,n}^1 - i dB_{s,n}^2)\,,$$

providing an explicit semi-martingale representation for $(I_t)_{t\geq 0}$. Therefore, applying the Itô formula to $p(I_t)$ we have

$$
\begin{aligned}
p(I_{t\wedge\zeta_{x_-,x_+}}) &= p(I_0) + \int_0^{t\wedge\zeta_{x_-,x_+}} p'(I_s)\, dI_s + \frac{1}{2}\int_0^{t\wedge\zeta_{x_-,x_+}} p''(I_s) d\langle I\rangle_s \\
&= p(I_0) + \int_0^{t\wedge\zeta_{x_-,x_+}} p'(I_s) \sum_{n\in\mathbb{Z}^*} a_n(I_s) R_n^{1/2}(I_s)\frac{1}{\sqrt{2}}(dB^1_{s,n} - i dB^2_{s,n}) \\
&\quad + \int_0^{t\wedge\zeta_{x_-,x_+}} p'(I_s)\mathbf{b}(I_s)\, ds \\
&\quad + \frac{1}{2}\int_0^{t\wedge\zeta_{x_-,x_+}} -2\frac{\mathbf{b}(I_s)}{\mathbf{a}(I_s)} p'(I_s) \sum_{n\in\mathbb{Z}^*} |a_n(I_s)|^2 R_n(I_s)\, ds \\
&= p(I_0) + \int_0^{t\wedge\zeta_{x_-,x_+}} p'(I_s) \sum_{n\in\mathbb{Z}^*} a_n(I_s) R_n^{1/2}(I_s)\frac{1}{\sqrt{2}}(dB^1_{s,n} - i dB^2_{s,n})
\end{aligned}
$$

thanks to (3.61) and (3.54). Note that the stochastic integral on the right-hand side of the last line is a martingale starting from 0, so by taking the expectation, and passing to the limit $t \to \infty$ we obtain

$$
p(I_0) = \boldsymbol{E}[p(I_{\zeta_{x_-,x_+}})] = p(x_-)\boldsymbol{P}(I_{\zeta_{x_-,x_+}} = x_-) + p(x_+)\boldsymbol{P}(I_{\zeta_{x_-,x_+}} = x_+)\,.
$$

Solving for $\boldsymbol{P}(I_{\zeta_{x_-,x_+}} = x_-)$ and $\boldsymbol{P}(I_{\zeta_{x_-,x_+}} = x_+)$ we obtain the desired result. □

With this lemma, we can now turn to the proof of Proposition 3.20.

Proof of Proposition 3.20. Let us start with the following remark. The term $\mathbf{b}$ can be written as

$$
\mathbf{b}(I) = \frac{1}{2}\partial_I \mathbf{a}(I)\,,
$$

so that

$$
\begin{aligned}
p(0_+) &= \lim_{x\to 0_+} \int_1^x \exp\left(-2\int_1^\xi \frac{\mathbf{b}(\nu)}{\mathbf{a}(\nu)} d\nu\right) d\xi \\
&= \lim_{x\to 0_+} \mathbf{a}(1)\int_1^x \exp\left(-\ln(\mathbf{a}(\xi))\right) d\xi \\
&= \lim_{x\to 0_+} \mathbf{a}(1)\int_1^x \frac{d\xi}{\mathbf{a}(\xi)}\,.
\end{aligned}
$$

Now, according to (3.54) together with Proposition 3.5, we have

$$
0 \leq \mathbf{a}(\xi) \leq C\,\xi
$$

since for any $n \in \mathbb{Z}^*$

$$\begin{aligned} 0 \le R_n(\xi) &= 2\int_0^\infty R(u)\cos(2\pi n\omega(\xi)u)\,du \\ &= 2\int_0^\infty \Big(\int_S r(p)e^{-\mu|p|^{2\beta}u}\,dp\Big)\cos(2\pi n\omega(\xi)u)\,du \\ &= 2\int_S dp\,r(p)\Big(\int_0^\infty \cos(2\pi n\omega(\xi)u)e^{-\mu|p|^{2\beta}u}\,du\Big) \\ &= \int_S dp\,r(p)\frac{2\mu|p|^{2\beta}}{\mu^2|p|^{4\beta}+4\pi^2n^2\omega^2(\xi)} \le \frac{C_{\omega_0}}{n^2}\int_S |p|^{2\beta}r(p)\,dp\,. \end{aligned} \tag{3.62}$$

As a result, we have

$$p(0_+) \le \lim_{x\to 0_+} \mathbf{a}(1)\int_1^x \frac{d\xi}{C\xi} = -\infty\,,$$

which implies $p(0_+) = -\infty$. To conclude the proof we consider two cases, that is $p(\infty) = \infty$ and $p(\infty) < \infty$. Following the same lines as [KS91, Proposition 5.22 p. 345], if $p(\infty) = \infty$ we directly have the desired result, and if $p(\infty) < \infty$ we have

$$\lim_{x_-\to 0_+}\lim_{x_+\to\infty} \boldsymbol{P}(I_{\zeta_{x_-,x_+}} = x_+) = \lim_{x_-\to 0_+}\lim_{x_+\to\infty} \frac{p(I_0)-p(x_-)}{p(x_+)-p(x_-)} = 1\,.$$

This implies $\boldsymbol{P}(\zeta_\infty = \zeta_{1,\infty}) = 1$ meaning that if I_t exits the interval $(0,\infty)$, it almost surely does so at ∞. Here

$$\zeta_{1,\infty} = \lim_{M\to\infty}\zeta_{1,M}\,.$$

To prove that $\boldsymbol{P}(\zeta_{1,\infty} = \infty) = 1$, and then conclude the proof of the Proposition 3.20, we need the following result.

Lemma 3.22. *For any $T > 0$ we have*

$$\lim_{M\to\infty} \boldsymbol{P}(0 \sup_{t\in[0,T]} I_{t\wedge\zeta_M} \ge M) = 0\,.$$

Indeed, writing

$$\begin{aligned} \boldsymbol{P}(\zeta_{1,\infty} < \infty) &= \lim_{T\to\infty}\boldsymbol{P}(\zeta_{1,\infty} \le T) \\ &= \lim_{T\to\infty}\lim_{M_0\to\infty}\boldsymbol{P}(\zeta_{1,\infty} \le T, \inf_{t\in[0,\zeta_\infty)} I_t > 1/M_0), \\ &= \lim_{T\to\infty}\lim_{M_0\to\infty}\lim_{M\to\infty}\boldsymbol{P}(\zeta_{1,M} \le T, \inf_{t\in[0,\zeta_M)} I_t > 1/M_0)\,. \end{aligned}$$

To justify the second equality, we note that if $\zeta_{1,\infty}$ is finite, then $\inf_{t\in[0,\zeta_\infty)} I_t$ must be positive since $\zeta_{1,\infty} = \zeta_\infty$ almost surely. We have for any $M > M_0$

$$\boldsymbol{P}\Big(\zeta_{1,M} \le T, \inf_{t\in[0,\zeta_M]} I_t > \frac{1}{M_0}\Big) \le \boldsymbol{P}(\zeta_{1,M} \le T,\, \zeta_M = \zeta_{1,M})$$
$$\le \boldsymbol{P}(\sup_{t\in[0,T]} I_{t\wedge\zeta_M} \ge M).$$

Therefore, Lemma 3.22 will imply $\boldsymbol{P}(\zeta_{1,\infty} < \infty) = 0$.

