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MEAN-FIELD MODELS AND NONLOCAL PROBLEMS

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Abstract

In this thesis, we were motivated by the stochastic bi-stable mean-field model initiated by *Dawson* et al. in [1] and the nonlocal-in-time problem developed by Du et al. in [2] and observed that formally the stochastic nonlocal-in-time reaction-diffusion problem gives a continuous case of the stochastic bi-stable mean-field model. We generalized the bi-stable mean-field models by introducing an inter-group interaction mechanism, analyzed the well-posedness of this generalization, and derived the asymptotic expression of equilibrium from a coupled nonlinear system of compatible conditions. For $h = 0$, we computed the probability of systemic transitions of both the homogeneous case and the heterogeneous cases, from which we concluded that due to the small heterogeneities on model parameters in different groups, the system will become more likely to fail.

Based on the results for linear nonlocal-in-time problem developed in [2], we established the well-posedness theorems, the local limit results and the semi-group properties of the linear nonlocal-in-time problems with integrable kernels. With the Lipschitz continuous nonlinearity added, we proved the global existence and uniqueness of the solution of the nonlinear nonlocal-in-time problem. In addition, considering the nonlinearity of $u^p|u|^\alpha$ type and with integrable kernels, we established local existence and uniqueness results for solution in both $L^\infty(0, t_0)$ and $W^{1,\infty}(0, t_0)$ spaces. In particular, for small initial data, we also had the global existence and uniqueness of solutions in $L^\infty(0, T)$. And for the non-integrable kernel case, we formulated the local existence of the solutions of nonlinear nonlocal-in-time problem. We then obtained the well-posedness result of a class of the deterministic nonlocal-in-time reaction-diffusion problems with integrable nonlocal kernels. Due to the limitation of the solution spaces we have studied on the nonlinear problems, the related results for stochastic nonlocal reaction-diffusion problems are still open. However, based on the definition of the nonlocal-in-time operator, we introduced the nonlocal versions of the Brownian motions and the it$\hat{o}$ integrals, from which we established the well-posedness and the localization results of a stochastic nonlocal-in-time problem.

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Chapter 1 Introduction

In this dissertation, motivated by the stochastic bi-stable mean-field model initiated by *Dawson* et al. in [1] and the nonlocal-in-time problem developed recently in [2,3] by Du et al., we considered a class of generalized stochastic bi-stable mean-field model and the stochastic nonlocal-in-time reaction-diffusion problem.

The prototype of the mean-field model is a single anharmonic oscillator driven by Gaussian noise and governed by a stochastic differential equation

$$\begin{cases} dX(t) = (X(t) - X^3(t))dt + \sigma dW(t), \quad t > 0, \\ X(0) = X_0, \end{cases} \tag{1.1}$$

where $\sigma > 0$, and $\{W(t)\}$ is a standard *Wiener* process [1]. Then a dynamical model of a collection of anharmonic oscillators in a double-well potential together with an attractive mean-field interaction has been considered. The model is given by the system of Itô's stochastic differential equations:

$$\begin{cases} dx_j(t) = h(-x_j^3(t) + x_j(t))dt + \theta(\bar{x}(t) - x_j(t))dt + \sigma dW_j(t), \quad t > 0, \\ x_j(0) = x_j^0, \end{cases} \tag{1.2}$$

where $j = 1, \ldots, N$, $\bar{x}(t) := N^{-1} \sum_{j=1}^{N} x_j(t)$ and $\theta > 0$.

This model is called a bi-stable, mean-field model which was initiated by *Dawson* et al. in 1982 [1]. The mathematical analysis by *Dawson* is motivated by the phenomena of dynamical phase transitions and metastability for ferromagnetic *Curie-Weiss* models with continuous spin [1, 4, 5] and includes the mean-field limit, the existence of multiple equilibria, and a fluctuation theory. This system provides a simple example of a cooperative interaction and it has been used to model muscle contraction [6]. Similar models have been proposed in chemical kinetics [7], statistical physics [8], and large economic systems [9]. Other mean-field models have been studied in [10–16].

In this dissertation we start from this stochastic model of interacting agents and systemic risk which has been studied by *Garnier* et al. with large deviations theory in [17]. The systemic risk is the risk that a large number of agents in an interconnected system fail simultaneously leading to the failure of the system [17]. In [17] they analytically and numerically analyzed the effects of cooperation between agents and the diversity of sensitivities on the systemic or overall risk. Using large-deviations theory for small h they derived the asymptotic probability of systemic transitions for large N and sufficiently small δ and h, also the effect of diversity of sensitivities on the transition probability when $h = 0$.

We generalized this model studied in [17] by introducing two groups of agents with an additional inter-group cooperative or competitive mechanism. We analyzed the well-posedness of this generalization, and derived the asymptotic expression of equilibrium from a coupled nonlinear system of compatibility conditions. For $h = 0$, we computed the probability of systemic transitions of both the homogeneous case and the heterogeneous cases, from which we concluded that due to the small heterogeneities on model parameters in different groups, the system will become more likely to fail.

Nonlocal models are becoming widespread in all areas of science and engineering. In recent years, research on nonlocal models has increased in many applications, from physics and biology to materials and social sciences, for instance phase transitions [18, 19], nonlocal heat conduction [20], nonlocal Dirichlet forms [21], kinetic equations [22], and even image analyses [23–25]. To account for nonlocal spatial interactions, nonlocal models take on integral equation forms that avoid using spatial derivatives explicitly and potentially allow solutions with spatial singularities that better describe the physical reality.

There has also been a rapid growth of interest in the peridynamic theory (PD), a nonlocal theory of continuum mechanics based on an integro-differential equation without spatial derivatives, introduced first by Silling in [26], which can be used to study cracks and materials failure, where discontinuities in the displacement field occur [27–30]. The introduction of PD has also motivated the nonlocal vector calculus developed by Du et al. in [31, 32] that provides a framework to analyze nonlocal operators, their connections via identities such as the nonlocal integration by parts formula, and their relationship to corresponding classical and local differential operators. The first work of the nonlocal-in-time problem that we studied here is given by Du et al. [2] where the well-posedness, smoothing properties, local limits and the transient solution behavior have all been studied.

In the stochastic bi-stable mean-field model, instead of the discrete particle system we considered a continuous case formally. We introduce a function $u(t,x)$ in $[0,T]\times\Omega$ instead of $\{x_k\}$ and replaced the group average $\bar{x}$ by a weighted integration of u over the x-neighborhood $[x-\sigma, x+\sigma]$. Also we considered the nonlocal-in-time operator instead of the classical derivative. Then the stochastic mean-field problem becomes the stochastic nonlocal reaction-diffusion problem:

$$\begin{cases} \mathcal{G}_t^\delta u(t,x) = \mathcal{L}_x^\sigma u(t,x) + f(u(t,x)) + \epsilon \mathcal{G}_t^\delta W(t,x), & (t,x)\in(0,T]\times\Omega, \\ u(t,x) = g(t,x), & t\in[-\delta,0]\times\Omega. \end{cases}$$

Here the definition of nonlocal operators is given in Chapter 2.

To study the deterministic and stochastic nonlocal reaction-diffusion problems, we first divide the above nonlocal problem into three sub-problems: the linear nonlocal-in-time problem in Banach spaces, the nonlinear nonlocal-in-time problem, and the stochastic nonlocal-in-time problem. The well-posedness of the linear nonlocal-in-time problem

$$\begin{cases} \mathcal{G}_\delta u(t) + \lambda u(t) = f(t), & t\in(0,T], \\ u(t) = g(t), & t\in[-\delta,0], \end{cases} \tag{1.3}$$

which can be considered as a nonlocal version of the first-order linear differential equation with constant coefficient, has been established for $\lambda \geq 0$ in [2]. By completing the study on three sub-problems we actually establish the related theories for the nonlocal versions of linear diffusion equation, the first-order nonlinear differential equation, and the first order linear stochastic differential equation with constant coefficients, respectively.

In Chapter 4, we first introduce a special version of the product rule for nonlocal operator $\mathcal{G}_\delta$ and the corresponding nonlocal integrating factor method, and based on them we extend the well-posedness theorem and the local limit result for the integrable kernels to the linear problem when the coefficient $-\lambda \in h_\delta(\mathbb{R})$. Moreover, for the linear nonlocal-in-time problem in Banach spaces

$$\begin{cases} \mathcal{G}_\delta u(t) + \lambda u(t) = Au(t), & t \in (0, T], \\ u(t) = g(t), & t \in [-\delta, 0], \end{cases} \tag{1.4}$$

under the same assumption of the operator A as in the *Hille-Yosida* theorem for integrable nonlocal kernels, we establish the well-posedness result and proved the corresponding semigroup property of the nonlocal problem.

In Chapter 5, by imposing a nonlinear term $f(u)$ we consider the nonlinear nonlocal-in-time problem,

$$\begin{cases} \mathcal{G}_\delta u(t) + \lambda u(t) = f(u(t)), & t \in (0, T], \\ u(t) = g(t), & t \in [-\delta, 0]. \end{cases} \tag{1.5}$$

First, with the Lipschitz continuity assumption made on the nonlinear term, we prove the global existence and uniqueness of the solution of the nonlinear nonlocal-in-time problem. Then considering the nonlinearity of $u^p|u|^\alpha$ type and with integrable kernels, we establish a local existence and uniqueness result for solution in $L^\infty(0, t_0)$ space for a some time interval $(0, t_0)$. Also for small initial data, we obtain a global existence and uniqueness result.

In addition, for non-integrable kernel case, using the same truncated kernel approach as introduced in [2], we first prove the local existence and uniqueness of the truncated problem

$$\begin{cases} \mathcal{G}_\delta^\epsilon u_\epsilon(t) + \lambda u_\epsilon(t) = u_\epsilon^p |u|_\epsilon^\alpha, & t \in (0, t_0], \\ u_\epsilon(t) = g_\epsilon(t), & t \in [-\delta, 0], \end{cases} \tag{1.6}$$

in $W^{1,\infty}(0, t_0)$ space with the ϵ-dependent initial data g_ϵ in $W^{1,\infty}(-\delta, 0)$ which converges to the initial data of the original problem, where ϵ is the truncation parameter. Since the space $W^{1,\infty}(0, t_0)$ is compactly embedded in the space $C^0(0, t_0)$, any bounded sequence in $W^{1,\infty}(0, t_0)$ has a convergent subsequence that converges in $C^0(0, t_0)$. The solution u_ϵ of the truncated problems has a uniform bound in $W^{1,\infty}(0, t_0)$. Thus there is a subsequence of u_ϵ converges to u in $C^0(0, t_0)$. Furthermore following the lemma in [2] we proved that this limit u satisfies the nonlinear nonlocal problem with non-integrable kernels. Therefore the local existence of the solutions of nonlinear nonlocal problem with non-integrable kernels was established.

Based on the results of the linear nonlocal-in-time problems in Banach spaces and the nonlinear nonlocal-in-time problems, we then obtained the well-posedness result of a class of the deterministic nonlocal-in-time reaction-diffusion problems with integrable nonlocal kernels. Unfortunately, due to the limitation of the solution spaces we have studied on the nonlinear problems, the related results for stochastic nonlocal reaction-diffusion problems are still open. However, based on the definition of the nonlocal-in-time operator, in Chapter 6 we developed the nonlocal versions of the Brownian motions and the It$\hat{o}$ integrals, from which we established the well-posedness of the stochastic nonlocal-in-time problem, or the nonlocal Langevin equation,

$$\begin{cases} \mathcal{G}_\delta u(t) = -\theta u(t) + \sigma \mathcal{G}_\delta W_{t\vee 0}, & t \in (0,T], \\ \quad u(t) = g(t), & t \in [-\delta, 0], \end{cases} \tag{1.7}$$

whose solution converges to the *Ornstein-Uhlenbeck* process in $\mathcal{S}_2(0,T)$ norm.

This thesis is organized as follows:

In Chapter 2, we give a brief introduction on stochastic bi-stable mean-field model and nonlocal calculus.

In Chapter 3, we generalize the stochastic bi-stable mean-field model studied in [17] with introducing two groups of agents with an additional inter-group cooperative or competitive mechanism. We analyze the well-posedness of this generalization and derive the asymptotic expression of equilibrium from a coupled nonlinear system of compatible conditions. For $h = 0$, we compute the probability of systemic transitions of both the homogeneous case and the heterogeneous cases.

In Chapter 4, we consider the first sub-problem starting from the basic linear nonlocal-in-time problem for ODE models. Based on the theory developed by Du et al. in [2], we will first extend the well-posedness results and the localization theory to the linear nonlocal-in-time problems with more general drift term $\lambda u(t)$, for $\lambda \in \mathbb{R}$, with the help of the integrating factor method for nonlocal-in-time operator. Then considering the linear nonlocal-in-time problems in Banach spaces with a linear unbounded dissipative operator A, we will establish the well-posedness theorem and the semigroup property.

In Chapter 5, we continue to study the second sub-problem, the nonlinear nonlocal-in-time problem, by introducing the nonlinearities $f(u)$ into the linear problems. We first consider the Lipschitz continuous nonlinearities and show the global existence and uniqueness of the nonlocal-in-time problem. Then for a class of autonomous nonlinearities in the form of $f(u) = u^p|u|^\alpha$, with $p \in \mathbb{N}^+$ and $\alpha \in [0,1)$, we establish the local existence and uniqueness of the solutions in both $L^\infty(0,t_0)$ and $W^{1,\infty}(0,t_0)$ spaces for integrable kernels, and only the local existence of the solutions in $C^0(0,t_0)$ space of the nonlinear nonlocal-in-time problems for non-integrable kernels.

In Chapter 6, we consider the last sub-problem, the stochastic nonlocal-in-time problem. Based on the definition of the nonlocal-in-time operator, we introduced the nonlocal versions of the Brownian motions and the *Itô* integrals, from which we establish the well-posedness and the localization results of a stochastic nonlocal-in-time problem. We show this stochastic nonlocal-in-time problem is well-posed and its solution converges to the *Ornstein-Uhlenbeck* process in $\mathcal{S}_2$ norm.

In Chapter 7, we summarize our results and draw some conclusions. We also outline open problems and future lines of investigation.

Chapter 2 Preliminaries

In this chapter we give some necessary background on stochastic bi-stable mean-field model and nonlocal calculus that we shall use in this thesis.

2.1 Stochastic Bi-Stable Mean-Field Model

The stochastic bi-stable mean-field model is a system of stochastic differential equations with mean-field interaction. Let $x_j(t)$ be the state of component j, taking real values. For $j = 1, \ldots, N$, the $x_j(t)$'s are modeled as continuous-time stochastic processes satisfying the system of Itô stochastic differential equations:

$$dx_j(t) = -hU(x_j(t))dt + \theta(\bar{x}(t) - x_j(t))dt + \sigma dw_j(t), \tag{2.1}$$

with given initial conditions. Here $-hU(y) = -hV'(y)$ is the restoring force, V is a potential which has two stable states, and $\{w_j(t)\}_{j=1}^N$ are independent, standard Brownian motions. The parameter h controls the level of intrinsic stabilization and σ the strength of the destabilizing random forces. The interaction is the mean reversion term with rate of mean reversion θ and with $\bar{x} := \frac{1}{N}\sum_{j=1}^N x_j(t)$ denoting the empirical mean of the processes. The evolution of the system is characterized by the initial conditions, the three parameters (h, θ, σ) and by the system size N.

Define the empirical probability measure process $X_N(t, dy) = \frac{1}{N}\sum_{j=1}^N \delta_{x_j(t)}(dy)$ and note that $X_N \in C([0,T], M_1(\mathbb{R}))$. The mean-field limit theorem for X_N, proved in [1], is as followes:

Theorem 2.1.1 (Dawson, 1983)**.** *X_N converges in law as $N \to \infty$ to a deterministic process with density $u(t, y)dy \in C([0,T], M_1(\mathbb{R}))$ satisfying the Fokker-Planck* *equation:*

$$\frac{\partial}{\partial t}u = h\frac{\partial}{\partial y}[U(y)u] - \theta\frac{\partial}{\partial y}\left\{\left[\int yu(t,y)dy - y\right]u\right\} + \frac{1}{2}\sigma^2\frac{\partial^2}{\partial y^2}u. \tag{2.2}$$

Here, $U(y) = y^3 - y$ as in [1]. Explicit solutions of the equation 2.2 are not available in general, but we can find equilibrium solutions. Assuming that the first order moment is ξ, then an equilibrium solution u^e_ξ satisfies

$$h\frac{d}{dy}[(y^3 - y)u^e_\xi] - \theta\frac{d}{dy}[(\xi - y)u^e_\xi] + \frac{1}{2}\sigma^2\frac{d^2}{dy^2}u^e_\xi = 0,$$

and has the form

$$u^e_\xi(y) = \frac{1}{Z_\xi\sqrt{2\pi\frac{\sigma^2}{2\theta}}}\exp\left\{-\frac{(y-\xi)^2}{2\frac{\sigma^2}{2\theta}} - h\frac{2}{\sigma^2}V(y)\right\}, \tag{2.3}$$

with Z_ξ the normalization constant:

$$Z_\xi = \int \frac{1}{\sqrt{2\pi\frac{\sigma^2}{2\theta}}}\exp\left\{-\frac{(y-\xi)^2}{2\frac{\sigma^2}{2\theta}} - h\frac{2}{\sigma^2}V(y)\right\}dy.$$

Now ξ must satisfy the compatibility or consistency condition:

$$\xi = m(\xi) := \int yu^e_\xi(y)dy. \tag{2.4}$$

Finding equilibrium solutions has thus been reduced to finding solutions of this equation.

For $U(y) = y^3 - y$, $\xi = 0$ is a solution for equation 2.4. With the same $U(y)$, it can be shown that there are two additional non-zero solutions $\pm\xi_b$ if and only if $\frac{d}{d\xi}m(0) > 1$, and for given h and θ, there exists a critical $\sigma_c(h,\theta) > 0$ such that $\frac{d}{d\xi}m(0) > 1$ if and only if $\sigma < \sigma_c(h,\theta)$.

For small h the solution $\pm\xi_b$ can be approximated to order $O(h)$ in [17] as follows.

Theorem 2.1.2 (Garnier et al., 2013)**.** *For small h, the critical value σ_c can be expanded as*

$$\sigma_c = \sqrt{\frac{2\theta}{3}} + O(h). \tag{2.5}$$

In addition, the non-zero solutions $\pm\xi_b$ are

$$\pm\xi_b = \pm\sqrt{1 - 3\frac{\sigma^2}{2\theta}}\left(1 + h\frac{6}{\sigma^2}\left(\frac{\sigma^2}{2\theta}\right)^2 \frac{1 - 2(\sigma^2/2\theta)}{1 - 3(\sigma^2/2\theta)}\right) + O(h^2). \tag{2.6}$$

2.2 Nonlocal Calculus

Nonlocal vector calculus has been first introduced in [31] as a framework to analyze nonlocal models. It involves some basic nonlocal operators and associated integral identities similar to those in classical, local calculus.

The nonlocal peridynamics (PD) theory, formulated in integro-differential equations, has attracted much attention as a possible alternative to classical partial differential equation formulation. The nonlocality in PD has been largely described by spatial nonlocal interactions but it is natural, from a modeling perspective, to consider nonlocal or memory effect as well. Although there have been some work concerning the related nonlocal gradient operator in $\mathbb{R}^d$ [24, 31, 33–35] in the form

$$\mathbb{G}_\delta(u)(x) = \int_{B_\delta(x)} (u(y) - u(x)) \frac{y - x}{|y - x|^2} \rho_\delta(|y - x|) dy, \tag{2.7}$$

where the non-negative radial function $\rho_\delta = \rho_\delta(|x|) \in L^1(\mathbb{R}^d)$ is compactly supported in the ball $B_\delta(0)$ with the horizon parameter δ, the theoretical study on the nonlocal-in-time problems involving the nonlocal-in-time operator $\mathcal{G}_\delta$ remains limited. In 2017, Du et al. established the first study of the nonlocal-in-time parabolic model in [2]. Associated with a given non-negative, symmetric kernel function $\rho_\delta(s) = \rho_\delta(|s|)$, the nonlocal-in-time operator $\mathcal{G}_\delta$ is defined by

$$\mathcal{G}_\delta v(t) = \lim_{\epsilon \to 0} \int_\epsilon^\delta (v(t) - v(t - s)) s \rho_\delta(s) ds, \tag{2.8}$$

whenever the limit exists in $L^2(0, T)$ for any function $v \in L^2(-\delta, T)$. ρ_δ has a compact support contained in $[-\delta, \delta]$, and a normalized second moment, i.e.,

$$\int_0^\delta s^2 \rho_\delta(s) ds = \frac{1}{2} \int_{-\delta}^\delta s^2 \rho_\delta(s) ds = 1.$$

In case $s\rho_\delta(s) \notin L^1(0, \delta)$, we assume further that the possible singularity of $s\rho_\delta(s)$ appears only at the origin, to ensure that the definition above makes sense.

A formula of nonlocal integration by parts has been derived in [2] to construct the solution space for the nonlocal-in-time problems.

Lemma 2.2.1 (Du et al., 2017)**.** *Suppose that* $u \in L^2(-\delta, T)$*, and* $\varphi \in C_c^\infty(-\delta, T)$ *with zero extension out of the interval* $(0, T)$*. Further, we suppose that*

$$\int_0^T \int_{-\delta}^t |u(t) - u(s)|(t-s)\rho_\delta(t-s)dsdt < \infty.$$

Then it holds that

$$\int_0^T \mathcal{G}_\delta u(t)\varphi(t)dt = -\int_{-\delta}^T u(t)(\mathcal{G}_\delta^* \varphi)(t)ds$$

with

$$\mathcal{G}_\delta^* \varphi(t) = \int_t^{T+\delta} (\varphi(s) - \varphi(t))(s-t)\rho_\delta(s-t)ds. \tag{2.9}$$

A nonlocal function space V_δ has been introduced in [2],

$$V_\delta = \{u \in L^2(-\delta, T) : \tilde{\mathcal{G}}_\delta u \in L^2(0, T)\}, \tag{2.10}$$

where $\tilde{\mathcal{G}}_\delta$ is a distributional nonlocal operator defined by

$$\langle \tilde{\mathcal{G}}_\delta u, \varphi \rangle = -\int_{-\delta}^T u(t)\mathcal{G}_\delta^* \varphi(t)dt, \quad \forall \varphi \in C_c^\infty(0, T).$$

The next result from [2] shows the well-posedness of the linear nonlocal-in-time problem.

$$\begin{cases} \mathcal{G}_\delta u(t) + \lambda u(t) = f(t), & t \in (0, T], \\ u(t) = g(t), & t \in [-\delta, 0], \end{cases} \tag{2.11}$$

with $\lambda \geq 0$ and the initial data $g \in L^\infty(-\delta, 0)$ given in a prehistoric time interval.

Theorem 2.2.1 (Du et al., 2017)**.** *Let* $g \in L^\infty(-\delta, 0)$ *and* $f \in L^2(0, T)$*, then there is a unique solution* $u \in V_\delta$ *with* $\lambda > 0$ *and it holds that*

$$\|u\|_{L^2(0,T)} \leq c(\lambda^{-1/2}\|g\|_{L^\infty(-\delta,0)} + \lambda^{-1}\|f\|_{L^2(0,T)}).$$

Moreover, if $f \in L^\infty(0, T)$*, the solution* $u \in L^\infty(0, T)$ *satisfies*

$$\|u\|_{L^\infty(0,T)} \leq c(\|g\|_{L^\infty(-\delta,0)} + \lambda^{-1}\|f\|_{L^\infty(0,T)}).$$

Theorem 2.2.2 (Du et al., 2017)**.** *Let $g \in L^\infty(-\delta, 0)$ and $f \in L^2(0,T)$, then there is a unique solution $u \in V_\delta$ with $\lambda = 0$ and it holds that*

$$\|u\|_{L^2(0,T)} \leq c(T^{1/2}\|g\|_{L^\infty(-\delta,0)} + \max(\delta, T)\|f\|_{L^2(0,T)}).$$

Further, if $f \in L^\infty(0,T)$, the solution $u \in L^\infty(0,T)$ satisfies

$$\|u\|_{L^\infty(0,T)} \leq c(\|g\|_{L^\infty(-\delta,0)} + \max(\delta, T)\|f\|_{L^\infty(0,T)}).$$

As the nonlocal horizon δ goes to zero, the localization theory of the nonlocal problem has been developed in [2].

Theorem 2.2.3 (Du et al., 2017)**.** *Let u_δ and u be the solutions of problems*

$$\begin{cases} \mathcal{G}_\delta u_\delta(t) + \lambda u_\delta(t) = f(t), & t \in (0,T], \\ u_\delta(t) = g(t), & t \in [-\delta, 0], \end{cases}$$

and

$$\begin{cases} u'(t) + \lambda u(t) = f(t), & t \in (0,T], \\ u(0) = g(0), \end{cases}$$

respectively, with $g \equiv 0$ and $f \in L^2(0,T)$. Then it holds that

$$\lim_{\delta \to 0} \|u_\delta - u\|_{L^2(0,T)} = 0.$$

Chapter 3
The Stochastic Mean-Field Model

In this chapter we shall start from a simple stochastic model of interacting agents and systemic risk which has been studied by *Garnier* et al. with large deviations theory in [17]. The systemic risk is the risk that a large number of agents in an interconnected system fails simultaneously leading to the failure of the system [17]. This model is called a bi-stable, mean-field model which was initiated by *Dawson* et al. in 1982 [1]. The mathematical analysis by *Dawson* is motivated by the phenomena of dynamical phase transitions and metastability for ferromagnetic *Curie-Weiss* models with continuous spin [1, 4, 5] and includes the mean-field limit, the existence of multiple equilibria, and a fluctuation theory.

The prototype of the mean-field model is a single anharmonic oscillator subject to a stochastic disturbance determined by a stochastic differential equation:

$$dx(t) = \left[-x^3(t) + x(t)\right] dt + \sigma dw(t), \tag{3.1}$$

where $\sigma > 0$, and $\{w(t) : t \geq 0\}$ is a standard *Wiener* process [1].

Then a dynamical model of a collection of anharmonic oscillators in a double-well potential together with an attractive mean-field interaction has been considered. The model is given by the system of Itô's stochastic differential equations:

$$dx_j = h(-x_j^3 + x_j)dt + \theta(\bar{x} - x_j)dt + \sigma dw_j(t), \quad j = 1, \ldots, N, \tag{3.2}$$

where $\bar{x}(t) := N^{-1} \sum_{j=1}^{N} x_j(t)$ and $\theta > 0$.

This system provides a simple example of a cooperative interaction and it has been used to model muscle contraction [6]. Similar models have been proposed in chemical kinetics [7], statistical physics [8], and large economic systems [9].

In [17] they analytically and numerically analyzed the effects of cooperation between agents and the diversity of sensitivities on the systemic or overall risk. Using large-deviations theory for small h they derived the asymptotic probability of systemic transitions for large N and sufficiently small δ and h, also the effect of diversity of sensitivities on the transition probability when $h = 0$.

Other mean-field models have been studied in [10–16] and large deviations results for various models can be found in [40–46].

In this chapter, we will generalize the stochastic bi-stable mean-field model studied in [17] with introducing two groups of agents with an additional inter-group cooperative or competitive mechanism. We analyzed the well-posedness of this generalization, and derived the asymptotic expression of equilibrium from a coupled nonlinear system of compatible conditions. For $h = 0$, we computed the probability of systemic transitions of both the homogeneous case and the heterogeneous cases, from which we conclude that due to the small heterogeneities on model parameters in different groups, the system will become more likely to fail.

3.1 Generalized Mean-Field Model

In the mean-field model of systemic risk (3.2), the individual risk processes tend to mean-revert to their empirical mean for $\theta > 0$, which is a simple but non-trivial form of cooperation.

As a possible generalization of this model, we would like to introduce competitions upon the cooperative interactions. Now let us consider two groups of agents, with the group 1 consisting of agents $\{x_1, x_2, \ldots, x_N\}$ and the group 2 consisting of agents $\{y_1, y_2, \ldots, y_N\}$. We then include the competition terms between agents in different groups in the following deterministic model:

$$\begin{cases} dx_i = -hU(x_i)dt + \theta(\bar{x} - x_i)dt + \zeta(\bar{y} - x_i)dt, & i = 1, \ldots, N, \\ dy_j = -hU(y_j)dt + \theta(\bar{y} - y_j)dt + \zeta(\bar{x} - y_j)dt, & j = 1, \ldots, N. \end{cases} \tag{3.3}$$

Here the terms $\theta(\bar{x} - x_i)$ and $\theta(\bar{y} - y_j)$ represent the in-group cooperations and the terms $\zeta(\bar{y} - x_i)$ and $\zeta(\bar{x} - y_j)$ give the inter-group cooperative interactions where $\zeta > 0$.

