INVESTIGATION OF 3-DIMENSIONAL CAUSTIC GENERATION
WITH APPLICATION TO OFF-TRACK SONIC BOOM FOCUSING

A Thesis in
Acoustics

by
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Abstract

A focused sonic boom or “superboom” is caused by the convergence of acoustic waves generated by a supersonic aircraft; this focusing occurs along a surface known as a caustic. The term “superboom” is fitting, as the resulting ground pressure signature is invariably louder than any other boom event associated with a given aircraft configuration. Unfortunately, when an aircraft accelerates to supersonic speeds, sonic boom focusing is unavoidable. Previous research has modeled the focus boom expected under the flight track, but relatively little is known about the off-track effects of focusing. In particular, the most popular model for sonic boom focusing, the lossy nonlinear Tricomi equation (LNTE), does not account for lateral diffraction. Therefore, the aim of this thesis is first, to present a comprehensive review of the history and current state of the art of sonic boom focusing theory and second, to apply this review to the development of a generalization of LNTE which explicitly accounts for such effects. Emphasis will be placed on the extension of the geometrical-acoustic approximation used outside of the caustic boundary layer and the resolution of an apparent conflict between existing two-dimensional and three-dimensional focusing models.
Contents

List of Figures viii
List of Tables xiv
List of Symbols xv
Acknowledgments xxiii

Chapter 1
Introduction 1
1.1 The sonic boom phenomenon 3
1.2 Basic sonic boom prediction 6
1.3 Cutoff and superboom 10
1.4 Problem statement 16
1.5 Outline of the thesis 17

Chapter 2
Linear Acoustics and Ray Theory 20
2.1 Introduction 20
2.2 Linear acoustics and the wave equation 20
  2.2.1 The fluid equations 20
  2.2.2 The acoustic approximation 21
  2.2.3 Acoustic equations in an inhomogeneous atmosphere with steady mean flow 22
  2.2.4 Acoustic equations in a quiescent, inhomogeneous atmosphere 24
  2.2.5 Acoustic equations in a quiescent, homogeneous atmosphere 25
2.3 Ray theory 26
  2.3.1 Assumptions 26
  2.3.2 Wavefronts 28
  2.3.3 The eikonal equation and the transport equation 29
  2.3.4 Rays 30
  2.3.5 The ray equations 31
  2.3.6 Solution of the transport equation 34
  2.3.7 Ray coordinates 36
A.1.7 Project Have BEARS, 1994 ........................................... 206
A.1.8 Water tank experiments, 2003 ..................................... 211
A.1.9 SCAMP, 2010-2012 .................................................. 213
A.1.10 Conclusion ........................................................... 216
A.2 Numerical solutions of the NTE ....................................... 218
A.2.1 Murman-Cole scheme ............................................... 218
A.2.2 Approximate analytical solution ................................ 219
A.2.3 Unsteady pseudospectral method ............................... 222
A.2.4 Conservation law schemes ........................................ 224
A.3 Other sonic boom focusing models .................................. 225
A.3.1 The Nonlinear Progressive Equation (NPE) ................. 226
A.3.2 Homogeneous One-Way Approximation for the Resolution of Diffraction (HOWARD) ......................... 229

Appendix B
The Inverse and Implicit Function Theorems .......................... 233
B.1 Introduction ............................................................ 233
B.2 Multivariable calculus ................................................ 233

Appendix C
Mathematica Code ........................................................ 240

Appendix D
Translation of Yu. L. Gazaryan, “О геометро-акустическом приближении поля в окрестности неособого участка каустики” .......................................................... 260
D.1 Sample page from original document ............................... 261
D.2 Translation ............................................................. 262

Appendix E
Translation of J. P. Guiraud, “Acoustique géométrique, bruit balistique des avions supersoniques et focalisation” ....................... 279
E.1 Sample page from original document ............................... 280
E.2 Translation ............................................................. 281

Appendix F
Translation of T. Auger, “Modélisation et simulation numérique de la focalisation d’ondes dechoc acoustiques en milieu en mouvement. Application à la focalisation du bang sonique en accélération.” .......................................................... 335
F.1 Sample page from original document ............................... 336
F.2 Translation ............................................................. 337

Bibliography ................................................................. 520
List of Figures

1.1 Digital rendering of NASA X-59 Quiet Supersonic Technology (QueSST) aircraft, [179] ......................................................... 2

1.2 Formation and geometry of a Mach cone .................................... 3

1.3 Airborne Background Oriented Schlieren Imaging (AirBOS) of two Northrop T-38 Talons, [59, 94] ............................................. 4

1.4 Shock coalescence and the ground N-wave, [156] ......................... 5

1.5 Sonic boom prediction; adapted from [145] ................................. 9

1.6 Sonic boom carpet, [127] .......................................................... 10

1.7 Mach cutoff; adapted from [124] ................................................. 11

1.8 Turn caustic ................................................................. 12

1.9 Acceleration caustic ............................................................. 12

1.10 Cusp forming maneuvers ...................................................... 13

1.11 Graphical illustration of the focus boom prediction scheme, [144, 169] ... 15

1.12 SCAMP measurement site layout, [144] ..................................... 17

2.1 The relationship between rays and wavefronts in a quiescent medium . . 30

2.2 A ray tube ................................................................. 34

2.3 Ray coordinates defined with respect to a reference wavefront; adapted from [10] ................................................................. 37
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.4</td>
<td>Ray coordinates for an accelerating supersonic aircraft</td>
<td>38</td>
</tr>
<tr>
<td>2.5</td>
<td>Wavefront area element and ray tube area</td>
<td>40</td>
</tr>
<tr>
<td>2.6</td>
<td>Approximate ray tube area in PCBoom; adapted from [156]</td>
<td>41</td>
</tr>
<tr>
<td>3.1</td>
<td>Matched asymptotic expansion; adapted from [10]</td>
<td>53</td>
</tr>
<tr>
<td>3.2</td>
<td>Relay race analogy</td>
<td>54</td>
</tr>
<tr>
<td>3.3</td>
<td>Curvilinear caustic coordinates; adapted from [107]</td>
<td>56</td>
</tr>
<tr>
<td>3.4</td>
<td>Local caustic coordinates; adapted from [107]</td>
<td>58</td>
</tr>
<tr>
<td>3.5</td>
<td>Centers of curvature; adapted from [200]</td>
<td>64</td>
</tr>
<tr>
<td>3.6</td>
<td>Interference and indistinguishability of incoming and outgoing waves near a caustic</td>
<td>68</td>
</tr>
<tr>
<td>3.7</td>
<td>Eikonal at a point ((x, y, z)) near (O)</td>
<td>69</td>
</tr>
<tr>
<td>3.8</td>
<td>Interpretation of (R_{c0}) and (R_{xz}); adapted from [169]</td>
<td>71</td>
</tr>
<tr>
<td>3.9</td>
<td>Incoming and outgoing waves at a caustic</td>
<td>75</td>
</tr>
<tr>
<td>3.10</td>
<td>Pressure and intensity of linear waves near a caustic</td>
<td>83</td>
</tr>
<tr>
<td>3.11</td>
<td>N-wave evolution in linear diffraction theory</td>
<td>88</td>
</tr>
<tr>
<td>4.1</td>
<td>Nonlinear steepening of waveform along a ray; adapted from [156]</td>
<td>96</td>
</tr>
<tr>
<td>4.2</td>
<td>Equal area rule; adapted from [156]</td>
<td>97</td>
</tr>
<tr>
<td>4.3</td>
<td>Sonic line versus geometrical caustic; adapted from [10]</td>
<td>101</td>
</tr>
<tr>
<td>4.4</td>
<td>Solution of linear (red) and nonlinear (blue) Tricomi equations for incident N-wave at (\bar{z} = 1) (edge of boundary layer), (\bar{z} = 0.5, \bar{z} = 0) (geometrical caustic), and (\bar{z} = -0.5) (shadow zone); adapted from [10]</td>
<td>103</td>
</tr>
<tr>
<td>4.5</td>
<td>Evolution of an N-wave near a caustic, linear versus nonlinear theory; adapted from [10]</td>
<td>104</td>
</tr>
</tbody>
</table>
A.3 Comparison of calculated and measured elapsed times and overpressures for a longitudinal acceleration at constant altitude; adapted from [112] 196

A.4 Normalized plot of measured sonic-boom overpressures at various locations along the ground track for three acceleration passes at $M \approx 0.9$ to 1.5 at an altitude of 11,339 m (37,200 ft); adapted from [88] 198

A.5 Sonic boom focusing recorded during Operation Jericho; adapted from [192] 200

A.6 Recordings in the vicinity of the cusp caustic/superfocus generated during turn entry; adapted from [192] 201

A.7 BREN Tower microphone placement and maneuvers; adapted from [81] 203

A.8 Experimental setup and comparison of measured and predicted shocks for the refraction (Mach cutoff) focusing ballistics experiment; adapted from [172] 204

A.9 Schlieren imaging of projectile fired in homogeneous (top) and stratified (bottom) mediums; [172] 205

A.10 Focal region waveforms recorded during acceleration passes under calm, thermally turbulent, and mechanically turbulent atmospheric conditions; adapted from [64] 207

A.11 Measured and predicted waveforms during a level acceleration pass under calm atmospheric conditions; adapted from [64] 208

A.12 Focal region waveforms recorded during dive, steady turn, and climbout-pushover maneuvers; adapted from [64] 210

A.13 Experimental setup for water tank focusing—a curved wavefront radiated by the piezoarray synthesizes a fold caustic 1 m away. The hydrophone is moved perpendicular to the caustic over 8 cm; adapted from [131] 211
(a) one period of the measured pressure field. (b) Comparison between measured field (solid line), linear (dashed line), and nonlinear (dotted line) numerical simulations at five distances from the geometrical caustic. The outgoing peak follows the incoming shock at $z \sim 0.011$ and 0.017. The two have merged by the time they reach $z \sim 0.005$ on the sonic line. The two merge again at $z \sim 0.023$, but this is an artifact of the periodic incident waveform. (c) Measured incoming signal used for numerical solutions; adapted from [131].

Comparison of LNTE predictions (blue) and SCAMP recordings (red) at $\varpi = 0.97$ (mic 71, near edge of nominal caustic boundary layer, N and U-waves fairly well separated), $\varpi = 0.67$ (mic 67, within boundary layer, N/U waves overlap and focusing begins to be noticeable), $\varpi = 0.085$ (mic 59, near geometrical caustic, N/U waves indistinguishable), and $\varpi = -0.22$ (mic 55, shadow zone, signal rounded and much weaker); adapted from [170].

Comparison at maximum focus between Gill-Seebass, NPE, and LNTE solutions for Gulfstream shaped-boom configuration; adapted from [144].

Computation grid for Murman-Cole scheme, [47, 176].

Incident step shock (blue) and resulting focused waveform predicted by the Gill-Seebass method (red); adapted from [159].

Results of Plotkin-Cantril method for incident N-wave; adapted from [158, 159].

Operator-splitting scheme for solving the nonlinear Tricomi equation; adapted from [11]. Note that some notation has been changed to match that in use here.

Comparison of third-order WENO and DG schemes to the analytical linear solution for an N-wave focusing at a caustic (3.127); adapted from [178].

Snapshots of the density perturbation $\rho(x,z)$ at fixed times. Time has been converted to range traversed by the moving grid since step 0. (a)–(d) linear case ($\beta = 0$). (e)–(g) nonlinear case ($\beta = 3.5$). Three-dimensional perspective plots are projected down onto contour plots with fixed contour interval $7.5 \times 10^{-5}$. The range direction is $x$ and depth is $z$; adapted from [134].
List of Tables

4.1 Geometric quantities in quiescent and moving media ............... 125

A.1 Compilation of transition flight test experiments; adapted from [126] . . 217
List of Symbols

The symbols most commonly used in the thesis are listed below.

$\text{Ai}(t), \text{Bi}(t)$  Airy functions of the first and second kind
$a_i, b_i, c_i, d_i, e_i, f_i, g_i$  Taylor coefficients in ray family expansion
$B_0$  Prandtl-Glauert factor $= \sqrt{M_0^2 - 1}$
$B/A$  Parameter of nonlinearity
$C_p$  Pressure coefficient
$c_0$  Small-signal sound speed
$c$  Effective sound speed in moving medium $\equiv c_0 + u_0 \cdot \hat{n}$
$c_\infty$  Frozen sound speed
$c_{NL}$  Sound speed corrected for nonlinearity
$c_p, c_v$  Specific heat capacities at constant pressure/volume
$D(u, v, \tau)$  Jacobian determinant of transformation from ray coordinates $(u, v, \tau)$ to spatial Cartesian coordinates $(x, y, z)$
$D/Dt$  Material derivative $= \partial/\partial t + u \cdot \nabla$
$D_0/Dt$  Linearized material derivative $= \partial/\partial t + u_0 \cdot \nabla$
$D$  Reference distance on the order of the characteristic boundary layer thickness $\delta(\omega_{ac})$
$d$  Airfoil thickness-to-chord ratio
$F(t)$  Dimensionless incident waveform profile
$\mathcal{F}\{}\cdot\mathcal{\}$  Fourier transform operator
$f(T)$  Initial reduced pressure waveform $= q(\sigma = 0, T)$
$f'_{\text{max}}$  Maximum slope of initial waveform
\( G(t) \)  Dimensionless waveform profile leaving caustic boundary layer

\( H(p_j, q_j) \)  Hamiltonian in generalized coordinates \( p_j \) and generalized momenta \( q_j \)

\( H(t) \)  Heaviside function = \( \begin{cases} 0, & t \leq 0, \\ 1, & t > 0. \end{cases} \)

\( \mathcal{H}\{\cdot\} \)  Hilbert transform operator

\( \mathcal{J} \)  Ray divergence

\( \mathcal{J} \)  Caustic divergence

\( K \)  Nonlinear similarity parameter \( \equiv \mu/\sqrt{\varepsilon} \)

\( \mathbf{k}(r) \)  Acoustic wavevector, \( |\mathbf{k}| = k = \omega/c_0 \)

\( L \)  Aircraft length;
   Generic length scale

\( \mathcal{L} \)  Lagrangian density

\( \ell \)  Characteristic variable;
   Airfoil chord length

\( M_{ac} \)  Acoustic Mach number

\( M_0 \)  Ambient flow Mach number \( \equiv u_0/c_0 \), where \( u_0 = |\mathbf{u}_0| \)

\( M_{fl} \)  Perturbed flow Mach number

\( M_X, M_Y, M_Z \)  Ambient flow (wind) Mach numbers along the \( OX, OY, \) and \( OZ \) axes of the local caustic coordinate system

\( m_\nu \)  Dispersion parameter for \( \nu \)-th relaxing species

\( \mathbf{n} \)  Ray principal normal vector

\( \mathbf{\hat{n}} \)  Wavefront normal vector

\( \mathbf{n} \)  Unit normal on the caustic surface

\( O \)  Origin of the local caustic coordinate system

\( \mathcal{O}(\cdot) \)  Big O asymptotic notation (Landau’s symbol)

\( P(r) \)  Geometrical acoustic pressure amplitude

\( P \)  Point along the caustic normal whose projection onto the osculating plane of the ray contacting \( O \) is \( Q \)

\( \mathbf{\tilde{P}} \)  Center of curvature of the normal section of the caustic passing through the \( v \)-line at \( O \)
\( P_c \)  Dimensional gauge for acoustic pressure in the caustic boundary layer 
\[ \equiv \rho_0 c_0^2 M_{ac} \]

\( PV\{\cdot\} \)  Cauchy principal value

\( \Pr \)  Prandtl number

\( \varphi \)  Slowness vector \( \equiv \nabla \tau \)

\( \varphi_u, \varphi_v \)  Conjugate slowness vector elements \( \equiv \partial \varphi / \partial u, \partial \varphi / \partial v \)

\( p \)  Fluid pressure

\( \bar{p} \)  Normalized acoustic pressure \( \equiv p / P_c \)

\( p_s(r) \)  (Complex) spatial pressure dependence

\( Q \)  Smooth reference surface;
Center of curvature of a ray touching the caustic at \( O \)

\( q(s,T) \)  Reduced pressure

\( R(u,v,\tau) \)  Parametric representation of ray family

\( R_0(u,v) \)  Parametric representation of reference surface for ray family

\( R_u, R_v \)  Wavefront tangent vectors \( \equiv \partial R / \partial u, \partial R / \partial v \)

\( R_\tau \)  Ray tangent vector \( \equiv \partial R / \partial \tau \bigg|_{u,v} \equiv \frac{d \mathbf{r}}{d \tau} \)

\( \tilde{R}(u,v) \)  Parametric representation of caustic surface

\( R \)  Relative radius of curvature between the ray and the caustic such that
\[ 1/R \equiv 1/OP - 1/O\tilde{P} \]

\( R_{xz}, R_{yz}, R_{zz} \)  Taylor coefficients such that \( 1/R_{x_i} \equiv c_0(O) \partial^2 \tau / \partial x_i \partial x_j \)

\( R_{XZ}, R_{YZ}, R_{ZZ} \)  ANALOGS of \( R_{xz}, R_{yz}, R_{zz} \) in moving media

\( R_{tot} \)  Analog of relative radius of curvature \( R \) in moving media

\( \mathbf{r} \)  Generic position vector;
Ray vector

\( \delta S \)  Differential ray tube area

\( s \)  Specific fluid entropy;
Arc length along a ray

\( \tilde{s} \)  Arc length along a \( \nu \)-line of the caustic

\( \text{sgn}(x) \)  Signum function
\[ \begin{cases} 1, & x > 0, \\ -1, & x < 0. \end{cases} \]
\( T \)  Period of an acoustic signal; Retarded time following a geometrical acoustic wavefront \( \equiv t - \tau \)

\( \mathcal{T} \)  (Normalized) half-duration of N-wave approaching a caustic

\( t \)  Time

\( t_\nu \)  Characteristic relaxation time of \( \nu \)-th relaxing species

\( U_{ex}, U_{ey}, U_{ez} \)  Dimensional gauges for boundary layer particle velocity components in the \((x, y, z)\) local caustic coordinate system

\( U_{cX}, U_{cY}, U_{cZ} \)  Dimensional gauges for boundary layer particle velocity components in the \((X, Y, Z)\) local caustic coordinate system

\( u \)  Fluid velocity

\((\bar{u}_x, \bar{u}_y, \bar{u}_z)\)  Dimensionless unit-order particle velocity components in the \((x, y, z)\) local caustic coordinate system

\((\bar{u}_X, \bar{u}_Y, \bar{u}_Z)\)  Dimensionless unit-order particle velocity components in the \((X, Y, Z)\) local caustic coordinate system

\((u_{OX}, u_{OY}, u_{OZ})\)  Components of the ambient flow (wind) vector in the \((X, Y, Z)\) local caustic coordinate system

\((u, v)\)  Curvilinear wavefront coordinates; Induced caustic coordinates

\((u_{1,2}, v_{1,2}, \tau_{1,2})\)  Ray coordinates of the two rays passing through a point \((x_c, y_c, z_c)\) near the caustic

\( v_{ray} \)  Ray (group) velocity vector

\( v_{wf} \)  Wavefront (phase) velocity vector

\((X, Y, Z)\)  Local caustic coordinates with origin \( O \) in moving medium

\( x = x_M(y, z) \)  Parametric representation of Mach conoid

\((x, y, z)\)  Local caustic coordinates with origin \( O \) in quiescent medium; Aircraft-fixed coordinates in stratified atmosphere

\((x_c, y_c, z_c)\)  Point near a caustic in local caustic coordinates
Greek

\[ \alpha_{tv} \quad \text{Absorption coefficient} \]
\[ \bar{\alpha} \quad \text{Normalized absorption coefficient} \]
\[ \beta \quad \text{Coefficient of nonlinearity} \]
\[ \beta_T \quad \text{Coefficient of thermal (volume) expansion} \]
\[ \gamma \quad \text{Magnitude of } Oy \text{ basis vector } \equiv |R_u|_O \]
\[ \gamma \quad \text{Ratio of specific heats } \equiv c_p/c_v \]
\[ \Delta \quad \text{Generic atmospheric heterogeneity scale on the order of } H_{p0}, H_{p0}, H_{s0} \]
\[ \Delta((\partial \tilde{x}/\partial \tilde{z})_0, \tilde{z}, \tilde{t}_0) \quad \text{Jacobian determinant of transformation from characteristic coordinates} \]
\[ \delta \quad \text{Caustic boundary layer thickness} \]
\[ \delta_{tv} \quad \text{Diffusivity of sound} \]
\[ \varepsilon \quad \text{Diffraction parameter } \equiv \lambda_{ac}/\delta \]
\[ \zeta \quad \text{Ratio of transverse length scales/diffraction strengths } \equiv \delta/L_y \]
\[ \theta \quad \text{Angle between ray tangent vector } dr/d\tau \text{ and wavefront normal } \hat{n} \]
\[ \bar{\theta}_\nu \quad \text{Normalized dispersion parameter} \]
\[ \tilde{\kappa}_n \quad \text{Normal curvature of caustic along v\textendash line at } O \]
\[ \kappa \quad \text{Curvature of ray touching caustic at } O; \]
\[ \quad \text{Coefficient of thermal conductivity} \]
\[ \kappa_r \quad \text{Relative curvature } \equiv 1/R = \kappa \cos \varphi - \tilde{\kappa}_n \]
\[ \lambda \quad \text{Acoustic wavelength } = 2\pi/k; \]
\[ \quad \text{Similarity parameter for accelerated flight} \]
\[ \lambda_{ac} \quad \text{Characteristic acoustic wavelength } = c_0/\omega_{ac} \]
\[ \mu \quad \text{Mach angle } \equiv \sin^{-1}(1/M); \]
\[ \quad \text{Nonlinearity factor } \equiv \beta M_{ac}/\varepsilon^2; \]
\[ \mu_B \quad \text{Bulk viscosity} \]
\[ \mu_{sh} \quad \text{Shear (dynamic) viscosity} \]
\[ \xi \quad \text{Coordinate following the Mach conoid } \equiv x - x_M \]
\[ \rho \quad \text{Fluid density} \]
\[ \delta \Sigma \quad \text{Wavefront area element} \]
\[ \sigma(s) \quad \text{Reduced path length} \]
\[ \bar{\sigma} \quad \text{Pseudotime variable} \]
\[ \varsigma \quad \text{Ratio of } Z-\text{wind to diffraction } \equiv M_Z/\varepsilon \]
\[ \tau(r) \quad \text{Acoustic eikonal } \equiv \Psi/\omega \]
\[ \Phi \quad \text{Dimensional velocity potential such that } \mathbf{u} = \nabla \Phi \]
\[ \bar{\phi} \quad \text{Reduced velocity potential} \]
\[ \varphi \quad \text{Angle between ray principal normal } \mathbf{n} \text{ and caustic normal } \tilde{\mathbf{n}} \]
\[ \tilde{\chi} \quad \text{Ratio of area element of caustic to reference ray tube area} \]
\[ \Psi(r) \quad \text{Spatial phase variable} \]
\[ \Omega \quad \text{Doppler factor } \equiv (1 + u_0 \cdot \hat{n}/c_0)^{-1} \]
\[ \omega \quad \text{Angular frequency} \]
\[ \omega_{ac} \quad \text{Characteristic acoustic frequency} \]
Subscripts, superscripts, and accents

Many symbols are one of the above modified by a particular subscript, superscript, or accent. These markings have the following meanings for a generic quantity $f$:

- $f_0$ Reference, initial, or ambient value of $f$
  - Examples: $p_0, u_0, \rho_0, s_0$

- $f'$ Acoustic perturbation of $f$
  - Examples: $p', u', \rho', s'$
  
  Derivative of $f$ with respect to its argument
  - Examples: $F'(t + 2z^{3/2}/3), M_2'(z)$

- Dummy variable of integration
  - Examples: $s', t'$

- Similarity variables
  - Examples: $\xi', \eta', \phi'$

- $\delta f$ Small shift in quantity $f$
  - Examples: $\delta u, \delta v, \delta \tau, \delta R$

- $f|_O, f(O)$ Value of $f$ at point $O$ on a caustic
  - Examples: $R_u|_O, D_r|_O, c_0(O), 1/R(O)$

- $f_*$ Value of $f$ at sonic line
  - Examples: $x_*, y_*, z_*, I_*$

- $f_{**}$ Value of $f$ at $\Delta = 0$
  - Examples: $x_{**}, y_{**}, z_{**}, g_{**}$

- $[f]$ Coefficient of $f$
  - Examples: $[v], [v^2], [x^2], [xy]$

- $\overline{f}$ $f$ normalized by a factor characteristic of the focal region
  - Examples: $\overline{p}, \overline{u_x}, \overline{F}, \overline{z}, \overline{t}$

- $\hat{f}$ Fourier transform of $\overline{f}$
Examples: $\hat{p} = \mathcal{F}\{\overline{p}\}$, $\hat{F} = \mathcal{F}\{\overline{F}\}$

$f$ normalized by aircraft length scale $L$
- Examples: $\hat{x}, \hat{y}, \hat{z}, \hat{\xi}, \hat{\phi}$

$\tilde{f}$ Quantity associated with a caustic
- Examples: $\hat{\mathcal{J}}, \hat{\mathbf{n}}, \hat{\mathbf{P}}, \hat{\mathcal{R}}, \hat{\delta\mathcal{S}}, \hat{s}, \hat{\chi}$

$f$ normalized by atmospheric heterogeneity scale $\Delta$
- Examples: $\hat{x}, \hat{y}, \hat{z}, \hat{\xi}, \hat{\phi}$

$f_w$ Quantity analogous to $f$ in a moving (windy) medium
- Examples: $D_w, \hat{n}_w, R_{f,w}, \mu_w, \zeta_w$

$f_\nu$ Quantity associated with $\nu$-th relaxing molecular species
- Examples: $c_{\nu,\nu}, m_\nu, s_\nu, T_\nu, t_\nu$

$f_{fr}$ Quantity associated with the frozen state of a relaxing fluid
- Examples: $c_{\nu,fr}, s_{fr}$

$L_f$ Characteristic length scale of $f$
- Examples: $L_P, L_{ki}, L_y, L_Y$

$H_f$ Characteristic length scale of $f$ along vertical aircraft-fixed coordinate
- Examples: $H_{p_0}, H_{\rho_0}, H_{s_0}$

$R_f$ Characteristic length scale of $f$ along caustic normal
- Examples: $R_{p_0}, R_{\epsilon_0}$

$K_f$ Transonic similarity parameter defined in terms of $f$
- Examples: $K_d, K_{L/\Delta}, K_\eta$
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The supersonic transport (SST) industry has a history not unlike the aircraft it develops. The industry’s “takeoff” began in the 1950s with the commencement of United States Air Force (USAF) and National Aeronautics and Space Administration (NASA) studies on the propagation, intensity, and predictability of the sonic boom [127]. The 60s saw a rapid acceleration in the number of flight tests and papers published on the subject, but progress leveled out in the 70s. This was due in part to the 1973 U.S. Code of Federal Regulations (CFR) [44] restricting supersonic flight to overwater routes save for certain military and research installations. The high point of the decade was the 1976 commercial release of the Anglo-French Concorde, a marvel of aeronautical engineering. Unfortunately, it also exemplified the problems associated with supersonic vehicles of the time. It was exceedingly loud on landing and takeoff, potentially environmentally unfriendly, not very fuel efficient, and, due in part to regulations like the U.S. CFR, had trouble remaining economically viable. For these reasons, the idea of supersonic transport vehicles began its slow descent back to earth through the late 70s and 80s. At this time, research efforts shifted away from the development of SSTs, instead focusing on the study of atmospheric effects on sonic booms and their psychoacoustic perception. Not long after its introduction, the Concorde became a luxury vehicle, existing only for the novel experience of supersonic flight rather than becoming the standard in commercial air travel that was hoped for. A short time after a fatal crash, the Concorde was retired from service, ending a rather turbulent period of civilian supersonic flight. However, the end of the Concorde also marked the beginning of a new era.

As computational fluid dynamics (CFD) technology advanced, new aircraft designs could be investigated without the cost and logistics associated with flight tests. This eventually resulted in innovative designs with carefully prescribed geometries allowing some level of control over the shape of the ground sonic boom signature. These shaped
sonic boom aircraft were expected to produce ground signatures with lower pressure amplitudes and longer rise times than their N-wave predecessors, but full-scale shaped boom aircraft were not shown to be viable until the Defense Advanced Research Projects Agency (DARPA)-funded Quiet Supersonic Platform (QSP). The QSP program emphasized improved efficiency and noise reduction of supersonic aircraft, eventually resulting in the 2003 flight of the Shaped Sonic Boom Demonstrator (SSBD), an F-5E aircraft modified to produce a shaped sonic boom [127, 148]. Early flight tests of the SSBD indicated the success of the shaped boom concept, further substantiated by a follow-on effort by NASA known as the Shaped Sonic Boom Experiment (SSBE) which found that the SSBD shaped boom signature is fairly robust under a variety of flight and atmospheric conditions. These flight tests marked the true rise of the shaped sonic boom concept, providing a potential avenue for the return of overland civilian supersonic flight. Innovation continues at a remarkable pace to this day, due in part to the FAA Reauthorization Act of 2018 [66] tasking the Federal Aviation Administration with the creation of regulations and standards for civil supersonic aircraft. 2022 will mark the first flight of the quietest supersonic aircraft to date, the NASA-developed X-59 Quiet Supersonic Technology (QueSST) aircraft, currently under construction by Lockheed Martin Skunk Works. A digital mockup of the aircraft is shown in Figure 1.1. While the X-59 promises

![Figure 1.1: Digital rendering of NASA X-59 Quiet Supersonic Technology (QueSST) aircraft, [179]](image)

...
kph (940 mph/Mach 1.4) [62, 77], there are a few key issues that need to be addressed before we can state with certainty that this aircraft will signal the return of civil overland supersonic flight.

In order to ensure that shaped sonic boom aircraft designs produce consistently quieter booms, highly accurate prediction tools are needed at all stages of aircraft noise production and propagation. After describing the sonic boom phenomenon in more detail, we will outline the process of predicting sonic boom propagation in 1.2. That said, some phenomena do not fit neatly into this basic mold and remain active areas of research. Of primary interest is sonic boom focusing, in which the noise generated by a supersonic aircraft converges along a surface known as a caustic. The thesis will be a study of this phenomenon, and it will be considered in more detail in Section 1.3. Once we have sufficiently described the problem posed by sonic boom focusing in Section 1.4, we will conclude the chapter by outlining the remainder of the thesis.

1.1 The sonic boom phenomenon

Informally, the basic mechanism for the sonic boom is a “bunching up” of sound waves along a surface known as a Mach cone, whose apex is aligned with the flight axis. As can be seen in the simplified schematic of Figure 1.2, this envelope of sound waves develops because an aircraft flying at a speed \( V \) faster than the local speed of sound \( c_0 \) overtakes the acoustic disturbances it generates.

![Figure 1.2: Formation and geometry of a Mach cone](image)

Applying some trigonometry, we see that the half-angle of the Mach cone’s apex, known as the Mach angle, is given by \( \mu = \sin^{-1} \left( \frac{c_0}{V} \right) = \sin^{-1} \left( \frac{1}{M} \right) \), where \( M \equiv \frac{V}{c_0} \) is the aircraft Mach number. Thus, as the aircraft speed \( V \) increases, the Mach angle \( \mu \)
decreases and the Mach cone narrows.

Figure 1.2 assumes the aircraft may be treated as a compact source. At sufficiently large propagation distances, this is a fair assumption; locally though, the aircraft behaves like a distributed line source. As a result, multiple Mach cones develop along its length with their respective Mach angles determined by the local speed of sound. Where there is significant bunching up of sound, shock waves characterized by nearly discontinuous jumps in pressure develop and propagate normal to the Mach cones. Shocks typically develop where there are sudden changes in geometry including the aircraft’s nose, canopy, inlet, wing, and tail [127]. Using Schlieren imaging techniques, these shock flow patterns can be visualized as in Figure 1.3.

Figure 1.3: Airborne Background Oriented Schlieren Imaging (AirBOS) of two Northrop T-38 Talons, [59, 94]

In general, the propagation speed of pressure waves is amplitude-dependent. For weaker waves this effect is miniscule, only becoming noticeable after the wave has propagated a great distance. However, for sufficiently strong waves including sonic booms, this feature can no longer be ignored. Higher pressure regions travel faster than the small-signal sound speed while lower pressure regions travel slower. As a result, the shocks generated by supersonic aircraft tend to merge as they propagate through the atmosphere, while smooth components of the waveform tend to distort, such that they may eventually steepen into shocks themselves. Over large distances, this shock coalescence effect transforms the complicated shock pattern seen near the aircraft into two primary shocks: one at the front (the bow shock), where the pressure suddenly increases from its ambient value, and one at the rear (the tail shock), where the pressure increases
from an expanded state back to its ambient value. In between, there is a nearly linear decrease in pressure. The overall waveform, commonly referred to as an N-wave, is the typical ground signature associated with a sonic boom. The development of the N-wave due to shock coalescence is depicted in Figure 1.4. The bottommost plot represents the human auditory response to an N-wave—two loud, closely spaced bangs similar to a gunshot\(^{(1)}\).

![Shock coalescence and the ground N-wave](image)

Typically, when a supersonic aircraft flies by, a stationary or slowly-moving observer will only hear a single sonic boom event (with two bangs). This has led to the common misconception that a sonic boom is only generated when an aircraft first exceeds the local speed of sound. In fact, a supersonic aircraft will generate a continuous stream of sonic booms as it flies, leading to an extended boom carpet along the flight track. The lateral extent of the carpet is limited by the atmospheric refraction of sound, but is still quite large. As a result, if an aircraft flies supersonically over a populated area, many people will be exposed to a sonic boom. This is a problematic feature, and one factor leading to the ban on overland civil supersonic flight.

\(^{(1)}\)A bullet is much shorter than an aircraft, so the two shocks present in its pressure signature arrive at an observer in extremely rapid succession. As a result, we typically only perceive one loud bang.
1.2 Basic sonic boom prediction

Here we will briefly outline the process of sonic boom prediction. Far more comprehensive reviews of the theory and its practical implementation can be found in Hayes [84], Hayes et al. [87], Onyeonwu [139–141], Maglieri et al [127], and Plotkin [156]. Specific implementations can be found in [34, 87, 95, 141, 145, 161].

Sonic boom prediction can be broken down into five pieces:

1. Flow near the aircraft
2. Propagation through the atmosphere
3. Nonlinear distortion
4. Ground interaction
5. Psychoacoustic response

The first step refers to the need to characterize the effect a supersonic aircraft has on the surrounding air. This behavior is governed by the equations of fluid dynamics and is exceedingly complex, depending on aircraft geometry, maneuvering, and atmospheric conditions. As a result, flight tests, wind tunnel tests, or numerical methods are typically required to extract detailed near-field flow behavior. Therefore, especially in the early days of sonic boom prediction, other approximate methods were developed.

Several body lengths away from the aircraft, many shocks have already coalesced, the pressure signature is substantially weaker, and near-field cross flow effects have largely died out. As a result, the flow behavior is much simpler and a linearized supersonic flow theory will often suffice to describe it. In this regime, the aircraft may be treated as a combination of line (fuselage) and surface (wing) distributions of source (monopole) and lifting (dipole) elements [84, 145]. If, in addition, the aircraft is sufficiently slender and an observer is sufficiently far from it, then for each azimuthal angle, the aircraft geometry may be replaced by a simpler equivalent body of revolution determined from the source and lift contributions. The acoustic pressure may then be expressed in terms of the area distribution of the equivalent body through the Whitham F-function [84, 156, 194]. In summary, the F-function is an asymptotic description of the aircraft near-field several body lengths away from the flight axis, and is typically one of the inputs provided to a sonic boom prediction code.

(2) In particular, its length must be much greater than the radius of any of its cross sections.
To propagate the near-field signature through the atmosphere to the ground, the method of *geometrical acoustics* is used. In this formulation, the pressure field is approximated as a set of locally plane waves which propagate along *rays* launched normal to the Mach cone. The trajectories of the rays depend on flight parameters and local atmospheric properties. Amplitude variations follow from the conservation of energy-like quantities along narrow bundles of rays, or *ray tubes*, with varying cross-sectional areas that may also be determined from flight and atmospheric parameters.

As already mentioned, the speed at which pressure waves propagate depends on their amplitude. Sonic booms, said to be of the *weakly nonlinear wave* type (to be defined later) are sufficiently strong that this effect cannot be ignored. It leads to the coalescence of shocks and the distortion of the propagating waveform, giving rise to the characteristic N-wave pressure signature observed on the ground. The geometrical theory must be modified to account for this *nonlinear distortion*, and doing so results in what is known as *weakly nonlinear geometrical acoustics* (WNGA). This modification affects the evolution of the geometrical acoustic amplitude and phase, but leaves the ray trajectories unchanged. Particularly in earlier sonic boom propagation codes (e.g. [87]), only the nonlinear distortion of the phase was accounted for while the linear geometrical acoustic theory was employed for the amplitude variation. This is sometimes referred to as the *quasi-linear* theory of geometrical acoustics.

In the lossless weakly nonlinear approximation, amplitude and phase variations are governed by an inviscid form of *Burgers’ equation*, the standard model for weakly nonlinear nondispersive lane progressive waves. Relative to the linear phase, the WNGA phase undergoes an advance proportional to the local pressure amplitude. The proportionality factor is known as the *age variable*. Over long distances, this *aging* of the waveform can render it multivalued. This is not a physical result, instead indicating the presence of shocks in the true signal. To render the waveform single-valued, the *equal area rule* is introduced. The rule, which follows from the *Rankine-Hugoniot relations* for mass, momentum, and energy conservation across shocks, states that shocks are located at points which cut out equal areas in multivalued portions of the amplitude function [83, 151, 197]. In practice, the age variable and shock location are computed concurrently with the geometrical acoustic solution so that the signal distorts as it propagates, saving memory and allowing the resulting waveform to be read off at any altitude. More recent sonic boom prediction codes (e.g. [95, 96, 145, 146, 161]) also account for atmospheric loss mechanisms. In lossy WNGA, the inviscid Burgers’ equation is replaced by the full viscous equation. The newly incorporated losses serve to balance out nonlinear steep-
enning effects, leading to “thickened” shocks with finite rise times (loosely, the time it takes a shock to reach its peak amplitude) rather than discontinuous jumps [156]. As a result, multivalued profiles can never develop, eliminating the need for the equal area rule.

Once a sonic boom has propagated through the atmosphere, we must consider how it interacts with the ground surface. The simplest case—a perfectly reflecting hard ground—corresponds to a doubling of the incident pressure signal. To empirically account for the fact that the ground will absorb some acoustic energy, a ground reflection factor of 1.9 is also common [87, 145]. In either case, no phase shift is introduced into the reflected signal. Depending on the application, a more realistic ground model which does include phase effects may be needed. Several ground impedance models exist [9, 58], as well as models accounting for variable topography [61]. A summary of these models can be found in [171].

When the sonic boom arrives at the ground, the last step is to determine an observer’s response to it. This is of course a subjective matter, but researchers have still attempted to quantitatively assess the average human response to sonic boom signatures. For this purpose, metrics including the Stevens Mark VII Perceived Level (PL) [182] and the A through E-weighted Sound Exposure Levels (SEls) [60] have been considered for both N-wave and shaped boom aircraft configurations [117, 119, 185]. Doebler and Sparrow [63] recently found that the BSEL metric is also particularly stable in the presence of simulated turbulence. A weighted sum of PL, ASEL, and CSEL that empirically accounts for low-frequency boom-induced rattle in structures, known as the Indoor Sonic Boom Annoyance Predictor (ISBAP) [118], has proved to be a reliable predictor of indoor listener annoyance.

The determination of response metrics completes the basic program of sonic boom prediction. In Figure 1.5, we summarize the process.
Figure 1.5: Sonic boom prediction; adapted from [145]
1.3 Cutoff and superboom

The preceding description provides a basic understanding of how sonic boom prediction is done. This is not to say that it is a completely solved problem. There remain various edge cases that do not fall neatly within this process and remain active areas of research. For instance, in an inhomogeneous atmosphere, temperature and wind gradients alter the local sound speed, causing sound to refract. In fact, atmospheric sound speed gradients can be so great that sound generated by supersonic aircraft will “turn around” before ever reaching the ground. This is sometimes referred to as cutoff, since it results in shadow zones on the ground where ray theory predicts a complete absence of sound. In reality, diffraction dominates in these regions, leading to weak (but observable) amplitudes [50, 142, 154]. In other words, ray theory is not a valid description of the sound field in shadow zones.

An example of the cutoff phenomenon is lateral cutoff, in which rays at large azimuthal launch angles refract back into the upper atmosphere. This results in a finite carpet width, delimiting the lateral extent of a sonic boom’s ground impact (see Figure 1.6). Another cutoff condition, known as Mach cutoff, occurs when an aircraft flies supersonically, but below a particular threshold Mach number (below about Mach 1.15 in the standard atmosphere) [46, 81, 89]. In this case, the “turnaround” of sound occurs roughly when the local speed of sound equals the aircraft speed, along a surface known as...
as the *sonic surface* (or sonic line in plane propagation) below which a shadow zone forms.

![Figure 1.7: Mach cutoff; adapted from [124]](image)

The shadow zone below the sonic surface suggests a noise mitigation strategy in which a flight is routed to consistently place the cutoff above the ground. Utilization of this concept for supersonic transport vehicles has been explored by at least one aerospace company [132]. However, much like the Mach cone, the sound arriving at the sonic surface tends to bunch up, leading to significant amplification of sound in a small neighborhood of the surface. This amplification effect, geometrically associated with the convergence of rays along a surface known as a *caustic* (in this case the sonic surface, Figure 1.7), is referred to as *sonic boom focusing*, while the maximum amplitude pressure signature is referred to as a *focus boom* or *superboom*[^3].

Caustics are a general wave phenomenon not specific to sonic boom theory. The term is borrowed from optics, and descends from the ancient Greek word καυστικός (kaustikós) [35], which translates to “burning,” due to the particularly intense electromagnetic field near light caustics. In terms of rays, a caustic is the locus of points at which initially differentially separated rays cross. It may also be seen as an envelope of a set of rays, such that the caustic is everywhere tangent to the rays it makes contact with. There are other equivalent definitions (see Section 2.4.1), but these two are sufficient to consider how a caustic might develop during sonic boom propagation.

In addition to Mach cutoff flight, certain maneuvers are particularly prone to sonic boom focusing. In tight turns, rays converge along (half of) a hyperboloid of one sheet whose radius remains smaller than that of the turn (Figure 1.8). Another focusing case occurs when an aircraft undergoes rectilinear acceleration, either transonically or at an already supersonic speed. Transonic focusing is sometimes referred to as a *transition*

[^3]: A similar phenomenon occurs for *secondary sonic booms*, caused by sound radiated from the top of the aircraft or reflected from the ground reversing direction in the upper atmosphere and returning to the ground. More information on focusing of secondary sonic boom can be found in, for instance, [102].
focus boom, but the focusing mechanism is essentially the same regardless of the speed range [127]. With reference to Figure 1.9, we see that as the aircraft’s speed increases, the Mach angle narrows and the launch angles of the rays generated normal to the Mach cone widen. This leads to the crossing of (differentially) adjacent rays, forming a caustic.

Figure 1.8: Turn caustic

Figure 1.9: Acceleration caustic

As in Mach cutoff, both turn and acceleration caustics separate a region into an insonified (ray-filled) portion and a shadow zone into which rays do not penetrate and the
The acoustic field becomes evanescent. This property distinguishes these caustics from other types which can develop. Namely, the caustics generated during Mach cutoff, rectilinear acceleration, and turns are all said to be of *fold* type or more colloquially, *smooth/simple* type [107, 160, 200]. Fold caustics are the simplest in a series of caustics, the next of which is variously referred to as the *cusp caustic* [24, 144, 160], *superfocus* [127, 192], or *arête* (French for “peak” or “ridge”) [151, 184], forming when two fold caustic sheets meet. Cusps develop during turn-entry, dive, and pushover maneuvers (Figure 1.10) and divide the acoustic field into regions containing one or three rays, such that no shadow zones develop [127, 192]. Along the two fold curves (surfaces in 3-D) making up the cusp, two differentially separated rays cross, while three cross at the cusp point (curve in 3-D). This triple-crossing leads to focused sonic booms even more intense than those observed at fold caustics, occasionally referred to as *super-superbooms* [147]. For comparison, the pressures of sonic booms focusing at fold caustics have been found to be 2 to 5 times greater than under non-focusing conditions [127, 192], while pressures at cusps are estimated to be anywhere from 9 to 13 times greater [172, 192].

As a rule of thumb, the more gradual the maneuver, the less intense the focusing effect, with the tradeoff that the focal zone will be spread over a larger ground area. In

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1. Turn-entry (top view); adapted from [192]
2. Dive; adapted from [160]

**Figure 1.10: Cusp forming maneuvers**

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(4) This series is classified by the mathematical discipline of *catastrophe theory* (see Chapters 3 and 6).

(5) Acceleration caustics are actually the lower branch of a cusp forming at the location of the aircraft when it first goes supersonic [138, 144]. However, the upper fold typically curves toward the upper atmosphere, so focused waves associated with the cusp and upper fold either never reach the ground or else are extremely attenuated.

(6) The truth is more complicated. For instance, past transition focus boom flight tests have variously
some cases, the maneuver may be modified to completely avoid focusing near the ground [82, 84, 125, 127, 192], but this is the exception and not the rule. In particular, the acceleration caustic is unavoidable [125, 127]. One way or another, to fly supersonically an aircraft must accelerate to supersonic speed. This inevitably leads to the convergence of rays, inevitably generating a superboom. For this reason, the fold caustic structure has received considerable attention in the sonic boom focusing literature. Higher-order caustics, while important in the complete characterization of sonic boom focusing, have remained secondary concerns\textsuperscript{(7)}. We take a similar stance here and will tend to use “a/the caustic” to refer to the fold caustic, only drawing a distinction for emphasis or comparison to other types of caustics.

To study the superboom phenomenon, tools beyond geometrical acoustics will be required. Though generally sufficient to study the geometrical properties of caustics, geometrical acoustics is not capable of providing accurate predictions of corresponding field amplitudes in a finite region surrounding a caustic. Inside this boundary layer, an asymptotic expansion of the nonlinear acoustic equations must be used. The resulting governing equation, first derived by J.-P. Guiraud in 1965 [79] is the \textit{nonlinear Tricomi equation} (NTE), so-named due to its similarity to the well-known \textit{Tricomi equation} from transonic aerodynamics [47, 67].

The NTE accounts for focusing, diffraction, and nonlinear steepening in the boundary layer of a fold caustic. It is essentially a 2-dimensional model, as it only accounts for the flow sufficiently near a particular normal section of the caustic. This is not merely a simplifying assumption, but a mathematical result. In particular, it can be shown that, at least for a caustic section taken along the flight track, the acoustic velocity components normal to and along the section dominate the transverse component such that it may be neglected. The equation is of \textit{hyperbolic} type in the insonified region and \textit{elliptic} type in the shadow zone. The two regions are separated by a line (surface in 3 dimensions) where the equation is \textit{parabolic}. In linear acoustic theory, the parabolic line coincides with the geometrically-constructed caustic, but there is a slight difference between the two when nonlinear effects are accounted for. In recent years, the Tricomi

\textsuperscript{(7)}The cusp has seen more interest in recent years, due in part to its role in the propagation of sonic booms through atmospheric turbulence [13, 14, 152].
model has been augmented by Auger [10, 11] to account for wind and by Salamone [169, 170] to incorporate atmospheric losses. Salamone’s model, known as the *lossy nonlinear Tricomi equation*, or LNTE, is perhaps the most popular variant, and has proven to be quite successful in predicting focused sonic booms [98, 144, 162].

A typical setup for focus boom prediction using the nonlinear Tricomi model (or its windy/lossy variant) is depicted in Figure 1.11. The near-field aircraft signature is propagated through the atmosphere along rays computed using a weakly nonlinear geometrical acoustic propagation code, accounting for wind and/or atmospheric losses as appropriate. Raytracing is also used to determine the caustic geometry, in this case arising from aircraft acceleration. Of primary interest is the vicinity of the location at which the caustic intersects the ground surface, as this region is expected to contain the highest pressures observed over the entire sonic boom carpet [144, 169]. The ray tangent to the caustic at the ground intercept is required to compute the local ray-caustic geometry defining the extent of the caustic boundary layer. Within the boundary layer, diffraction becomes non-negligible, geometrical acoustics fails to predict accurate pressures, and the nonlinear Tricomi model must be used. The matching from geometrical acoustics to a Tricomi-based code is accomplished by determining the ray tangent to the upper edge of the boundary layer\(^{(8)}\)—labelled the δ tangent ray because the thickness of the caustic boundary layer is usually denoted by δ—and using the geometrically

\[ \text{(8)In particular, the } \delta \text{ tangent ray is the ray that is tangent to the upper edge of the boundary layer while also lying along the caustic normal extending from the ground intercept location.} \]

Figure 1.11: Graphical illustration of the focus boom prediction scheme, [144, 169]
predicted pressure signature at the tangent point as the starting signature for the Tricomi code. The Tricomi code then computes the propagation of this incoming signature through the boundary layer, after which point a ground impedance model or reflection factor may be applied to compute pressure signatures throughout the ground focal region (i.e., the portion of the Tricomi code region cut out by the ground line in Figure 1.11). Implementations of this $\delta$ tangent ray focus boom prediction procedure are further discussed in [98, 144, 146, 162, 170].

1.4 Problem statement

One might wonder about the validity of applying the locally 2-D LNTE model to the 3-D sonic boom focusing problem. The model’s applicability boils down to two factors. First, as we have already pointed out, it may be shown that in a certain neighborhood of the caustic, any effects in the transverse direction are completely dominated by the flow normal to and along the caustic section. So locally, the flow behavior is two-dimensional. Second, the majority of flight testing for sonic boom focusing has taken place with the aircraft flying directly above and along a linear microphone array [64, 88, 112, 128]. One such array, used during the NASA-sponsored Superboom Caustic Analysis and Measurement Program (SCAMP) [144] is depicted in Figure 1.12. This placement corresponds to a normal section of the caustic containing the aircraft and the microphone array. Transverse flow effects are more substantial near lateral cutoff, so we would not expect such an array placement to capture them. Hence, we should not have any reason to doubt the applicability of the LNTE model for such cases.

If the array is instead laterally offset from the flight track, we will begin to see errors in predictions made with the LNTE model, because it assumes there will be no lateral variation in pressure whatsoever. For significant separation distances of the flight track and the array, this leads to large disagreement in the measured and predicted signatures [144]. Therefore, it is desirable to have a model that can account for this lateral variation.

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(9) The application of the usual ground reflection procedures to superbooms might seem suspect, but it may be shown that the incident and reflected fields are additive even in the nonlinear case [163, 165]. In fact, [163] uses the method of images to show that the incident and reflected fields in the focal region each satisfy separate and entirely uncoupled nonlinear Tricomi equations, leading to a relationship between the incident and reflected fields identical to the reflection coefficient for linear plane waves.

(10) To be clear, the SCAMP team did conduct several passes laterally offset from this array. In fact, these flights constitute the lion’s share of existing data on off-track acceleration focusing and demonstrate that LNTE degrades in accuracy off-track. The point here is that many previous flight test campaigns employed arrays similar to the SCAMP array, but did not consider laterally offset passes.
1.5 Outline of the thesis

The sonic boom focusing problem is highly interdisciplinary, drawing on results from transonic flow theory [47, 67, 113], seismology [36, 200], pure and applied mathematics [5, 33, 65, 92, 160], nonlinear wave theory [83, 197], optics [15, 16, 30, 31, 91, 107, 189, 202], and of course sonic boom theory itself. As a result, much of the thesis is review of the existing theory on sonic boom focusing. The review is more-or-less chronological, allowing us to point out where each iteration of the theory faltered and how it was subsequently corrected to eventually arrive at the model in use today.

In Chapter 2, we develop the basic theory of geometrical acoustics. This allows us to define and study caustics geometrically, leading to the conclusion that, while very useful in predicting the propagation paths, geometrical theory is not sufficient to predict field amplitudes near caustics.

In Chapter 3, we take a closer look at the caustic region requiring a different approximation of the acoustic equations. The result, known as the Tricomi equation, governs diffraction within a boundary layer surrounding the caustic, and may be solved analytically in terms of Airy functions. For an incident N-wave, this solution produces
waveforms qualitatively similar to those observed during sonic boom focusing. However, at discontinuities representative of shocks, the solution invariably becomes infinite, signaling the need for further modification of the theory.

Chapter 4 resolves the blowup at shocks by accounting for nonlinear steepening effects in the caustic boundary layer, leading to the celebrated nonlinear Tricomi equation. The properties of this equation and its analogy to the theory of transonic aerodynamics are then explored.

Chapter 5 concludes the review of sonic boom focusing with a brief discussion of the (limited) literature on fully 3-dimensional focusing in which the typically neglected lateral diffraction terms are retained. Synthesizing the results of the review, we begin the original work of the thesis in Section 5.4. We first modify the geometrical-acoustic description near the caustic to fully account for three-dimensional propagation of acoustic waves in the boundary layer. This will suggest a new set of inner variables with which to express the lossy nonlinear acoustic equations resulting in a new 3-dimensional lossy nonlinear Tricomi equation. We then demonstrate that under the appropriate circumstances, this equation reduces to every other (steady) form of Tricomi equation discussed in the literature review. We then close out the chapter by suggesting a numerical scheme with which to solve the new model equation amounting to a straightforward modification of an existing code.

In Chapter 6, we conclude the thesis by examining the gaps within our own work, as well as what we believe the next steps should be in the study of sonic boom focusing. Six appendices are included at the end of the thesis.

Appendix A discusses the results of several major sonic boom focusing experiments and numerical studies, providing further substantiation to the theoretical results recounted in the main body of the thesis.

Appendix B presents a brief primer on the inverse and implicit function theorems of multivariable calculus, as they are useful in the study of geometrical-acoustic behavior near caustics.

Appendix C contains Mathematica code which automates some of the rather extensive algebra required in the method we use to extend the geometrical-acoustic theory in Chapter 5.

The final three appendices are English translations of important works in the theory of sonic boom focusing. These translations were performed with Google Translate [78] and DeepL [57], with human intervention for wording, equation typesetting, document formatting, and figure editing. Permissions to include these translations in the
thesis have been obtained from the respective authors and/or publishers. We take full responsibility for any mistakes or misconceptions which may have been introduced in the translation process, but hope the translations will be of some value to readers.

Appendix D is a translation of a 1961 article by Yu. L. Gazaryan, [200]. In this article, Gazaryan provides a derivation of certain geometrical-acoustic quantities near a caustic and develops a local solution of the field amplitude for linear incident waves.

Appendix E translates the 1965 article in which J. P. Guiraud first derives the nonlinear Tricomi equation as well as the well-known scaling law bearing his name [79].

Finally, Appendix F translates the 2001 dissertation of T. Auger [10], in which the effects of wind are incorporated into the nonlinear Tricomi equation. Auger also develops a computational scheme to numerically solve the NTE that has become the basis for many modern focusing codes. Further, this work provides an excellent review of sonic boom focusing in general that we fall back on many times throughout the thesis.
Chapter 2
Linear Acoustics and Ray Theory

2.1 Introduction

Our review begins with the acoustic equations of motion. Three forms, differing based on the properties of the medium, will be presented. These equations can be simplified (and commonly are for sonic boom propagation codes) using a high-frequency asymptotic approximation known as the ray or eikonal approximation. We will present the resulting geometrical acoustic equations and their solutions, as well as several consequences that will be important later in the thesis. Ray concepts will then be used to present three equivalent definitions of a caustic. Applying these definitions will lead to the conclusion that, while essential in determining the paths along which sound propagates, ray theory ultimately fails to predict correct pressures in the vicinity of caustics.

2.2 Linear acoustics and the wave equation

In this section, we show how the linear acoustic equations may be derived under various assumptions on the behavior of the fluid medium. We will not dwell much on the details, and instead will give only a high-level overview of the major steps. For the most part, the presentation is based on the acoustics text by Pierce \cite{Pierce}.

2.2.1 The fluid equations

The standard route to arrive at the linear acoustic equations for fluids is to treat acoustic waves as small perturbations to the overall state of a fluid, governed by the equations of fluid dynamics. In an inviscid medium, we require equations representing the conservation of mass and momentum. We also need a thermodynamic equation of state in
order to relate the pressure and density. Lastly, sound is usually assumed to propagate adiabatically, which, as explained by Pierce, is consistent with assuming isentropic flow. These equations are provided without proof, but derivations can be found in standard texts on acoustics [76, 151].

1. Conservation of mass

\[
\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} = 0, \tag{2.1}
\]

2. Conservation of momentum (Euler’s equation)

\[
\frac{D\mathbf{u}}{Dt} + \frac{1}{\rho} \nabla p = 0, \tag{2.2}
\]

3. Thermodynamic equation of state

\[
p = p(\rho, s), \tag{2.3}
\]

4. Isentropic flow

\[
\frac{Ds}{Dt} = 0. \tag{2.4}
\]

In these equations, \( \rho \) represents the density of the fluid, \( \mathbf{u} \) the local fluid velocity, \( p \) the fluid pressure, and \( s \) the specific entropy of the fluid. In general, all are functions of position (\( \mathbf{r} \)) and time (\( t \)). We have neglected the effect of gravity in Euler’s equation, as its effect on acoustic waves is typically negligible. The operator \( D/Dt \equiv \partial/\partial t + \mathbf{u} \cdot \nabla \) is the convective or material derivative, representing the time rate of change of a quantity as measured by an observer moving with the fluid.

### 2.2.2 The acoustic approximation

We take acoustic disturbances to be small-amplitude, space and time dependent perturbations to an ambient state free of any disturbance. In particular, we assume that the fluid variables may be decomposed as

\[
p = p_0 + p' \quad \rho = \rho_0 + \rho' \quad \mathbf{u} = \mathbf{u}_0 + \mathbf{u}' \quad s = s_0 + s', \tag{2.5}
\]

where primes denote the acoustic perturbation to the ambient state variables, denoted by a 0-subscript. When all ambient variables are position-independent, the fluid medium
is said to be homogeneous. When all ambient variables are time-independent and the fluid initially at rest \((u_0 = 0)\), it is quiescent.

We require the acoustic perturbation to be small relative to the ambient state, such that for any fluid quantity \(f = f_0 + f'\) the perturbation \(f'\) satisfies

\[
\frac{f - f_0}{f_0} = \frac{f'}{f_0} \ll 1. \tag{2.6}
\]

The benefit of decomposition (2.5) is that it allows the fluid equations to be simplified in a way that retains the most important effects of the acoustic perturbations. This simplification can be done in a systematic way by introducing an ordering scheme that allows the relative sizes of different terms to be compared in a unit-independent way. A convenient choice of dimensionless ordering parameter is the acoustic Mach number \(M_{ac} = u'/c_0\), the ratio of the flow perturbation \(u'\) to the small-signal sound speed \(c_0\), the latter of which will be defined shortly. If all ambient quantities are taken to be of unit order, then by definition \(u' = O(M_{ac}) \ll 1\). Likewise, the remaining perturbation variables are taken to be of order \(M_{ac} \ll 1\) at most. The leading-order equations governing the perturbation variables are then found by neglecting any terms smaller than order \(M_{ac}\).

### 2.2.3 Acoustic equations in an inhomogeneous atmosphere with steady mean flow

The time required for a sonic boom to reach the ground is typically much shorter than the time scale for significant changes in the ambient atmospheric state. On the other hand, spatial gradients in atmospheric wind and temperature have been found to have a significant effect on the propagation and intensity of sonic booms [127]. As such, the atmosphere is commonly modeled as a time-independent, moving, inhomogeneous medium where the ambient flow represents the effect of a steady, spatially varying wind. In this case, decomposition (2.5) takes the form

\[
p = p_0(r) + p'(r, t) \quad \rho = \rho_0(r) + \rho'(r, t) \quad u = u_0(r) + u'(r, t) \quad s = s_0(r) + s'(r, t). \tag{2.7}
\]

Substituting (2.7) into equations (2.1)-(2.4) and neglecting terms of order \(M_{ac}^2\) or smaller (which correspond to products of two or more perturbation variables), we arrive at the linearized fluid equations, or acoustic equations for an inhomogeneous medium with a
steady mean flow:

\[ \frac{D_0\rho'}{Dt} + \mathbf{u}' \cdot \nabla \rho_0 + \rho' \nabla \cdot \mathbf{u}_0 + \rho_0 \nabla \cdot \mathbf{u}' = 0, \]  

\[ (2.8) \]

\[ \frac{D_0\mathbf{u}'}{Dt} + (\mathbf{u}' \cdot \nabla)\mathbf{u}_0 - \frac{\rho'}{\rho_0^2} \nabla p_0 + \frac{1}{\rho_0} \nabla p' = 0, \]  

\[ (2.9) \]

\[ p' = \left. \frac{\partial p}{\partial \rho} \right|_{\rho_0} \rho' + \left. \frac{\partial p}{\partial s} \right|_{s_0} s', \]  

\[ (2.10) \]

\[ \frac{D_0s'}{Dt} + \mathbf{u}' \cdot \nabla s_0 = 0, \]  

\[ (2.11) \]

The operator \( \frac{D_0}{Dt} \equiv \partial/\partial t + \mathbf{u}_0 \cdot \nabla \) is the linearized material derivative, representing the time rate-of-change following the ambient flow. The notation in (2.10) indicates that the derivative of the thermodynamic quantity (in this case \( p \)) is taken with all other state variable(s) (\( s \) or \( \rho \)) held constant. The second subscript indicates that the quantity should be evaluated at the ambient state, \( (\rho = \rho_0, s = s_0) \). The linearized speed of sound \( c_0 \) is defined by the first of these derivatives,

\[ c_0^2 \equiv \left. \frac{\partial p}{\partial \rho} \right|_{s_0,0}, \]  

\[ (2.12) \]

and is equal to \( \sqrt{\gamma p_0/\rho_0} \) for sound waves propagating adiabatically, where \( \gamma \) is the ratio of specific heats. In general, both the sound speed and the thermodynamic coefficient \( \partial p/\partial s \big|_{\rho_0} \) are functions of position. Using (2.10) and (2.11) to eliminate \( \rho' \) and \( s' \) reduces the system governing the inhomogeneous moving medium to just two coupled equations in the pressure and particle velocity perturbations,

\[ \frac{D_0p'}{Dt} + \mathbf{u}' \cdot \nabla p_0 + c_0^2 p' \nabla \cdot \mathbf{u}_0 + \rho_0 c_0^2 \nabla \cdot \mathbf{u}' = 0, \]  

\[ (2.13) \]

\[ \frac{D_0\mathbf{u}'}{Dt} + (\mathbf{u}' \cdot \nabla)\mathbf{u}_0 - \frac{p'}{(\rho_0 c_0)^2} \nabla p_0 + \frac{1}{\rho_0} \nabla p' = 0, \]  

\[ (2.14) \]

which could be further reduced to a single wave equation (in any one of the acoustic perturbation variables) for the moving, inhomogeneous medium if desired.
2.2.4 Acoustic equations in a quiescent, inhomogeneous atmosphere

In a quiescent atmosphere $u_0 = 0$, reducing $D_0/Dt$ to the usual partial derivative $\partial/\partial t$. Evaluating Euler’s equation (2.16) at the ambient state also implies that $\nabla p_0 = 0$ when $u_0 = 0$. With these simplifications, the acoustic equations (2.8)–(2.11) become

$$\frac{\partial \rho'}{\partial t} + \nabla \cdot (\rho_0 u') = 0,$$  \hspace{1cm} (2.15)

$$\frac{\partial u'}{\partial t} + \frac{1}{\rho_0} \nabla p' = 0,$$  \hspace{1cm} (2.16)

$$p' = c_0^2 \rho' + \frac{\partial p}{\partial s}_{s_0} \bigg|_{s'} \bigg( \rho \bigg),$$  \hspace{1cm} (2.17)

$$\frac{\partial s'}{\partial t} + u' \cdot \nabla s_0 = 0.$$  \hspace{1cm} (2.18)

The third and fourth equations may be combined by first taking the time derivative of (2.17) and substituting (2.18), yielding

$$\frac{\partial p'}{\partial t} = c_0^2 \frac{\partial \rho'}{\partial t} - \frac{\partial p}{\partial s}_{s_0} \bigg|_{s'} (u' \cdot \nabla s_0).$$  \hspace{1cm} (2.19)

Then, taking the gradient of the equation of state (2.3) for the ambient state ($p = p_0$, $\rho = \rho_0$, $s = s_0$),

$$\frac{\partial p}{\partial s}_{s_0} \bigg|_{s_0} = \nabla p_0 - c_0^2 \nabla \rho_0,$$  \hspace{1cm} (2.20)

and substituting this back into (2.19) (with $\nabla p_0 = 0$), we find:

$$\frac{\partial p'}{\partial t} = c_0^2 \left( \frac{\partial \rho'}{\partial t} + u' \cdot \nabla \rho_0 \right).$$  \hspace{1cm} (2.21)

Now, expanding the divergence operator in (2.15) and noting its similarity to the right-hand side of (2.21), we have

$$\frac{\partial \rho'}{\partial t} + \rho_0 c_0^2 \nabla \cdot u' = 0.$$  \hspace{1cm} (2.22)

We can then eliminate the particle velocity by taking the time derivative of (2.22), interchanging the gradient and time derivative acting on $u'$, and substituting (2.16),
resulting in a single equation governing the acoustic pressure,

\[ \rho_0 \nabla \cdot \left( \frac{\nabla p'}{\rho_0} \right) - \frac{1}{c_0^2} \frac{\partial^2 p'}{\partial t^2} = 0. \]  

(2.23)

This is the wave equation for the inhomogeneous, quiescent atmosphere. To put it in a more familiar form, we expand out the divergence operator to find

\[ \nabla^2 p' - \frac{1}{\rho_0} \nabla \rho_0 \cdot \nabla p' - \frac{1}{c_0^2} \frac{\partial^2 p'}{\partial t^2} = 0, \]  

(2.24)

where \( \nabla^2 \) denotes the Laplacian operator, given in Cartesian coordinates by \( \nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \).

### 2.2.5 Acoustic equations in a quiescent, homogeneous atmosphere

In the quiescent, homogeneous atmosphere, the ambient variables are position-independent and their gradients vanish, reducing the acoustic equations (2.15)–(2.18) to

\[ \frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot \mathbf{u'} = 0, \]  

(2.25)

\[ \frac{\partial \mathbf{u'}}{\partial t} + \frac{1}{\rho_0} \nabla p' = 0, \]  

(2.26)

\[ p' = c_0^2 \rho', \]  

(2.27)

\[ s = s_0 = \text{cst.} \implies s' = 0. \]  

(2.28)

As in the previous section, we can eliminate across the four equations to arrive at a wave equation for a quiescent, homogeneous medium. Alternatively, we could simply treat \( c_0 \) and \( \rho_0 \) as constants in (2.24). In either case, we arrive at the homogeneous wave equation,

\[ \nabla^2 p' - \frac{1}{c_0^2} \frac{\partial^2 p'}{\partial t^2} = 0, \]  

(2.29)

where the sound speed \( c_0 \) is now a constant, independent of position.

With that, we have the major equations governing linear acoustics in a variety of fluid mediums. In later usage, we will drop the primes from the acoustic variables. When the fluid perturbation becomes appreciable relative to the ambient state, these equations
will no longer suffice as the neglected terms may no longer be considered small. In this case, we will require the nonlinear acoustic equations, in which terms of order $M^2_{ac}$ are also retained. Their derivation is similar to the linear case, but the resulting equations are significantly more complex (see Section 4.2.1).

2.3 Ray theory

Ray theory was originally developed as a means of studying light [181]. Only much later was the so-called geometrical optics adapted to sound, resulting in the analogous theory of geometrical acoustics [151]. In its most basic form, ray theory is comprised of the rectilinear propagation of rays in homogeneous media, the law of reflection, and the law of refraction. However, this formulation is not sufficient for our purposes. Instead, we will view ray theory as a high-frequency asymptotic approximation of the acoustic equations, a view pioneered in electromagnetic theory by Sommerfeld and Runge [180].

2.3.1 Assumptions

The assumptions of geometrical acoustics are essentially identical to geometrical optics. A concise account is given in [107], and it is primarily this reference which we will follow. More detailed presentations can be found in [111, 136].

For time-harmonic waves (or the Fourier components of an arbitrary waveform) with time dependence $e^{i\omega t}$, the acoustic wave equation for homogeneous, quiescent media (2.29) reduces to the Helmholtz equation,

$$\nabla^2 p_s + k^2 p_s = 0,$$

(2.30)

where $p_s$ represents the spatial dependence of the acoustic pressure. That is,

$$p(r, t) = \text{Re}\{p_s(r)e^{i\omega t}\}.$$

(2.31)

The wavenumber $k = \omega/c_0$ is the ratio of the angular frequency $\omega$ to $c_0$, the (constant) speed of sound. In free space, the Helmholtz equation admits a simple plane wave solution of the form

$$p_s(r) = Pe^{-i\Psi(r)}, \quad \Psi(r) \equiv k \cdot r.$$

(2.32)

$P$ is the constant amplitude of the wave, and $\Psi(r)$ (really, $\omega t - \Psi(r)$) its phase, such that the wave vector $k = \nabla \Psi$ has constant magnitude $|k| = k$ and points in the direction of
The simplicity of the plane wave solution leads us to wonder whether more general solutions might be described in a similar way. If the properties of the wave change only slowly over a wavelength, we expect that the overall acoustic field will also vary slowly in space. This slow spatial scale suggests the existence of an “almost plane wave,” or locally plane wave solution of the form

\[ p_s(r) = P(r)e^{-i\Psi(r)}, \quad (2.33) \]

where the amplitude \( P(r) \) and the local wave vector \( k(r) \) now both vary, but only slowly, over a wavelength \( \lambda = 2\pi \cdot |k(r)|^{-1} \).

To quantify “slowly varying” we introduce the characteristic length, which Kravtsov and Orlov [106] define as “the distance over which the increment of a certain variable is comparable with it in value.” Say we have some scalar function of position, \( f(r) \). If we increment \( r \) by some amount \( \Delta r \), then the increment in \( f \) will be of magnitude \( |f(r + \Delta r) - f(r)| \). For small \( \Delta r \), the increment in \( f \) is well-approximated by the first term in its Taylor expansion, \( |\Delta r \cdot \nabla f(r)| \), which is bounded by \( |\Delta r||\nabla f| \). The characteristic length \( L_f \) of \( f \) is then the distance \( |\Delta r| \) over which this increment is comparable to \( f(r) \), or

\[ L_f |\nabla f| \approx |f| \implies L_f \approx \frac{|f|}{|\nabla f|}. \quad (2.34) \]

The function \( f \) is then said to be slowly varying over a wavelength if its characteristic length is much greater than the local wavelength \( \lambda = 2\pi \cdot |k(r)|^{-1} \).

Returning to the locally plane wave solution, we therefore require that the amplitude \( P(r) \) and the local wave vector \( k(r) \) satisfy

\[ L_P \equiv \left| \frac{P}{\nabla P} \right| \gg \lambda, \quad L_{k_i} \equiv \left| \frac{k_i}{\nabla k_i} \right| \gg \lambda, \quad (2.35) \]

where \( k_i \) are the components of the wave vector. These restrictions can also be interpreted as conditions on the principal radii of curvature of the wavefront so defined. Essentially by definition, a wavefront that fails to be locally planar will have non-negligible curvature, or equivalently a small local radius of curvature. Therefore, the radii of curvature of a locally plane wave should be large relative to the wavelength.

We can also extend the locally plane wave concept to inhomogeneous media. The Helmholtz-type equation corresponding to the inhomogeneous wave equation (2.24) is
given by
\[
\nabla^2 p_s - \frac{1}{\rho_0} \nabla \rho_0 \cdot \nabla p_s + k^2 p_s = 0, \tag{2.36}
\]
where now, \( c_0 = c_0(r) \) and \( \rho_0 = \rho_0(r) \) are taken to be given functions of \( r \). The locally plane wave solution will hold for (2.36) if, in addition to the requirements of (2.35), the medium is only weakly inhomogeneous. That is to say, the sound speed and density vary slowly over a wavelength:
\[
L \approx \frac{c_0}{\nabla c_0} \gg \lambda, \quad L \approx \frac{\rho_0}{\nabla \rho_0} \gg \lambda. \tag{2.37}
\]
Conditions (2.35) and (2.37) can be combined into the single inequality
\[
L \gg \lambda, \tag{2.38}
\]
where \( L = \min(L_P, L_{k_i}, L_{c_0}, L_{\rho_0}) \) is the smallest of the characteristic length scales of the problem. \( L \) may then be considered the defining characteristic length of the problem, which must be much larger than the local wavelength for the locally plane wave solution to apply. Therefore, the locally plane wave solution represents a short-wavelength, or high-frequency approximation of the governing equations. Moreover, it is an asymptotic solution, in that the approximation improves as the frequency increases and, in the limit \( k \to \infty \), constitutes an exact solution of the wave equation.

### 2.3.2 Wavefronts

Moving forward, we will take the usual acoustic convention in which \( \omega \) is factored out of the phase \( \Psi \), and instead treat \( \tau \equiv \Psi/\omega \) as the unknown phase function. The locally plane wave solution then takes the form
\[
p_s(r) = P(r)e^{-i\omega(\Psi(r)/\omega)} = P(r)e^{-i\omega \tau(r)}, \tag{2.39}
\]
where the to-be-determined phase function \( \tau \) has dimensions of time. The locally plane wave solution for a time-harmonic signal is then given by
\[
p(r, t) = \Re \{ P(r)e^{i\omega(t-\tau(r))} \}. \tag{2.40}
\]
Under the geometrical acoustic approximation, the surfaces of constant phase, or wavefronts, are given by the set \( \{r, t\} \) for which \( t - \tau(r) = \text{cst} \). The wavefront at any
particular time, say $t = t_0$, is thus the set of points $r$ for which $\tau(r) = t_0 + \text{cst.} = \text{cst.}$ Each choice of the constant corresponds to a different time and thus a different surface in space, so we can use $\tau$ to track the evolution of a wavefront. This description also allows us to relate the rays and wavefronts of the system, as we will see shortly.

### 2.3.3 The eikonal equation and the transport equation

The high-frequency asymptotic characterization of ray theory allows us to quickly find the governing equations of geometrical acoustics. Following [76], we first substitute the locally plane wave assumption $p_s = P(r) \exp(-i\omega \tau(r))$ into the inhomogeneous Helmholtz equation (2.36), resulting in

$$e^{-i\omega \tau} \left\{ \frac{\nabla^2 P - \frac{1}{\rho_0} (\nabla P \cdot \nabla \rho_0)}{\omega^2} - P \left[ (\nabla \tau)^2 - \frac{1}{c_0^2} \right] \right. $$

$$\left. - \frac{i}{\omega} \left[ 2 \nabla P \cdot \nabla \tau - \frac{P}{\rho_0} (\nabla \rho_0 \cdot \nabla \tau) + P \nabla^2 \tau \right] \right\} = 0,$$  

(2.41)

which may be separated into two equations, one each for its real and imaginary parts. For lossless, nondispersive media, we may assume that $P$ and $\tau$ are purely real quantities, so

$$\nabla^2 P - \frac{1}{\rho_0} (\nabla P \cdot \nabla \rho_0) - P \left[ (\nabla \tau)^2 - \frac{1}{c_0^2} \right] = 0,$$  

(2.42)

and

$$2 \nabla P \cdot \nabla \tau - \frac{P}{\rho_0} (\nabla \rho_0 \cdot \nabla \tau) + P \nabla^2 \tau = 0,$$  

(2.43)

We begin with (2.42). Its first term is a ratio of two quantities: spatial derivatives of the pressure amplitude and ambient density, and the square of the frequency. Under the assumptions of geometrical acoustics, this ratio is a small quantity divided by a very large quantity, so we may neglect it in comparison to the second term. Therefore, we require that

$$(\nabla \tau)^2 - \frac{1}{c_0^2} = 0.$$  

(2.44)

This is the well-known eikonal equation. The function $\tau$ is referred to as the eikonal function, or simply the eikonal\(^{(1)}\). From Section 2.3.2, we see that it governs the spatial evolution of the wavefronts.

\(^{(1)}\)In optics, the eikonal is instead defined as $W = c_r \tau$ where $c_r$ is a constant reference speed (typically the speed of light in vacuo). The eikonal equation then becomes $(\nabla W)^2 = n^2$, where $n = c_r/c_0$ is the index of refraction [181]. Moreover, $\tau$ has dimensions of time while $W$ has dimensions of length.
The second equation, \((2.43)\), is known as the transport equation. As we will show in Section 2.3.6, it governs local pressure amplitude variations along rays. To put \((2.43)\) in a more convenient form, we multiply by the integrating factor \(P/\rho_0\), note that \(2P\nabla P = \nabla P^2\) and \(-\rho_0^{-2}\nabla \rho_0 = \nabla \rho_0^{-1}\), and apply the product rule to find
\[
\nabla \cdot \left( \frac{P^2}{\rho_0} \nabla \tau \right) = 0.
\]
\((2.45)\)

This is the conservation form of the transport equation. Together, the eikonal equation and transport equation form the basis of the method of geometrical acoustics\(^{(2)}\).

### 2.3.4 Rays

Rather than trying to determine the propagation of wavefronts as a whole, it is simpler to first consider the trajectories of individual points lying on their surfaces. With these trajectories, or rays determined, the wavefront at any time may be constructed as the surface passing through a particular collection of points along the rays. In this sense, we can visualize wavefronts as surfaces \(\tau = \text{cst.} \) (in time) propagating along a set of rays at the speed of sound (see Figure 2.1).

![Figure 2.1: The relationship between rays and wavefronts in a quiescent medium](image)

It now remains to determine the equations governing the rays. This may be done us-

\(^{(2)}\)The eikonal and transport equations are the first of a more general recursive system of equations. This system is found by inserting an asymptotic expansion in inverse powers of the wavenumber, known as the ray expansion, into the Helmholtz equation. The higher-order equations resulting from this procedure represent amplitude corrections to the leading-order geometrical acoustic solution \([36, 107]\). However, these amplitude corrections also become infinite on caustics.
ing the eikonal equation (2.44) after a quick detour through the theory of differential equations.

2.3.5 The ray equations

The eikonal equation is an example of an equation of Hamilton-Jacobi type [107], which has the general form

\[ H \left( \frac{\partial \psi}{\partial q_1}, \frac{\partial \psi}{\partial q_2}, \ldots, \frac{\partial \psi}{\partial q_n}; q_1, q_2, \ldots, q_n \right) = 0, \] 

or

\[ H(\varphi_j, q_j) = 0, \quad \varphi_j = \frac{\partial \psi}{\partial q_j}. \] 

By analogy with classical mechanics, the function \( H(\varphi_j, q_j) \) is known as the Hamiltonian of the system. \( q_j (j = 1, 2, \ldots, n) \) are a set of “generalized coordinates,” \( \psi(q_1, q_2, \ldots, q_n) \) the function to be determined, and \( \varphi_j^{(3)} \) the derivatives of \( \psi \) with respect to the \( q_j \), sometimes called the generalized momenta.

Equations of Hamilton-Jacobi type can be solved by introducing a parameter \( \ell \) and applying the method of characteristics. This reduces the partial differential equation \( H = 0 \) to a set of ordinary differential equations in \( q_j, \varphi_j, \) and \( \psi \), known as the canonical equations. The solution \( q_j = q_j(\ell), \varphi_j = \varphi_j(\ell), \psi = \psi(\ell) \) of this system is known as the characteristic strip while the family of functions \( \varphi_j = \varphi_j(\ell), q_j = q_j(\ell) \) are said to be the characteristics of (2.46) [106].

To apply the Hamilton-Jacobi formalism to our problem, we take for \( q_j (j = 1, 2, 3) \) the usual Cartesian coordinates (i.e. \( q = r \) and \( \varphi = \nabla \psi \)). In this case, the canonical equations take the vector form

\[ \frac{dr}{d\ell} = \frac{\partial H}{\partial \varphi}, \quad \frac{d\varphi}{d\ell} = -\frac{\partial H}{\partial r}, \] 

\[ \frac{d\psi}{d\ell} = \varphi \cdot \frac{\partial H}{\partial \varphi}. \] 

Taking \( \psi = \tau \), and hence \( \varphi = \nabla \tau \), we can rearrange the eikonal equation (2.44) into

(3) We have used \( \varphi \) in favor of the traditional \( p \) to avoid confusion with the acoustic pressure.
the Hamilton-Jacobi form:

\[ H \equiv \frac{1}{2} \left[ \varphi^2 - \frac{1}{c_0^2(r)} \right] = 0. \]  \hfill (2.50)

The factor \( \frac{1}{2} \) has been introduced for mathematical convenience, and \( \varphi^2 \equiv \varphi \cdot \varphi = (\nabla \tau)^2 \). Substituting this expression into (2.48)–(2.49), the canonical equations for the eikonal equation follow as

\[ \frac{dr}{d\ell} = \varphi, \quad \frac{d\varphi}{d\ell} = -\frac{1}{c_0^2} \nabla c_0, \]  \hfill (2.51)

\[ \frac{d\tau}{d\ell} = \frac{1}{c_0}. \]  \hfill (2.52)

The characteristic strip of the eikonal equation is thus \( r = r(\ell), \varphi = \varphi(\ell), \tau = \tau(\ell) \), and its characteristics are the set \( r = r(\ell), \varphi = \varphi(\ell) \). The coordinate-space projection \( r = r(\ell) \) of a characteristic then yields a ray (though rays are occasionally identified with the characteristics themselves) \[106\]. In other words, the rays of geometrical acoustics are the space curves satisfying (2.51) while the eikonal determining the propagation of the wavefronts must satisfy (2.52).

We can clarify the physical significance of (2.51)–(2.52) by relating the parameter \( \ell \) to the arc length \( s \) along a ray. From the first equation in (2.48),

\[ ds^2 = dr \cdot dr = \left( \frac{\partial H}{\partial \varphi} \right)^2 d\ell^2, \]  \hfill (2.53)

with \( ds \) the differential arc length element. Therefore, the relationship between parameter \( \ell \) and the arc length \( s \) is given by

\[ d\ell = \frac{ds}{|\partial H/\partial \varphi|}, \]  \hfill (2.54)

or

\[ d\ell = \frac{1}{|\varphi|} ds = \frac{1}{|\nabla \tau|} ds = c_0 ds. \]  \hfill (2.55)

Thus, under the arc length parametrization, equations (2.51)–(2.52) become

\[ \frac{dr}{ds} = c_0 \nabla \tau, \]  \hfill (2.56)

\[ \frac{d}{ds}(\nabla \tau) = -\frac{1}{c_0^2} \nabla c_0, \]  \hfill (2.57)

32
\[
\frac{d\tau}{ds} = \frac{1}{c_0}.
\]  

(2.58)

(2.56) and (2.57) are known as the ray equations, or raytracing equations. The first determines the position \( r \) of a ray as it propagates. It also implies that the tangent vector to the ray \( dr/ds \) is always parallel to \( \nabla \tau \), meaning that the second ray equation (2.57) describes how a ray’s direction changes as it is traced out. Thus, to find the ray with direction \( \mathbf{\varphi}_0 = \mathbf{\varphi}(s_0) \) at point \( r_0 = r(s_0) \), the coupled system (2.56)–(2.57) must be solved. Note that \( \mathbf{\varphi} = \nabla \tau \) is sometimes called the slowness vector, because from (2.44) or (2.56) we see that \( |\nabla \tau| \) is the inverse of the sound speed \( c_0 \).

Once the rays of the system have been determined, the integration of (2.58) from \( s_0 \) to \( s \) along a particular ray gives the eikonal as a function of the arc length:

\[
\tau(s) - \tau_0 = \int_{s_0}^{s} \frac{ds'}{c_0(s')},
\]  

(2.59)

where \( s' \) is a variable of integration and \( \tau_0 \) represents any contribution to the eikonal associated with the propagation of the ray prior to reaching \( r_0 = r(s_0) \). Since rays propagate at the speed of sound, (2.59) indicates that the acoustic eikonal is equal to the travel time of sound along a ray. Moreover, since geometrical wavefronts are defined as the level sets of the eikonal,

\[
\tau = \text{cst.,}
\]  

(2.60)

we see that wavefronts are the surfaces of equal travel time along a set of rays, and in this way may be said to propagate along the rays.

Since the gradient of a function is always perpendicular to its level sets, another consequence of (2.60) is that the slowness vector \( \nabla \tau \) is always perpendicular to the wavefronts. Therefore, (2.56) implies that the rays are always perpendicular to the wavefronts\(^(4)\). This is not so in moving media, as we will see later. In closing, we note that when parametrized with respect to the the eikonal, the ray equations become

\[
\frac{dr}{d\tau} = c_0^2 \nabla \tau,
\]  

(2.61)

\[
\frac{d}{d\tau}(\nabla \tau) = -\frac{1}{c_0} \nabla c_0.
\]  

(2.62)

\(^{(4)}\) For this reason, optics literature often refers to a set of rays possessing a common wavefront as an orthotomic system or normal congruence of rays [181]. We will use the more general nomenclature of a ray family.
2.3.6 Solution of the transport equation

We now turn to the transport equation, (2.43), which may be solved by introducing the concept of a *ray tube*. Suppose the ray under consideration, which we term the *principal ray*, has a trajectory determined by the ray equations (2.56)–(2.57). The ray tube consists of narrow bundle of rays surrounding the principal ray as it propagates from some reference point \( r_0 \) to another point \( r(s) \) (see Figure 2.2). As the principal ray propagates, sound speed variations cause the surrounding bundle of rays to bunch up or spread out. As a result, the cross-sectional area \( S \) of the ray tube (the *ray tube area*) varies, sweeping out a volume \( V \) with end caps centered at \( r_0 \) and \( r \). Integrating (2.45) over \( V \) and applying the divergence theorem, we find that

\[
0 = \iiint_{V} \nabla \cdot \left( \frac{P^2}{\rho_0} \nabla \tau \right) \, dV = \iint_{\partial V} \left( \frac{P^2}{\rho_0} \nabla \tau \right) \cdot \hat{n}_V \, dS, \tag{2.63}
\]

where \( \partial V \) is the surface enclosing the volume \( V \), i.e. the surface of the ray tube, and \( \hat{n}_V \) is the outward unit normal at each point of \( \partial V \). The right-hand side may subsequently be decomposed into three integrals, one for each end cap of the ray tube and one for its

![Figure 2.2: A ray tube](image-url)
side walls. The latter vanishes, as $\nabla \tau$ is tangent to the rays making up the ray tube such that $\nabla \tau \cdot \hat{n}_V = 0$ everywhere along the side walls. At the end cap centered at $r_0$, the surface normals are all antiparallel to $\nabla \tau$, so that $\nabla \tau \cdot \hat{n}_V = -|\nabla \tau||\hat{n}_V| = -1/c_0$, where the last equality follows from the eikonal equation, (2.44). Similarly, at the end cap at $r$ the surface normals are parallel to the slowness vector and $\nabla \tau \cdot \hat{n}_V = 1/c_0$. Substituting these relationships into (2.63), we conclude that

\[ \iiint_{S(s)} \frac{P^2}{\rho_0 c_0} \, dS - \iiint_{S(s_0)} \frac{P^2}{\rho_0 c_0} \, dS = 0. \]  

Letting the ray tube shrink down about the principal ray results in an *infinitesimal ray tube*, across which the values of $P$, $\rho_0$, and $c_0$ cannot vary significantly, and so are essentially determined by their values on the principal ray. Therefore, with $\delta S$ the infinitesimal ray tube area, (2.64) becomes

\[ \left. \frac{P^2}{\rho_0 c_0} \delta S \right|_s = \left. \frac{P^2}{\rho_0 c_0} \delta S \right|_{s_0}. \]  

This equation implies that in a quiescent medium, acoustic energy is conserved along ray tubes, thereby reducing acoustic propagation to a simpler geometric problem. In moving media, this property generalizes to the conservation of *wave action*, an *adiabatic invariant* analogous to the *action* variable of classical mechanics [151]. Rearranging this expression leads to a formula for the pressure amplitude at any point along a ray,

\[ P(s) = \sqrt{\frac{\delta S(s_0) \rho_0(s) c_0(s)}{\delta S(s) \rho_0(s_0) c_0(s_0)}} P(s_0). \]  

Therefore, if we know the pressure amplitude $P$ at some reference point $r_0$ along a ray, we can find its value at any other point $r$ by evaluating the ray tube area $\delta S$, ambient density $\rho_0$, and sound speed $c_0$ at both $r$ and $r_0$. In theory, the ray tube area at any point can be found by constructing all the rays in the region of interest and then identifying sufficiently narrow ray tubes. In practice, this is unnecessary, as we will see in Section 2.3.9. Our point here is that to find the ray tube area at any point, it is sufficient that we be able to construct the rays using (2.56)–(2.57).
Lastly, we note that (2.66) is often written in the form

\[ P(s) = \frac{P(s_0)}{\sqrt{\mathcal{J}}}, \quad \text{where} \quad \mathcal{J}(s) = \frac{\delta S(s)}{\delta S(s_0)} \frac{\rho_0(s_0) c_0(s_0)}{\rho_0(s) c_0(s)}. \]  

(2.67)

\( \mathcal{J} \) is a measure of the stretching and squishing of a ray tube relative to some reference point along the principal ray due to both geometrical spreading and the heterogeneity of the medium. For this reason, it is sometimes referred to as the \textit{generalized ray divergence} [107].

\[ \textbf{2.3.7 Ray coordinates} \]

In order to predict the sound field in a given region, a large system of rays is required. We would like a convenient way to refer to and mathematically manipulate this system. For this purpose, we will introduce a curvilinear coordinate system known as the \textit{ray coordinates} [107].

Consider a smooth surface \( Q \) intersecting all the rays in some insonified (ray-filled) region. \( Q \) may be parametrized by two (not necessarily orthogonal) curvilinear coordinates, say \( u \) and \( v \). Far from caustics, closely spaced rays do not intersect, so a single ray will pass through each point of \( Q \). This means that each \((u, v)\) pair on \( Q \) can be identified with a unique ray. As long as rays do not intersect, we may continue to identify them by their \((u, v)\) pairs on \( Q \) even after they have passed through it. On the other hand, we have seen that any individual ray can be parametrized by the eikonal \( \tau \). Taken together, this means that sufficiently far from caustics, any point \( r \) in the insonified region can be uniquely identified by a parametric equation\(^{(5)}\)

\[ r = R(u, v, \tau), \]  

(2.68)

where \( R \) describes \( r \) in terms of its position (i.e., eikonal \( \tau \)) along a particular ray \((u, v)\).

It is often convenient to let \( Q \) be a wavefront. In this case, \( Q \) may be identified by the value the eikonal takes on it, say \( \tau = \tau_0 \). Then the set of points resulting from fixing \( \tau \) and varying \( u \) and \( v \) in (2.68) corresponds to a particular wavefront, while fixing \((u, v)\) and varying \( \tau \) corresponds to a particular ray (see Figure 2.3). Moreover, if \( Q \) is a wavefront, the direction of increasing \( \tau \) will be perpendicular to the directions of

\(^{(5)}\)Note that here, \( r \) refers to a generic position in space rather than the position vector for a particular ray. This overlap in notation is common in the literature, e.g. [107], but can be confusing if it is not clear which object is being referred to. We have attempted to make this distinction clear through context.
Figure 2.3: Ray coordinates defined with respect to a reference wavefront; adapted from [10]

increasing $u$ and $v$, at least in quiescent media.

Recall that the Mach cone is the envelope of wavefronts generated by a supersonic aircraft. Hence, we can speak of a complementary ray cone made up of rays perpendicular to the Mach cone, an angle $90^\circ - \mu$ from the flight axis (where $\mu \equiv \sin^{-1}(1/M)$ is the Mach angle associated with aircraft Mach number $M$). In this case, sensible choices for ray coordinates $u$ and $v$ might be the azimuthal exit angle $\phi_{ac}$ of the rays and the aircraft flight time with respect to some reference time $t_{ac}$ (e.g., right before a maneuver is performed) [87, 141, 156]\(^{(6)}\). Such a coordinate system is depicted in Figure 2.4 for an aircraft in rectilinear acceleration. As the aircraft accelerates, the Mach cone narrows and the ray cone widens. Each aircraft time $t_{ac}$ identifies a particular ray cone from which an individual ray can be selected by fixing $\phi_{ac}$. The position of a point along this ray is then identified by the eikonal $\tau$.

2.3.8 Initial conditions on the rays

To trace out all of the rays for a particular source in a given medium, we must integrate the ray equations (2.56)–(2.57). Doing so requires an initial position and direction for each ray which will vary depending on the source characteristics. In practical applica-

\(^{(6)}\)That said, $u$ and $v$ may be freely chosen so long as they serve to parametrize the wavefront surfaces. Their units depend on the parametrization chosen.
Figure 2.4: Ray coordinates for an accelerating supersonic aircraft

It is often convenient to specify these quantities on a surface some distance away from the source. For example, sonic boom prediction softwares often specify an initial pressure distribution on a cylinder several body lengths wide and centered on the flight axis, with rays initialized normal to its surface [145].

Let \( \mathbf{r} = \mathbf{R}_0(u, v) \) be a parametric equation for some such reference surface \( Q \) in ray coordinates \( u, v \). Away from caustics, each ray in the set \( \mathbf{R}(u, v, \tau) \) can be identified by a unique \( (u, v) \) pair on \( Q \), and passes through \( Q \) for some value \( \tau = \tau_0(u, v) \) of the eikonal (with \( \tau_0 \) a constant if \( Q \) is a wavefront). Therefore, an initial condition on the positions of the rays may be given by

\[
\mathbf{R}(u, v, \tau_0(u, v)) = \mathbf{R}_0(u, v). \tag{2.69}
\]

We can obtain initial conditions on the rays' directions, and in particular on \( \mathbf{\varphi} = \nabla \tau \), by differentiating the eikonal on \( Q \), \( \tau|_Q = \tau_0(u, v) \), with respect to \( u \) and \( v \). The result is

\[
\frac{\partial \tau_0}{\partial u} = \left[ \frac{\partial \tau}{\partial \mathbf{R}} \cdot \frac{\partial \mathbf{R}}{\partial u} \right]_{Q} = \mathbf{\varphi}|_Q \cdot \frac{\partial \mathbf{R}_0}{\partial u}, \tag{2.70}
\]
\[ \frac{\partial \tau_0}{\partial \nu} = \left[ \frac{\partial \tau}{\partial \mathbf{R}} \cdot \frac{\partial \mathbf{R}}{\partial \nu} \right]_Q = \varphi_0 \frac{\partial \mathbf{R}_0}{\partial \nu}. \tag{2.71} \]

We recognize the right-hand side of these expressions as the projection of \( \varphi \) onto the tangent plane of \( Q \) in the \( u \) and \( v \) directions (if \( Q \) is a wavefront, \( \nabla \tau \) is perpendicular to it and (2.70) and (2.71) vanish). The remaining component normal to \( Q \) may then be determined from the eikonal equation (2.44), completing the required set of initial conditions on the rays. The set of rays \( \mathbf{R}(u,v,\tau) \) determined by integrating the ray equations with these initial conditions is known as a ray family.

To find the amplitude variation along each ray, we also require an initial amplitude and ray tube area on \( Q \), \( P|Q = P_0(u,v) \) and \( \delta S|Q = \delta S_0(u,v) \). For supersonic aircraft, the initial pressure distribution is often given in terms of the Whitham F-function [84, 194] while the initial ray tube area depends on the radiation characteristics of the source.

### 2.3.9 Calculation of ray tube area

Suppose that the ray coordinates \( (u,v) \) parametrize an initial wavefront \( \tau = \tau_0 \). Then the trajectory of the rays leaving this wavefront can be determined via the ray equations, and later positions of the wavefront correspond to constant values \( \tau > \tau_0 \) within the ray family. The local ray tube area at any point along a particular ray \( (u,v) \) can then be expressed in terms of the wavefront area element. In a quiescent medium, we have

\[ \delta S(\tau) = \delta \Sigma(\tau) = |\mathbf{R}_u \times \mathbf{R}_v|dudv, \tag{2.72} \]

where \( \delta \Sigma \) is the area element of the wavefront \( \tau = \text{cst.} \) at the point parametrized by \( (u,v) \), \( \mathbf{R}_u \equiv \partial \mathbf{R}/\partial u \), and \( \mathbf{R}_v \equiv \partial \mathbf{R}/\partial v \). That is, the ray tube area is equivalent to the wavefront area element, as demonstrated in Figure 2.5. This is not so in moving media, though as we will see later, the two remain closely related. In one way or another, calculation of the ray tube area boils down to calculation or approximation of the quantity on the right-hand side of (2.72).
One of the earliest comprehensive sonic boom prediction codes [87] used a semi-analytical raytracing procedure in which both the ray trajectories and the ray tube area are expressed as quadratures with respect to the altitude variable. However, this procedure assumes a stratified atmosphere, determining the ray tube area based on its projection into horizontal cutting planes parallel to the ground. As a result, if a ray becomes horizontal, the code invariably computes a zero ray tube area. The ray tube area can vanish at such turning points, but this is not always the case [104].

Another method for determining the ray tube area used in older versions of the NASA/KBRwyle PCBoom software [145] amounts to estimating (2.72) for small differences in the ray coordinates. First, the three rays nearest the ray in question are determined (Figure 2.6). Taking the cross product of the vectors connecting the diagonals of these four rays yields a vector whose magnitude is proportional to the wavefront area element $\delta\Sigma$. Projecting this vector onto the ray of interest then yields a quantity proportional to the ray tube area $\delta S$. Since this method relies on the closeness of the four rays considered, its accuracy decreases as the discretization step for the ray coordinates increases. More recent versions of PCBoom [115, 146] circumvent this problem by using a method that requires only a single ray, made possible by the correspondence between the ray tube area and a particular Jacobian determinant to be discussed in the next section.

Lastly, a more general approach to the calculation of ray tube area may be found in Candel’s work [32]. From (2.72), we may infer that the variation in ray tube area along a ray depends on the spatial derivatives of $R_u$ and $R_v$. Just as equations for both the ray position vector $\mathbf{r}(\tau) = \mathbf{R}(u = \text{cst.}, v = \text{cst.}, \tau)$ and its “conjugate momentum”
\( \Phi = \nabla \tau \) are required to uniquely determine ray propagation, two vector equations are required to determine the variations of each wavefront tangent vector: one for the vector itself (\( R_u \) or \( R_v \)) and one for its “conjugate element” (\( \Phi_u \) or \( \Phi_v \)). Therefore, a complete determination of the rays and ray tube areas can be done via the simultaneous solution of 6 coupled vector differential equations, or equivalently 18 coupled scalar differential equations.

### 2.3.10 Ray tube area and the Jacobian

The positions of rays are determined by the parametric equation for the ray family (2.68). This equation may be thought of as a mapping of the ray coordinates \((u, v, \tau)\) to physical Cartesian coordinates \((x, y, z)\), describing the spatial positions of the rays as they propagate in terms of their relative positions on the initial wavefront. Therefore, to measure variations in the relative positions of rays in the family as the ray coordinates are varied, we should examine the derivative of \( R \). If all partial derivatives of the ray family components \( x, y, z \) with respect to the ray coordinates \( u, v, \tau \) exist and are continuous, then the ray family is continuously differentiable and its (total) derivative at any point
is given by the Jacobian matrix (See Appendix B),

\[
\mathbf{R}'(u, v, \tau) \equiv \begin{bmatrix}
\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial \tau} \\
\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial \tau} \\
\frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial \tau}
\end{bmatrix}.
\]  

(2.73)

The determinant of the Jacobian matrix, \(D(u, v, \tau) \equiv \det [\mathbf{R}']\), is a continuous function known as the Jacobian determinant, or simply the Jacobian. This determinant can be seen as a measure of how a volume element at a certain point in \((u, v, \tau)\) space, \(dV_{(u,v,\tau)} = du dv d\tau\), transforms when it is mapped into \((x, y, z)\) space:

\[
dV_{(x,y,z)} = dx dy dz = D \cdot dV_{(u,v,\tau)}.
\]  

(2.74)

In other words, the Jacobian tells us how the volumes traced out by small bundles of rays distort under small changes in the ray coordinates. Therefore, it should not be surprising that it is related to the ray tube area, which we will now make explicit.

We will eventually return to the usual ray coordinates \((u, v, \tau)\), but it will be more convenient to begin with the coordinates \((u, v, \ell)\), with \(\ell\) the characteristic parameter defined in (2.55). Under this parametrization, the Jacobian is given by:

\[
D(u, v, \ell) = \frac{\partial (x, y, z)}{\partial (u, v, \ell)} = \det \begin{bmatrix}
\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial \ell} \\
\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial \ell} \\
\frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial \ell}
\end{bmatrix}.
\]  

(2.75)

In this form, the Jacobian is cumbersome to work with. Noting that the scalar triple product of any three vectors \(\mathbf{a}, \mathbf{b}, \mathbf{c}\) in \(\mathbb{R}^3\) can be written as the determinant of a matrix of their components \(a_i, b_i, c_i\),

\[
(\mathbf{a}, \mathbf{b}, \mathbf{c}) \equiv (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \det \begin{bmatrix}
c_1 & c_2 & c_3 \\
a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3
\end{bmatrix},
\]  

(2.76)

and applying the cyclic shift property of scalar triple products,

\[
(\mathbf{a}, \mathbf{b}, \mathbf{c}) = (\mathbf{b}, \mathbf{c}, \mathbf{a}) = (\mathbf{c}, \mathbf{a}, \mathbf{b}),
\]  

(2.77)

it follows [107] that

\[
D(u, v, \ell) = (\mathbf{R}_u, \mathbf{R}_v, \mathbf{R}_\ell),
\]  

(2.78)
which is far more convenient for our purposes.

To relate the ray tube area and the Jacobian, we will solve the transport equation in terms of the Jacobian and equate the result to (2.66). From the results of 2.3.5, we can show that for any scalar quantity $f$ defined along a ray, the following relationship holds:

\[
\nabla f \cdot \nabla \tau = \frac{df}{ds} |\nabla \tau| = \frac{df}{d\ell} \frac{d\ell}{ds} \left( \frac{1}{c_0} \right) = \frac{df}{d\ell},
\]

(2.79)

where $d/ds$ again refers to differentiation along a ray, i.e. the directional derivative in the direction defined by $\nabla \tau/|\nabla \tau|$. Applying this relationship to (2.43), the transport equation becomes

\[
2 \frac{dP}{d\ell} - \frac{P}{\rho_0} \frac{d\rho_0}{d\ell} + P \nabla^2 \tau = 0.
\]

(2.80)

With the integrating factor $P/\rho_0$, this can be rewritten as

\[
\frac{d}{d\ell} \ln \left( \frac{P^2}{\rho_0} \right) = -\nabla^2 \tau,
\]

(2.81)

which can be integrated with respect to $\ell$ to find

\[
\frac{P^2(\ell)}{\rho_0(\ell)} = \frac{P^2(\ell_0)}{\rho_0(\ell_0)} \exp \left( - \int_{\ell_0}^{\ell} \nabla^2 \tau \, d\ell \right),
\]

(2.82)

where $P(\ell_0), \rho_0(\ell_0)$ are the initial values of the pressure amplitude and the ambient density. We then apply an identity known as the Liouville formula [107], often used in continuum mechanics\(^{(7)}\):

\[
\frac{d}{d\ell} \ln(D(\ell)) = \nabla^2 \tau,
\]

(2.83)

where $D(\ell)$ is the Jacobian defined in (2.75). Strictly, $D$ is a function of $u, v, \ell$ as above, but this derivative is taken along a ray, and hence for $(u,v)$ fixed. Substituting (2.83) into (2.82), we find that

\[
P(\ell) = \sqrt{\frac{D(\ell_0) \rho_0(\ell)}{D(\ell) \rho_0(\ell_0)}} P(\ell_0).
\]

(2.84)

We also have that

\[
D(s) = (R_u, R_v, R_s) = (R_u, R_v, R_\ell) \frac{d\ell}{ds} = c_0 D(\ell).
\]

(2.85)

\(^{(7)}\)Proofs can be found in [15, 36, 93].
Therefore, the pressure amplitude (2.84) can be written in terms of the arc length as

\[ P(s) = \sqrt{\frac{D(s_0) \rho_0(s) c_0(s)}{D(s_0) \rho_0(s_0) c_0(s_0)}} P(s_0). \]  

(2.86)

Comparing this to (2.66), we see that the Jacobian \( D(s) \) is related to the ray tube area by

\[ \frac{D(s_0)}{D(s)} = \frac{\delta S(s_0)}{\delta S(s)}. \]  

(2.87)

This is the property mentioned in 2.3.9 allowing the ray tube area to be computed in terms of a single ray [87, 114]. It may be expressed in terms of the eikonal \( \tau \) by noting that, by (2.59), \( D(s) = \frac{1}{c_0} D(\tau) \) so

\[ \frac{D(\tau_0)}{D(\tau)} \frac{c_0(\tau)}{c_0(\tau_0)} = \frac{\delta S(\tau_0)}{\delta S(\tau)}. \]  

(2.88)

Hence, in terms of the eikonal, the pressure amplitude along a given ray \((u, v)\) is given by

\[ P(\tau) = \sqrt{\frac{D(\tau_0) \rho_0(\tau) c_0^2(\tau)}{D(\tau) \rho_0(\tau_0) c_0^2(\tau_0)}} P(\tau_0) = \frac{P(\tau_0)}{\sqrt{J(\tau)}}, \]  

(2.89)

where the ray divergence is now written

\[ J(\tau) = \frac{D(\tau_0) \rho_0(\tau_0) c_0^2(\tau_0)}{D(\tau) \rho_0(\tau) c_0^2(\tau)}. \]  

(2.90)

### 2.3.11 Determination of the pressure field

Consider a region in which the local sound speed and ambient density are known functions of position. If the pressure field is known on some surface \( Q \) defined by \( r = R_0(u, v) \), then the ray family for this system can be constructed by integrating the ray equations (2.56)–(2.57) with initial conditions given by (2.44) and (2.69)–(2.71). The eikonals of the rays in the family may be determined using (2.59), and in theory the ray tube areas may then be computed. With this done, the pressure amplitude at any other point can be found using (2.66). The final result is that the pressure at any point within the ray
family \( R(u,v,\tau) \) is given by

\[
p(R(u,v,\tau), t) = \text{Re} \left\{ P(u,v,\tau) e^{i\omega(t-\tau)} \right\}
\]

\[
= \text{Re} \left\{ \frac{P_0(u,v)}{J(\tau)} \exp \left\{ i\omega \left[ t - \left( \tau_0(u,v) + \int_{s_0}^{s} \frac{ds'}{c_0(s')} \right) \right] \right\} \right\},
\]

(2.91)

where \( J \) is defined by (2.67) or (2.90).

For an arbitrary dimensionless waveform profile \( F(t) \), the geometrical acoustic ansatz (i.e., the locally plane wave solution (2.40)) is replaced by the more general expression

\[
p(r, t) = P(r) F(t - \tau(r)),
\]

(2.92)

where as before, \( P(r) \) is an amplitude factor with dimensions of pressure and \( \tau \) the eikonal [151]. Substitution of this expression into the linear acoustic equations (2.15)–(2.18) once again leads to the eikonal equation for \( \tau \) (2.44) and the transport equation for \( P \) (2.43). Therefore, if \( F \) is defined such that it equals 1 at \( \tau = \tau_0 \), the complete geometrical acoustic solution for an arbitrary waveform propagating along any ray \((u,v)\) is given by

\[
p(R(u,v,\tau), t) = P(u,v,\tau) F(t-\tau)
\]

\[
= \frac{P_0(u,v)}{\sqrt{J(\tau)}} F \left[ t - \left( \tau_0(u,v) + \int_{s_0}^{s} \frac{ds'}{c_0(s')} \right) \right].
\]

(2.93)

2.4 Caustics in geometrical acoustics

In the context of geometrical acoustics, at least three equivalent definitions of a caustic may be given. Examination of these definitions will lead to the conclusion that the geometrical-acoustic description of the pressure amplitude becomes inapplicable in some finite region surrounding a caustic.

2.4.1 Three definitions of a caustic

We have already pointed out that caustics may be defined as the locus of points at which differentially separated rays cross, or as an envelope of rays. Another definition based on ray tube areas may also be given. Their equivalence will now be demonstrated. The proof is identical to that given by Auger [10], which is primarily based on a discussion in Babic and Buldyrev [15]. A fourth equivalent definition in terms of wavefronts will
be introduced later when we recount the effects of a mean flow (wind) on focusing.

Claim: the following three definitions of a caustic are equivalent:

a) a caustic is the envelope surface of a set of rays

b) a caustic is the locus of points where the ray tube area vanishes

c) a caustic is the locus of points where two differentially separated rays cross

Proof:

a) $\implies$ b)

In ray coordinates, the envelope of the rays can be written as $\tau = \tau(u, v)$, where $\tau(u, v)$ is the value of the eikonal at the point along each ray $(u, v)$ where it is tangent to the caustic. The positions of points on the caustic are then identified by $\tilde{R}(u, v) \equiv \mathbf{R}(u, v, \tau(u, v))$. Differentiating with respect to $\tau$, we have

$$\frac{\partial \mathbf{R}}{\partial \tau}(u, v, \tau(u, v)) = \frac{\partial \mathbf{R}}{\partial s}(u, v, \tau(u, v)) \frac{\partial s}{\partial \tau} = c_0 \frac{\partial \mathbf{R}}{\partial s}(u, v, \tau(u, v)),$$ (2.94)

meaning that, as we already knew, $\mathbf{R}_\tau$ is tangent to the ray $(u, v)$. By definition of an envelope surface, $\mathbf{R}_\tau$ is also tangent to the caustic [107]. Since the caustic surface has the parametric representation $\tilde{\mathbf{R}}(u, v)$, its natural tangent vectors are given by $\tilde{\mathbf{R}}_u = \mathbf{R}_u + \mathbf{R}_\tau \tau_u$ and $\tilde{\mathbf{R}}_v = \mathbf{R}_v + \mathbf{R}_\tau \tau_v$, implying that $\mathbf{R}_u$ and $\mathbf{R}_v$ must also lie in the tangent plane of the caustic. The coplanarity of $\mathbf{R}_u$, $\mathbf{R}_v$, and $\mathbf{R}_\tau$ then implies that their scalar triple product vanishes:

$$(\mathbf{R}_u, \mathbf{R}_v, \mathbf{R}_\tau) = (\mathbf{R}_u \times \mathbf{R}_v) \cdot \mathbf{R}_\tau = 0. \quad (2.95)$$

Therefore, on the caustic, either $\mathbf{R}_u \times \mathbf{R}_v$ is orthogonal to the ray tangent vector $d\mathbf{r}/ds$ or $\mathbf{R}_u \times \mathbf{R}_v = 0$.

On the other hand, by definition of the ray coordinates $u$ and $v$, $\mathbf{R}_u$ and $\mathbf{R}_v$ are tangent to the wavefront. Therefore, if their cross product does not vanish on the caustic, it produces a vector normal to the wavefront, and so collinear to $\partial \mathbf{R}/\partial s = d\mathbf{r}/ds$. In combination with our previous observation, this would imply that $d\mathbf{r}/ds \perp d\mathbf{r}/ds$, which is of course impossible. Therefore, $\mathbf{R}_u \times \mathbf{R}_v$ must vanish on the caustic. But the ray tube area is given by $\delta S = |\mathbf{R}_u \times \mathbf{R}_v| du dv$, and is therefore zero at all points on the caustic.
b) $\implies$ a)

We now write $\tau = \tau(u, v)$ for the values of the eikonal at the points along the ray family at which the ray tube area vanishes, and $\tilde{R}(u, v) \equiv R(u, v, \tau(u, v))$ for the equation of the surface so defined. Then by virtue of (2.72), $R_u \times R_v = 0$ at every point of the caustic surface. Therefore, the scalar triple product $(R_u, R_v, R_\tau) = (\tilde{R}_u, \tilde{R}_v, \tilde{R}_\tau)$ vanishes. This implies that $\tilde{R}_u, \tilde{R}_v,$ and $R_\tau$ are coplanar, and thus that the ray tangent vector $R_\tau$ for every ray lies in the plane spanned by $(\tilde{R}_u, \tilde{R}_v)$, which is the caustic tangent plane. In other words, the rays are tangent to the caustic, and thus the caustic is an envelope to the set of rays, making contact at the points where the ray tube area vanishes. Thus we have a) $\iff$ b).

a) $\iff$ c)

In a) $\implies$ b), we demonstrated that the caustic is also equivalent to the locus of points at which $R_u, R_v,$ and $R_\tau$ are coplanar, meaning that the three vectors are not linearly independent. Therefore, at any such point $R(u, v, \tau(u, v))$, there exist quantities $\delta u, \delta v,$ $\delta \tau$ not all equal to zero and scaled as small as we wish such that

$$R_u \delta u + R_v \delta v + R_\tau \delta \tau = 0.$$  \hspace{1cm} (2.96)

In the limit $\delta u, \delta v, \delta \tau \to 0$, this yields $dR = 0$, or

$$R(u, v, \tau) = R(u + du, v + dv, \tau + d\tau).$$  \hspace{1cm} (2.97)

That is, the two differentially separated rays $(u, v)$ and $(u + du, v + dv)$ intersect at this point. The converse follows by Taylor expanding (2.97) and following the argument in reverse. This concludes the proof that the three definitions of a caustic given above are equivalent.

2.4.2 Caustics and the Jacobian

We have found that the Jacobian of the mapping from ray coordinates to spatial coordinates is proportional to the local ray tube area along any particular ray. Therefore, $\delta S = 0$ if and only if $D = 0$, meaning that the caustic is also the locus of points within the ray family for which $D = 0$. This definition is not fundamentally different from the ray tube area condition, but its use greatly simplifies certain calculations. For instance, Onyeonwu has used this property to analytically determine a focusing condition for supersonic aircraft in arbitrarily maneuvering flight in a windy, horizontally strat-
ified atmosphere [139, 141]. The expression is rather complicated, but may be shown to be equivalent to Hayes, Haefeli, and Kulsrud’s formula for the ray tube area for a supersonic aircraft [87] when \( \delta S = 0 \).

Another interpretation of \( D = 0 \) follows from the theorem that the determinant of a matrix is the product of its eigenvalues. In particular, \( D = \det [R'] = 0 \) implies that one or more of the eigenvalues of the Jacobian matrix are zero and the Jacobian matrix is not full-rank (in this case \( \text{rank}(R') < 3 \)). For points at which \( \text{rank}(R') = 2 \), the ray tube area element degenerates into a line. Such points are referred to as \textit{caustic points of the first order}. Therefore, a fold/Smooth caustic is a collection of first-order caustic points [36]. Other first-order caustics exist, including the cusp, and are classified by catastrophe theory (see [92, 160]). When \( \text{rank}(R') = 1 \), the ray tube area shrinks down to a \textit{caustic point of the second order} or \textit{focal point}, an unstable singularity commonly seen in optical apparatus [181]. These eigenvalues are also related to derivatives of the principal wavefront curvatures. On a smooth caustic, one of the curvatures reaches a local minimum resulting in a cusped wavefront. For a focal point, both principal curvatures of the wavefront vanish simultaneously [36].

The condition \( D = 0 \) can also be used to determine a parametric equation for the caustic. Following Gazaryan [200], we first note that in component form, the parametric equation for the ray family (2.68) is given by

\[
\begin{align*}
x &= x(u, v, \tau), \\
y &= y(u, v, \tau), \\
z &= z(u, v, \tau).
\end{align*}
\]  

Consider a point \( (x = x_0, y = y_0, z = z_0) \) lying on a ray \( (u = u_0, v = v_0) \) of the family, on some initial wavefront parametrized by \( \tau = \tau_0 \). Suppose this point is far from any caustics so that \( D \neq 0 \), and for concreteness, \( D > 0 \) (a ‘positive’ ray tube area). By the continuity of \( D \), \( D > 0 \) not only at \( (u_0, v_0, \tau_0) \) but at all points in some neighborhood of it (in the ray coordinate space). Therefore, by the \textit{inverse function theorem} (see Appendix B), (2.98) is uniquely solvable with respect to \( u, v \), and \( \tau \) in this neighborhood. In other words, at least in a section of the ray family near the initial wave surface, a single ray \( (u, v) \) passes through each point \( (x, y, z) \). Now suppose \( D \) vanishes at some point \( \tau_c > \tau_0 \) along ray \( (u_0, v_0) \), which we will refer to as the \textit{characteristic point} of the ray. Then by continuity, \( D = 0 \) at all points in some neighborhood of \( (u_0, v_0, \tau_c) \), and (2.98) cannot be uniquely solved at such points. Therefore, more than
one ray has passed through each point, and since $D = 0$, their spatial derivatives are equal—differentially separated rays have crossed at their characteristic points. Moreover, if $D_\tau \neq 0$ at $(u_0, v_0, \tau_c)$, the \textit{implicit function theorem} (see Appendix B) says that the equation $D(u, v, \tau) = 0$ defines a single-valued function $\tau = \tau(u, v)$. That is, the characteristic points of the ray family defines a surface in $(u, v)$–space with parametric equation $\tau = \tau(u, v)$—the caustic, whose position in physical space is then described by the parametric equation $r = R(u, v, \tau(u, v)) \equiv \hat{R}(u, v)$ [200].

The more mathematical approach taken here uncovers assumptions not explicitly addressed in earlier sections. First, in order for the Jacobian to be defined, the ray family must be continuously differentiable with respect to the ray coordinates. This is guaranteed as long as the variables describing the medium (sound speed and ambient density) and the initial wavefront (or at least some portion of it) are sufficiently smooth functions of the spatial variables. Second, use of the implicit function theorem to find the parametric equation for the caustic surface requires $D_\tau \neq 0$. In order for $D_\tau = (R_u, R_v, R_\tau) + (R_{uv}, R_{v\tau}, R_{\tau\tau})$ to vanish, $R_u$ and $R_v$ would have to vanish simultaneously on the caustic. However, this cannot happen at a caustic point of the first order, as it would require the vanishing of 2 or more eigenvalues in the Jacobian matrix. Therefore, the condition $D_\tau \neq 0$ always holds for smooth caustics [200]. In particular, with our choice of $D > 0$ away from the caustic, we must take $D_\tau < 0$ so that the ray tube area decreases and passes through zero on the caustic.

### 2.4.3 Breakdown of geometrical acoustics near caustics

Recalling the expression for the pressure variation along a ray (2.66), we see that if the infinitesimal ray tube area $\delta S$ vanishes at any point along a ray, an infinite pressure results. Since a caustic is the locus of points at which the ray tube area vanishes, geometrical acoustics predicts an infinite pressure everywhere on a caustic, and is therefore not a sufficient description of the pressure field near a caustic. This is not surprising if we recall the basic assumptions of geometrical acoustics from Section 2.3.1. In particular, as an acoustic wave approaches a caustic, the focusing effect leads to a rapid amplitude variation inconsistent with the slowly-varying wave assumption. Therefore, in order to remove the pressure singularity on the caustic, we require a more accurate description of the pressure field in the vicinity of the caustic than geometrical acoustics can provide.
2.5 Chapter summary

In this chapter, the linear acoustic equations and their high-frequency approximation, the geometrical acoustic equations, were reviewed. In the framework of geometrical acoustics, sound propagates along rays, the characteristic curves of the eikonal equation (2.44), while pressure amplitude variations are governed by the transport equation, (2.43). Solving the latter equation required the introduction of ray tubes, narrow bundles of rays whose cross-sectional area governs the pressure amplitude through an inverse square root relationship. An alternate solution to the transport equation in terms of the Jacobian of the ray family mapping was then derived, leading to a useful connection between the ray tube area and the Jacobian, (2.88). After defining appropriate initial conditions in terms of the ray coordinates (2.69)–(2.71), the complete geometrical acoustic solution (2.93) was presented. Three equivalent geometrical definitions of a caustic were then provided, leading to the conclusion that the geometrical acoustic solution becomes infinite on caustics, indicating the need for a more careful study of the field near the caustic.
Chapter 3  
The Focusing of Linear Waves

3.1 Introduction

In this chapter, we will see how the caustic pressure singularity predicted by geometrical acoustics may be alleviated using diffraction theory, which more accurately accounts for the wave nature of sound. In this framework, the region surrounding a caustic is known as a boundary layer, a transitional layer connecting two domains of distinctly different field behavior. Following a more careful study of the geometrical acoustic solution near the caustic, a mixed-type equation governing the pressure field in the boundary layer will be derived. With the proper boundary conditions this equation, known as the Tricomi equation, will be seen to admit a solution in terms of the Airy function of mathematical physics. This solution will result in a connection formula relating the form of a wave entering the caustic boundary layer to its form upon leaving the boundary layer. Unfortunately, we will also find that for incident signals containing discontinuities, this solution still admits singularities. In the context of sonic boom focusing where shocks are expected to be prevalent, this singularity result is unacceptable, indicating the need for further analysis. Through a quick order of magnitude study, we will see that nonlinear waveform steepening within the boundary layer must be considered for the weakly nonlinear waves typical of sonic booms.

3.2 Boundary layer theory

Recall the field behavior observed near a caustic. On one side, waves propagate freely (the insonified region); on the other, there is a rapid decay in pressure (the shadow zone). In between, the waveform amplitude undergoes rapid variations with a net effect
of significant pressure amplification. The caustic region thus connects two distinctly different field behaviors. On physical grounds, we expect the transition between these two regimes to be smooth and finite, though rapid. In the language of diffraction theory, the caustic region is therefore referred to as a transition zone or boundary layer [30, 31].

Outside the caustic boundary layer, the geometrical acoustic approximation is valid and is referred to as the outer expansion of the pressure field. Inside the boundary layer, diffraction effects are no longer negligible and we must use a different approximation of the acoustic equations, accordingly known as the inner expansion. Within a certain matching region, both the outer and inner descriptions of the pressure field are applicable, and must therefore agree, allowing any remaining unknowns of the two expansions to be determined. The procedure to do so is known as the method of matched asymptotic expansions, in which limiting forms of the outer and inner expansions are equated (see Figure 3.1). In some cases, the matching procedure leads to a connection formula, relating the form of an incident signal to that leaving the boundary layer after grazing the caustic. An analogous procedure can also be performed to determine the shape of the evanescent signal in the shadow zone on the other side of the caustic.

The asymptotic matching process may be more clear if thought of like a relay race (Figure 3.2). Outside of the boundary layer, the outer expansion is used to “run” the waveform down toward the caustic. Just outside of the boundary layer, the outer expansion must hand the waveform off to its teammate, the inner expansion. The inner expansion then runs the waveform through the boundary layer, grazes the caustic, and turns around, eventually exiting the boundary layer. At this point, it must hand the waveform back to the outer expansion to finish out the propagation. The asymptotic matching process ensures that the “runners” meet up properly in the matching region so that the hand-off of the waveform can happen. Given the state of the waveform at the first hand-off, the connection formula tells us what it will look like at the second hand-off, after touching the caustic. That is, it tells us the net effect of propagation through the boundary layer. Thus, if one only needs to know what happens to a waveform after it has grazed a caustic, the connection formula acts as a “shortcut,” removing the need for an explicit calculation of the waveform’s propagation through the boundary layer.

\(^{(1)}\)Modern boundary layer theory was first applied to the study of the wave field near (optical) caustics by Buchal and Keller [31]. Only linear waves in 2-dimensional homogeneous media were considered, but the theory was quickly extended to three-dimensional inhomogeneous media by Gazaryan [200]. Much of the exposition in this chapter is based on Gazaryan’s work.
3.3 Inner limit of the outer solution

We have already discussed the outer expansion, geometrical acoustics, at length. The next step, finding its inner limit, is facilitated by a description of the ray variables in terms of a local coordinate system situated on the caustic.

3.3.1 Caustic coordinates

To find the inner behavior of the geometrical acoustic solution, we will need to express the eikonal and ray tube area at a particular point in terms of the point’s relationship to the caustic. To do so, it will be useful to define a set of coordinates on the caustic surface. At least two coordinate systems, one curvilinear and one Cartesian, naturally follow from a particular choice of the u and v lines on the initial wavefront surface \[15, 107, 165, 200\].

Consider a ray family \( \mathbf{R} \) defined with respect to some reference wavefront parametrized by two arbitrary curvilinear coordinates \( u \) and \( v \). Call the value of the eikonal on this wavefront \( \tau = \tau_0 \), and consider a particular ray \( (u, v) = (u_0, v_0) \) on the wavefront surface. Moving in the direction \( (\delta u, \delta v) \) on the reference wavefront, we will encounter
another ray of the family at \((u_0 + \delta u, v_0 + \delta v)\), displaced an amount \(\delta R(u_0, v_0, \tau_0) = R_u(u_0, v_0, \tau_0)\delta u + R_v(u_0, v_0, \tau_0)\delta v\) from ray \((u_0, v_0)\). The displacement between these two rays on any later wavefront \(\tau > \tau_0\) is similarly given by \(\delta R(u_0, v_0, \tau) = R_u(u_0, v_0, \tau)\delta u + R_v(u_0, v_0, \tau)\delta v\). Were these rays to cross, say at the wavefront \(\tau = \tau_c\), we would then have \(\delta R(u_0, v_0, \tau_c) = R_u(u_0, v_0, \tau_c)\delta u + R_v(u_0, v_0, \tau_c)\delta v = 0\). Taking the dot product of this expression with \(R_u\) and rearranging, we find

\[
\frac{\delta u}{\delta v} = -\frac{R_u(u_0, v_0, \tau_c) \cdot R_v(u_0, v_0, \tau_c)}{|R_u(u_0, v_0, \tau_c)|^2}.
\] (3.1)

Therefore, in order for ray \((u_0 + \delta u, v_0 + \delta v)\) to cross ray \((u_0, v_0)\) at \(\tau = \tau_c\), it must satisfy the constraint on \(\delta u\) and \(\delta v\) imposed by (3.1).

In the limit \(\delta u, \delta v \to 0\), the intersection point \(\tau = \tau_c\) for ray \((u_0, v_0)\) lies on a caustic, as it is now the intersection of differentially separated rays. In this case, \(\tau_c\) is the characteristic point of the ray in question (i.e. \(D(u_0, v_0, \tau_c) = 0\)). Since \(R_u \times R_v\) vanishes at the characteristic point of a ray, we have \(R_u \parallel R_v\) at the intersection point so that \(R_u \cdot R_v = |R_u||R_v|\). Thus, for \(\delta u, \delta v \to 0\), (3.1) becomes a condition that two
differentially separated rays must satisfy if they are to intersect on a caustic:

\[
\frac{du}{dv} = -\frac{|R_v(u,v,\tau(u,v))|}{|R_u(u,v,\tau(u,v))|}.
\]  \tag{3.2}

Therefore, in order for an infinitesimally close ray to intersect ray \((u_0,v_0)\) at its characteristic point \(\tau_c = \tau(u_0,v_0)\), it must lie along the integral curve \(u(v)\) of (3.2) that contains \((u_0,v_0)\). In summary, we know from the definition of a caustic that a differentially separated ray will cross ray \((u,v)\) at its characteristic point, but (3.2) tells us which differentially separated ray does so. Moreover, (3.2) defines an entire family of curves on the initial wavefront surface with the property that the differentially separated rays lying along them will cross at caustics.

We now redefine the curvilinear \((u,v)\) coordinates on the initial wavefront such that the integral curves of (3.2) are the \(v\)-lines, with each curve indexed by a different value of \(u\). Therefore, the displacement vector between differentially separated rays reduces to \(d\mathbf{R} = R_v dv\) so that on the caustic where they cross, \(d\mathbf{R} = R_v dv = 0\), and thus

\[ R_v = 0 \]  \tag{3.3}

on the caustic. Hence, the differentially separated rays lying in each coordinate surface \(u = \text{cst.}\) will intersect on the caustic, but since \(R_u\) and \(R_v\) cannot vanish simultaneously, the coordinate surfaces themselves will not. Hence, the newly-defined curvilinear coordinates for the reference wavefront function equally well as coordinates for the caustic surface. In this sense, the reference wavefront coordinates naturally induce a coordinate system on the caustic surface.

To clarify the nature of the caustic coordinates, recall that a parametric equation for the caustic surface is given by

\[ \mathbf{R}(u,v) = (u,v,\tau(u,v)), \]  \tag{3.4}

where \(\tau = \tau(u,v)\) is the set of characteristic points of the ray family at which \(D(u,v,\tau) = 0\). The tangent vectors to the caustic surface may then be computed as

\[
\tilde{R}_u(u,v) = \frac{\partial}{\partial u}(R(u,v,\tau(u,v))) = R_u + \tau_u R_{\tau} \tag{3.5}
\]

\[
\tilde{R}_v(u,v) = \frac{\partial}{\partial v}(R(u,v,\tau(u,v))) = R_v + \tau_v R_{\tau} = \tau_v R_{\tau}, \tag{3.6}
\]
where (3.3) has been used. The second equation states that the rays are tangent to the chosen $v$–lines on the caustic. The $u$ coordinate can be chosen more freely, as the only requirement is that it remain constant along the $v$–lines. In particular, the coordinate lines need not be orthogonal. Of course, an orthogonal coordinate system is the most convenient choice, and is guaranteed if we choose the $u$ coordinate such that $\tau_u = 0$, because then $\vec{R}_u \cdot \vec{R}_v = \tau_v (\vec{R}_u \cdot \vec{R}_\tau) = 0$. Further, $\tau = \text{cst.}$ along the $u$–lines implies that the $u$–lines are the traces of the cusped wavefronts on the caustic surface. The resulting \textit{curvilinear caustic coordinates} are depicted in Figure 3.3.

![Figure 3.3: Curvilinear caustic coordinates; adapted from [107]](image)

To consider points \textit{near} the caustic, we need to be able to move away from its surface. Hence, we also compute a normal vector, given by the cross product of the caustic tangent vectors,

$$\vec{R}_u \times \vec{R}_v = (\vec{R}_u + \tau_u \vec{R}_\tau) \times (\tau_v \vec{R}_\tau) = \tau_v (\vec{R}_u \times \vec{R}_\tau).$$

(3.7)

Therefore, so long as $\tau_v \neq 0^{(2)}$, a normal vector may be uniquely defined. In particular, we take the positive direction of the caustic $v$–lines to be the direction of the rays as they touch the caustic so that $\tau_v > 0$.

\textsuperscript{(2)}The condition $\tau_v \neq 0$ ensures that the caustic is a \textit{regular surface} meaning, loosely, that it contains no sharp points or self-intersections, and a tangent plane can be uniquely defined at every point of it [33].
With the normal vector computed, we have a complete coordinate system. With reference to Figure 3.3, we see that any point near the caustic may be identified by three values: the \((u, v)\) coordinates of the point’s orthogonal projection onto the caustic surface and the length of its projection line. However, for calculations, it is simpler to work in a local Cartesian coordinate system on the caustic.

Consider the point \(O\) at which a particular ray makes contact with the caustic. From 2.4.1, we know that the ray’s tangent vector \(\mathbf{R}_\tau\) and the two wavefront vectors \(\mathbf{R}_u\) and \(\mathbf{R}_v\) lie in the caustic tangent plane at \(O\). Further, in quiescent media, the wavefront tangent vectors are perpendicular to the ray tangent vector. Therefore, an orthogonal basis with origin \(O\) on the caustic is given by \(\{\mathbf{R}_\tau|_O, \mathbf{R}_u|_O, \pm(\mathbf{R}_\tau \times \mathbf{R}_u)|_O\}\), where we have used the notation \((\cdot)|_O\) to denote that the quantity should be evaluated at point \(O\).

With our choice of \(v\)-lines, \(\mathbf{R}_\tau|_O\) is directed along the \(v\)-line passing through \(O\). \(\mathbf{R}_u|_O\) points along the \(u\)-line at \(O\), and thus along the trace of the cusped wavefront orthogonal to the ray touching \(O\). Lastly, since \(\mathbf{R}_u\) and \(\mathbf{R}_\tau\) both lie in the caustic tangent plane, \(\pm(\mathbf{R}_\tau \times \mathbf{R}_u)|_O\) is a normal to the caustic (which could also be inferred from (3.7)). Either sign may be used, but the positive sign is preferable because it will turn out that this orientation of the normal vector always points into the insonified region. Normalizing these vectors results in the desired orthonormal basis:

- Unit vector tangent to the ray touching the caustic at \(O\):
  \[
  \frac{\mathbf{R}_\tau}{|\mathbf{R}_\tau|}|_O, \tag{3.8}
  \]

- Unit vector along the wavefront cusp passing through \(O\):
  \[
  \frac{\mathbf{R}_u}{|\mathbf{R}_u|}|_O, \tag{3.9}
  \]

- Unit vector normal to the caustic at \(O\):
  \[
  \frac{\mathbf{n}}{|\mathbf{n}|}|_O = \left(\frac{\mathbf{R}_\tau \times \mathbf{R}_u}{|\mathbf{R}_\tau \times \mathbf{R}_u|}\right)|_O = \left(\frac{\mathbf{R}_\tau \times \mathbf{R}_u}{|\mathbf{R}_\tau||\mathbf{R}_u|}\right)|_O. \tag{3.10}
  \]

We may then define Cartesian coordinates with respect to this basis by taking the \(Ox\) direction along the first vector, the \(Oy\) direction along the second, and the \(Oz\) direction along the last. The resulting local caustic coordinates [200] are depicted atop
the curvilinear coordinates in Figure 3.4.

![Figure 3.4: Local caustic coordinates; adapted from [107]](image)

### 3.3.2 Eikonal along a caustic normal

Consider a point normal to the caustic a distance $z_c$ from $O$. In our newly defined local coordinate system, this point is described by $(x, y, z) = (0, 0, z_c)$. We wish to determine which ray(s) pass through this point and their corresponding eikonals. Following [200], we expand the ray family in a power series in $(u, v, \tau)$ about $O^{(3)}$. This requires that $(2.68)$ be analytic$^{(4)}$, which follows from the analyticity of the initial wave surface and the speed of sound, $c_0 = c_0(x, y, z)$ (supposing that it is nowhere zero so that its reciprocal is also analytic). Taking $O$ as the origin of the ray coordinate system so that $(u, v, \tau) = \ldots$

---

$^{(3)}$[200] actually uses the “optical eikonal” $W = \int n ds$ described earlier, where $n = c_r/c_0$ is the index of refraction relative to an arbitrary reference speed $c_r$. This definition implies that $\tau = W/c_r$, so a simple change of variables allows us to instead use the acoustic eikonal $\tau$.

$^{(4)}$A real function is said to be analytic if it possesses derivatives of all orders and agrees with its Taylor series in a neighborhood of every point [193] (see also [160, 167]).
\((x, y, z) = (0, 0, 0)\) at \(O\), the resulting expansion is given by

\[
\begin{align*}
R(u, v, \tau) &= \left. R_u \right|_O u + \left. R_\tau \right|_O \tau \\
&+ \frac{1}{2} \left. R_{uu} \right|_O u^2 + \frac{1}{2} \left. R_{vv} \right|_O v^2 + \frac{1}{2} \left. R_{\tau\tau} \right|_O \tau^2 \\
&+ \left. R_{uv} \right|_O uv + \left. R_{v\tau} \right|_O v\tau + \left. R_{u\tau} \right|_O u\tau \\
&+ \frac{1}{6} \left. R_{vvv} \right|_O v^3 + \cdots .
\end{align*}
\tag{3.11}
\]

The \(\left. R \right|_O\) term has vanished because \(O\) is the origin of both the \((x, y, z)\) and \((u, v, \tau)\) coordinate systems. The \(\left. R_u \right|_O\) term does not appear due to our choice of \(v-\)lines on the caustic, \((3.3)\). We can then find equations for the components of the ray family along each coordinate direction by projecting \((3.11)\) onto the coordinate directions:

\[
\begin{align*}
x(u, v, \tau) &= R(u, v, \tau) \cdot \left( \frac{\left. R_\tau \right|_O}{\left. |R_\tau| \right|_O} \right), \\
y(u, v, \tau) &= R(u, v, \tau) \cdot \left( \frac{\left. R_u \right|_O}{\left. |R_u| \right|_O} \right), \\
z(u, v, \tau) &= R(u, v, \tau) \cdot \left. \vec{n} \right|_O.
\end{align*}
\tag{3.12}
\]

To simplify the notation a bit, we first note that from the first ray equation \((2.61)\), \(\left. |R_\tau| \right|_O = c_0(O)\), the speed of sound at point \(O\). We will typically just write \(c_0\) with the understanding that if it appears in a series expansion term, it should be evaluated at point \(O\) on the caustic; at times we will use \(c_0(O)\) for emphasis. We also let \(\gamma \equiv \left. |R_u| \right|_O\), and define

\[
\begin{align*}
\xi_0 &\equiv \frac{\left. R_\tau \right|_O}{\left. |R_\tau| \right|_O} = \frac{1}{c_0(O)} \left. R_\tau \right|_O, \\
\xi_1 &\equiv \frac{\left. R_u \right|_O}{\left. |R_u| \right|_O} = \frac{1}{\gamma} \left. R_u \right|_O, \\
\xi_2 &\equiv \left. \vec{n} \right|_O.
\end{align*}
\tag{3.13}
\]

Since our basis is orthonormal, we have

\[
\xi_i \cdot \xi_j = \begin{cases} 
0, & i \neq j, \\
1, & i = j.
\end{cases}
\tag{3.14}
\]
Then, substituting (3.11) into (3.12) we see that the expansion along any coordinate $x_i$ (where $(x_0, x_1, x_2) = (x, y, z)$) can be written as

$$x_i = \left. (R_u \cdot \xi_i) \right|_o u + \left. (R_\tau \cdot \xi_i) \right|_o \tau$$
$$+ a_i u^2 + b_i v^2 + c_i \tau^2$$
$$+ d_i uv + e_i v\tau + f_i \tau u$$
$$+ g_i v^3 + \cdots,$$

where the expansion coefficients are given by

$$a_i = \frac{1}{2} \left. (R_{uu} \cdot \xi_i) \right|_o, \quad b_i = \frac{1}{2} \left. (R_{vv} \cdot \xi_i) \right|_o, \quad c_i = \frac{1}{2} \left. (R_{\tau\tau} \cdot \xi_i) \right|_o$$
$$d_i = \left. (R_{uv} \cdot \xi_i) \right|_o, \quad e_i = \left. (R_{v\tau} \cdot \xi_i) \right|_o, \quad f_i = \left. (R_{\tau u} \cdot \xi_i) \right|_o,$$
$$g_i = \frac{1}{6} \left. (R_{vvv} \cdot \xi_i) \right|_o.$$

To simplify (3.15), we note that the wavefront vectors are perpendicular to $R_\tau$ everywhere in the ray family, so in general $R_u \cdot R_\tau = R_v \cdot R_\tau = 0$. Differentiating this relationship with respect to the ray coordinates and evaluating at $O$ (where $R_v = 0$), we find

$$\left. (R_{vv} \cdot R_\tau) \right|_o = \left. (R_{vu} \cdot R_\tau) \right|_o = \left. (R_{vv} \cdot R_\tau) \right|_o = 0, \quad \left. (R_{uv} \cdot R_\tau) \right|_o = 0, \quad \left. (R_{vvv} \cdot R_\tau) \right|_o + 2 \left. (R_{vv} \cdot R_{\tau v}) \right|_o = 0.$$

Thus, substituting (3.14) and (3.17), (3.15) simplifies to the system

$$\begin{cases}
x = c_0 \tau + a_1 u^2 + c_1 \tau^2 + f_1 \tau u + g_1 v^3 + \cdots, \\
y = \gamma u + a_1 u^2 + b_1 v^2 + c_1 \tau^2 + d_1 uv + f_1 \tau u + g_1 v^3 + \cdots, \\
z = a_2 u^2 + b_2 v^2 + c_2 \tau^2 + d_2 uv + e_2 v\tau + f_2 \tau u + g_2 v^3 + \cdots,
\end{cases}$$

where blank spaces represent terms equal to zero and the 0–subscripts indexing the $x$ terms have been suppressed.

We can find the ray(s) $(u, v)$ passing through point $(0, 0, z_c)$ and their respective
eikonals by solving (3.18) for \((0, 0, z_c)\):

\[
\begin{align*}
\begin{cases}
  x(u, v, \tau) = 0, \\
y(u, v, \tau) = 0, \\
z(u, v, \tau) = z_c.
\end{cases}
\end{align*}
\]  

(3.19)

To do so, we first note that from (3.18),

\[
\frac{\partial(x, y)}{\partial(u, \tau)} \bigg|_o = \det \begin{bmatrix}
\frac{\partial x}{\partial u} & \frac{\partial x}{\partial \tau} \\
\frac{\partial y}{\partial u} & \frac{\partial y}{\partial \tau}
\end{bmatrix} \bigg|_o = -\gamma c_0 \neq 0.
\]  

(3.20)

Therefore, by the inverse function theorem (see Appendix B), the first two equations of (3.18) are uniquely solvable with respect to \(u\) and \(\tau\). To this end, we expand each in power series in \(v\):

\[
u(v) = \sum_{i=1}^{\infty} m_i v^i, \quad \tau(v) = \sum_{i=1}^{\infty} n_i v^i,
\]  

(3.21)

where \(m_i, n_i\) are expansion coefficients to be determined. Inserting these expansions into the first two equations of (3.18) and equating the coefficient of each power of \(v\) to zero results in a recursive system of equations from which the expansion coefficients in (3.21) may be determined\(^{(5)}\). The first few systems are given by

\[
[v]: \begin{cases}
c_0 n_1 = 0, \\
\gamma m_1 = 0,
\end{cases}
\]  

\[\implies m_1 = n_1 = 0, \]  

(3.22)

\[
[v^2]: \begin{cases}
c_0 n_2 + a m_1^2 + c n_1^2 + f m_1 n_1 = 0, \\
\gamma m_2 + a_1 m_1^2 + c_1 n_1^2 + f_1 m_1 n_1 + b_1 + d_1 m_1 = 0,
\end{cases}
\]  

\[\implies m_2 = -\frac{b_1}{\gamma}, \quad n_2 = 0, \]  

(3.23)

\[
[v^3]: \begin{cases}
c_0 n_3 + 2a m_1 m_2 + 2c n_1 n_2 + f (m_2 n_1 + m_1 n_2) + g = 0, \\
\gamma m_3 + 2a_1 m_1 m_2 + 2c_1 n_1 n_2 + d_1 m_2 + f_1 (m_2 n_1 + m_1 n_2) + g_1 = 0,
\end{cases}
\]  

\[\implies m_3 = \frac{b_1 d_1 - \gamma g_1}{\gamma^2}, \quad n_3 = -\frac{1}{c_0} g. \]  

(3.24)

\(^{(5)}\)The equations that would have resulted for \(m_0\) and \(n_0\) (i.e., the constant terms) are satisfied by \(m_0 = 0, n_0 = 0\), so they have been set to zero from the start to simplify subsequent equations.
where the notation $[\cdot]$ indicates that the corresponding system comes about by equating the coefficient of the bracketed quantity to 0. This process can be continued indefinitely, but the computed coefficients are sufficient for the moment. Substituting the results of (3.22)–(3.24) into (3.21), we find that $u(v)$ and $\tau(v)$ are given to leading order by

$$ u(v) = -\frac{b_1}{\gamma} v^2 + O(v^3), \quad \tau(v) = -\frac{1}{c_0} g v^3 + O(v^4). \quad (3.25) $$

These expressions can then be inserted into the $z-$expansion of (3.19), resulting in an equation of the form

$$ z_c = b_2 v^2 + O(v^3). \quad (3.26) $$

This equation yields two solutions which may be expressed as series in powers of $\pm z_c^{\frac{1}{2}}$, one for each branch of the square root:

$$ v_1 = -\sqrt{\frac{z_c}{b_2}} + O(z_c), \quad v_2 = \sqrt{\frac{z_c}{b_2}} + O(z_c). \quad (3.27) $$

Substituting this result back into (3.25) yields the other ray coordinates associated with point $(0, 0, z_c)$:

$$ u_{1,2} = -\frac{b_1}{\gamma b_2} z_c + O(z_c^{3/2}), \quad \tau_{1,2} = \pm \frac{1}{c_0(\mathcal{O})} g \left( \frac{z_c}{b_2} \right)^{3/2} + O(z_c^2). \quad (3.28) $$

Equation (3.29) gives the desired expression for the eikonal at a point along the caustic normal. To proceed further, we must examine the expansion coefficients appearing in it.