Proof of Lemma 3.22. First we remark that using Chebychev's inequality

$$\boldsymbol{P}(\sup_{t\in[0,T]} I_{t\wedge\zeta_M} \ge M) = \boldsymbol{P}(\sup_{t\in[0,T]} I^2_{t\wedge\zeta_M} \ge M^2)$$
$$\le \frac{1}{M^2}\boldsymbol{E} \sup_{t\in[0,T]} I^2_{t\wedge\zeta_{1,M}}\,. \tag{3.63}$$

Let $T' \le T$, we also have

$$\boldsymbol{E} \sup_{t\in[0,T']} I^2_{t\wedge\zeta_M} \le 3I_0 + 3\boldsymbol{E} \sup_{t\in[0,T']} N_t^2 + 3\boldsymbol{E} \sup_{t\in[0,T']} \Big(\int_0^{t\wedge\zeta_M} \mathbf{b}(I_s)ds\Big)^2$$
$$= 3I_0 + J_{T'} + K_{T'}\,,$$

and where, using (3.57)

$$N_{t\wedge\zeta_M} \stackrel{\text{def}}{=} I_{t\wedge\zeta_M} - I_0 - \int_0^{t\wedge\zeta_M} \mathbf{b}(I_s)\,ds = \int_{s=0}^{t\wedge\zeta_M} \int_{\tau=0}^{1} a(I_s,\tau)\,dW_s(I_s,\tau)\,d\tau$$

is a martingale with quadratic variation

$$\langle N\rangle_t = \int_0^{t\wedge\zeta_M} \mathbf{a}(I_s)\,ds\,.$$

For the term $J_{T'}$ we apply Burkholder-Davis-Gundy inequality [KS91, Theorem 3.28 p. 166] together with (3.62) from which we obtain

$$J_{T'} \le C\boldsymbol{E}\int_0^{T'\wedge\zeta_M} \mathbf{a}(I_s)\,ds \le C\int_0^{T'} \boldsymbol{E}[\mathbf{a}(I_{s\wedge\zeta_M})]\,ds$$
$$\le C_{\omega_0}\int_S |p|^{2\beta}|r(p)|\,dp \int_{s=0}^{T'}\int_{\tau=0}^{1} \boldsymbol{E}[|a(I_{s\wedge\zeta_M},\tau)|^2]\,d\tau\,ds\,.$$

Moreover, using Proposition 3.5, we have

$$\begin{aligned}\int_0^1 \boldsymbol{E}[|a(I_{s\wedge\zeta_M},\tau)|^2]\,d\tau &= \int_0^1 \Big(\boldsymbol{E}[|a(I_{s\wedge\zeta_M},\tau)|^2 \mathbf{1}_{(I_{s\wedge\zeta_M}\le r)}] \\ &\qquad + \boldsymbol{E}[|a(I_{s\wedge\zeta_M},\tau)|^2 \mathbf{1}_{(I_{s\wedge\zeta_M}> r)}]\Big)\,d\tau \\ &\le r\,C + C_r \boldsymbol{E} I^2_{s\wedge\zeta_M} \\ &\le r\,C + C_r \boldsymbol{E} \sup_{t\in[0,s]} I^2_{t\wedge\zeta_M}\,,\end{aligned}$$

and hence

$$J_{T'} \le C_1\,T + C_2 \int_0^{T'} \boldsymbol{E}\left(\sup_{t\in[0,s]} I^2_{t\wedge\zeta_M}\right) ds\,.$$

Now, for $K_{T'}$, we use Cauchy-Schwarz inequality

$$\begin{aligned}\boldsymbol{E} \sup_{t\in[0,T']} \Big(\int_0^{t\wedge\zeta_M} \mathbf{b}(I_s)\,ds\Big)^2 &\le T\boldsymbol{E} \sup_{t\in[0,T']} \int_0^{t\wedge\zeta_M} \mathbf{b}^2(I_s)\,ds \\ &\le T\boldsymbol{E} \int_0^{T'} \mathbf{b}^2(I_{s\wedge\zeta_M})\,ds\,.\end{aligned}$$

From Proposition 3.5, for $I \le r$, we again use the Cauchy-Schwarz inequality

$$\begin{aligned}|\mathbf{b}(I)| &\le C\,|\omega'(I)| \sqrt{\int_0^1 |a(I,\tau)|^2\,d\tau \int_0^1 |\partial_\tau a(I,\tau)|^2\,d\tau} \\ &\quad + C_{\omega_0} \sqrt{\int_0^1 |a(I,\tau)|^2\,d\tau \int_0^1 |\partial_I a(I,\tau)|^2\,d\tau} \\ &\le C.\end{aligned}$$

As a result,

$$\begin{aligned}\boldsymbol{E}\int_0^{T'} \mathbf{b}^2(I_{s\wedge\zeta_M})\,ds &= \boldsymbol{E}\int_0^{T'} \mathbf{b}^2(I_{s\wedge\zeta_M})\mathbf{1}_{(I_{s\wedge\zeta_M}\le r)}\,ds \\ &\quad + \boldsymbol{E}\int_0^{T'} \mathbf{b}^2(I_{s\wedge\zeta_M})\mathbf{1}_{(I_{s\wedge\zeta_M}> r)}\,ds \\ &\le C_1\,T + C_{2,r}\boldsymbol{E}\int_0^{T'} I^2_{s\wedge\zeta_M}\mathbf{1}_{(I_{s\wedge\zeta_M}> r)}\,ds \\ &\le C_1\,T + C_{2,r}\int_0^{T'} \boldsymbol{E}\Big(\sup_{t\in[0,s]} I^2_{t\wedge\zeta_M}\Big)\,ds\,,\end{aligned}$$

and

$$K_{T'} \le C_1\,T^2 + C_2\,T \int_0^{T'} \boldsymbol{E}\Big(\sup_{t\in[0,s]} I^2_{t\wedge\zeta_M}\Big)\,ds\,.$$

Finally, we have

$$\boldsymbol{E}\sup_{t\in[0,T']} I^2_{t\wedge\zeta_M} \le C_1(1+T+T^2) + C_2(1+T)\int_0^{T'} \boldsymbol{E}\Big(\sup_{t\in[0,s]} I^2_{t\wedge\zeta_M}\Big)\,ds$$

and then by Gronwall's inequality with $T' = T$

$$\boldsymbol{E}\sup_{t\in[0,T]} I^2_{t\wedge\zeta_M} \le C_1(1+T+T^2)e^{C_2(1+T)T}\,.$$

This concludes the proof of the Lemma by letting $M \to \infty$ in (3.63). Consequently, the proof of Proposition 3.20 is also complete. □

3.6.3 Uniqueness of solutions

The weak uniqueness property for equation (3.20) comes from the fact that this equation has strong uniqueness on $\mathcal{M}_{\zeta_{2,M}}$ for any M, and then weak uniqueness on $\mathcal{M}_{\zeta_{2,M}}$ for any M [KS91, Proposition 3.20 p. 309], since we avoid the lack of regularity at 0 for the action variable I. However, according to Proposition 3.20, we have $\lim_{M\to\infty} \boldsymbol{P}^0_I(\zeta_{2,M} \le T) = 0$ for any $T > 0$ which guarantees the weak uniqueness thanks to [SV06, Theorem 1.3.5 p. 34].

3.6.4 Convergence of $(I^\varepsilon, \psi^\varepsilon)_{\varepsilon>0}$

In this section, we will show how to use Propositions 3.8 and 3.9 to complete the proof of Theorem 3.6. The idea is to remove the cutoff M to show the convergence in distribution of $(I^\varepsilon, \psi^\varepsilon)_{\varepsilon>0}$ to $(I_t, \psi_t)_{t\ge0}$, where I, ψ are defined by (3.20)-(3.22). This is similar to Lemma 11.1.1 (p. 262) in [SV06], and we adapt the proof to our situation.