Denoting $X = \left[x_1, \ldots, x_N\right]^\top$, $Y = \left[y_1, \ldots, y_N\right]^\top$, $Z = \left[X^\top, Y^\top\right]^\top$ and the energy functional $E = E(Z)$, we have:

$$E(Z) = h\sum_{k=1}^{2N} V(z_k) - \frac{\theta}{2N}\left(\sum_{k=1}^{N} x_k\right)^2 - \frac{\theta}{2N}\left(\sum_{k=1}^{N} y_k\right)^2 + \frac{\theta+\zeta}{2}\sum_{k=1}^{2N} z_k^2 - \zeta N\bar{x}\bar{y}. \quad (3.4)$$

Then, the system above has the following gradient structure:

$$\frac{dZ}{dt} = -\nabla_Z E(Z). \quad (3.5)$$

When $\zeta > 0$, we can rewrite the two mean-field terms into an inhomogeneous averaging term and the equations become:

$$\begin{cases} dx_i = -hU(x_i)dt + \theta(\alpha\bar{x} + \beta\bar{y} - x_i)dt, & i = 1, \ldots, N, \\ dy_j = -hU(y_j)dt + \theta(\alpha\bar{y} + \beta\bar{x} - y_j)dt, & j = 1, \ldots, N, \end{cases} \quad (3.6)$$

where $\alpha, \beta > 0$ and $\alpha + \beta = 1$.

The corresponding stochastic model is

$$\text{(Model I)} \quad \begin{cases} dx_i = -hU(x_i)dt + \theta(\alpha\bar{x} + \beta\bar{y} - x_i)dt + \sigma dw_i(t), & i = 1, \ldots, N, \\ dy_j = -hU(y_j)dt + \theta(\alpha\bar{y} + \beta\bar{x} - y_j)dt + \sigma dw_{j+N}(t), & j = 1, \ldots, N. \end{cases} \quad (3.7)$$

Let $\theta = \xi + \zeta$, $\alpha = \frac{\xi}{\xi+\zeta}$, and $\beta = \frac{\zeta}{\xi+\zeta}$, the system becomes exactly

$$\begin{cases} dx_i = -hU(x_i)dt + \xi(\bar{x} - x_i)dt + \zeta(\bar{y} - x_i)dt, & i = 1, \ldots, N, \\ dy_j = -hU(y_j)dt + \xi(\bar{y} - y_j)dt + \zeta(\bar{x} - y_j)dt, & j = 1, \ldots, N. \end{cases} \quad (3.8)$$

Moreover, the deterministic system can be treated as a gradient flow:

$$\frac{dZ}{dt} = -\nabla_Z E(Z),$$

with the energy functional (or Hamiltonian):

$$E(Z) = \frac{h}{4}\sum_{k=1}^{2N} V(z_k) - \frac{\theta N}{2}(\alpha\bar{x}^2 + 2\beta\bar{x}\bar{y} + \alpha\bar{y}^2) + \frac{\theta}{2}\sum_{k=1}^{2N} z_k^2. \quad (3.9)$$

So, the corresponding dissipative energy law is

$$\frac{d}{dt}E(Z) = -\left|\frac{dZ}{dt}\right|^2. \tag{3.10}$$

And in fact this energy functional above is well-defined. Using *Cauchy-Schwartz* inequality, we can show that it is non-negative and the minimum value is 0 which can be obtained if and only if

$$z_1 = z_2 = \cdots = z_{2N} = 1 \text{ or } z_1 = z_2 = \cdots = z_{2N} = -1.$$

Moreover, when h is relatively small, the points above are the only local minimum of the energy functional. We have the following theorem:

Theorem 3.1.1 (Local Minimum of Deterministic Model)**.** *For $h < \beta\theta$ the energy functional $E(Z)$ defined above has only two local minima:*

$$z_1 = z_2 = \cdots = z_{2N} = 1 \text{ or } z_1 = z_2 = \cdots = z_{2N} = -1.$$

Proof. Let us first consider the simplest deterministic model corresponding to Model I (3.7) with $N = 1$:

$$\begin{cases} dx = -h(x^3 - x)dt + \beta\theta(y - x)dt, \\ dy = -h(y^3 - y)dt + \beta\theta(x - y)dt, \end{cases}$$

with energy functional defined as $E(x, y) = \frac{h}{4}(x^2 - 1)^2 + \frac{h}{4}(y^2 - 1)^2 + \frac{\beta\theta}{2}(x - y)^2$. We have

$$E_x = h(x^3 - x) + \beta\theta(x - y), \quad E_y = h(y^3 - y) + \beta\theta(y - x),$$

and

$$\begin{cases} E_{xx} = h(3x^2 - 1) + \beta\theta, \\ E_{xy} = -\beta\theta, \\ E_{yy} = h(3y^2 - 1) + \beta\theta. \end{cases}$$

Setting $E_x = E_y = 0$, we have

$$\begin{cases} h(x^3 - x) + \beta\theta(x - y) = 0, \\ h(y^3 - y) + \beta\theta(y - x) = 0. \end{cases}$$

Adding them up we have

$$h(x + y)(x^2 - xy + y^2 - 1) = 0,$$

and subtracting them we have

$$(x - y)[h(x^2 + xy + y^2 - 1) + 2\beta\theta] = 0.$$

If $x = y$, from $h(x^3 - x) = 0$, we have $x = y = 0$ or ± 1 and obviously $(1, 1)$ and $(-1, -1)$ are both local minima while $(0, 0)$ is not since the Hessian matrix is not positive definite.

If $x = -y$, then from $h(x^3 - x) + 2\beta\theta x = 0$, we have $x = y = 0$ or $x^2 = y^2 = 1 - \frac{2\beta\theta}{h}$. For the second case, while $2\beta\theta < h$, to guarantee the Hessian matrix to be positive definite, we have

$$E_{xx} = h\Big[3(1 - \frac{2\beta\theta}{h}) - 1\Big] = 2(h - 3\beta\theta) > 0,$$

and

$$E_{xx}E_{yy} - E_{xy}^2 = \Big[h\left(3(1 - \frac{2\beta\theta}{h}) - 1\right)\Big]^2 - (\beta\theta)^2 = (2h - 5\beta\theta)(2h - 7\beta\theta) > 0,$$

which imply that

$$\begin{cases} h > 3\beta\theta, \\ h > \frac{7}{2}\beta\theta \text{ or } h < \frac{5}{2}\beta\theta, \end{cases}$$

that is,

$$h > \frac{7}{2}\beta\theta.$$

So in this case $(x, y) = \left(\sqrt{1 - \frac{2\beta\theta}{h}}, \sqrt{1 - \frac{2\beta\theta}{h}}\right)$ or $\left(-\sqrt{1 - \frac{2\beta\theta}{h}}, -\sqrt{1 - \frac{2\beta\theta}{h}}\right)$ can be local minima for our problem.

However, for small h analysis, we can always assume that $h < \beta\theta$ so that $x = -y$

cannot give us any local minima.

If $h(x^2+xy+y^2-1)+2\beta\theta=0$, with small h analysis this cannot happen either. Actually

$$x^2+xy+y^2=1-\frac{2\beta\theta}{h},$$

with $h<\beta\theta$ the right-hand-side is negative while the left-hand-side is always positive.

If $x^2-xy+y^2=1$, we have

$$x^3-x=x(x^2-1)=x(xy-y^2)=xy(x-y),$$

and then from $E_x=h(x^3-x)+\beta\theta(x-y)=0$

$$(hxy+\beta\theta)(x-y)=0.$$

Thus for this case when $x\neq y$,

$$xy=-\frac{\beta\theta}{h},$$

which implies

$$x^2+y^2=1+xy=1-\frac{\beta\theta}{h},$$

similarly for small h, the right-hand-side is negative while the left-hand-side is positive.

After the discussion above, we can claim that for $h<\beta\theta$, the local minima are only $(1,1)$ and $(-1,-1)$.

For non-trivial case when $N>1$,

$$\begin{cases} dx_i=-h(x_i^3-x_i)dt+\theta(\alpha\bar{x}+\beta\bar{y}-x_i)dt, & i=1,\dots,N,\\ dy_j=-h(y_j^3-y_i)dt+\theta(\alpha\bar{y}+\beta\bar{x}-y_j)dt, & j=1,\dots,N,\end{cases}$$

where $\alpha,\beta>0$ and $\alpha+\beta=1$. The energy functional is defined as

$$E(X,Y)=\frac{h}{4}\left[\sum_{i=1}^{N}(x_i^2-1)^2+\sum_{j=1}^{N}(y_j^2-1)^2\right]-\frac{\theta N}{2}(\alpha\bar{x}^2+2\beta\bar{x}\bar{y}+\alpha\bar{y}^2)+\frac{\theta}{2}\left(\sum_{i=1}^{N}x_i^2+\sum_{j=1}^{N}y_j^2\right).$$

$\nabla E(X,Y) = \mathbf{0}$ implies that

$$\begin{cases} h(x_i^3 - x_i) - \theta(\alpha\bar{x} + \beta\bar{y} - x_i) = 0, & i = 1, \dots, N, \\ h(y_j^3 - y_i) - \theta(\alpha\bar{y} + \beta\bar{x} - y_j) = 0, & j = 1, \dots, N. \end{cases}$$

From the first N equations, for any $i \neq j$, we have

$$h(x_i^3 - x_i) + \theta x_i = \theta(\alpha\bar{x} + \beta\bar{y}) = h(x_j^3 - x_j) + \theta x_j,$$

which implies

$$(x_i - x_j)[h(x_i^2 + x_i x_j + x_j^2 - 1) + \theta] = 0.$$

Similar to the discussion above, with small h we have $x_i = x_j$, thus $x_1 = x_2 = \cdots = x_N$. Moreover, $y_1 = y_2 = \cdots = y_N$ can be obtained from the rest N equations. Now the system can be reduced into $N = 1$ case, so with small h the only local minimum is

$$x_1 = x_2 = \cdots = x_N = y_1 = y_2 = \cdots = y_N = 1,$$

and

$$x_1 = x_2 = \cdots = x_N = y_1 = y_2 = \cdots = y_N = -1.$$

□

Similarly, we can consider the competitive inter-group mean-field interaction

$$\begin{cases} dx_i = -hU(x_i)dt + \theta(\bar{x} - x_i)dt + \zeta(\bar{y} + x_i)dt, & i = 1, \dots, N, \\ dy_j = -hU(y_j)dt + \theta(\bar{y} - y_j)dt + \zeta(\bar{x} + y_j)dt, & j = 1, \dots, N, \end{cases} \tag{3.11}$$

and we can rewrite two mean-field terms into one inhomogeneous averaging term:

$$\begin{cases} dx_i = -hU(x_i)dt + \theta(\alpha\bar{x} - \beta\bar{y} - x_i)dt, & i = 1, \dots, N, \\ dy_j = -hU(y_j)dt + \theta(\alpha\bar{y} - \beta\bar{x} - y_j)dt, & j = 1, \dots, N, \end{cases} \tag{3.12}$$

where $\alpha, \beta > 0$ and $\alpha + \beta = 1$. And the corresponding stochastic model is

$$\text{(Model II)} \quad \begin{cases} dx_i = -hU(x_i)dt + \theta(\alpha\bar{x} - \beta\bar{y} - x_i)dt + \sigma dw_i(t), & i = 1, \dots, N, \\ dy_j = -hU(y_j)dt + \theta(\alpha\bar{y} - \beta\bar{x} - y_j)dt + \sigma dw_{j+N}(t), & j = 1, \dots, N, \end{cases} \tag{3.13}$$

where $\alpha, \beta > 0$ and $\alpha + \beta = 1$.

Similar discussion can be applied to the deterministic model of (3.13), and with small h the only local minimum is

$$x_1 = x_2 = \cdots = x_N = 1, y_1 = y_2 = \cdots = y_N = -1,$$

and

$$x_1 = x_2 = \cdots = x_N = -1, y_1 = y_2 = \cdots = y_N = 1.$$

3.2 Equilibrium of Stochastic System

With the discussions above about the local minima of the generalized deterministic models, we then can expect the equilibria of stochastic models to be around these local minima. In [1, 17], there are some discussions about the non-zero solutions of the compatible condition for the original mean-field model, $m(\xi) = \xi$. In [1], *Dawson* has studied the case when $h = 1$ and the result actually can be extended for any h.

Denote

$$p_a(x) = Z_a^{-1} \exp\{2a\sigma^{-2}\theta x + \sigma^{-2}(h - \theta)x^2 - \frac{h}{2}\sigma^{-2}x^4\}, \tag{3.14}$$

$$Z_a = \int \exp\{2a\sigma^{-2}\theta y + \sigma^{-2}(h - \theta)y^2 - \frac{h}{2}\sigma^{-2}y^4\}dy. \tag{3.15}$$

Z_a can be rewritten as $\int \exp(hy) \cdot \exp(-\gamma y^2)\rho(dy)$, where $h = 2a\sigma^{-2}\theta \geq 0$, $\gamma = \sigma^{-2}(h - \theta)$, and

$$\rho(dy) = \exp[-V(y)]dy = \exp[-\frac{h}{2}\sigma^{-2}y^4]dy.$$

Moreover, $V(x) = \frac{h}{2}\sigma^{-2}x^4$ satisfies

$$V(x) \text{ is even, continuous, } \lim_{x\to\pm\infty} V(x) = \infty, \tag{3.16}$$

and

$$V(x) = \int_0^x g(y)dy, \text{ with } g(x) = -8\sigma^{-2}x^3, g(0) = 0 \text{ and } g \text{ convex on } [0, \infty). \tag{3.17}$$

Then the ferromagnetic inequality given by *Griffiths*, *Hurst* and *Sherman* [1] is satisfied:

$$\frac{d^3}{dk^3}\left[\ln \int \exp(kx) \cdot \exp(-\gamma x^2)\rho(dx)\right] \leq 0, \tag{3.18}$$

for all $k \geq 0$, and for all values of γ (including negative values) for which $\exp(-\gamma x^2)\rho(dx)$ is a finite measure.

Denote $\Lambda(a) = \ln Z_a$, and let $v(h, \theta, \sigma)$ denote the variance of the equilibrium distribution $p_0(\cdot)dx$, where

$$p_0(x) = Z_0^{-1} \exp\{\sigma^{-2}(h-\theta)y^2 - \frac{h}{2}\sigma^{-2}y^4\},$$

$$v(h, \theta, \sigma) = m_2(h, \theta, \sigma) \text{ note that } m_1 = 0.$$

$$\zeta(h, \theta, \sigma) = \frac{2\theta}{\sigma^2} \cdot v(h, \theta, \sigma).$$

We have some relations:

$$m(a) = m_1(a) = \frac{\sigma^2}{2\theta} \cdot \left.\frac{d\Lambda(\xi)}{d\xi}\right|_{\xi=a}, \tag{3.19}$$

$$v(a) = m_2(a) - m_1^2(a). \tag{3.20}$$

Similar to the result in [1], we have the following theorem:

Theorem 3.2.1. *The equation $m(\xi) = \xi$ has a strictly positive solution if and only if*

$$\zeta(h, \theta, \sigma) > 1.$$

Proof.

$$m(\xi) = \int y u_\xi^e(y)dy$$

$$
\begin{aligned}
&= \frac{\int y \exp\{-\sigma^{-2}[\theta(y-\xi)^2 + 2hV(y)]\}dy}{\int \exp\{-\sigma^{-2}[\theta(y-\xi)^2 + 2hV(y)]\}dy} \\
&= \frac{\int y \exp\{2\sigma^{-2}\theta\xi y + \sigma^{-2}(h-\theta)y^2 - \frac{h}{2}\sigma^{-2}y^4\}dy}{\int \exp\{2\sigma^{-2}\theta\xi y + \sigma^{-2}(h-\theta)y^2 - \frac{h}{2}\sigma^{-2}y^4\}dy}.
\end{aligned}
$$

Since

$$m(a) = \frac{\sigma^2}{2\theta} \cdot \frac{d\Lambda(\xi)}{d\xi}\bigg|_{\xi=a},$$

we have

$$\frac{d\Lambda(\xi)}{d\xi}\bigg|_{\xi=a} = 2\theta\sigma^{-2}m(a), \quad \text{and}$$

$$\frac{d^2\Lambda(\xi)}{d\xi^2}\bigg|_{\xi=a} = (2\theta\sigma^{-2})^2 v(a) = (2\theta\sigma^{-2}) \cdot \frac{dm(\xi)}{d\xi}\bigg|_{\xi=a}.$$

By the *GHS* inequality in [1],

$$\frac{d^3\Lambda(\xi)}{d\xi^3} \leq 0 \text{ if } a \geq 0.$$

Therefore,

$$\frac{d^2m(\xi)}{d\xi^2}\bigg|_{\xi=a} = (2\theta\sigma^{-2})^{-1} \cdot \frac{d^3\Lambda(\xi)}{d\xi^3}\bigg|_{\xi=a} \leq 0, \text{ if } a \geq 0.$$

This implies that the function $\{m(\xi) : \xi \geq 0\}$ is concave and that for $a > 0$,

$$\frac{dm(\xi)}{d\xi}\bigg|_{\xi=a} \leq \frac{dm(\xi)}{d\xi}\bigg|_{\xi=0} = (2\theta\sigma^{-2}) \cdot v(h,\theta,\sigma) = \zeta(h,\theta,\sigma).$$

Therefore the equation $m(a) = a$ has a strictly positive solution if and only if

$$\frac{dm(\xi)}{d\xi}\bigg|_{\xi=0} = \zeta(h,\theta,\sigma) > 1$$

and the proof is complete. □

Thus, a triple (h, θ, σ_c) for which $\zeta(h, \theta, \sigma_c) = 1$ is said to be critical.

For the generalized models (3.7) and (3.13), similarly the compatible condition becomes

$$\begin{cases} m(\alpha\xi \pm \beta\eta) = \xi, \\ m(\alpha\eta \pm \beta\xi) = \eta. \end{cases}$$

Obviously, $\xi = \eta = 0$ is a trivial solution, moreover $\xi = \eta = \pm\xi_b$ (or $\xi = -\eta = \pm\xi_b$) are two additional solutions from the result that there are two additional solutions $\xi = \pm\xi_b$ for the equation $m(\xi) = \xi$.

Actually, we can show that for small h there is no more solution. We have the following theorem:

Theorem 3.2.2. *For small h, the critical value σ_c can be expanded as*

$$\sigma_c = \sqrt{\frac{2\theta}{3}} + \mathcal{O}(h). \tag{3.21}$$

Moreover, if $0 < \alpha, \beta < 1$, for the model (3.7), the non-trivial solutions of the compatible condition

$$\begin{cases} m(\alpha\xi + \beta\eta) = \xi \\ m(\alpha\eta + \beta\xi) = \eta \end{cases} \tag{3.22}$$

are only $\xi = \eta = \xi_b$ or $\xi = \eta = -\xi_b$, where

$$\xi_b = \sqrt{1 - \frac{3\sigma^2}{2\theta}} \left(1 + h\frac{6}{\sigma^2} \left(\frac{\sigma^2}{2\theta}\right)^2 \frac{1 - 2(\sigma^2/2\theta)}{1 - 3(\sigma^2/2\theta)} \right) + \mathcal{O}(h^2). \tag{3.23}$$

Similarly, for model (3.13), the non-trivial solutions of the compatible condition

$$\begin{cases} m(\alpha\xi - \beta\eta) = \xi \\ m(\alpha\eta - \beta\xi) = \eta \end{cases} \tag{3.24}$$

are only $(\xi = \xi_b, \eta = -\xi_b)$ or $(\xi = -\xi_b, \eta = \xi_b)$.

Proof. To prove this, we can follow the proof of the Proposition 2.2 in [17]: For small h, we view u_ξ^e as a perturbed Gaussian density function. Let $p_\xi(y)$ be the Gaussian density function with mean ξ and variance $\sigma/2\theta$, Y be the Gaussian random variable with the density p_ξ, and $\gamma = 2/\sigma^2$. By using the expansion

$\exp(-h\gamma V) = 1 - h\gamma V + h^2\gamma^2 V^2/2 + O(h^3)$, we have

$$Z_\xi = 1 - h\gamma \mathbf{E}_\xi V(Y) + \frac{1}{2}h^2\gamma^2 \mathbf{E}_\xi V^2(Y) + \mathcal{O}(h^3),$$

$$Z_\xi^{-1} = 1 + h\gamma \mathbf{E}_\xi V(Y) - \frac{1}{2}h^2\gamma^2 \mathbf{E}_\xi V^2(Y) + h^2\gamma^2(\mathbf{E}_\xi V(Y))^2 + \mathcal{O}(h^3).$$

Then calculating $m(\xi)$, we have:

$$m(\xi) = \xi - h\gamma\frac{\sigma^2}{2\theta}\mathbf{E}_\xi V_y(Y) + h^2\gamma^2\frac{\sigma^2}{2\theta}\mathbf{Cov}_\xi(V_y(Y), V(Y)) + \mathcal{O}(h^3).$$

So for model (3.7), the compatible conditions become

$$\begin{cases} \xi = \alpha\xi + \beta\eta - h\gamma\frac{\sigma^2}{2\theta}\mathbf{E}_{\alpha\xi+\beta\eta}V_y(Y) + h^2\gamma^2\frac{\sigma^2}{2\theta}\mathbf{Cov}_{\alpha\xi+\beta\eta}(V_y(Y), V(Y)) + \mathcal{O}(h^3), \\ \eta = \alpha\eta + \beta\xi - h\gamma\frac{\sigma^2}{2\theta}\mathbf{E}_{\alpha\eta+\beta\xi}V_y(Y) + h^2\gamma^2\frac{\sigma^2}{2\theta}\mathbf{Cov}_{\alpha\eta+\beta\xi}(V_y(Y), V(Y)) + \mathcal{O}(h^3). \end{cases}$$

Assuming that the solution $(\bar{\xi}, \bar{\eta})$ has the form $\bar{\xi} = \xi_0 + h\xi_1 + \mathcal{O}(h^2)$ and $\bar{\eta} = \eta_0 + h\eta_1 + \mathcal{O}(h^2)$, the $\mathcal{O}(1)$ terms in the conditions give

$$\begin{cases} \xi_0 = \alpha\xi_0 + \beta\eta_0, \\ \eta_0 = \alpha\eta_0 + \beta\xi_0, \end{cases}$$

that is

$$\xi_0 = \eta_0.$$

The $\mathcal{O}(h)$ terms give

$$\begin{cases} \xi_1 = \alpha\xi_1 + \beta\eta_1 - \gamma\frac{\sigma^2}{2\theta}\Big[(\alpha\xi_0 + \beta\eta_0)^3 + (3\frac{\sigma^2}{2\theta} - 1)(\alpha\xi_0 + \beta\eta_0)\Big], \\ \eta_1 = \alpha\eta_1 + \beta\xi_1 - \gamma\frac{\sigma^2}{2\theta}\Big[(\alpha\eta_0 + \beta\xi_0)^3 + (3\frac{\sigma^2}{2\theta} - 1)(\alpha\eta_0 + \beta\xi_0)\Big], \end{cases}$$

since $\xi_0 = \eta_0$, $\alpha\xi_0 + \beta\eta_0 = \alpha\eta_0 + \beta\xi_0 = \xi_0$, which implies that

$$\begin{cases} -\beta(\xi_1 - \eta_1) - \gamma\frac{\sigma^2}{2\theta}\Big[\xi_0^3 + (3\frac{\sigma^2}{2\theta} - 1)\xi_0\Big] = 0, \\ -\beta(\eta_1 - \xi_1) - \gamma\frac{\sigma^2}{2\theta}\Big[\xi_0^3 + (3\frac{\sigma^2}{2\theta} - 1)\xi_0\Big] = 0, \end{cases}$$

that is

$$\xi_1 = \eta_1 \quad \text{and} \quad \xi_0^3 + \left(3\frac{\sigma^2}{2\theta} - 1\right)\xi_0 = 0.$$

Then $\xi_0 = 0, \pm\sqrt{1 - 3\sigma^2/2\theta}$ if $3\sigma^2 < 2\theta$, or otherwise $\xi_0 = 0$. In order to obtain the non-trivial result, we suppose that $3\sigma^2 < 2\theta$ and ξ_0 takes $\pm\sqrt{1 - 3\sigma^2/2\theta}$ in the later calculations.

Note that $\mathbf{E}_\xi V_y(Y) = \xi^3 + (3\sigma^2/2\theta - 1)\xi = 2h\xi_0^2\xi_1 + \mathcal{O}(h^2)$, and

$$\mathbf{Cov}_\xi(V_y(Y), V(Y)) = \mathbf{E}_\xi[\frac{1}{4}Y^7 - \frac{3}{4}Y^5 + \frac{1}{2}Y^3] + \mathcal{O}(h).$$

Along with the identity $\xi_0^2 + 3\sigma^2/2\theta = 1$, we have

$$\mathbf{Cov}_\xi(V_y(Y), V(Y)) = 6\left(\frac{\sigma^2}{2\theta}\right)^2\left(1 - \frac{2\sigma^2}{2\theta}\right)\xi_0 + \mathcal{O}(h).$$

Since $\xi_0 = \eta_0$ and $\xi_1 = \eta_1$, the $\mathcal{O}(h^2)$ terms imply

$$\begin{cases} -\beta(\xi_2 - \eta_2) - 2\gamma\frac{\sigma^2}{2\theta}\xi_0^2\xi_1 + \gamma^2\frac{\sigma^2}{2\theta}6\left(\frac{\sigma^2}{2\theta}\right)^2\left(1 - \frac{2\sigma^2}{2\theta}\right)\xi_0 = 0, \\ -\beta(\eta_2 - \xi_2) - 2\gamma\frac{\sigma^2}{2\theta}\xi_0^2\xi_1 + \gamma^2\frac{\sigma^2}{2\theta}6\left(\frac{\sigma^2}{2\theta}\right)^2\left(1 - \frac{2\sigma^2}{2\theta}\right)\xi_0 = 0, \end{cases}$$

that is

$$\xi_2 = \eta_2 \quad \text{and} \quad \xi_1 = \eta_1 = \frac{3\gamma\left(\frac{\sigma^2}{2\theta}\right)^2\left(1 - \frac{2\sigma^2}{2\theta}\right)}{\xi_0}.$$

Actually for all $k = 1, 2, \ldots$, we will have $\xi_k = \eta_k$. So $(\bar{\xi}, \bar{\eta}) = (\pm\xi_b, \pm\xi_b)$ are the only non-zero solutions.

On the other hand, for model (3.13), the compatibility condition becomes

$$\begin{cases} \xi = \alpha\xi - \beta\eta - h\gamma\frac{\sigma^2}{2\theta}\mathbf{E}_{\alpha\xi-\beta\eta}V_y(Y) + h^2\gamma^2\frac{\sigma^2}{2\theta}\mathbf{Cov}_{\alpha\xi-\beta\eta}(V_y(Y), V(Y)) + \mathcal{O}(h^3), \\ \eta = \alpha\eta - \beta\xi - h\gamma\frac{\sigma^2}{2\theta}\mathbf{E}_{\alpha\eta-\beta\xi}V_y(Y) + h^2\gamma^2\frac{\sigma^2}{2\theta}\mathbf{Cov}_{\alpha\eta-\beta\xi}(V_y(Y), V(Y)) + \mathcal{O}(h^3). \end{cases}$$

Assuming that the solution $(\bar{\xi}, \bar{\eta})$ has the form $\bar{\xi} = \xi_0 + h\xi_1 + O(h^2)$ and $\bar{\eta} = \eta_0 + h\eta_1 + \mathcal{O}(h^2)$, the $\mathcal{O}(1)$ terms in the conditions give

$$\begin{cases} \xi_0 = \alpha\xi_0 - \beta\eta_0, \\ \eta_0 = \alpha\eta_0 - \beta\xi_0, \end{cases}$$

that is

$$\xi_0 = \eta_0 \text{ or } \xi_0 = -\eta_0.$$

The $\mathcal{O}(h)$ terms give

$$\begin{cases} \xi_1 = \alpha\xi_1 - \beta\eta_1 - \gamma\frac{\sigma^2}{2\theta}\Big[(\alpha\xi_0 - \beta\eta_0)^3 + (3\frac{\sigma^2}{2\theta} - 1)(\alpha\xi_0 - \beta\eta_0)\Big], \\ \eta_1 = \alpha\eta_1 - \beta\xi_1 - \gamma\frac{\sigma^2}{2\theta}\Big[(\alpha\eta_0 - \beta\xi_0)^3 + (3\frac{\sigma^2}{2\theta} - 1)(\alpha\eta_0 - \beta\xi_0)\Big], \end{cases}$$

since $\xi_0 = -\eta_0$, $\alpha\xi_0 - \beta\eta_0 = \xi_0, \alpha\eta_0 - \beta\xi_0 = \eta_0 = -\xi_0$, which implies that

$$\begin{cases} -\beta(\xi_1 + \eta_1) - \gamma\frac{\sigma^2}{2\theta}\Big[\xi_0^3 + (3\frac{\sigma^2}{2\theta} - 1)\xi_0\Big] = 0, \\ -\beta(\xi_1 + \eta_1) + \gamma\frac{\sigma^2}{2\theta}\Big[\xi_0^3 + (3\frac{\sigma^2}{2\theta} - 1)\xi_0\Big] = 0, \end{cases}$$

that is

$$\xi_1 = -\eta_1 \quad \text{and} \quad \xi_0^3 + (3\frac{\sigma^2}{2\theta} - 1)\xi_0 = 0.$$

Then $\xi_0 = 0, \pm\sqrt{1 - 3\sigma^2/2\theta}$ if $3\sigma^2 < 2\theta$, or otherwise $\xi_0 = 0$. In order to obtain the non-trivial result, we suppose that $3\sigma^2 < 2\theta$ and ξ_0 takes $\pm\sqrt{1 - 3\sigma^2/2\theta}$ in the later calculations.