Using (3.16), we see that $b_2$ and $g$ are given by

$$ b_2 = \frac{1}{2} (R_{vv} \cdot \widetilde{n}) \big|_o, \quad g = \frac{1}{6 c_0} (R_{vvv} \cdot R_\tau) \big|_o. \quad (3.30) $$

Applying the definition of $\widetilde{n}$ to the first expression yields

$$ b_2 = \frac{1}{2} (R_{vv} \cdot \widetilde{n}) \big|_o = \frac{1}{2} \left( \frac{R_{vv} \cdot (R_\tau \times R_u)}{|R_\tau||R_u|} \right) \big|_o = \frac{(R_u, R_{vv}, R_\tau) \big|_o}{2 \gamma c_0} = \frac{D_v \big|_o}{2 \gamma c_0}, \quad (3.31) $$

where the second to last equality follows from the cyclic shift property of scalar triple
products (2.77). The final equality in (3.17) implies that

\[ g = -\frac{1}{3c_0} (R_{vv} \cdot R_{r\tau}) \bigg|_O. \]  

(3.32)

This expression can be related to \( b_2 \) using \( e_2 \), which is given by

\[ e_2 = (R_{\tau \tau} \cdot \tilde{n}) \bigg|_O = \frac{(R_u, R_{vv}, R_{r\tau}) \bigg|_O}{\gamma c_0(O)} = \frac{D_r \bigg|_O}{\gamma c_0(O)}. \]  

(3.33)

In particular, we will use the following identity for a product of scalar triple products

\[ (a, b, c)(d, e, f) = \det \begin{bmatrix} a \cdot d & b \cdot d & c \cdot d \\ a \cdot e & b \cdot e & c \cdot e \\ a \cdot f & b \cdot f & c \cdot f \end{bmatrix}, \]  

which follows from the fact that the determinant of a matrix product is the product of their determinants. Applying this to \( b_2 \) and \( e_2 \), we find

\[ 2\gamma^2 c_0^2 b_2 e_2 = \left( (R_u, R_{vv}, R_{r\tau})(R_u, R_{vv}, R_{r\tau}) \right) \bigg|_O \]

\[ = \det \begin{bmatrix} R_u \cdot R_u & R_u \cdot R_{vv} & R_u \cdot R_{r\tau} \\ R_{vv} \cdot R_u & R_{vv} \cdot R_{vv} & R_{vv} \cdot R_{r\tau} \\ R_{r\tau} \cdot R_u & R_{r\tau} \cdot R_{vv} & R_{r\tau} \cdot R_{r\tau} \end{bmatrix} \bigg|_O = \gamma^2 c_0^2 (R_{vv} \cdot R_{r\tau}) \bigg|_O, \]  

(3.35)

where the cancellations follow from the orthogonality of the local caustic coordinates and the identities in (3.17). Comparing this to (3.32), we find that

\[ g = -\frac{2}{3c_0} b_2 e_2 = -\frac{(D_v D_{r\tau}) \bigg|_O}{3\gamma^2 c_0^2}. \]  

(3.36)

Substituting this expression and that found for \( b_2 \) into (3.29), the eikonal at any point \((0, 0, z)\) along the caustic normal at \( O \) is given (for \( z \) sufficiently small) by

\[ \tau_{1,2} = \pm \frac{2^{3/2}}{3c_0} \left( \frac{D_r^2}{\gamma c_0 D_v} \bigg|_O \right)^{1/2} z^{3/2}, \]  

(3.37)

where \( \sqrt{D_r^2} \bigg|_O = |D_r| \bigg|_O = -D_r \bigg|_O \) since \( D_r < 0 \) on the caustic. Interpreting this expression requires a closer look at the local ray-caustic geometry.
3.3.3 Local caustic geometry

To interpret (3.37), we must better understand the factor multiplying $z^{3/2}$, which may be seen as a measure of the degree to which a ray has propagated toward or away from the caustic by the time it has reached $(0, 0, z)$. Consider two examples of caustics. In a homogeneous atmosphere, the rays are straight and the caustic is curved. Therefore, the degree to which a ray departs from the caustic surface is solely determined by the caustic curvature. On the other hand, in horizontally stratified media, the rays are curved but the caustic is straight (as in Mach cutoff flight), so the degree of departure is determined by the ray curvature. Since (3.37) should apply to either case, the coefficient of $z^{3/2}$ must depend on the caustic curvature alone when the rays are straight and the ray curvature alone when the caustic is straight. This suggests the idea of a relative curvature between the ray and the caustic, which may be developed as follows.

With reference to Figure 3.5, let $\tilde{P}$ be the center of curvature of the normal section of the caustic formed by taking a planar slice perpendicular to its surface, directed along the v–line at $O$. Then line segment $O\tilde{P}$ lies along caustic normal $\vec{n}$ and has a length $|O\tilde{P}|$ equal to the radius of curvature of the normal section at $O$. Similarly, let $Q$ be the center of curvature of the ray touching the caustic at $O$, which by definition lies in

![Figure 3.5: Centers of curvature; adapted from [200]](image-url)
the ray’s osculating plane at $O$—the plane spanned by the ray’s tangent and (principal) normal vectors. Segment $OQ$ then has a length equal to the ray’s radius of curvature of the ray at $O$, and the ray normal vector $n$ satisfies $d^2R/ds^2|_O = n/OQ$ [33]. Finally, let $P$ be the point along the caustic normal whose projection onto the ray’s osculating plane is $Q$. The length of segment $OP$ is then equal to the projection of the radius of curvature of the ray at $O$ into the caustic normal plane. The relative radius of curvature $R$ between a ray and the caustic at $O$ is then defined as

$$\frac{1}{R(O)} = \frac{1}{OP} - \frac{1}{OP},$$

(3.38)

where the signs of $OP$ and $OP$ are determined relative to the chosen direction of $\hat{n}$ [200].

For calculations, it will be simpler to work in terms of curvatures, the reciprocals of the various radii of curvature (up to sign). In particular, let $\kappa$ be the ray curvature $1/OQ$, $\tilde{\kappa}_n = 1/O\hat{P}$ the normal curvature of the caustic in the direction of the $v$–line at $O$, and $\kappa_r = 1/R$ the relative curvature between the ray and the caustic at $O$. Then (3.38) can be written in the form

$$\kappa_r(O) = \kappa \cos \varphi - \tilde{\kappa}_n,$$

(3.39)

where $\cos \varphi = OQ/OP$ is the angle between $n$ and $\hat{n}$, or equivalently, the angle between the ray osculating plane and the caustic normal plane at $O$. Taking $\hat{n}$ directed into the insonified region, $\tilde{\kappa}_n$ is positive when the caustic normal section curves toward the rays (i.e. the rays and the caustic lie on the same side of the caustic tangent plane at $O$) and negative when it curves away (the two lie on opposite sides)\(^{(6)}\).

We first compute $\tilde{\kappa}_n$. Let $\tilde{s}$ be the arc length along the $v$–line at $O$. Then by definition [33]\(^{(7)}\),

$$\tilde{\kappa}_n = \frac{d^2\tilde{R}}{d\tilde{s}^2} \cdot \tilde{n}.$$  

(3.40)

\(^{(6)}\)This is opposite the convention used in [10, 11, 15], and is reflected in the alternate definition of the relative curvature given in these references, equivalent to $1/R = \kappa \cos \varphi + \tilde{\kappa}_n$.

\(^{(7)}\)By Meusnier’s theorem, $\tilde{\kappa}_n$ is the same for any planar section of the caustic surface containing $R_\tau$ at $O$. It is particularly convenient to compute $\tilde{\kappa}_n$ using the normal section because in this case, the geodesic component of the curvature vector vanishes and the curvature of the normal section is equivalent to $\tilde{\kappa}_n$ (hence the name “normal curvature”) [33].
In general, a small displacement along the caustic surface is given by\(^{(8)}\)

\[
d\tilde{R}(u, v) = dR(u, v, \tau(u, v)) = R_u du + R_v dv + R_\tau d\tau. \tag{3.41}
\]

But along a v-line \(R_u du + R_v dv = 0\), so this reduces to \(d\tilde{R} = R_\tau d\tau\). Similarly, it may be shown \([200]\) that

\[
d^2\tilde{R}(u, v) = d(d\tilde{R}) = R_\tau d\tau dv + R_\tau d^2\tau + R_{\tau\tau}(d\tau)^2 \tag{3.42}
\]

along the normal section. Using the fact that \(\left|\frac{d\tilde{R}}{d\tilde{s}}\right| = 1\) (as it is an arc-length parametrized tangent vector), we have \((d\tilde{s})^2 = d\tilde{R} \cdot d\tilde{R} = |R_\tau|^2(d\tau)^2\). Substituting this and (3.42) into (3.40) and simplifying, we find

\[
\vec{\kappa}_n = \frac{R_{\tau\tau} \cdot \hat{n}}{|R_\tau|^2} + \frac{R_{\tau\tau} \cdot \hat{n}}{|R_\tau|^2}. \tag{3.43}
\]

On the other hand, the ray curvature vector \(\kappa_n\) is given \([106, 200]\) by

\[
\kappa_n = \frac{d}{d\tilde{s}} \left( \frac{R_\tau}{|R_\tau|} \right) = \frac{1}{c_0} \frac{d}{d\tau} \left( \frac{R_\tau}{|R_\tau|} \right) = \frac{1}{c_0} \left( \frac{d}{d\tau} \left( \frac{1}{|R_\tau|} \right) R_\tau + \frac{1}{|R_\tau|} R_{\tau\tau} \right). \tag{3.44}
\]

Projecting this vector onto the caustic normal \(\hat{n}\) yields

\[
\kappa(n \cdot \hat{n}) = \kappa \cos \varphi = \frac{1}{c_0} \left( \frac{d}{d\tau} \left( \frac{1}{|R_\tau|} \right) (R_\tau \cdot \hat{n}) + \frac{1}{|R_\tau|} (R_{\tau\tau} \cdot \hat{n}) \right) = \frac{R_{\tau\tau} \cdot \hat{n}}{|R_\tau|^2}. \tag{3.45}
\]

The last equality follows since \(R_\tau\) is perpendicular to \(\hat{n}\) and \(c_0 = |R_\tau|\). Thus, substituting (3.43) and (3.45) into (3.39), we find that the relative curvature at \(O\) may be written

\[
\kappa_r(O) = \frac{1}{R(O)} = \kappa \cos \varphi - \vec{\kappa}_n = -\frac{(R_{\tau\tau} \cdot \hat{n})}{c_0^2(O) \tau_v}. \tag{3.46}
\]

But the numerator of the right-hand side is the Taylor coefficient \(e_2\), so we substitute (3.33) resulting in

\[
\frac{1}{R(O)} = -\frac{D_\tau}{\gamma c_0^3(O) \tau_v}. \tag{3.47}
\]

Finally, since \(D_\tau \neq 0\) on the caustic, the second part of the implicit function theorem

\(^{(8)}\)We could also view the normal section as a parametrized curve with parametric equation \(r = \tilde{R}(u(\tilde{s}), v(\tilde{s}))\) and apply the chain rule, but the differential approach is far less painful.
(see Appendix B) implies that
\[ \tau_v = -Dv/D\tau, \quad (3.48) \]
so (3.47) becomes\(^{(9)}\)
\[ \frac{1}{R(O)} = \frac{D\tau}{\gamma c_0^2 Dv} \bigg|_{O}. \quad (3.49) \]
Comparing (3.49) to (3.37), we finally find that the eikonal along a normal to the caustic is given by\(^{(10)}\)
\[ \tau_{1,2}(0, 0, z) = \pm \frac{1}{c_0(O)} \sqrt{\frac{8}{9R(O)} z^{3/2}}. \quad (3.50) \]

In light of this expression, we see that two rays pass through each point near the caustic, one for each sign of the eikonal [200]. The positive root indicates an outgoing ray that has already touched the caustic, while the negative corresponds to an incoming ray that has not yet reached the caustic, as can be seen in Figure 3.4. Moreover, using the definition of the relative curvature (3.39), one may show that (3.50) reduces to a function of the ray curvature alone when the caustic is straight and the caustic curvature alone when the rays are straight as expected.

Equation (3.50) can also be used to estimate the size of the caustic boundary layer. Since the acoustic eikonal is equal to the travel time along a ray, the difference \( \Delta t = \tau_1 - \tau_2 \) gives the time delay between the (presently unknown) signal propagating along the ray that has already grazed the caustic and that propagating along the ray which has not yet reached the caustic at \((0, 0, z)\). When \( z \) is large, so is \( \Delta t \), and the incoming and outgoing waves are easily distinguished. But as \( z \) decreases, the incoming and outgoing signals arrive closer and closer in time and eventually interfere. Some distance from the caustic, the two signals will overlap so significantly that it becomes impossible to distinguish the signals “belonging” to the incoming ray and outgoing rays (see Figure 3.6). As a result, in some neighborhood of the caustic surface, the notion of waves “propagating along the rays” loses its physical meaning [200]\(^{(11)}\). The increasing interference between the two waves as the distance from the caustic decreases is what leads to the rapid

\(^{(9)}\)Since \( D\tau, Dv \neq 0 \) on a fold caustic, (3.49) implies that \( 1/R \) is always nonzero. Therefore, the relative curvature cannot change sign, demonstrating that rays can only lie on one side of a fold caustic [200]. \( D\tau \) does however vanish for higher-order caustics including the cusp, indicating that the ray and caustic are not only tangent, but have equal curvatures (or even higher-order derivatives), and that rays are not restricted to one side of the caustic (see e.g. [107, 160]).

\(^{(10)}\)Other derivations of (3.50) have been given in [10, 15, 16, 30, 107, 151].

\(^{(11)}\)Even so, it can be mathematically convenient to decompose the acoustic field in the boundary layer as if it were the sum of waves propagating along incoming and outgoing rays. For instance, this approach is used to great effect in [107, 200, 202].
pressure fluctuations that render the geometrical acoustic solution invalid. Hence, the boundary layer may also be seen as the region in which the waves propagating along the incoming and outgoing rays are indistinguishable, and the boundary layer thickness as the distance from the caustic at which distinguishing the waves first becomes impossible. Following [107], we take this to be the distance at which their phases differ by an amount on the order of half the wavelength of the incident signal, $\lambda/2$, so that their arrival times differ by half the wave’s characteristic period $T$. Then (3.50) gives, up to a constant, the caustic boundary layer thickness $\delta$,

$$\tau_1 - \tau_2 = \frac{2}{c_0(O)} \sqrt{\frac{8}{9R(O)}} \delta^{3/2} \approx \frac{T}{2} \implies \delta(\omega) \approx \left( \frac{R(O) c_0^2(O)}{2 \omega^2} \right)^{1/3}. \quad (3.51)$$

A more accurate estimate of $\delta$ is also discussed in [107], amounting to a slightly different multiplicative constant in (3.51). We are mainly interested in the order of magnitude of the boundary layer thickness, and so do not need to be so precise$^{(12)}$.

Figure 3.6: Interference and indistinguishability of incoming and outgoing waves near a caustic

3.3.4 Eikonal at an arbitrary point near the caustic

Equation (3.50) only gives the eikonals of the rays passing through points $(0, 0, z)$ along the caustic normal at $O$. We would also like to be able to determine the eikonals of rays passing through points some distance away from this normal (i.e., with nonzero $x$)

$^{(12)}$The longitudinal scale of the boundary layer along a particular ray has also been estimated [8, 10].
and $y$ values) as in Figure 3.7\(^{(13)}\). To this end, we follow Auger [10] and Taylor expand

\[
\tau(x, y, z) \approx \tau(0, 0, z) + x \cdot \frac{\partial \tau}{\partial x} \bigg|_{(0,0,z)} + y \cdot \frac{\partial \tau}{\partial y} \bigg|_{(0,0,z)} + \mathcal{O}(x^2, y^2, z^2),
\]

then further approximate $\partial \tau / \partial x$ and $\partial \tau / \partial y$ at $(0, 0, z)$ by expanding about $O$,

\[
\left. \frac{\partial \tau}{\partial x} \right|_{(0,0,z)} \approx \left. \frac{\partial \tau}{\partial x} \right|_O + z \cdot \left. \frac{\partial^2 \tau}{\partial z \partial x} \right|_O + \mathcal{O}(z^2),
\]

\[
\left. \frac{\partial \tau}{\partial y} \right|_{(0,0,z)} \approx \left. \frac{\partial \tau}{\partial y} \right|_O + z \cdot \left. \frac{\partial^2 \tau}{\partial z \partial y} \right|_O + \mathcal{O}(z^2).
\]

Since the ray tangent to the caustic at $O$ is directed along $Ox$, we have $\partial \tau / \partial y = \partial \tau / \partial z = 0$ at $O$. Hence, from the eikonal equation (2.44), $\partial \tau / \partial x = 1 / c_0$ at $O$. Applying these observations to (3.53), then substituting the result and (3.50) back into (3.52), we have

\[
\tau_{1,2}(x, y, z) \approx \pm \frac{1}{c_0(O)} \sqrt{\frac{8}{9R(O)}} z^{3/2} + x \cdot \left( \frac{1}{c_0(O)} + z \cdot \left. \frac{\partial^2 \tau}{\partial z \partial x} \right|_O \right) + y z \cdot \left. \frac{\partial^2 \tau}{\partial z \partial y} \right|_O + \mathcal{O}(x^2, y^2, z^2).
\]

Note that the ray defining the local coordinate system (i.e., the ray touching the caustic at $O$) is not pictured to avoid crowding the figure.
To complete our description of the eikonal near the caustic, we will need to evaluate the partial derivatives in this expression. This may be done with the method used to find $\tau(0, 0, z)$ in Section 3.3.2, but [10] provides a more straightforward approach. To this end, we define the quantities $R_{xz}$, $R_{yz}$, and $R_{zz}$ by

$$\begin{align*}
\frac{1}{R_{xz}} &\equiv c_{0}(O) \left. \frac{\partial^{2} \tau}{\partial z \partial x} \right|_{O}, \\
\frac{1}{R_{yz}} &\equiv c_{0}(O) \left. \frac{\partial^{2} \tau}{\partial z \partial y} \right|_{O}, \\
\frac{1}{R_{zz}} &\equiv c_{0}(O) \left. \frac{\partial^{2} \tau}{\partial z^{2}} \right|_{O}.
\end{align*}$$

(3.55)

In local caustic coordinates, the gradient of (3.54) at $(0, 0, z)$ is then given by

$$\nabla \tau(0, 0, z) = \frac{1}{c_{0}(O)} \left( \frac{1 + \frac{z}{R_{xz}}}{\frac{z}{R_{yz}} + \frac{z}{R_{zz}}} \pm \sqrt{\frac{2z}{R(O)} + \frac{2z}{R_{zz}}} \right) + \mathcal{O}(z^{2}),$$

(3.56)

and hence, by Taylor expansion,

$$|\nabla \tau(0, 0, z)| = \frac{1}{c_{0}(O)} \left( 1 - \frac{z}{R_{c_{0}}} + \mathcal{O}(z^{2}) \right) + \mathcal{O}(z^{2}).$$

(3.57)

On the other hand, from the eikonal equation (2.44)

$$|\nabla \tau(0, 0, z)| = \frac{1}{c_{0}(0, 0, z)}.$$  

(3.58)

Approximating the sound speed by its value at $O$,

$$c_{0}(0, 0, z) = c_{0}(O) + \left. \frac{dc_{0}}{dz} \right|_{O} z + \mathcal{O}(z^{2}),$$

(3.59)

we then have

$$|\nabla \tau(0, 0, z)| = \frac{1}{c_{0}(O)} \left( 1 - \frac{z}{R_{c_{0}}} + \mathcal{O}(z^{2}) \right),$$

(3.60)

where $R_{c_{0}}$ is defined by

$$R_{c_{0}}(O) \equiv \left( \frac{1}{c_{0}(O)} \left. \frac{dc}{dz} \right|_{O} \right)^{-1}.$$  

(3.61)

For this expansion to agree with (3.57), we must have

$$\frac{1}{R_{zz}} = 0 \text{ and } \frac{1}{R_{xz}} = -\frac{1}{R(O)} - \frac{1}{R_{c_{0}}(O)}.$$  

(3.62)
In fact, $1/R_{xz}$ is the normal curvature $\bar{\kappa}_n$. To see this, first note that the ray curvature vector (3.44) satisfies

$$\kappa \mathbf{n} = \frac{1}{c_0} \frac{d}{d\tau} \left( \frac{R_{\tau}}{c_0} \right) = \frac{1}{c_0} \left( \frac{dc_0}{d\tau} \nabla \tau + c_0 \frac{d}{d\tau} (\nabla \tau) \right)$$

$$= \frac{1}{c_0} \left( \frac{dc_0}{ds} \frac{R_{\tau}}{c_0} - \nabla c_0 \right) = -\frac{1}{c_0} \nabla_\perp c_0,$$

where repeated use of the ray equations (2.61)–(2.62) has been made [106, 151]. $\nabla_\perp c_0$ represents the sound speed gradient perpendicular to the ray direction. Therefore, in the local caustic coordinate system at $O$ where $\nabla_\perp c_0 = (0, \partial c_0/\partial y, \partial c_0/\partial z)$, we have

$$\kappa \cos \varphi = \kappa (\mathbf{n} \cdot \tilde{\mathbf{n}}) = -\frac{1}{c_0} (\nabla_\perp c_0 \cdot \tilde{\mathbf{n}}) = -\frac{1}{c_0} \frac{\partial c_0}{\partial z},$$

or

$$\kappa \cos \varphi = -\frac{1}{R_{c_0}},$$

so that

$$\frac{1}{R_{xz}} = -\frac{1}{R(O)} - \frac{1}{R_{c_0}} = \kappa \cos \varphi - \frac{1}{R(O)} = \bar{\kappa}_n,$$

where the last equality follows from the definition of $R$, (3.46). $R_{c_0}$ and $R_{xz}$ are depicted in Figure 3.8.

Figure 3.8: Interpretation of $R_{c_0}$ and $R_{xz}$; adapted from [169]

Notably absent from (3.62) is $R_{yz}$, which cannot be determined at the order of approximation made. We will see later that the field variations in the $x$ direction dominate the variations in the $z$ direction, which in turn are far larger than in the $y$ direction. As
a result, it is admissible to neglect the $y$ terms in the eikonal expansion, at least in a first approximation. Therefore, eikonal (3.54) becomes

$$\tau_{1,2}(x, y, z) = \pm \frac{1}{c_0(O)} \sqrt{\frac{8}{9R(O)}} z^{3/2} + \frac{x}{c_0(O)} \left(1 + \frac{z}{R_{xx}(O)}\right) + O(x^2, y^2, z^2), \quad (3.67)$$

where $R_{xx}$ satisfies (3.62). Equation (3.67) is the desired eikonal for a point $(x, y, z)$ near the caustic surface, and is again seen to have two roots, one for each ray passing through $(x, y, z)$.

As a final note, recall that (3.67) assumes the ray coordinate origin coincides with the local caustic coordinate origin $O$. In practice, it is usually more convenient to define the ray coordinates with respect to some reference wavefront (Figure 2.3) or the source (Figure 2.4). This is a straightforward modification, amounting to an additive constant in (3.67)

$$\tau_{1,2}(x, y, z) = \tau(O) \pm \frac{1}{c_0(O)} \sqrt{\frac{8}{9R(O)}} z^{3/2} + \frac{x}{c_0(O)} \left(1 + \frac{z}{R_{xx}(O)}\right) + O(x^2, y^2, z^2). \quad (3.68)$$

Here, $\tau(O)$ is the value of the eikonal of the ray touching $O$ in the new ray coordinate system, computed according to (2.59).

### 3.3.5 Geometrical acoustic solution near the caustic

We now examine the asymptotic behavior of the ray divergence approaching the caustic. As in [200], we expand $\mathcal{J}$ in ray coordinates about point $O$:

$$\mathcal{J}(u, v, \tau) \approx \mathcal{J}(O) + \frac{\partial \mathcal{J}}{\partial u} \bigg|_O u + \frac{\partial \mathcal{J}}{\partial v} \bigg|_O v + \frac{\partial \mathcal{J}}{\partial \tau} \bigg|_O \tau. \quad (3.69)$$

The first term vanishes because the ray tube area goes to zero on the caustic. The relative sizes of the remaining terms can be compared by recalling (3.27)–(3.29) which indicate that $u \sim z$, $v \sim z^{1/2}$, and $\tau \sim z^{3/2}$ near the caustic. Therefore, for $z$ small, we
expect the \(v\)-term in (3.69) to dominate, resulting in

\[
\mathcal{J}_{1,2}(u, v, \tau) \approx \frac{\partial \mathcal{J}}{\partial v} \bigg|_O v = \mp \frac{\rho_0(\tau_0)c_0^2(\tau_0)}{D(u, v, \tau_0)} \frac{\partial}{\partial v} \left( \frac{D(u, v, \tau)}{\rho_0 c_0^2} \right) \bigg|_O \left( \frac{2\gamma c_0}{\frac{D(u, v, \tau)}{\rho_0}} \right)^{1/2} z^{1/2} + \mathcal{O}(z)
\]

\[
= \frac{\rho_0(\tau_0)c_0^2(\tau_0) \gamma |\tau_v(O)|}{D(u, v, \tau_0)} \sqrt{\frac{2z}{R(O)}} + \mathcal{O}(z),
\]

(3.70)

where we have used (2.67), (3.48), (3.27), (3.31), (3.49), and that \(D = 0\) on the caustic. Equation (3.70) indicates that to leading order, the ray divergence for either ray passing through a point near the caustic is the same. The scaling factor multiplying the square root term,

\[
\mathcal{J} = \frac{\rho_0(\tau_0)c_0(\tau_0)}{\rho_0(O)c_0(O)} \tilde{\chi},
\]

(3.71)

is completely specified by the source and atmospheric properties, and as shown in [200], can be interpreted as a “caustic divergence,” analogous to the ray divergence \(\mathcal{J}\). In particular, we have

\[
\tilde{\mathcal{J}} = \frac{\rho_0(\tau_0)c_0(\tau_0)}{\rho_0(O)c_0(O)} \tilde{\chi},
\]

(3.72)

where

\[
\tilde{\chi} = \frac{\delta \tilde{S}}{\delta S(\tau_0)}
\]

(3.73)

is the ratio of the area element of the caustic at \(O\),

\[
\delta \tilde{S} = \left| \tilde{R}_u \times \tilde{R}_v \right|_O = (|R_u||R_\tau||\tau_v|)|_O = \gamma c_0(O)|\tau_v(O)|,
\]

(3.74)

to the ray tube area \(\delta S\) at the reference point \(\tau = \tau_0\). Note that unlike the ray tube area and ray divergence, \(\delta \tilde{S}\) and \(\tilde{\mathcal{J}}\) remain nonzero on the caustic.

In summary, for an incident signal \(F(t)\), the asymptotic behavior of the geometrical acoustic solution (2.93) near a smooth/fold caustic is given by

\[
p(x, y, z, t) = \frac{P_0(u, v)}{\sqrt{\mathcal{J}(\tau)}} F(t - \tau) \approx \frac{P_0(x, y, z)}{\sqrt{\mathcal{J}(x, y, z)}} \left( \frac{R(O)}{2z} \right)^{1/4}
\]

\[
\times F \left( t - \left[ \tau(O) - \frac{1}{c_0(O)} \sqrt{\frac{8}{9R(O)}} z^{3/2} + \frac{x}{c_0(O)} \left( 1 + \frac{z}{R_{xz}(O)} \right) \right] \right).
\]

(3.75)

This expression is still singular on the caustic, but it makes the nature of the singularity explicit. The geometrical acoustic field near a smooth caustic diverges as \(z^{-1/4}\), where
is the normal distance from the caustic. Since two rays pass through each point near the caustic, the general solution for the field above the caustic is the superposition of (3.75) and the pressure along the outgoing ray, corresponding to a signal that has already tangented the caustic. In particular, let $G(t)$ be the (a priori unknown) outgoing waveform. Then, taking a point some distance $z = D$ from the caustic as the reference at which $P = P_0$, the total geometrical acoustic field near the caustic may be written as

$$p(x, y, z, t) \sim P_0 \left( \frac{D}{z} \right)^{1/4} \left\{ F \left( t - \left[ \tau(O) - \frac{1}{c_0(O)} \sqrt{\frac{8}{9R(O)}} \frac{z^{3/2}}{z} + \frac{x}{c_0(O)} \left( 1 + \frac{z}{R_{xx}(O)} \right) \right] \right) + G \left( t - \left[ \tau(O) + \frac{1}{c_0(O)} \sqrt{\frac{8}{9R(O)}} \frac{z^{3/2}}{z} + \frac{x}{c_0(O)} \left( 1 + \frac{z}{R_{xx}(O)} \right) \right] \right) \right\}.$$  

(3.76)

Note the (-) in the phase of the incoming signal and the (+) in the outgoing signal, as determined by (3.68).

As a reminder, (3.76) does not hold in the boundary layer because the amplitude fluctuates too rapidly and the signals propagating along incoming and outgoing rays are no longer distinguishable. Instead, use of (3.76) requires that $D = O(\delta(\omega_{ac}))$, where $\delta(\omega_{ac})$ is the boundary layer thickness at the characteristic acoustic frequency $\omega_{ac}$, the center frequency of the incident signal [10]. That is to say, this expression is only valid sufficiently near the caustic but still outside of the boundary layer, $z = \delta$.

This completes the description of the inner limit of the outer expansion, summarized in Figure 3.9. Since $G$ results from propagation through the caustic boundary layer, it cannot be determined using geometrical acoustics. Rather, we must push ahead and find the inner expansion governing the acoustic field within the boundary layer.

### 3.4 Inner solution and asymptotic matching

To find the inner expansion, we must return to the linear acoustic equations, recast in terms of a set of dimensionless inner variables appropriate to the description of the boundary layer. All physical variables will also need to be normalized to ensure that the

\(^{(14)}\)As can be seen in Figure 3.9, a wave grazing a caustic is not unlike reflection at an interface, and in particular, total internal reflection. The signal below the caustic is evanescent, while the signal leaving the boundary layer on the insonified side is akin to a reflected wave. Moreover, the incoming and outgoing portions of the rays satisfy the law of reflection at the caustic for linear waves while the law holds approximately for weakly nonlinear waves and shocks [134]. For this reason, the wave leaving the caustic boundary layer is sometimes referred to as a reflected wave [107].
relative sizes of different terms can be compared in a consistent, unit-independent way.

### 3.4.1 The linear Tricomi equation

We introduce the following dimensionless quantities for the inner variables [10]:

$$
\bar{z} \equiv \frac{z}{\delta(\omega_{ac})} = \left( \frac{2 \omega_{ac}^2}{R(O) c_0^2(O)} \right)^{1/3} z,
$$

(3.77)

the distance along the caustic normal at $O$, normalized by a characteristic boundary layer thickness $\delta(\omega_{ac})$ defined by (3.51) and

$$
\bar{t} \equiv \omega_{ac} \left[ t - \tau(O) - \frac{x}{c_0(O)} \left( 1 + \frac{z}{R_{xx}} \right) \right],
$$

(3.78)

a retarded time following the wavefront propagating along the ray tangent to the caustic at $O$, normalized by the characteristic duration of the incident signal $\omega_{ac}^{-1}$. In order to be able to match the inner solution to the outer (geometrical acoustic) solution (3.75), we see that it must depend on at least these two variables. In fact, due to a deep relationship between the caustics of geometrical acoustics and the *structurally stable catastrophes* of catastrophe theory, it may be shown that we only need two independent variables to describe the field near the caustic.

Pioneered by Thom, Arnold, Zeeman, and Berry [6, 7, 20, 21, 187, 199], catastrophe theory is loosely a study of how smooth changes in the input(s) to a system, known as *control parameters* can lead to sudden changes in the behavior, or *state* of the system
In our case $z$ is a control parameter, because varying it corresponds to passage through the caustic which clearly constitutes a dramatic change in the field behavior. Variable $\bar{t}$, on the other hand, is a *state variable*, telling us how the state of the system evolves for a fixed value of the control parameter(s).

Catastrophe theory tells us that through an appropriate choice of variables, only one control parameter and one state variable are needed to describe the evolution of the field sufficiently near a fold caustic [160]. Any other quantities that may enter into the problem are secondary, and may be (locally) “scaled out” of the problem through a smooth change of coordinates. Following this recommendation, we suppose that the acoustic field variables can be expressed in terms of only $z$ and $\bar{t}$ [10], and take

$$p(x, y, z, t) = P_{c} \bar{p}(z, \bar{t}),$$

$$u(x, y, z, t) = \begin{bmatrix} U_{cx} \bar{u}_{x}(z, \bar{t}) \\ U_{cy} \bar{u}_{y}(z, \bar{t}) \\ U_{cz} \bar{u}_{z}(z, \bar{t}) \end{bmatrix},$$

thereby reducing the number of independent variables from four to two. $P_{c}$, $U_{cx}$, $U_{cy}$, and $U_{cz}$ are the dimensional gauges for the dimensionless functions $\bar{p}$, $\bar{u}_{x}$, $\bar{u}_{y}$, and $\bar{u}_{z}$, representing the pressure and the projection of the particle velocity onto the local caustic coordinate axes $Ox$, $Oy$, and $Oz$ respectively. $P_{c}$ is taken to be $P_{c} \equiv P_{0} = \rho_{0}c_{0}^{2}M_{ac}$, where $M_{ac}$ is the acoustic Mach number at $z = D$, the normal distance to some reference point outside of the boundary layer. We also introduce the dimensionless *diffraction parameter* $\varepsilon$, defined by

$$\varepsilon \equiv \frac{\lambda_{ac}}{\delta} = \left( \frac{2}{R(O)} \frac{c_{0}(O)}{\omega_{ac}} \right)^{1/3} = \left( \frac{2}{R(O)} \lambda_{ac} \right)^{1/3},$$

where $\lambda_{ac}$ is the characteristic acoustic wavelength $c_{0}/\omega_{ac}$. By definition, $\varepsilon$ compares the scale of an acoustic wave to the size of the boundary layer. For a typical sonic boom, $\lambda_{ac}$ is on the order of 100 meters, while $R(O)$ can be 100 kilometers or more [10]. Hence, $\varepsilon$ is typically on the order of 0.1. We will therefore take $\varepsilon \ll 1$ as another ordering parameter (in addition to the acoustic Mach number $M_{ac}$) determining the strength of diffraction-related effects. It will turn out that within the boundary layer, the lowest-order (non-trivial) approximation of the acoustic equations results from retaining terms out to a relative order $\varepsilon^{2}$, so any terms smaller than this will be neglected along the way. The derivation will follow reference [10]^{15}.

\(^{15}\)In this reference, Auger cites an internal report by Coulouvrat [48] for the derivation of the inner asymptotic behavior of geometrical acoustics as well as the expansion procedure used to arrive at the
We begin with the linear acoustic equations in an inhomogeneous medium, (2.16) and (2.22), restated here for convenience:

\[
\frac{\partial u}{\partial t} + \frac{1}{\rho_0} \nabla p = 0, \\
\frac{\partial p}{\partial t} + \rho_0 c_0^2 \nabla \cdot u = 0.
\]  

(3.81)

The transformation from \((x, y, z, t)\) to \((\tilde{z}, \tilde{t})\) variables given by (3.77) and (3.78) implies the relations

\[
\frac{\partial}{\partial x} = -\frac{\omega_{ac}}{c_0(O)} \left(1 + \varepsilon^2 \frac{R(O)}{2R_{xz}} \right) \frac{\partial}{\partial \tilde{t}} + \mathcal{O}(\varepsilon^3), \\
\frac{\partial}{\partial y} = \mathcal{O}(\varepsilon^3), \\
\frac{\partial}{\partial z} = \varepsilon \frac{\omega_{ac}}{c_0(O)} \frac{\partial}{\partial \tilde{z}} + \mathcal{O}(\varepsilon^3), \\
\frac{\partial}{\partial \tilde{t}} = \omega_{ac} \frac{\partial}{\partial \tilde{t}}.
\]  

(3.82)

We will also approximate the ambient density and sound speed about point \(O\). For any function \(f\), we have

\[
f(x, y, z) \approx f(O) + x \left. \frac{\partial f}{\partial x} \right|_O + y \left. \frac{\partial f}{\partial y} \right|_O + z \left. \frac{\partial f}{\partial z} \right|_O \\
= f(O) \left(1 + \frac{z}{f(O)} \left. \frac{\partial f}{\partial z} \right|_O \right) + x \left. \frac{\partial f}{\partial x} \right|_O + y \left. \frac{\partial f}{\partial y} \right|_O \\
= f(O) \left(1 + \frac{z}{R_f} \right) + \mathcal{O}(\varepsilon^3) \\
= f(O) \left(1 + \varepsilon^2 \frac{R(O)}{2R_f} \tilde{z} \right) + \mathcal{O}(\varepsilon^3),
\]  

(3.83)

where

\[
R_f \equiv \left( \frac{1}{f(O)} \frac{df}{dz} \right)^{-1},
\]  

(3.84)

and \(\delta = \varepsilon^2 R/2\) has been used. Therefore, we approximate the sound speed and ambient density by

\[
c_0(x, y, z) = c_0(O) \left(1 + \varepsilon^2 \frac{R(O)}{2R_{c_0}} \tilde{z} \right) + \mathcal{O}(\varepsilon^3),
\]  

(3.85)

linear and nonlinear Tricomi equations. We do not have access to Coulouvrat’s report, but based on Auger’s comments and our own reading of Guiraud’s work [79], it seems that Coulouvrat significantly streamlined Guiraud’s original derivations.
\[ \rho_0(x, y, z) = \rho_0(O) \left( 1 + \varepsilon^2 \frac{R(O)}{2R_{p_0}} \right) + \mathcal{O}(\varepsilon^3), \quad (3.86) \]

where
\[ R_{c_0} = \left( \frac{1}{c_0(O)} \frac{dc_0}{dz} \right)_O^{-1}, \quad R_{p_0} = \left( \frac{1}{\rho_0(O)} \frac{d\rho_0}{dz} \right)_O^{-1}. \quad (3.87) \]

To simplify notation, we also define the quantities
\[ a = \frac{R(O)}{2R_{xz}}, \quad b = \frac{R(O)}{2R_{p_0}}, \quad e = \frac{R(O)}{2R_{c_0}}. \quad (3.88) \]

Projecting the momentum equation (2.16) onto the \(x, y,\) and \(z\) axes of the local caustic coordinate system, applying the change of variables formulas (3.82), and inserting expansions (3.85)–(3.87) yields
\[ U_{cx} \frac{\partial \pi_x}{\partial t} - \frac{P_c}{\rho_0(O)c_0(O)} \left[ 1 + \varepsilon^2(a - b)\bar{z} \right] \frac{\partial \bar{p}}{\partial t} = \mathcal{O}(\varepsilon^3 M_{ac}), \quad (3.89) \]
\[ U_{cy} \frac{\partial \pi_y}{\partial t} = \mathcal{O}(\varepsilon^3 M_{ac}), \quad (3.90) \]
\[ U_{cz} \frac{\partial \pi_z}{\partial t} + \varepsilon \frac{P_c}{\rho_0(O)c_0(O)} \frac{\partial \bar{p}}{\partial \bar{z}} = \mathcal{O}(\varepsilon^3 M_{ac}), \quad (3.91) \]
implying the following relationships between the dimensional gauges and field variables, consistent to order \(\varepsilon^2:\)
\[ U_{cx} = \frac{P_c}{\rho_0 c_0}, \quad \pi_x = \bar{p}, \]
\[ U_{cy} = \varepsilon^3 \frac{P_c}{\rho_0 c_0}, \]
\[ U_{cz} = \varepsilon \frac{P_c}{\rho_0 c_0}, \quad \frac{\partial \bar{u}_z}{\partial t} = -\frac{\partial \bar{p}}{\partial \bar{z}}. \quad (3.92) \]

Therefore, the acoustic particle velocity components form a hierarchy such that
\[ U_{cx} = \mathcal{O}(M_{ac}) \gg U_{cx} = \mathcal{O}(\varepsilon M_{ac}) \gg U_{cy} = \mathcal{O}(\varepsilon^3 M_{ac}), \quad (3.93) \]
regardless of the relative sizes of \(\varepsilon\) and \(M_{ac}\). Hence to the order retained, the particle velocity component in the \(y\)-direction may be neglected. Applying (3.92) to the remaining components of the momentum equation (3.89) results in
\[ \frac{\partial \pi_x}{\partial t} - \left[ 1 + \varepsilon^2(a - b)\bar{z} \right] \frac{\partial \bar{p}}{\partial t} = 0, \quad (3.94) \]
\[ \frac{\partial u_z}{\partial t} + \frac{\partial p}{\partial z} = 0. \]  
\hfill (3.95)

Similarly, in \((\xi, t)\) variables, the (dimensionless) mass conservation equation is given by

\[ \frac{\partial p}{\partial t} - \left[ 1 + \varepsilon^2 (b + a + 2e)\xi \right] \frac{\partial u_x}{\partial t} + \varepsilon^2 \frac{\partial p}{\partial z} = 0. \]  
\hfill (3.96)

To arrive at a single equation for the pressure, we must eliminate the particle velocity components between the previous three equations. Differentiating (3.96) with respect to \(t\) and interchanging the order of the derivatives (they commute at order \(\varepsilon^2\)), we have

\[ \frac{\partial^2 p}{\partial t^2} - \left[ 1 + \varepsilon^2 (b + a + 2e)\xi \right] \frac{\partial^2 u_x}{\partial t^2} + \varepsilon^2 \frac{\partial^2 p}{\partial z^2} = 0. \]  
\hfill (3.97)

Substituting in the momentum equations (3.94) then yields

\[ \frac{\partial^2 p}{\partial t^2} - \left[ 1 + 2\varepsilon^2 (a + e)\xi \right] \frac{\partial^2 p}{\partial t^2} - \varepsilon^2 \frac{\partial^2 p}{\partial z^2} = 0. \]  
\hfill (3.98)

Lastly, in Section 3.3.4 we found that \(R_{xx}^{-1} = -R^{-1} - R_{e_0}^{-1}\), implying that \(a + e = -1/2\). Simplifying, we arrive at the equation governing the acoustic pressure inside the caustic boundary layer:

\[ \frac{\partial^2 p}{\partial z^2} - \varepsilon^2 \frac{\partial^2 p}{\partial \xi^2} = 0. \]  
\hfill (3.99)

This equation is known as the \textit{linear Tricomi equation}, or simply the \textit{Tricomi equation}\footnote{Named after Italian mathematician Francesco Tricomi, due to his extensive study of the equation and other equations of mixed type in [67, 188].} [10, 47]. Because the \(y\)—component of the particle velocity was neglected, the Tricomi equation is an effectively planar model, only valid sufficiently near the \(x - z\) plane of the local caustic coordinate system. That is, near the normal section of the caustic passing through \(O\). The equation is hyperbolic for \(\xi > 0\) (the insonified region), elliptic for \(\xi < 0\) (the shadow zone), and parabolic on the surface \(\xi = 0\) separating the two regions (the caustic). Hence, (3.99) is said to be of \textit{mixed} type.

### 3.4.2 Boundary conditions

A complete boundary value problem for (3.99) requires an appropriate set of boundary conditions. The equation is second-order in both \(\xi\) and \(\xi\), and so requires two conditions...
on each independent variable. For \( \bar{t} \), we require that the field vanish at large times:

\[
\bar{p}(\bar{z}, \bar{t} \to \pm \infty) = 0. \tag{3.100}
\]

The field should also vanish deep into the shadow zone, such that

\[
\bar{p}(\bar{z} \to -\infty, \bar{t}) = 0. \tag{3.101}
\]

Lastly, we require that the inner solution agree with the geometrical acoustic solution (3.76) at the edge of the boundary layer, i.e. for large \( \bar{z} \). Recasting (3.76) in the inner variables, this matching condition may be written

\[
\bar{p}(\bar{z} \to +\infty, \bar{t}) = \frac{1}{z^{1/4}} \left[ F \left( \bar{t} + \frac{2}{3} \bar{z}^{3/2} \right) + G \left( \bar{t} - \frac{2}{3} \bar{z}^{3/2} \right) \right], \tag{3.102}
\]

where

\[
\omega_{\text{ac}}(t - \tau) = \bar{t} \pm \frac{2}{3} \bar{z}^{3/2}, \tag{3.103}
\]

is the geometrical acoustic phase in terms of the inner variables. Functions \( F \) and \( G \) arise from expressing the geometrical acoustic amplitude in the inner variables and are related to \( F \) and \( G \) by

\[
\begin{bmatrix}
F \\
G
\end{bmatrix} = \left( \frac{D}{\delta} \right)^{1/4} \begin{bmatrix}
F \\
G
\end{bmatrix}. \tag{3.104}
\]

Since \( G \) is not yet known, it is more convenient to express condition (3.102) in terms of \( F \) alone. Taking derivatives of and adding, \( G \) may be eliminated from (3.102) resulting in the equivalent radiation condition [10, 169]

\[
\bar{z}^{1/4} \frac{\partial \bar{p}}{\partial \bar{t}} + \frac{1}{\bar{z}^{3/4}} \frac{\bar{p}}{4} + \frac{1}{\bar{z}^{1/4}} \frac{\partial \bar{p}}{\partial \bar{z}} = 2 F' \left( \bar{t} + \frac{2}{3} \bar{z}^{3/2} \right), \quad \bar{z} \to \infty, \tag{3.105}
\]

where \( F' \) represents the derivative of \( F \) with respect to its argument.

### 3.4.3 Solution of the linear Tricomi equation

An analytical solution for the boundary value problem just introduced may be found. The boundedness of the pressure field at large times ensures the existence of its Fourier
transform. We will use the following convention for the transform:

\[ \hat{p}(z, \omega) = \mathcal{F}\{p\} = \int_{-\infty}^{\infty} p(z, t) \exp(-i \omega t) \, dt, \]

(3.106)

\[ \hat{p}(z, \omega) = \mathcal{F}^{-1}\{\hat{p}\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{p}(z, \omega) \exp(i \omega t) \, d\omega, \]

where \( \omega \) is the dimensionless angular frequency conjugate to retarded time \( \tilde{t} \). Transforming \( (3.99) \) yields

\[ \frac{d^2 \hat{p}}{dz^2} + \bar{\omega}^2 \hat{p} = 0. \]

(3.107)

Setting \( \eta = |\omega|^{2/3} z \), this equation becomes

\[ \frac{d^2 \hat{p}}{d\eta^2} + \eta \hat{p} = 0, \]

(3.108)

or, with \( \bar{m} = -\eta \),

\[ \frac{d^2 \hat{p}}{dm^2} - \bar{m} \hat{p} = 0. \]

(3.109)

(3.109) is the Airy differential equation, governing the inner solution for each time-harmonic component of the incident signal\(^{(17)}\). Two linearly independent solutions of this equation are given by\(^{(18)}\) the Airy function of the first kind (often just called the Airy function) \( \text{Ai} \) and the Airy function of the second kind \( \text{Bi} \),\(^{(19)}\), defined by the integrals

\[ \text{Ai}(m) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(mx + \frac{x^3}{3})} \, dx = \frac{1}{\pi} \int_{0}^{\infty} \cos \left( mx + \frac{x^3}{3} \right) \, dx, \]

\[ \text{Bi}(m) = \frac{1}{\pi} \int_{0}^{\infty} \left[ e^{(mx - \frac{x^3}{3})} + \sin \left( mx + \frac{x^3}{3} \right) \right] \, dx. \]

\( (17) \) The Airy equation may be seen as the leading-order equation resulting from a caustic expansion, much like the geometrical acoustic equations arise from the ray expansion. The caustic expansion is a particular type of boundary-layer expansion in fractional powers of the wavenumber as opposed to the integer powers used in the ray expansion. For more on this perspective, see \([30, 31, 107]\).

\( (18) \) Solutions of the Airy differential equation/linear Tricomi equation have also been given in terms of Bessel/Hankel functions of \( 1/3 \)rd order \([200]\), Legendre functions of \(-1/6\)th order \([17, 203]\), and more generally, confluent hypergeometric functions \([72, 73, 75, 85, 203]\).

\( (19) \) Named after G.B. Airy, who first showed that the field near a simple caustic could be described in terms of such functions in 1868 \([3]\). Airy used this result to propose a new model for the rainbow, far more accurate in its predictions than the ray model of Descartes and Newton \([2]\).
Therefore, a general solution to (3.107) is given by

\[
\hat{p}(\bar{z}, \bar{\omega}) = A(\bar{\omega}) \text{Ai}(\bar{m}) + B(\bar{\omega}) \text{Bi}(\bar{m})
\]

\[
= A(\bar{\omega}) \text{Ai}(-|\bar{\omega}|^{2/3} \bar{z}) + B(\bar{\omega}) \text{Bi}(-|\bar{\omega}|^{2/3} \bar{z}).
\]

(3.111)

The coefficients \(A\) and \(B\) are determined by applying the boundary conditions in \(\bar{z}\).

For \(\bar{z} \to -\infty\), the Airy function of the first kind tends to zero while the Airy function of the second kind grows exponentially. Therefore, for the field to vanish as \(\bar{z} \to -\infty\) we require \(B(\bar{\omega}) = 0\), so (3.111) simplifies to

\[
\hat{p}(\bar{z}, \bar{\omega}) = A(\bar{\omega}) \text{Ai}(-|\bar{\omega}|^{2/3} \bar{z}).
\]

(3.112)

To match this expression to the geometrical acoustic field as \(\bar{z} \to +\infty\), we first Fourier transform the boundary condition (3.105), yielding

\[
\left( i\bar{\omega} \bar{z}^{1/4} + \frac{1}{4\bar{\omega}^{5/4}} \right) \hat{p} + \frac{1}{\bar{z}^{1/4} d\bar{z}} = 2i\bar{\omega} \exp \left( \frac{2}{3} i\bar{\omega} \bar{z}^{3/2} \right) \hat{F}(\bar{\omega}),
\]

(3.113)

or in terms of \(q\),

\[
|\omega|^{5/6} \left( i\text{sgn}(\omega)q^{1/4} \hat{p} + \frac{1}{q^{1/4} d\eta} + \frac{1}{q^{5/4}} \frac{d\hat{p}}{d\eta} \right) = 2i\text{sgn}(\omega)|\omega| \exp \left( \frac{2}{3} i\text{sgn}(\omega)q^{3/2} \right) \hat{F}(\omega),
\]

(3.114)

where \(\text{sgn}(\cdot)\) is the signum function and \(\hat{F}\) is the Fourier transform of \(F\). We can neglect the \(\hat{p}/4q^{5/4}\) term, as for large \(q\) (large \(\bar{z}\)) it is far smaller than any other term appearing in (3.114). Determination of \(A\) then follows by substituting (3.112) into (3.114) and requiring that the left-hand side equal the right-hand side for \(\bar{z} \to +\infty\). This matching is simplified by use of an asymptotic expansion of the Airy function and its derivative for large argument [1, 15],

\[
\hat{p}(\eta, \omega) = A(\bar{\omega}) \text{Ai}(-\bar{\eta}) \sim \frac{A(\bar{\omega})e^{i\pi/4}}{2\sqrt{\pi}} - \bar{q}^{-1/4} \left[ \exp \left( -\frac{2}{3} i\bar{q}^{3/2} \right) - i \exp \left( \frac{2}{3} i\bar{q}^{3/2} \right) \right],
\]

\[
\frac{d\hat{p}}{d\eta} \sim \frac{A(\bar{\omega})e^{i\pi/4}}{2\sqrt{\pi}} \bar{q}^{-1/4} \left[ \exp \left( \frac{2}{3} i\bar{q}^{3/2} \right) - i \exp \left( -\frac{2}{3} i\bar{q}^{3/2} \right) \right].
\]

(3.115)

Applying these expansions to (3.114), we find that \(A\) is given by

\[
A(\bar{\omega}) = \sqrt{2\pi(1 + i\text{sgn}(\bar{\omega}))}|\omega|^{1/6} \hat{F}(\bar{\omega}).
\]

(3.116)
Substituting this back into (3.111) and inverse transforming, we find that the exact solution to the boundary value problem is given by [10, 75, 175]

\[ p(z,t) = \mathcal{F}^{-1}\left\{ \sqrt{2\pi(1 + i\text{sgn}(\omega))|\omega|^{1/6}}\text{Ai}(-|\omega|^{2/3}z)\hat{F}(\omega) \right\}, \tag{3.117} \]

The analogous result for time-harmonic signals can be found in [15, 31, 200].

Plotting the Airy function in Figure 3.10, we see how the acoustic pressure and intensity \((\propto \text{Ai}^2(-z))\) vary as a function of the normal distance \(z\) from the caustic. Recall that the caustic corresponds to \(z = 0\) (the red vertical axis in the figure) while the edge of the boundary layer in the insonified side region is at approximately \(z = 1\) (i.e., \(z = \delta\), the green dashed line to the right of the caustic). On the far right of the plots, the envelope of the amplitude evolves as predicted by geometrical acoustics. Continuing from right to left, we see that the amplitude oscillates smoothly, gradually increasing as the caustic is approached but all the while remaining finite. The oscillations come about due to interference of the incoming and outgoing waves near the caustic. The amplitude fluctuations become more rapid as the phase differences between the interfering waves decrease until eventually, the incoming and outgoing waves can no longer be clearly distinguished. Diffraction has become non-negligible and we have entered the boundary layer. The amplitude then reaches its maximum very near, but not on the caustic\(^{(20)}\).

\(^{(20)}\)Physically, this shift is another side effect of geometrical theory failing to capture the diffraction that occurs within the caustic boundary layer. That is to say, the geometrical caustic was never the location of the maximum pressure to begin with, but simply what was predicted by a high-frequency approximation.
Past this point, the field begins to decay, smoothly decreasing to near-zero levels in the shadow zone. In comparison, the geometrical acoustic prediction would predict a discontinuous jump from the infinite amplitude at the caustic straight to zero in the shadow zone. Clearly the Airy function, and hence the Tricomi equation, are far better at describing the acoustic field within the caustic boundary layer.

We emphasize that the linear Tricomi equation and the Airy function solution are only valid sufficiently close to the caustic boundary layer. While (3.117) asymptotes to geometrical acoustics outside of the boundary layer and to zero within the shadow zone as we expect, it does not do so in a uniform way. That is, the error between the field predicted by the Tricomi equation and that predicted by an exact solution of the wave equation is not uniformly bounded during either transition. For this reason, the Airy solution is referred to as a local or transitional asymptotic expansion of the (not necessarily known) exact solution [30, 107].

A uniform asymptotic expansion of the field, valid both inside and outside of the caustic boundary layer, was first found by Kravtsov in [202] for scalar (e.g. acoustic, unpolarized light) fields and [201] for vector (electromagnetic) fields. Kravtsov’s solution employs an ansatz in terms of the Airy function and its first derivative and shows that the amplitude and phase of the solution satisfy the geometrical acoustic equations exactly outside of the boundary layer. In this way, Kravtsov’s work represents a generalization of ray theory specifically adapted to caustics. A ray propagation model incorporating Kravtsov’s method for caustics is presented in [171].

Local and uniform asymptotic solutions for the field near higher-order caustics have also been developed. The local asymptotic solution for the cusp is given by the Pearcey function, so named because Pearcey carried out a local study of the cusp similar to that just recounted for the fold caustic [149]. The uniform expansion method developed by Kravtsov was independently discovered and extended to more general systems of partial differential equations and higher-order caustics by Ludwig [120, 121], motivated by a generalization of the method of stationary phase developed by Chester, Friedman, and Ursell [43]. Catastrophe theory allows even more general results to be derived, rendering the Airy and Pearcey functions the first in a series of standard diffraction integrals or special wave-catastrophe functions for higher-order caustics [22, 65, 107, 110, 160], which

of the true wave nature of sound. The linear Tricomi equation/Airy function are more accurate in this regard because they discard fewer terms in the full acoustic equations. That said, by including an additional phase shift of π/2 every time a ray grazes a caustic (see Section 3.4.4), the locations of amplitude maxima and minima can still be predicted fairly reliably via geometrical considerations [107]. Numerical simulations [10, 11, 144, 169, 170] suggest that this shift of the maximum pressure away from the geometrical caustic also persists in the nonlinear case discussed in Chapter 4.

84
have also been extended to more general diffraction phenomena [108, 109].

### 3.4.4 Caustic interaction as a Hilbert transform

Equation (3.117) allows us to find an explicit *connection formula* relating the waveform approaching the caustic boundary layer, $\mathcal{F}$, and the waveform leaving it, $\mathcal{G}$. Inserting the asymptotic expansion of the Airy function in (3.117), we see that at large $\bar{z}$, the solution to the Tricomi boundary value problem takes the form

\[
\bar{p}(\bar{z}, \bar{t}) \sim \frac{1}{\bar{z}^{1/4}} \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{F}(\omega) e^{i\omega(\frac{1}{3} \bar{z}^{3/2})} d\omega \right. \\
+ \frac{i}{2\pi} \left( \int_{0}^{\infty} \hat{F}(\omega) e^{i\omega(\frac{1}{3} \bar{z}^{3/2})} d\omega - \int_{-\infty}^{0} \hat{F}(\omega) e^{i\omega(\frac{1}{3} \bar{z}^{3/2})} d\omega \right) \right\} \quad (3.118)
\]

By construction, this must agree with (3.102), and thus $\mathcal{G}$ is related to $\mathcal{F}$ by

\[
\mathcal{G}(t) = \frac{1}{\pi} \text{Re} \left\{ \int_{0}^{\infty} \hat{F}(\omega) e^{i\omega t} d\omega \right\}. \quad (3.119)
\]

To clarify the meaning of this expression, we expand out the inner Fourier transform and swap the order of integration, yielding

\[
\mathcal{G}(t) = \frac{1}{\pi} \text{Re} \left\{ i \int_{-\infty}^{\infty} \hat{F}(\omega) e^{\omega t} d\omega \right\}. \quad (3.120)
\]

As is, this expression does not converge to a finite value, but as suggested by Pierce [151], it may be “tempered” by introducing the quantity $\theta > 0$ such that $t \to t + i\theta$, with the intent that $\theta$ will eventually go zero:

\[
\mathcal{G}(t) = \frac{1}{\pi} \lim_{\theta \to 0^+} \text{Re} \left\{ i \int_{-\infty}^{\infty} \hat{F}(t') dt' \int_{0}^{\infty} e^{\omega(t-t')} d\omega \right\}. \quad (3.121)
\]

Evaluating the latter integral and expanding out the real part yields

\[
\mathcal{G}(t) = \frac{1}{\pi} \lim_{\theta \to 0^+} \int_{-\infty}^{\infty} \hat{F}(t') \frac{t' - t}{\theta^2 + (t' - t)^2} dt'. \quad (3.122)
\]
But this is just a form of the Cauchy principal value (denoted \( \text{PV} \)) for the integral of the function \( F(t')/(t' - t) \):

\[
\mathcal{G}(t) = \frac{1}{\pi} \text{PV} \left\{ \int_{-\infty}^{\infty} \frac{F(t')}{t' - t} \, dt' \right\} = \frac{1}{\pi} \lim_{\theta \to 0^+} \left( \int_{-\infty}^{t-\theta} \frac{F(t')}{t' - t} \, dt' + \int_{t+\theta}^{\infty} \frac{F(t')}{t' - t} \, dt' \right). \tag{3.123}
\]

In turn, (3.123) is, by definition, the Hilbert transform of \( F \). Hence, for linear waves, the outgoing signal leaving the caustic is the Hilbert transform of the incoming signal arriving at the caustic,

\[
\mathcal{G} = \mathcal{H}\{F\}. \tag{3.124}
\]

Moreover, by a well-known property of the Hilbert transform, we have

\[
\hat{\mathcal{G}}(\omega) = \mathcal{F}\{\mathcal{H}\{F\}\} = e^{i\pi/2} \text{sgn}(\omega) \hat{F}(\omega), \tag{3.125}
\]

where \( \hat{\mathcal{G}} \) is the Fourier transform of \( \mathcal{G} \). Therefore, a wave approaching a caustic and the wave leaving it differ only by a constant phase shift in their component frequencies, as opposed to the blowup predicted by geometrical acoustics. In [89], this connection formula is used to approximate the behavior of sonic booms under Mach cutoff conditions.

From the point of view of ray theory, the \( \pi/2 \) increase\(^{(21)} \) in phase relative to geometrical acoustics can be interpreted as a phase “jump” on the caustic. This shift may be heuristically determined by examination of the geometrical acoustic solution, (2.91). Ignoring the singularity in the amplitude, we see that after touching the caustic the ray tube area has passed through zero, and so at least formally may be considered negative. Compared to the incident wave, this introduces a factor of -1 in the ray tube area, which in turn introduces a factor of \((-1)^{-1/2} = e^{\pm i\pi/2}\) into the pressure solution, which may be treated as an additional phase shift in (2.91)\(^{(22)} \) [151].

\(^{(21)}\) The sign of the phase shift is dependent upon the analytic representation of the signal \( F \). For time-harmonic waves with the \( \exp\{+i\omega t\} \) sign convention (\( \exp\{-i\omega t\} \) Fourier transform convention), a positive sign should be used as seen here. For time-harmonic waves with the \( \exp\{-i\omega t\} \) sign convention (\( \exp\{+i\omega t\} \) Fourier transform convention), a negative sign should be used [36].

\(^{(22)}\) As long as their boundary layers are well-separated, the phase shifts induced by grazing multiple caustics are additive. For instance, if a (linear) waveform grazes \( N \) fold caustic points, the resulting waveform will have undergone an additional phase shift of \( N\pi/2 \) relative to that predicted by the geometrical acoustic solution. The generalization of this result to any structurally stable caustic, as well as to focal points, is given in terms of what is known as the Maslov trajectory index [107], or KMAH index, after the work of Keller, Maslov, Arnold, and Hörmander [36].
3.4.5 N-wave focusing in linear diffraction theory

We are now equipped to examine how the Tricomi theory fares in predicting the behavior of sonic booms near caustics. Consider an N-wave of (normalized) duration $2\overline{T}$, defined by the piecewise function

$$F(\overline{t}) = \begin{cases} 
\overline{t}/\overline{T}, & \overline{t} < \overline{T}, \\
0, & \text{otherwise},
\end{cases}$$

(3.126)

such that there are discontinuous jumps representing shocks at $\overline{t} = \pm \overline{T}$. (3.117) can be used to show [10, 175] that for an incident wave given by (3.126), the (normalized) pressure on the caustic is given by

$$\overline{p}(\overline{t}, 0) = \frac{2\text{Ai}(0)\Gamma(1/6)}{\overline{T}\sqrt{2\pi}} \left\{ \frac{\text{sgn}(\overline{t})}{5} \sin \left( \frac{\pi}{12} \right) \left[ |\overline{T} - |\overline{t}|^{1/6} - (\overline{T} - |\overline{t}|)^{1/6} \right] 
+ |\overline{t}| \cos \left( \frac{\pi}{12} \right) \left[ |\overline{T} - |\overline{t}|^{1/6} - (\overline{T} + |\overline{t}|)^{1/6} \right] 
- \frac{1}{5} \cos \left( \frac{\pi}{12} \right) \left[ |\overline{T} + |\overline{t}|^{1/6} + \text{sgn}(\overline{T} - |\overline{t}|) |\overline{T} - |\overline{t}|^{1/6} \right] \right. 
\left. - \overline{t} \sin \left( \frac{\pi}{12} \right) \left[ (\overline{T} + |\overline{t}|)^{-1/6} + \text{sgn}(\overline{T} - |\overline{t}|) |\overline{T} - |\overline{t}|^{-1/6} \right] \right\}. $$

(3.127)

This function is plotted in Figure 3.11a. We see that there are still singularities (of order $(23) |\overline{T} - |\overline{t}||^{-1/6}$) on the caustic, but they are now confined to $\overline{t} = \pm \overline{T}$ and are therefore associated with the shocks in the incident signal. The waveform leaving the boundary layer can also be computed by taking the Hilbert transform of (3.126), yielding

$$\overline{G}(\overline{t}) = -\overline{T} \overline{\text{Ai}(0)} \Gamma(1/6) \frac{1}{\sqrt{2\pi}} \left\{ \frac{\text{sgn}(\overline{t})}{5} \sin \left( \frac{\pi}{12} \right) \left[ |\overline{T} - |\overline{t}|^{1/6} - (\overline{T} - |\overline{t}|)^{1/6} \right] 
+ |\overline{t}| \cos \left( \frac{\pi}{12} \right) \left[ |\overline{T} - |\overline{t}|^{1/6} - (\overline{T} + |\overline{t}|)^{1/6} \right] 
- \frac{1}{5} \cos \left( \frac{\pi}{12} \right) \left[ |\overline{T} + |\overline{t}|^{1/6} + \text{sgn}(\overline{T} - |\overline{t}|) |\overline{T} - |\overline{t}|^{1/6} \right] \right. 
\left. - \overline{t} \sin \left( \frac{\pi}{12} \right) \left[ (\overline{T} + |\overline{t}|)^{-1/6} + \text{sgn}(\overline{T} - |\overline{t}|) |\overline{T} - |\overline{t}|^{-1/6} \right] \right\}. $$

(3.128)

Plotting this waveform in Figure 3.11b, we see that the singularities at $\overline{t} = \pm \overline{T}$ persist, but they are now of logarithmic type and blow up more slowly. As can be seen in [81, 127, 128, 144, 192] as well as Appendix A, both waveforms are in qualitative agreement with the U-waves observed in the focusing of real sonic booms. Of course, a theory that still yields infinite amplitudes cannot be fully satisfactory and indicates the need for further examination.

$^{(23)}$More generally, if the incident waveform has a singularity of power $n$, then a singularity of power $n - 1/6$ will develop on the caustic. If the incident waveform is smooth, or contains a singularity no worse than order 1/6, singularities will not appear in the waveform on the caustic [85].

87
3.4.6 Augmenting the linear theory

Although we have found that linear diffraction theory is not entirely sufficient for predicting the focusing of typical sonic boom waveforms, it is still far more successful than simple geometrical acoustics. In particular only shocks, or discontinuities in the incident waveform become singular upon reaching the caustic. Hence, we would like to augment the linear theory rather than starting from scratch. To do so, we must first determine the reason for the singularities at the shocks.

Mathematically, singularities arise at shocks in the linear theory because of the frequency-dependence of the caustic boundary layer. Recalling the expression for the boundary layer thickness (3.51), we see that the boundary layer shrinks with increasing source frequency. Hence, the higher the frequency, the longer geometrical acoustics remains a valid description of the pressure field as the caustic is approached. This in turn implies greater amplification of the incident signal, as the waveform amplitude will follow the $z^{-1/4}$ trend of geometrical acoustics for longer. Since discontinuities have unbounded frequency spectra, the effective size of the boundary layer vanishes (at least in the present theory), once again leading to infinite pressures on the caustic. Physically, this is an indication that our current model lacks another critical limiting mechanism.
present during real sonic boom focusing.

Since thermoviscous absorption, vibrational relaxation of diatomic nitrogen and oxygen, higher-order diffraction effects, and nonlinear waveform steepening are all known to play roles in sonic boom propagation at large [83, 151, 156], we are presented with a few likely choices for the missing limiting mechanism. The question is then which effect is most important in the boundary layer, and which remain secondary (but would of course still improve the model).

As singularities only arise in the linear theory when shocks are present, we will compare the impact of the candidate effects at “high” frequencies, on the order of 10,000 to 20,000 Hz. The key ordering parameter in the boundary layer is the diffraction parameter \( \varepsilon \), so it will be convenient to express the relative “sizes” of these effects, governed by other dimensionless parameters, in terms of \( \varepsilon \). Reference [10] indicates that \( \alpha \lambda_{ac} \), the parameter characterizing the strength of thermoviscous absorption effects is on the order of \( \varepsilon^5 \), and according to [169, 170] remains roughly two orders of magnitude smaller than the lowest-order diffraction effects even at near-ultrasonic frequencies. The parameters \( m_N \) and \( m_O \), characterizing the vibrational relaxation effects of diatomic nitrogen and oxygen at their respective relaxation frequencies (\( \sim 300 \) and 30,000 Hz at 45% relative humidity) are \( 1.26 \times 10^{-4} \) and \( 6.71 \times 10^{-4} \) respectively, thus on the order of \( \varepsilon^4 - \varepsilon^5 \) [10]. By construction, the diffraction effects neglected in the Tricomi equation are \( O(\varepsilon^3) \) at most, and their strength will tend to decrease with increasing frequency. Lastly, at pressures typical of sonic boom waveforms, the acoustic Mach number \( M_{ac} = P/\rho_0 c_0^2 \) is on the order of \( 5 \times 10^{-3} \) [83], hence an order of \( \varepsilon^2 - \varepsilon^3 \), and this value will tend to increase as the caustic is approached and focusing takes hold. Therefore, we can expect that after the leading-order diffraction already accounted for in the Tricomi equation, nonlinear steepening is the next most important physical phenomenon within the boundary layer, and may in fact be the missing piece required to eliminate the infinite pressures at the shocks.

### 3.5 Chapter summary

In this chapter, we have taken a closer look at the behavior of the geometrical acoustic solution near the caustic by expressing it in a set of local caustic coordinates. It was found that the eikonal could be expressed in terms of a certain relative curvature between the caustic and the ray defining the local coordinate system. The specific nature of the geometrical acoustic amplitude singularity near a caustic, an algebraic singularity
of order $-1/4$, was also uncovered. This behavior suggested a rescaling (and more precisely, a \textit{boundary layer expansion}) of the acoustic equations which retained previously neglected diffraction-related terms that become important near caustics. The ultimate result of this expansion was the linear Tricomi equation, whose solution was then given in terms of the Airy function, resulting in a much more realistic evolution of the field amplitude near a caustic. In particular, if a smooth wave enters the boundary layer, the Tricomi equation predicts that the wave leaving the boundary layer will also be smooth, and that this waveform is simply the Hilbert transform of the incident waveform. However, on encountering discontinuities used to approximate weak shocks, the solution again became singular. It was determined that the most likely cause of this singularity was the lack of any consideration of nonlinear steepening effects inside the boundary layer, a process known to be important in other portions of the sonic boom propagation domain.
Chapter 4
The Focusing of Nonlinear Waves

4.1 Introduction

Recall the basic assumption of the linear acoustic equations: products of acoustic variables are negligible relative to linear terms. But sonic booms are far louder than typical acoustic sources, meaning that they produce much higher pressures in the surrounding air. Hence, the validity of this assumption becomes doubtful. It is completely refuted by the observation of amplitude-dependent propagation speeds and nonlinear steepening in sonic booms, as these effects simply cannot be predicted by a linear propagation theory. Therefore, in this chapter we explore the nonlinear acoustic, or finite amplitude equations resulting from retaining both linear and (quadratically) nonlinear perturbation terms in the fluid equations. These equations form the basis of the theory of weakly nonlinear acoustics (strong nonlinearity usually refers to sources which require that cubic terms be retained). Incorporating weak nonlinearity into the geometrical acoustic and boundary layer diffraction theories leads, respectively, to weakly nonlinear geometrical acoustics, commonly used to predict long-range propagation of sonic boom waveforms, and the nonlinear Tricomi equation, which will be seen to accurately predict the focusing of sonic booms at caustics. Unfortunately, unlike the linear Tricomi equation, no analytical solution is known for the nonlinear Tricomi equation when shocks are present in the incident signal. Still, several useful properties may be gleaned from it, including an important scaling law and a deep connection to transonic aerodynamics. After discussing these features, we will conclude the chapter by looking at two generalizations of the nonlinear Tricomi equation accounting for wind and atmospheric loss mechanisms.
4.2 Nonlinearity in sonic boom focusing

Recall our relay race analogy for the linear theory: the outer expansion (geometrical acoustics) runs a waveform down to the matching region, hands the waveform off to the inner expansion (the linear Tricomi equation) to run it to the caustic and back out of the boundary layer, at which point the waveform is handed back to the outer expansion. The asymptotic matching of the two expansions (the matching of the geometrical acoustic solution in local caustic coordinates to the Airy asymptotic) ensures that each hand-off goes smoothly. The connection formula (the Hilbert transform) tells us what happened between the two hand-offs. To adapt this process to weakly nonlinear waves, we need to consider the effect of nonlinearity on the outer expansion, the inner expansion, their limiting behavior, and the connection formula.

The foundation of nonlinear focusing theory can be traced back to a 1965 article by J. P. Guiraud [79]. Guiraud’s key result is that the appropriate outer expansion for nonlinear wave focusing at a caustic is provided by weakly nonlinear geometrical acoustics (WNGA), in which the transport equation is replaced by an inviscid Burgers equation, while the inner expansion is a generalization of the Tricomi equation, accounting for nonlinear effects within the caustic boundary layer. While not strictly necessary for smooth weakly nonlinear waves [91], this new nonlinear Tricomi equation (NTE) is critical for the study of weak shock wave focusing [165](2). This distinction is studied in detail in [165], where it is shown that the effective size of the caustic boundary layer differs for smooth waves and weak shocks, in turn suggesting different inner variables for each. However, the authors demonstrate that a hybrid model can be developed by using the smooth-wave inner variables and retaining nonlinear terms in the boundary layer expansion. The end result is simply Guiraud’s model, but [165] provides a more concise development, as well as new insight on the model’s significance and range of applicability. Since 1965, Guiraud’s theory has been validated many times over by both flight tests and lab-scale experiments, some of which are recounted in Appendix A.

To summarize, we introduce the new participants in our relay race. Weakly nonlinear geometrical acoustics runs the waveform away from the aircraft along the rays. Over this distance, the waveform undergoes nonlinear distortion according to the inviscid Burgers
equation. Upon reaching the matching region just outside of the caustic boundary layer, WNGA hands the waveform off to the nonlinear Tricomi equation according to an appropriate matching condition—we will see later that this matching condition is essentially that of the linear case. The nonlinear Tricomi equation then runs the waveform through the boundary layer to the caustic and back out of the boundary layer, at which point the waveform is handed back to the outer expansion. As the nonlinear Tricomi equation has no known analytical solution for incident shocks, an explicit connection formula is not known either. Hence, computation of focused nonlinear waveforms must instead be done numerically in all but the simplest cases. With that, we are ready to extend the results of Chapter 3 to weakly nonlinear waves.

4.2.1 Nonlinear acoustics

Weakly nonlinear waves are well-described by retaining terms out to order $M_{ac}^2$ in the acoustic perturbation of (2.1)–(2.4). In a quiescent, inhomogeneous medium, the resulting nonlinear or finite amplitude correction to the acoustic equations is given by [10, 83],

\[
\begin{align*}
\text{linear terms} & \quad \frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot \mathbf{u}' = 0, \\
& \quad \rho_0 \frac{\partial \mathbf{u}'}{\partial t} + \nabla p' + \rho_0 (\mathbf{u}' \cdot \nabla) \mathbf{u}' + \rho' \frac{\partial \mathbf{u}'}{\partial t} = -\mathbf{u}' \cdot \nabla \rho_0, \\
\text{nonlinearity} & \quad \rho' c_0^2 - \frac{1}{2} \frac{\partial^2 p}{\partial \rho^2} \bigg|_{s_0, \rho_0} s' + \frac{\partial^2 p}{\partial \rho \partial s} \bigg|_{\rho_0, \rho_0} s' + \frac{1}{2} \frac{\partial^2 p}{\partial s^2} \bigg|_{\rho_0, \rho_0} s'^2. \\
\text{heterogeneity} & \quad \rho_0 \frac{\partial s'}{\partial t} = -\mathbf{u}' \cdot \nabla s_0 - \mathbf{u}' \cdot \nabla s',
\end{align*}
\]

From these equations, one may deduce that progressive plane waves propagate at the phase speed

\[
c_{NL} = c_0 + \beta u,
\]

where $u = |\mathbf{u}|$ is the magnitude of the acoustic particle velocity and $c_0$ is the small-signal sound speed, the wave propagation speed (2.12) in the linear approximation. Hence, (4.2) recovers the amplitude-dependent propagation speed observed in sonic booms and other weakly nonlinear waves, and for $u$ vanishingly small, reduces to $c_0$. $\beta$ is known as the coefficient of nonlinearity, sometimes written as

\[
\beta = 1 + \frac{B}{2A},
\]
where \( B/A = (\rho_0/c_0^2)(\partial^2 p/\partial \rho^2)_{s,0} \) is known as the parameter of nonlinearity. In gases, it is equal to \( \gamma - 1 \), where \( \gamma \) is the ratio of specific heats, so that \( \beta = \frac{1}{2}(\gamma + 1) \).

We conclude this section by noting the following useful property: the substitution of first-order (i.e., linear in \( M_{ac} \)) acoustic relations into second-order (weakly nonlinear) terms yields equations which are correct to second-order, as the error introduced from the approximate substitution will be third-order [83]. For example, for plane progressive waves, we can substitute the linear impedance relation \( u = p/\rho_0 c_0 \) into (4.2) to express the nonlinear sound speed in terms of the acoustic pressure,

\[
c_{NL} = c_0 + \beta \frac{p}{\rho_0 c_0}.
\]

(4.4)

### 4.2.2 Weakly nonlinear geometrical acoustics

As in the linear case, the weakly nonlinear geometrical acoustic (WNGA) equations can be deduced as a high-frequency approximation of the governing acoustic equations, in this case the weakly nonlinear acoustic equations (4.1). However, due to the nonlinear steepening effect, the amplitude and phase evolution along a ray cannot be separated, and the linear ansatz (2.92) is no longer appropriate. Instead, we fall back to the more general geometrical acoustic ansatz\(^{(3)}\) [76, 83]

\[
p(r, t) = P(s, t - \tau(s)).
\]

(4.5)

It will also be convenient to take the retarded time \( T \equiv t - \tau(s) \) as an independent variable, as working in a frame following the (linear) wavefronts emphasizes the changes introduced by nonlinearity. Substituting (4.5) into (4.1) yields the following eikonal equation [116, 174],

\[
(\nabla \tau)^2 - \frac{1}{c_0^2} = 0,
\]

(4.6)

identical to the linear case, (2.44). Consequently, the ray equations, and thus the ray trajectories are identical to the linear case as well. That is, weakly nonlinear acoustic waves propagate along the same paths as linear waves, but their amplitudes and phases evolve differently along these paths. In particular, the transport equation is replaced by

\[^{(3)}(4.5)\text{ can also be used to derive the linear geometrical acoustic equations, but since they are independent of } T, \text{ it is simpler to separate variables from the start. This leads back to (2.92).}\]
a nonlinear transport equation for \( P = P(s,T) \),

\[
\nabla \cdot \left( \frac{P^2}{\rho_0} \nabla \tau \right) = \frac{2\beta}{3\rho_0 c_0^4} \frac{\partial P^3}{\partial T}.
\]

(4.7)

Expanding this expression out and using (2.79) and (2.83), we find

\[
\frac{\partial}{\partial s} \left( \mathcal{J} P^2 \right) = \frac{2\beta}{3\rho_0 c_0^3} \mathcal{J} \frac{\partial P^3}{\partial T},
\]

(4.8)

where \( \mathcal{J} \) is the generalized ray divergence (2.67). For \( P \) sufficiently small, (4.7) and (4.8) reduce, respectively, to the linear transport equation (2.45) and the conservation of energy along a ray, (2.66).

To put (4.8) into a more familiar form, we introduce the reduced pressure \( q = q(s,T) \) and the reduced path length \( \sigma = \sigma(s) \) [83], defined by

\[
q(s,T) \equiv \sqrt{\mathcal{J}(s) P(s,T)},
\]

\[
\sigma(s) \equiv \frac{\rho_0(s_0)c_0^3(s_0)}{\beta(s_0)} \int_{s_0}^{s} \frac{\beta(s')}{\rho_0(s')c_0^3(s')} ds' = \int_{s_0}^{s} \frac{\Lambda(s')}{\delta S(s_0)} \left( \frac{\delta S(s_0)}{\delta S(s')} \right)^{1/2} ds',
\]

(4.9)

where \( \Lambda(s) \equiv \beta \rho_0^{-1/2} c_0^{-5/2} \). Substituting these expressions into (4.8), we find

\[
\frac{\partial q}{\partial \sigma} = \left( \frac{\beta}{\rho_0 c_0^3} \right)_0 q \frac{\partial q}{\partial T},
\]

(4.10)

an inviscid form of Burgers’ equation, the basic model for weakly nonlinear plane progressive waves. Hence, by use of \( q \) and \( \sigma \), the propagation of locally plane waves along curved rays has been reduced to the problem of one-dimensional plane wave propagation. The 0-subscript indicates that the parenthesized quantities should be evaluated at some reference point, say \( \sigma = 0 \) (\( s = s_0 \)). Given an initial waveform \( q(\sigma = 0,T) = f(T) \) (e.g. an appropriately scaled aircraft F-function), this equation admits the implicit Earnshaw/Poisson solution [83]

\[
q(\sigma,T) = f \left( T + \left( \frac{\beta}{\rho_0 c_0^3} \right)_0 \sigma q \right).
\]

(4.11)

For vanishingly small \( q \), the second term in the argument of \( f \) drops out, leading to the linear geometrical acoustic solution (2.93), in which the initial waveform shape remains unchanged as it propagates. The second term is therefore a nonlinear correction to
the linear phase variable $T$. Relative to the linear phase, components with positive amplitude undergo a phase advance while those with negative amplitude experience a phase lag, reflecting the nonlinear steepening or *aging* observed in real sonic boom signatures. For this reason, the quantity $(\beta/\rho_0c_0^3)q\sigma$ is sometimes referred to as the *age variable*. Moreover, since the surfaces of constant phase are now given by

$$T + \left( \frac{\beta}{\rho_0c_0^3} \right)q\sigma = t - \left( \frac{\beta}{\rho_0c_0^3} \right)\sigma = \text{cst.},$$

we see that the travel time along a ray is corrected to read $\tau - (\beta/\rho_0c_0^3)q\sigma$. Hence, the eikonal $\tau = \int ds'/c_0$ loses its role as the acoustic travel time for waves of sufficiently high amplitude.

The nonlinear distortion term in (4.11) can eventually cause the propagating waveform to become multivalued. This effect is not physical, but mathematical. In reality, thermoviscous dissipation tends to limit nonlinear steepening leading to steep, but still single-valued *shocks*. Basic geometrical acoustics does not account for losses, and hence has no such limiting mechanism on the waveform steepness. Instead, whenever a multivalued profile develops, lossless theory approximates shocks by replacing the multivalued portion of the signal with a vertical discontinuity.

The *shock formation distance*, the propagation distance at which a discontinuity will first develop in a (lossless) steepening waveform, can be determined from (4.11).
Differentiating (4.11), we find that the slope of the waveform profile is given by

$$\frac{\partial q}{\partial T} = \frac{f'}{1 - f' \left( \frac{\beta}{\rho_0 c_0} \right)_0 \sigma},$$

(4.13)

where \( f' \) denotes differentiation with respect to the argument of \( f \). The right-hand side of this expression becomes infinite, signaling a vertical discontinuity, whenever

$$\sigma = \frac{(\rho_0 c_0^3/\beta)_0}{f'}.\quad (4.14)$$

Since the steepest part of the initial waveform, \( f'_{\text{max}} \), will steepen into a shock first, the earliest shock will therefore form at a (reduced) distance \( \sigma = \sigma_{\text{shock}} \) given by

$$\sigma_{\text{shock}} = \frac{(\rho_0 c_0^3/\beta)_0}{f'_{\text{max}}},$$

(4.15)

which may then be used to recover the physical shock formation with (4.9).

Shocks may be inserted into waveforms which have already become multivalued by using the equal area rule [83, 87, 151, 156, 166, 197]. A corollary of energy conservation, the rule states that shocks are located at the points which cut out equal areas in the multivalued portions of the waveform as in Figure 4.2. However, modern sonic boom

![Figure 4.2: Equal area rule; adapted from [156]](image)

propagation codes employ lossy geometrical theory, so the practical use of the equal area rule has become less prevalent in recent years [114, 198].
4.2.3 The nonlinear Tricomi equation (NTE)

Based on our study of the linear problem, we may conclude that the right-hand sides of the weakly nonlinear acoustic equations (4.1), associated with atmospheric heterogeneity, are $O(\varepsilon^3 M_{ac})$ within the caustic boundary layer. But in Section 3.4.6 we found that $M_{ac}$ is invariably larger than $\varepsilon^3$ in the boundary layer. Therefore, weakly nonlinear terms of order $M_{ac}^2$ become predominant before $O(\varepsilon^3 M_{ac})$ terms \[10\]. Thus, to consider the leading-order effects of nonlinearity on focused waveforms, we may neglect the right-hand sides of (4.1), simplifying the weakly nonlinear acoustic equations to

$$\frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot \mathbf{u}' + \nabla \cdot (\rho' \mathbf{u}') = O(\varepsilon^3 M_{ac}) + O(M_{ac}^3),$$

$$\rho_0 \frac{\partial \mathbf{u}'}{\partial t} + \nabla p' + \rho_0 (\mathbf{u}' \cdot \nabla) \mathbf{u}' + \rho' \frac{\partial \mathbf{u}'}{\partial t} = O(\varepsilon^3 M_{ac}) + O(M_{ac}^3),$$

$$\frac{\partial s'}{\partial t} = O(\varepsilon^3 M_{ac}) + O(M_{ac}^3),$$

$$p' - c_0^2 \rho' - \frac{1}{2} \frac{\partial^2 p}{\partial \rho^2} \bigg|_{s,0} \rho^2 = O(\varepsilon^3 M_{ac}) + O(M_{ac}^3).$$

This simplification implies that within the caustic boundary layer, the entropy fluctuations caused by a weakly nonlinear acoustic disturbance remain negligible and the effects of heterogeneity are locally weak. As a consequence, $\rho_0$ and $c_0$ may be treated as essentially constant, and thus freely moved in and out of derivative operators. This property is used implicitly in many results to follow.

Eliminating across (4.16) (using the fact that linear relations may consistently be substituted into nonlinear terms) leads to only two equations for the nonlinear acoustic field.

$$\frac{\partial p}{\partial t} + \rho_0 c_0^2 \nabla \cdot \mathbf{u} = -\nabla \cdot (p \mathbf{u}) + \frac{1}{\rho_0 c_0^3} \frac{B}{2A} \frac{\partial p^2}{\partial t} + O(\varepsilon^3 M_{ac}) + O(M_{ac}^3),$$

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{\rho_0} \nabla p = -(\mathbf{u} \cdot \nabla) \mathbf{u} - \frac{p}{\rho_0 c_0^2} \frac{\partial \mathbf{u}}{\partial t} + O(\varepsilon^3 M_{ac}) + O(M_{ac}^3),$$

where primes are once again dropped from acoustic variables. The definition of the parameter of nonlinearity, $B/A$, has also been applied.

Because the eikonal equation is the same for linear and nonlinear geometrical acoustics, so too is their asymptotic behavior near the caustic. Therefore, we once again introduce the inner variables $\tau$ (3.77) and $\tilde{t}$ (3.78), and the normalization of the pressure and particle velocity (3.79). It can then be shown [10] that to lowest order in $\varepsilon$, the
impendance relationships relating the velocity and pressure scales are identical to the linear case, (3.92):

\[
U_{cx} = \frac{P_c}{\rho_0 c_0}, \quad U_{cy} = \varepsilon^3 \frac{P_c}{\rho_0 c_0}, \quad \text{and} \quad U_{cz} = \varepsilon \frac{P_c}{\rho_0 c_0}. \quad (4.19)
\]

Hence, just as in the linear case, \(\bar{u}_x = \bar{p} + \mathcal{O}(\varepsilon^2)\), and

\[
U_{cx} = \mathcal{O}(M_{ac}) \gg U_{cz} = \mathcal{O}(\varepsilon M_{ac}) \gg U_{cy} = \mathcal{O}(\varepsilon^3 M_{ac}). \quad (4.20)
\]

Applying these results to (4.18) and dividing through by \(P_c\), the \(x\) and \(z\) components of the (dimensionless) nonlinear momentum equation become, to the order retained,

\[
\frac{\partial \bar{u}_x}{\partial t} - \frac{1 + \varepsilon^2(a - b)\bar{z}}{1 + \varepsilon^2(a - b)\bar{z}} \frac{\partial \bar{p}}{\partial t} = M_{ac}(\bar{u}_x - \bar{p}) \frac{\partial \bar{u}_x}{\partial t}, \quad (4.21)
\]

\[
\frac{\partial \bar{u}_z}{\partial t} + \frac{\partial \bar{p}}{\partial \bar{z}} = 0, \quad (4.22)
\]

while the nondimensional nonlinear conservation of mass equation is given by

\[
\frac{\partial \bar{p}}{\partial t} - \frac{1 + \varepsilon^2(b + a + 2e)\bar{z}}{1 + \varepsilon^2(b + a + 2e)\bar{z}} \frac{\partial \bar{u}_x}{\partial t} + \varepsilon^2 \frac{\partial \bar{u}_z}{\partial \bar{z}} = \beta M_{ac} \frac{\partial \bar{p}^2}{\partial t}. \quad (4.23)
\]

Substituting the linear relationship \(\bar{u}_x = \bar{p}\) into any nonlinear terms, (4.21)–(4.23) reduce to

\[
\frac{\partial \bar{u}_x}{\partial t} - \frac{1 + \varepsilon^2(a - b)\bar{z}}{1 + \varepsilon^2(a - b)\bar{z}} \frac{\partial \bar{p}}{\partial t} = 0, \quad (4.24)
\]

\[
\frac{\partial \bar{u}_z}{\partial t} + \frac{\partial \bar{p}}{\partial \bar{z}} = 0, \quad (4.25)
\]

\[
\frac{\partial \bar{p}}{\partial t} - \frac{1 + \varepsilon^2(b + a + 2e)\bar{z}}{1 + \varepsilon^2(b + a + 2e)\bar{z}} \frac{\partial \bar{u}_x}{\partial t} + \varepsilon^2 \frac{\partial \bar{u}_z}{\partial \bar{z}} = \beta M_{ac} \frac{\partial \bar{p}^2}{\partial t}, \quad (4.26)
\]

where in the last equation, we have used the definition of the coefficient of nonlinearity \(\beta\) (4.3). Differentiating (4.26) with respect to \(\bar{t}\) and interchanging \(\partial / \partial \bar{t}\) and \(\partial / \partial \bar{z}\), we have

\[
\frac{\partial^2 \bar{p}}{\partial \bar{t}^2} - \frac{1 + \varepsilon^2(b + a + 2e)\bar{z}}{1 + \varepsilon^2(b + a + 2e)\bar{z}} \frac{\partial^2 \bar{u}_x}{\partial \bar{t}^2} + \varepsilon^2 \frac{\partial}{\partial \bar{z}} \left( \frac{\partial \bar{u}_z}{\partial \bar{t}} \right) = \beta M_{ac} \frac{\partial^2 \bar{p}^2}{\partial \bar{t}^2}. \quad (4.27)
\]
Lastly, we define the dimensionless nonlinearity factor $\mu$ \cite{10, 85} by

$$\mu = \frac{\beta M_{ac}}{\epsilon^2} \tag{4.28}$$

The nonlinearity factor compares the relative strengths of nonlinearity and diffraction, and is equivalent to

$$\mu = \frac{\beta M_{ac}}{\epsilon^2} = \frac{\delta}{x_{\text{shock}} \lambda_{ac}}, \tag{4.29}$$

where $x_{\text{shock}} = c_0 / \beta \omega_{ac} M_{ac}$ is the shock formation distance for a plane wave \cite{83}. Since a typical shock formation distance is much larger than the thickness of the caustic boundary layer $\delta$ and diffraction effects dominate for $\lambda_{ac} \ll \delta$, $\mu$ is a product of a large quantity and a small quantity, and so can take on a wide range of values in real-world problems [10].

Substituting (4.28), (4.24) and (4.25) into (4.27) and simplifying, we arrive at

$$\frac{\partial^2 \bar{p}}{\partial z^2} - \zeta \frac{\partial^2 \bar{p}}{\partial t^2} + \mu \frac{\partial^2 \bar{p}^2}{\partial t^2} = 0. \tag{4.30}$$

We recognize the first two terms as the linear Tricomi equation, (3.99), while the third term represents cumulative nonlinear effects within the caustic boundary layer. For this reason, (4.30) is known as the nonlinear Tricomi equation (NTE), first derived in [79]\(^{(5)}\).

An important feature of (4.30) is that, while of mixed type, its type does not change where the linear Tricomi equation does. Rather, (4.30) is hyperbolic for $\bar{z} > 2\mu \bar{p}$, parabolic along the curve $\bar{z} = 2\mu \bar{p}$, and elliptic if $\bar{z} < 2\mu \bar{p}$ [10]. Therefore, the “caustic” of the nonlinear problem only agrees with the geometrical (linear) caustic $\bar{z} = 0$ when $\bar{p} = 0$ ($\mu = 0$ degenerates to the linear equation). As a result, (4.30) predicts that the cusps of nonlinear wavefronts are offset from the geometrical caustic in a direction determined by the sign of the local acoustic perturbation. This effect has also been observed in the focusing of real nonlinear acoustic waves and weak shocks [172, 184]), and is attributed to the phenomenon of self-refraction, wherein the amplitude-dependent propagation speed of nonlinear waves distorts their effective ray paths\(^{(6)}\) and hence the points

\(^{(4)}\)The nonlinearity factor is sometimes defined as $2\mu$ to eliminate the factor of 2 resulting from expanding out the derivative of $\bar{p}^2$, e.g. [169].

\(^{(5)}\)This article is translated in Appendix E.

\(^{(6)}\)Since this difference depends on the local acoustic pressure it is generally small, and particularly so away from focal regions. Hence, while self-refraction clearly plays an important role within the boundary layer, it is safe to ignore it in the asymptotic description of the geometrical theory outside the boundary layer.
at which adjacent rays cross \([83, 134]\). To distinguish the linear and nonlinear cases, the nonlinear “caustic” is usually referred to as the *sonic line*. The term is borrowed from aerodynamic theory \([47, 175, 190]\) and applies because superbooms generated during Mach cutoff flight occur at approximately the altitude for which the local sound speed is equal to the aircraft speed \([81, 127, 172]\). Figure 4.3 illustrates the noted differences between the geometrical caustic and the sonic line.

![Sonic line versus geometrical caustic](image)

Figure 4.3: Sonic line versus geometrical caustic; adapted from \([10]\)

### 4.2.4 Boundary conditions and solution of the nonlinear Tricomi problem

The conditions for the nonlinear Tricomi boundary value problem are, perhaps surprisingly, identical to the linear case:

1. Return to ambient state at large time

   \[
   p(z, t \to \pm \infty) = 0. \tag{4.31}
   \]

2. Decay of field into the shadow zone

   \[
   p(z \to -\infty, \bar{t}) = 0. \tag{4.32}
   \]
3. Passage to geometrical acoustics away from the caustic

\[ p(z \rightarrow +\infty, t) = \frac{1}{z^{1/4}} \left[ F \left( \frac{t}{z} + \frac{2}{3} z^{3/2} \right) + G \left( \frac{t}{z} - \frac{2}{3} z^{3/2} \right) \right], \]

or

\[ z^{1/4} \frac{\partial p}{\partial t} + \frac{1}{z^{1/4}} \frac{\partial p}{\partial z} = 2F' \left( \frac{t}{z} + \frac{2}{3} z^{3/2} \right), \quad z \rightarrow \infty. \]  

(4.33)

The fact that equations (4.31) and (4.32) still apply to the nonlinear problem should not be surprising, but (4.33) warrants some explanation. For \( z \) large and \( p \) sufficiently small, the second term of (4.30) dominates the third, such that the equation asymptotes to the linear Tricomi equation. Therefore, provided an incident waveform is specified sufficiently far from the caustic boundary layer, the linear matching condition may be used without much error.

Since (4.30) asymptotes to the linear equation, one might wonder why the Hilbert transform connection formula (3.124) does not hold for weak shocks. Recall that this formula is a consequence of the Airy function solution for the linear problem, which came about because we could neglect the Airy function of the second kind, Bi. In [165], Rosales and Tabak argue that discarding Bi is inadmissible for shocks, because doing so implicitly assumes that the linear Tricomi equation remains uniformly valid throughout the boundary layer. This property holds for linear and smooth weakly nonlinear waves, but clearly does not apply to shocks since their amplitudes become infinite on caustics in the linear theory. Hence, the Bi term remains and the Hilbert transform connection formula does not follow.

The correct connection formula for weak shocks is not known. Instead, the boundary value problem must generally be solved numerically\(^{(7)}\). Various methods have been developed to do so, some of which are discussed in Appendix A.2. It should be noted that in practice, a more accurate evanescence condition, derived in a matching procedure similar to that used to find (4.33), is used in place of (4.32) (see e.g. [10, 11, 169, 170]).

In Figure 4.4, a solution of (4.30) from [10] is compared to the linear equation, (3.99), for an incident N-wave at several \( z \) values. \( \mu \) is held fixed at 0.08, a value typical of focused sonic booms [10, 11]. We see that the additional nonlinear term in (4.30) has led to shocks that arrive earlier (resp. later) in time for positive (negative) acoustic pressures relative to the linear solution, consistent with the amplitude-dependent sound

\(^{(7)}\)For smooth and sufficiently weak nonlinear waves (i.e. small enough values of \( \mu \)), a solution follows from linearization of (4.30) using a Legendre/hodograph transform technique. See [69, 75, 158, 175].
speed of nonlinear theory. Moreover, both the smooth components and shocks present in the incident wave remain finite at all stages of propagation through the boundary layer. Thus, we have finally arrived at a model capable of predicting finite amplitudes for both smooth waves and weak shocks in the vicinity of caustics (sonic lines). The waveforms of Figure 4.4 are also in remarkable qualitative agreement with recorded superboom signatures found in, e.g., [81, 144, 192] and discussed further in Appendix A. To conclude, the key differences between the linear and nonlinear focusing theories (caustic versus sonic line, displaced wavefront cusps, infinite versus finite amplitudes at shocks) for an incident N-wave are summarized in Figure 4.5.
4.2.5 Velocity potential form of the nonlinear Tricomi equation

In [85], Hayes provides a heuristic derivation of the nonlinear Tricomi equation in terms of a velocity potential. Recall that in order for a velocity potential to be defined, the flow must be irrotational [151], which in general will not hold in the entire flow field. Instead, Hayes leverages the local nature of the problem, supposing that the characteristic length scale for any vorticity in the flow is at least of order $R$, the relative radius of curvature. Since $\delta = \varepsilon^2 R/2 \ll R$, this assumption assures that no vorticity develops within the boundary layer. The assumption is justified by the leading-order boundary layer conservation of momentum equation along the $z$ direction (4.25), since with $p \approx u_x$, this equation can be interpreted as a dimensionless statement of the irrotationality of the flow $(-\bar{u}_x, \bar{u}_z)$ in $(\bar{z}, \bar{t})$ coordinates. Accordingly, we define a (normalized) velocity potential $\Phi$ such that

$$\frac{\partial \Phi}{\partial \bar{t}} = - \bar{u}_x (\bar{z}, \bar{t}), \quad \frac{\partial \Phi}{\partial \bar{z}} = \bar{u}_z (\bar{z}, \bar{t}). \quad (4.34)$$

Substituting the potential into (4.30), integrating with respect to $\bar{t}$, and neglecting higher-order terms, we find the following equation for $\Phi$,

$$\frac{\partial^2 \Phi}{\partial \bar{z}^2} - \left( \bar{z} + 2\mu \frac{\partial \Phi}{\partial \bar{t}} \right) \frac{\partial^2 \Phi}{\partial \bar{t}^2} = 0. \quad (4.35)$$
Equation (4.35) (together with the definition of $\Phi$, (4.34)) is also often referred to as the nonlinear Tricomi equation, as in the papers by Seebass [175], Gill and Seebass [75], Hunter and Keller [91], and Plotkin and Cantril [158, 159].

### 4.2.6 Guiraud similitude

An important property of the nonlinear Tricomi equation is that it possesses a form of self-similarity. That is, under a certain change of variables (independent and dependent), its form remains essentially unchanged. In general, similarity transformations allow one to identify one or more similarity parameters corresponding to some symmetry (e.g. scale invariance or rotation invariance) of the problem. In some cases, these symmetries may be used to define new variables which reduce the dimensionality of the governing differential equation, occasionally yielding a more tractable problem [4]. In the case of the NTE, a scaling law for the amplification of step shocks may be developed. This scaling law was first demonstrated by Guiraud in [79] (the same paper in which (4.30) was derived) and is therefore known as the Guiraud similitude.

Though Guiraud does not employ a velocity potential, the scaling law is most commonly demonstrated starting with the velocity potential NTE (4.35) [75, 85, 158, 159]. As suggested by Hayes [8], it is even cleaner to work in terms of a “reduced” velocity potential $\psi = \varepsilon^{1/2}\Phi$ [85]. Substituting $\psi$ into (4.35) yields the following equation for the reduced potential in the caustic boundary layer:

$$\frac{\partial^2 \psi}{\partial \xi^2} - \left( \bar{z} + 2K \frac{\partial \psi}{\partial \bar{t}} \right) \frac{\partial^2 \psi}{\partial \bar{t}^2} = 0,$$

(4.36)

where

$$K \equiv \frac{\mu}{\sqrt{\varepsilon}} = \frac{\beta M_{ac}}{\varepsilon^{5/2}}.$$

(4.37)

We then introduce the similarity transformation [9]

$$\bar{\psi} = K\psi', \quad \bar{t} = K^{6/5} \xi', \quad \bar{z} = K^{4/5} \eta',$$

(4.38)

where the new variables are indicated by primes. Applying this transformation to (4.36)

[8] Hayes also uses an ordering parameter which differs from $\varepsilon$ by a factor of $2^{-1/3}$, but this does not affect the result in any meaningful way.

[9] If using the pressure form of the equation (4.30), the same scaling for $\bar{t}$ and $\bar{z}$ may be used with $K$ replaced by $\mu$ (since we did not scale the physical pressure by $\sqrt{\varepsilon}$ as we did for the potential), while the relationship between the pressure and velocity potential implies the transformation $p = \mu^{-1/5}p'$. This will eliminate $\mu$ from the pressure NTE.
then yields the canonical form of the nonlinear Tricomi equation,

\[
\frac{\partial^2 \psi'}{\partial \eta'^2} - \left( \eta' + 2 \frac{\partial \psi'}{\partial \xi'} \right) \frac{\partial^2 \psi'}{\partial \xi'^2} = 0,
\]

(4.39)

identical to (4.36) save for the absence of parameter \(K\). For the boundary condition (3.75) for the incoming wave, Hayes chooses a reference distance \(z = D = R\), resulting in

\[
\frac{p}{\rho_0 c_0^2} = M_{ac} \left( \frac{R}{z} \right)^{1/4} F \left( t - \left[ \tau(O) - \frac{1}{c_0(O)} \sqrt{\frac{8}{9R(O)}} z^{3/2} + \frac{x}{c_0(O)} \left( 1 + \frac{z}{R_{zz}(O)} \right) \right] \right).
\]

(4.40)

where \(M_{ac}\) is measured at \(z = R\). This expression can be reduced to the canonical form by first using (4.34) and (4.38) to find

\[
\bar{p} = -\frac{\partial \Phi}{\partial t} = -\frac{1}{\sqrt{\varepsilon}} \frac{\partial \bar{\psi}}{\partial t} = -\frac{K^{-1/5}}{\sqrt{\varepsilon}} \frac{\partial \psi'}{\partial t}
\]

(4.41)

Then, normalizing \(z (3.77), \) the geometrical phase (3.78), \(F (3.104), \) and \(p (3.79), \) and applying transformations (4.38) and (4.41) to (4.40), we have

\[
\frac{p}{\rho_0 c_0^2 M_{ac}} = \bar{p} = -\frac{K^{-1/5}}{\sqrt{\varepsilon}} \frac{\partial \psi'}{\partial \xi'} (\eta', \xi') \sim \left( \frac{R}{\delta K^{4/5} \eta'} \right)^{1/4} F \left( K^{6/5} (\xi' + \frac{2}{3} \eta'^{3/2}) \right)
\]

\[
\Rightarrow -\frac{\partial \psi'}{\partial \xi'} (\eta', \xi') \sim \frac{1}{\eta'^{1/4}} F \left( K^{6/5} (\xi' + \frac{2}{3} \eta'^{3/2}) \right).
\]

(4.42)

In contrast to (4.39), the similarity transformation has actually introduced a \(K\)-dependence into the boundary condition. But if \(F\) is the Heaviside step function defined by

\[
H(t) = \begin{cases} 
0, & t \leq 0, \\
1, & t > 0,
\end{cases}
\]

(4.43)

it will be invariant under the phase dilation induced by \(K\) in (4.42). Therefore, for an incident step shock, the entire boundary value problem is independent of \(K\), and hence so is its solution, \(\psi'(\eta', \xi')\). Since \(K\) was the only remaining parameter in the problem, \(\psi'(\eta', \xi')\) is universal in that once found, it may be rescaled using (4.38) to find the solution for any step shock approaching any smooth caustic (i.e., for any combination of \(\beta, M_{ac}, \lambda_{ac}, R\) and \(c_0\)). In particular, we find that for an incident step shock, the
acoustic pressure near the caustic varies as

\[
\frac{p}{\rho_0 c_0^2} = M_{ac} \bar{p} = -M_{ac} K^{-1/5} \frac{\partial \psi'}{\partial \xi'} (\eta', \xi') = -\beta^{-1/5} M_{ac}^{4/5} \frac{\partial \psi'}{\partial \xi'} (\eta', \xi'),
\]

(4.44)

where in the last equality we have used the definition of \( K \), (4.37). Therefore, the pressure near the caustic (including the reflected signal) varies as the \( 4/5 \)th power of the amplitude of the incident waveform, a nonlinear phenomenon. This is the Guiraud similitude law first derived in [79]. Unfortunately, an analytical solution of (4.39) is not known for incident shocks, so the scaling law is somewhat limited. However, it may be manipulated into a more useful form.

Following Plotkin and Cantril [158], we first define the position-dependent pressure coefficient \( C_p \), a dimensionless measure of wave amplitude, by

\[
C_p = \frac{p}{\frac{1}{2} \rho_0 V_\infty^2} \approx \frac{p}{\frac{1}{2} \rho_0 c_0^2},
\]

(4.45)

where \( V_\infty \approx c_0 \) is the free-stream flow velocity. We will define the focusing factor as the ratio of the maximum value of the pressure coefficient \( C_{p_{\text{max}}} \)—which will occur somewhere near but not directly on the sonic line—to a reference value \( C_{p_{\text{ref}}} \) at some point a distance \( z = z_{\text{ref}} > \delta(\omega_{ac}) \) from the sonic line for which focusing effects may be considered negligible. If the incident waveform is a step shock of strength \( M_{ac} \) a distance \( z = R \) from the sonic line, then from (4.40) the pressure coefficient at \( z_{\text{ref}} \) is given by

\[
\frac{1}{2} C_{p_{\text{ref}}} = M_{ac} \left( \frac{R}{z_{\text{ref}}} \right)^{1/4}.
\]

(4.46)

On the other hand, Guiraud’s similitude (4.44) indicates that the maximum pressure for any step shock will correspond to the universal (but unknown) point \((\eta', \xi')\) at which \(-\partial \psi'/\partial \xi'\) is maximum, hence

\[
\frac{1}{2} C_{p_{\text{max}}} = \beta^{-1/5} M_{ac}^{4/5} \max_{(\eta', \xi')} \left( -\frac{\partial \psi'}{\partial \xi'} \right).
\]

(4.47)

Eliminating \( M_{ac} \) across these two equations, we find that the focusing factor is given
by\(^{(10)}\)

\[
\frac{C_{p_{\text{max}}}}{C_{p_{\text{ref}}}} = \left( \frac{2}{\beta R C_{p_{\text{ref}}}} \right)^{1/5} \max_{(\eta', \xi')} \left( -\frac{\partial \psi'}{\partial \xi'} \right).
\]  

(4.48)

The quantity \(\max(-\partial \psi' / \partial \xi')\), while entirely independent of any physical parameter, is a priori unknown, and must be determined from a (numerical or analytical) solution to (4.39) for an incident step shock or inferred from experimental data. In the absence of such information, the focusing factor is only determined up to this unknown value, and so is often written in the form

\[
\frac{C_{p_{\text{max}}}}{C_{p_{\text{ref}}}} = \text{cst.} \left( \frac{z_{\text{ref}}}{2\beta R C_{p_{\text{ref}}}} \right)^{1/5}.
\]  

(4.49)

There is some freedom in the definition of the unknown constant, but the form above is commonly seen in the literature \([75, 158, 159, 172]\), and corresponds to a constant of \(2^{2/5} \max(-\partial \psi' / \partial \xi')\).

The focusing factor has consistently been measured to be between 2 and 5 for both flight tests and ballistics experiments (see e.g. \([127, 144, 172, 173, 192]\) and Appendix A). The value of the universal constant is a bit more controversial. An approximate analytical method developed by Gill and Seebass \([75]\) based on a hodograph transform of the nonlinear Tricomi equation leads to a value of 2.8. On the other hand, a theoretical study by the French Working Group \([186]\) determines a value of approximately 1.85, comparable with the value of 1.7 reported by Sanai et al. \([172]\) based on both ballistics experiments and flight test data\(^{(11)}\).

The rather significant discrepancy between Gill and Seebass’ result and other references is clarified to an extent by Plotkin and Cantril \([158]\). They first note that the relative radius of curvature used by Sanai et al. may not be consistent with their recorded data. Accounting for this leads Plotkin and Cantril to a constant of 2.1, still not completely accounting for the difference. The remaining disagreement is attributed to two effects. First, atmospheric viscosity and relaxation mechanisms limit the steepening of real shocks, leading to a finite shock thickness and a lower overall amplification due to focusing. The nonlinear Tricomi equation, and therefore the Guiraud similitude, do not account for these effects. Thus, they will tend to overpredict the focusing factor of real

\(^{(10)}\)Through heuristic physical arguments, Pierce developed a similar scaling law for the cusp caustic/arête \([153]\). A slight correction to Pierce’s result followed from a more rigorous study of nonlinear wave focusing at a cusp by Coulouvrat \([49]\).

\(^{(11)}\)The reported values come from Plotkin and Cantril’s report \([158]\) which assumes the constant \(2^{2/5} \max(-\partial \psi' / \partial \xi')\) implicit in (4.49), occasionally leading to different values than those found in the original references.
superbooms, meaning that the value of 2.8 found by Gill and Seebass is likely an upper bound for the effective value of the constant. Second, as noted in many of the flight tests, it is very difficult to capture the true maximum pressure using a discrete microphone array. This inevitably leads to an underprediction of the true maximum pressure coefficient $C_{p_{\text{max}}}$, which would be reflected as an underprediction of the universal constant. Taken together, these two effects serve to significantly reduce the perceived discrepancy between different sources, suggesting that the true value of the constant for real shock waves lies somewhere between 2.1 and 2.8.

More recent direct numerical simulations of the nonlinear Tricomi equation by Auger and Coulouvrat [10, 11] suggest that the similitude law does not fully apply to N-waves or more complicated signatures characteristic of real aircraft. In particular, for an incident N-wave, the maximum pressure predicted by Guiraud’s similitude is found to deviate from that found with the numerical solver by up to 13%, with the disagreement worsening for stronger nonlinearity (larger $\mu$ or $K$). The location of the maximum amplitude point determined from scaling (4.38) is even more suspect, with a deviation of 37% for the distance from the caustic and 73% for the phase variable. However, Auger and Coulouvrat’s code is of a pseudospectral nature, requiring periodic inputs. For example, an incident N-wave is instead represented by an incoming sawtooth wave. Unlike a true transient shock, periodic waveforms must satisfy certain phase constraints which, as the authors note, may serve to explain the particularly large discrepancy in the phase variable.

It should be remembered that the Guiraud similitude law assumes an incident step shock. As evidenced by Auger and Coulouvrat’s work, applying it to signatures more typical of sonic booms is not strictly valid to begin with. Nevertheless, the similitude was a very useful tool in early studies of sonic boom focusing. It is still used as a validation tool for modern focusing codes and, bearing in mind its limitations, can also serve as a useful sanity check on experimental results.

### 4.2.7 Analogy with transonic small disturbance theory

The Tricomi equation, sonic line, pressure coefficient, and hodograph transform, while clearly useful in the study of focusing, are concepts more familiar to aerodynamics than acoustic theory [47, 188, 190]. This may seem coincidental, but in fact underlies a deep connection between the focusing of weakly nonlinear acoustic waves and the theory of transonic aerodynamics. Here, we explore this connection.

The governing equation for weak disturbances in transonic flow is known, fittingly,
as the *transonic small disturbance equation* (TSDE). Its derivation begins with the basic fluid equations, \((2.1)-(2.4)\), supplemented by a requirement that energy be conserved across any shocks which may develop in the flow. With these equations, it may be shown (see e.g. \([18, 47, 67]\)) that the vorticity and entropy of a fluid flow are closely linked. In particular, if a flow is isentropic, it must be irrotational. On the other hand, conservation of energy requires that entropy increase across shocks, thereby inducing vorticity in the fluid. Even so, provided the shock is sufficiently weak, this entropy jump is third-order in the shock strength, as is the vorticity. Therefore, at least to second order in the shock strength, the flow may be considered irrotational and a velocity potential can be defined\(^{(12)}\). For steady flow in a perfect gas, these considerations lead to the following system in the (dimensional) velocity potential \(\Phi (u = \nabla \Phi)\),

\[
c^2 \nabla^2 \Phi - \frac{1}{2} \nabla \Phi \cdot \nabla (|\nabla \Phi|^2) = 0,
\]

\[
\frac{1}{2} |\nabla \Phi|^2 + \frac{c^2}{\gamma - 1} = \frac{1}{2} u_0^2 + \frac{c_0^2}{\gamma - 1}.
\]

(4.50)

The first equation represents the conservation of mass and the second is a form of *Bernoulli’s equation*, which relates static and dynamic pressures along a streamline and follows from the conservation of energy for isentropic flows. \(c = \sqrt{\gamma p/\rho}\) is the adiabatic sound speed\(^{(13)}\) with \(\gamma\) the ratio of specific heats and \(p, \rho\), the local (overall) pressure and density. \(u = |u| = |\nabla \Phi|\) is the flow speed, and \(0\)--subscripts indicate values for the undisturbed flow, for instance far upstream of the body under consideration. This system is exact (i.e. follows from \((2.1)-(2.4)\) without further approximation) for shock-free flows, and accurate to third order in the shock strength for weak shocks. Elimination of \(c\) between the two equations yields

\[
c_0^2 \left[ 1 + \frac{\gamma - 1}{2c_0^2} \left( u_0^2 - |\nabla \Phi|^2 \right) \right] \nabla^2 \Phi - \frac{1}{2} \nabla \Phi \cdot \nabla (|\nabla \Phi|^2) = 0,
\]

(4.51)

known as the *full potential equation* \([18]\). Supposing an immersed body generates only small disturbances to the overall flow, we may take \(\Phi = \Phi_0 + \Phi’\) with \(\Phi’\) small. Substi-

\(^{(12)}\)Note the similarity to Hayes’ argument for introducing a potential into the nonlinear Tricomi equation \([85]\).

\(^{(13)}\)\(c = \sqrt{\partial p/\partial \rho}\) accounts for the fact that a pressure wave changes the local pressure and density of a fluid as it propagates. For small disturbances this change is negligible and \(c\) may be approximated by the linearized/small-signal sound speed \(c_0 = \sqrt{\partial p/\partial \rho}_{s,0}\), but this effect should not be ignored for stronger waves. Note however that the actual propagation speed of a nonlinear wave is equal to \(c + u \approx c_0 + \beta u\), only equaling \(c\) where \(u = 0\), e.g. the ends of a transient pulse \([83]\).
Substituting this into (4.51) and neglecting all but the leading-order terms leads to a linear equation for the potential which (for two-dimensional flow) is given by

\[
\frac{\partial^2 \Phi'}{\partial z^2} - \left( M_0^2 - 1 \right) \frac{\partial^2 \Phi'}{\partial x^2} = 0.
\] (4.52)

\((x, z)\) are coordinates fixed with respect to the immersed body, with \(x\) along the direction of the ambient flow \(u_0\) and \(z\) orthogonal to it. \(M_0 \equiv u_0/c_0\) is the Mach number of the ambient flow. Like the linear Tricomi equation, (4.52) is of mixed type: hyperbolic when the ambient flow is supersonic \((M_0 > 1)\), elliptic when it is subsonic \((M_0 < 1)\), and parabolic along a curve (surface in 3-D) known as the sonic line, where \(M_0 = 1\).

Though generally sufficient for subsonic and supersonic flows, (4.52) predicts odd behavior in the transonic regime, \(M_0 \approx 1\). For \(M_0 > 1\), (4.52) is a form of the acoustic wave equation. But letting \(M_0 \to 1\), we see that its dependence on the streamwise \((x)\) direction becomes very weak, indicating that the disturbance caused by the body primarily propagates laterally (along the \(z\) direction) while remaining confined in a narrow region about the body in the streamwise direction. As a result, the disturbances generated by the body cannot leave it quickly enough and the pressure near it will gradually increase. With no mechanism in place to relieve this pressure buildup, it will grow without bound rather than reaching the expected steady state [47], indicating the failure of (4.52) to properly model flow in the transonic regime.

The pressure singularity predicted by (4.52) is analogous to that predicted by the linear Tricomi equation for shocks, and can be similarly eliminated by a more accurate approximation of (4.51) which accounts for nonlinear steepening. This leads to the transonic small disturbance equation (TSDE) [47, 175],

\[
\frac{\partial^2 \Phi'}{\partial z^2} - \left( M_0^2 - 1 + \frac{(\gamma + 1)M_0^2}{u_0} \frac{\partial \Phi'}{\partial x} \right) \frac{\partial^2 \Phi'}{\partial x^2} = 0,
\] (4.53)

or more generally,

\[
\frac{\partial^2 \Phi'}{\partial z^2} - \left( M_0^2 - 1 + \frac{2\beta M_0^2}{u_0} \frac{\partial \Phi'}{\partial x} \right) \frac{\partial^2 \Phi'}{\partial x^2} = 0,
\] (4.54)

with \(\beta = 1 + B/2A\) the coefficient of nonlinearity (recall that \(\beta = (\gamma + 1)/2\) for a perfect gas). (4.54) is also of mixed type, but just as for the nonlinear Tricomi equation, the nonlinear term slightly offsets the parabolic line from the true sonic line \(M_0 = 1\). However in this case, the linear and nonlinear parabolic lines are both commonly referred
to as sonic lines.

As demonstrated by Seebass [175], the precise link between the transonic small disturbance equation and the nonlinear Tricomi equation can be uncovered by examining a particular aerodynamic problem. Consider a steady inviscid flow of initially uniform speed $u_0$ past a slender airfoil (14). Suppose $c_0 = c_0(z)$, decreasing with $z$ (15), and choose the origin of the $(x, z)$ coordinate system fixed relative to the airfoil such that $c_0(0) = u_0$ (Figure 4.6). Then for $z$ sufficiently small, $M_0^2 = M_0^2(z)$ may be expanded about $z = 0$.

\[ \frac{\partial^2 \Phi}{\partial z^2} - \left( M_0^2(0) z + \frac{2\beta}{u_0} \frac{\partial \Phi}{\partial x} \right) \frac{\partial^2 \Phi}{\partial x^2} = 0, \tag{4.55} \]

where the prime on $\Phi$ has been dropped. For $z$ sufficiently large (but small enough that the expansion of $M_0^2$ is still valid), the first term in the parentheses dominates the second, so for boundary data we may specify the incoming disturbance generated by the airfoil (sitting at some $z = \text{cst.} > 0$) along a characteristic of the linear equation, (4.52).

---

(14) Equivalent to an airfoil moving through a quiescent medium at speed $u_0$.

(15) If the coordinate system so defined is parallel to the ground, this is consistent with a stratified atmosphere.
With the approximation of $M_0^2$, these characteristics are given by
\[ x \pm \frac{2}{3} \sqrt{M_0^2(0)} z^{3/2}. \] (4.56)

(+) specifies a signal propagating toward the sonic line (in the direction of decreasing $z$) and (-) a signal propagating away from it (increasing $z$). The (dimensionless) incident signal $F$, taken to be of characteristic length $\lambda_{ac}$ and amplitude $M_{fl}^{(16)}$ a distance $z = D$ from the origin, may be matched to the potential $\Phi$ for the flow near the sonic line by requiring
\[
\frac{1}{2} C_p(x, z) = -\frac{1}{u_0} \frac{\partial \Phi}{\partial x} \sim M_{fl} \left( \frac{D}{z} \right)^{1/4} F \left( x + \frac{2}{3} \sqrt{M_0^2(0)} z^{3/2} \right).
\] (4.57)

On the other hand, the flow disturbance $u'$ must decay and eventually vanish far up- and downstream of the body, as well as deep into the subsonic flow region $z < 0$ where the shocks degenerate into weak acoustic disturbances. These requirements may be combined into a single condition,
\[
\sqrt{\left( \frac{\partial \Phi}{\partial x} \right)^2 + \left( \frac{\partial \Phi}{\partial z} \right)^2} \to 0, \quad x^2 + z^2 \to \infty, \quad z < 0.
\] (4.58)

With this, all required conditions are specified and the boundary value problem may be uniquely solved. Seebass nondimensionalizes the problem by introducing the change of variables
\[
x = D \bar{x}, \quad z = \left( \frac{D^2}{M_0^2(0)} \right)^{1/3} \bar{z}, \quad \Phi = \frac{u_0}{\beta} D^{5/3} \left( M_0^2(0) \right)^{2/3} \bar{\Phi}.
\] (4.59)

At first glance, this set of transformations does not appear particularly similar to those used for the NTE, (3.77)–(3.79), but the connection is clarified by breaking the transformation of $\Phi$ down. Taking $\Phi = D u_0 M_{fl} \bar{\Phi}$, where $D u_0 M_{fl}$ may be thought of as a characteristic dimensional scale for the potential, (4.55) becomes
\[
\frac{\partial^2 \bar{\Phi}}{\partial z^2} - \left( \bar{z} + 2 \frac{\beta M_{fl}}{D M_0^2(0)} \right) \frac{\partial \bar{\Phi}}{\partial \bar{x}} \frac{\partial^2 \bar{\Phi}}{\partial x^2} = 0.
\] (4.60)

(16) The flow Mach number $M_{fl}$ is analogous to the acoustic Mach number $M_{ac}$. We distinguish the two because the considered problem takes an aerodynamic perspective instead of the usual acoustic perspective. As Hayes puts it [84], “the main difference is that for the aerodynamic solution the air is moving and the solution is steady while for an acoustic solution the air has little motion and the solution is unsteady...the difference is one of point of view.”
Comparing this to (4.35), (3.51), and (3.80), we see that $\beta M_{fl}(D M_0^2(0))^{-2/3}$ is analogous to nonlinearity factor $\mu$, $(D M_0^2(0))^{1/3}$ the diffraction parameter $\varepsilon$, $(M_0^2(0))^{-1}$ the relative radius of curvature $R$, and $(D^2/M_0^2(0))^{1/3}$ the boundary layer thickness $\delta$. The transformation to $\bar{\phi}$ in (4.59) is completed by a subsequent division by this nonlinearity factor, $\Phi = (D M_0^2(0))^{2/3}\bar{\phi}/\beta M_{fl}$, resulting in an equation free of any parameters:

$$\frac{\partial^2 \bar{\phi}}{\partial z^2} - \left( \bar{z} + 2 \frac{\partial \bar{\phi}}{\partial \bar{x}} \right) \frac{\partial^2 \bar{\phi}}{\partial \bar{x}^2} = 0. \quad (4.61)$$

On the other hand, the boundary condition on the incident signal becomes

$$-\frac{\partial \bar{\phi}}{\partial \bar{x}} \sim \frac{\alpha}{\bar{z}^{1/4}} F \left( \bar{x} + \frac{2}{3} \bar{z}^{3/2} \right), \quad (4.62)$$

where

$$\alpha = \frac{\beta M_{fl}}{(D M_0^2(0))^{7/12}}. \quad (4.63)$$

$\alpha$ may also seem unfamiliar, but writing it as

$$\alpha = \frac{\beta M_{fl}}{(D M_0^2(0))^{2/3} \left( \frac{D}{(D^2/M_0^2(0))^{1/3}} \right)^{1/4}}. \quad (4.64)$$

we see that $\alpha$ is analogous to the product $\mu(D/\delta)^{1/4}$ in the NTE variables. The latter factor would typically be absorbed into $\bar{F} = (D/\delta)^{1/4} F$ as in (3.104), but Seebass does not choose to do so, instead proceeding with the transformation

$$\xi = \alpha^{6/5} \xi', \quad \eta = \alpha^{4/5} \eta', \quad \bar{\phi} = \alpha^2 \phi'. \quad (4.65)$$

The end result is essentially the same, as (4.61) is invariant under this transformation while the incident signal becomes

$$-\frac{\partial \phi'}{\partial \xi'} \sim \frac{1}{\eta^{1/4}} F \left( \alpha^{6/5} \left( \xi' + \frac{2}{3} \eta^{3/2} \right) \right). \quad (4.66)$$

Comparing (4.61) and (4.66) to (4.39) and (4.42), we see that aside from variable names, the two boundary value problems are identical. In other words, the unsteady acoustic problem posed by the nonlinear Tricomi equation is equivalent to the steady aerodynamic problem posed by (a local approximation of) the transonic small disturbance equation. As we have already seen, this allows many tools and results from aerodynamic theory...
to be applied to the nonlinear Tricomi equation (and vice versa).

As another example of this correspondence, the transformations used to nondimensionalize the NTE, (3.77)–(3.79), as well as those in (4.59) are similar to a well-known transformation of the full TSDE, (4.54). In particular, let a thin airfoil be characterized by chord length \( \ell \) and thickness-to-chord ratio \( d \) at zero angle-of-attack with the oncoming flow (Figure 4.7). Consider the change of variables

\[
x = \ell \overline{x}, \quad z = \frac{\ell}{d^{1/3}} \overline{z}, \quad \Phi = \ell u_0 d^{2/3} \overline{\phi}.
\]  

(4.67)

Taking \( \ell = D \) (analogous to \( \lambda_{ac} \)) and \( d = DM_0^{2'}(0) \) (analogous to \( \lambda_{ac}/R \)), we see

\[u_0\]
\[t_{max}\]
\[d = t_{max}/\ell\]

![Figure 4.7: Airfoil dimensions](image)

that this transformation is (up to a division by \( \beta \)) equivalent to (4.59), and leads to a dimensionless form of (4.54) given by\(^{(17)}\)

\[
\frac{\partial^2 \overline{\phi}}{\partial \overline{z}^2} - \left( -K_d + 2\beta M_0^2 \frac{\partial \overline{\phi}}{\partial \overline{x}} \right) \frac{\partial^2 \overline{\phi}}{\partial \overline{x}^2} = 0,
\]  

(4.68)

where

\[K_d = \frac{1 - M_0^2}{d^{2/3}}\]  

(4.69)

is known as the transonic similarity parameter for thin airfoils. With \( M_0^2 \approx 1 + M_0^{2'}(0)z \), we see that \( K_d \approx -M_0^{2'}(0)z/d^{2/3} \). In the NTE analogy where \( d^{2/3} \) corresponds to \((\lambda_{ac}/R)^{2/3} \propto \varepsilon^2 \) and \((M_0^{2'}(0))^{-1} \) to \( R \), this reduces to a quantity proportional to \(-z/\delta = -\overline{\tau}\). Hence, the transonic similarity parameter is in some sense a generalization of the inner variable \( \overline{\tau} \), a property noted by Cheng and colleagues in [37, 39–41]\(^{(18)}\).

Much like the nonlinearity factor \( \mu \) (or \( K \)), the transonic similarity parameter is associated with a particular scaling law, or similarity rule. In particular, (4.68) and (4.69)\(^{(17)}\)Since \( M_0 \approx 1 \), one could additionally set \( M_0 = 1 \) in the term multiplying \( \partial \overline{\phi}/\partial \overline{x} \) (but not in \( K_d \), as this changes the type of the equation). The result is the form of the TSDE used in [47], where it is referred to as the von Kármán-Guderley (K-G) equation.

\(^{(18)}\)Similarly, it is well-known that the hodograph transform of the transonic small disturbance equation is the linear Tricomi equation [47], which may have motivated Gill and Seebass [75, 176] to develop their hodograph transform method to linearize the nonlinear Tricomi equation.
imply a similarity rule\textsuperscript{(19)} for \textit{affinely-related bodies}, bodies which can be mapped onto each other by a particular “stretching” transformation. The rule states that transonic flow across two affinely-related airfoils in the same medium will be similar (i.e., one may be calculated from the other) if their corresponding transonic similarity parameters $K_{d1}$, $K_{d2}$ are equal [47, 190]. The parallel to the Guiraud similitude—which implies that two step shocks will undergo similar amplification near smooth caustics when their corresponding nonlinearity factors $K_1$, $K_2$ are equal—is evident.

\section*{4.3 Effects of wind on focusing}

The generalization of the nonlinear Tricomi equation to moving media was established by Auger [10]. He finds that a (steady, spatially-dependent) mean flow has the following effects on the boundary layer procedure, to be demonstrated in subsequent sections.

In moving media, rays and wavefronts are no longer orthogonal and the ray tangent vector does not coincide with the wavefront normal vector. Consequently, the ray tube area $\delta S$ no longer agrees with the wavefront area element $\delta \Sigma$, and the invariant quantity used to determine pressure amplitude variations along rays changes. However, the wavefront area element becomes singular (a cusp develops) if and only if the ray tube area goes to zero, and the definitions of a caustic provided in 2.4.1 remain equivalent in moving media. Similar modifications hold for the weakly nonlinear geometrical theory which will not be recounted here, but may be found in e.g. [10, 116, 174].

Since wind offsets the direction of sound propagation from the ray direction, it becomes more convenient to orient a local caustic coordinate system with respect to the wavefront normal rather than the ray tangent vector. With crossflow neglected, this amounts to a rotation of the local $Oxyz$ coordinate system by some angle $\theta$, dependent on the direction of the wind relative to the wavefront normal at $O$. This affects various quantities associated with the study of the inner geometrical acoustic behavior including the various radii of curvature, the effective thickness of the caustic boundary layer, and consequently, the inner variables themselves.

For the inner expansion, the nonlinear acoustic equations must also be modified to account for a steady mean flow. This generates new terms which must be compared in the inner expansion, resulting in a new form of Tricomi equation. However, it turns out that this equation may be transformed back into the standard (quiescent) nonlinear

\textsuperscript{(19)}The similarity rule also depends on the \textit{flow tangency condition}, the requirement that the flow be tangent to the immersed body at every point along its surface [47].
Tricomi equation by a particular change of variables.

4.3.1 Geometrical acoustics in moving media

Earth’s atmosphere is often modeled as an inhomogeneous medium possessing a steady, spatially independent mean flow $u_0 = u_0(r)$ representative of wind [143, 151], which we further suppose to be subsonic relative to the small signal sound speed, i.e. $|u_0| < c_0$. This assumption is not particularly restrictive, as atmospheric winds are rarely more than 100 km/h $\approx 28$ m/s near the ground [10]. To capture the convective effects associated with wind, we must modify the geometrical-acoustic theory, starting with determining an appropriate eikonal equation for the wavefronts. For linear waves in an inhomogeneous, moving medium, the desired equation may be found as a high-frequency approximation of (2.13)–(2.14), with the end result [10, 143, 151] that

$$\left(\nabla \tau\right)^2 = \frac{\Omega^2}{c_0^2}, \quad (4.70)$$

where

$$\Omega = 1 - u_0 \cdot \nabla \tau = \frac{c_0}{c_0 + u_0 \cdot \hat{n}} \quad (4.71)$$

is a correction to account for the convective effect of the wind. In the absence of wind, $\Omega = 1$ and (4.70) reduces to the usual eikonal equation, (2.44). Since wavefronts are surfaces of constant phase $\tau = \text{cst.}$, a unit normal to a wavefront is given by $\hat{n} = \nabla \tau / |\nabla \tau|$, so (4.70) also implies that

$$\hat{n} = \frac{c_0}{\Omega} \nabla \tau,$$

or

$$\nabla \tau = \frac{\hat{n}}{c_0 + u_0 \cdot \hat{n}} \quad (4.72)$$

Therefore, wavefronts propagate at a speed

$$|v_{wf}| = \frac{1}{\nabla \tau \cdot \hat{n}} = c_0 + u_0 \cdot \hat{n} \quad (4.73)$$

normal to themselves. That is, with the velocity $v_{wf} = (c_0 + u_0 \cdot \hat{n})\hat{n}$.

Applying the method of characteristics to (4.70) results in the ray equations for a

\[\text{(20)}\text{Note that this is a condition on the wind, not the aircraft speed as in Section 4.2.7, and that the reference frames used for each differ.}\]
moving medium, which under the eikonal parametrization take the form

\[
\frac{dr}{d\tau} = u_0 + c_0 \hat{n}, \tag{4.74}
\]

\[
\frac{d}{d\tau} (\nabla \tau) = -\frac{\Omega}{c_0} \left[ (\nabla u_0) \cdot \hat{n} + \nabla c_0 \right], \tag{4.75}
\]

while the eikonal satisfies

\[
\frac{d\tau}{ds} = \frac{1}{|u_0 + c_0 \hat{n}|}, \tag{4.76}
\]

where \( s \) is, as usual, the arc length along a ray. If \( u_0 = 0 \), these reduce to the usual ray equations (2.61)–(2.62) and the canonical equation for the eikonal, (2.58). (4.74) states that the rays travel at a speed \( |v_{\text{ray}}| = |u_0 + c_0 \hat{n}| \), implying, with (4.76), that the eikonal still corresponds to the travel time along a ray (for linear waves). However, it also follows that the unit vector tangent to a ray, \( dr/ds = (u_0 + c_0 \hat{n})/|u_0 + c_0 \hat{n}| \), no longer coincides with the wavefront normal \( \hat{n} \). Rather, comparing (4.74) to (4.73), we see that the wavefront speed is the component of the ray velocity along the wavefront normal, and so is in general slower than the ray speed. Moreover, due to the convective effect of the ambient flow, neither the rays nor the wavefronts propagate at the adiabatic sound speed, and may be greater or less than \( c_0 \) depending on the angle between \( \hat{n} \) and \( u_0 \). These differences are depicted in Figure 4.8.

---

**Figure 4.8:** Rays and wavefronts in a moving medium; adapted from [10]

Since the wavefront normal and ray tangent vector are no longer parallel, the ray tube area \( \delta S \) and the area element of the wavefront, \( \delta \Sigma = |R_u \times R_v| dudv \), no longer agree either. Noting Figure 4.9, we see that the two are instead related by

\((21)\) Using the identity \( \nabla (A \cdot B) = A \times (\nabla \times B) + B \times (\nabla \times A) + (A \cdot \nabla)B + (B \cdot \nabla)A \), with \( A = u_0, B = \Omega \hat{n}/c_0 = \nabla \tau \), (4.75) takes the alternate form \( d(\nabla \tau)/d\tau = -\Omega \nabla c_0/c_0 - \nabla \tau \times (\nabla \times u_0) - (\nabla \tau \cdot \nabla) u_0 \) used in e.g. [151].
Therefore, the two only agree when \( \theta = 0 \), i.e. when the rays are normal to the wavefronts or equivalently, when the ray and wavefront speeds are equal. We also see that \( \delta S = 0 \) when either \( \delta \Sigma = 0 \) or \( \cos \theta = 0 \). However, the second-to-last last equality of (4.77) implies that the latter is impossible for \( |u_0| < c_0 \), so the ray tube area vanishes if and only if the wavefront area element does. But when \( \delta \Sigma = 0 \), \( \mathbf{R}_u \times \mathbf{R}_v = 0 \), hence \( \mathbf{R}_u \) and \( \mathbf{R}_v \) are collinear, corresponding to a singularity in the wavefront: a cusp. Thus, in addition to the three definitions in Section 2.4.1, a caustic may be defined as the locus of wavefront cusps. We have just proved the equivalence of the wavefront cusp and ray tube area definitions in moving media, and the proofs for the ray envelope and differential ray crossing definitions carry over nearly \((22)\) unchanged from the quiescent case [10].

\[^{22}\text{When proving } (a \implies b) \text{ in } 2.4.1, \text{ we first found that on an envelope of rays, } (\mathbf{R}_u \times \mathbf{R}_v) \perp \frac{d\mathbf{r}}{ds}. \text{ We then observed that } \mathbf{R}_u \times \mathbf{R}_v \text{ produces a vector normal to the wavefront, leading to the contradiction that it is both perpendicular and collinear to } \frac{d\mathbf{r}}{ds}, \text{ forcing us to take } \mathbf{R}_u \times \mathbf{R}_v = 0 \text{ and completing the proof. But in moving media, the wavefront normal is not collinear to the ray tangent vector. Instead, the contradiction follows because if } (\mathbf{R}_u \times \mathbf{R}_v) \perp \frac{d\mathbf{r}}{ds} \text{ and } |\mathbf{R}_u \times \mathbf{R}_v| \neq 0, \text{ we’d have } \cos \theta = 0, \text{ which we just found to be impossible when } u_0 \text{ is subsonic.}\]
Turning to the transport equation for moving media, we have
\[
\nabla \cdot \left[ \frac{P^2}{\rho_0} \left( \frac{c_0 \hat{n} + u_0}{c_0^2 \Omega} \right) \right] = 0, \tag{4.78}
\]
which can be integrated over a ray tube in exactly the same way as (2.45) [10]. The result is that, as the cross-sectional area of the ray tube goes to zero \((S \to \delta S)\), the amplitude \(P\) satisfies
\[
\left. \frac{P^2|c_0 \hat{n} + u_0| \delta S}{\rho_0 c_0^2 \Omega} \right|_s = \left. \frac{P^2|c_0 \hat{n} + u_0| \delta S}{\rho_0 c_0^2 \Omega} \right|_{s_0}. \tag{4.79}
\]
This relationship was first obtained by Blokhintzev in [27, 28]. Hence, the quantity conserved along ray tubes in moving media,
\[
\frac{P^2|c_0 \hat{n} + u_0| \delta S}{\rho_0 c_0^2 \Omega} = \text{cst.} \tag{4.80}
\]
is known as the Blokhintzev invariant. In light of this expression, it may be shown that the time-averaged acoustic energy now propagates at a group speed \(|c_0 \hat{n} + u_0|\) [28, 143, 151]. That is, the group speed corresponds to the ray speed. On the other hand, the phase speed may be computed as
\[
|v_{\text{phase}}| = \frac{\omega}{|k|} = \frac{1}{|\nabla \tau|} = c_0 + u_0 \cdot \hat{n}, \tag{4.81}
\]
where the second to last equality follows since \(|k| = |\nabla \Psi| = \omega |\nabla \tau|\). Therefore, wavefronts propagate at the phase speed while rays propagate at the group speed, and the phase velocity is the projection of the group velocity normal to the wavefronts. For this reason, the quantity \(c = c_0 + u_0 \cdot \hat{n}\) is referred to as the effective speed of sound in a moving medium [143, 151].

Rearranging (4.79), we find that the amplitude at any point along a ray is given by
\[
P(s) = \sqrt{\frac{\delta S(s_0)}{\rho_0(s_0)} \rho_0(s) \frac{|c_0(s_0) \hat{n}(s_0) + u_0(s_0)|}{|c_0(s) \hat{n}(s) + u_0(s)|} \frac{c_0^2(s) \Omega(s)}{c_0^2(s_0) \Omega(s_0)} \frac{\delta S(s)}{\rho_0(s_0)} P(s_0)}. \tag{4.82}
\]
Similar to (2.88), we may also relate the ray tube area to the Jacobian of the transformation from ray coordinates to spatial coordinates. Expanding out (4.78), we have
\[
\frac{P^2}{\rho_0 c_0^2 \Omega} \nabla \cdot (c_0 \hat{n} + u_0) + \nabla \left( \frac{P^2}{\rho_0 c_0^2 \Omega} \right) \cdot (c_0 \hat{n} + u_0) = 0. \quad (4.83)
\]

But from (4.74),
\[
c_0 \hat{n} + u_0 = \frac{dr}{d\tau} = |c_0 \hat{n} + u_0| \frac{dr}{ds}. \quad (4.84)
\]

Substituting this into the previous equation and rearranging yields
\[
\frac{1}{|c_0 \hat{n} + u_0|} \nabla \cdot \frac{dr}{d\tau} + \left( \frac{P^2}{\rho_0 c_0^2 \Omega} \right)^{-1} \left[ \nabla \left( \frac{P^2}{\rho_0 c_0^2 \Omega} \right) \cdot \frac{dr}{ds} \right] = 0. \quad (4.85)
\]

Meanwhile, the generalization of the Liouville formula (2.83) to isotropic (and in particular moving) media is given for any parameter \( \alpha \) defined along a ray by\(^{(23)}\) \([36]\)
\[
\nabla \cdot U = \frac{1}{D(\alpha)} \frac{d}{d\alpha} \left( \frac{d\alpha}{ds} U D(\alpha) \right), \quad (4.86)
\]

where \( U \) is the group velocity and \( U = |U| \) the group speed. Applying (4.86) with \( \alpha = s \) and \( U \) the ray velocity \( \frac{dr}{d\tau} \) and noting that \( \nabla f \cdot \frac{dr}{ds} = df/ds \), (4.85) becomes
\[
\frac{1}{|c_0 \hat{n} + u_0|} \frac{d}{ds} \left( |c_0 \hat{n} + u_0| D(s) \right) + \left( \frac{P^2}{\rho_0 c_0^2 \Omega} \right)^{-1} \frac{d}{ds} \left( \frac{P^2}{\rho_0 c_0^2 \Omega} \right) = 0, \quad (4.87)
\]
or
\[
\frac{d}{ds} \ln \left( \frac{P^2 |c_0 \hat{n} + u_0| D(s)}{\rho_0 c_0^2 \Omega} \right) = 0. \quad (4.88)
\]

Integration of this expression from \( s_0 \) to \( s \) yields
\[
P(s) = \sqrt{\frac{D(s_0) \rho_0(s) |c_0(s_0) \hat{n}(s_0) + u_0(s_0)| c_0^2(s_0) \Omega(s)}{D(s) \rho_0(s) |c_0(s) \hat{n}(s) + u_0(s)| c_0^2(s) \Omega(s)}} P(s_0). \quad (4.89)
\]

Therefore, the relationship between \( D(s) \) and \( \delta S \) is
\[
\frac{D(s_0)}{D(s)} = \frac{\delta S(s_0)}{\delta S(s)}, \quad (4.90)
\]
just as in the quiescent medium. On the other hand, since \( ds/d\tau = |c_0 \hat{n} + u_0| \), the

\[^{(23)}\](2.83) follows from (4.86) when we recall that for the quiescent medium, \( \nabla^2 \tau = \nabla \cdot (\nabla \tau) = \nabla \cdot (dr/d\ell) \).
pressure amplitude is given in terms of $\tau$ by

$$P(\tau) = \sqrt{\frac{D(\tau_0) \rho_0(\tau) c_0^2(\tau) \Omega(\tau)}{D(\tau) \rho_0(\tau_0) c_0^2(\tau_0) \Omega(\tau_0)}} \frac{P(\tau_0)}{\sqrt{J_w(\tau)}}$$

(4.91)

where

$$J_w(\tau) = \frac{D(\tau) \rho_0(\tau_0) c_0^2(\tau_0) \Omega(\tau_0)}{D(\tau_0) \rho_0(\tau) c_0^2(\tau) \Omega(\tau)}$$

(4.92)

is the generalization of the ray divergence (2.90) to moving media. Thus, the geometrical acoustic solution for the moving medium is given \[10, 143\] by

$$p(R(u, v, \tau), t) = P(u, v, \tau)F(t - \tau) = \frac{P_0(u, v)}{\sqrt{J_w(\tau)}} F\left[ t - \left( \tau_0(u, v) + \int_{s_0}^s \frac{ds'}{|c_0(s')\hat{n}(s') + u_0(s')|} \right) \right].$$

(4.93)

### 4.3.2 Local caustic coordinates with wind

Since the ray tangent vector $\frac{dr}{ds}$ no longer coincides with the wavefront normal $\hat{n}$, the direction of wavefront propagation will no longer be along the $Ox$ axis of the local caustic coordinate system. Therefore, it is desirable to establish a new coordinate system oriented relative to the wavefront normal at $O$. However, since the wavefront develops a cusp on the caustic, it does not technically possess a well-defined normal vector at $O$. Still, we can define a nominal wavefront normal vector rotated an angle $\theta = \cos^{-1}(|\mathbf{v}_{wfl}|/|\mathbf{v}_{ray}|)$ relative to the ray tangent vector. Using this vector, a new local caustic coordinate system, $OXYZ$, may be defined for the moving medium. Origin $O$ is still some point on the caustic, but $OX$ is now directed along the nominal wavefront normal, $OY$ along the wavefront cusp on the caustic (just like $Oy$), and $OZ$ mutually perpendicular to $OX$ and $OY$ \[10, 11\]. The unit vector along $OZ$ is then the normal to the wavefront normal, which we will call $\hat{n}_w$, rotated by the same angle $\theta$ relative to the $Oz$ axis, the caustic normal. The new $OXYZ$ coordinates are depicted alongside the old $Oxyz$ coordinates in Figure 4.10a. As in the windless case, it will turn out that the acoustic velocity component along $OY$ is negligible to a certain order, so the problem may be treated as locally 2-dimensional as in Figure 4.10b. It is in this sense that we speak of the angle $\theta$ between the ray tangent and wavefront normal vectors—$\theta$ is the angle between their projections in the $X - Z$ plane.
Using an analysis similar to the windless case\(^{(24)}\), Auger concludes that in the new \(OXYZ\) coordinate system, the eikonal near the caustic (3.68) generalizes to moving media as

\[
\tau_{1,2}(X, Y, Z) = \tau(O) \pm \frac{1}{c(O)} \sqrt{\frac{8}{9R_{\text{tot}}(O)}} Z^{3/2} + \frac{X}{c(O)} \left(1 + \frac{Z}{R_{XZ}(O)}\right) + \cdots, \tag{4.94}
\]

where \(c(O)\) is the effective speed of sound at \(O\),

\[
c(O) = c_0 + \mathbf{u}_0(O) \cdot \hat{n}(O). \tag{4.95}
\]

We are also met with two new quantities. \(R_{XZ}\), analogous to \(R_{xz}\) is defined by

\[
\frac{1}{R_{XZ}} = -\frac{1}{R_{\text{tot}}} - \frac{1}{R_{cw}}, \tag{4.96}
\]

where \(1/R_{cw} \equiv c^{-1} \frac{dc}{dZ}\bigg|_O\) and \(R_{\text{tot}}\) is an effective relative radius of curvature, defined by

\[
R_{\text{tot}}(O) = \frac{R_{w}^2(O)}{R(O) \cos \theta}. \tag{4.97}
\]

\(^{(24)}\)Auger’s approach [10] is more direct but less general than Gazaryan’s method [200] which we used for the windless case in Chapter 3, and will use again in Chapter 5.
In this expression, $R$ is the usual relative radius of curvature (3.49) while $R_w$ is an analogous quantity defined by

$$\frac{1}{R_w} = \kappa_w \cos \varphi_w - \kappa_w. \quad (4.98)$$

$k_w$, analogous to the ray curvature $\kappa$, is the curvature of the curve traced out by the wavefront normal vector $\hat{n}$, such that $d(\hat{n})/ds = \kappa_w n_w$, with $n_w$ the curve’s principal normal vector. $\kappa_w$, analogous to the normal curvature of the caustic $\kappa_n$, is defined as the curvature of the caustic section cut out by the plane spanned by $\hat{n}$ and $\hat{n}_w$. In the absence of crosswind, this plane agrees with that spanned by the ray tangent vector and the caustic normal, and $\kappa_w$ may be identified with the normal curvature of the caustic along the ray tangent direction. $\varphi_w = \cos^{-1}(n_w \cdot \hat{n}_w)$ is the angle between the osculating plane of the curve traced out by $\hat{n}$ (the $(\hat{n}, n_w)$ plane) and the $(\hat{n}, \hat{n}_w)$ plane.$^{(25)}$

Lastly, the expansion of the ray divergence is entirely analogous to the windless case, and is absorbed into a generic reference distance $Z = D_w = \mathcal{O}(\delta_w(\omega_{ac}))$. Therefore, just outside of the caustic boundary, the pressure field may be described by the geometrical acoustic solution

$$p(X, Y, Z, t) = P_0 \left( \frac{D_w}{Z} \right)^{\frac{1}{2}} \left\{ F \left( t - \left[ \tau(O) - \frac{1}{c(O)} \sqrt{\frac{8}{9R_{cel}(O)}} Z^{3/2} + \frac{X}{c(O)} \left( 1 + \frac{Z}{R_{XZ}(O)} \right) \right] \right) \right. + \\
\left. G \left( t - \left[ \tau(O) + \frac{1}{c(O)} \sqrt{\frac{8}{9R_{cel}(O)}} Z^{3/2} + \frac{X}{c(O)} \left( 1 + \frac{Z}{R_{XZ}(O)} \right) \right] \right) \right\}, \quad (4.100)$$

analogous to (3.76). The differences between the local geometric quantities in quiescent and moving media are summarized in Table 4.1.