In what follows, we denote by $\boldsymbol{P}^M$ and $\boldsymbol{P}^0$ the distribution of $Y^M = (I^M_t, \psi^M_t)_{t\ge0}$ (defined in Proposition 3.8) and $Y = (I_t, \psi_t)_{t\ge0}$ (defined in Proposition 3.6), respectively. Note that according to (3.33) and (3.22) the distributions $\boldsymbol{P}^M$ and $\boldsymbol{P}^0$ are completely determined by their first marginal $\boldsymbol{P}^M_I$ and $\boldsymbol{P}^0_I$. Also, in view of (3.32) and (3.33), it is straightforward to see that (with η_M defined in (3.29))

$$\boldsymbol{P}^0 = \boldsymbol{P}^M \qquad \text{on} \quad \mathcal{M}_{\eta_M}\,. \tag{3.64}$$

To prove this convergence, let Φ be a continuous bounded function on $\mathcal{C}([0,\infty),\mathbb{R}^2)$, and write

$$\begin{aligned}
&\boldsymbol{E}^{\boldsymbol{P}^\varepsilon}[\Phi(y_t, t\in[0,T])] - \boldsymbol{E}^{\boldsymbol{P}}[\Phi(y_t, t\in[0,T])] \\
&\quad= \boldsymbol{E}^{\boldsymbol{P}^\varepsilon}[\Phi(y_t, t\in[0,T])] - \boldsymbol{E}^{\boldsymbol{P}^{\varepsilon,M}}[\Phi(y_t, t\in[0,T])] \\
&\qquad+ \boldsymbol{E}^{\boldsymbol{P}^{\varepsilon,M}}[\Phi(y_t, t\in[0,T])] - \boldsymbol{E}^{\boldsymbol{P}^M}[\Phi(y_t, t\in[0,T])] \\
&\qquad+ \boldsymbol{E}^{\boldsymbol{P}^M}[\Phi(y_t, t\in[0,T])] - \boldsymbol{E}^{\boldsymbol{P}^0}[\Phi(y_t, t\in[0,T])]\,.
\end{aligned}$$

Considering the following decomposition

$$\begin{aligned}\boldsymbol{E}^{\boldsymbol{P}^\varepsilon}[\Phi(y_t, t\in[0,T])] = \boldsymbol{E}^{\boldsymbol{P}^\varepsilon}&[\Phi(y_t, t\in[0,T])\mathbf{1}_{\{\eta_M\le T\}}] \\ &+ \boldsymbol{E}^{\boldsymbol{P}^\varepsilon}[\Phi(y_t, t\in[0,T])\mathbf{1}_{\{\eta_M> T\}}]\end{aligned}$$

with

$$\begin{aligned}\boldsymbol{E}^{\boldsymbol{P}^\varepsilon}&[\Phi(y_t, t\in[0,T])\mathbf{1}_{\{\eta_M> T\}}] = \boldsymbol{E}^{\boldsymbol{P}^\varepsilon}[\Phi(y_{t\wedge\eta_M}, t\in[0,T])\mathbf{1}_{\{\eta_M> T\}}] \\ &= \boldsymbol{E}^{\boldsymbol{P}^{\varepsilon,M}}[\Phi(y_{t\wedge\eta_M}, t\in[0,T])\mathbf{1}_{\{\eta_M> T\}}] \\ &= \boldsymbol{E}^{\boldsymbol{P}^{\varepsilon,M}}[\Phi(y_t, t\in[0,T])\mathbf{1}_{\{\eta_M> T\}}] \\ &= \boldsymbol{E}^{\boldsymbol{P}^{\varepsilon,M}}[\Phi(y_t, t\in[0,T])] - \boldsymbol{E}^{\boldsymbol{P}^{\varepsilon,M}}[\Phi(y_t, t\in[0,T])\mathbf{1}_{\{\eta_M\le T\}}]\,.\end{aligned}$$

Here we used (3.30) in the second line. Using (3.30) again we note

$$\begin{aligned}\boldsymbol{E}^{\boldsymbol{P}^\varepsilon}[\Phi(y_t, t\in[0,T])] - \boldsymbol{E}^{\boldsymbol{P}^{\varepsilon,M}}&[\Phi(y_t, t\in[0,T])] \\ &\le \|\Phi\|_\infty(\boldsymbol{P}^\varepsilon(\eta_M\le T) + \boldsymbol{P}^{\varepsilon,M}(\eta_M\le T)) \\ &\le 2\|\Phi\|_\infty \boldsymbol{P}^{\varepsilon,M}(\eta_M\le T)\,.\end{aligned}$$

The same results hold for $\boldsymbol{P}^0$ and $\boldsymbol{P}^M$ instead of $\boldsymbol{P}^\varepsilon$ and $\boldsymbol{P}^{\varepsilon,M}$ respectively, but using (3.64) instead of (3.30). As a result, we obtain

$$\begin{aligned}\limsup_{\varepsilon\to0}|\boldsymbol{E}^{\boldsymbol{P}^\varepsilon}[\Phi(y_t, t\in[0,T])] - \boldsymbol{E}^{\boldsymbol{P}^0}&[\Phi(y_t, t\in[0,T])]| \\ &\le 2\|\Phi\|_\infty \limsup_{\varepsilon\to0}\boldsymbol{P}^{\varepsilon,M}(\eta_M\le T) \\ &+ 2\|\Phi\|_\infty \boldsymbol{P}^0(\eta_M\le T)\,.\end{aligned}$$

Notice that the stopping time η_M is lower semi-continuous on $\mathcal{C}([0,\infty),\mathbb{R}^2)$, so that $(\eta_M\le M)\in\mathcal{M}_{\eta_M}$ is a closed subset of $\mathcal{C}([0,\infty),\mathbb{R}^2)$. According to the Portmanteau theorem [Bil99, Theorem 2.1 p. 16] we have

$$\limsup_{\varepsilon\to0}|\boldsymbol{E}^{\boldsymbol{P}^\varepsilon}[\Phi(y_t, t\in[0,T])] - \boldsymbol{E}^{\boldsymbol{P}^0}[\Phi(y_t, t\in[0,T])]| \le 4\|\Phi\|_\infty \boldsymbol{P}^0(\eta_M\le T)\,,$$

for any M. Moreover, we have

$$\boldsymbol{P}^0(\eta_M\le T)\le \boldsymbol{P}^0(\zeta_M\le T) + \boldsymbol{P}^0(\zeta'_M\le T)\,.$$

where for any any $y=(y^1,y^2)\in\mathcal{C}([0,\infty),\mathbb{R}^2)$ we define

$$\zeta'_M(y)\stackrel{\text{def}}{=}\inf\{t\ge0: |y^2_t|\ge M\}\,.$$

By Proposition 3.20 we know

$$\boldsymbol{P}^0(\zeta_M\le T) = \boldsymbol{P}^0_I(\zeta_M\le T)\underset{M\to\infty}{\longrightarrow}0\,.$$

For the other term, let $0 < \eta < 1$ and write

$$\boldsymbol{P}^0(\zeta'_M \leq T) \leq \boldsymbol{P}\Big(\sup_{t\in[0,T]} |\psi_t| \geq M,\ \inf_{t\in[0,T]} I_t > \eta,\ \sup_{t\in[0,T]} I_t < \frac{1}{\eta}\Big) + \boldsymbol{P}\Big(\sup_{t\in[0,T]} I_t \geq \frac{1}{\eta}\Big) + \boldsymbol{P}\Big(\inf_{t\in[0,T]} I_t \leq \eta\Big)\,,$$

then using Markov inequality and (3.22), we obtain the following for some $C_\eta > 0$

$$\begin{aligned}
&\boldsymbol{P}\Big(\sup_{t\in[0,T]} |\psi_t| \geq M,\ \inf_{t\in[0,T]} I_t > \eta,\ \sup_{t\in[0,T]} I_t < \frac{1}{\eta}\Big) \\
&\quad \leq \frac{1}{M^2}\boldsymbol{E}\Big[\sup_{t\in[0,T]} |\psi_t|^2 \mathbf{1}_{\{\inf_{t\in[0,T]} I_t > \eta,\, \sup_{t\in[0,T]} I_t < \frac{1}{\eta}\}}\Big] \\
&\quad \leq \frac{2}{M^2}\boldsymbol{E}\Big[\sup_{t\in[0,T]} \Big|\int_{\tau=0}^1 \int_{s=0}^t b(I_s,\tau) dW_s(I_s,\tau) d\tau\Big|^2 \mathbf{1}_{\{\inf_{t\in[0,T]} I_t > \eta,\, \sup_{t\in[0,T]} I_t < \frac{1}{\eta}\}}\Big] \\
&\qquad + \frac{C_\eta}{M^2}\,.
\end{aligned}$$

Indeed, since we are on the event $\{\inf_{t\in[0,T]} I_t > \eta, \sup_{t\in[0,T]} I_t < \frac{1}{\eta}\}$ we avoid the singularity of the function b at $I = 0$ and keep I bounded. Therefore, we can directly bound the drift term in (3.22) by bounding continuous functions $a, b, \partial_I a, \partial_I b$ by some constant C_η. We will subsequently let C_η be a constant that only depends on η whose value may change from line to line. The details are as follows.

