MODELING ACOUSTICAL HORNS AS A CASCADE OF CONICAL TRANSMISSION LINE SEGMENTS TERMINATED WITH A SPHERICAL RADIATOR

A Thesis in
Acoustics
by
Max R. Pagnucco

© 2022 Max R. Pagnucco

Submitted in Partial Fulfillment of the Requirements for the Degree of

Master of Science

May 2022
The thesis of Max R. Pagnucco was reviewed and approved by the following:

Stephen C. Thompson  
Research Professor of Acoustics  
Thesis Advisor

Daniel A. Russell  
Teaching Professor of Acoustics

David C. Swanson  
Research Professor of Acoustics

Andrew R. Barnard  
Director of Graduate Program in Acoustics
Abstract

This thesis presents a method of modeling propagation in acoustical horns and tubes of varying cross-sectional area using a published method that seems to have been overlooked in previous literature. Acoustical tubes and horns are commonly modeled as a cascade of uniform cylinders where the accuracy is directly proportional to the number of elements used. This technique is highly effective when modeling cylindrical waveguides, but its accuracy and efficiency begin to break down when the waveguide radius changes along its length. The common solution is to add additional elements, but the resulting computational requirement then increases. Additionally, this type of model does not account for any curvature in the propagating wave shape which will occur in cones or flaring horns.

The alternative solution developed for both the wave propagation and radiation for conical and flaring horns uses a cascade of conical elements, rather than cylindrical. Additionally, a radiation impedance model for a spherical cap replaces the flat piston model that is often used. The model is designed as an acoustical-domain circuit in MATLAB’s Simscape, and is also verified numerically. Custom components are built to increase the capabilities of a recently developed “Acoustical” domain in Simscape, and they can be used in time and frequency domain models of loudspeakers, musical instruments, and any other multi-domain system.

Observing the input impedance and resonance frequencies from increasingly complex acoustical horn shapes, comparisons are made between the historical and proposed model in terms of efficiency and accuracy. Ultimately, the conical transmission line provides accurate results but shows no comprehensive advantages. Accuracy between the two, as a circuit and numerically, is similar, and both require a comparable computational cost to reach a point of convergence.
Table of Contents

List of Figures vi
List of Tables x
List of Symbols xii
Acknowledgments xv

Chapter 1
Introduction 1
  1.1 Background ................................................. 2
  1.2 Lumped Elements and Equivalent Circuits ................. 2
  1.3 Historical Modeling Techniques .......................... 3
    1.3.1 Uniform Cylinder Transmission Line .................. 3
    1.3.2 Radiation Impedance of a Piston .................... 7
  1.4 Objectives ................................................. 9
  1.5 Literature Review ........................................ 10

Chapter 2
Numerical Validation 13
  2.1 Cone Model ............................................... 13
    2.1.1 MATLAB Implementation ............................... 17
  2.2 Radiation Impedance Model ............................... 20
    2.2.1 MATLAB Implementation ............................... 21
    2.2.2 Comparison to Spherical and Planar Radiation ...... 22
    2.2.3 Optimization to Acoustical Domain Circuit .......... 24

Chapter 3
Methodology 26
  3.1 Simscape Multi-Domain Modeling .......................... 26
    3.1.1 Custom Components ................................. 27
  3.2 Integration with MATLAB ................................. 29
  3.3 Input Impedance Calculation ............................. 30
Chapter 4
Simscape Modeling
4.1 Acoustical Cone Sub-Circuit .......................... 31
4.2 Conical Element Transmission Line ..................... 32
4.3 Radiation Impedance Sub-Circuit ......................... 33
4.4 Far-Field Pressure Estimation ........................ 34
4.5 Finalized Simscape Model .............................. 36

Chapter 5
Results
5.1 Acoustical Waveguide Test-Cases ....................... 39
  5.1.1 Conical ........................................ 39
  5.1.2 Exponential .................................. 41
5.2 Musical Instrument Bores .............................. 43
  5.2.1 Oboe ........................................ 44
  5.2.2 Trumpet ..................................... 46
  5.2.3 Trombone ................................... 49
5.3 Analytical and Simscape Comparison ................... 51
5.4 Including Thermoviscous Damping ...................... 52