$^{(25)}$A substantially different notation is used in [10] and our translation of it in Appendix F. For comparison, use $\text{dr}/ds = \hat{\tau}$, $\kappa = 1/R_{ray}$, $n = \hat{\tau}_{ray}$, $\hat{n} = \hat{N}$, $\kappa_n = 1/R_{sec}$, $\varphi = \theta(\hat{\tau}_{ray}, \hat{N})$, $R = R_{cau}$, $\hat{n} = \hat{n}^{\omega}$, $\kappa_w = 1/R_{FO}$, $n_w = \hat{n}^{FO}$, $R_c = R_{cel}$, $\bar{n}_w = \hat{N}^{FO}$, $\theta = \theta(\hat{N}, \hat{N}^{FO})$, $\kappa_w = 1/R_{sec}$, $\varphi_w = \theta(\hat{n}^{FO}, \hat{N}^{FO})$, and $R_w = R_{cau}$. 

124
Table 4.1: Geometric quantities in quiescent and moving media

<table>
<thead>
<tr>
<th>Quiescent medium</th>
<th>Moving medium</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(x, y, z)$ coordinates</td>
<td>$(X, Y, Z)$ coordinates</td>
</tr>
<tr>
<td>$\frac{dr}{ds}$ Ray tangent vector</td>
<td>$\hat{n}$ Wavefront normal vector</td>
</tr>
<tr>
<td>$c_0$ Small-signal sound speed</td>
<td>$c$ Effective sound speed</td>
</tr>
<tr>
<td>$\delta S$ Ray tube area</td>
<td>$\delta \Sigma$ Wavefront area element</td>
</tr>
<tr>
<td>$n$ Ray principal normal vector</td>
<td>$n_w$ Principal normal of curve traced out by $\hat{n}$</td>
</tr>
<tr>
<td>$\kappa$ Ray curvature</td>
<td>$\kappa_w$ Curvature of curve traced out by $\hat{n}$</td>
</tr>
<tr>
<td>$\tilde{n}$ Caustic normal vector</td>
<td>$\tilde{n}_w$ Normal to wavefront normal vector</td>
</tr>
<tr>
<td>$\tilde{\kappa}_n$ Normal curvature of caustic in $x - z$ plane</td>
<td>$\tilde{\kappa}_w$ Normal curvature of caustic in $X - Z$ plane</td>
</tr>
<tr>
<td>$\varphi$ Angle between $n$ and $\tilde{n}$</td>
<td>$\varphi_w$ Angle between $n_w$ and $\tilde{n}_w$</td>
</tr>
<tr>
<td>$R$ Relative radius of curvature</td>
<td>$R_w$ Effective relative radius of curvature</td>
</tr>
<tr>
<td>$R_{co}$ Length scale for sound speed heterogeneity along $Oz$ axis</td>
<td>$R_{cw}$ Length scale for effective sound speed heterogeneity along $OZ$ axis</td>
</tr>
<tr>
<td>$R_{xz}$ $1/R_{xz} = -1/R - 1/R_{co}$</td>
<td>$R_{XZ}$ $1/R_{XZ} = -1/R_{tot} - 1/R_{cw}$</td>
</tr>
<tr>
<td>$\delta$ Caustic boundary layer thickness</td>
<td>$\delta_w$ Effective boundary layer thickness</td>
</tr>
<tr>
<td>$\mathcal{D}$ Reference distance on the order of $\delta(\omega_{ac})$</td>
<td>$\mathcal{D}<em>w$ Reference distance on the order of $\delta_w(\omega</em>{ac})$</td>
</tr>
</tbody>
</table>
4.3.3 The windy nonlinear Tricomi equation (WiNTE)

To develop Auger's windy nonlinear Tricomi equation (which we refer to as WiNTE), we first require the nonlinear acoustic equations for a steady mean flow. They are given by

\[
\begin{align*}
\frac{D_0\rho'}{Dt} + \rho_0 \nabla \cdot \mathbf{u}' &= +\nabla \cdot (\rho' \mathbf{u}')
\end{align*}
\]

\[
\rho_0 \frac{D_0\mathbf{u}'}{Dt} + \nabla p' + \rho_0 (\mathbf{u}' \cdot \nabla) \mathbf{u}' + \rho' \frac{D_0\mathbf{u}'}{Dt} = -\rho_0 (\mathbf{u}' \cdot \nabla) \mathbf{u}_0 - \rho' \frac{D_0\mathbf{u}_0}{Dt} - \rho' (\mathbf{u}' \cdot \nabla) \mathbf{u}_0,
\]

\[
\frac{D_0 s'}{Dt} = -\rho' \nabla \cdot \mathbf{s}_0 - \rho' \cdot \nabla \mathbf{s}',
\]

\[
p' - c_0^2 \rho' - \frac{c_0^2 B}{\rho_0} \rho'^2 = \frac{\partial p}{\partial s}\bigg|_{\rho,0} s' + \frac{\partial^2 p}{\partial \rho \partial s}\bigg|_0 \rho' s' + \frac{1}{2} \frac{\partial^2 p}{\partial s^2}\bigg|_{\rho,0} s'^2,
\]

where once again, \(D_0/Dt \equiv \partial/\partial t + u_0 \cdot \nabla\) is the material derivative following the ambient flow. For \(\mathbf{u}_0 = 0\), the material derivative reduces to a partial derivative, and these equations immediately simplify to the standard nonlinear acoustic equations (4.1). Just as in the windless case, we introduce a diffraction parameter, now defined by

\[
\varepsilon_w \equiv \frac{\lambda_{ac}}{\delta_w} = \left(\frac{2}{R_{tot}(O)} \frac{\lambda_{ac}}{\rho_0} \right)^{1/3} = \left(\frac{2}{R_{tot}(O)} \frac{c(O)}{\omega_{ac}} \right)^{1/3},
\]

such that terms of a relative order \(\varepsilon_w^3\) or smaller are discarded. As before, the heterogeneity terms on the right-hand side of (4.101) are negligible in the boundary layer, such that

\[
\begin{align*}
\frac{D_0\rho'}{Dt} + \rho_0 \nabla \cdot \mathbf{u}' + \nabla \cdot (\rho' \mathbf{u}') &= \mathcal{O}(\varepsilon_{w}^3 M_{ac}) + \mathcal{O}(M_{ac}^3),
\rho_0 \frac{D_0\mathbf{u}'}{Dt} + \nabla p' + \rho_0 (\mathbf{u}' \cdot \nabla) \mathbf{u}' + \rho' \frac{D_0\mathbf{u}'}{Dt} &= \mathcal{O}(\varepsilon_{w}^3 M_{ac}) + \mathcal{O}(M_{ac}^3),
\frac{D_0 s'}{Dt} &= \mathcal{O}(\varepsilon_{w}^3 M_{ac}) + \mathcal{O}(M_{ac}^3),
p' - c_0^2 \rho' - \frac{c_0^2 B}{\rho_0} \rho'^2 &= \mathcal{O}(\varepsilon_{w}^3 M_{ac}) + \mathcal{O}(M_{ac}^3).
\end{align*}
\]

Eliminating \(s'\) and \(\rho'\) in the first two equations using the latter two (and linear acoustic relations), we arrive equations in terms of the acoustic pressure and particle velocity.
Dropping primes, we have

\[
\frac{D_0 p}{Dt} + \rho_0 c_0^2 \nabla \cdot u = -\nabla \cdot (p u) + \frac{1}{\rho_0 c_0^2} \frac{B}{2A} \frac{D_0 p^2}{Dt} + \mathcal{O}(\varepsilon_w^3 M_{ac}) + \mathcal{O}(M_{ac}^3),
\]

(4.104)

\[
\frac{D_0 u}{Dt} + \frac{1}{\rho_0} \nabla p = -(u \cdot \nabla)u - \frac{p}{\rho_0 c_0^2} \frac{D_0 u}{Dt} + \mathcal{O}(\varepsilon_w^3 M_{ac}) + \mathcal{O}(M_{ac}^3).
\]

(4.105)

We then introduce the modified inner variables \( \bar{z}_w \) and \( \bar{t}_w \) [10] by

\[
\bar{z}_w = \frac{Z}{\delta_w} = \left( \frac{2}{R_{tot}(O)} \frac{\omega_{ac}^2}{c^2(O)} \right)^{1/3} Z,
\]

(4.106)

\[
\bar{t}_w = \omega_{ac} \left[ t - \tau(O) - \frac{X}{c(O)} \left( 1 + \frac{Z}{R_{XZ}} \right) \right],
\]

(4.107)

where \( \bar{z}_w \) is a dimensionless variable along the normal to the wavefront normal \( \bar{n}_w \) rather than along the caustic normal \( \bar{n} \). Similarly, we normalize the acoustic perturbation variables according to

\[
p(X, Y, Z, t) = P_c \bar{p}(\bar{z}_w, \bar{t}_w), \quad \text{and} \quad u(X, Y, Z, t) = \begin{bmatrix} U_{cX} \bar{u}_X(\bar{z}_w, \bar{t}_w) \\ U_{cY} \bar{u}_Y(\bar{z}_w, \bar{t}_w) \\ U_{cZ} \bar{u}_Z(\bar{z}_w, \bar{t}_w) \end{bmatrix}. \]

(4.108)

where \( u \) is now projected onto the \( OX, OY, \) and \( OZ \) axes. We then have the change of variables formulas

\[
\frac{\partial}{\partial X} = -\frac{\omega_{ac}}{c(O)} \left( 1 + \varepsilon_w^2 \frac{R_{tot}(O)}{2R_{XZ}} \right) \frac{\partial}{\partial \bar{t}_w} + \mathcal{O}(\varepsilon_w^3),
\]

\[
\frac{\partial}{\partial Y} = \mathcal{O}(\varepsilon_w^3),
\]

\[
\frac{\partial}{\partial Z} = \varepsilon_w \frac{\omega_{ac}}{c(O)} \frac{\partial}{\partial \bar{z}_w} + \mathcal{O}(\varepsilon_w^3),
\]

\[
\frac{\partial}{\partial t} = \omega_{ac} \frac{\partial}{\partial \bar{t}_w}.
\]

(4.109)

In addition to the sound speed and ambient density, we will need to expand the components of the wind vector, this time with respect to the \( OXYZ \) coordinate system. In
total, we have

\[ c(X, Y, Z) = c(O) \left( 1 + \varepsilon_w^2 \frac{R_{\text{tot}}(O)}{2R_{\text{cw}}} \varepsilon_w \right) + \mathcal{O}(\varepsilon_w^3), \]

\[ \rho_0(X, Y, Z) = \rho_0(O) \left( 1 + \varepsilon_w^2 \frac{R_{\text{tot}}(O)}{2R_{\text{pw}}} \varepsilon_w \right) + \mathcal{O}(\varepsilon_w^3), \]

\[ u_{OX}(X, Y, Z) = u_{OX}(O) \left( 1 + \varepsilon_w^2 \frac{R_{\text{tot}}(O)}{2R_{OX}} \varepsilon_w \right) + \mathcal{O}(\varepsilon_w^3), \]

\[ u_{OZ}(X, Y, Z) = u_{OZ}(O) \left( 1 + \varepsilon_w^2 \frac{R_{\text{tot}}(O)}{2R_{OZ}} \varepsilon_w \right) + \mathcal{O}(\varepsilon_w^3), \]

\begin{equation}
(4.110)
\end{equation}

where

\[ R_{\text{cw}} = \left( \frac{1}{c(O)} \frac{dc}{dZ} \right)_O^{-1}, \quad R_{\text{pw}} = \left( \frac{1}{\rho_0(O)} \frac{d\rho_0}{dZ} \right)_O^{-1}, \]

\[ R_{OX} = \left( \frac{1}{u_{OX}(O)} \frac{du_{OX}}{dZ} \right)_O^{-1}, \quad R_{OZ} = \left( \frac{1}{u_{OZ}(O)} \frac{du_{OZ}}{dZ} \right)_O^{-1}, \]

\begin{equation}
(4.111)
\end{equation}

and \( u_{OX}, u_{OZ} \) are the components of the wind vector along the \( OX \) and \( OZ \) axes (the \( Y \) component will be neglected shortly). Two ambient flow Mach numbers, \( M_X \) and \( M_Z \), may then be defined along the \( OX \) and \( OZ \) axes:

\[ M_X = \frac{u_{OX}(O)}{c(O)}, \quad M_Z = \frac{u_{OZ}(O)}{c(O)}. \]

\begin{equation}
(4.112)
\end{equation}

Note that these quantities measure the wind relative to the effective sound speed \( c = c_0 + u_0 \cdot \hat{n} \), not the small-signal sound speed \( c_0 \). Therefore at \( O \), where \( \hat{n} \) is directed along the \( OX \) axis, we have

\[ c_0(O) = c(O) - u_0(O) \cdot \hat{n}(O) = c(O)(1 - M_X). \]

\begin{equation}
(4.113)
\end{equation}

Since atmospheric wind speeds are rarely greater than \( \approx 28 \, \text{m/s} \) near the ground, \( M_X \) and \( M_Z \) are generally smaller than 0.08 [10]. Therefore, these quantities will be assumed to be \( \mathcal{O}(\varepsilon_w) \) at most. The material derivative following the ambient flow is then given by

\[ \frac{D_o}{Dt} = \omega_{ac} \left( (1 - M_X) \frac{\partial}{\partial \varepsilon_w} + \varepsilon_w M_Z \frac{\partial}{\partial \varepsilon_w} \right) + \mathcal{O}(\varepsilon_w^3). \]

\begin{equation}
(4.114)
\end{equation}
Applying (4.106)–(4.114) to (4.104) and (4.105) results in familiar relations between the pressure and the particle velocity components, now in terms of $\varepsilon_w$

\[
U_{cX} = \frac{P_c}{\rho_0 c_0}, \quad U_{cY} = \varepsilon_w^3 \frac{P_c}{\rho_0 c_0}, \quad \text{and} \quad U_{cZ} = \varepsilon_w \frac{P_c}{\rho_0 c_0},
\]

eventually resulting in a dimensionless equation for the pressure of the form

\[
\frac{\partial^2 \bar{p}}{\partial \bar{Z}^2} - \bar{Z} \frac{\partial^2 \bar{p}}{\partial \bar{T}^2} - 2\varsigma \frac{\partial^2 \bar{p}}{\partial \bar{Z} \partial \bar{T}} + \mu_w \frac{\partial^2 \bar{p}^2}{\partial \bar{T}^2} = 0.
\]

Parameters $\varsigma \equiv M_Z / \varepsilon_w$ and $\mu_w \equiv \beta M_{ac} / \varepsilon_w^2$ are respectively measures of wind strength and nonlinearity relative to diffraction within the effective caustic boundary layer of the moving medium. (4.116) is Auger’s windy nonlinear Tricomi equation [10, 11], or WiNTE, differing from the windless equation (4.30) by a cross term, $-2\varsigma \frac{\partial^2 \bar{p}}{\partial \bar{Z} \partial \bar{T}}$. However, Auger shows that through the change of variables

\[
\bar{T} = \bar{t}_w + \varsigma \bar{z}_w, \quad \bar{Z} = \bar{z}_w + \varsigma^2, \quad \bar{p}(\bar{z}_w, \bar{t}_w) = \bar{P}(\bar{Z}, \bar{T}),
\]

(4.116) becomes

\[
\frac{\partial^2 \bar{P}}{\partial \bar{Z}^2} - \bar{Z} \frac{\partial^2 \bar{P}}{\partial \bar{T}^2} + \mu_w \frac{\partial^2 \bar{P}^2}{\partial \bar{T}^2} = 0,
\]

eliminating the cross term and leaving an equation identical to that of the windless case, (4.30), but shifting the origin of the local coordinate system away from the caustic.

### 4.3.4 Boundary conditions

Since (4.116) can be transformed back into the standard nonlinear Tricomi equation, its associated boundary conditions are essentially the same as well. In particular, for $\bar{P} = \bar{P}(\bar{Z}, \bar{T})$, they are given by

1. Return to ambient state at large time

\[
\bar{P}(\bar{Z}, \bar{T} \to \pm \infty) = 0.
\]

2. Decay of field into the shadow zone

\[
\bar{P}(\bar{Z} \to -\infty, \bar{T}) = 0.
\]
3. Passage to geometrical acoustics away from the caustic

\[
\bar{P}(\bar{Z} \to +\infty, \bar{T}) = \frac{1}{(\bar{Z} - \zeta^2)^{1/4}} \left[ F \left( \bar{T} - \zeta \bar{Z} + \zeta^3 + \frac{2}{3}(\bar{Z} - \zeta^2)^{3/2} \right) 
+ G \left( \bar{T} - \zeta \bar{Z} + \zeta^3 - \frac{2}{3}(\bar{Z} - \zeta^2)^{3/2} \right) \right],
\]

(4.121)

where \((F, G) = (D_w/\delta_w)^{1/4}(F, G)\). Moreover, for \(M_Z = \mathcal{O}(\varepsilon), \zeta = M_Z/\varepsilon = \mathcal{O}(1)\), so at sufficiently large \(\bar{Z}\), (4.122) may be replaced by the standard far-field matching condition, (4.33) as

\[
\bar{P}(\bar{Z} \to +\infty, \bar{T}) = \frac{1}{\bar{Z}^{1/4}} \left[ F \left( \bar{T} + \frac{2}{3}\bar{Z}^{3/2} \right) + G \left( \bar{T} - \frac{2}{3}\bar{Z}^{3/2} \right) \right],
\]

(4.122)

subsequently leading to the typical radiation condition in (4.33).

Using Auger’s model and an adaptation [129] of a numerical scheme developed by Auger and Coulouvrat [10, 11], a statistical analysis of the effect of atmospheric variability on sonic boom focusing during both steady and accelerated flight is performed in companion articles [29] and [52]. Both geometrical (e.g. caustic geometry, ground impact zone) and loudness (maximum pressure, weighted sound exposure levels) characteristics of the superboom are considered. It is found that the caustic geometry and ground impact zone are particularly sensitive to wind, while its effect on the amplitude is secondary (though still significant). Depending on its direction relative to the flight path, wind tends to shrink (enlarge) the superboom ground impact zone, decreasing (increasing) its maximum amplitude relative to the quiescent case. Moreover, crosswinds are found to introduce lateral asymmetry in the caustic geometry. Of course, this short summary cannot fully do justice to the comprehensive quantitative study carried out in [29, 52] and a further look is recommended to the interested reader.

### 4.4 Effects of losses and vibrational relaxation

A further generalization of the nonlinear Tricomi equations to lossy, relaxing media was provided in 1977 [17]\(^{(26)}\), but the most well-known formulation followed in the 2013 dis-
ertation of Salamone [169, 170]. Salamone’s work has likely proved more popular in recent simulations of sonic boom focusing [98, 144, 162] because it also provides an extensively validated numerical scheme (a generalization of Auger and Coulouvrat’s method [10, 11]) with which to solve the equation. Details on the numerical implementation can be found in [144, 169, 170] and Appendix A, but here we concentrate on the derivation of the model equation, known as the lossy nonlinear Tricomi equation or LNTE. The process differs from that used to derive WiNTE in a few key ways.

First, the outer (geometrical acoustic) expansion must be generalized to lossy, relaxing media. This may be done for both linear [36, 106, 171] and weakly nonlinear [83, 114, 174] waves. That being said, the region in which geometrical acoustics and the inner boundary layer solution are matched is generally small compared to length scales typical of atmospheric absorption and dispersion [17, 165]. Thus, while important for long-range propagation up to this matching region, these effects may be neglected within the matching region. In particular, we may still use the lossless inner limit, inner variables, and matching condition for the lossy problem. A rigorous justification of this conclusion may be found in [165], but here we take it as given. In fact, we will forgo description of the effect of losses and dispersion on geometrical acoustics altogether, leaving that task to the references already mentioned. On the other hand, the inner expansion requires considerable modification.

As usual, the inner expansion begins with the full fluid equations. Until now we have used the lossless forms of these equations, but description of atmospheric losses requires that we revert to a more general lossy form. Incorporating these effects into the fluid equations leads, in the weakly nonlinear approximation, to an augmented form of the well-known Westervelt equation [83] with which to begin the expansion procedure.

In Chapter 3, we found that atmospheric losses are typically $O(\varepsilon^5)$. Therefore, atmospheric dissipation mechanisms are expected to be negligible in the boundary layer, at least to the order of accuracy maintained thus far. However, the strength of these mechanisms increases roughly quadratically with frequency [151], such that at the upper end of the audible frequency range ($\sim$10,000-20,000 Hz), dissipation effects come within two orders of magnitude of diffraction and nonlinearity-related terms [169]. Hence, while small, atmospheric dissipation may play a role in the human perception of focused sonic booms, especially for shocks. Therefore, Salamone compromises by retaining leading-order dissipation effects in the Westervelt equation, regardless of their size, while continuing to approximate all other terms to a relative order $\varepsilon^2$. Aside from this ad hoc step, the approximation procedure carries on as usual, eventually leading to the lossy nonlinear
To obtain the nonlinear acoustic equations for a lossy, relaxing medium, we must first return to the full fluid equations. As before, the conservation of mass, or continuity equation is given by

\[ \frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} = 0. \] (4.123)

However, the addition of viscosity requires that additional forces be considered in the conservation of momentum. In place of Euler’s equation, we must use the compressible Navier-Stokes equation [151],

\[ \rho \frac{D\mathbf{u}}{Dt} + \nabla p = \mu_{sh} \nabla^2 \mathbf{u} + \left( \mu_B + \frac{1}{3} \mu_{sh} \right) \nabla (\nabla \cdot \mathbf{u}), \] (4.124)

where \( \mu_{sh} \) is the dynamic, or shear viscosity and \( \mu_B \) the bulk viscosity. \( \nabla^2 \mathbf{u} \equiv \nabla(\nabla \cdot \mathbf{u}) - \nabla \times (\nabla \times \mathbf{u}) \) is the vector Laplacian. Similarly, though small, there is a nonzero heat conduction associated with the passage of an acoustic wave, particularly for finite amplitudes. In a polyatomic gas, the departure from thermodynamic equilibrium induced by the acoustic wave also leads to a cascade of molecular relaxation processes in which energy is exchanged between the translational, rotational, and vibrational modes of the medium until equipartition of energy has been achieved and a new equilibrium state reached [151]. The characteristic times over which these processes occur are known as the relaxation times, and differ significantly depending on the type of motion and molecular species (primarily diatomic nitrogen and oxygen in air). The rotational and translational modes of relaxation are accounted for by the bulk viscosity, while the vibrational modes are described by an apparent vibration temperature [83, 151]. Therefore, we may no longer assume that sound wave propagation is completely adiabatic, and must replace the isentropic flow condition by the more general entropy balance equation [151],

\[ \rho T \frac{Ds_{fr}}{Dt} + \sum_{\nu} \rho c_{v,\nu} \frac{DT_{\nu}}{Dt} - \kappa \nabla^2 T = 0, \] (4.125)

where terms related to viscous dissipation have already been neglected as they turn out to be third order in the acoustic perturbation variables [45, 151]. Variable \( \nu \) indexes over the relaxing molecular species of the medium and \( c_{v,\nu} \) and \( T_{\nu} \) are respectively the specific heat at constant volume and the apparent vibration temperature associated
with the relaxation process of the \( \nu \)-th species. \( T \) is the absolute temperature and \( \kappa \) the coefficient of thermal conductivity. \( s_{fr} \) is the entropy with the vibrational degrees of freedom frozen, such that the total (specific) entropy is

\[
s = s_{fr} + \sum_{\nu} s_{\nu}, \tag{4.126}\]

where \( s_{\nu} \) is the change in entropy associated with the vibrational relaxation of the \( \nu \)-th species. The apparent vibration temperature for each relaxing species satisfies a relaxation equation

\[
\frac{D T_{\nu}}{D t} = \frac{1}{t_{\nu}} (T - T_{\nu}), \tag{4.127}\]

where \( t_{\nu} \) is the characteristic relaxation time of the \( \nu \)-th species. To close the system of equations, we now require two equations of state for the thermodynamic state variables, given by\(^{(27)}\) \([151, 169]\)

\[
p = p(\rho, s_{fr}), \tag{4.128}\]

and

\[
T = T(p(\rho, s_{fr}), s_{fr}) = T(\rho, s_{fr}). \tag{4.129}\]

We may then introduce the acoustic decomposition

\[
p = p_{0}(r) + p'(r, t), \quad \rho = \rho_{0}(r) + \rho'(r, t), \quad u = u_{0}(r) + u'(r, t), \quad s_{fr} = s_{fr,0}(r) + s'_{fr}(r, t), \quad T = T_{0}(r) + T'(r, t), \tag{4.130}\]

\[
T_{\nu} = T(r, t) + T'_{\nu}(r, t) = T_{0}(r) + T'(r, t) + T'_{\nu}(r, t),
\]

where as usual, all primed terms are taken to be on the order of the acoustic Mach number or smaller (In fact, away from solid boundaries, the entropy perturbation \( s'_{fr} \) may be shown to be a second-order quantity \([151]\)). Substituting (4.130) into (4.123)–(4.129) and neglecting terms beyond second order in the acoustic perturbation, we arrive at the lossy nonlinear acoustic equations for a (steady) moving medium:

\[
\frac{D_{0} \rho'}{D t} + \rho_{0} \nabla \cdot u' + \nabla \cdot (\rho' u') = O(\varepsilon^{3_{w}} M_{ac}) + O(M_{ac}^{3}), \tag{4.131}\]

\[
\rho_{0} \frac{D_{0} u'}{D t} + \nabla p' + \rho_{0} (u' \cdot \nabla) u' + \rho' \frac{D_{0} u'}{D t} \tag{4.132}\]

\[
- \mu_{sh} \nabla^{2} u' - (\mu_{B} + \frac{1}{3} \mu_{sh}) \nabla (\nabla \cdot u') = O(\varepsilon^{3_{w}} M_{ac}) + O(M_{ac}^{3}),
\]

\(^{(27)}\)Relaxation could have instead been incorporated into the first equation of state as in \([17, 45, 83, 166]\), but the resulting Westervelt equation is the same either way.
\[
\rho_0 T_0 \frac{D_o s'_{fr}}{Dt} + \sum_{\nu} \rho_0 c_{\nu,\nu} \frac{D_o T'_\nu}{Dt} - \kappa \nabla^2 T' = O(\varepsilon_w^3 M_{ac}) + O(M_{ac}^2),
\]

(4.133)

\[
\frac{D_o T'_\nu}{Dt} + \frac{D_o T'_\nu}{Dt} + \frac{1}{t'_\nu} T'_\nu = O(\varepsilon_w^3 M_{ac}) + O(M_{ac}^2),
\]

(4.134)

\[
p' - c_0^2 \rho' - \frac{c_0^2 B}{\rho_0} 2A \rho^2 - \rho_0 c_0^2 \beta_t T_0 \frac{s'_{fr}}{c_p} = O(\varepsilon_w^3 M_{ac}) + O(M_{ac}^2),
\]

(4.135)

\[
T' - \frac{\beta_t T_0}{\rho_0 c_p} p' - \frac{T_0}{c_p} s'_{fr} = O(\varepsilon_w^3 M_{ac}) + O(M_{ac}^3),
\]

(4.136)

where neglected terms are related to weak heterogeneity, third-order thermoviscous dissipation and relaxation, and cubic nonlinearity. The following thermodynamic and acoustic relationships have been used in the two equations of state (see e.g. [151]),

\[
\frac{\partial p}{\partial \rho}_{s_{fr},0} = c_0^2, \quad \frac{\partial^2 p}{\partial \rho^2}_{s_{fr},0} = \frac{c_0^2 B}{\rho_0} 2A, \quad \frac{\partial p}{\partial s}_{\rho_0,0} = \rho_0 c_0^2 \beta_t T_0 \frac{1}{c_p},
\]

(4.137)

Here, \(c_0\) is the small signal sound speed and \(B/A\) the parameter of nonlinearity. \(T_0\) is the ambient temperature, \(c_p = T(\partial s/\partial T)_{p,0}\) the specific heat capacity at constant pressure, and \(\beta_t = \rho(\partial (1/\rho)/\partial T)_{p,0}\) the coefficient of thermal expansion. Strictly, these coefficients should be evaluated at the frozen state, but the error resulting from approximating them by their equilibrium values is slight [151]. At this stage we once again drop the primes from the acoustic variables.

We begin the elimination of variables by solving (4.134) for the apparent vibrational temperature \(T'_\nu\). Multiplying through by the integrating factor \(\exp\left(\int dt'/t'_\nu\right)\) and integrating over all time, we find that

\[
T'_\nu(r, t) = - \int_{-\infty}^{t} \exp\left(\frac{t' - t}{t'_\nu}\right) \frac{D_o T(r, t')}{Dt'} dt',
\]

(4.138)

where the integral should be interpreted as a line integral along a streamline of the ambient flow \(u_0\). Alternatively, (4.134) is commonly “solved” by a formal manipulation.
in which the derivative operators are treated as algebraic quantities \([45, 169]\), as
\[
\frac{D_0 T}{Dt} + \frac{D_0 T}{Dt} + \frac{1}{t} T = 0 \implies \left(1 + t \frac{D_0}{Dt}\right) T = -t \frac{D_0 T}{Dt}
\]
\[
\implies T = \frac{-t \left\{ \frac{D_0 T}{Dt} \right\}}{1 + t \frac{D_0}{Dt}}.
\]
(4.139)
Comparing this “solution” to the analytical solution (4.138), we see that the operator appearing in the latter expression, which we refer to as the relaxation operator, may be defined by
\[
\frac{t \left\{ f(t) \right\}}{1 + t \frac{D_0}{Dt}} = \int_{-\infty}^{t} \exp\left(\frac{t' - t}{t}\right) f(t') dt',
\]
(4.140)
for any function \(f\) for which the integral converges. The formal manipulation used to arrive at the relaxation operator solution can be made rigorous through the theory of linear differential operators, but here it is essentially a notational shorthand. For instance, using (4.134) and further formal manipulations, we may find \(\frac{D_0 T}{Dt}\):
\[
\frac{D_0 T}{Dt} = -\frac{1}{t} T - \frac{D_0 T}{Dt} = -\frac{1}{t} \left\{ \frac{-t \left\{ \frac{D_0 T}{Dt} \right\}}{1 + t \frac{D_0}{Dt}} \right\} - \frac{D_0 T}{Dt}
\]
\[
= \frac{\frac{D_0 T}{Dt}}{1 + t \frac{D_0}{Dt}} = \frac{\left(1 + t \frac{D_0}{Dt}\right) \frac{D_0 T}{Dt}}{1 + t \frac{D_0}{Dt}} = -\frac{t \left\{ \frac{D_0^2 T}{Dt^2} \right\}}{1 + t \frac{D_0}{Dt}}.
\]
(4.141)
Applying the definition of the relaxation operator then yields the correct result,
\[
\frac{D_0 T}{Dt} = -\frac{t \left\{ \frac{D_0^2 T}{Dt^2} \right\}}{1 + t \frac{D_0}{Dt}} = -\int_{-\infty}^{t} \exp\left(\frac{t' - t}{t}\right) \frac{D_0^2 T(r, t')}{Dt^2} dt',
\]
(4.142)
which would have otherwise required an integration by parts. For the sake of space, we proceed by placing the function operated on in the numerator of the relaxation operator, sans curly brackets.

As usual, the error incurred by substituting any first-order (linear) relationship between the acoustic variables into a second-order expression is third-order \([83]\). Therefore, the temperature may be eliminated in favor of the pressure by substituting the linearized form of (4.136) into (4.142), yielding
\[
\frac{D_0 T}{Dt} = -\frac{\beta T_0}{\rho_0 c_p} \frac{t \frac{D_0^2 p}{Dt^2}}{1 + t \frac{D_0}{Dt}} \quad \text{and} \quad \nabla^2 T = \frac{\beta T_0}{\rho_0 c_p} \nabla^2 p,
\]
(4.143)
where gradients of ambient quantities are once again neglected due to the assumption of weak heterogeneity. Substituting (4.143) into the entropy equation (4.133) then yields

\[
\rho_0 T_0 \frac{D_s s_{fr}}{Dt} - \frac{\beta_t T_0}{c_p} \sum_\nu c_{\nu,\nu} \frac{t_\nu D_p^2 p}{1 + t_\nu D_0^2} - \frac{\beta_t T_0 \kappa}{\rho_0 c_p} \nabla^2 p = \mathcal{O}(\varepsilon_w^3 M_{ac}) + \mathcal{O}(M_{ac}^3). \tag{4.144}
\]

To eliminate the entropy, we solve the first equation of state, (4.135), for \(s_{fr}\)

\[
s_{fr} = \frac{c_p}{\beta_t T_0 \rho_0} \left( \frac{p}{c_0^2} - \frac{1}{\rho_0} \frac{B}{2A} \rho^2 - \rho \right) + \mathcal{O}(\varepsilon_w^3 M_{ac}) + \mathcal{O}(M_{ac}^3),
\]

\[
= \frac{c_p}{\beta_t T_0 \rho_0} \left( \frac{p}{c_0^2} - \frac{1}{\rho_0 c_0^4} \frac{B}{2A} \rho^2 - \rho \right) + \mathcal{O}(\varepsilon_w^3 M_{ac}) + \mathcal{O}(M_{ac}^3), \tag{4.145}
\]

so that (4.144) becomes

\[
\frac{D_s p}{Dt} = \frac{1}{c_0^2} \frac{D_s p}{Dt} - \frac{1}{\rho_0 c_0^4} \frac{B}{2A} \frac{D_s p^2}{Dt} - \frac{\beta_t^2 T_0 \kappa}{\rho_0 c_p} \nabla^2 p + \mathcal{O}(\varepsilon_w^3 M_{ac}) + \mathcal{O}(M_{ac}^3). \tag{4.146}
\]

The density can then be eliminated between this equation and (4.131). After some rearranging and another use of the first-order relation \(\rho = p/c_0^2\), the result is

\[
\frac{D_s p}{Dt} + \frac{\rho_0 c_0^2}{c_p} \nabla \cdot u = -\nabla(pu) + \frac{1}{\rho_0 c_0^4} \frac{B}{2A} \frac{D_s p^2}{Dt} - \frac{\beta_t^2 T_0 \kappa}{\rho_0 c_p} \nabla^2 p + \mathcal{O}(\varepsilon_w^3 M_{ac}) + \mathcal{O}(M_{ac}^3). \tag{4.147}
\]

Lastly, we substitute \(\rho = p/c_0^2\) into the lossy nonlinear momentum equation, (4.132), to find

\[
\frac{D_s u}{Dt} + \frac{1}{\rho_0} \nabla p = -(u \cdot \nabla)u - \frac{p}{\rho_0 c_0^4} \frac{D_s u}{Dt} + \frac{\mu_{sh}}{\rho_0} \nabla^2 u + \frac{(\mu_B + \frac{1}{3} \mu_{sh})}{\rho_0} \nabla(\nabla \cdot u) + \mathcal{O}(\varepsilon_w^3 M_{ac}) + \mathcal{O}(M_{ac}^3). \tag{4.148}
\]

Once again, we have reduced the acoustic equations to two coupled equations in the acoustic pressure and particle velocity. We could begin applying the usual boundary layer approximations here, but Salamone chooses to eliminate one more equation first.
To arrive at a single equation for the pressure, we first note the vector identities

\[ \nabla(\nabla \cdot u) = \nabla^2 u + \nabla \times (\nabla \times u), \]
\[ (u \cdot \nabla)u = \frac{1}{2} \nabla(u^2) - u \times \nabla \times u, \]  

(4.149)

where \( u^2 = u \cdot u \). Sufficiently far from any boundaries, the latter terms on the right-hand sides of each equation are negligible relative to the former \[151\]. Therefore, we substitute the approximate relationships

\[ \nabla(\nabla \cdot u) \approx \nabla^2 u, \]
\[ (u \cdot \nabla)u \approx \frac{1}{2} \nabla(u^2), \]  

(4.150)

into the momentum equation, (4.148), yielding

\[ \rho_0 \frac{D_0 u}{Dt} + \nabla p + \frac{1}{2} \rho_0 \nabla(u^2) + \frac{p}{c_0^2} \frac{D_0 u}{Dt} - (\mu_B + \frac{4}{3} \mu_{sh}) \nabla(\nabla \cdot u) = \mathcal{O}(\varepsilon_\omega^3 M_{ac}) + \mathcal{O}(M_{ac}^3). \]

(4.151)

We then apply the linear acoustic relations

\[ \frac{D_0 u}{Dt} = -\frac{\nabla p}{\rho_0}, \quad \nabla \cdot u = -\frac{1}{\rho_0} \frac{D_0 \rho}{Dt} = -\frac{1}{\rho_0 c_0^2} \frac{D_0 p}{Dt}, \]  

(4.152)

((2.8) and (2.9) with heterogeneity-related terms neglected) resulting in

\[ \rho_0 \frac{D_0 u}{Dt} + \nabla p + \frac{1}{2} \rho_0 \nabla(u^2) - \frac{1}{2} \rho_0 c_0^2 \nabla \left( \frac{D_0 p}{Dt} \right) = \mathcal{O}(\varepsilon_\omega^3 M_{ac}) + \mathcal{O}(M_{ac}^3), \]

(4.153)

or, after generous application of the weak heterogeneity assumption,

\[ \rho_0 \frac{D_0 u}{Dt} + \nabla p + \nabla \mathcal{L} + \frac{(\mu_B + \frac{4}{3} \mu_{sh})}{\rho_0 c_0^2} \nabla \left( \frac{D_0 p}{Dt} \right) = \mathcal{O}(\varepsilon_\omega^3 M_{ac}) + \mathcal{O}(M_{ac}^3), \]

(4.154)

where the quantity

\[ \mathcal{L} = \frac{1}{2} \rho_0 u^2 - \frac{1}{2} \frac{p^2}{\rho_0 c_0^2}, \]  

(4.155)

is the Lagrangian density, the difference between the kinetic and potential energy densities associated with an acoustic wave. For progressive plane waves, \( \mathcal{L} = 0 \) at \( \mathcal{O}(M_{ac}^2) \) \[151\], but it may be discarded in general if cumulative nonlinear effects dominate any local effects of nonlinearity \[83\]. We will see shortly that \( \mathcal{L} \) may be consistently be
neglected in the boundary layer. A Lagrangian density term may also be pulled out of (4.147) by considering the $-\nabla(pu)$ term. We have

$$-\nabla(pu) = -\nabla p \cdot u - p \nabla \cdot u = - \left( -\rho_0 \frac{D_0 u}{Dt} \right) \cdot u - p \left( -\frac{1}{\rho_0 c_0^2} \frac{D_0 p}{Dt} \right)$$

$$= \frac{1}{2} \rho_0 \frac{D_0 u^2}{Dt} + \frac{1}{2 \rho_0 c_0^2} \frac{D_0 p^2}{Dt} = \frac{1}{2} \rho_0 \frac{D_0 u^2}{Dt} - \frac{1}{2 \rho_0 c_0^2} \frac{D_0 p^2}{Dt} + \frac{1}{\rho_0 c_0^2} \frac{D_0 p^2}{Dt}$$

$$= \frac{D_0 L}{Dt} + \frac{1}{\rho_0 c_0^2} \frac{D_0 p^2}{Dt},$$

where we have used the linear acoustic relations (4.152) in the second equality. Applying this result to (4.147) yields

$$\frac{D_0 p}{Dt} + \rho_0 c_0^2 \nabla \cdot u - \frac{D_0 L}{Dt} - \frac{1}{\rho_0 c_0^2} \left( 1 + \frac{B}{2A} \right) \frac{D_0 p^2}{Dt}$$

$$- \frac{\beta^2 T_0 c_0^2}{c_p^2} \sum_{\nu} \frac{c_{v,\nu} t_\nu \frac{D_0 p}{Dt}}{1 + t_\nu \frac{D_0 p}{Dt}} - \frac{\beta^2 T_0 c_0^2}{\rho_0 c_p^2} \nabla^2 p = \mathcal{O}(\varepsilon_w^3 M_{ac}) + \mathcal{O}(M_{ac}^3).$$

We once again recognize the coefficient of nonlinearity, $\beta = 1 + B/2A$ in this expression. We will also apply the thermodynamic identity [151]

$$\frac{\beta^2 T_0 c_0^2}{c_p^2} = \gamma - 1,$$

where $\gamma = c_p/c_{v,fr}$ is the ratio of specific heats in the frozen state, resulting in

$$\frac{D_0 p}{Dt} + \rho_0 c_0^2 \nabla \cdot u - \frac{D_0 L}{Dt} - \frac{\beta}{\rho_0 c_0^2} \frac{D_0 p}{Dt}$$

$$- (\gamma - 1) \sum_{\nu} \frac{c_{v,\nu}}{c_p} \frac{t_\nu \frac{D_0 p}{Dt}}{1 + t_\nu \frac{D_0 p}{Dt}} - \frac{(\gamma - 1) \kappa}{\rho_0 c_p} \nabla^2 p = \mathcal{O}(\varepsilon_w^3 M_{ac}) + \mathcal{O}(M_{ac}^3).$$

To combine this equation with (4.154), we divide (4.159) by $c_0^2$, take its (ambient) ma-
terial derivative, and subtract the divergence of \((4.154)\) resulting in

\[
\frac{1}{c_0^2} \frac{D^2_p}{Dt^2} + \rho_0 \frac{D_p}{Dt} (\nabla \cdot u) - \frac{1}{c_0^2} \frac{D^2_L}{Dt^2} - \frac{\beta}{\rho_0 c_0^4} \frac{D_p^2}{Dt^2} - \frac{(\gamma - 1)}{c_0^2} \sum_{\nu} \frac{c_{v,\nu}}{c_p} \frac{t_{\nu} D^2_p}{Dt} - \frac{(\gamma - 1) \kappa}{\rho_0 c_0^2 c_p} \frac{D_p}{Dt} (\nabla^2 p) 
\]

\[
- \rho_0 \nabla \cdot \left( \frac{D_p u}{Dt} \right) - \nabla^2 p - \nabla^2 L - \frac{t_{\nu} D^3_p}{Dt^3} + \frac{1}{\rho_0 c_0^2} \frac{1}{\mu_{sh} c_p} \nabla^2 \left( \frac{D_p}{Dt} \right) = O(\varepsilon_w^3 M_{ac}) + O(M_{ac}^3).
\]

(4.160)

To the order retained, the spatial and material derivatives commute, so aside from the Lagrangian density, the terms containing \(u\) cancel and the terms related to thermal and viscous effects may be combined, simplifying this expression to

\[
\nabla^2 p - \frac{1}{c_0^2} \frac{D^2_p}{Dt^2} + \frac{\beta}{\rho_0 c_0^4} \frac{D_p^2}{Dt^2} + \frac{(\gamma - 1)}{c_0^2} \sum_{\nu} \frac{c_{v,\nu}}{c_p} \frac{t_{\nu} D^3_p}{Dt^3} + \frac{\mu_{sh}}{\rho_0 c_0^2} \frac{4}{3} + \frac{\mu_B}{\mu_{sh}} + \frac{(\gamma - 1) \kappa}{\mu_{sh} c_p} \frac{D_p}{Dt} (\nabla^2 p) 
\]

(4.161)

\[
= - \left( \nabla^2 L + \frac{1}{c_0^2} \frac{D^2_L}{Dt^2} \right) + O(\varepsilon_w^3 M_{ac}) + O(M_{ac}^3).
\]

We may replace the Laplacian in the thermoviscous term with a material derivative by substituting the linear relationship \(\nabla^2 p = c_0^{-2} D^2_p / Dt^2\) (an approximate wave equation for a moving, inhomogeneous medium) yielding

\[
\frac{D_p}{Dt} (\nabla^2 p) = \frac{D_p}{Dt} \left( \frac{1}{c_0^2} \frac{D^2_p}{Dt^2} \right) = \frac{1}{c_0^2} \frac{D^3_p}{Dt^3}.
\]

(4.162)

Similar to the coefficient of nonlinearity, it will be useful to define various dimensionless parameters characterizing the relative sizes of thermal, viscous, and vibrational relaxation effects. We first introduce the Prandtl number \(Pr\),

\[
Pr = \frac{\mu_{sh} c_p}{\kappa},
\]

(4.163)

a measure of the relative importance of viscosity and heat conduction. We also have the dispersion parameter \(m_{\nu}\),

\[
m_{\nu} = \frac{c_{\nu}^2 - c_0^2}{c_0^2} \approx \frac{(\gamma - 1) c_{v,\nu}}{c_p},
\]

(4.164)
characterizing changes in phase speed induced by vibrational relaxation. $c_\infty$ is the frozen sound speed, the phase speed in the high-frequency limit, and $c_0$ the usual equilibrium sound speed \[151\]. Lastly, we introduce the diffusivity of sound $\delta_{tv}$,

$$
\delta_{tv} = \frac{\mu_{sh}}{\rho_0} \left( \frac{4}{3} + \frac{\mu_B}{\mu_{sh} c_p} \right) = \frac{\mu_{sh}}{\rho_0} \left( \frac{4}{3} + \frac{\mu_B}{\mu_{sh} c_p} + \frac{\gamma - 1}{\text{Pr}} \right),
$$

(4.165)
governing the attenuation of a waveform due to thermoviscous absorption \[28\] \[83\]. It is related to the thermoviscous absorption coefficient $\alpha_{tv}$ [Nepers/m] by

$$
\alpha_{tv}(\omega) = \frac{\omega^2}{2c_0^3} \delta_{tv}.
$$

(4.166)
Inserting these quantities into (4.161) finally yields an equation governing the propagation of weakly nonlinear waves in a weakly heterogeneous, lossy, relaxing, moving medium:

$$
\nabla^2 p - \frac{1}{c_0^2} \frac{D_0^2 p}{Dt^2} + \frac{\beta}{\rho_0 c_0^4} \frac{D_0^2 p^2}{Dt^2} + \sum_\nu \frac{m_\nu t_\nu}{c_0^2} \frac{D_0^3 p}{Dt^3} + \frac{\delta_{tv}}{c_0^4} \frac{D_0^3 p}{Dt^3} = -\left( \nabla^2 \mathcal{L} + \frac{1}{c_0^2} \frac{D_0^2 \mathcal{L}}{Dt^2} \right) + \mathcal{O}(\varepsilon_{w}^3 M_{ac}) + \mathcal{O}(M_{ac}^3).
$$

(4.167)
\(4.167\), first derived in \[169\], generalizes an equation developed by Cleveland \[45\] to moving media. In turn, Cleveland’s equation generalizes the well-known Westervelt equation \[83\] of nonlinear acoustics to relaxing media. Hence, (4.167) is referred to as the augmented Westervelt equation.

### 4.4.2 The lossy nonlinear Tricomi equation (LNTE)

We are now ready to perform the inner expansion procedure to arrive at a Tricomi equation. As losses do not affect the inner variables, we once again introduce $\tilde{z}_w$ (4.106) and $\tilde{t}_w$ (4.107),

$$
\tilde{z}_w = \frac{Z}{\delta_w} = \left( \frac{2}{R_{tot}(O) c^2(O)} \right)^{1/3} Z,
$$

(4.168)

$$
\tilde{t}_w = \omega_{ac} \left[ t - \tau(O) - \frac{X}{c(O)} \left( 1 + \frac{Z}{R_{xz}} \right) \right].
$$

(4.169)

\(\text{Following Cleveland} \ [45\], one could additionally define a relaxation “coefficient” in terms of the relaxation operator to consolidate all terms containing a third-order material derivative. Salamone does not do so in order to make the boundary layer approximations to follow more explicit.\)
as well as the familiar normalizations for the acoustic variables (4.108),

\[ p(X, Y, Z, t) = P_c \overline{\rho}(\overline{z}_w, \overline{t}_w), \quad \text{and} \quad \mathbf{u}(X, Y, Z, t) = \begin{bmatrix} U_{cX} \overline{n}_X(\overline{z}_w, \overline{t}_w) \\ U_{cY} \overline{n}_Y(\overline{z}_w, \overline{t}_w) \\ U_{cZ} \overline{n}_Z(\overline{z}_w, \overline{t}_w) \end{bmatrix}. \] (4.170)

We also continue to use \( \varepsilon_w (4.102) \) as a dimensionless ordering parameter for diffraction effects. Hence, the approximations of Section 4.3.3 hold equally well here, as do the leading-order relationships between the pressure and particle velocity scales (4.115),

\[ U_{cX} = \frac{P_c}{\rho_0c_0}, \quad U_{cY} = \varepsilon_w^3 \frac{P_c}{\rho_0c_0}, \quad \text{and} \quad U_{cZ} = \varepsilon_w \frac{P_c}{\rho_0c_0}. \] (4.171)

We now apply these results to (4.167), starting with the Lagrangian density:

\[ \mathcal{L} = \frac{1}{2} \rho_0 u^2 - \frac{1}{2} \frac{p^2}{\rho_0c_0^2} = \frac{1}{2} \rho_0 \left[ (U_{cX} \overline{n}_X)^2 + (U_{cY} \overline{n}_Y)^2 + (U_{cZ} \overline{n}_Z)^2 \right] - \frac{1}{2} \frac{(P_c \overline{\rho})^2}{\rho_0c_0^2} \] (4.172)

\[ = \frac{1}{2} \frac{P_c^2}{\rho_0c_0^2} \left[ \overline{n}_X^2 + \varepsilon_w^2 \overline{n}_Y^2 + \varepsilon_w^4 \overline{n}_Z^2 \right] - \frac{1}{2} \frac{\varepsilon_w^2 \overline{\rho}^2}{\rho_0c_0^2}. \]

To leading order, \( \overline{n}_X = \overline{\rho} \), so the first and last terms of this expression cancel. Therefore, the dominant term in the Lagrangian density is of order \( P_c^2 \varepsilon_w^2 = \mathcal{O}(M_{ac}^2 \varepsilon_w^2) \) within the boundary layer, hence negligible at order \( \varepsilon^2 M_{ac} \).

Applying (4.172) and the approximations of 4.3.3 to (4.167) results in

\[
\left( 1 + 2a_w \varepsilon_w^2 \overline{z}_w \right) \frac{\partial^2 \overline{n}}{\partial t_w^2} + \varepsilon_w^2 \frac{\partial^2 \overline{n}}{\partial z_w^2} - \left( 1 - 2 \varepsilon_w^2 \overline{z}_w \right) \left( 1 - 2M_X + M_X^2 \right) \frac{\partial^2 \overline{n}}{\partial t_w^2} + 2\varepsilon_w M_Z \frac{\partial^2 \overline{n}}{\partial t_w \partial z_w}
\]

\[
+ \left( \omega_{ac} \frac{\delta_{t_w}}{c_0^2(O)} \left( 1 - 4 \varepsilon_w \overline{z}_w \right) \right) + \left( 1 - 2 \varepsilon_w \overline{z}_w \right) \sum_{\nu} \frac{\omega_{ac} m_{\nu} t_{\nu}}{1 + \omega_{ac} t_{\nu} \left( \frac{M_X}{\partial t_w} + \varepsilon_w M_Z \frac{\partial}{\partial \varepsilon_w} \right)}
\]

\[
\times \left( 1 - 3M_X + M_X^2 \right) \frac{\partial^2 \overline{n}}{\partial t_w^2} + 3\varepsilon_w M_Z \frac{\partial^2 \overline{n}}{\partial t_w \partial z_w}
\]

\[
+ \beta M_{ac} \left( 1 - \varepsilon_w^2 b_w \overline{z}_w \right) \left( 1 - 4 \varepsilon_w \overline{z}_w \right) \left( 1 - 2M_X + M_X^2 \right) \frac{\partial^2 \overline{n}}{\partial t_w^2} + 2\varepsilon_w M_Z \frac{\partial^2 \overline{n}}{\partial t_w \partial z_w} = 0,
\] (4.173)

where

\[ a_w \equiv \frac{R_{tot}(O)}{2R_{XZ}}, \quad b_w \equiv \frac{R_{tot}(O)}{2R_{pow}}, \quad \varepsilon_w \equiv \frac{R_{tot}(O)}{2R_{cw}}. \] (4.174)
In this expression, we see the usual dimensionless parameters representing the strength of diffraction ($\varepsilon_w$), nonlinearity ($M_{ac}$), and wind ($M_X, M_Z$) effects. In order to determine the strength of thermoviscous absorption and relaxation, similar dimensionless parameters must be defined. By inspection of (4.173), we see that a natural choice for the thermoviscous loss term is the parameter $\alpha$, defined by

$$\alpha \equiv \omega_{ac}\delta_{tv}/c_0(O)^2.$$  \hspace{1cm} (4.175)

Comparing with (4.166), we see that $\alpha$ is equal to $2\alpha_{tv}(\omega_{ac})\lambda_{ac}$, and so is a dimensionless measure of the energy dissipated by thermoviscous effects over a characteristic acoustic wavelength [151]. Similarly, we can define a dimensionless relaxation time by

$$\bar{t}_\nu \equiv \omega_{ac}t_\nu,$$  \hspace{1cm} (4.176)

so that the quantity

$$\bar{\theta}_\nu \equiv m_\nu\bar{t}_\nu$$  \hspace{1cm} (4.177)

may be interpreted as a measure of the dispersive effect of vibrational relaxation over a single characteristic period $\omega_{ac}^{-1}$. At the end of Chapter 3, we noted that $\varepsilon_w, M_{ac},$ and $M_Z$ are typically on the order of 0.1, 0.005, and 0.08 respectively. In comparison, $\alpha$ and $\bar{t}_\nu$ are typically on the order of $10^{-4}$ to $10^{-5}$ for most of the audible frequency range [169]. Therefore, the terms in (4.173) associated with thermoviscous absorption and relaxation are expected to be negligible compared to the retained $O(\varepsilon_w^2)$ terms, and perhaps even third-order terms. But, as mentioned in the introduction, Salamone argues that these terms are still important in the characterization of focused signatures of sufficiently high frequencies. Hence, we retain the leading-order thermoviscous absorption and relaxation terms, regardless of their size, while maintaining the usual $O(\varepsilon_w^2)$ prescription for all other terms. The result is Salamone’s lossy nonlinear Tricomi equation $^{(29)}$ [169, 170],

$$\frac{\partial^2 \tilde{p}}{\partial z^2} - \tau \frac{\partial^2 \tilde{p}}{\partial t^2} - 2\varsigma \frac{\partial^2 \tilde{p}}{\partial t \partial z} + \left(\frac{2M_X - M_Z^2}{\varepsilon_w^2}\right) \frac{\partial^2 \tilde{p}}{\partial t^2}$$

$$+ \frac{1}{\varepsilon_w^2} \left(\alpha + \sum_{\nu} \frac{\bar{\theta}_\nu}{1 + \bar{t}_\nu \frac{\partial}{\partial t}}\right) \frac{\partial^2 \tilde{p}}{\partial t^2} + \mu_w \frac{\partial^2 \tilde{p}^2}{\partial t^2} = 0,$$  \hspace{1cm} (4.178)

$^{(29)}$In [169], the term containing $M_Z$ has $\varepsilon_w^2$ in the denominator rather than the $\varepsilon_w$ seen here (within $\varsigma$). Comparing to (4.173) and Auger’s result (4.116), we see that this is a minor algebra mistake on Salamone’s part when dividing through by $\varepsilon_w^2$ (which does not appear in the companion article [170]).
with $\zeta \equiv M_Z/\varepsilon_w$ and $\mu_w \equiv \beta M_{ac}/\varepsilon_w^2$ as in (4.116). With that, the model for the focusing of weakly nonlinear acoustic waves approaching smooth caustics has been generalized to moving, lossy, relaxing media.

A curiosity of (4.178) is that the term containing $M_X$ and its square, associated with the ambient flow along a wavefront normal at the caustic, does not appear in Auger’s windy nonlinear Tricomi equation (4.116). This term arises from expressing the second-order ambient material derivative $D_0^2/Dt^2$ in $(\bar{z}_w, \bar{t}_w)$ coordinates, a step required as a result of combining the mass and momentum equations before applying the usual boundary layer approximations. In comparison, Auger approximates the two equations separately, then combines them [10]. Hence, the operations of combining the acoustic equations and applying the boundary layer approximations do not appear to commute, at least when material derivatives are involved. In the absence of a formal proof, we suspect Salamone’s procedure to be the correct one, as it postpones the boundary layer approximation for longer. This is supported by the appearance of similar terms in an analogous generalization of of the KZK equation [51], as Salamone notes [169].

In Figure 4.11, the lossy (LNTE) and lossless (NTE) models are compared for an incident N-wave a distance $\tau = 0.5$ from the geometrical caustic (winds are not considered, so $\bar{z}_w = \bar{\tau}$). For reference, the profile of the incident wave after propagation through a lossy medium is also included. The incident profile has been scaled and phased according to the incoming wave portion of (4.122) to compare how favorably each solution matches with the geometrical acoustic field outside of the boundary layer [169]. We see that the effect of losses is to round and attenuate the waveform, tending to decrease the amplitude-dependent phase speed of the waveform. On the other hand, relaxation-induced dispersion tends to make high-frequency components of the waveform (e.g. shocks) travel faster. Dispersion wins out (consistent with the order-of-magnitude analysis at the end of Chapter 3), and the lossy waveform arrives slightly earlier in time than the lossless one. We also note an increased rise time, turning the idealized vertical discontinuity into a smooth (but very steep) jump in pressure more descriptive of real shock waves. These effects are small on the scale of the overall waveform (note the time scale), but still noticeable as Salamone proposed. We also see that the lossy focused waveform is qualitatively similar to the incident waveform, indicating a smoother matching when both the inner and outer expansions account for losses.

Figure 4.12 depicts the lossy and lossless focused waveforms, as well as a waveform with thermoviscous losses only, a distance $\bar{\tau} = 0.15$ from the geometrical caustic. We see that closer to the caustic, the conclusions of the previous paragraph still hold, but to
Figure 4.11: Comparison of focused waveforms with thermoviscous losses and relaxation (blue) and no losses or relaxation (green) at $\tau = 0.5$. The red waveform depicts the incident N-wave, scaled and phased according to geometrical matching condition (4.122), [169]

a lesser degree. As diffraction and focusing effects take hold, thermoviscous absorption does little to counteract nonlinearity, leading to waveforms nearly identical to the lossless case save for marginally lower amplitudes, longer rise times, and later arrival of shocks. The dispersive and dissipative effects of relaxation are more pronounced, resulting in notably lower amplitudes, increased rise times, and earlier arrival of shocks.

4.5 Chapter summary

In this chapter, we have recounted the role nonlinearity plays in the focusing of sonic booms. The higher amplitudes of weakly nonlinear waves required the inclusion of quadratic nonlinear terms in the acoustic equations, resulting in a weakly nonlinear geometrical acoustic theory for the outer solution. The rays remained unchanged, but the amplitude and phase were found to evolve according to a nonlinear transport equation equivalent to an inviscid Burgers’ equation. Fortunately, this result was similar enough to the linear case that the same limiting behavior could be used for the sake of
Figure 4.12: Comparison of focused waveforms with thermoviscous losses and relaxation (blue), thermoviscous losses only (red), and no losses (green) at \( \tau = 0.15 \), [169]

determining the inner variables, leading to the nonlinear Tricomi equation. Unlike in the linear case, no analytical solution is known for this equation\(^{(30)}\) when the incident wave contains shocks, but numerical simulations indicate that its solution remains finite for both smooth and weak shock waves, owing to a careful balance between focusing and nonlinear steepening effects. The equation was then recast in terms of a velocity potential from which the Guiraud scaling law for step shocks was developed. The potential equation also revealed an intimate connection between the nonlinear Tricomi equation and the steady transonic small-disturbance equation of aerodynamics. Finally, the generalizations of the nonlinear Tricomi equation to both moving and lossy, relaxing media were presented. It was found that wind tends to affect the caustic geometry and ground footprint of the superboom while losses and relaxation serve to decrease focused waveform amplitudes, increase the rise time of shocks, and hasten the arrival time of superbooms.

\(^{(30)}\) As discussed in Appendix A, the linearization method of Gill and Seebass [75, 175] does produce analytical solutions for smooth and sufficiently weak waves, but their solution is only approximate for weak shocks.
Chapter 5
A 3-D Lossy Nonlinear Tricomi Equation

5.1 Introduction

In every Tricomi equation considered thus far, the acoustic velocity in the transverse ($Oy$ or $OY$) direction was found to be an order $\varepsilon^3$ smaller than the longitudinal ($Ox$ or $OX$) component, and hence neglected. As a result, these equations are effectively 2-dimensional models. This behavior is well-substantiated near the flight track of Mach cutoff and acceleration focusing flights (see Appendix A), but far enough from this section of the caustic (e.g. approaching lateral cutoff), one would expect a non-negligible change in the amplitude of the acoustic perturbation. The standard Tricomi equations cannot predict this decay, and would instead imply that there is no change in acoustic pressure whatsoever when moving laterally away from the flight track (or any other section of the caustic). Therefore, in order to have a complete picture of the acoustic field created by a focusing sonic boom, we must incorporate lateral flow effects into the Tricomi equation. The goal of this chapter is to propose how such a model equation might come about and what questions arise in the process.

We begin with a review of existing literature on fully 3-dimensional sonic boom focusing, though references are rather scarce in this regard. The key articles come from a small group of authors, and study the 3-D focusing of sonic booms generated during both steady and unsteady supersonic flight in a lossless, quiescent, stratified atmosphere. Interestingly, a basic assumption of the 3-D focusing model will be seen to contradict the key result on which the locally 2-D model is founded. Namely, the formal development of a 3-D focusing equation requires that the characteristic length scales of the boundary layer be the same for both transverse ($Oy$ and $Oz$) directions.
Otherwise, at least from an order-of-magnitude perspective, the equation reduces to the 2-D model. But equivalent length scales imply comparable particle velocity strengths in the \( O_y \) and \( O_z \) directions, in direct contrast to the velocity hierarchy (3.93) leading to the 2-D model.

The original work of this thesis begins in Section 5.4, with an attempt to resolve the apparent contradiction between the 2-D and 3-D focusing models. This will be done by applying an argument similar to that used by Salamone to justify retaining losses in LNTE. We then extend the ray coordinate expansion used in Chapter 3. In particular, we will amend the asymptotic (inner) description of the windless geometrical-acoustic solution to better account for the behavior of the eikonal along the \( O_y \) axis of the local caustic coordinate system. This will suggest a modified set of inner variables with which to express the nonlinear acoustic equations, leading to a 3-dimensional nonlinear Tricomi equation (3DNTE) that reduces to existing model equations in the appropriate circumstances. We then apply the knowledge gained from this exercise and the results of earlier chapters to propose a marriage of the 3-dimensional and lossy focusing models. The result is a form of 3-dimensional lossy nonlinear Tricomi equation (3DLNTE) similar to an augmented form of the well-known \textit{KZ equation} governing the propagation of finite-amplitude acoustic beams.

We emphasize at the outset that the resulting 3DLNTE has several undetermined parameters requiring further investigation. In particular, our solution for the contradiction between the 2-D and 3-D models requires a reinterpretation of the characteristic length scale for the \( O_y \) direction which we do not calculate here, and which requires clearer physical interpretation. Further, we do not perform an analogous extension of the asymptotic geometrical-acoustic solution to moving media. Therefore, while it remains straightforward to determine the basic form of the inner variables for moving media, we have not calculated the generalizations of the coefficients (various radii of curvature) appearing therein. That said, the expansion procedure employed should readily generalize to moving media, and in the absence of wind the undetermined coefficients must reduce to the expressions we have determined. We conclude the chapter by addressing how the 3DLNTE model may be numerically solved once the noted gaps have been addressed.
5.2 The 3-D nonlinear Tricomi equation

Since the 2-D variants of the Tricomi equation are generally successful in predicting focused sonic boom waveforms undertrack, it is rare to see the cross-flow \((y \text{ or } Y)\) terms retained. However, a few references have considered their effects, concluding that they may become significant in certain regions of the sonic boom carpet. Of primary interest are the series of conference papers by Cheng, Lee, Hafez, and Guo \([39, 41, 80]\) in which a 3-D nonlinear Tricomi equation is derived and numerically solved\(^{(1)}\). The derivation of this equation is the subject of this section.

5.2.1 Mid-field potential equation

Cheng et al. begin by considering a slender body of length \(L\) in steady horizontal motion at a speed \(u_0\) in a perfect gas \([39, 41]\). In body-fixed coordinates, this may instead be thought of as a steady flow of speed \(u_0\) passing a stationary body. \(x\) is taken to be the streamwise coordinate, \(z\) directed toward the ground, and \(y\) the remaining lateral coordinate, as depicted in Figure 5.1. Further, the gas is taken to be both gravitationally

\(\text{Figure 5.1: Coordinate system used in [39]}\)

\(^{(1)}\)Cheng, Lee, Hafez, and Guo derive and numerically solve the 3-D nonlinear Tricomi equation in \([41]\), as well as addressing related problems in the matching of the sonic boom mid-field description to an improved (relative to the F-function) near-field description and the consideration of sonic boom penetration into a wavy ocean. This study then branches out into two more detailed articles—one numerical, by Guo and Hafez \([80]\), and one theoretical, by Cheng and Lee \([39]\). There is significant overlap of results between references, but the exposition presented here primarily follows \([39]\).
and thermally stratified in the $z$ direction, such that the ambient pressure, density, and temperature (and hence entropy) vary with altitude. Scale heights for variations in pressure, density, and entropy are therefore introduced as $H_{p0}$, $H_{\rho_0}$, and $H_{s0}$ respectively. In general, atmospheric density depends on both the pressure and the entropy, so the scale heights are coupled. In a perfect gas, they are related \cite{39, 47, 151} by

$$\frac{1}{H_{\rho_0}} = \frac{1}{\gamma} \left( \frac{1}{H_{p0}} - \frac{1}{H_{s0}} \right). \tag{5.1}$$

In our discussion of the transonic small disturbance equation in Section 4.2.7, we noted that a nonzero entropy gradient in the fluid implies nonzero vorticity (known as \textit{baroclinic vorticity}), violating the assumption of flow irrotationality and invalidating the use of a velocity potential to describe the entire flow. A similar problem arises here, but for the ambient state rather than the perturbation. Cheng et al. address this issue by arguing that, at least to the order of approximation to be made, the perturbation velocity $u'$ can be decomposed as the sum of a velocity potential term and a correction term for the baroclinic vorticity:

$$u' = \nabla \Phi' - \frac{\Phi'}{\gamma H_{s0}} \hat{k}, \tag{5.2}$$

where $\hat{k}$ is a unit vector in the $z$ direction. This amounts to decomposing the velocity field into an irrotational component (the gradient of the velocity potential) and a divergence-free component (the baroclinic vorticity term), a process sometimes known as \textit{Helmholtz decomposition}.

To simplify the fluid equations, two small perturbation parameters are used. The first is the Mach number of the flow perturbation caused by the body, which we again denote $M_{fl}$. The ratio $L/\Delta$, where $\Delta$ is a quantity comparable to the atmospheric scale heights, is taken as the second small parameter (and will turn out to be related to the diffraction parameter $\varepsilon$). That is, the aircraft is presumed to be much shorter than a typical scale for atmospheric heterogeneity. The fluid equations will be approximated in the distinguished limit in which the two parameters are taken to be of a similar order, such that their ratio is held fixed:

$$M_{fl} \frac{\Delta}{L} = \mathcal{O}(1). \tag{5.3}$$

Substituting (5.2) into the fluid equations (2.1)–(2.4) (with gravitational effects retained) and discarding terms smaller than second order in $M_{fl}$ and $L/\Delta$ then results in an equation for the mid-field in terms of the perturbation potential $\Phi'$. Introducing the
nondimensional (hat) variables
\[ \hat{x} \equiv \frac{x}{L}, \quad \hat{y} \equiv \frac{y}{L}, \quad \hat{z} \equiv \frac{z}{L}, \quad \hat{\phi} \equiv \frac{\Phi'}{L u_0 M_{fl}}, \]
(5.4)

this equation takes the form\(^{(2)}\)
\[ \frac{\partial^2 \hat{\phi}}{\partial \hat{y}^2} + \frac{\partial^2 \hat{\phi}}{\partial \hat{z}^2} + (1 - M_0^2) \frac{\partial^2 \hat{\phi}}{\partial \hat{x}^2} = M_{fl} M_0^2 \frac{\partial}{\partial \hat{x}} \left[ \frac{\gamma + 1}{2} \left( \frac{\partial \hat{\phi}}{\partial \hat{x}} \right)^2 + \left( \frac{\partial \hat{\phi}}{\partial \hat{y}} \right)^2 + \left( \frac{\partial \hat{\phi}}{\partial \hat{z}} \right)^2 \right] - \frac{L}{H \rho_0} \frac{\partial \hat{\phi}}{\partial \hat{z}}, \]
(5.5)

where \( M_0 \equiv u_0/c_0 \) is a function of \( \hat{z} \) due to the temperature stratification. Retaining only the left-hand side of this equation results in
\[ \frac{\partial^2 \hat{\phi}}{\partial \hat{y}^2} + \frac{\partial^2 \hat{\phi}}{\partial \hat{z}^2} + (1 - M_0^2) \frac{\partial^2 \hat{\phi}}{\partial \hat{x}^2} = 0, \]
(5.6)
a 3-dimensional analog of the linearized potential equation (4.52), while neglecting only the last term on the right of (5.5) results in a form of 3-dimensional transonic small disturbance equation analogous to\(^{(3)}\) (4.54). This term combines the terms associated with gravitational and thermal stratification using (5.1) and accounts for the baroclinic vorticity effect.

### 5.2.2 Characteristics of the Mach conoid

At this point, we would typically reduce the governing equation, (5.5), to a set of geometrical acoustic equations with which to determine the inner variables. However, geometrical acoustics is not the only possible geometrical description of the sonic boom mid-field. We could also speak in terms of the Mach cone itself (really, the Mach conoid, as heterogeneity distorts its shape from a perfect cone).

Recall that the rays for a supersonic aircraft are initialized along a cone/conoid (called the ray cone) complementary to the Mach cone. The characteristics of this cone are the rays. Similarly, the Mach conoid has a certain set of characteristic curves lying on

\(^{(2)}\)In Cheng et al.’s work, the factor multiplying \((\partial \hat{\phi}/\partial \hat{x})^2\) is \(1 + \frac{\gamma - 1}{2} M_0^2\). This does not appear to be consistent with the full potential equation, (4.51). We have instead substituted what we believe to be the correct expression, \((\gamma + 1)/2\). This change will also be carried through in potential equations to follow. It should however be noted that in the transonic regime the two expressions are nearly equal.

\(^{(3)}\)The last two terms in brackets can often be discarded through other arguments, as in \([18]\).
its surface, perpendicular to the rays (see Figure 5.2). Cheng et al. choose to describe the

mid-field in terms of these characteristics rather than the rays. Just as for geometrical
acoustics, the development of a singularity in this description will indicate the need for a
more accurate approximation, again leading to a boundary layer description of the field
near a caustic.

Cheng et al. first consider the Mach conoid associated with the linearized mid-field
equation, (5.6). A parametric equation for this surface, say $\hat{x} = \hat{x}_M(\hat{y}, \hat{z})$, may be found
by applying the method of characteristics to (5.6). The result is

$$\left(\frac{\partial \hat{x}_M}{\partial \hat{y}}\right)^2 + \left(\frac{\partial \hat{x}_M}{\partial \hat{z}}\right)^2 = M_0^2(\hat{z}) - 1. \quad (5.7)$$

The individual characteristic curves lying on the surface of the Mach conoid (known as
the bicharacteristics\(^{(5)}\) of (5.6)) are then found by another application of the method of
characteristics to (5.7), leading to a coupled system of ordinary differential equations
in $\hat{x}_M, \hat{y}, \hat{z}, \partial \hat{x}_M/\partial \hat{y},$ and $\partial \hat{x}_M/\partial \hat{z}$. Integrating these equations leads to the conclusion

\(^{(4)}\)For this reason, the Mach conoid is known as the characteristic surface of (5.6). Similarly, wavefronts
are the characteristic surfaces of the Helmholtz equation (2.30) (note the similarity of (5.7) to the eikonal
equation (2.44)) \(^{[107]}\).

\(^{(5)}\)Similarly, the rays defined by (2.56), (2.57) are the bicharacteristics of the Helmholtz equation \(^{[106]}\).
that, at least for the linear system (5.6),

$$\frac{\partial \hat{x}_M}{\partial \hat{y}} = \left. \frac{\partial \hat{x}_M}{\partial \hat{y}} \right|_0 = \text{cst.}, \quad \frac{\partial \hat{x}_M}{\partial \hat{z}} = \pm \sqrt{M_0^2(\hat{z}) - 1 - \left(\frac{\partial \hat{x}_M}{\partial \hat{y}} \right)_0^2}. \quad (5.8)$$

The first expression indicates that $\partial \hat{x}_M/\partial \hat{y}$ does not vary along a characteristic curve, instead remaining equal to its initial value at the apex of the Mach conoid. Hence, in a stratified atmosphere, each characteristic remains in a fixed plane defined by its initial direction on the Mach conoid.

The second equation indicates that each characteristic curve will reach a maximum (i.e., nearest to the ground) value of $\hat{z}$ when $\partial \hat{x}_M/\partial \hat{z} = 0$, at which point the characteristic will refract back toward the upper atmosphere. Provided the sound speed decreases with altitude, the altitude of these turning points will increase with $|\partial \hat{x}_M/\partial \hat{y}|$, such that only the undertrack ($\partial \hat{x}_M/\partial \hat{y} = 0$) characteristic actually reaches the sonic line. Therefore, if $u_0$ is chosen such that $M_0 = u_0/c_0 = 1$ above the ground, no characteristics will ever reach the ground. This is the Mach cutoff phenomenon, now described in terms of the characteristics of the Mach conoid.

Equation (5.8) may also be used to determine the lateral extent of the sonic boom carpet by thinking of the ground as a (geometric) plane slicing through the Mach conoid surface. If the plane is moved closer to the aircraft, it will intersect characteristics associated with larger azimuthal angles. Moving the plane down leads to fewer intersections, and none at all once the plane goes below the sonic line. Therefore, the lateral extent of the sonic boom carpet is tied to the position of the sonic line relative to the ground, shrinking to zero under Mach cutoff conditions. With this in mind, one would think the lateral pressure gradient would be significant in Mach cutoff focusing, apparently contradicting the basic premise of the locally 2-D focusing model. Cheng et al. use this argument to propose that the lateral field behavior should not be neglected when considering the superboom phenomenon. However, these results are based on the linear theory, and the nonlinear terms in (5.5) will need to be considered for a complete picture.

---

(6) Again, the condition $M_0 = 1$ actually defines a surface in three dimensions, but sonic line gets the point across.
5.2.3 Mach conoid coordinates and nonlinearity

Cheng et al. now shift to a coordinate system affixed to the linear Mach conoid defined by (5.7). As a result, any nonlinear flow effects on the Mach conoid will appear as a departure from this reference surface\(^{(7)}\). The transverse coordinates are also renormalized with respect to \(\Delta\) rather than \(L\) in order to emphasize the distinctly different sizes of the two length scales. The Mach conoid variables \((\hat{\xi}, \tilde{y}, \tilde{z})\) so defined are given by

\[
\hat{\xi} \equiv \hat{x} - \hat{x}_M(\tilde{y}, \tilde{z}) = \frac{\Delta}{L} [\hat{x} - \hat{x}_M(\tilde{y}, \tilde{z})],
\]

\[
\tilde{y} \equiv \frac{L}{\Delta} \hat{y}, \quad \tilde{z} \equiv \frac{L}{\Delta} \hat{z}.
\]

In addition to \(\hat{\phi}\), these variables are taken to be of unit order in the mid-field. Recasting potential equation (5.5) in terms of the Mach conoid variables (5.9) yields a reduced equation for the mid-field of the form

\[
\Gamma \hat{u} \frac{\partial \hat{u}}{\partial \xi} + \left( \frac{\partial \hat{x}_M}{\partial \tilde{y}} \frac{\partial \hat{u}}{\partial \tilde{y}} + \frac{\partial \hat{x}_M}{\partial \tilde{z}} \frac{\partial \hat{u}}{\partial \tilde{z}} \right) + \frac{1}{2} \left( \frac{\partial^2 \hat{x}_M}{\partial \tilde{y}^2} + \frac{\partial^2 \hat{x}_M}{\partial \tilde{z}^2} + \frac{\Delta}{H_\rho} \frac{\partial \hat{x}_M}{\partial \tilde{z}} \right) \hat{u} = \mathcal{O}((M_{fl} + L/\Delta)^2),
\]

where

\[
\hat{u} \equiv \frac{\partial \hat{\phi}}{\partial \xi}, \quad \Gamma(\tilde{z}) \equiv M_{fl} \frac{\Delta \gamma + 1}{2} M_0.\]

Further application of the method of characteristics yields the Mach conoid, characteristic curves, and geometrical solution of (5.10). Comparing to the linear results, Cheng et al. find that nonlinearity displaces the Mach conoid (and the characteristic curves lying on it) windward by an amount proportional to the local field strength \(\hat{u}\), recovering the amplitude-dependent phase speed of weakly nonlinear waves. On the other hand, the projections of the characteristics into planes \(x = \text{cst.}\) are unaffected by nonlinearity. The field strength in question is given by

\[
\hat{u}(\hat{\xi}, \tilde{y}, \tilde{z}) = \hat{u}(\hat{\xi}_0, \tilde{y}_0, \tilde{z}_0) \left( \frac{\rho_0(\hat{\xi}_0, \tilde{y}_0, \tilde{z}_0)}{\rho_0(\hat{\xi}, \tilde{y}, \tilde{z})} \right)^{1/2} \exp \left\{ - \frac{1}{2} \int_{\tilde{z}_0}^{\tilde{z}} \tilde{A} \, d\tilde{z} \right\},
\]

where

\[
\tilde{A} \equiv \left( \frac{\partial \hat{x}_M}{\partial \tilde{z}} \right)^{-1} \left( \frac{\partial^2 \hat{x}_M}{\partial \tilde{y}^2} + \frac{\partial^2 \hat{x}_M}{\partial \tilde{z}^2} \right).
\]

\(^{(7)}\)This is similar to the use of the retarded time frame in Section 4.2.2.
and \((\hat{\xi}_0, \bar{y}_0, \bar{z}_0)\) are the values of \(\hat{\xi}, \bar{y}, \bar{z}\) at some reference point, e.g. near the apex of the Mach conoid. Note that \(\bar{x}_M\) still refers to the reference conoid of the linear problem.

### 5.2.4 Singularity in the mid-field

Much like the geometrical acoustic solution, (5.12) becomes singular at certain points in the mid-field, indicating a nonuniformity in the chosen variables such that they may no longer be considered unit order. From the definition of \(\tilde{A}\) we see that a singularity occurs whenever \(\partial \bar{x}_M / \partial \bar{z} = 0\) or, returning to (5.8), when the quantity \(R\) defined by

\[
R \equiv \left( \frac{\partial \bar{x}_M}{\partial \bar{z}} \right)^2 = M_0^2(\bar{z}) - 1 - \left( \frac{\partial \bar{x}_M}{\partial \bar{y}} \right)_{\bar{y}=0}^2
\]

vanishes. When \(\partial \bar{x}_M / \partial \bar{y}\) is small, \(R = 0\) corresponds approximately to the sonic condition \(M_0 = 1\). In fact, the mid-field ceases to be hyperbolic for any initial value of \(\partial \bar{x}_M / \partial \bar{y}\) when the flow becomes sonic. Therefore, to consider the limiting behavior of the mid-field as the flow approaches sonic speeds from the supersonic (hyperbolic) side, Cheng et al. restrict consideration to characteristics with vanishingly small initial \(\partial \bar{x}_M / \partial \bar{y}\)—a narrow range of launch angles about the under-track direction.

Much like Gill and Seebass [75], Cheng et al. examine the singularity in the mid-field solution by expanding the Mach number in a neighborhood of the sonic line. They do so through the quantity \(B_0^2 = M_0^2 - 1\) rather than \(M_0^2\) directly, such that

\[
B_0^2(\bar{z}) \approx \left( \frac{dB_0^2}{d\bar{z}} \right)_* (\bar{z} - \bar{z}_*),
\]

where the \(\text{*}\) subscript signifies the value at the sonic condition \(B_0^2 = 0\). The asymptotic behavior of the reference Mach conoid \(\bar{x}_M\) as \(\bar{z} \rightarrow \bar{z}_*\) is studied first. The authors find that under approximation (5.15), the equation satisfied by the reference Mach conoid (5.7) admits a self-similar solution of the form

\[
\bar{x}_M - \bar{x}_M* = \mathcal{G}(\nu)|\bar{z} - \bar{z}_*|^{3/2}, \quad \nu \equiv \frac{\bar{y}}{\bar{z} - \bar{z}_*},
\]

where \(\mathcal{G}\) satisfies an ordinary differential equation in \(\nu\). This result has several important implications. First, for \(\bar{z}\) near the sonic line, \(\bar{x}_M\) vanishes as \(|\bar{z} - \bar{z}_*|^{3/2}\), just as for the acoustic eikonal near the caustic (3.68). Second, holding \(\bar{z} - \bar{z}_*\) fixed and approximating
(5.16) about $\tilde{y} = 0$ ($\nu = 0$), Cheng et al. find that

$$\tilde{x}_M - \tilde{x}_{M*} \sim G(0)|\tilde{z} - \tilde{z}_s|^3/2 + \left(\frac{dG}{d\zeta}\right)_{\nu=0} \tilde{y}|\tilde{z} - \tilde{z}_s|^{1/2} + \cdots. \quad (5.17)$$

For $\tilde{y} \neq 0$ and $\tilde{z}$ sufficiently near the sonic line, the second term in this expansion dominates the first. Geometrically, this means that near the sonic line, the projection of the Mach conoid in any plane $\tilde{y} \neq 0$ is a parabola. Only in the $\tilde{y} = 0$ plane can the first term dominate, yielding the equation for a cusp (a semicubical parabola). Further analysis confirms the same trend in the characteristic curves themselves. In short, Cheng et al. have demonstrated that the characteristics of the Mach conoid are generally parabolas, only sharpening into a cusp in the symmetry plane $\tilde{y} = 0$ for which the characteristic reaches the sonic line (see Figure 5.3). Moreover, this trend is not disturbed by nonlinear steepening effects.

![Figure 5.3: Characteristics of the (linear) Mach conoid at azimuthal angles in the vicinity of the undertrack plane (90°); adapted from [39]](image-url)
A final consequence of (5.16) is that as $\bar{z} \to \bar{z}_*$,

$$
\frac{\partial \bar{x}_M}{\partial \bar{y}}, \frac{\partial \bar{x}_M}{\partial \bar{z}} \sim |\bar{z} - \bar{z}_*|^{1/2},
\frac{\partial^2 \bar{x}_M}{\partial \bar{y}^2} + \frac{\partial^2 \bar{x}_M}{\partial \bar{z}^2} \sim |\bar{z} - \bar{z}_*|^{-1/2},
$$

(5.18)

Applying the last relation to (5.12) leads to the conclusion that near the sonic line,

$$
\tilde{u} \sim |\bar{z} - \bar{z}_*|^{C/2},
$$

(5.19)

for some constant $C$ of the same sign as $\tilde{A}$. Through numerical calculations, Cheng et al. find that $\tilde{A}$, and hence $C$, become negative as $B^2_0 \to 0$, consistent with the expected blowup of $\tilde{u}$ in the boundary layer. Based on (3.75) we would expect $C = -1/2$, but the authors do not propose an explicit value.

To summarize their work thus far, Cheng et al. have derived a first-order quasilinear potential equation (5.10) governing the mid-field of a sonic boom propagating in a stratified, perfect gas. This equation admits the solution (5.12) which becomes singular in the vicinity of the sonic line $B^2_0 = M^2_0 - 1 = 0$. This singularity may be traced back to the spatial derivatives of the Mach conoid in this region, (5.18), and is not resolved by accounting for nonlinearity alone. Therefore, (5.12) is too loose an approximation of the fluid equations in the boundary layer.

5.2.5 Patching up the singularity

Cheng et al. continue by noting that the singularity in the mid-field solution indicates that one or more variables used in the description of the mid-field are no longer of unit order in some vicinity of the sonic line. This nonuniformity must be addressed by recasting the governing equation in variables better suited to description of the boundary layer. The authors reason that the appropriate rescaling of the variables may be inferred from (5.16) and (5.18). In particular, provided $\xi$ is still of unit order, $\bar{x}_M - \bar{x}_M*$ is of order $L/\Delta$ by (5.9). Therefore, by (5.16), the vertical scale of the boundary layer $\bar{z} - \bar{z}_*$ must be of order $(L/\Delta)^{2/3}$, with the scales of the derivatives of $\bar{x}_M$ following immediately from (5.18). A new set of unit-order independent variables for the boundary layer may
then be introduced as
\[ \xi \equiv \hat{\xi}, \quad \pi_M \equiv \left( \frac{L}{\Delta} \right)^{-1} (\bar{x}_M - \bar{x}_{M^*}) \quad (\bar{y}, \bar{z}) \equiv \left( \frac{L}{\Delta} \right)^{-2/3} (\bar{y} - \bar{y}_*, \bar{z} - \bar{z}_*). \] (5.20)

Therefore, using (5.4), (5.9), and (5.20), Cheng et al. conclude that the physical (i.e. dimensional) length scales describing the extent of the boundary layer transverse to the flight path are each given by \( \Delta (L/\Delta)^{2/3} = L(\Delta/L)^{1/3} \). Since \( L/\Delta \) is presumed small, this result indicates that both transverse scales of the boundary layer are much larger than the scale of the aircraft, but much smaller than the characteristic length scale for atmospheric heterogeneity. Further, comparing \( L(\Delta/L)^{1/3} \) to (3.51) with \( L \sim \lambda_{ac} \) and \( \Delta \sim R \), we see that length scale \( L(\Delta/L)^{1/3} \) is analogous to the boundary layer thickness \( \delta \). Thus, the variables in (5.20) are a three-dimensional analog of the usual inner variables \( \bar{z} \) (3.77) and \( \bar{t} \) (3.78).

Backtracking to a more general (i.e., less approximate) form of (5.10) and applying (5.20), Cheng et al. arrive at a potential equation describing the boundary layer, given to leading order by

\[
\frac{\partial}{\partial \xi} \left( \frac{1}{2} \left( \frac{\partial \phi}{\partial \xi} \right)^2 \right) + 2 \left( \frac{\partial \pi_M}{\partial \bar{y}} \frac{\partial}{\partial \bar{y}} \left( \frac{\partial \phi}{\partial \xi} \right) + \frac{\partial \pi_M}{\partial \bar{z}} \frac{\partial}{\partial \bar{z}} \left( \frac{\partial \phi}{\partial \xi} \right) \right) + \frac{\partial^2 \pi_M}{\partial \bar{y}^2} + \frac{\partial^2 \pi_M}{\partial \bar{z}^2} = \frac{\partial^2 \phi}{\partial \bar{y}^2} + \frac{\partial^2 \phi}{\partial \bar{z}^2} + O((L/\Delta)^{2/3}),
\] (5.21)

where \( \phi \) is a reduced potential variable defined by
\[ \phi \equiv \frac{(\gamma + 1) M_{fl}}{(L/\Delta)^{2/3}} \phi. \] (5.22)

Recalling (3.80), we see that the quantity \( (L/\Delta)^{1/3} \) used to order terms in (5.21) is analogous to diffraction parameter \( \varepsilon \), while the scaling factor in (5.22) is analogous to nonlinearity factor \( \mu = \beta M_{ac}/\varepsilon^{2(8)} \).

Boundary layer equation (5.21) differs from the equation for the mid-field (5.10) in two key ways. First, the stratification term is negligibly small in the boundary layer, consistent with the neglect of weak heterogeneity in previous Tricomi equations. Therefore, (5.2) may be reduced to the standard definition of a potential, such that \( \phi \) is equal to the gradient of some (reduced) velocity field in the boundary layer. On the other hand,
the rescaled variables have “promoted” the transverse Laplacian of $\phi$ on the right-hand side of (5.21) to the same order as the left-hand side. These terms govern transverse diffraction and as in all previous Tricomi equations, their reinstatement removes the singularity caused by the caustic/sonic line. The difference is that in this formulation, the diffraction terms in both the $y$ and $z$ directions have been retained due to the use of the same rescaling for each in (5.20).

As it stands, (5.21) doesn’t appear very similar to previous Tricomi equations. However, Cheng et al. show that returning from Mach conoid fixed coordinates $(\xi, \tilde{y}, \tilde{z})$ to the $(x, y, z)$ coordinates defined with respect to the sonic line transforms (5.21) into

$$\frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} - \left( \frac{dB_0^2}{dz} \right) \frac{\partial \phi}{\partial \tilde{z}} \frac{\partial^2 \phi}{\partial \tilde{z}^2} = 0,$$

(5.23)
a 3-D form of the (velocity potential) nonlinear Tricomi equation, (4.36) or (4.61). Note also that the first term in parentheses is a form of transonic similarity parameter, as

$$\left( \frac{dB_0^2}{dz} \right) \frac{\partial \phi}{\partial \tilde{z}} \sim B_0^2 \left( \frac{\Delta}{L} \right)^{2/3} = \frac{M_0^2 - 1}{(L/\Delta)^{2/3}} \equiv -KL/\Delta,$$

(5.24)
consistent with (4.69) for a “thickness ratio” of $L/\Delta^{(9)}$.

References [38, 41, 80] solve (5.23) for an incident “patch” of uniformly spaced N-waves approaching a sonic line coinciding (approximately) with a hard, flat ground surface. In Figure 5.4a, pressure coefficients computed using the 3-D model (solid line) are compared to the standard nonlinear Tricomi equation (4.35) (dashed line) as the waveform propagates toward the sonic line in the undertrack plane ($y = 0$)\(^{(10)}\). We see that the 3-dimensional model tends to predict slightly lower amplitudes (in absolute value). This is attributed to a 3-dimensional relieving effect, in which the extra dimension allows the pressure field to spread out laterally rather than being confined to the undertrack plane [38, 190]. However, the difference is slight, substantiating the use of a 2-D formulation for undertrack focusing.

Figure 5.4b explores the lateral variation in the pressure coefficient by considering three values of lateral coordinate $y$. The far-left plot depicts signatures in the undertrack

\(^{(9)}\)If we take the heterogeneity length scale $\Delta$ as that of $M_0^2$: $M_0^2/(dM_0^2/dz) \approx (dM_0^2/dz)^{-1}$, (5.23) reduces identically to (a 3-dimensional form of) (4.61) up to the factor of 2 explained in the previous footnote.

\(^{(10)}\)Note that the $z$ axis has been reversed such that it now increases with altitude. Further, the pressure coefficient is plotted with respect to position ($x$), not time, resulting in plots that appear flipped compared to other references (e.g. Figure 7 in [192]).
plane $y = 0$ while the far-right plot corresponds to a lateral distance far from the flight track, $y >> 1$. The middle plot is a distance halfway between these two extremes, several boundary layer thicknesses away from the flight track. We see that the field tends to decay with increasing lateral distance, eventually becoming evanescent far from the flight track.

The 3-D character of the superboom region is further illuminated by Figure 5.4c, which compares the pressure contours in the undertrack plane (left) to those far from the flight track (right). While weaker, the field far from the flight track exhibits a shock structure clearly reminiscent of the undertrack plane, indicating that the superboom region is in fact spread over a large lateral extent.

Figure 5.4: Numerical results for the three-dimensional nonlinear Tricomi equation (5.23) developed by Cheng et al.

As expected from the flow hierarchy (3.93), the 2-D model is largely successful in predicting the pressure behavior near the undertrack plane. In fact, the similarity of the midspan and undertrack waveforms in Figure 5.4b suggests that the 2-D model may
remain a reasonable approximation for an extended lateral distance. Nevertheless, the standard 2-D nonlinear Tricomi model (4.35) is not capable of predicting the decay apparent in the final plots of Figures 5.4b and 5.4c. It is in this region nearing lateral cutoff that the utility of the 3-D model becomes clear. While the 2-D model is generally satisfactory in characterizing the superboom field near the flight track, a complete description of the focal region requires a fully 3-dimensional model.

5.3 Unsteady 3-D nonlinear Tricomi equation

Thus far, we have implicitly supposed that the Tricomi model applies wholesale to wave propagation near a fold caustic, regardless of whether this caustic is generated by atmospheric refraction or aircraft maneuvering. This hinges on the assumption that any unsteady effects associated with maneuvering are negligible in the boundary layer surrounding the caustic, which numerical results suggest generally holds for typical maneuver profiles [64, 144]{11}. However, the question still arises as to whether the nonlinear Tricomi equation remains valid when both focusing mechanisms are simultaneously at play. This situation is inherent to the transition focus boom associated with acceleration through Mach 1, as it will only reach the ground once the aircraft has exceeded the threshold Mach number. Ground recordings taken during transition flight generally agree favorably with Tricomi-based prediction tools (see Appendix A), but it is still worthwhile to consider what effect, if any, aircraft acceleration has on the superboom generated in this case.

With the transition focus boom in mind, Cheng, Lee, Hafez, and Guo appear to be the first authors to explicitly account for rectilinear aircraft acceleration in the Tricomi model, leading to an unsteady 3-dimensional nonlinear Tricomi equation [37, 38, 40, 42] for a perfect thermally and gravitationally stratified gas{12}. In the limit of vanish-

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{11} Catastrophe theory would also suggest, at least for the linear theory, that parameters like aircraft acceleration and climb are auxiliary in the determination of the basic structure of the wave field near a caustic [22, 160].

{12} Cheng, Lee, Hafez, and Guo develop a Tricomi equation for an aircraft undergoing a gradual climb and/or horizontal acceleration [42], but this equation does not appear to be equivalent to the unsteady Tricomi equation discussed in later works by Cheng and Hafez [37, 38] and Cheng and Lee [40] when terms related to aircraft climb are neglected. This could be attributable to a difference in the definition of parameters or some asymptotic equivalence of terms, but the three later works curiously make little to no reference to [42], and never do so with regards to its account of the acceleration case, so the result therein may be in error. As a precaution, our discussion hinges on the later references, and primarily the first article by Cheng and Hafez [38] ([40] provides more-or-less the same information and [37] uses the resulting equation to examine the acceleration superboom of a particular launch vehicle).
ing aircraft acceleration, this equation reduces to the 3-dimensional nonlinear Tricomi equation (5.23) just discussed, but for acceleration-dominated flight, we will see that it asymptotes to a form of the transonic small disturbance equation. The analysis is similar to the steady problem, but significantly more complicated in its mathematical details. Therefore, in lieu of more detailed derivations we will focus on highlighting the major differences between the steady and unsteady theories.

5.3.1 Unsteady mid-field

To model accelerated flight, Cheng et al begin with an unsteady form of the full potential equation (whose steady form is given by (4.51)). A small-amplitude approximation of this equation in terms of the (pseudo)potential defined by (5.2) then leads to an unsteady generalization of (5.5), further simplified by assuming the aircraft is in rectilinear motion along the $x$ axis at all times. We refer to the resulting equation as the unsteady mid-field equation.

Cheng et al next determine the reference Mach conoid associated with a linearized form of the unsteady mid-field equation. Since the aircraft speed $u_0 = u_0(t)$ now varies with time, so too does the Mach conoid $x = x_M(y, z, t)$. In particular, since the apex of the Mach conoid is defined by a Mach angle $\sin^{-1}(u_0(t)/c_0(0))$, the Mach conoid will tend to narrow about the flight axis as the aircraft accelerates (recall Figure 1.9). This may be thought of as the aircraft generating a new Mach conoid at each point along its trajectory, and hence a new set of characteristic curves at each moment in time. The fixed time $t = t_0$ associated with any particular Mach conoid $x = x_M(y, z, t_0)$ is therefore referred to as the aircraft time, and in addition to the azimuthal angle of a characteristic on the Mach conoid $\partial \tilde{x}/\partial \tilde{y}|_0$ and the (normalized) altitude $\tilde{z}$, may be used to specify any point along any characteristic generated during the flight of the aircraft. Hence, Cheng et al refer to this triplet as the characteristic coordinates. Unlike in the steady case, $\partial \tilde{x}_M/\partial \tilde{y}$ is not restricted to small values, allowing the behavior of characteristics at larger azimuthal launch angles to be examined.

Recasting the potential equation in terms of the Mach conoid, normalizing by the usual scales ((5.4) and (5.9)) and discarding terms beyond the leading order leads to a reduced equation for the mid-field analogous to (5.10). Integrating this equation along its

---

(13) This is entirely analogous to the use of the use of aircraft time, ray launch angle, and eikonal for ray coordinates depicted in Figure 2.4, the only differences being the use of the Mach conoid/characteristics rather than the ray conoid/rays, and the altitude rather than the eikonal. The latter simply leverages the stratification of the medium to simplify calculations.
characteristic curves (the characteristics of the Mach conoid with nonlinearity accounted for) leads to the following expression for the field strength:

\[ \hat{u}(\hat{\xi}, \tilde{y}, \tilde{z}, \tilde{t}) = \hat{u}(\hat{\xi}_0, \tilde{y}_0, \tilde{z}_0, \tilde{t}_0) \left( \frac{\rho_0(\hat{\xi}_0, \tilde{y}_0, \tilde{z}_0, \tilde{t}_0)}{\rho_0(\hat{\xi}, \tilde{y}, \tilde{z}, \tilde{t})} \right)^{1/2} \exp \left\{ -\frac{1}{2} \int_{\tilde{z}_0} \left( \tilde{A} + \frac{1}{u_0} \frac{du_0}{dt_0} \tilde{B} \right) d\tilde{z} \right\}, \]  

(5.25)

where

\[ \tilde{A} \equiv \left( \frac{\partial \tilde{x}_M}{\partial \tilde{z}} \right)^{-1} \left( \frac{\partial^2 \tilde{x}_M}{\partial \tilde{y}^2} + \frac{\partial^2 \tilde{x}_M}{\partial \tilde{z}^2} \right), \quad \tilde{B} \equiv \left( \frac{\partial \tilde{x}_M}{\partial \tilde{z}} \right)^{-1} \left( \frac{u_0 c_0(0)}{c_0^2} \right) \frac{\partial \tilde{t}_0}{\partial t}, \]  

(5.26)

and \( u_0 \equiv u_0(\tilde{t}_0) \) refers to the aircraft speed for a given (normalized) aircraft time \( \tilde{t}_0 \equiv c_0(0) t_0 / \Delta \). Compared to (5.12), we see that (5.25) contains a new term, \( \tilde{B} \), accounting for aircraft acceleration. As a result, (5.25) not only generalizes (5.12) to unsteady flight, but presents a new opportunity for singularities (and more generally nonuniformities) to arise in the mid-field description.

5.3.2 Singularities of the unsteady mid-field equation

By studying the characteristic curves of the time-dependent reference Mach conoid, Cheng et al find that the transition boundary for Mach cutoff (the sonic line) is still defined by the vanishing of the quantity \( \mathcal{R} \) in (5.14), although the Mach number \( M_0 \) now depends on both time and altitude. Moreover, just as in the steady case, (5.25) may become singular when \( \mathcal{R} = 0 \) due to the vanishing of \( \partial \tilde{x}_M / \partial \tilde{z} \). However, the authors identify another possible singularity in (5.14), occurring when the Jacobian determinant \( \Delta \) (note the difference from the unbolded atmospheric length scale \( \Delta \)) of the transformation from characteristic coordinates \( (\partial \tilde{x}_M / \partial \tilde{y})_0, \tilde{z}, \tilde{t}_0 \) to aircraft-fixed coordinates \( (\tilde{x}, \tilde{y}, \tilde{z}) \) vanishes. This condition indicates the crossing of differentially separated characteristics, and can only occur during accelerated flight \( du_0 / d\tilde{t}_0 > 0^{(14)} \). Therefore, one may anticipate not one, but three possible breakdown scenarios for mid-field solution (5.25):

1. \( \Delta = 0 \) and \( \mathcal{R} \) is far from zero (acceleration focusing alone)

\( ^{(14)} \)This parallels the vanishing of the determinant \( D \) of the transformation from ray coordinates to local caustic coordinates, indicating the crossing of differentially separated rays. However, since it is proportional to the ray tube area, \( D \) is expected to vanish for both Mach cutoff and acceleration focusing. It appears that \( \Delta \) vanishes for accelerated flight alone due to the particular choice of characteristic coordinates made by Cheng et al.

162
2. $\mathcal{R} = 0$ and $\Delta$ is far from zero (Mach cutoff)

3. $\Delta = 0$ and $\mathcal{R}$ is small (acceleration focusing in the vicinity of the sonic line)

The authors indicate that the first type of singularity can only occur for particularly large accelerations, $|u_0^{-1}du_0/d\tilde{t}_0| = \mathcal{O}(1)$, uncommon even in combat maneuvers. Therefore, more interest is paid to cases 2 and 3, requiring $|u_0^{-1}du_0/d\tilde{t}_0| \ll \mathcal{O}(1)$. Since $\Delta \neq 0$ in case 2, aircraft acceleration need not be considered, but steady ($du_0/d\tilde{t}_0 = 0$) and decelerating ($du_0/d\tilde{t}_0 < 0$) flight may still affect the field behavior in the vicinity of the sonic line\(^{(15)}\). As a result, case 3 is the only breakdown situation which must be considered for accelerated flight.

To study each case, the authors first introduce the subscripts $^{(*)}$ and $^{(**)}$, denoting the value of a quantity when $\mathcal{R} = 0$ and $\Delta = 0$ respectively. Case 2 then concerns the behavior of \((5.25)\) at the points $\tilde{z}$ for which

$$0 \leq \tilde{z}_* - \tilde{z} \ll 1,$$

for both steady and decelerating flight. For case 3, \((5.25)\) must instead be examined in the vicinity of $\tilde{z} = \tilde{z}_*$, but only for small values of $\mathcal{R}$. This condition may be written as

$$0 \leq \tilde{z}_* - \tilde{z} \leq \mathcal{O}(\tilde{z}_* - \tilde{z}) \ll 1,$$

and only needs to be considered for accelerated flight.

For steady flight in case 2, Cheng et al find that in the vicinity of the sonic line $\mathcal{R} = 0$ (where $\tilde{z} = \tilde{z}_*$), \((5.25)\) depends on $\tilde{z} - \tilde{z}_*$ through a factor of the form

$$\sqrt{\tilde{z}_* - \tilde{z}} + \frac{2}{I_*} \left( \frac{\partial \tilde{x}_M}{\partial \tilde{y}} \right)_0^2 \left( \frac{d\mathcal{R}}{d\tilde{z}} \right)_*^{-3/2} \tilde{z}_*^{-1/2},$$

\((5.29)\)

where $I_*$ is a constant depending on the characteristic under consideration. For vanishing $\partial \tilde{x}_M/\partial \tilde{y}|_0$, \((5.29)\) must reduce to \((5.19)\), confirming that the undetermined constant in Section 5.2.4 is in fact $C = -1/2$. Curiously, \((5.29)\) also implies that $\hat{u}$ actually remains bounded when $\tilde{z} = \tilde{z}_*$, provided $\partial \tilde{x}_M/\partial \tilde{y} \neq 0$. Nevertheless, the undertrack singularity remains, indicating a nonuniformity in the mid-field description.

\(^{(15)}\) Note that since $\partial \tilde{x}_M/\partial \tilde{y}$ can now take on non-negligible values, $\mathcal{R} = 0$ cannot generally be simplified to the sonic condition $B_0^2 = M_0^2 - 1 = 0$ as it was in Section 5.2. Nevertheless, we will continue to refer to the surface $\mathcal{R} = 0$ as the sonic line.

\(^{(16)}\) Cheng et al actually use $^*$ to denote the initial value of a quantity, $^{**}$ for its value at $\mathcal{R} = 0$, and $^{***}$ for $\Delta = 0$, but we have attempted to make the notation consistent with the rest of the thesis.
For a decelerating aircraft in case 2, it is determined that \((5.29)\) instead depends on \(\tilde{z} - \tilde{z}_*\) through a factor of the form

\[
|\tilde{z}_* - \tilde{z}|^{-1/4} \exp \left\{ -Q \ln |\tilde{z}_* - \tilde{z}| - \frac{1}{2} S |\tilde{z}_* - \tilde{z}|^{-1/2} \right\},
\]

(5.30)

where \(Q\) and \(S\) depend on the deceleration rate and the characteristic under consideration. Once again, \((5.30)\) turns out to remain bounded at \(\tilde{z} = \tilde{z}_*\), and in fact tends to decay exponentially for decelerating flight. However, its complicated form also leads to rapid variations in \(\hat{u}\) in the immediate vicinity of the sonic line, still indicative of a nonuniformity in the mid-field description.

For case 3, an acceleration-induced focus\(^{(17)}\) in the vicinity of the sonic line, \((5.29)\) is found to have a \(\tilde{z}\)–dependence of the form

\[
|\tilde{z}_* - \tilde{z}|^{-1/4} \left( 1 - \sqrt{\frac{\tilde{z}_* - \tilde{z}_{**}}{\tilde{z}_* - \tilde{z}}} \right)^{-1+\epsilon},
\]

(5.31)

where \(\epsilon\) is a function of the aircraft acceleration and characteristic under consideration. In the limit of vanishing acceleration, \((5.31)\) reduces to the steady case \((5.29)\), but for acceleration dominated flight, \((5.31)\) takes on an entirely different algebraic behavior characterized by a value of \(\tilde{z}_* - \tilde{z}\) comparable to \(\tilde{z}_* - \tilde{z}_{**}\). Hence, the asymptotic behavior of \(\hat{u}\) is not uniformly bounded during aircraft acceleration.

### 5.3.3 Rescaling the mid-field equation

With the asymptotic behavior of the mid-field solution found in all cases, the appropriate rescaling of the independent variables and the corresponding equation governing wave propagation in the focal region may be determined. For steady flight, the rescaled variables are readily inferred from the breakdown behavior \((5.29)\) and are given by \((5.20)\), leading to the 3-D nonlinear Tricomi equation \((5.23)\) already discussed. In contrast, the more complicated algebraic behavior of \(\hat{u}\) for accelerated flight (given instead by \((5.31)\)) hampers immediate determination of the proper rescaling.

To find the required rescaling, Cheng et al first introduce two gauges for the independent variables—\(d\), characterizing the (presumed equal) transverse scales of the focal region and \(q = q(\tilde{z}_{**} - \tilde{z})\), the scale for \(\partial \hat{x}_M / \partial \tilde{z}\)—and subsequently determine their forms

---

\(^{(17)}\)Cheng et al liken the line/surface \(\tilde{z} = \tilde{z}_{**}\) along which this focusing condition occurs to a *limit line* in the theory of compressible fluid flow, as it arises from the vanishing of a Jacobian determinant \((\Delta = 0)\), just as limit lines are the curves along which the Jacobian determinant of the transformation from hodograph (velocity) to physical variables vanishes (see e.g. \([47, 67]\)).
from the requirement that the transverse Laplacian of the potential variable emerge at the leading order in the rescaled equation for the focal region\(^{(18)}\). The dependent variable \(\hat{u} = \partial \hat{\phi} / \partial \hat{\xi}\) is also decomposed as \(\hat{u} \sim g_{**}(\tilde{z}_{**} - \tilde{z})\), where \(\nu\) is the \(\tilde{z}_{**} - \tilde{z}\) dependence of \(\hat{u}\) given by (5.31) and \(g_{**}\) encapsulates any remaining factors independent of \(\tilde{z}_{**} - \tilde{z}\), as the behavior of \(\nu\) is what determines the amplification of the wave approaching the caustic.

As a result of the foregoing procedure, the authors conclude that \(d\) may be determined from a cubic equation of the form

\[
k^{-3} + k^{-2} = m^{-2} n^{-3},
\]

(5.32)

where

\[
k \equiv \frac{\tilde{z}_* - \tilde{z}_{**}}{d}, \quad m \equiv \frac{dR}{dz_*}, \quad n \equiv \frac{\tilde{z}_* - \tilde{z}_{**}}{(L/\Delta)^{2/3}},
\]

(5.33)

and as before, \(L\) is the length scale of the aircraft (and hence the flow disturbance) and \(\Delta\) the length scale for atmospheric heterogeneity. \(k\) is the ratio of the separation between the sonic line (\(\tilde{z} = \tilde{z}_*\)) and acceleration caustic (\(\tilde{z} = \tilde{z}_{**}\)) relative to the size of the focal region \(d\), \(m\) is the rate at which the sonic line is approached, and \(n\) is the separation between the sonic line and acceleration caustic relative to the (scaled) boundary layer thickness \((L/\Delta)^{2/3}\) expected for the steady case in which the two coincide (\(\tilde{z}_* = \tilde{z}_{**}\)). Once \(d\) is known, \(q\) and \(\nu\) follow immediately. Namely, \(q\) must satisfy the condition

\[
dq(d) = \frac{L}{\Delta}
\]

(5.34)

within the focal region, consistent with supposing that \(\tilde{x}_M = O(L/\Delta)\) as in Section 5.2.5, while \(\nu\) is simply evaluated at the (normalized) distance \(\tilde{z}_{**} - \tilde{z} = d\).

As expected, for gradual accelerations for which \(\tilde{z}_* - \tilde{z}_{**} \ll (L/\Delta)^{2/3}\) (i.e., the acceleration caustic falls within the nominal boundary layer of the sonic line), this scale set reduces to the steady state set (5.20),

\[
d = \left(\frac{L}{\Delta}\right)^{2/3}, \quad q(d) = \left(\frac{L}{\Delta}\right)^{1/3}, \quad \nu = \left(\frac{L}{\Delta}\right)^{-1/6}.
\]

(5.35)

On the other hand, for acceleration-dominated flight with \(\tilde{z}_* - \tilde{z}_{**} \gg (L/\Delta)^{2/3}\) (the ac-

\(^{(18)}\) As we have seen in previous Tricomi equations, the reinstatement of this diffraction term (or its \(z\)-component in the 2-D case) is what removes the singularity at the caustic, so this is a reasonable, if not essential requirement on the focal region variables.
celeration caustic and the sonic line are well-separated), the scale set instead asymptotes to
\[ d = \left( \frac{L}{\Delta} \right) (\bar{z}_* - \bar{z}_{**})^{-1/2}, \quad q(d) = (\bar{z}_* - \bar{z}_{**})^{1/2}, \quad \nu = \left( \frac{L}{\Delta} \right)^{-1} (\bar{z}_* - \bar{z}_{**})^{5/4}. \] (5.36)

In any case, the scale set defined by (5.31)–(5.33) allows a new set of unit-order focal region variables generalizing (5.20) to be introduced:
\[ \xi = \hat{\xi}, \quad \bar{x}_M \equiv (dq)^{-1}(\bar{x}_M - \bar{x}_{M*}), \quad (\bar{y}, \bar{z}) \equiv d^{-1}(\bar{y} - \bar{y}_{**}, \bar{z} - \bar{z}_{**}), \quad \hat{\phi} = (G_{**}\nu(d))^{-1}\hat{\phi}, \] (5.37)

where \( G_{**} \equiv \int g_{**}d\hat{\xi}. \) Rescaling the unsteady mid-field equation according to (5.37) then yields Cheng et al.’s \textit{unsteady 3-D nonlinear Tricomi equation} explicitly accounting for aircraft acceleration,
\[ \frac{\partial^2 \hat{\phi}}{\partial \bar{y}^2} + \frac{\partial^2 \hat{\phi}}{\partial \bar{z}^2} - \left( \frac{M_0^2 - 1}{q^2} \right) + \lambda \frac{\partial \hat{\phi}}{\partial \bar{\tau}} \frac{\partial^2 \hat{\phi}}{\partial \bar{\tau}^2} = 0, \] (5.38)

where
\[ M_0 \equiv \frac{u_0(\hat{t}_0)}{c_0(\bar{z})}, \quad \lambda \equiv \frac{p\nu}{q} \left( \frac{M_{fl} \Delta}{L} \right) \frac{\gamma + 1}{2} \frac{u_0 u_0(\hat{t}_0 = 0)}{c_{0**}^2} G_{**}, \] (5.39)

with \( \hat{t}_0 = 0 \) some reference aircraft time, e.g. at the onset of aircraft acceleration.

### 5.3.4 Properties of the unsteady 3-D nonlinear Tricomi equation

The key difference between (5.38) and previous nonlinear Tricomi equations stems from the similarity parameters \( \lambda \) and \( (M_0^2 - 1)/q^2 \), so let us examine each in more detail.

First, we see that in the limit of vanishing acceleration, \( \lambda \) becomes
\[ \lambda = \left( \frac{\gamma + 1}{2} \right) M_{fl} \frac{u_0 u_0(\hat{t}_0 = 0)}{c_{0**}^2} G_{**} \nu \to \left( \frac{\gamma + 1}{2} \right) \frac{M_{fl}}{(L/\Delta)^{2/3}} G_{**} \nu, \] (5.40)

where we have used (5.35) and the fact that \( M_0^2 \approx 1 \) near the sonic line. The parenthesized quantity on the right-hand side is a form of the nonlinearity factor \( \mu \) (4.28), so the quantity \( \lambda/G_{**}\nu \) may be thought of as a generalization of the nonlinearity factor to the
unsteady case. Therefore, if we define $\bar{\phi}$ by (19)

$$\bar{\phi} = \frac{\lambda}{G_*} \hat{\phi} = \lambda \hat{\phi},$$  

(5.41)

where the last equality follows from the definition of $\hat{\phi}$ (5.37), then (5.38) reduces to a form directly comparable to (5.23),

$$\frac{\partial^2 \bar{\phi}}{\partial \overline{\eta}^2} + \frac{\partial^2 \bar{\phi}}{\partial \overline{\zeta}^2} - \left( \frac{M_0^2 - 1}{q^2} + \frac{\partial \hat{\phi}}{\partial X} \right) \frac{\partial^2 \bar{\phi}}{\partial \overline{\tau}^2} = 0,$$  

(5.42)

hence why we used the alternative check notation for the potential in (5.38).

The other parameter in (5.38), $(M_0^2 - 1)/q^2$, may be recognized as a transonic similarity parameter $-K_q$ with “thickness ratio” $q^3$, such that near the sonic line,

$$-K_q \equiv \frac{M_0^2 - 1}{q^2} \approx \frac{m(\tilde{z}_* - \tilde{z})}{q^2} = \frac{\bar{z} + k}{1 + k}.$$  

(5.43)

Thus, in the limit $k \to 0$ corresponding to a vanishing aircraft acceleration, $-K_q \to \bar{z}$, and (5.42) reduces to the steady 3-D nonlinear Tricomi equation, (5.23), confirming that the nonlinear Tricomi equation remains a valid model for the transition boom provided aircraft acceleration is sufficiently gradual. On the other hand, in the acceleration-dominated limit $k \to \infty$, $-K_q \to 1$ and (5.42) becomes a 3-dimensional form of the transonic small disturbance equation (4.68) with transonic similarity parameter $K_q = -1$, a behavior not borne out by the standard steady Tricomi models.

The influence of acceleration on transition focusing is also apparent in the location of the transition from hyperbolic (supersonic) to elliptic (subsonic) wave field behavior. While (5.42) is of mixed type like other Tricomi/transonic small disturbance equations, its transition boundary now lies along the surface

$$\bar{z} = -k - \lambda(1 + k) \frac{\partial \hat{\phi}}{\partial X} = -k - (1 + k) \frac{\partial \tilde{\phi}}{\partial X}.$$  

(5.44)

A nonlinearity-induced offset of the transition boundary from the geometrical caustic $\bar{z} = 0$ (indicated by the $\lambda \partial \hat{\phi}/\partial X$ term) arises as expected, but two additional shifts are also evident—one due to acceleration alone (through $k$) and one arising from a coupling of acceleration and nonlinear effects (through $k \partial \hat{\phi}/\partial X$). Only in the small-acceleration limit $k \to 0$ does this boundary coincide with the sonic line of the steady (2-D or 3-D)

\footnote{Compare to (5.22) for the steady case.}
nonlinear Tricomi equation, $\bar{z} = 2\mu \bar{p}$. Hence, the steady Tricomi equations correspond to the special case in which the sonic line and the acceleration caustic nearly coincide, which only occurs for particularly gradual accelerations.

### 5.3.5 Limits of the steady theory

The fact that the transition focal region depends on the proximity of the acceleration caustic to the sonic line allows one to estimate the range of accelerations for which a steady model (i.e., every previous Tricomi equation) should hold. The condition leading to the steady scale set \((5.35)\), $\bar{z}_* - \bar{z}_{**} \ll (L/\Delta)^{2/3}$, may be read as the requirement that the acceleration be gradual enough to produce a separation between the acceleration caustic $\bar{z} = \bar{z}_{**}$ and sonic line $\bar{z} = \bar{z}_*$ much smaller than the size of the (normalized) steady-state boundary layer, $(L/\Delta)^{2/3}$. In turn, Cheng et al show that this separation $\bar{z}_* - \bar{z}_{**}$ goes as the square of the quantity $u_0^{-1}du_0/d\bar{t}_0$, and hence the square of the acceleration. Thus, provided that

$$\left(\frac{1}{u_0} \frac{du_0}{d\bar{t}_0}\right)^2 \ll \left(\frac{L}{\Delta}\right)^{2/3}, \quad (5.45)$$

the usual steady Tricomi equations are expected to remain suitable models of transition focusing in their respective domains of applicability. The authors indicate that the left-hand side of \((5.45)\) is on the order of $10^{-3}$ for a typical supersonic transport and $10^{-1}$ for a fighter aircraft in a high-speed dash. Therefore, taking $(L/\Delta)^{1/3} = \varepsilon \approx 0.1$ (i.e., taking the heterogeneity scale $\Delta$ to be the relative radius of curvature $R$), we can anticipate that a steady model is likely sufficient for gradual accelerations typical of transport vehicles, but a model like \((5.42)\) may be more suited to combat maneuvers. The question, then, is how one would go about solving \((5.38)\).

Though the generalized nonlinearity parameter $\lambda$ may be scaled out of \((5.38)\) to obtain \((5.42)\)—a form more-or-less the same as the steady 3-D Tricomi equation \((5.23)\)—the matching of this inner description to the inner limit of the outer solution is fundamentally altered by acceleration. In particular, since the asymptotic behavior of the outer solution \((5.31)\) only reduces to the usual $z^{-1/4}$ result under steady conditions, there is an approximation inherent in using the typical matching procedure for accelerated flight. Cheng et al do however indicate that, provided the modified parameters and matching condition are accounted for, a numerical solution of \((5.38)\) could proceed in much the same way as the steady case. Namely, since \((5.42)\) and \((5.31)\) only depend on time
through the similarity parameters $\lambda$ and $(M_0^2 - 1)/q^2$, the associated boundary value problem may be solved as a quasi-steady one for each aircraft time. Unfortunately, beyond a brief qualitative account in [37], the unsteady model does not appear to have been numerically investigated in the literature so little can be said about its applicability to real flight data for the time being\(^{(20)}\).

5.4 A 3-dimensional lossy nonlinear Tricomi equation

The remainder of the thesis summarizes our own work on the problem of sonic boom focusing. In light of its apparent applicability to transport vehicles, we will return to the steady theory, and will instead concentrate our efforts on wedding the 3-dimensional and lossy aspects of sonic boom focusing. Along the way, we will show how the asymptotic expansion of the geometrical acoustic solution near a caustic may be generalized to a fully three-dimensional expression not found in the literature. This will motivate the introduction of a new set of inner variables for the caustic boundary layer, leading to a three-dimensional nonlinear Tricomi equation similar to (5.23), though expressed in terms of a normalized pressure variable. We will then augment this equation by applying results from our discussion on the lossy nonlinear Tricomi equation, leading to the desired lossy 3-dimensional model for sonic boom propagation near a fold caustic.

5.4.1 Extending the geometrical-acoustic theory near the caustic

To find the leading-order $y$-dependence of the inner limit of geometrical acoustic solution (2.93), we will extend the method used by Gazaryan in [200] and recounted in Section 3.3. We begin with the expansion of the ray family $R$ about point $O$ on the caustic (3.18):

\[
\begin{align*}
    x &= c_0 \tau + au^2 + c_1 \tau^2 + fru + gv^3 + \cdots \\
    y &= \gamma u + a_1 u^2 + b_1 v^2 + c_1 \tau^2 + d_1 uv + f_1 \tau u + g_1 v^3 + \cdots \\
    z &= a_2 u^2 + b_2 v^2 + c_2 \tau^2 + d_2 uv + e_2 v \tau + f_2 \tau u + g_2 v^3 + \cdots
\end{align*}
\]

As before, $u$ and $v$ are the curvilinear coordinates parametrizing the wavefronts and the caustic surface, $\tau$ is the acoustic eikonal, and the origin of the ray coordinates has been chosen to coincide with that of the local caustic coordinate system (i.e., $(u, v, \tau) = \ldots$).

\(^{(20)}\)That said, since the steady/unsteady nonlinear Tricomi equations are essentially approximations of steady/unsteady transonic small disturbance equations, one could simply fall back to a fuller transonic equation solver with the appropriate source and atmospheric characteristics specified.
\((x, y, z) = (0, 0, 0)\) at \(O\). The Taylor coefficients \(a_i - g_i\) remain as defined by (3.16), but the first expansion must be supplemented by terms of the form \(h v^2 u + j v^2 \tau\), where

\[
h \equiv \frac{1}{2} \left( \frac{R_{vu} \cdot R_\tau}{c_0} \right) \bigg|_O, \quad j \equiv \frac{1}{2} \left( \frac{R_{v\tau} \cdot R_\tau}{c_0} \right) \bigg|_O,\tag{5.47}
\]

as these terms turn out to be relevant to the leading-order \(y\) behavior of the eikonal near the caustic\(^{21}\). Then to determine which rays pass through an arbitrary point \((x, y, z) = (x_c, y_c, z_c)\) near the caustic, we must solve

\[
\begin{aligned}
x(u, v, \tau) &= x_c, \\
y(u, v, \tau) &= y_c, \\
z(u, v, \tau) &= z_c.
\end{aligned}
\tag{5.48}
\]

Based on the results of Section 3.3, we suppose that the ray coordinates may be expressed as power series in \(x_c, y_c, \pm \sqrt{z_c}\):

\[
u(x_c, y_c, z_c) = \sum_{i,j,k} a_{ijk} x_c^i y_c^j (\pm \sqrt{z_c})^k, \\
v(x_c, y_c, z_c) = \sum_{i,j,k} b_{ijk} x_c^i y_c^j (\pm \sqrt{z_c})^k, \tag{5.49}
\]

\[
\tau(x_c, y_c, z_c) = \sum_{i,j,k} c_{ijk} x_c^i y_c^j (\pm \sqrt{z_c})^k.
\]

where the constant terms \(a_{000}, b_{000}, c_{000}\) may be set to zero since the ray coordinate and local caustic coordinate origins coincide. Substituting these expansions into (5.46) with the noted modification and applying the result to (5.48) then yields a system of equations from which the expansion coefficients may be determined. The algebra is rather involved, so we have used Mathematica 12.0 to carry it out. The code was written in a “by-hand” manner to ensure that no steps were obfuscated within built-in Mathematica functions, and is provided in Appendix C.

Before discussing the results, it is worth noting that certain constraints must be met for three of the coefficient systems to be solved. In particular, the systems arising from matching the coefficients of \(x^2\), \(xy\), and \(y^2\) in (5.48) are, with \(c-\)subscripts dropped from

\(^{21}\)All terms of third order in the ray variables were included in the Mathematica code to be discussed, but it was found that only \(h\) and \(j\) appear at the order of approximation made.
the expansion variables,

\[
\begin{align*}
[x^2] : & \quad \frac{1}{c_0} + c_0c_{200} = 0, \\
& \quad \frac{1}{c_0} c_1 + \frac{b_1 e_2^2}{4b_2^2} + \gamma a_{200} = 0, \\
& \quad \frac{1}{c_0} (c_2 - \frac{e_2^2}{4b_2}) = 0, \\
\end{align*}
\]

\[
\begin{align*}
[xy] : & \quad \frac{1}{c_0 \gamma} + c_0c_{110} = 0, \\
& \quad \frac{1}{c_0 \gamma} (f_1 + \frac{d_2 (b_1 d_2 - b_2 d_1)}{2b_2}) + \gamma a_{110} = 0, \\
& \quad \frac{1}{c_0 \gamma} (f_2 - \frac{d_2 e_2}{2b_2}) = 0, \\
\end{align*}
\]

\[
\begin{align*}
[y^2] : & \quad \frac{1}{\gamma} + c_0c_{020} = 0, \\
& \quad \frac{1}{\gamma} (a_1 + \frac{d_2 (b_1 d_2 - 2b_2 d_1)}{4b_2}) + \gamma a_{020} = 0, \\
& \quad \frac{1}{\gamma} (a_2 - \frac{d_2^2}{4b_2}) = 0.
\end{align*}
\]

The first two equations of each system are readily solved for their respective expansion coefficients, but no such coefficient appears in the final equation of each. Therefore, for these equations to be satisfied, the following identities must hold:

\[
c_2 - \frac{e_2^2}{4b_2} = 0, \quad f_2 - \frac{d_2 e_2}{2b_2} = 0, \quad a_2 - \frac{d_2^2}{4b_2} = 0.
\]

(5.53)

Unfortunately, these relations do not actually hold identically. For instance, it may be shown that the left-hand side of the first expression is half the caustic normal curvature \(\tilde{\kappa}_n\), clearly nonzero in general. Nevertheless, supposing the equations in (5.53) are satisfied and continuing with the expansion leads to results that are both physically reasonable and reduce to the correct expressions when \(y = 0\). Therefore, we will treat (5.53) as “approximate identities” in the sense that they are satisfied to a certain order of approximation, presumed sufficient for the expansions considered\(^{(22)}\). We are then lead to the following expressions for \(u, v,\) and \(\tau\):

\[
u_{1,2} = \frac{1}{\gamma} y - \frac{b_1}{\gamma b_2} z \mp \frac{b_1 e_2}{\gamma c_0 b_2^{3/2}} x \sqrt{z} \mp \frac{b_1 d_2 - b_2 d_1}{\gamma^2 b_2^{3/2}} y \sqrt{z}
\]

\[
\quad \mp \frac{\gamma (b_1 g_2 - b_2 g_1) - b_1 (b_1 d_2 - b_2 d_1) z^{3/2}}{\gamma^2 b_2^{5/2}} + \cdots .
\]

\(^{(22)}\)This amounts to restricting consideration to caustics with sufficiently gradual curvatures, but we have not explicitly determined the limit of this approximation at present.
\[
\begin{align*}
\nu_{1,2} &= \mp \frac{1}{\sqrt{b_2}} \sqrt{z} - \frac{e_2}{2c_0 b_2} x - \frac{d_2}{2 \gamma b_2} y - \frac{\gamma g_2 - b_1 d_2}{2 \gamma b_2^2} z + \cdots, \\
\tau_{1,2} &= \frac{1}{c_0} x \pm \frac{g}{c_0 b_2^{3/2}} z^{3/2} - \frac{f}{\gamma c_0^2} x y \\
&+ \frac{2f b_1 b_2 + 3\gamma g e_2 - 2\gamma j b_2}{2 \gamma c_0^2 b_2^2} x z + \frac{4a b_1 b_2 + 3\gamma g d_2 - 2\gamma h b_2}{2 \gamma^2 c_0 b_2^2} y z + \cdots,
\end{align*}
\]

where subscripts 1 and 2 refer to each of the two rays passing through the point \((x, y, z) = (x_c, y_c, z_c)\), the first having already tangented the caustic and the second having not yet reached it.

To check our result, we note that (5.56) should reduce to (3.68) when \(y = 0\). The agreement of the first two terms has already been shown, so it remains to demonstrate that the coefficient of \(xz\) is equivalent to \((c_0 R_{xz})^{-1} = -c_0^{-1}(1/R + 1/R_c)\). First, note that

\[
\begin{align*}
\frac{2f b_1 b_2 + 3\gamma g e_2 - 2\gamma j b_2}{2 \gamma c_0^2 b_2^2} &= \frac{fb_1}{\gamma c_0^2 b_2^2} + 3 \left( \frac{-2}{3c_0^2 b_2^2} e_2 \right) - j \\
&= -\frac{1}{\gamma c_0^4 D_v} (2D_r^2 + \gamma^2 c_0^2 (R_{vv} \cdot R_r) - c_0^2 (R_{ru} \cdot R_r) (R_{vv} \cdot R_u)) \bigg|_o,
\end{align*}
\]

where we have used (3.36) for \(g\), (3.31) for \(b_2\), (3.33) for \(e_2\), and (3.16) for the remaining coefficients. Then, differentiating \(R_u \cdot R_r = 0\) and \(R_v \cdot R_r = 0\), we have the identities

\[
(R_{ru} \cdot R_r) \bigg|_o = -(R_{rr} \cdot R_u) \bigg|_o, \quad (R_{vv} \cdot R_r) \bigg|_o = -(R_{rv} \cdot R_{rr}) \bigg|_o - (R_{vr} \cdot R_{rr}) \bigg|_o.
\]

Using (3.34), we also find that

\[
D_r^2 \bigg|_o = (\gamma c_0 (R_{v \tau} \cdot \vec{n}) \bigg|_o)^2 = \gamma^2 c_0^2 (R_{v \tau} \cdot R_{v \tau}) \bigg|_o.
\]

Applying (5.58) and (5.59) to (5.57) yields

\[
-\frac{1}{\gamma c_0^4 D_v} \left( D_r^2 - \left[ \gamma^2 c_0^2 (R_{vv} \cdot R_{rr}) - c_0^2 (R_{rr} \cdot R_u) (R_{vv} \cdot R_u) \right] \right) \bigg|_o.
\]

We also have

\[
\gamma^2 c_0^2 (R_{vv} \cdot R_{rr}) - c_0^2 (R_{rr} \cdot R_u) (R_{vv} \cdot R_u) = \gamma^2 c_0 D_v (R_{rr} \cdot \vec{n}) = \gamma c_0 D_v (R_{rr} \cdot \vec{n}),
\]

172
as can be verified by another application of (3.34). Therefore, (5.60) becomes

\[
-\frac{1}{c_0} \left( \frac{D^2}{\gamma c_0^3 D_v} - \frac{\mathbf{R}_{xx} \cdot \mathbf{n}}{c_0^2} \right) \bigg|_\circ = -\frac{1}{c_0} \left( \frac{1}{R(O)} - \kappa \cos \varphi \right)
\]

\[
= -\frac{1}{c_0} \left( \frac{1}{R(O)} + \frac{1}{R_{co}} \right) = \frac{1}{c_0} \frac{1}{R_{xx}},
\]

(5.62)

where (3.45), (3.49), (3.62), and (3.65) have been used. Therefore, when \( y = 0 \), eikonal (5.56) reduces to (3.68) as expected.

For the coefficient of \( xy \), we have

\[
\frac{1}{c_0} \frac{1}{R_{xy}} = -\frac{f}{\gamma c_0^2} = -\frac{(\mathbf{R}_{\tau u} \cdot \mathbf{R}_\tau)|_\circ}{\gamma c_0^2} = \frac{(\mathbf{R}_{\tau \tau} \cdot \mathbf{R}_u)|_\circ}{\gamma c_0^2},
\]

(5.63)

where we have used (3.16) and (5.58). On the other hand, recalling the expression for the ray curvature vector \( \kappa \mathbf{n} \) (3.44), we see that

\[
\kappa \left( \frac{\mathbf{n} \cdot \mathbf{R}_u}{\gamma} \right) = \frac{\mathbf{R}_{\tau \tau} \cdot \mathbf{R}_u}{\gamma c_0^2}. 
\]

(5.64)

Therefore, we have

\[
\frac{1}{c_0} \frac{1}{R_{xy}} = -\frac{f}{\gamma c_0^2} = \frac{1}{c_0} \kappa \left( \frac{\mathbf{n} \cdot \mathbf{R}_u}{\gamma} \right) = \frac{\kappa \sin \varphi}{c_0},
\]

(5.65)

where as before, \( \varphi \) is the angle between the caustic normal plane and the osculating plane of the ray at \( O \). Thus, the coefficient of \( xy \) is the projection of the ray curvature into the caustic tangent plane at \( O \), as illustrated in Figure 5.5.

Since the acoustic field near the caustic primarily propagates along the \( Ox \) axis, we expect the \( yz \) term of (5.56) to be much smaller than the \( xy \) and \( xz \) terms. Therefore, we will neglect it moving forward. In passing, we note that it is equivalent to

\[
\frac{1}{c_0} \frac{1}{R_{yz}} = \frac{4ab_1b_2 + 3\gamma gd_2 - 2\gamma hb_2}{2\gamma^2 c_0^2 b_2^2} = -\frac{1}{c_0} \left( \frac{D_u D_\tau}{\gamma^2 c_0^4 D_v} - \frac{\mathbf{R}_{\tau u} \cdot \mathbf{n}}{\gamma c_0} \right) \bigg|_\circ,
\]

(5.66)

whose similarity to (5.62) suggests that this term is related to the normal curvature of the caustic in the \( y - z \) rather than the \( x - z \) plane.

Collecting results and shifting to an arbitrary ray coordinate origin such that \( \tau = \tau(O) \) on the caustic, we find that the reinstatement of leading-order \( y \) behavior results
in an eikonal of the form

$$\tau_{1,2} = \tau(O) + \frac{1}{c_0(O)}x \pm \frac{1}{c_0(O)}\sqrt{\frac{8}{9R(O)}}z^{3/2} + \frac{\kappa \sin \varphi}{c_0(O)}xy + \frac{\kappa n}{c_0(O)}xz + \cdots \quad (5.67)$$

Using (2.73) and expansion (3.70), the three-dimensional asymptotic behavior of the ray divergence can also be found. With extensive algebra, the Jacobian determinant $D$ can be shown to take the form

$$D(u, v, \tau) \approx \sqrt{z}(A + B\sqrt{z} + Cx + Ey + Hz), \quad (5.68)$$

where $A$, $B$, $C$, $E$, and $H$ are combinations of the Taylor coefficients (3.16). Therefore, the ray divergence takes the form

$$\frac{1}{\sqrt[F]{s}} \propto \frac{1}{\sqrt{D}} \approx \frac{1}{\sqrt{A}z^{1/4}} \left(1 - \frac{B\sqrt{z} + Cx + Ey + (H - 3B^2/4A)z}{2A}\right). \quad (5.69)$$

Hence, the geometrical acoustic solution still experiences a $z^{-1/4}$ singularity near caustics and the importance of the $x$ and $y$ terms remains secondary in the determination of the amplitude. As a result, we will continue to use (3.70) for the expansion of the
ray divergence, but we will modify the phase variable according to (5.67), yielding the following expression for the geometrical acoustic solution near a 3-dimensional caustic:

\[
p(x, y, z, t) \approx P_0(x, y, z) \left( \frac{R(O)}{2z} \right)^{1/4} \frac{1}{\mathcal{J}(x, y, z)} \times F \left( t - \left[ \tau(O) - \frac{1}{c_0(O)} \sqrt{\frac{8}{9R(O)}} z^{3/2} + \frac{x}{c_0(O)} (1 + (\kappa \sin \varphi)y + \kappa_n z) \right] \right). \tag{5.70}
\]

Note that the previous results assume a quiescent medium and should be modified if the medium is in motion. This may be accomplished by applying the procedure just used with the basis vectors reoriented along the \(OXYZ\) coordinate system introduced in 4.3.2, or by generalizing Auger’s procedure \[10\] to 3 dimensions, but we will not do so here.

### 5.4.2 The discrepancy between two- and three-dimensional nonlinear Tricomi equations

Before we can apply the results of the previous section to the development of a Tricomi equation, we must address an apparent inconsistency between the existing 2-D and 3-D focusing models. To this end, let \(L_y\) be the length scale for acoustic pressure variations along the \(y\) direction of the local caustic coordinate system and \(\zeta\) the ratio of the length scale for the pressure gradient in the \(z\) direction (the boundary layer thickness \(\delta\)) to \(L_y\).

The 2-dimensional natures of LTE, NTE, WiNTE, and LNTE all rely on the hierarchy of the acoustic particle velocity components (3.93), which suggests

\[
\zeta = \frac{\delta}{L_y} \propto \left( \frac{U_{cz}}{U_{cy}} \right)^{-1} = \varepsilon^2, \tag{5.71}
\]

implying that \(L_y = \mathcal{O}(\varepsilon^{-3})\). If this is so, then \(\partial^2 p/\partial y^2\) is of order \(\varepsilon^6 M_{ac}\), negligible in comparison to any other terms in the 2-D Tricomi equations.

On the other hand, the 3-D nonlinear Tricomi equation (5.23) derived by Cheng et al. assumes that the transverse length scales of the boundary layer are identical, hence

\[
\zeta = \frac{\delta}{L_y} = \frac{\delta}{\delta} = 1. \tag{5.72}
\]

In fact, were \(L_y\) any smaller than \(\delta\), the \(\partial^2 \phi/\partial y^2\) term in (5.23) would formally be of
a smaller order than the dominant terms and hence neglected, reducing (5.23) to the usual NTE\(^{(23)}\).

Comparing (5.71) to (5.72), we see that the basic result used to derive all two-dimensional Tricomi appears to directly contradict the key assumption required for a consistent three-dimensional Tricomi equation. To resolve this conflict, it seems reasonable to simply retain the leading-order cross-flow diffraction terms regardless of their size relative to other terms while maintaining the \(O(\varepsilon^2)\) bound for all other phenomena. Physically, one could argue that these terms become more important as we depart from the flight track and approach the lateral cutoff where the transverse gradient is clearly non-negligible. This is similar to Salamone’s argument that loss terms, while typically small, become important when the source frequency is sufficiently high and so should be retained. Perhaps a more mathematically satisfying explanation is to suppose that \(L_y\) varies as one moves away from the flight track. In particular, we will assume that \(L_y\) is on the order of \(\varepsilon^{-3}\) undertrack (\(\zeta = O(\varepsilon^2)\)), but decreases with increasing \(y\), eventually becoming comparable to \(\delta\) (\(\zeta = O(1)\)) near lateral cutoff. Then we may treat \(\zeta\) as an \(O(1)\) quantity (at most) in expressions to follow.