Write the Fourier expansions of a and b as

$$a(I,\tau) = \sum_{n\in\mathbb{Z}^*} e^{2i\pi n\tau} a_n(I), \qquad b(I,\tau) = \sum_{n\in\mathbb{Z}} e^{2i\pi n\tau} b_n(I)\,.$$

Considering only the first integrand in the drift term in (3.22) and substituting the above expansions, we obtain

$$\begin{aligned}
&\int_{u=0}^\infty R(u) \int_{\tau=0}^1 \partial_I\big(b(I,\tau+\omega(I)u)\big)_{|I=I_t} a(I_t,\tau)\, d\tau\, du \\
&\quad = \int_{u=0}^\infty R(u) \int_{\tau=0}^1 \Big[\sum_{n\in\mathbb{Z}} \Big((\partial_I b)_n(I_t) + 2i\pi n\omega'(I_t) u b_n(I_t)\Big) e^{2i\pi n(\tau+\omega(I_t)u)} \\
&\qquad\qquad \cdot \Big(\sum_{n'\in\mathbb{Z}^*} a_{n'}(I_t) e^{2i\pi n'\tau}\Big)\Big]\, d\tau\, du \\
&\quad = \int_{u=0}^\infty R(u) \sum_{n\in\mathbb{Z}^*} \Big((\partial_I b)_n(I_t) + 2i\pi n\omega'(I_t) u b_n(I_t)\Big) a_{-n}(I_t) e^{2i\pi n\omega(I_t)u}\, du \qquad (3.65) \\
&\quad = R_1 + R_2\,, \qquad (3.66)
\end{aligned}$$

where

$$R_1 \overset{\text{def}}{=} \int_{u=0}^{\infty} R(u) \sum_{n\in\mathbb{Z}^*} (\partial_I b)_n(I_t) a_{-n}(I_t) \cos(2\pi n\omega(I_t)u)\, du$$
$$R_2 \overset{\text{def}}{=} -\int_{u=0}^{\infty} R(u) \sum_{n\in\mathbb{Z}^*} (\partial_I b)_n(I_t) a_{-n}(I_t) \cdot 2\pi n\omega'(I_t) u \sin(2\pi n\omega(I_t)u)\, du\,.$$

Here (3.65) followed by expanding the double sum in n and n', integrating in τ, and observing that the only terms that do not vanish are those with $n + n' = 0$. The last equality (3.66) followed since the expression is real-valued and so must equal its real part.

Using (3.62) to bound the integral in u and bounding $(\partial_I b)_n(I_t)$, $a_{-n}(I_t)$ by some constant C_η, we have that

$$|R_1| \leq C_\eta \sum_{n\in\mathbb{Z}^*} \frac{1}{n^2} \leq C_\eta\,.$$

Similarly, we can estimate that

$$\begin{aligned}
&\int_{u=0}^{\infty} R(u) \cdot 2\pi n u \sin(2\pi n\omega(I_t)u)\, du \\
&\quad = \int_{u=0}^{\infty} \Big(\int_S r(p) e^{-\mu|p|^{2\beta}u}\, dp \Big) \cdot 2\pi n u \sin(2\pi n\omega(I_t)u)\, du \\
&\quad = \int_S dp\, r(p) \Big(\int_{u=0}^{\infty} 2\pi n u \sin(2\pi n\omega(I_t)u) e^{-\mu|p|^{2\beta}u}\, du \Big) \\
&\quad = 2\int_S dp\, r(p) \frac{(2\pi n)^2 \omega(I_t)\mu|p|^{2\beta}}{(\mu^2|p|^{4\beta} + 4\pi^2 n^2\omega^2(I_t))^2} \leq \frac{C_{\omega_0}}{n^2} \int_S |p|^{2\beta} r(p)\, dp\,.
\end{aligned}$$

Therefore, we also have

$$|R_2| \leq C_\eta \sum_{n\in\mathbb{Z}^*} \frac{1}{n^2} \leq C_\eta\,.$$

Combining these estimates we have

$$\begin{aligned}
\Big| \int_{u=0}^{\infty} R(u) \int_{\tau=0}^{1} & \partial_I \big(b(I, \tau + \omega(I)u) \big)_{|I=I_t} a(I_t, \tau) \\
& + b(I_t, \tau + \omega(I_t)u) \partial_I a(I_t, \tau)\, d\tau\, du \Big| \leq C_\eta \sum_{n\in\mathbb{Z}^*} \frac{1}{n^2} \leq C_\eta\,.
\end{aligned}$$

For the stochastic integral term, we introduce a smooth function b^η such that for any $\tau \in (0,1)$ we have

$$b^\eta(I,\tau) = b(I,\tau) \quad \text{if} \quad I > \eta \qquad \text{and} \qquad b^\eta(I,\tau) = 0 \quad \text{if} \quad I < \frac{\eta}{2}\,.$$

Then

$$\begin{aligned}&\boldsymbol{E}\Big[\sup_{t\in[0,T]}\Big|\int_{\tau=0}^{1}\int_{s=0}^{t}b(I_s,\tau)dW_s(I_s,\tau)d\tau\Big|^2\mathbf{1}_{\{\inf_{t\in[0,T]}I_t>\eta,\,\sup_{t\in[0,T]}I_t<\frac{1}{\eta}\}}\Big]\\&=\boldsymbol{E}\Big[\sup_{t\in[0,T]}\Big|\int_{\tau=0}^{1}\int_{s=0}^{t}b^{\eta}(I_s,\tau)dW_s(I_s,\tau)d\tau\Big|^2\mathbf{1}_{\{\inf_{t\in[0,T]}I_t>\eta,\,\sup_{t\in[0,T]}I_t<\frac{1}{\eta}\}}\Big]\\&\le C_\eta\,,\end{aligned}$$

where the last inequality follows from the Burkholder-Davis-Gundy inequality to bound the supremum of a continuous local martingale. Finally, for any $0<\eta<1$ we have

$$\limsup_{M\to\infty}\boldsymbol{P}^0(\zeta_M'\le T)\le\boldsymbol{P}\Big(\sup_{t\in[0,T]}I_t\ge\frac{1}{\eta}\Big)+\boldsymbol{P}\Big(\inf_{t\in[0,T]}I_t\le\eta\Big),$$

and

$$\begin{aligned}&\lim_{\eta\to0}\boldsymbol{P}\Big(\inf_{t\in[0,T]}I_t\le\eta\Big)=\boldsymbol{P}\Big(\inf_{t\in[0,T]}I_t=0\Big)=0\,,\\&\lim_{\eta\to0}\boldsymbol{P}\Big(\sup_{t\in[0,T]}I_t\ge\frac{1}{\eta}\Big)=0\qquad\text{(by Lemma 3.22)}\,.\end{aligned}$$

As a result, we obtain

$$\lim_{\varepsilon\to0}\boldsymbol{E}^{\boldsymbol{P}^\varepsilon}[\Phi(y_t,t\in[0,T])]=\boldsymbol{E}^{\boldsymbol{P}^0}[\Phi(y_t,t\in[0,T])]\,,$$

which concludes the proof of Theorem 3.6.

3.7 Proofs of useful lemmas

In this section, we will prove several lemmas that are used in the previous sections.