Note that $\mathbf{E}_\xi V_y(Y) = \xi^3 + (3\sigma^2/2\theta - 1)\xi = 2h\xi_0^2\xi_1 + \mathcal{O}(h^2)$, and

$$\mathbf{Cov}_\xi(V_y(Y), V(Y)) = \mathbf{E}_\xi[\frac{1}{4}Y^7 - \frac{3}{4}Y^5 + \frac{1}{2}Y^3] + \mathcal{O}(h).$$

Along with the identity $\xi_0^2 + 3\sigma^2/2\theta = 1$, we have

$$\mathbf{Cov}_\xi(V_y(Y), V(Y)) = 6\left(\frac{\sigma^2}{2\theta}\right)^2\left(1 - \frac{2\sigma^2}{2\theta}\right)\xi_0 + \mathcal{O}(h).$$

Since $\xi_0 = \eta_0$ and $\xi_1 = \eta_1$, the $\mathcal{O}(h^2)$ terms imply

$$\begin{cases} -\beta(\xi_2 + \eta_2) - 2\gamma\frac{\sigma^2}{2\theta}\xi_0^2\xi_1 + \gamma^2\frac{\sigma^2}{2\theta}6\left(\frac{\sigma^2}{2\theta}\right)^2\left(1 - \frac{2\sigma^2}{2\theta}\right)\xi_0 = 0, \\ -\beta(\xi_2 + \eta_2) + 2\gamma\frac{\sigma^2}{2\theta}\xi_0^2\xi_1 - \gamma^2\frac{\sigma^2}{2\theta}6\left(\frac{\sigma^2}{2\theta}\right)^2\left(1 - \frac{2\sigma^2}{2\theta}\right)\xi_0 = 0, \end{cases}$$

that is

$$\xi_2 = -\eta_2 \quad \text{and} \quad \xi_1 = \frac{3\gamma\left(\frac{\sigma^2}{2\theta}\right)^2\left(1 - \frac{2\sigma^2}{2\theta}\right)}{\xi_0}.$$

Moreover, for all $k = 1, 2, \ldots$, we will have $\xi_k = -\eta_k$. So $(\bar{\xi}, \bar{\eta}) = (\pm\xi_b, \mp\xi_b)$ are the only non-zero solutions. $\square$

Actually, assuming $\xi < \eta$, we have $\xi < \alpha\xi + \beta\eta, \alpha\eta + \beta\xi < \eta$, from Figure 3.1, we can find that $m(\alpha\xi + \beta\eta) = \xi$ can only happen while $\alpha\xi + \beta\eta > \xi_b$. Similar discussion will show that $\alpha\eta + \beta\xi < \xi_b$. Thus, except $\xi = \eta = \xi_b$, non-trivial solutions will appear only if $\frac{dm(\xi)}{d\xi}\Big|_{\xi=\xi_b} < 0$, and that case looks like Figure 3.2.

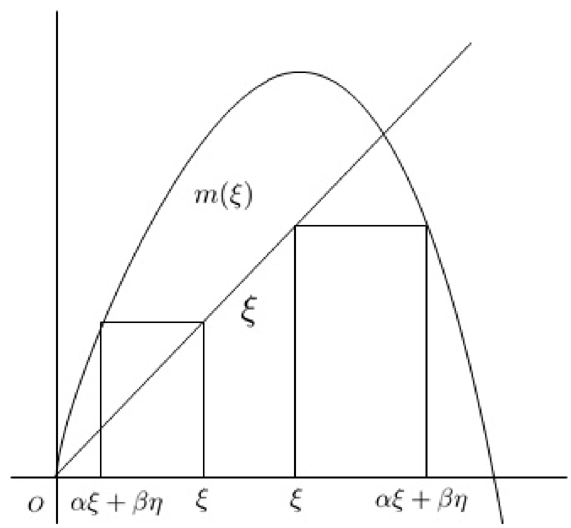

Figure 3.1. $m(\alpha\xi + \beta\eta) = \xi$ can only happen while $\alpha\xi + \beta\eta > \xi_b$

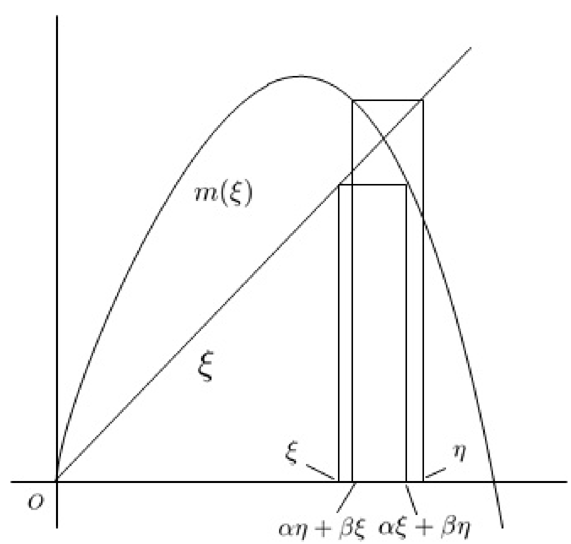

Figure 3.2. The non-trivial solutions appear when $\frac{dm(\xi)}{d\xi}\Big|_{\xi=\xi_b} < 0$

3.3 Heterogeneous Model

We then consider the heterogeneous models with diversities of model parameters in different groups. For instance, a heterogeneous 2-group cooperative problem with diversities of ratios between in-group and inter-group interactions is given as

$$\begin{cases} dx_i = -hU(x_i)dt + \theta(\alpha_1\bar{x} + \beta_1\bar{y} - x_i)dt + \sigma dw_i^{(1)}(t), \quad i = 1, \dots, N, \\ dy_j = -hU(y_j)dt + \theta(\alpha_2\bar{y} + \beta_2\bar{x} - y_j)dt + \sigma dw_j^{(2)}(t), \quad j = 1, \dots, N, \end{cases} \tag{3.25}$$

where $(\alpha, \beta) = (\alpha_1, \beta_1)$ and (α_2, β_2) in two different groups.

Denoting $Z = \begin{bmatrix} X \\ Y \end{bmatrix}$, and defining the energy functional as

$$\begin{aligned} E(Z) \;=\; & h\Big[\beta_2 \sum_{i=1}^{N} V(x_i) + \beta_1 \sum_{j=1}^{N} V(y_j)\Big] - \frac{\theta N}{2}\left[\alpha_1\beta_2\bar{x}^2 + 2\beta_1\beta_2\bar{x}\bar{y} + \alpha_2\beta_1\bar{y}^2\right] \\ & + \frac{\theta}{2}\left[\beta_2 \sum_{i=1}^{N} x_i^2 + \beta_1 \sum_{j=1}^{N} y_j^2\right], \end{aligned}$$

the deterministic system has the gradient structure:

$$\frac{dZ}{dt} = -A\nabla_Z E(Z),$$

where the matrix $A = \begin{bmatrix} \beta_2^{-1} I_N & O \\ O & \beta_1^{-1} I_N \end{bmatrix}$. Moreover, for the corresponding stochastic model, similarly we can consider the mean-field limits and to find the equilibrium solutions to the *McKean-Vlasov* equations is equivalent to find ξ and η satisfying the compatible conditions:

$$\begin{cases} \xi = m(\alpha_1\xi + \beta_1\eta), \\ \eta = n(\alpha_2\eta + \beta_2\xi). \end{cases} \tag{3.26}$$

Similar discussion can be applied to show the uniqueness of positive solution $\xi = \eta = \xi_b$ with the assumption $\frac{dm(\xi)}{d\xi}\Big|_{\xi=\xi_b} < 0$, and the expansion of ξ_b is the same for small h which does not depend on α_i and β_i.

We can also consider the heterogeneous models with diversities of group sizes in different groups. For instance, a heterogeneous 2-group cooperative problem with diversities of group sizes is given as

$$\begin{cases} dx_i = -hU(x_i)dt + \theta(\alpha\bar{x} + \beta\bar{y} - x_i)dt + \sigma dw_i^{(1)}(t), & i = 1, \dots, M, \\ dy_j = -hU(y_j)dt + \theta(\alpha\bar{y} + \beta\bar{x} - y_j)dt + \sigma dw_j^{(2)}(t), & j = 1, \dots, N. \end{cases} \tag{3.27}$$

We will then compute the transition probability of the system for $h = 0$ case and study the influences of the model parameters on the transition probability.

3.4 Computation of Transition Probability for $h = 0$

3.4.1 Homogeneous Case

After finding the equilibrium of stochastic system, now we can study the transition probabilities between these equilibrium. Generally, the stochastic system is nonlinear, but when $h = 0$ we have a linear system of stochastic differential equations:

$$\begin{cases} dx_i = \theta(\alpha\bar{x} \pm \beta\bar{y} - x_i)dt + \sigma dw_i(t), & i = 1, \dots, N, \\ dy_j = \theta(\alpha\bar{y} \pm \beta\bar{x} - y_j)dt + \sigma dw_{j+N}(t), & j = 1, \dots, N. \end{cases}$$

For the cooperative model (3.7), by taking averages of $\{x_i\}$ and $\{y_j\}$, we have:

$$\begin{cases} d\bar{x} = \theta(\alpha\bar{x} + \beta\bar{y} - \bar{x})dt + \frac{\sigma}{\sqrt{N}} d\bar{w}_x(t) = \beta\theta(\bar{y} - \bar{x})dt + \frac{\sigma}{\sqrt{N}} d\bar{w}_x(t), \\ d\bar{y} = \theta(\alpha\bar{y} + \beta\bar{x} - \bar{y})dt + \frac{\sigma}{\sqrt{N}} d\bar{w}_y(t) = \beta\theta(\bar{x} - \bar{y})dt + \frac{\sigma}{\sqrt{N}} d\bar{w}_y(t), \end{cases}$$

which is,

$$d\begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} = \beta\theta \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} dt + \frac{\sigma}{\sqrt{N}} d\begin{bmatrix} \bar{w}_x \\ \bar{w}_y \end{bmatrix} = M \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} dt + \frac{\sigma}{\sqrt{N}} d\begin{bmatrix} \bar{w}_x \\ \bar{w}_y \end{bmatrix},$$

where $M = \beta\theta \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$.
If $\bar{x}(0) = \bar{y}(0) = \xi_b$, we have

$$M \begin{bmatrix} \bar{x}(0) \\ \bar{y}(0) \end{bmatrix} = M \begin{bmatrix} \xi_b \\ \xi_b \end{bmatrix} = \xi_b \beta\theta \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which implies that

$$e^{Mt} \begin{bmatrix} \xi_b \\ \xi_b \end{bmatrix} = \begin{bmatrix} \xi_b \\ \xi_b \end{bmatrix}.$$

Thus, the solution of the coupled linear stochastic system of $\bar{x}$ and $\bar{y}$ is

$$\begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} = \begin{bmatrix} \xi_b \\ \xi_b \end{bmatrix} + \frac{\sigma}{\sqrt{N}} \int_0^t e^{M(t-s)} d \begin{bmatrix} \bar{w}_x \\ \bar{w}_y \end{bmatrix}.$$

Denoting $\bar{z} = (\bar{x} + \bar{y})/2$, we have

$$\bar{z} = \xi_b + \frac{\sigma_b}{\sqrt{N}} \int_0^t \rho^\top e^{M(t-s)} d \begin{bmatrix} \bar{w}_x \\ \bar{w}_y \end{bmatrix}, \tag{3.28}$$

where $\rho = \left[\frac{1}{2}, \frac{1}{2}\right]^\top$. Thus, $\bar{z}(T)$ is a Gaussian random variable with mean ξ_b and variance $\sigma_T^2 = Var(\bar{z}(T))$, given by

$$\sigma_T^2 = \frac{\sigma^2}{N} \int_0^T \rho^\top e^{Ms} (e^{Ms})^\top \rho ds.$$

Then we can compute the transition probability after computing the above integral.

Similarly, if $\bar{x}(0) = \bar{y}(0) = -\xi_b$, $\bar{z}(T)$ is a Gaussian random variable with mean $-\xi_b$ and variance σ_T^2 as above.

Now, we need to calculate e^{Mt}, where $M = \beta\theta \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$. By simple computation, we have

$$M^n = (\beta\theta)^n (-2)^{n-1} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}.$$

So,

$$e^{Mt} = \frac{1}{2} \begin{bmatrix} 1 + e^{-2\beta\theta t} & 1 - e^{-2\beta\theta t} \\ 1 - e^{-2\beta\theta t} & 1 + e^{-2\beta\theta t} \end{bmatrix}.$$

Moreover,

$$\rho^\top e^{Mt}(e^{Mt})^\top \rho = \frac{1}{2},$$

which implies that the variance

$$\sigma_T^2 = \frac{\sigma^2}{N}\int_0^T \frac{1}{2} ds = \frac{\sigma^2 T}{2N}. \tag{3.29}$$

Then for both cases we can deduce that the transition probability is

$$p_T \approx \exp\left(-\frac{4N\xi_b^2}{\sigma^2 T}\right). \tag{3.30}$$

3.4.2 Heterogeneous Cases

And when the heterogeneities are introduced, we have the following results about the transition probability p_T.

Prop 3.4.1. *For a 2-group cooperative problem with heterogeneity in group size,*

$$\begin{cases} dx_i = -hU(x_i)dt + \theta(\alpha_1\bar{x} + \beta_1\bar{y} - x_i)dt + \sigma dw_i^{(1)}(t), & i = 1, \dots, N, \\ dy_j = -hU(y_j)dt + \theta(\alpha_2\bar{y} + \beta_2\bar{x} - y_j)dt + \sigma dw_j^{(2)}(t), & j = 1, \dots, N, \end{cases}$$

the minimum transition probability is obtained when $M = N$.

Prop 3.4.2. *For a 2-group cooperative problem with heterogeneity in ratio between in-group and inter-group interactions,*

$$\begin{cases} dx_i = -hU(x_i)dt + \theta(\alpha\bar{x} + \beta\bar{y} - x_i)dt + \sigma dw_i^{(1)}(t), & i = 1, \dots, M, \\ dy_j = -hU(y_j)dt + \theta(\alpha\bar{y} + \beta\bar{x} - y_j)dt + \sigma dw_j^{(2)}(t), & j = 1, \dots, N. \end{cases}$$

the minimum transition probability is obtained when $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$.

Considering heterogenities in both group size and the ratio between interactions, we have the following theorem.

Theorem 3.4.3. *For a 2-group cooperative problem with heterogeneities in both* *group size and the ratio between in-group and inter-group interactions,*

$$\begin{cases} dx_i = -hU(x_i)dt + \theta(\alpha_1\bar{x} + \beta_1\bar{y} - x_i)dt + \sigma dw_i^{(1)}(t), & i = 1, \dots, M, \\ dy_j = -hU(y_j)dt + \theta(\alpha_2\bar{y} + \beta_2\bar{x} - y_j)dt + \sigma dw_j^{(2)}(t), & j = 1, \dots, N. \end{cases}$$

the minimum transition probability is obtained when $\beta_1 M = \beta_2 N$.

Proof. Following the computations for the homogeneous model, we have the averaging equations

$$\begin{cases} d\bar{x} = \beta_1\theta(\bar{y} - \bar{x})dt + \frac{\sigma}{\sqrt{M}}d\bar{w}_x \\ d\bar{y} = \beta_2\theta(\bar{x} - \bar{y})dt + \frac{\sigma}{\sqrt{N}}d\bar{w}_y \end{cases}$$

$$\frac{d}{dt}\begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} = \theta \begin{bmatrix} -\beta_1 & \beta_1 \\ \beta_2 & -\beta_2 \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} + \sigma \begin{bmatrix} \frac{1}{\sqrt{M}} & \\ & \frac{1}{\sqrt{N}} \end{bmatrix} \frac{d}{dt} \begin{bmatrix} \bar{w}_x \\ \bar{w}_y \end{bmatrix}$$

Here, $M = \theta \begin{bmatrix} -\beta_1 & \beta_1 \\ \beta_2 & -\beta_2 \end{bmatrix}$, Denote $M = \rho_1 S$, $N = \rho_2 S$, where $\rho_1 > 0$, $\rho_2 > 0$ and $\rho_1 + \rho_2 = 1$. Given $\rho = \begin{bmatrix} \frac{M}{M+N} \\ \frac{N}{M+N} \end{bmatrix} = \begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix}$, $R = \begin{bmatrix} \frac{M}{M+N} & \\ & \frac{N}{M+N} \end{bmatrix} = \begin{bmatrix} \rho_1 & \\ & \rho_2 \end{bmatrix}$, and $R^{-1} = (M+N) \begin{bmatrix} \frac{1}{M} & \\ & \frac{1}{N} \end{bmatrix} = \begin{bmatrix} \rho_1^{-1} & \\ & \rho_2^{-1} \end{bmatrix}$.

$$\rho^T e^{Mt} R^{-1} (e^{Mt})^T \rho = 1 + \frac{1}{\rho_1\rho_2} \frac{(\rho_1\beta_1 - \rho_2\beta_2)^2}{(\beta_1 + \beta_2)^2} \left(1 - 2e^{-\theta(\beta_1+\beta_2)t} + e^{-2\theta(\beta_1+\beta_2)t}\right)$$

$$\begin{aligned} \sigma_T^2 &= \frac{\sigma^2}{S} \int_0^T \rho^T e^{Mt} R^{-1} (e^{Mt})^T \rho dt \\ &= \frac{\sigma^2 T}{S} + \frac{\sigma^2}{S} \frac{1}{\rho_1\rho_2} \frac{(\rho_1\beta_1 - \rho_2\beta_2)^2}{(\beta_1 + \beta_2)^2} \left[T - \frac{3}{2\theta(\beta_1 + \beta_2)} + \left(2e^{-\theta(\beta_1+\beta_2)T} - \frac{1}{2}e^{-2\theta(\beta_1+\beta_2)T}\right)\right] \end{aligned}$$

Then we have $\sigma_T^2 \geq \frac{\sigma^2 T}{S}$, if $T \geq \frac{3}{2\theta(\beta_1+\beta_2)}$. And the minimum can be obtained when $\rho_1\beta_1 = \rho_2\beta_2$, or equivalently, $\beta_1 M = \beta_2 N$. □

3.5 Conclusions

In this chapter, a generalized bi-stable, mean-field model has been established by introducing a group interactive mechanism. The stochastic equilibrium of the system has been studied and we give an asymptotic expression of this equilibrium by small h analysis. Moreover we also studied generalized models with heterogeneous ratios between in-group and inter-group interactions and heterogeneous group sizes in different groups. For $h = 0$ the probabilities of systemic transition for these hetergeneous models have been computed and we can conclude that due to the small heterogeneities on model parameters in different groups, the system will become more likely to fail.

There are many questions remain to be studied. A nonlinear mean-field interaction might be considered, and the large deviations theory and perturbation theory might help us to study the systemic transition probabilities in these cases. Moreover, as mentioned in [47], we can generalize this bi-stable mean-field model by using a general symmetric potential instead of the double-well potential. With numerical simulations, we can see the phenomenon of phase transitions still remains. However one of the obstacles is that the convexity of first order moment function $m(\xi)$ might get lost.

Chapter 4 The Linear Nonlocal-in-Time Problems

In the previous chapter we have generalized the following stochastic bi-stable mean-field model with introducing inter-group interactions.

$$\begin{cases} dx_k(t) = \theta(\bar{x}(t) - x_k(t))dt + f(x_k(t))dt + \sigma dW_k(t), & t > 0, \\ x_k(0) = g_k, \end{cases}$$

for $k = 1, 2, \ldots, N$. In this stochastic bi-stable mean-field model, instead of the discrete particle system we now consider a continuous case formally. We introduce a function $u(t, x)$ in $[0, T] \times \Omega$ instead of $\{x_k\}$ and replace the group average $\bar{x}$ by a weighted integration of u over the x-neighborhood $[x - \sigma, x + \sigma]$. Also we consider the nonlocal-in-time operator instead of the classical derivative. Then the stochastic mean-field problem becomes the stochastic nonlocal reaction-diffusion problem:

$$\begin{cases} \mathcal{G}_t^\delta u(t, x) = \mathcal{L}_x^\sigma u(t, x) + f(u(t, x)) + \sigma \mathcal{G}_t^\delta W(t, x), & \text{in } (0, T] \times \Omega, \\ u(t, x) = g(t, x), & \text{in } [-\delta, 0] \times \Omega. \end{cases}$$

In general, we can substitute the nonlocal operator $\mathcal{L}_x^\sigma$ by an abstract operator A in Banach space and obtain formally a stochastic nonlocal-in-time reaction-dffusion problem in Banach spaces,

$$\begin{cases} \mathcal{G}_\delta u(t) = Au(t) + f(u(t), t) + \sigma \mathcal{G}_\delta W_t, & t \in (0, T], \\ u(t) = g(t), & t \in [-\delta, 0]. \end{cases} \tag{4.1}$$

In order to study this nonlocal-in-time problem, we shall decompose it into three sub-problems: the linear nonlocal-in-time problem in Banach spaces, the nonlinear nonlocal-in-time problem, and the stochastic nonlocal-in-time problem. In this chapter, we will consider the first sub-problem starting from the basic linear nonlocal-in-time problem for ODE models. Based on the theory developed by Du et al. in [2], we shall first extend the well-posedness results and the localization theory to the linear nonlocal-in-time problems with more general drift term $\lambda u(t)$, for $\lambda \in \mathbb{R}$, with the help of a special version of the product rule for nonlocal-in-time operator. Then considering a linear unbounded dissipative operator A, we will establish the well-posedness theorems and the semigroup property of the linear nonlocal-in-time problems in Banach spaces.

4.1 Linear Nonlocal-in-Time Problems for ODE Models

We start from the deterministic linear nonlocal-in-time problem in the form of

$$\begin{cases} \mathcal{G}_\delta u(t) + \lambda u(t) = f(t), & t \in (0, T], \\ u(t) = g(t), & t \in [-\delta, 0]. \end{cases} \tag{4.2}$$

For $\lambda > 0$ and $\lambda = 0$ cases, the well-posedness with the a priori estimate of the solution has been given in [2] [48], as Theorem 2.2.1 and Theorem 2.2.2 in Chapter 2.

In the classical ODE theory, applying the integrating factor method in this following initial value problem with a drift term $\lambda u(t)$,

$$\begin{cases} \dfrac{d}{dt} u(t) + \lambda u(t) = 0, & t \in (0, T], \\ u(0) = u_0, \end{cases}$$

we can introduce a new function $v(t) = e^{\lambda t} u(t)$, such that

$$\frac{d}{dt} v(t) = e^{\lambda t} \left(\frac{d}{dt} u(t) + \lambda u(t) \right) = 0.$$

Then $v(t)$ satisfies the new problem without the drift term,

$$\begin{cases} \dfrac{d}{dt}v(t) = 0, & t \in (0, T], \\ \quad v(0) = u_0, & t = 0, \end{cases}$$

For the nonlocal-in-time problem (4.2) with $\lambda < 0$, by applying a nonlocal version of the integrating factor method, we can transform the problem into an equivalent nonlocal-in-time problem without the drift term.

Lemma 4.1.1. *The nonlocal-in-time derivative of exponential functions $e^{\lambda t}$, $\lambda \in \mathbb{R}$, is given as*

$$\mathcal{G}_\delta(e^{\lambda t}) = \lambda_\delta e^{\lambda t},$$

where

$$\lambda_\delta = h_\delta(\lambda) := \int_0^\delta s\rho_\delta(s)(1 - e^{-\lambda s})ds = \int_0^\delta s^2 \rho_\delta(s) \frac{1 - e^{-\lambda s}}{s} ds = \lambda + \mathcal{O}(\delta),$$

with the same sign as λ, converges to λ, as $\delta \to 0$.

Proof. By the definition of the nonlocal-in-time operator $\mathcal{G}_\delta$,

$$\mathcal{G}_\delta(e^{\lambda t}) = \int_0^\delta s\rho_\delta(s)(e^{\lambda t} - e^{\lambda(t-s)}) = \lambda_\delta e^{\lambda t}.$$

In addition, by the definition of h_δ, $h_\delta(0) = 0$ and λ_δ has the same sign as λ. By the Taylor's expansion of e^{-x},

$$\int_0^\delta s\rho_\delta(s)(1 - e^{-\lambda s})ds = \int_0^\delta s\rho_\delta(s)(\lambda s + \mathcal{O}(s^2))ds = \lambda + \mathcal{O}(\delta). \qquad \square$$

Lemma 4.1.2. *The function $h_\delta(x) = \int_0^\delta s\rho_\delta(s)(1 - e^{-sx})ds$ is a continuously differentiable and invertible function from $\mathbb{R}$ to its range $h_\delta(\mathbb{R})$, which is $\mathbb{R}$ when the nonlocal kernel $s\rho_\delta(s)$ is non-integrable and is $(-\infty, b_\delta)$ for integrable kernels.*

Proof. The function h_δ is continuously differentiable on the domain $\mathbb{R}$ with strictly positive derivatives,

$$\frac{d}{dt}h_\delta(x) = \int_0^\delta s^2 \rho_\delta(s) e^{-sx} ds \in (0, e^{-\min(0,x)\cdot\delta}).$$

Therefore by the inverse function theorem, we know h_δ is a invertible function from the domain $\mathbb{R}$ to its range $h_\delta(\mathbb{R})$ and the inverse function of h_δ, $h_\delta^{-1}(y)$, is a continuously differentiable function on $h_\delta(\mathbb{R})$. When the kernel $s\rho_\delta(s)$ is integrable, $h_\delta(+\infty) = b_\delta$. Therefore $h_\delta(\mathbb{R}) = (-\infty, b_\delta)$. When the kernel is non-integrable, the range of h_δ is $\mathbb{R}$ since $h_\delta(+\infty) = +\infty$. □

Remark. *Furthermore, since $h_\delta(0) = 0$ and $h_\delta(-\infty) = -\infty$, h_δ is also one-to-one as a map from $\mathbb{R}^-$ to itself. Therefore the inverse function $h_\delta^{-1}(y)$ is always well-defined for $y < 0$ whether the nonlocal kernel is integrable or not.*

With the invertibility of function h_δ, we then have the following proposition.