### 5.4.3 A modified 3-dimensional nonlinear Tricomi equation

The form of the phase variable in (5.70) suggests that we define the following normalized variables for the inner expansion:

\[
\bar{y} \equiv \frac{y}{L_y}, \quad \bar{z} = \frac{z}{\delta}, \quad \bar{t} = \omega ac \left[ t - \tau(O) - \frac{x}{c_0(O)} \left( 1 + \frac{y}{R_{xy}} + \frac{z}{R_{xz}} \right) \right],
\]

and suppose that the acoustic perturbation variables of the inner solution may be expressed as

\[
\begin{align*}
    p(x, y, z, t) &= P_c \bar{p}(\bar{y}, \bar{z}, \bar{t}), \\
    u(x, y, z, t) &= \begin{bmatrix} U_{cx} \bar{u}_x(\bar{y}, \bar{z}, \bar{t}) \\
                             U_{cy} \bar{u}_y(\bar{y}, \bar{z}, \bar{t}) \\
                             U_{cz} \bar{u}_z(\bar{y}, \bar{z}, \bar{t}) \end{bmatrix}.
\end{align*}
\]

\(^{(23)}\)The numerical results of Guo and Hafez \cite{80} discussed in Section 5.2 (specifically, Figure 5.4) suggest that the the cross-flow terms in (5.23) do in fact become small undertrack. However, the authors exploit the stratified medium assumption by imposing the symmetry condition \(\partial \phi / \partial y = 0\) at \(y = 0\), allowing only half the computational domain to be considered. This condition appears to more-or-less force the cross flow to zero in the symmetry plane and would not necessarily hold in a general atmosphere.
Then we have the following change of variable formulas

\[
\begin{align*}
\frac{\partial}{\partial x} &= -\frac{\omega_{ac}}{c_0(O)} \left(1 + \varepsilon^2 \frac{R(O)}{2R_{xz}} z + \frac{\varepsilon^2}{\zeta} \frac{R(O)}{2R_{xy}} y\right) \frac{\partial}{\partial \tilde{t}}, \\
\frac{\partial}{\partial y} &= \varepsilon \zeta \frac{\omega_{ac}}{c_0(O)} \frac{\partial}{\partial \tilde{y}}, \\
\frac{\partial}{\partial z} &= \varepsilon \frac{\omega_{ac}}{c_0(O)} \frac{\partial}{\partial \tilde{z}}, \\
\frac{\partial}{\partial \tilde{t}} &= \omega_{ac} \frac{\partial}{\partial \tilde{t}},
\end{align*}
\]

(5.75)

where we have used

\[
\tilde{y} = \frac{\delta}{L_y}, \quad \tilde{z} = \frac{\zeta}{\delta} y,
\]

(5.76)

with \( \zeta = \delta/L_y \) as before. The sound speed and ambient density may then be expanded as

\[
\begin{align*}
c_0(x, y, z) &\approx c_0(O) \left(1 + \varepsilon^2 \frac{R(O)}{2R_{c_0,z}} z + \frac{\varepsilon^2}{\zeta} \frac{R(O)}{2R_{c_0,y}} y\right), \\
\rho_0(x, y, z) &\approx \rho_0(O) \left(1 + \varepsilon^2 \frac{R(O)}{2R_{\rho_0,z}} z + \frac{\varepsilon^2}{\zeta} \frac{R(O)}{2R_{\rho_0,y}} y\right),
\end{align*}
\]

(5.77)

where

\[
\begin{align*}
R_{c_0,y} &\equiv \left(\frac{1}{c_0(O)} \frac{\partial c_0}{\partial y}\right)_o^{-1}, & R_{c_0,z} &\equiv \left(\frac{1}{c_0(O)} \frac{\partial c_0}{\partial z}\right)_o^{-1}, \\
R_{\rho_0,y} &\equiv \left(\frac{1}{\rho_0(O)} \frac{\partial \rho_0}{\partial y}\right)_o^{-1}, & R_{\rho_0,z} &\equiv \left(\frac{1}{\rho_0(O)} \frac{\partial \rho_0}{\partial z}\right)_o^{-1}.
\end{align*}
\]

(5.78)

Applying (5.74)–(5.78) to nonlinear acoustic equations (4.17) and (4.18) leads to the following relationships between the dimensional pressure and particle velocity gauges:

\[
\frac{\partial}{\partial \tilde{t}} \left[1 + \varepsilon^2 \left(\frac{1}{\zeta} (a_y - c_y) \tilde{y} + (a_z - b_z) \tilde{z}\right)\right] \frac{\partial \tilde{p}}{\partial \tilde{t}} = M_{ac} (\tilde{u}_x - \tilde{p}) \frac{\partial \tilde{p}}{\partial \tilde{t}},
\]

(5.80)

\[
\begin{align*}
\frac{\partial \tilde{p}}{\partial \tilde{t}} + \frac{\partial \tilde{p}}{\partial \tilde{y}} &= 0, \\
\frac{\partial \tilde{p}}{\partial \tilde{t}} + \frac{\partial \tilde{p}}{\partial \tilde{z}} &= 0,
\end{align*}
\]

(5.81)

Substituting (5.79) back into (4.17) and (4.18) then reduces the components of the momentum equation to

\[
\begin{align*}
\frac{\partial \pi_x}{\partial \tilde{t}} - \left[1 + \varepsilon^2 \left(\frac{1}{\zeta} (a_y - c_y) \tilde{y} + (a_z - b_z) \tilde{z}\right)\right] \frac{\partial \tilde{p}}{\partial \tilde{t}} = M_{ac} (\tilde{u}_x - \tilde{p}) \frac{\partial \tilde{p}}{\partial \tilde{t}}, \\
\frac{\partial \pi_y}{\partial \tilde{t}} + \frac{\partial \tilde{p}}{\partial \tilde{y}} &= 0, \\
\frac{\partial \pi_z}{\partial \tilde{t}} + \frac{\partial \tilde{p}}{\partial \tilde{z}} &= 0.
\end{align*}
\]

(5.82)
and the mass conservation equation to

\[
\frac{\partial \bar{p}}{\partial t} - \left[ 1 + \varepsilon^2 \left( \frac{1}{\zeta} (b_y + a_y + 2e_y)\bar{y} + (b_z + a_z + 2e_z)\bar{z} \right) \right] \frac{\partial \bar{u}_x}{\partial t} + (\zeta \varepsilon)^2 \frac{\partial \bar{y}}{\partial \bar{y}} + \varepsilon^2 \frac{\partial \bar{z}}{\partial \bar{z}} = M_{ac} \frac{\partial}{\partial t} (\bar{p} \bar{u}_x) + \frac{B}{2A} \frac{\partial \bar{p}^2}{\partial \bar{t}^2},
\]

(5.83)

where

\[
\begin{align*}
  a_y & \equiv \frac{R(O)}{2R_{xy}}, & b_y & \equiv \frac{R(O)}{2R_{p0,y}}, & e_y & \equiv \frac{R(O)}{2R_{c0,y}}, \\
  a_z & \equiv \frac{R(O)}{2R_{xz}}, & b_z & \equiv \frac{R(O)}{2R_{p0,z}}, & e_z & \equiv \frac{R(O)}{2R_{c0,z}}.
\end{align*}
\]

(5.84)

(5.80) and (5.83) may then be eliminated to arrive at a single equation in the pressure just as in the 2-dimensional case. Namely, we first substitute the linear relationship \(u_x = p + O(\varepsilon^2)\) into any nonlinear terms in (5.80) and (5.83). We then differentiate the resulting mass conservation equation with respect to \(t\), interchange the order of spatial and time derivatives, and substitute (5.80) to eliminate any remaining particle velocity components in favor of the pressure. The end result is an equation of the form

\[
\zeta^2 \frac{\partial^2 \bar{p}}{\partial \bar{y}^2} + \frac{\partial^2 \bar{p}}{\partial \bar{z}^2} + \left[ \frac{2}{\zeta} (a_y + e_y)\bar{y} - \bar{z} \right] \frac{\partial^2 \bar{p}}{\partial \bar{t}^2} + \mu \frac{\partial^2 \bar{p}^2}{\partial \bar{t}^2} = 0,
\]

(5.85)

where \(\mu\) is the usual nonlinearity factor, (4.28). The \(a_y\) and \(e_y\) coefficients may be related by noting that from (3.63),

\[
\kappa \sin \varphi = \kappa \left( \frac{\mathbf{n} \cdot \mathbf{R_u}}{\gamma} \right) = -\frac{1}{c_0} \left( \nabla \cdot c_0 \mathbf{n} \right) \cdot \mathbf{R_u} = -\frac{1}{c_0} \frac{\partial c_0}{\partial y}.
\]

(5.86)

Hence, using (5.65),

\[
\begin{align*}
a_y + e_y &= \frac{R(O)}{2} \left( \frac{1}{R_{xy}} + \frac{1}{R_{c0,y}} \right) = \frac{R(O)}{2} \left( \frac{1}{R_{xy}} + \frac{1}{c_0} \frac{\partial c_0}{\partial y} \right) \\
 &= \frac{R(O)}{2} \left( \kappa \sin \varphi + \frac{1}{c_0} \frac{\partial c_0}{\partial y} \right) = \frac{R(O)}{2} \left( -\frac{1}{c_0} \frac{\partial c_0}{\partial y} + \frac{1}{c_0} \frac{\partial c_0}{\partial y} \right) = 0.
\end{align*}
\]

(5.87)

Therefore, (5.85) reduces to

\[
\zeta^2 \frac{\partial^2 \bar{p}}{\partial \bar{y}^2} + \frac{\partial^2 \bar{p}}{\partial \bar{z}^2} - \bar{z} \frac{\partial^2 \bar{p}}{\partial \bar{t}^2} + \mu \frac{\partial^2 \bar{p}^2}{\partial \bar{t}^2} = 0.
\]

(5.88)
With the proposed interpretation of $L_y$, we have that for large $y$, $\zeta = 1$ and (5.88) reduces to (the pressure form of) Cheng et al.’s 3-D NTE (5.23). On the other hand, near the flight track $y = 0$, $L_y$ has grown such that $\zeta = O(\varepsilon^2)$ and the first term in (5.88) becomes negligible, leading to the usual nonlinear Tricomi equation, (4.30).

The approach just considered is still somewhat ad hoc, and at least two questions arise from it. First, what physical quantity does $L_y$ represent? The conditions imposed on it limit the possibilities, but it remains to be seen whether a characteristic scale that decreases with distance from the flight track naturally arises from a more detailed analysis. One possibility would be the distance $y_{co}$ to lateral cutoff, but this is a global quantity which seems somewhat odd in the local context considered for most of the analysis. The second curiosity is the disappearance of one of the $\overline{y}$ terms in (5.85) due to the relationship between $a_y$ and $e_y$. In the transonic analogy, this term would correspond to variations in the ambient flow Mach number along the $y$ coordinate. Unless the caustic is parallel to the ground and the atmosphere is horizontally stratified (as in the case considered by Cheng et al.), such variations are not generally negligible. It is possible that the neglected $1/R_{yz}$ term of the eikonal expansion should be incorporated into the inner variable set to account for “more” $y$ behavior, but this also requires further consideration. For the time being, we will suppose that such a length scale $L_y$ does exist and that the $\overline{y}$ behavior retained is sufficient for the study of lateral diffraction effects and move on to the lossy, moving medium.

### 5.4.4 A 3-dimensional lossy nonlinear Tricomi equation

At present, the asymptotic behavior of the geometrical acoustic solution near a caustic has only been generalized to moving media in the locally 2-dimensional case considered by Auger [10]. Nevertheless, we may conjecture that the difference between the fully 3-dimensional quiescent and moving mediums is the same as the 2-dimensional case. Namely, that the basic form of the solutions is the same, but with respect to different coordinate systems that result in slightly different expansion coefficients. If so, then the eikonal for the 3-dimensional moving medium follows readily from Auger’s result (4.94),

$$
\tau_{1,2}(X,Y,Z) = \tau(O) \pm \frac{1}{c(O)} \sqrt{\frac{8}{9R_{tot}(O)}} Z^{3/2} + \frac{X}{c(O)} \left(1 + \frac{Y}{R_{XY}(O)} + \frac{Z}{R_{XZ}(O)}\right) + \cdots,
$$

(5.89)
where \( c(O), R_{\text{tot}}, \) and \( R_{XZ} \) are defined in (4.95)–(4.97) while \( R_{XY} \) is the (unknown) generalization of \( R_{xy} \) (5.65) to moving media\(^{24}\). This in turn suggests a modification of the inner variables for the moving medium analogous to (5.73),

\[
\bar{Y} \equiv \frac{y}{L_Y}, \quad \bar{z}_w \equiv \frac{Z}{\delta_w}, \quad \bar{t}_w \equiv \omega_{ac} \left[ t - \tau(O) - \frac{X}{c(O)} \left( 1 + \frac{Y}{R_{XY}} + \frac{Z}{R_{XZ}} \right) \right], \tag{5.90}
\]

where \( L_Y \), analogous to \( L_y \), is the (unknown) transverse length scale for the moving medium and \( \delta_w \) is the effective caustic boundary layer thickness defined by (4.99). We then assume that the acoustic perturbation variables may be expressed in the following form in the boundary layer

\[
p(X, Y, Z, t) = P_c \bar{p}(\bar{y}_w, \bar{z}_w, \bar{t}_w), \quad \text{and} \quad \mathbf{u}(X, Y, Z, t) = \begin{bmatrix} U_{cX} \bar{u}_X(\bar{y}_w, \bar{z}_w, \bar{t}_w) \\ U_{cY} \bar{u}_Y(\bar{y}_w, \bar{z}_w, \bar{t}_w) \\ U_{cZ} \bar{u}_Z(\bar{y}_w, \bar{z}_w, \bar{t}_w) \end{bmatrix}, \tag{5.91}
\]

and determine the change of variable formulas

\[
\begin{align*}
\frac{\partial}{\partial X} & = -\frac{\omega_{ac}}{c(O)} \left( 1 + \varepsilon_w^2 \frac{R_{\text{tot}}(O)}{2R_{XZ}} \bar{z}_w + \frac{\varepsilon_w^2 R_{\text{tot}}(O)}{\zeta_w 2R_{XY}} \bar{y}_w \right) \frac{\partial}{\partial \bar{t}_w}, \\
\frac{\partial}{\partial Y} & = \varepsilon_w \zeta_w \omega_{ac} \frac{\partial}{\partial \bar{y}_w}, \\
\frac{\partial}{\partial Z} & = \varepsilon_w \frac{\omega_{ac}}{c(O)} \frac{\partial}{\partial \bar{z}_w}, \\
\frac{\partial}{\partial t} & = \omega_{ac} \frac{\partial}{\partial \bar{t}_w},
\end{align*} \tag{5.92}
\]

where \( \zeta_w \equiv \delta_w / L_Y \). We then expand the effective sound speed and and ambient density with respect to the \( OXYZ \) coordinates,

\[
c(X, Y, Z) = c(O) \left( 1 + \varepsilon_w^2 \frac{R_{\text{tot}}(O)}{2R_{cw,Z}} \bar{z}_w + \frac{\varepsilon_w^2 R_{\text{tot}}(O)}{\zeta_w 2R_{cw,Y}} \bar{y}_w \right), \tag{5.93}
\]

\[
\rho_0(X, Y, Z) = \rho_0(O) \left( 1 + \varepsilon_w^2 \frac{R_{\text{tot}}(O)}{2R_{\rho_0 W,Z}} \bar{z}_w + \frac{\varepsilon_w^2 R_{\text{tot}}(O)}{\zeta_w 2R_{\rho_0 W,Y}} \bar{y}_w \right).
\]

\(^{24}\)Given the similarity of the results for ray and wavefront-based coordinates seen in Section 4.3.3, we expect that \( 1/R_{XY} = \kappa_w \sin \varphi_w \), the \( OY \) component of the curvature of the curve traced out by the wavefront normal vector passing through \( O \), but this requires further investigation.
as well as the components of the ambient flow (wind) vector

\[ u_{OX}(X,Y,Z) = u_{OX}(O) \left( 1 + \varepsilon_w^2 \frac{R_{tot}(O)}{2R_{OX,Z}} z_w + \frac{\varepsilon_w^2 R_{tot}(O)}{2R_{OX,Y}} y_w \right), \]

\[ u_{OY}(X,Y,Z) = u_{OY}(O) \left( 1 + \varepsilon_w^2 \frac{R_{tot}(O)}{2R_{OY,Z}} z_w + \frac{\varepsilon_w^2 R_{tot}(O)}{2R_{OY,Y}} y_w \right), \]

\[ u_{OZ}(X,Y,Z) = u_{OZ}(O) \left( 1 + \varepsilon_w^2 \frac{R_{tot}(O)}{2R_{OZ,Z}} z_w + \frac{\varepsilon_w^2 R_{tot}(O)}{2R_{OZ,Y}} y_w \right), \]

(5.94)

where

\[ R_{cw,Y} = \left( \frac{1}{c(O) \, dY} \right)_{O}^{-1}, \quad R_{\rho_0w,Y} = \left( \frac{1}{\rho_0(O) \, dY} \right)_{O}^{-1}, \]

\[ R_{cw,Z} = \left( \frac{1}{c(O) \, dZ} \right)_{O}^{-1}, \quad R_{\rho_0w,Z} = \left( \frac{1}{\rho_0(O) \, dZ} \right)_{O}^{-1}, \]

(5.95)

and

\[ R_{OX,Y} = \left( \frac{1}{u_{OX}(O) \, dY} \right)_{O}^{-1}, \quad R_{OX,Z} = \left( \frac{1}{u_{OX}(O) \, dZ} \right)_{O}^{-1}, \]

\[ R_{OY,Y} = \left( \frac{1}{u_{OY}(O) \, dY} \right)_{O}^{-1}, \quad R_{OY,Z} = \left( \frac{1}{u_{OY}(O) \, dZ} \right)_{O}^{-1}, \]

\[ R_{OZ,Y} = \left( \frac{1}{u_{OZ}(O) \, dY} \right)_{O}^{-1}, \quad R_{OZ,Z} = \left( \frac{1}{u_{OZ}(O) \, dZ} \right)_{O}^{-1}. \]

(5.96)

We then define the ambient flow Mach numbers at \( O \),

\[ M_X = \frac{u_{OX}(O)}{c(O)}, \quad M_Y = \frac{u_{OY}(O)}{c(O)}, \quad M_Z = \frac{u_{OZ}(O)}{c(O)}, \]

(5.97)

with \( c(O) \) the effective sound speed at \( O \), related to \( c_0(O) \) as before by

\[ c_0(O) = c(O) - u_0(O) \cdot \hat{n}(O) = c(O)(1 - M_X). \]

(5.98)

As in Section 4.3.3, we suppose that \( M_X, M_Y, \) and \( M_Z \) are \( O(\varepsilon_w) \) at most. The material derivative following the ambient flow is then approximated by

\[ \frac{D_0}{Dt} = \omega_{ac} \left( 1 - M_X \right) \frac{\partial}{\partial t_w} + \varepsilon_w \zeta_w M_Y \frac{\partial}{\partial y} + \varepsilon_w M_Z \frac{\partial}{\partial z}. \]

(5.99)
Carrying (5.90)–(5.99) through (4.104) and (4.105) (the nonlinear acoustic equations with a mean flow) leads to the following relationships between the gauge variables:

\[ U_{cX} = \frac{P_c}{\rho_0 c_0}, \quad U_{cY} = \varepsilon_w \zeta_w P_c, \quad \text{and} \quad U_{cZ} = \varepsilon_w \frac{P_c}{\rho_0 c_0}. \]  

(5.100)

Substituting (5.90)–(5.100) into the augmented Westervelt equation (4.167) and retaining terms out to a relative order \( \varepsilon^2 \) (save for the caveat for the loss terms already discussed) then results in the following expression for a 3-dimensional lossy nonlinear Tricomi equation:

\[
\zeta_w^2 \frac{\partial^2 \bar{p}}{\partial y^2} + \frac{\partial^2 \bar{p}}{\partial z^2} + 2 \zeta_w \frac{M_Y}{\varepsilon_w} \frac{\partial^2 \bar{p}}{\partial t \partial y} - 2 M_Z \frac{\partial^2 \bar{p}}{\partial t \partial z} + \frac{1}{\varepsilon_w^2} \sum_{\nu} \left( \bar{\theta}_\nu \frac{\theta}{\bar{\gamma}_w} \right) \frac{\partial^3 \bar{p}}{\partial t^3} + \mu_w \frac{\partial^2 \bar{p}^2}{\partial t^2} = 0,
\]

(5.101)

where

\[
a_Y \equiv \frac{R_{tot}(O)}{2R_{XY}}, \quad e_Y \equiv \frac{R_{tot}(O)}{2R_{cw,Y}}.
\]

(5.102)

As before, \( \mu_w = \frac{\beta M_{ac}}{\varepsilon_w^2} \) is the nonlinearity factor for the moving medium, while \( \bar{\tau}, \bar{t}_\nu, \) and \( \bar{\theta}_\nu \) are the dimensionless parameters associated with thermoviscous absorption and vibrational relaxation defined in (4.175)–(4.177).

Since we have not explicitly determined the form of \( R_{XY} \), we cannot state with certainty that \( a_Y + e_Y = 0 \) as in the quiescent medium. Therefore, (5.101) is the final form of 3-dimensional lossy nonlinear Tricomi equation, or 3DLNTE, that we will consider here. Under the appropriate circumstances, it reduces to all (steady) Tricomi equations discussed thus far. In particular, if we neglect all \( \bar{y} \) terms, consistent with consideration of points near the flight track where \( \zeta_w = O(\varepsilon_w^2), \) (5.101) reduces to the usual lossy nonlinear Tricomi equation, (4.178). In turn, neglecting thermoviscous absorption and vibrational relaxation terms in (4.178) reduces it to the windy nonlinear Tricomi equation (4.116), which becomes the nonlinear Tricomi equation (4.115) in a quiescent medium, finally reducing to the linear Tricomi equation (3.99) for sufficiently weak waves. On the other hand, if we neglect losses, wind, and relaxation, and consider points far from the flight track, (5.101) reduces to Cheng et al.’s 3-dimensional nonlinear Tricomi equation (5.23). This hierarchy is summarized in Figure 5.6.
Figure 5.6: Hierarchy of Tricomi equations
5.5 Suggestions for the numerical implementation of a 3-D LNTE

As it stands, the only numerical method devised for the express purpose of solving the lossy nonlinear Tricomi equation is the operator-splitting scheme developed by Salamone\textsuperscript{(25)} [169, 170] based on the work of Auger and Coulouvrat [10, 11] (see also Appendix A). This method has only been applied to essentially two-dimensional equations but should not face additional difficulty for an equation of the form (5.101). To see why, let us quickly recount how the method works for the standard LNTE.

Following Salamone [169, 170], the first step in the operator splitting method is the introduction of an unsteady term in the pseudotime variable \( \sigma \) to the left-hand side of (4.178), yielding

\[
\frac{\partial^2 \bar{p}}{\partial \sigma \partial t_w} = \frac{\partial^2 \bar{p}}{\partial z_w^2} - z_w \frac{\partial^2 \bar{p}}{\partial t_w^2} + \left( \frac{2M_X - M_X^2}{\varepsilon_w^2} \right) \frac{\partial^2 \bar{p}}{\partial t_w^2} - 2M_Z \frac{\partial^2 \bar{p}}{\partial t_w \partial \sigma_w} + \frac{1}{\varepsilon_w^2} \frac{\partial^3 \bar{p}}{\partial \sigma_w^3} + \mu_w \frac{\partial^2 \bar{p}}{\partial t_w^2}. 
\]

(5.103)

Over small time steps (in \( \bar{t} \)), this equation can be split into a set of sub-equations to be solved independently and then added. Though the splitting can be chosen any way we wish, it is convenient to decompose (5.103) into the following three equations:

- **Z-Diffraction and Z-wind**

\[
\frac{\partial^2 \bar{p}}{\partial \sigma \partial t_w} = \frac{\partial^2 \bar{p}}{\partial z_w^2} - 2M_Z \frac{\partial^2 \bar{p}}{\partial t_w \partial \sigma_w} - z \frac{\partial^3 \bar{p}}{\partial \sigma_w^3}, \tag{5.104}
\]

- **Absorption, relaxation, and X-wind**

\[
\frac{\partial^2 \bar{p}}{\partial \sigma \partial t_w} = \left( \frac{2M_X - M_X^2}{\varepsilon_w^2} \right) \frac{\partial^2 \bar{p}}{\partial t_w^2} + \frac{1}{\varepsilon_w^2} \left( \alpha + \sum_{\nu} \frac{\theta_{\nu}}{1 + t_{\nu} \frac{\partial}{\partial \sigma_w}} \right) \frac{\partial^3 \bar{p}}{\partial \sigma_w^3}, \tag{5.105}
\]

\( \textbf{Footnote:} \) An independent numerical implementation of the LNTE model has been developed by Kanamori, Takahashi, and Makino of the Japan Aerospace Exploration Agency (JAXA) [98]. Their implementation uses the same splitting scheme as Salamone, with the primary difference being the use of a potential variable rather than the acoustic pressure. This potential formulation, first suggested by Marchiano et al. [129] (see also Appendix A) for solving the (lossless) nonlinear Tricomi equation, significantly simplifies the handling of shocks in the inviscid Burgers’ equation resulting from the splitting scheme.
Nonlinearity
\[
\frac{\partial^2 p}{\partial \sigma \partial t_w} = \mu_w \frac{\partial^2 p}{\partial t_w^2}.
\]  
(5.106)

The first equation may then be solved in the frequency domain using a tridiagonal matrix algorithm (see for instance Chapter 3 and Appendix I of [10]). The second, (5.105), can be solved exactly in the frequency domain. Finally, when integrated with respect to \( t \), (5.106) becomes an inviscid Burgers’ equation of the same form as (4.10), and can therefore be solved in the time domain using the implicit Poisson solution (4.11). This splitting process is continued until the unsteady term becomes sufficiently small (according to some predetermined error bound), indicating that a steady-state has been reached. Since the left-hand side of (5.103) is now negligibly small, this steady state solution must be a solution of the original equation (4.178).

Let us now return to the question of numerically solving the 3-D LNTE. Introducing the unsteady pseudotime term into (5.101) results in

\[
\frac{\partial^2 p}{\partial \sigma \partial t_w} = \frac{\zeta^2_w \partial^2 p}{\partial \eta_w^2} + \frac{\partial^2 p}{\partial \sigma^2} + \left[ \frac{2}{\zeta_w (a_Y + e_Y) \bar{y}_w - \bar{z}_w} \right] \frac{\partial^2 p}{\partial t_w^2} \]
\[
+ \left( \frac{2M_X - M^2_Y}{\varepsilon_w^2} \right) \frac{\partial^2 p}{\partial \eta_w^2} - 2\zeta_w \frac{M_Y}{\varepsilon_w} \frac{\partial^2 p}{\partial t_w \partial \eta_w} - 2M_Z \frac{M^2}{\varepsilon_w} \frac{\partial^2 p}{\partial t_w \partial \zeta_w} 
\]
\[
+ \frac{1}{\varepsilon_w^2} \left( \alpha + \sum_{\nu} \frac{\bar{y}_\nu}{1 + \bar{t}_\nu \frac{\partial}{\partial t_w}} \right) \frac{\partial^3 p}{\partial t_w^3} + \mu_w \frac{\partial^2 p}{\partial t_w^2}. 
\]  
(5.107)

Splitting (5.104)–(5.106) is also suitable for (5.107) provided an additional equation is introduced for \( Y \)-diffraction and \( Y \)-wind:

- **Y-Diffraction and Y-wind**

\[
\frac{\partial^2 p}{\partial \sigma \partial t_w} = \zeta^2_w \frac{\partial^2 p}{\partial \eta_w^2} - 2\zeta_w \frac{M_Y}{\varepsilon_w} \frac{\partial^2 p}{\partial t_w \partial \eta_w} + \frac{2}{\zeta_w (a_Y + e_Y) \bar{y}_w} \frac{\partial^2 p}{\partial t_w^2} 
\]  
(5.108)

This equation is identical in form to (5.104), and can therefore also be solved using the standard tridiagonal matrix algorithm provided the appropriate boundary conditions are specified. The most straightforward choice would be the requirement that the field decay to zero as \( \bar{y} \to \pm \infty \), reflecting the rapid attenuation of the field observed beyond lateral cutoff. This condition is for instance used by Guo and Hafez in [80]. Of course, a boundary condition arising from a more realistic lateral cutoff model is desirable. Such a model is provided Coulouvrat [50], who finds that at lateral cutoff, nonlinear waves
are governed by an unsteady Tricomi equation of the form

\[
\frac{\partial^2 p}{\partial t \partial x} = \frac{\partial^2 p}{\partial z^2} - z \frac{\partial^2 p}{\partial t^2} + \mu \frac{\partial^2 p}{\partial t^2},
\]

(5.109)

where the normalized inner variables are given by

\[
x = \frac{x}{(2\lambda_a R^2)^{1/3}}, \quad z = \left(\frac{2}{\lambda_a^2 R}\right)^{1/3} z, \quad \tilde{t} = \omega \left(\frac{t - x}{c_0}\right).
\]

(5.110)

In this case, the “caustic” is the ground surface while the tangent ray is the limiting ray whose contact point with the ground defines the lateral extent of the sonic boom carpet. Hence, for a flat ground where \(1/R_{xz} = 0\), the relative radius of curvature \(R\) is the projection of the radius of curvature of the limiting ray into the plane perpendicular to the ground, as depicted in Figure 5.7. \(x\) is the distance from the grazing point of the ray, \(z\) the altitude, and \(t\) the time. Note that the unsteady term in (5.109) is physical rather than numerical, arising from the unsteady behavior of the acoustic field as it penetrates deeper into the shadow zone. The similarity of this model to the focusing model suggests that near the lateral edges of the boundary layer, a “hand-off” could occur between the two based on an asymptotic matching principle. The exact nature of this hand-off remains an open question for now.

![Figure 5.7: Lateral cutoff geometry; adapted from [50]](image)

We conclude by noting the similarity of (5.107) to another well-known equation in sonic boom propagation. Recall that the Khoklov-Zabalotskaya (KZ) equation is the standard model for the focusing of nonlinear waves at a cusp caustic [49]. Unlike every variant of the Tricomi equation, the KZ equation is strictly hyperbolic, reflecting the fact that rays propagate on every side of a cusp caustic [107]. In fact, Auger [10] cites the
KZ equation as motivation for the choice of the unsteady term in the solution method just discussed as it renders the initially mixed-type nonlinear Tricomi equation strictly hyperbolic and thus simpler to solve in general. Because of its relative simplicity, the KZ equation and its lossy variant, the Khoklov-Zabalotskaya-Kuznetsov (KZK) equation have been applied to numerous problems in both high-intensity focused ultrasound and outdoor sound propagation \[83\]. The KZK equation has further been generalized to heterogeneous, moving media \[12\] and relaxing media \[45\], as well as being coupled to various turbulence models \[13, 26\]. The end result is an augmented KZK equation modeling the propagation of weakly nonlinear waves near caustic cusps in turbulent\(^{(26)}\), lossy, relaxing, heterogeneous, moving media, recently used to study the propagation of sonic booms through turbulent structures in the atmospheric boundary layer (ABL) above the earth’s surface\(^{(27)}\) \[13, 97, 183\]. Integrating \((5.107)\) with respect to \(\tilde{t}_w\) results in an equation strikingly similar to the augmented KZK equation. The only real difference is the absence of the term

\[
\frac{2}{\zeta} (a_Y + e_Y) \tilde{\sigma}_w - \tilde{\tau}_w \frac{\partial \tilde{p}}{\partial \tilde{t}_w},
\]

\((5.111)\)

as it is unique to the fold caustic structure assumed by the Tricomi equation. Therefore, we suspect that \((5.107)\) and thus \((5.101)\) would benefit greatly from the application of numerical methods originally developed for the KZ equation and its variants, both in speed and accuracy.

### 5.6 Chapter summary

In this chapter, the effects of cross-flow diffraction in the sonic boom focusing problem have been considered explicitly. We first reviewed the existing literature, in which it was found that the lateral terms become particularly significant near lateral cutoff, which is in turn intimately linked with Mach cutoff focusing. The effects of aircraft acceleration were also discussed, leading to a bound on the acceleration for which it remains acceptable to use the steady model provided by the NTE. In either case, the conclusion was that

\(^{(26)}\)An alternative model for sonic boom propagation through turbulence based on the transonic small disturbance equation (more-or-less the nonlinear Tricomi equation) has also been proposed \[74, 168\].

\(^{(27)}\)The motivation for use of the KZ equation in this context is the observation that sonic boom waveforms that have passed through the ABL exhibit random spiking and rounding as well as U-wave profiles \[127, 155\]. This behavior has been conjectured to be associated with stochastic focusing and defocusing of waveforms due to fine-scale temperature and wind fluctuations in the ABL, leading to folded wavefronts consistent with passage through a cusp caustic \[152, 155\].
both transverse length scales must be identical, as otherwise the lateral terms become negligible in the ordering scheme. This appeared to contradict a basic result of the standard nonlinear Tricomi equation, but was resolved by considering the lateral length scale to be dependent on distance from the flight track, or equivalently, to retain the cross-flow diffraction terms regardless of their size relative to other terms. After returning to geometrical acoustics to ascertain the form of certain terms appearing in the fully three-dimensional eikonal, we derived an equation which extends the LNTE model to include cross-flow diffraction. A simple modification of an existing code was then proposed for the numerical solution of this equation, as well as both simple and more accurate boundary conditions for the transverse coordinate.
Chapter 6  
Conclusions and Future Work

6.1 Conclusions

The basic model for the focusing of sonic booms is the nonlinear Tricomi equation, a second-order partial differential equation of mixed type. Since its derivation, much of the literature has been dedicated to generalizing this equation to account for a broader range of physical phenomena including wind, thermoviscous losses, and vibrational relaxation. Though derived for a three-dimensional medium, the model and its generalizations are effectively two-dimensional due to the relatively slow pressure variations found to occur in the direction transverse to the rays which tangent the caustic. However, near cutoff conditions, transverse diffraction becomes important and a fully 3-dimensional model becomes a necessity. Hence, a synthesis of these models was desired.

By generalizing the geometrical-acoustic description of the field approaching a caustic and leveraging results from existing models, we developed a new equation governing the effects of caustic focusing, 3-dimensional diffraction, atmospheric heterogeneity, wind, thermoviscous absorption, and vibrational relaxation on weakly nonlinear waves characteristic of sonic booms. This equation was found to reduce to all previous forms of the Tricomi equation in their appropriate regimes of applicability, yielding a quite flexible approach to the study of sonic boom focusing.

Though work remains to be done on the proposed model, we believe that the comprehensive literature review provided here will facilitate further work by placing the relevant results in one place. We are also confident that in addition to their historical interest, the translations provided in the appendices will provide further understanding and appreciation of the intricacies of this problem and the ingenuity of the researchers involved.
6.2 The future of the superboom problem

In order to substantiate the ordering scheme proposed in Chapter 5, the lateral length scale $L_y$ must be interpreted in terms of the ray-caustic geometry. The three-dimensional geometrical acoustic description near the caustic will also need to be adapted to a moving medium for complete determination of the newly proposed inner variables ($5.90$)\(^{(1)}\). With this done, the relevant inputs to a numerical solver can be computed. Of course, such a solver must still be developed and validated against experiment. Unfortunately, the data for off-track focused booms is limited. Of the superboom flight tests discussed in Appendix A, only the Operation Jericho-Carton [192], BREN Tower [81], and SCAMP [144] missions employed microphones laterally offset from the flight track. SCAMP possessed the densest array of the three, and so will likely be the most useful dataset for validation purposes. Of the 17 laterally offset focusing passes attempted during the project, 5 captured the entire focal region. These recordings will provide a promising start, but further flight tests and/or laboratory-scale experiments may be needed for comprehensive validation.

Even with a 3-D LNTE solver in hand, much work can still be done on the modeling side. For instance, while Cheng and colleagues have succeeded in extending the lossless model to accelerated flight and gradual vertical motion, it remains to be seen what role losses and wind play in such maneuvers. We suspect that the steady focusing term in ($5.101$) could simply be replaced by its unsteady analog in ($5.42$), but this requires more rigorous justification. With enough work, the model could even be adapted to the study of the focusing induced by combinations of maneuvers. In reality, the most limiting feature of the Tricomi model is its critical dependence on the fold caustic structure. Dives, turn entries, and more generally unsteady non-rectilinear maneuvers can result in the development of cusp caustics (arêtes). The Tricomi equation cannot hope to accurately model the pressure field for such situations, and will tend to underpredict the field strength due to the higher focusing factor at cusps. Instead, the archetype for focusing at cusps in two and three spatial dimensions is the KZ equation [49, 53–55]. Comparison with experiment shows good agreement [130], but much like the Tricomi equation, the KZ equation is only a local description. In the linear case, uniform asymptotic solutions capable of describing the entire problem domain are known [105, 107, 120, 121], but we are not aware of such results for weakly nonlinear waves. Therefore, if

\footnote{Note that $L_y$ is also expected to change due to wind ($L_y \rightarrow L_Y$), though this change may be insignificant provided the component of wind along the new $Y$ axis is small enough.}
one is interested in the entire focal region of a cusp-forming maneuver or a wave meeting multiple caustics, several layers of matching conditions would be required to stitch together the local solutions, potentially lowering accuracy and efficiency. Therefore, a different procedure entirely may be required for sufficiently complex maneuvers. The solution to this problem may follow from catastrophe theory.

Though not necessary for simple cases, the benefit of catastrophe theory becomes clear when studying physical systems corresponding to higher-order catastrophes. For example, the theory has been extensively applied to optics [20–25] leading to highly accurate predictions of interference patterns for quite complicated systems. In the language of catastrophe theory, caustics can be viewed as the bifurcation sets of a particular set of elementary, or structurally stable catastrophes [5, 22, 160]. This relationship allows very general conclusions to be drawn on the focusing configurations which are likely to develop during routine maneuvers. For instance, using catastrophe theory one can quickly conclude that the only caustics expected to occur in the ground plane are folds and cusps, and that they must do so as curves and points respectively [160]. Catastrophe theory also provides a general method for computing the appropriate phase variables for any order of caustic, as well as the corresponding diffraction integral (Airy, Pearcey, etc.) for the field amplitude in the linear case. In particular, the phase variable determined from geometric considerations in 3.3 corresponds to the universal unfolding of the fold catastrophe, and the same can be said for any order of caustic.

It is just as straightforward to find that in addition to the fold (as a surface) and cusp (as a curve), the higher-order swallowtail, elliptic umbilic, and hyperbolic umbilic caustic points can develop in three-dimensional space. Remarkably, this list is exhaustive. No other stable singularities can occur in 3-D. These higher-order caustics may lead to isolated “airbursts” that could be cause for concern in a city or mountainous area as their focusing factors are even higher than that of the cusp [160]. Therefore, a complete treatment of the superboom problem requires a model for such events. However, since these events can only occur at isolated points in 3-D space, we can also rest assured that they are less pressing than comprehensive understanding of the fold and cusp morphologies.

Lastly, it is interesting to note that each elementary catastrophe is formed by a nondegenerate configuration of lower-order catastrophes. Two folds meet at a cusp, two cusps form a swallowtail, etc. [160]. Similarly, a nearly planar wave meeting a caustic will typically fold in such a way that it resembles the next highest-order caustic. A wave meeting a fold caustic leaves as a cusp, a wave meeting a cusp leaves as a
swallowtail, etc. (this can be seen in Figures 1.9 and 1.10). It may be possible to leverage this subordinance of caustics to develop a focusing code which can actively adapt to arbitrary focusing conditions by only computing terms relevant to the singularity under consideration. Hence, catastrophe theory not only tells us where to go next with the superboom problem, but how we might finish it.
Appendix A
Experimental and Numerical Studies of Sonic Boom Focusing

A.1 Experimental studies

Over the past six decades, numerous experimental studies of sonic boom focusing have been carried out at both aircraft and laboratory scales. The most thoroughly researched superbooms are the transition focus boom—due to its unavoidability—and the Mach cutoff boom—as a byproduct of studying the potential noise reduction benefits of cutoff flight. Focused signatures generated during other maneuvers, while generally more difficult to measure, have also been captured. Detailed descriptions of every major superboom experiment may be found in Chapters 3 and 9 of Maglieri et al.’s comprehensive text [127], so we content ourselves with a relatively brief treatment here.

A.1.1 AGARD flight tests, 1959

As part of a series of tests to measure the ground intensity and potential structural damage caused by sonic booms, the NATO Advisory Group for Aerospace Research and Development (AGARD) produced some of the earliest recordings of transition focus booms [103]. The transition flights were conducted with a Fairey Delta 2 aircraft at a relatively low altitude of 10,000 feet between Mach numbers of 0.98 and 1.2 (Figure A.1). Measuring stations were placed along one mile of track in the region where the ground boom was expected to be loudest. A house outfitted with accelerometers was situated at the far end of this mile to measure boom-induced structural vibrations. Two distinct booms were measured within about 0.2 seconds of each other by the house accelerometers, indicative of an incident N-wave and a U-wave resulting from an earlier N-wave grazing
the acceleration caustic. Observers reported similar findings, with no boom heard before the initial superboom touchdown about 7 miles down-track, and one or more booms (with two bangs each) heard beyond this point. Pressures up to 4 times higher than the cruise boom were measured, but the maximum pressure expected near (but not directly at) where the geometrical caustic intersects the ground was not recorded.

Figure A.1: Time history of accelerated flight performed during AGARD experiment; adapted from [103]

A.1.2 NASA/USAF flight tests, 1961

The next set of focus boom tests was undertaken by NASA and the United States Air Force (USAF) in 1961 [112, 128]. A Lockheed F-104 Starfighter conducted longitudinal accelerations (Figure A.2a), circular turns (A.2b), pullup-climb-pushover (A.2c) (PCP), and pushover-dive-pullout (PDP) maneuvers (A.2d) at various Mach numbers and altitudes along a linear ground microphone array. The Mach number range for the acceleration tests (4 flights) was roughly 0.98–1.2 at an altitude of approximately 14,000 feet. Turns (four 90° turns and one 360° turn) were conducted at altitudes of 32–34,000 feet at constant Mach numbers of 1.4–1.5. The sole PCP maneuver climbed from 36,000 to 40,000 feet at a nearly constant Mach number of 1.4. Four PDP maneuvers beginning at a Mach number of 1.2 and altitude of 40,000 feet, increasing to Mach 1.5 during the dive phase, and bottoming out at 10,000 (1 of 4 flights) or 20,000 (3 of 4 flights) feet, were performed. Two theoretical sonic boom propagation methods were then compared to
recordings by plotting measured and predicted ground shock arrival times and pressure amplitudes as a function of distance along the flight track. The first method assumes a linear sound speed gradient and no wind and was applied to all flights. The second method allows for arbitrary sound speed/wind gradients and was only applied to one case each from the acceleration and PDP maneuvers.

![Diagram of maneuvers](image)

Figure A.2: Maneuvers performed during NASA/USAF flight tests; [112]

Anywhere from 1 to 3 complete sonic boom signatures (2 to 6 bangs) were measured depending on the maneuver and observer location\(^1\). Both theoretical methods were capable of predicting the number of bangs and successfully determined their arrival times to within a few seconds. In regions far from wavefront cusps and associated superbooms (which were found to have pressures 2–4 times higher than cruise conditions for all maneuvers), pressure predictions provided good estimates of the average strength of the first (and typically stronger) boom while slightly overshooting the later, weaker boom(s). The theoretical methods do predict cusps and substantial increases in pressure, but the predicted cusps lie about 2 miles down-track of the observed ones\(^2\), and the pressures

\(^1\)In theory, 1 sonic boom (2 bangs) is associated with a cruise boom, superboom, or super-superboom, 2 booms (4 bangs) with the post-focus region after a superboom/fold caustic, and 3 booms (6 bangs) with the post-focus region of a super-superboom/cusp caustic. In reality, the temporal spacing of these bangs—on the order of 100-250 ms for the cruise boom, even less for interacting N/U wave combinations—may be too close for each to be clearly distinguished by a ground observer, resulting in fewer perceived bangs.

\(^2\)These are interpreted as the points of maximum pressure, but the maximum pressure occurs some
predicted near the cusps are not accurate, as can be seen from the plots for the fourth acceleration pass, Figure A.3. The circles and solid lines represent, respectively, the measured and predicted values for the first shock, the bow shock of an incident N-wave that has not reached the caustic. The squares and dashed lines correspond to measured and predicted values for the second shock, the bow shock of a U-wave resulting from an earlier N-wave grazing the caustic (recall Figure 3.6).

![Figure A.3: Comparison of calculated and measured elapsed times and overpressures for a longitudinal acceleration at constant altitude; adapted from [112]](image)

The first method is geometric in nature and, by virtue of the crossing of adjacent rays, predicts an infinite pressure at the caustic. The second, developed by Friedman, Kane, and Sigalla [68], employs a fully nonlinear geometrical theory that is in a sense an overcorrection. Namely, their theory suggests that as a shock front approaches a caustic, the amplitude-dependent propagation speed will cause the portion of the focusing front nearest the caustic to accelerate and eventually overtake the outer portions, corresponding to a change in wavefront concavity and divergence in the rays that ultimately yields distance away from the geometrical caustic and is further offset by nonlinearity, so some discrepancy is to be expected at the outset.
a finite ray tube area. The apparent contradiction lies in the fact that Friedman, Kane, and Sigalla’s theory is based on the shock-dynamic theory of Whitham [195, 196], which only holds for particularly strong shocks not typical of sonic booms\(^{(3)}\).

### A.1.3 NASA flight tests, 1964

Further NASA-sponsored flight tests followed in 1964 [88]. Five accelerated flight passes of an F-104 fighter aircraft were performed at an altitude of 37,200 feet between Mach numbers of 0.9 and 1.4-1.6. Five more flights were conducted at or near grazing conditions associated with Mach cutoff flight \((M \approx 1.15\) in the standard atmosphere\)), three at 37,500 feet and two at 33,500 feet. An attempt was also made to better define the lateral extent of the primary sonic boom carpet during steady flight. The acceleration passes exhibited behavior similar to the 1961 tests: significant waveform amplification (up to 2.5 times cruise conditions) followed by two booms within a short time interval, the latter being of U-wave character. Multiple booms were also observed near Mach cutoff conditions and some amplification of the waveform is suggested, but this effect tends to be counteracted by the reduced ground reflection factor associated with grazing incidence, yielding pressures comparable with steady non-cutoff flight. At microphone locations prior to the initial ground-shock intersection of the longitudinal acceleration runs and Mach numbers below that associated with a ground cutoff, low-amplitude “rumbling” signatures consistent with the formation of a shadow zone were observed. Peak overpressures recorded during three acceleration runs are plotted against distance along the ground track in Figure A.4. The data is plotted such that the maximum recorded overpressure of each run associated with the superboom phenomenon is aligned at zero. The shadow zone to the left of zero, the large amplification associated with the superboom at zero, and the two-wave region to the right of zero are evident. Note that the “2d N-wave” label in the diagram refers to the U-wave leaving the caustic.

\[^{(3)}\]That said, the work of Sturtevant and Kulkarny [184] demonstrates that the distinctly different focusing behaviors observed for weak and strong shocks are part of a continuum. In particular, it is found that a shock approaching a focus (be it fold, cusp, or perfect) will evolve according to the same set of complex nonlinear processes no matter its strength, with different post-focus behaviors developing as a result of the crossing (or lack thereof) of a set of diffracted shocks generated during this process.
Figure A.4: Normalized plot of measured sonic-boom overpressures at various locations along the ground track for three acceleration passes at $M \approx 0.9$ to 1.5 at an altitude of 11,339 m (37,200 ft); adapted from [88]

**A.1.4 Operation Jericho, 1966–1969**

Beginning in 1966, a comprehensive study of the focusing of sonic booms was undertaken by the French Working Group on sonic booms [191, 192]. The project, known as Operation Jericho, included an assessment of the existing theory on sonic boom focusing and an increasingly precise series of experimental measurements. The experimental leg consisted of four stages: Jericho-Instrumentation—test planning and choice of recording apparatus, Jericho-Focalisation—low-altitude (2,000 feet) acceleration focusing tests, Jericho-Virage—high-altitude (36,000 feet) turn and acceleration focusing, and Jericho-Carton—a study of the effect of variations in acceleration and lateral spread on the focusing factor as well as an attempt to measure the “superfocus” associated with the cusp caustic that develops during turn entry (recall Figure 1.10a). All runs were performed by a Dassault Mirage III or Mirage IV aircraft. A relatively dense microphone array and a set of manometric sensors transverse to the flight track were used, allowing for more precise placement of the superboom and determination of its focusing factor than previously possible. The low-altitude Jericho-Focalisation flights also allowed for
reliable optical guidance of the aircraft and significantly reduced signature variability caused by atmospheric scattering.

Signatures recorded in the vicinity of the acceleration and turn superbooms are shown in Figures A.5a and A.5b. The ground shock intersections for each maneuver are also sketched above the signatures, making the connection between the recorded waveforms and the local wavefront geometry clear. The ground touchdown of the superbooms—at which focusing factors of up to 5 were determined—can be seen at microphone 11 for the acceleration case and 15 for the turn. Beyond these microphones, the waveform gradually separates into two “sheets,” the first an N-wave and the second a weaker U-wave. Signature rounding prior to the superboom can also be observed in microphones 11–14 of the turn maneuver, indicating a shadow zone.

A similar set of plots for the turn entry super-superboom can be seen in Figure A.6. The signatures above microphone 18 (mics 19, 1, ...) correspond to the fold caustic curving toward the the outside of the turn—which the authors refer to as the pseudofocus—that only develops during the entry phase. The signatures below (mics 17, 16, ...) are associated with the fold on the other side of the cusp point interior to the turn (the focus), which remains when the turn reaches a steady state. Superbooms are clearly generated at both the focus (around mic 15) and the pseudofocus (around 19), and can be seen to interact with a third incident N-wave. Moving toward the cusp point lying near microphone 18, we see that this system of waves merges into an extremely steep super-superboom signature, which the authors estimate to have a focusing factor of 9, though the two peaks at the front of the signal indicate that microphone 18 does not lie exactly at the superfocus. Moving away in either direction, the system gradually separates into three waves: the incident N wave and an N/U combination that quickly spreads, weakens, and rounds off. To close out the study, the authors propose mitigation strategies for all maneuvers except the transition focus boom since as noted previously, it is the only unavoidable maneuver-induced superboom.
Figure A.5: Sonic boom focusing recorded during Operation Jericho; adapted from [192]
Figure A.6: Recordings in the vicinity of the cusp caustic/superfocus generated during turn entry; adapted from [192]

**A.1.5 BREN Tower, 1970**

Another joint NASA/U.S. Air Force project on superbooms was carried out in Nevada [81] in 1970. Fifteen microphones were placed at 100 foot intervals up the 1,529 ft BREN (Bare Reactor Experiment, Nevada) Tower, fourteen along the ground parallel to the flight path, and two perpendicular to it (Figure A.7a). This layout recorded sonic boom signatures as a function of altitude for the first time, leading to detailed vertical profiles of focusing occurring during longitudinal accelerations and Mach/lateral cutoff conditions (Figures A.7b–A.7d). The main goals of the study were to interpret the focused signatures in light of the test conditions and to determine the accuracy of sonic boom prediction methods available at the time.

79 passes were flown in the vicinity of the threshold Mach number at which a sonic boom first reaches the ground. The data was generally good, indicating rumbling signatures consistent with a shadow zone below the threshold Mach number, focused U-like waves up to 1.8 times stronger and 40% longer in duration than cruising flight at the threshold condition, and multiple booms at higher speeds. Away from cutoff conditions, overpressure and time predictions were generally quite good with ground arrival of shocks predicted to within ±1 second under normal atmospheric conditions, a trend which carried through to the acceleration and lateral cutoff cases. However, the sensitivity of
Mach cutoff focusing to the aircraft ground speed occasionally led to spurious focused signatures associated with inadvertent aircraft accelerations as the pilot attempted to maintain a constant speed. The results of this study also allowed the authors to suggest both “safe speeds” and “safe altitudes” for which no boom will reach the ground.

Over the 19 longitudinal acceleration passes flown during the project, the focusing factors were determined to be between 2 and 5, tending to increase with greater accelerations. A novel result of the study is an attempt to quantify the distance from the caustic at which the geometrical theory first becomes invalid—that is, the caustic boundary layer thickness where diffraction and nonlinearity become predominant. The authors find that geometrical theory gives reasonable results up to about 1300 ft down-track of the caustic-ground intersection and about 500 feet vertically above the caustic, though these distances are somewhat dependent on the acceleration magnitude, tending to increase as the acceleration magnitude decreases. The authors also provide an estimate of the region in which the incident N-wave and the outgoing U-wave leaving the caustic are additive, i.e. superimpose linearly. They find that superposition is valid up to about 100 feet vertically above the caustic while incoming and outgoing waves closer to the caustic interact nonlinearly. Note that this does not imply that the N and U waves themselves evolve linearly within this region. In fact, a study we will discuss shortly shows that as a focusing condition (fold, cusp, or otherwise) is approached, nonlinear effects tend to dominate before diffraction if the incident wave is a shock, no matter its strength.

Under lateral cutoff conditions (9 passes), signatures similar in shape to those observed during acceleration and Mach cutoff focusing were captured—two waves before cutoff, an amplified (relative to nearby locations) U-wave at cutoff, and rumbling acoustic disturbances beyond cutoff. Unlike in the acceleration case, the amplitudes of the cutoff waves were significantly lower than the undertrack cruise boom. This is attributed to the greater distance traveled by signatures at the lateral edges of the sonic boom carpet and the reduced ground reflection coefficient associated with grazing incidence, measured to be about 1.2 at cutoff and anywhere from 0.86 to 1.08 at greater lateral distances. It is noted that this result may also be affected by the presence of significant crosswinds during several of the lateral cutoff passes. Theoretical calculations allowed the lateral cutoff to be placed to within about ±1.0 km.
Figure A.7: BREN Tower microphone placement and maneuvers; adapted from [81]

### A.1.6 Ballistics experiments, 1974

In 1974, Sanai, Toong, and Pierce simulated Mach cutoff flight in a stratified atmosphere at the laboratory scale by firing supersonic projectiles into a carbon dioxide-air mixture [172] \(^{(4)}\). The resulting flow patterns were captured using schlieren imaging techniques and subsequently compared to flows in a simulated homogeneous (air-only) atmosphere. The experimental setup is depicted on the left-hand side of Figure A.8, while the right-hand side is a blown-up schematic of a typical result for the shock flow pattern. The waveforms and caustic computed using geometrical acoustics are superimposed for comparison. While there are clearly cusps in the imaged shock pattern, they are slightly displaced relative to both the linear wavefronts and the geometrical caustic, consistent with the nonlinear distortion of the sonic line predicted by the nonlinear Tricomi equation \((4.30)\). A result not predicted by the Tricomi equation \([10]\) is the presence of *triple-points* at the cusp points, where three shocks are seen to meet: the incident shock, the out-

\(^{(4)}\) As noted by the authors, a similar experiment was carried out during Operation Jericho and briefly recounted in [192].
going shock, and a *Mach stem*, more typically seen in shock waves reflecting at solid boundaries [19]. We will see shortly that this effect is only prominent for particularly strong shocks and is not generally expected to occur for focused sonic booms.

In Figure A.3, Schlieren images and pressure-time histories are compared for projectiles of similar Mach numbers in homogeneous and stratified mediums. The focusing effects of refraction are clear—no cusps form in the homogeneous medium, but they are ubiquitous in the stratified runs. Based on these recordings, Sanai et al. determine a maximum focus factor of 1.7, roughly consistent with flight tests. At the time of this paper Guiraud’s work on the nonlinear Tricomi equation was becoming well known, so the authors also attempt to infer the universal constant in Guiraud’s scaling law, resulting in a value of 1.3. Comparison of this value to that implied by the BREN Tower tests shows very good agreement. A comparison was also made to theoretical calculations of the constant, and while their result was in line with the value of 1.4 computed by Thery, Lecomte, and Reggiani [186], it was significantly different from the value of 2.6 found by Gill and Seebass [75], leading to a call for further investigation\(^{(5)}\).

\(^{(5)}\)As noted in Section 4.2.6, this discrepancy was later resolved by Plotkin and Cantril [158, 159].
Figure A.9: Schlieren imaging of projectile fired in homogeneous (top) and stratified (bottom) mediums; [172]

In a companion article [173], Sanai et al. simulated the transition focus boom by firing constant-speed projectiles into a gas mixture for which the sound speed decreases along the flight axis, thereby increasing the effective Mach numbers of the projectiles. Interestingly, wavefront folding consistent with passage through a cusp caustic is observed at lower shock strengths\(^{(6)}\), but as the size and speed of the projectile are increased, the folds tend to disappear. As a result, the authors conjecture that the wavefront folding mechanism and lack thereof are complementary processes which occur at distinct (though not yet clearly delineated) shock strengths. Scaling their results to dimensions typical of an accelerating aircraft in the atmosphere, the authors determine focusing factors of 6.1 for the smooth caustic and 12.7 for the cusp/arête. The computed factor for the fold caustic is at the upper end of values reported during flight tests, but is within a reasonable margin of error. The cusp value is significantly larger than the value of 9

\(^{(6)}\) The acceleration caustic is actually the lower sheet of a cusp caustic forming at the aircraft position at the onset of acceleration, so cusp-like wavefront folding is possible during aircraft acceleration. However, by the time the lower fold caustic intersects the ground, the upper fold is typically so far away that the crossing of the cusped wavefronts is not noticeable, or even measurable by a ground array.
reported by Wanner et al. [192], but since their recording was taken some distance away from the true superfocus, this value may still be reasonable.

A.1.7 Project Have BEARS, 1994

The 1994 USAF Have BEARs project [64] was the first focusing-oriented flight test campaign since BREN Tower. The name of the project stems from the use of boom event analyzer recorders (BEARs), instrumentation specially designed to digitally record sonic boom signatures. In addition to the transonic accelerations more common to civilian aircraft, Have BEARS produced superboom recordings generated by an F-16B fighter aircraft during low-altitude (10,000–20,000 feet) combat maneuvers including accelerating dives, tight high-g turns, and climbout-pushover maneuvers. Flight planning was conducted using PCBoom’s raytracing module, allowing an approximate target point for focusing (appearing to be the expected location of the first boom to reach the ground) to be set and subsequently backtracked to determine where each stage of the maneuver should begin relative to the ground BEAR array. The goal of the study was to test pilots’ ability to control the placement of the focal zone within the array during both externally guided and unguided maneuvers, validate current superboom prediction methods, and examine the effect of atmospheric turbulence on focusing.

The majority of the runs conducted were level accelerations, which were performed under calm, thermally turbulent, and mechanically turbulent (gusting wind) atmospheric conditions. This maneuver was emphasized as it proved to be the simplest for pilots to acclimate to for producing controlled booms in unguided runs. The consistency of the flight profiles then allowed any major differences between passes to be attributed to turbulent scattering. Representative waveforms gathered under each type of atmospheric condition are depicted in Figure A.10. The waveforms are plotted with distance along the flight track where the zero point is the nominal target point for the focus and positive values indicate points downtrack of the target. The waveforms are also time-aligned with respect to their leading shocks. The signatures recorded during calm conditions, Figure A.10b, are as expected—little to no noise followed by a greatly amplified superboom signature, reaching a maximum pressure of 19.1 psf (3.8 times higher than the 5.0 psf predicted under cruise conditions) a short distance beyond the first ground impact. Beyond this point, the waveform gradually separates into an incident N-wave and postfocus U-wave. In contrast, thermal turbulence (Figure A.10c) has rounded and distorted the signatures so significantly that no clear focal region is discernible. The maximum recorded pressure of 10.0 is also only twice as high as the cruise boom, and falls much
Figure A.10: Focal region waveforms recorded during acceleration passes under calm, thermally turbulent, and mechanically turbulent atmospheric conditions; adapted from [64].

Further down the flight track than in the calm atmosphere. Mechanical turbulence (Figure A.10d) leads to yet a third case in which the signatures are significantly spiked in character, associated with decreased shock rise times and greater peak amplitudes. Moreover, two distinct focal regions have formed, the first occurring shortly uptrack of the target where the focusing factor reaches 4.2 and the second nearly 7,000 feet down-track with a focusing factor of 3.4. Hence, depending on the character of the random atmospheric fluctuations experienced, turbulence can have either a strong focusing or defocusing effect on already focused signatures(7).

Another effort of the project was comparison of the recorded signatures to those predicted by a focusing code based on the approximate analytical solution to the nonlinear Tricomi equation developed by Gill and Seebass [75, 175]. An output of the code,

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(7) This has been attributed to the development of cusp caustics and folded wavefronts from the (effective) sound speed heterogeneities induced by random temperature and wind fluctuations characteristic of turbulent media [13, 14, 71, 152, 155].
computed using a flight profile and atmospheric conditions gathered from the calm level acceleration case of Figure A.10b, is shown on the right in Figure A.11. The measured signatures are reproduced on the left for comparison. As discussed by the authors, the predicted focus appears too far up-track of the target (bearing in mind the difference in the vertical axes of the measured and predicted plots) and the N-U wave separations are larger than measured. However, it is also pointed out that the nonlinear distortion of the caustic (i.e., the sonic line) predicted by the nonlinear Tricomi equation shifts the effective position of the focus, the maximum overpressure point, and so on. Applying a correction for this distortion shifts the predicted focus much closer to the measured one (the two differ by about 1,500 feet post-correction). On the other hand, the maximum predicted overpressure is about 37% lower than what is measured. The authors attribute this to the sensitivity of the Gill-Seebass solution to small errors, claiming good agreement overall.

Figure A.11: Measured and predicted waveforms during a level acceleration pass under calm atmospheric conditions; adapted from [64]

The other aircraft maneuvers performed during the project were primarily carried out under calm atmospheric conditions. The recorded signatures are qualitatively similar to the level acceleration case, and can be seen in Figure A.12. The recordings from the dive (Figure A.12a) indicate that the postfocus region for the dive is narrower than for
the level acceleration, as the postfocus U-wave moves away from the incident N-wave much more quickly. The focusing factor, 2.3, is also seen to be lower than for the level case. For the turn (A.12b), an attempt was made to capture only the steady (non-cusp) portion of the focal region. This so-called “focal line” is very narrow, as focusing is only found to occur within a band of 2500 feet or so about the target point. The resulting focusing factor is found to be 4 when compared to cruise booms measured prior to the initiation of the turn. Unlike the level acceleration and the dive, the pushover maneuver (Figure A.12c) was executed in a plane laterally offset from the ground array. This was done to account for a predicted offset of the focus boom relative to the centerline of the flight track. The superboom is seen to occur about 2000 feet downtrack from the target point, and the maximum focusing factor is found to be 3.4. Interestingly, beyond this point, only weak disturbances are seen to trail behind the incident N-waves rather than the usual U-waves, indicating an extremely narrow focal region.

In total, aircrews were successful in placing a focus boom within the ground array in 37 out of the 49 passes conducted during the project. 27 of these passes placed the focus within ±3000 feet of the target point, an impressive feat considering that the entire focal region of any given pass was found to be only 100–1,000 feet wide. Unsurprisingly, placement was most accurate under calm atmospheric conditions and runs in which the pilot received real-time course corrections from the ground crew, but pilots were still quite capable of placing the focus autonomously.

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(8) Comparing the maneuver in Figure A.12b to Figure 1.10a (both overhead views), it appears that Downing et al. primarily captured focusing occurring along the “pseudofocus” of Operation Jericho, as waves propagating along the fold interior to the turn (the “focus” of Operation Jericho) would tend to move away from the target point. Of course, the pseudofocus is still a fold caustic all the same.

(9) We presume that this offset is due to crosswind/asymmetric lateral refraction present during the experiment. Otherwise, it is unclear how this maneuver, which is restricted to a vertical plane, could lead to a laterally offset focusing condition.

(10) The definition of the width of the focal region is not completely clear from the article, but appears to be defined by some predetermined separation distance between the leading shocks of the N and U-waves. If so, the estimated focal region width is a constant multiple of the caustic boundary layer thickness $\delta$ ($3.51$). For instance, the SCAMP project defined the focal region as the ground projection of the caustic boundary layer, $2\delta/\sin \theta$, where $\theta$ is the angle at which the caustic grazes the ground [144].
Figure A.12: Focal region waveforms recorded during dive, steady turn, and climbout-pushover maneuvers; adapted from [64]
A.1.8 Water tank experiments, 2003

One of the first studies to compare direct numerical solutions of the nonlinear Tricomi equation to recorded superboom signatures was carried out by Marchiano, Thomas, and Coulouvrat in 2003 [131]. Though conducted at the laboratory scale, the experiment was carefully designed such that all relevant physical quantities scaled at a 1:100,000 ratio to values typical of aircraft-generated superbooms while important similitude parameters\(^{(11)}\) were held fixed at an order of magnitude comparable to that expected at the aircraft scale. In this way, it was assured that the NTE predictions would be compared to signatures representative of realistic, yet easily reproducible focusing conditions.

The setup used to accomplish the 1:100,000 scaling is depicted in Figure A.13. A transducer array generates a series of ultrasonic waves phased such that they focus at a fold caustic one meter away. A hydrophone connected to a set of stepper motors then collects pressure measurements at various locations relative to the caustic which are subsequently read off from a digital oscilloscope and saved to a PC for later processing.

![Experimental setup for water tank focusing](image)

Figure A.13: Experimental setup for water tank focusing—a curved wavefront radiated by the piezoarray synthesizes a fold caustic 1 m away. The hydrophone is moved perpendicular to the caustic over 8 cm; adapted from [131]

\(^{(11)}\)In particular, the quantities $\beta M_{ac}$ (associated with nonlinearity) and $\lambda_{ac}/R$ (associated with diffraction) were held fixed, resulting in a value of the nonlinearity factor $\mu$ (4.28) comparable to that seen in an aircraft-generated superboom. The ratio of the characteristic duration of the incident signal to the leading shock rise time was also controlled to emulate the slightly attenuated incident sonic boom expected after long-range propagation through the atmosphere.
After ensuring that their experimental setup synthesized the expected caustic field in the linear regime (i.e., the Airy function), the authors moved into the nonlinear regime by increasing the amplitude of the focusing fronts until they steepen into weak shocks. Care is taken to ensure that the shocks are well-developed by the time they reach the synthesized caustic so that a direct comparison can be made with the focusing of aircraft-generated sonic booms. The only difference is that the incident N-waves expected for real sonic booms are replaced by a periodic sawtooth configuration, a constraint imposed by the limited bandwidth of the transducer array.

The field measured over one acoustic period along a control line normal to the caustic (see Figure A.13) is shown in part (a) of Figure A.14. The cusped shock fronts dividing regions of high and low pressure are evident, and can be seen to touch the distorted sonic line (highlighted by a dotted line) dividing the wave-filled and evanescent regions of the field as we have come to expect. In part (b) of the figure, numerical solutions of the linear (3.99) and nonlinear (4.30) Tricomi equations (found via a numerical solver developed by Auger and Coulouvrat [10, 11] to be discussed in Section A.2.3) are compared to the measured field at five different distances from the (geometrical) caustic. The incident signal $F$ input into the solvers is provided in part (c) of the figure, and is found by applying the radiation condition (4.33) to the measured field. The linear solution is in clear disagreement with the measured field in terms of amplitude, phase, and rise time, but the nonlinear solution performs satisfactorily in all regards. The authors do note that the outgoing signal is slightly smoother in the numerical solution, and conjecture that this may be due either to unsteady behavior resulting from the finite size of the array or the approximate nature of the matching condition used to determine the incident waveform\(^{(12)}\). Nevertheless, the success of the nonlinear Tricomi equation in predicting the pressure field created by weak shocks focusing at a fold caustic is apparent, as is the effectiveness of the experimental setup used to make the comparison. As a result, a very similar set of experiments was carried out for cusp caustics [70, 130], for which the model equation is the Khokhlov-Zabolotskaya (KZ) equation [49].

\(^{(12)}\)As a reminder, condition (4.33) is approximate because it replaces the nonlinear phase variable of the incoming waveform by the linear phase (governed by the acoustic eikonal), a step justified by the fact that the nonlinear solution must eventually asymptote to the linear solution some distance from the caustic.
Figure A.14: (a) one period of the measured pressure field. (b) Comparison between measured field (solid line), linear (dashed line), and nonlinear (dotted line) numerical simulations at five distances from the geometrical caustic. The outgoing peak follows the incoming shock at $z \sim 0.011$ and 0.017. The two have merged by the time they reach $z \sim 0.005$ on the sonic line. The two merge again at $z \sim 0.023$, but this is an artifact of the periodic incident waveform. (c) Measured incoming signal used for numerical solutions; adapted from [131]

A.1.9 SCAMP, 2010-2012

The most recent and largest focused boom flight test campaign to date was the Superboom Caustic Analysis and Measurement Program (SCAMP) [144] conducted over 27 months from 2010 to 2012. The program was sponsored by NASA, but also received significant contributions from Wyle Laboratories (now KBRwyle), Penn State, Gulfstream Aeronautics, Eagle Aeronautics, Central Washington University, and numerous other collaborators. The aim of the project was to build an extensive library of focused sonic boom signatures with which to develop, validate, and compare various numerical models of sonic boom focusing which could then be applied to the prediction of superboom signatures of future shaped-boom aircraft.

Over the course of the program, 61 superboom-generating acceleration and pushover passes of an F-18B aircraft were recorded by an array denser than any previous flight test campaign. 32 of the focusing passes successfully placed the entire focal region within
the array, and 5 additional passes captured at least the maximum focus, resulting in a
suite of signatures characterizing the entire focusing process. A TG-14 powered glider
was also flown during several of the passes to capture airborne focused boom events
above the atmospheric boundary layer, resulting in “clean” signatures free of turbulent
scattering effects.

The extensive dataset gathered during the flight test portion of SCAMP was sub-
sequently applied to the development and validation of three focus boom prediction
tools (which we will discuss in more detail in later sections): an approximate solution
of the (velocity potential) nonlinear Tricomi equation (4.35) developed by Gill and
Seebass [75, 175] and subsequently implemented into PCBoom by Plotkin and Cantril
[158, 159], Salamone’s implementation of the lossy nonlinear Tricomi equation (4.178)
[169, 170], and Piacsek’s numerical solution of the nonlinear progressive equation (NPE)
[150], a time-domain parabolic approximation of the exact nonlinear wave equation de-
rived by McDonald and Kuperman [134] (13). Though all three focus boom prediction
methods showed notable success, LNTE exhibited particularly remarkable agreement
with recorded signatures throughout the entire focal region. An example of this con-
sensus is illustrated in Figure A.15, where LNTE outputs are compared to signatures
recorded during a particular acceleration pass at four different (normalized) distances
from the geometrical caustic. Aside from slight discrepancies in the peak amplitudes
and the absence of small-scale “wobbles,” the predicted waveforms are seen to match
the measured signatures almost identically, particularly in the insonified region above
the caustic. Both differences are likely attributable to turbulent scattering associated
with propagation through the atmospheric boundary layer, a phenomenon not considered
in the LNTE model.

Near the end of the project, the newly validated focusing codes were applied to the
prediction of superbooms generated by four theoretical shaped-boom aircraft configura-
tions. The data preparation process is broken down into three steps. First, PCBoom
was run for a predetermined transonic acceleration profile to determine the flight condi-
tions at which a focus would be generated. This data was then provided to partners to
compute near-field pressure signatures for each aircraft configuration. Finally, PCBoom
was used to propagate these signatures down to the edge of the caustic boundary layer
and compute the relevant inputs for each focusing model. An example output is shown
in Figure A.16, which compares the maximum-pressure focused signatures predicted by

(13) A numerical solution of the NTE developed by Kandil and Zheng [100, 101] was also considered,
but was ultimately dismissed as a special case of the LNTE model.
Figure A.15: Comparison of LNTF predictions (blue) and SCAMP recordings (red) at $\zeta = 0.97$ (mic 71, near edge of nominal caustic boundary layer, N and U-waves fairly well separated), $\zeta = 0.67$ (mic 67, within boundary layer, N/U waves overlap and focusing begins to be noticeable), $\zeta = 0.085$ (mic 59, near geometrical caustic, N/U waves indistinguishable), and $\zeta = -0.22$ (mic 55, shadow zone, signal rounded and much weaker); adapted from [170]

Each method for the Quiet Supersonic Jet/QSJ, a theoretical shaped-boom configuration developed by Gulfstream.

Evidently, the three methods produce substantially different outputs. At the same time, each is seen to contain steeper shock-like portions and slope discontinuities, features not generally expected of shaped sonic booms. Hence, while it is not necessarily clear what the resulting signature will look like, focusing is expected to distort shaped booms away from their intended design point, potentially negating loudness reductions gained by the shaping process.

Though the exact form of the focused signatures generated by the four hypotheti-
Figure A.16: Comparison at maximum focus between Gill-Seebass, NPE, and LNTE solutions for Gulfstream shaped-boom configuration; adapted from [144]

...cal shaped boom configurations could not be determined, the SCAMP team conjectures that the LNTE model would produce the most credible loudness calculations of the three models because it is the only model to properly account for molecular relaxation effects. For this reason, the LNTE model alone was applied to the final leg of the project, an exploration of the effect of the flight profile on the focusing characteristics of three of the four shaped-boom configurations mentioned. Though transonic accelerations generally fall outside of the optimal design envelope of shaped-boom aircraft, it is found that some shaping features do persist through the focusing process. Further, both lower acceleration rates and higher altitudes invariably reduce the perceived loudness of focused shaped booms, as they evolve closer to their intended design point in these regimes. That said, lower accelerations and higher altitudes also tend to extend the focal region over a larger ground footprint, potentially exposing a larger population to focused shaped booms.

A.1.10 Conclusion

While largely representative of the experimental study of sonic boom focusing, the studies discussed are not an exhaustive list. For studies prior to 1986, a more complete bibliography can be found in a report by Hubbard, Maglieri, and Stephens [90], which
also collects references on sonic boom minimization and the effect of sonic booms on communities, structures, wildlife, and terrain. For references published after this report, we recommend the final report for the SCAMP project [144] and chapters 3 and 9 of Maglieri et al.’s text [127]. The latter reference for instance provides Table A.1, a handy summary of every transition focus boom flight test conducted since 1959:

Table A.1: Compilation of transition flight test experiments; adapted from [126]

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<td>9,850</td>
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<td>None</td>
<td>To 6.2 mi One Side</td>
<td>0.34 Mile to Each Side</td>
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<td>5.6</td>
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<td>2.0 - 5.0</td>
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<td>1 of 7</td>
<td>2 of 5</td>
<td>12 of 12</td>
<td>4 of 5</td>
<td>1 of 23</td>
<td>15/26</td>
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<tr>
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<td>None</td>
<td>2 of 2</td>
<td>None</td>
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</table>

\[14]\) Note that some of these tests conducted other maneuvers as well. The numbers provided are strictly for the level acceleration passes.
A.2 Numerical solutions of the NTE

Numerous methods have been developed to numerically solve the nonlinear Tricomi equation and its variants. Here we describe their basic structure, leaving more detailed discussions of their efficacy to the source material.

A.2.1 Murman-Cole scheme

In any problem, it is desirable to use a numerical scheme suited to the local flow behavior—hyperbolic schemes for supersonic flow, parabolic schemes for sonic flow, and elliptic schemes for subsonic flow. Clearly, this task becomes more complicated when the equation to be solved is of mixed type. Worse still, the sonic (parabolic) line of the nonlinear Tricomi equation separating the various flow regimes depends on the local pressure, and so cannot be determined without a solution to the equation. In summary, to solve the equation we need to know the local flow behavior, but to find the local flow behavior we need to solve the equation. This apparent paradox is resolved by use of the iterative Murman-Cole scheme originally developed for the numerical simulation of transonic flows [47, 137] and later adapted to the solution of the NTE by Seebass, Murman and Krupp [176] and eventually to the three-dimensional nonlinear Tricomi equation (5.23) in [38, 80].

The basic process of the Murman-Cole scheme goes as follows. First, the solution to the linear Tricomi equation (given analytically by (3.117)) is computed. Second, the problem domain is discretized according to the flow regions determined by the linear solution—the elliptic (shadow zone) region is covered by a uniform grid while the hyperbolic (insonified) region is spanned by the characteristics (wavefronts) of the linear equation, as seen in A.17. The nonlinear Tricomi equation is then discretized according to this result, with a first-order implicit scheme used for (nominally) hyperbolic points and a a second-order centered scheme for (nominally) elliptic points, resulting in a set of nonlinear algebraic difference equations. Using the Airy function solution of the linear equation as an initial “guess,” these equations can be solved iteratively at each point in the problem domain, resulting in a new trial solution. The process then repeats, beginning with determination of the nominal flow regions. This procedure is repeated until a (more-or-less user defined) convergence criterion is met.
A.2.2 Approximate analytical solution

An approximate analytical solution of the nonlinear Tricomi equation has been derived by Gill and Seebass [75, 175]. The solution is founded on a type of hodograph transformation, a technique well known in the theory of transonic aerodynamics [47]. Applying this transformation to the NTE and its associated boundary conditions results in a linear Tricomi equation in the hodograph variables. This equation can then be solved in terms of the Airy function, and the hodograph transform inverted to obtain a solution in the physical plane. Given this, one might wonder why the method is only approximate. The devil is in the details of the inverse transformation. In particular, when the incident waveform contains shocks, the inverse transformation becomes singular at certain points in the hodograph plane, leading to gaps and multivalued regions in the physical solution at which expansion fans and shocks must be inserted. Usually, these shocks would be determined by a set of jump conditions arising from mass and momentum conservation arguments, but this cannot in general be done for this problem, as the matching of one condition actually makes it impossible to meet the other [11, 75, 86, 158]. Therefore,

\[\text{(15)}\]

This method appears to have been independently rediscovered in the Russian literature by Manukyan [203].
choice must be made in which condition to meet, meaning that either mass or momentum conservation will be violated by the computed “solution,” though the discrepancy is typically small [158]. As a result, the Gill-Seebass method is only an approximate solution to the nonlinear Tricomi equation if the incident waveform contains weak shocks. In a way this is comforting, as it further supports the notion that nonlinearity is an essential feature in the focusing of weak shocks (a conclusion first suggested when it was found that weak shocks lead to pressure discontinuities in the linear focusing theory, and further confirmed by experiment [184]).

In [158, 159], Plotkin and Cantril present a focusing code based on the Gill-Seebass method and in particular, the approximate solution for the step shock shown in Figure A.18.

![Figure A.18: Incident step shock (blue) and resulting focused waveform predicted by the Gill-Seebass method (red); adapted from [159]](image)

Recalling the Guiraud similitude (4.44), we see that the solution for a step shock of any strength focusing at any fold caustic follows immediately by a simple rescaling of the red waveform in A.18. The problem is that this scaling law only applies for step shocks, and so cannot strictly be used for more realistic sonic boom signatures such as the N-wave. However, the authors argue that because the higher frequencies associated with shocks focus more strongly (due to the narrowing of the boundary layer with increasing frequency), their focusing behavior is expected to dominate the overall focused signature. Given this, the behavior of the linear expansion portion of the N-wave near the focus

\[\text{(16)}\]

Of course, the Gill-Seebass method can be applied to essentially arbitrary incident signals if desired. See for example [69].
would not be nearly as important as that of the two shocks enclosing it, each of which can be approximated by a step shock. Provided these shocks are sufficiently separated in time, it is also reasonable to assume that their focusing processes are more-or-less independent. In short, Plotkin and Cantril have reduced the computation of a focused N-wave (and more generally, any signal whose shocks are sufficiently well-separated) to the computation of the focusing of an independent set of step shocks, in turn reducing to a simple variable rescaling by the Guiraud similitude.

Plotkin and Cantril test their method by applying it to two typical maneuvers: a level acceleration and a turn. First, the caustic geometry and incident signature are determined via raytracing. The incident signature is then shifted and scaled such that the step shock solution can be separately applied to each shock, and the resulting focused shocks recombined into a complete focused signature. The result of this procedure for each type of maneuver is shown in Figure A.19, which plots the predicted focused signatures at the distances from their respective caustics for which the pressure reaches a maximum.

![Figure A.19: Results of Plotkin-Cantril method for incident N-wave; adapted from [158, 159]](image)

The qualitative features of each signature are in good agreement with what we have come to expect of a focused N-wave, but an adverse side effect of the method is also apparent. Treating the incident shocks independently has led to a nonphysical jump discontinuity between the resulting focused shocks. The greater the mismatch (in absolute value) between the peaks of the incident bow and tail shocks, the larger this jump will tend to be. As a result, certain physical characteristics of the focused boom cannot be reliably predicted using this method. Of particular concern are loudness metrics, as they tend
to be quite sensitive to discontinuities. This was noted during the SCAMP project, and led to the quick dismissal of Plotkin and Cantril’s implementation of the Gill-Seebass method in the prediction of the loudness of focused shaped sonic booms [144]. That said, the method has proven to be a fast and reliable predictor of peak overpressures of real focused superbooms [64, 144] and remains a legacy option in NASA’s PCBoom prediction software [145].

A.2.3 Unsteady pseudospectral method

In [10, 11] Auger and Coulouvrat propose an iterative method for solving the nonlinear Tricomi equation (4.30), and by extension the windy nonlinear Tricomi equation (4.116) (through transformation (4.117)). In this method, an associated unsteady nonlinear Tricomi equation (UNTE) is first introduced, replacing the zero in the nonlinear Tricomi equation with an unsteady term of the form $\frac{\partial^2 p}{\partial \sigma \partial t}$ where $\sigma$ is an artificial “pseudo-time” variable. This term serves two purposes. First, it renders the unsteady equation strictly hyperbolic, as opposed to the usual mixed nature of the standard nonlinear Tricomi equation. Therefore, unlike in the Murman-Cole scheme, the discretization of the equation does not depend on the point considered (aside from boundary points of course). Second, since the unsteady term must asymptote to zero at large values of the pseudotime, a solution for the standard nonlinear Tricomi equation may be obtained as a large (pseudo)time steady state of the unsteady equation.

The unsteady equation is solved using an operator-splitting scheme summarized in Figure A.20. For each pseudotime iteration, the solution is divided into two substeps. In the first, the term associated with nonlinearity is discarded, resulting in an unsteady linear Tricomi equation (ULTE) that can be solved in the frequency domain (of the retarded time variable) by a standard tridiagonal matrix algorithm. After inverse transformation, the result is fed into the second step in which the diffraction term is neglected and the nonlinear term retained. This leads to an equation equivalent to the inviscid Burgers equation which may be solved for each value of the $\tau$ coordinate using a shock-capturing scheme. The two-step process is repeated until convergence—determined as some prescribed small difference between successive pseudotime iterations—is achieved.

The choice of splitting used in this method is not unique, but is often made on physical grounds as seen here. The benefit of such a splitting is two-fold. First, it allows the various physical effects involved to be coded as separate subroutines, and thus considered independently. For example, if only the first step of the above procedure is carried out, a solution to the linear Tricomi equation, and hence the focusing of linear waves
at a fold caustic, is obtained. Second, given their physical significance, the resulting sub-equations are often well-studied and occasionally possess analytical solutions. If the splitting is chosen to maximize the availability of such solutions, computation time can be drastically decreased.

The modularity and simplicity of the Auger-Coulouvrat has made it quite popular, leading to a variety of offshoots and generalizations. Marchiano et al. [129] refine the scheme by solving for an intermediate variable proportional to the velocity potential, resulting in a shock-fitting procedure for the inviscid Burgers equation that is both simpler and far more efficient than methods typically used to render the pressure single-valued. Kandil and Zheng [100, 101] compare the performance of various solution methods for the subequations resulting from the splitting scheme. Kandil and Khasdeo [99] apply one of these methods to a parametric study in which the effects of variations in the size

Figure A.20: Operator-splitting scheme for solving the nonlinear Tricomi equation; adapted from [11]. Note that some notation has been changed to match that in use here.
of the computational domain and the characteristics of the incident signal are considered. Perhaps the most widely used adaptation of the Auger-Coulouvrat scheme (see e.g. [162]) is that proposed by Salamone for his lossy nonlinear Tricomi equation (4.178) [169, 170], which introduces an additional subequation for wind, losses, and relaxation effects. More recently, Kanamori, Takahashi, and Makino have applied Salamone’s splitting scheme to a potential variable form of LNTE (the potential suggested by Marchiano et al.) with further improvements in the handling of shocks arising from the inviscid Burgers’ subequation [98]. These authors also used their code to study the effects of aircraft acceleration and atmospheric humidity on the Perceived Levels (PLs) and maximum overpressures of focused signatures associated with incident N-wave, flattop, and ramp pressure signatures.

A.2.4 Conservation law schemes

Sescu, Afjeh, and Sescu [177] and Sescu and Afjeh [178] propose a solution for the NTE in terms of the unsteady equation which does not employ operator splitting. Instead, after modifying the radiation boundary condition (4.33) by assuming the signal approaching the caustic is of finite duration (as is the case for focused sonic booms), the authors show that the unsteady nonlinear Tricomi equation can be decomposed into a system of first-order hyperbolic conservation equations (17), opening the door to a variety of methods devised for the solution of hyperbolic conservation laws.

In [177], Sescu, Afjeh, and Sescu solve the hyperbolic system by applying a weighted essentially non-oscillatory (WENO) scheme in (z, t) space and a total-variation diminishing (TVD) Runge-Kutta scheme in pseudotime. The WENO scheme involves the use of a weighted sum of numerical stencils where the weights adapt to the local smoothness of the numerical solution. Namely, any stencils spanning large gradients and discontinuities receive very little weight, avoiding artificial oscillations and smoothing of discontinuities while in smooth regions, the weights are determined to ensure uniformly high-order accuracy.

An alternative solution to the conservation law form of the unsteady nonlinear Tricomi equation is provided by Sescu and Afjeh [178], who use a discontinuous Galerkin (DG) computation scheme for the spatial discretization of the problem (a TVD Runge-Kutta method is still used for the time discretization). In general, the Galerkin scheme finds a solution to a weak formulation of the partial differential equation under con-

\[^{(17)}\] The analogous decomposition of the steady equation is studied along with more general systems of hyperbolic conservation laws in [91].
sideration with respect to a particular set of basis functions. Its discontinuous variant allows the basis functions representing the solution (here the Legendre polynomials) to be discontinuous from one discretized element of the domain to the next, but requires some additional handling to control the artificial oscillations introduced by these discontinuities.

[178] provides a comparison of third-order DG and WENO schemes to the analytical linear solution for an incident N-wave (given by (3.127)), reproduced in Figure A.21. Two features are of note. First, the (more-or-less identical) computed waveforms appear to be completely free of the artificial oscillations occasionally observed in pseudospectral focusing codes [10, 169]. The unfortunate tradeoff is considerable rounding at amplitude and slope discontinuities, significantly reducing predicted peak amplitudes.

Figure A.21: Comparison of third-order WENO and DG schemes to the analytical linear solution for an N-wave focusing at a caustic (3.127); adapted from [178]

A.3 Other sonic boom focusing models

We conclude with a brief qualitative comparison between the NTE model and two other models proposed for the study of sonic boom focusing: the Nonlinear Progressive Equation (NPE) and the Homogeneous One-Way Approximation for the Resolution of Diffraction (HOWARD).
A.3.1 The Nonlinear Progressive Equation (NPE)

The NPE, first introduced by McDonald and Kuperman [134], is a time-domain small-angle approximation of the lossless nonlinear wave equation analogous to the well-known parabolic equation model of linear acoustics (see e.g. [171]). The model treats the pressure field as a perturbation about a dominant state, characterized by a linear acoustic wave propagating unidirectionally at some constant reference speed (perhaps a mean value) \( c_0 = c_0,\text{ref} \). The ambient pressure and density \( p_0, \rho_0 \) are also presumed constant, but the (small-signal) sound speed \( c_0 \) is free to vary so long as its fluctuation about \( c_0,\text{ref} \) is small. In a frame moving with the nominal acoustic disturbance, the NPE for a two-dimensional medium is given by

\[
\left( \frac{\partial}{\partial t} + c_0,\text{ref} \frac{\partial}{\partial x} \right) \tilde{p} = -\frac{\partial}{\partial x} \left( c' \tilde{p} + \frac{\beta c_0,\text{ref}^2 \tilde{p}^2}{2} \right) - \frac{c_0,\text{ref}^2}{2} \int_{x_0}^{x} \frac{\partial^2 \tilde{p}}{\partial z^2} \, dx',
\]

where \( \tilde{p} \equiv \rho' / \rho_0 \) is the normalized density fluctuation, \( x \) the primary propagation direction of the disturbance, \( z \) the transverse coordinate perpendicular to the \( x \) axis, and \( c' = c_0 - c_0,\text{ref} \) the fluctuation of the local sound speed about reference speed \( c_0,\text{ref} \). The left-hand side of (A.1) represents the forward propagation of the reference plane wave at a speed \( c_0 \) along the \( x \)-axis. The right-hand side corrects this result to account for the effects of refraction, nonlinearity, and diffraction. The use of a wave-following reference frame is motivated by numerical considerations, as it allows only a small region surrounding the dominant acoustic disturbance to be considered at any given time. As a result, a high-resolution grid may be used to study the phenomenon of interest—the fine wave structure of the focusing disturbance—with little computation time being wasted on quiescent regions not yet reached (or already passed) by the waveform.

To study the effects of nonlinearity on focusing waveforms, McDonald and Kuperman apply both the linear (\( \beta = 0 \)) and nonlinear (\( \beta \neq 0 \)) forms of (A.1) to an N-wave focusing at a caustic generated by refraction in a medium with parameters typical of fresh water. The sound speed \( c_0 = c_0(z) \) is taken to decrease linearly with depth until a point \( z = z_0 \) beyond which it remains constant, ensuring that a source situated below \( z = z_0 \) will generate initially spherical wavefronts. A large incident amplitude is chosen to make any differences due to nonlinear focusing effects apparent.

Contour plots\(^{(18)}\) of the resulting density perturbations in the region surrounding the

\(^{(18)}\)Note the similarity of the nonlinear plots to those produced by Guo and Hafez [80] and Cheng and Hafez [38] for the 3-D nonlinear Tricomi equation, Figure 5.4.
caustic are shown in Figure A.22. The left column corresponds to the linear case and the right to the nonlinear case, while time (equivalently the distance of the dominant disturbance from the upper part of the domain) evolves downward in each column. For reference, the geometrical (i.e., linear) caustic lies along a line at 500 m depth which the incident waveform approaches from the bottom right of the grid. The upper and rightmost boundaries of the domain are pressure release surfaces.
Figure A.22: Snapshots of the density perturbation $\bar{\rho}(x,z)$ at fixed times. Time has been converted to range traversed by the moving grid since step 0. (a)–(d) linear case ($\beta = 0$). (e)–(g) nonlinear case ($\beta = 3.5$). Three-dimensional perspective plots are projected down onto contour plots with fixed contour interval $7.5 \times 10^{-5}$. The range direction is $x$ and depth is $z$; adapted from [134]
The difference between the linear and nonlinear cases becomes noticeable by step 200 of the propagation code (row 2 of the figure), where positive (resp. negative) contours in the nonlinear case have shifted forward (backward) relative to the linear case. By step 500, the roughly symmetrical development of the wavefront cusps in the linear case is seen to be disturbed by the development of shocks (in addition to those present in the incident signal) in the nonlinear case. The shock structure to the left of the figure is particularly interesting, as a Mach stem can be seen forming a triple-point with the incoming and outgoing fronts associated with the tail shock of the N-wave—a feature often (but not always) observed in experimental studies of shock wave focusing [131, 172, 184, 192]. Nonlinear distortion of the caustic (i.e., a sonic line) is also suggested, as the extremities of the shocks are offset to either side of the geometrical caustic according to the sign of the local density perturbation. Finally, the waveform leaving the caustic has the expected U shape for both the linear and nonlinear cases, but the U-wave of the nonlinear case is both weaker and significantly longer in duration than in the linear case—features noted in real focused U-waves [81].

The NPE has seen several generalizations since its inception, including a wide-angle modification analogous to that often used for the standard parabolic equation [133] and the addition of terms related to density stratification [135] and atmospheric losses/turbulence [150]. The wide-angle form of the lossy, turbulent model was used during the SCAMP project as a means of studying the focusing of sonic booms generated by both standard and shaped boom aircraft configurations [144].

A.3.2 Homogeneous One-Way Approximation for the Resolution of Diffraction (HOWARD)

HOWARD is a numerical method developed by Dagrau et al. based on a one-way approximation of the lossless nonlinear wave equation in a weakly heterogeneous medium [56]. The governing equation is derived by retaining all terms appearing in the homogeneous nonlinear wave equation exactly while simplifying any terms related to heterogeneity by a wide-angle parabolic approximation. Taking \( x \) as the primary propagation direction presumed by the parabolic approximation, the so-called HOWARD equation is given by
\[
2c_{0,\text{ref}} \frac{\partial^2 \psi}{\partial x \partial T} - c_{0,\text{ref}}^2 \left( \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} \right) - \left( 1 - \frac{c_{0,\text{ref}}^2}{c_0^2} \right) \frac{\partial^2 \psi}{\partial T^2} \\
+ \frac{c_0^2}{\rho_0} \left( \left[ \frac{\partial \psi}{\partial x} - \frac{1}{c_{0,\text{ref}}} \frac{\partial \psi}{\partial T} \right] \frac{\partial \rho_0}{\partial x} + \frac{\partial \psi}{\partial y} \frac{\partial \rho_0}{\partial y} + \frac{\partial \psi}{\partial z} \frac{\partial \rho_0}{\partial z} \right) \\
= \frac{\beta}{\rho_0 c_0^2} \frac{\partial}{\partial T} \left[ \left( \frac{\partial \psi}{\partial T} \right)^2 \right],
\]

(A.2)

where \( \psi \equiv \int_{-\infty}^t p \, dt \) is a potential variable and \( T \equiv t - x/c_{0,\text{ref}} \) is the retarded time for a frame moving at the mean sound speed \( c_{0,\text{ref}} \). In order of appearance, the terms of (A.2) correspond to wave propagation, homogeneous diffraction, sound speed heterogeneity, density heterogeneity, and nonlinearity.

The HOWARD method solves (A.2) by operator splitting, with one sub-equation for each physical phenomenon. The heterogeneity and nonlinearity equations are solved exactly while the solution of the diffraction equation relies on a one-way approximation that neglects any backscattered components. This mixed approximation method (wide-angle parabolic heterogeneity terms, one-way homogeneous diffraction terms) preserves the dispersion relation of the exact linear wave equation for all forward-propagating waves. Therefore, while the standard parabolic approximation is considered valid for waves propagating within \( \pm 15^\circ \) of the primary propagation direction and its wide-angle extension holds out to about \( \pm 30^\circ \), the HOWARD formulation is expected to hold out to \( \pm 90^\circ \)—that is, for any waves which propagate toward positive \( x \), regardless of their behavior along the transverse axes.

As an example of their method, Dagrua et al. examine the focusing of an N-wave at an “acoustical lens,” a localized sound speed heterogeneity leading to the development of a cusp caustic. Figure A.23a depicts the pressure distributions predicted by (A.2) for this scenario at various distances from the center of the acoustical lens. The lens is seen to introduce a curvature into the initially planar shock fronts (a), causing each to converge and eventually focus at cusp points (b), from which they emerge as folded “swallowtail” fronts (c), continuing to expand in space as they propagate (d). On-axis pressure-time histories corresponding to distances (b)–(d) are shown in Figure A.23b. The focused U-wave expected at the cusp is apparent, as is its subsequent splitting into N and U waves. However, as evidenced by the final signature extracted from a slightly off-axis point at lateral distance (d), there are not one, but two N-waves in addition to the U-wave, resulting in six distinct shocks as the waves separate. This is consistent with the three-ray/wave cusp caustic structure associated with the “superfocus” condition.
developing during turn entry and pushover/dive maneuvers (recall figures 1.10), and similar six-shock signatures have been noted in flight tests for such maneuvers \cite{112}.

(a) Pressure (Pa) versus time (s) and transverse distance (normalized by wavelength $\lambda$ of incident N-wave) at (a) 0, (b) 24 (geometrical focus), (c) 39, and (d) 95 wavelengths from lens center.

(b) Pressure waveforms (Pa) versus time (s) along the axis at (a) 24 (geometrical focus), (b) 39, and (c) 95 wavelengths from lens center and (d) off-axis ($x = 95\lambda, y = 0.9\lambda$).

Figure A.23: Wavefront folding and focused pressure signatures associated with propagation of an N-wave through an acoustical lens; adapted from \cite{56}
In more recent years, the HOWARD model has been generalized to systems with mean flow (wind) [164] and atmospheric absorption, relaxation, and turbulence [122, 123]. The latter two references also apply their generalized model to prediction of the acoustic field associated with a sonic boom at lateral cutoff in both calm and turbulent atmospheres, yielding several interesting results on the focusing of weak shocks in turbulent media.
Appendix B
The Inverse and Implicit Function Theorems

B.1 Introduction

In this appendix, we outline the mathematical background necessary for understanding of the inverse and implicit function theorems. The information comes, nearly verbatim, from Rudin’s analysis textbook [167] with the occasional clarifying remark from Poston and Stewart’s text on catastrophe theory [160].

B.2 Multivariable calculus

Definition B.2.1. [Linear transformation] A mapping $A$ of a vector space $X$ into a vector space $Y$ is said to be a linear transformation or linear map if

$$A(x_1 + x_2) = A(x_1) + A(x_2), \quad A(cx) = cA(x)$$

for all $x, x_1, x_2 \in X$ and all scalars $c$.

Note that one often writes $Ax$ instead of $A(x)$ if $A$ is linear. Linear transformations of $X$ into $X$ are often called linear operators on $X$.

Definition B.2.2 (Inverse of a linear operator). A linear operator $A : X \to X$ is said to be invertible or non-singular if

1. $A$ is one-to-one
2. $A$ maps $X$ onto $X$. 
In this case, we can define an *inverse* linear operator $A^{-1}$ on $X$ such that $A^{-1}(Ax) = A(A^{-1}x) = x$ for all $x \in X$.

**Definition B.2.3 (Differentiability at a point).** Suppose $E$ is an open set in $\mathbb{R}^n$, $f : E \to \mathbb{R}^m$, and $x \in E$. If there exists a linear transformation $A$ of $\mathbb{R}^n$ into $\mathbb{R}^m$ such that

$$\lim_{h \to 0} \left( \frac{\|f(x + h) - f(x) - Ah\|}{\|h\|} \right) = 0,$$

then we say that $f$ is *differentiable at* $x$ and we write

$$f'(x) = A.$$

for the *derivative* of $f$ at $x$.

In the definition above, $\|\cdot\| : \mathbb{R}^k \to \mathbb{R}$ is the usual Euclidean norm,

$$\|x\| = \|(x_1, \ldots, x_n)\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_k^2}.$$

As it is an element of $\mathbb{R}^m$, the numerator of the expression inside the limit has $k = m$, while the denominator has $k = n$.

$f'(x)$ is sometimes called the *total derivative* or *differential* of $f$ at $x$, and generalizes the interpretation of the derivative as the best linear approximation of $f : \mathbb{R} \to \mathbb{R}$ at $x \in \mathbb{R}$—the tangent line at $x$—to the best approximating linear map of $f : \mathbb{R}^n \to \mathbb{R}^m$ at $x \in \mathbb{R}^n$—a tangent hyperplane. This property is illustrated for a function $f : \mathbb{R}^2 \to \mathbb{R}$ in Figure B.1.

![Figure B.1: Geometric interpretation of the total derivative; adapted from [160].](image)

**Definition B.2.4 (Differentiability/Derivative).** If a function $f : E \subset \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at every $x \in E$, we say that $f$ is *differentiable in* $E$. Its *derivative* is then
the function $f' : E \to L(\mathbb{R}^n, \mathbb{R}^m)$, where $L(\mathbb{R}^n, \mathbb{R}^m)$ is the set of all linear maps $\mathbb{R}^n \to \mathbb{R}^m$. The value of $f'$ at $x$, $f'(x)$, is the linear map defined in B.2.3.

**Definition B.2.5** (Partial derivative). Consider a function $f$ that maps an open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^m$. Let $\{e_1, \ldots, e_n\}$ and $\{u_1, \ldots, u_m\}$ be the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$. The components of $f$ are the real functions $f_1, \ldots, f_m$ defined by

$$f(x) = \sum_{i=1}^{m} f_i(x)u_i \quad (x \in E),$$

or, equivalently, by $f_i(x) = f(x) \cdot u_i$, $1 \leq i \leq m$.

For $x \in E$, $1 \leq i \leq m$, $1 \leq j \leq n$, we define

$$(D_j f_i)(x) = \lim_{t \to 0} \frac{f_i(x + te_j) - f_i(x)}{t},$$

provided the limit exists. Writing $f_i(x_1, \ldots, x_n)$ in place of $f_i(x)$, we see that $D_j f_i$ is the derivative of $f_i$ with respect to $x_j$, keeping the other variables fixed. The notation

$$\left. \frac{\partial f_i}{\partial x_j} \right|_x, \quad \text{or} \quad \frac{\partial f_i}{\partial x_j}(x) = \frac{\partial f_i}{\partial x_j}(x_1, \ldots, x_j, \ldots, x_n).$$

is therefore often used in place of $D_j f_i$, and $D_j f_i$ is called a partial derivative. $D_j f_i(x)$ is the slope of the tangent line to the graph of $f_i$ in the $x_j$-direction, which is the line in which the tangent hyperplane to the graph of $f_i$ meets the coordinate plane $x_k = \text{constant} \ (k \neq j)$ (See Figure B.2)

![Figure B.2: Geometric interpretation of the partial derivative; adapted from [160].](image-url)
By composing the partial derivatives of a function along each coordinate surface, an explicit coordinate expression for the total derivative of the function may also be developed:

**Theorem B.2.6.** Suppose \( f \) maps an open set \( E \subset \mathbb{R}^n \) into \( \mathbb{R}^m \), and \( f \) is differentiable at a point \( x \in E \). Then the partial derivatives \( (D_j f_i)(x) \) exist, and

\[
f'(x)e_j = \sum_{i=1}^{m} (D_j f_i)(x)u_i \quad (1 \leq j \leq n).
\]

If we let \([f'(x)]\) be the matrix that represents \( f'(x) \) with respect to our standard bases, then \( f'(x)e_j \) is the \( j \)th column vector of \([f'(x)]\). The previous theorem then shows that the number \((D_j f_i)(x)\) occupies the spot in the \( i \)th row and \( j \)th column of \([f'(x)]\). Thus

\[
[f'(x)] = \begin{bmatrix}
(D_1 f_1)(x) & \cdots & (D_n f_1)(x) \\
\vdots & \ddots & \vdots \\
(D_1 f_m)(x) & \cdots & (D_n f_m)(x)
\end{bmatrix}.
\]

That is, in coordinates, \( f'(x) \) can be identified with the Jacobian matrix of \( f \) at \( x \).

**Definition B.2.7** (Continuous differentiability). A differentiable mapping \( f \) of an open set \( E \subset \mathbb{R}^n \) into \( \mathbb{R}^m \) is said to be continuously differentiable in \( E \) if \( f' \) is a continuous mapping of \( E \) into \( L(\mathbb{R}^n, \mathbb{R}^m) \). If this is so, we also say that \( f \) is a \( C^1 \) (or \( C' \)) mapping, or that \( f \in C^1(E) \).

**Theorem B.2.8.** Suppose \( f \) maps an open set \( E \subset \mathbb{R}^n \) into \( \mathbb{R}^m \). Then \( f \in C^1(E) \) if and only if the partial derivatives \( D_j f_i \) exist and are continuous on \( E \) for \( 1 \leq i \leq m, 1 \leq j \leq n \).

We are now ready to state the inverse function theorem. Loosely speaking, it says that a continuously differentiable mapping \( f \) is invertible in a neighborhood of any point \( x \) at which the linear transformation \( f'(x) \) is invertible.

**Theorem B.2.9** (Inverse Function Theorem). Suppose \( f \) is a \( C^1 \)—mapping of an open set \( E \subset \mathbb{R}^n \) into \( \mathbb{R}^n \), \( f'(a) \) is invertible for some \( a \in E \), and \( b = f(a) \). Then

(a) there exist open sets \( U \) and \( V \) in \( \mathbb{R}^n \) such that \( a \in U, b \in V, f \) is one-to-one on \( U \), and \( f(U) = V \);

(b) if \( g \) is the inverse of \( f \) (which exists, by (a)), defined in \( V \) by

\[
g(f(x)) = x \quad (x \in U),
\]
then \( \mathbf{g} \in C^1(V) \).

In practice, we must usually work in a particular set of coordinates. To this end, we instead express the equation \( \mathbf{y} = \mathbf{f}(\mathbf{x}) \) in component form as \( y_i = f_i(x_1, \ldots, x_n) \) (\( 1 \leq i \leq n \)) and note that \( \mathbf{f}'(\mathbf{a}) \) is invertible if and only if the Jacobian determinant, or simply Jacobian \( \mathbf{D}(\mathbf{a}) \) is nonzero,

\[
\mathbf{D}(\mathbf{a}) = \det [\mathbf{f}'(\mathbf{a})] = \det \left[ \frac{\partial f_i}{\partial x_j} \right] \neq 0, \quad (1 \leq i, j \leq n).
\]

The inverse function theorem then states: if \( \mathbf{f} \) is continuously differentiable and the Jacobian does not vanish at \( \mathbf{a} = (a_1, \ldots, a_n) \), then, provided we restrict \( \mathbf{x} \) and \( \mathbf{y} \) to small enough neighborhoods of \( \mathbf{a} \) and \( \mathbf{b} \), the system of \( n \) equations

\[
y_i = f_i(x_1, \ldots, x_n) \quad (1 \leq i \leq n)
\]
can be uniquely solved for \( x_1, \ldots, x_n \) in terms of \( y_1, \ldots, y_n \), and the solution is continuously differentiable.

Before introducing the implicit function theorem, we will establish the following notation.

If \( \mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n \) and \( \mathbf{y} = (y_1, \ldots, y_m) \in \mathbb{R}^m \), let us write \((\mathbf{x}, \mathbf{y})\) for the point (or vector)\(^{(1)}\)

\[
(x_1, \ldots, x_n, y_1, \ldots, y_m) \in \mathbb{R}^{n+m}.
\]

In what follows, the first entry in \((\mathbf{x}, \mathbf{y})\) or in a similar symbol will always be a vector in \( \mathbb{R}^n \), the second will be a vector in \( \mathbb{R}^m \).

**Theorem B.2.10** (Implicit Function Theorem). *Let \( \mathbf{f} \) be a \( C^1 \)-mapping of an open set \( E \subset \mathbb{R}^{n+m} \) into \( \mathbb{R}^m \), such that \( \mathbf{f}(\mathbf{a}, \mathbf{b}) = \mathbf{0} \) for some point \((\mathbf{a}, \mathbf{b})\) \( \in E \).*

If the Jacobian matrix

\[
\left[ \frac{\partial f_i}{\partial y_j} (\mathbf{a}, \mathbf{b}) \right]
\]

is invertible, then there exist open sets \( U \subset \mathbb{R}^{n+m} \) and \( W \subset \mathbb{R}^n \), with \((\mathbf{a}, \mathbf{b}) \in U \) and \( \mathbf{a} \in W \), having the following property:

To every \( \mathbf{x} \in \mathbf{W} \) corresponds a unique \( \mathbf{y} \) such that

\[
(\mathbf{x}, \mathbf{y}) \in U \quad \text{and} \quad \mathbf{f}(\mathbf{x}, \mathbf{y}) = \mathbf{0}.
\]

\(^{(1)}\)We have interchanged the roles of \( \mathbf{x} \) and \( \mathbf{y} \) (and all associated symbols) with respect to Rudin’s statement of the theorem in order to put it in a more suitable form for our purposes.
If this \( y \) is defined to be \( g(x) \), then \( g \) is a \( C^1 \)-mapping of \( W \) into \( \mathbb{R}^m \), \( g(a) = b \),

\[
f(x, g(x)) = 0 \quad (x \in W),
\]

and

\[
g'(a) = - \left[ \frac{\partial f_i}{\partial y_j}(a, b) \right]^{-1} \left[ \frac{\partial f_i}{\partial x_k}(a, b) \right].
\]

Thus, the function \( g \) is “implicitly” defined by the equation \( f(x, g(x)) = 0 \), hence the name of the theorem.

Less formally, if the straight approximation (tangent line, plane, etc.) to the set \( S \) of solutions of an equation

\[
f(x, y) = 0
\]

is the graph of a function

\[
y = y(x)
\]

then so, locally, is the set of solutions itself [160]. For example, let

\[
f(x, y) = x^2 + y^2 - 1,
\]

so that the set \( S \) is a unit circle at the origin (Figure B.3).

![Figure B.3: \( f(x, y) = x^2 + y^2 - 1 \); adapted from [160].](image)

At point \( P \), we have \( \partial f/\partial y = 2y \neq 0 \). Therefore, the tangent at \( P \) is the graph of
a function, and hence so is \( S \) around \( P \). In particular, \( S \) is locally the graph of the function

\[
y = +\sqrt{1 - x^2},
\]

and the slope of the tangent line at \( P \) is given by

\[
\left. \frac{dy}{dx} \right|_P = -\left( \left. \frac{\partial f}{\partial y} \right|_P \right)^{-1} \left. \frac{\partial f}{\partial x} \right|_P = -\frac{x}{y} \bigg|_P.
\]

On the other hand, the tangent lines at \((1, 0)\) and \((-1, 0)\) are not graphs, and neither is \( S \) around these points.

Note that when the conditions of the implicit function theorem are not fulfilled, \( S \) may still locally be a graph. For example, if

\[
f(x, y) = x - y^3,
\]

and \( P \) is the origin, then \( \partial f / \partial y = -3y^2 = 0 \). But the solution set \( S \) shown in Figure B.4 is the graph of \( y = x^{1/3} \) at \( P \) (and everywhere else). But since the tangent line is vertical at \( P \), \( S \) is not the graph of a differentiable function.

Figure B.4: \( f(x, y) = x - y^3 \); adapted from [160].
Appendix C
Mathematica Code

Here, we provide the Mathematica code used to compute the ray coordinate expansions (5.54)–(5.56) about an arbitrary point \((x_c, y_c, z_c)\) near a fold caustic, generalizing the expansion method developed by Gazaryan \([200]\) to arbitrary order. The code is written in a “by-hand” style for clarity, but a more experienced Mathematica user could easily convert it to a compact iterative form if desired. Similar results can also be obtained with Mathematica 12.0 and above’s \textit{AsymptoticSolve} function.

\begin{verbatim}
In[1]:= (* MultiPolyTruncate[func_,varList_,maxOrder_] truncates multinomial
func with variables varList to maximum order maxOrder, where
order=sum of exponents of all variables in varList *)

In[2]:= MultiPolyTruncate[func_,varList_,maxOrder_]:=FromCoefficientRules[
Cases[Flatten[CoefficientRules[func,varList]],
Rule[a_,b_]/;(Total[a]≤maxOrder),
varList

240
\end{verbatim}
(* poly[{x_,y_,z_},n_,a_] Defines a sum in powers of x^i*y^j*z^(k/2) with
coefficients a_i,j,k up to maximum order n. To find the ray coordinate
expansions in the negative branch of Sqrt[z] (i.e., the coordinates
for the other ray passing through (x_c,y_c,z_c)), replace z^(k/2) by
(-Sqrt[z])^k (this is the only change required in the code). *)

poly[{x_,y_,z_},n_,a_]:=Sum[a_i,j,k x^i y^j z^(k/2),
{i,0,n},{j,0,n-i},{k,0,2*(n-i-j)}];

(* Express u, v, \tau as power series in x, y, Sqrt[z] *)

Tayldeg=2; (* Maximum order for u, v, and \tau expansions *)

u_c = u \rightarrow poly[{x_c,y_c,z_c},Tayldeg,m]

u \rightarrow m_{0,0,0} + m_{0,0,1} \sqrt{z_c} + m_{0,0,2} z_c + m_{0,0,3} z_c^{3/2} + m_{0,0,4} z_c^2 + m_{0,1,0} y_c
+ \text{m}_{0,1,1} y_c \sqrt{z_c} + \text{m}_{0,1,2} y_c^2 + \text{m}_{1,0,0} x_c + \text{m}_{1,0,1} x_c \sqrt{z_c}

+ \text{m}_{1,0,2} x_c z_c + \text{m}_{1,1,0} x_c y_c + \text{m}_{2,0,0} x_c^2

\text{In}[8]:= 
\text{u}_c = \text{u}_c / . \text{m}_{0,0,0} \rightarrow 0; (* \text{Set constant term to zero} *)

\text{In}[9]:= 
\text{v}_c = \text{v} \rightarrow \text{poly}\{\{x_c, y_c, z_c\}, \text{Tayldeg, n}\}

\text{Out}[9] = 
\text{v} \rightarrow n_{0,0,0} + n_{0,0,1} \sqrt{z_c} + n_{0,0,2} z_c + n_{0,0,3} z_c^{3/2} + n_{0,0,4} z_c^2 + n_{0,1,0} x_c

+ n_{0,1,1} y_c \sqrt{z_c} + n_{0,1,2} y_c^2 + n_{0,2,0} x_c + n_{1,0,0} x_c \sqrt{z_c}

+ n_{1,0,1} x_c z_c + n_{1,1,0} x_c y_c + n_{2,0,0} x_c^2

\text{In}[10]:= 
\text{v}_c = \text{v}_c / . \text{n}_{0,0,0} \rightarrow 0;

\text{In}[11]:= 
\text{\tau}_c = \text{\tau} \rightarrow \text{poly}\{\{x_c, y_c, z_c\}, \text{Tayldeg, p}\}

\text{Out}[11] = 
\text{\tau} \rightarrow p_{0,0,0} + p_{0,0,1} \sqrt{z_c} + p_{0,0,2} z_c + p_{0,0,3} z_c^{3/2} + p_{0,0,4} z_c^2 + p_{0,1,0} y_c

+ p_{0,1,1} y_c \sqrt{z_c} + p_{0,1,2} y_c^2 + p_{0,2,0} x_c + p_{1,0,0} x_c \sqrt{z_c}

+ p_{1,0,1} x_c z_c + p_{1,1,0} x_c y_c + p_{2,0,0} x_c^2

\text{In}[12] =
\( \tau_c = \tau_c / . p_{0,0,0} \rightarrow 0; \)

\begin{verbatim}
In[13]:= (* Expand x, y, and z in u, v, \tau *)

In[14]:= xTayl = c_0 \tau + a u^2 + c \tau^2 + f \tau u + g v^3 + h v^2 u + j v^2 \tau + k v^4
+ q u^2 v + r u^3 + s u v \tau + t u^2 \tau + w \tau^2 v + \alpha \tau^2 u + \beta \tau^3;

In[15]:= yTayl = \gamma u + a_1 u^2 + b_1 v^2 + c_1 \tau^2 + d_1 u v + f_1 \tau u + g_1 v^3
+ h_1 v^2 u + j_1 v^2 \tau + k_1 v^4 + q_1 u^2 v + r_1 u^3 + s_1 u v \tau + t_1 u^2 \tau
+ w_1 \tau^2 v + \alpha_1 \tau^2 u + \beta_1 \tau^3;

In[16]:= zTayl = a_2 u^2 + b_2 v^2 + c_2 \tau^2 + d_2 u v + e_2 v \tau + f_2 \tau u + g_2 v^3
+ h_2 v^2 u + j_2 v^2 \tau + k_2 v^4 + q_2 u^2 v + r_2 u^3 + s_2 u v \tau + t_2 u^2 \tau
+ w_2 \tau^2 v + \alpha_2 \tau^2 u + \beta_2 \tau^3;

In[17]:= (* Expand x in u, v, \tau *)
\end{verbatim}
In[18]:= 
    xSub = Expand[xTayl/.{\tau_c, u_c, v_c}];

In[19]:= 
    x_K = xSub/.{z_c \rightarrow K^2}; (* MultiPolyTruncate has trouble with noninteger 
    powers, so we temporarily convert powers of Sqrt[z] to powers of K *)

In[20]:= 
    x_K = PowerExpand[x_K]; (* Replaces all Sqrt[K^2] with K's *)

In[21]:= 
    x_K = MultiPolyTruncate[x_K,{K,x_c,y_c},4] ;

In[22]:= 
    x_K = MultiPolyTruncate[x_K,{x_c,y_c},2] ;

In[23]:= 
    x_z = Collect[x_K/.K \rightarrow Sqrt[z_c],{z_c,x_c,y_c}]; (* Return to Sqrt[z] *)

In[24]:= 
    (* Repeat for y *)

In[25]:= 
    ySub = Expand[yTayl/.{\tau_c, u_c, v_c}];

In[26]:= 
    y_K = ySub/.{z_c \rightarrow K^2};
\[ y_k = \text{PowerExpand}[y_k]; \]

\[ y_k = \text{MultiPolyTruncate}[y_k,\{K,x_c,y_c\},4]; \]

\[ y_k = \text{MultiPolyTruncate}[y_k,\{x_c,y_c\},2]; \]

\[ y_z = \text{Collect}[y_k/.K \to \sqrt{z_c},\{z_c,x_c,y_c\}]; \]

\[ (\ast \text{Repeat for } z \ast) \]

\[ z_{\text{Sub}} = \text{Expand}[z_{\text{Tayl}}/.\{r_c,u_c,v_c\}]; \]

\[ z_K = z_{\text{Sub}}/.\{z_c \to K^2\}; \]

\[ z_K = \text{PowerExpand}[z_K]; \]

\[ z_K = \text{MultiPolyTruncate}[z_K,\{K,x_c,y_c\},4]; \]

\[ z_K = \text{MultiPolyTruncate}[z_K,\{x_c,y_c\},2]; \]
\[ z_z = \text{Collect}[z_k/.K\rightarrow \text{Sqrt}[z_c],\{z_c,x_c,y_c\}]; \]

\[ (\ast \text{Equate Coefficients} \ast) \]

\[ (\ast \text{Order } \text{Sqrt}[z] \ast) \]

\[ x\text{Coeff} = \text{Coefficient}[x_z, z_c^{1/2}]/.\{x_c\rightarrow 0,y_c\rightarrow 0\}; \]

\[ y\text{Coeff} = \text{Coefficient}[y_z, z_c^{1/2}]/.\{x_c\rightarrow 0,y_c\rightarrow 0\}; \]

\[ z\text{Coeff} = \text{Coefficient}[z_z, z_c^{1/2}]/.\{x_c\rightarrow 0,y_c\rightarrow 0\}; \]

\[ \text{Sols} = \text{Flatten}[\text{Solve}[x\text{Coeff}=0 \&\& y\text{Coeff}=0 \&\& z\text{Coeff}=0,\{p_{0,0,1},m_{0,0,1}\}]]; \]

\[ (\ast \text{Order } x \ast) \]
xCoeff = Coefficient[x, x]/.{yc -> 0, zc -> 0};

yCoeff = Coefficient[y, x]/.{yc -> 0, zc -> 0};

zCoeff = Coefficient[z, x]/.{yc -> 0, zc -> 0};

AppendTo[Sols, Solve[xCoeff == 1 && yCoeff == 0 && zCoeff == 0, {p1, 0, 0, m1, 0, 0}]];

Sols = Flatten[Sols];

(* Order y *)

xCoeff = Coefficient[x, y]/.{xc -> 0, zc -> 0};

yCoeff = Coefficient[y, y]/.{xc -> 0, zc -> 0};

zCoeff = Coefficient[z, y]/.{xc -> 0, zc -> 0};

AppendTo[Sols, Solve[xCoeff == 0 && yCoeff == 1 && zCoeff == 0, {p0, 1, 0, m0, 1, 0}]];

247
In[55] :=
    Sols = Flatten[Sols];

(* Order z *)

In[56] :=
    xCoeff = Coefficient[x, z].{x \rightarrow 0, y \rightarrow 0};

In[57] :=
    yCoeff = Coefficient[y, z].{x \rightarrow 0, y \rightarrow 0};

In[58] :=
    zCoeff = Coefficient[z, z].{x \rightarrow 0, y \rightarrow 0};

In[59] :=
    zCoeff = zCoeff/.Sols;

In[60] :=
    AppendTo[Sols,

        Solve[xCoeff==0 \&\& yCoeff==0 \&\& zCoeff==1, {p_{0,0,2}, n_{0,0,1}, m_{0,0,2}}][[1]]];

In[62] :=
    Sols = Flatten[Sols];

(* Order x Sqrt[z] *)
xCoeff = Coefficient[x[z], x c Sqrt[z c]]/.{y c \rightarrow 0};

xCoeff = xCoeff/.Sols;

yCoeff = Coefficient[y[z], x c Sqrt[z c]]/.{y c \rightarrow 0};

yCoeff = yCoeff/.Sols;

zCoeff = Coefficient[z[z], x c Sqrt[z c]]/.{y c \rightarrow 0};

zCoeff = zCoeff/.Sols;

AppendTo[Sols,

    Solve[xCoeff==0 && yCoeff==0 && zCoeff==0,\{p_{1,0,1},n_{1,0,0},m_{1,0,1}\}[[1]]];

Sols = Flatten[Sols];

(* Order y Sqrt[z] *)
In[73]:= 
xCoeff = Coefficient[x, y Sqrt[z] /. {x -> 0};

In[74]:= 
xCoeff = xCoeff/.Sols;

In[75]:= 
yCoeff = Coefficient[y, y Sqrt[z] /. {x -> 0};

In[76]:= 
yCoeff = yCoeff/.Sols;

In[77]:= 
zCoeff = Coefficient[z, y Sqrt[z] /. {x -> 0};

In[78]:= 
zCoeff = zCoeff/.Sols;

In[79]:= 
AppendTo[Sols, Solve[xCoeff==0 && yCoeff==0 && zCoeff==0, {p0,1,1,n0,1,0,m0,1,1}][[1]]];

In[80]:= 
Sols = Flatten[Sols];

In[81]:= 
(* Order z^{3/2} *)
In[82]:= xCoeff = Coefficient[x, \(z^{3/2}\)]/.\{x \rightarrow 0, y \rightarrow 0\};

In[83]:= xCoeff = xCoeff/.Sols;

In[84]:= yCoeff = Coefficient[y, \(z^{3/2}\)]/.\{x \rightarrow 0, y \rightarrow 0\};

In[85]:= yCoeff = yCoeff/.Sols;

In[86]:= zCoeff = Coefficient[z, \(z^{3/2}\)]/.\{x \rightarrow 0, y \rightarrow 0\};

In[87]:= zCoeff = zCoeff/.Sols;

In[88]:= AppendTo[Sols,

Solve[xCoeff==0 && yCoeff==0 && zCoeff==0, \{\(p_{0,0,3}, n_{0,0,2}, m_{0,0,3}\}\}][[1]]];

In[89]:= Sols = Flatten[Sols];

In[90]:= (* Order x^2 *)
In[91]:=  
xCoeff = Coefficient[x, \(x^2\)]/.\(\{y \rightarrow 0, z \rightarrow 0\}\);

In[92]:=  
xCoeff = xCoeff/.Sols;

In[93]:=  
yCoeff = Coefficient[y, \(x^2\)]/.\(\{y \rightarrow 0, z \rightarrow 0\}\);

In[94]:=  
yCoeff = yCoeff/.Sols;

In[95]:=  
zCoeff = Coefficient[z, \(x^2\)]/.\(\{y \rightarrow 0, z \rightarrow 0\}\);

In[96]:=  
zCoeff = zCoeff/.Sols;

In[97]:=  
AppendTo[Sols,Solve[xCoeff==0 && yCoeff==0,\(\{p_{2,0,0},m_{2,0,0}\}\)\[[1]]]];

In[98]:=  
Sols = Flatten[Sols];

In[99]:=  
(* Order \(y^2\) *)
In[100]:= 
xCoeff = Coefficient[x, y^2] /. {x \rightarrow 0, z \rightarrow 0};

In[101]:= 
xCoeff = xCoeff/.Sols;

In[102]:= 
yCoeff = Coefficient[y, y^2] /. {x \rightarrow 0, z \rightarrow 0};

In[103]:= 
yCoeff = yCoeff/.Sols;

In[104]:= 
zCoeff = Coefficient[z, y^2] /. {x \rightarrow 0, z \rightarrow 0};

In[105]:= 
zCoeff = zCoeff/.Sols;

In[106]:= 
AppendTo[Sols,Solve[xCoeff==0 && yCoeff==0,{p0,2,0,m0,2,0}][[1]]];

In[107]:= 
Sols = Flatten[Sols];

In[108]:= 
(* Order z^2 *)

In[109]:= 
xCoeff = Coefficient[x, z^2] /. {x \rightarrow 0, y \rightarrow 0};
xCoeff = xCoeff/.Sols;

yCoeff = Coefficient[yz, \(z_c^2\)]/.\{x_c \to 0, y_c \to 0\};

yCoeff = yCoeff/.Sols;

zCoeff = Coefficient[zz, \(z_c^2\)]/.\{x_c \to 0, y_c \to 0\};

zCoeff = zCoeff/.Sols;

AppendTo[Sols, 
  Solve[xCoeff==0 && yCoeff==0 && zCoeff==0, \{p_{0,0,4}, n_{0,0,3}, m_{0,0,4}\}][[1]]];

Sols = Flatten[Sols];

(* Order xy *)

xCoeff = Coefficient[xz, x_c y_c]/.\{z_c \to 0\};
In[19]:= 
   xCoeff = xCoeff/.Sols;

In[20]:= 
   yCoeff = Coefficient[y, x y]/.{z→ 0};

In[21]:= 
   yCoeff = yCoeff/.Sols;

In[22]:= 
   zCoeff = Coefficient[z, x y]/.{z→ 0};

In[23]:= 
   zCoeff = zCoeff/.Sols;

In[24]:= 
   AppendTo[Sols,Solve[xCoeff===0 && yCoeff===0, {p1,1,0,m1,1,0}][[1]]];

In[25]:= 
   Sols = Flatten[Sols];

In[26]:= 
   (* Order xz *)

In[27]:= 
   xCoeff = Coefficient[x, x z]/.{y→ 0};

In[28]:= 
   xCoeff = xCoeff/.Sols;
\textbf{In[129]}:=
\[ y\text{Coeff} = \text{Coefficient}[y_x, x_c z_c] /. \{y_c \rightarrow 0\}; \]
\textbf{In[130]}:=
\[ y\text{Coeff} = y\text{Coeff} /. \text{Sols}; \]
\textbf{In[131]}:=
\[ z\text{Coeff} = \text{Coefficient}[z_x, x_c z_c] /. \{y_c \rightarrow 0\}; \]
\textbf{In[132]}:=
\[ z\text{Coeff} = z\text{Coeff} /. \text{Sols}; \]
\textbf{In[133]}:=
\[ \text{AppendTo[} \text{Sols, Solve[xCoeff==0 \&\& yCoeff==0 \&\& zCoeff==0, \{p_{1,0,2}, m_{1,0,2}, n_{1,0,1}\}] \text{[[1]]]}\text{];} \]
\textbf{In[134]}:=
\[ \text{Sols = Flatten[Sols];} \]
\textbf{In[135]}:=
\[ (* \text{Order } yz *) \]
\textbf{In[136]}:=
\[ x\text{Coeff} = \text{Coefficient}[x_z, y_c z_c] /. \{x_c \rightarrow 0\}; \]
\textbf{In[137]}:=
\[ x\text{Coeff} = x\text{Coeff} /. \text{Sols}; \]
In[138]:= yCoeff = Coefficient[y, y c z c] /._{x c \rightarrow 0};

In[139]:= yCoeff = yCoeff /._{Sols};

In[140]:= zCoeff = Coefficient[z, y c z c] /._{x c \rightarrow 0};

In[141]:= zCoeff = zCoeff /._{Sols};

In[142]:= AppendTo[Sols, Solve[xCoeff == 0 && yCoeff == 0 && zCoeff == 0, \{p_{0,1,2}, m_{0,1,2}, n_{0,1,1}\}[[1]]];

In[143]:= Sols = Flatten[Sols];

In[144]:= (* Evaluate the ray coordinates with the newly determined coefficients *)

In[145]:= \tau /._{Sols}

Out[145]=
\[\tau \rightarrow \frac{x_c}{c_0} + \frac{g z_{c2}^{2/3}}{b_2^{3/2} c_0} - \frac{f x_c y_c}{\gamma c_0^2} - \frac{(2 j b_2 - 2 f b_{12} b_2 - 3 g \gamma e_2 x_c z_c)}{2 \gamma b_2^2 c_0^2}\]
\[-\frac{(2 \, h \, \gamma \, b_2 - 4 \, a \, b_1 \, b_2 - 3 \, g \, \gamma \, d_2) \, y_c \, z_c}{2 \, \gamma^2 \, b_2^2 \, c_0} - \frac{c \, x_c^2}{c_0^3} - \frac{a \, y_c^2}{\gamma^2 \, c_0}\]

\[-\frac{(2 \, k \, \gamma^2 \, b_2 - 2 \, h \, \gamma \, b_1 \, b_2 + 2 \, a \, b_1^2 \, b_2 + 3 \, g \, \gamma \, b_1 \, d_2 - 3 \, g \, \gamma^2 \, g_2) \, z_c^2}{2 \, \gamma^2 \, b_2^3 \, c_0}\]

\[\text{In[146]} := u_c ./ \text{Sols}\]

\[\text{Out[146]} =\]

\[u \rightarrow \frac{y_c}{\gamma} - \frac{b_1 \, z_c}{\gamma \, b_2} - \frac{b_1 \, e_2 \, x_c \, \sqrt{z_c}}{\gamma \, b_2^{3/2} \, c_0} - \frac{(-b_2 \, d_1 + b_1 \, d_2) \, y_c \, \sqrt{z_c}}{\gamma^2 \, b_2^{3/2}}\]

\[-\frac{(b_1 \, b_2 \, d_1 - b_1^2 \, d_2 - \gamma \, b_2 \, g_1 + \gamma \, b_1 \, g_2) \, z_c^{3/2}}{\gamma^2 \, b_2^{5/2} \, c_0} + \frac{(b_2 \, d_1 \, e_2 - b_1 \, d_2 \, e_2 - 2 \, b_2^2 \, f_1 \, x_c \, y_c}{2 \, \gamma^2 \, b_2^3 \, c_0}\]

\[-\frac{1}{2 \, \gamma^2 \, b_2^2 \, c_0} \left(3 \, b_1 \, b_2 \, d_1 \, e_2 - 4 \, b_1^2 \, d_2 \, e_2 - 2 \, b_1 \, b_2^2 \, f_1 + 2 \, b_1^2 \, b_2 \, f_2 - 3 \, \gamma \, b_2 \, e_2 \, g_1 - 2 \, \gamma \, b_1 \, e_2 \, g_2 + 2 \, \gamma \, b_2^2 \, j_1 - 2 \, \gamma \, b_1 \, b_2 \, j_2 \right) x_c \, z_c\]

\[-\frac{1}{2 \, \gamma^3 \, b_2^3} \left(4 \, a_2 \, b_1^2 \, b_2 - 4 \, a_1 \, b_1 \, b_2^2 - 2 \, b_2^2 \, d_1^2 + 6 \, b_1 \, b_2 \, d_1 \, d_2 - 4 \, b_1^2 \, d_2^2 - 3 \, \gamma \, b_2 \, d_2 \, g_1\right)\]

\[-\gamma \, b_2 \, d_1 \, g_2 + 4 \, \gamma \, b_1 \, d_1 \, g_2 + 2 \, \gamma \, b_2^2 \, h_1 - 2 \, \gamma \, b_1 \, b_2 \, h_2 \right) y_c \, z_c\]

\[-\frac{4 \, b_2^2 \, c_1 + b_1 \, e_2^2}{4 \, \gamma \, b_2^2 \, c_0^2} x_c^2 + \frac{(-4 \, a_1 \, b_2^2 + 2 \, b_2 \, d_1 \, d_2 - b_1 \, d_2^2) \, y_c^2}{4 \, \gamma^3 \, b_2^2}\]

\[-\frac{1}{2 \, \gamma^3 \, b_2^4 \, c_0} \left(-2 \, a_2 \, b_1^3 \, b_2 \, c_0 + 2 \, a_1 \, b_1^2 \, b_2 \, c_0 + 2 \, b_1 \, b_2^2 \, c_0 \, d_1^2 - 5 \, b_1^2 \, b_2 \, c_0 \, d_1 \, d_2\right)\]

\[+ 3 \, b_1^3 \, c_0 \, d_2^2 + 2 \, g \, \gamma^2 \, b_1 \, b_2 \, e_2 - 2 \, \gamma \, b_2^2 \, c_0 \, d_1 \, g_1 + 5 \, \gamma \, b_1 \, b_2 \, c_0 \, d_2 \, g_1\]

\[+ 3 \, \gamma \, b_1 \, b_2 \, c_0 \, d_1 \, g_2 - 6 \, \gamma \, b_1^2 \, c_0 \, d_2 \, g_2 - 3 \, \gamma^2 \, b_2 \, c_0 \, g_1 \, g_2 + 3 \, \gamma^2 \, b_1 \, c_0 \, g_2^2\]

\[-2 \, \gamma \, b_1 \, b_2^2 \, c_0 \, h_1 + 2 \, \gamma \, b_1^2 \, b_2 \, c_0 \, h_2 + 2 \, \gamma^2 \, b_2^2 \, c_0 \, k_1 - 2 \, \gamma^2 \, b_1 \, b_2 \, c_0 \, k_2 \right) z_c^2\]

\[\text{In[147]} := v_c ./ \text{Sols}\]

258
\( v \to - \frac{\sqrt{z_c}}{b_2} - \frac{e_2 \; x_c}{2 \; b_2 \; c_0} - \frac{d_2 \; y_c}{2 \; b_2} - \frac{(- b_1 \; d_2 + \gamma \; g_2 \; z_c)}{2 \; \gamma \; b_2^2} \\
- \frac{(-3 \; b_1 \; d_2 \; e_2 + 2 \; b_1 \; b_2 \; g_2 + 3 \; \gamma \; e_2 \; g_2 - 2 \; \gamma \; b_2 \; j_2) \; x_c \sqrt{z_c}}{4 \; \gamma \; b_2^{5/2} \; c_0} \\
- \frac{(4 \; a_2 \; b_1 \; b_2 + 2 \; b_2 \; d_1 \; d_2 - 3 \; b_1 \; d_2^2 + 3 \; \gamma \; d_2 \; g_2 - 2 \; \gamma \; b_2 \; h_2) \; y_c \sqrt{z_c}}{4 \; \gamma^2 \; b_2^{5/2}} \\
- \frac{1}{8 \; \gamma^2 \; b_2^{7/2} \; c_0} (-4 \; a_2 \; b_1 \; b_2 \; c_0 - 4 \; b_1 \; b_2 \; c_0 \; d_1 \; d_2 + 5 \; b_1^2 \; c_0 \; d_2^2 + 4 \; \gamma^2 \; b_2 \; e_2 \\
+ 4 \; \gamma \; b_2 \; c_0 \; d_2 \; g_1 - 10 \; \gamma \; b_1 \; c_0 \; d_2 \; g_2 + 5 \; \gamma^2 \; c_0 \; g_2^2 + 4 \; \gamma \; b_1 \; b_2 \; c_0 \; h_2 \\
- 4 \; \gamma^2 \; b_2 \; c_0 \; k_2) \; z_c^{3/2} \\
+ n_{1,1,0} \; x_c \; y_c + n_{1,0,2} \; x_c \; z_c + n_{0,1,2} \; y_c \; z_c + n_{2,0,0} \; x_c^2 + n_{0,2,0} \; y_c^2 + n_{0,0,4} \; z_c^2) \)
Appendix D

Translation of Yu. L. Gazaryan, “О геометро-акустическом приближении поля в окрестности неособого участка каустики”

In this appendix we provide a translation of reference [200], the article “О геометро-акустическом приближении поля в окрестности неособого участка каустики” (On the geometrical-acoustic approximation of the field in the vicinity of a nonsingular caustic) by Yuri L. Gazaryan, published in Russian in 1961 by the Leningrad State University Press (now the Publishing House of St. Petersburg University) in volume 5 of the journal Voprozy dinamicheskoĭ teorii rasprostraneniiia seĭsmicheskikh (Problems in the dynamic theory of seismic wave propagation). The first page of the original document is also provided for reference.

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Translation was assisted by use of the machine-learning translation tools Google Translate and DeepL, while equation typesetting, document formatting, and figure editing were done by hand. We take full responsibility for any technical or typographical errors that may have been introduced during this process. We have attempted to emulate the formatting of the original document, but differences in word/character length from one language to another have made it difficult to maintain consistent page numbering. The upper right page numbers are (approximately) those of the original work, while the bottom center numbers are the page number within the present document.

The translation may be cited as:

Ю. Л. Газарян

О ГЕОМЕТРО-АКУСТИЧЕСКОМ ПРИБЛИЖЕНИИ ПОЛЯ В ОКРЕСТНОСТИ НЕОСОБОГО УЧАСТКА КАУСТИКИ

1. Для вычисления звукового поля вблизи каустики обычно берется приближенное интегральное представление поля, построенное с помощью того или иного геометро-акустического приближения, и затем вычисляется асимптотическое по частоте значение интegraла (см., например, [1], [2]). Между тем здесь также возможно последовательное проведение метода геометрической акустики, что и является целью настоящей работы.*

Как известно, метод геометрической акустики заключается в том, что лучам исходящих из излучателя геометрических лучей ставится в соответствие "распространяющаяся вдоль этих лучей" волна. Фаза волны определяется "акустической длиной пути"; квадрат амплитуды изменяется вдоль луча пропорционально волновому сопротивлению среды и обратно пропорционально площади поперечного сечения бесконечно тонкой лучевой трубы. Если через некоторую точку проходит несколько лучей, то поле в точке представляется соответственно в виде суммы нескольких волн. Однако в этом случае обязательно имеются точки пересечения бесконечно близких лучей, образующих каустику, вблизи которой, как известно, обычное геометро-акустическое приближение неприменимо. Наличие каустики приводит также к тому, что геометро-акустическое приближение, по крайней мере в некоторой части области его применимости, не удается найти обычным методом. Для пояснения рассмотрим плоский случай, изображенный на рис. 1. Здесь через каждую точку, лежащую над каустикой (K) не проходит ни одного луча (зона тени); через каждую точку,

ON THE GEOMETRICAL-ACOUSTIC APPROXIMATION OF THE FIELD IN THE VICINITY OF A NONSINGULAR CAUSTIC

1. In order to calculate the sound field near caustics, an approximate integral representation of the field is usually constructed using one or another geometrical-acoustic approximation and the frequency-asymptotic value of the integral is calculated (cf., for example, [1], [2]). Meanwhile, it is also possible to consistently carry out the method of geometrical acoustics here, which is the purpose of this work.*

As is known, the method of geometrical acoustics lies in the fact that the bundle of geometric rays emanating from the source is associated with a wave “propagating along these rays.” The phase of the wave is determined by the “acoustic path length”; the square of the amplitude varies along the ray in proportion to the wave impedance of the medium and inversely proportional to the cross-sectional area of an infinitely thin ray tube. If several rays pass through a certain point, then the field at the point is represented, respectively, as the sum of several waves. In this case, there are necessarily intersection points of infinitely close rays that form a caustic, near which, as is known, the usual geometrical-acoustic approximation is inapplicable. The presence of caustics also leads to the fact that the geometrical-acoustic approximation, at least in some part of the region of its applicability, cannot be found by the usual method. For clarification, consider the planar case, shown in Fig. 1. Here, no ray passes through each point above the caustic (K) (the shadow zone); through each point

*A method, very similar to the one presented below, was applied to calculate the field in the vicinity of the caustic for the case of the plane propagation problem in a homogeneous medium in the work: R. N. Buchal, J. B. Keller. Boundary layer problems in diffraction theory. Commun. op pure and appl. math. 13, 85-114 (1960).
lying under (K) two rays pass: coming to the caustic from the source and leaving from the caustic (the illuminated region). Let the segment $AA'$ of the wavefront of the incoming portions of the rays (Fig. 2) and the segment $BB'$ of the wavefront of the outgoing portions (Fig. 3) be so far from the caustic that before $AA'$ and after $BB'$ the geometrical-acoustic approximation is applicable.

![Fig. 1.](image1)

Then this approximation gives a wave propagating along the incoming portions of the rays from the source to $AA'$. For the outgoing parts of the rays lying beyond $BB'$, the geometrical-acoustic approximation can be expressed through its value on $BB'$; however, this value itself remains unknown. Thus, the method of geometrical acoustics needs to be modified, allowing, firstly, to find the field near the caustic and, secondly, to find a connection between the field of a “wave propagating along the incoming sections” and the field of a “wave propagating along the outgoing sections of the rays.”

The problem formulated in this form has actually been solved in the available literature for a system of rays corresponding to the generalization of a plane wave to the case of layered inhomogeneous media (the caustic here will be some plane parallel to the layers of the medium). The WKB-method shows that the amplitude in the outgoing sections of the ray will be the same as if it were obtained according to the rule given above (ignoring the fact that the intersection of infinitely close rays occurs at intermediate points of the ray tube), but in the phase shift\(^{†}\)

\(^{†}\) Gazaryan instead uses the phrase “phase incursion,” which seems to be common in the Russian literature.
we observe a loss of a quarter wavelength compared to geometrical acoustics.* The disadvantage of the WKB method is that it does not produce a field near caustics. In this respect, a modification of the WKB method given by Langer [3], which allows one to obtain a field on caustics for the same particular case, is of extreme interest. In [3], the separation of the solution into the field of the “incoming” and the field of the “outgoing” wave from the caustic is not used, but the expression obtained there can be easily split into two such components. It should be noted that this separation near the caustic loses its physical meaning, which is especially clearly seen when considering the change in the phase of a wave propagating along the ray. It turns out that when approaching the caustic, we observe an additional increase in phase shift compared to the geometrical-acoustic shift, reaching $\pi/12$ at the caustic; during the transition to the outgoing part of the ray, the phase abruptly changes by $-2\pi/3$ (the amplitude is continuous), and when moving away from the caustic again, we again observe an increase in phase by $\pi/12$. Thus, consideration of a wave “propagating” along the ray has no physical meaning here, because the phase of this wave turns out to be discontinuous on the caustic. However, as will be seen later, the consideration of such a wave turns out to be an extremely convenient mathematical method.

Below we will find formulas that allow us, for a more general case than in [3], to extend the geometrical-acoustic field of a wave “propagating along a given family of rays” up to caustics, and then, having such generalizations of the geometrical-acoustic field for the system of incoming and outgoing sections of rays, we get a connection between these two fields. In this case, we restrict ourselves to the case when the considered bundle of rays touches a nonsingular section of the caustic.

* The same relationship for the amplitude and phase was obtained in the case of a point source in a layered inhomogeneous medium.
and also exclude from consideration the rays whose order of tangency with the caustic is higher than the first.

2. Let the medium be characterized by the density \( \rho \) and the sound propagation speed \( c \), which we will consider as sufficiently smooth functions of the points in space \((x, y, z)\), and let a section of the initial wave surface be sufficiently smooth. Let us choose on this section some curvilinear coordinates \( u, v \) and pass through each point of the section an orthogonally outgoing ray (i.e., the extremal of the corresponding variational problem). Let us introduce the index of refraction \( \mu(x, y, z) = a/c \), where \( a \) — some constant, having the dimension of velocity, and the eikonal along the ray \( \sigma = \int \mu ds \), where \( s \) — the arc length along the ray; the constant value of the eikonal on the initial wave surface is taken to be equal to \( \sigma_1 \). The ray equation can then be written in the form

\[
\begin{align*}
x &= x(u, v, \sigma), \\
y &= y(u, v, \sigma), \\
z &= z(u, v, \sigma),
\end{align*}
\]

or in vector form \( \Gamma = \Gamma(u, v, \sigma) \). Under our conditions, \( \Gamma \) is a single-valued and sufficiently differentiable function of all arguments.\(^*\) Surfaces defined by the condition \( \sigma = \text{const.} \) are wave fronts; due to the isotropy of the medium, they are orthogonal to the rays. We introduce the Jacobian \( D = \frac{\partial(x, y, z)}{\partial(u, v, \sigma)} \); it can be written as a mixed product: \( D = \Gamma_u \Gamma_v \Gamma_\sigma \). Clearly, for \( \sigma \) sufficiently close to \( \sigma_1 \), \( D \) will be nonzero; we will assume for definiteness that in this case \( D > 0 \). Consider some ray \((u, v)\) and follow the behavior of the ray bundle close to it. As long as \( D > 0 \) is satisfied on our ray, this condition is also satisfied at points close to it, i.e., (1) is uniquely solvable with respect to \( u, v, \sigma \) and, therefore, if we restrict ourselves to considering the initial sections of the rays, a single ray passes through each point. Now let \( D \) vanish at some point of the ray under consideration (the characteristic point of the ray). We will assume that at the characteristic point \( D_\sigma \neq 0 \) (i.e. \( D_\sigma < 0 \), since \( D \) decreases), in this case, the equation \( D(u, v, \sigma) = 0 \) defines a single-valued function \( \sigma = \sigma(u, v) \) i.e. \( D \) vanishes not only on the ray under consideration, but also on all adjacent ones, and the characteristic points of these rays form a surface — the caustic. Since \(|\Gamma_\sigma| = 1/\mu \neq 0 \) and \( \Gamma_\sigma \) is orthogonal to \( \Gamma_u \) and \( \Gamma_v \), the condition \( D = 0 \) implies collinearity of \( \Gamma_u \) and \( \Gamma_v \). These two vectors cannot vanish simultaneously (because then \( D_\sigma = 0 \)); we will assume for definiteness that at the characteristic point \( \Gamma_u \) is nonzero.