Chapter 6
Conclusion
6.1 Summary ........................................... 54
6.2 Future Work ....................................... 55

Appendix A
Numerical Simulation (MATLAB) .............................. 57
A.1 Transfer Matrix Script ................................ 57
A.2 Radiation Impedances and Cost Function ............... 61

Appendix B
Acoustical Components (Simscape) ......................... 64
B.1 Acoustical Converter ................................ 64
B.2 Single Conical Element ............................... 64
B.3 Conical Transmission Line .............................. 66

Appendix C
Final Model (MATLAB/Simscape) ............................ 68
C.1 Simscape Companion Script ........................... 68
C.2 Simscape Referencing Function ......................... 69

Bibliography ............................................. 72
# List of Figures

1.1 Electrical equivalent circuit component used in future circuits to represent mechanical and acoustical domain quantities. .............................................. 3

1.2 Uniform cylindrical segment of length $\ell$, radius $a$, and cross-sectional area $A$. ................................................................. 3

1.3 Acoustical equivalent circuit representing a cylindrical element of air, as seen in Figure 1.2. Model includes two inductors, each for half of the air mass, and a capacitor for the compliance of the air cavity. The shape is that of a 'T-model' circuit, commonly cascaded in transmission lines. Also referred to as a Basic Acoustic Element (BAE) by Gabrielson [1]. . . 4

1.4 A exponential curve $y = e^{3x}$ (red), representing the bore geometry of a flaring horn, estimated using $N$ equally spaced cylindrical elements. As $N$ increases from 5 to 50, the accuracy greatly increases as indicated by the black curve largely covering the red. .............................................. 6

1.5 Beranek [2] circuit structure for the acoustic analogy of the radiation impedance of a piston in an infinite baffle or for a piston in the end of a long tube. ................................................................. 7

1.6 Comparison of an analytical (Eq. 1.7) and equivalent circuit model (Figure 1.5) for the radiation impedance of a baffled piston in a tube, from Beranek [2]. ................................................................. 8

2.1 A exponential curve $y = e^{3x}$ (red), representing the bore geometry of a flaring horn, estimated using equally spaced cylindrical elements (left) and equally spaced conical elements (right). Both black curves use 10 elements along the length and are superimposed on top of a red curve. . 14
2.2 Conical waveguide element of length $\ell$ measured on the outer edge, radii $a_0$ and $a_e$ at the throat and mouth, respectively, and the distance to an imaginary apex on the throat side, $x_0$. A coordinate system along the center line is provided as well. Figure recreated from Benade [3].

2.3 Acoustical equivalent circuit structure recreated from Benade [3]. The diagram appends three components to a uniform line (T-model): inertances $M_0$ and $M_e$, as well as an acoustical-to-acoustical domain transformer with a ratio $r = a_e/a_0$. The uniform line can include one or more cylindrical elements, like the one seen in Figure 2.3.

2.4 Geometry of a circular horn with a linear change in radius, or cone.

2.5 Input impedance of the cone in Figure 5.1 using three models for validation: a transmission line of cylindrical elements, a transmission line of conical elements, and an analytical solution from Stewart and Lindsay [4] (S&L). All three models use the radiation impedance of an unflanged cylindrical pipe from Levine and Schwinger [5].

2.6 Radiation impedance of a horn where the bell is approximated as a sphere, $S$, where only the portion $S_0$ radiates. So, the radiating face is a spherical cap set in a spherical baffle. The sphere has a radius $r_0$ and the half angle of the cap is denoted by $\theta_0$.

2.7 Visualization of $S$, $S_0$, $r_0$, and $\theta_0$ for manual inspection to confirm model is set up according to Figure 2.6 for the given horn geometry. In this case, the horn, shown in red, is a Salmon horn with a flare rate of $m = 4.5$.

2.8 Coefficients of the $\theta_0$-dependent polynomial expansions used in Eq. 2.11. Graphic borrowed from Eveno et. al. [6].

2.9 Comparison of complex radiation impedance of a spherical radiator and the spherical cap model from Hélie and Rodet using equal radii for the spheres and $\theta_0 = \pi/2$.