3.7.1 Conditional expectation identities

First, to prove (3.4) it suffices to show, for all $n\ge1$ and $0\le t_1\le\cdots\le t_{n+1}$, that

$$\boldsymbol{E}\big[V(t_{n+1},dp)|V(t_1,\cdot),\ldots,V(t_n,\cdot)\big]=e^{-\mu|p|^{2\beta}(t_{n+1}-t_n)}V(t_n,dp).$$

For $n=1$ and $0\le t_1\le t_2$, we write

$$V(t_2,dp)=e^{-\mu|p|^{2\beta}(t_2-t_1)}V(t_1,dp)+Y,$$

where Y and $V(t_1, dp)$ are independent. In fact, since they are mean-zero Gaussian variables, we have

$$\begin{aligned}\boldsymbol{E}[Y(\varphi)V(t_1,\psi)] &= \boldsymbol{E}[V(t_2,\varphi)V(t_1,\psi)] - \boldsymbol{E}[V(t_1,\varphi_{t_2-t_1})V(t_1,\psi)] \\ &= \int dp\, r(p)\varphi(p)\psi(p)(e^{-\mu|p|^{2\beta}(t_2-t_1)} - e^{-\mu|p|^{2\beta}(t_2-t_1)}) \\ &= 0,\end{aligned}$$

for all φ, ψ bounded continuous functions where $\varphi_s(p) = e^{-\mu|p|^{2\beta}s}\varphi(p)$. As a result, we have

$$\boldsymbol{E}\big[V(t_2, dp)|V(t_1,\cdot)\big] = e^{-\mu|p|^{2\beta}(t_2-t_1)}V(t_1, dp).$$

Now, let us fix $n \geq 2$ and assume that for all family $(s_j)_{j\in\{1,\dots,n\}}$ such that $0 \leq s_1 \leq \cdots \leq s_n$

$$\boldsymbol{E}\big[V(s_n, dp)|V(s_1,\cdot),\dots,V(s_{n-1},\cdot)\big] = e^{-\mu|p|^{2\beta}(s_n-s_{n-1})}V(s_{n-1}, dp),$$

Then, we write

$$V(t_{n+1}, dp) = e^{-\mu|p|^{2\beta}(t_{n+1}-t_n)}V(t_n, dp) + Y,$$

where Y and $V(t_n, dp)$ are independent as explained above, so that

$$\begin{aligned}&\boldsymbol{E}\big[V(t_{n+1}, dp)|V(t_1,\cdot),\dots,V(t_n,\cdot)\big] \\ &\quad= e^{-\mu|p|^{2\beta}(t_{n+1}-t_n)}V(t_n, dp) + \boldsymbol{E}\big[Y|V(t_1,\cdot),\dots,V(t_{n-1},\cdot)\big] \\ &\quad= e^{-\mu|p|^{2\beta}(t_{n+1}-t_n)}V(t_n, dp) + \boldsymbol{E}\big[V(t_{n+1}, dp)|V(t_1,\cdot),\dots,V(t_{n-1},\cdot)\big] \\ &\qquad - e^{-\mu|p|^{2\beta}(t_{n+1}-t_n)}\boldsymbol{E}\big[V(t_n, dp)|V(t_1,\cdot),\dots,V(t_{n-1},\cdot)\big] \\ &\quad= e^{-\mu|p|^{2\beta}(t_{n+1}-t_n)}V(t_n, dp) \\ &\qquad + (e^{-\mu|p|^{2\beta}(t_{n+1}-t_{n-1})} - e^{-\mu|p|^{2\beta}(t_{n+1}-t_n)}e^{-\mu|p|^{2\beta}(t_n-t_{n-1})})V(t_{n-1}, dp) \\ &\quad= e^{-\mu|p|^{2\beta}(t_{n+1}-t_n)}V(t_n, dp),\end{aligned}$$

which concludes the proof of (3.4) by induction.

Second, to prove (3.5) it suffices to show that for all $n \geq 1$, $0 \leq t_1 \leq \cdots \leq t_{n+1} \leq \tilde{t}_{n+1}$ and φ, ψ bounded continuous functions, that

$$\begin{aligned}&\boldsymbol{E}\big[V(\tilde{t}_{n+1},\varphi)V(t_{n+1},\psi)|V(t_1,\cdot),\dots,V(t_n,\cdot)\big] \\ &\quad= \boldsymbol{E}\big[V(\tilde{t}_{n+1},\varphi)|V(t_1,\cdot),\dots,V(t_n,\cdot)\big]\boldsymbol{E}\big[V(t_{n+1},\psi)|V(t_1,\cdot),\dots,V(t_n,\cdot)\big] \\ &\qquad + \int dp\, r(p)\varphi(p)\psi(p)\Big(e^{-\mu|p|^{2\beta}(\tilde{t}_{n+1}-t_{n+1})} \\ &\qquad\qquad - e^{-\mu|p|^{2\beta}(\tilde{t}_{n+1}-t_n)}e^{-\mu|p|^{2\beta}(t_{n+1}-t_n)}\Big).\end{aligned}$$

This last relation is a consequence of the following lemma, which is a consequence of [Roz87, Theorem 10.1 and Theorem 10.2].

Lemma 3.23. *Let $(X, Y, Z_1, \dots, Z_n)$ be a Gaussian vector on a probability space $(\Omega, \mathcal{F}, \boldsymbol{P})$, and $\mathcal{G} = \sigma(Z_1, \dots, Z_n)$ be the σ-field generated by $Z_1, \dots, Z_n$. Then, the couple $(X - \boldsymbol{E}[X|\mathcal{G}], Y - \boldsymbol{E}[Y|\mathcal{G}])$ is independent of $\mathcal{G}$, and*

$$\boldsymbol{E}[XY|\mathcal{G}] = \boldsymbol{E}[X|\mathcal{G}]\boldsymbol{E}[Y|\mathcal{G}] + \boldsymbol{E}[(X - \boldsymbol{E}[X|\mathcal{G}])(Y - \boldsymbol{E}[Y|\mathcal{G}])]. \tag{3.67}$$

Proof of Lemma 3.23. The proof of the independence of $(X - \boldsymbol{E}[X|\mathcal{G}], Y - \boldsymbol{E}[Y|\mathcal{G}])$ with respect to $\mathcal{G}$ is a consequence of Theorem 10.1 and Theorem 10.2 of [Roz87]. Consequently,

$$\boldsymbol{E}\Big[\Big(X - \boldsymbol{E}[X|\mathcal{G}]\Big)\Big(Y - \boldsymbol{E}[Y|\mathcal{G}]\Big)\Big|\mathcal{G}\Big] = \boldsymbol{E}\Big[\Big(X - \boldsymbol{E}[X|\mathcal{G}]\Big)\Big(Y - \boldsymbol{E}[Y|\mathcal{G}]\Big)\Big],$$

and then

$$\begin{aligned}\boldsymbol{E}[XY|\mathcal{G}] &= \boldsymbol{E}\Big[\Big(X - \boldsymbol{E}[X|\mathcal{G}] + \boldsymbol{E}[X|\mathcal{G}]\Big)\Big(Y - \boldsymbol{E}[Y|\mathcal{G}] + \boldsymbol{E}[Y|\mathcal{G}]\Big)\Big|\mathcal{G}\Big] \\ &= \boldsymbol{E}[X|\mathcal{G}]\boldsymbol{E}[Y|\mathcal{G}] + \boldsymbol{E}\Big[\Big(X - \boldsymbol{E}[X|\mathcal{G}]\Big)\Big(Y - \boldsymbol{E}[Y|\mathcal{G}]\Big)\Big] \\ &\quad + \boldsymbol{E}[Y|\mathcal{G}]\underbrace{\boldsymbol{E}\big[\big(X - \boldsymbol{E}[X|\mathcal{G}]\big)\big|\mathcal{G}\big]}_{=0} + \boldsymbol{E}[X|\mathcal{G}]\underbrace{\boldsymbol{E}\big[\big(Y - \boldsymbol{E}[Y|\mathcal{G}]\big)\big|\mathcal{G}\big]}_{=0}.\end{aligned}$$