\(^*\)The speed of sound is assumed to be everywhere nonzero.
Restricting, if necessary, to a section of the initial wave surface, we can assume that for all rays of the bundle, at the characteristic points the conditions \( D_\sigma < 0, \Gamma_u \neq 0 \) are satisfied. It is easy to see that the choice of \( v \)-lines of the curvilinear coordinate system on the initial wave surface can be realized so that at the characteristic points \( \Gamma_v \) would not only be collinear to \( \Gamma_u \), but would also vanish. Indeed, if we move on the surface \( \sigma_1 \) from the point \((u, v)\) in the direction \((\delta u, \delta v)\), then the point of intersection of the ray with the surface \( \sigma \) experiences a displacement \( \delta \Gamma = \Gamma_u \delta u + \Gamma_v \delta v \).

Considering that at characteristic points \( \Gamma_u \parallel \Gamma_v \), it is easy to see that the family of curves on the surface \( \sigma_1 \), defined by the differential equation \( \frac{du}{dv} = -\frac{\Gamma_v(u, v, \sigma(u, v))}{\Gamma_u(u, v, \sigma(u, v))} \), has the property that, when displaced along a curve of the family, the condition \( \delta \Gamma = 0 \) is satisfied at the characteristic point of the corresponding ray. We will take the curves of this family as new \( v \)-lines, then the condition \( \Gamma_v = 0 \) is satisfied at the characteristic points of the bundle. Hence it follows that in the case under consideration, the infinitely close rays lying in each coordinate surface \( u = \text{const} \) intersect each other on the caustic, while the surfaces \( u = \text{const} \) themselves do not intersect in the vicinity of the caustic. The caustic equation will be \( \tilde{\Gamma} = \Gamma(u, v, \sigma(u, v)) \), that is, \( u, v \) form a curvilinear coordinate system on the caustic. We have \( \tilde{\Gamma}_u = \Gamma_u + \sigma_u \Gamma_\sigma, \quad \tilde{\Gamma}_v = \sigma_v \Gamma_\sigma \). From this it can be seen that the rays touch \( v \)-lines on the caustic. Note that if \( \sigma_v \neq 0 \), then \( \left[ \tilde{\Gamma}_u \tilde{\Gamma}_v \right] = 0 \), i.e. the considered caustic point is nonsingular. We will consider this condition to be fulfilled for all rays of the bundle.

Let us say a few words about the above condition \( D_\sigma \neq 0 \) on the caustic. As we have seen, it excludes, first of all, the simultaneous vanishing of \( \Gamma_u, \Gamma_v \), which corresponds to a ray intersecting at its characteristic point with all rays infinitely close to it (the “principal ray” in optical terminology). As shown in the Appendix, if we restrict ourselves to the assumption that curves \( u, v \) on the caustic form a coordinate grid and \( \sigma_v \neq 0 \), then the condition \( D_\sigma = 0 \) means that the ray has a tangency with the caustic of order higher than the first.

3. Let us now proceed to consider a wave propagating along the incoming portions of the rays, assuming its field to be given on the initial wave surface. Moreover, in view of the fact that on the arriving sections of rays \( D > 0 \), equations (1) are uniquely solvable, i.e., any function of \((u, v, \sigma)\) can be considered as a single-valued function of \((x, y, z)\). It is easy
to obtain from (1) that for any function \( F(u,v,\sigma) \) (scalar or vector) we have

\[
\frac{\partial F}{\partial x} = \frac{1}{D} \begin{vmatrix}
F_u & F_v & F_\sigma \\
y_u & y_v & y_\sigma \\
z_u & z_v & z_\sigma
\end{vmatrix}
\] (2)

and similar expressions for derivatives with respect to \( y \) and \( z \).

This also implies

\[
\nabla F = \frac{1}{D} \{ F_u [\Gamma_v \Gamma_\sigma] + F_v [\Gamma_\sigma \Gamma_u] + F_\sigma [\Gamma_u \Gamma_v] \}. \tag{3}
\]

In order to calculate the required field, we pre-transform the wave equation to the form from which the geometric-acoustic approximation is usually obtained. Assuming that the time dependence is given by a factor of \( \exp(-i\omega t) \), which we will further discard, we have the following equation for the sound pressure \( p \):

\[
\Delta p - \nabla p \nabla \ln \rho + \omega^2 c^2 p = 0.
\]

The solution to this equation is sought in the form \( p = P(u,v,\sigma) \cdot \exp(i\omega \sigma a) \), considering that \( \nabla \sigma \parallel \Gamma_\sigma \) and that \( (\nabla \sigma)^2 = \mu^2 \), we get, substituting,

\[
\frac{i\omega}{a} \mu^2 \left[ 2 \frac{\partial P}{\partial \sigma} + P \left( \frac{\Delta \sigma}{\mu^2} - \frac{\partial \ln \rho}{\partial \sigma} \right) \right] + \Delta P - \nabla P \nabla \ln \rho = 0. \tag{4}
\]

Notice, that \( \Delta \sigma = \nabla (\nabla \sigma) = \nabla (\mu^2 \Gamma_\sigma) = \frac{\partial \mu^2 \Gamma_\sigma}{\partial \sigma} + \mu^2 \nabla \Gamma_\sigma \). Using (3), we get

\[
\nabla \Gamma_\sigma = \frac{\{ \Gamma_u \Gamma_v \Gamma_\sigma \} \sigma}{D} = \frac{D_\sigma}{D}, \quad \text{i.e.}
\]

\[
\frac{\Delta \sigma}{\mu^2} - \frac{\partial \ln \rho}{\partial \sigma} = \frac{\partial}{\partial \sigma} \ln \frac{D_\mu^2}{\rho}. \tag{5}
\]

We transform this expression by introducing the ratio \( \kappa \) of the cross-sectional area cut out in the current wavefront \( \sigma \) by an infinitely thin ray tube to the area of the corresponding cross-section of the original wavefront \( \sigma_1 \). We have \( \kappa = \sqrt{J/J_1} \geq 0 \), where

\[
J = \begin{vmatrix}
\Gamma^2_u & \Gamma_u \Gamma_v \\
\Gamma_u \Gamma_v & \Gamma^2_v
\end{vmatrix}.
\]

the subscript 1 indicates that the value is taken at \( \sigma = \sigma_1 \). It is easy to see

\[
D^2 = \begin{vmatrix}
\Gamma^2_u & \Gamma_u \Gamma_v & 0 \\
\Gamma_u \Gamma_v & \Gamma^2_v & 0 \\
0 & 0 & \Gamma^2_\sigma
\end{vmatrix} = \frac{1}{\mu^2 J}.
\]

Taking into account that \( D > 0 \), we obtain \( \kappa = D_\mu/D_1 \mu_1 \). We introduce another value \( \eta \) by the formula
$$\eta = \kappa \frac{\rho_1 c_1}{\rho c} = \frac{D}{\rho c^2} \frac{\rho_1 c_1^2}{D_1} = \frac{D \mu^2}{\rho} \frac{\rho_1}{D_1 \mu_1^2}. \quad (6)$$

Taking into account that $\rho_1, D_1, \mu_1$ do not depend on $\sigma$, we get that the right-hand side of (5) is reduced to $\frac{\partial \ln \eta}{\partial \sigma}$. So, (4) can be written in the following form:

$$2 \frac{\partial P}{\partial \sigma} + P \frac{\partial \ln \eta}{\partial \sigma} + \frac{a}{i \omega \mu^2} \{\Delta P - \nabla P \nabla \ln \rho\} = 0. \quad (7)$$

Letting $\omega$ here tend to infinity and integrating the resulting equation, we find the usual geometric-acoustic approximation for $P$:

$$P(u, v, \sigma) = \frac{P(u, v, \sigma_1)}{\sqrt{\eta}}. \quad (8)$$

To find out the limits of applicability of this expression, it should be substituted in (7) and then require the smallness of the last term in comparison with one of the first two. Moreover, if everywhere $\eta \geq \varepsilon > 0$, where $\varepsilon$ is fixed, then the curly brace is finite and $\omega$ can be chosen so large that the required condition is satisfied.

However, it follows from (6) that, when approaching the caustic, $\eta$ together with $D$ tends to zero, and neglecting the curly brace in (7) becomes inadmissible. To accommodate the curly brace, we will proceed as follows. We will seek a solution to equation (7) in the form

$$P = A(u, v) \cdot B(\eta). \quad (9)$$

If we consider the result of substitution of this expression in (7) on a fixed ray, we obtain for the unknown function $B(\eta)$ a linear differential equation with coefficients depending on $\eta$ and $\sigma$. It is not difficult to see from (6), under our conditions $\sigma$ is, on a given ray in sufficient proximity to the caustic, a single-valued function of $\eta$, therefore the same is true for the coefficients of the differential equation. Taking into account that on the caustic $\eta = 0$, we can get a fairly good approximation for the terms taken from the curly brace if we replace their coefficients with the principal terms of the expansions in powers of $\eta$. The resulting equation will be more accurate the smaller the maximum value of $\eta$ in the ray section, where it is impossible to neglect the curly brace. Since this section can be made smaller the higher the frequency, and as the section decreases, the maximum value of $\eta$ on it tends to zero, then by choosing a sufficiently high frequency, the approximate equation can be made arbitrarily close to the exact one along the entire length of the incoming ray section from the initial wavefront to the caustic.
Therefore, we can expect that with increasing frequency, the solution to the approximate equation will uniformly tend to a solution of the exact equation over the entire incoming portion of the ray.

Proceeding to the execution of the stated program, let us preliminarily write (7) in the following form:

\[
2B' + \frac{1}{\eta}B + \frac{a}{i\omega\mu^2} \frac{\Delta P - \nabla P\nabla \ln \rho}{A \frac{\partial \eta}{\partial \sigma}} = 0. \tag{10}
\]

Using (2) and taking into account (9), (6), we obtain

\[
\frac{\partial P}{\partial x} = AB' \frac{\partial \eta}{\partial x} + \frac{B}{D} \begin{vmatrix} A_u & A_v & 0 \\ y_u & y_v & y_\sigma \\ z_u & z_v & z_\sigma \end{vmatrix},
\]

\[
\frac{\partial^2 P}{\partial x^2} = AB'' \left( \frac{\partial \eta}{\partial x} \right)^2 + B' \left( A \frac{\partial^2 \eta}{\partial x^2} + 2 \frac{\partial \eta}{\partial x} \frac{\partial A}{\partial x} \right) + B \frac{\partial}{\partial x} \begin{vmatrix} 1 \\ A_u & A_v & 0 \\ y_u & y_v & y_\sigma \\ z_u & z_v & z_\sigma \end{vmatrix},
\]

\[
\frac{1}{\eta} \frac{\partial \eta}{\partial x} = - \frac{\partial}{\partial x} \ln \rho c^2 + \frac{1}{D \rho_1 c_1^2} \begin{vmatrix} (\rho_1 c_1^2)_u & (\rho_1 c_1^2)_v & 0 \\ y_u & y_v & y_\sigma \\ z_u & z_v & z_\sigma \end{vmatrix},
\]

\[
- \frac{1}{DD_1} \begin{vmatrix} D_{1u} & D_{1v} & 0 \\ y_u & y_v & y_\sigma \\ z_u & z_v & z_\sigma \end{vmatrix} + \frac{1}{D^2} \begin{vmatrix} D_u & D_v & D_\sigma \\ y_u & y_v & y_\sigma \\ z_u & z_v & z_\sigma \end{vmatrix},
\]

\[
\frac{\partial^2 \eta}{\partial x^2} = - \frac{\eta}{D^4} \begin{vmatrix} D_u & D_v & D_\sigma \\ y_u & y_v & y_\sigma \\ z_u & z_v & z_\sigma \end{vmatrix} + \cdots,
\]

where the dots denote terms of a higher order of smallness.

Using these relations we get:

\[
\Delta P - \nabla P\nabla \ln \rho = AB'' \left( \eta \frac{M}{D^4} + \cdots \right) - AB' \left( \eta \frac{M}{D^4} + \cdots \right) - B \left( \frac{N}{D^3} + \cdots \right),
\]

where

\[
M = \begin{vmatrix} D_u & D_v & D_\sigma \\ y_u & y_v & y_\sigma \\ z_u & z_v & z_\sigma \end{vmatrix}^2 + \begin{vmatrix} x_u & x_v & x_\sigma \\ D_u & D_v & D_\sigma \\ z_u & z_v & z_\sigma \end{vmatrix}^2 + \begin{vmatrix} x_u & x_v & x_\sigma \\ D_u & D_v & D_\sigma \\ z_u & z_v & z_\sigma \end{vmatrix}^2,
\]
The values $M, N$ must be calculated at the point of contact between the ray and the caustic. Since at this point $x_v = y_v = z_v = 0$, we get

$$M = D_v^2 \Gamma_u \Gamma_\sigma = D_v^2 \Gamma_u^2 \Gamma_\sigma^2, \quad N = D_v A_u \Gamma_u^2 \Gamma_\sigma^2$$

(note that on the caustic $D_v = -D_\sigma \sigma_v \neq 0$). Further, we have from (6)

$$D = \eta D_1 \rho \mu \rho_1 \mu^2 + \cdots, \quad \frac{1}{\eta} \frac{\partial \eta}{\partial \sigma} = \frac{D_\sigma}{D} + \cdots,$$

where $\rho, \mu, D_\sigma$ are taken on the caustic. Substituting into (10), we obtain

$$2B' + \frac{1}{\eta} B + \frac{a}{i \omega \mu} \left( \frac{\mu^2 \rho_1}{\rho_1 \mu^2} \right)^3 \frac{D_v^2 \Gamma_u^2 \Gamma_\sigma^2}{D_\sigma} \frac{1}{\eta^3} \times \left\{ B''(\eta + \cdots) - B'(1 + \cdots) - B \left( \frac{A_v D_1}{A} \frac{\mu^2 \rho}{D_v \mu^2 \rho_1} + \cdots \right) \right\} = 0,$$

(11)

where all coefficients must be taken at the point of contact between the ray and the caustic. Omitting in this equation the terms of higher orders denoted by dots, we get the desired approximate equation.

Its coefficients can be given a simple geometric meaning. Let us first of all introduce the ratio $\tilde{\kappa}$ of the area of the caustic section, filled with the contact points of the rays of an infinitely thin ray tube to the cross-sectional area of this tube of the initial wave surface ($\tilde{\kappa}$ characterizes the density of rays on the caustic in the same way as $\kappa$ characterizes the density of rays on the wavefront); we introduce the corresponding quantity $\tilde{\eta} = \tilde{\kappa} \rho_1 c_1 / \rho c$, where $\rho c$ is taken on the caustic.

We denote by $\tilde{n}$ the normal to the caustic, by $\tilde{P}$ — the center of curvature of the normal section of the caustic by the plane tangent to the $v$-line (i.e., the corresponding ray), we also mark on the caustic normal the point $P$, the projection of which onto the osculating plane of the ray taken at the tangency point with the caustic $O$, is the center of curvature $Q$ of the ray for point $O$ (cf. Fig. 4). Let us give the lengths of segments $OP, O\tilde{P}$ from the base of the normal a sign corresponding to the established direction $\tilde{n}$. 

270
It turns out (cf. the Appendix), that in our case the value of \( \frac{1}{R} = \frac{1}{OP} - \frac{1}{O\tilde{P}} \) is nonzero.

Since the center of curvature of the caustic section by the osculating plane of the ray is obtained by projecting \( \tilde{P} \) onto the osculating plane, this means that the sections under consideration lie entirely on one side of the caustic (the "illuminated region").

We will direct \( \tilde{n} \), as shown in Fig. 4, into the illuminated region, then \( \frac{1}{R} > 0 \).

Note further that \( A(u,v) \) can be regarded as a function of the caustic point \((u,v)\). In particular, denoting by \( \tilde{s} \) the arc length of the \( v \)-line on the caustic and assuming that the positive direction coincides with the positive direction of the rays, we can form \( \frac{\partial A}{\partial \tilde{s}} \).

Finally, we put \( k = \omega/c \), where \( c \) is taken on the caustic, and introduce the notation \( \alpha = \frac{R}{\tilde{\eta}^3} \). Then (cf. the Appendix) our equation will take the following form:

\[
\beta = R \frac{\partial \ln A}{\partial \tilde{s}} / \tilde{\eta}. \text{ Then (cf. the Appendix) our equation will take the following form:}
\]

\[ \eta B'' - (1 + 2ik\alpha \eta^3)B' - (\beta + ik\eta^2)B = 0. \]

Setting \( B(\eta) = \eta^{\frac{1}{2}} \exp\left(\frac{ik\alpha}{3} \eta^3\right) C(\eta) \), we get the following equation for \( C \):

\[ C'' - \left( \frac{3}{4\eta^2} + \frac{\beta}{\eta} - k^2 \alpha^2 \eta^4 \right) C = 0. \]

Hence it is clear that with the accuracy we have adopted, the term with \( \beta \) can be neglected. Indeed, if the frequency is sufficiently high, then for \( \eta \) not too small, the second term in the parenthesis can be neglected in comparison with the third, and for small \( \eta \) — in comparison with the first. Dropping this term, we get that \( C \) is expressed in terms of a cylindrical function:
\[ C = \eta^2 Z_{1/3} \left( \frac{k\alpha}{3} \eta^3 \right) \cdot \text{const.,} \]

where the constant, generally speaking, depends on the ray, i.e. on \((u, v)\).

We put \(\frac{k\alpha}{3} = \gamma^3\).

So, we get

\[ P = \eta e^{i\gamma^3 \eta^3} Z_{1/3} \left( \gamma^3 \eta^3 \right) a(u,v). \quad (12) \]

In order to determine the form of the cylindrical function and the value of \(a(u, v)\), we observe, that far from the caustic, i.e., at large \(\gamma \eta\), we must obtain the usual geometric-acoustic approximation \((8)\), i.e. asymptotically with respect to \(\gamma \eta\), \(P \sim P(u, v, \sigma_1) \eta^{-\frac{1}{2}}\) must be fulfilled. Using asymptotic expansions of cylindrical functions, we obtain from this

\[ Z_{1/3} = H_{1/3}^{(2)}, \quad P = e^{-i\frac{\pi}{12} \sqrt{\frac{\pi \gamma}{2}}} P(u, v, \sigma_1) \gamma \eta e^{i\gamma^3 \eta^3} H_{1/3}^{(2)} \left( \gamma^3 \eta^3 \right). \quad (13) \]

Note now that, in contrast to the original assumption, \(P\) does not have the form \((9)\), but the following:

\[ P = A(u, v) \cdot B(\gamma \eta), \quad (14) \]

where \(\gamma\) depends on the ray, i.e. \(\gamma = \gamma(u, v)\).

However, it is easy to see that if in \((7)\) we substitute not \((9)\), but \((14)\), then the leading terms will not contain derivatives of \(\gamma\), i.e. the result will not change. So, \((13)\) is indeed the sought-for asymptotic approximation of the field of a wave propagating along the incoming portions of the rays. Quite similarly to \((12)\), we obtain for a wave propagating along outgoing sections,

\[ P = \eta e^{-i\gamma^3 \eta^3} H_{1/3}^{(1)} \left( \gamma^3 \eta^3 \right) \cdot b(u, v). \quad (15) \]

4. Our next task is to find a connection between functions \(a(u, v)\) and \(b(u, v)\) from \((12)\) and \((15)\). To do this, let us first calculate the general field at some points normal to the caustic. We will assume that the base of the normal is the origin of our rectangular coordinate system, the \(x\)-axis is directed along \(\Gamma_\sigma\), the \(y\)-axis — along \(\Gamma_u\), and the \(z\)-axis — along \(\tilde{n}\) (see Fig. 4). We choose the positive direction of the \(v\)-lines so that on the caustic it coincides with the direction of the rays. Then on the caustic \(\sigma_v > 0\), i.e. \(D_v > 0\).

The origin of the curvilinear coordinate system is chosen as the origin of \(\sigma\) so that point \(O\) corresponds to \(u = v = \sigma = 0\). Consider the normal point, a distance of \(\nu\) from the base. We will look for the field as a series in powers of \(\nu\);
for this to be possible, we require the analyticity of the initial wave surface and the function 
\( \mu(x, y, z) \). In this case, the functions (1) are also analytic. From the fact that \( \Gamma_u \Gamma_\sigma \equiv \Gamma_v \Gamma_\sigma \), we 
obtain, differentiating, that at point \( O \) the following relations hold:

\[
\begin{align*}
\Gamma_{v\sigma} \Gamma_\sigma &= \Gamma_{v\sigma} \Gamma_\sigma = \Gamma_{v\sigma} = 0, \quad \Gamma_u \Gamma_{\sigma v} = -\Gamma_{u\sigma} \Gamma_\sigma = 0, \\
\Gamma_{vv} \Gamma_\sigma + 2\Gamma_{vv} \Gamma_{\sigma v} &= 0.
\end{align*}
\]

With their help, and also taking into account the choice of the coordinate system, we have, expanding 
(1) in the vicinity of point \( O \):

\[
\begin{align*}
x &= \frac{\sigma}{\mu} + au^2 + c\sigma^2 + f\sigma u + gv^3 + \cdots; \\
y &= \delta u + a_1u^2 + b_1v^2 + c_1\sigma^2 + d_1uv + f_1\sigma u + g_1v^3 + \cdots; \\
z &= a_2u^2 + b_2v^2 + c_2\sigma^2 + d_2uv + e_2\sigma v + f_2\sigma u + g_2v^3 + \cdots; \\
\end{align*}
\]

(16)

where \( \delta = |\Gamma_u|, \ g = \frac{2}{3}\mu b_2 e_2 \). Note further, that at point \( O \) we have \( D_\sigma = \Gamma_u \Gamma_{v\sigma} \Gamma_\sigma = \frac{\delta}{\mu} e_2, \)

\( D_v = \Gamma_u \Gamma_{vv} \Gamma_\sigma = \frac{2\delta}{\mu} b_2, \) i.e. \( b_2 = \frac{\mu D_v}{2|\Gamma_u|}, \) \( g = -\frac{\mu^3 D_v D_\sigma}{3|\Gamma_u|^2} \).

To calculate the field at the point under consideration, it is necessary first of all to find the 
rays passing through it, i.e. solve the following system of equations:

\[
\begin{align*}
x(u, v, \sigma) &= 0, \\
y(u, v, \sigma) &= 0, \\
z(u, v, \sigma) &= \nu.
\end{align*}
\]

(17)

From (16) it follows that \( \left. \frac{D(x, y)}{D(u, \sigma)} \right|_0 \neq 0 \), i.e. the first two equations can be solved uniquely 
with respect to \( u, \sigma \): \( u = u(v), \ \sigma = \sigma(v) \). Calculating, we get \( u = -\frac{b_1}{|\Gamma_u|} v^2 + \cdots, \ \sigma = -\mu g v^3 + \cdots \).

Substituting these expressions in the third equation (17), we get, taking into account (16),

\[
\nu = b_2 v^2 + \cdots = \frac{\mu D_v}{2|\Gamma_u|} v^2 + \cdots
\]

(18)

As is known, such an equation has two solutions, which can be represented in the form of series 
in powers of \( \nu^{\frac{1}{2}} \), and each solution corresponds to a certain choice of the branch of the two-valued 
function \( \nu^{\frac{1}{2}} \). So, solving (18), we have

\[
v_1 = -\left(2\frac{|\Gamma_u|}{\mu D_v}\right)^{\frac{1}{2}} \nu^{\frac{1}{2}} + \cdots, \quad v_2 = +\left(2\frac{|\Gamma_u|}{\mu D_v}\right)^{\frac{1}{2}} \nu^{\frac{1}{2}} + \cdots,
\]

273
where the arithmetic values of the roots are implied. Substituting these expressions in \( u(v), \sigma(v) \), we get the parameters that determine the two rays passing through the point under consideration. In particular, for \( \sigma \) we obtain, taking into account (25),

\[
\sigma_1 = -\frac{\mu^4 D_v D_\sigma}{3} \left( \frac{2|\Gamma_u|}{\mu D_v} \right)^{\frac{3}{2}} \nu^{\frac{3}{2}} + \cdots + \frac{2}{3} \mu \left( \frac{2}{R} \right)^{\frac{1}{2}} \nu^{\frac{1}{2}} + \cdots ;
\]

the leading term of the expansion for \( \sigma_2 \) will be obtained by changing the sign.

As follows from geometric considerations, for a bundle of outgoing rays, the eikonal should increase with increasing \( \nu \), i.e., the first root corresponds to the outgoing and the second to the incoming ray to the caustic. Fig. 5 shows both rays passing through our point; here \( M_1 \) and \( M_2 \) have coordinates \( u_1, v_1 \) and \( u_2, v_2 \), respectively. Using (12), (15), we obtain for the general field

\[
p = a(u_2, v_2) \eta_2 \exp \left[ i \left( \frac{\omega}{a} \sigma_2 + \gamma_2^3 \eta_2^3 \right) \right] H^{(2)}_{\frac{1}{2}} \left( \gamma_2^3 \eta_2^3 \right) + b(u_1, v_1) \eta_1 \exp \left[ i \left( \frac{\omega}{a} \sigma_1 - \gamma_1^3 \eta_1^3 \right) \right] H^{(1)}_{\frac{1}{2}} \left( \gamma_1^3 \eta_1^3 \right). \tag{19}
\]

For small \( \nu \) we get a fairly good approximation for \( p \), if we keep in (19) only the leading terms of the expansions in \( \nu \). To do this, we find the leading term of the expansion for \( \eta_1, \eta_2 \). We have:

\[
\eta = \left( \frac{\partial \eta}{\partial u} \right)_0 u + \left( \frac{\partial \eta}{\partial v} \right)_0 v + \left( \frac{\partial \eta}{\partial \sigma} \right)_0 \sigma + \cdots = \left( \frac{\partial \eta}{\partial v} \right)_0 v + \cdots , \quad \text{because} \ u, \sigma \ \text{are of a higher order of smallness, than} \ v. \text{Using (6), where for the outgoing ray it is necessary to change the sign of the right-hand side, we get} \eta_1 = \frac{\rho \mu^2}{\rho \mu^2 D_1} D_v \left( \frac{2|\Gamma_u|}{\mu D_v} \right)^{\frac{1}{2}} \nu^{\frac{1}{2}} + \cdots = \tilde{\eta} \left( \frac{2}{R} \right)^{\frac{1}{2}} \nu^{\frac{1}{2}} + \cdots ; \quad \text{the leading term of the expansion for} \ \eta_2 \ \text{will be the same as for} \ \eta_1. \text{It follows from the expressions obtained that the expansion of the arguments of the exponentials in (19) begins with a term of order} \ \nu^2, \ \text{and since we neglect terms of this order in the argument of the Hankel function, the exponentials} \]
can be set equal to unity. (Note that our consideration does not exclude the case of large arguments of the Hankel function: by increasing the frequency and decreasing $\nu$, one can make the first term in the expansion of the argument arbitrarily large and the second arbitrarily small.)

So, we have approximately

$$p \approx \left\{a(0, 0)H_{\frac{3}{4}}^{(2)}(w) + b(0, 0)H_{\frac{3}{4}}^{(1)}(w)\right\}\tilde{\eta}\left(\frac{2}{R}\right)^{\frac{3}{2}}\nu^{\frac{1}{2}},$$

where $w = \frac{2}{3}k\left(\frac{2}{R}\right)^{\frac{3}{2}}\nu^{\frac{3}{2}}$.

In a neighborhood of $\nu = 0$, this expression is a holomorphic function of $\nu$, i.e., it admits an analytic continuation to the region of negative values of $\nu$. Such an extension, as can be seen from (17), is an analytic continuation of the calculated field into the shadow region, that is, it gives, to the accuracy we have accepted, the field in the shadow zone.

We now require that the field in the shadow zone does not increase exponentially away from the caustic. From this condition and the properties of cylindrical functions, we obtain the desired relation between functions $a(u, v)$ and $b(u, v)$: $b = ae^{i\frac{\pi}{3}}$. Using this relation and (12), (13), (15), we obtain the following expression for the field of a wave propagating along the outgoing sections of the rays:

$$P = e^{-i\frac{\pi}{12}}\sqrt{\frac{\pi\gamma}{2}}P(u, v, \sigma_1)\gamma\eta e^{-i\gamma^3\eta^3}H_{\frac{3}{4}}^{(1)}(\gamma^3\eta^3).$$

Formulas (13), (21) give the solution to the problem. They can also be rewritten in a slightly different way. Using the fact that $P(u, v, \sigma_1)\eta^{-\frac{1}{2}} = P_{\text{geom.}}$, where $P_{\text{geom.}}$ — the usual geometric-acoustic approximation for the pressure amplitude on the current wave surface, one can, for example, write formula (13) in the following form:

$$P = P_{\text{geom.}}e^{-i\frac{\pi}{12}}\sqrt{\frac{\pi\xi}{2}}\xi e^{i\xi^3}H_{\frac{3}{4}}^{(2)}(\xi^3), \quad \text{where} \quad \xi = \frac{\eta}{\tilde{\eta}}\left(\frac{kR}{3}\right)^{\frac{1}{3}}.$$ (22)

It is easy to see that $\xi$ does not depend on the choice of the initial wave surface, i.e., expression (22) depends only on the current point and on the point of contact of the ray with the caustic.

In applications, the case of a point source is often encountered. In this case, it is convenient to determine the value of $\kappa$ not in the way it was done above, but as the ratio of the cross-sectional
area of an infinitely thin ray tube to its exit solid angle. If we introduce the values $\eta, \tilde{\eta}$ according to this new definition, then the form of $\xi$ will not change. The value of $P_{geom.}$ in this case is equal to $\frac{F(u, v)}{\sqrt{\eta}}$, where the parameters $(u, v)$ characterize the exit angle of the ray and $F(u, v)$ is characteristic of the directivity of the source (i.e., near the source we have $P_{geom.} \approx \frac{F(u, v)}{r}$, where $r$ — the distance from the source).

With this notation we have for the field of the “wave arriving at the caustic”

$$P = \frac{F(u, v)}{\sqrt{\eta}} \sqrt{\frac{\pi \xi}{2}} \exp \left\{ i \left( \frac{\omega}{a} \sigma + \frac{5\pi}{12} \right) \right\} H^{(2)}_{\frac{1}{2}}(\xi^3),$$

and for the “outgoing wave” field —

$$P = \frac{F(u, v)}{\sqrt{\eta}} \sqrt{\frac{\pi \xi}{2}} \exp \left\{ i \left( \frac{\omega}{a} \sigma - \frac{\pi}{12} \right) \right\} H^{(1)}_{\frac{1}{2}}(\xi^3).$$

The total field is given by the sum of these two expressions, and in each of them the values of $u, v, \eta, \xi, \sigma$ for the corresponding ray should be substituted.

We also present an approximate formula (20) for the total field near the caustic. Expressing the combination of cylindrical functions in (20) in terms of the Airy function $v(t)$, the definition of which is given in [4], we obtain

$$p \approx \frac{F(u, v)}{\sqrt{\tilde{\eta}}} \exp \left( \frac{i\omega}{a} \tilde{\sigma} - \frac{\pi}{4} \right) 2 \left( \frac{kR}{2} \right)^{\frac{1}{2}} v \left( - \left( \frac{2}{kR} \right)^{\frac{1}{2}} k\nu \right),$$

where $\tilde{\sigma}$ — the value of the eikonal at the base of the normal to the caustic at which the point under consideration lies, and $\nu$ — the distance to the caustic; the values of $u, v, R, k, \tilde{\eta}$ are taken here for a ray passing through the base of the normal.

As indicated, this expression gives the field near the caustic and for the shadow zone; in this case, the value $\nu$ should be considered negative. A similar formula for a less general case is given in [5].

APPENDIX

Let $\tilde{K}$ — the curvature of the normal section of the caustic in the direction of the $v$-line. We have $\tilde{K} = \tilde{n} d^2 \tilde{\Gamma} / (d\Gamma)^2$, where $\tilde{n}$ — the normal to the caustic. With a displacement along the caustic $d\tilde{\Gamma} = d\Gamma + \Gamma_\sigma d\sigma$, where $d\Gamma$ is taken at $\sigma = \text{const}$. But along the $v$-line of the caustic we have $d\Gamma \equiv 0$, i.e. in this case we get $d\tilde{\Gamma} = \Gamma_\sigma d\sigma$, $d^2 \tilde{\Gamma} = \Gamma_\sigma d\sigma dv + \Gamma_{\sigma\nu} d\sigma dv + \Gamma_\sigma d^2 \sigma$, whence
\[
\tilde{K} = \frac{\hat{n}\Gamma_{\sigma\sigma}}{\Gamma_{\sigma}^2} + \frac{\hat{n}\Gamma_{\sigma\nu}}{\Gamma_{\sigma}^2\sigma_v}. \tag{23}
\]

Let us find further the curvature \( K \) of the ray at its point of tangency with the caustic. Denoting by \( s \) the arc length of the ray, by \( t \) — the unit tangent vector and by \( n \) — the principal normal of the ray, we have \( t = \mu\Gamma_{\sigma}, \ K n = \Gamma_{\sigma}\mu_{\sigma} + \Gamma_{\sigma\sigma\mu_{\sigma}}^2, \) whence \( K \cos \varphi = \hat{n}\Gamma_{\sigma\sigma}\mu_{\sigma}^2, \) where \( \varphi \) — the angle formed by the normal to the caustic and the principal normal of the ray.

Comparing with (23), we get
\[
K \cos \varphi - \tilde{K} = -\frac{\hat{n}\Gamma_{\sigma\sigma}\mu_{\sigma}^2}{\Gamma_{\sigma}^2}. \tag{24}
\]

Taking into account that on the caustic \( D_{\sigma} = \Gamma_u \Gamma_{\sigma\sigma} \Gamma_{\sigma} \neq 0 \) and \( [\Gamma_{\sigma} \Gamma_{\sigma}] \parallel \hat{n}, \) we obtain that the right-hand side here is nonzero, i.e., the left-hand side of (24) retains its sign on the considered section of the caustic. Hence it follows that the ray sections under consideration lie entirely on one side of the caustic. Assuming \( \hat{n} \) directed in the same direction, we find that the right-hand side of (24) is positive. Further we have
\[
|\hat{n}\Gamma_{\sigma\nu}| = \frac{|\Gamma_u \Gamma_{\sigma\sigma}\Gamma_{\sigma\nu}|}{|\Gamma_u| \cdot |\Gamma_{\sigma}|} = \frac{|D_{\sigma}|}{|\Gamma_u| \cdot |\Gamma_{\sigma}|}
\]
and, given that \( D_{\sigma} < 0, \) we get
\[
\frac{1}{R} = K \cos \varphi - \tilde{K} = -\frac{D_{\sigma}\mu_{\sigma}^3}{|\Gamma_u| \cdot |\sigma_v|} = \frac{D_{\sigma}^2\mu_{\sigma}^3}{|\Gamma_u| \cdot |D_v|}. \tag{25}
\]

Hence it is seen that if there were \( D_{\sigma} = 0, \) then the ray would have tangency with the caustic of order higher than the first. Further we have
\[
\hat{\kappa} = \sqrt{\hat{\beta}/J_1} \geq 0, \quad \text{where} \quad \hat{\beta} = \begin{vmatrix} \hat{\Gamma}_u^2 & \hat{\Gamma}_u \hat{\Gamma}_v \\ \hat{\Gamma}_u \hat{\Gamma}_v & \hat{\Gamma}_v^2 \end{vmatrix}.
\]
But \( \hat{\Gamma}_u = \Gamma_u + \Gamma_{\sigma\sigma} u, \ \hat{\Gamma}_v = \Gamma_{\sigma\sigma} v, \) whence \( \hat{\beta} = \Gamma_u^2 \Gamma_{\sigma\sigma}^2 \), i.e.
\[
\hat{\eta} = \hat{\kappa} \frac{\rho_{1\sigma}}{\rho c} = \frac{|\Gamma_u| \cdot |\sigma_v| \rho_1}{D_1 \mu_{\sigma}^3 \rho} \tag{26}
\]
Finally, taking into account that \( D_v^2 = D_{\sigma}^2 \sigma_{\nu}^2, \) we obtain
\[
\frac{1}{\mu^2} \left( \frac{\rho_{1\mu^2}}{\rho_{\mu_1^2} D_1} \right)^3 \cdot \frac{D_v^2 \Gamma_u^2 \Gamma_{\sigma\sigma} \nu_{\sigma}}{D_{\sigma}} = \frac{1}{\mu^2} \cdot \frac{\hat{\eta}^3 \mu^6}{|\Gamma_u|^3 \cdot |\sigma_v|^3} \cdot \frac{\Gamma_{\sigma\sigma}^2 D_{\sigma}}{\mu^2} = -\hat{\eta}^3 \mu R.\]
From this we obtain, using the above notation, that the coefficient at the curly brace in (11) is — \(-1/ik\alpha\).

Passing to the calculation of the second coefficient, we obtain from (25), (26)
\[
\frac{\hat{\eta}}{R} = -\frac{D_{\sigma} \rho_{1\mu^2}}{\rho_{\mu_1^2} D_1}. \tag{27}
\]
Further, we have on the caustic

\[ \frac{\partial \ln A}{\partial v} \frac{D_\sigma}{D_v} = -\frac{\partial \ln A}{\partial v} \frac{1}{\sigma_v} = -\frac{\partial \ln A}{\partial \sigma}, \]

where differentiation by \( \sigma \) is done along the \( v \)-line on the caustic. Denoting by \( \tilde{s} \) the arc length of this line and taking into account that with a displacement along it \( d\tilde{\Gamma} = \Gamma_\sigma d\sigma \), i.e. \( \mu d\tilde{s} = d\sigma \), we obtain, taking into account (27), that the second coefficient in (11) is equal to \( \beta \).

**Literature**


[V. A. Fok. Airy function tables. M., 1946.]

Appendix E

Translation of J. P. Guiraud, “Acoustique géométrique, bruit balistique des avions supersoniques et focalisation”

In this appendix we provide a translation of reference [79], the article “Acoustique géométrique, bruit balistique des avions supersoniques et focalisation” (Geometric acoustics, ballistic noise of supersonic aircraft and focusing) by Jean-Pierre Guiraud, published in French in 1965 in volume 4, number 2 of the journal formerly known as Journal de Mécanique (Journal of Mechanics), now the European Journal of Mechanics - B/Fluids. The first page of the original document is also provided for reference.

This work is the intellectual property of Jean-Pierre Guiraud and the Elsevier publishing company. Permissions to include its translation in the thesis were obtained from Elsevier via the Senior Copyrights Coordinator, Roopa Lingayath, on September 2, 2021.

Translation was assisted by use of the machine-learning translation tools Google Translate and DeepL, while equation typesetting, document formatting, and figure editing were done by hand. We take full responsibility for any technical or typographical errors introduced during this process. We have attempted to emulate the formatting of the original document, but differences in word/character length from one language to another have made it difficult to maintain consistent page numbering. The upper right page numbers are (approximately) those of the original work, while the bottom center numbers are the page number within the present document.

The translation may be cited as:

Acoustique géométrique, 
bruit balistique des avions supersoniques 
et focalisation

par

Jean-Pierre GUIRAUD.

SOMMAIRE. — On analyse de quelle manière se déforme le N balistique provoqué par le vol supersonique d'un avion, lorsque au cours de sa propagation le N en question s'approche d'une caustique.

1. Introduction.

La figure 1 schématisé la situation dont l'étude fait l'objet du présent Mémoire. En haut à droite est figurée la trajectoire d'un avion en vol supersonique, représenté par un point parce que sa longueur est très petite à l'échelle du phénomène qu'il s'agit d'étudier. Le comportement asymptotique des perturbations provoquées par le vol est très différent dans les trois régions indiquées sur la figure. La terminologie adoptée est celle établie aux États-Unis par Lagerstrom [17] et son école, s'agissant en fait du meilleur correspondant en langue française, soit : comportement asymptotique distal pour « outer asymptotic behavior » et comportement asymptotique proximal pour « inner asymptotic behavior ». La région marquée distal 1 est celle où est valable un premier comportement asymptotique ; ce serait la seule région à considérer si le vol était subsonique. Le comportement asymptotique en question est celui d'un champ acoustique présentant sur la trajectoire de l'avion une singularité de type déterminé, dépendant des caractéristiques de l'avion et du vol. Dès qu'on raisonne sur des écoulements tridimensionnels non

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SUMMARY. – We analyze how an N wave caused by the supersonic flight of an aircraft is deformed, when during its propagation the N in question approaches a caustic.

1 Introduction.

Figure 1 shows the situation whose study is the subject of the present Memoir. At the top right of the figure is the trajectory of an aircraft in supersonic flight, represented by a point because its length is very small on the scale of the phenomenon to be studied. The asymptotic behavior of the disturbances caused by the flight is very different in the three regions indicated in the figure. The terminology adopted is that established in the United States by Lagerstrom [17] and his school, as he is regarded as the best correspondent in French, namely: distal asymptotic behavior for “outer asymptotic behavior” and proximal asymptotic behavior for “inner asymptotic behavior.” The region marked distal 1 is that where asymptotic behavior is first valid; it would be the only region to consider if the flight was subsonic. The asymptotic behavior in question is that of an acoustic field producing a singularity of a determined type on the trajectory of the aircraft, depending on the characteristics of the aircraft and the flight. As soon as we reason on three-dimensional flows
that are not stationary, as is the case here, it is essential to do so in terms of sound rays. Therefore, let us say that the distal 1 region is that obtained by moving away from the aircraft indefinitely in the direction of a non-characteristic ray of sound. The characteristic sound rays are located, for each point of the trajectory, on the complementary cone of the classical Mach cone. When the sound ray approaches a characteristic direction, the distal 1 asymptotic behavior becomes singular and it is then necessary to describe the disturbance field using a new asymptotic behavior which we must name, in the terminology in use, proximal 1. The asymptotic behavior in question is that of a Mach wave train of very small width, spread along the Mach wave associated with the flight of the point-source aircraft. Just as the distal 1 asymptotic behavior is singular when the sound ray approaches the characteristic direction, so too does the proximal 1 asymptotic behavior become singular when the wave train approaches the caustic to which the considered sound ray is tangent. Specifically, on approach of the caustic, the signal of the wave train does not change significantly in shape, but its amplitude, on the other hand, increases indefinitely as the inverse square root of the distance to the point of contact, evaluated along the sound ray. The neighborhood of the caustic must then be the subject of a separate study in order to construct a new asymptotic behavior which is to be called proximal 2 and which must be matched with the old asymptotic behavior proximal 1, which in fact becomes the asymptotic behavior of distal 2 in its relation to the new proximal. How to carry out the detailed study of the proximal 2 region using a mathematical microscope is precisely the subject of this Memoir. The situation to be studied is that of figure 2: an N wave approaches a caustic along the sound ray in the figure. The aircraft is very far away at a distance which is of the order \( L \), and the width of the N wave is \( l \), assumed to be much less than \( L \); the influence of the

![Fig. 1.](image-url)
caustic begins to be felt from a distance from the latter which is of the order of magnitude \((\frac{1}{3})^{\frac{1}{3}}\) times the distance \(L\). Our study is therefore essentially intended to describe how the \(N\) wave is deformed when it enters the zone of influence of the caustic, as has been drawn in dashed lines in figure 2. We would like beforehand, however, to recall the elements of the usual ballistic noise theory.

2 Ballistic and geometric acoustic noise.

2.1 ASYMPTOTIC BEHAVIOR OF THE ACOUSTIC FIELD PRODUCED BY THE FLIGHT OF A SUPERSONIC AIRCRAFT IN A HOMOGENEOUS ATMOSPHERE.

We have indicated elsewhere ([7], [8]) how it is possible to find the acoustic field of a supersonic aircraft when we know its trajectory, its geometric definition and the distribution of lift on its skeleton. The starting point for this study is constituted by the result so obtained and also by the special form it takes when we study its asymptotic behavior in the Mach wave train in Figure 3, which is spread around the Mach wave \(\Sigma(t)\) – this is the spatial section of the space-time wave – associated for example with the nose of the fuselage.
It has long been known from the work of Hayes ([14], [15]), Whitham [30], and Rao [24] that, from the point of view of the asymptotic behavior in question, the aircraft can be replaced by an equivalent body of revolution whose law of the areas of normal sections $S_e(\xi; R)$ would be dependent on the abscissa $\xi$ of the section and on the sound ray $R$ considered. Recall that a sound ray is obtained if one moves by following a wave in the direction which is defined by the speed of propagation $V_0 + c_0 n$ of the latter, where $V_0$ denotes the wind speed\(^\text{(1)}\) and $c_0$ the speed of sound, while $n$ denotes the unit vector normal to the wave. The sound rays $R$ considered here, and in particular those that are characteristics, are those described by the points of contact of the central waves (emitted by the successive positions $x_a(\tau)$ of the nose of the fuselage) with their envelope $\Sigma(t)$.

If we accept this equivalence with a body of revolution it is not difficult to find the formula giving the asymptotic behavior mentioned above. Let us consider for this purpose the function

\[
\Gamma(t, x; \tau, \xi) = (t - \tau)^2 - c_0^{-2}|x - x_a(\tau) - \xi e_1(\tau)|^2,
\]

which was introduced by Hadamard and Riesz and named by these authors the square of the geodesic distance between the instant-point of emission $\tau, x_a(\tau) + \xi e_1(\tau)$, where $e_1$ designates the unit vector associated with the axis of the fuselage, and the instant-point of observation $t, x$. Denote by $\tau_i(t, x; \xi)$ the roots of the equation, in $\tau$, $\Gamma = 0$; it is not difficult to show that $U_i(t, x; \tau, \xi)$ is a solution to the wave equation. It suffices to do the calculations, for example for $\xi = 0$, but we may proceed more elegantly. It is known\(^\text{(2)}\) that $\delta(\Gamma)$ is a distribution solution of the wave equation if $\delta$ denotes the Dirac mass unit. Under these conditions, if $f(\tau)$ is an arbitrary function and if $\langle \cdot , \cdot \rangle_\tau$ characterizes the integration, in the sense of distributions, over the variable $\tau$, then

\[
\langle f(\tau), \delta(\Gamma) \rangle_\tau = \sum_i f(\tau_i) U_i,
\]

is a solution of the wave equation. The given result is then made evident by a suitable choice of function $f(\tau)$. The reader will be able to verify that, if the point $x_a(\tau)$ is in uniform supersonic motion, we have

\[
U_1 = U_2 = \frac{c_0}{2} \left\{ x^2 - \beta^2 r^2 \right\}^{-\frac{1}{2}},
\]

with the notations of figure 4.

\(^{(1)}\)In fact, in this section, we will assume that $V_0 = 0$.

\(^{(2)}\)We may for example go through the intermediary of the formula of retarded potentials; see also [27].
If we now suppose that the instant-point $t, x$ is located on the Mach wave train introduced above, and is very far from the instant-point $\tau, x_a$ but close to the characteristic ray $R$ emanating from it, then there exists a system of values $\tau_0, \xi_0$ which makes $\Gamma$ and $\Gamma_x$ vanish simultaneously, since it expresses that $t, x$ is on the Mach wave corresponding to the point $\xi = \xi_0$ of the axis of the fuselage and, more precisely, on a characteristic sound ray coming from $\tau_0, x_a(\tau_0) + \xi_0 e_1(\tau_0)$. The function $\Gamma$ may be, asymptotically, replaced by a Taylor expansion

$$\Gamma \approx \frac{1}{2} \Gamma_{\tau_0 \xi_0} (\tau - \tau_0)^2 + \Gamma_{\xi_0} (\xi - \xi_0),$$

the justification essentially being due to the fact that any derivative with respect to $\tau$ or $\xi$ equates, from an order of magnitude point of view, to a division by $c_0^{-1}L$ or by $L$ and we have $|\xi - \xi_0| \ll L$, $c_0(\tau - \tau_0) \ll L$. Under these conditions it is immediate that we have

$$U_1(t, x; \xi) \approx U_2(t, x; \xi) \approx \sqrt{2 \Gamma_{\tau_0 \xi_0} \Gamma_{\xi_0} (\xi_0 - \xi)},$$

and the asymptotic behavior of the pressure perturbation is given by

$$\Delta p = p - p_0 \approx \frac{2F(\xi; R)}{\sqrt{2|\Gamma_{\tau_0 \xi_0} \Gamma_{\xi_0}|}},$$

with

$$F(\xi; R) = \frac{1}{2\pi} \int_{-\infty}^{\xi_0} \frac{S''(\xi; R)}{\sqrt{\xi_0 - \xi}} d\xi,$$

$$F(\xi; R) = \frac{1}{2\pi} \int_{\xi_0}^{\infty} \frac{S''(\xi; R)}{\sqrt{\xi - \xi_0}} d\xi.$$

We must take $U_1 = U_2 = 0$ if the quantity under the radical is negative.
We leave aside, for the moment, the study of the neighborhood of the caustic, the locus of instant-points where $\Gamma_{0t0} = 0$. Recall that $F(\xi; R)$ denotes the Whitham function.

2.2 GEOMETRIC ACOUSTICS IN AN INHOMOGENEOUS ATMOSPHERE.

10 Equation of acoustic energy. — Let us use the usual notations $p, \rho, e, s, \mathbf{V}$ for pressure, density, specific internal energy, specific entropy, and the velocity vector, and assign to the index zero the same quantities relative to the unperturbed atmosphere in which we will assume the exchanges of energy, either internal or with the outside (solar radiation), are negligible. Let $\mathcal{E}, \mathbf{M}, \mathcal{C}$ be the first members of the equations of energy, momentum and mass, respectively. The combination

$$\mathcal{E} - \mathbf{M} \cdot \mathbf{V}_0 + \mathcal{C} \left( e_0 + \frac{|\mathbf{V}_0|^2}{2} \right) = 0,$$

leads to the energy perturbation equation

$$\frac{\partial}{\partial t} \left\{ \rho(e - e_0) + \frac{\rho |\mathbf{V} - \mathbf{V}_0|^2}{2} \right\}$$

$$+ \nabla \cdot \left\{ \rho \left( e - e_0 + \frac{|\mathbf{V} - \mathbf{V}_0|^2}{2} \right) \mathbf{V} + p(\mathbf{V} - \mathbf{V}_0) \right\}$$

$$+ \rho(\mathbf{V} - \mathbf{V}_0) \cdot (\nabla \mathbf{V}_0) \cdot (\mathbf{V} - \mathbf{V}_0) + \frac{\rho(\mathbf{V}_0 - \mathbf{V}) \cdot \nabla p_0}{\rho_0}$$

$$+ \frac{\rho_0 p - \rho p_0}{\rho_0} \mathbf{V} \cdot \mathbf{V}_0 + \rho(\mathbf{V} - \mathbf{V}_0) \cdot \nabla e_0 = 0.$$

If $f$ now designates any one of the quantities $p, \rho, \cdots$, we set $f = f_0 + f_1$, and if we carry this out in the previous equation, by systematically neglecting the cubic terms in $f_1$, we obtain the acoustic energy equation:

$$\frac{\partial \mathcal{E}_a}{\partial t} + \nabla \cdot \mathbf{W}_a + \Pi_a = 0,$$

where

$$\mathcal{E}_a = \frac{1}{2} \left( \rho_0 |\mathbf{V}_1|^2 \right) + \frac{c_0^2 \rho_1^2}{\rho_0} + \frac{g_{ss} \rho_1 s_1}{\rho_0},$$

$$\mathbf{W}_a = \rho_1 \mathbf{V}_1 + \mathcal{E}_a \mathbf{V}_0,$$

$$\Pi_a = \mathbf{V}_1 \cdot (\rho_0 \nabla \mathbf{V}_0) \cdot \mathbf{V}_1$$

$$+ \left\{ \frac{\rho_1^2}{\rho_0} \left( c_0^2 - \frac{p_0}{\rho_0} \right) + \frac{1}{2} g_{ss} \rho_1 s_1 \right\} \nabla \cdot \mathbf{V}_0 + \frac{1}{2} \rho_1 s_1 \mathbf{V}_1 \cdot \nabla \rho_0$$

$$+ \frac{1}{2} \rho_1^2 \left( \frac{\partial}{\partial t} + \mathbf{V}_0 \cdot \nabla \right) \left\{ \frac{1}{\rho_0} \left( \frac{p_0}{\rho_0} - c_0^2 \right) \right\} + \frac{1}{2} \rho_1 s_1 \left( \frac{\partial}{\partial t} + \mathbf{V}_0 \cdot \nabla \right) \frac{T_0 - 2g_{ss}}{\rho_0},$$

286
respectively denote the (volume) acoustic energy density, the (surface) acoustic energy flux density vector and the (volume) density of acoustic energy production. To perform this calculation we have used

\[ e_1 \approx \frac{p_0}{\rho_0^2} \rho_1 + T_0 s_1. \quad p_1 = \frac{c_0^2}{\rho_1} \rho_1 + g_{s_0} s_1, \]

where \( p = g(\rho, s) \) is the equation of state of the gas and \( T \) its temperature.

20 Acoustic equations in characteristic variables. — Let us consider, as indicated in figure 5, a family of surfaces moving with time, defined by the parametric representation

\[ t = x_0, \quad x = x_\Sigma(x_0, a), \]

and letting \( n(x_0, a) \) be the unit vector on the surface which arrives at \( x \) at the instant \( t = x_0 \), we can obtain a four dimensional representation of instant-points by setting

\[ t = x_0, \quad x = x_\Sigma + x_1 n. \]

![Fig. 5.](image_url)

We will suppose here that the family of surfaces \( x_1 = 0 \) defines an acoustic wave, the trajectory of \( x_\Sigma \), with \( a \) fixed, being a sound ray.

This leads to

\[ \frac{\partial x_\Sigma}{\partial x_0} = V_0 + c_0 n, \quad \frac{\partial n}{\partial x_0} = -Dc_0 - (DV_0) \cdot n, \]

under the condition that \( D \) designates the gradient operator in the plane tangent to \( x_\Sigma \) on the wave surface. We introduce the curvature tensor \( K \) such that, for a displacement \( d_\Sigma x_\Sigma \) on the wave surface, we have

\[ dn = d_\Sigma x_\Sigma \cdot K, \]

\[ \text{† Translator's Note: In other words, the expressions in Eq. (8) are respectively the acoustic energy density, the instantaneous acoustic intensity, and the rate of acoustic energy production per unit volume.} \]
and, finally, denote by $\mathbf{H}$ the tensor of the plane tangent to $\mathbf{x}_\Sigma$ such that, if $\mathbf{1}_T$ is the unit tensor of this same tangent plane, we have

(14) \hspace{1cm} \mathbf{H} \cdot (\mathbf{1}_T + x_1 \mathbf{K}) = (\mathbf{1}_T + x_1 \mathbf{K}) \cdot \mathbf{H} = \mathbf{1}_T,

With this system of notations, it is possible to establish the following change of variables formulas

(15) \hspace{1cm} \begin{cases} \frac{\partial}{\partial t} = \frac{\partial}{\partial x_0} - (c_0 + \mathbf{V}_0 \cdot \mathbf{n}) \frac{\partial}{\partial x_1} - (\mathbf{V}_0 - x_1[\mathbf{D}c_0 + (\mathbf{DV}_0) \cdot \mathbf{n}] \cdot \mathbf{H} \cdot \mathbf{D}, \\ \nabla = \mathbf{n} \frac{\partial}{\partial x_1} + \mathbf{H} \cdot \mathbf{D}, \end{cases}

if $\nabla$ designates the vector gradient in space. If we propose to write the acoustic equations in variables $x_0, \mathbf{a}, x_1$, we find, after having set

(16) \hspace{1cm} \mathbf{V}_1 = \mathbf{v}_1 \mathbf{n} + \mathbf{U}_1, \hspace{1cm} \mathbf{n} \cdot \mathbf{U}_1 = 0

and, after some calculations,

(17) \hspace{1cm} \begin{cases} c_0 \frac{\partial s_1}{\partial x_1} - \frac{\partial s_1}{\partial x_0} - v_1 \frac{\partial s_0}{\partial x_1} - \mathbf{U}_1 \cdot \mathbf{H} \cdot \mathbf{D} s_0 - x_1(\mathbf{D}c_0 + (\mathbf{DV}_0) \cdot \mathbf{n}) \cdot \mathbf{H} \cdot \mathbf{D} s_1 = 0, \\ \frac{\partial}{\partial x_1}(\rho_0 v_1 - \rho_1 c_0) + \frac{\partial \rho_1}{\partial x_0} + \rho_1 \frac{\partial c_0}{\partial x_1} + \rho_0 v_1 \mathbf{H} : \mathbf{K} + \mathbf{U}_1 \cdot \mathbf{H} \cdot \mathbf{D} \rho_0 \\
\rho_0 c_0 \frac{\partial \mathbf{U}_1}{\partial x_1} + \rho_0 v_1 \left(\mathbf{D}c_0 + (\mathbf{DV}_0) \cdot \mathbf{n} - \frac{\partial \mathbf{V}_0}{\partial x_1} \cdot \mathbf{1}_T\right) - \rho_1 \mathbf{H} \cdot \mathbf{D} \rho_1^2 - s_1 \mathbf{H} \cdot \mathbf{D} s_0 \\
+ \frac{\rho_1}{\rho_0}(c_0^2 \mathbf{H} \cdot \mathbf{D} \rho_0 + g_{s_0} \mathbf{H} \cdot \mathbf{D} s_0) - c_0^2 \mathbf{H} \cdot \mathbf{D} \rho_1 - g_{s_0} \mathbf{H} \cdot \mathbf{D} s_1 \\
- \rho_0 \frac{\partial \mathbf{U}_1}{\partial x_0} \cdot \mathbf{1}_T - \rho_0 \mathbf{U}_1 \cdot \mathbf{H} \cdot (\mathbf{DV}_0) \cdot \mathbf{1}_T \\
\rho_0 c_0 \frac{\partial \mathbf{V}_1}{\partial x_0} + \frac{\partial \rho_0}{\partial x_0} + \frac{\partial \mathbf{s}_0}{\partial x_0} + \frac{\partial \mathbf{s}_0}{\partial x_0} \frac{\partial \mathbf{U}_1}{\partial x_0} \\
+ \frac{\partial \mathbf{V}_0}{\partial x_1} \cdot \mathbf{n} + \rho_0 c_0^2 \mathbf{H} : \mathbf{K} + c_0^2 \frac{\partial \rho_0}{\partial x_1} + g_{s_0} \frac{\partial \mathbf{s}_0}{\partial x_1} \\
- \frac{\rho_1}{\rho_0} \left(c_0^2 \frac{\partial \rho_0}{\partial x_1} + c_0 g_{s_0} \frac{\partial \mathbf{s}_0}{\partial x_1} \right) + \frac{c_0^2 \rho_1}{\partial x_1} \nabla \cdot \mathbf{V}_0 + c_0 \rho_1 \frac{\partial \mathbf{V}_0}{\partial x_1} + c_0 \frac{\partial g_{s_0}}{\partial x_1} \\
+ \mathbf{U}_1 \{\rho_0 c_0(\mathbf{D}c_0 + (\mathbf{DV}_0) \cdot \mathbf{n} + \mathbf{H} \cdot (\mathbf{DV}_0) \cdot \mathbf{n}) \\
+ c_0^2 \mathbf{H} \cdot \mathbf{D} \rho_0 + g_{s_0} \mathbf{H} \cdot \mathbf{D} s_0 \} + \rho_0 c_0^2 (\mathbf{H} \cdot \mathbf{D}) \cdot \mathbf{U}_1 \\
+ x_1 \{\rho_0 c_0(\mathbf{D}c_0 + (\mathbf{DV}_0) \cdot \mathbf{n} \cdot \mathbf{H} \cdot (\mathbf{DV}_1 - \mathbf{K} \cdot \mathbf{U}_1) \\
+ c_0^2 (\mathbf{D}c_0 + (\mathbf{DV}_0) \cdot \mathbf{n} \cdot \mathbf{H} \cdot \mathbf{D} \rho_1 \\
+ g_{s_0} (\mathbf{D}c_0 + (\mathbf{DV}_0) \cdot \mathbf{n} \cdot \mathbf{H} \cdot \mathbf{D} s_1)\} = 0, \end{cases}
A : B designating the complete scalar product (A\(_{ij}\)B\(_{ij}\)) of the two tensors A and B.

30 Acoustic field concentrated on a surface. — The equations (17) admit solutions of the following type

\[
\begin{align*}
V_1 &= c_0 \Phi n \frac{dK \delta(x_1)}{dx_1^k} + \cdots = v_1' \frac{dK \delta(x_1)}{dx_1^k} + \cdots, \\
\rho_1 &= \rho_0 \Phi \frac{dK \delta(x_1)}{dx_1^k} + \cdots = \rho_1' \frac{dK \delta(x_1)}{dx_1^k} + \cdots, \\
s_1 &= \cdots,
\end{align*}
\]

where \(\cdots\) designates a sum of terms which have the structure of a function in \(x_1\), or a distribution of type \(d_l \delta x_1^l\) with \(l < K\). The quantities \(v_1'\), \(\rho_1'\), \(s_1'\) are functions of \(x_0\) and \(a\), that is to say again of \(x_0\) and of the position of \(x_\Sigma\) on the wave surface. We have the relations

\[
s_1' = 0, \quad \rho_0 v_1' = c_0 \rho_1',
\]

which respectively express the isentropy and the equipartition of energy between kinetic and potential components, while \(v_1'\) satisfies the following evolution equation along the sound ray

\[
\begin{align*}
2 \frac{\partial v_1'}{\partial x_0} + v_1' \left( c_0 K + \frac{c_0}{\rho_0} \frac{\partial}{\partial x_0} \left( \frac{\rho_0}{c_0} \right) + \frac{1}{c_0} \frac{\partial c_0^2}{\partial x_1} + \nabla \cdot V_0 + \frac{\partial V_0}{\partial x_1} \cdot n \right) &= 0,
\end{align*}
\]

\(K = \text{Trace}(K)\),

and this equation must be applied at \(x_1 = 0\). By forming an obvious combination of the mass equation and the entropy equation for the unperturbed atmosphere, applied at \(x_1 = 0\), we can obtain

\[
\frac{\partial c_0^2}{\partial x_0} - c_0 \frac{\partial c_0^2}{\partial x_1} + 2c_0^2(\Gamma_0 - 1) \nabla \cdot V_0 = 0 \text{ at } x_1 = 0,
\]

with

\[
\Gamma = \frac{1}{c} \left( \frac{d(cp)}{d\rho} \right)_{s=Cst} = \left( \frac{\gamma + 1}{2} \right) \text{ for a perfect gas}.
\]

It is advantageous to introduce, at this stage, energy concepts and form the following expressions:

\[
\begin{align*}
\mathcal{E}' &= \frac{1}{2} \left( \rho_0 v_1'^2 + \frac{c_0^2 \rho_1'^2}{\rho_0} \right) = \rho_0 v_1'^2 = \frac{c_0^2 \rho_1'^2}{\rho_0}, \\
W' &= \mathcal{E}'(V_0 + c_0 n), \\
\Pi' &= \mathcal{E}' \{ n \cdot (\nabla V_0) \cdot n + (\Gamma_0 - 1) \nabla \cdot V_0 \},
\end{align*}
\]

which are functions of \(x_0\) and \(a\) whose relationship to \(\mathcal{E}, W, \Pi\) is evident. Let us note only that, to arrive at the expression for \(\Pi'\) we made
use of the isentropic nature of the unperturbed atmosphere

\[ \frac{\partial s_0}{\partial t} + \mathbf{V}_0 \cdot \nabla s_0 = 0 \]

and the relation

\[ (24) \quad 1 - \frac{p_0}{\rho_0 c_0^2} - \frac{p_0^2}{2 \rho_0} \frac{\partial}{\partial \rho_0} \left\{ \frac{1}{\rho_0} \left( \frac{p_0}{\rho_0} - c_0^2 \right) \right\} = \Gamma_0 - 1. \]

With these notations, relation (20) takes the form

\[ (25) \quad \frac{\partial c_0 E'}{\partial x_0} + c_0 E' (c_0 K + \Gamma_0 \nabla \cdot \mathbf{V}_0) + c_0 \Pi' = 0, \]

but this formula lacks elegance. It is necessary to replace \( c_0 E' \) by the energy flux density of the acoustic field flowing along a tube of sound rays. For that we define a function \( A \) such that \( \int_{\Omega} A \, d\mathbf{a} \) remains equal, when \( x_0 \) varies, to the area of the section of the wave surface \( x_1 = 0 \) cut out by the tube of sound rays characterized by the condition that \( \mathbf{a} \) is in the domain \( \Omega \). We easily verify by calculation the relations

\[ (26) \quad \begin{cases}
  \frac{1}{\mathcal{A}} \frac{\partial \mathcal{A}}{\partial x_0} = \nabla \cdot \mathbf{V}_0 - \mathbf{n} \cdot (\nabla \mathbf{V}_0) \cdot \mathbf{n} + c_0 K, \\
  - \frac{\mathbf{V}_0 \cdot \mathbf{n}}{c_0} \frac{\partial c_0}{\partial x_0} + \frac{\partial (\mathbf{V}_0 \cdot \mathbf{n})}{\partial x_0} \\
  - (c_0 + \mathbf{V}_0 \cdot \mathbf{n}) \{ \mathbf{n} \cdot (\nabla \mathbf{V}_0) \cdot \mathbf{n} + (\Gamma_0 - 1) \nabla \cdot \mathbf{V}_0 \} = \frac{\partial c_0}{\partial t} + \frac{\partial \mathbf{V}_0}{\partial t} \cdot \mathbf{n},
\end{cases} \]

so that (25) takes the form

\[ (27) \quad \frac{\partial}{\partial x_0} (\mathcal{A} W' \cdot \mathbf{n}) + \mathcal{A} \Pi' (c_0 + \mathbf{V}_0 \cdot \mathbf{n}) - \mathcal{A} E' \left( \frac{\partial \mathbf{V}_0}{\partial t} \cdot \mathbf{n} + \frac{\partial c_0}{\partial t} \right) = 0. \]

If the unperturbed atmosphere is independent of time, the preceding relation has an elegant interpretation: the variation between two instants of the acoustic energy flux of the primed field through a section of a sound ray tube cut out by the wave surface is compensated by the energy produced in the volume swept out between the two instants by the section considered. If the atmosphere depends on time, this statement no longer holds, but we may put formula (27) in a form that remains elegant. Letting \( \psi(t, \mathbf{x}) = 0 \) be the equation of the wave \( x_1 = 0 \) in spatio-temporal variables, we set

\[ (28) \quad \left( \frac{\partial \psi}{\partial t} \right)_{\psi=0} = p_0(x_0, \mathbf{a}), \]

we may show that we have

\[ (29) \quad \frac{1}{p_0} \frac{\partial p_0}{\partial x_0} = \frac{\partial c_0}{\partial t} + \frac{\partial \mathbf{V}_0}{\partial t} \cdot \mathbf{n}, \]
so that, if we set

\[
(30) \quad \mathcal{C} = \frac{A}{p_0},
\]

it follows that

\[
(31) \quad \frac{\partial}{\partial x_0} \left( \mathcal{C} W' \cdot n \right) + \mathcal{C} \Pi' \left( c_0 + V_0 \cdot n \right) = 0,
\]

whose interpretation can be seen if we note that \( \mathcal{E} W' \cdot n = \mathcal{A} \mathcal{E}' \frac{dn}{d\psi} \), if \( dn \) is the normal distance between the surfaces \( \psi = 0 \) and \( \psi = d\psi \). We will note that if we replace \( \frac{dK}{dx} \) by \( Y(x_1) \), where \( Y(x_1) \) denotes the Heaviside function, the preceding considerations remain, and, in this case, the acoustic field \( (v'_1 n' Y, \rho'_1 Y, 0) \) is an acoustic shock treated by JB Keller in [18].

40 Acoustic wave train of very small width. — Let us again consider a regular acoustic field and suppose that this field is strongly concentrated in the vicinity of the wave \( x_1 = 0 \). More precisely, suppose that \( \frac{E_{\max}}{|x_1|} \) is very small for \( |x_1| > l \) and suppose that \( \frac{1}{L} \) and \( \frac{1}{H} \) are both \( \ll 1 \), if \( L \) denotes the order of magnitude of the length traveled along the sound rays by this acoustic field since its creation and if \( H \) denotes a scale of inhomogeneity for the undisturbed atmosphere; then equations (17) indicate that with a relative error \( \mathcal{O}(\frac{1}{L} + \frac{1}{H}) \) we have

\[
(32) \quad s_1 = \rho_0 v_1 - \rho_1 c_0 = |U_1| = 0,
\]

and that \( v_1 \), which this time depends on \( x_1 \) as well as on \( x_0 \) and \( a \), satisfies the same equation as \( v'_1 \) in the previous paragraph. This means that \( v_1 \) is the product of a function of \( x_1 \) and a function of \( x_0 \) and \( a \) which is precisely equal to \( v'_1 \). An acoustic wave train whose width is very small compared to the distance traveled, as well as the scale of inhomogeneity of the atmosphere, and which is spread around a particular acoustic wave, is equal to the product of a signal of arbitrary shape, a function of the distance to the wave surface, by an amplitude which varies as the amplitude of an acoustic shock, or of an acoustic field concentrated on the reference wave. So

\[
(33) \quad v_1 = f(x_1)v'_1(x_0, a),
\]

where \( v'_1 \) satisfies (20). If with \( v_1, \rho_1 \) we form \( \mathcal{E}, W, \Pi \), equation (31) is satisfied, \( W' \) and \( \Pi' \) being replaced by \( W \) and \( \Pi \).
2.3 TRAIN OF NONLINEAR SOUND WAVES OF VERY SMALL WIDTH AND VERY LOW AMPLITUDE.

10 Generalities. — We suppose, so as not to make the technique too cumbersome, that the unperturbed atmosphere is homogeneous and windless, and we will use the same coordinates as in paragraph 2.2.20, but this time we consider the exact nonlinear equations of motion, without dissipation effects, and we denote by \( \mathbf{V} = v_1 \mathbf{n} + \mathbf{V}_T \) the velocity vector. We have

\[
\begin{align*}
\frac{\partial \rho}{\partial x_0} + (v_1 - c_0) \frac{\partial \rho}{\partial x_1} + \mathbf{V}_T \cdot \mathbf{H} \cdot \mathbf{D} \rho \\
+ \rho \left( \frac{\partial v_1}{\partial x_1} + v_1 \mathbf{H} : \mathbf{K} + (\mathbf{H} \cdot \mathbf{D}) \cdot \mathbf{V}_T \right) &= 0, \\
\frac{\partial s}{\partial x_0} + (v_1 - c_0) \frac{\partial s}{\partial x_1} + \mathbf{V}_T \cdot \mathbf{H} \cdot \mathbf{D} s &= 0, \\
\rho \left\{ \frac{\partial v_1}{\partial x_0} + (v_1 - c_0) \frac{\partial v_1}{\partial x_1} + \mathbf{V}_T \cdot \mathbf{H} \cdot \mathbf{D} v_1 - \mathbf{V}_T \cdot \mathbf{H} \cdot \mathbf{K} \cdot \mathbf{V}_T \right\} + \frac{\partial p}{\partial x_1} &= 0, \\
\rho \left\{ \frac{\partial \mathbf{V}_T}{\partial x_0} + (v_1 - c_0) \frac{\partial \mathbf{V}_T}{\partial x_1} + \mathbf{V}_T \cdot (\mathbf{H} \cdot \mathbf{D} \mathbf{V}_T) \cdot \mathbf{1}_T + v_1 \mathbf{V}_T \cdot (\mathbf{H} \cdot \mathbf{K}) \right\} + \mathbf{H} \cdot \mathbf{D} p &= 0, \\
\left( \frac{\partial}{\partial x_0} + (v_1 - c_0) \frac{\partial}{\partial x_1} + \mathbf{V}_T \cdot \mathbf{H} \cdot \mathbf{D} \right) \left( \rho \left( e + \frac{v_1^2 + |\mathbf{V}_T|^2}{2} \right) \right) \\
+ \frac{\partial}{\partial x_1} \left\{ \rho \left( h + \frac{v_1^2 + |\mathbf{V}_T|^2}{2} \right) v_1 \right\} \\
+ \rho \left( h + \frac{v_1^2 + |\mathbf{V}_T|^2}{2} \right) v_1 \mathbf{H} : \mathbf{K} \\
+ \mathbf{H} \cdot \mathbf{D} \cdot \left\{ \rho \left( h + \frac{v_1^2 + |\mathbf{V}_T|^2}{2} \right) \mathbf{V}_T \right\} &= 0,
\end{align*}
\]

which are respectively the equations of continuity, entropy, longitudinal and tangential momentum, and energy.

Let \( L \) be a length scale for the global propagation of the sound phenomenon and \( l \) a length scale for the width of the wave train. We introduce the following non-dimensionalization

\[
\begin{align*}
x_0 &= c_0^{-1} L \bar{x}_0, \quad x_1 = l \bar{x}_1, \quad \mathbf{K} = L^{-1} \mathbf{K}, \quad \mathbf{D} = L^{-1} \mathbf{D}, \\
\mathbf{H} &= \mathbf{1}_T + \sum_{n=1}^{\infty} \left( \frac{1}{L} \right)^n \bar{x}_1^n \mathbf{H}_n, \\
p &= p_0 + \rho_0 c_0^2 \hat{\rho}, \quad \rho = \rho_0 (1 + \hat{\rho}), \quad s = s_0 + c_0 \hat{s}, \\
\mathbf{V} &= c_0 (\hat{v}_1 \mathbf{n} + \hat{\mathbf{V}}_T), \quad \rho e = \rho_0 e_0 + \rho_0 c_0^2 \hat{\rho} e, \quad \rho h = \rho_0 h_0 + \rho_0 c_0^2 \hat{\rho} h,
\end{align*}
\]

we want to study the situation in which \( \varepsilon = \frac{l}{L} \) and the \( \hat{f} \) are small but without knowing \textit{a priori} the relative orders of magnitude of these various
The equations of motion have the following structure:

$$
\mathcal{L}(\hat{f}) + \Omega(\hat{f}) + \varepsilon \{ \mathcal{L}(\hat{f}) + \Omega(\hat{f}) \} + \sum_{n=1}^{\infty} \varepsilon^{n+1} \{ \mathcal{L}(\hat{f}) + \Omega(\hat{f}) \} = 0,
$$

where $\mathcal{L}(\hat{f})$ and $\Omega(\hat{f})$ designate, generically, collections of terms which are respectively linear or quadratic with respect to $\hat{f}$. If the $\hat{f}(\eta)$ tend towards non-zero finite limits when $\varepsilon$ and $\eta = \sup(|\hat{f}|)$ approach zero, then these limits satisfy $\mathcal{L}(\hat{f}) = 0$; this gives

$$
\begin{align*}
\hat{p} &= \hat{\rho} = \hat{v}_1 = \frac{\varepsilon^2}{h_0} \hat{p} \hat{e}, \\
\hat{s} &= 0, \quad \hat{V}_T = 0,
\end{align*}
$$

but we still do not know what the relative order of magnitude of $\varepsilon$ and $\hat{v}_1$ are, and we do not have an equation regulating, at this stage, the evolution of $\hat{v}_1$. It is necessary to push further and write

$$
\begin{align*}
\hat{p} &= \hat{v}_1 (1 + \hat{p}), \quad \hat{\rho} = \hat{v}_1 (1 + \hat{\rho}), \quad \hat{s} = \hat{v}_1 \hat{s}, \quad \hat{V}_T = \hat{v}_1 \hat{V}_T \\
\hat{\rho} \hat{h} &= \hat{p} \hat{e} + \hat{\rho} \hat{e} = \frac{h_0}{\varepsilon} \hat{v}_1 (1 + \hat{\rho} \hat{e}),
\end{align*}
$$

to take $\hat{v}_1$ and $\varepsilon$ as infinitesimally small to leading order and to suppose that the $\hat{f}$ approach zero in a way to be determined. Substituting in the equations of motion and rearranging, equations which have the following structure follow:

$$
\begin{align*}
\Omega(\hat{v}_1, \hat{f}) + \mathcal{C}(\hat{v}_1, \hat{f}) + \varepsilon \{ \mathcal{L}(\hat{v}_1) + \Omega(\hat{v}_1, \hat{f}) + \mathcal{C}(\hat{v}_1, \hat{f}) \} \\
+ \sum_{n=1}^{\infty} \varepsilon^{n+1} \{ \mathcal{L}(\hat{v}_1) + \Omega(\hat{v}_1, \hat{f}) + \mathcal{C}(\hat{v}_1, \hat{f}) \} = 0,
\end{align*}
$$

where $\mathcal{L}$, $\Omega$, $\mathcal{C}$ denote collections of linear, quadratic, and cubic terms with respect to the arguments. Moreover, in the terms independent of $\varepsilon$, we have $\Omega = \hat{v}_1 \mathcal{L}(\hat{v}_1, \hat{f})$ and $\mathcal{C} = \hat{v}_1^2 \mathcal{L}(\hat{v}_1, \hat{f})$, so that $\mathcal{C} \ll \Omega$ without prejudging the relative order of magnitude of $\hat{v}_1$ and the $\hat{f}$'s. To find the limit behavior when $\varepsilon$, $\hat{v}_1$ and the $\hat{f}$'s tend towards zero, we are therefore led to reduce equations (38) to the form

$$
\Omega(\hat{v}_1, \hat{f}) + \varepsilon \mathcal{L}(\hat{v}_1) = 0,
$$

(4) $\hat{f}$ is in fact a vector and $\sup \hat{f}$ is read as the supremum of its components.
which greatly simplifies the writing work, and we arrive at

\[
\begin{align*}
(a) & \quad 2\dot{v}_1 \frac{\partial \dot{v}_1}{\partial x_1} - \frac{\partial \rho_1}{\partial x_1} + \varepsilon \left\{ \frac{\partial \dot{v}_1}{\partial x_0} + \dot{v}_1 K \right\} = 0, \\
(b) & \quad \frac{\partial \rho_1}{\partial x_1} + \varepsilon \frac{\partial \dot{v}_1}{\partial x_0} = 0, \\
(c) & \quad \frac{\partial \dot{v}_1 \tilde{s}}{\partial x_1} - \varepsilon \frac{\partial \tilde{s}}{\partial x_0} = 0, \\
(d) & \quad \frac{\partial \dot{v}_1 \tilde{v}}{\partial x_1} - \varepsilon \tilde{D} \dot{v}_1 = 0, \\
(e) & \quad \left(2\frac{h_0}{c_0^2} + 1\right) \dot{v}_1 \frac{\partial \dot{v}_1}{\partial x_1} - \frac{h_0}{c_0^2} \frac{\partial \dot{v}_1 \tilde{\rho}}{\partial x_0} + \varepsilon \frac{h_0}{c_0^2} \left( \frac{\partial \dot{v}_1}{\partial x_0} + \dot{v}_1 K \right) = 0.
\end{align*}
\]

Moreover, if we make use of the equation of state for gases, we find

\[
\begin{align*}
\tilde{p} &= \Gamma_0 \dot{v}_1 + \tilde{\rho} + \frac{c_0 \rho_1}{\varepsilon \rho \nu c_0} \left( \frac{\partial \tilde{s}}{\partial x_0} \right)_0 \tilde{s} + \cdots, \\
\tilde{\rho e} &= \tilde{\rho} + \frac{1}{2} \frac{c_0^2}{\rho_0} \dot{v}_1 + \frac{c_0 T_0}{h_0} \tilde{s} + \cdots.
\end{align*}
\]

It should also be noted that, as we are in non-dissipative fluid and we are taking nonlinear effects into account, it is necessary to admit the presence of shocks. For such shocks to be compatible with the narrow-width wave train structure, they must be approximately parallel to the surface of the reference wave. Let \( x_1 = x_{1s}(x_0, a) \) be one of these shocks.†

By applying the Rankine-Hugoniot conditions to the work which we carried out on the equations of motion, we arrive at the following shock conditions:

\[
\begin{align*}
\left[ \dot{v}_1 \right] &= 0, \\
\left[ \dot{v}_1 \left( \Gamma_0 \dot{v}_1 - 2\varepsilon \frac{\partial \tilde{v}_1}{\partial x_0} \right) \right] = 0, \\
\left[ \dot{v}_1 (\tilde{V}_T + \varepsilon \tilde{D}) \right] = 0, \\
\left[ \dot{v}_1 \tilde{s} \right] &= 0,
\end{align*}
\]

with the usual notation \([f]\) to designate the discontinuity of \( f \) across a shock.

We can now see without difficulty that all of the quantities \( \dot{v}_1, \tilde{p}, \tilde{\rho}, \tilde{V}_T, \varepsilon \) are of the same order and that \( \tilde{s} \) is of a higher order. We leave it to the reader to note the following two points: (i) we may form a combination of the equations \([40]\) \( a, b, \) and \( c \) which, taking into account \( \tilde{s} = 0 \), contains only \( \dot{v}_1 \) and similarly we may form a combination of the shock conditions which contains only \( v_1 \) and \( x_{1s} \); (ii) equation \((40 e)\) is a consequence of all the other equations of \((40)\) and of \((41)\).

† Translator’s Note: Guiraud uses the subscript \( c \) to refer to the shock (‘choc’), so we have substituted the corresponding English subscript, \( s \).
We may state: suppose that, in a calm and homogeneous atmosphere, a train of nonlinear waves, but of small amplitude and small width, propagates in the vicinity of an acoustic wave obtained by making $x_1 = 0$ in the representation (11), with $l$ the length scale measuring the width of the wave train and $L$ the length scale of the overall propagation phenomenon, we first take $\varepsilon = \frac{l}{L}$ as infinitesimally small to leading order. If the wave train in question admits a limit structure as $\varepsilon \to 0$, it is given as follows: $n$ designating the unit vector normal to the wave, the perturbations $p - p_0$ in $\rho_0 c_0^2$, $\rho - \rho_0$ in $\frac{\rho}{\rho_0}$, $\nabla n$ are of order $\varepsilon$ while $c_0^{-1} V_T = \frac{V - (\nabla n)n}{c_0}$ is of order $\varepsilon^2$ and $s - s_0$ is of order $\varepsilon^3$. More precisely, designating generically by $R$ one of the sound rays which generate the acoustic wave and by $x_0$ the time variable which describes the propagation, the first order approximation for the structure of the wave train is defined by the formulas

$$
\begin{align*}
V &= v_1(x_0, x_1; R)n(x_0; R), \\
p &= p_0 + \rho_0 c_0^2 v_1, \\
\rho &= \rho_0 + \rho_0 c_0^{-1} v_1, \\
s &= s_0,
\end{align*}
$$

(43)

while $v_1$ satisfies the equation

$$\Gamma_0 v_1 \frac{\partial v_1}{\partial x_1} + \frac{\partial v_1}{\partial x_0} + \frac{1}{2} v_1 K = 0,$$

(44)

where $K$ is the mean curvature of the wave surface $x_1 = 0, x_0 = \text{Cst}$, counted positively if the convexity is in the direction of propagation; the function $v_1$ is discontinuous on the shock surfaces and, if one of them is defined by

$$x_1 = x_{1s}(x_0; R),$$

(45)

then the discontinuity of $v_1$ must satisfy the shock condition

$$\left[ v_1 \left( \Gamma_0 v_1 - 2 \frac{\partial x_{1s}}{\partial x_0} \right) \right] = 0.$$

(46)

With the first approximation thus obtained we may improve the description thanks to formulas of the type

$$
\begin{align*}
V &= v_1 n + v_1 \tilde{V}_T, \\
p &= p_0 + \rho_0 c_0^2 v_1 (1 + \tilde{p}), \\
\rho &= \rho_0 + \rho_0 c_0^{-1} v_1 (1 + \tilde{\rho}), \\
s &= s_0 + c_0 c_0^{-1} v_1 \tilde{s},
\end{align*}
$$

(47)
and, if we agree to designate by \( v_1 \) and \( v_1 V_T \) the true components of the speed we may then calculate the limiting values taken by \( \tilde{p}, \tilde{\rho}, \tilde{V}_T, \tilde{s} \), using only the first approximation obtained for \( v_1 \); for that it is necessary to use the differential system

\[
\begin{cases}
(a) & \frac{\partial v_1 \tilde{p}}{\partial x_1} + \frac{\partial v_1}{\partial x_0} = 0, \\
(b) & \frac{\partial v_1 \tilde{V}_T}{\partial x_1} - Dv_1 = 0, \\
(c) & \frac{\partial v_1 \tilde{s}}{\partial x_1} = 0, \quad \tilde{p} = \tilde{\rho} + (\Gamma_0 - 1)c_0^{-1}v_1,
\end{cases}
\]

and the shock conditions

\[
\begin{cases}
\left[v_1 \left( \tilde{\rho} + c_0^{-1} \frac{\partial v_1}{\partial x_0} \right) \right] = 0, \\
\left[v_1 (c_0 D x_1 s + \tilde{V}_T) \right] = 0, \\
\left[c_v T_0 v_1 \tilde{s} - \frac{\Gamma_0}{6} v_1^3 \right] = 0.
\end{cases}
\]

Supposing now that the atmosphere is inhomogeneous with wind of speed \( V_0 \), and that \( H \) is the inhomogeneity scale of the atmosphere, if \( \frac{H}{H} \ll 1 \), the preceding first approximation remains valid on the condition of modifying equation (44) as follows

\[
\Gamma_0 v_1 \frac{\partial v_1}{\partial x_1} + \frac{\partial v_1}{\partial x_0} + \frac{1}{2} v_1 \left( c_0 K + \frac{1}{\rho_0 c_0} \frac{\partial \rho_0 c_0}{\partial x_0} + (2\Gamma_0 - 1) \nabla \cdot V_0 + \frac{\partial V_0}{\partial x_1} \cdot n \right) = 0.
\]

Finally, if we wish to take viscosity effects into account, the shock condition must be removed and (50) must be replaced by

\[
\Gamma_0 v_1 \frac{\partial v_1}{\partial x_1} + \frac{\partial v_1}{\partial x_0} + \frac{1}{2} v_1 \left( c_0 K + \frac{1}{\rho_0 c_0} \frac{\partial \rho_0 c_0}{\partial x_0} + (2\Gamma_0 - 1) \nabla \cdot V_0 + \frac{\partial V_0}{\partial x_1} \cdot n \right)
= \frac{\mu_v}{2\rho_0} + \frac{2}{3} \mu + k_0 (g_{so})^2 \rho_0^{-1} c_0^{-2} T_0^{-1} \frac{\partial^2 v_1}{\partial x_1^2},
\]

where \( \mu \) and \( \mu_v \) are the usual viscosity coefficients and \( k \) is the coefficient of thermal conductivity.

20 Reduction to the canonical Burgers’ form. Ballistic noise. — Let us define a function \( \Sigma(x_0; R) \) by the condition

\[
\frac{1}{\Sigma} \frac{\partial \Sigma}{\partial x_0} = c_0 K + \frac{1}{\rho_0 c_0} \frac{\partial \rho_0 c_0}{\partial x_0} + (2\Gamma_0 - 1) \nabla \cdot V_0 + n \cdot (\nabla V_0) \cdot n,
\]
which specifies $\Sigma$ to within a numerical factor depending only on $R$, and perform the
following changes of variables and functions

\[
\begin{align*}
\mathcal{T} &= \int_{x_0^2(R)}^{x_0} \Gamma_0 \Sigma^{-\frac{1}{2}} \, dx_0, \\
\mathcal{X} &= x_1, \\
W &= \Sigma^{\frac{1}{2}} v_1,
\end{align*}
\]

we can easily see that (51) takes the form

\[
W \frac{\partial W}{\partial \mathcal{X}} + \frac{\partial W}{\partial \mathcal{T}} = \nu_e \frac{\partial^2 W}{\partial \mathcal{X}^2},
\]

with

\[
\nu_e = \frac{\mu v_0 + \frac{4}{3} \mu_0 + k_0 \rho_c^0 c_0^{-2} \rho_0^{-1} \Sigma_0^{-1}}{2 \Gamma_0 \rho_0} \Sigma^{\frac{1}{2}}.
\]

As was given by Lighthill [22] in 1956, without carrying out the detailed analysis
above, the structure of a quasi-plane, quasi-acoustic wave train is governed by a Burgers’
equation with a kinematic viscosity coefficient dependent on time. If we neglect $\nu_e$
it is necessary to tolerate shocks and, if one of them develops at $\mathcal{X} = \mathcal{X}_s(\mathcal{T}; R)$, the
corresponding shock condition is

\[
\left[ W \left( W - \frac{\partial \mathcal{X}_s}{\partial \mathcal{T}} \right) \right] = 0.
\]

We will not discuss here the Burgers’ equation which was the subject of the work of
Hopf [16], Cole [2], Lighthill [22], Hayes [15], and Germain [4], and will content ourselves
with an application of it to the problem of ballistic noise caused by supersonic flight.
We first verify that, if the atmosphere is homogeneous and windless, the asymptotic
behavior (43) is in agreement with the statement of paragraph 2.2.40, the wave train
being, this time, a train of Mach waves spread around the Mach wave associated with
a point on the aircraft. More precisely, select the characteristic sound rays, in the sense
of section 2.1, to define the parametric representation of paragraph 2.2.20 and replace
the variable $x_0$ by $T$, the travel time of the Mach wave along the sound ray from the
aircraft to the point considered such that, if the aircraft passes $x_0(\tau)$ at the instant $\tau$,
the instant considered is $t = \tau + T$. The component $v_1$ of the perturbation speed normal
to the Mach wave is, according to the acoustic approximation (4), (5),

\[
v_1 = V_a \mathcal{M}_a^3 (\mathcal{M}_a^2 - 1)^{-\frac{1}{2}} (2c_0 a T)^{-\frac{1}{2}} \left( 1 - \frac{T}{\tau} \right)^{-\frac{1}{2}} F(-\mathcal{M}_a x_1; \mathcal{R}),
\]
with the following notations: \( V_a \) is the speed of the aircraft at the instant \( \tau \) and \( M_a \) is the corresponding Mach number, \( F \) is the Whitham function, defined in (5), and finally \( T^* \) is the instant of focusing of the sound ray \( R \). The atmosphere being homogeneous, we may use

\[
\frac{1}{2} \Gamma_{\tau \tau} = (1 - M_a^2) \left( 1 - \frac{T}{T^*} \right). 
\]

If the atmosphere is not homogeneous we may either directly generalize the analysis of section 2.1, as was done in [8], or use the statement of paragraph 2.2.40. If we may find a length scale \( L \) satisfying the double condition \( l \ll L \ll H \), formula (4) will be valid with \( c_0 T = O(L) \). For larger values of \( T \) the effects of inhomogeneity must be taken into account, but this is simple as shown in the statement of paragraph 2.2.40. Specifically, (57) should be replaced by

\[
v_1 = V_a M_a^2 (M_a^2 - 1)^{-\frac{1}{2}} (2c_0 T)^{-\frac{1}{2}} B^{-1}(T; R) F(-M_a x_1; R),
\]

with a function \( B \) defined by

\[
B(T; R) = \left( \frac{\rho_0 c_0}{\rho_{0a} c_{0a}} \right)^{\frac{1}{2}} \times \exp \left\{ \frac{1}{2} \int_0^T \left( c_{01} K_1 - \frac{1}{T_1} + (2\Gamma_0 - 1) \nabla \cdot V_01 + n_1 \cdot (\nabla V_01) \cdot n_1 \right) dT_1 \right\}.
\]

Note that, if we suppose that the gas is perfect, we have \( 2\Gamma_0 - 1 = \gamma \). The index \( a \) specifies that it is a quantity evaluated at the position of the aircraft at the instant \( \tau \), while the index 1 specifies that it is at the instant-point running the length of the sound ray. In a homogeneous atmosphere, without wind, it is clear that \( \Sigma = T B^2 \).

We now come to the effects of nonlinearity which are described by the Burgers’ equation (54) with \( \nu_e = 0 \). If we can find a length scale \( L \) such that \( \frac{v_1}{c_0} L \ll l \ll L \), which is possible if the perturbations which have been made by the aircraft are small enough, then the effects of nonlinearity are unimportant for \( c_{0a} T = O(L) \) and the acoustic behavior (59) may be used as an initial condition for the Burgers’ equation; we thus have

\[
W(0, X) = V_a M_a^2 (M_a^2 - 1)^{-\frac{1}{2}} (2l_a)^{-\frac{1}{2}} F(-M_a X).
\]
We are concerned here only with the behavior of ballistic noise for large values of $T$, when the effects of nonlinearity have distorted the acoustic signal, given by $W(0, X)$, into an N wave. The way in which this wave is obtained is classical and we limit ourselves here to recalling the result. Let

$$I = \sup_{X} \int_{-\infty}^{X} W(0, X_1) \, dX_1 = \sup_{X} \left\{ -\int_{X}^{\infty} W(0, X_1) \, dX_1 \right\},$$

we find, for $T \to \infty$, that the asymptotic behavior of $W(T, X)$ is given by the following formula

$$W(T, X) \sim \sqrt{2} \sqrt{I} T X \sqrt{2} \sqrt{I} T \left\{ Y(X + \sqrt{2IT}) - Y(X - \sqrt{2IT}) \right\},$$

so that the asymptotic signal of $W$ is an N of width $2 \sqrt{2IT}$ and amplitude $\sqrt{2} \sqrt{I} T$.

We now have all the elements to determine how formula (59) must be modified to account for the effects of nonlinearity. Let us define factors of the form $K_{F}^{\text{vol}}(R)$ and $K_{F}^{\text{lift}}(R)$, respectively for the effects of volume and lift, by setting

$$K_{F}^{\text{vol}} = \frac{P_{a}}{\rho_{0a} V_{a}^{2} l_{a}^{2}} = \max_{\xi} \int_{-\infty}^{\xi} S_{e}^{\prime} e_{2}(\xi_{1}; R) \sqrt{\xi - \xi_{1}} \, d\xi_{1},$$

if $V_{a}$ denotes the total volume of the aircraft and $P_{a}$ denotes its total lift, $l_{a}$ being its length; we may then state the following rule: with the notations used in the statement of paragraph 2.3.10, the component $v_{1}$ of the speed, in the case where the perturbations are produced by the supersonic flight of an aircraft, is given by the formula

$$v_{1} = c_{0a} \Lambda x_{1} \left\{ Y(x_{1} + \Lambda) - Y(x_{1} - \Lambda) \right\},$$

with

$$\Lambda = (\gamma + 1) \frac{3}{2} \pi^{\frac{1}{2}} 2^{-\frac{1}{2}} \underbrace{\rho_{0}^{\frac{3}{2}} (\rho_{0}^{2} - 1)}_{\text{vol}} \left\{ K_{F}^{\text{vol}} V_{a} + K_{F}^{\text{lift}} \right\} \left( \int_{0}^{T} \frac{dT}{T^{2}} \right)^{\frac{1}{2}}$$

and

$$J = (\gamma + 1)^{\frac{1}{2}} \pi^{\frac{1}{2}} 2^{-\frac{1}{2}} \underbrace{\rho_{0}^{\frac{3}{2}} (\rho_{0}^{2} - 1)}_{\text{vol}} \left\{ K_{F}^{\text{vol}} V_{a} + K_{F}^{\text{lift}} \right\} \left( \int_{0}^{T} \frac{dT}{T^{2}} \right)^{\frac{1}{2}}.$$
The function \( B(T; R) \) must be obtained by the quadrature defined in (60), carried out along the characteristic sound ray. The relative pressure perturbation is \( \frac{\Delta p}{p_0} = \gamma \frac{v_1}{c_0} \); it is constituted by a signal in the form of an \( N \), with amplitude \( \gamma c_0 a c^{-1} \) and width \( 2\Lambda \). Let \( J_{W_h} \) and \( \Lambda_{W_h} \) be the values of \( J \) and \( \Lambda \) corresponding to the flight of the same aircraft, in a homogeneous atmosphere, without wind, undergoing rectilinear and uniform motion, corresponding to the same travel time, along a characteristic sound ray of the same initial direction; we account for the effects of inhomogeneity of the atmosphere, most simply, via the formulas

\[
\begin{align*}
J &= J_{W_h} \frac{T^{\frac{1}{2}}}{B(T; R)} \left( \int_0^T \frac{dT}{2T^{\frac{3}{2}}B} \right)^{-\frac{1}{2}}, \\
\Lambda &= \Lambda_{W_h} \left( T^{-\frac{1}{2}} \int_0^T \frac{dT}{2T^{\frac{3}{2}}B} \right)^{\frac{1}{2}}.
\end{align*}
\]

The classical theory of ballistic noise in an inhomogeneous atmosphere, as we have presented it, emerged gradually and followed the work of many authors. Whitham ([29], [30]), in two reports, which served as a starting point for further studies, studied the case of a stationary body of revolution in a homogeneous atmosphere, but the underlying developments of the technique of proximals and distals emerged much later. In the first report, Whitham [29] systematically studies the structure of a supersonic flow around a body of revolution far from the axis of symmetry and manages to identify a rule which seems to be an application of the general technique of Lighthill, universally known under the name “P. L. K. method” [23]. The rule in question is the following: the asymptotic behavior of the supersonic flow around a stationary body of revolution far from the axis, is a modified acoustic field which propagates along the Mach lines of the perturbed flow, while the unmodified acoustic field would propagate along the Mach lines of the unperturbed state. The comparison of equations (57) and (65) makes it possible to show that the extension of this rule to the most general case is as follows: the perturbation field due to a supersonic flight is, in the first approximation, perturbative far from the aircraft, and in the vicinity of the Mach wave, a geometric acoustic wave train, modified by propagation along privileged sound rays with the actual velocity of the acoustic waves in the perturbed fluid, while the unmodified acoustic field would travel along the same sound rays with the propagation speed of the acoustic waves in the unperturbed atmosphere. It should be noted that the speed in question is not equal to what is commonly called the speed of sound when there is wind. Rao [24], then Warren and Randall [28] have used
the rule thus stated in an isothermal atmosphere for the case of an accelerated flight, given without proof. Whitham [31], then Friedman et al. [3] treated the problem as a non-stationary one-dimensional flow problem in a tube of sound rays and arrived at an equation similar to (20), but Whitham neglected the convection of sound energy by the atmospheric wind through the walls of the tube, so that his expression for Σ is identified with ours only in a windless atmosphere. Friedman et al. [3] took this effect into account but arrived, for Σ, at an equation formally different from (52), when there is wind. They give, instead of (52), the following equation:

$$
\frac{2}{\Sigma} \frac{d\Sigma}{dT} + \frac{1}{A} \frac{dA}{dT} - \frac{1}{T} + \frac{1}{p_0} \frac{dp_0}{dT} - \frac{1}{c_0} \frac{dc_0}{dT} + \frac{2}{w + c_0} \frac{dw_0}{dT} - \frac{(\gamma - 1)w_0}{w_0 + c_0} \frac{d\rho_0}{dT} = 0,
$$

(68)

where $w_0$ is the component of the wind velocity normal to the Mach wave and $A$ is a function of $T$ and $\mathcal{R}$ which is proportional, on each $\mathcal{R}$, to the area of the section of a tube of sound rays – infinitely thin – cut out by the Mach wave, unless it is the area of the cross-section because they are not, on this point, as explicit as one might wish and so we have interpreted in view of the equations. In a windless atmosphere, the sound rays are normal to the Mach surfaces and we do not meet any difficulty, both in the interpretation of (68) and in the effective verification of the identity of (52) and (68), but the question remains open in the case of the atmosphere with wind and we believe that these equations are effectively irreducible to one another. Let us continue this review of previous works by noting that Guiraud [8] studied the behavior of a representation of the acoustic solution in an inhomogeneous atmosphere in proximal variables and obtained a result identical to (57) in the case of a windless atmosphere, only technical difficulties having for the moment prevented us from providing an actual proof of identity between the two formulations, when there is wind. In the same report we also find a justification of Whitham’s rule, by a process quite different from that mentioned here and more difficult to implement. Note that Lighthill [22] was written in 1956, but without proof and without specifying the form of Σ in the most general case, equation (54). We should also mention here the works of Hayes ([14], [15]) and also those of Kristianovich [20] and Goubkin [6], who also used the writing of equations in characteristic variables, and Rijov ([25], [26]) who considered the effects of inhomogeneity of the atmosphere. Finally, the classic treatise by Landau and Lifschitz [21] devotes a few pages to this question.
3 Ballistic noise and focusing.

3.1 DISTAL APPROACH BEHAVIOR.

10 Geometric and acoustic preliminaries. — Let \( \Gamma(t,x;\tau) \) be the solution of the equation

\[
\left( \frac{\partial \Gamma}{\partial t} + V_0 \cdot \nabla \Gamma \right)^2 - c_0^2 |\nabla \Gamma|^2 = 4\Gamma,
\]

which admits, in the vicinity of the aircraft, the structure

\[
\Gamma \equiv (t - \tau)^2 - c_0^{-2} |x - x_a(\tau) - V_0a(t - \tau)|^2,
\]

the relation \( \Gamma = 0 \) describes a wave centered at the instant-point \((\tau, x_a(\tau))\). According to paragraph 2.2.30 there is therefore an acoustic field including a singular part \( V_1' = c_0 n \delta(\Gamma) \) and the function \( \delta(\Gamma) = |\nabla \Gamma|^{-1} c_0 \Phi \) satisfies equation (20) because we have \( \delta(\Gamma) = |\nabla \Gamma|^{-1} \delta(x_1) \). The Mach wave passing through \( t, x \) is clearly obtained by elimination of \( \tau \) between \( \Gamma = 0 \) and \( \Gamma_{\tau \tau} = 0 \). Let \( A_{0C} \) and \( A_{0M} \) then be, respectively, the values of the function \( A \) for the central wave and the Mach wave having a contact along the spatio-temporal bicharacteristic associated with the characteristic sound ray \( R \). It is clear, according to geometric acoustics, that along the sound ray we have,

\[
\frac{c_0 \Phi}{2 |\nabla \Gamma|} = \frac{c_0^3}{B \sqrt{c_0a \Gamma}} \sqrt{\frac{A_{0M}}{A_{0C}}},
\]

if we specify \( \Phi \) and \( A \) by the condition that \( \Phi \to 1, c_0^{-2} T^{-2} A_{0C} \to 1 \) and \( c_0^{-1} T^{-1} A_{0M} \to 1 \) when \((t, x) \to (\tau, x_a)\).

In a homogeneous atmosphere, without wind, \( \Phi \equiv 1 \) and it is to be expected that, if we wish to know how the acoustic behavior [eq. (57)] is modified by the effects of inhomogeneity, it suffices to multiply the result by \( \Phi \) and to use, for \( \Gamma \), the function which has just been defined. We exploit this remark in seeking to specify \( \Gamma_{\tau \tau} \) and \( \Gamma_{\xi} \). The theory of first order partial differential equations and the calculus of variations allow us to show that the function of the two instant-points \( \Sigma \) and \( \mathfrak{M} \) defined by

\[
\left\{ \begin{array}{l}
\Gamma(\Sigma; \mathfrak{M}) = \text{sgn} \int_{\Sigma}^{\mathfrak{M}} |\mathcal{I}(dt, dx)|^{\frac{1}{2}} \ , \\
\mathcal{I}(u^0, u) = (u^0)^2 - c_0^{-2} |u - u^0 V_0|^2 \ ,
\end{array} \right.
\]

satisfies (69) relative to both \( \Sigma \) and \( \mathfrak{M} \) if the integral is evaluated along a path which renders it extremal. To obtain \( \Gamma(t, x; \tau) \) it

\[\text{[6]} \text{The gradient is evaluated with respect to the coordinates } x.\]

302
is sufficient to take for \( L \) the instant-point \((\tau, x_a(\tau))\). The search for the extremals in question is classical, we define \( q^0 \) and \( q \) by

\[
(73) \quad u^0 = q^0 - V_0 \cdot q, \quad u = (q^0 - V_0 \cdot q)V_0 + c_0^2 q,
\]

and the function \( \mathcal{H}(q^0, q) \) by

\[
(74) \quad \mathcal{T}(u^0, u) \equiv \mathcal{H}(q^0, q) \equiv (q^0 - V_0 \cdot q)^2 - c_0^2 |q|^2;
\]

the differential equations of the extremals are, with a suitable choice of parameter \( s \),

\[
(75) \quad \begin{cases}
\frac{dt}{ds} = u^0 = q^0 - V_0 \cdot q; \\
\frac{dx}{ds} = u = (q^0 - V_0 \cdot q)V_0 + c_0^2 q; \\
\frac{dq^0}{ds} = (q^0 - V_0 \cdot q) \frac{\partial V_0}{\partial t} \cdot q + |q|^2 c_0 \frac{\partial c_0}{\partial t}; \\
\frac{dq}{ds} = -(q^0 - V_0 \cdot q)(\nabla V_0) \cdot q - |q|^2 c_0 \nabla c_0
\end{cases}
\]

and, with this parametrization, \( \mathcal{T}\left(\frac{dt}{ds}, \frac{dx}{ds}\right) = \mathcal{H}(q^0, q) \) is constant along the extremal.

The bicharacteristics of the acoustic equations, in the sense of Hadamard, are the extremals for which \( \mathcal{H} = 0 \); In particular, the sound rays are the spatial projections of the bicharacteristics. To evaluate the differential of the function \( \Gamma \) we use the principle of stationarity by choosing a parametrization for which \( s_m - s_L = 1 \) when \( \mathcal{M} \) and \( \mathcal{L} \) move, and it follows that

\[
(76) \quad d\Gamma = (q^0 dt - q \cdot dx)|_m - (q^0 dt - q \cdot dx)|_L,
\]

\( q^0 \) and \( q \) satisfy the system (75) with \( s_L = 0 \) and \( s_m = 1 \). Along the wave \( \Gamma = 0 \) we have \( q^0 = c_0|q| + V_0 \cdot q \) and, for the function \( \Gamma(t, x; \tau, \zeta) \), along this wave it follows that

\[
(77) \quad \Gamma_\xi = |q_a|n_a \cdot e_1, \quad \Gamma_\tau = |q_a|(V_{ar} \cdot n_a - c_{0a}),
\]

if \( V_{ar} = V_a - V_{0a} \) is the velocity vector of the aircraft relative to the local wind. On the Mach wave, at \( \Gamma = \Gamma_\tau = 0 \), we have

\[
(78) \quad \Gamma_{\tau\tau} = c_{0a}|q_a| \frac{d}{d\tau} \left( \frac{V_{ar} \cdot n_a}{c_{0a}} \right),
\]

in terms of a derivative in \( \tau \), the instant-point \( t, x \) being fixed. Furthermore, equations (75) and (21) show that

\[
(79) \quad c_{0a}|q_a| = c_0|q| \exp \left\{ \int_0^T \left( \frac{\gamma - 1}{2} \nabla \cdot V_0 + n \cdot (\nabla V_0) \cdot n \right) \, d\tau \right\},
\]
and this makes it possible to obtain the relation

\[ \Phi B \frac{\sqrt{\nabla \Gamma}}{2} = \sqrt{\frac{d}{d\tau} \left( \frac{V_{a_r} \cdot n_a}{c_{0a}} \right) - (\nabla^2 - 1)c_{0a}^2 T} \exp \left\{ -\int_0^T \left( \frac{\gamma - 1}{2} \nabla \cdot V_0 + n \cdot (\nabla V_0) \cdot n \right) d\tau \right\}, \]

demonstrating that the solution (59) is identical to that obtained by substituting the values presently found for \( \Gamma_\xi \) and \( \Gamma_\tau \) in (4) and multiplying the result by \( \Phi \). Now, we remark that, according to geometric acoustics, we have opted to use “strength” to refer to pressure jumps across shocks.

\[ \frac{\Phi \sqrt{A_{0c}}}{2 \sqrt{\nabla \Gamma}} = c_{0a}^2 \sqrt{\frac{p_0 a c_{0a}}{p_0 c_0}} \exp \left\{ -\int_0^T \left( \frac{\gamma - 1}{2} \nabla \cdot V_0 + n \cdot (\nabla V_0) \cdot n \right) d\tau \right\}, \]

and we finally obtain

\[ B = \sqrt{\frac{p_0 c_0}{p_0 a c_{0a}}} \left\{ \frac{A_{0c} \frac{d}{d\tau} \left( \frac{V_{a_r} \cdot n_a}{c_{0a}} \right)}{(\nabla^2 - 1)c_{0a}^2 T} \right\}^{\frac{1}{2}} \times \exp \left\{ \int_0^T (\gamma - 1) \nabla \cdot V_0 + 2n \cdot (\nabla V_0) \cdot n \right\} \]

We now suppose that the instant-point \( t, \mathbf{x} \) moves along the bicharacteristic associated with the characteristic sound ray \( \mathcal{R} \) considered in the study of ballistic noise, so that \( \Gamma_\tau \) approaches zero and \( B^{-1} \) increases indefinitely. We can specify the nature of this singularity in the ballistic noise strength\(^\dagger\) when approaching the focus. For this purpose we must know how to evaluate the derivative which appears in the expression of \( B \), and this calculation results from the following considerations: consider the characteristic sound ray \( (\mathbf{x}_a, \mathbf{x}^*) \) as well as the neighboring sound ray \( (\mathbf{x}_a', \mathbf{x}^*) \), since \( \mathbf{x}^* \) is the focal point of ray \( (\mathbf{x}_a, \mathbf{x}^*) \); letting \( \delta \mathbf{n}_a' \) be the variation of the vector \( \mathbf{n}_a \) in moving from \((\mathbf{x}_a', \mathbf{x}^*)\) to \((\mathbf{x}_a, \mathbf{x})\) and letting \( \delta \tau \) be the time interval spanning the passage of the aircraft from \( \mathbf{x}_a \) to \( \mathbf{x}_a' \) respectively, we have

\[ \frac{d}{d\tau} \left( \frac{V_{a_r} \cdot n_a}{c_{0a}} \right) = \lim_{\delta \tau \to 0} \frac{V_{a_r}' \cdot \delta \mathbf{n}_a'}{c_{0a} \delta \tau} \]

as shown in figure 6. To explain the second expression it is necessary to know how to perform variations on elements related to sound rays. Let us therefore consider a generic sound ray joining the points \( \mathbf{x}_1, \mathbf{x}_2 \), traversed by a wave between the instants \( t_1 \) and \( t_2 = t_1 + T \); let \( \Theta_1 \) and \( \Theta_2 \) be the unit vectors tangent to this ray at \( \mathbf{x}_1 \) and \( \mathbf{x}_2 \), oriented in the direction of travel from one point to the other; \( \mathbf{x}_1 \) being fixed we have

\[ d\mathbf{x}_2 = \sqrt{c_{01}c_{02}} T F_{21} \cdot d\Theta_1; \quad d\Theta_1 = \frac{1}{\sqrt{c_{01}c_{02}}} T G_{12} \cdot d\mathbf{x}_2, \]

\(^\dagger\) Translator’s Note: A direct translation would have used “intensity.” To avoid confusion with the acoustic intensity, we have opted to use “strength” to refer to pressure jumps across shocks.
for a displacement of $x_2$ in the plane tangent to the wave; analogously, $x_2$ being fixed, we have

$$d x_1 = \sqrt{c_{01} c_{02}} T \mathbf{F}_{12} \cdot d \Theta_2 ; \quad d \Theta_2 = \frac{1}{\sqrt{c_{01} c_{02}}} T \mathbf{G}_{21} \cdot d x_1 .$$

(85)

Placing $x_1$ at $x_a$ and $x_2$ at $x^*$, and denoting $\mathbf{F}_{21} = \mathbf{F}^*$, $\mathbf{F}_{12} = \mathbf{F}_a$ with an analogous notation for $\mathbf{G}^* = \mathbf{G}_{21}$, $\mathbf{G}_a = \mathbf{G}_{12}$, $\Theta^* = \Theta_2$, $\Theta_a = \Theta_1$, we find that the displacement from $x_a$ to $x_a'$ corresponds to $d x_1 = (V_{a_r} - c_{0a} n_a) \delta \tau$, so that the displacement from $x$ to $x'$ can easily be obtained and, after it, the value of $\delta n_a'$. With all calculations done, we have

$$\begin{cases}
\lim_{T \to T^*} B(T; R) (1 - \frac{T}{T^*})^{-\frac{1}{2}} = B^*(R), \\
B^*(R) = \sqrt{\frac{\rho_0^* c_{0a}^2 A_{0a}^* D_{0a}^2}{\rho_0 c_{0a} c_{0a}^2}} \exp \left\{ \int_{0}^{T} [(\gamma - 1) \nabla \cdot V_0 + 2n \cdot (\nabla V_0) \cdot n] dT \right\},
\end{cases}$$

(86)

with

$$\begin{align*}
W_0^* \cdot n^* W_{0a} \cdot n_a (M_{0a}^2 - 1) \mathcal{D}^* &= |W_{0a}| V_{a_r} \cdot (B_a W_{0a} - W_{0a} B_a) \cdot n_a, \\
B_a &= G_a \cdot (A^* W_0^* - W_0^* A^*) \cdot n^*, \\
A^* &= G^* \cdot (V_{a_r} - c_{0a} n_a), \\
W_0 &= c_0 n + V_0.
\end{align*}$$

(87)

We may verify that $\mathcal{D}^*$ is equal to unity in an isothermal atmosphere, without wind. Moreover, if $\mathbf{u}_1$ and $\mathbf{u}_2$ are two unit vectors forming an orthonormal reference frame with $\Theta_a$, we have

$$\frac{A_{0c}^*}{c_{0a}^2 T^{*2}} = \frac{c_0^*}{c_{0a}} (\mathbf{F}^* \cdot \mathbf{u}_1) \wedge (\mathbf{F}^* \cdot \mathbf{u}_2).$$

(88)

At the end of this study we may state: approaching the focus, the usual theory of ballistic noise predicts a pressure signal in the form of an $N$ wave whose amplitude behaves as follows

$$\left( \frac{\Delta p}{\rho_0^* c_{0}^{*2}} \right)_{\text{Max}} \simeq \varepsilon \left( 1 - \frac{T}{T^*} \right)^{-\frac{1}{2}},$$

(89)
while its width approaches a finite limit

\begin{equation}
(90) \quad \varepsilon c_0^* T^* \Lambda = \Lambda,
\end{equation}

and we have

\begin{equation}
(91) \quad \begin{cases}
\varepsilon = 2^{-\frac{1}{2}} - \frac{1}{2} (\gamma + 1)^{-\frac{1}{2}} M_a^2 (\gamma - 1)^{-\frac{1}{2}} \left( K_F^{\text{vol}}(\mathcal{R}) \mathcal{V}_a + K_F^{\text{lift}}(\mathcal{R}) \frac{P_a l_a}{\rho_0 V_a^2} \right)^{1/2} \\
\varepsilon c_0^*(l_a c_0) T^* - \frac{1}{2} \int_0^{T^*} \frac{dT}{B^*(\mathcal{R})} \left( \frac{1}{2} B(T; \mathcal{R}) \right)^{1/2} \int_0^{T^*} \frac{dT}{2 T^* B(T; \mathcal{R})}.
\end{cases}
\end{equation}

We give the name of caustic to the two-dimensional variety in space-time which is defined by

\begin{equation}
(92) \quad \Gamma = \Gamma_\tau = \Gamma_{\tau_\tau} = 0.
\end{equation}

It follows from this definition that the caustic enjoys several characteristic geometric properties. As we have already done, we will designate the corresponding instant-points with an asterisk. If \((t^*, x^*)\) is on the caustic there exists an instant-point \((\tau, x_a(\tau))\) such that the sound ray joining \(x_a\) to \(x^*\) is a stationary characteristic with respect to \(\tau\). This property makes it possible to solve the following problem: given a characteristic sound ray \(\mathcal{R}\), coming from \(x_a(\tau)\), find the acceleration to impart to the aircraft at the instant \(\tau\) such that one of the points of \(\mathcal{R}\) is on the corresponding caustic. That is to say, the sound ray in question focuses at this point. Another useful property is the following: the caustic is the envelope of the family of conoids of characteristic sound rays having their apexes along the trajectory, it being understood that the associated instant is that of the passage of the aircraft \((7)\). The line of contact of a given conoid is also the line of intersection with the conoid associated with an infinitesimally close instant. Finally the Mach conoid has a cuspidal edge on the caustic entirely along a line which, in general, does not coincide with the contact curve of the conoid of characteristic sound rays. It should be noted that the tangent plane of the cusp is not normal to the caustic if there is wind.

20 Equations of motion in the vicinity of the caustic.

--- a. Generalities. --- We will use the equations of section 2.2 as a point of departure, but we will expressly specify the definition of the sound ray \(\mathcal{R}\) in relation to the caustic.

\footnote{This is important if the atmosphere is time dependent.}
Let
\[ t^\star = t^\star(x_2, x_3), \quad x^\star = x^\star(x_2, x_3), \]
be the instant-point of contact, the coordinates \( x_2 \) and \( x_3 \) chosen such that the lines \( x_2 = \text{Cst} \) coincide with the contact curves of the conoids of characteristic sound rays and the curves \( x_3 = \text{Cst} \) are tangents to the directions of contact of the sound rays. Under these conditions, it is possible to take \( x_2 = \tau \) and the unit vector \( \Theta_a = \Theta_a(\tau, x_3) \) is a function of \( x_2 \) and \( x_3 \) for the characteristic sound ray. To complete the representation, we use \( x_1 \) and \( \sigma = t^\star - \tau - T \). The Mach wave is therefore defined by \( x_1 = 0 \) and, if its parametric representation is
\[ t = t^\star - \sigma, \quad x = x_\text{er}(x_2, x_3, \sigma), \]
the representation of the wave train is as indicated in figure 7 and we have
\[ h_2 e_2 \, dx_2 + h_3 e_3 \, dx_3 - W_0 \, d\sigma = dx_\text{er}, \]
where \( e_2 \) and \( e_3 \) designate unit vectors. We establish the relation
\[ \begin{cases} h_2 |e_2 \wedge e_3| = \frac{|W_{0a}| \sqrt{\frac{\gamma m^2_{a\tau}}{c_0^2 c_{0a}^2} - 1} |u \wedge G_a \cdot e_3|}{\sqrt{c_0 e_0^* T^*}} A_{0\text{er}}, \\ V_{a_\tau} - c_0 a n_a = c_0 a \sqrt{\frac{\gamma m^2_{a\tau}}{c_0^2} - 1} u, \quad |u| = 1, \end{cases} \]
and, when \( \sigma \) approaches zero we have, provided that \( V_0^* = 0 \) at the instant-point \( t^\star, x^\star \), which we may always suppose to be realized via a local Galilean transformation,
\[ \begin{cases} K^* = \frac{|W_{0a}|}{c_0 a} \sqrt{\frac{c_0 a}{c_0^2} \sqrt{\frac{\gamma m^2_{a\tau}}{c_0^2 c_{0a}^2} - 1} |u \wedge G_a \cdot e_3| \frac{A_0}{c_0^2 T^*}} c_{0a}^2 \xi \times \exp \left\{ \int_{0}^{T^*} \left[ (\gamma - 1) \nabla \cdot V_0 + 2 n \cdot (\nabla V_0) \cdot n \right] \, dT \right\}. \end{cases} \]
If the atmosphere is isothermal and windless, we can give a very simple geometric interpretation of \( K^* \):
\[ K^* = |W_0^*| \frac{\partial t^*}{\partial x_2} T^* K^* = |W_0^*| \left( 1 + \frac{\partial T^*}{\partial \tau} \right) T^* K^*, \]
where $K^*$ is the curvature of the normal section of the caustic tangent to the sound ray; naturally, in this case it is necessary to take $|W_0^*| = c_0^*$, although the preceding formula is general, only the interpretation of $K^*$ having to change. In the general case, consider two plane curves traced out in the previous plane which are one, the section of the caustic, and the other, the projection of the sound ray, then $K^*$ is the difference of the curvatures of these two curves, but it is necessary to specify that the caustic and the sound ray have been traced out in the Galilean reference frame for which $V_0^* = 0$ at the instant-point considered.

In the vicinity of $\sigma = 0$, but under the condition $x_1 \ll c_0^* \sigma$, the formulas for the change of coordinates are

$$
\begin{align*}
\frac{\partial}{\partial x_0} &= -\frac{\partial}{\partial \sigma}, & \frac{\partial}{\partial x_1} &= \frac{\partial}{\partial x_1}, \\
& & \frac{\partial}{\partial x_2} &= \left(1 + \frac{\partial T^*}{\partial \tau}\right) \frac{\partial}{\partial \sigma}, \\
& & \frac{\partial}{\partial x_3} &= \frac{\partial t^*}{\partial x_3} \frac{\partial}{\partial \sigma},
\end{align*}
$$

provided we calculate $h_2 e_2$ and $h_3 e_3$ at $\sigma - c_0^{-1} x_1$, and, keeping only the first term, we get the result

$$
\begin{align*}
H \cdot D &\cong \frac{T^* N^*}{K^*(c_0^* \sigma - x_1)} \left\{ \frac{\partial}{\partial x_2} + \left(1 + \frac{\partial T^*}{\partial \tau}\right) \frac{\partial}{\partial \sigma} \right\},
\end{align*}
$$

if $N^*$ designates the unit vector normal to the caustic (drawn in the Galilean reference frame used above) which points to the region swept by the sound rays. Finally, the formulas for the change of coordinates
\((t, x) \rightarrow (\sigma, x_1, x_2, x_3)\) are

\[
\left\{ \frac{\partial}{\partial t} \approx - \frac{\partial}{\partial \sigma} - c^*_0 \frac{\partial}{\partial x_1}, \right.
\]

\[
\nabla \approx n^* \frac{\partial}{\partial x_1} + \frac{T^*N^*}{K^*(c^*_0 \sigma - x_1)} \left\{ \frac{\partial}{\partial x_2} + \left( 1 + \frac{\partial T^*}{\partial \tau} \right) \frac{\partial}{\partial \sigma} \right\},
\]

to limit ourselves to the approximation which will be subsequently useful.

Rather than using (101) from now on to transcribe the equations of motion in the coordinate system used, it is preferable to first treat the case of a homogeneous atmosphere without wind because it is then easier to control the nature of the approximations made.

**b. Homogeneous atmosphere.** — We adopt here the representation

\[
t = t^* - \sigma, \quad x = x^* - (c^*_0 \sigma - x_1)n^*,
\]

with the orthogonal curvilinear coordinates \(x_2\) and \(x_4\) on the caustic and we set

\[
\left\{ \begin{array}{l}
dx^* = c^*_0 T^* x^* d\bar{x}_2 + c^*_0 T^* x^*_4 d\bar{x}_4, \quad E^*_2 = -n^*, \\
E^*_4 = N^* \land E^*_2,
\end{array} \right.
\]

\[
\begin{aligned}
c^*_0 T^* dN^* &= dx^* \cdot \bar{K}^*, \\
\bar{K}^* &= \sum_i \sum_j K^*_{ij} E^*_i E^*_j,
\end{aligned}
\]

\[
x_1 = \varepsilon c^*_0 T^* \bar{x}_1, \quad x_2 = c^*_0 T^* \bar{x}_2, \quad x_4 = c^*_0 T^* \bar{x}_4, \quad \sigma = T^* \bar{x},
\]

\[
\begin{aligned}
p &= p^*_0 + \varepsilon \rho^*_0 c^* \bar{p}, \\
\rho &= \rho^*_0 + \varepsilon \rho^*_0 \bar{\rho}, \\
V &= c^*_0 \{ \varepsilon \bar{m} n^* + \varepsilon^2 \bar{m} N^* + \varepsilon^2 \bar{m} E^*_4 \},
\end{aligned}
\]

\(T^*\) being a reference time assumed constant.

By using the change of variables formulas

\[
\left\{ \begin{array}{l}
T^* \frac{\partial}{\partial t} = - \frac{\partial}{\partial \sigma} - \frac{1}{\varepsilon} \frac{\partial}{\partial x_1}, \\
c^*_0 T^* \nabla = \frac{n^*}{\varepsilon} \frac{\partial}{\partial x_1}
\end{array} \right.
\]

\[
+ \frac{(\sigma - \varepsilon \bar{x}_1) \bar{H}^*_2 \bar{K}^*_2}{(\sigma - \varepsilon \bar{x}_1) \bar{H}^*_2 \bar{H}^*_4 \bar{K}^*_2 + (\sigma - \varepsilon \bar{x}_1)^2 \left( \bar{K}^*_2 \frac{\partial \bar{H}^*_2}{\partial \bar{x}_2} + \bar{K}^*_2 \frac{\partial \bar{H}^*_2}{\partial \bar{x}_4} \right) - \bar{H}^*_4}{(\sigma - \varepsilon \bar{x}_1) \bar{H}^*_2 \bar{H}^*_4 \bar{K}^*_2 + (\sigma - \varepsilon \bar{x}_1)^2 \left( \bar{K}^*_2 \frac{\partial \bar{H}^*_2}{\partial \bar{x}_2} + \bar{K}^*_2 \frac{\partial \bar{H}^*_2}{\partial \bar{x}_4} \right)} N^*,
\]

309
it is possible to form the equations of motion that we confine ourselves to writing at a
certain order of approximation in $\varepsilon$, sufficient for the following:

\[
\begin{align*}
(a) \quad & \frac{\partial(\bar{u} - \bar{p})}{\partial \bar{x}_1} + \varepsilon \left( \frac{\partial \bar{u}}{\partial \bar{\sigma}} - \bar{u} \frac{\partial \bar{u}}{\partial \bar{x}_1} + \bar{p} \frac{\partial \bar{u}}{\partial \bar{x}_1} \right) + O(\varepsilon^2) = 0, \\
(b) \quad & \frac{\partial(\bar{v} - \bar{p})}{\partial \bar{x}_1} + \varepsilon \left( \frac{\partial \bar{v}}{\partial \bar{\sigma}} - \bar{v} \frac{\partial \bar{v}}{\partial \bar{x}_1} + \gamma \bar{p} \frac{\partial \bar{v}}{\partial \bar{x}_1} - \bar{p} \frac{\partial \bar{v}}{\partial \bar{x}_1} \right) + O(\varepsilon^2) = 0, \\
(c) \quad & \bar{u} \frac{\partial \bar{p}}{\partial \bar{x}_1} + \bar{u} \frac{\partial \bar{p}}{\partial \bar{x}_1} + \gamma \bar{p} \frac{\partial \bar{p}}{\partial \bar{x}_1} - \frac{\partial(\bar{v} + \bar{v})}{\partial \bar{\sigma}} - \bar{p} \frac{\partial \bar{p}}{\partial \bar{x}_1} - \frac{\bar{\sigma} - 2\bar{\sigma}}{\bar{\sigma} - \bar{\sigma}} \\
& \quad \varepsilon \left( \frac{\bar{H}_2^* \frac{\partial \bar{w}}{\partial \bar{\sigma}}}{\bar{\Delta}_0} + \cdots \right) + O(\varepsilon^2) = 0, \\
(d) \quad & \frac{\partial \bar{w}}{\partial \bar{x}_1} - \left\{ \frac{\bar{H}_2^* \bar{K}_{24}^*}{\bar{\Delta}_0} \left( \frac{\partial \bar{p}}{\partial \bar{\sigma}} - \frac{1}{\bar{H}_2^*} \frac{\partial \bar{p}}{\partial \bar{x}_1} \right) \right. \\
& \quad + \left. \frac{\bar{H}_2^* \bar{K}_{22}^* \frac{\partial \bar{p}}{\partial \bar{x}_4}}{\bar{\Delta}_0} \right\} + O(\varepsilon) = 0, \\
(e) \quad & \frac{\partial \bar{w}}{\partial \bar{x}_1} - \left\{ \frac{\bar{H}_2^* \bar{K}_{24}^*}{\bar{\Delta}_0} \left( \frac{\partial \bar{p}}{\partial \bar{\sigma}} - \frac{1}{\bar{H}_2^*} \frac{\partial \bar{p}}{\partial \bar{x}_1} \right) \right. \\
& \quad - \left. \frac{\bar{\sigma} \frac{\partial \bar{H}_2^*}{\partial \bar{\sigma}} \frac{\partial \bar{p}}{\partial \bar{x}_4}}{\bar{\Delta}_0} \frac{\partial \bar{H}_2^*}{\partial \bar{x}_4} \right\} + O(\varepsilon) = 0,
\end{align*}
\]

where we have set

\[
\begin{align*}
\bar{\Delta}_0 &= \bar{H}_2^* \bar{K}_{22}^* \left( 1 - \frac{\bar{\sigma}}{\lambda} \right), \\
- \frac{1}{\lambda} &= \frac{1}{\bar{H}_2^*} \frac{\partial \bar{H}_2^*}{\partial \bar{x}_2} + \frac{\bar{K}_{24}^*}{\bar{K}_{22}^*} \frac{1}{\bar{H}_2^*} \frac{\partial \bar{H}_2^*}{\partial \bar{x}_4}.
\end{align*}
\]

If, when $\varepsilon$ approaches zero, the functions $\bar{u}, \bar{v}, \bar{w}, \bar{p}, \bar{\sigma}$ tend towards the limits $\bar{u}_0, \bar{v}_0, \bar{w}_0, \bar{p}_0, \bar{\sigma}_0$, these limits are given by the following highly decoupled system

\[
\begin{align*}
(a) \quad & \bar{u}_0 = \bar{p}_0 = \bar{p}_0, \\
(b) \quad & 2 \frac{\partial \bar{p}_0}{\partial \bar{\sigma}} + \frac{\bar{u}_0(1 - \frac{2\bar{\sigma}}{\lambda})}{\bar{\sigma}(1 - \frac{\bar{\sigma}}{\lambda})} - (\gamma + 1) \bar{u}_0 \frac{\partial \bar{u}_0}{\partial \bar{x}_1} = 0, \\
(c) \quad & \frac{\partial \bar{v}_0}{\partial \bar{x}_1} - \left\{ \frac{\bar{H}_2^* \bar{K}_{24}^*}{\bar{\Delta}_0} \left( \frac{\partial \bar{p}_0}{\partial \bar{\sigma}} - \frac{1}{\bar{H}_2^*} \frac{\partial \bar{p}_0}{\partial \bar{x}_1} \right) + \frac{\bar{H}_2^* \bar{K}_{22}^* \frac{\partial \bar{p}_0}{\partial \bar{x}_4}}{\bar{\Delta}_0} \frac{\partial \bar{p}_0}{\partial \bar{x}_4} \right\} = 0, \\
(d) \quad & \frac{\partial \bar{w}_0}{\partial \bar{x}_1} - \left\{ \frac{\bar{H}_2^* \bar{K}_{24}^*}{\bar{\Delta}_0} \left( \frac{\partial \bar{p}_0}{\partial \bar{\sigma}} - \frac{1}{\bar{H}_2^*} \frac{\partial \bar{p}_0}{\partial \bar{x}_1} \right) - \frac{\bar{\sigma} \frac{\partial \bar{H}_2^*}{\partial \bar{\sigma}} \frac{\partial \bar{p}_0}{\partial \bar{x}_4}}{\bar{\Delta}_0} \frac{\partial \bar{H}_2^*}{\partial \bar{x}_4} \right\} = 0.
\end{align*}
\]
We may check that the equation (107b) is, except for the notations, identical to equation (44) and that the equations (107c, d) are equivalent to the vector equation (48b). In other words, the current analysis is in perfect conformity with that of the previous chapter and it has the advantage of specifying the connection with the caustic. In the vicinity of $\sigma = 0$ the solution of the system (107) is singular:

\[
\begin{align*}
\bar{u}_0 &\sim (\bar{\sigma})^{-\frac{1}{2}} \frac{\bar{x}_1}{\Lambda} \{ y(\bar{x}_1 + \Lambda) - y(\bar{x}_1 - \Lambda) \}, \\
\bar{v}_0 &\sim \mathcal{O}\left( \frac{\bar{u}_0}{\bar{\sigma}} \right), \\
\bar{w}_0 &\sim \mathcal{O}\left( \frac{\bar{u}_0}{\bar{\sigma}^2} \right),
\end{align*}
\]

(108)

in conformity with the result stated at the end of paragraph 3.1.10 provided that the small parameter $\varepsilon$ is chosen as indicated in equation (91) with $B \equiv 1$ since the atmosphere is assumed to be homogeneous.

If we now relax the hypothesis of homogeneity of the atmosphere, the replacement system for (105) is extremely cumbersome writing, but it is not necessary to have this complete system to continue because only the vicinity of $\sigma = 0$ is of interest to us here for obtaining behavior analogous to (108). Indeed, the behavior in question is governed by the system

\[
\begin{align*}
2 \frac{\partial \bar{u}_0}{\partial \bar{\sigma}} + \frac{\bar{u}_0}{\bar{\sigma}} = 0, \\
\frac{\partial \bar{v}_0}{\partial \bar{x}_1} - \frac{\vec{K}_{24}^*}{\vec{K}_{22}^*} \frac{\partial \bar{u}_0}{\partial \bar{\sigma}} = 0, \\
\frac{\partial \bar{w}_0}{\partial \bar{x}_1} - \frac{1}{\vec{K}_{22}^* \bar{\sigma}} \frac{\partial \bar{u}_0}{\partial \bar{\sigma}} = 0,
\end{align*}
\]

(109)

which is obtained by replacing the operator $\nabla$ with the approximation

\[
c_0^* T^* \nabla \sim \frac{n^*}{\varepsilon} \frac{\partial}{\partial \bar{x}_1} + \frac{\vec{K}_{24}^*}{\vec{K}_{22}^*} n^* \wedge N^* \frac{\partial}{\partial \bar{\sigma}} + \frac{N^*}{\vec{K}_{22}^* \bar{\sigma}} \frac{\partial}{\partial \bar{\sigma}},
\]

(110)

whose analog, in the inhomogeneous atmosphere, is

\[
c_0^* T^* \nabla = \frac{n^*}{\varepsilon} \frac{\partial}{\partial \bar{x}_1} + \frac{n^* \wedge N^*}{h_5^*} \frac{\partial t^*}{\partial \bar{x}_3} \frac{\partial}{\partial \bar{\sigma}} + \frac{N^*}{\vec{K}_{22}^* \bar{\sigma}} \left( 1 + \frac{\partial T^*}{\partial \tau} \right) \frac{\partial}{\partial \bar{\sigma}},
\]

(111)

provided we assume that the scale of inhomogeneity is very large in comparison with $c_0^* T^* \bar{\sigma}$. It thus results, in the general case and in
place of (109),

\[
\begin{align*}
2 \frac{\partial \bar{u}_0}{\partial \bar{\sigma}} + \frac{\bar{u}_0}{\bar{\sigma}} &= 0, \\
\frac{\partial \bar{v}_0}{\partial \bar{x}_1} - \frac{1}{\hbar^3} \frac{\partial \bar{u}_0}{\partial \bar{x}_3} \\
\frac{\partial \bar{w}_0}{\partial \bar{x}_1} - \frac{1 + \frac{\partial \bar{T}^*}{\partial \bar{x}}}{\mathcal{K}^*} \frac{\partial \bar{u}_0}{\partial \bar{\sigma}} &= 0,
\end{align*}
\] (112)

the meaning of \( K^* \) having been specified in paragraph 3.1.20 a. We will not seek to clarify

the meaning of \( 1 \hbar^3 \frac{\partial \bar{u}_0}{\partial \bar{x}_3} \) because we will not have to use the corresponding equation in

what follows.

3.2 PROXIMAL APPROACH BEHAVIOR.

10 Differential system describing the approach. — The approach is made in a
certain neighborhood of the caustic, that is to say in a neighborhood of \( \bar{\sigma} = 0 \). It is
to be expected that the extent of this neighborhood depends on \( \varepsilon \) and we are led to
translate this fact by setting

\( \bar{\sigma} = \varepsilon^{2\alpha} \hat{\sigma}, \quad \bar{x}_1 = \hat{x}, \) (113)

with the exponent \( \alpha \) provisionally indeterminate, chosen such that \( \hat{\sigma} \) remains finite in
the neighborhood in question when \( \varepsilon \to 0 \). The distal approach behavior (108) suggests
that we set

\[
\begin{align*}
\bar{p} &= \varepsilon^{-\alpha} \hat{p}, \\
\bar{\rho} &= \varepsilon^{-\alpha} \hat{\rho}, \\
\bar{u} &= \varepsilon^{-\alpha} \hat{u}, \\
\bar{v} &= \varepsilon^{-3\alpha} \hat{v}, \\
\bar{w} &\equiv \varepsilon^{-5\alpha} \hat{w}
\end{align*}
\] (114)

enforcing that, for \( \alpha \) suitably chosen, \( \hat{p}, \cdots \) approach \( \hat{p}_0, \cdots \) when \( \varepsilon \to 0 \). The connection
with the behavior in question imposes the initial conditions

\[
\begin{align*}
\lim_{\bar{\sigma} \to \infty} \hat{\sigma}^{\frac{1}{2}} \hat{u}_0 &= \frac{\bar{x}_1}{\Lambda} \{ y(\bar{x}_1 + \Lambda) - y(\bar{x}_1 - \Lambda) \}, \\
\lim_{\bar{\sigma} \to \infty} \hat{\sigma}^{\frac{1}{2}} \frac{\partial \hat{v}_0}{\partial \bar{x}_1} &= -\frac{1}{2 \hbar^3} \frac{\partial \hat{T}^*}{\partial \bar{x}_3} \frac{\bar{x}_1}{\Lambda} \{ y(\bar{x}_1 + \Lambda) - y(\bar{x}_1 - \Lambda) \}, \\
\lim_{\bar{\sigma} \to \infty} \hat{\sigma}^{\frac{1}{2}} \frac{\partial \hat{w}_0}{\partial \bar{x}_1} &= -\frac{11 + \frac{\partial \hat{T}^*}{\partial \bar{x}}}{\mathcal{K}^*} \frac{\bar{x}_1}{\Lambda} \{ y(\bar{x}_1 + \Lambda) - y(\bar{x}_1 - \Lambda) \}.
\end{align*}
\] (115)

It remains to discover the value of \( \alpha \) and to determine the equations which govern
the evolution of \( \hat{u}_0, \hat{v}_0, \hat{w}_0 \). The two operations are related and are performed with the
help of equations (105). By examining all possibilities and adopting as a rule the need
to obtain a non-trivial system for \( \hat{u}_0, \hat{v}_0, \hat{w}_0 \), compatible with (115), we may
be convinced that $\alpha$ is determined by the following condition: after substitution of (113) and (114), the two terms in equation (105) given below

$$\frac{\partial(p + w)}{\partial \sigma} + \varepsilon \frac{\partial w}{K_{22} \partial \sigma},$$

must be of the same order in $\varepsilon$. This identification immediately gives

$$\alpha = \frac{1}{6},$$

and the equations governing $\hat{p}, \hat{\rho}_0, \hat{u}_0, \hat{v}_0, \hat{w}_0$ are then

$$\hat{p}_0 = \hat{\rho}_0 = \hat{u}_0,$$

$$\begin{cases}
2 \frac{\partial \hat{u}_0}{\partial \hat{\sigma}} + \hat{u}_0 \frac{\hat{u}_0}{\hat{\sigma}} - \frac{1 + \frac{\partial \hat{T}^*}{\partial \hat{\tau}}}{\hat{K}^*} \frac{1}{\hat{\sigma}} \frac{\partial \hat{w}_0}{\partial \hat{\sigma}} = 0, \\
\frac{\partial \hat{w}_0}{\partial \hat{x}} - \frac{1 + \frac{\partial \hat{T}^*}{\partial \hat{\tau}}}{\hat{K}^*} \frac{1}{\hat{\sigma}} \frac{\partial \hat{u}_0}{\partial \hat{\sigma}} = 0,
\end{cases}$$

$$\begin{cases}
\frac{\partial \hat{v}_0}{\partial \hat{x}} - \frac{1}{h_3} \frac{\partial \hat{T}^*}{\partial \hat{\tau}} \frac{\partial \hat{u}_0}{\partial \hat{\sigma}} = 0.
\end{cases}$$

Thus, to a first approximation, the approach to the caustic is a phenomenon exhibiting remarkable properties: it is governed by linear equations, it is isentropic and obeys the equipartition of energy, which is expressed in (118); it is a locally two-dimensional phenomenon in the plane normal to the caustic passing through the sound ray, which translates the fact that the determination of $\hat{v}_0$ is not coupled with that of $\hat{u}_0$ and $\hat{w}_0$.

It may be verified that the system (119) admits the characteristic curves

$$\hat{x} = \text{Cst}; \quad \hat{x} + \frac{2}{3} \left( \frac{\hat{K}^*}{1 + \frac{\partial \hat{T}^*}{\partial \hat{\tau}}} \right)^2 \hat{\sigma}^3 = \text{Cst},$$

![Fig. 8.](image-url)
which are, in $\hat{x}$, $\hat{\sigma}$ coordinates, the images of the Mach waves of the wave train considered, before and after reflection on the caustic, as can be seen in figure 8.