Prop 4.1.3. *For any $\lambda \in h_\delta(\mathbb{R})$, with h_δ defined in Lemma 4.1.1,*

$$\mathcal{G}_\delta\left(e^{h_\delta^{-1}(\lambda)t}\right) = \lambda e^{h_\delta^{-1}(\lambda)t}.$$

Lemma 4.1.4 (Product Rules for Nonlocal-in-Time Gradient)**.** *Given functions u, v, and $\lambda \in \mathbb{R}$, such that $\mathcal{G}_\delta u$, $\mathcal{G}_\delta v$, $\mathcal{G}_\delta(uv)$ and $\mathcal{G}_\delta(e^{\lambda t}u(t))$ are well-defined, we have that*

$$\begin{aligned}\mathcal{G}_\delta(u\cdot v)(t) &= v(t)\mathcal{G}_\delta u(t) + u(t)\mathcal{G}_\delta v(t) - \int_0^\delta s\rho_\delta(s)(u(t)-u(t-s))(v(t)-v(t-s))ds\\ &= v(t)\mathcal{G}_\delta u(t) + \int_0^\delta s\rho_\delta(s)u(t-s)(v(t)-v(t-s))ds.\end{aligned}$$

In addition, taking $v(t)$ to be exponential functions, we have that

$$\mathcal{G}_\delta(e^{\lambda t}u(t)) = \lambda_\delta e^{\lambda t}u(t) + e^{\lambda t}C_\delta^\lambda \mathcal{G}_\delta^\lambda u(t),$$

where

$$\mathcal{G}_\delta^\lambda u(t) := (C_\delta^\lambda)^{-1}\int_0^\delta s\rho_\delta(s)e^{-\lambda s}(u(t)-u(t-s))ds,$$

and

$$C_\delta^\lambda := \int_0^\delta s^2\rho_\delta(s)e^{-\lambda s}ds \to 1, \quad \text{as } \delta \to 0.$$

Proof. The proof is straightforward by the definition of the nonlocal-in-time operator:

$$\begin{aligned}
\mathcal{G}_\delta(u(t)v(t)) &= \int_0^\delta s\rho_\delta(s)(u(t)v(t) - u(t-s)v(t-s))ds \\
&= \int_0^\delta s\rho_\delta(s)(u(t)v(t) - u(t)v(t-s))ds + \int_0^\delta s\rho_\delta(s)(u(t)v(t) - u(t-s)v(t))ds \\
&\quad + \int_0^\delta s\rho_\delta(s)(-u(t)v(t) + u(t)v(t-s) + u(t-s)v(t) - u(t-s)v(t-s))ds \\
&= v(t)\mathcal{G}_\delta u(t) + u(t)\mathcal{G}_\delta v(t) - \int_0^\delta s\rho_\delta(s)(u(t) - u(t-s))(v(t) - v(t-s))ds \\
&= v(t)\mathcal{G}_\delta u(t) + \int_0^\delta s\rho_\delta(s)u(t-s)(v(t) - v(t-s))ds.
\end{aligned}$$

$$\begin{aligned}
\mathcal{G}_\delta(e^{\lambda t}u(t)) &= \int_0^\delta s\rho_\delta(s)\left[e^{\lambda t}u(t) - e^{\lambda(t-s)}u(t-s)\right]ds \\
&= \int_0^\delta s\rho_\delta(s)\left[e^{\lambda t}u(t) - e^{\lambda(t-s)}u(t)\right]ds \\
&\quad + \int_0^\delta s\rho_\delta(s)\left[e^{\lambda(t-s)}u(t) - e^{\lambda(t-s)}u(t-s)\right]ds \\
&= u(t)\mathcal{G}_\delta(e^{\lambda t}) + e^{\lambda t}\int_0^\delta s\rho_\delta(s)e^{-\lambda s}(u(t) - u(t-s))ds \\
&= \lambda_\delta e^{\lambda t}u(t) + e^{\lambda t}C_\delta^\lambda \mathcal{G}_\delta^\lambda u(t).
\end{aligned}$$

□

Remark. *The nonlocal operator* $\mathcal{G}_\delta^\lambda$ *gives a new nonlocal-in-time gradient since*

$$(C_\delta^\lambda)^{-1}\int_0^\delta s^2\rho_\delta(s)e^{-\lambda s}ds = 1.$$

Now let us introduce a nonlocal version of the integrating factor method with this product rule. Given $-\lambda \in h_\delta(\mathbb{R})$, replacing $u(t)$ by $e^{h_\delta^{-1}(-\lambda)t}v(t)$ in the nonlocal-in-time problem with drift term (4.2), we have that

$$\begin{aligned}
\mathcal{G}_\delta u(t) + \lambda u(t) &= \mathcal{G}_\delta(e^{h_\delta^{-1}(-\lambda)t}v(t)) + \lambda e^{-h_\delta^{-1}(-\lambda)t}v(t) \\
&= -\lambda e^{h_\delta^{-1}(-\lambda)t}v(t) + e^{h_\delta^{-1}(-\lambda)t}C_\delta^{h_\delta^{-1}(-\lambda)}\mathcal{G}_\delta^{h_\delta^{-1}(-\lambda)}v(t) + \lambda e^{-h_\delta^{-1}(-\lambda)t}v(t) \\
&= e^{h_\delta^{-1}(-\lambda)t}C_\delta^{h_\delta^{-1}(-\lambda)}\mathcal{G}_\delta^{h_\delta^{-1}(-\lambda)}v(t)
\end{aligned}$$

So

$$\mathcal{G}_\delta^{h_\delta^{-1}(-\lambda)}v(t) = (C_\delta^{h_\delta^{-1}(-\lambda)})^{-1}e^{-h_\delta^{-1}(-\lambda)t}f(t)$$

Hence the function $v(t) = e^{-h_\delta^{-1}(-\lambda)t}u(t)$ satisfies a new nonlocal-in-time problem with no drift term.

$$\begin{cases} \tilde{\mathcal{G}}_\delta v(t) = \tilde{f}(t), & t \in (0,T], \\ \quad v(t) = \tilde{g}(t), & t \in [-\delta, 0]. \end{cases} \tag{4.3}$$

Here,

$$\begin{aligned} \tilde{\mathcal{G}}_\delta v(t) &= \mathcal{G}_\delta^{h_\delta^{-1}(-\lambda)} v(t), \\ \tilde{f}(t) &= (C_\delta^{h_\delta^{-1}(-\lambda)})^{-1} e^{-h_\delta^{-1}(-\lambda)t} f(t) \in L^2(0,T), \\ \tilde{g}(t) &= e^{-h_\delta^{-1}(-\lambda)t} g(t) \in L^\infty(-\delta, 0). \end{aligned}$$

From Theorem 2.2.2, we know that there is a unique solution $v(t) \in V_\delta$ with the a priori estimate,

$$\|v\|_{L^2(0,T)} \le c(T^{1/2} \|\tilde{g}(t)\|_{L^\infty(-\delta,0)} + \max(\delta, T) \|\tilde{f}(t)\|_{L^2(0,T)}).$$

Moreover, when $\lambda < 0$ and $-\lambda \in h_\delta(\mathbb{R})$,

$$\begin{aligned} \|\tilde{f}(t)\|_{L^2(0,T)} &= \left\| (C_\delta^{h_\delta^{-1}(-\lambda)})^{-1} e^{-h_\delta^{-1}(-\lambda)t} f(t) \right\|_{L^2(0,T)} \le (C_\delta^{h_\delta^{-1}(-\lambda)})^{-1} \|f(t)\|_{L^2(0,T)} \\ \|\tilde{g}(t)\|_{L^\infty(-\delta,0)} &= \left\| e^{-h_\delta^{-1}(-\lambda)t} g(t) \right\|_{L^\infty(-\delta,0)} \le \|g(t)\|_{L^\infty(-\delta,0)}, \\ (C_\delta^{h_\delta^{-1}(-\lambda)})^{-1} &= \left(\int_0^\delta s^2 \rho_\delta(s) e^{-h_\delta^{-1}(-\lambda)s} ds \right)^{-1} \le e^{h_\delta^{-1}(-\lambda)\delta}, \end{aligned}$$

The correspondence $u(t) = e^{h_\delta^{-1}(-\lambda)t} v(t)$ is one-to-one in V_δ, so the nonlocal-in-time problem with the drift term (4.2) for $-\lambda \in h_\delta(\mathbb{R})$ has a unique solution $u(t) \in V_\delta$ satisfying the a priori estimate

$$\begin{aligned} \|u(t)\|_{L^2(0,T)} &= \|e^{h_\delta^{-1}(-\lambda)t} v(t)\|_{L^2(0,T)} \\ &\le e^{h_\delta^{-1}(-\lambda)T} \|v\|_{L^2(0,T)} \\ &\le c e^{h_\delta^{-1}(-\lambda)T} (T^{1/2} \|\tilde{g}(t)\|_{L^\infty(-\delta,0)} + \max(\delta,T) \|\tilde{f}(t)\|_{L^2(0,T)}) \\ &\le c e^{h_\delta^{-1}(-\lambda)T} \left(T^{1/2} \|g(t)\|_{L^\infty(-\delta,0)} + \max(\delta,T) e^{h_\delta^{-1}(-\lambda)\delta} \|f(t)\|_{L^2(0,T)} \right). \end{aligned}$$

Theorem 4.1.1. *Let $g \in L^\infty(-\delta,0)$, $f \in L^2(0,T)$ and $-\lambda \in h_\delta(\mathbb{R})$, then there is a unique solution $u \in V_\delta$ of the nonlocal-in-time problem (4.2) and it holds that*

$$\|u\|_{L^2(0,T)} \le c e^{h_\delta^{-1}(-\lambda)T} \left(T^{1/2} \|g(t)\|_{L^\infty(-\delta,0)} + \max(\delta,T) e^{h_\delta^{-1}(-\lambda)\delta} \|f(t)\|_{L^2(0,T)} \right).$$

where the invertible function $h_\delta(x)$ is defined in the Lemma 4.1.1.

Moreover, as the nonlocal horizon δ goes to zero, the localization theory in [2], Theorem 2.2.3, can also be extended to the cases when $\lambda < 0$ and $-\lambda \in h_\delta(\mathbb{R})$ as follows.

Prop 4.1.5. *Suppose that $\lambda < 0$ and $-\lambda \in h_\delta(\mathbb{R})$. Let u_δ and u be the solutions of problems*

$$\begin{cases} \mathcal{G}_\delta u_\delta(t) + \lambda u_\delta(t) = f(t), & t \in (0,T], \\ u_\delta(t) = g(t), & t \in [-\delta, 0], \end{cases}$$

and

$$\begin{cases} u'(t) + \lambda u(t) = f(t), & t \in (0,T], \\ u(0) = g(0), \end{cases}$$

respectively, with $g \equiv 0$ and $f \in L^2(0,T)$. Then it holds that

$$\lim_{\delta \to 0} \|u_\delta - u\|_{L^2(0,T)} = 0.$$

Proof. By the nonlocal integrating factor method we have developed in this section, let $\tilde{u}_\delta = e^{-h_\delta^{-1}(-\lambda)t} u_\delta(t)$ and $\tilde{u}(t) = e^{\lambda t} u(t)$, then $\tilde{u}_\delta$ and $\tilde{u}$ satisfies the following two problems

$$\begin{cases} \tilde{\mathcal{G}}_\delta \tilde{u}_\delta(t) = \tilde{f}(t), & t \in (0,T], \\ \tilde{u}_\delta(t) = 0, & t \in [-\delta, 0], \end{cases}$$

and

$$\begin{cases} \tilde{u}'(t) = e^{\lambda t} f(t), & t \in (0,T], \\ \tilde{u}(0) = 0, \end{cases}$$

respectively, where,

$$\tilde{\mathcal{G}}_\delta u(t) = \mathcal{G}_\delta^{h_\delta^{-1}(-\lambda)} u(t) \text{ and } \tilde{f}(t) = (C_\delta^{h_\delta^{-1}(-\lambda)})^{-1} e^{-h_\delta^{-1}(-\lambda)t} f(t).$$

In addition, let $\tilde{v}_\delta$ satisfy the problem

$$\begin{cases} \tilde{\mathcal{G}}_\delta \tilde{v}_\delta(t) = e^{\lambda t} f(t), & t \in (0,T], \\ \tilde{v}_\delta(t) = 0, & t \in [-\delta, 0]. \end{cases}$$

By Theorem 2.2.3, we know that $\lim_{\delta \to 0} \|\tilde{v}_\delta - \tilde{u}\|_{L^2(0,T)} = 0$. And by Theorem 2.2.2, we have that

$$\|\tilde{u}_\delta - \tilde{v}_\delta\|_{L^2(0,T)} \le c \max(\delta, T) \|\tilde{f}(t) - e^{\lambda t} f(t)\|_{L^2(0,T)}.$$

Since $(C_\delta^{h_\delta^{-1}(-\lambda)})^{-1} e^{-h_\delta^{-1}(-\lambda)t}$ converges to $e^{\lambda t}$ in $L^\infty(0,T)$ as $\delta \to 0$, we then have $\lim_{\delta \to 0} \|\tilde{u}_\delta - \tilde{v}_\delta\|_{L^2(0,T)} = 0$.Therefore,

$$\lim_{\delta \to 0} \|\tilde{u}_\delta - \tilde{u}\|_{L^2(0,T)} = 0. \tag{4.4}$$

In addition, from the definition of $\tilde{u}_\delta$ and $\tilde{u}$, we have that

$$\|u_\delta - u\|_{L^2(0,T)} = \|e^{h_\delta^{-1}(-\lambda)t} \tilde{u}_\delta(t) - e^{-\lambda t} \tilde{u}(t)\|_{L^2(0,T)},$$

The convergence $e^{h_\delta^{-1}(-\lambda)t} \to e^{-\lambda t}$ in $L^\infty(0,T)$ as $\delta \to 0$ then implies the convergence

$$\lim_{\delta \to 0} \|u_\delta - u\|_{L^2(0,T)} = 0. \qquad \square$$

4.2 Linear Nonlocal-in-Time Problem in Banach Spaces for PDE Models

In 2017, *Du* et al. established the first study of the nonlocal-in-time parabolic model in [2] in the form of

$$\begin{cases} \mathcal{G}_\delta u(x,t) - \Delta u(x,t) = f(x,t), & \text{in } \Omega \times [0,T], \\ u(x,t) = 0, & \text{in } \partial\Omega \times [0,T], \\ u(x,t) = g(x,t), & \text{in } \Omega \times [-\delta, 0), \end{cases} \tag{4.5}$$

where Ω is a bounded convex polygonal domain in $\mathbb{R}^d$ $(d = 1, 2, 3)$ with a boundary $\partial\Omega$, g and f are given functions and $T > 0$ is a positive terminal time.

By the means of the nonlocal integrating factor method we developed in the previous section, we have the well-posedness theorems for linear nonlocal-in-time problem with and without a drift term. Now, let us take one more step and look at the linear nonlocal-in-time problem in Banach spaces with integrable kernels,

$$\begin{cases} \mathcal{G}_\delta u(t) = Au(t), & t \in (0,T], \\ u(t) = g(t), & t \in [-\delta, 0]. \end{cases} \tag{4.6}$$

Here we assume that the linear (unbounded) dissipative operator A satisfies the conditions in Theorem 4.2.3 (*Hille-Yosida*) for some Banach space B.

Considering only the integrable kernels, the integral form of the nonlocal-in-time problem (4.6) is given as

$$u(t) = (b_\delta I - A)^{-1} \left[\int_0^t s\rho_\delta(s) \mathbb{1}_{[0,\delta]}(s) u(t-s) ds + \int_t^\delta s\rho_\delta(s) \mathbb{1}_{[0,\delta]}(s) g(t-s) ds \right], \tag{4.7}$$

where $b_\delta = \int_0^\delta s\rho_\delta(s)ds$ and $\mathbb{1}_{[0,\delta]}(s)$ is a characteristic function. And the integrations involved in the nonlocal-in-time problems in Banach spaces are *Bochner* integrals.

4.2.1 Well-posedness of the nonlocal problems in Banach spaces

Instead of the nonlocal equation in (4.6), we consider the mild solutions of the integral form of this nonlocal-in-time problem (4.7). To prove the well-posedness of this integral equation, we introduce the map $\mathcal{K}$ defined by

$$\mathcal{K}u(t) = (b_\delta I - A)^{-1} \left[\int_0^t s\rho_\delta(s) \mathbb{1}_{[0,\delta]}(s) u(t-s) ds + \int_t^\delta s\rho_\delta(s) \mathbb{1}_{[0,\delta]}(s) g(t-s) ds \right].$$

Then

$$\begin{aligned} (\mathcal{K}u - \mathcal{K}v)(t) &= (b_\delta I - A)^{-1} \int_0^t s\rho_\delta(s) \mathbb{1}_{[0,\delta]}(s)(u-v)(t-s) ds, \\ \|\mathcal{K}u - \mathcal{K}v\| &\le b_\delta \|(b_\delta I - A)^{-1}\| \cdot \|u - v\|. \end{aligned}$$

For the dissipative operator A, we have that $b_\delta \|(b_\delta I - A)^{-1}\| \le 1$. To exploit the Banach fixed-point theorem, a drift term $\lambda u(t)$ with $\lambda > 0$ added to the left-hand-side of the nonlocal-in-time problem (4.6) is then required in order to guarantee the map $\mathcal{K}$ a contraction mapping, which results in the following modified nonlocal-in-time problem

$$\begin{cases} \mathcal{G}_\delta u(t) + \lambda u(t) = Au(t), & t \in (0,T], \\ u(t) = g(t), & t \in [-\delta, 0], \end{cases} \tag{4.8}$$

with the corresponding integral form

$$u(t) = ((b_\delta+\lambda)I-A)^{-1}\left[\int_0^t s\rho_\delta(s)\mathbb{1}_{[0,\delta]}(s)u(t-s)ds + \int_t^\delta s\rho_\delta(s)\mathbb{1}_{[0,\delta]}(s)g(t-s)ds\right]. \tag{4.9}$$

With the help of the nonlocal integrating factor method we have developed in the previous section, we can draw the equivalence between two nonlocal-in-time problems in Banach spaces (4.6) and (4.8).

First, given $u(t)$ satisfies the problem with drift term (4.8), let $v(t) = e^{-h_\delta^{-1}(-\lambda)t}u(t)$, then

$$\begin{aligned}
\mathcal{G}_\delta u(t) &= \mathcal{G}_\delta(e^{h_\delta^{-1}(-\lambda)t}v(t)) \\
&= v(t)\mathcal{G}_\delta\left(e^{h_\delta^{-1}(-\lambda)t}\right) + e^{h_\delta^{-1}(-\lambda)t}\int_0^\delta s\rho_\delta(s)e^{-h_\delta^{-1}(-\lambda)s}(v(t)-v(t-s))ds \\
&= -\lambda v(t)e^{h_\delta^{-1}(-\lambda)t} + e^{h_\delta^{-1}(-\lambda)t}\int_0^\delta s\rho_\delta(s)e^{-h_\delta^{-1}(-\lambda)s}(v(t)-v(t-s))ds \\
&= -\lambda u(t) + e^{h_\delta^{-1}(-\lambda)t}\int_0^\delta s\rho_\delta(s)e^{-h_\delta^{-1}(-\lambda)s}(v(t)-v(t-s))ds,
\end{aligned}$$

which implies that $v(t,\cdot)$ satisfies the nonlocal problem

$$\begin{cases}
\int_0^\delta s\rho_\delta(s)e^{-h_\delta^{-1}(-\lambda)s}(v(t)-v(t-s))ds = Av(t), & t\in(0,T], \\
v(t) = e^{-h_\delta^{-1}(-\lambda)t}g(t), & t\in[-\delta,0].
\end{cases}$$

Even though the left-hand-side integral is not identical to the nonlocal-in-time gradient we considered, i.e.,

$$\int_0^\delta s\rho_\delta(s)e^{-h_\delta^{-1}(-\lambda)s}(v(t)-v(t-s))ds \neq \mathcal{G}_\delta v(t),$$

From Lemma 4.1.4,

$$\mathcal{G}_\delta^{h_\delta^{-1}(-\lambda)}v(t) = (C_\delta^{h_\delta^{-1}(-\lambda)})^{-1}\int_0^\delta s\rho_\delta(s)e^{-h_\delta^{-1}(-\lambda)s}(v(t)-v(t-s))ds,$$

gives a new nonlocal-in-time gradient operator with the kernel $(C_\delta^{h_\delta^{-1}(-\lambda)})^{-1} s\rho_\delta(s) e^{-h_\delta^{-1}(-\lambda)s}$. Thus $v(t)$ satisfies a nonlocal-in-time problem without the drift term,

$$\begin{cases} \tilde{\mathcal{G}}_\delta v(t) = \tilde{A} v(t), & t \in (0,T], \\ \quad v(t) = \tilde{g}(t), & t \in [-\delta, 0], \end{cases}$$

with a modified nonlocal-in-time gradient $\tilde{\mathcal{G}}_\delta = \mathcal{G}_\delta^{h_\delta^{-1}(-\lambda)}$, a modified operator $\tilde{A} = (C_\delta^{h_\delta^{-1}(-\lambda)})^{-1} A$, and a modified initial data $\tilde{g}(t) = g^{h_\delta^{-1}(-\lambda)}(t) = e^{-h_\delta^{-1}(-\lambda)t} g(t)$.

Conversely, if $u(t)$ satisfies the nonlocal-in-time problem in Banach spaces without drift (4.6). Let $v(t) = e^{-\lambda t} u(t)$ for some constant $\lambda > 0$, then $v(t)$ satisfies

$$\begin{cases} \mathcal{G}_\delta^\lambda v(t) + (C_\delta^\lambda)^{-1} h_\delta(\lambda) v(t) = (C_\delta^\lambda)^{-1} A v(t), & t \in (0,T], \\ \qquad\qquad\qquad\qquad v(t) = e^{-\lambda t} g(t), & t \in [-\delta, 0], \end{cases}$$

with $(C_\delta^\lambda)^{-1} h_\delta(\lambda) > 0$. Thus the function $v(t)$ satisfies a nonlocal-in-time problem with a drift term on the left-hand-side with positive coefficient.

Theorem 4.2.1. *Let B be a Banach space and nonlocal kernel $s\rho_\delta(s) \in L^1(0,\delta)$. Given initial data $g \in L^\infty([-\delta,0];B)$, $\lambda > 0$, and the dissipative linear (unbounded) operator A satisfies the conditions in* Hille-Yosida *Theorem 4.2.3, there is a unique mild solution $u \in L^\infty([0,T];B)$ which satisfies the integral form*

$$u(t) = ((b_\delta + \lambda) I - A)^{-1} \left[\int_0^t s\rho_\delta(s) \mathbb{1}_{[0,\delta]}(s) u(t-s) ds + \int_t^\delta s\rho_\delta(s) \mathbb{1}_{[0,\delta]}(s) g(t-s) ds \right],$$

of the nonlocal-in-time problem in Banach spaces with integrable kernel (4.8) and it holds that $\|u\|_{L^\infty([0,T];B)} \le \|g\|_{L^\infty([-\delta,0];B)}$.

Proof. First we define the map $\mathcal{K}$ by

$$\mathcal{K}u(t) := ((b_\delta + \lambda) I - A)^{-1} \left[\int_0^t s\rho_\delta(s) \mathbb{1}_{[0,\delta]}(s) u(t-s) ds + \int_t^\delta s\rho_\delta(s) \mathbb{1}_{[0,\delta]}(s) g(t-s) ds \right],$$

where $b_\delta = \int_0^\delta s\rho_\delta(s) ds$. Here $g \in L^\infty([0,T];B)$ and $\mathcal{D}(A) \subseteq B$ imply that $\mathcal{K}u(t) \in B$ for $t \in [0,T]$. Since the conditions in *Hille-Yosida* Theorem are satisfied by the dissipative operator A, we have that

$$b_\delta \left\| [(b_\delta + \lambda)I - A]^{-1} \right\| \le \frac{b_\delta}{b_\delta + \lambda} < 1.$$

For $u \in L^\infty([0,T];B)$, such that $\|u\|_{L^\infty([0,T];B)} \le \|g\|_{L^\infty([-\delta,0];B)}$, we have that

$$\begin{aligned}
(\mathcal{K}u - \mathcal{K}v)(t) &= ((b_\delta + \lambda)I - A)^{-1} \int_0^t s\rho_\delta(s)\mathbb{1}_{[0,\delta]}(s)(u-v)(t-s)ds, \\
\|\mathcal{K}u - \mathcal{K}v\|_{L^\infty([0,T];B)} &\le b_\delta \left\| ((b_\delta + \lambda)I - A)^{-1} \right\| \cdot \|u - v\|_{L^\infty([0,T];B)} \\
&\le \frac{b_\delta}{b_\delta + \lambda} \|u - v\|_{L^\infty([0,T];B)}.
\end{aligned}$$

Here $\frac{b_\delta}{b_\delta+\lambda} < 1$, so the map $\mathcal{K}$ is contractive.

$$\begin{aligned}
\|\mathcal{K}u(t)\|_B &\le \left\| ((b_\delta + \lambda)I - A)^{-1} \right\| \cdot \left(b_\delta(t) \|u\|_{L^\infty([0,T];B)} + (b_\delta - b_\delta(t)) \|g\|_{L^\infty([-\delta,0];B)} \right) \\
&\le b_\delta \left\| ((b_\delta + \lambda)I - A)^{-1} \right\| \cdot \max\left(\|u\|_{L^\infty([0,T];B)}, \|g\|_{L^\infty([-\delta,0];B)} \right), \\
\|\mathcal{K}u\|_{L^\infty([0,T];B)} &\le b_\delta \left\| ((b_\delta + \lambda)I - A)^{-1} \right\| \cdot \|g\|_{L^\infty([-\delta,0];B)} \\
&\le \frac{b_\delta}{b_\delta + \lambda} \|g\|_{L^\infty([0,T];B)} < \|g\|_{L^\infty([-\delta,0];B)}.
\end{aligned}$$

This implies that $\mathcal{K}$ maps from the Banach space

$$V = \{u \in L^\infty([0,T];B) : \|u\|_{L^\infty([0,T];B)} \le \|g\|_{L^\infty([-\delta,0];B)}\}, \tag{4.10}$$

into itself. Thus $\mathcal{K}$ is a contraction mapping on V. By Banach fixed-point theorem, there is a unique solution $u \in V$ satisfies the integral form

$$u(t) = ((b_\delta + \lambda)I - A)^{-1} \left[\int_0^t s\rho_\delta(s)\mathbb{1}_{[0,\delta]}(s)u(t-s)ds + \int_t^\delta s\rho_\delta(s)\mathbb{1}_{[0,\delta]}(s)g(t-s)ds \right],$$

of the nonlocal-in-time problem in Banach spaces (4.8). □

Lemma 4.2.1. *For $c > 0$, the operator A satisfies the conditions in* Hille-Yosida *Theorem 4.2.3 is equivalent to the operator cA satisfies those conditions.*

Proof. We only need to show that

$$\forall \lambda > 0, \|R(\lambda : A)\| \le \frac{1}{\lambda} \quad \Longleftrightarrow \quad \forall \lambda > 0, \|R(\lambda : cA)\| \le \frac{1}{\lambda}.$$

For “$\Longrightarrow$":

$$\|R(\lambda : cA)\| = \left\|(\lambda\mathbb{I} - cA)^{-1}\right\| = c^{-1}\left\|(c^{-1}\lambda\mathbb{I} - A)^{-1}\right\| \le c^{-1}\frac{1}{c^{-1}\lambda} = \frac{1}{\lambda}.$$

For “$\Longleftarrow$":

$$\|R(\lambda : A)\| = \left\|(\lambda\mathbb{I} - A)^{-1}\right\| = c\left\|(c\lambda\mathbb{I} - cA)^{-1}\right\| \le c\frac{1}{c\lambda} = \frac{1}{\lambda}. \qquad \square$$

With this lemma and the equivalence of two nonlocal-in-time problems (4.6) and (4.8), we then have the following proposition for the nonlocal-in-time problems in Banach spaces without drift (4.6).

Prop 4.2.2. *Let B be a Banach space and nonlocal kernel $s\rho_\delta(s) \in L^1(0,\delta)$. Given initial data $g \in L^\infty([-\delta,0];B)$, and the dissipative linear (unbounded) operator A satisfies the conditions in Theorem 4.2.3, there is a unique mild solution $u \in L^\infty([0,T];B)$ which satisfies the integral form*

$$u(t) = (b_\delta I - A)^{-1}\left[\int_0^t s\rho_\delta(s)\mathbb{1}_{[0,\delta]}(s)u(t-s)ds + \int_t^\delta s\rho_\delta(s)\mathbb{1}_{[0,\delta]}(s)g(t-s)ds\right],$$

of the nonlocal-in-time problem in Banach spaces with integrable kernel (4.6) and it holds that $\|u\|_{L^\infty([0,T];B)} \le \|g\|_{L^\infty([-\delta,0];B)}$.

Proof. Set λ to be a positive constant and $v(t) = e^{-\lambda t}u(t)$, the function $u(t)$ solves the nonlocal-in-time problem

$$\begin{cases} \mathcal{G}_\delta u(t) = Au(t), & t \in (0,T], \\ u(t) = g(t), & t \in [-\delta,0], \end{cases} \tag{4.11}$$

is equivalent to that the function $v(t)$ solves the nonlocal-in-time problem

$$\begin{cases} \mathcal{G}_\delta^\lambda v(t) + (C_\delta^\lambda)^{-1}h_\delta(\lambda)v(t) = (C_\delta^\lambda)^{-1}Av(t), & t \in (0,T], \\ v(t) = e^{-\lambda t}g(t), & t \in [-\delta,0]. \end{cases} \tag{4.12}$$

By Lemma 4.2.1, the operator $(C_\delta^\lambda)^{-1}A$ satisfies the conditions in Theorem 4.2.3, and by Theorem 4.2.1, there is a unique mild solution $v \in L^\infty([0,T];B)$ which satisfies the nonlocal-in-time problem (4.12) and it holds that $\|v\|_{L^\infty([0,T];B)} \le \|e^{-\lambda t}g(t)\|_{L^\infty([-\delta,0];B)}$. Therefore, from the equivalence of two nonlocal-in-time problems, we know there is a unique mild solution u in $L^\infty([0,T];B)$ satisfies the nonlocal-in-time problem (4.11) and it holds that

$$\begin{aligned}\|u\|_{L^\infty([0,T];B)} &= \|e^{\lambda t}v(t)\|_{L^\infty([0,T];B)} \leq e^{\lambda T}\|v\|_{L^\infty([0,T];B)} \\ &\leq e^{\lambda T}\|e^{-\lambda t}g(t)\|_{L^\infty([-\delta,0];B)} \leq e^{\lambda T}\|g\|_{L^\infty([-\delta,0];B)}.\end{aligned}$$

By taking the value of $\lambda > 0$ approaching to 0, we have that

$$\|u\|_{L^\infty([0,T];B)} \leq \|g\|_{L^\infty([-\delta,0];B)}.$$

□

Furthermore, following the similar approach above, we can show the well-posedness of nonlocal-in-time problems in $C^0([-\delta,T];B)$ given the initial data $g \in C^0([-\delta,0];B)$. Here, the continuity of the solution at zero is required, i.e.,

$$\lim_{t\downarrow 0} u(t) = u(0^+) = g(0),$$

so we need to furthermore assume that this initial data g satisfies the continuity condition

$$g(0) = ((b_\delta + \lambda)I - A)^{-1} \int_0^\delta s\rho_\delta(s)g(-s)ds. \tag{4.13}$$

Theorem 4.2.2. *Let B be a Banach space and nonlocal kernel $s\rho_\delta(s) \in L^1(0,\delta)$. Given the initial data $g \in C^0([-\delta,0];B)$ satisfying the continuity condition,*

$$g(0) = ((b_\delta + \lambda)I - A)^{-1} \int_0^\delta s\rho_\delta(s)g(-s)ds,$$

the coefficient $\lambda > 0$, and the dissipative linear (unbounded) operator A satisfies the conditions in Theorem 4.2.3, there is a unique mild solution $u \in C^0([-\delta,T];B)$ which satisfies the integral form

$$u(t) = ((b_\delta+\lambda)I-A)^{-1}\left[\int_0^t s\rho_\delta(s)\mathbb{1}_{[0,\delta]}(s)u(t-s)ds + \int_t^\delta s\rho_\delta(s)\mathbb{1}_{[0,\delta]}(s)g(t-s)ds\right],$$

of the nonlocal-in-time problem in Banach spaces (4.8) and it holds that $\|u\|_{L^\infty([-\delta,T];B)} \leq \|g\|_{L^\infty([-\delta,0];B)}$.