□

As a result, for all φ, ψ bounded continuous functions, with

$$X = V(\tilde{t}_{n+1}, \varphi) \qquad \text{and} \qquad Y = V(t_{n+1}, \psi),$$

we have

$$\begin{aligned}&\boldsymbol{E}\big[V(\tilde{t}_{n+1}, \varphi)V(t_{n+1}, \psi)|V(t_1, \cdot), \dots, V(t_n, \cdot)\big] \\ &\qquad = \boldsymbol{E}\big[V(\tilde{t}_{n+1}, \varphi)|V(t_1, \cdot), \dots, V(t_n, \cdot)\big] \\ &\qquad\qquad \cdot \boldsymbol{E}\big[V(t_{n+1}, \psi)|V(t_1, \cdot), \dots, V(t_n, \cdot)\big] \\ &\qquad + P,\end{aligned}$$

where, using (3.4),

$$\begin{aligned}P &= \boldsymbol{E}\Big[\Big(X - \boldsymbol{E}\big[X|V(t_1, \cdot), \dots, V(t_n, \cdot)\big]\Big)\Big(Y - \boldsymbol{E}\big[Y|V(t_1, \cdot), \dots, V(t_n, \cdot)\big]\Big)\Big] \\ &= \boldsymbol{E}\Big[\Big(V(\tilde{t}_{n+1}, \varphi) - V(t_n, \varphi_{\tilde{t}_{n+1}-t_n})\Big)\Big(V(t_{n+1}, \psi) - V(t_n, \psi_{t_{n+1}-t_n})\Big)\Big] \\ &= \int dp\, r(p)\varphi(p)\psi(p)(e^{-\mu|p|^{2\beta}(\tilde{t}_{n+1}-t_{n+1})} - e^{-\mu|p|^{2\beta}(\tilde{t}_{n+1}-t_n)}e^{-\mu|p|^{2\beta}(t_{n+1}-t_n)}),\end{aligned}$$

which concludes the proof of (3.5).

3.7.2 Estimates for the noise

We will prove Lemma 3.4. Let us start with the following remark. For any $(t, \varphi) \in D_{k,M}$, it is straightforward that $\varphi(t, \cdot) \in W_{k,M}$, so that

$$\boldsymbol{E} \sup_{(t,\varphi)\in D_{k,M}} \left|V\Big(\frac{t}{\varepsilon^2},\varphi(t,\cdot)\Big)\right| \le \boldsymbol{E} \sup_{(t,\varphi)\in \tilde{D}_{k,M}} \left|V\Big(\frac{t}{\varepsilon^2},\varphi\Big)\right|,$$

with

$$\tilde{D}_{k,M} \stackrel{\text{def}}{=} [0,T]\times W_{k,M}.$$

As a result, to prove inequality (3.6), we only need to prove that

$$\boldsymbol{E} \sup_{(t,\varphi)\in \tilde{D}_{k,M}} \left|V\Big(\frac{t}{\varepsilon^2},\varphi\Big)\right| \le C + \frac{C(\varepsilon)}{\varepsilon},$$

where $C>0$ and $C(\varepsilon)>0$ goes to 0 as $\varepsilon\to 0$. To this end we first remark that there is a compact embedding of $\tilde{D}_{k,M}$ into $[0,T]\times C^0(S)$ equipped with the metric

$$\|(t,\varphi)\|_\varepsilon := \sqrt{|t|}/\varepsilon + \|\varphi\|_\infty . \tag{3.68}$$

Let us also recall [Bre11, Theorem 8.8] that there exists $C_k>0$ such that for any $\varphi\in W^{1,k}(S)$, we have

$$\|\varphi\|_\infty \le C_k \|\varphi\|_{W^{1,k}} .$$

We also remark that for the pseudometric

$$d_{V,\varepsilon}((t,\varphi),(s,\psi)) = \boldsymbol{E}[|V(t/\varepsilon^2,\varphi) - V(s/\varepsilon^2,\psi)|^2]^{1/2},$$

for which

$$\begin{aligned} d^2_{V,\varepsilon}((t,\varphi),(s,\psi)) &= \int_S dp\, r(p)(\varphi^2(p)+\psi^2(p) - 2\varphi(p)\psi(p)e^{-\mu|p|^{2\beta}|t-s|/\varepsilon^2}) \\ &= \int_S dp\, r(p)(\varphi^2(p)+\psi^2(p))(1-e^{-\mu|p|^{2\beta}|t-s|/\varepsilon^2}) \\ &+ \int_S dp\, r(p)(\varphi(p)-\psi(p))^2 e^{-\mu|p|^{2\beta}|t-s|/\varepsilon^2} \\ &\le C_{k,M,r}\|(t,\varphi)-(s,\psi)\|^2_\varepsilon , \end{aligned} \tag{3.69}$$

we have

$$diam_{d_{V,\varepsilon}}(\tilde{D}_{k,M}) \le \sqrt{2C_{k,M,r}(T+M)} \stackrel{\text{def}}{=} c_{k,M,r} .$$

Note now that it is not easy to see if $\tilde{D}_{k,M}$ is relatively compact equipped with $d_{V,\varepsilon}$, while it is required to apply [AT07, Theorem 1.5.1 p. 41]. The problem is that $d_{V,\varepsilon}$ is only a pseudometric. For instance, $d_{V,\varepsilon}$ does not separate well points, since we have for any s and t

$$d_{V,\varepsilon}((t,0),(s,0)) = 0 .$$

However, according to (3.69), the application $d_{V,\varepsilon}$ provides a true metric on $\tilde{D}_{k,M}\backslash[0,T]\times\{0\}$. Then, instead of working directly with $\tilde{D}_{k,M}$ let us consider now the increasing sequence of $\|\cdot\|_\varepsilon$-relatively compact subset of $\tilde{D}_{k,M}$

$$D^n_{k,M} \stackrel{\text{def}}{=} \tilde{D}_{k,M} \cap [0,T] \times \left\{\varphi \in W^{1,k}(S) : \|\varphi\|_\infty \geq 1/n\right\}.$$

Now, since $d_{V,\varepsilon}$ is a metric on every $D^n_{k,M}$, it is straightforward to see from (3.69) that all the $D^n_{k,M}$ are also $d_{V,\varepsilon}$-relatively compact from the sequential characterization of compactness.

We can now apply [AT07, Theorem 1.5.1 p. 41 and Lemma 1.5.2 p. 44] over all the $\overline{D^n_{k,M}}^{d_{V,\varepsilon}}$, we have

$$\boldsymbol{E} \sup_{(t,\varphi)\in D^n_{k,M}} V\Big(\frac{t}{\varepsilon^2},\varphi\Big) \leq K\Big(c_{k,M,r}\ln(c_{k,M,r}) + \int_0^{c_{k,M,r}} \sqrt{\ln(N_\varepsilon(r))}dr\Big),$$

where $N_\varepsilon(r)$ is the smallest number of balls covering $D^n_{k,M}$ with radius r, which are defined by

$$B(X,r) = \{Y \in D^n_{k,M} : d_{V,\varepsilon}(X,Y) < r\}.$$

Because of (3.69), it is clear that

$$N_\varepsilon(r) \leq \mathcal{N}_\varepsilon\big(r/C^{1/2}_{k,M,r}\big),$$

where $\mathcal{N}_\varepsilon(u)$ stands for the smallest number of balls with radius u associated to the metric defined by the norm $\|\cdot\|_\varepsilon$. Since $D^n_{k,M}$ is defined as a product space, one can determine the smallest number of balls with radius r that cover each of its components. For each parts, the metric is defined by the corresponding parts of the r.h.s of (3.68). Therefore, the smallest number of balls with radius r that cover $[0,T]$ is of order $1/(r^2\varepsilon^2)$, and for $\{\varphi \in W^{1,k}(S) : \quad \|\varphi\|_{W^{1,k}} \leq M\}$ it is of order $\exp(1/r)$ [BS67, Theorem 5.2 p. 311]. As a result, for any $n \in \mathbb{N}^*$

$$\begin{aligned}\boldsymbol{E} \sup_{(t,\varphi)\in D^n_{k,M}} V\Big(\frac{t}{\varepsilon^2},\varphi\Big) &\leq C_1 + C_2 \int_0^{c_{k,M,r}} \sqrt{\ln\Big(\frac{C\exp(1/r)}{r^2\varepsilon^2}\Big)}\, dr\,,\\ &\leq C_1' + C_2' \int_0^{c_{k,M,r}} \frac{dr}{\sqrt{r}} + \frac{C_3'}{\varepsilon}\int_0^{\varepsilon c_{k,M,r}} \sqrt{\ln\Big(\frac{1}{u}\Big)}\, du\,.\end{aligned}$$

Finally, setting

$$C(\varepsilon) \stackrel{\text{def}}{=} C_3' \int_0^{\varepsilon c_{k,M,r}} \sqrt{\ln\Big(\frac{1}{u}\Big)}\, du\,,$$

and using the monotone convergence theorem, we have

$$E_1 \stackrel{\text{def}}{=} \boldsymbol{E} \sup_{(t,\varphi)\in \tilde{D}_{k,M}} V\Big(\frac{t}{\varepsilon^2},\varphi\Big) = \lim_{n\to\infty} \boldsymbol{E} \sup_{(t,\varphi)\in D^n_{k,M}} V\Big(\frac{t}{\varepsilon^2},\varphi\Big) \leq C_1'' + \frac{C(\varepsilon)}{\varepsilon}.$$

We conclude the proof of the bound (3.6) using that

$$\boldsymbol{E} \sup_{(t,\varphi)\in\tilde{D}_{k,M}} \left|V\Big(\frac{t}{\varepsilon^2},\varphi\Big)\right| \le 2E_1$$

by symmetry.