20 Discussion of the differential system. — By Fourier transformation

\begin{equation}
U = \int e^{\zeta \hat{x}} \hat{u}_0 \, d\hat{x}; \quad W = \int e^{\zeta \hat{x}} \hat{w}_0 \, d\hat{x}; \quad \zeta = \xi + i\eta,
\end{equation}

the system (119) becomes

\begin{equation}
\begin{cases}
2 \frac{dU}{d\hat{\sigma}} + \frac{U}{\hat{\sigma}} - \frac{\hat{R}}{\hat{\sigma}} \frac{dW}{d\hat{\sigma}} = 0, \\
\zeta W - \frac{\hat{R}}{\hat{\sigma}} \frac{dU}{d\hat{\sigma}} = 0,
\end{cases}
\end{equation}

with

\begin{equation}
\hat{R} = \frac{1 + \frac{\partial T}{\partial \tau}}{X^*}.
\end{equation}

The general solution of this new system is

\begin{equation}
\begin{cases}
U = \mathcal{F}_1(\zeta) \hat{\sigma} \exp \left( \frac{\zeta \hat{\sigma}^3}{3R^2} \right) \mathcal{C}_{\frac{1}{3}} \left( \frac{-\zeta \hat{\sigma}^3}{3iR^2} \right), \\
W = \zeta^{-1} \mathcal{F}_1(\zeta) \hat{\sigma}^{-1} \frac{d}{d\sigma} \left\{ \hat{\sigma} \exp \left( \frac{\zeta \hat{\sigma}^3}{3R^2} \right) \mathcal{C}_{\frac{1}{3}} \left( \frac{-\zeta \hat{\sigma}^3}{3iR^2} \right) \right\},
\end{cases}
\end{equation}

with, for $\mathcal{F}_1(\zeta)$, an arbitrary function, and, for $\mathcal{C}_{\frac{1}{3}}$, a Bessel function of index $\frac{1}{3}$. Thanks to the following integral representation

\begin{equation}
\mathcal{C}_{\frac{1}{3}} \left( \frac{z}{i} \right) = \text{Cst} \int_L \exp \left\{ i \frac{t}{3} - z \sin t \right\} \, dt,
\end{equation}

with, for $L$, a suitable path in the complex plane $t$, we obtain, by return to the original system,

\begin{equation}
\hat{u}_0 = \frac{\text{Cst} \hat{\sigma}}{2i\pi} \int_{\xi - i\infty}^{\xi + i\infty} \mathcal{F}_1(\zeta) \, d\zeta \int_L \, dt \exp \left\{ i \frac{t}{3} + \zeta \left[ \hat{x} + \frac{\hat{\sigma}^3}{3R^2} (1 + \sin t) \right] \right\}.
\end{equation}

Setting

\begin{equation}
X = \frac{1 + \sin t}{2}, \quad Y = -\frac{1}{3} \frac{\cos t}{\cos \frac{t}{3}},
\end{equation}

results in

\begin{equation}
27(Y^3 - Y^2) = 16(X^2 - X),
\end{equation}

and, if we define

\begin{equation}
G(X) = \sum_k |Y_k(X)|^{-1},
\end{equation}

314
where the $Y_k$ are the real roots of (61), we find

$$ (131) \quad \int_{-\infty}^{\infty} e^{-i\hat{x}\eta} G\left(-\frac{3\hat{x}\hat{R}^2}{2\hat{\sigma}^3}\right) d\hat{x} = -\frac{\hat{\sigma}^3}{\hat{R}^2} \int_L \exp\left\{\frac{i}{3} + i\eta \frac{T^2}{3\hat{R}^2}(1 + \sin t)\right\} dt, $$

the contour $L$ being, this time, that which is specified in figure 9. From these considerations it follows that the function

$$ (132) \quad \hat{u}_0 = \frac{\sqrt{2} \hat{R}}{3\pi \hat{\sigma}^3} \int_{-\infty}^{\infty} g(\xi) G\left(-\frac{3(\hat{x} - \xi)\hat{R}^2}{2\hat{\sigma}^3}\right) d\xi, $$

and the function $\hat{w}_0$ which is associated with it by

$$ (133) \quad \frac{\partial \hat{w}_0}{\partial \hat{x}} = \hat{R} \frac{\partial \hat{u}_0}{\partial \hat{\sigma}} $$

resolve system (119). Furthermore, taking the properties of the function $G$ into account, it is easily verified that we have

$$ (134) \quad \lim_{\hat{\sigma} \to \infty} \hat{\sigma}^{\frac{1}{2}} \hat{u}_0 = \frac{1}{\pi} \int_{-\infty}^{\hat{x}} f(\xi) \frac{g(\xi)}{\sqrt{\hat{x} - \xi}} d\xi = \frac{1}{\pi} \int_{\hat{x}}^{\infty} g(\xi) \frac{f(\xi)}{\sqrt{\xi - \hat{x}}} d\xi, $$

so that, if $f$ is chosen such that

$$ (135) \quad \frac{1}{\pi} \int_{-\infty}^{\hat{x}} \frac{f(\xi)}{\sqrt{\hat{x} - \xi}} d\xi = \frac{\hat{x}}{\hat{\lambda}} \{y(\hat{x} + \hat{\lambda}) - y(\hat{x} - \hat{\lambda})\}, $$

the formula (132) solves the problem. We find

$$ (136) \begin{cases} \overline{\lambda} f(\xi) = \frac{2\xi + \hat{\lambda}}{\sqrt{\xi + \hat{\lambda}}} \overline{y}(\xi + \hat{\lambda}) - \frac{2\xi - \hat{\lambda}}{\sqrt{\xi - \hat{\lambda}}} \overline{y}(\xi - \hat{\lambda}), \\ \overline{\lambda} g(\xi) = \frac{2\xi - \hat{\lambda}}{\sqrt{\hat{\lambda} - \xi}} \overline{y}(\hat{\lambda} - \xi) - \frac{2\xi + \hat{\lambda}}{\sqrt{-\xi - \hat{\lambda}}} \overline{y}(-\hat{\lambda} - \xi), \end{cases} $$

30 Pressure behavior during approach. — In the vicinity of the caustic the pressure is given by the formula

$$ (137) \quad p - p_0^* \simeq \varepsilon^2 \rho_0 c_0^* \hat{u}_0 \left(\frac{x_1}{\varepsilon c_0^* T^*}, \frac{\sigma}{\varepsilon^3 T^*}\right), $$

315
so that we may state: consider a sound ray which touches the caustic at a point where the characteristics of the unperturbed atmosphere are \( p^*_0, \rho^*_0, c^*_0 \) at the instant when the Mach wave created by the aircraft arrives there; let \( \tau + T^*(\tau) \) be the instant in question if the sound ray considered comes from the point reached by the aircraft at the instant \( \tau \); also let

\[
R = \frac{c^*_0 T^*}{K^*} \left( 1 + \frac{\partial T^*}{\partial \tau} \right)
\]

\( K^* \) being expression (97 b) with the system of notations of paragraph 3.1.10; if we suppose that the incident \( N \) wave is defined by a temporal variable \( T = t - \tau \) and the spatial variable \( x \) and if it is described, during the approach and according to the usual ballistic noise theory, by the formula

\[
p - p^*_0 \cong \varepsilon \rho^*_0 c^*_0 \left( 1 - \frac{T}{T^*} \right)^{-\frac{1}{2}} x \frac{1}{A} \left\{ Y(x + \Lambda) - Y(x - \Lambda) \right\},
\]

with \( \varepsilon \) and \( \Lambda \) given by (90), (91), then the true behavior during the approach, accounting for the focusing phenomenon, is given by

\[
p - p^*_0 \cong \rho^*_0 c^*_0 \varepsilon \frac{2}{3} \left[ c_0^*(T - T^*) \right]^{\frac{3}{2}} H \left( \frac{3xR^2}{2(c^*_0(T - T^*))^3}, \frac{3AR^2}{2(c^*_0(T^* - T))^3} \right),
\]

if the function \( H(z, z_0) \) is defined by the formula

\[
H(z, z_0) = \frac{1}{\pi} \int_{-\infty}^{-z_0} \left( \frac{2\zeta - z_0}{\sqrt{z_0 - \zeta}} - \frac{2\zeta + z_0}{\sqrt{-\zeta - z_0}} \right) G(\zeta - z) \, d\zeta
\]

\[
+ \frac{1}{\pi} \int_{-z_0}^{z_0} \frac{2\zeta - z_0}{\sqrt{z_0 - \zeta}} G(\zeta - z) \, d\zeta,
\]

(141)

\[
H(-z, z_0) = -\frac{1}{\pi} \int_{-\infty}^{-z_0} \frac{2\zeta + z_0}{\sqrt{\zeta + z_0}} G(z - \zeta) \, d\zeta
\]

\[
+ \frac{1}{\pi} \int_{z_0}^{\infty} \left( \frac{2\zeta + z_0}{\sqrt{\zeta + z_0}} - \frac{2\zeta - z_0}{\sqrt{-\zeta - z_0}} \right) G(z - \zeta) \, d\zeta,
\]

where

\[
G(X) = \sum_k |Y_k(X)|^{-1},
\]

(142)

\( Y_k \) are the real roots of \( 27(Y^3 - Y^2) = 16(X^2 - X) \),

\(^{(8)}\) \( \frac{1}{\pi} \) is, if you like, the difference of the curvatures of two plane curves: one is the section of the caustic by a normal plane passing through the tangent to the sound ray, the other is the projection of the sound ray onto this plane, the caustic and the ray being drawn in a frame where the wind vanishes at the instant-point considered.
The extent of the portion of the sound ray involved in the approach phenomenon is given by the condition

\[(143) \quad c_0^*(T^* - T) = \mathcal{O}(\varepsilon^{\frac{1}{2}} c_0^* T^*)\]

The graph of the function \(G(X)\) has been drawn in figure 10 where we see that this function is singular at \(X = 0\) and at \(X = 1\); the first singularity gives rise to two acoustic shocks at \(x = \pm \Lambda\), with strength \(\varepsilon \rho_0^* c_0^{*2} \left(1 - \frac{T}{T^*}\right)^{-\frac{1}{2}}\); these acoustic shocks come from head and tail shocks in the incident \(N\) wave and the theory presented here can only retain them because it is based on a linear differential system having the lines \(x = \text{Cst}\) as characteristics.

Fig. 10.

The singularity in \(\left(1 - \frac{T}{T^*}\right)^{-\frac{1}{2}}\) from the usual theory is found here in the shocks of the incident wave. The second singularity of \(G(X)\), at \(X = 1\) gives rise to a singularity along the curves

\[(144) \quad x = \pm \Lambda - \frac{2}{3} \frac{[c_0^*(T^* - T)]^3}{R^2} = x_{\text{RS}}^\pm\]

which are the images of the preceding shocks after reflection on the caustic; the singularity in question is logarithmic

\[(145) \quad p - p_0^* \simeq \rho_0^* c_0^{*2} \frac{\varepsilon}{\pi \sqrt{1 - \frac{T}{T^*}}} \log \left|\frac{1}{|x - x_{\text{RS}}^\pm|}\right|,\]

and we see that its coefficient is itself singular on the caustic.

---

Translator’s Note: Guiraud originally used the notation \(x_{\text{CR}}^\pm\) here. In view of the fact that this singularity arises from the shock reflected off the caustic, we interpret his subscript to be an abbreviation of ‘choc réfléchi,’ or ‘reflected shock,’ and so have substituted the corresponding English abbreviation for clarity.
On the caustic itself, formula (132) leads to a finite limit for \( \hat{u}_0 \), that is

\[
\hat{u}_0(0, \hat{x}) = 2^{-\frac{1}{3}} 3^{-\frac{2}{3}} \pi^{-1} \hat{R}^{-\frac{1}{3}} \int_{-\infty}^{\infty} \frac{f(\xi)}{|\hat{x} - \xi|^{\frac{2}{3}}} \, d\xi,
\]

hence, for the form of the pressure signal on the caustic itself

\[
\frac{p - p_0^*}{\rho_0^* c_0^*} \approx -2^{-\frac{1}{3}} 3^{-\frac{2}{3}} \varepsilon \left( \frac{\Lambda}{c_0^* T^*} \right)^{-\frac{1}{6}} \left( \frac{c_0^* T^*}{\hat{R}} \right)^{\frac{1}{3}} C\left(-\frac{x}{\Lambda}\right),
\]

with

\[
C(z) = \frac{1}{\pi} \int_{-1}^{1} \frac{2\zeta + 1}{\sqrt{\zeta + 1}} \frac{d\zeta}{|z - \zeta|^{\frac{2}{3}}} + \frac{1}{\pi} \int_{1}^{\infty} \frac{2\zeta + 1}{\sqrt{\zeta + 1}} \frac{d\zeta}{|z - \zeta|^{\frac{2}{3}}}.
\]

The graph of the function \( C(z) \) is drawn in figure 11\(^{(9)}\). This function is singular at \( z = \pm 1 \) and its singularity is of the following nature

\[
\begin{cases}
\lim_{z \to \pm 1 - 0} |z - 1|^{\frac{1}{3}} C(z) = -\frac{1}{\pi} \int_{0}^{\infty} \frac{du}{u^{\frac{1}{3}}(u + 1)^{\frac{1}{3}}}, \\
\lim_{z \to \pm 1 - 0} |z - 1|^{\frac{1}{3}} C(z) = -\frac{1}{\pi} \int_{0}^{\infty} \frac{du}{u^{\frac{1}{3}}(u - 1)^{\frac{1}{3}}}.
\end{cases}
\]

40 Return to acoustics. — It is instructive to note that, the system (119) being linear, it has the same structure as that which we could obtain by performing, on the acoustic equations, the sequence of operations which we performed on the complete equations. On the other hand, we know that this system is not affected, in its form, by the inhomogeneity of the atmosphere. Now, the general solution of the acoustic equations, corresponding to supersonic flight, is given in section 2.1; therefore, if we perform the same sequence of operations on this solution, we must find a solution of the system (119). This is indicated in the flowchart in figure 12. We see that the system (119) is common to acoustics and to the nonlinear theory of ballistic noise, when we approach the focus. In both cases we must look for a solution of this system satisfying the condition

\[
\lim_{\hat{\sigma} \to \infty} \hat{\sigma}^{\frac{1}{2}} \hat{u}_0 = F(\hat{x}),
\]

\(^{(9)}\)Figure 2 given in reference [10] and also Figure 7 given in reference [11] are incorrect. The function depicted in these figures mistakenly represents the difference and not the sum of the two integrals.
but the function $F(\hat{x})$ differs from one case to another; if it is acoustic it is necessary to identify $F(\hat{x})$ with the Whitham function, whereas in the present case we must take for $F(\hat{x})$ an N wave.

Let us return to acoustics, in a homogeneous atmosphere, and consider the function $U_i(t, \mathbf{x}; \xi) = |\Gamma_{\tau_{\xi}}|^{-1}$ introduced in section 2.1, with the asymptotic representation, within the Mach wave train,

\[
\Gamma_{\tau_{\tau_{\tau}}} = \frac{1}{2} \text{\gamma}(\tau_0) \cdot \mathbf{n}(\tau_0) \cdot (T - T^*).
\]

Let us suppose that the instant-point $t, \mathbf{x}$ is in the vicinity of an instant-point of the caustic, situated on the sound ray coming from the instant-point $\tau^*, \mathbf{x}_a(\tau^*)$, it follows

\[
\Gamma \approx \frac{1}{6} \Gamma_{\tau\tau\tau}(\tau - \tau_0)^3 + \frac{1}{2} \Gamma_{\tau_0\tau_0}(\tau - \tau_0)^2 + \Gamma_{\xi}(\xi - \xi_0),
\]
the pair \((τ_0, ξ_0)\) having, for the instant-point \(t, x\), the meaning given in paragraph 2.1.
If \(γ\) is constant we easily verify the relation

\[
(153) \quad c_0^\star T^{\gamma^2} |Γ_{τ^\gamma τ^\gamma τ^\gamma}| = 2(\mathcal{M}_a^2 - 1)^{\frac{3}{2}} R,
\]

\(R\) being the radius of curvature of the section of the caustic by a normal plane passing through the sound ray. Let us adopt the approximation (152) and set

\[
(154) \quad X = \frac{3}{2} \frac{(ξ_0 - ξ)R^2}{\mathcal{M}_a σ^3}, \quad Y = \frac{c_0}{3(\mathcal{M}_a^2 - 1)^{\frac{1}{2}}} \frac{R}{σ^2} Γ_{τ^γ},
\]

\(Γ_{τ^γ}\) being evaluated at one of the roots of \(Γ = 0\), we find

\[
(155) \quad 27(Y^3 - Y^2) = 16(X^2 - X).
\]

It is clear that the expression

\[
(156) \quad p - p_0 \approx \frac{ρ_0 V_a^2}{6π(\mathcal{M}_a^2 - 1)^{\frac{1}{2}}} \frac{R}{σ^2} \int_{-∞}^{∞} S''(ξ)G \left( \frac{3}{2} \frac{(ξ_0 - ξ)R^2}{\mathcal{M}_a σ^3} \right) dξ
\]

gives the pressure field created by a fuselage with area law \(S(ξ)\) in the vicinity of the caustic. Therefore, taking into account the remarks presented,
we have just verified that (132) solves the system (119). The diagram in figure 13 explains, geometrically, why it is necessary to involve a third degree equation in the vicinity of the caustic.

50 Analogy with transonic flow. — Let us imagine that an observer moves on the caustic by following the wave train and that this observer examines the phenomena on the scale of the width of the train; the caustic will therefore appear flat to his uninformed senses, but if he observes the sound phenomenon he cannot help but notice that this phenomenon reveals a certain curvature of the caustic. More precisely, he will observe that the wave train progresses slower than him above the caustic and that the train no longer exists, as waves, below. He will be tempted to believe that the phenomenon in question occurs in an environment where the speed of sound is a linear function of the distance to the caustic. As he moves with the train, the phenomenon is stationary but, as the speed of the waves, relative to him, change sign across the caustic, he will be tempted to think that he is observing a transonic phenomenon. Seen in another way we may say that, deceived by the quasi-planarity of the caustic, our observer mistakes as Galilean the Darboux-Ribaucourt reference frame linked to an integral curve of the direction field.
defined on the caustic by the sound rays, whose origin moves on this curve, with the speed of sound. However, due to the weak curvature of the caustic and the non-isothermal character of the atmosphere, it results that the points of this reference frame located above the caustic — on the side of the convexity — move with a speed greater than the speed of sound and that the situation is the opposite below. There is every reason to think that this phenomenon is described by a Tricomi equation

\[ \zeta \frac{\partial^2 U}{\partial \xi^2} - \frac{\partial^2 U}{\partial \zeta^2} = 0. \]

Indeed, the elimination of \( \hat{w}_0 \) between the equations of the system (119) leads to the equation

\[ \frac{\partial}{\partial \hat{x}} \left( 2 \frac{\hat{u}_0}{\hat{\sigma}} + \hat{u}_0 \right) - \frac{\hat{R}}{\hat{\sigma}} \frac{\partial}{\partial \hat{\sigma}} \left( \frac{\hat{R}}{\hat{\sigma}} \frac{\partial \hat{u}_0}{\partial \hat{\sigma}} \right) = 0 \]

and we recover equation (89) if we set

\[ \zeta = \frac{\dot{\sigma}^2}{(2\hat{R}^2)^{3/2}}, \quad \xi = \hat{x} + \frac{\dot{\sigma}^3}{3\hat{R}^2}, \quad \hat{u}_0 = U. \]

We can recover this result without going through the detour of the \( x_1, \sigma \) coordinates used previously. Let us therefore choose coordinates

\[ x_0 = c_0 T, \quad x_1, \quad x_2, \quad x_3 \text{ with, for } x_1 \text{ and } x_2, \text{ orthogonal curvilinear coordinates on the caustic, the lines } x_2 = \text{Cst being tangent to rectilinear sound rays, and for } x_3, \text{ the distance to the caustic. Perform the change of variables} \]

\[ x_0 = \tau, \quad x_1 = X(\tau) + s, \quad x_2 = y, \quad x_3 = n, \]

\[ \text{Fig. } 1', \]
in such a way that by fixing \(s\) and \(y\) and making \(n = 0\), we move on the caustic with the wave, normal to it. Let \(e_1, e_2, e_3\) be the unit vectors of the tangents to the coordinate lines and set

\[
V = v_1e_1 + v_2e_2 + v_3e_3.
\]

If we define \(H_1, H_2, K_{11}, K_{12}, K_{22}\) via the relation

\[
dx = \{H_1(1 + x_3K_{11})e_1 + x_3H_1K_{12}e_2\} \, dx_1 \\
+ \{x_3H_2K_{12}e_1 + H_2(1 + x_3K_{22})e_2\} \, dx_2 + e_3 \, dx_3,
\]

we easily arrive at the change of variables formulas

\[
\nabla = \frac{e_1}{\Delta} \left( B \frac{\partial}{\partial s} - D \frac{\partial}{\partial y} \right) + \frac{e_2}{\Delta} \left( -A \frac{\partial}{\partial s} + C \frac{\partial}{\partial y} \right) + e_3 \frac{\partial}{\partial n},
\]

\[
\frac{1}{c_0} \left( \frac{\partial}{\partial t} + V \cdot \nabla \right) = \frac{\partial}{\partial \tau} - \frac{1}{H_2} \frac{\partial}{\partial s} + \frac{v_1B - v_2A}{c_0\Delta} \frac{\partial}{\partial s}
\]

\[
+ \frac{-v_1D + v_2C}{c_0\Delta} \frac{\partial}{\partial y} + \frac{v_3}{c_0} \frac{\partial}{\partial n},
\]

with

\[
A = nH_2K_{12}, \quad B = H_2(1 + nK_{22}), \\
C = H_2(1 + nK_{11}), \quad D = nH_1K_{12}, \\
\Delta = H_1H_2\{1 + n(K_{11} + K_{22}) + n^2(K_{11}K_{22} - K_{12}^2)\}.
\]

If we now use the relations

\[
\begin{align*}
\frac{\partial e_1}{\partial s} &= -\frac{1}{H_2} \frac{\partial H_1}{\partial x_2} e_2 - H_1K_{11}e_3, \\
\frac{\partial e_2}{\partial s} &= \frac{1}{H_2} \frac{\partial H_1}{\partial x_2} e_1 - H_1K_{12}e_3, \\
\frac{\partial e_3}{\partial s} &= H_1K_{11}e_1 + H_1K_{12}e_2, \\
\frac{\partial e_1}{\partial y} &= \frac{1}{H_1} \frac{\partial H_2}{\partial x_1} e_2 - H_2K_{12}e_3, \\
\frac{\partial e_2}{\partial y} &= -\frac{1}{H_1} \frac{\partial H_2}{\partial x_1} e_1 - H_2K_{22}e_3, \\
\frac{\partial e_3}{\partial y} &= H_2K_{12}e_1 + H_2K_{22}e_2, \\
\frac{\partial e_i}{\partial n} &= 0 \quad (i = 1, 2, 3).
\end{align*}
\]

It is easy to write the equations of motion. The result occupies a lot of space and numerous terms are entirely useless,
so we are only going to write what may be called the dominant equations; with the usual notations

\[
\begin{align*}
\rho &= \rho_0^* + \varepsilon\rho_0^* c_0^* \overline{\rho}, \\
p &= p_0^* + \varepsilon p_0^* \overline{\rho}, \\
v_1 &= \varepsilon c_0^* v_1, \\
v_2 &= \varepsilon^2 c_0^* v_2, \\
v_3 &= \varepsilon^2 c_0^* v_3, \\
s &= \varepsilon c_0^* T^* \pi, \\
y &= c_0^* T^* y, \\
n &= c_0^* T^* n, \\
\tau &= c_0^* T^* \tau, \\
K_{ij} &= (c_0^* T^*)^{-1} K_{ij},
\end{align*}
\]

(166)

it follows

\[
\begin{align*}
\frac{\partial (\overline{v}_1 - \overline{p})}{\partial \xi} - \pi K_{11} \frac{\partial \overline{v}_1}{\partial \xi} + \varepsilon \left( \frac{\partial \overline{v}_1 \overline{p}}{\partial \xi} + H_1 \frac{\partial \overline{p}}{\partial \tau} \right) + \varepsilon^2 H_1 \frac{\partial \overline{v}_3}{\partial \eta} + \varepsilon^3 H_1 \overline{v}_3 \frac{\partial \overline{p}}{\partial \eta} + \cdots &= 0, \\
\frac{\partial (\overline{p} - \overline{v}_1)}{\partial \xi} - \pi K_{11} \frac{\partial \overline{v}_1}{\partial \xi} + \varepsilon \left( (\overline{v}_1 - \overline{p}) \frac{\partial \overline{v}_1}{\partial \xi} + H_1 \frac{\partial \overline{v}_1}{\partial \tau} \right) + \varepsilon^2 H_1 \overline{v}_3 \frac{\partial \overline{v}_1}{\partial \eta} + \cdots &= 0, \\
\frac{\partial \overline{v}_3}{\partial \xi} - H_1 \frac{\partial \overline{p}}{\partial \eta} + \varepsilon \left( (\overline{p} - \overline{v}) \frac{\partial \overline{v}_3}{\partial \xi} - H_1 \frac{\partial \overline{v}_3}{\partial \tau} \right) - \varepsilon^2 H_1 \overline{v}_3 \frac{\partial \overline{v}_3}{\partial \eta} + \cdots &= 0, \\
\frac{\partial \overline{v}_2}{\partial \xi} + \varepsilon^{-1} \pi K_{12} \frac{\partial \overline{p}}{\partial \xi} - \frac{H^2_1}{H_2} \frac{\partial \overline{p}}{\partial \eta}
\end{align*}
\]

(167)

\[
\begin{align*}
\frac{\partial \overline{v}_3}{\partial \xi} - H_1 \frac{\partial \overline{p}}{\partial \eta} + \varepsilon \left( (\overline{p} - \overline{v}) \frac{\partial \overline{v}_3}{\partial \xi} - H_1 \frac{\partial \overline{v}_3}{\partial \tau} \right) - \varepsilon^2 H_1 \overline{v}_3 \frac{\partial \overline{v}_3}{\partial \eta} + \cdots &= 0, \\
\frac{\partial \overline{v}_2}{\partial \xi} + \varepsilon^{-1} \pi K_{12} \frac{\partial \overline{p}}{\partial \xi} - \frac{H^2_1}{H_2} \frac{\partial \overline{p}}{\partial \eta}
\end{align*}
\]

The origin of these equations is, in order, the following: continuity equation, momentum equation projected onto \( e_1 \), entropy equation, momentum equations projected onto \( e_2 \) and \( e_3 \) respectively. Let us also note the following combination of the first three equations

\[
\begin{align*}
-\pi K_{11} \frac{\partial (\overline{v}_1 + \overline{p})}{\partial \xi} + \varepsilon \left( (\overline{v}_1 - \overline{p}) \frac{\partial \overline{v}_1}{\partial \xi} + \gamma \overline{p} \frac{\partial \overline{v}_1}{\partial \xi} + \overline{v}_1 \frac{\partial \overline{v}_1}{\partial \xi} + H_1 \frac{\partial (\overline{v}_1 + \overline{p})}{\partial \tau} \right)
\end{align*}
\]

(168)

\[
\begin{align*}
+ \varepsilon^2 H_1 \frac{\partial \overline{v}_3}{\partial \eta} + \varepsilon^3 H_1 \overline{v}_3 \frac{\partial (\overline{v}_1 + \overline{p})}{\partial \eta} + \cdots &= 0.
\end{align*}
\]

The written equations are dominant in the sense that no reasonable change of scale translating the behavior of the various quantities in the vicinity of the caustic can cause a neglected term to become preponderant. To be convinced of this fact it is necessary to carry out a detailed analysis of the complete equations. Note that we have developed \( \frac{A}{\Delta}, \cdots \) in powers of \( \pi \) and that we have each time limited ourselves to the first term occurring as the coefficient of a derivative.
Now we carry out the change of scale

\begin{equation}
\begin{aligned}
&\bar{s} = \hat{s}, \quad \bar{\pi} = \varepsilon^{4\alpha} \hat{n}, \quad \bar{\tau} = \hat{\tau}, \quad \bar{y} = \hat{y}, \\
&\bar{v}_1 = \varepsilon^{-\alpha} \hat{u}, \quad \bar{p} = \varepsilon^{-\alpha} \hat{u}(1 + \tilde{p}), \quad \bar{p} = \varepsilon^{-\alpha} \hat{u}(1 + \tilde{p}), \\
&\bar{v}_2 = \varepsilon^{-3\alpha} \hat{v}, \quad \bar{v}_3 = \varepsilon^{-5\alpha} \hat{w},
\end{aligned}
\end{equation}

as suggested by the distal approach behavior, as well as by the study of paragraph 3.2.10, which also indicates that it is necessary to take \( \alpha = \frac{1}{5} \). By substitution we find

\begin{equation}
\begin{aligned}
\frac{\partial \hat{w}}{\partial \hat{n}} - \frac{2K_{11} \hat{n} \partial \hat{u}}{H_1} + \frac{(\gamma + 1)\varepsilon^{\frac{1}{2}}}{H_1} \frac{\partial \hat{u}}{\partial \hat{s}} + 2\varepsilon^{\frac{1}{4}} \frac{\partial \hat{u}}{\partial \hat{t}} + 2\varepsilon^{\frac{3}{5}} \frac{\partial \hat{w}}{\partial \hat{n}} + \cdots = 0, \\
\frac{1}{H_1} \frac{\partial \hat{w}}{\partial \hat{n}} - \frac{\partial \hat{u}}{\partial \hat{n}} - \frac{\varepsilon^{\frac{3}{6}} \hat{u} \partial \hat{w}}{H_1} - \varepsilon^{\frac{3}{10}} \frac{\partial \hat{w}}{\partial \hat{n}} - \varepsilon^{\frac{3}{6}} \frac{\partial \hat{w}}{\partial \hat{t}} + \cdots = 0;
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
\frac{\partial \hat{w}}{\partial \hat{s}} + nK_{12} \frac{\partial \hat{u}(1 + \tilde{p})}{\partial \hat{s}} - \frac{\varepsilon^{\frac{1}{2}}}{H_2} \frac{\partial \hat{u}}{\partial \hat{y}} + \varepsilon^{\frac{3}{6}} \frac{\hat{u} \partial \hat{w}}{\partial \hat{s}} - \varepsilon \frac{\hat{u} \partial \hat{w}}{\partial \hat{t}} - \varepsilon^{\frac{3}{10}} \frac{\partial \hat{w}}{\partial \hat{n}} + \cdots = 0,
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
- \frac{\partial \hat{p}}{\partial \hat{s}} + \varepsilon^{\frac{3}{6}} \left( H_1 \frac{\partial \hat{w}}{\partial \hat{n}} - nK_{11} \frac{\partial \hat{u}}{\partial \hat{s}} \right) + 2\varepsilon^{\frac{3}{5}} \frac{\partial \hat{u}}{\partial \hat{s}} + \varepsilon H_1 \frac{\partial \hat{u}}{\partial \hat{t}} + \varepsilon^{\frac{3}{5}} H_1 \frac{\partial \hat{w}}{\partial \hat{n}} + \cdots = 0, \\
\frac{\partial \hat{u}(\hat{p} - \tilde{p})}{\partial \hat{s}} + \varepsilon^{\frac{3}{6}} (\gamma - 1) \hat{u} \frac{\partial \hat{w}}{\partial \hat{s}} + \cdots = 0.
\end{aligned}
\end{equation}

If we assume that \( \hat{u}, \hat{v}, \hat{w} \) approach finite limits \( \hat{u}_0, \hat{v}_0, \hat{w}_0 \) when \( \varepsilon \) approaches zero, we see that \( \hat{p} \) and \( \tilde{p} \) approach zero and we have, to determine \( \hat{u}_0, \hat{w}_0 \) the differential system

\begin{equation}
\begin{aligned}
\frac{\partial \hat{w}_0}{\partial \hat{n}} - \frac{2K_{11} \hat{n} \partial \hat{u}_0}{H_1} = 0, \\
\frac{1}{H_1} \frac{\partial \hat{w}_0}{\partial \hat{s}} - \frac{d \hat{u}_0}{d \hat{n}} = 0,
\end{aligned}
\end{equation}

which leads to the well-known Tricomi equation (157) if we set

\begin{equation}
H_1 \hat{s} = \xi, \quad (2K_{11})^{\frac{1}{3}} \hat{n} = \zeta, \quad \hat{u}_0 = U,
\end{equation}

again such that

\begin{equation}
\hat{\sigma}^2 = 2\hat{R} \hat{n}, \quad \hat{x} = H_1 \hat{s} - \frac{2\sqrt{2} \hat{n}^{\frac{3}{2}}}{\hat{R}^{\frac{1}{2}}},
\end{equation}
that is to say, returning to dimensioned variables,

\[(176) \quad \sigma^2 = 2Rn, \quad x_1 = H_1s - \frac{2\sqrt{2} n^3}{3 R^3},\]

relations for which the geometric interpretation is evident.

### 3.3 STUDY OF THE IMMEDIATE VICINITY OF THE CAUSTIC.

**10 Reinterpretation of the proximal approach behavior.** — The analysis presented in paragraph 3.2.10 was well suited to the description of the proximal approach behavior, in relation to its distal behavior, but it is impossible to attain, through this process, the side of the caustic which is not swept out by the characteristic sound rays. The same criticism is not valid for the analysis of paragraph 3.2.50. As a consequence, the description of the immediate vicinity of the caustic must be done in two stages: first, it is necessary to reinterpret the solution of paragraph 3.2.30 in order to deduce a solution expressed in the variables \(x_0, x_1, x_2, x_3\) of paragraph 3.2.50; this solution, in reduced variables \(\xi, \zeta\), is necessarily a solution of the Tricomi equation and if it is a solution which vanishes for \(\zeta \to -\infty\) we will have suitably described the degradation from relatively intense noise into

relative silence when we traverse the caustic, passing from the region swept out by characteristic sound rays to that which is not; the second step requires that we do a special study of the neighborhood of the shocks which escape description by the Tricomi equation. To obtain the appropriate solution of this equation, it must be remembered that this solution results from the fundamental solution \(\delta(\Gamma)\), of the wave equation. Now, in the vicinity of the caustic, \(\Gamma = 0\) is, in variables \(\xi, \zeta\), a tangent to one of the characteristics

\[(177) \quad \xi \pm \frac{2}{3} \zeta^\frac{3}{2} = \text{Cst}\]
of the Tricomi equation. We may easily verify the following formula:

\[(178) \Gamma \cong 2\varepsilon T^{*2} \left( \xi - \xi_0 + \lambda \zeta - \frac{\lambda^3}{3} \right),\]

and we can deduce that the function \(U(\xi, \zeta)\) defined by

\[(179) \quad U = |\zeta - \lambda^2|^{-1}, \quad \xi + \lambda \xi - \frac{\lambda^3}{3} = 0,\]

is a solution of the Tricomi equation, which can further be verified by calculation. Starting from this observation, we can easily check that, if \(G^*(t)\) is the function defined by

\[(180) \quad G^*(t) = \sum_k |U_k(t)|,\]

the sum depending on the real roots of the equation

\[(181) \quad (2U + 1)^2(U - 1) = 9U^3,\]

the expression

\[(182) \quad \hat{u}_0 = \frac{1}{2^\frac{1}{2} \pi R^\frac{1}{2} |\xi|} \int_{-\infty}^{\infty} f(\xi_0)G^* \left\{ \frac{(\xi - \xi_0)^2}{\xi^3} \right\} d\xi_0,\]

is a solution of the Tricomi equation, and that, if the change of variables (159) is performed, the condition (134) is satisfied, since it is a limit attained at fixed \(\hat{x}\). Hence it follows that (182) is just a retranscription of solution (132) and that this formula now describes the behavior of perturbations on both sides of the caustic. We may note that (182) gives back (146) by passage to the limit \(\zeta \to 0\) and, by means of the relation

\[(183) \quad \int_{-\infty}^{\infty} f(\xi_0) d\xi_0 = 0,\]

we see that \(\hat{u}_0\) approaches zero as \(\zeta^{-\frac{13}{12}}\) when \(\zeta \to -\infty\). More precisely, returning to dimensioned variables, we find that the pressure field decays as indicated by the formula

\[(184) \quad \frac{p - p_0^*}{\rho_0^* c_0^* T^*} \cong \varepsilon \left( \frac{A}{c_0^* T^*} \right)^2 \left( \frac{R}{c_0^* T^*} \right)^{\frac{3}{2}} \left( \frac{c_0^* T^*}{\rho_0^* c_0^* T^*} \right)^{\frac{13}{12}} \frac{1}{2^{\frac{13}{12}} \pi} \int_0^{\infty} \frac{G^*(-t)}{t^{\frac{13}{12}}} dt,\]

but the rate of this decrease is no doubt illusory.

20 Study of the neighborhood of the shocks. — The formula (182), just like its analog (132), is not valid in the vicinity of shocks. To describe

\[\text{Translator’s Note:} \int \text{indicates the Hadamard finite part ("partie finie") of the integral, a generalization of the Cauchy principle value. The application of the finite part, or Hadamard regularization, is a way of assigning a finite value to a divergent integral and arises in the theory of hypersingular integral equations.}\]
such a neighborhood requires a local study, but it is clear that the dominant equations are, in variables $\bar{x}$, $\bar{\sigma}$

\[
\begin{align*}
2 \frac{\partial \bar{\pi}}{\partial \bar{\sigma}} + \frac{\pi}{\bar{\sigma}} - \varepsilon \frac{\bar{R}}{\bar{\sigma}} \frac{\partial \bar{\pi}}{\partial \bar{\sigma}} - (\gamma + 1) \pi \frac{\partial \bar{\pi}}{\partial \bar{x}} &= 0, \\
\frac{\partial \bar{w}}{\partial \bar{x}} - \frac{\bar{R}}{\bar{\sigma}} \frac{\partial \bar{u}}{\partial \bar{\sigma}} &= 0,
\end{align*}
\]

and the following changes of variables and functions are needed

\[
\begin{align*}
\bar{x} &= \pm \Lambda + 2^{-\frac{2}{5}} \gamma + 1) \frac{6}{5} \varepsilon \frac{1}{5} \bar{x}, \\
\bar{\sigma} &= 2^{-\frac{4}{5}} (\gamma + 1) \frac{6}{5} \varepsilon \frac{2}{5} \bar{\sigma}, \\
\bar{u} &= 2 \bar{u} (\gamma + 1) \frac{2}{5} \varepsilon \frac{4}{5} \bar{u}, \\
\bar{w} &= 2^{-\frac{2}{5}} (\gamma + 1) \frac{4}{5} \varepsilon \frac{4}{5} \bar{w},
\end{align*}
\]

whereby we obtain the following equation for $\bar{u}(\bar{\sigma}, \bar{x})$:

\[
\frac{\partial}{\partial \bar{x}} \left( 2 \frac{\partial \bar{u}}{\partial \bar{\sigma}} + \frac{\bar{u}}{\bar{\sigma}} - 2 \bar{u} \frac{\partial \bar{u}}{\partial \bar{x}} \right) - \frac{1}{\bar{\sigma}} \frac{\partial}{\partial \bar{\sigma}} \left( \frac{1}{\bar{\sigma}} \frac{\partial \bar{u}}{\partial \bar{\sigma}} \right) = 0,
\]

for which we seek a solution satisfying the following condition:

\[
\lim_{\bar{\sigma} \to \infty} \sqrt{\bar{\sigma}} \bar{u} = \begin{cases} -y(\bar{x}), & \text{tail shock}, \\
-y(\bar{x}) + 1, & \text{head shock}, \end{cases}
\]

expressing the connection with the distal approach behavior. To express the connection with the proximal approach behavior, let us instead use the dominant equations (167)-(168) with the change of variables and functions

\[
\begin{align*}
H_1 \bar{x} &= \pm \Lambda + 2^{-\frac{2}{5}} (\gamma + 1) \frac{6}{5} \varepsilon \frac{1}{5} \bar{\xi}, \\
\bar{\pi} &= 2^{-\frac{13}{15}} (\gamma + 1) \frac{4}{5} \varepsilon \frac{4}{5} \bar{\zeta}, \\
\bar{u} &= 2 \bar{u} (\gamma + 1) \frac{2}{5} \varepsilon \frac{4}{5} \bar{u},
\end{align*}
\]

which lead to the following equation:

\[
\frac{\partial^2 \bar{u}}{\partial \bar{\zeta}^2} - \bar{\zeta} \frac{\partial^2 \bar{u}}{\partial \bar{\xi}^2} + \frac{\partial}{\partial \bar{\xi}} \left( \bar{u} \frac{\partial \bar{u}}{\partial \bar{\xi}} \right) = 0.
\]

This equation tolerates the presence of shocks with the condition

\[
[\bar{u}] + \left( \frac{d \bar{\zeta}}{d \bar{\xi}} \right)^2 \left[ \frac{\bar{u}^2}{2} - \bar{\zeta} \bar{u} \right] = 0,
\]
if \([f]\) designates the discontinuity of \(f\) across the shock in question, whose slope is \(\frac{d\zeta}{d\xi}\).

The connection with the proximal approach behavior requires that, for large values of \(\tilde{\xi}\) and \(\tilde{\zeta}\), we have

\[
\begin{cases}
\tilde{u} \approx -\frac{1}{2\pi} \Phi\left(\frac{\tilde{\xi}}{|\tilde{\zeta}|^{3/2}}, \text{sgn } \tilde{\zeta}\right), \\
\Phi = \int_0^{\infty} G^*(\Theta^2 \text{sgn } \tilde{\zeta}) \left\{ \frac{y\left(\frac{\xi}{|\tilde{\zeta}|^{3/2}} + \Theta\right)}{\sqrt{\frac{\xi}{|\tilde{\zeta}|^{3/2}} + \Theta}} - \frac{y\left(\frac{\xi}{|\tilde{\zeta}|^{3/2}} - \Theta\right)}{\sqrt{\frac{\xi}{|\tilde{\zeta}|^{3/2}} - \Theta}} \right\} d\Theta.
\end{cases}
\]  

(192)

Indeed, when \(\tilde{\xi}\) and \(\tilde{\zeta}\) tend towards infinity, \(\tilde{u}\) tends toward zero and approaches a homogeneous solution of the Tricomi equation, whose general form is \(\tilde{\xi}^n F\left(\frac{\xi^2}{\zeta^3}\right)\), where here the exponent \(n\) is equal to \(-\frac{1}{4}\). Moreover, it is simple to check that an expression of this type cannot be a solution of (190). The function \(\Phi\) is discontinuous along the curve \(\tilde{\xi} = \frac{2}{3}|\tilde{\zeta}|^{3/2}\) for \(\tilde{\zeta} > 0\), this discontinuity clearly being characteristic of the acoustic shock that we encountered in the solution of the Tricomi equation. The amplitude of the discontinuity is easily calculated: \([\Phi] = \pi\). We also note that \(\Phi\) is singular along \(\tilde{\xi} = -\frac{2}{3}|\tilde{\zeta}|^{3/2}\), still on \(\tilde{\zeta} > 0\). This singularity is logarithmic, that is

\[
\Phi \sim 2 \log \left(\left|\frac{\tilde{\xi}}{\tilde{\zeta}^{3/2}} + \frac{2}{3}\right|^{-1}\right),
\]

(193)

in perfect conformity with (145). Furthermore, we also find

\[
\lim_{|\tilde{\xi}| \to \infty} 2^{1/2} 3^{3/4} |\tilde{\xi}|^{-1/2} \Phi = \begin{cases} 
-\frac{1}{\pi} \int_0^{\infty} \frac{d\tau}{\tau^{1/2}(\tau-1)^{3/4}} & \text{if } \tilde{\xi} > 0, \\
-\frac{1}{\pi} \int_0^{\infty} \frac{d\tau}{\tau^{1/2}(\tau+1)^{3/4}} & \text{if } \tilde{\xi} < 0,
\end{cases}
\]

(194)

in full agreement with (149), that is to say again

\[
\lim_{|\tilde{\xi}| \to \infty} |\tilde{\xi}|^{-1/2} \tilde{u} = \text{Cst}
\]

(195)
Finally, the connection with the distal approach behavior imposes the conditions

\[
\lim_{\zeta \to \infty} \tilde{u}(\tilde{x} + \frac{2}{3} \zeta^{\frac{3}{2}}, \zeta) \zeta^{\frac{1}{4}} = \begin{cases} 
-2^{\frac{1}{3}} y(\tilde{x}), & \text{head shock}, \\
2^{\frac{1}{3}} \left(1 - y(\tilde{x})\right), & \text{tail shock}.
\end{cases}
\]

It is reasonable to conjecture that the configuration realized is that of figure 16. The vicinity of point A may undoubtedly be studied via the technique implemented by Germain in [5].

![Diagram](image)

It may be worth pointing out that the error incurred is \(O(\varepsilon^{\frac{1}{30}})\).

The search for a function \(\tilde{u}\) satisfying equation (190), the shock condition (191), the matching conditions (192), (195), (196) and the vanishing condition at \(\zeta = -\infty\) is undoubtedly very difficult. If we conjecture the existence and uniqueness of such a function, it provides a universal solution, asymptotic to all the problems of ballistic noise focusing. *Let us suppose that, in the vicinity of the focus, the ballistic noise of a supersonic aircraft is given by (139); if \(\varepsilon^{\frac{1}{5}}\) may be considered small, the maximum noise strength in the focal zone is given by the formula*

\[
(197) \quad \left(\frac{|p - p_0|}{p_0^6 c_0^2}\right)_{\text{Max}} = \text{Cst}(\gamma + 1)^{-\frac{1}{5}} \left(\frac{c_0^* T^*}{R}\right)^{\frac{2}{5}} \varepsilon^{\frac{4}{5}},
\]

*where the unknown numerical constant is a universal constant independent of the aircraft, the trajectory, the atmosphere, and the shape of the caustic.*
3.4 DESCRIPTION OF THE FAR-FIELD.

The far-field is obtained, in formula (132), by taking $\sigma \to -\infty$. The result of this passage to the limit is clearly

\[
\lim_{\sigma \to -\infty} |\sigma|^\frac{1}{2} \hat{u}_0 = \frac{1}{\pi} \int_{\hat{x}}^{\infty} \frac{f(\xi)}{\sqrt{\xi - \hat{x}}} \, d\xi.
\]

Returning to dimensioned variables we find

\[
\frac{p - p_0^*}{\rho_0^* c_0^*} = \varepsilon \left| 1 - \frac{T}{T^*} \right|^{-\frac{1}{2}} E\left( -\frac{x}{\Lambda} \right),
\]

with

\[
\pi E(z) = z \log \left| \frac{z - 1}{z + 1} \right| + 2
\]

The graph of the function $\pi E(z)$ is shown in figure 18; it has a logarithmic singularity at $z = \pm 1$. It is precisely the singularity that we have already encountered in (145). To remove this singularity it would be necessary to know how to solve the problem of paragraph 3.3.20.

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{figure18.png}
\caption{Graph of $\pi E(z)$}
\end{figure}

4 Conclusion.

The present Memoir was intended to present a theoretical contribution to the problem of ballistic noise from a supersonic aircraft approaching a focus. Because of the practical interest of the problem,
it is no doubt advisable to emphasize the results obtained. To apply the theory presented here, engineers must perform some preliminary calculations. It may be necessary to obtain $\varepsilon$, $\Lambda$, and $R$ which are defined by the formulas (18), (19), (23) of chapter 2. The aircraft is only involved in $\varepsilon$, while the sound ray and trajectory are factors in $\varepsilon$, $\Lambda$, and $R$, and the caustic is involved in $R$, whose geometric interpretation is clear: it is the relative radius of curvature of the caustic and the sound ray in a plane normal to the caustic. The wording of paragraph 3.2.30 indicates how an N wave is distorted when approaching the caustic and the formulas (147) and (148) indicate the strength and shape of the pressure signal on the caustic itself if we exclude the neighborhood of the cusp points. Because of its simplicity, the result of section 3.3 is probably the most significant of the entire study. Without providing a complete solution since the value of the numerical constant is not known, it provides a similitude rule. In particular, for a given aircraft, this rule permits us, via a parametric study, to search for the most favorable trajectory or trajectories from the point of view of ballistic noise focusing. It is nevertheless appropriate to draw the reader’s attention to one disappointing aspect: the condition $\varepsilon^{-1/5} \ll 1$ is in principle required for the validity of the result, but this condition is equivalent to $\varepsilon^{-1/5} \gg 1$, which translates to a very high amplification of noise on the caustic. In practice, the amplification remains moderate, so that the results of the present study are limited in scope.

In fact, the value, if any, of this study lies rather in the heuristic indications that it provides on the asymptotic structure of the solutions of a quasi-linear hyperbolic system of a particular type.

**BIBLIOGRAPHY.**


Appendix F


In this appendix we provide a translation of reference [10], the French-language doctoral dissertation of Thierry Auger, “Modélisation et simulation numérique de la focalisation d’ondes de choc acoustiques en milieu en mouvement. Application à la focalisation du bang sonique en accélération.” (Modeling and numerical simulation of the focusing of acoustic shock waves in a moving medium. Application to the focusing of sonic boom during acceleration.), defended in 2001 at the Sorbonne Université (formerly Université Pierre et Marie Curie, Paris VI). The first page of the original document is also provided for reference.

This work is the intellectual property of Thierry Auger and the Sorbonne Université. Permissions to include its translation in the thesis were obtained from Thierry Auger on August 27, 2021. A thanks is also owed to François Coulouvrat—the advisor of Auger’s dissertation—for providing the original document and supporting materials.

Translation was assisted by use of the machine-learning translation tools Google Translate and DeepL, while equation typesetting, document formatting, and figure editing\(^{(1)}\) were done by hand. We take full responsibility for any technical or typographical errors introduced during this process. We have attempted to emulate the formatting of the original document, but differences in word/character length from one language to another have made it difficult to maintain consistent page numbering. The upper right page numbers are (approximately) those of the original work, while the bottom center numbers are the page number within the present document.

The translation may be cited as:


\(^{(1)}\)Figure translations were carried out with a raster editing program, unfortunately (but necessarily) lowering the quality of the original vector graphics.
THÈSE de DOCTORAT DE L'UNIVERSITÉ PARIS VI

Spécialité : MÉCANIQUE

présenté par

Thierry AUGER

pour obtenir le grade de DOCTEUR de L'Université Paris VI

Sujet de thèse :

MODÉLISATION ET SIMULATION NUMÉRIQUE DE LA FOCALISATION D'ONDES DE CHOC ACOUSTIQUES EN MILIEU EN MOUVEMENT. APPLICATION À LA FOCALISATION DU BANG SONIQUE EN ACCÉLÉRATION.

Soutenue le 26 janvier 2001

devant le jury composé de :

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R. ETCHEVEST
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336
DOCTORAL THESIS FROM L'UNIVERSITÉ PARIS VI

Specialty: MECHANICS

presented by

Thierry AUGER

to obtain the degree of DOCTOR from L'Université Paris VI

Subject of thesis:

MODELING AND NUMERICAL SIMULATION OF THE FOCUSING OF ACOUSTIC SHOCK WAVES IN A MOVING MEDIUM. APPLICATION TO THE FOCUSING OF SONIC BOOM DURING ACCELERATION.

Submitted the 26th of January 2001

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M. ROUSSEAU  Examiner
S. ZALESKI  President
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Résumé

L’objectif de cette étude est de modéliser et de simuler numériquement la focalisation d’ondes de choc acoustiques dans un milieu en mouvement afin d’estimer la gêne acoustique au sol induite par la focalisation du bang sonique en accélération. La propagation classique du bang sonique de l’avion jusqu’au sol est modélisée par l’acoustique géométrique. Celle-ci tombe en défaut au voisinage des caustiques, qui correspondent à des zones d’amplification du champ acoustique. La théorie de la focalisation du bang sonique a été établie par J. P. Guiraud en 1965. Au voisinage de la caustique, le champ de pression est décrit à travers une théorie de couche limite et se raccorde à l’acoustique géométrique loin de la caustique. La forme générale des signaux est conforme à celle observée lors des essais en vol mais les chocs conduisent à des pics infinis. Pour obtenir des amplitudes finies, il est nécessaire de prendre en compte les effets non linéaires. Il en résulte l’équation de Tricomi non linéaire. La présence d’un vent de vitesse inférieure à 100km/h, hypothèse convenablement vérifiée près du sol, modifie essentiellement l’épaisseur de la couche limite de diffraction. Pour résoudre l’équation de Tricomi non linéaire, un algorithme itératif, basé sur une version instationnaire de cette équation, est utilisé. Cet algorithme est inspiré du code pseudo-spectral utilisé pour résoudre l’équation KZ. La robustesse de la méthode numérique est démontrée à travers différents tests et la validation du code est effectuée à l’aide de la loi de similitude de Guiraud. Les simulations numériques indiquent que, sous l’influence des nonlinéarités, les pics des signaux reflétés sont transformés en chocs finis tandis que la position du maximum de pression est déplacée. L’application au bang sonique en accélération montre que celleci a une influence faible et conduit à un coefficient d’amplification de l’ordre de 3.

Abstract

The aim of the work is to model and numerically simulate the focusing of weak shock waves in moving media, so as to estimate the annoyance produced on the ground by the focusing of sonic boom in acceleration. In classical theory, sonic boom propagation from the plane down to the ground is described by geometrical acoustics. This approximation fails near caustics, which correspond to zones of signal amplification. The theory of focusing was established by J. P. Guiraud in 1965. Near the caustic, the pressure field is described through a boundary layer theory that is matched to geometrical acoustics far from the caustic. However, weak shock waves lead to infinite pressure peaks. To recover finite amplitudes, it is necessary to take nonlinearities into account. The resulting equation to be solved is the so-called nonlinear Tricomi equation. Adding wind with velocity less than 100 km/h, a well satisfied assumption near the ground, mostly modifies the thickness of the diffraction boundary layer. To solve the nonlinear Tricomi equation, an iterative algorithm, based on an unsteady version of the equation, is used. The algorithm is a modification of the pseudo-spectral code used for solving the KZ equation. The robustness of the numerical scheme has been demonstrated through several tests and the convergence has been validated using Guiraud’s scaling law. The simulations indicate that, under the influence of nonlinear effects, the peaks of reflected signals exhibit finite-amplitude shocks and the position of maximum amplitude peaks is moved away from the geometrical caustic. Application to sonic boom acceleration shows that its value has a small influence and lead to an amplification coefficient roughly equal to 3.
Table of Contents

INTRODUCTION .................................................................................................................. 7

Chapter 1 GEOMETRICAL ACOUSTICS IN A MOVING FLUID

INTRODUCTION .................................................................................................................. 16

1.1 RAY THEORY IN A MOVING FLUID ................................................................. 16
   1.1.1 The linear acoustic equations in a heterogeneous atmosphere with wind .... 16
   1.1.2 The assumptions of linear geometrical acoustics ...................................... 19
   1.1.3 The eikonal equation .............................................................................. 21
   1.1.4 Solutions of the eikonal equation – Fermat’s principle ......................... 22
   1.1.5 The equation of acoustic energy in a moving fluid ............................... 27
   1.1.6 The Blokhintzev invariant .................................................................... 29
   1.1.7 “Ray” Coordinates .............................................................................. 30
   1.1.8 Calculation of the ray tube area ............................................................ 32
   1.1.9 Conclusion ........................................................................................... 34

1.2 REMINDERS ON THE INFLUENCE OF NONLINEAR EFFECTS DURING PROPA GATION ........................................................................................... 34
   1.2.1 Speed of sound in nonlinear acoustics .................................................. 35
   1.2.2 Burgers’ equation ................................................................................ 35
   1.2.3 Burgers’ equation in a stratified atmosphere with wind ...................... 35

1.3 SONIC BOOM FOCUSING ................................................................................. 39
   1.3.1 The focusing phenomenon for a supersonic aircraft ............................. 42

Chapter 2 LOCAL STUDY OF THE ACOUSTIC FIELD NEAR A CAUSTIC

INTRODUCTION .................................................................................................................. 45

2.1 BEHAVIOR OF THE GEOMETRICAL ACOUSTIC SOLUTION NEAR A CAUSTIC ................................................................. 46
   2.1.1 Definition of a caustic .......................................................................... 46
   2.1.2 Curvilinear coordinates associated with the caustic ......................... 47
   2.1.3 Asymptotic expansion of the eikonal function near the caustic ............ 48
   2.1.4 Expression for the geometric divergence near the caustic .................. 59
   2.1.5 The geometric acoustic solution near a caustic ................................... 60
   2.1.6 Conclusion : influence of wind ............................................................. 61
2.1.7 Table summarizing the variables introduced in section 2.1 ................. 62

2.2 THE LINEAR ACOUSTIC FIELD NEAR A CAUSTIC ............................... 63
  2.2.1 Assumptions for the acoustic field near the caustic ....................... 64
  2.2.2 The linear Tricomi equation .......................................................... 65
  2.2.3 Boundary conditions ....................................................................... 71
  2.2.4 Solutions of the linear Tricomi equation ......................................... 71
  2.2.5 Field reflected by the caustic ........................................................ 73
  2.2.6 Insufficiency of the linear Tricomi equation in the case of a sonic boom 74
  2.2.7 Conclusion: general summary ....................................................... 76
  2.2.8 Table summarizing the variables introduced in section 2.2 ............... 79

2.3 EFFECTS OF NONLINEARITY NEAR A CAUSTIC ................................. 80
  2.3.1 The nonlinear Tricomi equation .................................................. 80
  2.3.2 Conclusion ................................................................................... 84

Chapter 3  PRESENTATION OF THE NUMERICAL SCHEME

INTRODUCTION .................................................................................... 85

3.1 UNSTEADY TRICOMI EQUATION .................................................. 86
  3.1.1 Justification of the choice of the unsteady Tricomi equation ............. 87
  3.1.2 Initial conditions ........................................................................... 90
  3.1.3 Mathematical nature of the nonlinear Tricomi equation ................... 90

3.2 PRINCIPLE OF THE PSEUDO-SPECTRAL METHOD ....................... 92

3.3 SOLUTION OF THE STATIONARY LINEAR TRICOMI EQUATION ....... 94
  3.3.1 Fourier Transform of the stationary linear Tricomi equation .......... 94
  3.3.2 Fourier Transform of the boundary condition at positive infinity .... 95
  3.3.3 Boundary condition at negative infinity ....................................... 95
  3.3.4 Discretization of the problem ....................................................... 96
  3.3.5 Matrix representation of the problem ............................................ 96
  3.3.6 Inversion via the Thomas algorithm ............................................. 98
  3.3.7 Choice of calculation domain ...................................................... 98
  3.3.8 Validation of the method in the stationary linear case ..................... 100

3.4 VALIDATION OF RESULTS IN THE LINEAR STATIONARY CASE ....... 101
  3.4.1 Comparison of results with analytic solutions ............................... 101
  3.4.2 Influence of the number of frequencies ....................................... 104
  3.4.3 Influence of the number of discretization points of the z axis .......... 107
  3.4.4 Improvement of the boundary conditions at plus and minus infinity 112
  3.4.5 Conclusion .................................................................................. 113

3.5 SOLUTION OF THE UNSTEADY TRICOMI EQUATION .................... 116
  3.5.1 Fourier Transform of the unsteady linear Tricomi equation and the boundary conditions ......................................................... 116
  3.5.2 Discretization of the problem ....................................................... 117
  3.5.3 Matrix representation of the problem ......................................... 117

341
3.5.4 Validation of the method in the unsteady linear case ........................................ 118
3.5.5 Solution of Burgers’ equation ........................................................................... 119
3.5.6 Definition of the iteration step and the convergence criterion for the numerical code .................................................................................. 122
3.5.7 Solution of the nonlinear Tricomi equation .................................................. 122
3.5.8 Particularities of the nonlinear case ............................................................... 123

3.6 VALIDATION OF THE METHOD IN THE NONLINEAR CASE .......... 124
3.6.1 Comparison of solutions obtained in the linear and nonlinear case ............... 125
3.6.2 Influence of the value of the coefficient of nonlinearity \( \mu \) ....................... 130
3.6.3 Influence of the initial estimate ...................................................................... 130
3.6.4 Influence of a new distribution of the terms of the Tricomi equation in the time and frequency domains ...................................................... 136
3.6.5 Influence of the number of frequencies ....................................................... 138
3.6.6 Guiraud’s similitude law ............................................................................. 144
3.6.7 Conclusion ............................................................................................... 148

3.7 CONCLUSION OF THE CHAPTER .................................................. 148

Chapter 4 APPLICATION TO SONIC BOOM FOCUSING

INTRODUCTION ........................................................................................................ 149

4.1 STUDY OF THE INFLUENCE OF PARAMETERS ............................................. 149
4.1.1 Influence of the nonlinearity factor \( \mu \) ..................................................... 149
4.1.2 Influence of rise time on the results ........................................................... 152
4.1.3 Influence of an incident signal possessing three shocks .............................. 155

4.2 APPLICATION TO UNDER-TRACK ACCELERATION FOCUSING ........ 158
4.2.1 Description of the propagation code ............................................................... 158
4.2.2 General description of results ..................................................................... 159
4.2.3 Dimensioned pressure at the caustic .......................................................... 160
4.2.4 Conclusion .................................................................................................. 164

CONCLUSION ........................................................................................................ 164

BIBLIOGRAPHICAL REFERENCES ........................................................................ 169

APPENDIX I THE THOMAS ALGORITHM ...................................................... 173

APPENDIX II THE McDONALD AND AMBROSIANO SCHEME .......... 175
INTRODUCTION

THE PHYSICAL PHENOMENON

The sonic boom, also called ballistic detonation, is a natural acoustic phenomenon, associated with any vehicle moving in the atmosphere at a speed greater than the local speed of sound. This manifestation is linked to the aircraft throughout its supersonic flight (and not only, contrary to popular opinion, just passing the sound barrier). The sonic boom results from the propagation in the atmosphere of the acoustic waves created by the movement of the plane and their accumulation on a privileged surface, which is the Mach cone. In the case of a rectilinear and uniform supersonic flight, in a homogeneous, quiescent atmosphere, the disturbed domain is located inside the Mach cone. The half-angle $\theta_{av}$ of this cone is determined by the Mach number of the aircraft: $\sin \theta_{av} = 1/M_{av}$. The sonic boom, as perceived by an observer at ground level, is the culmination of a complex process, consisting essentially of two distinct phases. The first corresponds to the creation of the wave as described above; in the vicinity of the aircraft, the pressure signal has a complex shape linked to the details of the geometry of the aircraft. During the second phase, this complex form changes during its progression under the action of the mechanisms of atmospheric refraction and nonlinear propagation. Far from the aircraft, the signal takes the regular form of an “N”; the profile of the wave is then preserved throughout the propagation, but its amplitude decreases.

On the ground, there are different regions depending on the nature of the sonic boom observed (Figure 0.1, p. 3). First of all, the primary carpet extends on either side of the projection of the trajectory on the ground (the trace), over a width of 10 to 100 km (depending on the flight conditions); it is a corridor on which the classic boom is heard, corresponding to the propagation of waves coming directly from the aircraft. In this region, the acoustic signal generally presents 2 shocks (sometimes more), responsible for the discomfort caused, similar to that of a nearby thunderclap. The signal strength decreases as we approach the edges of the carpet (cutoffs). Outside this carpet is the shadow zone. In this area, the noise comes from the propagation of creeping waves along the ground, coming from the boom at the cutoff. The boom remains audible for a few kilometers inside the shadow zone, but its amplitude decreases and the shock waves disappear, which reduces the discomfort considerably. Finally, several hundred kilometers beyond the cutoff, extends the secondary carpet corresponding to the waves that have propagated in the upper atmosphere located above the aircraft. The secondary boom is inaudible but can induce vibrations of light structures. Atmospheric variability has a strong impact on the sonic boom. Wind and temperature gradients as well as the turbulence on the ground influence, on the one hand, the position and the extent of the primary carpet, and on the other hand, the amplitude and the discomfort (rise time) of the acoustic signals.

Finally, certain aircraft maneuvers (acceleration or turn) can lead to local amplification (focusing) of the boom at the edge of the carpet, by a factor of up to 3 or more; the shape of the signal is also considerably changed. Unlike the primary boom, the “focused” boom can potentially cause damage to structures (breakage of windows). Focusing in turns may be avoided by prohibiting weakly Mach supersonic maneuvers. On the other hand, the acceleration focus cannot be avoided.
when the aircraft slightly exceeds Mach 1 (between Mach 1.15 and 1.2), over a crescent zone with
a length of a few kilometers and a width of about a hundred meters. It is this latter phenomenon,
associated with the sonic boom, which will be the subject of the present study.
Various summaries on the sonic boom are presented by Carlson and Maglieri (1972), Plotkin (1989)

BIBLIOGRAPHY

Modeling of the source and propagation
The modeling of the pressure field in the vicinity of a slender profile, of varying cross-section, and
its wake, in a supersonic flow, was established by Whitham (1952). He then added the wings to
his modeling (1956). Rao (1956) extended the study to an unsteady flow for a slender profile and
Walkden widened it to a “fuselage + wing” system in 1958. This modeling of the pressure field is a
linearized theory, valid for a slender body. According to this theory, the airplane (fuselage + wing)
is replaced by its “skeleton” on which mass flows and force densities are distributed, resulting,
on the one hand, from the air masses set in motion by the airplane when it enters the air and,
on the other hand, forces applied in reaction to the lift forces exerted on it. The sound field is
then described by a wave equation, with line source terms (for the fuselage) and surface sources
(for the wings) representing the aircraft. The equation can be solved by the method of retarded
potentials. Far from the aircraft (at approximately a wingspan), we then show that the acoustic
field propagates along acoustic rays, normal to the Mach cone, the time signature being given by
the “F” function, the so-called Whitham function. This theory is correct for a flight at Mach 2 but
is not suitable for transonic and hypersonic flights for which the Euler equation must be used.

The study of the propagation of the sonic boom from the aircraft to the ground is provided by
geometrical acoustics. This approximate theory was adapted to the study of sonic boom by Hayes
(1968a) through a numeric code, still used in the USA (Hayes et al. 1969). It is an asymptotic
method valid at high frequencies which assumes that the acoustic perturbations emitted by the
airplane propagate along acoustic rays normal to the Mach cone. The rays are influenced by
atmospheric refraction associated with the presence of a vertical gradient of temperature and wind.
The ray tracing is obtained by Fermat’s principle, and the amplitude of the acoustic disturbance,
by conservation of the acoustic intensity along a ray tube according to the Blokhintzev invariant
(1946). Throughout the propagation of these acoustic disturbances, nonlinear effects, inseparable
from intense signals and therefore from sonic booms, are taken into account, by correcting the
speed of sound by the instantaneous amplitude of the pressure signal. This code allows fairly
precise modeling of the primary carpet but does not provide results for the limiting rays because
these are parameterized by altitude. Candel (1977) proposed a general method for calculating ray
tube areas by parameterizing acoustic rays by the eikonal function. Geometrical acoustics may
then be extended to the limiting rays of the carpet.
However, the geometrical acoustics approximation fails in the vicinity of caustics which correspond
to regions where there is convergence of acoustic rays. Indeed, this theory furnishes an infinite
pressure in the vicinity of this region, which is physically unacceptable: there is a singularity.
In these regions, the effects of diffraction, not taken into account in the geometrical acoustic
approximation, can no longer be neglected.
Figure 0.1: The sonic boom on the ground

**Focused Sonic Boom**
- Amplified, "U" wave
- Up to 500 Pa

**Primary sonic boom**
- "N" wave
- ~10 to 100 Pa

**Shadow zone and silence**
- Smooth wave, amplitude with distance
- <10 Pa

**Secondary boom**
- Weak low frequency wave
- ~0.1 Pa
Theory of focusing

The caustics associated with the boom are the singularities of geometrical acoustics. Thus, they fall within the general framework of catastrophe theory (Berry, 1981) which classifies structurally stable caustics (catastrophes). We will limit this study to the case of the simplest catastrophe which is the “fold caustic” (the first in catastrophe theory) which is also called an “Airy caustic”, as represented by the Airy function in the linear case. In the sixties, two different theories were proposed to describe the focusing of shock waves. Whitham (1957) supposed that nonlinear effects, appearing when rays are about to converge, prevent the creation of the line or point of focus, by bending the rays away from each other. It is therefore a totally nonlinear theory. The second theory is the purely linear classical theory of diffraction. Formalized by Buchal and Keller (1960) in the framework of the theory of matched expansions, it permits us to describe the acoustic field using the Airy function, and is part of the more general framework of catastrophe theory.

The link between these two theories was established by Guiraud in 1965. In the vicinity of the caustic, he showed that the dominant phenomenon was linear diffraction. By analogy with transonic flows, he established that the pressure field, in the vicinity of the caustic, satisfies the so-called Tricomi equation. However, the presence of shocks in the signals leads to infinite peaks. Thus according to an expression of Guiraud himself, for the Tricomi equation “to tolerate the presence of shock”, it is necessary to add a nonlinear term to it. The link between Whitham’s completely nonlinear theory and linear diffraction theory was established. Guiraud also deduced a nonlinear similarity law which links the maximum overpressure on the caustic to the amplitude of the incident signal. These results were confirmed by Hayes (1968b).

In order to treat the supersonic flow around a moving wing profile in a two-dimensional atmosphere, in the presence of a weak wind gradient, Pechuzal and Kevorkian (1977) recovered Guiraud’s results through a simpler formalism using the principle of matched asymptotic expansions.

Coulouvrat (1997) extended the previous results to a heterogeneous three-dimensional atmosphere.

Rosales and Tabak (1997) also recovered the nonlinear Tricomi equation and Guiraud’s similitude law in the case of a homogeneous two-dimensional atmosphere. In addition, they put forward the possibility of a new singularity of the solutions of the Tricomi equation near the maximum pressure: the triple point. This consists of the meeting of incident and reflected shocks at a point merging into a single shock extending over a finite distance, then forming a “Y”. Very fine numerical simulations had previously drawn their attention to this fact (1994). They have shown that the existence of a triple point is incompatible with the nonlinear Tricomi equation, although it had been observed in experimental simulations (Sanai, 1976). This paradox is similar to that of Von Neumann relating to the reflection of oblique shocks (1963).

In addition, Lipkens and Blanc-Benon (1994) and Blanc-Benon et al. (1995), through their work on the propagation of weakly nonlinear acoustic waves in a turbulent medium, have shown, experimentally and numerically, that random focusing due to the heterogeneity of the medium played a central role in the propagation of the boom. In particular, the presence of several rays connecting the aircraft and the observation point on the ground, some of which may have tangented caustics, explained the complex shape of some of the observed signals, as well as their finite rise time.
Numerical solution of the nonlinear Tricomi equation

Inspired by transonic aerodynamics, Seebass (1971) and Gill and Seebass (1973), with the aim of finding solutions to the nonlinear Tricomi equation, developed an approximate analytical solution for an incident step function. This solution was obtained by applying the so-called hodograph transform to the equation in order to reduce it to the linear Tricomi equation. This transformation is therefore a nonlinear process. However, the linear boundary conditions associated with the nonlinear Tricomi equation are transposed as such in hodograph space; the problem associated with these boundary conditions is therefore approached (Gill, 1974). By Fourier transform, they found a solution which they then evaluated numerically. Returning to the physical domain, the hodograph transform leads to the appearance of multivalued or, on the contrary absent portions of curves. This double difficulty is overcome by introducing either shocks or simple expansion waves. This rather complex procedure only works well for low amplitude signals. Signals of high amplitude are deduced by applying Guiraud’s similitude law. This technique was extended by Fung (1980) to a wider range of incident signals and was also used by Plotkin and Cantril (1976) who applied it to the case of sonic booms.

Direct numerical simulations of the nonlinear Tricomi equation were undertaken using finite difference algorithms. Thereby Seebass et al. (1971) implemented a “shock capturing” technique on a mesh aligned with the linear wavefronts in the vicinity of the caustic. This technique is an extension of that used by Murman and Cole (1971) to numerically simulate transonic flows. In the vicinity of the shocks, a numerical viscosity is introduced in order to regularize the numerical solution. The numerical scheme resulting from this method is stable and convergent. However, the numerical solutions obtained do not give a satisfactory representation of the discontinuities, which are too spread out. The peaks resulting from the focusing of the shock waves therefore have an amplitude reduced in an uncontrolled manner by the numerical viscosity.

Another method is that of “shock fitting” originally proposed by Moretti (1972) which assumes, a priori, the existence of discontinuities; thus, the partial differential equation is discretized so as never to “cross” them. This procedure was applied by Yu and Seebass (1974) for solving the nonlinear Tricomi equation. It makes it possible to determine, at the same time, the position of the shocks and their amplitude with reasonable accuracy. It therefore requires a smaller number of discretization points and shorter computation time in comparison with the “shock capturing” method. However, the position of shocks is to be determined and for a complex signal it may be numerically costly. Finally, this procedure is not unconditionally stable, which leaves doubt about the validity of the results.

A recent method for simulating the focusing of shock waves has been proposed by McDonald and Kuperman (1987) using the NPE equation (“Nonlinear Progressive wave Equation”). This equation appears as an extension, in the time domain, of the parabolic approximation used to treat linear time-periodic signals in frequency space. The NPE equation is similar to the KZK equation (Zabolotskaya and Khokhlov, 1969). However, in the case of the NPE equation, the evolution is temporal and not spatial. The NPE equation contains separate terms for the following physical processes: refraction, nonlinear steepening, radial propagation and diffraction. It thus allows the study of these phenomena taken individually or collectively, without going into the frequency domain. The wave considered propagates as a plane wave in one dimension, the transverse phenomena (such as diffraction) being assimilated into small disturbances. This method allows one to monitor the propagation of the signal, not on a fixed domain in which there is “numerical waste” for regions at rest, but following a moving frame in the direction for which the change
in the acoustic disturbance is the strongest. In their article, McDonald and Kuperman (1987) present the phenomenon of focusing weak shock waves on a simple caustic as an application of the NPE equation. The results they present are in qualitative agreement (transformation of the wave into “N” into a wave in “U” and displacement of the maximum pressure above the caustic) with Guiraud’s theory but no study or systematic validation has been undertaken. Piacek (1997) adopted this model, adding dissipation effects. He wrote a numerical code with finite differences, modeling the behavior of a step function in the vicinity of a “cusped” caustic. He was thus able to observe how the mechanisms of nonlinearity and diffraction controlled the focusing. However, the validation of his numerical code is partial (linear or one-dimensional case).

We may also cite the work of Liang (1995) and Inoue (1995) concerning the use of finite volumes on the Euler equation through a numerical scheme called TVD (Total Variation Diminishing) to simulate the focusing of reflected signals respectively on a parabolic surface and in a logarithmic spiral duct.

Test campaigns

In parallel to this theoretical work, test campaigns related to sonic boom focusing were carried out. From a practical point of view, these tests were intended to give an approximation of the noise level associated with the focusing of the sonic boom. For this the coefficient of focusing amplification was defined as the ratio between the maximum overpressure of the “focused” signal and the amplitude of the incident signal, coming directly from the aircraft, at a certain (unspecified) distance from the caustic. We quickly notice that it is difficult to obtain a correct amplification coefficient. In fact, accuracy is linked to three limiting factors:

- weather variability makes it difficult to locate the geometric caustic,
- an insufficient number of sensors on the ground which will prevent, in the majority of tests, obtaining the maximum value of the “focused” signal,
- the location of the reference signal coming directly from the airplane must be close enough to the caustic without being influenced by it.

These tests did however enable us to obtain the range in which the amplification of the “focused” signal is located.

The first test campaigns were organized by the USA in 1961. From that date, tests have been carried out episodically. Here, we will cite the results of 4 American test campaigns and a French program comprising 3 test campaigns with interesting results.

Maglieri et al. (1966) present tests which aimed to obtain, on the one hand, the distribution of pressure levels across the primary carpet and, on the other hand, the location and the pressure levels of the focus during acceleration. The amplification coefficient for these tests was 2.5.

Haglund and Kane (1974) summarize the tests carried out at the Edwards AFB base in 1970, concerning the measurement of focusing at altitude with an instrumented tower (BREN Tower) placed in the scanning area of a supersonic aircraft. This device made it possible to analyze the creation of caustics for an airplane flying at the critical Mach number. The focusing amplification coefficient in this case was 1.8. The latter approached 3 during uncontrolled accelerations of the aircraft. Furthermore, the amplification coefficient obtained for an accelerated longitudinal flight varied from 2 to 5 for a flight at Mach 1.2.
Plotkin (1990) comments on tests performed in the USA, with an SR71 and 3 smaller fighter planes, which aimed to analyze the focus induced by tactical maneuvers. One of the most interesting results is the fact that the focus created by high “g” turns draws smaller surfaces on the ground than those induced by smoother maneuvers of a larger plane. In addition, these high “g” maneuvers lead to amplification coefficients of 2 to 3 while the acceleration performed by the SR71 leads to an amplification coefficient of 5.

The last campaign to date was executed in 1994 and is the result of a collaboration between “USAF Armstrong Laboratory” and “USAF Test Pilot School.” These trials were the subject of an article by Downing et al. (1998). The amplification factor, defined above, was not constant from one trial to another but was always between 2 and 4.5. Furthermore, it was observed that turbulence had a “defocusing” effect on the signals, leading to a reduction in the strength of the overpressures.

We may also cite, on the French side, the STÀ e-CEV program, which is still a benchmark today for the quality of the results obtained and which included 3 test campaigns:
- Operation Jéricho Focus (1966),
- Operation Jéricho Virage (1967),

These tests were the subject of a very complete article (Wanner et al. 1972). During this campaign, different types of focus due to aircraft maneuvers were observed and recorded: acceleration at low and high altitudes, turning and turn entry with superfocus and stationary rectilinear flight (associated with refractive caustics). The main observation is that, whatever the type of focusing, the amplification factor is always equal to 5. The latter is equal to 9 in the case of superfocusing. Furthermore, these experiments have shown that acceleration has a very small influence on the value of this ratio.

Plotkin and Cantril (1976) analyzed the different types of focusing occurring during particular maneuvers of a supersonic aircraft from a geometrical (rays and caustics) and theoretical point of view. In doing so, they showed that a finite rise time had to be added to the theoretical shock waves in order for the test results and theory to agree.

**Experimental simulations of focusing in the laboratory**

Since test campaigns are expensive and difficult to implement, focusing has mainly been studied in the laboratory. We may distinguish the experiments intended to exactly reproduce the focus of a supersonic aircraft from those with a less industrial scope.

Concerning the focusing of a supersonic aircraft, Beasley et al. (1969) organized experiments with “N” waves produced by a spark for the purpose of identifying the theory best suited to the sonic boom phenomenon. They thus showed that Whitham’s theory was unsuitable for modeling focusing.

The remarks concerning the triple point, evoked above, were inspired to their authors by the results of experiments of Sturtevant and Kulkarny (1976) on the focusing of weak shock waves.
The aim of these experiments was to study the transition between linear and totally nonlinear focal behavior, by comparing the Whitham model with classical geometrical acoustics. It appears that weak shock waves can be described by the linear theory of diffraction while strong shocks must be approached by the completely nonlinear theory of Whitham. The authors have thus shown the need to take into account the mechanisms associated with nonlinearity to avoid the formation of very pronounced unrealistic peaks, which is in line with Guiraud’s theoretical conclusions.

For the purpose of studying refraction caustics, Sanai et al. (1976) performed tests with rifle bullets fired into a tank filled with a stratified mixture of air and carbon dioxide. This study made it possible to give a value to the constant involved in Guiraud’s similitude law. However, the value found by these experiments is two times smaller than the theoretical value found by Gill and Seebass (1975). Furthermore, the presence of triple shocks was also observed during these experiments.

OBJECTIVE OF THE STUDY AND ORGANIZATION OF THE THESIS

In order to best estimate the “focused” noise of a signal emitted by a supersonic aircraft, at ground level, the study, which this work summarizes, aims to complete the current state of knowledge on the focusing of weak shock waves, summarized in the following paragraph. Thus, the two objectives of this study are, on the one hand, to complete Guiraud’s results by widening his study to the case with wind and, on the other hand, to write and to validate, including in the nonlinear case, a numerical code solving the nonlinear Tricomi equation. This code will eventually be integrated into the general sonic boom prediction code developed by EADS Airbus SA. In the first two chapters, the theoretical aspect of the problem with the addition of the wind will be studied, while the following two will be concerned with its numerical aspects.

In the first chapter, we will recall the theory of geometrical acoustics, in the presence of wind, used for the study of the classical propagation of the sonic boom. We will also recall the main results concerning consideration of nonlinear effects during the propagation of a signal from the aircraft. Next we will describe the different types of caustics caused by supersonic flights.

The second chapter concerns the study of the pressure field in the vicinity of these caustics, in the presence of wind. With the help of the method of matched asymptotic expansions, we will introduce a boundary layer in the vicinity of the caustic, inside which diffraction phenomena, neglected by geometrical acoustics, will be taken into account. The presence of shocks in sonic boom signals will force us to introduce nonlinearities to limit the amplitude of the field. Finally, we will show that in the presence of a low speed wind, the pressure field near a caustic always satisfies the nonlinear Tricomi equation, up to a change of variables.

The numerical development of this equation will be the subject of a third section. The equation being nonlinear, an iterative scheme is necessary. We will base it on an unsteady equation which will be constructed from the nonlinear Tricomi equation. The choice of the unsteady term will be discussed in the first section. This new equation will be solved by a pseudo-spectral method: in the same iterative step, the term associated with diffraction will be treated in the spectral domain while the nonlinear portion is calculated in the time domain through the classical Burgers’ equation.
Finally, these two processes will be repeated at each iteration until the code converges. The solution of the Tricomi equation will first be described and validated in the linear case, then in the nonlinear case. The validation of the whole code (nonlinearity + diffraction) will be carried out using Guiraud’s similitude law.

In the last chapter, we will focus more specifically on the focusing of the sonic boom, for which this study was intended. From this perspective, we will first analyze the influence of three parameters on the results obtained: variation of the coefficient of nonlinearity, addition of a rise time in the “N” signals, and the presence of more than 2 shocks in the incident signal. Finally, in the second and last section we will seek to quantify the amplification of the focused signal in acceleration under-track, for the case of a civil supersonic transport aircraft of Concorde type, by coupling the nonlinear Tricomi equation solution code with the boom simulation code currently developed by EADS Airbus SA.
Chapter 1

GEOMETRICAL ACOUSTICS
IN A MOVING FLUID

INTRODUCTION

In general, the propagation of the sonic boom from the aircraft to the ground is modeled by geometrical acoustics. In this first chapter, we are going to recall the general principles: it is an asymptotic theory valid at high frequencies which supposes that the wave propagates locally like a plane wave, whose amplitude and direction vary slowly along curves called acoustic rays. Thus in the first section, the theory of rays in a moving fluid will be presented. We will then see that the ray tracing is obtained by Fermat’s principle and the amplitude of the signal is given by the Blokhintzev invariant (1946). In a second section, we will recall the main results concerning the consideration of nonlinear effects during the propagation of a signal coming from an aircraft. Finally, in a last section, we will describe the different types of caustics caused by supersonic flights; these being regions of space where there is amplification of noise and on which we may no longer apply geometrical acoustics.

1.1 RAY THEORY IN A MOVING FLUID

In this first section we will establish the equations of linear acoustics in a weakly stratified medium with wind. To do this, we will proceed in the manner of Pierce (1989), by generalizing the geometrical acoustic approximation given by Coulouvrat (1997) in the windless case. We will then be led to more precisely introduce the notion of acoustic ray, then of ray tube. We will see that these two notions are then sufficient to calculate the linear acoustic field.

1.1.1 The linear acoustic equations in a heterogeneous atmosphere with wind

In this study, we will only be interested in the propagation of the boom while neglecting the effects due to the absorption of the atmosphere. These complementary hypotheses allow us to neglect the contribution of mass caused by the source as well as the external forces that it applies to the environment. Since we will not take atmospheric absorption into account, the propagation medium, although made up of different gases, can be considered a single gas. This is why we will consider a perfect, compressible and heterogeneous fluid, stratified under the action of gravitational forces.
We recall the balance laws of fluid mechanics, translating the general principles of conservation of mass, momentum and energy:

**Equation of continuity** \( \frac{d\rho}{dt} + \rho \nabla \cdot \vec{u} = 0 \),

**Momentum balance** \( \rho \frac{d\vec{u}}{dt} = -\nabla p + \rho \vec{g} \),

**Energy balance** \( \frac{ds}{dt} = 0 \).

In these equations, \( \rho \) denotes the density of the medium, \( \vec{u} \) the velocity vector, \( p \) the fluid pressure, \( \vec{g} \) the gravitational acceleration vector, \( e = e(s, \rho) \) the specific internal energy of the medium, \( s \) its specific entropy, and \( T \) its temperature. The notation \( \frac{d}{dt} = \frac{\partial}{\partial t} + \vec{u} \cdot \nabla \) is used for the particle (material) derivative of a quantity whose motion we follow. The Gibbs relationship defines the temperature and the pressure of the fluid from its internal energy:

\[
de = T ds + \frac{p}{\rho^2} d\rho \text{ where } T = \left( \frac{\partial e}{\partial s} \right)_\rho (s, \rho) \text{ and } p = \rho^2 \left( \frac{\partial e}{\partial \rho} \right)_s (s, \rho).
\]

For a perfect fluid, the energy balance equation states that the entropy of each fluid "particle" is constant: \( s = s_0(\vec{a}) \) where \( \vec{a} \) represents the position of the particle in Lagrangian coordinates.

We first assume that the fluid moves with a constant speed \( \vec{u}(\vec{x}) \). The equations of the undisturbed state are then written:

**Equation of continuity** \( \frac{d\rho_0}{dt} + \rho_0 \nabla \cdot \vec{u}_0 = 0 \),

**Momentum balance** \( \rho_0 \frac{d\vec{u}_0}{dt} = -\nabla p_0 + \rho_0 \vec{g} \),

**Energy balance** \( \frac{ds_0}{dt} = 0 \).

With \( p_0 = p(s_0, \rho_0) \).

We now suppose that this same fluid is subjected to a small acoustic disturbance. The quantities associated with this acoustic disturbance are indicated by "a", so we write:

\[
p = p_0 + p_a
\]
\[
\rho = \rho_0 + \rho_a
\]
\[
\vec{u} = \vec{u}_0 + \vec{u}_a
\]
\[
s = s_0 + s_a.
\]

By inserting these definitions into the balance equations, we have:
\[
\frac{\partial (\rho_0 + \rho_a)}{\partial t} + (\rho_0 + \rho_a) \nabla \cdot (\vec{u}_0 + \vec{u}_a) + (\vec{u}_0 + \vec{u}_a) \cdot \nabla (\rho_0 + \rho_a) = 0,
\]

\[
(\rho_0 + \rho_a) \frac{\partial (\vec{u}_0 + \vec{u}_a)}{\partial t} + (\rho_0 + \rho_a) [(\vec{u}_0 + \vec{u}_a) \cdot \nabla] (\vec{u}_0 + \vec{u}_a) = -\nabla (p_0 + p_a) + (\rho_0 + \rho_a) \vec{g},
\]

\[
\frac{\partial (s_0 + s_a)}{\partial t} + (\vec{u}_0 + \vec{u}_a) \nabla (s_0 + s_a) = 0.
\]

By subtracting from these equations those of the undisturbed state and retaining only the linear terms compared to the acoustic disturbances, we get the following linearized system:

\[
\frac{d_0 \rho_a}{dt} + \vec{u}_a \cdot \nabla \rho_0 + \rho_a \nabla \cdot \vec{u}_0 + \rho_0 \nabla \cdot \vec{u}_a = 0,
\]

\[
\frac{\rho_0 d_0 \vec{u}_a}{dt} + \rho_0(\vec{u}_a \cdot \nabla) \vec{u}_0 = -\nabla p_a + \rho_a \vec{g} - \rho_a \frac{d_0 \vec{u}_0}{dt},
\]

\[
\frac{d_0 s_a}{dt} = -\vec{u}_a \cdot \nabla s_0,
\]

with \( d_0 = \frac{\partial}{\partial t} + \vec{u}_0 \cdot \nabla \), the particle derivative associated with the mean flow.

By taking the momentum equation of the undisturbed state and multiplying it by \( \rho_a \), we have:

\[
\rho_a \vec{g} - \rho_a \frac{d_0 \vec{u}_0}{dt} = \frac{\rho_a}{\rho_0} \nabla p_0.
\]

The preceding system is then written

\[
\frac{d_0 \rho_a}{dt} + \vec{u}_a \cdot \nabla \rho_0 + \rho_a \nabla \cdot \vec{u}_0 + \rho_0 \nabla \cdot \vec{u}_a = 0,
\]

\[
\frac{d_0 \vec{u}_a}{dt} + (\vec{u}_a \cdot \nabla) \vec{u}_0 + \frac{1}{\rho_0} \nabla p_a - \frac{\rho_a}{\rho_0} \nabla p_0 = 0,
\]

\[
\frac{d_0 s_a}{dt} = -\vec{u}_a \cdot \nabla s_0,
\]

The equation of state \( p_0 + p_a = p(s_0 + s_a, \rho_0 + \rho_a) \), after linearization, has the following form:

\[
p_a - c_0^2 \rho_a = \left( \frac{\partial p}{\partial s} \right)_{\rho_0} s_a, \text{ with } c_0^2 = \left( \frac{\partial p}{\partial \rho} \right)_{s_0} \text{ the square of the speed of sound.}
\]

Yet, \( c_0 \) and \( \left( \frac{\partial p}{\partial s} \right)_{\rho_0} \) depend only on the position. As a consequence, by calculating the particle derivative (following the mean flow) of the equation of state, we may write:

\[
c_0^2 \left( \frac{d_0 \rho_a}{dt} + \rho_a \nabla \cdot \vec{u}_0 \right) = \frac{d_0 p_a}{dt} - \left( \frac{\partial p}{\partial s} \right)_{\rho_0} \frac{d_0 s_a}{dt} + c_0^2 \rho a \nabla \cdot \left( \vec{u}_0 \right) - c_0^2 s_a \nabla \cdot \left[ \frac{c_0^2}{c_0^2} \left( \frac{\partial p}{\partial s} \right)_{\rho_0} \right].
\]

Furthermore, the gradient of the equation of state gives us:

\[
\left( \frac{\partial p}{\partial s} \right)_{\rho_0} \nabla s_0 = \nabla p_0 - c_0^2 \nabla \rho_0.
\]
Substituting this in the linearized energy equation, we get:

\[- \left( \frac{\partial p}{\partial s} \right)_{\rho,0} \frac{d_0 s_a}{dt} = \bar{u}_a \cdot \nabla p_0 - c_0^2 \bar{u}_a \cdot \nabla \rho_0.\]

Multiplying the linearized mass conservation equation by \( c_0^2 \) and replacing the different terms by their expression above, we have:

\[\frac{d_0 p_a}{dt} + \rho_0 c_0^2 \nabla \cdot \bar{u}_a = -\bar{u}_a \cdot \nabla p_0 - c_0^2 p_a \nabla \cdot \left( \frac{\bar{u}_0}{c_0^2} \right) + s_a c_0^2 \nabla \cdot \left[ \frac{\bar{u}_0}{c_0^2} \left( \frac{\partial p}{\partial s} \right)_{\rho,0} \right].\]

In the linearized equation of momentum, we replace \( p_a \) by \( \frac{p_a}{c_0^2} - \frac{1}{c_0^2} \left( \frac{\partial}{\partial s} \right)_0 s_a \), then we have the following equation:

\[\frac{d_0 \bar{u}_a}{dt} + \frac{1}{\rho_0} \nabla p_a = -(\bar{u}_a \cdot \nabla) \bar{u}_0 + \frac{p_a}{\rho_0 c_0^2} \nabla p_0 - \frac{s_a}{\rho_0 c_0^2} \left( \frac{\partial p}{\partial s} \right)_{\rho,0} \nabla p_0.\]

In the above equations, on the left hand side, we find the terms associated with acoustics in a homogeneous fluid. The medium being weakly heterogeneous, the size of the heterogeneities \( L_{atm} \) (defined by \( L_{atm} = T/|\nabla T| \approx 34 \text{ km} \) for the standard atmosphere) is large compared to the wavelength \( \lambda_{ac} \) (which we may assume equals 80 m); the terms of the right hand side are therefore at least of order \( \lambda_{ac}/L_{atm} \) compared to the terms of the left hand side. However by the same definition of \( s_a \), as written above:

\[- \left( \frac{\partial p}{\partial s} \right)_{\rho,0} \frac{d_0 s_a}{dt} = \bar{u}_a \cdot \nabla p_0 - c_0^2 \bar{u}_a \cdot \nabla \rho_0,\]

\( s_a \) is itself of order \( \lambda_{ac}/L_{atm} \). As a consequence, the terms \( s_a c_0^2 \nabla \cdot \left[ \bar{u}_0/c_0^2 (\partial p/\partial s)_{\rho,0} \right] \) and \( s_a/\rho_0 c_0^2 (\partial p/\partial s)_{\rho,0} \nabla p_0 \) in the equations above are of order \( \lambda_{ac}/L_{atm} \), and so are perfectly negligible.

The system of equations describing linear acoustic disturbances in a perfect heterogeneous and moving fluid can be written as follows after simplifications

\[
\begin{align*}
\frac{d_0 p_a}{dt} + \rho_0 c_0^2 \nabla \cdot \bar{u}_a &= -\bar{u}_a \cdot \nabla p_0 - c_0^2 p_a \nabla \cdot \left( \frac{\bar{u}_0}{c_0^2} \right) \\
\frac{d_0 \bar{u}_a}{dt} + \frac{1}{\rho_0} \nabla p_a &= -(\bar{u}_a \cdot \nabla) \bar{u}_0 + \frac{p_a}{\rho_0 c_0^2} \nabla p_0.
\end{align*}
\]

Conclusion: On the left side are the usual terms of acoustics in a homogeneous fluid, and on the right side the terms which come from the heterogeneity of the medium. As we come to see it, the medium being weakly heterogeneous, the terms of the right side are small (of order \( \lambda_{ac}/L_{atm} \)) compared to those of the left side (of order 1). Locally, the influence of stratification is weak.

1.1.2 The assumptions of linear geometrical acoustics

Geometrical acoustics is an approximate method of estimating the acoustic field, based on the basic assumption that the sound field propagates locally like a plane wave. We recall that the plane wave is a particular solution of the wave equation \( \partial^2 p_a/\partial t^2 - c_0^2 \Delta p_a = 0 \), which is written in the form: \( p_a(t, \vec{x}) = F(t - \vec{x} \cdot \vec{n}/c_0) \). In this expression, \( \vec{n} \) is a unit vector determining the direction of propagation of the wave, and \( F \) is any scalar function. The plane wave solution has the distinction of being identical at every point of any plane normal to the direction of propagation. This justifies the name plane wave. We also have the remarkable property that the solution at time \( t \) in a plane orthogonal to \( \vec{n} \), is deduced from the solution in another plane by a simple time shift (phase shift).
If we imagine an observer moving along a line parallel to the direction $\vec{n}$ with a speed equal to $c_0$, this observer will constantly measure the same acoustic pressure. In this sense, $c_0$ clearly defines the speed of propagation of the signal. The acoustic pressure is therefore constant and uniform in any plane orthogonal to $\vec{n}$ and moving with a speed $c_0$: $p_a(t, \vec{x} = \vec{x}_0 + c_0 t \vec{n}) = \text{cst}$. Such a surface moving in space is called a wavefront, and any line parallel to the direction $\vec{n}$ is called an acoustic ray.

In practical cases for which the wavefront may be locally assimilated to a plane wave, the acoustic pressure and the particle velocity vary during the propagation of these wavefronts, if the medium is heterogeneous or mobile. However, if the speed of propagation of the wavefronts is much greater than the variation in acoustic pressure on the latter, two observers located at a distance not too great from each other (i.e. the wavelength) will practically see the same acoustic pressure on each wavefront.

This is what happens in an inhomogeneous medium: the acoustic pressure varies, but slowly enough that we can assume that locally, the acoustic field propagates as a plane wave. This assumption will only be valid if the acoustic wavelength is small compared to the characteristic length of the heterogeneity; which is satisfied in our problem as previously stated. Formally, this amounts to searching for solutions of the above linear system of the form:

$$
\hat{p}_a(\vec{x}, t) = \hat{p}_a[\vec{x}, \hat{\tau} = t - \psi(\hat{x})],
\hat{u}_a(\vec{x}, t) = \hat{u}_a[\vec{x}, \hat{\tau} = t - \psi(\hat{x})].
$$

where the function $\psi(\vec{x})$, called the eikonal function, is for the moment indefinite. The variable $\hat{\tau}$, called the phase variable of the signal, generalizes the notion of retarded time. The wavefronts are defined as the (moving) surfaces over which the phase variable $\hat{\tau}$ is constant. The change of variables $(t, \vec{x}) \to (\hat{\tau}, \vec{x})$ leads to the following expressions:

$$
\frac{\partial f}{\partial t} = \frac{\partial \hat{f}}{\partial \hat{\tau}}, \quad \nabla f = \nabla \hat{f} - \frac{\partial \hat{f}}{\partial \hat{\tau}} \nabla \psi, \quad \nabla \cdot \vec{f} = \nabla \cdot \hat{\vec{f}} - \frac{\partial \hat{\vec{f}}}{\partial \hat{\tau}} \cdot \nabla \psi,
\frac{d_0}{dt} = (1 - \vec{u}_0 \cdot \nabla \psi) \frac{\partial}{\partial \hat{\tau}} + \vec{u}_0 \cdot \nabla.
$$

In the case of a plane wave, in a homogeneous fluid, only the derivatives with respect to the phase variable are nonzero, the derivatives with respect to the spatial variable being zero. In the geometrical acoustic framework, we suppose that these derivatives are no longer zero, but nevertheless remain small, that is to say:

$$
|\nabla \hat{p}_a| \ll \left| \frac{\partial \hat{p}_a}{\partial \hat{\tau}} \nabla \psi \right| \text{ i.e. } |\nabla \hat{p}_a| = \mathcal{O}\left( \frac{\lambda_{ac}}{L_{atm}} \right) \left| \frac{\partial \hat{p}_a}{\partial \hat{\tau}} \nabla \psi \right|,
|\nabla \cdot \hat{\vec{u}}_a| \ll \left| \frac{\partial \hat{\vec{u}}_a}{\partial \hat{\tau}} \nabla \psi \right| \text{ i.e. } |\nabla \cdot \hat{\vec{u}}_a| = \mathcal{O}\left( \frac{\lambda_{ac}}{L_{atm}} \right) \left| \frac{\partial \hat{\vec{u}}_a}{\partial \hat{\tau}} \nabla \psi \right|.
$$

This amounts to saying that the wave behaves locally like a plane wave, the deviation with respect to an exact plane wave only becoming significant over large distances compared to the wavelength. In this sense, we speak also of short waves.
1.1.3 The eikonal equation

By inserting these new variables into the system of equations describing the propagation, it follows:

\[
\begin{align*}
(1 - \tilde{u}_0 \cdot \nabla \psi) \frac{\partial \hat{\psi}}{\partial \tilde{\tau}} - \rho_0 c_0^2 \frac{\partial \hat{\psi}}{\partial \tilde{\tau}} \nabla \psi &= -\tilde{u}_0 \cdot \nabla \hat{p}_a - \rho_0 c_0^2 \nabla \cdot \hat{u}_a - \hat{u}_a \cdot \nabla p_0 - \frac{c_0^2}{\rho_0^2} \nabla \cdot \left( \frac{\hat{u}_a}{c_0^2} \right) \\
(1 - \tilde{u}_0 \cdot \nabla \psi) \frac{\partial \hat{u}_a}{\partial \tilde{\tau}} - \frac{1}{\rho_0} \frac{\partial \hat{p}_a}{\partial \tilde{\tau}} \nabla \psi &= -\hat{u}_a \cdot \nabla \tilde{u}_a - \frac{1}{\rho_0} \nabla \hat{p}_a - \hat{u}_a \cdot \nabla \tilde{u}_a(\tilde{u}_0 + \frac{\bar{p}}{\rho_0^2 c_0^2} \nabla \tilde{p}_0)
\end{align*}
\]

where the terms on the right side are of order \(\lambda_{ac}/L_{atm}\) compared to those of the left side of order 1.

At first order, we keep only the terms on the left of each of the two equations above. If we multiply the first equation by \((1 - \tilde{u}_0 \cdot \nabla \psi)\) and the second by \(\rho_0 c_0^2 \nabla \psi\) we have, by adding the two expressions:

\[
[(1 - \tilde{u}_0 \cdot \nabla \psi)^2 - c_0^2(\nabla \psi)^2] \frac{\partial \hat{p}_a}{\partial \tilde{\tau}} = 0,
\]

which leads to the eikonal equation satisfied by the eikonal function:

\[
[(1 - \tilde{u}_0 \cdot \nabla \psi)^2 - c_0^2(\nabla \psi)^2] = 0.
\]

This equation admits the solutions: \(\nabla \psi = \alpha \tilde{n}\), \(\tilde{n}\) being the unit vector normal to the wavefront, of the form:

\[
\nabla \psi = \frac{\tilde{n}}{\tilde{n} \cdot \tilde{u}_0 + c_0} \quad \text{and} \quad \nabla \tilde{u}_a = \frac{\tilde{n} \cdot \tilde{u}_0 - c_0}{\tilde{n} \cdot \tilde{u}_0 + c_0}.
\]

However if in the second form we replace \(\tilde{n}\) by \(-\tilde{n}\), which amounts to changing the direction of propagation, we recover the first, which is thus the only one to remember. So the slow (slowness) vector, for a moving fluid, has the expression:

\[
\tilde{s} = \nabla \psi = \frac{\tilde{n}}{\tilde{n} \cdot \tilde{u}_0 + c_0}.
\]

By inserting this last result into either of the first-order equations of the system established above, we have:

\[
\begin{align*}
\left[ 1 - \frac{\tilde{u}_0 \cdot \tilde{n}}{(c_0 + \tilde{n} \cdot \tilde{u}_0)} \right] \frac{\partial \tilde{u}_a}{\partial \tilde{\tau}} - \frac{1}{\rho_0} \frac{\tilde{n} \cdot \tilde{u}_0}{(c_0 + \tilde{n} \cdot \tilde{u}_0)} \frac{\partial \hat{p}_a}{\partial \tilde{\tau}} &= \mathcal{O}\left( \frac{\lambda_{ac}}{L_{atm}} \right);
\end{align*}
\]

where:

\[
\tilde{u}_a = \frac{\hat{p}_a \tilde{n}}{\rho_0 c_0} \left[ 1 + \mathcal{O}\left( \frac{\lambda_{ac}}{L_{atm}} \right) \right]
\]

We remark that we obtain the same result as in a fluid at rest. The fact that the fluid is in motion does not disturb the impedance relationship between velocity and pressure. Through this relationship, we find the main hypothesis of geometrical acoustics which is to locally identify the acoustic wave with a plane wave; only the component of the acoustic velocity in the direction of propagation is retained. The fundamental assumption underlying geometrical acoustics is to suppose that the influence of the transverse components of the sound velocity, i.e. diffraction effects, is negligible compared to the longitudinal components. This assumption is satisfied except in the case where there would be convergence of acoustic rays; the notion of acoustic ray will be clarified later. Such a region of convergence is called a caustic. In the latter case, it seems pertinent to assume that the acoustic field varies significantly in the direction transverse to the direction of propagation, and therefore that the diffraction effects are no longer negligible. We will return later to the notion of caustic.
1.1.4 Solutions of the eikonal equation – Fermat’s principle

The solutions of the eikonal equation are obtained by application of Fermat’s principle, according to which the eikonal function is equal to the travel time along an acoustic ray from the source to the point of observation. An acoustic ray is, among all possible curves joining these two points, the one that extremizes this travel time. In the case of a windy atmosphere, the speed of propagation of the acoustic waves along the rays is \( c_0 \vec{n} \) where \( \vec{n} \) is the unit normal to the wavefront. The entrainment velocity of the mean flow must be added. We therefore have: \( \vec{v}_{\text{ray}} = c_0 \vec{n} + \vec{u}_0 \) (Pierce, 1989). As a consequence, the unit vector tangent to the acoustic ray \( \hat{t} = \vec{n}_{\text{ray}} / v_{\text{ray}} \) is no longer coincident with the normal \( \vec{n} \) to the wavefront. The rays are no longer normal to the wavefronts.
From this postulate, we will show that we can then express the speed of propagation along the ray as a function of its geometric characteristics, noting that $v_{\text{ray}}\vec{r} - \vec{u}_0 = c_0\vec{n}$, that is, by squaring: $v_{\text{ray}}^2 - 2v_{\text{ray}}\vec{r} \cdot \vec{u}_0 + u_0^2 - c_0^2 = 0$. The positive solution to this 2nd degree equation is (for a subsonic mean flow): $v_{\text{ray}}(\vec{x},\vec{t}) = \vec{r} \cdot \vec{u}_0(\vec{x}) + \sqrt{c_0^2(\vec{x}) - u_0^2(\vec{x}) + (\vec{r} \cdot \vec{u}_0)^2}$.

Fermat’s principle is expressed in the following way. Let 2 points A and B be fixed. We consider a regular curve, oriented from A to B, with curvilinear abscissa $l$ and tangent vector $\vec{t}$. The travel time along this curve is defined by:

$$T_C = \int_{l_A}^{l_B} \frac{dl}{v_{\text{ray}}(\vec{x}(l),\vec{t}(l))}.$$ 

Fermat’s Principle leads to the following result:

1) we define an acoustic ray joining points A and B as a curve which makes this travel time extremal. The rays obey the following differential system:

$$\frac{d\vec{x}_{\text{ray}}}{dl} = \frac{\vec{u}_0 + c_0\vec{s}/s}{|\vec{u}_0 + c_0\vec{s}/s|}$$

$$\frac{d\vec{s}}{dl} = -s \frac{\nabla c_0 + (\nabla \vec{u}_0) \cdot \vec{s}/s}{|\vec{u}_0 + c_0\vec{s}/s|},$$

2) Let A be a fixed point and B a moving point with coordinates $\vec{x}$. Then the function $T_C(\vec{x})$, the travel time along an acoustic ray joining A to B, is a solution of the eikonal equation. The unit slowness vector $\vec{s}$ is interpreted as the gradient of the eikonal function. The unit vector $\vec{n} = \vec{s}/s$ is interpreted as the normal vector to the wavefront with: $\vec{s} = \vec{n}/(c_0 + \vec{u}_0 \cdot \vec{n})$.

Proof:
To establish the differential equations satisfied by the rays, we apply results from the calculus of variations (Smirnov, 1964) similarly to the windless atmosphere (Coulouvrat 1997, Appendix I).
To do this, we parameterize all curves joining the 2 fixed points A and B by a fixed parameter independent of the choice of the curve:
\[ \ddot{x} = \bar{x}(a) \quad a_1 \leq a \leq a_2. \]

In this case, by denoting \( \ddot{x} = \frac{d\bar{x}}{da} \) we have:
\[ dl = \sqrt{\ddot{x}^2(a)} \frac{da}{da} \]
\[ \ddot{t} = \frac{\ddot{x}(a)}{\sqrt{\ddot{x}^2(a)}}, \]
so that:
\[ T_C(\ddot{x}, \ddot{x}) = \int_{a_1}^{a_2} \frac{\ddot{x}^2(a) \frac{da}{da}}{\dot{x} \cdot \ddot{u}_0(\bar{x}) + \sqrt{\left( \ddot{x} \cdot \ddot{u}_0(\bar{x}) \right)^2 + \ddot{x}^2(c_0^2(\ddot{x}) - u_0^2(\ddot{x}))}} = \int_{a_1}^{a_2} \Phi(\ddot{x}, \ddot{x}) \frac{da}{da}. \]

The endpoints being fixed, the bounds of integration \( a_1 \) and \( a_2 \) are also fixed. Under these conditions, the calculus of variations tells us that the travel time \( T_C \) is extremal provided that \( \Phi \) is a solution of:
\[ \frac{d}{da} \left( \frac{\partial \Phi}{\partial \ddot{x}} \right) = \frac{\partial \Phi}{\partial \ddot{x}}, \]
with \( \frac{\partial \Phi}{\partial \ddot{x}} = \left( \frac{\partial \Phi}{\partial \ddot{x}_1}, \frac{\partial \Phi}{\partial \ddot{x}_2}, \frac{\partial \Phi}{\partial \ddot{x}_3} \right) \) and \( \frac{\partial \Phi}{\partial \dot{x}} = \left( \frac{\partial \Phi}{\partial \dot{x}_1}, \frac{\partial \Phi}{\partial \dot{x}_2}, \frac{\partial \Phi}{\partial \dot{x}_3} \right). \)

Let us explain this condition. By definition of \( \Phi \), we have:
\[ \frac{\partial \Phi}{\partial \ddot{x}} = \frac{2\ddot{x} - \ddot{x} \cdot \ddot{u}_0 + \sqrt{(\ddot{x} \cdot \ddot{u}_0)^2 + \ddot{x}^2(c_0^2 - u_0^2)}}{\left( \ddot{x} \cdot \ddot{u}_0 + \sqrt{(\ddot{x} \cdot \ddot{u}_0)^2 + \ddot{x}^2(c_0^2 - u_0^2)} \right)^2} \cdot \left( \ddot{u}_0 + \frac{(\ddot{x} \cdot \ddot{u}_0)\ddot{u}_0 + (c_0^2 - u_0^2)\ddot{x}}{\sqrt{(\ddot{x} \cdot \ddot{u}_0)^2 + \ddot{x}^2(c_0^2 - u_0^2)}} \right). \]

If we now return to the definition of \( v_{\text{ray}} \), we can easily see that:
\[ \sqrt{(\ddot{x} \cdot \ddot{u}_0)^2 + \ddot{x}^2(c_0^2 - u_0^2)} = \sqrt{\ddot{x}^2(v_{\text{ray}} - \ddot{t} \cdot \ddot{u}_0)} \]
so that
\[ \frac{\partial \Phi}{\partial \ddot{x}} = \frac{2\ddot{t}}{v_{\text{ray}}} - \frac{1}{v_{\text{ray}}^2} \left( \frac{v_{\text{ray}}\ddot{u}_0 + (c_0^2 - u_0^2)\ddot{t}}{v_{\text{ray}} \cdot \ddot{u}_0 - \ddot{t}} \right), \]

Or, by reducing to the same denominator and using the second degree equation satisfied by \( v_{\text{ray}} \):
\[ \frac{\partial \Phi}{\partial \ddot{x}} = \frac{v_{\text{ray}}\ddot{t} - \ddot{u}_0}{v_{\text{ray}}(v_{\text{ray}} \cdot \ddot{u}_0 - \ddot{t})}. \]

We define \( \ddot{n} = (v_{\text{ray}}(\ddot{t} - \ddot{u}_0(\bar{x}))/c_0(\ddot{x}) \). Given the second degree polynomial whose root is \( v_{\text{ray}} \), it is easy to verify that \( \ddot{n} \) is unitary. We also verify: \( v_{\text{ray}} = \ddot{t} \cdot \ddot{u}_0 + c_0 \ddot{n} \) and therefore:
\[ v_{\text{ray}} = \ddot{t} \cdot \ddot{u}_0 + c_0 \ddot{n} \cdot \ddot{t}, \]
so that:
\[ \frac{\partial \Phi}{\partial \ddot{x}} = \frac{\ddot{n}}{v_{\text{ray}}\ddot{t} - \ddot{u}_0} = \frac{\ddot{n}}{v_{\text{ray}} \ddot{n} \cdot \ddot{t}} = \frac{\ddot{n}}{c_0 + \ddot{u}_0 \cdot \ddot{n}} = \ddot{s}. \]
We note that \( \vec{t} = \frac{\vec{v}_{ray}}{v_{ray}} = \frac{\vec{u}_0 + c_0\vec{n}}{|\vec{u}_0 + c_0\vec{n}|} \) so that the derivative of the position along a curve is written:

\[
\frac{d\vec{x}_{ray}}{dl} = \frac{\vec{u}_0 + c_0\vec{s}/s}{|\vec{u}_0 + c_0\vec{s}/s|}
\]

We now compute the vector \( \frac{\partial \Phi}{\partial \vec{x}} \).

\[
\frac{\partial \Phi}{\partial \vec{x}} = -\frac{\vec{x}^2}{v_{ray}^2} \left( \frac{\vec{x} \cdot \vec{t} + (\vec{t} \cdot \vec{u}_0)(\nabla \vec{u}_0) \cdot \vec{t} + (c_0 \nabla c_0 - (\nabla \vec{u}_0) \cdot \vec{u}_0)}{v_{ray} - \vec{t} \cdot \vec{u}_0} \right),
\]

where the notation \( \nabla \vec{a} \cdot \vec{b} \) denotes the vector whose \( i \)-th component equals \( b_j \frac{\partial a_i}{\partial x_j} \). Returning to the variables \( \vec{t} \) and \( v_{ray} \) as before, we have:

\[
\frac{\partial \Phi}{\partial \vec{x}} = -\frac{\sqrt{\vec{x}^2}}{v_{ray}^2} \frac{\vec{v}_{ray}(\nabla \vec{u}_0) \cdot \vec{t} + (c_0 \nabla c_0 - (\nabla \vec{u}_0) \cdot \vec{u}_0)}{v_{ray} - \vec{t} \cdot \vec{u}_0}.
\]

But \( v_{ray} = \vec{t} \cdot \vec{u}_0 + c_0\vec{n} \cdot \vec{t} \) so that \( v_{ray} - \vec{t} \cdot \vec{u}_0 = c_0\vec{n} \cdot \vec{t} = c_0\vec{n} \cdot \vec{u}_{ray}/v_{ray} = c_0(c_0 + \vec{u}_0 \cdot \vec{n})/v_{ray} \), and thus:

\[
\frac{\partial \Phi}{\partial \vec{x}} = -\frac{dl}{da} \frac{v_{ray}(\nabla \vec{u}_0) \cdot \vec{t} + c_0 \nabla c_0 - (\nabla \vec{u}_0) \cdot \vec{u}_0}{v_{ray}c_0(c_0 + \vec{u}_0 \cdot \vec{n})} = -\frac{dl}{da} \frac{(\nabla \vec{u}_0) \cdot \vec{v}_{ray} + c_0 \nabla c_0 - (\nabla \vec{u}_0) \cdot \vec{u}_0}{v_{ray}c_0(c_0 + \vec{u}_0 \cdot \vec{n})}.
\]

Since \( \vec{v}_{ray} = \vec{u}_0 + c_0\vec{n} \), we have, finally:

\[
\frac{\partial \Phi}{\partial \vec{x}} = -\frac{dl}{da} \frac{\nabla c_0 + (\nabla \vec{u}_0) \cdot \vec{n}}{|\vec{u}_0 + c_0\vec{n}|(c_0 + \vec{u}_0 \cdot \vec{n})}.\]

Taking these calculations into account, the condition for the curve considered to be an acoustic ray, that is to say render the travel time extremal, can be written as:

\[
\frac{d\vec{s}}{da} = -\frac{dl}{da} \frac{\nabla c_0 + (\nabla \vec{u}_0) \cdot \vec{n}}{|\vec{u}_0 + c_0\vec{n}|}(c_0 + \vec{u}_0 \cdot \vec{n})',
\]

or:

\[
\frac{d\vec{s}}{dl} = -\frac{\nabla c_0 + (\nabla \vec{u}_0) \cdot \vec{n}}{|\vec{u}_0 + c_0\vec{n}|}(c_0 + \vec{u}_0 \cdot \vec{n})',
\]

with \( \vec{s} = \frac{\vec{n}}{c_0 + \vec{u}_0 \cdot \vec{n}} \) and therefore \( \vec{n} = \frac{s}{\vec{s}} \).

In conclusion, the acoustic ray equations are:

\[
\begin{align*}
\frac{d\vec{x}_{ray}}{dl} &= \frac{\vec{u}_0 + c_0\vec{s}/s}{|\vec{u}_0 + c_0\vec{s}/s|} \\
\frac{d\vec{s}}{dl} &= \frac{\nabla c_0 + (\nabla \vec{u}_0) \cdot \vec{s}/s}{|\vec{u}_0 + c_0\vec{s}/s|}.\\
\end{align*}
\]
Knowing the position of the source \( \vec{x}_{\text{ray}}(0) \) and the initial wavefront normal \( \vec{n}(0) \), we deduce \( \vec{s}(0) = \frac{\vec{n}(0)}{c_0 + \vec{u}_0 \cdot \vec{n}(0)} \). Given these initial conditions, the differential system governing the rays can be solved, either analytically in some simple cases, or in general by a standard numerical procedure. In the case of the sonic boom, the initial position \( \vec{x}_{\text{ray}}(0) \) is determined only by the position of the aircraft at the transmission time \( t_{av} \). The wavefront normal is uniquely determined by the orientation of the aircraft (a function of \( t_{av} \) if the aircraft is turning or climbing), by the Mach number of the aircraft (itself a function of \( t_{av} \) if the aircraft accelerates or decelerates) at this position and by the azimuthal angle \( \Phi_{av} \) marking the position of the normal on the normal cone. The set of acoustic rays will therefore describe a family of curves in space, identified by the 2 parameters \( (t_{av}, \Phi_{av}) \).

2) Now consider the travel time as a function of \( \vec{x} \), the position of the observation point \( B \), the source being fixed at \( A \). When we vary point \( B \) along a ray, we have, according to a formula from the calculus of variations (with the notations of Coulouvrat 1997, Appendix I):

\[
\delta T_C = \left( \Phi(\vec{x}, \dot{\vec{x}}) - \frac{\partial \Phi}{\partial \vec{x}} (\vec{x}, \dot{\vec{x}}) \cdot \dot{\vec{x}} \right) \delta a_2 + \frac{\partial \Phi}{\partial \vec{x}} (\vec{x}, \dot{\vec{x}}) \cdot \delta \vec{x} + \int_{a_1}^{a_2} \left[ \frac{\partial \Phi}{\partial \vec{x}} - \frac{d}{da} \left( \frac{\partial \Phi}{\partial \vec{x}} \right) \right] da.
\]

Along an acoustic ray, the last term on the right hand side is zero. The function \( \Phi \) being homogeneous and first degree with respect to the variables \( \dot{\vec{x}} \), we deduce that \( \Phi(\vec{x}, \dot{\vec{x}}) = \frac{\partial \Phi}{\partial \vec{x}} (\vec{x}, \dot{\vec{x}}) \cdot \dot{\vec{x}} \) so that the first term is also zero. Given the calculation of \( \frac{\partial \Phi}{\partial \vec{x}} = \vec{s} \), we therefore have: \( \delta T_C = \nabla T_C \cdot \delta \vec{x} = \vec{s} \cdot \delta \vec{x} \) whatever the variation \( \delta \vec{x} \). Hence \( \nabla T_C = \vec{s} \). But \( \vec{s} = \frac{\vec{n}}{c_0 + \vec{u}_0 \cdot \vec{n}} \). By replacement, it is easy to verify that \( T_C \) is indeed a solution of the eikonal equation \((1 - \vec{u}_0 \cdot \nabla \psi)^2 - c_0^2 (\nabla \psi)^2 = 0\). It follows that the eikonal function is indeed equal to the travel time along the acoustic rays from

![Mach cone and normal cone: definitions of \( \Phi_{av} \) and \( \theta_{av} \)](image-url)

Figure 1.1.3: Mach cone and normal cone: definitions of \( \Phi_{av} \) and \( \theta_{av} \)
the source. So the vector \( \vec{s} \) (called the slowness vector), parallel to the unit vector \( \vec{n} \), is equal to the gradient of the eikonal function.

**Remark**

The ray equations were established by parameterizing the ray by its curvilinear abscissa \( l \). They may also be parameterized by the travel time along the ray (at the speed \( v_{ray} = |\vec{u}_0 + c_0 \vec{n}| \)), that is:

\[
\frac{d\vec{x}_{ray}}{d\psi} = \frac{d\vec{x}_{ray}}{dl} = v_{ray} \frac{d\vec{x}_{ray}}{dl} = \vec{u}_0 + c_0 \vec{n} = \vec{u}_0 + c_0 \vec{s}/s
\]

\[
\frac{d\vec{s}}{d\psi} = \frac{d\vec{s}}{dl} = v_{ray} \frac{d\vec{s}}{dl} = -s \nabla c_0 - (\nabla \vec{u}_0) \cdot \vec{s}
\]

These equations are therefore written:

\[
\frac{d\vec{x}_{ray}}{d\psi} = \vec{u}_0(x) + c_0(x) \vec{s}/s = \vec{f}(\vec{x}, \vec{s})
\]

\[
\frac{d\vec{s}}{d\psi} = -s \nabla c_0 - (\nabla \vec{u}_0) \cdot \vec{s} = \vec{g}(\vec{x}, \vec{s}).
\]

These will be useful in what follows to calculate the ray tube areas. Note that, by a classical relation: \( (\nabla \vec{u}_0) \cdot \vec{s} = \vec{s} \times (\nabla \times \vec{u}_0) + (\vec{s} \cdot \nabla) \vec{u}_0 \) we recover Pierce’s formulas, 1989 (Eqs. 8-1.10a & b, p.375).

The solution of the eikonal equation provided us with the ray tracing to obtain the acoustic path followed by an acoustic wave during its propagation. However, it does not provide information on the noise level in the insonified area; this is determined by an energy argument, detailed in the next section.

### 1.1.5 The equation of acoustic energy in a moving fluid

Let us again take the system of equations, established in paragraph 1.1.1, describing the linear acoustic disturbances in a heterogeneous, moving fluid. If we multiply the first equation by \( p_a/\rho_0 c_0^2 \) and scale the second by \( \rho_0 \vec{u}_a \), the addition of these equations gives us the generalized Kirchhoff equation:

\[
\frac{d\vec{u}_a}{dt} \left( \frac{\rho_0 \vec{u}_a^2}{2} + \frac{p_a^2}{2\rho_0 c_0^2} \right) + \nabla \cdot (p_a \vec{u}_a) =
\]

\[
\vec{u}_a^2(\vec{u}_0 \cdot \nabla)\left( \frac{\rho_0}{2} \right) + p_a^2(\vec{u}_0 \cdot \nabla)\left( \frac{1}{2\rho_0 c_0^2} \right) - \rho_0 \vec{u}_a \cdot [(\vec{u}_0 \cdot \nabla)\vec{u}_0] - \frac{1}{\rho_0} p_a^2 \nabla \cdot \left( \frac{\vec{u}_0}{c_0^2} \right) + \mathcal{O} \left( \frac{\lambda_{ac}}{L_{atm}} \right)^2,
\]

which gives the acoustic energy volume density \( E_a = \frac{\rho_0 \vec{u}_a^2}{2} + \frac{p_a^2}{2\rho_0 c_0^2} \), and the acoustic intensity vector \( \vec{I}_a = p_a \vec{u}_a \).

In the above equation, the terms on the right side are of order \( \lambda_{ac}/L_{atm} \), with respect to terms on the left side. We may therefore substitute in these terms the impedance relationship approach found at the end of paragraph 1.1.3: \( \vec{u}_a \approx \rho_0 \vec{n}/\rho_0 c_0 \) without modifying the precision.

So, by noting that \( p_a^2 \approx \rho_0 c_0^2 E_a \), \( \vec{u}_a^2 \approx E_a/\rho_0 \) and \( \vec{I}_a \approx c_0 E_a \vec{n} \) we get:

\[
\frac{d\vec{u}_a}{dt} E_a + \nabla \cdot \vec{I}_a = E_a \left[ \frac{1}{\rho_0} (\vec{u}_0 \cdot \nabla) \left( \frac{\rho_0}{2} \right) + \rho_0 \vec{u}_0^2 (\vec{u}_0 \cdot \nabla) \left( \frac{1}{2\rho_0 c_0^2} \right) \right] - E_a \vec{n} \cdot [(\vec{n} \cdot \nabla)\vec{u}_0] - c_0 E_a \nabla \cdot \left( \frac{\vec{u}_0}{c_0^2} \right).
\]

We introduce: \( \Omega = 1 - \nabla \psi \cdot \vec{u}_0 = \frac{c_0}{c_0 + \vec{n} \cdot \vec{u}_0} \).
Let us detail some preliminary calculations.

First of all, note that: 
\[-\frac{1}{\rho_0} \nabla \left( \frac{\rho_0}{2} \right) = \nabla \left( \ln \rho_0^{-\frac{1}{2}} \right),\]

and:
\[-\rho_0 c_0^2 \nabla \left( \frac{1}{2 \rho_0 c_0^2} \right) = \nabla \left[ \ln \left( \rho_0 c_0^2 \right)^{\frac{1}{2}} \right].\]

Let us calculate:
\[
\nabla \psi \cdot \left[ (\nabla \psi \cdot \nabla) \tilde{u}_0 \right] = \nabla \psi \cdot \nabla \left( \nabla \psi \cdot \tilde{u}_0 \right) = \nabla \psi \cdot \nabla \left( 1 - \Omega \right) - \tilde{u}_0 \cdot \left[ \frac{1}{2} \nabla (\nabla \psi)^2 \right] \]

But by definition \( \nabla \psi = \frac{\Omega \tilde{n}}{c_0} \); the above may also be written:
\[
\frac{\Omega^2}{c_0^2} \tilde{n} \cdot \left[ (\tilde{n} \cdot \nabla) \tilde{u}_0 \right] = -\frac{\Omega \tilde{n}}{c_0} \cdot \nabla \Omega - \tilde{u}_0 \cdot \left[ \frac{1}{2} \nabla \left( \frac{\Omega^2}{c_0^2} \right) \right],
\]

so that we have:
\[
\tilde{n} \cdot [(\tilde{n} \cdot \nabla) \tilde{u}_0] = \Omega c_0 \tilde{n} \cdot \nabla \left( \frac{1}{\Omega} \right) + \frac{\Omega}{c_0} \tilde{u}_0 \cdot \nabla \left( \frac{c_0}{\Omega} \right) = \Omega c_0 \tilde{n} \cdot \nabla \left( \frac{1}{\Omega} \right) + \tilde{u}_0 \cdot \nabla \left[ \ln \left( \frac{c_0}{\Omega} \right) \right].
\]

Given these results, the generalized Kirchhoff equation has the following form:
\[
\frac{d_0 E_a}{dt} + \nabla \cdot \tilde{I}_a = E_a \tilde{u}_0 \cdot \nabla \left[ \ln \rho^{-\frac{1}{2}} + \ln \left( \rho_0 c_0^2 \right)^{\frac{1}{2}} + \ln \left( \frac{c_0}{\Omega} \right) \right] + \Omega E_a c_0 \tilde{n} \cdot \nabla \left( \frac{1}{\Omega} \right) + c_0^2 E_a \nabla \cdot \left( \frac{\tilde{u}_0}{c_0^2} \right) = 0.
\]

Now, \( \tilde{I}_a = E_a c_0 \tilde{n} \left[ 1 + \mathcal{O} \left( \frac{\lambda_{ac}}{L_{atm}} \right) \right] \), so that after simplifying the logarithms, we have, to the same precision:
\[
\frac{d_0 E_a}{dt} + \nabla \cdot \tilde{I}_a = E_a \tilde{u}_0 \cdot \nabla \left[ \ln \left( \frac{c_0^2}{\Omega} \right) \right] + \Omega \tilde{I}_a \cdot \nabla \left( \frac{1}{\Omega} \right) + c_0^2 E_a \nabla \cdot \left( \frac{\tilde{u}_0}{c_0^2} \right) = 0.
\]

By dividing, now, this last equation by \( \Omega \) we obtain:
\[
\frac{\partial}{\partial t} \left( \frac{E_a}{\Omega} \right) + \frac{1}{\Omega} \tilde{u}_0 \cdot \nabla E_a + \nabla \cdot \left( \tilde{I}_a \right) + \frac{E_a \tilde{u}_0}{c_0^2} \cdot \nabla \left( \frac{c_0^2}{\Omega} \right) + \frac{1}{\Omega} c_0^2 E_a \nabla \cdot \left( \frac{\tilde{u}_0}{c_0^2} \right) = 0,
\]

which can also be written:
\[
\frac{\partial}{\partial t} \left( \frac{E_a}{\Omega} \right) + \tilde{u}_0 \cdot \nabla E_a + \nabla \cdot \left( \frac{\tilde{I}_a}{\Omega} \right) + E_a \nabla \cdot \left( \frac{\tilde{u}_0}{\Omega} \right) = 0,
\]

so that finally
\[
\frac{\partial}{\partial t} \left( \frac{E_a}{\Omega} \right) + \nabla \cdot \left( \tilde{I}_a + \tilde{u}_0 E_a \right) = 0.
\]

Returning to the approximations of \( E_a \) and \( \tilde{I}_a \), we have, at first order:
\[
\frac{\partial}{\partial t} \left( \frac{p_a^2}{\rho_0 c_0^2 \Omega} \right) + \nabla \cdot \left( \frac{p_a^2 (c_0 \tilde{n} + \tilde{u}_0)}{\rho_0 c_0^2 \Omega} \right) = \mathcal{O} \left( \frac{\lambda_{ac}}{L_{atm}} \right).
\]
1.1.6 The Blokhintzev invariant

Let us return to the hypotheses of geometrical acoustics, by the change of variable:

\[ p_a(\vec{x}, t) = \hat{p}_a[\vec{x}, \hat{\tau} = t - \psi(\vec{x})]. \]

As we have seen previously:

\[ \nabla \cdot \vec{f} = \nabla \cdot \hat{\vec{f}} - \frac{\partial \hat{\vec{f}}}{\partial \hat{\tau}} \cdot \nabla \psi. \]

The expression found in the previous paragraph becomes:

\[
\frac{\partial}{\partial t} \left( \frac{\hat{p}_a^2}{\rho_0 c_0^2 \Omega} \right) + \nabla \cdot \left[ \frac{\hat{p}_a^2 (c_0 \vec{n} + \vec{u}_0)}{\rho_0 c_0^2 \Omega} \right] - \frac{(c_0 \vec{n} + \vec{u}_0)}{\rho_0 c_0^2 \Omega} \frac{\partial \hat{p}_a^2}{\partial \hat{\tau}} \nabla \psi = 0.
\]

But \( \nabla \psi = \frac{\vec{n}}{c_0 + \vec{n} \cdot \vec{u}_0} \), so that \( (c_0 \vec{n} + \vec{u}_0) \cdot \nabla \psi = 1 \).

Consequently, the first and the last term of the above equation are eliminated; thus results the transport equation:

\[
\nabla \cdot \left[ \frac{\hat{p}_a^2 (c_0 \vec{n} + \vec{u}_0)}{\rho_0 c_0^2 \Omega} \right] = 0.
\]

The solution of the eikonal equation only provides part of the information on the acoustic field. More precisely, the ray tracing only determines the path followed by an acoustic wave during its propagation, which makes it possible to estimate the insonified regions, and the zones of silence which will be practically unaffected by the sound field. However, it does not provide any information on the noise level in the insonified area. To get this level, we must solve the transport equation.

For this purpose, consider a particular ray (principal ray) and a ray tube, that is to say the volume enveloped by a set of rays close to this principal ray (Figure 1.1.4). We denote by \( l \) the curvilinear coordinate along the principal ray, by \( \vec{l}(l) \) the vector tangent to the principal ray, by \( \vec{n}_{ext} \) the outward normal to the ray tube and \( S(l) \) the cross-sectional ray tube area at the point of the curvilinear abscissa \( l \). Note that the outward normal is equal to the tangent vector on the section \( S(l) \), and its opposite on the section \( S(0) \), and that it is orthogonal to the adjacent ray on the lateral surface of the ray tube.

![Figure 1.1.4: Ray tube](image)

By integrating the transport equation over the volume bounded by the ray tube, and applying the
differential equation, we have:

\[
0 = \int \int \int_V \nabla \cdot \left[ \hat{p}^2 \frac{(c_0 \vec{n} + \vec{u}_0)}{\rho_0 c_0^2 \Omega} \right] dV = \int \int_{\partial V} \frac{\hat{p}^2}{\rho_0 c_0^2 \Omega} (c_0 \vec{n} + \vec{u}_0) \cdot \vec{n}_{ext} dS = \int \int_{\partial V} \frac{\hat{p}^2}{\rho_0 c_0^2 \Omega} (c_0 \vec{n} + \vec{u}_0) \cdot \vec{t} \cdot \vec{n}_{ext} dS,
\]

with \( \vec{t} = \frac{\vec{u}_0 + c_0 \vec{n}}{|\vec{u}_0 + c_0 \vec{n}|} \).

On the lateral walls \( \vec{t} \perp \vec{n}_{ext} \), so we have:

\[
\int \int_{S(l)} \frac{\hat{p}^2}{\rho_0 c_0^2 \Omega} (c_0 \vec{n} + \vec{u}_0) \cdot \vec{n}_{ext} dS - \int \int_{S(0)} \frac{\hat{p}^2}{\rho_0 c_0^2 \Omega} (c_0 \vec{n} + \vec{u}_0) \cdot \vec{t} \cdot \vec{n}_{ext} dS = 0.
\]

By making the area of the cross sections of the ray tube approach zero, which amounts to considering an infinitesimal ray tube of section \( \delta S \), the above formula gives:

\[
\frac{\delta S(0) |c_0(0) \vec{n}(0) + \vec{u}_0(0)| \hat{p}^2(\vec{\tau}, 0)}{\rho_0(0) c_0^2(0) \Omega(0)} = \frac{\delta S(l) |c_0(l) \vec{n}(l) + \vec{u}_0(l)| \hat{p}^2(\vec{\tau}, l)}{\rho_0(l) c_0^2(l) \Omega(l)},
\]

that is:

\[
\hat{p}_a(\vec{\tau}, l) = \left[ \frac{\delta S(0) |c_0(0) \vec{n}(0) + \vec{u}_0(0)|}{\delta S(l) |c_0(l) \vec{n}(l) + \vec{u}_0(l)|} \right]^{\frac{1}{2}} \hat{p}_a(\vec{\tau}, 0).
\]

This result may also be interpreted as the fact that the acoustic energy flux (the acoustic intensity) through a cross section of the ray tube is constant along the ray. This is translated by means of Blokhintzey’s invariant (1946):

\[
\frac{\hat{p}^2}{\rho_0 c_0^2 \Omega} |c_0 \vec{n} + \vec{u}_0| \delta S = \text{cst.}
\]

The Blokhintzey invariant therefore makes it possible to calculate the level of the acoustic pressure field in a heterogeneous fluid using only geometric quantities which are the rays and the ray tubes. The propagation of the sonic boom from the aircraft to the ground is conventionally modeled by this principle. The ray calculation allows us to determine the geometric carpet on the ground, that is, the region in which the classical sonic boom is heard, while the Blokhintzey invariant provides the strength of the signal not distorted by the nonlinear effects associated with the propagation.

The above result demonstrates the limit of the validity of the geometrical acoustics hypothesis. Indeed, when the area \( \delta S \) along a ray tends to 0, the sound pressure goes to infinity. Near the point where the section vanishes, the acoustic energy is concentrated in an area narrower than the ray tube: there is focusing. The set of points in space where the ray tube area vanishes defines a surface called a caustic. A caustic can also be interpreted as the intersection of two infinitely close rays. As we saw earlier, the assumptions of geometrical acoustics are no longer valid in the vicinity of caustics. The transverse variations of the field are no longer negligible, and these are precisely the diffraction terms which limit the amplification of the field.

1.1.7 “Ray” Coordinates

A point \( M \) in space is generally identified by its Cartesian coordinates \((x, y, z)\) in an orthonormal reference frame. However, in the context of geometrical acoustics, it is preferable to identify this
point in another, more suitable curvilinear coordinate system. Indeed, a point which belongs to the insonified area (i.e. the area touched by the set of rays emitted by the source) is obviously touched by an acoustic ray, and it occupies a position on this ray identified by the curvilinear coordinate \( l \), or in an equivalent manner, by the eikonal function \( \psi \) (which measures the travel time of sound along this ray). So we can see that it is possible to identify the position of point \( M \) in space by \( \vec{x}_{\text{ray}}(\alpha, \beta, \psi) \) with:

1) 2 coordinates, denoted \( \alpha, \beta \), which will identify the particular ray which touches point \( M \),
2) the coordinate \( \psi \) which locates the position of this point along this ray.

The parameters \((\alpha, \beta, \psi)\) are the “ray” coordinates identifying the point \( M \). The lines \( \alpha = \text{cst.}, \beta = \text{cst.} \) determine a particular ray, for a point source, while the surfaces \( \psi = \text{cst.} \) define the wavefront at the given moment (Figure 1.1.5).

Figure 1.1.5 : Ray coordinates

In the particular case of the supersonic aircraft, there are two coordinates particularly adapted to identifying the different rays coming from the aircraft: They are:

1) the time \( t_{av} \) which identifies the set of conditions (position, velocity, climb angle and turn angle) of the aircraft at the moment when the considered ray was emitted;

2) the azimuthal angle \( \Phi_{av} \) which identifies the angle, normal to the Mach cone, at which the ray has been emitted.

In the case of the sonic boom, we will therefore use in a privileged way the “ray” coordinates \((t_{av}, \Phi_{av}, \psi)\).
1.1.6 : Coordinates initializing the calculation of rays

Let us return to the results of the paragraph “Solutions of the eikonal equation - Fermat’s principle”; we thus have:

\[
\frac{d\vec{x}_{\text{ray}}}{d\psi} = \vec{u}_0(\vec{x}) + c_0(\vec{x})\vec{s}/s = \vec{f}(\vec{x}, \vec{s})
\]

\[
\frac{d\vec{s}}{d\psi} = -s\nabla c_0 - (\nabla \vec{u}_0) \cdot \vec{s} = \vec{g}(\vec{x}, \vec{s}).
\]

The acoustic rays are emitted from the aircraft. The initial ray emission conditions therefore depend solely on the aircraft flight characteristics. So, \(\vec{x}_{\text{ray}}(0) = \vec{x}_{av}(t_{av})\) is only a function of \(t_{av}\). Similarly, as we saw in the above paragraph:

\[
\vec{s}(0) = \frac{\vec{n}}{\vec{u}_0(\vec{x}_{av}) + c_0(\vec{x}_{av})\vec{n}}.
\]

But, the normal \(\vec{n}(0)\) at the wavefront (normal to the Mach cone), at emission is only a function of the azimuth angle \(\Phi_{av}\) and the flight characteristics of the aircraft, which is written: \(\vec{n} = \mathcal{J}[\Phi_{av}, M_{av}(t_{av}), \Gamma_{av}(t_{av}), \theta_{av}(t_{av})]\), with \(M_{av}(t_{av})\) the Mach number, \(\Gamma_{av}(t_{av})\) the climb angle and \(\theta_{av}(t_{av})\) the turn angle (relative to a given direction).

1.1.8 Calculation of the ray tube area

We will now, in the manner of Candel (1977), demonstrate that the calculation of ray tube areas is reduced to the resolution of a differential system which completes the ray equations.
By observing the figure above (Figure 1.1.7), we note that $\delta S = \delta \Sigma \cos(\vec{t}, \vec{n}) = (\vec{t} \cdot \vec{n}) \delta \Sigma$. Moreover, outside a caustic, the area element of the front, in ray coordinates, with the notation $(\alpha, \beta, \psi)$, is written:

$$\delta \Sigma = \left[ \left( \frac{\partial \vec{x}_{ray}}{\partial \alpha} \right)_{\beta,\psi} \times \left( \frac{\partial \vec{x}_{ray}}{\partial \beta} \right)_{\alpha,\psi} \right] d\alpha d\beta,$$

with $\frac{\partial \vec{x}_{ray}}{\partial \alpha}_{\beta,\psi}$ the first vector tangent to the wavefront and $\frac{\partial \vec{x}_{ray}}{\partial \beta}_{\alpha,\psi}$ the second vector tangent to the wavefront.

Let us write now, the equations allowing us to find these vectors. Differentiating the system of 6 differential equations governing the ray with respect to $\alpha$, we have:

$$\frac{\partial}{\partial \alpha} \left( \frac{d\vec{x}_{ray}}{d\psi} \right) = \frac{d}{d\psi} \left( \frac{d\vec{x}_{ray}}{d\alpha} \right) = \frac{\partial \vec{f}}{\partial \vec{x}} \cdot \frac{\partial \vec{x}_{ray}}{\partial \alpha} + \frac{\partial \vec{f}}{\partial \vec{s}} \cdot \frac{\partial \vec{s}_{ray}}{\partial \alpha}.$$ 

If we set $\vec{x}_\alpha = \frac{\partial \vec{x}_{ray}}{\partial \alpha}$ and $\vec{s}_\alpha = \frac{\partial \vec{s}_{ray}}{\partial \alpha}$, we get

$$\frac{d\vec{x}_\alpha}{d\psi} = \frac{\partial \vec{f}}{\partial \vec{x}} \cdot \vec{x}_\alpha + \frac{\partial \vec{f}}{\partial \vec{s}} \cdot \vec{s}_\alpha,$$

where $\frac{\partial \vec{f}}{\partial \vec{x}}$ and $\frac{\partial \vec{f}}{\partial \vec{s}}$ are known (Jeanjean 1998).

Similarly, we write:

$$\frac{\partial}{\partial \alpha} \left( \frac{d\vec{s}}{d\psi} \right) = \frac{d}{d\psi} \left( \frac{d\vec{s}}{d\alpha} \right) = \frac{\partial \vec{g}}{\partial \vec{x}} \cdot \frac{\partial \vec{x}_{ray}}{\partial \alpha} + \frac{\partial \vec{g}}{\partial \vec{s}} \cdot \frac{\partial \vec{s}_{ray}}{\partial \alpha},$$

so that: $\frac{d\vec{s}_\alpha}{d\psi} = \frac{\partial \vec{g}}{\partial \vec{x}} \cdot \vec{x}_\alpha + \frac{\partial \vec{g}}{\partial \vec{s}} \cdot \vec{s}_\alpha$, where $\frac{\partial \vec{g}}{\partial \vec{x}}$ and $\frac{\partial \vec{g}}{\partial \vec{s}}$ are known (Jeanjean 1998).
The notation $\frac{\partial a_i}{\partial b_j}$ designates a 2nd order tensor which in indicial notation is written: $\frac{\partial a_i}{\partial b_j}$. We have the same proofs for $\beta$. The system to solve is therefore the following:

$$\begin{align*}
\frac{d\vec{x}_{\text{ray}}}{d\psi} &= \vec{f}(\vec{x}, \vec{s}) \\
\frac{d\vec{s}}{d\psi} &= \vec{g}(\vec{x}, \vec{s}) \\
\frac{d\vec{x}_\alpha}{d\psi} &= \frac{\partial \vec{f}}{\partial \vec{x}} \cdot \vec{x}_\alpha + \frac{\partial \vec{f}}{\partial \vec{s}} \cdot \vec{s}_\alpha \\
\frac{d\vec{x}_\beta}{d\psi} &= \frac{\partial \vec{f}}{\partial \vec{x}} \cdot \vec{x}_\beta + \frac{\partial \vec{f}}{\partial \vec{s}} \cdot \vec{s}_\beta \\
\frac{d\vec{s}_\alpha}{d\psi} &= \frac{\partial \vec{g}}{\partial \vec{x}} \cdot \vec{x}_\alpha + \frac{\partial \vec{g}}{\partial \vec{s}} \cdot \vec{s}_\alpha \\
\frac{d\vec{s}_\beta}{d\psi} &= \frac{\partial \vec{g}}{\partial \vec{x}} \cdot \vec{x}_\beta + \frac{\partial \vec{g}}{\partial \vec{s}} \cdot \vec{s}_\beta
\end{align*}$$

The system consists of 18 equations with 18 unknowns to be determined. The first 6 characterize the propagation of the acoustic rays and the others form a system which makes it possible to calculate the ray tube areas. By determining this system of 18 equations and by finding the 18 initial conditions necessary to close the problem, we may numerically solve it by a standard method. We have thus entirely determined the rays and their propagation in the case of a windy atmosphere. The resolution of this system was carried out by Jeanjean (1998) and resulted in the writing of a Matlab program modeling the propagation of the boom in the atmosphere in the presence of wind. The purpose of this code was ultimately to replace the program of Hayes et al. (1969), which used an inappropriate definition of the ray tube, which resulted in a large imprecision of the calculations when the acoustic rays became horizontal (limiting cutoff rays).

### 1.1.9 Conclusion

In this section, we recalled the results of geometrical acoustics in a moving fluid. This approximation allows us to correctly model the propagation of the sonic boom from the aircraft to the ground. However, we have seen the limits of this when there is convergence of acoustic rays. This approximation is therefore not suitable for studying the focusing of the sonic boom.

In addition, geometrical acoustics does not take into account the nonlinear effects, inseparable from the propagation of the sonic boom. In the next section, we will demonstrate how to reintegrate them during the propagation along the acoustic rays.

### 1.2 REMINDERS ON THE INFLUENCE OF NONLINEAR EFFECTS DURING PROPAGATION

In the previous section, we saw that the propagation of the sonic boom of the aircraft down to the ground was correctly modeled by geometrical acoustics; the path traveled by the acoustic disturbance is obtained thanks to Fermat’s principle and the signal amplitude is given by the Blokhintzev invariant. However, nonlinear effects, inseparable from the propagation of the sonic
boom, are not taken into account in this approximation. They are locally weak, but their influence becomes significant over fairly large distances: they are cumulative. In this section we will show how to add nonlinearities to the geometrical acoustics results, in the presence of the wind, in order to take them into account when propagating the sonic boom. We will also recall classical results of nonlinear acoustics, which can be found in the books of Rudenko and Soluyan (1977) and Hamilton and Blackstock (1998).

1.2.1 Speed of sound in nonlinear acoustics

We recall the expression for the speed of sound in the nonlinear case:

$$c_{\text{actual}} = c_0 + \beta \frac{p_a}{\rho_0 c_0},$$

where $\beta = 1 + B/2A$,

with $B = \frac{\rho_0}{c_0^2} \left( \frac{\partial^2 p}{\partial \rho^2} \right) (\rho_0, s_0)$, is the parameter of nonlinearity introduced by Beyer (1974).

Additionally:

$$\beta = \frac{1 + \gamma}{2} = 1.2 \text{ when the carrying fluid is air.}$$

The different parts of the profile of an acoustic wave of finite amplitude propagate with slightly different speeds, which depend on the “local” value of the acoustic disturbance. In particular, compression waves propagate faster than expansion waves, so that, as the propagation progresses, the initial profile will deform.

1.2.2 Burgers’ equation

The wave equation, in one spatial dimension $x$ is written:

$$\frac{1}{c_0^2} \frac{\partial^2 p_a}{\partial t^2} - \frac{\partial^2 p_a}{\partial x^2} = \left( \frac{1}{c_0} \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right) \left( \frac{1}{c_0} \frac{\partial}{\partial t} - \frac{\partial}{\partial x} \right) p_a.$$

If we only consider the waves progressing towards positive $x$, we get:

$$\frac{1}{c_0} \frac{\partial p_a}{\partial t} + \frac{\partial p_a}{\partial x} = 0.$$

In the nonlinear case, the latter equation may be written:

$$\frac{1}{c_0 + \beta \frac{p_a}{\rho_0 c_0}} \frac{\partial p_a}{\partial t} + \frac{\partial p_a}{\partial x} = 0.$$

1.2.3 Burgers’ equation in a stratified atmosphere with wind

We may take nonlinearities into account, in a heuristic way, by replacing the linear speed of sound $c_0$ by the nonlinear speed of sound. We then obtain the following equation:

$$\frac{1}{c_0} \frac{\partial p_a}{\partial t} + \frac{\partial p_a}{\partial x} = \frac{\beta}{\rho_0 c_0^3} \frac{p_a}{\partial t}. $$
The right side corresponds to the nonlinear correction. We introduce the following dimensionless variables:

\[ T = \omega \left( t - \frac{x}{c_0} \right), \]
\[ X = \frac{\beta \omega P_0}{c_0^2} x, \]
\[ p_a(x, t) = \rho_0 c_0 P_0 P(X, T), \]

where \( \omega \) is the frequency and \( P_0 \) is a reference pressure.