Proof. First we define the map $\mathcal{K}$ by

$$\mathcal{K}u(t) := \begin{cases} ((b_\delta+\lambda)I-A)^{-1}\Big[\int_0^t s\rho_\delta(s)\mathbb{1}_{[0,\delta]}(s)u(t-s)ds \\ \qquad + \int_t^\delta s\rho_\delta(s)\mathbb{1}_{[0,\delta]}(s)g(t-s)ds\Big], & t\in(0,T], \\ g(t), & t\in[-\delta,0], \end{cases}$$

where $b_\delta = \int_0^\delta s\rho_\delta(s)ds$. Here $g\in C^0([-\delta,0];B)$ and $\mathcal{D}(A)\subseteq B$ imply that $\mathcal{K}u(t)\in B$ for $t\in[0,T]$. In addition, for $u\in C^0([-\delta,T];B)$ with $u(t)=g(t)$ in $[-\delta,0]$, the expression

$$((b_\delta+\lambda)I-A)^{-1}\left[\int_0^t s\rho_\delta(s)\mathbb{1}_{[0,\delta]}(s)u(t-s)ds + \int_t^\delta s\rho_\delta(s)\mathbb{1}_{[0,\delta]}(s)g(t-s)ds\right]$$

is continuous in $t\in(0,T]$. With the continuity condition satisfied by g, we have that

$$\lim_{t\downarrow 0}\mathcal{K}u(t) = ((b_\delta+\lambda)I-A)^{-1}\int_0^\delta s\rho_\delta(s)g(-s)ds = g(0).$$

Hence $\mathcal{K}u(t)$ is continuous in t at $t=0$, which implies that $\mathcal{K}u\in C^0([-\delta,T];B)$. Moreover, since the conditions in *Hille-Yosida* Theorem are satisfied by the dissipative operator A, we have that

$$b_\delta\left\|[(b_\delta+\lambda)I-A]^{-1}\right\| \le \frac{b_\delta}{b_\delta+\lambda} < 1.$$

For $u,v\in C^0([-\delta,T];X)$ with imposed initial data $u(t)=v(t)=g(t)$ in $[-\delta,0]$, such that $\|u\|_{L^\infty([-\delta,T];B)}, \|v\|_{L^\infty([-\delta,T];B)} \le \|g\|_{L^\infty([-\delta,0];B)}$, we have that

$$(\mathcal{K}u-\mathcal{K}v)(t) = ((b_\delta+\lambda)I-A)^{-1}\int_0^t s\rho_\delta(s)\mathbb{1}_{[0,\delta]}(s)(u-v)(t-s)ds,$$

$$\begin{aligned}\|\mathcal{K}u-\mathcal{K}v\|_{L^\infty([0,T];B)} &\le b_\delta\left\|((b_\delta+\lambda)I-A)^{-1}\right\|\cdot\|u-v\|_{L^\infty([0,T];B)} \\ &\le \frac{b_\delta}{b_\delta+\lambda}\|u-v\|_{L^\infty([0,T];B)}.\end{aligned}$$

Here $\frac{b_\delta}{b_\delta+\lambda} < 1$, so the map $\mathcal{K}$ is contractive.

$$\begin{aligned}
\|\mathcal{K}u(t)\|_B &\leq \left\|((b_\delta+\lambda)I - A)^{-1}\right\| \cdot \Big(b_\delta(t)\|u\|_{L^\infty([0,T];B)} + (b_\delta - b_\delta(t))\|g\|_{L^\infty([-\delta,0];B)}\Big) \\
&\leq b_\delta \left\|((b_\delta+\lambda)I - A)^{-1}\right\| \cdot \|u\|_{L^\infty([-\delta,T];B)}, \\
\|\mathcal{K}u\|_{L^\infty([0,T];X)} &\leq b_\delta \left\|((b_\delta+\lambda)I - A)^{-1}\right\| \cdot \|u\|_{L^\infty([-\delta,T];B)} \\
&\leq \frac{b_\delta}{b_\delta+\lambda}\|g\|_{L^\infty([-\delta,0];B)} < \|g\|_{L^\infty([-\delta,0];B)}.
\end{aligned}$$

This implies that $\mathcal{K}$ maps from the subset

$$V = \{u \in C^0([-\delta,T];B) : u(t) = g(t), t \in [-\delta,0] \text{ and } \|u\|_{L^\infty([-\delta,T];B)} \leq \|g\|_{L^\infty([-\delta,0];B)}\}, \tag{4.14}$$

into itself. Thus $\mathcal{K}$ is a contraction mapping on V. By Banach fixed-point theorem, there is a unique solution $u \in V$ satisfies the integral form

$$u(t) = ((b_\delta+\lambda)I - A)^{-1}\left[\int_0^t s\rho_\delta(s)\mathbb{1}_{[0,\delta]}(s)u(t-s)ds + \int_t^\delta s\rho_\delta(s)\mathbb{1}_{[0,\delta]}(s)g(t-s)ds\right],$$

of the nonlocal-in-time problem in Banach spaces (4.8). □

Remark. *The subspace V in the proof of Theorem 4.2.2 is a Banach space. Actually, given a Cauchy sequence $\{u_n\} \subseteq V \subseteq C^0([-\delta,T];B)$, it converges to a function $u \in C^0([-\delta,T];B)$ by the completeness of $C^0([-\delta,T];B)$. Moreover, $u_n = g$ in $t \in [-\delta,0]$ implies $u = g$ in $t \in [-\delta,0]$. So u is also in the subspace V. Then V is a complete subspace of $C^0([-\delta,T];B)$ and therefore a Banach space equipped with the norm of space $C^0([-\delta,T];B)$.*

Similar to Theorem 4.2.2, using the nonlocal integrating factor method we have the following proposition for the nonlocal-in-time problems in Banach spaces without drift (4.6) in $C^0([-\delta,T];B)$.

Prop 4.2.3. *Let B be a Banach space and nonlocal kernel $s\rho_\delta(s) \in L^1(0,\delta)$. Given the initial data $g \in C^0([-\delta,0];B)$ satisfying the continuity condition,*

$$g(0) = (b_\delta I - A)^{-1}\int_0^\delta s\rho_\delta(s)g(-s)ds,$$

and the dissipative linear (unbounded) operator A satisfies the conditions in Theorem 4.2.3, there is a unique mild solution $u \in C^0([-\delta,T];B)$ which satisfies the integral *form*

$$u(t) = (b_\delta I - A)^{-1} \left[\int_0^t s\rho_\delta(s) \mathbb{1}_{[0,\delta]}(s) u(t-s) ds + \int_t^\delta s\rho_\delta(s) \mathbb{1}_{[0,\delta]}(s) g(t-s) ds \right],$$

of the nonlocal-in-time problem in Banach spaces (4.6) and it holds that $\|u\|_{L^\infty([-\delta,T];B)} \leq \|g\|_{L^\infty([-\delta,0];B)}$.

4.2.2 Semigroup properties of the nonlocal problems in Banach spaces

From the above discussion, the well-posedness theorems of the nonlocal-in-time problem in Banach spaces (4.6) have been established. We now continue to study the semigroup properties of this nonlocal-in-time problem in Banach spaces.

Semigroup theory has been intensively used in PDE theory. Under the semigroup framework, many results in PDE can be re-phrased and re-proved in beautiful way. *Evans* used one chapter to give a brief introduction on semigroup theory and its connections to PDEs in [36]. *Pazy* presented this theory and its applications to PDEs systematically in [37], while *Lunardi* and *Amann* focused more on analytic semigroups and optimal regularity in [38] and [39] respectively.

Definition 4.2.1 (Semigroup [37])**.** *Let X be a Banach space. A one parameter family $\{T(t)\}_{t\geq 0}$ of bounded linear operators mapping X into X is called a semigroup of bounded linear operators on X if the following two conditions are satisfied.*

1. $T(0) = I$, *(I is the identity operator on X).*
2. $T(s+t) = T(t)T(s)$, *for every $t, s \geq 0$ (the semigroup property).*

$\{T(t)\}_{t\geq 0}$ *is called a semigroup of class C_0 or simply a C_0-semigroup if*

$$\lim_{t\downarrow 0} T(t)x = x, \qquad \textit{for every } x \in X.$$

The linear operator A defined by

$$D(A) = \left\{ x \in X : \lim_{t\downarrow 0} \frac{T(t)x - x}{t} \text{ exists} \right\}$$

and

$$Ax = \lim_{t\downarrow 0} \frac{T(t)x - x}{t}, \text{ for } x \in D(A)$$

is the infinitesimal generator of the semigroup $T(t)$, $D(A)$ is the domain of A.

Recall that if A is a linear, not necessarily bounded, operator in X, the resolvent set $\rho(A)$ of A is the set of all complex numbers λ for which $\lambda I - A$ is invertible, i.e., $(\lambda I - A)^{-1}$ is a bounded linear operator in X. The family $R(\lambda : A) = (\lambda I - A)^{-1}$, $\lambda \in \rho(A)$ of bounded linear operators is called the resolvent of A.

Theorem 4.2.3 (Hille-Yosida [37])**.** *A linear (unbounded) operator A is the infinitesimal generator of a C_0-semigroup of contractions $\{T(t)\}_{t\geq 0}$ if and only if*

1. *A is closed and $\overline{D(A)} = X$.*

2. *The resolvent set $\rho(A)$ of A contains $\mathbb{R}^+$ and for every $\lambda > 0$*

$$\|R(\lambda : A)\| \leq \frac{1}{\lambda}.$$

Based on the solution $u_g \in C^0([-\delta, T]; B)$ of the nonlocal-in-time problem in Banach space

$$\begin{cases} \mathcal{G}_\delta u(t) = Au(t), & t \in (0, T], \\ u(t) = g(t), & t \in [-\delta, 0], \end{cases} \tag{4.15}$$

corresponding to the initial data $g \in C^0([-\delta, 0]; B)$ satisfies the continuity condition

$$g(0) = (b_\delta I - A)^{-1} \int_0^\delta \tau \rho_\delta(\tau) g(-\tau) d\tau,$$

we can introduce a map $T^\delta(t)$, for $t \geq 0$, defined on the set

$$X_\delta = \left\{ g \in C^0([-\delta, 0]; B) : g(0) = (b_\delta I - A)^{-1} \int_0^\delta \tau \rho_\delta(\tau) g(-\tau) d\tau \right\}, \tag{4.16}$$

by

$$(T^\delta(t)g)(s) := u_g(t + s),$$

for $s \in [-\delta, 0]$. In fact, given $g \in X_\delta$, the mild solution $u_g \in C^0([-\delta, T]; B)$ satisfies the integral equation

$$u_g(t) = (b_\delta I - A)^{-1} \int_0^\delta \tau \rho_\delta(\tau) u_g(t - \tau) d\tau.$$

Then by the definition of map $(T^\delta(t)g)(s)$,

$$(T^\delta(t)g)(0) = (b_\delta I - A)^{-1} \int_0^\delta \tau\rho_\delta(\tau)(T^\delta(t)g)(-\tau)d\tau,$$

which implies that for $t \geq 0$ the map $T^\delta(t)$ maps from X_δ into itself.
From this definition, we can immediately obtain the following two properties of T^δ:

- $T^\delta(0) = I$;
- $(T^\delta(t+h)g)(s) = (T^\delta(t)g)(s+h)$, for t, s, h such that $t, t+h \geq 0, s, s+h \leq 0$.

Now let us compare $(T^\delta(t_1)(T^\delta(t_2)g))(s)$ and $(T^\delta(t_1+t_2)g)(s)$ for $t_1, t_2 \geq 0, s \in [-\delta, 0]$.
First, when $t_1 + s \leq 0$, applying the second property above we have that

$$\begin{aligned}
&(T^\delta(t_1+t_2)g)(s) - (T^\delta(t_1)(T^\delta(t_2)g))(s) \\
=&(T^\delta(t_2)g)(t_1+s) - (T^\delta(0)(T^\delta(t_2)g))(t_1+s) \\
=&(T^\delta(t_2)g)(t_1+s) - (T^\delta(t_2)g)(t_1+s) \\
=&0.
\end{aligned}$$

Second, when $t_1 + s > 0$, again applying the second property, we have that

$$(T^\delta(t_1+t_2)g)(s) - (T^\delta(t_1)(T^\delta(t_2)g))(s) = (T^\delta(t_1+t_2+s)g)(0) - (T^\delta(t_1+s)(T^\delta(t_2)g))(0).$$

So we only need to show that for all $t_1, t_2 \geq 0$,

$$(T^\delta(t_1+t_2)g)(0) = (T^\delta(t_1)(T^\delta(t_2)g))(0).$$

For any fixed $t_1 \geq 0$, let

$$v_{t_1}(t) := (T^\delta(t+t_1)g)(0) - (T^\delta(t)(T^\delta(t_1)g))(0), \quad t \geq 0.$$

By the definition of $T^\delta(t)$ and that u_g satisfies the integral form

$$u_g(t) = (b_\delta I - A)^{-1} \int_0^\delta \tau\rho_\delta(\tau)u_g(t-\tau)d\tau,$$

we have that

$$
\begin{aligned}
(T^\delta(t+t_1)g)(0) &= (b_\delta I - A)^{-1}\int_0^\delta \tau\rho_\delta(\tau)(T^\delta(t+t_1)g)(-\tau)d\tau \\
&= (b_\delta I - A)^{-1}\Big[\int_0^t \tau\rho_\delta(\tau)\mathbb{1}_{[0,\delta]}(\tau)(T^\delta(t+t_1)g)(-\tau)d\tau \\
&\quad + \int_t^\delta \tau\rho_\delta(\tau)\mathbb{1}_{[0,\delta]}(\tau)(T^\delta(t+t_1)g)(-\tau)d\tau\Big],
\end{aligned}
$$

and similarly

$$
\begin{aligned}
(T^\delta(t)(T^\delta(t_1)g))(0) &= (b_\delta I - A)^{-1}\int_0^\delta \tau\rho_\delta(\tau)(T^\delta(t)(T^\delta(t_1)g))(-\tau)d\tau \\
&= (b_\delta I - A)^{-1}\Big[\int_0^t \tau\rho_\delta(\tau)\mathbb{1}_{[0,\delta]}(\tau)(T^\delta(t)(T^\delta(t_1)g))(-\tau)d\tau \\
&\quad + \int_t^\delta \tau\rho_\delta(\tau)\mathbb{1}_{[0,\delta]}(\tau)(T^\delta(t)(T^\delta(t_1)g))(-\tau)d\tau\Big].
\end{aligned}
$$

Taking the difference of the above two equalities, we obtain that

$$
\begin{aligned}
v_{t_1}(t) &= (T^\delta(t+t_1)g)(0) - (T^\delta(t)(T^\delta(t_1)g))(0) \\
&= (b_\delta I - A)^{-1}\Big[\int_0^t \tau\rho_\delta(\tau)\mathbb{1}_{[0,\delta]}(\tau)[(T^\delta(t+t_1)g)(-\tau) - (T^\delta(t)(T^\delta(t_1)g))(-\tau)]d\tau \\
&\quad + \int_t^\delta \tau\rho_\delta(\tau)\mathbb{1}_{[0,\delta]}(\tau)[(T^\delta(t+t_1)g)(-\tau) - (T^\delta(t)(T^\delta(t_1)g))(-\tau)]d\tau\Big] \\
&= (b_\delta I - A)^{-1}\Big[\int_0^t \tau\rho_\delta(\tau)\mathbb{1}_{[0,\delta]}(\tau)[(T^\delta(t-\tau+t_1)g)(0) - (T^\delta(t-\tau)(T^\delta(t_1)g))(0)]d\tau \\
&\quad + \int_t^\delta \tau\rho_\delta(\tau)\mathbb{1}_{[0,\delta]}(\tau)[(T^\delta(t_1)g)(t-\tau) - (T^\delta(0)(T^\delta(t_1)g))(t-\tau)]d\tau\Big] \\
&= (b_\delta I - A)^{-1}\Big[\int_0^t \tau\rho_\delta(\tau)\mathbb{1}_{[0,\delta]}(\tau)[(T^\delta(t-\tau+t_1)g)(0) - (T^\delta(t-\tau)(T^\delta(t_1)g))(0)]d\tau\Big] \\
&= (b_\delta I - A)^{-1}\int_0^t \tau\rho_\delta(\tau)\mathbb{1}_{[0,\delta]}(\tau)v_{t_1}(t-\tau)d\tau,
\end{aligned}
$$

which implies that the function $v_{t_1}(t)$ satisfies the integral form of the nonlocal-in-time problem in Banach space with the trivial initial data $g \equiv 0$,

$$
\begin{cases}
\mathcal{G}_\delta u(t) = Au(t), & t \in (0, T], \\
u(t) = 0, & t \in [-\delta, 0].
\end{cases}
$$

From Theorem 4.2.3, it implies that $v_{t_1}(t) \equiv 0$ for $t \in [-\delta, T]$. So now we have that

$$(T^\delta(t+t_1)g)(0) \equiv (T^\delta(t)(T^\delta(t_1)g))(0),$$

for all $t, t_1 \geq 0$, which guarantees the semi-group property for the case when $t_1 + s > 0$.

Finally, the continuity condition

$$g(0) = (b_\delta I - A)^{-1} \int_0^\delta s\rho_\delta(s) g(-s) ds$$

satisfied by the initial data g, guarantees that the semigroup $\{T^\delta(t)\}_{t\geq 0}$ is strongly continuous, i.e.,

$$\lim_{t\downarrow 0} \|T^\delta(t)g - g\|_{L^\infty([-\delta,0];B)} = 0, \quad \forall g \in X_\delta.$$

As a summary, we have the following theorem which can be seen as the nonlocal version of one direction of the *Hille-Yosida* Theorem.

Theorem 4.2.4. *Assume that the nonlocal-in-time gradient operator $\mathcal{G}_\delta$ is defined as*

$$\mathcal{G}_\delta u(t) = \int_0^\delta s\rho_\delta(s)(u(t) - u(t-s)) ds.$$

with the corresponding integrable nonlocal kernel $s\rho_\delta(s) \in L^1(0, \delta)$. A linear (unbounded) operator A is the "nonlocal generator" corresponding to the nonlocal-in-time gradient $\mathcal{G}_\delta$ of a strongly continuous semigroup of contractions $\{T^\delta(t)\}_{t\geq 0}$, where

$$T^\delta(t) : X_\delta \to X_\delta, \quad \forall t \geq 0,$$

and

$$X_\delta = \left\{ g \in C^0([-\delta, 0]; B) : g(0) = (b_\delta I - A)^{-1} \int_0^\delta s\rho_\delta(s) g(-s) ds \right\},$$

if the operator A satisfies the conditions in Theorem 4.2.3.

4.3 Conclusions

In this chapter, based the on the theory of linear nonlocal-in-time problem developed by *Du* et al. in [2], we first extended the well-posedness results and the localization theory to the linear nonlocal-in-time problems with more general drift term $\lambda u(t)$, for $\lambda \in \mathbb{R}$, with the development of the integrating factor method for nonlocal-in-time operator. Then considering the linear nonlocal-in-time problems in Banach spaces with a linear unbounded dissipative operator A, we have established the well-posedness theorems in both $L^\infty(0,T)$ and $C^0([-\delta,T])$ spaces, and the semigroup properties corresponding to the mild solutions in $C^0([-\delta,T])$ of the integral form of the nonlocal problems.

There are many questions remain to be studied. The non-integrable nonlocal kernel case of this linear nonlocal-in-time problem in Banach spaces has not been considered in this thesis, but it could be extended directly from the results for integrable kernel case following a similar approach as in [2] where a sequence of solutions u^ϵ have been constructed, corresponding to a family of truncated nonlocal operators $\mathcal{G}_\delta^\epsilon$ with integrable kernels, which converges to a weak solution of the nonlocal problem with non-integrable kernel. In addition, a nonlocal version of one direction of the *Hille-Yosida* Theorem was given at the end of this chapter, which gives a sufficient condition for the strongly continuous semigroup of contractions. In the proof of this theorem, the mild solution of the nonlocal-in-time problem has been used to define the map $T^\delta(t)$ from X_δ into itself. Conversely, given a C^0-semigroup, to construct a "nonlocal generator" A is more challenging.

Chapter 5
The Nonlinear Nonlocal-in-Time Problems

In Chapter 4, we have established the well-posedness theorems and the semigroup properties of the linear nonlocal-in-time problem in Banach spaces. In this chapter, we shall continue to study the second sub-problem, the nonlinear nonlocal-in-time problem, by taking the autonomous nonlinearities $f(u)$ into consideration.

$$\begin{cases} \mathcal{G}_\delta u(t) + \lambda u(t) = f(u(t)), & t \in (0, T], \\ u(t) = g(t), & t \in [-\delta, 0]. \end{cases} \tag{5.1}$$

Here, in particular, we consider the Lipschitz continuous nonlinearities first and show the global existence and uniqueness of the nonlocal-in-time problem (5.1). Then for a class of autonomous nonlinearities in the form of $f(u) = u^p|u|^\alpha$, for $p \in \mathbb{N}^+$ and $\alpha \in [0, 1)$, we establish the local existence and uniqueness of the solutions in both $L^\infty(0, t_0)$ and $W^{1,\infty}(0, t_0)$ spaces for integrable kernels, and only the local existence of the solutions in $C^0(0, t_0)$ space for non-integrable kernels.

5.1 Nonlocal-in-Time Problems with Lipschitz Continuous Nonlinearities

Given the integrable nonlocal kernel $s\rho_\delta(s) \in L^1(0, \delta)$, we first consider solutions of the problem (5.1) in $L^\infty(0, T)$ with initial data $g \in L^\infty(-\delta, 0)$ and Lipschitz continuous nonlinearity $f(u(t))$,

$$|f(u) - f(v)| \leq C_f |u - v|.$$

Consider the following map $\mathcal{K}$ that maps from $L^\infty(0,T)$ into itself,

$$\mathcal{K}u(t) = \frac{1}{b_\delta + \lambda}\left[\int_0^t \tau\rho_\delta(\tau)u(t-s)d\tau + \int_t^\delta \tau\rho_\delta(\tau)g(t-s)d\tau + f(u(t))\right].$$

Given $u, v \in L^\infty(0,T)$ satisfying that

$$\|u\|_{L^\infty(0,T)}, \|v\|_{L^\infty(0,T)} \leq R,$$

where

$$R := (\lambda - C_f)^{-1}|f(0)| + \|g\|_{L^\infty(-\delta,0)}.$$

With the Lipschitz continuity assumption, we have that

$$|f(u(t))| \leq |f(0)| + |f(u(t)) - f(0)| \leq |f(0)| + C_f|u(t)|.$$

$$\begin{aligned}
|\mathcal{K}u(t)| &\leq \frac{1}{b_\delta + \lambda}\left[\int_0^t \tau\rho_\delta(\tau)|u(t-s)|d\tau + \int_t^\delta \tau\rho_\delta(\tau)|g(t-s)|d\tau + |f(u(t))|\right] \\
&\leq \frac{1}{b_\delta + \lambda}\left[\int_0^t \tau\rho_\delta(\tau)d\tau\|u\|_{L^\infty} + \int_t^\delta \tau\rho_\delta(\tau)d\tau\|g\|_{L^\infty} + |f(0)| + C_f\|u\|_{L^\infty}\right] \\
&\leq \frac{1}{b_\delta + \lambda}\left[b_\delta(t)\|u\|_{L^\infty} + (b_\delta - b_\delta(t))\|g\|_{L^\infty} + |f(0)| + C_f\|u\|_{L^\infty}\right] \\
&\leq \frac{1}{b_\delta + \lambda}\left[(b_\delta(t) + C_f)r_0 + (b_\delta - b_\delta(t))\|g\|_{L^\infty} + |f(0)|\right] \\
&\leq \frac{1}{b_\delta + \lambda}\Big[(\lambda - C_f)^{-1}(b_\delta(t) + C_f)|f(0)| + (b_\delta(t) + C_f)\|g\|_{L^\infty} \\
&\qquad + (b_\delta - b_\delta(t))\|g\|_{L^\infty} + |f(0)|\Big] \\
&= \frac{1}{b_\delta + \lambda}\left[(\lambda - C_f)^{-1}(b_\delta(t) + \lambda)|f(0)| + (b_\delta + C_f)\|g\|_{L^\infty}\right] \\
&\leq (\lambda - C_f)^{-1}|f(0)| + \frac{b_\delta + C_f}{b_\delta + \lambda}\|g\|_{L^\infty} \\
&= R - \frac{\lambda - C_f}{b_\delta + \lambda}\|g\|_{L^\infty},
\end{aligned}$$

$$
\begin{aligned}
|\mathcal{K}u(t)-\mathcal{K}v(t)| &\le \frac{1}{b_\delta+\lambda}\left[\int_0^t \tau\rho_\delta(\tau)|u(t-s)-v(t-s)|d\tau + |f(u(t))-f(v(t))|\right] \\
&\le \frac{1}{b_\delta+\lambda}\left[b_\delta\|u-v\|_{L^\infty} + C_f\|u-v\|_{L^\infty}\right] \\
&= \frac{b_\delta+C_f}{b_\delta+\lambda}\|u-v\|_{L^\infty},
\end{aligned}
$$

where $b_\delta(t) := \int_0^t \tau\rho_\delta(\tau)d\tau$. Thus, when $\lambda > C_f$ the map $\mathcal{K}$ is a contraction mapping and maps the closed ball $B_R(0)$ into itself. By the Banach fixed-point theorem, we know there is a unique fixed point u in the subset

$$
V = \left\{u \in L^\infty(0,T) : \|u\|_{L^\infty(0,T)} \le (\lambda - C_f)^{-1}|f(0)| + \|g\|_{L^\infty(-\delta,0)}\right\},
$$

satisfying that

$$
u(t) = \mathcal{K}u(t),
$$

that is, satisfying the nonlinear nonlocal-in-time problem (5.1). Therefore, we have obtained the global well-posedness in $L^\infty(0,T)$ of the nonlocal-in-time problem with the Lipschitz continuous nonlinearities.

Theorem 5.1.1 (Well-posedness in L^∞, Lipschitz continuous nonlinearities)**.** *Given the integrable nonlocal kernel $s\rho_\delta(s) \in L^1(0,\delta)$, the initial data $g \in L^\infty(-\delta,0)$, and the Lipschitz continuous nonlinearity $f(u)$ with the Lipschitz constant $C_f < \lambda$. The above nonlinear nonlocal-in-time problem (5.1) has a unique solution $u \in L^\infty(0,T)$ and it holds that*

$$
\|u\|_{L^\infty(0,T)} \le (\lambda - C_f)^{-1}|f(0)| + \|g\|_{L^\infty(-\delta,0)}.
$$

In addition, similarly to the linear problems, we can prove the following localization result of the nonlinear nonlocal-in-time problems with Lipschitz nonlinearities (5.1).