For proving (3.7), we first remark that by stationarity in t we have

$$\boldsymbol{E} \sup_{\varphi\in W_{k,M}} \left|V\Big(\frac{t}{\varepsilon^2},\varphi\Big)\right|^n = \boldsymbol{E} \sup_{\varphi\in W_{k,M}} |V(0,\varphi)|^n \,,$$

so that following the same lines as above by removing the t-dependence we have for the first order moment,

$$\boldsymbol{E} \sup_{\varphi\in W_{k,M}} |V(0,\varphi)| \le 2\boldsymbol{E} \sup_{\varphi\in W_{k,M}} V(0,\varphi) \le C_1 \,.$$

Finally, for the arbitrary order moments we write

$$\begin{aligned} E'_n \stackrel{\text{def}}{=} \boldsymbol{E} \sup_{\varphi\in W_{k,M}} |V(0,\varphi)|^n &= \int_0^\infty du\, \boldsymbol{P}\Big(\sup_{\varphi\in W_{k,M}} |V(0,\varphi)|^n > u\Big) \\ &= \int_0^\infty du\, \boldsymbol{P}\Big(\sup_{\varphi\in W_{k,M}} |V(0,\varphi)| > u^{1/n}\Big) \\ &\le 2\int_0^\infty du\, \boldsymbol{P}\Big(\sup_{\varphi\in W_{k,M}} V(0,\varphi) > u^{1/n}\Big) \,. \end{aligned}$$

Denoting

$$E''_1 = \boldsymbol{E} \sup_{\varphi\in W_{k,M}} V(0,\varphi) \,,$$

we have

$$\begin{aligned} E'_n &\le 2E''^n_1 + 2\int_{E''^n_1}^\infty du\, \boldsymbol{P}\Big(\sup_{\varphi\in W_{k,M}} V(0,\varphi) - E''_1 > u^{1/n} - E''_1\Big) \\ &\le 2E''^n_1 + 2\int_{E''^n_1}^\infty du\, e^{-(u^{1/n}-E''_1)^2/(2\sigma_D^2)} \,, \end{aligned}$$

where the last line is given by [AT07, Theorem 2.1.1 p. 50] and

$$\sigma_D^2 \stackrel{\text{def}}{=} \sup_{\varphi\in W_{k,M}} \boldsymbol{E}V(0,\varphi)^2 = \sup_{\varphi\in W_{k,M}} \int dp\, r(p)\varphi(p)^2 \le C_k^2 M^2 \int dp\, r(p) \,.$$

As a result,

$$\begin{aligned} E'_n &\le 2E''^n_1 + 2(n-1)\int_{E''_1}^\infty dv\, v^{n-1}\, e^{-(v-E''_1)^2/(2\sigma_D^2)} \\ &\le 2E''^n_1 + 2(n-1)\int_0^\infty dv\, v^{n-1}\, e^{-v^2/(2\sigma_D^2)} \,, \end{aligned}$$

which concludes the proof of the bound (3.7). □

3.7.3 Estimates for the action-angle transformation

We will prove Lemma 3.5. The estimates for a, $\partial_\theta a$, and $\partial_I a$ are based on the following estimates on the inverse map φ^{-1}.

Lemma 3.24. *There exist $r > 0$ small enough, and a constant $C_r > 0$ such that for any $I \in (0, r)$*

$$\sup_{\theta\in\mathbb{T}} \Big(\|\varphi^{-1}(I,\theta)\| + \|\partial_\theta \varphi^{-1}(I,\theta)\| \Big) \le C_r \sqrt{I}\,,$$

and

$$\sup_{\theta\in\mathbb{T}} \|\partial_I \varphi^{-1}(I,\theta)\| \le \frac{C_r}{\sqrt{I}}\,.$$

The proof of this lemma is postponed to the end of this section. In what follows we denote by

$$B_r = \varphi^{-1}((0,r)\times\mathbb{T})$$

with $r > 0$ small enough so that B_r describes a small neighborhood of $(0,0) \in \mathbb{R}^2$.

- *Bound for a.* Let us remark that

$$a(I,\theta) = e_2 \cdot \nabla\varphi_1(\varphi^{-1}(I,\theta)) = \partial_y \varphi_1(\varphi^{-1}(I,\theta))\,,$$

so that differentiating w.r.t. y the relation $K(I(x,y)) = H(x,y)$ we have

$$\partial_y I(x,y) = \partial_y \varphi_1(x,y) = \partial_y H(x,y)/\omega(I)\,.$$

Now, using that $\partial_y H(0,0) = 0$, we have for any $I \in (0,r)$

$$\begin{aligned} |a(I,\theta)| &\le \omega_0^{-1} |\partial_y H(\varphi^{-1}(I,\theta)) - \partial_y H(0,0)| \\ &\le \omega_0^{-1} \sup_{B_r} \|\nabla\partial_y H\| \cdot \|\varphi^{-1}(I,\theta)\| \le C_r\sqrt{I}. \end{aligned}$$

- *Bound for $\partial_\theta a$.* Using that

$$a(I,\theta) = \partial_y H(\varphi^{-1}(I,\theta))/\omega(I)\,, \tag{3.70}$$

and now differentiating w.r.t. θ, we obtain

$$\partial_\theta a(I,\theta) = \partial_\theta \varphi^{-1}(I,\theta) \cdot \nabla\partial_y H(\varphi^{-1}(I,\theta))/\omega(I)\,.$$

Then, by the Cauchy-Schwarz inequality

$$|\partial_\theta a(I,\theta)| \le \omega_0^{-1} \|\partial_\theta \varphi^{-1}(I,\theta)\| \sup_{B_r} \|\nabla \partial_y H\| \le C_r \sqrt{I}\,.$$

- *Bound for $\partial_I a$.* Differentiating (3.70) w.r.t. I, we obtain

$$\begin{aligned}\partial_I a(I,\theta) &= -\frac{\omega'(I)}{\omega^2(I)}\partial_y H(\varphi^{-1}(I,\theta)) + \frac{1}{\omega(I)}\partial_I\varphi^{-1}(I,\theta)\cdot\nabla\partial_y H(\varphi^{-1}(I,\theta))\\ &\overset{\text{def}}{=} T_1 + T_2\,.\end{aligned}$$

  For T_1, using (3.19) and proceeding as for the function a to deal with the term $\partial_y H$ we have

$$|T_1| \le \frac{C_r}{\sqrt{I}}\,.$$

  For T_2, applying the Cauchy-Schwarz inequality together with Lemma 3.24 yields

$$|T_2| \le \frac{C'}{\sqrt{I}}\,,$$

  which concludes the proof of Lemma 3.5.

We finally give a proof of Lemma 3.24].