2.10 Comparison of complex radiation impedance of a piston radiator and the spherical cap model from Hélie and Rodet using equal area for both models and $\theta_0 = \pi/6$.

2.11 Real and imaginary components of radiation impedance using the results of Eq. 2.9 as well as the Beranek structure with optimized values. After the optimization, the real and imaginary curves match with high accuracy.
3.1 Sub-circuit built in Simscape code for the T-model structure. The ports of a two-port network are created, three foundational components are called, and connections are made attaching the circuit together and with the external ports. ......................................................... 28

4.1 Re-creation of Figure 2.3 in Simscape’s acoustical domain. Also this model includes the two inertances and compliance for the cylindrical uniform line as well as a custom transformer component. .......................... 32

4.2 Conical transmission line Simscape component block parameters. Inputs include the uniformly spaced radii along the center axis as well as the total length of the waveguide. ......................................................... 33

4.3 Spherical cap radiation impedance Simscape component block parameters. Inputs include values for two resistances, one capacitance, one inductance, and the distance for a far-field pressure estimation. ...................... 34

4.4 Frequency dependent pressure radiation in dB from a spherical cap on a sphere as calculated by Eq. 4.3 for a sinusoidal input. ......................... 35

4.5 Completed horn model in Simscape. Model includes an acoustical source and pressure measurement device to drive the conical transmission line and spherical cap radiation impedance. ............................................ 36

5.1 Geometry of a circular horn with a linear change in radius, or cone. ... 40

5.2 Input impedance of the waveguide in Figure 5.1 modeled in Simscape with a conical transmission line and spherical cap radiation impedance. Resonance peaks found in Table 5.1. ................................. 40

5.3 Geometry of an exponentially flaring horn as calculated by Eq. 5.1 with \( m = 4.5 \). ................................................................. 42

5.4 Input impedance of the waveguide in Figure 5.3 modeled in Simscape with a conical transmission line and spherical cap radiation impedance. Resonance peaks found in Table 5.2. ................................. 42

5.5 Geometry of an oboe as defined by Plitnik and Strong [7]. .............. 44

5.6 Input impedance of the oboe shown in Figure 5.5 modeled in Simscape with a conical transmission line and spherical cap radiation impedance. Peak (resonance) frequencies found in Table 5.3. ......................... 45

viii
5.7 Geometry of a straight trumpet used for experimental analysis by Eveno et. al. [6]. Radii taken using the smallest sample spacing available in DataThief [8]. ................................................................. 47

5.8 Input impedance of the straight trumpet shown in Figure 5.7 modeled in Simscape with a conical transmission line and spherical cap radiation impedance. Peak (resonance) frequencies found in Table 5.5. ................. 47

5.9 Geometry of a tenor trombone used for experimental analysis by Eveno et. al [6]. Radii taken using the smallest sample spacing available in DataThief [8]. ................................................................. 49

5.10 Input impedance of the oboe shown in Figure 5.9 modeled in Simscape with a conical transmission line and spherical cap radiation impedance. Peak (resonance) frequencies found in Table 5.7. ......................... 50
List of Tables

2.1  Resonance frequencies of a conical horn modeled both as a conical and cylindrical transmission line with N=100, as well as Stewart and Lindsay’s analytical solution [4]. .................................................. 20

5.1  Resonance frequencies of a cone using with a conical/spherical model (top) vs a cylindrical/piston model (bottom) for increasing N. ................. 41

5.2  Resonance frequencies of an exponentially flaring horn using with a conical/spherical model (top) vs a cylindrical/piston model (bottom) for increasing N. .................................................. 43

5.3  Resonance frequencies of an oboe bore modeled with a conical transmission line and spherical cap radiation impedance for increasing values of N. Experimental results from Plitnik [7]. ................. 45

5.4  Percent error between calculated frequencies and experimental results found in Table 5.3 for the oboe. Average percent error across all resonances for each value of N is provided. ............................. 46