Theorem 5.1.2 (Local limit, Lipschitz continuous nonlinearities)**.** *Suppose that u_δ satisfies the nonlocal problem*

$$
\begin{cases}
\mathcal{G}_\delta u_\delta(t) + \lambda u_\delta(t) = f(u_\delta(t)), & t \in (0,T], \\
u_\delta(t) = g_\delta(t), & t \in [-\delta,0],
\end{cases}
$$

and that u satisfies the differential equation

$$\begin{cases} \dfrac{d}{dt}u(t) + \lambda u(t) = f(u(t)), & t \in (0,T], \\ u(t) = g_0, & t \in [-\delta, 0], \end{cases}$$

with the compatibility condition on the initial data $g_\delta \in C([-\delta,0])$ satisfied, that is,

$$\lim_{\delta \to 0} \left| \int_0^\delta \gamma_\delta(\tau) g_\delta(-\tau) d\tau - g_0 \right| = 0,$$

and f is Lipschitz continuous in u, that is,

$$|f(u) - f(v)| \le C_f |u - v|, \quad \textit{for } u, v \in \mathbb{R}.$$

Let δ tend to zero, we have the following convergence in L^2 norm,

$$\lim_{\delta \to 0} \|u_\delta - u\|_{L^2(0,T)} = 0.$$

Proof. First, let us write both equations into the integral form.

$$\overline{u_\delta}(t) = \overline{g_\delta}(0) - \lambda \int_0^t u_\delta(s) ds + \int_0^t f(u_\delta(s)) ds,$$
$$u(t) = g_0 - \lambda \int_0^t u(s) ds + \int_0^t f(u(s)) ds,$$

where $\overline{u_\delta}(t)$ is defined as

$$\overline{u_\delta}(t) = \int_0^\delta \gamma_\delta(\tau) u_\delta(t - \tau) d\tau.$$

By taking the difference of two integral equations, we have that

$$\overline{u_\delta}(t) - u(t) = \overline{g_\delta}(0) - g_0 - \lambda \int_0^t (u_\delta(s) - u(s)) ds + \int_0^t (f(u_\delta(s)) - f(u(s))) ds.$$

So,

$$\begin{aligned}|u_\delta(t) - u(t)| &\le |u_\delta(t) - \overline{u_\delta}(t)| + |\overline{u_\delta}(t) - u(t)| \\ &\le |u_\delta(t) - \overline{u_\delta}(t)| + |\overline{g_\delta}(0) - g_0| + \lambda \int_0^t |u_\delta(s) - u(s)| ds \\ &\quad + \int_0^t |f(u_\delta(s)) - f(u(s))| ds \\ &\le |u_\delta(t) - \overline{u_\delta}(t)| + |\overline{g_\delta}(0) - g_0| + (\lambda + C_f) \int_0^t |u_\delta(s) - u(s)| ds\end{aligned}$$

By the Grönwall's inequality,

$$|u_\delta(t) - u(t)| \le \alpha(t) + \int_0^t \alpha(s)(\lambda + C_f)e^{(\lambda+C_f)(t-s)} ds,$$

where

$$\alpha(t) = |u_\delta(t) - \overline{u_\delta}(t)| + |\overline{g_\delta}(0) - g_0|.$$

$$\begin{aligned}\|u_\delta(t) - u(t)\|_{L^\infty} &\le \|\alpha(t)\|_{L^\infty} + (\lambda + C_f)\|e^{(\lambda+C_f)t}\|_{L^1}\|\alpha(t)\|_{L^\infty} \\ &= \left[1 + (\lambda + C_f)\|e^{(\lambda+C_f)t}\|_{L^1}\right] \|\alpha(t)\|_{L^\infty} \\ &= e^{(\lambda+C_f)T}\|\alpha(t)\|_{L^\infty}\end{aligned}$$

$$\begin{aligned}u_\delta(t) - \overline{u_\delta}(t) &= \int_0^\delta \gamma_\delta(\tau)(u(t) - u(t-\tau)) d\tau \\ |u_\delta(t) - \overline{u_\delta}(t)| &\le \int_0^\delta \gamma_\delta(\tau)|u(t) - u(t-\tau)| d\tau \\ &\le \int_0^\delta \gamma_\delta(\tau) d\tau \sup_{\tau\in[0,\delta]} |u(t) - u(t-\tau)| \\ &= \sup_{\tau\in[0,\delta]} |u(t) - u(t-\tau)|\end{aligned}$$

Thus, $u \in C([0,T])$ implies that

$$\lim_{\delta\to 0} \|u_\delta(t) - u(t)\|_{L^\infty} \le e^{(\lambda+C_f)T} \lim_{\delta\to 0} \|\alpha(t)\|_{L^\infty} = 0. \qquad \square$$

5.2 Nonlocal-in-Time Problems with $u^p|u|^\alpha$-type Nonlinearities

5.2.1 L^∞ solutions for integrable kernel case

Now, let us study the local existence and uniqueness of the solution of the nonlinear nonlocal-in-time problems (5.1) with a more general continuous nonlinear term $f(u) = u^p|u|^\alpha$, for $p \in \mathbb{N}^+$, and $\alpha \in [0, 1)$. In this case, we mainly consider the nonlocal-in-time problem on a small time interval $[0, t_0]$ for some small $t_0 < \delta$. First, as usual we introduce a map $\mathcal{K}$ corresponding to the problem (5.1) by

$$\mathcal{K}u(t) = \frac{1}{b_\delta + \lambda}\left[\int_0^t s\rho_\delta(s)u(t-s)ds + \int_t^\delta s\rho_\delta(s)g(t-s)ds + u^p|u|^\alpha(t)\right]. \quad (5.2)$$

We need to determine that for what value of R we have the local existence and uniqueness of the solution in the set

$$V = \left\{u \in L^\infty(0, t_0) : \|u\|_{L^\infty(0,t_0)} \le R\right\}.$$

Given $u, v \in V$,

$$\begin{aligned}
|\mathcal{K}u(t)| &\le \frac{1}{b_\delta + \lambda}\left[\int_0^t s\rho_\delta(s)|u(t-s)|ds + \int_t^\delta s\rho_\delta(s)|g(t-s)|ds + |u^p|u|^\alpha(t)|\right] \\
&\le \frac{1}{b_\delta + \lambda}\left[b_\delta(t)\|u\|_{L^\infty(0,t_0)} + (b_\delta - b_\delta(t))\|g\|_{L^\infty(-\delta,0)} + \|u\|^{p+\alpha}_{L^\infty(0,t_0)}\right] \\
&\le \frac{1}{b_\delta + \lambda}\left[R^{p+\alpha} + b_\delta(t)R + (b_\delta - b_\delta(t))\|g\|_{L^\infty(-\delta,0)}\right],
\end{aligned}$$

$$\begin{aligned}
|\mathcal{K}u(t) - \mathcal{K}v(t)| &\le \frac{1}{b_\delta + \lambda}\left[\int_0^t s\rho_\delta(s)|u(t-s) - v(t-s)|ds + |u^p|u|^\alpha(t) - v^p|v|^\alpha(t)|\right] \\
&\le \frac{1}{b_\delta + \lambda}\left[b_\delta(t)\|u - v\|_{L^\infty(0,t_0)} + \|u^p|u|^\alpha - v^p|v|^\alpha\|_{L^\infty(0,t_0)}\right].
\end{aligned}$$

The existence of the fixed point of the map $\mathcal{K}$ requires the map to be non-expansive and maps a closed ball into itself. Furthermore, the uniqueness of the fixed point requires the map to be contractive in addition.

To estimate the term $\|u^p|u|^\alpha - v^p|v|^\alpha\|_{L^\infty(0,t_0)}$ above, we need the following lemmas.

Lemma 5.2.1. *Given $\|u\|_{L^\infty(0,t_0)}, \|v\|_{L^\infty(0,t_0)} \le R$, $p \in \mathbb{N}^+$,*

$$\|u^p - v^p\|_{L^\infty(0,t_0)} \le pR^{p-1}\|u-v\|_{L^\infty(0,t_0)}.$$

Proof.

$$u^p - v^p = (u-v)\sum_{k=0}^{p-1} u^{p-1-k}v^k,$$

and for $k = 0, \ldots, p-1$,

$$\|u^{p-1-k}v^k\|_{L^\infty(0,t_0)} \le \|u\|^{p-1-k}_{L^\infty(0,t_0)}\|v\|^k_{L^\infty(0,t_0)} \le R^{p-1},$$

Thus,

$$\|u^p - v^p\|_{L^\infty(0,t_0)} \le pR^{p-1}\|u-v\|_{L^\infty(0,t_0)}.$$ □

Lemma 5.2.2. *Given $\|u\|_{L^\infty(0,t_0)}, \|v\|_{L^\infty(0,t_0)} \le R$, $\alpha \in (0,1)$,*

$$\||u|^\alpha u - |v|^\alpha v\|_{L^\infty(0,t_0)} \le (1+\alpha)R^\alpha\|u-v\|_{L^\infty(0,t_0)}.$$

Proof. Similar to the previous lemma, we want to rewrite the difference $|u|^\alpha u - |v|^\alpha v$ as an expression in terms of $u - v$.

$$\frac{d}{dx}\left(|x|^\alpha x\right) = \frac{d}{dx}\left(|x|^\alpha\right)\cdot x + |x|^\alpha = \alpha|x|^{\alpha-1}\frac{d}{dx}|x|\cdot x + |x|^\alpha = (1+\alpha)|x|^\alpha,$$
$$\frac{d}{ds}\left[|x|^\alpha x\big|_{x=su+(1-s)v}\right] = \frac{d}{dx}\left(|x|^\alpha x\right)\Big|_{x=su+(1-s)v}\cdot(u-v) = (1+\alpha)(u-v)|su+(1-s)v|^\alpha,$$

$$\begin{aligned}\int_0^1 |su+(1-s)v|^\alpha\,(su+(1-s)v)ds &= \int_0^1 |s(u-v)+v|^\alpha\,(s(u-v)+v)ds\\ &= \frac{1}{1+\alpha}\frac{1}{u-v}\left[s(u-v)+v\right]^{1+\alpha}\Big|_{s=0}^{s=1}\\ &= \frac{1}{(1+\alpha)(u-v)}\left[u^{1+\alpha}-v^{1+\alpha}\right],\end{aligned}$$

Therefore, we have that

$$\begin{aligned}|u|^\alpha u - |v|^\alpha v &= \left[|su+(1-s)v|^\alpha(su+(1-s)v)\right]\Big|_{s=0}^{s=1}\\ &= \left[|x|^\alpha x\big|_{x=su+(1-s)v}\right]\Big|_{s=0}^{s=1}\\ &= (1+\alpha)\int_0^1 (u-v)|su+(1-s)v|^\alpha ds,\end{aligned}$$

and

$$\begin{aligned}\||u|^\alpha u-|v|^\alpha v\|_{L^\infty(0,t_0)} &\le (1+\alpha)\|u-v\|_{L^\infty(0,t_0)}\int_0^1 (s\|u\|_{L^\infty}+(1-s)\|v\|_{L^\infty(0,t_0)})^\alpha dt\\ &\le (1+\alpha)R^\alpha\|u-v\|_{L^\infty(0,t_0)}.\end{aligned}$$ □

In general, for the case when $p\in\mathbb{N}^+$, and $\alpha\in(0,1)$, we have the following lemma.

Lemma 5.2.3. *Given $\|u\|_{L^\infty(0,t_0)},\|v\|_{L^\infty(0,t_0)}\le R$, $p\in\mathbb{N}^+$, and $\alpha\in(0,1)$, it holds that*

$$\||u|^\alpha u^p-|v|^\alpha v^p\|_{L^\infty(0,t_0)}\le(p+\alpha)R^{p-1+\alpha}\|u-v\|_{L^\infty(0,t_0)}.$$

Lemma 5.2.4. *Let the function $f(x,y)$ be defined by*

$$f(x,y)=\frac{(x+\lambda)y-y^p}{x},$$

and on the domain

$$\mathcal{D}=((p-1)\lambda,b_\delta]\times\left(\left(\frac{\lambda}{p}\right)^{\frac{1}{p-1}},\left(\frac{b_\delta+\lambda}{p}\right)^{\frac{1}{p-1}}\right],$$

then the maximum value of f on $\mathcal{D}$ is obtained at $x=b_\delta$, $y=\left(\frac{b_\delta+\lambda}{p}\right)^{\frac{1}{p-1}}$.

Proof. The partial derivatives of f are

$$f_x(x,y)=\frac{y^p-\lambda y}{x^2},\quad f_y(x,y)=\frac{x+\lambda-py^{p-1}}{x},$$

Therefore,

$$f(x,y)\le f\left(x,\left(\frac{x+\lambda}{p}\right)^{\frac{1}{p-1}}\right)=\frac{p-1}{x}\left(\frac{x+\lambda}{p}\right)^{\frac{p}{p-1}}<\frac{p-1}{b_\delta}\left(\frac{b_\delta+\lambda}{p}\right)^{\frac{p}{p-1}},$$

where $h(x):=\frac{p-1}{x}\left(\frac{x+\lambda}{p}\right)^{\frac{p}{p-1}}$ is a strictly increasing function for $x>(p-1)\lambda$. □

For the case when $p\in\mathbb{N}^+$, $p\ge2$, we have the following local existence and uniqueness theorem.

Theorem 5.2.5 (Local existence and uniqueness in $L^\infty(0,t_0)$, $f(u)=u^p$, $p\in\mathbb{N}^+$, $p\ge2$)**.** *Given the integrable non-local kernel $s\rho_\delta(s)\in L^1(0,\delta)$ with horizon δ small* *enough such that $b_\delta > (p-1)\lambda$, and the initial data $g \in L^\infty(-\delta,0)$ satisfying that*

$$\|g\|_{L^\infty(-\delta,0)} < \frac{p-1}{b_\delta}\left(\frac{b_\delta+\lambda}{p}\right)^{\frac{p}{p-1}}.$$

For small $t_0 < \delta$ and small $\epsilon > 0$ such that, $b_\delta - b_\delta(t_0) - \epsilon > (p-1)\lambda$, there is a unique solution in the subset

$$V = \left\{u \in L^\infty(0,t_0) : \|u\|_{L^\infty(0,t_0)} \leq \left(\frac{b_\delta - b_\delta(t_0) + \lambda - \epsilon}{p}\right)^{\frac{1}{p-1}}\right\},$$

satisfying the nonlinear nonlocal-in-time problem (5.1) with nonlinearity $f(u) = u^p$.

Proof. Given the initial data $g \in L^\infty(-\delta,0)$ satisfying that

$$\|g\|_{L^\infty(-\delta,0)} < \frac{p-1}{b_\delta}\left(\frac{b_\delta+\lambda}{p}\right)^{\frac{p}{p-1}},$$

by the previous lemma, we know there is a small $t_0 \in (0,\delta)$ and a small $\epsilon > 0$ such that $b_\delta - b_\delta(t_0) - \epsilon > (p-1)\lambda$ and

$$\|g\|_{L^\infty(-\delta,0)} \leq f\left(b_\delta - b_\delta(t_0), R\right) = \frac{(b_\delta - b_\delta(t_0) + \lambda)R - R^p}{b_\delta - b_\delta(t_0)}, \tag{5.3}$$

where $R = \left(\frac{b_\delta - b_\delta(t_0)+\lambda-\epsilon}{p}\right)^{\frac{1}{p-1}}$. Define the map $\mathcal{K}$ as,

$$\mathcal{K}u(t) = \frac{1}{b_\delta+\lambda}\left[\int_0^t s\rho_\delta(s)u(t-s)ds + \int_t^\delta s\rho_\delta(s)g(t-s)ds + u^p(t)\right],$$

For $u, v \in V = \left\{u \in L^\infty(0,t_0) : \|u\|_{L^\infty(0,t_0)} \leq R\right\}$,

$$|\mathcal{K}u(t)| \leq \frac{1}{b_\delta+\lambda}\left[R^p + b_\delta(t)R + (b_\delta - b_\delta(t))\|g\|_{L^\infty(-\delta,0)}\right]$$

$$\begin{aligned}
|\mathcal{K}u(t) - \mathcal{K}v(t)| &\leq \frac{1}{b_\delta+\lambda}\left[b_\delta(t)\|u-v\|_{L^\infty(0,t_0)} + \|u^p - v^p\|_{L^\infty(0,t_0)}\right] \\
&\leq \frac{1}{b_\delta+\lambda}\left[b_\delta(t)\|u-v\|_{L^\infty(0,t_0)} + pR^{p-1}\|u-v\|_{L^\infty(0,t_0)}\right] \\
&= \frac{b_\delta(t) + pR^{p-1}}{b_\delta+\lambda}\|u-v\|_{L^\infty(0,t_0)}
\end{aligned}$$

From (5.3), we have $\|\mathcal{K}u\|_{L^\infty(0,t_0)} \leq R$ which means $\mathcal{K}u \in V$. In addition, $R = \left(\frac{b_\delta - b_\delta(t_0)+\lambda-\epsilon}{p}\right)^{\frac{1}{p-1}}$ implies that for $t \in [0, t_0]$,

$$b_\delta(t) + pR^{p-1} = b_\delta(t) + b_\delta - b_\delta(t_0) + \lambda - \epsilon < b_\delta + \lambda,$$

so the map $\mathcal{K}$ is a contraction mapping. By the Banach fixed-point theorem, there is a unique solution $u \in V$ satisfies the nonlinear nonlocal-in-time problem (5.1) with nonlinearity $f(u) = u^p$. □

Following the same approach, in general for the case when $p \in \mathbb{N}^+$ and $\alpha \in (0,1)$, we have the well-posedness of nonlocal-in-time problem (5.1) in $L^\infty(0,t_0)$ with nonlinearity $f(u) = u^p|u|^\alpha$.

Theorem 5.2.6 (Local existence and uniqueness in $L^\infty(0,t_0)$, $f(u) = u^p|u|^\alpha$, $p \in \mathbb{N}^+$, $\alpha \in (0,1)$)**.** *Given the integrable nonlocal kernel $s\rho_\delta(s) \in L^1(0,\delta)$ with horizon δ small enough such that $b_\delta > (p-1+\alpha)\lambda$, and the initial data $g \in L^\infty(-\delta,0)$ satisfying that*

$$\|g\|_{L^\infty(-\delta,0)} < \frac{p+\alpha-1}{b_\delta}\left(\frac{b_\delta+\lambda}{p+\alpha}\right)^{\frac{p+\alpha}{p+\alpha-1}}.$$

For small $t_0 < \delta$ and small $\epsilon > 0$ such that, $b_\delta - b_\delta(t_0) - \epsilon > (p-1+\alpha)\lambda$, there is a unique solution in the subset

$$V = \left\{u \in L^\infty(0,t_0) : \|u\|_{L^\infty(0,t_0)} \leq \left(\frac{b_\delta+\lambda}{p+\alpha}\right)^{\frac{1}{p+\alpha-1}}\right\},$$

satisfying the nonlinear nonlocal-in-time problem (5.1) with nonlinearity $f(u) = u^p|u|^\alpha$.

Remark. *The above theorems give the local existence and uniqueness of the solutions in $L^\infty(0,t_0)$. Here, $t_0 < \delta$ implies that the existence interval is δ-dependent and tends to zero when $\delta \to 0$. However, by taking coefficient λ in the drift term large and the initial data g small, the global existence and uniqueness of the solution in $L^\infty(0,T)$ for an interval $[0,T]$ independent of δ can also be proved.*

Theorem 5.2.1 (Global existence and uniqueness in $L^\infty(0,T)$, $f(u) = u^p|u|^\alpha$, $p \in \mathbb{N}^+$, $\alpha \in (0,1)$)**.** *Given $p \in \mathbb{N}^+$, $\alpha \in (0,1)$, the integrable nonlocal kernel $s\rho_\delta(s) \in L^1(0,\delta)$ and the initial data $g \in L^\infty(-\delta,0)$ satisfying that $\|g\|_{L^\infty(-\delta,0)} <$*

$\left(\frac{\lambda}{p+\alpha}\right)^{\frac{1}{p+\alpha-1}}$, *there is a unique solution* $u \in L^\infty(0,T)$ *satisfies the nonlinear nonlocal-in-time problem (5.1) with nonlinearity* $f(u) = u^p|u|^\alpha$, *and it holds that*

$$\|u\|_{L^\infty(0,T)} \leq \|g\|_{L^\infty(-\delta,0)}.$$

Proof. With the same definition of map $\mathcal{K}$ as in (5.2) defined on the subset

$$V = \left\{u \in L^\infty(0,T) : \|u\|_{L^\infty(0,T)} \leq \|g\|_{L^\infty(-\delta,0)}\right\},$$

and with the assumption on g that $g \in L^\infty(-\delta, 0)$, we have that for all $u \in V$,

$$\|\mathcal{K}u\|_{L^\infty(0,T)} \leq \|g\|_{L^\infty(-\delta,0)},$$
$$\|\mathcal{K}u - \mathcal{K}v\|_{L^\infty(0,T)} \leq \frac{b_\delta + (p+\alpha)\|g\|^{p+\alpha-1}_{L^\infty(-\delta,0)}}{b_\delta + \lambda}\|u - v\|_{L^\infty(0,T)}.$$

$\mathcal{K}$ is a contraction mapping from the closed subset V to itself, therefore there is a unique solution $u \in L^\infty(0,T)$ satisfies the nonlinear nonlocal-in-time problem. □

5.2.2 $W^{1,\infty}$ solutions for integrable kernel case

Given the integrable kernel $s\rho_\delta(s) \in L^1(0,\delta)$ and the initial data $g \in W^{1,\infty}(-\delta, 0)$ satisfying the continuity condition

$$\alpha g^p(0) - \lambda g(0) - \int_0^\delta s\rho_\delta(s)(g(0) - g(-s))ds = 0.$$

and equipped with the $W^{1,\infty}$ norm,

$$\|g\|_{W^{1,\infty}(-\delta,0)} = \|g\|_{L^\infty(-\delta,0)} + \|Dg\|_{L^\infty(-\delta,0)},$$

we now consider the nonlocal problem with solutions in $W^{1,\infty}(0,t_0)$ space,

$$\begin{cases} \mathcal{G}_\delta u(t) + \lambda u(t) = \alpha u^p(t), & t \in (0, t_0], \\ u(t) = g(t), & t \in [-\delta, 0]. \end{cases}$$

Similar to the linear problems, we introduce a map $\mathcal{K}$

$$\mathcal{K}u(t) = (b_\delta + \lambda)^{-1}\left[\int_0^t s\rho_\delta(s)u(t-s)ds + \int_t^\delta s\rho_\delta(s)g(t-s)ds + \alpha u^p(t)\right].$$

With the continuity condition of g satisfied, we know that given $u(0^+) = \lim_{t\downarrow 0} u(t) = g(0)$ then

$$\lim_{t\downarrow 0}\mathcal{K}u(t) = (b_\delta + \lambda)^{-1}\left[\int_0^\delta s\rho_\delta(s)g(-s)ds + \alpha g^p(0)\right] = g(0).$$

The derivative of $\mathcal{K}u(t)$ is

$$\begin{aligned}(\mathcal{K}u)'(t) &= (b_\delta + \lambda)^{-1}\Big[t\rho_\delta(t)u(0^+) - t\rho_\delta(t)g(0) + \int_0^t s\rho_\delta(s)u'(t-s)ds \\ &\quad + \int_t^\delta s\rho_\delta(s)g'(t-s)ds + \alpha p u^{p-1}(t)u'(t)\Big] \\ &= (b_\delta + \lambda)^{-1}\left[\int_0^t s\rho_t(s)u'(t-s)ds + \int_t^\delta s\rho_\delta(s)g'(t-s)ds + \alpha p u^{p-1}(t)u'(t)\right].\end{aligned}$$

Then we calculate the difference $\mathcal{K}u(t) - \mathcal{K}v(t)$ and $(\mathcal{K}u)'(t) - (\mathcal{K}v)'(t)$,

$$\begin{aligned}\mathcal{K}u(t) - \mathcal{K}v(t) &= (b_\delta + \lambda)^{-1}\left[\int_0^t s\rho_\delta(s)(u(t-s) - v(t-s))ds + \alpha u^p(t) - \alpha v^p(t)\right] \\ (\mathcal{K}u)'(t) - (\mathcal{K}v)'(t) &= (b_\delta + \lambda)^{-1}[\int_0^t s\rho_\delta(s)(u'(t-s) - v'(t-s))ds \\ &\quad + \alpha p u^{p-1}(t)u'(t) - \alpha p v^{p-1}(t)v'(t)]\end{aligned}$$

$$\begin{aligned}|\mathcal{K}u(t)| &\le (b_\delta + \lambda)^{-1}\left[\int_0^t s\rho_\delta(s)|u(t-s)|ds + \int_t^\delta s\rho_\delta(s)|g(t-s)|ds + |\alpha||u(t)|^p\right] \\ &\le (b_\delta + \lambda)^{-1}\left[b_\delta(t)\|u\|_{L^\infty} + (b_\delta - b_\delta(t))\|g\|_{L^\infty} + |\alpha|\|u\|_{L^\infty}^p\right] \\ |(\mathcal{K}u)'(t)| &\le (b_\delta + \lambda)^{-1}[\int_0^t s\rho_t(s)|u'(t-s)|ds \\ &\quad + \int_t^\delta s\rho_\delta(s)|g'(t-s)|ds + |\alpha|p|u(t)|^{p-1}|u'(t)|] \\ &\le (b_\delta + \lambda)^{-1}\left[b_\delta(t)\|u'\|_{L^\infty} + (b_\delta - b_\delta(t))\|g'\|_{L^\infty} + |\alpha|p\|u\|_{L^\infty}^{p-1}\|u'\|_{L^\infty}\right]\end{aligned}$$

$$
\begin{aligned}
|\mathcal{K}u(t)-\mathcal{K}v(t)| &\le (b_\delta+\lambda)^{-1}\left[\int_0^t s\rho_\delta(s)|u(t-s)-v(t-s)|ds+|\alpha||u^p(t)-v^p(t)|\right]\\
&\le (b_\delta+\lambda)^{-1}\left[b_\delta(t)\|u-v\|_{L^\infty}+|\alpha||u^p(t)-v^p(t)|\right]\\
|(\mathcal{K}u)'(t)-(\mathcal{K}v)'(t)| &\le (b_\delta+\lambda)^{-1}[\int_0^t s\rho_\delta(s)|u'(t-s)-v'(t-s)|ds\\
&\quad +|\alpha||pu^{p-1}(t)u'(t)-pv^{p-1}(t)v'(t)|]\\
&\le (b_\delta+\lambda)^{-1}[b_\delta(t)\|u'-v'\|_{L^\infty}+|\alpha||pu^{p-1}(t)u'(t)-pv^{p-1}(t)v'(t)|]
\end{aligned}
$$

$$
\begin{aligned}
&|\mathcal{K}u(t)|+|(\mathcal{K}u)'(t)|\\
\le&(b_\delta+\lambda)^{-1}\left[b_\delta(t)\|u\|_{W^{1,\infty}}+(b_\delta-b_\delta(t))\|g\|_{W^{1,\infty}}+|\alpha|\|u\|_{L^\infty}^p+|\alpha|p\|u\|_{L^\infty}^{p-1}\|u'\|_{L^\infty}\right]\\
&|\mathcal{K}u(t)-\mathcal{K}v(t)|+|(\mathcal{K}u)'(t)-(\mathcal{K}v)'(t)|\\
\le&(b_\delta+\lambda)^{-1}\left[b_\delta(t)\|u-v\|_{W^{1,\infty}}+|\alpha||u^p(t)-v^p(t)|+|\alpha||pu^{p-1}(t)u'(t)-pv^{p-1}(t)v'(t)|\right]
\end{aligned}
$$

Consider solution in the subset

$$V_g=\{u\in W^{1,\infty}(0,t_0):\|u\|_{W^{1,\infty}}\le R,\lim_{t\downarrow 0}u(t)=g(0)\}.$$

In order to show the map $\mathcal{K}$ has a unique fixed point in V_g, with the Banach fixed point theorem we only need that for some $0<t_0\le\delta$ and $0<c<1$, for all $t\in[0,t_0]$

$$
\begin{cases}
\quad b_\delta(t)\|u\|_{W^{1,\infty}(0,t_0)}+(b_\delta-b_\delta(t))\|g\|_{W^{1,\infty}(-\delta,0)}+|\alpha|p\|u\|_{W^{1,\infty}(0,t_0)}^p\\
\le(b_\delta+\lambda)\|u\|_{W^{1,\infty}(0,t_0)},\\
\quad b_\delta(t)\|u-v\|_{W^{1,\infty}(0,t_0)}+|\alpha||u^p(t)-v^p(t)|+|\alpha||pu^{p-1}(t)u'(t)-pv^{p-1}(t)v'(t)|\\
\le(1-c)(b_\delta+\lambda)\|u-v\|_{W^{1,\infty}(0,t_0)}.
\end{cases}
$$

For some $0<t_0\le\delta$ and $0<c<1$, for all $t\in[0,t_0]$,

$$
\begin{cases}
b_\delta(t)\|u\|_{W^{1,\infty}}+(b_\delta-b_\delta(t))\|g\|_{W^{1,\infty}}+|\alpha|p\|u\|_{W^{1,\infty}}^p\le(b_\delta+\lambda)\|u\|_{W^{1,\infty}},\\
b_\delta(t)\|u-v\|_{W^{1,\infty}}+|\alpha|p\max\{\|u\|_{W^{1,\infty}}^{p-1},\|v\|_{W^{1,\infty}}^{p-1}\}\|u-v\|_{W^{1,\infty}}\le(1-c)(b_\delta+\lambda)\|u-v\|_{W^{1,\infty}}.
\end{cases}
$$

For some $0<t_0\le\delta$, for all $t\in[0,t_0]$,

$$
\begin{cases}
\|g\|_{W^{1,\infty}(-\delta,0)}\le(b_\delta-b_\delta(t))^{-1}[(b_\delta-b_\delta(t)+\lambda)R-|\alpha|pR^p],\\
|\alpha|pR^{p-1}<b_\delta+\lambda.
\end{cases}
$$

Define $F(R)$ as

$$F_t(R) := (b_\delta - b_\delta(t))^{-1}[(b_\delta - b_\delta(t) + \lambda)R - |\alpha| p R^p],$$

We have that

$$\|g\|_{W^{1,\infty}(-\delta,0)} \leq \min_{t\in[0,t_0]} \max_{R<\left(\frac{(b_\delta+\lambda)}{|\alpha|p}\right)^{1/(p-1)}} F_t(R),$$

The critical point of $F_t(R)$ is

$$R_c = \left(\frac{b_\delta - b_\delta(t) + \lambda}{|\alpha|p^2}\right)^{\frac{1}{p-1}} < \left(\frac{(b_\delta + \lambda)}{|\alpha|p}\right)^{\frac{1}{p-1}},$$

for $p > 1$. Thus,

$$\max_{R<\left(\frac{(b_\delta+\lambda)}{|\alpha|p}\right)^{1/(p-1)}} F_t(R) = F_t(R_c) = \frac{p-1}{p}\left(\frac{1}{p^2}\right)^{\frac{p}{p-1}} \frac{(b_\delta - b_\delta(t) + \lambda)^{\frac{p}{p-1}}}{b_\delta - b_\delta(t)}.$$

Then let

$$G(z) = (z+\lambda)^{\frac{p}{p-1}}/z, \quad z \in [b_\delta - b_\delta(t_0), b_\delta]$$

With the assumption that

$$b_\delta - b_\delta(t_0) \geq (p-1)\lambda,$$

The minimum value of G is obtained at $z_{t_0} = b_\delta - b_\delta(t_0)$. Therefore

$$\|g\|_{W^{1,\infty}(-\delta,0)} \leq \frac{p-1}{p}\left(\frac{1}{p^2}\right)^{\frac{p}{p-1}} \frac{(b_\delta - b_\delta(t_0) + \lambda)^{\frac{p}{p-1}}}{b_\delta - b_\delta(t_0)}.$$

Thus, as a conclusion we have the following well-posedness theorem in $W^{1,\infty}(0,\delta)$ for nonlinear nonlocal problem with integrable kernel:

Theorem 5.2.2 (Well-posedness of problem with integrable kernels)**.** *Assume that the nonlocal kernel $s\rho_\delta(s) \in L^1(0,\delta)$, $p \geq 2$, and nonlocal horizon δ is small such that $b_\delta > (p-1)\lambda$. Given the initial data g in $W^{1,\infty}(-\delta, 0)$ satisfying that the* *continuity condition*

$$\alpha g^p(0) - \lambda g(0) - \int_0^\delta s\rho_\delta(s)(g(0) - g(-s))ds = 0,$$

and

$$\|g\|_{W^{1,\infty}(-\delta,0)} < \frac{p-1}{p}\left(\frac{1}{p^2}\right)^{\frac{p}{p-1}} \frac{(b_\delta + \lambda)^{\frac{p}{p-1}}}{b_\delta}.$$

There is a $t_0 < \delta$ *such that*

$$\begin{cases} b_\delta - b_\delta(t_0) \geq (p-1)\lambda, \\ \|g\|_{W^{1,\infty}(-\delta,0)} \leq \frac{p-1}{p}\left(\frac{1}{p^2}\right)^{\frac{p}{p-1}} \frac{(b_\delta - b_\delta(t_0) + \lambda)^{\frac{p}{p-1}}}{b_\delta - b_\delta(t_0)}. \end{cases}$$

there is a unique solution $u \in W^{1,\infty}(0, t_0)$ *of the nonlocal-in-time problem*

$$\begin{cases} \mathcal{G}_\delta u(t) + \lambda u(t) = \alpha u^p(t), & t \in (0, t_0), \\ u(t) = g(t), & t \in [-\delta, 0]. \end{cases}$$

satisfying that

$$\|u\|_{W^{1,\infty}(0,t_0)} \leq \left[\frac{1}{|\alpha| p}(b_\delta - b_\delta(t_0) + \lambda)\right]^{\frac{1}{p}},$$

and

$$\lim_{t \downarrow 0} u(t) = g(0).$$

Remark. *Further discussion about the continuity condition of the initial data* g *is absolutely necessary. First,* $g \equiv 0$*, the trivial zero initial data, satisfies the condition and the corresponding trivial solution of the nonlocal problem is* $u \equiv 0$*. Second, another trivial initial data satisfies the continuity condition is* $g \equiv \lambda^{\frac{1}{p-1}}$ *which corresponds to the constant trivial solution* $u \equiv \lambda^{\frac{1}{p-1}}$*.*
Finally, any $W^{1,\infty}(-\delta, 0)$ *function* $g(t)$ *can be rewritten as* $g(t) = g(0) + (g(t) - g(0))$*. Let constant* $x = g(0)$ *and* $W^{1,\infty}(-\delta, 0)$ *function* $f(t) = g(t) - g(0)$ *satisfying* $f(0) = 0$*. Therefore, the continuity condition for* g *is equivalent to the following condition for* f *and* x*:*

$$x^p - \lambda x + \int_0^\delta s\rho_\delta(s) f(-s)ds = 0.$$

We must have that

$$\int_0^\delta s\rho_\delta(s) f(-s) ds < +\infty.$$

Since we know that

$$\int_0^\delta s^2 \rho_\delta(s) ds = 1,$$

thus f satisfies that

$$\lim_{t\to 0^-} \frac{-t}{|f(t)|} < +\infty.$$

When p is odd numbers, for any constant C, there is at least one solution x of the polynomial equation

$$x^p - \lambda x + C = 0.$$

When p is even numbers, by simple computations we can see that the polynomial equation has at least solution x for constant C satisfying that

$$C \le (p-1)\left(\frac{\lambda}{p}\right)^{\frac{p}{p-1}}.$$

In particular, for $p = 2$, we need $C \le \lambda^2/4$.