The Burgers equation follows in the following form:

\[ \frac{\partial P}{\partial X} - P \frac{\partial P}{\partial T} = 0. \]

with the associated boundary conditions:

\[ P(X = 0, T) = F(T) \text{ and } P(X, T = 0) = 0. \]

The solution of the Burgers’ equation is implicit:

\[ P(X, T) = F(\phi) \text{ with } T = \phi - X F(\phi). \]

Burgers’ equation describes the first-order influence of cumulative nonlinear effects. It appears as a nonlinear partial differential equation, in which the slow variation term of the unknown function compensates for the cumulative nonlinear effects.

The Burgers’ equation leads to the formation of discontinuities. Indeed, if the change of variable \( \phi \to T \) is invertible, the solution remains perfectly defined. Otherwise, beyond a distance called the shock formation distance \( X \geq X_{\text{shock}} = 1/\max(F'(\phi)) \) the implicit solution is multivalued, that is to say that for certain values of time (that is to say of \( T \)), there are several possible values of \( \phi \), and therefore of pressure. This being physically unacceptable, the existence of discontinuities, i.e. weak shocks, is assumed: at time \( T_c(X) \), the acoustic pressure \( P \) instantly changes from the value \( P_-(X) \) to the value \( P_+(X) \). On both sides of \( T_c \), the acoustic pressure is still given by the implicit solution of the Burgers equation. The introduction of a discontinuity therefore makes it possible to link together different regions of the implicit solution where it remains single-valued.

The instant of the shocks \( T_c(X) \) is a solution of the differential equation:

\[ W = \frac{dT_c(X)}{dX} = -(P_-(X) + P_+(X))/2 \text{ with } P_-(X) < P_+(X), \]

\( W \) being the dimensionless “speed” of a shock. This is illustrated in Figure 1.2.1. The condition \( P_-(X) < P_+(X) \) is a consequence of the second principle of thermodynamics. This law is also interpreted geometrically by the law of equal areas, according to which the algebraic area of the multivalued portion of the wave profile on either side of the shock is zero.
According to the value of $X$ with respect to the shock formation distance, $X_{\text{shock}}$, we distinguish three regions in which the nonlinear effects affect the shape of the signal differently:

1. $X < X_{\text{shock}}$ DISTORTION OF THE WAVE
   - transfer of energy from low frequencies to high frequencies
   - reversible process.

2. $X > X_{\text{shock}}$ FORMATION OF SHOCKS
   - appearance of a sawtooth profile
   - loss of energy across shocks (irreversible process)

3. $X \gg X_{\text{shock}}$ FUSION OF SHOCKS
   - transfer of energy from high frequencies to low frequencies (creation of the “N” wave)
   - irreversible process.

Figure 1.2.2 recalls the properties of the linear case; the signal emitted by the source does not change during propagation.
Figure 1.2.3 illustrates the phenomena described above concerning the influence of nonlinearity on a signal during its propagation. The selected signal recalls the pressure profile that we find near an aircraft in supersonic flight and the $x$-axis the acoustic ray on which this acoustic disturbance travels. As the signal progresses, it becomes distorted until, beyond the shock formation distance, shocks appear. If the propagation continues, these shocks merge into a front and rear shock forming an “N” wave. This is exactly the profile of a sonic boom signal that we may observe at ground level.

Figure 1.2.3: propagation of a wave in the linear case
1.3 SONIC BOOM FOCUSING

In the geometrical acoustics approximation, the level of the sound pressure field in a heterogeneous fluid, in the presence of wind is obtained by means of the Blokhintzev invariant (cf. Chapter 1, paragraph 1.1.6):

\[ \hat{p}_a(\hat{\tau},l) \sqrt{B(l)} = G(\hat{\tau}), \text{ with } B(l) = \frac{|c_0\vec{n} + \vec{u}_0|\delta S}{\rho_0 c_0^2 \Omega} \text{ and } G(\hat{\tau}) = \hat{p}_a(\hat{\tau},l = 0) \sqrt{B(l = 0)}. \]

with \( \Omega = 1 - \nabla \psi \cdot \vec{u}_0 = \frac{\rho_0}{c_0 + \vec{n} \cdot \vec{u}_0} \).

The vector tangent to the acoustic ray is written:

\[ \vec{t} = \frac{\vec{v}_{ray}}{v_{ray}}, \]

where \( v_{ray} = \vec{t} \cdot \vec{u}_0 + c_0 \vec{n} \cdot \vec{t}. \)

We will now rewrite Blokhintzev’s invariant in the form of a partial differential equation. By including nonlinear effects in a way analogous to the plane wave, this equation will be modified then transformed into Burgers’ equation.

Figure 1.2.3: propagation of a wave in the nonlinear case
By differentiating the Blokhintzev invariant, we obtain the following equation between the pressure and the parameter $B(l)$:

$$\frac{\partial \hat{p}_a}{\partial l} + \frac{1}{2B(l)} \frac{dB}{dl} \hat{p}_a = 0.$$ 

But: $p_a(\vec{x}, t) = \hat{p}_a[\vec{x}, \hat{\tau} = t - \psi(\vec{x})]$ and $\frac{d\psi}{dl} = \frac{1}{v_{ray}} = \frac{1}{c_0 \vec{n} + \vec{u}_0} = \frac{\vec{t} \cdot \vec{n}}{c_0 + \vec{u}_0 \cdot \vec{n}}$.

So:

$$\frac{\partial \hat{p}_a}{\partial l} = \frac{\partial p_a}{\partial l} + \frac{\partial p_a}{\partial t} \cdot \frac{d\psi}{dl} = \frac{\vec{t} \cdot \vec{n}}{c_0 + \vec{u}_0 \cdot \vec{n}} \frac{\partial p_a}{\partial t}.$$ 

It follows:

$$\frac{\vec{t} \cdot \vec{n}}{c_0 + \vec{u}_0 \cdot \vec{n}} \frac{\partial p_a}{\partial t} + \frac{\partial p_a}{\partial l} + \frac{1}{2B(l)} \frac{dB}{dl} \hat{p}_a = 0.$$ 

By adopting the variables introduced in the first section, the thermodynamic definition of the linear sound speed in the presence of wind is written as:

$$c_0 = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_s (\rho_0, s_0)} + \vec{u}_0 \cdot \vec{n}.$$ 

In reality, when there is wave propagation, the fluid, initially in motion, is subjected to a small acoustic disturbance. The quantities associated with this acoustic disturbance are indexed by “a”.

It then follows:

$$p = p_0 + p_a$$
$$\rho = \rho_0 + \rho_a$$
$$\vec{u} = \vec{u}_0 + \vec{u}_a$$
$$s = s_0 + s_a.$$ 

The density is therefore not equal to $\rho_0$, but to $\rho_0 + \rho_a$. The actual speed of sound is thus written:

$$c_{NL} = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_s (\rho_0 + \rho_a, s_0) + \vec{u}_0 \cdot \vec{n} + \vec{u}_a \cdot \vec{n}}.$$ 

Now $\rho_a \ll \rho_0$, so by Taylor expansion the nonlinear speed of sound has the following form:

$$c_{NL} = c_0 \left[1 + \frac{1}{2} \left(\frac{1}{c_0^2}\right) \left(\frac{\partial^2 p}{\partial \rho^2}\right)(\rho_0, s_0)\right] + \vec{u}_0 \cdot \vec{n} + \vec{u}_a \cdot \vec{n}.$$ 

In the case of a plane wave, for which we have the following expressions:

$$p_a = c_0^2 \rho_a \rho_0$$
$$\vec{u}_a = \vec{p}_a \frac{\vec{n}}{\rho_0 c_0},$$

we may rewrite the expression for the nonlinear speed of sound in the following form:

$$c_{NL} = c_0 + \vec{u}_0 \cdot \vec{n} + \beta \frac{p_a}{\rho_0 c_0}.$$
In the previous equation between the pressure and the parameter $B(l)$, let us add the nonlinear effects as written above. We then have:

$$\frac{\vec{t} \cdot \vec{n}}{c_0 + \vec{u}_0 \cdot \vec{n} + \beta \frac{p_a}{\rho_0 c_0}} \frac{\partial p_a}{\partial t} + \frac{\partial p_a}{\partial l} + \frac{1}{2B} \frac{dB}{dl} p_a = 0,$$

which becomes by Taylor expansion:

$$\frac{\vec{t} \cdot \vec{n}}{c_0 + \vec{u}_0 \cdot \vec{n}} \frac{\partial p_a}{\partial t} - \frac{\beta \vec{t} \cdot \vec{n}}{\rho_0 c_0 (c_0 + \vec{u}_0 \cdot \vec{n})^2} \frac{p_a}{\partial t} + \frac{\partial p_a}{\partial l} + \frac{1}{2B} \frac{dB}{dl} p_a = 0.$$

Returning to the variables $(\hat{\tau}, l)$, we get the following expression:

$$\frac{\partial \hat{p}_a}{\partial l} + \frac{1}{2B} \frac{dB}{dl} \hat{p}_a - \frac{\beta \vec{t} \cdot \vec{n}}{\rho_0 c_0 (c_0 + \vec{u}_0 \cdot \vec{n})^2} \frac{\hat{p}_a}{\partial \hat{\tau}} \frac{\partial \hat{p}_a}{\partial \hat{\tau}} = 0.$$

We now introduce the function $q(\hat{\tau}, l)$, such that:

$$\hat{p}_a = \sqrt{\frac{B(l_0)}{B(l)}} q(\hat{\tau}, l).$$

After simplification, we have:

$$\frac{\partial q}{\partial l} = \frac{\beta \vec{t} \cdot \vec{n}}{\rho_0 c_0 (c_0 + \vec{u}_0 \cdot \vec{n})^2} \sqrt{\frac{B(l_0)}{B(l)}} \frac{\partial q}{\partial \hat{\tau}}.$$

We now introduce the variable $\sigma$, such that $q(l, \hat{\tau}) = \hat{q}(\sigma, \hat{\tau})$. In this new coordinate system, we have the following relationship:

$$\frac{\partial q}{\partial l} = \frac{\partial \hat{q}}{\partial \sigma} \frac{\partial \sigma}{\partial l} = \frac{\beta \vec{t} \cdot \vec{n}}{\rho_0 c_0 (c_0 + \vec{u}_0 \cdot \vec{n})^2} \sqrt{\frac{B(l_0)}{B(l)}} \frac{\partial \hat{q}}{\partial \hat{\tau}}.$$

We choose $\sigma$ so as to bring us back to the Burgers’ equation without a coefficient. From the previous relationship, we then extract the following differential equation:

$$\frac{d\sigma}{dl} = \frac{\beta \vec{t} \cdot \vec{n}}{\rho_0 c_0 (c_0 + \vec{u}_0 \cdot \vec{n})^2} \sqrt{\frac{B(l_0)}{B(l)}},$$

where $\sigma$ defines the age variable describing the cumulative distortion along the ray. The Burgers’ equation then follows:

$$\frac{\partial \hat{q}}{\partial \sigma} = \hat{q} \frac{\partial \hat{q}}{\partial \hat{\tau}},$$

with $\frac{d\sigma}{dl} = \frac{\beta \vec{t} \cdot \vec{n}}{\rho_0 c_0 (c_0 + \vec{u}_0 \cdot \vec{n})^2} \sqrt{\frac{B(l_0)}{B(l)}}$.

If we parametrize the acoustic ray by the eikonal function $\psi$, the problem modeling the nonlinear distortion is then the following:

$$\frac{d\sigma}{d\psi} = \frac{d\sigma}{dl} \frac{dl}{d\psi} = \frac{\beta}{\rho_0 c_0 (c_0 + \vec{u}_0 \cdot \vec{n})^2} \sqrt{\frac{B(l_0)}{B(l)}}.$$

The nonlinear effects in sonic boom modeling can be written with the Burgers’ equation and permit us to explain the formation of shock waves and the “N” shape of signals measured on the ground away from the aircraft.
In general, the study of the propagation of the sonic boom from the aircraft to the ground is assured by geometrical acoustics. This assumes, as we saw in the previous section, that the acoustic disturbances emitted by the aircraft propagate in the atmosphere along curves called acoustic rays. The path followed by these rays is established by Fermat’s principle. The pressure amplitude along these acoustic rays is obtained by the conservation of energy density along a ray tube through the Blokhintzev relation. Nonlinearities are also taken into account throughout the propagation by Burgers’ equation without the dissipative term. However, this approximation is no longer satisfied in the vicinity of caustics which correspond to areas of convergence of the acoustic rays. The Blokhintzev invariant provides a pressure tending to infinity, which is not physically acceptable: there is a singularity.

1.3.1 The focusing phenomenon for a supersonic aircraft

These caustics are produced during particular maneuvers of the aircraft, in acceleration or turn, and can lead to a local amplification (focusing) of the boom at the edge of the carpet, of a factor to be determined, but which is considered to be between 3 and 10. There is a third type of caustic, which is the caustic of refraction which can appear at altitude in transonic flight. Unlike the primary boom, the focused boom may eventually cause damage to structures (broken windows). Focusing in turns can be avoided by prohibiting weakly supersonic Mach maneuvers. However, focusing during acceleration cannot be avoided when the aircraft slightly exceeds Mach 1 (between Mach 1.15 and 1.2), over a crescent area a few kilometers long, and about a hundred meters wide. Figures 1.3.1, 1.3.2 and 1.3.3, inspired by those of Maglieri and Plotkin (1995) illustrate these 3 types of caustic.

Figure 1.3.1 shows the acceleration caustic for a rectilinear flight in an ideal atmosphere without a wind or temperature gradient. The Mach number increases with the acceleration of the aircraft, and the opening angle of the Mach cone decreases. The rays, emitted normal to the Mach cone, will therefore, at a certain distance from the aircraft, converge to the form of a curve in 2 dimensions, and to a a surface in 3 dimensions. In a real atmosphere, for any acceleration up to Mach 2, there is always a focus on the ground.

Figure 1.3.2 illustrates the creation of a caustic in the case of an aircraft performing a turn at constant speed. In this situation, it is the trajectory of the aircraft that is the origin of the creation of the caustic. In an ideal atmosphere, in 2 dimensions, the caustic draws a smaller circle, concentric with that of the trajectory. In 3 dimensions, a cylinder of variable diameter appears. Due to refraction, the caustic does not always reach the ground for large turn angles.

In Figure 1.3.3, we can observe the creation of a refraction caustic. This appears in transonic flight; the Mach number being close to 1, the rays leaving normal to the Mach cone are almost horizontal. When the Mach number is less than 1.15 (for a standard atmosphere without wind) they do not touch the ground; they turn around at altitude thus forming the refraction caustic.

According to catastrophe theory (Berry 1981), these caustics correspond to regular caustics (fold caustics) and are therefore structurally stable.
Figure 1.3.1: caustic from acceleration

Figure 1.3.2: caustic from turning
There are other types of caustics that we will not study here. For instance the turn entry of an aircraft in supersonic flight creates caustic arêtes which are represented by a point in two dimensions and by a line in three dimensions. These caustics correspond to the phenomenon of superfocusing studied by Cramer and Seebass (1978) and very recently by Coulouvrat (2000) and Marchiano (2000).

In chapter 2, we will proceed by doing a local study near a regular caustic to determine the pressure field in the presence of wind. We will then show that this pressure field satisfies the nonlinear Tricomi equation. Chapter 3 will outline a method of numerical solution for this equation and the validation of it. The last chapter will be devoted to the exposition of particular results obtained with this numerical code, in the case of the sonic boom.

Figure 1.3.3: caustic from refraction
Chapter 2

LOCAL STUDY OF THE ACOUSTIC FIELD NEAR A CAUSTIC

INTRODUCTION

In the previous chapter, we saw that the assumptions of geometrical acoustics are no longer satisfied when there is convergence of acoustic rays. Indeed, when the area of a ray tube vanishes, the Blokhintzev invariant furnishes a pressure tending to infinity, which is physically unacceptable: there is a singularity. Neglecting the transverse component of acoustic velocity, the main assumption of geometrical acoustics, is no longer suitable in this type of situation where diffraction effects must be taken into account.

These regions of space, places of amplification of noise, are the caustics. In this work, we will only be interested in regular caustics (fold caustics) which, according to catastrophe theory (Berry, 1981), are the simplest of the structurally stable caustics. Generally, they are produced by the uniform acceleration or turning flight of an aircraft maintaining a constant speed.

To obtain the pressure field in the vicinity of these caustics, we will use the method of matched asymptotic expansions; so we will describe the pressure field by an asymptotic boundary layer representation in the manner of Buchal and Keller (1960). According to this method, there is an outer solution, which here will be geometrical acoustics, and an inner solution, which is to be determined. The matching condition implies that the outer limit of the inner solution is identical to the inner limit of outer expansion. This matching condition is determined by the local study (inner limit) of the geometrical acoustic field (outer expansion) in the proximity of the caustic (cf. Figure 2.0). Therefore, we will first (2.1) study the solution of geometrical acoustics in the vicinity of the caustic, through the asymptotic behavior of the eikonal function. This local study will allow us to determine the adequate variables required for the reintroduction of diffraction in the inner region. In a second section (2.2), we will show that, in this system of variables, the inner solution, that is to say the sound field, in the presence of wind, locally obeys the linear Tricomi equation, the boundary conditions at infinity expressing the matching with geometrical acoustics far from the caustic. The Tricomi equation could be explicitly solved by Fourier transformation. However, we will demonstrate that, in the case of a signal with shocks incident on the caustic, such as the “N” wave of the sonic boom, the Tricomi equation leads to “U” solutions in qualitative agreement with experiments, but containing non-physical peaks of infinite amplitude. In order to limit the amplitude of these peaks, it is advisable to include an additional “limiting” mechanism, namely nonlinearities, which will be the subject of the next section (2.3). The approach will be similar to that of Coulouvrat (1997) but here we will introduce the influence of wind.
2.1 BEHAVIOR OF THE GEOMETRICAL ACOUSTIC SOLUTION NEAR A CAUSTIC

2.1.1 Definition of a caustic

In this section we will present the caustic from a geometric and mathematical point of view. For this we will use the notations and definitions introduced by Coulouvrat (1997) and generalize them when we take wind into account.

There are several definitions of a caustic. We can first consider a caustic as an area of convergence of acoustic rays, or as the locus of points such that the cross-section of an infinitesimal ray tube vanishes. We also define a caustic as an envelope of rays, that is to say a surface which, at any point, is tangent to the acoustic ray. We will show that these definitions are equivalent.

Theorem: The following four definitions of a caustic are equivalent:
1) a caustic is the envelope surface of the rays;
2) a caustic is the locus of points where the cross-section of a ray tube vanishes;
3) a caustic is the locus of points where the wavefront is singular;
4) a caustic is the locus of points where two infinitely close rays intersect.

Figure 2.0: Use of matched asymptotic expansions for the study of the focus
Proof:
Let us first consider a caustic as an envelope surface of rays. In ray coordinates, the equation of this surface can be written \( \psi = \Psi(\alpha, \beta) \), where \( \Psi(\alpha, \beta) \) is the value of the eikonal function at the point along the acoustic ray \((\alpha, \beta)\) where this ray is tangent to the caustic. The position of a caustic point is therefore identified by \( \vec{r} = \vec{r}(\alpha, \beta, \Psi(\alpha, \beta)) = \vec{R}(\alpha, \beta) \).

If we differentiate this vector by \( \psi \), it follows:
\[
\vec{r}_\psi = \frac{\partial \vec{r}}{\partial \psi}(\alpha, \beta, \Psi(\alpha, \beta)) = \frac{\partial \vec{r}}{\partial \psi}(\alpha, \beta, \Psi(\alpha, \beta)) \frac{\partial l}{\partial \psi} = v_{ray} \frac{\partial \vec{r}}{\partial l}(\alpha, \beta, \Psi(\alpha, \beta)).
\]

For an envelope surface of rays, the vector \( \vec{r}_\psi \) tangent to the acoustic ray \((\alpha, \beta)\) is also tangent to the caustic. According to the caustic equation, the vectors \( \vec{R}_\alpha = \vec{r}_\alpha + \vec{r}_\psi \Psi_\alpha \) and \( \vec{R}_\beta = \vec{r}_\beta + \vec{r}_\psi \Psi_\beta \) are tangent vectors to the caustic. Consequently, the three vectors \( \vec{r}_\alpha, \vec{r}_\beta, \) and \( \vec{r}_\psi \) are all three in the plane tangent to the caustic. They are therefore coplanar and their mixed (triple) product is null:
\[
(\vec{r}_\alpha, \vec{r}_\beta, \vec{r}_\psi) = (\vec{r}_\alpha \times \vec{r}_\beta) \cdot \vec{r}_\psi = 0.
\]
This signifies that the vector \( \vec{r}_\alpha \times \vec{r}_\beta \) is orthogonal to the tangent vector to the ray, \( \vec{l} \). On the other hand, by definition of the ray coordinates, the vectors \( \vec{r}_\alpha \) and \( \vec{r}_\beta \) are tangent to the wavefront, and therefore their vector product is either zero or directed along the normal to the wavefront, and therefore collinear to \( \vec{n} \). If it were not zero, we would deduce that \( \vec{l} \perp \vec{n} \), which is impossible for a subsonic flow for which \( |\vec{u}| < c_0 \). Therefore, we have \( \vec{r}_\alpha \times \vec{r}_\beta = 0 \).

Now, the area of a ray tube is written:
\[
\delta S = \delta \Sigma \cos(\vec{l}, \vec{n}) = (\vec{l} \cdot \vec{n}) \delta \Sigma \text{ (cf. section 1.1.7), or else:}
\]
\[
\delta S = (\vec{l} \cdot \vec{n})(\vec{r}_\alpha \times \vec{r}_\beta)\, d\alpha d\beta.
\]
We therefore deduce that the area of the infinitesimal ray tube is zero at all points of the caustic, which proves 1) to 2).

Conversely, if we write \( \psi = \Psi(\alpha, \beta) \), the value of the eikonal function at the point along the acoustic ray \((\alpha, \beta)\) where the area of the infinitesimal ray tube vanishes, and \( \vec{r} = \vec{r}(\alpha, \beta, \Psi(\alpha, \beta)) = \vec{R}(\alpha, \beta) \) the equation of the surface so defined, then at any point of this surface, \( \vec{r}_\alpha \times \vec{r}_\beta = 0 \). Consequently, the mixed product \( (\vec{r}_\alpha, \vec{r}_\beta, \vec{r}_\psi) = (\vec{R}_\alpha, \vec{R}_\beta, \vec{R}_\psi) = 0 \), which means that the three vectors \( \vec{R}_\alpha, \vec{R}_\beta \) and \( \vec{r}_\psi \) are coplanar, and therefore that the vector tangent to the ray \( \vec{r}_\psi \) is in the plane \( (\vec{R}_\alpha, \vec{R}_\beta) \) tangent to the caustic. The ray \((\alpha, \beta)\) is tangent to the caustic and it is therefore an envelope of the ray. The equivalence of the definitions 1)\( \iff \)2) is well established. The equivalence 2)\( \iff \)3) is an immediate consequence of the definition of \( \delta \Sigma \) (area element of the wavefront). At the point where the ray tube area vanishes, the vectors \( \vec{r}_\alpha \) and \( \vec{r}_\beta \) are collinear, and the wavefront is singular: the normal to the wavefront is no longer defined, the wavefront presents an arête.

To demonstrate the equivalence with the fourth point, it suffices to note that we have already shown that a caustic can also be defined as the locus of points where the three vectors \( \vec{r}_\alpha, \vec{r}_\beta \) and \( \vec{r}_\psi \) are coplanar. This is equivalent to saying that there exist three quantities of \( d\alpha, d\beta \) and \( d\psi \) (that we may choose as small as we wish) such that \( \vec{r}_\alpha d\alpha + \vec{r}_\beta d\beta + \vec{r}_\psi d\psi = 0 \), or:
\[
\vec{r}(\alpha, \beta, \psi) = \vec{r}(\alpha + d\alpha, \beta + d\beta, \psi + d\psi); \quad \text{the two infinitely close rays (} \alpha, \beta \text{ and } (\alpha + d\alpha, \beta + d\beta) \text{ intersect at this point.}
\]

### 2.1.2 Curvilinear coordinates associated with the caustic

In order to study the acoustic field near a regular caustic, it is necessary to introduce a system of orthogonal curvilinear coordinates, describing the geometry of the surface. To do this, we use the curvilinear coordinate system \((\sigma, q)\) introduced by Babić and Buldyrev (1991). These coordinates
are chosen as follows. Let $C$ be a point of the caustic; the caustic being by definition an envelope of rays, a ray tangent to the caustic passes through point $C$. We choose for $\sigma$ the curvilinear abscissa along the line tangent to this ray on the caustic. The second curvilinear coordinate $q$ is chosen orthogonal to $\sigma$ on the caustic.

The position of a point of the caustic is identified by a function $\vec{x}_{cau}(\sigma, q)$. On the caustic, the tangent vectors are written:

$$\vec{t}_c(\sigma, q) = \frac{\partial \vec{x}_{cau}}{\partial \sigma} \quad \text{and} \quad \vec{q}_c(\sigma, q) = \frac{\partial \vec{x}_{cau}}{\partial q},$$

and the normal directed toward the insonified side equals:

$$\vec{N} = \pm \frac{\vec{t}_c \times \vec{q}_c}{|\vec{t}_c \times \vec{q}_c|}.$$

We suppose that the caustic is regular. Therefore the vector product $\vec{t}_c \times \vec{q}_c$ is never zero.

Outside the caustic, we denote as before the vector tangent to the acoustic ray $\vec{t}(\alpha, \beta, \psi)$ as well as $\vec{n}(\alpha, \beta, \psi)$, the vector normal to the wavefront, $l$ being the curvilinear abscissa along the ray considered. For a point of the caustic, thanks to the choice of the variable $\sigma$, we clearly have $\vec{t}_c = \vec{t}$.

### 2.1.3 Asymptotic expansion of the eikonal function near the caustic

In this section we will establish an approximate expression for the eikonal function near a particular point of the caustic. This expression will allow us to estimate how far from the caustic the geometric approximation remains valid. It will also lead us to introduce the variables determining the acoustic field inside the boundary layer, which will then allow us to obtain the equation satisfied by the acoustic field in this region. We will assume that the caustic is regular which prohibits the nullity of the radii of curvature.

#### 2.1.3.1 Geometry and notations

Denote by $O$ a particular point of the caustic. We introduce an associated local orthonormal coordinate system $(Oxyz)$ such that the $Ox$ axis is oriented along the ray tangent to the caustic at $O$, the $Oy$ axis is directed normal to $Ox$ in the plane tangent to the caustic and the $Oz$ axis follows the normal to the caustic, directed toward the insonified side. We call $M$ a current point near the caustic on the insonified side, $C$ the point of contact with the caustic of one of the two rays which pass through $M$, and $P$ the projection of $M$ onto the caustic, the distance $MP$ being called $h$. Finally, at any point of the caustic, we denote by $\vec{N}$ the normal to the caustic directed toward the insonified side and by $\vec{n}_c(\sigma, q)$ the normal to the wavefront from a point of the caustic (Figure 2.1.1).
2.1.3.2 “Incident” and “reflected” rays

We write, first, that the points $C$ and $M$ are on the same acoustic ray. The equation of an acoustic ray is written:

$$ \frac{d\vec{x}_{\text{ray}}}{dl} = \vec{t}. $$

By Taylor expansion, we have the following expression:

$$ \vec{x}(M) = \vec{x}(C) + \vec{t}(C) \delta l + \frac{d\vec{t} \delta l^2}{dl^2} + O(\delta l^3), $$

$\delta l$ being the difference between the curvilinear abscissas of points $M$ and $C$ along the acoustic ray.

The radius of curvature of an acoustic ray, $R_{\text{ray}}$, is defined by:

$$ \frac{d\vec{x}_{\text{ray}}}{dl} = \vec{v}_{\perp \text{ray}}(C), $$

where $\vec{v}_{\perp \text{ray}}$ is the principal normal to the acoustic ray. It then follows:

$$ \vec{x}(M) = \vec{x}(C) + \vec{t}(C) \delta l + \frac{\vec{v}_{\perp \text{ray}}(C) \delta l^2}{R_{\text{ray}}(C) \ 2} + O(\delta l^3). $$

At point $P$, we have by definition of the normal $\vec{N}$:

$$ \vec{x}(M) - \vec{x}(P) = h\vec{N}(P), $$

$$ (\vec{x}(M) - \vec{x}(P)) \cdot \vec{t}(P) = 0, $$

$$ (\vec{x}(M) - \vec{x}(P)) \cdot \vec{q}(P) = 0. $$
By Taylor expansion along the caustic, we have the following expressions:

\[
\begin{align*}
\vec{x}(P) &= \vec{x}(C) + \vec{t}(C)\delta\sigma + \vec{q}(C)\delta q + \frac{\partial \vec{t}}{\partial \sigma}(C) \frac{\delta\sigma^2}{2} + \frac{\partial \vec{q}}{\partial q}(C) \frac{\delta q^2}{2} + \frac{\partial \vec{t}}{\partial q}(C) \delta\sigma\delta q + \mathcal{O}(\delta\sigma^3, \delta q^3), \\
\vec{N}(P) &= \vec{N}(C) + \frac{\partial \vec{N}}{\partial \sigma}(C) \delta\sigma + \frac{\partial \vec{N}}{\partial q}(C) \delta q + \mathcal{O}(\delta\sigma^2, \delta q^2), \\
\vec{t}(P) &= \vec{t}(C) + \frac{\partial \vec{t}}{\partial \sigma}(C) \delta\sigma + \frac{\partial \vec{t}}{\partial q}(C) \delta q + \mathcal{O}(\delta\sigma^2, \delta q^2), \\
\vec{q}(P) &= \vec{q}(C) + \frac{\partial \vec{q}}{\partial \sigma}(C) \delta\sigma + \frac{\partial \vec{q}}{\partial q}(C) \delta q + \mathcal{O}(\delta\sigma^2, \delta q^2),
\end{align*}
\]

Let us make explicit the difference \(\vec{x}(M) - \vec{x}(P)\):

\[
\begin{align*}
\vec{x}(M) - \vec{x}(P) &= \vec{t}(C)\delta l + \frac{\vec{v}_{\text{ray}}(C)}{R_{\text{ray}}(C)} \frac{\delta l^2}{2} - \vec{t}(C)\delta\sigma - \vec{q}(C)\delta q \\
&\quad - \frac{\partial \vec{t}}{\partial \sigma}(C) \frac{\delta\sigma^2}{2} - \frac{\partial \vec{q}}{\partial q}(C) \frac{\delta q^2}{2} - \frac{\partial \vec{t}}{\partial q}(C) \delta\sigma\delta q + \mathcal{O}(\delta\sigma^3, \delta q^3).
\end{align*}
\]

The relations \((\vec{x}(M) - \vec{x}(P)) \cdot \vec{t}(P) = 0, \ (\vec{x}(M) - \vec{x}(P)) \cdot \vec{q}(P) = 0\), the expansion of \(\vec{t}(P)\) and the fact that \(\vec{t}(C) = \vec{t}(C)\) lead to the following relations:

\[
\begin{align*}
\delta\sigma &= \delta l + \mathcal{O}(\delta l^2), \\
\delta q &= \mathcal{O}(\delta l^2).
\end{align*}
\]

This signifies that the two quantities drawn in red below (Figure 2.1.2), corresponding on the one hand to the curvilinear abscissa along the ray between the point of contact with the caustic and the point of observation, and on the other hand, the curvilinear abscissa along the caustic between the point of contact and the projection of the observation point are identical to within order \(\mathcal{O}(\delta l^2)\).

![Figure 2.1.2: Identification of distances \(\sigma\) and \(\delta\) to order \(\delta l^2\)](image)

Furthermore, we may make explicit the value of \(h\) in terms of \(\delta l\). We recall the expression for \(h\):

\[
h = (\vec{x}(M) - \vec{x}(P)) \cdot \vec{N}(P).
\]
Replacing $\delta \sigma$ and $\delta q$ by their respective expressions in terms of functions of $\delta l$ in the previous equation, we obtain:

$$h = \frac{\delta l^2}{2} \left[ \frac{\bar{v}_{\text{ray}}(C)}{R_{\text{ray}}(C)} - \frac{\partial \overline{\mathbf{r}}^\bot}{\partial \sigma} \right] \cdot \overline{N}(C) + \mathcal{O}(\delta l^3).$$

By definition, at any point of the caustic and in particular at $C$, the vectors $\overline{t}$ and $\overline{N}$ are orthogonal. So we have $\overline{N} \cdot \overline{t} = 0$, and, by differentiating with respect to $\sigma$:

$$\overline{N} \cdot \frac{\partial \overline{t}}{\partial \sigma} = -\overline{t} \cdot \frac{\partial \overline{N}}{\partial \sigma}.$$

Using classic results from the theory of surfaces, as was presented by Smirnov (1971), it follows:

$$\overline{t} \cdot \frac{\partial \overline{N}}{\partial \sigma} = \frac{1}{R_{\text{sec}}},$$

with $R_{\text{sec}}$ the radius of curvature of the plane curve, called the normal section, formed by the intersection of the caustic with the $(\overline{t}, \overline{N})$ plane. This radius of curvature is assigned the sign “+” if the caustic and the rays are located on opposite sides of the plane tangent to the caustic, and the sign “−” if they are on the same side. With this sign convention, $h$ is written as follows:

$$h = \frac{\delta l^2}{2} \left[ \cos \theta \left( \frac{\overline{v}_{\text{ray}}^\bot}{R_{\text{ray}}(C)}, \overline{N} \right) + \frac{1}{R_{\text{sec}}(C)} \right] + \mathcal{O}(\delta l^3),$$

where $\theta \left( \frac{\overline{v}_{\text{ray}}^\bot}{R_{\text{ray}}(C)}, \overline{N} \right)$ is the angle formed by the vectors $\frac{\overline{v}_{\text{ray}}^\bot}{R_{\text{ray}}(C)}$ and $\overline{N}$ at the contact point $C$ (cf. Figure 2.1.3).
Figure 2.1.3: Radii of curvature associated with the caustic according to $\tilde{r}^\epsilon$
At the point $C$ of the caustic, we will therefore denote by $R_{cau}$ the quantity defined by:

$$\frac{1}{R_{cau}(C)} = \frac{1}{R_{sec}(C)} + \frac{\cos \theta(\vec{v}_{\perp \text{ray}}, \vec{N})}{R_{ray}(C)}$$

$R_{cau}$ appears as the radius of curvature of the ray ($R_{ray}$) relative to the caustic ($R_{sec}$). In two dimensions, $\vec{v}_{\perp \text{ray}}$ and $\vec{N}$ coincide but in 3 dimensions, they can form an angle that should be taken into account, as noted by Babić and Buldyrev (1991).

Furthermore: $R_{cau}(P) = R_{cau}(C) + \frac{\partial R_{cau}(C)}{\partial \sigma} \delta \sigma + \mathcal{O}(\delta \sigma^2, \delta \phi^2)$.

So we can rewrite $h$ in the following way:

$$h = \frac{\delta l^2}{2R_{cau}(P)} + \mathcal{O}(\delta l^3).$$

By inverting the relationship, it follows:

$$\delta l = \pm \sqrt{2hR_{cau}(P) + \mathcal{O}(\delta l^{3/2})}.$$

The “+” sign corresponds to a positive curvilinear distance $CM$, that is to say that the point $M$ is located after the contact point of the ray; on the contrary, the “-” sign indicates that point $M$ is located before the point of contact of the ray with the caustic. The first case corresponds to a “reflected” ray which has “tangented” the caustic before reaching point $M$, while the second corresponds to a ray “incident” to the caustic, which reached point $M$ before “tangent” the caustic (Figure 2.1.4).

Figure 2.1.4 : “Incident” and “reflected” rays
2.1.3.3 Characteristic thickness of the refractive boundary layer

Let us calculate the tangent vector to the ray, at point \( M \):

\[
\vec{t}(M) = \hat{t}(C) + \frac{d\hat{t}}{dl}(C) \delta l + O(\delta l^2),
\]

so that by definition:

\[
\vec{t}(M) = \hat{t}(C) + \frac{\vec{n}_{ray}(C)}{R_{ray}(C)} \delta l + O(\delta l^2).
\]

Furthermore, we have the following expression:

\[
\hat{t}(C) - \hat{t}(P) = -\frac{\partial \hat{t}(P)}{\partial \sigma} \delta l + O(\delta l^2).
\]

Thus, the tangent vector at \( M \), a function of the point \( P \), is written:

\[
\vec{t}(M) = \hat{t}(P) + \left[ -\frac{\partial \hat{t}(P)}{\partial \sigma} + \frac{\vec{n}_{ray}(P)}{R_{ray}(P)} \right] \delta l + O(\delta l^2).
\]

These relationships will allow us to express the value of the eikonal function at a point \( M \) in the neighborhood of the caustic, as a function of its value at point \( P \), the projection of \( M \) onto the caustic.

The gradient of the eikonal function written at point \( M \) is given by the eikonal equation:

\[
\nabla \psi = \frac{\vec{n}(M)}{c_0(M) + \vec{u}_0(M) \cdot \vec{n}(M)} = \frac{\vec{n}(M)}{c(M)},
\]

where we have introduced the effective speed of sound \( c \), taking into account the movement of the fluid:

\[
c = c_0 + \vec{u}_0 \cdot \vec{n}.
\]

The normal to the wavefront at point \( M \) as a function of point \( C \) is written:

\[
\vec{n}(M) = \vec{n}(C) + \frac{d\vec{n}}{dl}(C) \delta l + O(\delta l^2).
\]

Written at point \( P \), the normal to the wavefront has the following form:

\[
\vec{n}^c(P) = \vec{n}^c(C) + \frac{d\vec{n}^c}{d\sigma}(C) \delta \sigma + O(\delta l^2) = \vec{n}(C) + \frac{d\vec{n}^c}{d\sigma}(C) \delta l + O(\delta l^2)
\]

We may therefore write the normal to the wavefront at point \( M \) in the following form:

\[
\vec{n}(M) = \vec{n}^c(P) + \left[ \frac{d\vec{n}}{dl}(C) - \frac{d\vec{n}^c}{d\sigma}(C) \right] \delta l + O(\delta l^2) = \vec{n}^c(P) + \left[ \frac{d\vec{n}}{dl}(P) - \frac{d\vec{n}^c}{d\sigma}(P) \right] \delta l + O(\delta l^2).
\]

Taylor expansions of the speed of sound and the wind speed at point \( M \) as a function of point \( P \) are written:

\[
c_0(M) = c_0(P)(1 + O(h)) = c_0(P)(1 + O(\delta l^2)),
\]

\[
\vec{u}_0(M) = \vec{u}_0(P)(1 + O(h)) = \vec{u}_0(P)(1 + O(\delta l^2)).
\]
The gradient of the eikonal function then takes the following form:

$$\nabla \psi = \frac{\vec{n}^c(P) + \delta l \left[ \frac{\partial \vec{n}^c}{\partial t} (P) - \frac{\partial \vec{n}^c}{\partial \sigma} (P) \right] + \mathcal{O}(\delta l^2)}{c(P) + \vec{u}_0(P) \cdot \left[ \frac{\partial \vec{n}^c}{\partial t} (P) - \frac{\partial \vec{n}^c}{\partial \sigma} (P) \right] \delta l + \mathcal{O}(\delta l^2)},$$

or else:

$$\nabla \psi = \frac{\vec{n}^c(P)}{c(P)} + \delta l \left( \frac{\left[ \frac{\partial \vec{n}^c}{\partial t} (P) - \frac{\partial \vec{n}^c}{\partial \sigma} (P) \right] \vec{n}^c(P)}{c(P)} - \left( \vec{u}_0(P) \cdot \left[ \frac{\partial \vec{n}^c}{\partial t} (P) - \frac{\partial \vec{n}^c}{\partial \sigma} (P) \right] \right) \vec{n}^c(P) \right) + \mathcal{O}(\delta l^2),$$

We define $\vec{N}^{FO}(P)$, the unit vector of the ($\vec{n}^c(P)$, $\vec{N}(P)$) plane, normal to the wavefront normal at point $P$, $\vec{n}^c(P)$ (Figure 2.1.5). In addition, we denote by $\theta(\vec{N}, \vec{N}^{FO})$, the angle between the vectors $\vec{N}$ (normal to the caustic) and $\vec{N}^{FO}(P)$ (normal to the wavefront normal).

By projecting the expression for $\nabla \psi$ above in the direction $\vec{N}^{FO}$, we obtain:

$$\nabla \psi \cdot \vec{N}^{FO}(P) = \delta l \left( \frac{\left[ \frac{\partial \vec{n}^c}{\partial t} (P) - \frac{\partial \vec{n}^c}{\partial \sigma} (P) \right] \vec{n}^c(P)}{c(P)} \right) \cdot \vec{N}^{FO}(P) + \mathcal{O}(\delta l^2).$$

We denote by $H$ the distance to the caustic measured in the direction $\vec{N}^{FO}$, such that $H = \frac{h}{\cos \theta(\vec{N}, \vec{N}^{FO})}$, it then follows:

$$\nabla \psi \cdot \vec{N}^{FO}(P) = \frac{d\psi}{dH}.$$

By analogy with the radius of curvature of the acoustic ray, we define the following quantity:

$$\frac{d\vec{n}(P)}{dl} = \frac{\vec{v}^{FO}(P)}{R^{FO}(P)},$$

where $\vec{v}^{FO}$ is the principal normal to the curve described by the wavefront normal $\vec{n}$ and $R^{FO}$ the radius of curvature of this curve.

Figure 2.1.5 : Definition of $\vec{N}^{FO}$ and $H$
In addition, $\vec{n}(P)$ and $\vec{N}^{FO}(P)$ are orthogonal so $\vec{n}(P) \cdot \vec{N}^{FO}(P) = 0$, thus:

$$-\frac{d\vec{n}(P)}{d\sigma} \cdot \vec{N}^{FO}(P) = \vec{n}(P) \cdot \frac{d\vec{N}^{FO}(P)}{d\sigma} = \frac{1}{R^{FO}(P)}.$$

where $R^{FO}(P)$ is the radius of curvature of the plane curve, formed by the intersection of the caustic with the $(\vec{n}, \vec{N})$ plane.

So that:

$$\frac{d\psi}{dH} = \pm \sqrt{2hR_{cau}(P)} \frac{1}{c(P)} \left[ \cos \theta\left(\vec{n}^{FO}(P), \vec{N}^{FO}(P)\right) \frac{R^{FO}(P)}{R^{FO}(P)} + \frac{1}{R^{FO}_{sec}(P)} \right] + O(\delta l^2).$$

At a point $P$ of the caustic, we denote by $R_{cau}^{FO}(P)$ the quantity defined by:

$$\frac{1}{R_{cau}^{FO}(P)} = \frac{\cos \theta\left(\vec{n}^{FO}(P), \vec{N}^{FO}(P)\right)}{R^{FO}(P)} + \frac{1}{R^{FO}_{sec}(P)}.$$

This definition is analogous to that obtained in the windless case and introduced in 2.1.3.2:

$$\frac{1}{R_{cau}(P)} = \frac{\cos \theta\left(\vec{n}(P), \vec{N}(P)\right)}{R^{ray}(P)} + \frac{1}{R^{sec}(P)}.$$

but the quantities associated with the vector $\vec{t}$ tangent to the ray, for $R_{cau}$, have been replaced by quantities associated with the wavefront normal $\vec{n}$, $R_{cau}^{FO}$. In the windless atmosphere case, the vectors $\vec{n}$ and $\vec{t}$ coincide, and the radii of curvature $R_{cau}^{FO}$ and $R_{cau}$ are identical.

By considering the relationship linking $h$ and $H$, we may rewrite the previous expression for $\frac{d\psi}{dH}$ in the following way:

$$\frac{d\psi}{dH} = \pm \frac{1}{c(P)} \sqrt{\frac{2H}{R_{tot}(P)}} + O(H),$$

where we have defined:

$$R_{tot}(P) = \frac{\left(R_{cau}^{FO}(P)\right)^2}{R_{cau}(P)} \frac{1}{\cos \theta\left(\vec{N}, \vec{N}^{FO}\right)}.$$

In the windless case, the radii of curvature $R_{cau}^{FO}$ and $R_{cau}$ being identical and $\theta\left(\vec{N}, \vec{N}^{FO}\right)$ being zero, we simply have: $R_{tot}(P) = R_{cau}(P)$.

By integrating $d\psi/dH$ in the direction $\vec{N}^{FO}$, we obtain:

$$\psi(Q) = \psi(P) \pm \frac{1}{c(P)} \sqrt{\frac{8H^3}{9R_{tot}(P)}} + O(H^2).$$
This expression already makes it possible to determine the characteristic thickness over which the influence of diffraction effects is important.

Indeed, the diffraction effects will be significant when the difference between the arrival times of the incident and reflected rays is less than an amount on the order of magnitude of the period of the acoustic signal, which is, in this case:

\[
\frac{1}{c(P)} \frac{\omega \delta^{3/2}}{R_{\text{tot}}^{1/2}} \approx 1, \quad \text{or} \quad \delta(\omega) \approx c^{2/3} R_{\text{tot}}^{1/3}/\omega^{2/3}.
\]

In the windless case, we find the same relation where \( R_{\text{cau}} \) is replaced by \( R_{\text{tot}} \):

Below this distance, the arrival times of the incident and reflected signals are sufficiently close to each other so that they can no longer be distinguished from each other (we speak of the "indiscernability" of the rays). The notion of acoustic ray then loses its physical meaning. Diffraction is no longer negligible, we are inside the boundary layer. This characteristic distance is going to serve as a characteristic scale to dimension the equations of motion within the diffraction boundary layer. Furthermore, it can be seen that the higher the frequency considered, the thinner the boundary layer. In the case of the sonic boom, the signals generally have shocks. From a numerical point of view, it will be necessary to discretize sufficiently in order to capture the boundary layers corresponding to the highest frequencies.

2.1.3.4 Eikonal function near the caustic

We apply the previous results to the case of a point located near \( O \). Let us consider an orthonormal reference frame \((OXYZ)\), such that the \((OZ)\) axis is directed along \( \vec{N}^F O \) (\( O \)) and the \((OX)\) axis is following \( \vec{n}^c(O) \) (Figure 2.1.6).

![Figure 2.1.6: Geometry around the new orthonormal reference \((OXYZ)\)](image)

By Taylor expansion of the eikonal function, we have:

\[
\psi(X,Y,Z) = \psi(0,0,Z) + \frac{\partial \psi}{\partial X}(0,0,Z) X + \frac{\partial \psi}{\partial Y}(0,0,Z) Y + \mathcal{O}(X^2,Y^2,XY),
\]
with, from the previous result:

$$\psi(0, 0, Z) = \psi(O) \pm \sqrt{\frac{8Z^3}{9c^2(O)R_{tot}(O)}} + \mathcal{O}(Z^2),$$

$$\frac{\partial \psi}{\partial X}(0, 0, Z) = \frac{\partial \psi}{\partial X}(0) + \mathcal{O}(Z) = \frac{1}{c(O)} + \mathcal{O}(Z),$$

$$\frac{\partial \psi}{\partial Y}(0, 0, Z) = \frac{\partial \psi}{\partial Y}(0) + \mathcal{O}(Z) = \mathcal{O}(Z).$$

In total, we have:

$$\psi(Q) = \psi(O) \pm \sqrt{\frac{8Z^3}{9c^2(O)R_{tot}(O)}} + \frac{X}{c(O)} + \mathcal{O}\left(\frac{X^2}{cR_{tot}}\right).$$

Here we find the formula given by Babić and Buldyrev (1991), for asymptotic expansion of the eikonal function in the neighborhood of a point of the caustic to the second order, generalized to the case of a moving atmosphere. In fact, this expression is not quite sufficient for further calculations. Let us write the unspecified term as follows:

$$\mathcal{O}\left(\frac{X^2}{cR_{tot}}\right) = \frac{1}{c(O)} \left(\frac{XZ}{R_{XZ}} + \frac{XY}{R_{XY}} + \frac{YZ}{R_{Y}Z} + \frac{X^2}{R_{XX}} + \frac{Y^2}{R_{YY}} + \frac{Z^2}{R_{ZZ}}\right) + \mathcal{O}\left(\frac{X^3}{c(O)R_{tot}(O)}\right).$$

We will see later that it is necessary to know the quantity $R_{XZ}$. To do this, let us calculate the gradient of the eikonal function:

$$\nabla \psi(0, 0, Z) = \frac{1}{c(O)} \left(\frac{1 + \frac{Z}{R_{XZ}}}{\pm \sqrt{\frac{2Z}{R_{tot}(O)} + \frac{2Z}{R_{XX}}}}\right) + \mathcal{O}\left(\frac{Z^2}{c(O)R_{tot}}\right).$$

We deduce, by Taylor expansion:

$$|\nabla \psi(0, 0, Z)| = \frac{1}{c(O)} \left(1 + \frac{Z}{R_{XZ}} + \frac{Z}{R_{tot}(O)} \pm \frac{(2Z)^{3/2}}{R_{tot}^{1/2}(O)R_{ZZ}} + \mathcal{O}\left(\frac{Z^2}{R_{tot}^2}\right)\right).$$

Now, according to the eikonal equation:

$$|\nabla \psi(0, 0, Z)| = \frac{1}{c(0, 0, Z)} = \frac{1}{c(Z)},$$

with $c(Z) = c(O) + \frac{dc}{dZ}(O)Z + \mathcal{O}(Z^2)$.

We define $R_{cel}$ as follows:

$$R_{cel}(O) = \left[\frac{1}{c(O)} \frac{dc}{dZ}(O)\right]^{-1},$$

which is a characteristic distance for the heterogeneity of the medium relative to the speed.
So:

$$|\nabla \psi(0,0,Z)| = \frac{1}{c(O)} \left[ 1 - \frac{Z}{R_{cel}(O)} + O(Z^2) \right].$$

By identifying the two expressions of $|\nabla \psi(0,0,Z)|$, we obtain the following relations:

$$\frac{1}{R_{ZZ}} = 0 \quad \text{and} \quad \frac{1}{R_{XZ}} = -\frac{1}{R_{tot}(O)} - \frac{1}{R_{cel}(O)}.$$

The expansion of the eikonal function close to a point of the caustic is finally written:

$$\psi(Q) = \psi(O) \pm \sqrt{\frac{8Z^3}{9c^2(O)R_{tot}(O)}} + \frac{X(1 + Z/R_{XZ}(O))}{c(O)} + O\left(\frac{X^2, Y^2, XY, YZ}{c(O)R_{tot}}\right).$$

### 2.1.4 Expression for the geometric divergence near the caustic

We may show (cf. Coulouvrat 1997, p. 29-31 & Appendix II) that the ray tube area can be identified with the Jacobian associated with the change of variables $(X,Y,Z) \rightarrow (\alpha,\beta,\psi)$:

$$J = \frac{D(X,Y,Z)}{D(\alpha,\beta,\psi)} = \det \begin{bmatrix} \frac{\partial X}{\partial \alpha} & \frac{\partial X}{\partial \beta} & \frac{\partial X}{\partial \psi} \\ \frac{\partial Y}{\partial \alpha} & \frac{\partial Y}{\partial \beta} & \frac{\partial Y}{\partial \psi} \\ \frac{\partial Z}{\partial \alpha} & \frac{\partial Z}{\partial \beta} & \frac{\partial Z}{\partial \psi} \end{bmatrix}.$$

In order to evaluate this quantity, we will express the ray coordinates $(\alpha,\beta,\psi)$ as a function of the Cartesian coordinates of the new reference frame $(X,Y,Z)$ at a point $Q$ near the caustic. Now knowing the eikonal function, it remains to establish the expressions for $(\alpha,\beta)$ near the caustic.

By definition, we have:

$$(\alpha(M), \beta(M)) = (\sigma(C), q(C)) = (\sigma(P) - \delta \sigma, q(P) - \delta q).$$

The point $P$ being close to the origin, it follows:

$$\bar{x}(P) = \bar{x}_{cau}(\sigma(P), q(P)) = \bar{x}_{cau}(O) + \bar{t}(\sigma(P) - \sigma(O)) + \bar{q}(q(P) - q(O)) + O(\sigma^2),$$

$$\bar{x}(P) = \bar{t}(\sigma(P) - \psi(O)) + \bar{q}(q(P) - q(O)) + O(\sigma^2).$$

On the other hand, by definition: $\bar{x}(Q) = \bar{x}(P) + H \bar{N}^{FO}(P) = \bar{x}(P) + H\bar{\sigma}Z + O(H\sigma)$. By identification it follows:

$$\sigma(P) = \psi(O) + X + O(\bar{X}^2),$$

$$q(P) = q(O) + Y + O(\bar{X}^2),$$

$$H = Z + O(\bar{X}^2).$$

In fact, these relationships amount to neglecting the curvature of the caustic and the rays at the first order. We have also previously shown that:

$$\delta \sigma = \delta l = \pm R^{FO}_{cau}(P) \sqrt{2H/\bar{R}_{tot}(P)} + O(\delta l^{3/2}),$$

$$\frac{1}{R_{ZZ}} = 0 $$

$$\frac{1}{R_{XZ}} = -\frac{1}{R_{tot}(O)} - \frac{1}{R_{cel}(O)}$$

The expansion of the eikonal function close to a point of the caustic is finally written:

$$\psi(Q) = \psi(O) \pm \sqrt{\frac{8Z^3}{9c^2(O)R_{tot}(O)}} + \frac{X(1 + Z/R_{XZ}(O))}{c(O)} + O\left(\frac{X^2, Y^2, XY, YZ}{c(O)R_{tot}}\right).$$

2.1.4 Expression for the geometric divergence near the caustic

We may show (cf. Coulouvrat 1997, p. 29-31 & Appendix II) that the ray tube area can be identified with the Jacobian associated with the change of variables $(X,Y,Z) \rightarrow (\alpha,\beta,\psi)$:

$$J = \frac{D(X,Y,Z)}{D(\alpha,\beta,\psi)} = \det \begin{bmatrix} \frac{\partial X}{\partial \alpha} & \frac{\partial X}{\partial \beta} & \frac{\partial X}{\partial \psi} \\ \frac{\partial Y}{\partial \alpha} & \frac{\partial Y}{\partial \beta} & \frac{\partial Y}{\partial \psi} \\ \frac{\partial Z}{\partial \alpha} & \frac{\partial Z}{\partial \beta} & \frac{\partial Z}{\partial \psi} \end{bmatrix}.$$
and thus:
\[ \delta \sigma = \pm R_{cau}^{FO}(P) \sqrt{2Z/R_{tot}(P)} + O(\| \vec{X} \|), \]
from which we have:
\[ \alpha(Q) = \psi(O) = R_{cau}^{FO}(P) \sqrt{2Z/R_{tot}(P)} + O(\| \vec{X} \|), \]
\[ \beta(Q) = q(O) + Y + O(Z) + O(\vec{X}^2). \]
Finally, based on the results of the previous section, \( \psi(Q) = \psi(O) + X/c(O) + O(Z^{3/2}). \)

These results lead to the expression of the determinant:
\[
\frac{D(\alpha, \beta, \psi)}{D(X, Y, Z)} = \det \begin{bmatrix}
O(1) & O(1) & \mp R_{cau}^{FO}(O) \sqrt{1/2ZR_{tot}(O)} + O(1) \\
O(|\vec{X}|) & 1 + O(|\vec{X}|) & O(1) \\
1/c(O) + O(|\vec{X}|) & O(|\vec{X}|) & O(\sqrt{Z})
\end{bmatrix},
\]
that is:
\[
\frac{D(\alpha, \beta, \psi)}{D(X, Y, Z)} = \pm R_{cau}^{FO}(O) \sqrt{1/2ZR_{tot}(O)} + O(1).
\]
We deduce the expression for the geometric divergence:
\[
J = \left( \frac{D(\alpha, \beta, \psi)}{D(X, Y, Z)} \right)^{-1} = c(O) \frac{1}{R_{cau}^{FO}(O)} \sqrt{2ZR_{tot}(O)} + O(Z).
\]
We recover the fact that the ray tube area vanishes on the caustic at \( Z = 0 \). However this result is more precise because it shows that this area varies as a function of the distance to the caustic like \( \sqrt{Z} \).

2.1.5 The geometric acoustic solution near a caustic

gEometrical acoustics expresses the sound field along a ray in the form:
\[
\hat{p}_u(\mathbf{\tau}, l) = \left[ J(0)[c_0(0)\bar{\mathbf{n}}(0) + \vec{u}_0(0)]/\rho_0(l)c_0^2(l) \right]^{1/2} \hat{p}_u(\mathbf{\tau}, l = 0).
\]
By substituting the previous results in the above expression, we finally obtain the first-order asymptotic expansion of geometrical acoustics close to a point \( O \) of the caustic:
\[
\frac{p_u(\vec{X}, t)}{\rho_0(0)c_0^2} = M_{ac}\left( \frac{D}{Z} \right)^{1/4} \left( F\left\{ t - \left[ \psi(O) - \frac{1}{c(O)} \sqrt{\frac{2Z^3}{R_{tot}(O)}} + \frac{X(1+Z/R_{XZ}(O))}{c(O)} \right] + O\left( \frac{X^2/Y^2/XY/YZ}{c(O)} \right) \right\} \right)
\[
G\left\{ t - \left[ \psi(O) + \frac{1}{c(O)} \sqrt{\frac{2Z^3}{R_{tot}(O)}} + \frac{X(1+Z/R_{XZ}(O))}{c(O)} \right] + O\left( \frac{X^2/Y^2/XY/YZ}{c(O)} \right) \right\} \right).
\]
In this expression, the functions \( F(t) \) and \( G(t) \) are dimensionless. The distance \( D \) is the characteristic distance at which we evaluate the geometric approximation at a certain distance from the caustic. In practice, \( D \) is of the order of \( \delta(\omega_{ac}) \), the thickness of the boundary layer at the central frequency \( \omega_{ac} \) of the spectrum of the incident signal. The amplitude of the field is measured by the small parameter \( M_{ac} = P_{ac}/\rho_0 c_0^2 \), which is the acoustic Mach number of the incident field at the distance \( Z = D \). The field amplitude increases as \( Z^{-1/4} \) near the caustic. The contributions of each of the two incident ("-" sign) and reflected ("+" sign) rays are represented by the functions \( F \) and \( G \) which determine the temporal form of the signal along the ray. This expression is clearly only valid sufficiently far from the caustic, at a distance on the order of the characteristic distance \( \delta(\omega) \).
In practice, the function $F$ representing the field “incident” to the caustic is determined by identifying the approximate expression above with the geometrical acoustics solution calculated numerically via ray tracing. On the other hand, the function $G$, which represents the “reflected” field is necessarily affected by the behavior of the solution in the vicinity of the caustic, where the geometrical acoustics approximation is no longer valid. It therefore cannot be determined \textit{a priori} from geometrical acoustics alone. To evaluate it, it is necessary to study the field more precisely near the caustic, at small distances ahead of the characteristic distance $\delta(\omega)$, in the region where the influence of diffraction effects can no longer be neglected.

![Diagram: Incident Wave, Caustic, Reflected Wave, Diffraction Boundary Layer](image)

\textbf{Figure 2.1.7 : incident wave and reflected wave}

### 2.1.6 Conclusion : influence of wind

We recall the expressions laid out by François Coulouvrat (1997) in the windless case:

$$
\psi(Q) = \psi(O) \pm \sqrt{\frac{8z^3}{9c^3(\omega)R_{cau}(O)}} \cdot \frac{x(1 + z/R_{xx}(O))}{c(O)}, \text{ in the } (x, y) \text{ reference frame.}
$$

We note that, in the case with wind, the expressions obtained here are similar, but with the following differences:

- the speed of sound is replaced by the apparent speed of sound $c = c_0 + \vec{u}_0 \cdot \vec{n}$,

- the appropriate coordinate system undergoes a rotation of $\theta\left(\vec{N}, \vec{N}_{FO}\right)$, due to the fact that the tangent vector and the wavefront normal no longer coincide,

- $R_{cau}$ is replaced by $R_{tot} = \frac{(R_{FO})^2}{R_{cau} \cos \theta(\vec{N}, \vec{N}_{FO})}$.

In the case of the near-ground atmosphere, the wind speed will rarely be greater than 100 km/h, or more than 28 m/s, so $\parallel \vec{u}_0 \parallel \ll c_0$. As a consequence, $\vec{t} \approx \vec{n}, c \approx c_0$ and $\theta(\vec{N}, \vec{N}_{FO}) \approx 0$. On the other hand, still close to the ground, the wind speed and sound speed gradients may be of the same order of magnitude: $\nabla \vec{u}_0 = O(\nabla c_0)$. Thus we may no longer identify $d\vec{t}/dl$ and $d\vec{n}/dl$ nor $R_{FO}$ and $R_{cau}$.

In practice, close to the ground, the wind therefore has a weak influence on the speed of sound (1.) and on the rotation of the coordinate system (2.), but a significant influence on the value of $R_{tot}$ (3.) and therefore on the thickness of the boundary layer.
2.1.7 Table summarizing the variables introduced in section 2.1

\[(\alpha, \beta, \psi) \] ray coordinates.
\[\psi\] eikonal function.
\[\vec{t}(\alpha, \beta, \psi)\] the vector tangent to the acoustic ray outside of the caustic.
\[\vec{n}(\alpha, \beta, \psi)\] the vector normal to the wavefront outside of the caustic.
\[l\] curvilinear abscissa along the ray considered.
\[\sigma\] curvilinear abscissa along the line tangent to the ray considered on the caustic.
\[q\] the 2nd curvilinear coordinate orthogonal to \[\sigma\] on the caustic.
\[\vec{x}_{cau}(\sigma, q)\] position of a point of the caustic.
\[\vec{t}_{c}(\sigma, q) = \frac{\partial \vec{x}_{cau}}{\partial \sigma}\] the vector tangent to the caustic in the direction of the incident ray.
\[\vec{n}_{c}\] the wavefront normal on the caustic.
\[\vec{q}_{c}(\sigma, q) = \frac{\partial \vec{x}_{cau}}{\partial q}\] the vector tangent to the caustic and normal to \[\vec{t}_{c}\].
\[\vec{N} = \pm \frac{\vec{t}_{c} \times \vec{q}_{c}}{|\vec{t}_{c} \times \vec{q}_{c}|}\] the normal to the caustic directed toward the insonified side.
\[c = c_{0} + \vec{u}_{0} \cdot \vec{n}\] apparent speed of sound.

Variables associated with tangent vector \[\vec{t}_{c}\]

\(O\) is any point of the caustic and \((Oxyz)\) is an associated local orthonormal reference frame, such that:

- the \(Ox\) axis is oriented along the ray tangent to the caustic at \(O\), that is to say along \[\vec{t}_{c}(O)\],
- the \(Oy\) axis is directed normal to \(Ox\) in the plane tangent to the caustic
- the \(Oz\) axis follows the normal to the caustic, \(\vec{N}\), directed toward the insonified side.

\(M\) is a current point in the vicinity of the caustic on the insonified side,
\(C\) the point of contact with the caustic of one of the two rays which pass through \(M\),
\(P\) the projection of \(M\) on the caustic, the distance \(MP\) being called \(h\).

\[\delta l\] the difference between the curvilinear abscissas of points \(M\) and \(C\) along the acoustic ray.

\[R_{ray}\] the radius of curvature of the acoustic ray defined by:
\[
\frac{d\vec{t}}{dl} = \frac{\vec{v}_{ray}(C)}{R_{ray}(C)}.
\]

\[\vec{v}_{ray}\] the principal normal to the acoustic ray.

\[R_{sec}\] the radius of curvature of the plane curve, called the normal section, formed by the intersection of the caustic with the \((\vec{t}_{c}, \vec{N})\) plane, such that:
\[
h = \frac{\delta l^{2}}{2} \left[ \frac{\cos \theta(\vec{v}_{ray}^{\perp}, \vec{N})}{R_{ray}(C)} + \frac{1}{R_{sec}(C)} \right] + O(\delta l^{3}).
\]

\[\theta(\vec{v}_{ray}^{\perp}, \vec{N})\] the angle formed by the vectors \(\vec{v}_{ray}^{\perp}\) and \(\vec{N}\) at the contact point \(C\).

\[R_{cau}\] the radius of curvature of the ray \((R_{ray})\) relative to the caustic \((R_{sec})\) such that
\[
\frac{1}{R_{cau}(C)} = \frac{1}{R_{sec}(C)} + \frac{\cos \theta(\vec{v}_{ray}^{\perp}, \vec{N})}{R_{ray}(C)}.
\]
Variables associated with the tangent vector $\vec{n}^c$

The point $O$ coincides with the point $P$ and $OXYZ$ is the associated orthonormal reference frame such that:

the $OZ$ axis is directed along $\vec{N}^{FO}$,

the $OX$ axis is oriented along $\vec{n}^c$,

and the $OY$ axis is unchanged from the $Oy$ axis.

$\vec{N}^{FO}$ the unit vector of the $\vec{n}^c, \vec{N}$ plane normal to the wavefront normal.

$\theta(\vec{N}, \vec{N}^{FO})$ the angle between the vectors $\vec{N}$ and $\vec{N}^{FO}$.

$H$ the distance to the caustic measured in the direction $\vec{N}^{FO}$, such that

$H = \frac{h}{\cos\theta(\vec{N}, \vec{N}^{FO})}$.

$R^{FO}$ the radius of curvature of the curve drawn by $\vec{n}$, such that: $\frac{d\vec{n}}{dl} = \vec{v}^{FO} R^{FO}$.

$\vec{v}^{FO}$ the principal normal to the curve described by the wavefront normal $\vec{N}$.

The radii of curvature $R^{FO}, R^{sec},$ and $R^{cau}$ are respectively defined as the radii $R_{ray}, R_{sec}, R_{cau}$, introduced previously following the tangent vector $\vec{t}$. However, $R_{tot}$ generalizes the radius $R_{cau}$ from the previous section:

$$\frac{1}{R_{cau}^{FO}} = \frac{\cos\theta(\vec{v}^{FO}, \vec{N}^{FO})}{R^{FO}} + \frac{1}{R_{sec}^{FO}},$$

$$R_{tot} = \frac{(R_{cau}^{FO})^2}{R_{cau}} \frac{1}{\cos\theta(\vec{N}, \vec{N}^{FO})}.$$

$R_{cel}$ the characteristic radius of the heterogeneity of the medium relative to the speed.

$R_{XZ}$ the radius of curvature defined as $\frac{1}{R_{XZ}} = -\frac{1}{R_{tot}} - \frac{1}{R_{cel}}$.

### 2.2 THE LINEAR ACOUSTIC FIELD NEAR A CAUSTIC

Neglecting diffraction effects, the fundamental assumption of geometrical acoustics, is no longer valid near a caustic, because it is a region of convergence of rays. The rays are too close to each other not to influence each other. In the neighborhood of a caustic, geometrical acoustics leads to pressures tending towards infinity; it is the diffraction which will be the main mechanism for limiting the sound field in this region.

After having studied, in the previous section, the behavior of geometrical acoustics near a caustic, in the presence of wind, it is now a question of estimating the acoustic field. The general idea, following the principle of Buchal and Keller (1960), is to introduce a boundary layer behavior. For this, we will use the principle of matched asymptotic expansions.

The objective of this section is to establish, in the presence of wind, the equation satisfied by the pressure inside the boundary layer, the latter being the region of thickness $\delta$ located in the vicinity
of the caustic, in which diffraction effects can no longer be neglected. In the sense of matched asymptotic expansions this region is the inner solution. Sufficiently far from the caustic, we have to find the outer solution which is here geometrical acoustics. The matching condition being the outer limit of the inner solution which is equated with the inner limit of the outer expansion (expression of geometrical acoustics in the vicinity of the caustic).

Concretely, in this section, we will, from previously established gauges, redimension the acoustic equations satisfied by the linear sound field in a heterogeneous, moving fluid, in order to account for the diffraction neglected by geometrical acoustics in the vicinity of the caustic. This will allow us to obtain in this region, on the one hand, the hierarchy of acoustic velocities in the different spatial directions and, on the other hand, the equation satisfied by the pressure.

### 2.2.1 Assumptions for the acoustic field near the caustic

Let us recall the geometrical acoustic behavior near a caustic with the assumption $OM \ll R_{tot}$:

\[
\frac{p_a(\vec{X},t)}{\rho_0(0)c^2} = M_{ac}\left(\frac{D}{Z}\right) \frac{1}{2} \left\{ F\left\{ t - \left[ \psi(O) - \frac{1}{c(O)} \sqrt{\frac{8Z^3}{gR_{tot}(O)}} + \frac{X(1+Z/R_{XZ}(O))}{c(O)} + \mathcal{O}\left(\frac{X^2,Y^2,XY,YZ}{c(O)R_{tot}}\right) \right] \right\} \
G\left\{ t - \left[ \psi(O) + \frac{1}{c(O)} \sqrt{\frac{8Z^3}{gR_{tot}(O)}} + \frac{X(1+Z/R_{XZ}(O))}{c(O)} + \mathcal{O}\left(\frac{X^2,Y^2,XY,YZ}{c(O)R_{tot}}\right) \right] \right\}
\]

This expression will allow us to determine the change of variables suitable for the study of the inner solution. We will now work with dimensionless variables. These new dimensionless variables will be identified by their name “covered” with a bar. For this, we introduce $\omega_{ac}^{-1}$, the characteristic duration of the incident signal, $\omega_{ac}$ being the center frequency of the spectrum of the incident signal.

---

**Figure 2.2.1**: Boundary layer in the vicinity of the caustic
Previously, we saw that the influence of diffraction was important in a region of characteristic thickness $\delta$ around the caustic in the direction $OZ$. Therefore we are going to choose the dimensioning of the variable $Z$ as follows:

$$\bar{z} = \frac{Z}{\delta} = \left[ \frac{2\omega_{ac}^2}{c^2(O)R_{tot}(O)} \right]^{\frac{1}{3}} Z.$$  

Still from the geometrical acoustics expression, we see that it is also suitable to introduce the modified retarded time variable:

$$\tau = \omega_{ac}\left[ t - \psi(O) - \frac{X(1 + Z/R_{XZ}(O))}{c(O)} \right].$$

From the matching with geometrical acoustics, the inner solution must depend at least on these two variables. Catastrophe theory (Berry 1981) specifies that in the presence of a simple caustic, the inner solution should only depend on one control parameter, here $\bar{z}$, and the time variable $\tau$. In accordance with this theory, we will therefore assume that in the vicinity of the caustic, the field depends only on the two variables $\bar{z}$ and $\tau$:

$$p_a(X,Y,Z,t) = P_a\bar{p}_a(\bar{z},\tau)$$
$$\vec{u}_a(X,Y,Z,t) = \begin{bmatrix} U_{aX} \bar{u}_{aX}(\bar{z},\tau) \\ U_{aY} \bar{u}_{aY}(\bar{z},\tau) \\ U_{aZ} \bar{u}_{aZ}(\bar{z},\tau) \end{bmatrix},$$

where $P_a, U_{aX}, U_{aY},$ and $U_{aZ}$ are the gauges, respectively for the pressure and the projections of the velocity vector onto the 3 axes $X,Y,$ and $Z$.

We choose $P_a$ in the following way:

$$P_a = \rho_0 c^2 M_{ac},$$

where $M_{ac}$ is calculated for the distance $D = \delta(\omega_{ac})$.

The matching with geometrical acoustics at infinity also provides us with a relationship between $P_a$ and $U_{aX}$:

$$U_{aX} = \frac{P_a}{\rho_0 c_0},$$

to satisfy the impedance relationship between pressure and velocity in the tangential direction.

### 2.2.2 The linear Tricomi equation

Let us recall the equations satisfied by the linear acoustic field in a heterogeneous fluid, in the presence of wind:

$$\frac{d_0 p_a}{dt} + \rho_0 c_0^2 \nabla \cdot \vec{u}_a = -\vec{u}_a \cdot \nabla p_0 - c_0^2 p_a \nabla \cdot \left( \frac{\vec{u}_0}{c_0^2} \right),$$

$$\frac{d_0 \vec{u}_a}{dt} + \frac{1}{\rho_0} \nabla p_a = -\left(\vec{u}_a \cdot \nabla\right)\vec{u}_0 + \frac{p_a}{\rho_0^2 c_0^2} \nabla p_0,$$

with $\frac{d_0}{dt} = \frac{\partial}{\partial t} + \vec{u}_0 \cdot \nabla$.  

401
Let us introduce the dimensionless parameter $\varepsilon$ such that $\varepsilon = \lambda_{ac}/\delta$, with $\lambda_{ac} = c(O)/\omega_{ac}$, a measure of the characteristic wavelength of the incident signal. By definition, it follows:

$$\varepsilon = \left[ \frac{2c(O)}{\omega_{ac}R_{tot}} \right]^{\frac{1}{3}} = \left[ \frac{2\lambda_{ac}}{R_{tot}} \right]^{\frac{1}{3}}.$$

The characteristic wavelength of an acoustic signal in the case of a sonic boom is about $\lambda_{ac} = 100m$, while the approximate size of a characteristic radius of curvature is one hundred kilometers. Under these conditions, the dimensionless parameter $\varepsilon$ is of the order of $10^{-1}$.

We will now substitute the expression of the field near the caustic in the previous equations. The change of variables chosen implies the following expressions for the partial derivatives:

$$\frac{\partial}{\partial t} = \omega_{ac} \frac{\partial}{\partial \tau},$$
$$\frac{\partial}{\partial X} = -\omega_{ac} \frac{c(O)}{2\lambda_{ac}} \left[ 1 + \varepsilon^2 \frac{R_{tot}(O)}{2R_{XZ}(O)z} \right] \frac{\partial}{\partial \tau} + O(\varepsilon^3),$$
$$\frac{\partial}{\partial Y} = O(\varepsilon^3),$$
$$\frac{\partial}{\partial Z} = \varepsilon \omega_{ac} \frac{c(O)}{2\lambda_{ac}} \frac{\partial}{\partial Z} + O(\varepsilon^3).$$

We must also use Taylor expansions of the speed of sound and density around point $O$, it then follows:

$$c(X, Y, Z) \approx c(O) + \nabla c(O) \cdot \vec{X}.$$

We define $R_{cel}$, the characteristic distance associated with the speed of sound, in the following way:

$$R_{cel}(O) = \left[ \frac{1}{c(O)} \frac{dc}{dZ}(O) \right]^{-1},$$
so that:

$$c(X, Y, Z) = c(O) \left[ 1 + \varepsilon^2 \frac{R_{tot}(O)}{2R_{cel}(O)z} \right] + O(\varepsilon^3).$$

In the same manner:

$$\rho_0(X, Y, Z) = \rho_0(O) \left[ 1 + \varepsilon^2 \frac{R_{tot}(O)}{2R_{den}(O)z} \right] + O(\varepsilon^3),$$
where $R_{den}$ is a distance characteristic of the heterogeneous in density of the medium, with:

$$R_{den}(O) = \left[ \frac{1}{\rho_0(O)} \frac{d\rho_0}{dZ}(O) \right]^{-1},$$

Regarding the wind, it is possible to decompose, in a similar way, the velocity field along the $X$ and $Z$ axes, as follows:

$$u_{0X}(X, Y, Z) = u_{0X}(O) \left[ 1 + \varepsilon^2 \frac{R_{tot}(O)}{2R_{0X}(O)z} \right] + O(\varepsilon^3),$$
$$u_{0Z}(X, Y, Z) = u_{0Z}(O) \left[ 1 + \varepsilon^2 \frac{R_{tot}(O)}{2R_{0Z}(O)z} \right] + O(\varepsilon^3).$$
where $R_{0X}$ and $R_{0Z}$ are the characteristic distances of the heterogeneities in wind of the medium along the directions $X$ and $Z$.

$$R_{0X}(O) = \left[ \frac{1}{u_{0X}(O)} \frac{du_{0X}}{dZ}(O) \right]^{-1},$$

and $R_{0Z}(O) = \left[ \frac{1}{u_{0Z}(O)} \frac{du_{0Z}}{dZ}(O) \right]^{-1},$

We introduce the following parameters, all assumed to be of order 1:

$$a = \frac{R_{\text{tot}}(O)}{2R_{XZ}(O)}, \quad e = \frac{R_{\text{tot}}(O)}{2R_{\text{cel}}(O)}, \quad f = \frac{R_{\text{tot}}(O)}{2R_{0X}(O)},$$

$$b = \frac{R_{\text{tot}}(O)}{2R_{\text{den}}(O)}, \quad g = \frac{R_{\text{tot}}(O)}{2R_{0Z}(O)}.$$ 

Furthermore, in the previous chapter, we saw that $\frac{1}{R_{XZ}} = -\frac{1}{R_{\text{tot}}(O)} - \frac{1}{R_{\text{cel}}(O)}$. The coefficients $a$ and $e$ are therefore linked by the following relation:

$$a + e = -1/2.$$

The Mach numbers of the flow compared to $c$ instead of $c_0$, in the $X$ and $Z$ directions, are defined by:

$$M_X = \frac{u_{0X}(O)}{c(O)} \quad \text{and} \quad M_Z = \frac{u_{0Z}(O)}{c(O)}.$$ 

We assumed above that at ground level the wind speed remained subsonic. In recognizing that it remains between 0 and 100 km/h, the associated Mach numbers $M_X$ and $M_Z$ are at most 0.08, and will therefore be assumed to be of order $O(\varepsilon)$ at most.

Projecting the linearized momentum equation on the $OX$, $OY$ and $OZ$ axes:

**On $OY$:** $U_{aY} \left[ (1 - M_X) \frac{\partial}{\partial \bar{\tau}} + \varepsilon M_Z \frac{\partial}{\partial \bar{z}} \right] \bar{u}_{aY} = O(\varepsilon^3),$

**On $OX$:** $U_{aX} \left[ (1 - M_X) \frac{\partial}{\partial \bar{\tau}} + \varepsilon M_Z \frac{\partial}{\partial \bar{z}} \right] \bar{u}_{aX} - \frac{P_a}{\rho_0 c(O)} \left[ 1 + \varepsilon^2(a - b)\bar{\varepsilon} \right] \frac{\partial \bar{\rho}_a}{\partial \bar{\tau}} = O(\varepsilon^3),$

**On $OZ$:** $U_{aZ} \left[ (1 - M_X) \frac{\partial}{\partial \bar{\tau}} + \varepsilon M_Z \frac{\partial}{\partial \bar{z}} \right] \bar{u}_{aZ} + \varepsilon \frac{P_a}{\rho_0 c(O)} \frac{\partial \bar{\rho}_a}{\partial \bar{\tau}} = O(\varepsilon^3).$

The conservation of mass equation is written:

$$P_a \left[ (1 - M_X) \frac{\partial}{\partial \bar{\tau}} + \varepsilon M_Z \frac{\partial}{\partial \bar{z}} \right] \bar{p}_a + \rho_0 c_0^2 \left[ 1 + \varepsilon^2(b + 2c)\bar{\varepsilon} \right] \left[ - \frac{U_{aX}}{c(O)} (1 + \varepsilon^2 a \bar{z}) \frac{\partial \bar{u}_{aX}}{\partial \bar{\tau}} + \varepsilon \frac{U_{aZ}}{c(O)} \frac{\partial \bar{u}_{aZ}}{\partial \bar{\tau}} \right] = O(\varepsilon^3).$$

To the order $\varepsilon$, the projection of the momentum onto the $X$ axis gives us the following relation:

$$U_{aX} (1 - M_X) = \frac{P_a}{\rho_0(O)c(O)} \left[ 1 + O(\varepsilon^2) \right].$$
Now \((1 - M_X)c = c_0\), it then follows:

\[
U_{aX} = \frac{P_a}{\rho_0 c_0} \quad \text{and} \quad \nabla_{aX} = \nabla_a.
\]

We recover the impedance relationship between the acoustic velocity in the direction tangential to the ray and the pressure from geometrical acoustics.

Similarly, on the \(Z\) axis, we have the expression:

\[
U_{aZ}(1 - M_X)\frac{\partial \nabla_{aZ}}{\partial \tau} = -\varepsilon \left(U_{aZ}M_Z \frac{\partial \nabla_{aZ}}{\partial z} + \frac{P_a}{\rho_0 c(O)} \frac{\partial \nabla_a}{\partial z} \right) + \mathcal{O}(\varepsilon^2).
\]

According to the assumptions on the order of magnitude of the Mach numbers, we may neglect here the first term of the right hand side, which leads to the following choice of gauge:

\[
U_{aZ} = \varepsilon \frac{P_a}{\rho_0 c_0}
\]

and the equation \(\frac{\partial \nabla_{aZ}}{\partial \tau} = -\frac{\partial \nabla_a}{\partial z}\).

Finally, on \(OY\), we have:

\[
U_{aY} = \mathcal{O}(\varepsilon^3).
\]

These relationships indicate how the hierarchy of acoustic velocities is established in the different spatial directions, near the caustic:

\[
U_{aX} = \mathcal{O}(1) \gg U_{aZ} = \mathcal{O}(\varepsilon) \gg U_{aY} = \mathcal{O}(\varepsilon^3).
\]

We recover the results of the windless case remembering however that we are in a reference frame where the vector \(X\) is directed according to the wavefront normal and not along the tangent to the ray. At first order, we recover geometrical acoustics; diffraction occurs at the order \(\varepsilon\) in the new normal direction (along \(\vec{N}_{FO}\)). The velocity in the \(OY\) direction is negligible.

In the windless case, at order \(\varepsilon^2\), the pressure satisfies the Tricomi equation; We will show that taking the wind into account does not perturb this result.

We divide the above equations by \((1 - M_X)\). Taking the relationships between the scales into account, the following system of equations follows:

Euler’s equation on the axes,

\[
OX : \left( \frac{\partial}{\partial \tau} + \varepsilon M_Z \frac{\partial}{\partial z} \right) \nabla_{aX} - \left[1 + \varepsilon^2 \pi(a - b)\right] \frac{\partial \nabla_a}{\partial \tau} = \mathcal{O}(\varepsilon^3),
\]

\[
OZ : \left( \frac{\partial}{\partial \tau} + \varepsilon M_Z \frac{\partial}{\partial z} \right) \nabla_{aZ} + \frac{\partial \nabla_a}{\partial z} = \mathcal{O}(\varepsilon^3),
\]

and the continuity equation:

\[
\left( \frac{\partial}{\partial \tau} + \varepsilon M_Z \frac{\partial}{\partial z} \right) \nabla_a - \left[1 + \varepsilon^2 \pi(b + a + 2e)\right] \frac{\partial \nabla_{aX}}{\partial \tau} + \varepsilon^2 \frac{\partial \nabla_{aZ}}{\partial z} = \mathcal{O}(\varepsilon^3).
\]

We introduce the particle derivative associated with the motion, such that:

\[
\frac{d}{d \tau} = \frac{\partial}{\partial \tau} + \varepsilon M_Z \frac{\partial}{\partial z}.
\]

It then follows:
Euler on \( OX \): \[ \frac{du_aX}{d\tau} - \left[ 1 + \epsilon^2(a - b)z \right] \frac{\partial p_a}{\partial \tau} = O(\epsilon^3), \]

Euler on \( OZ \): \[ \frac{du_aZ}{d\tau} + \frac{\partial p_a}{\partial z} = O(\epsilon^3), \]

Continuity: \[ \frac{dp_a}{d\tau} - \left[ 1 + \epsilon^2(b + a + 2e)z \right] \frac{\partial u_aX}{\partial \tau} + \epsilon^2 \frac{\partial u_aZ}{\partial \tau} = O(\epsilon^3). \]

Differentiating the last expression by \( \tau \): \[ \frac{d^2p_a}{d\tau^2} - \left[ 1 + \epsilon^2(b + a + 2e)z \right] \frac{d}{d\tau} \left( \frac{\partial u_aX}{\partial \tau} \right) + \epsilon^2 \frac{d}{d\tau} \left( \frac{\partial u_aZ}{\partial \tau} \right) = O(\epsilon^3). \]

We have the following relationships between the operators:
\[ \frac{d}{d\tau} \frac{\partial}{\partial \tau} = \frac{\partial}{\partial \tau} \frac{d}{d\tau}, \]
\[ \frac{d}{d\tau} \frac{\partial}{\partial z} = \frac{\partial}{\partial z} \frac{d}{d\tau} + O(\epsilon^3). \]

To the order of precision retained, the operators \( \frac{\partial}{\partial \tau} \) and \( \frac{\partial}{\partial z} \) commute with \( \frac{d}{d\tau} \); the previous equation is then written:
\[ \frac{d^2p_a}{d\tau^2} - \left[ 1 + \epsilon^2(b + a + 2e)z \right] \frac{d}{d\tau} \left( \frac{\partial u_aX}{\partial \tau} \right) + \epsilon^2 \frac{d}{d\tau} \left( \frac{\partial u_aZ}{\partial \tau} \right) = O(\epsilon^3). \]

Using that:
\[ \frac{d u_aX}{d\tau} = \left[ 1 + \epsilon^2(a - b)z \right] \frac{\partial u_a}{\partial \tau}, \]
\[ \frac{d u_aZ}{d\tau} = - \frac{\partial u_a}{\partial z}, \]
as well as the relation \( a + e = -\frac{1}{2} \), the following dimensionless scalar equation satisfied by the acoustic pressure is reached:
\[ \frac{d^2p_a}{d\tau^2} - (1 - \epsilon^2z) \frac{\partial^2 p_a}{\partial \tau^2} - \epsilon^2 \frac{\partial^2 p_a}{\partial z^2} = 0. \]

In the case without wind, \( \frac{d}{d\tau} = \frac{\partial}{\partial \tau} \); the terms of order 1 vanish; there are only terms of order \( \epsilon^2 \) left, which leads to the following equation:
\[ \epsilon^2 \frac{\partial^2 p_a}{\partial \tau^2} + \frac{\partial^2 p_a}{\partial z^2} = 0. \]

We recover the usual Tricomi equation in the case without wind.

In the case with wind, we must expand \( \frac{d^2p_a}{d\tau^2} \); it then follows:
\[ \frac{d^2}{d\tau^2} = \frac{\partial^2}{\partial \tau^2} + 2\epsilon M_z \frac{\partial^2}{\partial z \partial \tau} + O(\epsilon^3). \]

The pressure equation in the case with wind, \( \frac{d^2p_a}{d\tau^2} - (1 - \epsilon^2z) \frac{\partial^2 p_a}{\partial \tau^2} - \epsilon^2 \frac{\partial^2 p_a}{\partial z^2} = 0 \), is then written:
\[ 2\epsilon \frac{\partial^2 p_a}{\partial z \partial \tau} + z \frac{\partial^2 p_a}{\partial \tau^2} = O(\epsilon), \]

with \( \zeta = \frac{M_z}{\epsilon} \), which measures the influence of the effect of wind.
We see with respect to the Tricomi equation obtained without wind \( \frac{\partial^2 p_a}{\partial \tau^2} - \frac{\partial^2 p_a}{\partial z^2} = 0 \), that there is an additional cross term: \( \frac{\partial^2 p_a}{\partial z \partial \tau} \). We will now show that it is possible, by a suitable change of variables, to come back to the Tricomi equation. For this we set:

\[ T = \tau + \xi z \quad \text{and} \quad Z = z + \zeta, \]

so that:

\[ \frac{\partial}{\partial \tau} = \frac{\partial}{\partial T} \quad \text{and} \quad \frac{\partial}{\partial z} = \frac{\partial}{\partial Z} + \xi \frac{\partial}{\partial T}. \]

The second derivatives are then written:

\[ \begin{align*}
\frac{\partial^2}{\partial \tau^2} &= \frac{\partial^2}{\partial T^2}, \\
\frac{\partial^2}{\partial z^2} &= \frac{\partial^2}{\partial Z^2} + 2\xi \frac{\partial}{\partial T} \frac{\partial}{\partial Z} + \xi^2 \frac{\partial^2}{\partial T^2}, \\
\frac{\partial^2}{\partial z \partial \tau} &= \frac{\partial^2}{\partial T \partial Z} + \xi \frac{\partial^2}{\partial T^2}.
\end{align*} \]

By denoting the pressure associated with this change of variables, \( P(Z, T) \), the previous pressure equation is then written:

\[
-Z \frac{\partial^2 P}{\partial Z^2} + \left( \frac{\partial}{\partial T} \right)^2 P + 2(\zeta - \xi) \frac{\partial^2 P}{\partial T \partial Z} + 2(\zeta - \xi - \zeta^2) \frac{\partial^2 P}{\partial T^2} = 0.
\]

To eliminate the cross terms and the second derivatives of \( T \) not multiples of \( Z \), the following relations have to be satisfied:

\[ \zeta - \xi = 0 \quad \text{and} \quad -\zeta + 2\zeta \xi - \zeta^2 = 0, \]

that is:

\[ \xi = \zeta \quad \text{and} \quad \zeta = \zeta^2. \]

\( M_Z \) being of the same order of magnitude as \( \varepsilon \), we have \( \xi = O(1) \) and \( \zeta = O(1) \), at most.

This completes the demonstration that, even in the case with wind, and by means of the change of variables \( (\tau, z) \rightarrow (T, Z) \), the pressure field inside the boundary layer always satisfies the Tricomi equation:

\[
Z \frac{\partial^2 P}{\partial T^2} - \frac{\partial^2 P}{\partial Z^2} = 0.
\]

In the case with wind, the Tricomi equation is no longer defined from an origin placed on the geometric caustic but following an origin situated in the direction \( OZ \).
2.2.3 Boundary conditions

In order to solve the Tricomi equation, knowledge of the boundary conditions, satisfied by the acoustic field, is required. The equation being second-order in $Z$ and $\bar{T}$, there are 4 conditions, two on $Z$ and two on $\bar{T}$. The boundary conditions on $\bar{T}$ are simply that the field at large times must be zero, that is:

$$\bar{P}(\bar{T} \to \pm \infty, Z) = 0.$$ 

Far into the geometric zone of silence, there is attenuation of the acoustic field, thus:

$$\bar{P}(\bar{T}, Z \to -\infty) = 0.$$ 

On the insonified side, the sound field must be matched far from the caustic, at infinity, with the geometrical acoustic solution, which is written:

$$\bar{P}(\bar{T}, Z \to +\infty) = (Z - \varsigma^2)^{-\frac{1}{4}} \left[ F\left(\bar{T} - \varsigma Z + \varsigma^3 + \frac{2}{3}(Z - \varsigma^2)^{\frac{3}{2}}\right) + G\left(\bar{T} - \varsigma Z + \varsigma^3 - \frac{2}{3}(Z - \varsigma^2)^{\frac{3}{2}}\right) \right].$$

When $Z$ tends towards $+\infty$, this amounts to writing this matching condition as follows:

$$\bar{P}(\bar{T}, Z \to +\infty) = Z^{-\frac{1}{4}} \left[ F\left(\bar{T} + \frac{2}{3}Z^\frac{3}{2}\right) + G\left(\bar{T} - \frac{2}{3}Z^\frac{3}{2}\right) \right].$$

The functions $\bar{F}$ and $\bar{G}$ are identical to the functions $F$ and $G$ introduced in section 2.1.5, up to a scale factor closely related to the dimensioning of $\varsigma$:

$$\begin{bmatrix} F \\ G \end{bmatrix} = \rho_0c_0^2M_{ac}D^{1/4}\delta^{-1/4} \begin{bmatrix} F \\ G \end{bmatrix}.$$ 

In the boundary conditions above, the function $\bar{F}$ describes the incident field far from the caustic, given by geometrical acoustics, while $\bar{G}$ describes the reflected field which has been affected by the passage of rays in the vicinity of the caustic. It is therefore $a priori$ indeterminate.

2.2.4 Solutions of the linear Tricomi equation

The linear Tricomi equation with the above boundary conditions can be solved by Fourier analysis. The boundary conditions in $\bar{T}$ ensure that the Fourier transform of the pressure exists. We take as our definition of the Fourier transform the following expression:

$$\hat{P}(\bar{Z}, k) = FT(\bar{P}) = \int_{-\infty}^{\infty} \bar{P}(\bar{T}, \bar{Z}) e^{-i\omega\bar{T}} d\bar{T},$$

with $\omega$ the conjugate variable of $\bar{T}$, which is the dimensionless angular frequency.

The inverse transform is written:

$$\bar{P}(\bar{T}, \bar{Z}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{P}(\bar{Z}, \omega) e^{i\omega\bar{T}} d\omega.$$
The Fourier transform of the linear Tricomi equation is then written:
\[
\frac{d^2 \hat{P}}{dZ^2} + \omega^2 \hat{P} = 0,
\]
By setting \( U = |\omega|^{2/3} Z \), this equation becomes:
\[
\hat{U} \hat{P} + \frac{d^2 \hat{P}}{dU^2} = 0.
\]
The solutions of this second-order linear differential equation form a 2-dimensional vector space, whose basis is given by \( Ai(-U) \) and \( Bi(-U) \), two solutions of Airy’s differential equation (Coulouvrat 1997, Appendix III):
\[
\hat{P}(Z) = a(\omega) Ai(-U) + b(\omega) Bi(-U) = a(\omega) Ai(-|\omega|^{2/3} Z) + b(\omega) Bi(-|\omega|^{2/3} Z).
\]
At positive infinity \( Ai(-U) \) tends toward 0, while \( Bi(-U) \) grows exponentially. In order to ensure the condition at \( Z \to -\infty \), we must take \( b(\omega) = 0 \).
It remains to satisfy the boundary condition for \( Z \to +\infty \). Its Fourier transform is written:
\[
\left( Z^{-1/4} \frac{d\hat{P}}{dZ} + i\omega Z^{1/4} \hat{P} \right) \sim 2i\omega \exp \left( \frac{2i\omega Z^{3/2}}{3} \right) FT(F).
\]
that is:
\[
a(\omega) = |\omega|^{5/6} \left( U^{-1/4} \frac{d\hat{P}}{dU} + isgn(\omega) U^{1/4} \hat{P} \right) \sim 2isgn(\omega) |\omega| \exp \left( \frac{2isgn(\omega) U^{3/2}}{3} \right) FT(F).
\]
Now, from the expression for the asymptotic expansion of the Airy function at infinity (Abramowitz and Stegun, 1965), it follows:
\[
\hat{P}(U) = a(\omega) Ai(-U) \sim a(\omega) e^{i\pi/4} U^{-1/4} \left[ \exp \left( -\frac{2}{3} i U^{3/2} \right) - i \exp \left( \frac{2}{3} i U^{3/2} \right) \right],
\]
\[
\frac{d\hat{P}(U)}{dU} \sim a(\omega) e^{i\pi/4} U^{-1/4} \left[ \exp \left( \frac{2}{3} i U^{3/2} \right) - i \exp \left( -\frac{2}{3} i U^{3/2} \right) \right],
\]
Carrying this over, it follows:
\[
\omega > 0 \quad a(\omega) = 2i\sqrt{\pi} e^{-i\pi/4} \omega^{1/6} FT(F)
\]
\[
\omega < 0 \quad a(\omega) = 2\sqrt{\pi} e^{-i\pi/4} |\omega|^{1/6} FT(F),
\]
that is: \( a(\omega) = \sqrt{2\pi}(1 + isgn(\omega)) |\omega|^{1/6} FT(F) \).
We thus obtain the expression of the field near the caustic as a function of the incident field (Seebass 1971):
\[
\overline{P}(\overline{T}, \overline{Z}) = FT^{-1} \left[ \sqrt{2\pi}(1 + isgn(\omega)) |\omega|^{1/6} Ai(-|\omega|^{2/3} Z) FT(F) \right].
\]
2.2.5 Field reflected by the caustic

Far from the caustic, when \( Z \) tends toward infinity, the integral above restores geometrical acoustics. When we replace the Airy function with its expansion, we have:

\[
P(T, Z) \approx \frac{Z^{-1/4}}{2\pi} \left\{ \int_{-\infty}^{+\infty} FT(F) e^{i\omega(T + \frac{2}{3}Z^{3/2})} d\omega + i \left( \int_{0}^{+\infty} FT(F) e^{i\omega(T - \frac{2}{3}Z^{3/2})} d\omega - \int_{-\infty}^{0} FT(F) e^{i\omega(T - \frac{2}{3}Z^{3/2})} d\omega \right) \right\},
\]

or again, using the properties of Fourier transforms:

\[
P(T, Z \to +\infty) = Z^{-1/4} \left[ FT(T + \frac{2}{3}Z^{3/2}) G(T - \frac{2}{3}Z^{3/2}) \right],
\]

with:

\[
G(t) = \frac{1}{\pi} \text{Re} \left[ i \int_{0}^{+\infty} d\omega e^{i\omega t} \int_{0}^{+\infty} \left( \int_{-\infty}^{0} F(t') e^{-i\omega t'} dt' \right) \right].
\]

We recover the rule according to which the acoustic field undergoes a phase shift of \( +\pi/2 \), for the positive frequencies, after having “tangented” the caustic. According to Pierce (1989), it is possible to make the function \( G \) explicit by calculating:

\[
G(t) = \frac{1}{\pi} \text{Re} \left[ i \int_{0}^{+\infty} d\omega e^{i\omega t} \int_{-\infty}^{+\infty} \left( \int_{0}^{+\infty} F(t') e^{-i\omega t'} dt' \right) \right].
\]

so that by introducing an imaginary part \( \theta \) such that \( t \to t + i\theta \), destined to go to 0, it follows:

\[
G(t) = \lim_{\theta \to 0+} \frac{1}{\pi} \text{Re} \left[ i \int_{-\infty}^{+\infty} d\omega e^{i\omega t(t'-t)} \right].
\]

that is:

\[
G(t) = \lim_{\theta \to 0+} \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{F(t')}{\theta^2 + (t' - t)^2} dt'.
\]

This limit is also expressed as the principal value (denoted \( PV \)) of the integral of \( F(t')/(t' - t) \), that is:

\[
G(t) = \frac{1}{\pi} PV \left( \frac{F(t')}{t' - t} dt' \right) = \frac{1}{\pi} \lim_{\varepsilon \to 0+} \left( \int_{t-\varepsilon}^{t} \frac{F(t')}{t' - t} dt' + \int_{t+\varepsilon}^{+\infty} \frac{F(t')}{t' - t} dt' \right),
\]

Which is just the Hilbert transform of the function \( F \).
2.2.6 Insufficiency of the linear Tricomi equation in the case of a sonic boom

If we consider the pressure field for an incident “N” wave of total duration $2T$, of the form:

$$F(T) = \begin{cases} -T/T & \text{if } |T| < T \\ 0 & \text{otherwise} \end{cases}$$

the Hilbert transform may be given explicitly without difficulty in the following form:

$$G(T) = -\frac{T}{\pi} \left[ 2 + \frac{T}{T} \ln \frac{T - T}{T + T} \right].$$

By plotting the function $G$, (Figure 2.2.2), we observe that the temporal signal reflected on the caustic has a characteristic “U” shape with two logarithmic singularities at $t = \pm T$, associated with each of the discontinuities of the incident signal.

Likewise, still for an incident “N” wave, we may also explicitly specify the pressure field on the caustic (cf. Seebass 1971):

$$P(T, 0) = \frac{2Ai(0)\Gamma(1/6)}{T\sqrt{2\pi}} \left\{ \frac{\text{sgn}(T)}{5} \left[ |T - |T|| - \frac{1}{2} + (T - |T||)^{\frac{1}{6}} \right] - \frac{T}{2} \cos \left( \frac{\pi}{12} \right) \left[ |T + |T|| - \frac{1}{2} + \text{sgn}(T - |T||) |T - |T||^{\frac{1}{6}} \right] \right\}.$$

This time the field presents a singularity in the amplitude at $|T - |T||^{-\frac{1}{6}}$ associated with each of the discontinuities of the incident “N” wave. This singularity is more marked on the caustic itself than for the reflected field for which there is a logarithmic singularity, as we can see in Figure 2.2.3.

Figures 2.2.4 and 2.2.5 show the shape of the pressure field obtained by Fourier transformation of the Airy function at $Z = 0.5$ (Figure 2.2.4) and $Z = 1$ (Figure 2.2.5) using 8192 points. The discrete Fourier transform retaining only a finite number of frequencies, the numerically computed field does not present any singularity, but simply a high maximum amplitude. In reality, these signals present logarithmic-type singularities (Seebass, 1971).

By comparing with existing experimental results (Wanner et al., 1972) with the exception of singularities, we note that, at least qualitatively, the forms of the measured and calculated signals agree.
Figure 2.2.2: Reflected field for an "N" wave

Figure 2.2.3: Pressure field at the caustic for an "N" wave

Figure 2.2.4: Pressure field at $\bar{Z} = 0.5$ for an "N" wave

Figure 2.2.5: Pressure field at $\bar{Z} = 1$ for an "N" wave
These results indicate that diffraction is the main “limiting” mechanism of the amplitude of the field near a caustic. However, we have seen that the characteristic distance $\delta(\omega_{ac})$ below which geometrical acoustics is no longer valid varies as $\omega_{ac}^{2/3}$. This signifies that, for high frequencies, geometrical acoustics will be valid much closer to the caustic and therefore will lead to an even greater amplification. For a signal with a discontinuity, and therefore an infinite frequency spectrum, the field will have a singularity associated with each of the discontinuities. This means that the diffraction, to the order retained, is not sufficient to remove, by itself, the whole of the singularity in the case of a discontinuous signal. It is necessary to introduce a secondary “limiting” mechanism. This mechanism may be:

- higher order diffraction (e.g. velocity in the other transverse direction),
- nonlinearities,
- molecular relaxation,
- thermoviscous absorption.

In order to estimate which of these mechanisms is dominant, it is necessary to estimate a priori their respective orders of magnitude. We have seen that the first neglected diffraction terms were of order $\varepsilon^3$, that is of order $10^{-3}$, according to the definition of the small parameter $\varepsilon$ introduced at the beginning of the chapter. In the sonic boom case, the signals considered are essentially shocks which mainly involve high frequencies; $\varepsilon^3$ will therefore always be much less than $10^{-3}$.

Furthermore, the acoustic Mach number, $M_{ac} = P_{ac}/\rho_0 c_0^2$, in the case of a classical sonic boom propagation, is equal to about $5 \cdot 10^{-3}$. Near a caustic, the shocks lead to signals with infinite peaks. By its definition, we therefore note that the acoustic Mach will always be greater than $5 \cdot 10^{-3}$.

The small parameters $m_N$ and $m_O$, measuring the order of magnitude of the relaxation phenomena of nitrogen and oxygen at their relaxation frequency (Coulouvrat, 1996), are, under the usual conditions, respectively worth $1.26 \cdot 10^{-4}$ and $6.71 \cdot 10^{-4}$, the relaxation frequencies of nitrogen and oxygen being on the order of 300 and 30000 Hertz respectively (for 45% relative humidity). Finally, the thermoviscous absorption in air is negligible, except at ultrasonic frequencies.

This quick study of orders of magnitude indicates that the main secondary “limiting” mechanism can only be nonlinearities.

2.2.7 Conclusion: general summary

The previous section was devoted to the study of the behavior of geometrical acoustics in the vicinity of the caustic. This led to the determination of the gauges necessary for the change of variables for the study of the pressure field near the caustic.

This information allowed us to obtain in this section, the acoustic field in the vicinity of a caustic by means of matched asymptotic expansion. For this we have enlarged the neighborhood of the caustic through the following change of variables:

$$z = \varepsilon \omega_{ac} Z/c(O)$$

and $$\tau = \omega_{ac}[t - \psi(O) - X(1 + Z/R_{XZ}(O))/c(O)].$$

These two quantities being necessary and sufficient according to catastrophe theory (Berry 1981), to analyze the pressure field near a simple caustic.
After inserting these expressions into the system of equations satisfied by the acoustic field in an inhomogeneous, moving fluid, we obtained, on the one hand, the order of magnitude of the acoustic velocity in the 3 spatial directions and, on the other hand, an equation for the pressure. A suitable change of variables \((T = \tau + \varsigma z)\) and \((Z = \bar{z} + \varsigma^2)\) allowed us to recover the linear Tricomi equation, an equation which, in the windless case is satisfied by the pressure near the caustic. In the presence of wind, the origin is located outside of the geometric caustic.

The entire mathematical process is presented in the diagram below (Figure 2.2.6).

However, we found that diffraction was not sufficient to remove the whole of the singularity in the case of a discontinuous signal. A secondary “limiting” mechanism must be introduced which will be the nonlinearities; this is the subject of the next section.
Analysis of the geometric acoustic behavior in the vicinity of the caustic

\[ \varepsilon = \frac{\lambda_{ac}}{\delta} \]

\( \lambda_{ac} \): wavelength
\( \delta \): thickness of the boundary layer

\[ \bar{z} = \frac{\varepsilon \lambda_{ac}}{c(O)} \]

\[ \bar{r} = \lambda_{ac} \left[ t - \psi(O) - \frac{X(1 + \tilde{Z}/R_{XZ}(O))}{c(O)} \right] \]

System of equations satisfied by the acoustic field in a perfect, heterogeneous fluid in motion

\[ U_{ax} = O(1) \gg U_{az} = O(\varepsilon) \gg U_{ar} = O(\varepsilon^3) \]

geometric acoustics
diffraction
negligible

\[ 2\varepsilon \frac{\partial^3 p_a}{\partial z \partial r^2} - \frac{\partial^2 p_a}{\partial z^2} + z \frac{\partial^2 p_a}{\partial r^2} = O(\varepsilon^2) \]

TRICOMI EQUATION

\[ \frac{\partial^2 p}{\partial T^2} - \frac{\partial^2 p}{\partial Z^2} = 0 \]

\[ \bar{Z} = \bar{z} + \varepsilon^2 \]
\[ \bar{r} = \bar{r} + \varepsilon \bar{z} \]

Figure 2.2.6: Summary of the study of the acoustic field in the vicinity of a caustic
2.2.8 Table summarizing the variables introduced in section 2.2

Outer reference frame

The orthonormal reference frame $OXYZ$ introduced previously in section 2.1.

Inner reference frame

It depends on only 2 dimensionless variables:

- the normalized distance to the caustic:

$$\bar{z} = \frac{Z}{\delta} = \left[ \frac{2\omega_{ac}^2}{c^2(O)R_{tot}(O)} \right]^{\frac{1}{3}} Z.$$

- the phase variable:

$$\tau = \omega_{ac} \left[ t - \psi(O) - \frac{X(1 + Z/R_{XZ}(O))}{c(O)} \right].$$

$\omega_{ac}$ characteristic angular frequency.

$\delta$ boundary layer thickness in which we may no longer neglect diffraction.

$P_a$ dimensional scale for the pressure.

$U_{aX}$ dimensional scale for the velocity on the $X$ axis.

$U_{aY}$ dimensional scale for the velocity on the $Y$ axis.

$U_{aZ}$ dimensional scale for the velocity on the $Z$ axis.

$\bar{U}_{aX}$ dimensionless acoustic velocity in the $X$ direction.

$\bar{U}_{aY}$ dimensionless acoustic velocity in the $Y$ direction.

$\bar{U}_{aZ}$ dimensionless acoustic velocity in the $Z$ direction.

$\bar{p}_a$ dimensionless acoustic pressure.

$\varepsilon = \left[ \frac{2c(O)}{\omega_{ac} R_{tot}} \right]^{\frac{1}{3}}$ small parameter.

$\lambda_{ac} = c(O)/\omega_{ac}$ a measure of the characteristic wavelength of the incident signal.

$R_{cel}$ the characteristic distance associated with the speed of sound.

$R_{den}$ the characteristic distance of the heterogeneity in density of the medium.

$R_{0X}$ the characteristic distance of the heterogeneity of the medium, in wind along $X$.

$R_{0Z}$ the characteristic distance of the heterogeneity of the medium, in wind along $Z$.

$M_X$ the Mach number of the flow compared to $c$ instead of $c_0$ along $X$.

$M_Z$ the Mach number of the flow compared to $c$ instead of $c_0$ along $Z$.

The dimensionless relations associated with the distances above are written:

$$a = \frac{R_{tot}(O)}{2R_{XZ}(O)}, \quad e = \frac{R_{tot}(O)}{2R_{cel}(O)}, \quad f = \frac{R_{tot}(O)}{2R_{0X}(O)},$$

$$b = \frac{R_{tot}(O)}{2R_{den}(O)}, \quad g = \frac{R_{tot}(O)}{2R_{0Z}(O)}.$$
\[
\frac{d\eta}{d\tau} = \frac{\partial}{\partial \tau} + \epsilon M Z \frac{\partial}{\partial z}
\]

the particle derivative associated with motion.

\[
T = \tau + \varsigma z
\]

phase variable resulting from the change of variables used to find the linear Tricomi equation.

\[
Z = \varsigma + \zeta^2
\]

distance to the caustic resulting from the change of variables used to find the linear Tricomi equation.

\[
\varsigma = \frac{M \zeta}{\epsilon}
\]

coefficient measuring the influence of wind.

### 2.3 EFFECTS OF NONLINEARITY NEAR A CAUSTIC

It was seen in the previous section (2.2), how to account for diffraction effects near a caustic in order to lift the singularity imposed by geometrical acoustics. We show that in this region the pressure satisfies the Tricomi equation, if the wind speed remains quite small. However, as we have also seen, if the incident signal has shocks, the reflected signal has infinite peaks. It is therefore necessary to introduce an additional “limiter” mechanism: “nonlinearities.” By doing so, we obtain the nonlinear Tricomi equation established by Guiraud in 1965. We will show that we recover this result in the case with wind.

#### 2.3.1 The nonlinear Tricomi equation

Here we resume the calculation carried out to establish the Tricomi equation accounting for the nonlinearities to 1st order. To do this, let us recall the equations of motion in a perfect, compressible and heterogeneous fluid, expressed in the first chapter:

- **Continuity equation**
  \[
  \frac{d\rho}{dt} + \rho \nabla \cdot \vec{u} = 0,
  \]

- **Momentum balance**
  \[
  \rho \frac{d\vec{u}}{dt} = -\nabla p + \rho \vec{g}
  \]

- **Energy balance**
  \[
  \frac{ds}{dt} = 0.
  \]

Gibbs’ relationship defines the temperature and pressure of the fluid from its internal energy:

\[
\text{de} = T ds + \frac{p}{\rho^2} d\rho \quad \text{where} \quad T = (\frac{\partial e}{\partial s})_\rho (s, \rho) \quad \text{and} \quad p = \rho^2 \left(\frac{\partial e}{\partial \rho}\right)_s (s, \rho)
\]

We suppose, as before, that the fluid is moving at a constant velocity \(\vec{u}_0(x)\).

We introduce the acoustic perturbations in the expressions above. After expansion, by retaining only the linear and quadratic terms, it follows:

<table>
<thead>
<tr>
<th>Linear terms</th>
<th>Nonlinear terms</th>
<th>Terms associated with heterogeneity</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\frac{d\rho_a}{dt}) + (\rho_0 \nabla \cdot \vec{u}_a)</td>
<td>(+ \nabla \cdot \rho_a \vec{u}_a)</td>
<td>(- \rho_a \nabla \cdot \vec{u}_0 - \vec{u}_a \cdot \nabla \rho_0)</td>
</tr>
<tr>
<td>(\rho_0 \frac{d\vec{u}_a}{dt}) + (\nabla p_a)</td>
<td>(+ \rho_0 (\vec{u}_a \cdot \nabla) \vec{u}_a + \rho_0 \frac{d\rho_a}{dt})</td>
<td>(- \rho_0 (\vec{u}_a \cdot \nabla) \vec{u}_0 - \rho_a \frac{d\rho_0}{dt}) - (\rho_a (\vec{u}_a \cdot \nabla) \vec{u}_0 + \rho_0 \vec{g})</td>
</tr>
<tr>
<td>(\frac{ds_a}{dt})</td>
<td>(- \vec{u}_a \cdot \nabla s_0 - \vec{u}_a \cdot \nabla s_a)</td>
<td>(- \vec{u}_a \cdot \nabla s_0 - \vec{u}_a \cdot \nabla s_a)</td>
</tr>
</tbody>
</table>
Above, we have grouped on the left, the linear and nonlinear terms associated with acoustic propagation and to the right those associated with the heterogeneity of the medium. The latter are of order \( \varepsilon^3 \), while the former are of order 1 for the linear terms and of order \( M_{ac} \) for the nonlinear terms.

In the previous section, we saw that the dimensionless parameter \( \varepsilon \) was estimated at \( 10^{-1} \). In the case of the sonic boom, the signals considered are essentially shocks which mainly involve high frequencies; \( \varepsilon \) evolving as the cube root of the wavelength, \( \varepsilon \) will always be less than \( 10^{-1} \). In addition, the acoustic Mach number, \( M_{ac} = P_{ac}/\rho_0 c_0^2 \), in the case of classical propagation of the sonic boom, is worth approximately \( 5 \cdot 10^{-3} \). In the vicinity of a caustic, the shocks lead to signals with infinite peaks. By its definition, we therefore note that the acoustic Mach number will always be greater than \( 5 \cdot 10^{-3} \).

Thus, we may say that in the vicinity of the caustic the order \( M_{ac} \) terms are predominant before those of order \( \varepsilon^3 \), as a result of which it is legitimate to neglect the right hand side. It then follows:

\[
\begin{align*}
\frac{d_0 \rho_a}{dt} + \rho_0 \nabla \cdot \vec{u}_a + \nabla \cdot \rho_a \vec{u}_a &= O(\varepsilon^3) + O(M_{ac}^2), \\
\rho_0 \frac{d_0 \vec{u}_a}{dt} + \nabla p_a + \rho_0 (\vec{u}_a \cdot \nabla) \vec{u}_a + \rho_a \frac{d_0 \vec{u}_a}{dt} &= O(\varepsilon^3) + O(M_{ac}^2), \\
s_a &= O(\varepsilon^3) + O(M_{ac}^2), \\
p_a - c_0^2 \rho_a - \frac{1}{2} \left( \frac{\partial^2 p}{\partial \rho^2} \right)_{s,0} \rho_a^2 &= O(\varepsilon^3) + O(M_{ac}^2).
\end{align*}
\]

To the order retained the entropy perturbations are negligible. We may therefore, thanks to the last relationship, eliminate the acoustic density perturbation:

\[
\rho_a = \frac{1}{c_0^2} p_a - \frac{1}{2c_0^6} \left( \frac{\partial^2 p}{\partial \rho^2} \right)_{s,0} \rho_a^2 + O(\varepsilon^3) + O(M_{ac}^2),
\]

The system is then written:

\[
\begin{align*}
\frac{1}{c_0^2} \frac{d_0 p_a}{dt} - \frac{1}{2c_0^6} \left( \frac{\partial^2 p}{\partial \rho^2} \right)_{s,0} \frac{d_0 \rho_a^2}{dt} + \rho_0 \nabla \cdot \vec{u}_a + \nabla \cdot \left( \frac{p_a}{c_0^2} \vec{u}_a + O(M_{ac}^2) \right) + O(\varepsilon^3) + O(M_{ac}^2), \\
\rho_0 \frac{d_0 \vec{u}_a}{dt} + \nabla p_a + \rho_0 (\vec{u}_a \cdot \nabla) \vec{u}_a + \frac{p_a \rho_0}{c_0^2} \frac{d_0 \vec{u}_a}{dt} + O(\varepsilon^3) + O(M_{ac}^2).
\end{align*}
\]

so that all that remains is:

\[
\begin{align*}
\frac{d_0 p_a}{dt} + \rho_0 c_0^2 \nabla \cdot \vec{u}_a &= -\nabla \cdot (p_a \vec{u}_a) + \frac{1}{2c_0^6} \left( \frac{\partial^2 p}{\partial \rho^2} \right)_{s,0} \rho_0 \frac{d_0 \rho_a^2}{dt} + O(\varepsilon^3) + O(M_{ac}^2), \\
\frac{d_0 \vec{u}_a}{dt} + \frac{1}{\rho_0} \nabla p_a &= -(\vec{u}_a \cdot \nabla) \vec{u}_a - \frac{p_a}{\rho_0 c_0^2} \frac{d_0 \vec{u}_a}{dt} + O(\varepsilon^3) + O(M_{ac}^2).
\end{align*}
\]

Let us now introduce the variables \( \tau \) and \( \zeta \) associated with the geometry of the caustic as well as the pressure and velocity as defined previously by:

\[
p_a(X, Y, Z, t) = P_a \tilde{p}_a(\zeta, \tau) \quad \text{and} \quad \vec{u}_a(X, Y, Z, t) = \begin{bmatrix} U_{aX} \tilde{u}_{aX}(\zeta, \tau) \\ U_{aY} \tilde{u}_{aY}(\zeta, \tau) \\ U_{aZ} \tilde{u}_{aZ}(\zeta, \tau) \end{bmatrix},
\]
where \( P_a, U_{aX}, U_{aY} \) and \( U_{aZ} \) are the dimensional scales, respectively for the pressure and the projections of the velocity vector on the 3 axes \( X, Y \) and \( Z \). The impedance relationships linking the velocity scales to the pressure scale are identical to the linear case:

\[
U_{aX} = \frac{P_a}{\rho_0 c_0}, \quad U_{aZ} = \varepsilon \frac{P_a}{\rho_0 c_0} \quad \text{and} \quad U_{aY} = \varepsilon^3 \frac{P_a}{\rho_0 c_0}.
\]

Likewise the relationship \( \pi_{aX} = \bar{\rho}_a [1 + \mathcal{O}(\varepsilon^2 M_ac)] \) is conserved.

The calculations are absolutely similar to those performed to obtain the linear Tricomi equation. We already know how the hierarchy of acoustic velocities in the different spatial directions is established near the caustic:

\[
U_{aX} = \mathcal{O}(1) \gg U_{aZ} = \mathcal{O}(\varepsilon) \gg U_{aY} = \mathcal{O}(\varepsilon^3).
\]

In addition, we will retain the dimensionless variables introduced in the previous section, namely:

\[
a = \frac{R_{\text{tot}}(O)}{2R_{XZ}(O)}, \quad e = \frac{R_{\text{tot}}(O)}{2R_{\text{cel}}(O)}, \quad f = \frac{R_{\text{tot}}(O)}{2R_{X}(O)}, \quad b = \frac{R_{\text{tot}}(O)}{2R_{\text{den}}(O)}, \quad g = \frac{R_{\text{tot}}(O)}{2R_{OZ}(O)}.
\]

\[
M_X = \frac{u_{0X}(O)}{e(O)} = \mathcal{O}(\varepsilon) \quad \text{and} \quad M_Z = \frac{u_{0Z}(O)}{e(O)} = \mathcal{O}(\varepsilon).
\]

Let us also introduce \( M_{ac} = U_{aX}/c(O) \), the acoustic Mach number with respect to \( c \). Similarly, in considering the relation \( (1 - M_X)c = c_0 \), we may project the equation of motion on the \( OX \) and \( OZ \) axes as follows:

**On \( OX \):**

\[
\left[ (1 - M_X) \frac{\partial}{\partial \tau} + \varepsilon M_Z \frac{\partial}{\partial \tau} \right] \pi_{aX} - \left[ 1 + \varepsilon^2 (a - b) \right] \frac{\partial \eta_a}{\partial \tau} =
\]

\[
M_{ac} \pi_{aX} \frac{\partial \pi_{aX}}{\partial \tau} - M_{ac} \eta_a \frac{\partial \pi_{aX}}{\partial \tau} + \mathcal{O}(\varepsilon^3) + \mathcal{O}(M_{ac}^2) + \mathcal{O}(\varepsilon^2 M_{ac}),
\]

**On \( OZ \):**

\[
\left[ (1 - M_X) \frac{\partial}{\partial \tau} + \varepsilon M_Z \frac{\partial}{\partial \tau} \right] \pi_{aZ} + \frac{\partial \eta_a}{\partial \tau} = \mathcal{O}(\varepsilon, M_{ac}, \varepsilon^2 M_{ac}).
\]

The conservation of mass equation is written:

\[
\left[ (1 - M_X) \frac{\partial}{\partial \tau} + \varepsilon M_Z \frac{\partial}{\partial \tau} \right] \rho_a + \left[ 1 + \varepsilon^2 (b + 2) \right] \frac{\partial \pi_{aX}}{\partial \tau} + \varepsilon^2 \frac{\partial \pi_{aZ}}{\partial \tau} =
\]

\[
M_{ac} \rho_0 \frac{\partial}{\partial \tau} \left( \frac{\partial^2 p}{\partial \tau^2} \right)_{s,0} + M_{ac} \frac{\partial \pi_{aX}}{\partial \tau} + \mathcal{O}(\varepsilon, M_{ac}^2, \varepsilon^2 M_{ac}).
\]

By substituting the relation \( \pi_{aX} = \bar{\rho}_a \) in the nonlinear terms above, those associated with the momentum cancel each other out at the order retained. We also introduce the usual parameter of nonlinearity in nonlinear acoustics:

\[
\beta(\bar{x}) = 1 + \frac{\rho_0(\bar{x})}{2c_0^2(\bar{x})} \left( \frac{\partial^2 p}{\partial \tau^2} \right)_{s,0} (\bar{x}).
\]

By returning to the variables introduced in the previous section, as well as the particle derivative defined as follows:

\[
\frac{d}{d \tau} = \frac{\partial}{\partial \tau} + \varepsilon M_Z \frac{\partial}{\partial \tau},
\]

the system is rewritten as follows:

418
Euler on $OX$: \[ \frac{d\bar{p}_a}{d\tau} - (1 + \varepsilon^2(a-b)z) \frac{d\bar{p}_a}{d\tau} = \mathcal{O}(\varepsilon^3, M_{ac}^2, \varepsilon^2 M_{ac}), \]

Euler on $OZ$: \[ \frac{d\bar{p}_a}{d\tau} + \frac{d\bar{p}_a}{d\tau} = \mathcal{O}(\varepsilon^3, M_{ac}^2, \varepsilon^2 M_{ac}), \]

Continuity: \[ \frac{d\bar{p}_a}{d\tau} - [1 + \varepsilon^2(z(b + a + 2\varepsilon))] \frac{d\bar{p}_a}{d\tau} + \varepsilon^2 \frac{d\bar{p}_a}{d\tau} = M_{ac} \beta(O) \frac{\partial(\bar{p}_a^2)}{\partial\tau} + \mathcal{O}(\varepsilon^3, M_{ac}^2, \varepsilon^2 M_{ac}). \]

Differentiating the equation associated with the conservation of mass:

\[ \frac{d^2\bar{p}_a}{d\tau^2} - [1 + \varepsilon^2(b + a + 2\varepsilon)] \frac{d\bar{p}_a}{d\tau} + \varepsilon^2 \frac{d\bar{p}_a}{d\tau} = M_{ac} \beta(O) \frac{\partial(\bar{p}_a^2)}{\partial\tau} + \mathcal{O}(\varepsilon^3, M_{ac}^2, \varepsilon^2 M_{ac}). \]

As before, the operators satisfy the following relationships:

\[ \frac{d}{d\tau} \frac{\partial d}{\partial\tau} = \frac{\partial d}{\partial\tau} \frac{d}{d\tau}, \]

\[ \frac{d}{d\tau} \frac{\partial d}{\partial\tau} = \frac{\partial d}{\partial\tau} \frac{d}{d\tau} + \mathcal{O}(\varepsilon^3). \]

The previous equation is then written:

\[ \frac{d^2\bar{p}_a}{d\tau^2} - [1 + \varepsilon^2(b + a + 2\varepsilon)] \frac{d\bar{p}_a}{d\tau} + \varepsilon^2 \frac{d\bar{p}_a}{d\tau} = M_{ac} \beta(O) \frac{\partial(\bar{p}_a^2)}{\partial\tau} + \mathcal{O}(\varepsilon^3, M_{ac}^2, \varepsilon^2 M_{ac}). \]

In reality, in the expression above, not all $\mathcal{O}(\varepsilon^2 M_{ac})$ terms have been neglected. Indeed, with respect to the definition of the particle derivative $d/d\tau$, there remain terms of order $\varepsilon^2 M_{ac}$ in the nonlinear term. By neglecting them, it follows:

\[ \frac{d^2\bar{p}_a}{d\tau^2} - [1 + \varepsilon^2(b + a + 2\varepsilon)] \frac{d\bar{p}_a}{d\tau} + \varepsilon^2 \frac{d\bar{p}_a}{d\tau} = M_{ac} \beta(O) \frac{\partial(\bar{p}_a^2)}{\partial\tau} + \mathcal{O}(\varepsilon^3, M_{ac}^2, \varepsilon^2 M_{ac}). \]

Replacing $\frac{d\bar{p}_a}{d\tau}$ and $\frac{d\bar{p}_a}{d\tau}$ by their respective expressions, the following dimensionless equation then follows, the neglected terms having been indicated above:

\[ \frac{d^2\bar{p}_a}{d\tau^2} - (1 - \varepsilon^2) \frac{\partial^2\bar{p}_a}{\partial\tau^2} - \varepsilon^2 \frac{\partial^2\bar{p}_a}{\partial\tau^2} = M_{ac} \beta(O) \frac{\partial^2(\bar{p}_a^2)}{\partial\tau^2}. \]

In the windless case $\frac{d}{d\tau} = \frac{\partial}{\partial\tau}$; the terms of order 1 are canceled, and all that remains is:

\[ \varepsilon^2 \frac{\partial^2\bar{p}_a}{\partial\tau^2} - \frac{\partial^2\bar{p}_a}{\partial\tau^2} = M_{ac} \beta(O) \frac{\partial^2(\bar{p}_a^2)}{\partial\tau^2}. \]

By setting:

\[ \mu = \frac{M_{ac} \beta(O)}{\varepsilon^2}, \]

we recover the nonlinear Tricomi equation, of the windless case, established by Guiraud in 1965:

\[ \varepsilon^2 \frac{\partial^2\bar{p}_a}{\partial\tau^2} - \frac{\partial^2\bar{p}_a}{\partial\tau^2} = \mu \frac{\partial^2(\bar{p}_a^2)}{\partial\tau^2}. \]

This equation involves a parameter $\mu$, measuring the amplitude of the nonlinear effects compared to diffraction. This parameter equals:

\[ \mu = \frac{M_{ac} \beta(O)}{\varepsilon^2} = \frac{\delta}{L_{\text{shock}}} \cdot \frac{\delta}{\lambda_{ac}}, \]

where $L_{\text{shock}} = c(O)/\beta(O)\omega_{ac} M_{ac}$, is the shock formation distance of a plane wave.
The ratio $\delta/\lambda_{ac}$ is large, while $\delta/L_{\text{shock}}$ is small; the product of these two ratios is therefore small, medium or large.

In the case with wind, the following equation results:

$$\frac{\partial^2 \bar{p}_a}{\partial \tau^2} + 2\varepsilon M_Z \frac{\partial^2 \bar{p}_a}{\partial z \partial \tau} - \left(1 - \varepsilon^2 z\right) \frac{\partial^2 \bar{p}_a}{\partial \tau^2} - \varepsilon^2 \frac{\partial^2 \bar{p}_a}{\partial z^2} = M_{ac} \beta(O) \frac{\partial^2 (\bar{p}_a^2)}{\partial \tau^2},$$

and after simplification:

$$2 \frac{M_Z}{\varepsilon} \frac{\partial^2 \bar{p}_a}{\partial z \partial \tau} - \frac{\partial^2 \bar{p}_a}{\partial \tau^2} + \frac{\partial^2 \bar{p}_a}{\partial z^2} = \mu \frac{\partial^2 (\bar{p}_a^2)}{\partial \tau^2}.$$

We note as in the linear case, that there is an additional cross term $\partial^2 \bar{p}_a/\partial z \partial \tau$ compared to the equation obtained without wind. By returning to the change of variables, introduced in the previous section:

$$T = \tau + \varsigma z \quad \text{and} \quad Z = z + \varsigma^2,$$

and by denoting the pressure associated with this change of variables, $\bar{P}(Z, T)$, the previous pressure equation is written:

$$-\frac{\partial^2 \bar{P}}{\partial Z^2} + Z \frac{\partial^2 \bar{P}}{\partial T^2} = \mu \frac{\partial^2 (\bar{P}^2)}{\partial T^2}.$$

We recover the nonlinear Tricomi equation of the windless case exactly, with a shift of the reference identical to the linear case.

### 2.3.2 Conclusion

In the previous section, we have shown that the pressure field in the vicinity of a caustic in the presence of wind always satisfied the Tricomi equation. This was obtained by accounting for the diffraction neglected by geometrical acoustics in this region of space. This phenomenon is not sufficient to model the pressure behavior near a caustic; the incident shocks lead to signals with infinite peaks. In this section, we have added nonlinear effects to the diffraction effects. This study has shown that these two phenomena are predominant in the vicinity of a caustic. In assuming a wind speed much lower than 100 km/h, we saw, in section 2.1, that this had little influence on the speed of sound and the rotation of the coordinate system but could have a significant influence on the thickness of the boundary layer if its gradient was large. In this section, we have shown that the nonlinear Tricomi equation is preserved in the presence of wind, but in a reference frame whose origin is no longer on the geometric caustic.
Chapter 3

PRESENTATION OF THE NUMERICAL SCHEME

INTRODUCTION

In the previous chapters, the physical problem, as well as the associated mathematical model have been presented; we have thus demonstrated that the pressure field near a caustic, with wind taken into account, always satisfied the nonlinear Tricomi equation when the wind speed remained sufficiently weak with respect to that of sound (in practice, this hypothesis is well satisfied on the ground). To date and to our knowledge, no theoretical results concerning the existence and uniqueness of the solution(s) of this equation have been produced. The only results obtained come from numerical simulations taking into account various simplifying assumptions. A code solving the nonlinear Tricomi equation in full has thus been written, in order to obtain the complete pressure field on a considered domain for any complex signal. The first idea had been to discretize this equation as it is by means of finite differences, but the presence of signals with very fine peaks required too many discretization points to obtain convincing results. To solve this equation we have therefore taken inspiration from the scheme of Bergen (Frøysa, Tjøtta and Berntsen, 1993), used to simulate solutions of the KZ equation (Khokhlov-Zabolotskaya, 1969). For this we have, for the first time, constructed an unsteady equation from the nonlinear Tricomi equation based on existing equations (3.1). Then, following Bergen’s scheme, we have developed a code allowing us to numerically solve this equation by a pseudo-spectral method (3.2). The linear case was first studied (3.3) and then validated by comparison with the analytical solutions (3.4). In a second step, we have solved the Tricomi equation in the nonlinear case (3.5). The absence of known analytical solutions in this particular case has led to validating the code by various indirect tests (3.6).

Since the equation is equivalent with or without wind, to simplify writing, we will return, in this section, to the notations of the nonlinear Tricomi equation written in the windless case, that is:

\[
\frac{\partial^2 \tilde{p}_a}{\partial \tilde{z}^2} - \tilde{z} \frac{\partial^2 \tilde{p}_a}{\partial \tilde{\tau}^2} + \mu \frac{\partial^2}{\partial \tilde{\tau}^2} \left( \frac{\tilde{p}_a^2}{2} \right) = 0,
\]

with:

\( \tilde{p}_a \): the dimensionless pressure,
\( \tilde{z} \): the dimensionless distance to the caustic,
\( \tilde{\tau} \): the dimensionless phase variable (retarded time),
and \( \mu \): the dimensionless parameter which compares the effects of nonlinearity and diffraction,

The boundary conditions associated with the problem are recalled:
1. We consider that before and after the passage of an acoustic wave the domain is not disturbed by any acoustic phenomenon, therefore the acoustic field for large times at positive or negative infinity vanish:
\[ \bar{p}_a(\tau \to \pm \infty, \tau) = 0. \]

2. Far into the zone of silence, there is attenuation of the acoustic field, so that:
\[ \bar{p}_a(\tau, \tau \to -\infty) = 0. \]

3. On the “propagative” side, the sound field must match the geometrical acoustic solution far from the caustic, which is written:
\[ \bar{p}_a(\tau, \tau \to +\infty) = \frac{1}{4} \left[ F\left( \frac{\tau}{3} - \frac{2}{3}\zeta^2 \right) + G\left( \frac{\tau}{3} - \frac{2}{3}\zeta^2 \right) \right]. \]

The function \( F \) represents the incident field before it has “tangented” the caustic and the function \( G \) the reflected field. The latter being indeterminate, we may eliminate it in a way inspired by the Sommerfeld radiation condition, for the far field in free space. We calculate the following two derivatives:
\[ \frac{\partial \bar{p}_a}{\partial \tau} = \frac{1}{4} \left[ F\left( \frac{\tau}{3} + \frac{2}{3}\zeta^2 \right) + G\left( \frac{\tau}{3} - \frac{2}{3}\zeta^2 \right) \right], \]
\[ \frac{\partial \bar{p}_a}{\partial \zeta} = -\frac{1}{4} \left[ F\left( \frac{\tau}{3} + \frac{2}{3}\zeta^2 \right) + G\left( \frac{\tau}{3} - \frac{2}{3}\zeta^2 \right) \right] + \frac{1}{4} \left[ F\left( \frac{\tau}{3} + \frac{2}{3}\zeta^2 \right) - G\left( \frac{\tau}{3} - \frac{2}{3}\zeta^2 \right) \right]. \]

By combining the two previous expressions, we can eliminate \( G \) and \( G' \), and write the boundary condition at positive infinity in the form of a radiation condition at infinity in the direction normal to caustic:
\[ \frac{\tau}{4} \frac{\partial \bar{p}_a}{\partial \tau} + \frac{1}{4} \frac{\partial \bar{p}_a}{\partial \zeta} + \frac{\tau}{4} \frac{\bar{p}_a}{4} = 2F'\left( \frac{\tau}{3} + \frac{2}{3}\zeta^2 \right) \text{ at } \zeta \to \infty. \]

The 3rd term in the expression above is of order \( \zeta^{-\frac{1}{2}} \) with respect to the first two, of order 1. It therefore appears negligible in theory, but from a numerical point of view, it is important to retain it to increase the precision of the discretization in \( \zeta \).

### 3.1 Unsteady Tricomi Equation

The problem to be solved — the Tricomi equation and the associated boundary conditions — is nonlinear. The numerical solution of this type of system may only be carried out by means of an iterative scheme. The idea here is to base this iterative scheme on a generalized unsteady equation which will converge, for very large “pseudo-times”, towards the nonlinear Tricomi equation. To find the appropriate iterative term to add to the nonlinear Tricomi equation, we will draw inspiration from existing equations, some of which are used in the study of sonic boom. The unsteady equation that we pose is as follows:

\[ \bar{p}_a(\tau, \zeta) = \lim_{\tau \to +\infty} \bar{p}_a(\tau, \zeta, \bar{t}) \]

where \( \bar{p}_a(\tau, \zeta, \bar{t}) \) is a solution of
\[ \frac{\partial^2 \bar{p}_a}{\partial \bar{t}^2} = \frac{\partial^2 \bar{p}_a}{\partial \tau^2} - \zeta \frac{\partial^2 \bar{p}_a}{\partial \zeta^2} + \mu \frac{\partial^2}{\partial \tau^2} \left( \frac{\bar{p}_a}{2} \right). \]

In this equation, on the right hand side, we thus have the nonlinear Tricomi equation and on the left hand side the postulated “unsteadiness” term.
3.1.1 Justification of the choice of the unsteady Tricomi equation

In this section, we will present existing unsteady equations, modeling acoustic phenomena in which, as is the case with a caustic, diffraction and nonlinear effects are involved. This analysis will allow us to find the unsteady term used in the construction of the unsteady nonlinear Tricomi equation. Furthermore, as efficient numerical schemes exist for solving the Burgers’ equation, a classical equation modeling nonlinear effects, we will attempt to make it appear in the unsteady Tricomi equation.

3.1.1.1 KZ Equation

The so-called KZ (Khokhlov-Zabolotskaya) equation, in two dimensions, models a nonlinear paraxial sound field radiated by a piston (Zabolotskaya and Khokhlov, 1969). It is written:

\[
\frac{\partial^2 p_a}{\partial \tau \partial \tau} = \frac{\partial^2 p_a}{\partial z^2} + \mu \frac{\partial^2}{\partial \tau^2} \left( \frac{p_a^2}{2} \right).
\]

The choice of normalization is as follows:

\[
\bar{z} = \frac{z}{a}, \quad \bar{x} = \frac{x}{R} \quad \text{and} \quad \bar{\tau} = \omega_{ac} \left( \tau - \frac{x}{c_0} \right),
\]

\(a\) being the half width of the emitter, \(R = ka^2/2\) the Rayleigh distance determining the transition between the near field and the far field and \(\omega_{ac}\) the characteristic angular frequency of the signal. The nonlinearity parameter \(\mu\) is the ratio between the Rayleigh distance and the shock formation distance of a monochromatic signal.

We may note that the right hand side of this equation is identical to the nonlinear Tricomi equation, save for the term \(\bar{z} \frac{\partial^2 p_a}{\partial \tau^2}\). The “unsteady” term associated with it, \(\frac{\partial^2 p_a}{\partial x \partial \tau}\), takes the propagation of the signal into account as it moves away from the source.
3.1.1.2 Equation for the propagation of sonic boom in the shadow zone

The equation expressing the propagation of the sonic boom in the shadow zone (Coulouvrat, 1998) has the following form:

\[
\frac{\partial^2 p_a}{\partial \pi \partial \tau} = \frac{\partial^2 p_a}{\partial z^2} - \bar{z} \frac{\partial^2 p_a}{\partial \tau^2} + \mu_G \frac{\partial^2 p_a^2}{\partial \tau^2},
\]

where \( p_a \) is the normalized field amplitude, \( \pi \) the distance parallel to the ground normalized by the creeping wave attenuation distance, \( z \) the vertical distance normalized by the thickness of the diffraction boundary layer and \( \tau \) the retarded time normalized by the period of the signal. The dimensionless parameter \( \mu_G \) compares the order of magnitude of nonlinear effects and diffraction in the shadow zone. We note that the right side is identical to the nonlinear Tricomi equation.

![Diagram of sonic boom in the shadow zone](image_url)

**Figure 3.1.2:** Sonic boom in the shadow zone

In this particular case, the unsteady term, \( \partial^2 p_a / \partial \pi \partial \tau \), has physical meaning: it describes the evolution of the creeping wave as it enters the interior of the shadow zone.

3.1.1.3 Equation modeling the pressure in the vicinity of an arête of a caustic

Consider, in 2 dimensions, a wave propagating in a perfect homogeneous fluid with speed \( c_0 \). If the wave has a concave wavefront and the radius of curvature \( R(x) \) of this wavefront has a minimum \( R_0 \), a “cusped” caustic then appears (following the terminology of catastrophe theory, Berry 1981), as drawn below:
Cramer and Seebass (1978) have shown that the pressure field around it satisfies the Khokhlov-Zabolotskaya (KZ) equation. Coulouvrat (2000) expressed the boundary conditions associated with this equation sufficient to describe the phenomenon.

By introducing $a = \frac{9}{8} R_0^2 R_0''$, the only parameter determining the geometry of the problem, $\tau$, the dimensionless retarded time, $k$ the wavenumber and $\varepsilon = (27/ka)^{1/2}$ the dimensionless parameter measuring the effects of diffraction, the following change of scale, different in both directions, results:

$$\bar{z} = \frac{27}{2\varepsilon} \frac{(z-R_0)}{a} \quad \text{and} \quad \bar{x} = \frac{27}{\varepsilon^{3/2}} \frac{x}{a}.$$  

The pressure field near a point of the caustic then satisfies the KZ equation:

$$\frac{\partial^2 \bar{p}_a}{\partial \bar{z} \partial \tau} = \frac{\partial^2 \bar{p}_a}{\partial \bar{x}^2} + \mu \frac{\partial^2}{\partial \bar{\tau}^2} (\bar{p}_a^2).$$

The dimensionless parameter $\mu$ compares the order of magnitude of nonlinear effects and diffraction. The propagation direction of the shock is parallel to the $\bar{z}$ axis (tangent to the point of the caustic) and the $\bar{x}$ axis is normal to the direction of propagation.

The associated boundary conditions are written:

$$\bar{p}_a(\bar{\tau}, \bar{z}, \bar{x}) \xrightarrow{\bar{z}^2 + \bar{x}^2 \to \infty} \frac{1}{\sqrt{6\bar{\tau}^2 - \bar{z}}} \overline{F}_{\text{cusp}}(\bar{\tau} + \alpha \bar{z} + \alpha^2 \bar{x} - \bar{z}^4),$$

$\overline{F}_{\text{cusp}}$ being the incident signal and $\alpha$ the unique real root of $\bar{x} + 2\alpha \bar{z} - 3\alpha^3 = 0.$
The physics associated with this equation, in the particular case of the “cusped” point, is different from that of a field radiated by a piston for which the KZ equation was originally written. The boundary conditions are different. In this particular case, they correspond to matching with geometrical acoustics which appear here as the outer solution of the problem, in the sense of matched asymptotic expansions. However, this problem is, by the nature of the phenomenon described and by the method of solution chosen, very close to that which interests us.

3.1.1.4 Unsteady nonlinear Tricomi equation

We have just presented three “unsteady” equations modeling acoustic phenomena which involve, as is the case with a caustic, diffraction and nonlinearities. We have remarked that their stationary part was either identical or had very little difference from the nonlinear Tricomi equation. We also observed that their unsteady term was always the same, \( \frac{\partial^2 p_a}{\partial t \partial \tau} \), and that the associated unsteady variable was always a distance variable expressing a progression in a certain spatial progression.

The boundary conditions associated with the problems modeled by the equations above, are however different from those which interest us. Furthermore, unlike these equations, only the “converged” solution is important to us here.

In addition, if we add the same unsteady term, \( \frac{\partial^2 p_a}{\partial t \partial \tau} \), to Tricomi’s equation to constitute an unsteady equation, we remark that when the diffraction terms are neglected, we recover the Burgers’ equation that we know how to solve perfectly, either analytically, or numerically.

The unique form of the unsteady term, as well as the numerical perspectives that we can bring to this choice, lead us to write unsteady the nonlinear Tricomi equation in the following form:

\[
\frac{\partial^2 p_a}{\partial t \partial \tau} = \frac{\partial^2 p_a}{\partial \tau^2} - \tau \frac{\partial^2 p_a}{\partial \tau^2} + \mu \frac{\partial^2}{\partial \tau^2} \left( \frac{p_a^2}{2} \right),
\]

with \( t \) the iterative variable. We could have called it “pseudo-distance” in reference to acoustics, but the method being inspired by aerodynamics, we will call it “pseudo-time”.

The boundary conditions are unchanged with respect to the stationary case.

3.1.2 Initial conditions

The initialization of the iterative calculation, \( p_a(\tau, \tau, t = 0) \) will be done, \( a \ priori \), by any function. The choice of this will be discussed later when we address the influence of initial conditions on the solution, after convergence.

3.1.3 Mathematical nature of the nonlinear Tricomi equation

We recall the following result (Courant and Hilbert, 1962).

Let

\[
A_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + B = 0
\]

be the second-order quasilinear partial differential equation satisfied by \( u(x_1, \cdots, x_n) \), where \( (x_i)_{1 \leq i \leq n} \) denote \( n \) independent variables and where \( A_{ij} = A_{ji} \) and \( B \) are functions of \( x_1, \cdots, x_n, u, \frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_n} \).
The partial differential equation is hyperbolic if the eigenvalues of the matrix $A$ are all of the same sign save one, irrespective of $x_1, \ldots, x_n, u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}$. It is elliptic if all eigenvalues of the matrix $A$ are of the same sign.

**Application:**

1) Let the stationary nonlinear Tricomi equation be:

$$\frac{\partial^2 p}{\partial \bar{z}^2} + (\mu \bar{p} - \bar{z}) \frac{\partial^2 p}{\partial \bar{\tau}^2} + \mu \left( \frac{\partial p}{\partial \bar{\tau}} \right)^2 = 0.$$  

We have: $n = 2$, $x_1 = \bar{z}$, $x_2 = \bar{\tau}$, $A = \begin{pmatrix} 1 & 0 \\ 0 & \mu \bar{p} - \bar{z} \end{pmatrix}$ and $B = \mu \left( \frac{\partial p}{\partial \bar{\tau}} \right)^2$.

The eigenvalues $\lambda$ of the matrix $A$ are equal to $1$ and $\mu \bar{p} - \bar{z}$. They are of opposite sign if $\bar{z} > \mu \bar{p}$ (hyperbolic equation) and of the same sign otherwise (elliptic equation). As a consequence, the nonlinear Tricomi equation is an equation of mixed type.