Proof of Lemma 3.24. Before going into the proof, we remark that H admits a unique minimum at $(0,0)$ so that $\nabla^2 H(0,0)$ is a positive definite matrix

$$\nabla^2 H(0,0) = \begin{pmatrix}\lambda_1 & \lambda_{12}\\ \lambda_{12} & \lambda_2\end{pmatrix}$$

with λ_1, $\lambda_2 > 0$. In this case, the inner product corresponding to this matrix defines a norm on $\mathbb{R}^2$ which is equivalent to the Euclidean norm $|\cdot|$. In other words, there exist two constants λ, $\bar\lambda > 0$ such that for any $X = (x,y)^T \in \mathbb{R}^2$

$$\lambda|X|^2 \le X^T\nabla^2 H(0,0)X \le \bar\lambda|X|^2\,. \tag{3.71}$$

- *Bound for φ^{-1}.* Since $(0,0)$ is the unique minimum to H, a Taylor expansion at the second order gives us

$$K(I) = H(X) = \frac{1}{2}X^T\nabla^2 H(0,0)X + o(|X|^2)\,, \tag{3.72}$$

  with $X = (x,y)^T$. For any $X \in B_r$ we can have using (3.71) that

$$K(I) \ge \tilde\lambda|X|^2 \ge \tilde\lambda y^2 \tag{3.73}$$

  with $\tilde\lambda = \lambda/2 - \mu > 0$ for some small μ, yielding directly

$$|\varphi_2^{-1}(I,\theta)| = |y| \le \sqrt{K(I)/\tilde\lambda} = C\sqrt{K(I)-K(0)} \le C\sup_{J\in(0,r)}\sqrt{\omega(J)}\sqrt{I} = C'\sqrt{I}.$$

  In the same way, we also have

$$|\varphi_1^{-1}(I,\theta)| \le C_r'\sqrt{I}\,.$$

- *Bound for* $\partial_\theta \varphi^{-1}$. Differentiating w.r.t. x and then y the relation $K(I(X)) = H(X)$ we have

$$\partial_x \varphi_1(X) = \partial_x H(X)/\omega(I) \qquad \text{and} \qquad \partial_y \varphi_1(X) = \partial_y H(X)/\omega(I)\,,$$

so that using (3.17) we obtain

$$\partial_\theta \varphi_2^{-1}(I,\theta) = \frac{\partial_x H(\varphi^{-1}(I,\theta))}{\omega(I)} \quad \text{and} \quad \partial_\theta \varphi_1^{-1}(I,\theta) = \frac{-\partial_y H(\varphi^{-1}(I,\theta))}{\omega(I)}\,. \tag{3.74}$$

Therefore, we have

$$\|\partial_\theta \varphi^{-1}(I,\theta)\| \leq \|\nabla H(\varphi^{-1}(I,\theta))\|/\omega_0\,,$$

with

$$\begin{aligned}\|\nabla H(\varphi^{-1}(I,\theta))\| &= \|\nabla H(\varphi^{-1}(I,\theta)) - \nabla H(0,0)\| \\ &\leq \sup_{B_r} \|\nabla^2 H\| \cdot \|\varphi^{-1}(I,\theta)\| \leq C_r \sqrt{I}\,.\end{aligned} \tag{3.75}$$

- *Bound for* $\partial_I \varphi^{-1}$. Differentiating (3.74) w.r.t. I we obtain

$$\begin{aligned}\partial^2_{I\theta} \varphi_2^{-1}(I,\theta) &= -\frac{\omega'(I)}{\omega^2(I)} \partial_x H(\varphi^{-1}(I,\theta)) + \frac{1}{\omega(I)} \partial_I \varphi^{-1}(I,\theta) \cdot \nabla \partial_x H(\varphi^{-1}(I,\theta)) \\ \partial^2_{I\theta} \varphi_1^{-1}(I,\theta) &= -\frac{\omega'(I)}{\omega^2(I)} \partial_y H(\varphi^{-1}(I,\theta)) + \frac{1}{\omega(I)} \partial_I \varphi^{-1}(I,\theta) \cdot \nabla \partial_y H(\varphi^{-1}(I,\theta))\,,\end{aligned}$$

so that using (3.11) and (3.75)

$$|\partial_I \varphi_1^{-1}(I,\theta)| \leq \frac{C_1}{\sqrt{I}} + \frac{1}{\omega_0} \sup_{B_r} \|\nabla^2 H\| \int_0^\theta \|\partial_I \varphi^{-1}(I,\theta')\| d\theta'\,,$$

and in the same way for φ_2^{-1}

$$|\partial_I \varphi_2^{-1}(I,\theta)| \leq \frac{C_2}{\sqrt{I}} + |\partial_I \varphi_2^{-1}(I,0)| + \frac{1}{\omega_0} \sup_{B_r} \|\nabla^2 H\| \int_0^\theta \|\partial_I \varphi^{-1}(I,\theta')\|\, d\theta'.$$

Letting

$$U(I,\theta) \stackrel{\text{def}}{=} |\partial_I \varphi_1^{-1}(I,\theta)| + |\partial_I \varphi_2^{-1}(I,\theta)|\,,$$

and gathering the last two inequalities, we have

$$U(I,\theta) \leq \frac{C}{\sqrt{I}} + |\partial_I \varphi_2^{-1}(I,0)| + C' \int_0^\theta U(I,\theta')\, d\theta'\,.$$

Then, applying the Gronwall lemma, we obtain

$$\sup_{\theta\in(0,1)} U(I,\theta) \leq \Big(\frac{C}{\sqrt{I}} + |\partial_I \varphi_2^{-1}(I,0)|\Big) e^{C'} .$$

Moreover, differentiating w.r.t. I the relation

$$H(\varphi^{-1}(I,\theta)) = K(I)$$

we have

$$\partial_I \varphi_1^{-1}(I,\theta)\partial_x H(\varphi^{-1}(I,\theta)) + \partial_I \varphi_2^{-1}(I,\theta)\partial_y H(\varphi^{-1}(I,\theta)) = \omega(I)$$

and setting $\theta = 0$ we have from (3.10) and (3.11)

$$\partial_I \varphi_2^{-1}(I,0)\partial_y H(0,\varphi_2^{-1}(I,0)) = \omega(I) .$$

Also, using that

$$\partial_y H(0,y) = \partial_y H(0,0) + \partial_{yy}^2 H(0,0) y + o(y) = \lambda_2 y + o(y) ,$$

so that for any $(0,y) \in B_r$ we have, for some small μ,

$$|\partial_y H(0,y)| \geq (\lambda_2 - \mu)|y| .$$

As a result,

$$|\partial_I \varphi_2^{-1}(I,0)| \leq \frac{\omega(I)}{(\lambda_2 - \mu)|\varphi_2^{-1}(I,0)|} ,$$

and from (3.10) together with (3.72), we have

$$K(I) = H(0,\varphi_2^{-1}(I,0)) \leq \bar{\lambda}_2 \big(\varphi_2^{-1}(I,0)\big)^2,$$

with $\bar{\lambda}_2 = \lambda_2/2 + \mu$ so that

$$|\varphi_2^{-1}(I,0)| \geq \sqrt{K(I)/\bar{\lambda}_2} \geq \sqrt{c_1 I/\bar{\lambda}_2} .$$

Finally, we obtain

$$\sup_{\theta\in\mathbb{T}} |\partial_I \varphi_1^{-1}(I,\theta)| + |\partial_I \varphi_2^{-1}(I,\theta)| \leq \frac{C_r}{\sqrt{I}} ,$$

which gives the desired result and concludes the proof of the lemma.

□

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Vita
Hai Le

Education

The Pennsylvania State University State College, Pennsylvania 2016–2022
Ph.D. in Mathematics, expected in August 2022
Area of Specialization: Probability Theory, PDE.

University of Texas at San Antonio San Antonio, TX 2012–2016
B.S. in Mathematics

Awards and Honors

Jack and Eleanor Pettit Scholarship in Science, Penn State University 2017–2018
Third Prize, Vietnam Mathematical Olympiad 2011

Publications

Submitted

(i). (with Christophe Gomez, Gautam Iyer, and Alexei Novikov) An Oscillator Driven by Slowly Decorrelating Noise (2021)

Teaching Experience at Pennsylvania State University

- Instructor: MATH 231 Fall 2021–Spring 2022
- Instructor: MATH 22 Fall 2017–Spring 2018