5.2.3 C^0 solutions for non-integrable kernel case

When the nonlocal kernel is not integrable, i.e., $s\rho_\delta(s) \notin L^1(0,\delta)$, we consider the truncated nonlocal operator defined by

$$\mathcal{G}_\delta^\epsilon u(t) := \int_\epsilon^\delta s\rho_\delta(s)(u(t) - u(t-s)) ds,$$

with integrable kernel. The new nonlocal problem in terms of this integrable kernel is given by

$$\begin{cases} \mathcal{G}_\delta^\epsilon u_\epsilon(t) + \lambda u_\epsilon(t) = \alpha u_\epsilon^p(t), & t \in (0, t_0), \\ u_\epsilon(t) = g_\epsilon(t), & t \in [-\delta, 0]. \end{cases}$$

Let $b_\delta^\epsilon = \int_\epsilon^\delta s\rho_\delta(s) ds < +\infty$, and

$$b_\delta^\epsilon(t) = \begin{cases} \displaystyle\int_\epsilon^t s\rho_\delta(s) ds, & t \in (\epsilon, \delta], \\ 0, & t \in [0, \epsilon]. \end{cases}$$

Following the proof above, the truncated kernel problem has a unique solution $u_\epsilon \in W^{1,\infty}(0, t_0)$ satisfying that

$$\|u_\epsilon\|_{W^{1,\infty}(0,t_0)} \le R_\epsilon = \left[\frac{1}{|\alpha| p}(b_\delta^\epsilon - b_\delta^\epsilon(t_0) + \lambda)\right]^{\frac{1}{p-1}} = \left[\frac{1}{|\alpha| p}\left(\int_{t_0}^{\delta} s\rho_\delta(s)ds + \lambda\right)\right]^{\frac{1}{p-1}},$$

with the ϵ-dependent initial data $g_\epsilon \in W^{1,\infty}(0, t_0)$ satisfying that

$$\|g_\epsilon\|_{W^{1,\infty}(-\delta,0)} \le \frac{p-1}{p}\left(\frac{1}{p^2}\right)^{\frac{p}{p-1}} \frac{(b_\delta^\epsilon - b_\delta^\epsilon(t_0) + \lambda)^{\frac{p}{p-1}}}{b_\delta^\epsilon - b_\delta^\epsilon(t_0)} = \frac{p-1}{p}\left(\frac{1}{p^2}\right)^{\frac{p}{p-1}} \frac{\left(\int_{t_0}^{\delta} s\rho_\delta(s)ds + \lambda\right)^{\frac{p}{p-1}}}{\int_{t_0}^{\delta} s\rho_\delta(s)ds},$$

and the continuity condition

$$\alpha g_\epsilon^p(0) - \lambda g_\epsilon(0) - \int_\epsilon^\delta s\rho_\delta(s)(g_\epsilon(0) - g_\epsilon(-s))ds = 0,$$

for each ϵ. Here, both upper bounds are independent of the truncation parameter ϵ. Moreover, we require that the initial data g_ϵ converges to the initial data g in $W^{1,\infty}(-\delta, 0)$ satisfying the continuity condition

$$\alpha g^p(0) - \lambda g(0) - \int_0^\delta s\rho_\delta(s)(g(0) - g(-s))ds = 0.$$

With $\{u_\epsilon\} \in W^{1,\infty}(0, t_0)$ having the uniform bound in $W^{1,\infty}(0, t_0)$ norm, since $W^{1,\infty}(0, t_0)$ is compactly embedded in $C^0(0, t_0)$, there is a subsequence of $\{u_\epsilon\}$ strongly converges to u in $C^0(0, t_0)$. We still need to check that this strong limit $u \in C^0(0, t_0)$ satisfies the non-local problem with non-integrable kernel,

$$\mathcal{G}_\delta^\epsilon u + \lambda u = u^p.$$

Taking $\varphi \in C_c^\infty(0, t_0)$ with zero extension out of the interval and multiplying φ on both sides, we have the following after integrating from 0 to t_0 and interchanging the order of integration,

$$(u_\epsilon^p, \varphi) = (\mathcal{G}_\delta^\epsilon u_\epsilon, \varphi) + (\lambda u_\epsilon, \varphi).$$

Following the proof of Lemma 3.3 in [2] we have that

$$(u^p, \varphi) = \lim_{\epsilon \to 0} (u^p_\epsilon, \varphi) = (\tilde{\mathcal{G}}_\delta u, \varphi) + (\lambda u, \varphi).$$

Thus,

$$\tilde{\mathcal{G}}_\delta u = u^p - \lambda u \in C^0(0, t_0),$$

and

$$\tilde{\mathcal{G}}_\delta u = \mathcal{G}_\delta u \in C^0(0, t_0).$$

Hence, u satisfies the nonlocal-in-time equation

$$\mathcal{G}_\delta u + \lambda u = u^p, \tag{5.4}$$

almost everywhere which gives the local existence of the solution of the nonlinear nonlocal problem for non-integrable kernel.

Theorem 5.2.3 (Local existence for non-integrable kernel case)**.** *Assume that* $s\rho_\delta(s) \notin L^1(0,\delta)$, $p \geq 2$, *and* $t_0 < \delta$ *and small* δ *such that* $\int_{t_0}^{\delta} s\rho_\delta(s)ds \geq (p-1)\lambda$. *Given the initial data* g *in* $W^{1,\infty}(-\delta, 0)$ *satisfying that*

$$\|g\|_{W^{1,\infty}(-\delta,0)} \leq \frac{p-1}{p} \left(\frac{1}{p^2}\right)^{\frac{p}{p-1}} \frac{\left(\int_{t_0}^{\delta} s\rho_\delta(s)ds + \lambda\right)^{\frac{p}{p-1}}}{\int_{t_0}^{\delta} s\rho_\delta(s)ds},$$

and the continuity condition

$$\alpha g^p(0) - \lambda g(0) - \int_0^\delta s\rho_\delta(s)(g(0) - g(-s))ds = 0,$$

there exists a solution $u \in C^0(0, t_0)$ *of the nonlocal problem satisfying that*

$$\|u\|_{L^\infty(0,t_0)} \leq \left[\frac{1}{|\alpha| p}\left(\int_{t_0}^{\delta} s\rho_\delta(s)ds + \lambda\right)\right]^{\frac{1}{p-1}}.$$

Remark. *Again, we need to show that we can construct such* $\{g_\epsilon\} \subseteq W^{1,\infty}(-\delta, 0)$ *converges to* g *in* $W^{1,\infty}(-\delta, 0)$ *as* $\epsilon \to 0$. *Let* $g_\epsilon(t) = x_\epsilon + f(t)$, $C = \int_0^\delta s\rho_\delta(s)f(-s)ds$, *and* $C_\epsilon = \int_\epsilon^\delta s\rho_\delta(s)f(-s)ds$. *Then we have that* $C_\epsilon \to C$ *as* $\epsilon \to 0$. *From the continuity conditions for* g *and* g_ϵ, x *and* x_ϵ *solve the quadratic equations* $\alpha x^p - \lambda x + C = 0$ *and* $\alpha x_\epsilon^p - \lambda x_\epsilon + C_\epsilon = 0$ *respectively,* x_ϵ *can be chosen as the solution of* *the polynomial equations and converges to x as $\epsilon \to 0$. Therefore $g_\epsilon(t) = x_\epsilon + f(t)$ converges to $g(t) = x + f(t)$ in $W^{1,\infty}(-\delta, 0)$ as $\epsilon \to 0$.*

5.3 A Class of Nonlocal-in-Time Reaction-Diffusion problems

In this section, we consider the nonlocal version of the deterministic reaction-diffusion problem

$$\begin{cases} \mathcal{G}_\delta u(t) = u(t) - u^3(t) + Au(t), & t \in (0, t_0], \\ \quad u(t) = g(t), & t \in [-\delta, 0], \end{cases}$$

where the linear (unbounded) operator A satisfies the conditions in the *Hille-Yosida* Theorem 4.2.3 and $\mathcal{D}(A) \subseteq B$ for some Banach space B. For the sake of simplicity, by the nonlocal integrating factor method approach (Here we need to assume that 1 is in the range $h_\delta(\mathbb{R})$ which is true for small δ.), we only need to consider the problem with λu term added on the left-hand-side and for some positive coefficient λ,

$$\begin{cases} \mathcal{G}_\delta u(t) + \lambda u(t) = -u^3(t) + Au(t), & t \in (0, t_0], \\ \qquad u(t) = g(t), & t \in [-\delta, 0], \end{cases}$$

where $g \in L^\infty([-\delta, 0]; B)$ gives the initial data and $s\rho_\delta(s) \in L^1(0, \delta)$ is an integrable kernel. The integral form of the above problem is

$$u(t) = [(b_\delta + \lambda)I - A]^{-1} \left[\int_0^t \tau\rho_\delta(\tau)\mathbb{1}_{[0,\delta]}(\tau)u(t-\tau)d\tau + \int_t^\delta \tau\rho_\delta(\tau)\mathbb{1}_{[0,\delta]}(\tau)g(t-\tau)d\tau - u^3(t) \right].$$

The map T is defined as

$$Tu(t) = [(b_\delta + \lambda)I - A]^{-1} \left[\int_0^t s\rho_\delta(s)\mathbb{1}_{[0,\delta]}(s)u(t-s)ds + \int_t^\delta s\rho_\delta(s)g(t-s)ds - u^3(t) \right],$$

$$(Tu - Tv)(t) = [(b_\delta + \lambda)I - A]^{-1} \left[\int_0^t s\rho_\delta(s)\mathbb{1}_{[0,\delta]}(s)(u-v)(t-s)ds - (u^3(t) - v^3(t)) \right],$$

$$\|Tu(t)\|_B \le (b_\delta + \lambda)^{-1} \left[b_\delta(t)\|u\|_{L^\infty([0,t_0];B)} + (b_\delta - b_\delta(t))\|g\|_{L^\infty([-\delta,0];B)} + \|u\|^3_{L^\infty([0,t_0];B)} \right],$$

$$\|Tu(t) - Tv(t)\|_B \le (b_\delta + \lambda)^{-1} \left[b_\delta(t)\|u - v\|_{L^\infty([0,t_0];B)} + \|u^3(t) - v^3(t)\|_B \right].$$

Suppose that $\|u\|_{L^\infty([0,t_0];B)}, \|v\|_{L^\infty([0,t_0];B)} \le R$, then for all $t \in [0, t_0]$ and for some $0 < c < 1$, with the operator A satisfying the *Hille-Yosida* conditions, we have that

$$\begin{cases} \quad \|Tu(t)\|_B \\ \le (b_\delta + \lambda)^{-1} \left[b_\delta(t) R + (b_\delta - b_\delta(t)) \|g\|_{L^\infty([-\delta,0];B)} + R^3 \right] \le R, \\ \quad \|Tu(t) - Tv(t)\|_B \\ \le (b_\delta + \lambda)^{-1} \left[b_\delta(t) \|u - v\|_{L^\infty([0,t_0];B)} + 3R^2 \|u - v\|_{L^\infty([0,t_0];B)} \right] \le (1 - c) \|u - v\|_{L^\infty([0,t_0];B)}. \end{cases}$$

It implies that given the initial data g, for all $t \in [0, t_0]$ there is a R that satisfies the inequalities,

$$\begin{cases} R < \left[\dfrac{1}{3} (b_\delta - b_\delta(t) + \lambda) \right]^{1/2} \\ \|g\|_{L^\infty([-\delta,0];B)} \le (b_\delta - b_\delta(t))^{-1} ((b_\delta - b_\delta(t) + \lambda) R - R^3). \end{cases}$$

Thus, we must have that

$$\begin{aligned} \|g\|_{L^\infty([-\delta,0];B)} &\le \min_{t \in [0,t_0]} \frac{2}{3\sqrt{3}} (b_\delta - b_\delta(t))^{-1} (b_\delta - b_\delta(t) + \lambda)^{3/2} \\ &= \frac{2}{3\sqrt{3}} (b_\delta - b_\delta(t_0))^{-1} (b_\delta - b_\delta(t_0) + \lambda)^{3/2} \\ &= \frac{2(b_\delta - b_\delta(t_0) + \lambda)^{3/2}}{3^{3/2}(b_\delta - b_\delta(t_0))} \end{aligned}$$

Theorem 5.3.1 (Local existence of mild solutions)**.** *Assume that the linear (unbounded) operator A satisfies the conditions in Theorem 4.2.3 (*Hille-Yosida*), $\mathcal{D}(A) \subseteq B$ for some Banach space B. Assume that the nonlocal kernel $s\rho_\delta(s) \in L^1(0, \delta)$ and $0 < t_0 < \delta$ such that $b_\delta - b_\delta(t_0) > 2\lambda$. Given the initial data $g \in L^\infty([-\delta, 0]; B)$ satisfying that*

$$\|g\|_{L^\infty([-\delta,0];B)} \le \frac{2(b_\delta - b_\delta(t_0) + \lambda)^{3/2}}{3^{3/2}(b_\delta - b_\delta(t_0))},$$

the nonlocal problem with integrable kernel

$$\begin{cases} \mathcal{G}_\delta u(t) + \lambda u(t) = -u^3(t) + Au(t), & t \in (0, t_0], \\ \qquad u(t) = g(t), & t \in [-\delta, 0], \end{cases}$$

has a unique mild solution $u \in L^{\infty}([0,t_0];B)$ *satisfying that*

$$\|u\|_{L^{\infty}([0,t_0];B)} \leq \left[\frac{1}{3}(b_\delta - b_\delta(t_0) + \lambda)\right]^{1/2}.$$

5.4 Conclusions

In this chapter, by introducing the nonlinearities $f(u)$ into the linear nonlocal-in-time problems, we considered the nonlinear nonlocal-in-time problem (5.1). First, for the Lipschitz continuous nonlinearities, we have shown the global existence and uniqueness of the solution in $L^{\infty}(0,T)$ of the nonlinear nonlocal-in-time problem. The localization result has also been proved. Then for a class of autonomous nonlinearities in the form of $f(u) = u^p|u|^\alpha$, with $p \in \mathbb{N}^+$ and $\alpha \in [0,1)$, we have established the local existence and uniqueness of the solutions in both $L^{\infty}(0,t_0)$ and $W^{1,\infty}(0,t_0)$ spaces for integrable kernels, and only the local existence of the solutions in $C^0(0,t_0)$ space of the nonlinear nonlocal-in-time problems for non-integrable kernels. Here the existence interval $(0,t_0)$ is small since we assume that $t_0 < \delta$. Therefore, no localization theory was discussed in these sections. Finally, we have obtained the local existence and uniqueness of the mild solutions in $L^{\infty}([0,t_0];B)$ of the deterministic nonlocal-in-time reaction-diffusion problems in Banach spaces for integrable nonlocal kernels.

We have studied the nonlinear nonlocal-in-time problems for a class of autonomous nonlinearities in the form of $f(u) = u^p|u|^\alpha$ in this thesis because of the nonlinear terms in the stochastic bi-stable mean-field models and the stochastic Allen-Cahn/Cahn-Hilliard equations are usually considered in this type. The nonlocal problems with general nonlinearities could be considered, from which a more general theory of the nonlocal-in-time problems for the nonlinear ODE models could be established.

Chapter 6
The Stochastic Nonlocal-in-Time Problems for SDE Models

In the previous two chapters, we have studied the deterministic linear and nonlinear nonlocal-in-time problems. In this chapter, we shall continue to study the last sub-problem, the stochastic nonlocal-in-time problem, or the nonlocal Langevin equation,

$$\begin{cases} \mathcal{G}_\delta u(t) = -\theta u(t) + \sigma \mathcal{G}_\delta W_{t\vee 0}, & t \in (0, T], \\ u(t) = g(t), & t \in [-\delta, 0]. \end{cases} \tag{6.1}$$

Here the nonlocal-in-time gradient operator $\mathcal{G}_\delta$ requires the definition of W_t on $[-\delta, T]$, so we use $W_{t\vee 0}$ instead by assuming a zero extension of the standard Brownian motion W_t in $[-\delta, 0]$, where $t \vee 0 = \max(t, 0)$.

In the theory of stochastic processes, a random process X is called mean-square integrable from a to b if $\mathbb{E}[\int_a^b X^2(t)dt]$ is finite. The class of all such processes will be written $\mathcal{S}_2(a, b)$ and the norm of a process $X \in \mathcal{S}_2(a, b)$ is $\|X\|_{\mathcal{S}_2} = \left\{\mathbb{E}[\int_a^b X^2(t)dt]\right\}^{1/2}$. In this chapter, we shall study the solutions of (6.14) in this space. Given W_t the standard Brownian motion, the *Ornstein-Uhlenbeck* process X_t is defined by the stochastic differential equation

$$\begin{cases} dX_t = -\theta X_t dt + \sigma dW_t, & t > 0, \\ X_t = x_0, & t = 0, \end{cases} \tag{6.2}$$

where $\theta > 0$ and $\sigma > 0$. We will show the stochastic nonlocal-in-time problem (6.14) is well-posed and moreover its solution converges to the *Ornstein-Uhlenbeck* process in $\mathcal{S}_2$ norm.

6.1 Nonlocal-in-Time Problems with Random Force

We begin the study on the stochastic nonlocal-in-time problems with a linear nonlocal-in-time problem with random force,

$$\begin{cases} \mathcal{G}_\delta u(t) + \lambda u(t) = f(t), & t \in (0,T], \\ u(t) = g(t), & t \in [-\delta, 0]. \end{cases} \tag{6.3}$$

In the case of integrable kernels $s\rho_\delta(s) \in L^1(0,\delta)$, we have the well-posedness of the stochastic nonlocal-in-time problem (6.3) with the a priori estimate of the solution in $\mathcal{S}_2$-norm. When f is deterministic, the nonlocal-in-time problem (6.3) is a deterministic linear problem which has been studied in [2]. Here we use the similar approach to establish the related results for stochastic problems.

Theorem 6.1.1. *Suppose that the random force $f \in \mathcal{S}_2(0,T)$, the deterministic initial data $g(t) \in L^\infty(-\delta, 0)$, and the nonlocal kernel $s\rho_\delta(s) \in L^1(0,\delta)$. Then there is a unique (almost surely) solution $u \in \mathcal{S}_2(0,T)$ which satisfies the stochastic nonlocal-in-time problem with random force (6.3) and it holds that*

$$\|u\|_{\mathcal{S}_2(0,T)} \leq \lambda^{-1/2}\|g\|_{L^\infty(-\delta,0)} + \lambda^{-1}\|f\|_{\mathcal{S}_2(0,T)}.$$

Proof. Let the operator $\mathcal{K}$ be defined as

$$\mathcal{K}u(t) = \frac{1}{b_\delta + \lambda}\left(\int_0^\delta s\rho_\delta(s)u(t-s)ds + f(t)\right), \quad t \in (0,T],$$

together with $\mathcal{K}u(t) = g(t)$ in $[-\delta, 0]$, where $b_\delta = \int_0^\delta s\rho_\delta(s)ds$. Now we denote a subset $V \subseteq \mathcal{S}_2(0,T)$,

$$V = \left\{ u \in \mathcal{S}_2(0,T) \,\middle|\, \begin{array}{l} u = g \text{ almost surely in } [-\delta, 0], \\ \|u\|_{\mathcal{S}_2(0,T)} \leq \lambda^{-1/2}\|g\|_{L^\infty(-\delta,0)} + \lambda^{-1}\|f\|_{\mathcal{S}_2(0,T)} \end{array} \right\}. \tag{6.4}$$

Given $u \in V$, by the Hölder's inequality, we have that

$$\begin{aligned} &\|\mathcal{K}u(t)\|_{\mathcal{S}_2(0,T)} \\ \leq &\frac{1}{b_\delta + \lambda}\left(\int_0^\delta s\rho_\delta(s)\|u(t-s)\|_{\mathcal{S}_2(0,T)}ds + \|f\|_{\mathcal{S}_2(0,T)}\right) \end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{b_\delta+\lambda}\left[\left(\int_0^\delta s\rho_\delta(s)ds\right)^{1/2}\left(\int_0^\delta s\rho_\delta(s)\|u(t-s)\|^2_{\mathcal{S}_2(0,T)}ds\right)^{1/2}+\|f\|_{\mathcal{S}_2(0,T)}\right]\\
&\le \frac{1}{b_\delta+\lambda}\left[b_\delta^{1/2}\left(\int_0^\delta s\rho_\delta(s)\left(\|g\|^2_{\mathcal{S}_2(-s,0)}+\|u\|^2_{\mathcal{S}_2(0,T)}\right)ds\right)^{1/2}+\|f\|_{\mathcal{S}_2(0,T)}\right]\\
&\le \frac{1}{b_\delta+\lambda}\left[b_\delta^{1/2}\left(\int_0^\delta s\rho_\delta(s)\left(s\|g\|^2_{L^\infty(-\delta,0)}+\|u\|^2_{\mathcal{S}_2(0,T)}\right)ds\right)^{1/2}+\|f\|_{\mathcal{S}_2(0,T)}\right]\\
&= \frac{1}{b_\delta+\lambda}\left[b_\delta^{1/2}\left(\|g\|^2_{L^\infty(-\delta,0)}+b_\delta\|u\|^2_{\mathcal{S}_2(0,T)}\right)^{1/2}+\|f\|_{\mathcal{S}_2(0,T)}\right]\\
&\le \frac{1}{b_\delta+\lambda}\left[b_\delta^{1/2}\left(\|g\|^2_{L^\infty(-\delta,0)}+b_\delta(\lambda^{-1/2}\|g\|_{L^\infty(-\delta,0)}+\lambda^{-1}\|f\|_{\mathcal{S}_2(0,T)})^2\right)^{1/2}+\|f\|_{\mathcal{S}_2(0,T)}\right]\\
&\le \frac{1}{b_\delta+\lambda}\left[b_\delta^{1/2}\left((b_\delta\lambda^{-1}+1)^{1/2}\|g\|_{L^\infty(-\delta,0)}+b_\delta^{1/2}\lambda^{-1}\|f\|_{\mathcal{S}_2(0,T)}\right)+\|f\|_{\mathcal{S}_2(0,T)}\right]\\
&= \frac{1}{b_\delta+\lambda}\left[b_\delta^{1/2}(b_\delta\lambda^{-1}+1)^{1/2}\|g\|_{L^\infty(-\delta,0)}+(b_\delta\lambda^{-1}+1)\|f\|_{\mathcal{S}_2(0,T)}\right]\\
&\le \lambda^{-1/2}\|g\|_{L^\infty(-\delta,0)}+\lambda^{-1}\|f\|_{\mathcal{S}_2(0,T)}.
\end{aligned}
$$

Furthermore, we have that for $u, v \in V$,

$$
\begin{aligned}
\|\mathcal{K}u(t)-\mathcal{K}v(t)\|_{\mathcal{S}_2(0,T)} &= \frac{1}{b_\delta+\lambda}\left(\int_0^\delta s\rho_\delta(s)\|u(t-s)-v(t-s)\|_{\mathcal{S}_2(0,T)}ds\right)\\
&\le \frac{b_\delta}{b_\delta+\lambda}\|u(t)-v(t)\|_{\mathcal{S}_2(0,T)},
\end{aligned}
$$

with $\frac{b_\delta}{b_\delta+\lambda}<1$. So $\mathcal{K}$ is a contraction mapping which maps from V into itself. Then applying the Banach fixed-point theorem, there is a unique solution $u \in \mathcal{S}_2(0,T)$ satisfying the a priori estimate $\|u\|_{\mathcal{S}_2(0,T)} \le \lambda^{-1/2}\|g\|_{L^\infty(-\delta,0)}+\lambda^{-1}\|f\|_{\mathcal{S}_2(0,T)}$. □

Remark. *In the case of non-integrable kernels i.e., $s\rho_\delta(s) \notin L^1(0,\delta)$, which is due to the weak singularity of the kernel. We consider a sequence of nonlocal-in-time problems for $\{u^\epsilon\}_{\epsilon>0}$ given by*

$$
\begin{cases}
\mathcal{G}_\delta^\epsilon u^\epsilon(t)+\lambda u^\epsilon(t)=f(t), & t\in(0,T],\\
u^\epsilon(t)=g(t), & t\in[-\delta,0],
\end{cases}
\tag{6.5}
$$

with the truncated nonlocal operator $\mathcal{G}_\delta^\epsilon$ introduced in [2] and defined by

$$
\mathcal{G}_\delta^\epsilon u(t) := \int_\epsilon^\delta s\rho_\delta(s)(u(t)-u(t-s))ds.
$$

With the singularity removed, argument given above becomes applicable which implies that there is a unique solution $u^\epsilon \in \mathcal{S}_2(0,T)$ satisfying the uniform bound

$$\|u^\epsilon\|_{\mathcal{S}_2(0,T)} \leq \lambda^{-1/2}\|g\|_{L^\infty(-\delta,0)} + \lambda^{-1}\|f\|_{\mathcal{S}_2(0,T)}.$$

Following the similar approach as in [2], there is a subsequence of u^ϵ weakly converges to u in $\mathcal{S}_2(0,T)$, u is the unique (almost surely) solution of the nonlocal-in-time problem with non-integrable kernels and it holds that

$$\|u\|_{\mathcal{S}_2(0,T)} \leq \lambda^{-1/2}\|g\|_{L^\infty(-\delta,0)} + \lambda^{-1}\|f\|_{\mathcal{S}_2(0,T)}.$$

6.2 Nonlocal Stochastic Integrals

We have shown in the previous section the well-posedness of the stochastic nonlocal-in-time problems with random force in $\mathcal{S}_2(0,T)$. Now we will specify the random force as a nonlocal version of the white noise. Given the integrable nonlocal kernel $s\rho_\delta(s) \in L^1(0,\delta)$ and W_t the standard Brownian motion, we define the nonlocal Brownian motion W_t^δ as the following weighted integral

$$W_t^\delta := \int_0^\delta \gamma_\delta(s) W_{(t-s)\vee 0} ds, \tag{6.6}$$

with the function $\gamma_\delta(t)$ defined on $(0,\delta)$ as

$$\gamma_\delta(t) = \int_t^\delta s\rho_\delta(s)ds, \tag{6.7}$$

which is a non-negative, decreasing and continuous function in $(0,\delta)$, and satisfies that $\int_0^\delta \gamma_\delta(s)ds = 1$. In addition, for the stochastic nonlocal-in-time problems, we requires γ_δ to be in $L^2(0,\delta)$ so that the above nonlocal Brownian motion (6.6) is well-defined and in $\mathcal{S}_2(0,T)$. Equivalently, this definition of W_t^δ also gives a nonlocal version of the white noise as $\mathcal{G}_\delta W_t$. In general, we introduce the definition of the nonlocal version of the classical stochastic integral $\int_0^t \sigma(x_s,s)dW_s$ as follows.