This peculiarity reflects the fact that the problem, and therefore the solution, changes radically in nature on either side of the sonic line located at $\bar{z} - \mu \bar{p}_a = 0$.

Indeed, when $\bar{z} - \mu \bar{p}_a > 0$, the equation is hyperbolic. The associated domain is the insonified region. This corresponds to a “propagative” region: an incident wave coming from positive infinity, along characteristic curves of the Tricomi equation, which are the wave fronts, tangents the caustic and then goes back toward infinity in the form of a reflected wave.

For $\bar{z} - \mu \bar{p}_a < 0$, the problem becomes elliptic. The associated domain is the zone of silence. This corresponds to a “diffusive” region, in which the sound attenuates rapidly at a certain distance from the caustic (cf. Figure 3.3.3).

2) Let the unsteady nonlinear Tricomi equation be:

$$\frac{\partial^2 p}{\partial \bar{z}^2} + (\mu \bar{p} - \bar{z}) \frac{\partial^2 p}{\partial \bar{\tau}^2} - \frac{\partial^2 p}{\partial \bar{\tau} \partial \bar{\tau}} + \mu \left( \frac{\partial p}{\partial \bar{\tau}} \right)^2 = 0.$$  

We have: $n = 3$, $x_1 = \bar{z}$, $x_2 = \bar{\tau}$, $x_3 = \bar{t}$, $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu \bar{p} - \bar{z} & -1/2 \\ 0 & -1/2 & 0 \end{pmatrix}$ and $B = \mu \left( \frac{\partial p}{\partial \bar{\tau}} \right)^2$.

The eigenvalues $\lambda$ of the matrix $A$ are the roots of the polynomial: $(1 - \lambda) \left( \lambda^2 + (\bar{z} - \mu \bar{p}) \lambda - \frac{1}{4} \right)$.

The roots of the polynomial $\lambda^2 + (\bar{z} - \mu \bar{p}) \lambda - \frac{1}{4}$ are always real and their product equals $-1/4$. As a consequence the matrix $A$ always has $2$ positive eigenvalues and one negative eigenvalue, so that the unsteady nonlinear Tricomi equation is hyperbolic.

The solution of this equation, a priori more difficult in character because of the additional term, will be easier because the difficulty linked to the mixed nature of the stationary Tricomi equation is removed. This also reinforces the idea that the unsteady term chosen previously is perfectly appropriate in our problem.
3.2 PRINCIPLE OF THE PSEUDO-SPECTRAL METHOD

To solve this new equation we use a pseudo-spectral method; the diffraction terms are treated in the frequency domain and the nonlinear terms in the temporal domain. This brings us back to a sequence of one-dimensional problems. This type of solution is inspired by the numerical scheme of Bergen (Frøysa, Tjøtta and Berntsen 1993) used for solving the so-called KZ equation, presented above.

First of all, we are going to solve the equation corresponding to the “diffraction” part of the problem in the first half-step of time, that is:

\[
\frac{\partial^2 \tilde{p}_a}{\partial \tau \partial t} - \frac{\partial^2 \tilde{p}_a}{\partial z^2} = \frac{\tau}{2} \frac{\partial^2 \tilde{p}_a}{\partial \tau^2}.
\]

In spectral space (after using the Fourier transform), at fixed frequency, we obtain an equation that we discretize by finite differences. To avoid any numerical stability problems, an implicit scheme is chosen leading to the obtention of a tridiagonal matrix. Its inversion provides the solutions of these differential equations. By inverse Fourier transformation on these solutions, we return to physical (temporal) space.

Then, in a second half-step of time, at fixed \(z\), we solve the Burgers’ equation:

\[
\frac{\partial \tilde{p}_a}{\partial t} = \frac{\partial}{\partial \tau} \left( \frac{\mu \tilde{p}_a^2}{2} \right),
\]

by means of the hybrid “shock capturing” scheme of McDonald and Ambrosiano (1984).

Finally, for each iteration, the two steps above are repeated until the convergence of the scheme.

Remark: it is possible to distribute the terms of the unsteady nonlinear Tricomi equation differently in the frequency and time domains. We may, for example, move the term \(\tau \frac{\partial^2 \tilde{p}_a}{\partial \tau^2}\) of the unsteady linear Tricomi equation to add it to the Burgers’ equation. This new arrangement would then lead us to solve the following equation in Fourier space:

\[
\frac{\partial^2 \tilde{p}_a}{\partial \tau \partial t} = \frac{\partial^2 \tilde{p}_a}{\partial z^2}.
\]

As for the equation relating to nonlinear effects, it would find itself augmented by a term in the following way:

\[
\frac{\partial \tilde{p}_a}{\partial t} = \frac{\partial}{\partial \tau} \left( \frac{\mu \tilde{p}_a^2}{2} - \tau \tilde{p}_a \right).
\]

The discretization techniques, in this new arrangement, are identical to the previous case. The two arrangements will be compared in point 3.6.4.

We may summarize the preceding via the following diagram (Figure 3.2.1):
PRESENTATION OF THE NUMERICAL SCHEME

**Figure 3.2.1: Principle of the pseudospectral method**

**Physical Space**
- $p^n$  
- $p^{n+1/2}$  
- $p^{n+1}$

**Spectral Space**
- $\hat{P}^n$  
- $\hat{P}^{n+1/2}$

**Transformations**
- $\text{FT}$
- $\text{FT}^{-1}$

**Methods**
- **Burgers**
  - Finite Differences (fixed $\bar{z}$)

**Equations**
- $p^n = \bar{p}_a(\tau, \bar{z}, n\Delta t)$
- $\hat{P}^n = TF(p^n)$

**Additional Information**
- Unsteady Linear Tricomi
- Finite Differences (fixed frequency)
3.3 SOLUTION OF THE STATIONARY LINEAR TRICOMI EQUATION

We will now detail the method for solving the unsteady nonlinear Tricomi equation. The resulting code was constructed in stages. We will retain these for improved clarity and because the validation of the code could only be done completely in the linear case, the only case where analytical solutions are known. So in this first section (3.3), we will present the method of solution for the stationary linear Tricomi equation, as well as its validation by comparison with analytical results (3.4). Then in the next section (3.5), we will present the method for solving this same equation augmented by the unsteady term and analyze the convergence of this scheme as it iterates to the previous solutions. In this same section (3.5), we will present the method of solution for the Tricomi equation when taking nonlinear effects into account. Finally, in a last section (3.6), we will validate the code in the nonlinear case.

In the stationary linear case, the problem to solve is the following:

\[
\frac{\partial^2 p_a}{\partial z^2} - z \frac{\partial^2 p_a}{\partial \tau^2} = 0
\]

with the following expressions as boundary conditions:

\[
p_a(\tau \to \pm \infty, z) = 0,
\]
\[
p_a(\tau, z \to -\infty) = 0,
\]
\[
\frac{z^4}{2} \frac{\partial^2 p_a}{\partial \tau \partial z} - \frac{z^5}{2} p_a + \frac{z^5}{4} p_a = 2 F' \left( \tau + \frac{2}{3} \frac{z^3}{2} \right) \quad \text{at} \quad z \to +\infty.
\]

The numerical method consists of calculating the Fourier transform of the stationary linear Tricomi equation in the direction \( \tau \) and, then to solve the new equation obtained by finite differences.

3.3.1 Fourier Transform of the stationary linear Tricomi equation

We take the following expression as the definition of the Fourier transform:

\[
\hat{p}_a(z, \omega) = \int_{-\infty}^{+\infty} p_a(\tau, z) e^{-i\omega \tau} d\tau, \quad \text{with} \quad \omega \text{ the conjugate variable of} \quad \tau.
\]

The inverse transform is written:

\[
p_a(\tau, z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{p}(z, \omega) e^{i\omega \tau} d\omega.
\]

The Fourier transform of the Stationary Linear Tricomi Equation (FSLTE) is then written:

\[
\frac{\partial^2 \hat{p}}{\partial z^2} + \bar{z} \omega^2 \hat{p} = 0.
\]
3.3.2 Fourier Transform of the boundary condition at positive infinity

By taking the Fourier transform of the boundary condition at positive infinity we obtain the following expressions:

\[
\int_{-\infty}^{\infty} \left( \frac{1}{2} \frac{\partial P_a}{\partial \tau} + \frac{1}{4} \frac{\partial P_a}{\partial z} + \frac{1}{2} \frac{P_a}{4} \right) e^{-i\omega \tau} d\tau = \int_{-\infty}^{\infty} \left[2\tilde{F}'(\tau + \frac{2}{3} \frac{\omega}{z})\right] e^{-i\omega \tau} d\tau \quad \text{as } \tau \to +\infty,
\]

or else:

\[
i\omega \frac{1}{4} \tilde{P} + \frac{1}{4} \frac{\partial \tilde{P}}{\partial \tau} + \frac{1}{2} \frac{\tilde{P}}{4} = \int_{-\infty}^{+\infty} \left[2\tilde{F}'(\tau + \frac{2}{3} \frac{\omega}{z})\right] e^{-i\omega \tau} d\tau \quad \text{as } \tau \to +\infty.
\]

If we set: \( u = \tau + \frac{2}{3} \frac{\omega}{z} \) and \( \hat{F}(\omega) = \int_{-\infty}^{+\infty} \tilde{F}(u)e^{-i\omega u} du \), the Fourier transform of the incident signal, we finally have:

\[
i\omega \frac{1}{4} \tilde{P} + \frac{1}{4} \frac{\partial \tilde{P}}{\partial \tau} + \frac{1}{2} \frac{\tilde{P}}{4} = 2i\omega e^{\frac{1}{6} i\omega \frac{2}{3} \tau} \hat{F}(\omega), \quad \text{as } \tau \to +\infty.
\]

3.3.3 Boundary condition at negative infinity

We may replace the boundary conditions at negative infinity, not by a zero term, but by an approximation of the pressure in this region. We will call this new numerical boundary condition the “evanescence” condition. This is exact in the linear case. In the nonlinear case, the pressure tending towards 0, the nonlinearities become negligible and the condition is still valid.

We recall the solution of the linear Tricomi equation (cf. paragraph 2.2.6):

\[
p(\tau, z) = FT^{-1}\left[\sqrt{2\pi}(1 + isgn(\omega))|\omega|^\frac{1}{6} Ai\left(-|\omega|^\frac{1}{3} z\right) \hat{F}(\omega)\right].
\]

The Fourier transform of the solutions of the Tricomi equation is thus noted:

\[
\hat{P}(\tau) = FT^{-1}\left[\sqrt{2\pi}(1 + isgn(\omega))|\omega|^\frac{1}{6} Ai\left(-|\omega|^\frac{1}{3} z\right) \hat{F}(\omega)\right] = a(\omega)Ai\left(-|\omega|^\frac{1}{3} \tau\right),
\]

with \( a(\omega) = \sqrt{2\pi}(1 + isgn(\omega))|\omega|^\frac{1}{6} \hat{F}(\omega) \). Now, according to Abramowitz and Stegun (1965), the expansion of the Airy function at infinity for \( \xi > 0 \) is written:

\[
Ai(\xi) \sim \frac{1}{2\sqrt{\pi}} \xi^{-\frac{1}{6}} \left[ \exp\left(-\frac{2}{3} \xi^{\frac{1}{2}}\right) \right].
\]

By setting \( \xi = |\omega|^\frac{1}{3} |\tau| \), we have the expression for the Fourier transform of the pressure in the evanescent part:

\[
\hat{P}(\tau) \approx A(\omega)|\tau|^{-\frac{1}{6}} \left[ \exp\left(-\frac{2}{3} |\omega||\tau|^\frac{1}{2}\right) \right] \text{ with } A(\omega) = \frac{a(\omega)}{2\sqrt{\pi}} |\omega|^\frac{1}{6},
\]

We deduce after differentiation:

\[
\frac{d\hat{P}(\tau)}{d\tau} = \left[ \frac{1}{4|\tau|} + |\omega||\tau|^{\frac{1}{3}} \right] A(\omega)|\tau|^{-\frac{1}{6}} \left[ \exp\left(-\frac{2}{3} |\omega||\tau|^\frac{1}{2}\right) \right] = \left[ \frac{1}{4|\tau|} + |\omega||\tau|^{\frac{1}{3}} \right] \hat{P}(\tau),
\]

431
or else that:
\[
\frac{d\hat{P}}{d\bar{\tau}} - \left[\frac{1}{4}|\bar{\tau}|^{-1} + |\omega||\bar{\tau}|^{\frac{1}{2}}\right] \hat{P} = 0, \quad \text{at} \quad \bar{\tau} \to -\infty.
\]

The boundary condition above, known as evanescence, thus makes it possible to impose an exponential decrease of the pressure in the shadow zone.

Remark: This expression is to be compared to that which we have for the boundary conditions at plus infinity, that is to say for the matching with geometrical acoustics:
\[
\frac{d\hat{P}}{d\bar{\tau}} + \left(i\omega\bar{\tau}^{\frac{1}{2}} + \bar{\tau}^{-1}\right) \hat{P} = \bar{\tau}^{\frac{1}{4}} 2i\omega e^{\frac{i}{2}\omega\bar{\tau}^{\frac{3}{2}}} \hat{F}(\omega).
\]

### 3.3.4 Discretization of the problem

The stationary case, being the limit of the unsteady case when the “pseudo-time” \( \bar{t} \) tends towards infinity, we consider the the variables to be independent of \( \bar{t} \). For the same reasons, matrices introduced in this section are indexed by the mathematical sign \( \infty \).

We will now discretize the partial derivatives along the \( \bar{\tau} \) axis, the index \( j \) taking the values from 1 to \( M \) on this axis; we then write: \( \tilde{P}_j = \tilde{P}(\omega, z_j) \), where \( z_j = z_0 + j\Delta z \). In [FSLTE], the second derivative in \( \bar{\tau} \) will therefore be discretized according to a second-order accurate scheme:
\[
\frac{d^2\hat{P}_j}{d\bar{\tau}^2} = \frac{\hat{P}_{j-1} - 2\hat{P}_j + \hat{P}_{j+1}}{\Delta\bar{\tau}^2} + O(\Delta\bar{\tau}^2).
\]

For the boundary conditions at positive infinity, the partial derivatives are necessarily discretized according to an uncentered scheme since we are at the edge of the computational domain. We have studied the case of a 1st order discretization (with 2 points):
\[
\frac{d\hat{P}_M}{d\bar{\tau}} = \frac{\hat{P}_M - \hat{P}_{M-1}}{\Delta\bar{\tau}} + O(\Delta\bar{\tau}),
\]
as well as a 2nd order accurate discretization (with 3 points):
\[
\frac{d\hat{P}_M}{d\bar{\tau}} = \frac{3\hat{P}_M - 4\hat{P}_{M-1} + \hat{P}_{M-2}}{2\Delta\bar{\tau}} + O(\Delta\bar{\tau}^2).
\]

If we discretize the evanescence condition to first order, we have:
\[
\frac{\hat{P}_1 - \hat{P}_0}{\Delta\bar{\tau}} = \left[\frac{1}{4|\bar{\tau}|} + |\omega||\bar{\tau}|^{\frac{1}{2}}\right] \hat{P}_0 + O(\Delta\bar{\tau}).
\]

### 3.3.5 Matrix representation of the problem

Finally, the system to be solved is written, for each frequency, in matrix form:
\[
A_\infty(\omega) \hat{P} = B_\infty,
\]
with \( A_\infty \) a tri-(or tetra-) diagonal matrix, of size \( M \times M \), decomposing in the following way:
\[
A_\infty = A_\infty^{(0)} + \omega A_\infty^{(1)} + \omega^2 A_\infty^{(2)},
\]
with
The discretized boundary conditions at negative infinity give the coefficients:

\[
\alpha = -\left\{ 1 + \Delta z \left[ \frac{1}{4|z_1|} + |\omega||z_1|^{\frac{1}{2}} \right] \right\} \quad \text{and} \quad \beta = 1.
\]

The boundary conditions at positive infinity discretized to 1st order lead to the coefficients:

\[
\varepsilon = \frac{z_M^{-\frac{1}{4}}}{\Delta z} + \frac{z_M^{-\frac{5}{4}}}{4}, \quad \delta = -\frac{z_M^{-\frac{1}{4}}}{\Delta z} \quad \text{and} \quad \gamma = 0.
\]

At 2nd order, the discretization gives us:

\[
\varepsilon = \frac{3z_M^{-\frac{1}{4}}}{\Delta z} + \frac{z_M^{-\frac{5}{4}}}{4}, \quad \delta = -\frac{2z_M^{-\frac{1}{4}}}{\Delta z} \quad \text{and} \quad \gamma = \frac{z_M^{-\frac{1}{4}}}{2\Delta z}.
\]

The other matrices constituting \( A \) are written:

\[
A^{(1)}_{\infty} = \begin{bmatrix}
0 & 0 & \cdots & \cdots & \cdots & 0 \\
0 & 0 & \cdots & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \cdots & \cdots & 0
\end{bmatrix},
\]

\[
A^{(2)}_{\infty} = \begin{bmatrix}
0 & 0 & \cdots & \cdots & \cdots & 0 \\
0 & 0 & \cdots & \cdots & \cdots & 0 \\
0 & z_2 & \cdots & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & 0
\end{bmatrix}.
\]
Finally, the right hand side has the following form:

\[
B_\infty = \begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
2i\omega e^{\frac{2i\omega}{3}M^2/2} \hat{F}(\omega)
\end{bmatrix}.
\]

### 3.3.6 Inversion via the Thomas algorithm

The matrix \( A_\infty \) being tri (or tetra-) diagonal, we have used the Thomas algorithm (Appendix I) to invert it.

The direct and inverse Fourier transforms are calculated by the Fast Fourier Transform algorithm (FFT, Brigham, 1973).

### 3.3.7 Choice of calculation domain

The choice of the calculation domain is linked to the incident signals which we are going to study; we will not use the same computational box for a periodic and non-periodic signal. Indeed if, in the first case, the size of the box is determined by the period of the signal this is more complex in the second case. For this chapter, we have used two signals: a periodic sawtooth and an “N” wave (Figures 3.3.1-3.3.2). The second choice is linked to the fact that in the sonic boom case, far from the plane the acoustic disturbance frequently has the shape of an “N”. However, this signal will be artificially “periodized” because the scheme used is pseudo-spectral.

![Figure 3.3.1: Periodic sawtooth](image1)

![Figure 3.3.2: “N” wave](image2)
The periodic nature of the sawtooth determines the choice of the computational box on the phase axis \( \tau \); we will therefore take, on this axis, the following domain:

\[ [\tau_{\text{min}} = -1 \quad ; \quad \tau_{\text{max}} = 1]. \]

In regard to the “N” wave, it is necessary to determine the appropriate “computational box” to best capture the signal. The upper edge of the “computational box”, \( z_{\text{max}} \), will be located in the hyperbolic part, the lower edge, \( z_{\text{min}} \), in the elliptic part. The time discretization domain is chosen in order to capture all incident and reflected characteristics associated with the “N” wave. We have therefore chosen the following domain on the \( \tau \) axis as a computational box:

\[
\left[ \tau_{\text{min}} = -\frac{2}{3}z_{\text{max}}^3 - 2 \quad ; \quad \tau_{\text{max}} = \frac{2}{3}z_{\text{max}}^3 + 3 \right].
\]

The choice of the term \( \frac{2}{3}z_{\text{max}}^3 \) is due to the fact that, in the linear case, in the vicinity of the caustic, the incident and reflected fields propagate along characteristic curves with respective equations:

\[
\begin{align*}
\tau_{\text{incident}} &= -\frac{2}{3}z_{\text{max}}^3 + \text{cst} \\
\tau_{\text{reflected}} &= \frac{2}{3}z_{\text{max}}^3 + \text{cst}
\end{align*}
\]

The asymmetry of the computational box comes from the difference in the nature of the curves representing the incident field and the reflected field. In the case of the “N” wave, at the level of the first shock, we go directly from a zero value of pressure to its maximum value. In contrast, the reflected field a “U” shape and the decrease in pressure beyond the maximum amplitude of the second shock is done according to a logarithmic curve, and thus more slowly; We must go further in time to capture the entirety of the signal.

We may summarize the above by the following diagram:
3.3.8 Validation of the method in the stationary linear case

In the linear case, it is possible to compare these results with the analytical solutions obtained via Fourier transform.

\[
\tilde{p}_a(\tau, \zeta) = FT^{-1}\left[\sqrt{2\pi}(1 + \text{sgn}(\omega))|\omega|^{\frac{1}{6}}Ai\left(-|\omega|^{\frac{2}{3}}\right)\hat{F}(\omega)\right].
\]

If we consider the pressure field for an incident “N” wave of total duration \(2T\), of the form:

\[
F(\tau) = \begin{cases} 
-\tau/T & \text{if } |\tau| < T \\
0 & \text{otherwise,}
\end{cases}
\]

An exact solution of this equation exists on the caustic (cf. Seebass 1971):

\[
\tilde{p}_a(\tau, 0) = \frac{2Ai(0)\Gamma(1/6)}{T\sqrt{2\pi}}\left\{\frac{\text{sgn}(\tau)}{5} \sin\left(\frac{\pi}{12}\right)\left[T - |\tau|^\frac{2}{3} - (T - |\tau|)^\frac{2}{3}\right] + |\tau| \cos\left(\frac{\pi}{12}\right)\left[T - |\tau|^{-\frac{1}{6}} - (T + |\tau|)^{-\frac{1}{6}}\right] \right. \\
-\left. \frac{1}{5} \cos\left(\frac{\pi}{12}\right)\left[|T + |\tau||^\frac{2}{3} + \text{sgn}(T - |\tau|)|T - |\tau||^\frac{2}{3}\right] - \tau \sin\left(\frac{\pi}{12}\right)\left[(T + |\tau|)^{-\frac{1}{6}} + \text{sgn}(T - |\tau|)|T - |\tau||^{-\frac{1}{6}}\right] \right\}.
\]
These two analytical solutions will allow us to validate the linear stationary case, which is the subject of the next section.

### 3.4 VALIDATION OF RESULTS IN THE LINEAR STATIONARY CASE

In this section we will present results obtained with the numerical code in the linear case. We will commence the validation of the code with the two signals that we presented in section 3.3.7. In the case of the "N" wave, we will expose certain particularities to improve the results for the rest of the study.

Unless otherwise indicated, the figures in this section represent the dimensionless pressure as a function of the dimensionless phase variable at fixed $\tau$.

#### 3.4.1 Comparison of results with analytic solutions

##### 3.4.1.1 “N” Wave

Figures 3.4.1 and 3.4.2 compare the results of the numerical code with the analytical solutions at $\tau = 0$ and $\tau = 1$. The calculations were performed with 3500 discretization points over the interval [-1, 1] of the $\tau$ axis and 1024 frequencies over the interval [-8/3, 11/3] of the $\tau$ axis, or 3,584,000 degrees of freedom. On the caustic (at $\tau = 0$), we compare the solution of the numerical code (in blue) with Seebass’ exact solution (in green). We see that the curves are almost identical. At $\tau = 1$, the comparison is made with the Fourier transform of the Airy function (in red); the curves are also very close to each other. We also observe small oscillations upstream of the shocks in the numerical results; They are the Gibbs oscillations that result from the discrete Fourier transform (FFT) of a discontinuous signal.

In figures 3.4.3-3.4.8, we have enlarged the comparison between the analytical solution (in red), obtained with the Fourier transform of the Airy function, and the solutions of the numerical code (in blue). For this, we have widened the interval of the $\tau$ axis and the phase variable $\tau$: $\tau \in [-1, 1.5]$ and $\tau \in [-3.225, 4.225]$. However, the number of discretization points in $\tau$ and $\tau$ has been retained. We note that for the chosen values of $\tau$, the results almost completely overlap on the caustic and on either side of it.

On these same curves, we may also observe the expected physics of the phenomenon. Thus, at $\tau = 1.5$, the incident “N” signal and the reflected “U” signal (Hilbert transform of the incident signal) are well differentiated. While the “N” wave has shocks, the reflected signal possesses very narrow finite peaks. In reality, this signal is truncated by the discretization of the Fourier space which filters the high frequencies; it corresponds well with the expected reflected signal with infinite peaks, according to linear theory. As we move closer to the caustic (cf. curves associated with $\tau = 1$ and $\tau = 0.5$), we observe the coalescence of the two signals until they do not form more than a single signal on the caustic (at $\tau = 0$). For negative $\tau$ values, the curves are very regular - there are no more shocks or peaks - and we may observe the decrease in the amplitude of the signal as we move away from the caustic; we are well into the shadow zone in the presence of an evanescent wave.
Figure 3.4.1 and 3.4.2: Code validation in the linear case at $\mathcal{E} = 0$ and at $\mathcal{E} = 1$

- **Blue**: Numerical solution
- **Green**: Exact solution of Seebass
- **Red**: Analytical solution (Fourier transform of the Airy function)
Figures 3.4.3-3.4.8: Code validation in the linear case for an “N” wave, at:

\( \bar{\zeta} = -0.8 \quad \bar{\zeta} = -0.5 \quad \bar{\zeta} = 0 \quad \bar{\zeta} = 0.5 \quad \bar{\zeta} = 1 \quad \bar{\zeta} = 1.5 \)

Numerical solution
Analytical solution
3.4.1.2 Periodic sawtooth Figures 3.4.9-3.4.14 compare the results from the numerical code (in blue) with the analytic solution derived from the Fourier transform of the Airy function (in red). The calculations were performed with 6000 discretization points over the interval [-1, 1] of the $z$ axis and 1024 frequencies over the interval [-1, 1] of the $\tau$ axis, or 6,144,000 degrees of freedom. As for the “N” wave, we see that the analytical and numerical solutions overlap completely for all distances to the caustic $z$ shown. We may also observe the physics of the phenomenon on these curves. At $z = 1$, the incident signal, the periodic sawtooth, is well separated from the reflected signal which here is shaped like a very narrow “I”. As we get closer to the caustic, these two signals coalesce to form, on the caustic, a single signal. Finally, as in the the “N” wave case, for $z = -0.5$, we observe a smooth signal, of much weaker amplitude than for the other dimensions, which corresponds to the evanescent wave in the shadow zone.

In the case of two complex signals, with 1 or 2 shocks, the numerical simulation perfectly reproduces the analytical solution, so we can say that the numerical code is fully validated in the linear case.

3.4.2 Influence of the number of frequencies

Figures 3.4.15a and 3.4.15b demonstrate the influence of the number of frequencies chosen on the shape of the pressure signal at $z = 0$ (that is to say exactly on the caustic). The calculation have been performed over the interval [-1, 1] of the $z$ axis and over the interval [-8/3, 11/3] of the $\tau$ axis. We remark, first of all, that at a fixed number of discretization points (12,000 points) of the $z$ axis, the number of frequencies chosen influences the amplitude of the shocks. The value of each maximum pressure on the caustic is indicated in table 3.1. We can see that at 512 frequencies (in blue), the first peak is at 3.5 whereas at 1024 frequencies (in red), the value of this peak is a little over 4. If we double the number of frequencies again (in green) the pressure almost reaches the value of 5 while at 4096 frequencies (in black) it is slightly greater than this value. Thus as we increase the number of frequencies, the amplitude of the peaks of the “U” become more significant; this is consistent with the theory which predicts peaks of infinite amplitude for an infinite frequency spectrum. However, with the discretization of the Fourier space filtering highs frequencies, numerical signals will always be finite. We note, furthermore, that the maximum pressure is not on the caustic but just above it in the hyperbolic region. However, in view of the table, we note that the more the number of frequencies increases, the more the position of this maximum pressure approaches the caustic. This leads us to suppose that if we had an infinite frequency range, this maximum would be found exactly on the caustic. This is consistent with the finding of Seebass (1971) that the signal possesses singularities of logarithmic type over the entire computational domain, with the exception of the caustic on which it has a maximum singularity of power -1/6. Furthermore, by observing the maximum pressure values over the entire calculation domain, we note that these are on average one unit greater than the corresponding maximum values on the caustic.
Figures 3.4.9-3.4.14: Code validation in the linear case for a periodic sawtooth wave, at:

\[ \eta = -0.5 \quad \eta = 0 \quad \eta = 0.3 \],  
\[ \eta = 0.5 \quad \eta = 0.7 \quad \eta = 1 \]

- Numerical solution
- Analytical solution
Figures 3.4.15a and b: Influence of the number of frequencies on the form of the signal at $\Xi = 0$
- black: 4096 frequencies and 12000 points on the $\Xi$ axis
- green: 2048 frequencies and 12000 points on the $\Xi$ axis
- red: 1024 frequencies and 12000 points on the $\Xi$ axis
- blue: 512 frequencies and 12000 points on the $\Xi$ axis

Figures 3.4.16 and 3.4.17: Position of maximum pressure for 1024 frequencies (to the left) and 4096 frequencies (to the right)
- red: pressure signature at the distance for which the maximum pressure was found
- blue: pressure signature on the caustic
Figures 3.4.16 and 3.4.17 compare the form of the pressure signal on the caustic (in blue) and the distance from it for which the maximum pressure was found (in red), for 1024 (left) and 4096 (right) frequencies. These graphs confirm, for the two numbers of frequencies chosen, the results of the table concerning the difference between the maximum pressure on the whole computational domain and on the caustic. We note, furthermore, that the shape of the pressure signal for the two distances, is a little different for 1024 frequencies whereas the two pressure profiles for 4096 are nearly identical. This confirms the fact that the more we increase the number of frequencies, the closer the maximum pressure gets to the caustic.

### 3.4.3 Influence of the number of discretization points of the \( z \) axis

Figures 3.4.18a and 3.4.18b demonstrate the influence of the number of discretization points of the \( z \) axis on the shape of the signals. The calculations were performed on the interval \([-1, 1]\) of the \( z \) axis and on the interval \([-8/3, 11/3]\) of the \( \tau \) axis. First of all, we can see from these figures, that the number of \( z \) points has only a very slight effect on the amplitude of the signals. However, the number of points does influence the more or less marked oscillations of the pressure profiles. We also note that it is not enough to increase the number of points to make the oscillations disappear. So for 1024 frequencies, we observe in figure 3.4.18b, that the amplitude of the oscillations is as high for 2500 points (in blue) as for 4500 points (in green) while this is reduced by more than half for 3500 points (in red).

The amplitude of these oscillations indicating, in a way, an unoptimized discretization choice, we can assume that this could have consequences on the speed of convergence of the code in the nonlinear case. In light of these observations, the optimal number of discretization points on the \( z \) axis which could minimize these oscillations was sought.

In the case of an incident “N” wave, the minimum of the temporal pressure signal on the caustic is found between the two shocks. For 5 numbers of frequencies (256, 512, 1024, 2048 and 4096), we have plotted four pressure differences: \( \Delta p_i = |\bar{p}_a(\tau_{i+1}) - \bar{p}_a(\tau_i)| \), between two consecutive instants surrounding the minimum pressure, as a function of the number of discretization points of the \( z \) axis. Near the minimum pressure, the curve being close to the horizontal, we can consider these differences to be proportional to the amplitude of the oscillations. The approach taken for this study is described in figure 3.4.19 and its results can be found in figures 3.4.20 to 3.4.24. The variation of the maximum pressure over the whole computational domain as a function of the number of discretization points (the gray curve) has been added to these graphs. For the 5 numbers of frequencies chosen, we note that after an apparently random evolution that is different for each of the oscillations, the amplitude follows exactly the same evolution, which suggests that these

<table>
<thead>
<tr>
<th>number of frequencies</th>
<th>maximum pressure on the caustic</th>
<th>maximum pressure over the entire domain</th>
<th>distance to the caustic of the maximum pressure</th>
</tr>
</thead>
<tbody>
<tr>
<td>512</td>
<td>3.469</td>
<td>4.449</td>
<td>3.910^-3</td>
</tr>
<tr>
<td>1024</td>
<td>4.071</td>
<td>5.177</td>
<td>2.510^-2</td>
</tr>
<tr>
<td>2048</td>
<td>4.799</td>
<td>6.079</td>
<td>1.510^-2</td>
</tr>
<tr>
<td>4096</td>
<td>5.089</td>
<td>6.251</td>
<td>1.010^-2</td>
</tr>
</tbody>
</table>

Table 3.1: influence of the number of frequencies on the maximum pressure
oscillations are not numerical noise. The amplitude of these oscillations passes through minima whose number increases with the number of frequencies.

So for 256, 512 and 1024 frequencies, there are 2 minima, while for 2048 frequencies, there are 3 clearly defined minima. The problem then is to choose the appropriate minimum. Having no reference on this subject, we suppose that it must be as close as possible to the maximum of the pressure maxima (gray curve on the 5 graphs). We remark that for 256, 512, 1024 and 2048 frequencies, this maximum pressure is always very close to the last of the amplitude oscillation minima. For each frequency, the number of points on the $\tau$ axis, associated with the maximum pressure, $Nb\tau^{max}(p_a)$, (3rd column of table 3.2), is close to the number of points on the $\tau$ axis associated with the last minimum of the amplitude of the oscillations, $Nb\tau^{min(oisol.)}$ (5th column). For each number of frequencies, this last quantity therefore appears as the optimal number of points to choose on the $\tau$ axis. For 4096 frequencies, it would seem that there are not enough points to obtain all the existing minima and the maximum pressure variation curve does not seem complete; we cannot conclude for this number of frequencies.

In addition, taking inspiration from the law linking the boundary layer thickness of the $\delta$ to the angular frequency $\omega$, $\delta(\omega) \approx c^{2/3}R_1^{1/3} \omega^{2/3}$, we have attempted to demonstrate that the optimal discretization step, $\Delta\tau$, of the $\tau$ axis, was linked to the discretization step $\Delta\tau$, of the $\tau$ axis, by a 3/2 power law. To show this, we have calculated the ratio $\Delta\tau/\Delta\tau^{3/2}$ from previous results; the discretization step $\Delta\tau$, along the $\tau$ axis, being obtained from the numbers of frequencies, 256, 512, 1024 and 2048, over the interval $[-8/3, 11/3]$. The discretization step along the $\tau$ axis, $\Delta\tau$, for this calculation was chosen, on the one hand, from the maximum of the maximum pressure over the whole domain, $\Delta\tau^{max}(p_a)$, and on the other hand, from the last minimum of the oscillations, $\Delta\tau^{min(oisol.)}$. These ratios are the subject of the last two columns of Table 3.2. We see that these two ratios, for each number of frequencies, are almost equal to 1; Therefore, there is indeed a relationship of the form, $\Delta\tau \propto \Delta\tau^{3/2}$, linking the discretization steps of space and time.

<table>
<thead>
<tr>
<th>nbr freq</th>
<th>$\Delta\tau^{3/2}$</th>
<th>$Nb\tau^{max}(p_a)$</th>
<th>$\Delta\tau^{max}(p_a)$</th>
<th>$Nb\tau^{min(oisol.)}$</th>
<th>$\Delta\tau^{min(oisol.)}$</th>
<th>$\Delta\tau^{max}(p_a)/(\Delta\tau^{3/2})^{3/2}$</th>
<th>$\Delta\tau^{min(oisol.))/(\Delta\tau^{3/2})^{3/2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>256</td>
<td>0.0247396</td>
<td>506</td>
<td>0.0039526</td>
<td>500</td>
<td>0.004</td>
<td>1.0158</td>
<td>1.0279</td>
</tr>
<tr>
<td>512</td>
<td>0.0123998</td>
<td>1300</td>
<td>0.0015935</td>
<td>1250</td>
<td>0.0016</td>
<td>1.1183</td>
<td>1.163</td>
</tr>
<tr>
<td>1024</td>
<td>0.0061849</td>
<td>3500</td>
<td>0.0005714</td>
<td>3500</td>
<td>0.0006</td>
<td>1.1747</td>
<td>1.1747</td>
</tr>
<tr>
<td>2048</td>
<td>0.0030024</td>
<td>9600</td>
<td>0.0002083</td>
<td>10000</td>
<td>0.0002</td>
<td>1.2113</td>
<td>1.163</td>
</tr>
</tbody>
</table>

Table 3.2 : synthesis of the study on the search for the optimal number of discretization points on the $\tau$ axis

In this section, we have seen that the number of points on the $\tau$ axis should not be chosen at random. Indeed, the optimal discretization step $\Delta\tau$, of the space variable, and the discretization step $\Delta\tau$ of the phase variable, are correlated by a 3/2 power law, in accordance with the evolution law of the diffraction boundary layer thickness as a function of frequency.
Figures 3.4.18a and b: Influence of the number of points chosen on the \( \Xi \) axis, at \( \Xi = 0 \)

- **4500 points on the \( \Xi \) axis and 1024 frequencies**
- **3500 points on the \( \Xi \) axis and 1024 frequencies**
- **2500 points on the \( \Xi \) axis and 1024 frequencies**

Figure 3.4.19: Definition of \( \Delta p_i \) for the study of the optimum number of points in the \( \Xi \) direction
Figure 3.4.20: Influence of the number of $z$ axis points for 256 frequencies
- on the amplitude of the oscillations formed by 4 pairs of points located near the minimum of the “U” wave
- on the maximum pressure value of the whole domain

Figure 3.4.21: Influence of the number of $z$ axis points for 512 frequencies
- on the amplitude of the oscillations formed by 4 pairs of points located near the minimum of the “U” wave
- on the maximum pressure value of the whole domain
Figure 3.4.22: Influence of the number of \( \bar{z} \) axis points for 1024 frequencies
- on the amplitude of the oscillations formed by 4 pairs of points located near the minimum of the “U” wave
- on the maximum pressure value of the whole domain

Figure 3.4.23: Influence of the number of \( \bar{z} \) axis points for 2048 frequencies
- on the amplitude of the oscillations formed by 4 pairs of points located near the minimum of the “U” wave
- on the maximum pressure value of the whole domain
3.4.4 Improvement of the boundary conditions at plus and minus infinity

In this section, we will show how a better discretization of the boundary conditions can improve the results. The calculations have been performed with 3500 discretization points over the interval \([-1, 1]\) of the \(z\) axis and 1024 frequencies over the interval \([-8/3, 11/3]\) of the \(\tau\) axis. The curves associated with this section all represent the pressure field on the caustic as a function of the phase variable.

### 3.4.4.1 Boundary conditions at positive infinity

Figures 3.4.26a-c compare the results obtained by discretizing the boundary condition at positive infinity, on the one hand, to 1\(^{st}\) order and, on the other hand, to 2\(^{nd}\) order. The red curve corresponds to a 1\(^{st}\) order discretization. We remark that this curve has 2 entirely localized artifacts, which we have enlarged in figures 3.4.26b and c. They have a completely identical profile and seem to appear on the curve according to a particular law. In figure 3.4.25, the incident and reflected characteristics associated with the front and rear shock of the "N" wave are plotted in solid black line. The dotted curves are associated with artificially reflected characteristics on the upper edge of the computational box. We note that the intersections of these 2 characteristics with the caustic coincide exactly with the appearance of the two artifacts just described above. This shows that these artifacts are closely related to stray reflection of outgoing shock waves, stray reflection from insufficient discretization of the boundary condition on the upper edge of the computational domain.
The amplitude of these artifacts is drastically reduced with the use of 2nd order boundary conditions at positive infinity (in blue), which is consistent with the precision of the scheme in the rest of the domain. So, when we discretize the boundary conditions more precisely at the edge of the computational box, there are fewer waves numerically reflected towards the caustic, which decreases or eliminates the presence of numerical oscillations.

3.4.4.2 Boundary conditions at negative infinity

Figures 3.4.27a and 3.4.27b compare the results obtained, on the one hand, with the boundary condition at negative infinity, called the evanescence condition, with, on the other hand, the zero boundary condition.

In view of these figures, we can see that the pressure field on the caustic, obtained with the evanescence condition (in blue) applied on the interval [-1, 1], is almost identical to the curve (in red) corresponding to a calculation without the evanescence condition over the interval [-2, 1]. In the case without the evanescence condition, the discretization interval is wider for the same number of points, which explains the presence of oscillations with respect to the case where we have used the evanescence condition. The green curve corresponds to the result of a calculation carried out on the computational box [-1, 1] without the evanescence condition; the signal looks different from the two previous curves: we do not go far enough into the shadow zone to completely capture the evanescent signal.

Finally, we note that the evanescence condition allows us to reduce the boundary conditions at negative infinity to about half of the distance to the caustic, for the same computational precision. So with the same number of points on the \( \tau \) axis, we have a precision two times greater in the computational box, that is to say the shadow zone is reduced by half.

3.4.5 Conclusion

In this section, we have first shown that the numerical code was fully validated in the linear case. For this, we have compared the results of the numerical code, in the case of 2 incident signals with one and two shocks, with the corresponding analytical solution. We have thus seen that there was perfect agreement between the solutions.

Next, we have shown that a correlation between the discretization step of the distance to the caustic and that of the phase variable axis exists, according to the relationship \( \Delta \tau \propto \Delta \tau^{3/2} \), in agreement with the theory.

Finally, we have shown that discretizing the boundary condition at positive infinity to 2nd order and adding an evanescence condition at negative infinity significantly improved the quality of the results.
Figure 3.4.25 (at the top) : Characteristics associated with the wave front in the hyperbolic part
Figures 3.4.26a, b and c : Influence of the discretization order of the boundary condition at positive infinity on the form of the signal

- 2nd order
- 1st order
PRESENTATION OF THE NUMERICAL SCHEME

Figure 3.4.27a and b: Influence of the evanescence condition on the form of the signal

- Red: [-2; 1] interval on the $\tilde{z}$ axis
- Green: [-1; 1] interval on the $\tilde{z}$ axis
- Blue: Evanescence condition on the [-1; 1] interval of the $\tilde{z}$ axis
3.5 SOLUTION OF THE UNSTEADY TRICOMI EQUATION

In this second section, we will present the method for solving the unsteady Tricomi equation. The construction of an iterative scheme is not used to obtain particular results in the linear case, because the scheme must converge to the stationary solution obtained in the previous section, but to prepare to solve the nonlinear problem.

As in the previous section, we will first calculate the Fourier transform of the unsteady linear Tricomi equation in the \( \tau \) direction and then solve the new equation obtained by finite differences. In the unsteady linear case, the problem to solve is then the following:

\[
\frac{\partial^2 p_a}{\partial t \partial \tau} = \frac{\partial^2 p_a}{\partial z^2} - \frac{z}{2} \frac{\partial^2 p_a}{\partial \tau^2},
\]

with boundary conditions identical to the stationary case:

\[
p_a(\tau \to \pm \infty, z) = 0,
\]

\[
p_a(\tau, z \to -\infty) = 0,
\]

\[
\frac{z^1}{4} \frac{\partial p_a}{\partial \tau} + \frac{z^2}{1} \frac{\partial p_a}{\partial z} + \frac{z^3}{4} \frac{p_a}{4} = 2F' \left( \tau + \frac{2}{3} \frac{z^1}{2} \right) \text{ as } z \to +\infty.
\]

3.5.1 Fourier Transform of the unsteady linear Tricomi equation and the boundary conditions

By adopting the definitions of the Fourier transform and its inverse transform introduced in the previous section:

\[
P(\tau, z, \omega) = \int_{-\infty}^{\infty} p_a(\tau, \tau, z) e^{-i\omega \tau} d\tau \quad \text{and} \quad p_a(\tau, \tau, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{P}(\tau, z, \omega) e^{i\omega \tau} d\omega,
\]

the Fourier transform of the stationary linear Tricomi equation is then written in the following way:

\[
i\omega \frac{\partial \hat{P}}{\partial \tau} = \frac{\partial^2 \hat{P}}{\partial z^2} + \frac{z}{2} \omega^2 \hat{P}.
\]

The Fourier transform of the boundary conditions at plus and minus infinity are identical to the stationary case, therefore it is:

\[
i\omega \frac{z^1}{4} \hat{P} + \frac{z^2}{1} \frac{\partial \hat{P}}{\partial z} + \frac{z^3}{4} \frac{\hat{P}}{4} = 2i\omega e^{\frac{2}{3}i\omega \frac{z^1}{2}} \hat{F}(\omega) \text{ as } z \to +\infty,
\]

\[
\frac{\partial \hat{P}}{\partial z} - \left[ \frac{1}{4} |z|^{-1} + |\omega||z| \right] \hat{P} = 0 \text{ as } z \to -\infty.
\]
3.5.2 Discretization of the problem

Let us discretize [FULTE] to obtain an iterative scheme in time. For this we will replace the partial derivative with respect to $t$ by an uncentered first-order accurate finite difference. To avoid stability problems, we choose an implicit scheme. We then write [FULTE] as follows:

$$i\omega \left( \frac{\hat{P}_n^{n+1} - \hat{P}_n^n}{\Delta t} \right) = \frac{d^2 \hat{P}_n^{n+1}}{d\bar{z}^2} + \bar{z} \omega^2 \hat{P}_n^{n+1},$$

which gives:

$$\left( \frac{d^2}{d\bar{z}^2} - \frac{i\omega}{\Delta t} + \omega^2 \bar{z} \right) \hat{P}_n^{n+1} = -\frac{i\omega}{\Delta t} \hat{P}_n^n,$$

where $\hat{P}_n = \hat{P}(\omega, \bar{z}, n\Delta t)$, with $\Delta t$ the iteration time step.

We will now discretize the partial derivatives along the $\bar{z}$ axis, the index $j$ taking the values from 1 to $M$ on this axis; we then write: $\hat{P}_j^n = \hat{P}(\omega, z_j, n\Delta t)$, where $z_j = z_0 + j\Delta z$. In [FULTE], the second derivative in $\bar{z}$ will always be discretized according to a centered second-order accurate scheme:

$$\frac{d^2 \hat{P}_j}{d\bar{z}^2} = \frac{\hat{P}_{j-1} - 2\hat{P}_j + \hat{P}_{j+1}}{\Delta z^2} + O(\Delta z^2).$$

For the boundary conditions at plus and minus infinity, the expressions found in the stationary case remain valid for an iterative scheme, thus:

$$\frac{d\hat{P}_j}{d\bar{z}} = \frac{3\hat{P}_j - 4\hat{P}_{j-1} + \hat{P}_{j-2}}{2\Delta z} + O(\Delta z^2) \text{ at positive infinity}$$

and

$$\frac{\hat{P}_{j+1} - \hat{P}_j}{\Delta z} = \left[ \frac{1}{4|\bar{z}|} + |\omega||\bar{z}|^{1/2} \right] \hat{P}_j + O(\Delta z), \text{ at negative infinity.}$$

3.5.3 Matrix representation of the problem

Finally, the system to be solved is written in the form of a matrix system:

$$A(k) \hat{P}_n^{n+1} = B \left( \hat{P}_n^n \right), \quad (3.1)$$

with $A$ a tetradiagonal matrix of size $M \times M$ that decomposes as follows:

$$A = A^{(0)} + \omega A^{(1)} + \omega^2 A^{(2)}. \quad (3.2)$$

The matrices $A^{(0)}$ and $A^{(2)}$ are unchanged with respect to the linear case, only the index “∞” of these matrices has disappeared.
The new matrix $A^{(1)}$ is written:

$$A^{(1)} = i\begin{bmatrix}
0 & 0 & \cdots & \cdots & \cdots & 0 \\
0 & -1/\Delta t & \cdots & \cdots & \cdots & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & 0
\end{bmatrix} \frac{1}{z_M}$$

The right hand side has the following form:

$$B\left(\hat{P}^n\right) = \begin{bmatrix}
0 \\
-\frac{i\omega}{\Delta t} \hat{P}_2^n \\
\vdots \\
-\frac{i\omega}{\Delta t} \hat{P}_j^n \\
\vdots \\
-\frac{i\omega}{\Delta t} \hat{P}_{M-1}^n \\
2i\omega e^{\frac{i\omega}{\Delta t} z_M^2} \hat{F}(\omega)
\end{bmatrix}.$$ 

To solve the system $A(\omega) \hat{P}^{n+1} = B\left(\hat{P}^n\right)$, we use, as before, the Thomas algorithm (Appendix I) which still seems to be the most suitable for this problem.

Finally, to find the solution $p^{n+1} = p^{n+1}(\tau, z)$ in physical space, we must calculate the inverse Fourier transform of $\hat{P}^{n+1} = A^{-1}(\omega)B\left(\hat{P}^n\right)$, that is: $p^{n+1} = FT^{-1}\left[A^{-1}(\omega)B\left(\hat{P}^n\right)\right]$.

### 3.5.4 Validation of the method in the unsteady linear case

In this section, we will demonstrate that the iterative scheme, constructed from the stationary linear Tricomi equation, is validated. For this, we have, for each frequency, calculated the 2-norm of the difference between the stationary and unsteady solution in Fourier space. The calculations have been performed with 3500 discretization points over the interval $[-1, 1]$ of the $z$ axis and 1024 frequencies over the interval $[-8/3, 11/3]$ of the $\tau$ axis. The iteration time step, $\Delta t$, in the nonlinear case, is imposed by the stability condition of the McDonald and Ambrosiano scheme (1984), used to solve the Burgers' equation. The linear case is, in contrast, unconditionally stable. So for the results presented below, the iteration step is arbitrarily chosen as:

$$\Delta t = \Delta \tau / 0.9.$$ 

The results of these calculations are shown in figure 3.5.1. We can see that, starting from an initial zero solution, the iterative scheme converges toward the stationary linear solution. We note that
each norm, linked to a frequency between 1 and 508 Hertz (in blue), converges fairly regularly toward a value between $10^{-8}$ and $10^{-11}$, corresponding to the precision of finite-difference schemes. The set of norms associated with these different frequencies converges identically with the same speed. However, for high frequencies, for which the discretization is less precise, convergence is slower.

![Figure 3.5.1: convergence of the linear scheme](image)

### 3.5.5 Solution of Burgers’ equation

In this paragraph, let us present the solution method for the “nonlinear” part of the Tricomi equation in the temporal space.

On each line, that is to say at $\tau = \text{constant}$, we are led to solve the following equation:

$$\frac{\partial \bar{p}_a}{\partial \tau} = \frac{\partial}{\partial \tau} \left( \frac{\mu \bar{p}_a^2}{2} \right),$$

via the McDonald and Ambrosiano scheme (Appendix II).

It is an uncentered finite difference scheme. It is hybrid, 1st order accurate (dissipative in nature) near shocks and 2nd order accurate (dispersive in nature) everywhere else. The scheme is conservative and monotone to 1st order which, according to Lax’s theorem (Dautray and Lions 1985), if it is convergent, ensures it will converge to a physically acceptable solution. The numerical dissipation introduced by the 1st order scheme stabilizes shocks without being too significant, since it is placed just in the vicinity of the shocks. The passage between 1st and 2nd order is done by means of the
numerical filter of Boris and Book (1973), which is intended to prevent the formation of numerical oscillations. The scheme is also explicit, which subjects it to a Courant Friedrichs Levy (CFL) type stability condition, imposing the value of the iteration step $\Delta t$. We will take it to be identical for each $\tau = \text{constant}$ and it will be determined from the linear solution in the following way: 

$$\Delta t = \Delta \tau / 2 \mu \max(p_a).$$

Figures 3.5.2 and 3.5.3 show how two pressure profiles initializing Burgers’ equation evolve. In reality, this will allow us to observe the influence of nonlinear effects on two different pressure profiles.

Figure 3.5.2 shows the evolution of a “U” (in blue) initially having 2 infinite peaks. We can see that with the increase in the number of iterations, the infinite peaks gradually transform into finite-amplitude shocks. So after 1200 iterations we obtain an “N” (in red).

When we start with a sinusoid (in blue), as in figure 3.5.3, we observe the signal gradually steepening (cf. 200 iterations) then after a certain number of iterations, the signal is “shocked.” Beyond the shock formation distance, the signal tends towards a stable sawtooth shape whose amplitude decreases as the signal propagates.

At this stage of the study let us observe two characteristics of nonlinear effects:

- transformation of “infinite” peaks into shocks,

- “shocking” of continuous signals.

Figures 3.5.4a and 3.5.4b show the difference between a purely 1st order scheme (in blue) and the mixed scheme of McDonald and Ambrosiano (in red), after 600 iterations. We can clearly see the main characteristic of the 1st order scheme in figure 3.5.4b, that is to say the numerical dissipation, since with this scheme the amplitude of the shocks is a bit weaker than with that of McDonald and Ambrosiano, and the shocks are spread over a greater number of points. In conclusion, the 2nd order scheme is more appropriate for modeling shocks.
Evolution of solutions of the Burgers equation with an:

**Figure 3.5.2 : “N” wave**

**Figure 3.5.3 : Sinusoidal signal**

- blue: initial solution
- green: 200 iterations
- yellow: 100 iterations
- red: 1200 iterations
- light blue: 600 iterations

**Figures 3.5.4a and b : Influence of the choice of the discretization of the Burgers equation on the form of a shock**

- blue: 1st order discretization
- red: 2nd order discretization
3.5.6 Definition of the iteration step and the convergence criterion for the numerical code

3.4.6.1 Definition of the iteration step $\Delta t$

For all calculations performed in the nonlinear case, the iteration time step, $\Delta t$, was chosen in order to comply with the Courant Friedrich Levy type numerical stability constraint imposed by the McDonald and Ambrosiano scheme. This iteration step is therefore linked to the discretization step of the phase variable in the following way:

$$\Delta t = \Delta \tau / 2 \mu \max (\bar{p}_a).$$

3.4.6.2 Convergence criterion

We will estimate that convergence will be attained numerically if the difference between two successive iterations is less than a given value. We will choose this one:

$$\max |p^n - p^{n-1}| < 0.001.$$

3.5.7 Solution of the nonlinear Tricomi equation

Finally, we can summarize the resolution of the unsteady nonlinear Tricomi equation through the following diagram, by initializing the iterative calculation with the noniterative linear Tricomi solution:
3.5.8 Particularities of the nonlinear case

For the nonlinear case, we will use the signals from the linear case ("N" signal and sawtooth periodic), in the same associated calculation boxes. However, the figure 3.3.3 set out in the previous section is modified in the nonlinear case. Indeed the hyperbolic and elliptical parts are no longer separated by the caustic but by the sonic line with equation $z - \mu p_a = 0$. So the figure associated with the nonlinear case becomes:
In the previous sections, we first validated the code in the stationary linear case and then in the iterative linear case. We have thus seen that the stationary linear solution completely overlapped the analytical solutions obtained by Fourier transform the Airy function. Then, we were able to observe that the unsteady solution converged toward the stationary solution in a regular way.

In this section, we will show that the numerical code is validated in the nonlinear case. As there are no known analytical solutions with which we could have compared the results of the code in the nonlinear case, we will use two indirect methods of validation. The first will be rather qualitative;
PRESENTATION OF THE NUMERICAL SCHEME

It will consist, on the one hand, of analyzing the results to see if they are consistent with what is expected from the influence of nonlinear effects, and, on the other hand, of varying certain parameters of the code to see how the results evolve. The second, more rigorous method will consist in applying Guiraud’s similitude law (1965) to incident signals with only one shock.

As in the linear case, unless otherwise indicated, all figures in this section represent dimensionless pressure as a function of the dimensionless phase variable at fixed \( \tau \). In part 3.6.1, we will work with the two signals that we have presented in part 3.3.7, that is to say the “N” wave and the periodic sawtooth. In the following parts, only the results obtained with the “N” wave will be presented. However, in the last paragraph, we will use the periodic sawtooth as well as another periodic signal with only one shock, during the test of Guiraud’s similitude law.

3.6.1 Comparison of solutions obtained in the linear and nonlinear case

3.6.1.1 “N” Wave

Figures 3.6.2-3.6.7 compare the results of the numerical code in the nonlinear case (in blue) with the solutions obtained in the linear case (in red) for different distances from the caustic. The calculations have been carried out with 3500 discretization points over the interval \([-1, 1.5]\) of the \( \tau \) axis and 1024 frequencies over the interval \([-3.225, 4.225]\) of the \( \tau \) axis. The nonlinear case has been calculated with a coefficient of nonlinearity \( \mu = 0.08 \).

In light of these results, we first note that for the values of \( \tau \) chosen on either side of the caustic and on it, the pressure field in the nonlinear case has the same general appearance as in the linear case. The physics of the phenomenon described in section 3.4.1 is thus perfectly recovered in the nonlinear case. So, at \( \tau = 1.5 \), the incident “N” signal and the reflected “U” signal are well-differentiated.

As we get closer to the caustic (cf. curves associated with \( \tau = 1 \) and \( \tau = 0.5 \)), we observe the coalescence of the two signals until a single signal forms on the caustic (at \( \tau = 0 \)). For negative values of \( \tau \), the curves are very regular and we may observe the decrease in the amplitude of the signal as we move away from the caustic; this is characteristic of the presence of an evanescent wave, in the elliptic region of the Tricomi equation.

We may however observe three particularities associated with nonlinear results. First, we note that, for all distances to the caustic presented on these graphs, the amplitude of the pressure field in the nonlinear case is always lower than that of the corresponding signals in the linear case. The linear solution introduces signals with infinite peaks; the latter, through the introduction of nonlinearities, are transformed into shocks, sometimes very sharp but of finite amplitude.

The second characteristic of nonlinear signals lies in the fact that the finite peak associated with the front shock of the incident “N” wave is always temporally in advance of its linear counterpart. This is consistent with classical nonlinearity phenomena. Thus by adopting the definition of the speed of sound, corrected for nonlinear effects: \( c_{\text{actual}} = c_{\text{linear}} + \beta P_{ac}/\rho_0 c_0 \), \( \beta \) being the parameter of nonlinearity, we can see that the high amplitude portions of the signals have a greater speed than the small amplitude parts. Thus the upstream, high amplitude peaks propagate faster than the rest of the signal.

Finally, the third particularity of nonlinear effects is to displace the pressure maximum into the linear hyperbolic region while it is found on the caustic in the linear case. In the case presented here the maximum pressure of the nonlinear solution is found at a distance to the caustic equal to 0.2 and its amplitude is equal to almost 4 whereas on the caustic it equals 1.5.

This displacement of the maximum pressure is linked to the new spatial distribution of the hyperbolic and elliptic regions in the nonlinear case. They are no longer delimited by the caustic (in red on
figure 3.6.1) but by the sonic line (in black on the same figure) with equation $\overline{\tau} - \mu \overline{\rho}_a = 0$. By its definition, and as we can see in figure 3.6.1, its profile has the shape of the pressure field that we observe on the caustic (cf. Figure 3.6.5). The nonlinear elliptic region (in blue) encroaches on the linear hyperbolic region (red striped area). The maximum pressure over the entire computational domain is always associated with one of the shocks, front or rear, of the incident signal depending on its amplitude; it will therefore continuously be carried by one of the characteristics of the signal. In the case of the “N” wave that we have chosen, the maximum pressure is associated with the front shock, and is therefore carried by the characteristic to which it is attached. Its profile (yellow curve) will be similar in the linear and nonlinear case, when $\mu$ is small. However, this curve will be shorter in the second case, because it stops when it touches the sonic line. The elliptic part being diffusive by definition, the pressure maximum in the nonlinear case can only be found in the nonlinear hyperbolic region or on the sonic line, but always following the characteristic just defined above; this pressure maximum can therefore only be found on the yellow curve drawn in solid line. In reality, the value of the maximum pressure having been found at a distance $\overline{\tau} = 0.2$, it seems that this maximum is indeed on the sonic line. The displacement of the maximum pressure of the entire computational box is therefore consistent with the addition of nonlinear effects.

![Diagram showing distribution of hyperbolic and elliptic regions in the linear and nonlinear case](image)

**Figure 3.6.1 : Distribution of hyperbolic and elliptic regions in the linear and nonlinear case**

Furthermore, the Gibbs oscillations observed in the linear case have completely disappeared in the nonlinear case due to the numerical viscosity introduced by McDonald and Ambrosiano’s scheme. We note also that there are no differences between the linear and nonlinear cases in the shadow zone, which justifies the use of the evanescence condition even in the nonlinear case.

3.6.1.2 Periodic sawtooth Figures 3.6.8-3.6.13 compare the results of the numerical code in the nonlinear case (in red) with the solutions obtained in the linear case (in blue) for different distances to the caustic. Calculations have been performed with 6000 discretization points over the interval [-1, 1] of the $\overline{\tau}$ axis and 1024 frequencies over the interval [-1, 1] of the $\overline{\tau}$ axis. The nonlinear case has been calculated with a coefficient of nonlinearity, $\mu$, equal to 0.05.

As for the “N” wave, we see that the linear and nonlinear solutions look the same for all distances to the caustic $\overline{\tau}$ presented. The physics of the phenomenon described part 3.4.1 is thus, as for the “N” wave case, perfectly recovered in the nonlinear case. At $\overline{\tau} = 1$, the incident signal, the periodic sawtooth, is well separated from the reflected signal which here has the shape of an “I”. As we approach the caustic these two signals coalesce to form, on the caustic, a single signal.
The three particularities, associated with the nonlinear case, observed and described for the “N” wave, are even more glaring in the sawtooth case. Indeed, for all distances to the caustic considered, the amplitude of the nonlinear field is always less than that corresponding to the linear case. Furthermore, the upstream shock in the nonlinear case is always advanced in time with respect to its linear counterpart for all distances to the caustic presented, and the maximum pressure is not on the caustic but at a dimensionless distance $\bar{z} = 0.17$.

Finally, for the two signals presented above, the general shape of the pressure profiles in the nonlinear case is identical to that of the profiles of the linear case. However, three things particularly characterize the nonlinear case:

- the infinite peaks of the signals are transformed into lower-amplitude shocks,
- the front shock is always advanced in time with respect its linear counterpart,
- the maximum pressure has moved away from the caustic.

In view of these initial results, we can already say that they are consistent with what we expected in the nonlinear case.
Figures 3.6.2-3.6.7: Comparison of the linear and nonlinear cases for an “N” wave with $\mu = 0.08$ for the following distances to the caustic:

$\overline{z} = -0.5 \quad \overline{z} = 0 \quad \overline{z} = 0.2 \quad \overline{z} = 0.5 \quad \overline{z} = 1 \quad \overline{z} = 1.5$

- Nonlinear solution
- Linear solution
Figures 3.6.8-3.6.13: Comparison of the linear and nonlinear cases for a periodic sawtooth, with $\mu = 0.05$ for the following distances:

- $\bar{z} = 0$
- $\bar{z} = 0.17$
- $\bar{z} = 0.3$
- $\bar{z} = 0.5$
- $\bar{z} = 0.7$
- $\bar{z} = 1$

- Linear solution
- Nonlinear solution
3.6.2 Influence of the value of the coefficient of nonlinearity \( \mu \)

Figures 3.6.14 and 3.6.15 demonstrate the influence of the coefficient \( \mu \) on the shape of the signals at \( z = 0 \) and at \( z = 1 \). The calculations have been performed with 1250 discretization points over the interval \([-1, 1]\) of the \( z \) axis and 512 frequencies over the interval \([-8/3, 11/3]\) of the \( \tau \) axis. The blue curve corresponds to the linear case. We remark that the more the value of \( \mu \) increases, the further the calculated solution deviates from the linear solution. Thus when \( \mu = 0.01 \) (in green), the amplitudes of the “U” peaks are still quite strong and they are positioned in almost the same place as those in the linear case. On the other hand, for \( \mu = 0.1 \) (in red), the peaks have clearly decreased in amplitude and have moved away from those associated with the weaker or zero \( \mu \). This result extends the qualitative validation of the code in the nonlinear case, since the larger the coefficient \( \mu \), the greater the influence of the nonlinear effects on the signals, as described in the previous section. In the next chapter, devoted to the exploitation of the code, the action of \( \mu \) on the solutions will be detailed further.

3.6.3 Influence of the initial estimate

The calculations in this part have all been carried out with 1250 discretization points over the interval \([-1, 1]\) of the \( z \) axis and 512 frequencies over the interval \([-8/3, 11/3]\) of the \( \tau \) axis. The coefficient of nonlinearity, \( \mu \), is, here, equal to 0.1.

Figures 3.6.16-3.6.19 show the evolution of the pressure profile, on the caustic (on the left) and at \( z = 1 \) (on the right), when we take two different initial conditions: the zero solution (top) and the linear solution (bottom). We first remark that, in the case where we take as initial solution the linear solution, that the amplitude of the peaks decreases as the iterations increase. Furthermore, we see that the front shock gradually deviates from the initial signal. When we choose as the initial solution the zero solution, the evolution of the pressure profile is more turbulent; starting from a zero value, the pressure passes through a maximum, before going back down to stabilize. Additionally, the boundary condition at positive infinity influences the shape of the pressure profile from the first iterations, because at 50 (in yellow) and 100 (in green) iterations, it already has the form of an “N”. Finally, as we can see from the four graphs, the curves corresponding to 500 and 800 iterations can hardly be distinguished. The code has converged toward a stable solution, starting from two very different initial conditions.

Figures 3.6.20 and 3.6.21 superimpose the results obtained starting from the two initial conditions mentioned above: the linear solution and the zero solution, on the caustic and at \( z = 1 \), after convergence. The curves coincide completely. However, the number of iterations obtained for convergence is very different. The convergence criterion being as follows:

\[
\max |p_a^n - p_a^{n-1}| < 0.001,
\]

when we start from the linear solution, convergence is obtained after 444 iterations, whereas initializing the calculation with the zero solution, we need 704 iterations. Starting from an initial profile close to the final solution accelerates the convergence of the code and therefore decreases the computation time, as one might expect.

Figures 3.6.22-3.6.25, compare, on the caustic, the results of the code obtained after convergence (in blue) with the solutions that initialized the calculation (in red). We have chosen 4 very distinct functions:
- the linear solution
- the opposite of the linear solution
- the zero function
- an arbitrary function defined in the following way:

\[
\begin{align*}
\overline{p}_a(\tau, \z) &= 1 \quad \text{if} \quad |\tau| < 0.5, \\
\overline{p}_a(\tau, \z) &= 0 \quad \text{otherwise}
\end{align*}
\]

We can see, as before, that the final solutions are all completely identical. However by preserving the convergence criterion just used above, we note that we obtain convergence of the numerical code after 760 iterations if we initialize the code with the opposite of the linear solution and 5149 iterations if we initializes the calculation with the arbitrary function above.

These three studies have highlighted:

- first of all, that the code always converges to the same solution for, \textit{a priori}, any initial solution, whether it has compression or expansion shocks or infinite peaks,
- that it seems judicious to initialize the calculation with the linear solution, the latter having a profile similar to the solution after convergence.

These tests therefore demonstrate the absolute robustness of the code.
Figures 3.6.14 and 3.6.15: Influence of the value of $\mu$ on the signal at the caustic and at $\mathcal{F} = 1$

- **linear solution**
- $\mu = 0.01$
- $\mu = 0.1$
Figures 3.6.16-3.6.19: Comparison of results with two different initial conditions, on the caustic (to the left) and at $z = 1$ (to the right). The curves at the top correspond to a zero initial condition and the curves at the bottom to an initial solution equal to the linear solution.

- **Blue** line: 10 iterations or linear solution for the curves at the bottom
- **Yellow** line: 50 iterations
- **Green** line: 100 iterations
- **Light blue** line: 500 iterations
- **Red** line: 800 iterations
Figures 3.6.20 and 3.6.21: Influence of the initial condition on the form of the curve at $\mathcal{F} = 0$ and $\mathcal{F} = 1$

- initial solution = linear solution
- initial solution = zero solution
\[
\begin{cases}
\bar{p}(\bar{t}, \bar{z}) = 1 \quad \text{if} \quad |\bar{t}| < 0.5, \\
\bar{p}_a(\bar{t}, \bar{z}) = 0 \quad \text{otherwise}.
\end{cases}
\]

Figures 3.6.22-3.6.25: Influence of the initial condition on the form of the signal at \( z = 0 \)

- **Initial solution**
- **Final solution**
3.6.4 Influence of a new distribution of the terms of the Tricomi equation in the time and frequency domains

In this part we will compare the results obtained by the classical use of the numerical code with those obtained by redistributing the terms of the nonlinear Tricomi equation in the frequency and time domains. Therefore, we have removed the term $z\frac{\partial^2 p_a}{\partial \tau^2}$ from the unsteady linear Tricomi equation to add it to the Burgers’ equation. This new arrangement then leads us to solve the following equation in Fourier space:

$$\frac{\partial^2 p_a}{\partial \tau \partial \tau} = \frac{\partial^2 p_a}{\partial z^2}.$$

As for the equation relating to nonlinear effects, it is augmented by a term as follows:

$$\frac{\partial p_a}{\partial \tau} = \frac{\partial}{\partial \tau} \left( \mu \frac{p_a^2}{2} - z p_a \right).$$

The results are the subject of figures 3.6.26 and 3.6.27; the red curves correspond to the new arrangement while the blue curves correspond to the initial arrangement. The calculations have been performed with 3500 discretization points over the interval [-1, 1] of the $z$ axis and 1024 frequencies over the interval [-8/3, 11/3] of the $\tau$ axis, at $z = 0$ and $z = 1$.

We can see, in the light of these figures, that the rearrangement does very little to disturb the final solution; the curves overlap almost perfectly. We only note that, in the case of the new arrangement, the amplitude of the shocks is lower, especially at $z = 1$. This is explained by the new definition of the augmented Burgers equation. The term $\mu \frac{p_a^2}{2}$ is small because it comes from a product with $\mu$, when the value of $z$ increases, $zp_a$ becomes dominant before the term $\mu \frac{p_a^2}{2}$, so that far from the caustic, the term has a tendency to $zp_a$ numerically “squash” the term $\mu \frac{p_a^2}{2}$; this explains the drop in the pressure amplitude in the case of the new arrangement, which turns out to be a bit less precise.

These results once again confirm the robustness of the pseudo-spectral scheme and the convergence of the method based on the unsteady nonlinear Tricomi equation.
Figures 3.6.26 and 3.6.27: Influence of the arrangement on the signal at \( \bar{z} = 0 \) and at \( \bar{z} = 1 \)

- classic arrangement
- novel arrangement
3.6.5 Influence of the number of frequencies

Figures 3.6.28 (a and b)-3.6.30 (a and b) demonstrate the influence of the number of frequencies on the shape of the signals at \( \bar{z} = 0 \), \( \bar{z} = 0.5 \) and \( \bar{z} = 1 \), while figures 3.6.31a-c exhibit this influence at the distance to the caustic for which the maximum pressure was found. The calculations have been performed with 4 different numbers of frequencies over the interval \([-8/3, 11/3]\) of the \( \tau \) axis and with the optimal number of associated discretization points over the interval \([-1, 1]\) of the \( z \) axis. The discretizations chosen are as follows:

- 256 frequencies and 640 points on the \( \bar{z} \) axis (in black),
- 512 frequencies and 1250 points on the \( \bar{z} \) axis (in green),
- 1024 frequencies and 3500 points on the \( \bar{z} \) axis (in red),
- 2048 frequencies and 10000 points on the \( \bar{z} \) axis (in blue).

All calculations have been performed with a coefficient of nonlinearity \( \mu = 0.1 \).

By observing Figures 3.6.28a-3.6.30a (left figures), we first see that the signals obtained at the three caustic distances are not very sensitive to the number of frequencies chosen, except 256. Indeed, for this number of frequencies, the amplitude of the signals at the 3 distances to the caustic is much lower than that associated with the other numbers of frequencies. We can, furthermore, compare the results on the caustic with those obtained in the linear case of section 3.4. Thus, by observing figure 3.4.15, we remark that the amplitude of the signal clearly increases with the number of frequencies whereas for the nonlinear case the amplitudes are nearly identical for 512, 1024 and 2048 frequencies.

At \( \bar{z} = 0.5 \) and \( \bar{z} = 1 \), the red and blue curves are very close to each other but not perfectly superimposed as on the caustic (cf. figures 3.6.28b-3.6.30b, on the right). The perfect convergence of the numerical code at \( \bar{z} = 0 \) can be explained by the fact that the caustic in the nonlinear case, as we have seen in section 3.6.1, is found in the elliptic region. This region being a zone of rapid attenuation sound, the convergence of the numerical code there is facilitated compared to the hyperbolic region.

However, in general, it is clear that nonlinear effects are essential in limiting the amplitude of the signals. Convergence is almost attained for a number of frequencies equal to 1024. These results are in complete agreement with what was expected from the influence of nonlinear effects on the amplitude of the signal; this continues to perfect the validation of our digital code.

Figures 3.6.31a-c represent the results of the numerical code, at the distance to the caustic for which the maximum pressure over the entire computational domain was found. We remark that, unlike the distances to the caustic presented above, in this particular case, the number of frequencies has a significant influence on the amplitude of the signal. In figures 3.6.31b-c, we can see that the red (1024 frequencies) and blue (2048 frequencies) curves are well-overlapped for the 2\(^{nd}\) shock but not quite for the first shock while the green and black curves are well below. The results in table 3.3, below, come to confirm these results. It would therefore seem that there is no convergence of results at this distance. This appears to be related to the fact that we are at the level of the triple point observed by Sturtevant and Kulkarny (1976) and Tabak and Rosales (1994). Indeed, according to these authors, the nonlinear case would generate another highly localized singularity at the level of the maximum pressure: the incident and reflected shocks meet at a point constituting a single shock which continues to propagate towards the sonic line over a finite distance. This then forms a “Y” in the vicinity of the sonic line, as depicted in figure 3.6.32.
According to Tabak and Rosales (1993), this phenomenon is incompatible with the nonlinear Tricomi equation, but has been observed in several cases by the aforementioned authors; this fact is similar to Von Neumann’s paradox relating to the reflection of oblique shocks (1963). On figure 3.6.33a-b, we have plotted the evolution of the pressure profiles of the two shocks from the distance to the caustic for which the maximum amplitude was found, up to a distance corresponding to 120 discretization points of the $\tau$ axis, towards the top of the computational box. The calculations have been performed with 1024 frequencies on the $\tau$ axis and 3500 discretization points on the $z$ axis, respectively, on the intervals described above. The pressure profiles have been plotted in groups of 20 caustic distances, the color becoming darker as we move away from the sonic line. On each of the red curves, we can observe two shocks identified by the presence of a break in slope; we may therefore suppose that it is at this distance to the caustic that the triple point appears. For the first shock it is found at $\tau = 0.2778$, while the maximum pressure has been found at $\tau = 0.2266$, while for the second shock it is at $\tau = 0.1662$ for a maximum pressure located at $\tau = 0.1157$. These particular points are thus very close to the position of the pressure maxima (albeit slightly above). It is therefore possible that a potential singularity in the vicinity of the triple points prevents the convergence of the code at these points. A definitive answer would nevertheless require an in-depth theoretical study, which remains to be done.

However, if no convergence is possible, the divergence is much slower than in the linear case. It is interesting to observe the influence of the number of frequencies on the amplitude of the second shock; this shock is just above the sonic line, therefore in the hyperbolic part and unlike what we have observed for $\tau = 0.5$ and $\tau = 1$, two distances to the caustic lying within the hyperbolic region, the amplitude value is identical for 1024 and 2048 frequencies. The convergence of the numerical code is thus ensured for this shock.
Apart from the triple point, for which nothing can be concluded, the results obtained have two meanings: firstly, unlike the linear case, there exists a number of frequencies for which the amplitude hardly changes any more. This is positive since it amounts to saying that for an infinite number of frequencies the results remain bounded, which is in agreement with the nonlinear theory. On the other hand, we see that it is still necessary to take a large number of frequencies to properly model the nonlinear case.

In practice, numerical costs force us to limit ourselves to 1024 frequencies for real cases, which seems to be, at first glance, a good compromise between fineness of the results and numerical computation time. Indeed, taking the negative frequencies and the duration of \(19/3\) (cf. § 3.3.7) of the (dimensionless) computational time interval into account, if we consider an incident signal with a duration of 0.27s (typical of a Concorde sonic boom), the maximum frequency considered is:

\[
f_{a}^{\text{max}} = \left[\frac{1024}{2}\right]/\left(\frac{19}{3}\right) \times \left(\frac{1}{0.27}\right) = 300 \text{ Hz}.
\]

This frequency is associated with a period of 3 ms, comparable to the actual rise time of a signal from a classic sonic boom. However, the presence of a non-zero rise time at the level of a shock is linked to absorption effects (by molecular relaxation, turbulence or ground porosity) which filters the high frequencies; thus, in practice, limiting the frequency has no consequence on the results in real cases. Consequently, the choice of 1024 frequencies over the interval \([-8/3, 11/3]\) of the \(\tau\) axis, is a good compromise between the exact representation of the physical phenomenon and the numerical constraints.

<table>
<thead>
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<th>nbr freq.</th>
<th>max((\bar{p}_{a}^{\text{caustic}}))</th>
<th>max((\bar{p}_{a}))</th>
<th>(\bar{\tau}(\text{max}(\bar{p}_{a})))</th>
<th>(\bar{z}(\text{max}(\bar{p}_{a})))</th>
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</tr>
<tr>
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<tr>
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<td>1.23026</td>
<td>3.57495</td>
<td>-1.1135</td>
<td>0.23212</td>
</tr>
</tbody>
</table>

Table 3.3: influence of the number of frequencies on the amplitude of the signal
Figures 3.6.28(a and b)-3.6.30(a and b) : Influence of the number of frequencies at :
\[ \bar{\zeta} = 1, \bar{\zeta} = 0.5 \text{ and } \bar{\zeta} = 0 \]

256 frequencies and 640 points on the \( \bar{\zeta} \) axis
512 frequencies and 1250 points on the \( \bar{\zeta} \) axis
1024 frequencies and 3500 points on the \( \bar{\zeta} \) axis
2048 frequencies and 1000 points on the \( \bar{\zeta} \) axis
Figures 3.6.31a-c: Influence of the number of frequencies at $\bar{z} = \bar{z}(\bar{r}_{\text{NRE}})$

- Black: 256 frequencies and 640 points on the $\bar{z}$ axis
- Green: 512 frequencies and 1250 points on the $\bar{z}$ axis
- Red: 1024 frequencies and 3500 points on the $\bar{z}$ axis
- Blue: 2048 frequencies and 1000 points on the $\bar{z}$ axis
Figures 3.6.33a-b: triple point at the level of the front shock (top) and the rear shock (bottom)
3.6.6 Guiraud’s similitude law

In the previous sections, we validated the nonlinear numerical code in an indirect way. In this section, we will test Guiraud’s similitude law on results obtained for an input signal with only one shock.

In 1965, Guiraud demonstrated that with the following change of variables:

\[ p_a = \mu^{-\frac{1}{5}}\Pi, \quad \tau = \mu^{\frac{6}{5}}\Theta \quad \text{and} \quad \bar{z} = \mu^{\frac{4}{5}}Z, \]

the nonlinear Tricomi equation:

\[
\frac{\partial^2 p_a}{\partial \bar{z}^2} - \bar{z} \frac{\partial^2 p_a}{\partial \tau^2} + \mu \frac{\partial^2}{\partial \tau^2} \left( \frac{p_a^2}{2} \right) = 0,
\]

as well as the associated boundary condition:

\[ p_a(\tau, \bar{z} \to +\infty) = \bar{z}^{-\frac{1}{4}} \left[ F\left( \tau + \frac{2}{3}Z^2 \right) + \mathcal{G}\left( \tau - \frac{2}{3}Z^2 \right) \right], \]

were written in the following way:

\[
\frac{\partial^2 \Pi}{\partial Z^2} - Z \frac{\partial^2 \Pi}{\partial \Theta^2} + \frac{\partial^2}{\partial \Theta^2} \left( \frac{\Pi^2}{2} \right) = 0,
\]

\[ \Pi(\Theta, Z \to +\infty) = Z^{-\frac{1}{4}} \left[ F\left( \mu^{\frac{6}{5}}\left( \Theta + \frac{2}{3}Z^2 \right) \right) + \mathcal{G}\left( \mu^{\frac{6}{5}}\left( \Theta - \frac{2}{3}Z^2 \right) \right) \right]. \]

In this new system, we can see that the Tricomi equation is independent of the coefficient of nonlinearity \( \mu \), which is no longer the case for the boundary condition at positive infinity. However, when we choose a “step” function (Figure 3.6.34) as the incident signal, it is invariant under the transformation \( F(\tau) \to F(\mu^{6/5}\tau) \). Thus, via Guiraud’s transformation, the field in the vicinity of a caustic for a “step” function varies as the \( \frac{1}{5} \)th power of the parameter \( \mu \). As a dimensioned variable, we deduce that the signal in the vicinity of the caustic (including the reflected signal) varies as the \( \frac{4}{5} \)th power of the amplitude of the incident signal, which is clearly a nonlinear law. This is Guiraud’s similitude law. It therefore appears to be a relevant test in the validation of the nonlinear case. Recovering this law at the level of numerical simulations would therefore allow the nonlinear part of the code to be validated.

The difficulty stems from the pseudo-spectral nature of the code, which makes it necessary to artificially periodize the signal. It is thus not possible to exactly simulate an incident “step” function. Nevertheless, when the nonlinearity parameter \( \mu \) is small (\( \mu \ll 1 \)), we expect that nonlinear effects mainly play a part at the level of the shocks, which, in linear theory, leads to infinite peaks. The rest of the signal will be minimally affected by nonlinearities. If the signal can be locally likened to a step, we then expect that locally (and in particular at the maximum pressure level) the field obeys Guiraud’s similitude law. This will only be satisfied if the intrinsic duration of the local nonlinear effects around the shock (which varies as \( \mu^{6/5} \)) is small compared to the total duration of the signal, that is \( \mu^{6/5} \ll 1 \). We may therefore suppose that there is a universal numerical constant \( C_G \), independent of all parameters of the problem, such that the maximum of the pressure field in the vicinity of the caustic and the position of this maximum satisfy power laws of the form:

\[ p_a^{\text{max}} = C_G \mu^{-1/5} \quad \tau(p_a^{\text{max}}) = C_G \mu^{6/5} \quad \text{and} \quad \bar{z}(p_a^{\text{max}}) = C_G \mu^{4/5}. \]

It is these laws that Plotkin and Cantril (1976) use to simulate the focusing of sonic boom, based on the approximate semi-analytical solution established by Seebass (1971) in one particular case.
To test the similarity law, we will therefore first use the periodic sawtooth as described in figure 3.6.35. For a certain number of values of $\mu$, we will thus plot the following ratios:

$$\frac{p_{a\text{ max}}}{\mu^{-1/5}}, \frac{\tilde{\tau}(p_{a\text{ max}})}{\mu^{6/5}}, \frac{\tilde{z}(p_{a\text{ max}})}{\mu^{4/5}} \text{ and } \frac{p_{a\text{ max caust.}}}{\mu^{-1/5}}.$$ 

The last ratio corresponds to the maximum pressure on the caustic and not over the entire computational domain.

For each value of $\mu$, we have chosen the size of the computational domain, the number of frequencies and the number of discretization points on the $\tilde{z}$ axis which seemed best suited for optimal capture of the pressure signal. The results are shown in Figure 3.6.36 below. We observe that the similarity law is fairly well satisfied for the maximum pressure $p_{a\text{ max}}$: the associated curve plateaus over a wide range of $\mu$ from which it deviates notably for large or small values. For large values of $\mu$, this is explained by the fact that Guiraud’s similitude law is no longer satisfied. For small values of $\mu$, the discretization is not precise enough to capture the high frequencies which play an important role for these values of $\mu$. The agreement is moderately satisfactory for the distance to the caustic $\tilde{\tau}(p_{a\text{ max}})$ and for the pressure on the caustic $p_{a\text{ max caust.}}$. Finally, the results are not convincing for the phase variable.
We have just seen that the results obtained with the periodic sawtooth were moderately satisfactory. This may be related to the fact that the sawtooth has a slope on either side of the shock, which could locally influence the focusing of the shock and therefore interfere with the similarity law. We have thus repeated the test with a “pseudo-step” signal, as described in figure 3.6.37. This new incident signal makes it possible to recover the upstream and downstream plateaus of a “step”, while still retaining the periodic aspect, a constraint linked to the use of a pseudo-spectral scheme.

Figures 3.6.36: Traces of the following ratios as a function of \( \mu \):

- \( \frac{\overline{P_a}^{\text{max}}}{\mu^{-1/3}} \)
- \( \frac{\overline{\overline{P_a}^{\text{max}}}}{\mu^{4/5}} \)
- \( \frac{\overline{P_a}^{\text{max, const.}}}{\mu^{-1/3}} \)

We have just seen that the results obtained with the periodic sawtooth were moderately satisfactory. This may be related to the fact that the sawtooth has a slope on either side of the shock, which could locally influence the focusing of the shock and therefore interfere with the similarity law. We have thus repeated the test with a “pseudo-step” signal, as described in figure 3.6.37. This new incident signal makes it possible to recover the upstream and downstream plateaus of a “step”, while still retaining the periodic aspect, a constraint linked to the use of a pseudo-spectral scheme.
The results obtained with this new signal are shown in figure 3.6.38. We note that they are much more satisfactory than the previous results; for the pressure amplitudes (global and on the caustic) and the distance of the maximum to the caustic, the law of similarity is very well satisfied. As before, only the phase variable does not satisfactorily verify Guiraud’s similitude law, which is likely a trace of the periodic nature of the incident signal.
These results therefore provide us with real validation of the code in the nonlinear case, and allow us to conclude that the code has been fully validated. They also provide interesting results with regard to the practical application of Guiraud’s similitude law: in practice, the presence of gradients upstream and downstream of the incident shock interferes with the similarity law, which therefore cannot be satisfactorily applied in the case, for example, of an “N” wave. This runs counter to the ideas of Seebass (1971), and Plotkin and Cantril (1976), which assumed that we could use Guiraud’s similitude law for an “N” wave from a sonic boom. Furthermore, this reinforces the need to have a numerical code solving the nonlinear Tricomi equation, the only way to obtain the noise levels of a “focused” boom on the ground.

3.6.7 Conclusion

In this section we have validated the code in the nonlinear case. For this we have first qualitatively analyzed the results obtained with it. This allowed us to see that they were consistent with what was expected from nonlinear effects: reduction in the amplitude of the signals, temporal advance of the front shock and displacement of the maximum pressure into the linear hyperbolic region. We have also found that the code correctly responded to a variation of the nonlinearity coefficient $\mu$. We have also observed that the code was robust in the sense that the final solution was independent of the initial solution used to begin the iteration. Furthermore, the rearrangement of the terms of the Tricomi equation in the time and frequency domains does very little to disturb the final solutions, which reinforces the robustness of the pseudo-spectral scheme. In addition, the code converges to a stable solution for a finite number of frequencies (except perhaps at the level of the maximum pressure of the entire computational box, where there could be a new singularity linked to the presence of a triple point). Finally, the results perfectly satisfy Guiraud’s similitude law, in the case of a signal approaching the “step” function, that is to say a signal possessing downstream and upstream plateaus. We can therefore say that the numerical code, used to solve the nonlinear Tricomi equation, is completely validated.

3.7 CONCLUSION OF THE CHAPTER

In this chapter we have outlined a numerical method to solve the nonlinear Tricomi equation. We have shown that the solutions, obtained by the associated numerical code, were completely validated in the linear and nonlinear cases. This code does not seem to have any usage constraint except on the coefficient $\mu$ which must remain less than unity, which is always satisfied in our physical problem. This numerical code therefore appears to be more easily usable than that tied to the hodograph transformation method presented by Seebass (1971), Cramer and Seebass (1978) and Fung (1980), the latter being unusable for signals with high amplitudes. Furthermore, as we have seen in the last section, the use of Guiraud’s similitude law to find the amplitude of a focused signal appears to be unsuitable for complex signals of the sonic boom type, which goes against the ideas of Seebass (1971) and Plotkin and Cantril (1976), but reinforces the usefulness of the present code. In the next chapter, we will demonstrate and analyze particular results obtained with our numerical code: we will exhibit, among other things, results obtained with incident signals closer to the sonic booms actually observed.
Chapter 4

APPLICATION TO SONIC BOOM FOCUSING

INTRODUCTION

In the previous chapter we have detailed a numerical method which furnishes solutions of the nonlinear Tricomi equation for a given incident signal and a coefficient of nonlinearity $\mu$. We have validated the associated code in the linear and nonlinear cases. The nonlinear Tricomi equation models the pressure field in the vicinity of a caustic. In the particular case of our study, we are interested only in a caustic resulting from the acceleration of an airplane in supersonic flight. We will, in the first part, analyze the behavior of solutions obtained from theoretical signals for which we vary only one parameter:

- the coefficient of nonlinearity $\mu$,
- the replacement of shocks by a linear pressure variation (addition of a rise time),
- the presence of a 3rd intermediary shock in the incident signal.

In the second part we will examine results corresponding to more realistic incident signals, that is to say representative of an existing aircraft. These signals will be provided by a classical sonic boom propagation code.

4.1 STUDY OF THE INFLUENCE OF PARAMETERS

4.1.1 Influence of the nonlinearity factor $\mu$

At this stage of the study, we do not know how the nonlinearity parameter $\mu$ varies as a function on the flight conditions of the aircraft and the atmosphere of the given day (in particular the wind). Nevertheless, it is important to know the influence of its value on the intensity and shape of the “focused” signals. In the previous chapter, we had remarked that an increase in the value of $\mu$ was associated with a decrease in the dimensionless amplitude of the signals, and with a displacement of the maximum pressure. In this part, we will describe the influence of this parameter on the numerical results in more detail.

Figures 4.1.1-4.1.3 demonstrate the influence of a variation of $\mu$ on the shape of the signals as they appear at $\overline{z} = 0$, at $\overline{z} = 1$ and at the distance at which the pressure maximum is found. The calculations have been performed with 3500 discretization points over the interval $[-1, 1]$ of the $\overline{z}$ axis and 1024 frequencies over the interval $[-8/3, 11/3]$ of the $\overline{\tau}$ axis. The values chosen for $\mu$ are as follows: 0.1 0.2 0.3 0.4 and 0.5.
They have been chosen to highlight the phenomenon of nonlinearity while avoiding numerical difficulties.

For these 5 values, we have also drawn the sonic line over the entire computational domain and placed the two pressure maxima associated with the front and rear shocks of the signal (Figure 4.1.4).

For the 3 distances to the caustic presented here, we observe what we had already seen in chapter 3, especially the drop in the amplitude of the shocks as $\mu$ increases; nonlinear effects (dissipation across the shocks) are more important the greater the amplitude of the signal. In table 4.1, we thus observe that between $\mu = 0.1$ and $\mu = 0.5$, the maximum pressure of the first shock went from 3.327 to 2.196.

Furthermore, as we have also seen previously, nonlinear effects have a tendency to shift the front shock of the incident “N” signal upstream. In figures 4.1.1-4.1.3, we can see that this displacement seems to be proportional to the increase in $\mu$. This is consistent with classical nonlinearity phenomena. Thus by adopting the definition of the speed of sound, corrected for nonlinear effects: $c_{actual} = c_{linear} + \beta \frac{P_{ac}}{\rho_0 c_0}$, $\beta$ being the parameter of nonlinearity, we see that this speed increases with the increase in the value of the signal amplitude. Thus, the front shock being large, the shocks upstream will propagate faster than the rest of the signal, and all the more quickly as the $\mu$ parameter becomes large. The situation is reversed for the rear shock as can be seen in figure 4.1.1. We note that, in figure 4.1.2, the first shock corresponds to the incident signal which is clearly not affected by nonlinear effects. Let us also observe that, in the plots in figure 4.1.3, the rear shocks are affected by the superposition of the incident and reflected signals, as well as by the displacement of the sonic line.

Finally, the third particularity of nonlinear effects is to displace the pressure maximum into the linear hyperbolic region while it is found on the caustic in the linear case. This displacement is linked to the new spatial distribution of the hyperbolic and elliptic regions in the nonlinear case (cf. chapter 3). They are no longer delimited by the caustic but by the sonic line of equation $\overline{z} - \mu \overline{p}_a = 0$. Its profile has approximately the form of the pressure field which we observe on the caustic. Thus in figure 4.1.4, we can see that, the more the factor $\mu$ increases, the more the sonic line lies in the computational domain: the separation between the elliptic and hyperbolic region becomes more and more deformed. It reaches, in the linear hyperbolic region, a maximum of about $\overline{z} = 0.8$ for $\mu = 0.5$. This means that when we increase the value of $\mu$, we must extend the domain towards positive $\overline{z}$ to prevent the sonic line from touching the upper edge of the computational box. We have seen, in chapter 3, that the maximum pressure was found on the sonic line; gradually changing as $\mu$ increases, the pressure maximum is displaced along a line directed both upstream in time and upwards in the computational box. This is consistent with Guiraud’s similitude law, even if it is no longer strictly satisfied for such values of $\mu$. On the other hand, the second shock maintains a practically fixed position.

These results are consistent with what was expected from nonlinear effects. Furthermore, they highlight the fact that a fairly large computational box must be chosen, at least in the direction of positive $\overline{z}$, when the nonlinearity coefficient $\mu$ is large, in order to capture the deformation of the sonic line.
Figures 4.1.1-4.1.3: Influence of $\mu$ on the form of the signal:

- on the caustic (at the top left)
- at $\bar{z} = 1$ (at the top right)
- at the distance to the caustic for which the maximum pressure was found (to the left)

(the correspondence of the colors with the different values of $\mu$ can be found in the table below)

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>FIRST coordinates</th>
<th>SHOCK pressure</th>
<th>SECOND coordinates</th>
<th>SHOCK pressure</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>(-1.106 ; 0.227)</td>
<td>3.327</td>
<td>(1.029 ; 0.116)</td>
<td>2.673</td>
</tr>
<tr>
<td>0.2</td>
<td>(-1.199 ; 0.359)</td>
<td>2.916</td>
<td>(1.066 ; 0.142)</td>
<td>2.223</td>
</tr>
<tr>
<td>0.3</td>
<td>(-1.28 ; 0.457)</td>
<td>2.602</td>
<td>(1.103 ; 0.149)</td>
<td>1.871</td>
</tr>
<tr>
<td>0.4</td>
<td>(-1.354 ; 0.547)</td>
<td>2.364</td>
<td>(1.147 ; 0.132)</td>
<td>1.685</td>
</tr>
<tr>
<td>0.5</td>
<td>(-1.422 ; 0.628)</td>
<td>2.196</td>
<td>(1.19 ; 0.127)</td>
<td>1.508</td>
</tr>
</tbody>
</table>

Table 4.1: Influence of $\mu$ on the positions of pressure maxima and their values (dimensionless)
Conclusion.

The nonlinear effects described in the previous chapter clearly appear when the coefficient of nonlinearity $\mu$ is increased. The amplitude of the signals decreases, the front shock is shifted upstream in time and the maximum pressure is directed towards the top of the computational box. This displacement is linked to the sonic line which describes a surface which gradually increases as the value of $\mu$ grows, which may pose a problem for large values of $\mu$. Indeed, as the sonic line “spreads” more and more over the surface of the computational box, the upper edge of the box can be reached and the computational domain will have to be modified accordingly.

4.1.2 Influence of rise time on the results

The shocks of a real boom signature are not perfect but have a non-zero rise time linked to absorption effects from molecular relaxation, atmospheric turbulence or ground porosity. We will therefore study, in this part, the influence of a non-zero rise time on the amplitude of a modified “N” wave. For this we will add gradients to the front and rear shocks, as shown in the figure below (Figure 4.1.5).
The results can be found in Figures 4.1.6-4.1.9. The calculations have been performed with 3500 discretization points over the interval \([-1, 1]\) on the \(\bar{z}\) axis and 1024 frequencies on the \(\bar{\tau}\) axis. The nonlinear coefficient \(\mu\) has been fixed at 0.1. The rise times chosen for the calculations, in dimensionless values, vary from 0.05 for the lowest, to 0.5 (or 1/4 of the signal period). Figure 4.1.8 represents an enlargement of the incident signals used for the different calculations. Figures 4.1.6 and 4.1.7 show the dimensionless pressure as a function of the dimensionless phase variable, for the positions defined by \(\bar{z} = 0\) and by the pressure maximum for each rise time. Figure 4.1.9 shows the evolution of the pressure maximum as a function of the rise time (in red) as well as its displacement normal to the caustic in the computational domain (in blue).

As we can see in figure 4.1.6, adding a gradient to the signal does very little to change its shape at the caustic level; the signal gradually rounds off with an increase in rise time. However, its increase has a greater impact on the amplitude of the maximum pressure. Indeed, as we can see in figure 4.1.7, it goes from a dimensionless value of 3.327 to a value less than 2 for a rise time going from 0 to 1/4 of the signal period (cf. also figure 4.1.9).

In reality, the more we increase the rise time of the incident signal, the more its period widens and the more its spectrum is shifted towards low frequencies. The high frequencies disappearing, the reflected signals will thus have a finite amplitude. The nonlinear effects will therefore be weaker. Thus, the drop in the amplitude of the signals is no longer attributable to the nonlinear effects but quite simply to the shape of the incident signal. For long rise times, we are at the limit beyond which, nonlinear effects are no longer to be taken into account.

However, in the case of the sonic boom produced by a Concorde type aircraft, the duration of the signal is approximately 200 to 300 ms, for a rise time of a few ms; we may therefore suppose that the ratio between the rise time and the period of the signal is approximately 2 to 5%. In the dimensionless case treated here, this means that the rise time of 0.1 (in dimensionless value), chosen for our calculations, appears to be the maximum realistic value that we can have for a sonic boom. This rise time being very small, we may suppose that it is below the limit introduced above. For this value of the rise time, we have a dimensionless pressure value of 3.18 (figure 4.1.9) while for a zero rise time the pressure is 3.327. The percentage drop in amplitude is then 4.62%. Since in most cases the rise time is less than the dimensionless value 0.1, we can thus say that the addition of a rise time only slightly changes the maximum amplitude of a focused boom signal. This is in agreement with the results obtained by Gill (1974).
APPLICATION TO SONIC BOOM FOCUSING

\[ \Delta = 0 \quad \Delta = 0.1 \quad \Delta = 0.2 \quad \Delta = 0.3 \quad \Delta = 0.4 \quad \Delta = 0.5 \]

Figures 4.1.6-4.1.7: Influence of the rise time on the form of the signal at the level of the caustic (to the left) and at the distance from the caustic for which the maximum pressure was found (to the right).

Figure 4.1.8: Signal profile as a function of increasing rise time

Figure 4.1.9: Influence of the rise time on the maximum pressure rise time and on its position with respect to the caustic
4.1.3 Influence of an incident signal possessing three shocks

Far from the aircraft, the signal associated with the sonic boom takes the regular form of an “N”. However, for low Mach or low-altitude supersonic flights, the propagation distances are shorter. The cumulative nonlinear distortion is weaker and the incident signal has not attained its final “N” profile. All of the shocks may not have merged, and the signal reaching the ground may have a more complex shape, with shocks other than those directly associated with the nose and tail of the aircraft.

Figures 4.1.10 and 4.1.11 show the distribution of the pressure levels over the computational domain for an incident wave with 3 shocks, in the linear and nonlinear cases. Dark blue corresponds to high pressures and white to lower pressures. The calculation has been done with 5250 points over the interval [-1, 2] of the \( z \) axis and 1024 frequencies over the interval [-3.88, 4.88] of the \( \tau \) axis. Figures 4.1.12-4.1.15 come to detail the two previous graphs since they represent the pressure as a function of the phase variable \( \tau \), for the following distances to the caustic: \( z = 2, \ z = 0.5, \ z = \bar{z}(p_{a}^{\max}) \) and \( z = 0 \), for the linear (in red) and nonlinear (in blue) cases. The coefficient of nonlinearity \( \mu \) has been chosen equal to 0.1.

In Figure 4.1.12 we observe the shape of the incident signal and as a consequence, the location of the 3 shocks associated with it. In the same figure we find the reflected signal. We can see that the latter has an additional peak inside the “classic” “U” corresponding to the second shock which we may observe at the front of the incident signal. In the linear case as in the nonlinear case, we can perfectly see the characteristics associated with the three shocks of the incident signal and the three peaks (linear case) or shocks (nonlinear case) attached to the reflected signal. We remark, however, that those associated with the reflected signal are thicker and less dark, and therefore more spread out in the nonlinear case; the very fine peaks of the linear case have been changed to more blunt shocks.

The presence of characteristics in the linear and nonlinear cases makes it possible to clearly differentiate the hyperbolic part from the elliptic part. The elliptic parts are characterized by the presence of evanescent waves which, far from the pressure maxima, disappear to leave a region with very little color contrast: the pressure becomes uniform as we move away from the line separating the two regions. We also note that, as in the previous cases, the sonic line of equation \( \bar{z} = \mu \bar{p}_{a} \) (black curve in the bottom figure) has a profile similar to that of the pressure field that we observe on the caustic (cf. figure 4.1.15).

However, unlike the cases studied previously, there are not two pressure maxima but three, whether in the linear or nonlinear case. The 1st and 3rd maxima are located on the sonic line, at the reflection point of the front and rear shocks as in the case of the “N” boom. On the other hand, the 2nd maximum is positioned at the intersection of the 1st reflected shock with the second incident shock (Figure 4.1.11 and 4.1.12). Consequently, it is not located on the sonic line, but in the hyperbolic region and corresponds to a quadruple point (intersection of two shocks). In the nonlinear case, as for the other two maxima, this maximum is displaced toward the top of the computational box at a distance to the caustic approximately equal to \( z = 0.5 \); this can be seen in Figure 4.1.13. We see that its amplitude value is high since it is 3, while on the caustic the maximum pressure is 1.5 and over the whole box it reaches almost 4 (front shock on the line sonic).

We therefore conclude that, if the incident signal has more than 2 shocks, the maxima associated with the front and rear shocks will be on the sonic line while the other maxima associated with this particular combinations of shocks will be distributed above the sonic line. This result demonstrates that it may be difficult to locate the pressure maxima for complex incident signals, such as those related to the sonic boom, especially in the presence of a more or less reflecting ground.
Figures 4.1.10 and 4.1.11: Distribution of pressure levels over the calculation domain in the linear (top) and nonlinear (bottom) case, for an incident wave containing 3 shocks.
Figures 4.1.12-4.1.15: Plot of the pressure as a function of the phase variable at $\bar{Z} = 2$, $\bar{Z} = 0.5$, $\bar{Z} = \bar{Z}(\bar{P}_a^{\text{max}})$, and $\bar{Z} = 0$, for an incident wave containing 3 shocks.

- **linear solution**
- **nonlinear solution**
4.2 APPLICATION TO UNDER-TRACK ACCELERATION FOCUSING

So far, we have only been interested in dimensionless incident signals. In this part we will present dimensioned results, in order to provide a first assessment of the impact of acceleration focusing in terms of noise pollution.

The study of focusing is done through a local approach of the phenomenon; the history of the signal propagation, from the aircraft up to the caustic, is contained, on one hand, in the coefficient of nonlinearity $\mu$ of the dimensionless nonlinear Tricomi equation, on the other hand, in the form of the incident signal coming to “tangent” this caustic. To obtain these two essential parameters for the initialization of our computational code, we will use another existing program. The code models the propagation of the aircraft’s sonic boom to the ground or to the vicinity of the caustics by adopting the geometrical acoustics principles presented in the first chapter.

4.2.1 Description of the propagation code

This code, developed within EADS Airbus SA in collaboration with l’Université Pierre et Marie Curie (Paris 6), has been written in Matlab; it is called “Boomray”. It replaces the code of Hayes et al. (1969), long used by the USA and France, but which does not give satisfactory results near the the limiting rays.

In this study, we will only be interested in acceleration caustics, under-track, in the standard atmosphere.

To initialize the “Boomray” propagation code, we need to know the pressure field near the aircraft. As we have seen in the introduction, this is given by the Whitham function. To this end, we will use a result obtained elsewhere for the Concorde at Mach 1.3, and retransposed to the under-track focusing Mach number (about 1.2).

To determine the input parameters of the program for solving the nonlinear Tricomi equation, it is necessary to successively calculate, for a given altitude and acceleration:

- the focusing Mach number, corresponding to the emission of the ray (under-track, *i.e.* for a zero azimuthal angle) which will tangent the acceleration caustic at its intersection point with the ground,
- the corresponding ray according to the laws of geometrical acoustics detailed in chapter 1 (including its radius of curvature and the associated ray tube area),
- the incident signal near the caustic by solving the Burgers equation,
- the curvature of the caustic, by determining the points where two neighboring rays also tangent the caustic.

These data make it possible to deduce the diffraction boundary layer thickness DBLT (in meters, normalized by the length of the incident signal between the 1st and the last shock) according to the theory described in Chapter 2, and the amplitude of the incident signal $P_0$ (in Pascals) at this distance. This results in the nonlinearity parameter $\mu$ and the dimensionless incident signal, input parameters for the nonlinear Tricomi equation solution code. The main values are given in blue in table 4.2.
It also presents (in red) the principal results of the Tricomi solution code, namely:

- the normalized maximum amplitude $\tilde{p}_a^{\text{max}}$ (in other words the amplification coefficient),

- the corresponding pressure value $P^{\text{max}} = P_0 \tilde{p}_a^{\text{max}}$ in Pascals, the value that will determine the noise level,

- the distance the distance $\tau(\tilde{p}_a^{\text{max}})$ from this maximum to the geometric caustic, normalized by the diffraction boundary layer thickness.

Note that the signals from the “Boomray” propagation code cannot be directly used by ours. Indeed, in this code the return to zero of the pressure after the rear shock is very slow, which stretches the signals quite far in time, while the interesting parts of the signal are upstream of it. Furthermore, our code being pseudo-spectral, it is necessary to readjust the signal in a window adapted to the signal’s shape and the discretization necessary for the use of an FFT. To interpolate, we have used smoothing modules (cubic “spline” functions). Finally, it is necessary to readjust the signal so that it has a zero time average.

### 4.2.2 General description of results

Within the propagation code “Boomray”, we have taken 3 altitudes (11500, 12000 and 12500 m), for which we have varied the acceleration as follows:

0.2 0.4 0.6 0.8 and 1 $m \cdot s^{-2}$,

which corresponds to realistic operating values for a civil supersonic transport aircraft of the Concorde or ATSF type.

We can see, first of all, that all values obtained with the “Boomray” propagation code vary very little with altitude and acceleration. Thus, the value of the coefficient of nonlinearity $\mu$ maintains an almost constant value lying between 0.07 and 0.08 in most cases. This uniformity is also found in the profiles of the incident signals and by consequence in the numerical results of our computational code. We may observe this regularity in figures 4.2.1-4.2.4, for which we have plotted 4 signals obtained with the following flight conditions:

- in black: altitude 12000 m, acceleration 1 $m \cdot s^{-2}$
- in red: altitude 12000 m, acceleration 0.4 $m \cdot s^{-2}$
- in green: altitude 11,500 m, acceleration 0.6 $m \cdot s^{-2}$
- in blue: altitude 12,500 m, acceleration 0.2 $m \cdot s^{-2}$

We therefore conclude that it is the Whitham function, that is to say the aerodynamic characteristics of the aircraft, which completely determine the pressure profile in the vicinity of the caustic; the altitude and acceleration have a negligible influence, at least within the realistic range of parameters explored in a windless atmosphere. We remark, furthermore, that the Mach numbers found remain consistent with the Whitham function used, since they are between 1.16 and 1.21.

We will linger on the results obtained for an altitude of 12000 m and an acceleration of $1 m \cdot s^{-2}$. This corresponds to figures 4.2.5-4.2.8. The calculations have been carried out with 3500 discretization points over the interval [-1, 1] of the $\tau$ axis and 1024 frequencies over the interval [-2.61, 3.66] of the $\tilde{\tau}$ axis. The coefficient of nonlinearity $\mu$ is, in this specific case, equal to 0.0773.
Figure 4.2.5 shows the incident signal after it has been readjusted so that it may be used by our numerical code. The return to zero of the rear shock is faster than for the signal from the propagation code, however the peculiarities of the signal shape as well as its amplitude are unchanged. We first see that this signal has two high-amplitude shocks and a very low-amplitude shock at the front. The rear shock has an amplitude almost double that of each of the two principal front shocks. Figures 4.2.6-4.2.8 present the solutions of the numerical code respectively at $\tau = 1$, on the caustic, and at the distance to the caustic for which the maximum pressure has been found. The red curves correspond to the linear case and the blue curves to the nonlinear case. As always, we see that the shape of the signals in these two cases is fairly close. Furthermore, the three characteristics of the nonlinear phenomena are, as before, clearly visible (reduction of the amplitude of the signal in the nonlinear case, temporal advance of the front shocks, maximum pressure on the sonic line instead of the caustic).

However, compared to all the previous cases, the maximum amplitude is not linked to the front shock but to the rear shock. This is explained by the high amplitude value of this shock compared to the two front shocks. Indeed, these two shocks will not add up to form the first (finite amplitude) peak of the “U” of the reflected signal, but will constitute two well-separated peaks, and thus will have a lower amplitude than the second peak of the traditional “U” associated with the rear shock. The 3$^{rd}$ very low amplitude shock, placed in front of the incident signal, completely disappears in the nonlinear case.

### 4.2.3 Dimensioned pressure at the caustic

<table>
<thead>
<tr>
<th>altitude (m)</th>
<th>acceleration (m,s$^{-2}$)</th>
<th>Mach</th>
<th>$\mu$</th>
<th>$\bar{P}^{\text{max}}$</th>
<th>$P_0$ (Pa)</th>
<th>$P^{\text{max}}$ (Pa)</th>
<th>$z(P^{\text{max}})$</th>
<th>DBLT (m)</th>
</tr>
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<tr>
<td>11500</td>
<td>0.2</td>
<td>1.161</td>
<td>0.0751</td>
<td>2.9047</td>
<td>51.5933</td>
<td>149.8631</td>
<td>0.1026</td>
<td>500.3414</td>
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<td>0.4</td>
<td>1.1742</td>
<td>0.0745</td>
<td>2.8053</td>
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<td>145.7992</td>
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<td>465.1803</td>
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</table>

DBLT: diffraction boundary layer thickness in meters

Table 4.2: Undertrack acceleration focusing in a standard atmosphere. Influence of the acceleration and the altitude
In view of Table 4.2, we can see, first of all, that the maximum pressure on the ground varies little, around 140 Pa, or 280 Pa if we take account for the reflection coefficient on the ground. This result is important for estimating the level of discomfort caused. Furthermore, the ratio between the maximum amplitude of the “focused” signal and the maximum of the incident wave is quite low, since it has a value close to 3. This value brings to mind the amplification coefficient defined during the test campaigns (cf. the bibliography from the introduction). However, the definition we have given to it in this study is more precise since the reference distance is clearly defined as being $z = 1$. This coefficient is lower than that which is obtained for an “N” wave of the same period, for which it equals 3.488 with a coefficient of nonlinearity $\mu = 0.075$.

We remark, furthermore, that the boundary layer thickness being on the order of 500 m, the maximum pressure is found, for all cases considered here, at about 50 meters ($z(\overline{p}_d^{\text{max}}) \times DBLT$) from the geometric caustic. This result is important to take into account during test campaigns in order to correctly position the pressure sensors on the ground.
Figures 4.2.1-4.2.4: Pressure profiles, incident, at $\tilde{z} = 1$, $\tilde{z} = 0$ and at $\tilde{z} = \tilde{z}(\overline{P}_o^{\text{max}})$ for

- altitude 12000 m, acceleration 1 $m.s^{-2}$
- altitude 12000 m, acceleration 0.4 $m.s^{-2}$
- altitude 11500 m, acceleration 0.6 $m.s^{-2}$
- altitude 12500 m, acceleration 0.2 $m.s^{-2}$
Figures 4.2.5-4.2.8: Pressure profiles, incident, at $\bar{z} = 1$, $\bar{z} = 0$ and at $\bar{z} = \bar{z}(\bar{P}^{\text{max}})$ for an altitude of 12000 m at an acceleration of 1 $m.s^{-2}$.

- **Linear case**
- **Nonlinear case**
4.2.4 Conclusion

In this part, we have sought to calculate the maximum pressure of a “focused” signal, in the case of under-track acceleration focusing of a supersonic aircraft. The information necessary for the initialization of our code has been obtained by means of the “Boomray” program modeling the classical propagation of the sonic boom in standard atmosphere. This code was itself initialized by a Whitham function calculated for a Concorde-type aircraft at Mach 1.3 and retransposed to the focusing Mach. We first found that the wave profile depended above all on the Whitham function, since the latter, in dimensionless values, hardly varies for all altitudes and accelerations considered. Furthermore, the coefficient of nonlinearity also remains nearly constant, sitting between 0.07 and 0.08. The profile obtained is interesting since we observe particularities that we had not sought to highlight previously. With the front shock having been separated into two shocks of the same amplitude, they produce two independent finite peaks in the reflected signal. The maximum pressure is therefore no longer associated with the front shock, but with the rear shock, and its value is less than the amplitude of an equivalent “N” signal. In addition, this maximum pressure makes it possible to determine an amplification coefficient for the focusing of the sonic boom with a precise reference since defined at a dimensionless distance $\zeta = 1$. In the case of an “N” wave, the amplification coefficient for a coefficient of nonlinearity $\mu = 0.075$ is equal to a little less than 3.5, whereas for a signal composed of two shocks having nearly identical amplitude, this coefficient is almost 3. These values remain lower than those obtained by Wanner et al. (1972), which indicated a maximum value of 5. However, these test results are to be taken with caution, as we have shown in the bibliography, given the uncertainties about the experimental conditions and the definition of the amplification coefficient. Finally, these results suggest that it is preferable to have a signal that is not yet in the shape of a fully formed “N” so that the “focused” noise is reduced. They therefore open up prospects for the possibility of reducing the discomfort caused by the acceleration focusing of a boom.

CONCLUSION

The aim of this study was to estimate, at ground level, the “focused” noise of a signal emitted by a supersonic aircraft. In regards to this objective, two parallel paths have been explored. Firstly, extending the results of Guiraud (1965) to an atmosphere with a wind gradient and, on the other hand, developing a numerical tool to solve the nonlinear Tricomi equation for any type of incident signal.

Theoretical Results

In the first chapter, we found the main results of geometrical acoustics in the presence of wind in the manner of Candel (1977) and Coulouvrat (1997). Accounting for nonlinear effects, through the Burgers equation, when propagating the sonic boom along an acoustic ray, was also recalled.

The geometric approximation provides, at the level of caustics, a pressure tending towards infinity: there is therefore a singularity in this region. In reality, this method assumes the transverse field of the acoustic velocity (diffraction) is negligible compared to the longitudinal field when propagating an acoustic disturbance along a ray. This approximation becomes false when there is convergence of acoustic rays and is therefore unsuitable for study of the focusing phenomenon.
In order to apply the method of matched asymptotic expansions in the vicinity of this region, we first studied the geometrical acoustics behavior near a caustic. Compared to the windless case (Coulouvrat 1997), there are 3 changes:

- The speed of sound is replaced by an apparent speed of sound involving the wind speed.

- We show that at first order, diffraction does not occur in the direction normal to the caustic but in a direction offset from it, at an angle equal to that between the wavefront normal and the tangent vector of the acoustic ray considered. The associated coordinate system then undergoes a rotation of the same angle.

- The relative radius of curvature of a ray with respect to the caustic is modified.

In the case of the atmosphere near the ground, the wind speed will rarely be greater than 100 km/h, so that the angle that the normal to the caustic makes with the tangent to the acoustic ray will remain small. However, still close to the ground, gradients of wind speed and the speed of sound can be of the same order of magnitude; the wind will have a significant influence on the relative radius of curvature of a ray with respect to the caustic and therefore on the thickness of the boundary layer. The study of the geometrical acoustic behavior near the caustic allowed us to determine the characteristic thickness of the region close to the caustic, in which diffraction effects can no longer be overlooked. In the sense of matched asymptotic expansions, this region is the inner solution. Sufficiently far from the caustic, we have to find the outer solution which is here geometrical acoustic. The matching condition between the two regions is therefore given by the of geometrical acoustic expression in the vicinity of the caustic.

Finally, from this characteristic thickness, we have rescaled the equations satisfied by the linear sound field in a heterogeneous moving fluid, so as to take into account the diffraction neglected by geometrical acoustics near the caustic. In this region, we thus obtained on the one hand, the hierarchy of acoustic speeds in the different spatial directions and, on the other, an equation satisfied by the pressure. By changing the variables of the latter, which has the effect of moving the reference frame above the caustic, we retrieve the Tricomi equation that we had in the windless case. However, a new singularity in the solutions appears when the incident signals, coming to “tangent” the caustic, include shocks: the reflected signals have infinite peaks. To limit the amplitude of the reflected field, we are forced to introduce nonlinear effects. Finally, by making the same change of variables as that used in the linear case, we show that in the presence of a low speed wind, the pressure field in the neighborhood of a caustic always satisfies the nonlinear Tricomi equation established by Guiraud in 1965.

**Numerical Results**

In a third chapter, we presented a numerical solution method for this equation. It is based on an unsteady equation constructed from the nonlinear Tricomi equation. The choice of the unsteady term was established by analogy with equations modeling acoustic phenomena in which diffraction and nonlinearities are involved, as is the case for the caustic. This new equation has been solved by a pseudo-spectral method: on the same iterative step, the terms related to diffraction are treated in the spectral domain while the nonlinear part is calculated in the time domain via the Burgers equation. Finally, these two processes are repeated, at each iteration, until the numerical code converges.
The construction of the latter was done in steps. We maintained the same guideline in preparing this report. The code was first written in the stationary linear case, then in the unsteady linear case and finally in the unsteady nonlinear case.

The linear case was validated by comparison with the analytical solution of the Tricomi equation constructed from the Airy function. The unsteady linear case was studied in order to observe the convergence of the unsteady linear solution to its stationary solution. We remarked that the convergence, for each frequency used in the spectral domain, was regular.

The study of the linear case allowed us to demonstrate the existence of a relationship linking the optimal number of discretization points of the phase variable axis with that of the axis normal to the caustic. In addition, we saw the utility of a second-order discretization of the boundary conditions at infinity, corresponding to the matching with geometrical acoustics, making it possible to reduce oscillations due to artificial reflection of the reflected signal on the upper edge of the calculation domain. In addition, to reduce the size of the computational box, we have added an evanescence condition on the shadow zone side in order to make the diffusive signal tend towards zero more quickly.

Since in the nonlinear case, there are no analytical solutions with which we would have been able to compare our solutions, we set out to validate the numerical code in two different ways. The first method consisted, on the one hand, in analyzing the results obtained to see if these were consistent with what was expected from the influence of nonlinear effects, and, on the other hand, to examine the relevance of the results when the numerical code was subject to variation of certain parameters. The second method, more rigorous, consisted in applying Guiraud’s (1965) law of similitude to incident signals with only one shock. Three particularities of the nonlinear effects on the results were highlighted:

- decrease in signal amplitude,
- temporal advance of the front shock by the sound speed corrected for nonlinear effects, which accelerates high pressure areas at the expense of the rest of the signal,
- displacement of the maximum pressure towards the top of the computational box, following the displacement of the sonic line which no longer coincides with the geometric caustic.

Furthermore, a study of the influence of the initial solution on the final solution has shown that the code always converges to the same solution, a priori, for any initial solution, that has compression shocks, expansion shocks or infinite peaks. In addition, this analysis has revealed that the convergence was faster when the calculation is initialized by the linear solution, the latter having a profile close to the solution after convergence.

The robustness of the code was also highlighted by the quasi-invariance of the results obtained when the terms of the nonlinear Tricomi equation are distributed differently in the frequency and time domains.

We have also studied the influence of the number of frequencies on the amplitude of the signal. As expected in the linear case, an increase in the number of frequencies leads to an increase in the amplitude value. In the nonlinear case, there are a number of frequencies for which the amplitude of the signals no longer changes; there is therefore a convergence of code in this case. However, this may no longer be verified at a distance from the caustic for which the pressure is maximum. It would therefore seem that there is no convergence of results at this distance. This seems to be related to the fact that we are at the triple point level observed experimentally by Sturtevant and
Kulkarny (1976) and numerically by Tabak and Rosales (1994). Indeed, according to these authors, the nonlinear case would generate another singularity highly localized at the maximum pressure level: incident and reflected shocks meet in one point constituting a single shock spread over a finite distance and then forming a “Y” at near the sonic line. This phenomenon is incompatible with the nonlinear Tricomi equation, but has been observed in a number of cases by the aforementioned authors. A detailed study near this point for the two shocks in an “N” signal has been performed on a result obtained with our numeric code; we observe a slope discontinuity which suggests that there is indeed a triple point preventing the convergence of the numerical code at this point. Furthermore, numerical costs force us to limit ourselves to 1024 frequencies for real cases, which seems, at first glance, to be a good compromise between fineness of results and numerical calculation time. This is consistent with the presence, in the real case, of a nonzero rise time at a shock; this is linked to absorption effects (by molecular relaxation, turbulence or porosity of the ground) filtering the high frequencies.

After the set of observations which allowed us to partially validate the numerical code in the nonlinear case, we applied Guiraud’s law of similitude to our results. For that we used an incident signal constructed by adding upstream and downstream plateaus to a periodic sawtooth in order to get closer to the “step” function for which Guiraud’s similitude law is established. The results obtained with our numerical code perfectly satisfy Guiraud’s similitude law for pressure. However, this study also showed that it was not enough not to apply this law to any signal with a shock to find the amplitude of the focused signal; it is important to keep the characteristics of the “step” function. This goes to against the ideas of Seebass (1971), and Plotkin and Cantril (1976), which supposed that one could use Guiraud’s similitude law for a sonic boom “N” wave.

In the last chapter, we were particularly interested in sonic boom focusing, for which this study was intended. To this end, we have in a first section, analyzed the influence of three parameters on the results obtained. We have, thus, demonstrated that a sharp increase in the value of $\mu$, strongly amplified the nonlinear phenomena described above. Next, we observed, like Gill (1974), that the presence of a real rise time, that is to say short compared to the duration of the signal concerned, had only very little influence on the amplitude of the “focused” signal. Finally we have shown that the presence of more than 2 shocks (associated with the nose and tail of the aircraft) at the level of the incident signal, led to the presence of pressure maxima above the sonic line, which could make the search for pressure maxima in a focusing region more difficult.

Finally in the last section we obtained under-track acceleration results from a pressure signal and a nonlinearity coefficient $\mu$ given by a classical sonic boom propagation code. The Whitham function being fixed, we varied the acceleration and altitude. The propagation code always gave a value of $\mu$ between 0.07 and 0.08 and the incident signal initializing our computer code also had two shocks of the same amplitude at the front. We found that each of these shocks produced a particular finite peak in the reflected signal. The maximum amplitude always associated with the front shock in the case of an “N” wave, was in this case linked to the rear shock. In addition, for all the scenarios calculated, the amplitude of the “focused” signal over the associated incident signal, calculated at a dimensionless distance from the caustic equal to 1, was always equal to almost 3. This value is less than 3.5 which is that obtained from an N wave calculated for a value of $\mu = 0.075$. This coefficient appears as the amplification coefficient introduced by different authors and experimenters (Wanner et al., 1972). However, it is defined more precisely here since the reference pressure is chosen at a dimensionless distance from the caustic equal to unity.
General Conclusion and Outlook

From a theoretical point of view, we have studied the phenomenon of focusing in a moving atmosphere. We have thus recovered the nonlinear Tricomi equation, by making the assumption that the wind speed was not too high, that is to say a speed lower than $100 \text{ km/h}$. This is perfectly satisfied for the usual climatic conditions on the ground. However, it would be interesting to extend the study in the presence of a stronger wind, for focusing at altitude.

Furthermore, as we saw in the 3rd chapter, there is no perfect numerical convergence at the maximum pressure level. Indeed, at this point, the pressure amplitude does not stabilize with an increase in the number of frequencies (discretization points of the phase axis) as for the other points of the calculation domain. This may be related to the presence of a triple point; it would therefore be interesting to explore this phenomenon more deeply theoretically, numerically or experimentally.

From a numerical point of view, the code that we wrote appears, currently, sufficiently complete to solve the nonlinear Tricomi equation. However, like any initial software version, it needs to be improved; different avenues would be interesting to explore to increase its performance. So, as we have seen, current numerical constraints limit the calculation to 1024 frequencies. This number is just enough to model the phenomenon. It would be interesting to create two calculation boxes associated with low and high frequencies in order to explore a wider range of frequencies. It remains, furthermore, to integrate our numeric code in the “Boomray” propagation program, so that it can model all the focal points likely to appear during particular maneuvers of the aircraft and quantify the influence of the wind. Furthermore, within this propagation code, it would be interesting improve the source function, currently provided by the Whitham function; it is notoriously inaccurate for Mach numbers close to 1.2, Mach numbers at which the acceleration caustic usually occurs. It therefore appears necessary to supplement the Whitham function with a direct numerical simulation of the “near” field of the plane.

Finally, to validate the numerical code more completely, it appears essential to perform shock wave focusing experiments in the laboratory in order to compare them with our results and to validate both the theoretical and numerical bases of this study.
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APPENDIX I

THE THOMAS ALGORITHM

In the case of the first order boundary conditions at positive infinity, for both the iterative and noniterative cases, we have to solve a problem of the form $AX = Y$, with $A$ a square tridiagonal matrix:

$$
\begin{bmatrix}
 b_1 & c_1 & 0 & \cdots & \cdots & \cdots & 0 \\
 a_2 & b_2 & c_2 & & & & \\
 0 & \ddots & \ddots & \ddots & & & \\
 \vdots & a_i & b_i & c_i & & & \\
 \vdots & & \ddots & \ddots & \ddots & & \\
 0 & \cdots & \cdots & \cdots & \cdots & a_{n-1} & b_{n-1} & c_{n-1} \\
 & & \cdots & \cdots & \cdots & \cdots & a_n & b_n
\end{bmatrix}
\begin{bmatrix}
 x_1 \\
 x_2 \\
 \vdots \\
 x_i \\
 \vdots \\
 x_{n-1} \\
 x_n
\end{bmatrix}
\begin{bmatrix}
 y_1 \\
 y_2 \\
 \vdots \\
 y_i \\
 \vdots \\
 y_{n-1} \\
 y_n
\end{bmatrix}
$$

The Thomas algorithm replaces the matrix problem with three sequences. Indeed the system may be rewritten in the following form:

$$
\begin{align*}
 b_1 x_1 + c_1 x_2 &= y_1 \\
 a_2 x_1 + b_2 x_2 + c_2 x_3 &= y_2 \\
 a_3 x_2 + b_3 x_3 + c_3 x_4 &= y_3 \\
 & \vdots \\
 a_i x_{i-1} + b_i x_i + c_i x_{i+1} &= y_i \quad (\ast) \\
 & \vdots \\
 a_{n-1} x_{n-2} + b_{n-1} x_{n-1} + c_{n-1} x_n &= y_{n-1} \\
 a_n x_{n-1} + b_n x_n &= y_n
\end{align*}
$$

We seek a recurrence relation of the form:

$$
x_{i-1} = A_{i-1} x_i + B_{i-1} \quad (\ast\ast).
$$

For $i = 2$, $x_1 = \frac{-c_1}{b_1} + \frac{y_1}{b_1}$, thus $A_1 = \frac{-c_1}{b_1}$ and $B_1 = \frac{y_1}{b_1}$.

For any $i$, by substituting $(\ast\ast)$ into $(\ast)$, we obtain:

$$
a_i A_{i-1} x_i + a_i B_{i-1} + b_i x_i + c_i x_{i+1} = y_i,
$$
so that:

\[ x_i = A_i x_{i+1} + B_i \]

where \[ A_i = \frac{-c_i}{b_i + a_i A_{i-1}} \] and \[ B_i = \frac{y_i - a_i B_{i-1}}{b_i + a_i A_{i-1}} \].

Furthermore, we see that in the last equation \( c_n = 0 \), thus according to the previous relation:

\[ x_n = B_n. \]

Thus the previous system \( AX = Y \) may be replaced by the following three sequences:

\[ A_i = \frac{-c_i}{b_i + a_i A_{i-1}} \] with \( A_1 = \frac{-c_1}{b_1} \) and \( i \) varying from 2 to \( n \),

\[ B_i = \frac{y_i - a_i B_{i-1}}{b_i + a_i A_{i-1}} \] with \( B_1 = \frac{y_1}{b_1} \) and \( i \) varying from 2 to \( n \),

and the sequence \( x_{i-1} = A_{i-1} x_i + B_{i-1} \) with \( x_n = B_n = \frac{y_n - a_n B_{n-1}}{b_n + a_n A_{n-1}} \), \( i \) varying from \( n-1 \) to 1.

In reality, we can adopt the Thomas algorithm for 4 diagonals and since the second order condition at infinity only involves the last term of the second subdiagonal, only the initial condition of the third sequence is affected. So, if we add \( \gamma \) into the matrix we have:

\[ x_n = B_n = \frac{y_n - a_n B_{n-1} - \gamma(A_{n-2} B_{n-1} + B_{n-2})}{b_n + a_n A_{n-1} + \gamma A_{n-1} A_{n-2}}. \]
APPENDIX II

THE McDONALD AND AMBROSIANO SCHEME

We present below the detail of the McDonald and Ambrosiano scheme (1984) for solving a scalar nonlinear hyperbolic equation. The reader may also refer to Leveque (1992) for useful supplements on scalar hyperbolic equations, numerical finite difference schemes and their properties (conservative schemes, monotone schemes, at 1st and 2nd order, ...).

A1.1 Scalar hyperbolic evolution equation

Let the scalar hyperbolic evolution equation to be solved be of the form:

\[ \frac{\partial u}{\partial t} + \frac{\partial}{\partial x} \Phi(u) = 0, \]

\[ u(x, t = 0) = u_0(x), \]

the flux function being a regular function of \( u \). This equation may also be written in a nonconservative form:

\[ \frac{\partial u}{\partial t} + f(u) \frac{\partial u}{\partial x} = 0, \]

\[ f(u) = \frac{d\Phi}{du}. \]

The solutions of this equation are obtained by the method of characteristics, which are the lines in \( x, t \) space defined by: \( \frac{dx}{dt} = f(u) \) with \( u = \text{cst} \). Indeed, along these lines:

\[ \frac{du}{dt} = 0 = \frac{\partial u}{\partial t} + \frac{\partial x}{\partial t} \frac{du}{dt} = \frac{\partial u}{\partial t} + f(u) \frac{\partial u}{\partial x} \]

and thus \( u \) is a solution of the hyperbolic equation. We therefore deduce the solution of the equation in implicit form: \( x = f(u) t + \text{cst} \), the integration constant being fixed by the initial condition: \( x = x_0 + f(u_0(x_0)) t \), where again: \( u(x, t) = u_0(x_0) \) with \( x_0 = x - f(u_0(x_0)) t \). It is clear that the implicit solution becomes multivalued at a given time when the change of variable \( x \rightarrow x_0 \) is no longer invertible, that is, as soon as \( t \geq \max_{x_0} \left( \frac{f'(u_0(x_0))}{d\Phi}{\frac{du_0}{dx_0}} \right) \).

In this case, it is necessary to assume that the solution is discontinuous along the curve \( x_c(t) \), with a discontinuity speed \( w = \frac{dx_c}{dt} \). The qualitative appearance of the characteristics is illustrated in figure A1.1 below.
The speed \( w \) of a shock is given by the Rankine-Hugoniot relations: 

\[
    w = \frac{[\Phi]}{[w]},
\]

where \([\cdots]\) denotes the jump of a quantity on either side of the shock.

A1.2 The first-order scheme

We seek a discretized solution \( u^n_i \) of the evolution equation, where \( u^n_i \) denotes an approximation of the solution evaluated at the point \( x_i = i\Delta x \) and at the instant \( n\Delta t \), \( \Delta x \) and \( \Delta t \) denoting the space and time steps respectively.

The discretized scheme for solving the evolution equation is a first-order, explicit and conservative scheme in time and space, in the form:

\[
\frac{u^{n+1}_i - u^n_i}{\Delta t} + \frac{F^n_i - F^n_{i-1}}{\Delta x} = 0,
\]

the first term approaching the derivative of \( u \) with respect to time and the second that of \( \Phi \) with respect to space. The numerical flux \( F^n_i \) is evaluated by noting that, according to the characteristics,
the information comes from upstream if the characteristic speed \( \frac{dx}{dt} = f(u) \) is positive, and from downstream if it is negative (Figure A1.2).

Figure A1.2: determination of the decentering of the scheme

But \( \text{sgn} \left( \frac{dx}{dt} \right) = \text{sgn}(f(u)) \approx \text{sgn}(\Phi_{i+1}^n - \Phi_i^n) = \text{sgn}((\Phi_{i+1}^n - \Phi_i^n)(u_{i-1}^n - u_i^n)) \). As a consequence, the following form is a convenient choice for the numerical fluxes:

\[
\begin{align*}
  w_i^n &= (\Phi_{i+1}^n - \Phi_i^n)(u_{i-1}^n - u_i^n) \quad \text{(MDA.1)} \\
  F_i^n &= \begin{cases} \\
    \Phi_i^n & w_i^n \geq 0 \\
    \Phi_{i+1}^n & w_i^n < 0.
  \end{cases} \quad \text{(MDA.2)}
\end{align*}
\]

**A1.3 Monotonicity condition**

We then show that the scheme may be rendered monotone (and therefore stable) if the ratio \( \frac{\Delta t}{\Delta x} \) is small enough. To do this, let us examine the four possibilities:

**Case n°1**: \( w_{i-1} \geq 0 \) and \( w_i \geq 0 \).

In this case, the definition of numerical flux leads to an upstream decentralized scheme:

\[
  u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}(\Phi_i^n - \Phi_{i-1}^n) = u_i^n - \frac{\Delta t}{\Delta x} \frac{\Phi_i^n - \Phi_{i-1}^n}{u_i^n - u_{i-1}^n}(u_i^n - u_{i-1}^n).
\]

Now, according to the mean value theorem, if \( g(x) \) is a continuous function over the interval \([a, b]\), there exists a value \( x_{mean} \) in the same interval such that: \( \frac{1}{b-a} \int_a^b g(x)dx \). By applying this result to the function \( f(u) = \frac{\Phi_i^n}{u_i^n} \) between \( u_{i-1}^n \) and \( u_i^n \), we deduce that there exists a value \( u_{mean} \) such that:

\[
  u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x} f(u_{mean})(u_i^n - u_{i-1}^n) = u_i^n(1 - \varepsilon_{mean}) + \varepsilon_{mean} u_{i-1}^n,
\]

with: \( \varepsilon_{mean} = \frac{\Delta t}{\Delta x} f(u_{mean})(\geq 0 \text{ because } w_{i-1} \geq 0) \).
We then see that if $\varepsilon - \text{mean} < 1$, the scheme will be monotone: the new value $u_i^{n+1}$ will be intermediary between the two values $u_i^n$ and $u_{i-1}^n$.

Case n°2: $w_{i-1} < 0$ and $w_i < 0$.
In this case, the definition of numerical flux leads to a downstream decentralized scheme:

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x} (\Phi_{i+1}^n - \Phi_i^n) = u_i^n - \frac{\Delta t}{\Delta x} \frac{\Phi_{i+1}^n - \Phi_i^n}{u_{i+1}^n - u_i^n} (u_{i+1}^n - u_i^n).$$

By application of the mean value theorem, there exists $u_{\text{mean}}$ between $u_i^n$ and $u_{i+1}^n$ such that:

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x} f(u_{\text{mean}})(u_{i+1}^n - u_i^n) = u_i^n (1 - \varepsilon_{\text{mean}}) + \varepsilon_{\text{mean}} u_{i+1}^n,$$

with: $\varepsilon_{\text{mean}} = -\frac{\Delta t}{\Delta x} f(u_{\text{mean}})$ ($\geq 0$ because $w_i < 0$).
Again, the scheme will be monotone if $\varepsilon_{\text{mean}} < 1$.

Case n°3: $w_{i-1} < 0$ and $w_i \geq 0$.
In this case, the definition of numerical flux leads to the scheme:

$$u_i^{n+1} = u_i^n,$$

which is clearly monotone.

Note that the condition $w_{i-1} < 0$ and $w_i \geq 0$ implies that the function $f(u)$ vanishes somewhere in the interval $[x_{i-1}, x_{i+1}]$. The numerical scheme therefore amounts to evaluating the function $f(u)$ at this point, which demonstrates that the scheme retains precision of order 1.

Case n°4: $w_{i-1} \geq 0$ and $w_i < 0$.
The evaluation of the flux leads to:

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x} (\Phi_{i+1}^n - \Phi_i^n) = u_i^n - \frac{\Delta t}{\Delta x} (\Phi_{i+1}^n - \Phi_i^n) - \frac{\Delta t}{\Delta x} (\Phi_i^n - \Phi_{i-1}^n).$$

Calculations similar to those carried out for cases n°1 & 2 give:

$$u_i^{n+1} = u_i^n (1 - \varepsilon^-_{\text{mean}} + \varepsilon^+_{\text{mean}}) + \varepsilon^+_{\text{mean}} u_{i+1}^n + \varepsilon^-_{\text{mean}} u_{i-1}^n.$$  

This time, the monotonicity condition is given by $\varepsilon^+_{\text{mean}} < 1/2$ and $\varepsilon^-_{\text{mean}} < 1/2$. In conclusion, the scheme will therefore be monotone if the following condition is satisfied:

$$\left| \frac{\Delta t}{\Delta x} f(u) \right| \leq 1/2.$$

Note that this condition is 2 times more restrictive than the CFL stability condition of the linearized equation.
A1.4 Entropy condition

Let us study the case where the initial condition has a discontinuity:

\[ u_i^0 = \begin{cases} 
  u_- & i \leq 0 \\
  u_+ & i > 0
\end{cases} \]

with \( \Phi(u_-) = \Phi(u_+) \).

Then it is clear that the numerical solution will lead to an invariant solution, that is to say to a stationary shock. But two types of shocks occur: “compression” shocks and “expansion” shocks. The first are characterized by the fact that the characteristics converge towards the shocks, and the second by the fact that the characteristics diverge from the shocks (Figure A1.3). It is easy to see that the latter are “unstable”: a small perturbation of the shock (for example replacing the discontinuity with a steep slope) leads to an entirely different solution, of the simple wave type. Consequently, expansion shocks are to be avoided: this is the entropy condition, according to which in the presence of a discontinuity, it must be of the compression type; that is to say that the characteristics must converge toward the shocks. A formulation of the so-called entropy condition is therefore that, in the \((x, t)\) diagram, the slope of the upstream characteristic (\(dt/dx = 1/f_-\)) is greater than that of the shock (\(dt/dx = 1/w\)), which is greater than that of the downstream characteristic (\(dt/dx = 1/f_+\)) (Figure A1.4), that is: \(f_- > w > f_+\).
Figure A1.3: compression shock, expansion shock

Figure A1.4: entropy condition across the shock
In order to eliminate the formation of stationary expansion shocks in the difference scheme, we will therefore impose that in the case where the entropy condition is violated, that is to say if: 

\[ f(u^n_i) < 0 \text{ and } f(u^n_{i+1}) > 0, \]

then the flux is modified as follows:

\[ F_i^n \rightarrow \frac{1}{2}(F^n_{i+1} + F^n_{i-1} - (u^n_{i+1} - u^n_i)(f^n_{i+1} - f^n_i)). \] 

(MDA.3)

A polynomial approximation shows that the new flux considered approaches the quantity:

\[ F^n_i \approx \frac{1}{2}(\Phi^n_{i+1} + \Phi^n_{i-1} - (u^n_{i+1} - u^n_i)(f^n_{i+1} - f^n_i)) = \Phi_i + \Delta x^2\left(\frac{\partial^2 \Phi}{\partial x^2} - \frac{\partial u \partial f}{\partial x \partial x}\right)_i^n + \mathcal{O}(\Delta x^3). \]

But \( \frac{\partial \Phi}{\partial x} = f(u) \frac{\partial u}{\partial x} \) and thus \( \frac{\partial^2 \Phi}{\partial x^2} = f(u) \frac{\partial^2 u}{\partial x^2} + \frac{\partial f}{\partial x} \frac{\partial u}{\partial x}, \) so that:

\[ F^n_i = \Phi_i + \Delta x^2 f(u^n_i)\left(\frac{\partial^2 u}{\partial x^2}\right)_i^n + \mathcal{O}(\Delta x^3). \]

The order \( \Delta x^2 \) term thus added is similar to a viscosity term, which is very dissipative because \( f(u^n_i) < 0. \) This term will thus force the solution to converge towards the authorized solution, which is the limit of the viscous solution when the viscosity tends to 0.

### A1.5 The second-order scheme

Accounting for the equation satisfied by the flux \( \left( \frac{\partial \Phi}{\partial t} = f(u) \frac{\partial u}{\partial t} = -f(u) \frac{\partial \Phi}{\partial x}\right) \), we may define the new numerical flux predicted by the first-order scheme over a half-step of time:

\[ \Phi_i^{n+1/2} = \Phi_i^n - \frac{\Delta t}{2\Delta x} f_i^n (F^n_i - F^n_{i-1}). \] 

(MDA.4)

Let us now search for a second-order scheme. In the case of an upstream decentralized scheme, it is easy to see that:

\[ \frac{\partial \Phi}{\partial x} (x_i) = \frac{1}{\Delta x} \left(\frac{3}{2} \Phi_i - 2 \Phi_{i-1} + \frac{1}{2} \Phi_{i-2}\right) + \mathcal{O}(\Delta x^2) = \frac{1}{\Delta x} \left[\left(\frac{3}{2} \Phi_i - \frac{1}{2} \Phi_{i-1}\right) - \left(\frac{3}{2} \Phi_{i-1} - \frac{1}{2} \Phi_{i-2}\right)\right], \]

whereas for a downstream decentralized scheme:

\[ \frac{\partial \Phi}{\partial x} (x_i) = \frac{1}{\Delta x} \left(-\frac{3}{2} \Phi_i + 2 \Phi_{i+1} - \frac{1}{2} \Phi_{i+2}\right) + \mathcal{O}(\Delta x^2) = \frac{1}{\Delta x} \left[\left(\frac{3}{2} \Phi_i + \frac{1}{2} \Phi_{i+1}\right) - \left(\frac{3}{2} \Phi_{i+1} - \frac{1}{2} \Phi_{i+2}\right)\right]. \]

We will therefore have to define the following numerical fluxes, according to the slope of the characteristics:

\[ G^n_i = \begin{cases} 
\frac{3}{2} \Phi_i^{n+1/2} - \frac{1}{2} \Phi_{i-1}^{n+1/2} & w^n_i \geq 0 \\
\frac{3}{2} \Phi_i^{n+1/2} - \frac{1}{2} \Phi_{i+1}^{n+1/2} & w^n_i < 0
\end{cases} \] 

(MDA.5)

In the case of an expansion shock, we will perform the same procedure as for the first-order scheme, in order to ensure the entropy condition:

if \( f(u^n_i) < 0 \) and \( f(u^n_{i+1}) > 0, \) then \( G^n_i \rightarrow \frac{1}{2}(G^n_{i+1} + G^n_{i-1} - (u^n_{i+1} - u^n_i)(f^n_{i+1} - f^n_i)). \) 

(MDA.6)
A1.6 The hybrid scheme

The solution obtained with the first-order scheme would be:

$$u_i^{n+1/2} = u_i^n - \frac{\Delta t}{\Delta x} (F_i^n - F_{i-1}^n).$$  \hspace{1cm} (MDA.7)

That obtained with the second-order scheme would be:

$$u_i^{n+1} = u_i^{n+1/2} - \frac{\Delta t}{\Delta x} \left( (G_i^n - F_i^n) - (G_{i-1}^n - F_{i-1}^n) \right).$$

This allows us to define the flux correction provided by the order 2 scheme with respect to the order 1 scheme:

$$\delta_i^n = G_i^n - F_i^n$$  \hspace{1cm} (MDA.8)

However, a second-order scheme would be non-monotone. In order to obtain a monotone scheme, it is necessary to filter the flux correction, so as to prevent the formation of new extremas. This can be achieved using the filter of Boris and Book (1973):

$$\delta_i^n \rightarrow \text{sgn}(\delta_i^n) \max \left\{ 0, \min \left\{ |\delta_i^n|, \text{sgn}(\delta_i^n)(u_{i+2}^{n+1/2} - u_{i+1}^{n+1/2}), \text{sgn}(\delta_i^n)(u_i^{n+1/2} - u_{i-1}^{n+1/2}) \right\} \right\}. \hspace{1cm} (MDA.9)$$

The hybrid scheme is then:

$$u_i^{n+1} = u_i^{n+1/2} - \frac{\Delta t}{\Delta x} (\delta_i^n - \delta_{i-1}^n)$$  \hspace{1cm} (MDA.10)

A time iteration thus requires the sequence of operations (MDA.1) to (MDA.10), carried out in this order.
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523


527


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537