Definition 6.2.1 (Nonlocal Stochastic Integral)**.** *Let $\gamma_\delta(s) \in L^2(0,\delta)$ be defined as (6.7) and $x_t \in \mathcal{S}_2(-\delta,T)$. Given the* Lipschitz *continuous function $\sigma(x,t)$ with*

Lipschitz *continuous constant L such that,*

$$|\sigma(x,t)| \le 1 + L|x|, \quad |\sigma(x,t) - \sigma(y,t)| \le L|x-y|,$$

for $t \in [-\delta, T]$ and $x, y \in \mathbb{R}$. The nonlocal stochastic integral is defined as

$$\int_0^\delta \gamma_\delta(s) \int_0^{(t-s)\vee 0} \sigma(x_\tau, \tau) dW_\tau ds. \tag{6.8}$$

In particular, taking the function $\sigma(x,t)$ to be constant, i.e., $\sigma \equiv 1$, it gives a nonlocal version of the standard Brownian motion as (6.6).

Remark. *In this thesis, we only consider the Itô-type nonlocal stochastic integrals with the assumption that the inner integral in (6.8) is a Itô integral. In general, other types of classical stochastic integrals can be considered for this inner integral here and the nonlocal expression in (6.8) will then give the corresponding nonlocal versions in those cases.*

One important benefit directly from this definition of nonlocal stochastic integrals is that we can differentiate it and obtain a nonlocal version of the multiplicative noise which is well-defined in $\mathcal{S}_2(0,T)$ while we know that the *Wiener* process is nowhere differentiable.

Definition 6.2.2. *Let $\gamma_\delta(s) \in L^2(0,\delta)$ and the stochastic process $u(t) \in \mathcal{S}_2(-\delta, T)$, we introduce a process $u^\delta(t)$ in $[0,T]$ as a weighted averaging expression corresponding to nonlocal-in-time gradient operator $\mathcal{G}_\delta$ as following.*

$$u^\delta(t) := \int_0^\delta \gamma_\delta(s) u(t-s) ds. \tag{6.9}$$

The following lemma shows that when u is *Hölder* continuous, u^δ converges to u in $L^\infty(0,T)$, as $\delta \to 0$.

Lemma 6.2.1. *Given $u \in C^{0,\beta}(-\delta, T)$ for $\beta > 0$, we have that*

$$\lim_{\delta \to 0} \|u^\delta - u\|_{L^\infty(0,T)} = 0.$$

Proof.

$$\begin{aligned}
|u(t)-u^\delta(t)| &= \left|\int_0^\delta \gamma_\delta(s)(u(t)-u(t-s))ds\right| \\
&\le \int_0^\delta \gamma_\delta(s)|u(t)-u(t-s)|ds \\
&\le \int_0^\delta \gamma_\delta(s)s^\beta ds\|u\|_{C^{0,\beta}(-\delta,T)} \\
&\le \delta^\beta\|u\|_{C^{0,\beta}(-\delta,T)}, \\
\lim_{\delta\to 0}\|u^\delta-u\|_{L^\infty(0,T)} &\le \lim_{\delta\to 0}\sup_{t\in[0,T]}\delta^\beta\|u\|_{C^{0,\beta}(-\delta,T)} \\
&= \lim_{\delta\to 0}\delta^\beta\|u\|_{C^{0,\beta}(-\delta,T)} \\
&= 0.
\end{aligned}$$

□

Lemma 6.2.2 (Nonlocal Fundamental Theorem)**.** *Let $\gamma_\delta(s) \in L^2(0,\delta)$ and the stochastic process $u(t) \in \mathcal{S}_2(-\delta,T)$, then it holds that*

$$\int_a^b \mathcal{G}_\delta u(t)dt = u^\delta(b) - u^\delta(a),$$

almost surely. Equivalently,

$$\frac{d}{dt}u^\delta(t) = \mathcal{G}_\delta u(t),$$

almost surely.

Proof. By the definition of the nonlocal-in-time gradient operator $\mathcal{G}_\delta$,

$$\begin{aligned}
\int_a^b \mathcal{G}_\delta u(t)dt &= \int_a^b\int_0^\delta s\rho_\delta(s)(u(t)-u(t-s))dsdt \\
&= \int_a^b\int_0^\delta s\rho_\delta(s)u(t)dsdt - \int_a^b\int_0^\delta s\rho_\delta(s)u(t-s)dsdt \\
&= \int_0^\delta s\rho_\delta(s)\int_a^b u(t)dtds - \int_0^\delta s\rho_\delta(s)\int_{a-s}^{b-s} u(t)dtds \\
&= \int_0^\delta s\rho_\delta(s)\int_{b-s}^b u(t)dtds - \int_0^\delta s\rho_\delta(s)\int_{a-s}^{a} u(t)dtds \\
&= \int_{b-\delta}^b u(t)\int_{b-t}^\delta s\rho_\delta(s)dsdt - \int_{a-\delta}^a u(t)\int_{a-t}^\delta s\rho_\delta(s)dsdt \\
&= \int_0^\delta u(b-\tau)\int_\tau^\delta s\rho_\delta(s)dsd\tau - \int_0^\delta u(a-\tau)\int_\tau^\delta s\rho_\delta(s)dsd\tau \\
&= \int_0^\delta \gamma_\delta(\tau)u(b-\tau)d\tau - \int_0^\delta \gamma_\delta(\tau)u(a-\tau)d\tau \\
&= u^\delta(b)-u^\delta(a).
\end{aligned}$$

□

In particular, by taking $u(t) = \int_0^{t\vee 0} \sigma(x_s, s)dW_s$, we then have the following proposition which gives a nonlocal version of the multiplicative noise corresponding to the nonlocal-in-time gradient $\mathcal{G}_\delta$.

Prop 6.2.3. *Let $\gamma_\delta(s)$, $\sigma(x,t)$ and x_t be the same as in Definition* 6.2.1.

$$\frac{d}{dt}\int_0^\delta \gamma_\delta(s)\int_0^{(t-s)\vee 0} \sigma(x_\tau, \tau)dW_\tau ds = \mathcal{G}_\delta \int_0^{t\vee 0} \sigma(x_s, s)dW_s, \tag{6.10}$$

almost surely. Moreover, the nonlocal multiplicative noise $\mathcal{G}_\delta \int_0^{t\vee 0} \sigma(x_s, s)dW_s$ is in $\mathcal{S}_2(0,T)$.

Proof. With the Lipschitz assumption on $\sigma(x,t)$ and the assumption that $x_t \in \mathcal{S}_2(-\delta, T)$, we have $\sigma(x_t, t) \in \mathcal{S}_2(-\delta, T)$.

$$\begin{aligned}
\mathcal{G}_\delta \int_0^{t\vee 0} \sigma(x_s, s)dW_s &= \int_0^\delta s\rho_\delta(s)\int_{(t-s)\vee 0}^t \sigma(x_\tau, \tau)dW_\tau ds \\
&= \int_{(t-\delta)\vee 0}^t \sigma(x_\tau, \tau)\int_{t-\tau}^\delta s\rho_\delta(s)dsdW_\tau \\
&= \int_{(t-\delta)\vee 0}^t \sigma(x_\tau, \tau)\gamma_\delta(t-\tau)dW_\tau
\end{aligned}$$

$$\begin{aligned}
\left\|\mathcal{G}_\delta \int_0^{t\vee 0} \sigma(x_s, s)dW_s\right\|^2_{\mathcal{S}_2(0,T)} &= \mathbb{E}\int_0^T \left[\int_{(t-\delta)\vee 0}^t \sigma(x_s, s)\gamma_\delta(t-s)dW_s\right]^2 dt \\
&= \mathbb{E}\int_0^T \int_{(t-\delta)\vee 0}^t (\sigma(x_s, s)\gamma_\delta(t-s))^2 dsdt \\
&= \mathbb{E}\int_0^T \int_0^{t\wedge\delta} (\sigma(x_{t-s}, t-s)\gamma_\delta(s))^2 dsdt \\
&\leq \mathbb{E}\int_0^T \int_0^\delta (\sigma(x_{t-s}, t-s)\gamma_\delta(s))^2 dsdt \\
&\leq \mathbb{E}\int_0^\delta \int_{-\delta}^T \sigma^2(x_t, t)dt\gamma_\delta^2(s)ds \\
&= \|\sigma(x_t, t)\|^2_{\mathcal{S}_2(-\delta,T)}\|\gamma_\delta\|^2_{L^2(0,\delta)} \\
&< +\infty.
\end{aligned}$$

□

With the definition of the nonlocal stochastic integrals above, we now introduce the stochastic nonlocal-in-time problems as follows.

$$\begin{cases}\mathcal{G}_\delta u(t) = b(u(t), t) + \mathcal{G}_\delta \int_0^{t\vee 0} \sigma(u(s), s)dW_s, & t \in (0, T], \\ u(t) = g(t), & t \in [-\delta, 0].\end{cases} \tag{6.11}$$

In particular, by taking $b(u(t), t) = -\theta u(t)$ and $\sigma(u(t), t) \equiv \sigma$ for parameters $\theta > 0$ and $\sigma > 0$, we have a nonlocal version of the Langevin equation,

$$\begin{cases} \mathcal{G}_\delta u(t) = -\theta u(t) + \sigma \mathcal{G}_\delta W_{t\vee 0}, & t \in (0, T], \\ \quad u(t) = g(t), & t \in [-\delta, 0]. \end{cases} \tag{6.12}$$

Since $\mathcal{G}_\delta W_{t\vee 0}$ is in $\mathcal{S}_2(0, T)$, by Theorem 6.1.1 we have the above nonlocal Langevin equation with deterministic initial data $g \in L^\infty(-\delta, 0)$ is well-posed in $\mathcal{S}_2(0, T)$.

6.3 Localization of Nonlocal Langevin Equations

Given W_t the standard Brownian motion, the *Ornstein-Uhlenbeck* process x_t is defined by the stochastic differential equation

$$\begin{cases} dX_t = -\theta X_t dt + \sigma dW_t, & t > 0, \\ \quad X_t = x_0, & t = 0, \end{cases} \tag{6.13}$$

where $\theta > 0$ and $\sigma > 0$. The *Ornstein-Uhlenbeck* process is formally also written as a *Langevin* equation of the form

$$\frac{dX_t}{dt} = -\theta X_t + \sigma \eta(t), \tag{6.14}$$

where $\eta(t)$, also known as white noise, stands in for the supposed derivative dW_t/dt of the Wiener process. The Wiener process W_t is nowhere differentiable but from the previous section we can see that the nonlocal white noise $\mathcal{G}_\delta W_{t\vee 0}$ is in $\mathcal{S}_2(0, T)$.

In this section, we will study the local limit of the nonlocal Langevin equation (6.12) as $\delta \to 0$, i.e., to show that $\lim_{\delta \to 0} \|u(t) - X_t\|_{\mathcal{S}_2(0,T)} \to 0$, where u is the solution of the nonlocal Langevin equation (6.12) and X_t is the *Ornstein-Uhlenbeck* process solves (6.13). The *Ornstein-Uhlenbeck* process has the analytic expression as

$$X_t = x_0 e^{-\theta t} + \sigma \int_0^t e^{-\theta(t-s)} dW_s.$$

Let $w(t) = u(t) - X_{t\vee 0}$, then

$$\begin{aligned}\mathcal{G}_\delta w(t) &= \mathcal{G}_\delta u(t) - \mathcal{G}_\delta X_{t\vee 0} \\ &= -\theta u(t) + \sigma \mathcal{G}_\delta W_{t\vee 0} - \mathcal{G}_\delta X_{t\vee 0} \\ &= -\theta w(t) - \theta X_t + \sigma \mathcal{G}_\delta W_{t\vee 0} - \mathcal{G}_\delta X_{t\vee 0}\end{aligned}$$

Let $f(t) = -\theta X_t + \sigma \mathcal{G}_\delta W_{t\wedge 0} - \mathcal{G}_\delta X_{t\vee 0}$, then $w(t)$ satisfies the nonlocal-in-time problem

$$\begin{cases}\mathcal{G}_\delta w(t) = -\theta w(t) + f(t), & t \in (0, T], \\ \quad w(t) = g(t) - x_0, & t \in [-\delta, 0].\end{cases}$$

Based on the explicit expression of the *Ornstein-Uhlenbeck* process X_t, we can compute $f(t)$.

$$\begin{aligned}f(t) &= -\theta X_t + \sigma \mathcal{G}_\delta W_{t\vee 0} - \mathcal{G}_\delta X_{t\vee 0} \\ &= -\theta \left[x_0 e^{-\theta t} + \sigma \int_0^t e^{-\theta(t-s)} dW_s\right] + \sigma \mathcal{G}_\delta W_{t\vee 0} - \mathcal{G}_\delta \left[x_0 e^{-\theta(t\vee 0)} + \sigma \int_0^{t\vee 0} e^{-\theta(t-s)} dW_s\right] \\ &= -\theta \left(x_0 e^{-\theta t}\right) - \mathcal{G}_\delta \left(x_0 e^{-\theta(t\vee 0)}\right) - \theta\sigma \int_0^t e^{-\theta(t-s)} dW_s \\ &\qquad + \sigma \mathcal{G}_\delta W_{t\vee 0} - \sigma \mathcal{G}_\delta \left[e^{-\theta(t\vee 0)} \int_0^{t\vee 0} e^{\theta s} dW_s\right] \\ &= I + \sigma II\end{aligned}$$

Here, we define

$$\begin{aligned}I &= -\theta \left(x_0 e^{-\theta t}\right) - \mathcal{G}_\delta \left(x_0 e^{-\theta(t\vee 0)}\right), \\ II &= -\theta \int_0^t e^{-\theta(t-s)} dW_s + \mathcal{G}_\delta W_{t\vee 0} - \mathcal{G}_\delta \left[e^{-\theta(t\vee 0)} \int_0^{t\vee 0} e^{\theta s} dW_s\right]\end{aligned}$$

By the product rules (4.1.4) we introduced in Chapter 4, we can expand and simplify these two terms as follows.

$$\begin{aligned}
I &= -x_0\left[\theta e^{-\theta t} + \mathcal{G}_\delta(e^{-\theta(t\vee 0)})\right], \\
II &= -\theta\int_0^t e^{-\theta(t-s)}dW_s + \mathcal{G}_\delta W_{t\vee 0} - \mathcal{G}_\delta\left[e^{-\theta(t\vee 0)}\int_0^{t\vee 0} e^{\theta s}dW_s\right] \\
&= -\theta e^{-\theta t}\int_0^t e^{\theta s}dW_s - \mathcal{G}_\delta(e^{-\theta(t\vee 0)})\int_0^t e^{\theta s}dW_s \\
&\qquad + \int_0^\delta s\rho_\delta(s)\int_{(t-s)\vee 0}^t dW_\tau ds - \int_0^\delta s\rho_\delta(s)e^{-\theta((t-s)\vee 0)}\int_{(t-s)\vee 0}^t e^{\theta\tau}dW_\tau ds \\
&= -\left[\theta e^{-\theta t} + \mathcal{G}_\delta(e^{-\theta(t\vee 0)})\right]\int_0^t e^{\theta s}dW_s + \int_{(t-\delta)\vee 0}^t\int_{t-\tau}^\delta (1-e^{\theta(\tau-(t-s)\vee 0)})s\rho_\delta(s)dsdW_\tau
\end{aligned},$$

Thus,

$$\begin{aligned}
\|f\|_{\mathcal{S}_2(0,T)} &\le \|I\|_{\mathcal{S}_2(0,T)} + \sigma\|II\|_{\mathcal{S}_2(0,T)} \\
&\le |x_0|\left\|\theta e^{-\theta t} + \mathcal{G}_\delta(e^{-\theta(t\vee 0)})\right\|_{L^\infty(0,T)} \\
&\quad + \sigma\left\|\theta e^{-\theta t} + \mathcal{G}_\delta(e^{-\theta(t\vee 0)})\right\|_{L^\infty(0,T)}\left\|\int_0^t e^{\theta s}dW_s\right\|_{\mathcal{S}_2(0,T)} \\
&\quad + \sigma\left\|\int_{(t-\delta)\vee 0}^t\int_{t-\tau}^\delta (1-e^{\theta(\tau-(t-s)\vee 0)})s\rho_\delta(s)dsdW_\tau\right\|_{\mathcal{S}_2(0,T)} \\
&= \left(|x_0| + \sigma\left\|\int_0^t e^{\theta s}dW_s\right\|_{\mathcal{S}_2(0,T)}\right)\left\|\theta e^{-\theta t} + \mathcal{G}_\delta(e^{-\theta(t\vee 0)})\right\|_{L^\infty(0,T)} \\
&\quad + \sigma\left\|\int_{(t-\delta)\vee 0}^t\int_{t-\tau}^\delta (1-e^{\theta(\tau-(t-s)\vee 0)})s\rho_\delta(s)dsdW_\tau\right\|_{\mathcal{S}_2(0,T)}
\end{aligned}$$

Here,

$$|x_0| + \left\|\int_0^t e^{\theta s}dW_s\right\|_{\mathcal{S}_2(0,T)} = |x_0| + \left[\int_0^T\int_0^t e^{2\theta s}dsdt\right]^{1/2} < +\infty,$$

$$
\begin{aligned}
&\left\| \int_{(t-\delta)\vee 0}^{t} \int_{t-\tau}^{\delta} (1 - e^{\theta(\tau-(t-s)\vee 0)}) s\rho_\delta(s) ds dW_\tau \right\|_{\mathcal{S}_2(0,T)} \\
=&\left[\int_0^T \mathbb{E} \left(\int_{(t-\delta)\vee 0}^{t} \int_{t-\tau}^{\delta} (1 - e^{\theta(\tau-(t-s)\vee 0)}) s\rho_\delta(s) ds dW_\tau \right)^2 dt \right]^{1/2} \\
=&\left[\int_0^T \int_{(t-\delta)\vee 0}^{t} \left(\int_{t-\tau}^{\delta} (1 - e^{\theta(\tau-(t-s)\vee 0)}) s\rho_\delta(s) ds \right)^2 d\tau dt \right]^{1/2} \\
=&\left[\int_0^T \int_0^{t\wedge\delta} \left(\int_{\tau}^{\delta} (1 - e^{\theta((s-\tau)\wedge(t-\tau))}) s\rho_\delta(s) ds \right)^2 d\tau dt \right]^{1/2} \\
\le&\left[\int_0^T \int_0^{t\wedge\delta} \left(\int_{\tau}^{\delta} (1 - e^{\theta(s-\tau)}) s\rho_\delta(s) ds \right)^2 d\tau dt \right]^{1/2} \\
\le&\left[\int_0^T \int_0^{t\wedge\delta} \left(\int_{\tau}^{\delta} (e^{\theta s} - 1) s\rho_\delta(s) ds \right)^2 d\tau dt \right]^{1/2} \\
\le&\left[\int_0^T \int_0^{t\wedge\delta} \left(\int_{\tau}^{\delta} (e^{\theta\delta} - 1) s\rho_\delta(s) ds \int_0^{\delta} (e^{\theta s} - 1) s\rho_\delta(s) ds \right) d\tau dt \right]^{1/2} \\
=&(e^{\theta\delta} - 1)^{1/2} \left(\int_0^{\delta} (e^{\theta s} - 1) s\rho_\delta(s) ds \right)^{1/2} \left[\int_0^T \int_0^{t\wedge\delta} \int_{\tau}^{\delta} s\rho_\delta(s) ds d\tau dt \right]^{1/2} \\
\le&(e^{\theta\delta} - 1)^{1/2} \left(\int_0^{\delta} (e^{\theta s} - 1) s\rho_\delta(s) ds \right)^{1/2} T^{1/2} \\
\le&(\mathcal{O}(\delta))^{1/2} (\theta + \mathcal{O}(\delta))^{1/2} T^{1/2} \\
\to& 0, \quad \text{as } \delta \to 0.
\end{aligned}
$$

Following the similar proof as in Lemma 4.1.4, we can show that

$$
\left\| \theta e^{-\theta t} + \mathcal{G}_\delta(e^{-\theta(t\vee 0)}) \right\|_{L^\infty(0,T)} \to 0, \quad \text{as } \delta \to 0.
$$

Therefore, as $\delta \to 0$, $\|f\|_{\mathcal{S}_2(0,T)} \to 0$. Finally, by Theorem 6.1.1, we then have the following theorem of the localization theory of the nonlocal Langevin equations.

Theorem 6.3.1. *Given parameters $\theta > 0$ and $\sigma > 0$. Let u_δ be the solution of the nonlocal Langevin equation with deterministic initial data $g(t) \in L^\infty(-\delta, 0)$,*

$$
\begin{cases} \mathcal{G}_\delta u_\delta(t) = -\theta u_\delta(t) + \sigma \mathcal{G}_\delta W_{t\vee 0}, & t \in (0, T], \\ u_\delta(t) = g(t), & t \in [-\delta, 0]. \end{cases} \tag{6.15}
$$

and X_t *be the* Ornstein-Uhlenbeck *process satisfying the stochastic differential equation with the initial value* x_0*,*

$$\begin{cases} dX_t = -\theta X_t dt + \sigma dW_t, & t > 0, \\ X_t = x_0, & t = 0. \end{cases} \tag{6.16}$$

If we assume that $\gamma_\delta \in L^2(0, \delta)$ *and*

$$\lim_{\delta \to 0} \|g(t) - x_0\|_{L^\infty(-\delta,0)} = 0, \tag{6.17}$$

Then it holds that

$$\lim_{\delta \to 0} \|u_\delta - X_t\|_{\mathcal{S}_2} = 0.$$

6.4 Conclusions

In this chapter, by considering the external force f a random force, we considered the stochastic nonlocal-in-time problem (6.3). First, we followed the similar approach as in [2] and in Chapter 4 for linear nonlocal-in-time problems and established the well-posedness theorem for the nonlocal-in-time problems with random force. Then based on the definition of the nonlocal-in-time operator $\mathcal{G}_\delta$, we have introduced a nonlocal version of the Brownian motions and the stochastic integrals and the nonlocal fundamental theorem gives us a connection between the local and nonlocal gradients. Moreover, with the introduction of this nonlocal version of stochastic integrals, the nowhere-differentiable Wiener process W_t has a nonlocal white noise $\mathcal{G}_\delta W_{t\vee 0}$ in $\mathcal{S}_2(0, T)$.

Finally, with the nonlocal version of the standard Brownian motions introduced, we established the path-wise existence and uniqueness as well as the localization of the solutions of the third sub-problem, the stochastic nonlocal Langevin equation in the space of $\mathcal{S}_2(0, T)$ where many classical results, like the *Itô*'s isometry, from stochastic calculus could be applied.

Chapter 7
Conclusions and Future Work

In this thesis, motivated by a class of stochastic bi-stable mean-field model, we imposed the inter-group mechanism, analyzed the well-posedness of the generalized models, and derived the asymptotic expression of equilibrium from a coupled nonlinear system of compatibility conditions. In addition, we computed the probability of systemic transitions of both the homogeneous case and the heterogeneous cases, from which we concluded that due to the small heterogeneities on model parameters in different groups, the system will become more likely to fail. From the observation that formally the stochastic nonlocal-in-time reaction-diffusion problem gives a continuous case of the stochastic bi-stable mean-field model, the primary purpose of this thesis is trying to formulate the theory of this stochastic nonlocal-in-time reaction-diffusion equation while the stochastic bi-stable mean-field model, which was initiated in 1981, has been thoroughly studied.

To study the stochastic nonlocal-in-time reaction-diffusion equation, we decomposed it into three sub-problems. Based on the results for linear nonlocal-in-time problem developed in [2] recently, we established the well-posedness theorems, the localization results and the semigroup properties of the first sub-problem, the linear nonlocal-in-time problems in Banach spaces with integrable kernels. A nonlocal version of one direction of the *Hille-Yosida* Theorem was given at the end of this chapter, which gives a sufficient condition for the strongly continuous semigroup of contractions. In the proof of this theorem, the mild solution of the nonlocal-in-time problem has been used to define the map $T^\delta(t)$ from X_δ into itself. Conversely, given a C^0-semigroup, to construct a "nonlocal generator" A is more challenging and more interesting to study.

The second sub-problem is the nonlocal-in-time problem with nonlinearities.

With the Lipschitz continuous nonlinearity added, we have proved the global existence and uniqueness of the solution of the nonlinear nonlocal-in-time problem in $L^\infty(0,T)$. In addition, considering the nonlinearity of $u^p|u|^\alpha$ type, we have established local existence and uniqueness results for solutions in both $L^\infty(0,t_0)$ and $W^{1,\infty}(0,t_0)$ spaces with integrable nonlocal kernels. In particular, for small initial data, we also had the global existence and uniqueness of solutions in $L^\infty(0,T)$. Moreover, for the non-integrable kernel case, we have also formulated the local existence of the solutions of nonlinear nonlocal-in-time problem in $C^0(0,t_0)$. Finally, we have obtained the well-posedness result of the deterministic nonlocal-in-time reaction-diffusion problems in Banach spaces with integrable nonlocal kernels in $L^\infty([0,t_0];B)$.

We have studied the nonlinear nonlocal-in-time problems for a class of autonomous nonlinearities in the form of $f(u) = u^p|u|^\alpha$ in this thesis because of the nonlinear terms in the stochastic bi-stable mean-field models and the stochastic Allen-Cahn/Cahn-Hilliard equations are usually considered in this type. The nonlocal problems with general nonlinearities could be considered, from which a more general theory of the nonlocal-in-time problems for the nonlinear ODE models could be established.

Unlike the linear nonlocal-in-time problems for which we might be able to extend the L^∞ results into L^2 results, the extension to L^2 is not trivial for the nonlinear nonlocal-in-time problems. Here, when estimating the difference $\mathcal{K}f(u) - \mathcal{K}f(v)$ for the $u^p|u|^\alpha$ type nonlinearities we considered in Chapter 5, the L^∞ norm is obviously the most convenient one to use. Replacing by L^2 norm, the same map $\mathcal{K}$ that was introduced in the proof of well-posedness theorem will then not be a contraction mapping unless further assumptions are made, for instance, a uniform upper bound in L^∞. A potentially alternative approach for seeking the L^p solutions of the nonlinear nonlocal-in-time problems could be to consider the nonlocal function space $\mathcal{S}(\Omega)$ which was developed together with a L^p compactness criteria in [49].

Furthermore, based on the definition of the nonlocal-in-time operator, we have introduced a nonlocal version of the Brownian motions and the stochastic integrals, from which we have established the path-wise existence and uniqueness as well as the localization of our final sub-problem, the nonlocal Langevin equation in the space of $\mathcal{S}_2(0,T)$ where many results, like the itô's isometry, from the classical stochastic calculus could be applied. However, due to the limitation of the solution spaces in our study of the nonlinear nonlocal-in-time problems, the related results for stochastic nonlocal reaction-diffusion problems still remain unanswered. Here, as we have mentioned above, the nonlocal function space $\mathcal{S}(\Omega)$ might be the key if the theory of nonlinear nonlocal-in-time problems could be extended for solutions in L^p norms. In addition, with the introduction of the nonlocal stochastic integrals, more general stochastic nonlocal-in-time problems could be considered,

$$\begin{cases} \mathcal{G}_\delta u(t) = b(u(t),t) + \mathcal{G}_\delta \left[\int_0^{t\vee 0} \sigma(u(s),s)dW_s \right], & t \in (0,T], \\ \quad u(t) = g(t), & t \in [-\delta, 0], \end{cases}$$

where the coefficients b and σ are Lipschitz continuous.

We have seen benefits from the nonlocal equations. For instance, with the introduction of the nonlocal version of stochastic integrals, the nowhere-differentiable Wiener process W_t has a nonlocal white noise $\mathcal{G}_\delta W_{t\vee 0}$ in $\mathcal{S}_2(0,T)$. However, it comes with its own challenges at the same time. By replacing the classical differentials by the nonlocal-in-time gradients in the SDEs, to our knowledge, the classical *Itô*'s Lemma in stochastic calculus could not be applied directly, which is the same as the chain rule for the deterministic nonlocal-in-time problems. From the product rule of nonlocal-in-time gradient in Lemma 4.1.4 we have already observed the difference between the local and nonlocal rules. The formulation of nonlocal version of *Itô*'s Lemma could extend more classical results in stochastic calculus to the stochastic nonlocal-in-time problem.

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