5.5  Resonance frequencies of a straight trumpet bell modeled with a conical transmission line and spherical cap radiation impedance for increasing values of N. Experimental results from an identical, straight trumpet bell from Eveno et. al. [6] are provided for comparison. ................. 48

5.6  Percent error between calculated frequencies and experimental results found in Table 5.5 for the straight trumpet bell. Average percent error across all resonances for each value of N is provided. ......................... 48

5.7  Resonance frequencies of a tenor trombone bell modeled with a conical transmission line and spherical cap radiation impedance for increasing values of N. Experimental results from an identical, tenor trombone bell from Eveno et. al. [6] are provided for comparison. ................................. 50
5.8 Percent error between calculated frequencies and experimental results found in Table 5.7 for the tenor trombone bell. Average percent error across all resonances for each value of N is provided. 51

5.9 Resonance frequencies of a straight trumpet bell modeled numerically with ABDC matrices from Equation 1.6. Model still uses a conical transmission line and spherical cap radiation impedance, but without the Simscape solver. 52

5.10 Percent error between calculated frequencies and experimental results found in Table 5.9 for the straight trumpet bell using the MATLAB solver, compared to Simscape. Average percent error across all resonances for each value of N is provided. 52

5.11 Resonance frequencies of a straight trumpet bell modeled with a conical transmission line that includes thermoviscous losses. Radiation impedance is that of a spherical cap. Experimental results from an identical, straight trumpet bell from Eveno et. al. [6] are provided for comparison. 53
List of Symbols

$l$ Length of a cylindrical segment of pipe, p. 3
$a$ Radius of a cylindrical segment of pipe, p. 3
$A$ Cross sectional area of a cylindrical element, p. 3
$P$ Acoustic pressure, p. 4
$U$ Acoustic particle velocity, p. 4
$L$ Total length of a waveguide, p. 4

$M_{acs}$ Acoustic mass, p. 4
$\rho$ Density of the fluid medium, p. 4

$C_{acs}$ Acoustic compliance, p. 4

$j$ Imaginary unit, p. 5
$\omega$ Angular frequency, p. 5

$Z_{acs}$ Acoustical impedance, p. 5
$Z_s$ Impedance of a series circuit element, p. 5
$Z_p$ Impedance of a parallel circuit element, p. 5
$N$ Number of elements cascaded in a transmission line, p. 6

$J_1$ First order Bessel function, p. 7
$K_1$ Modified Bessel function of the second kind, p. 7
$k$ Acoustic wavenumber, p. 9

$R_{a1}$ First resistance element for modeled acoustic radiation, p. 9
$R_{a2}$ Second resistance element for modeled acoustic radiation, p. 9
\( M_{a1} \) Mass element for modeled acoustic radiation, p. 9
\( C_{a1} \) Compliance element for modeled acoustic radiation, p. 9
\( c \) Sound speed of the fluid medium, p. 9
\( r \) Acoustical to acoustical transformer element ratio, p. 14
\( a_e \) Radius of a cone at the mouth, p. 14
\( a_0 \) Radius of a cone at the throat, p. 14
\( x_0 \) Distance from a conical segment’s throat to where the apex would be, p. 14
\( x_e \) Distance from the apex to mouth of a cone, p. 14
\( M_e \) Conicity inertance at the throat of a cone, p. 15
\( M_0 \) Negative conicity inertance at the mouth of a cone, p. 15
\( T_s \) Transfer matrix of a series circuit element, p. 17
\( T_p \) Transfer matrix of a parallel circuit element, p. 17
\( T \) Transfer matrix for an entire circuit system, p. 17
\( k a \) Acoustic size, p. 18
\( Z_{rad} \) Radiation impedance, p. 18
\( Z_{in} \) Input impedance, p. 18
\( S \) Surface of a complete sphere, p. 21
\( S_0 \) Cap-shaped surface taken from a sphere, p. 21
\( r_0 \) Radius of a sphere, p. 21
\( \theta_0 \) Angle of a horn opening as measured from the central axis, p. 21
\( P_\alpha \) First low order polynomial for radiation impedance calculation, p. 21
\( P_\xi \) Second low order polynomial for radiation impedance calculation, p. 21
\( P_\nu \) Third low order polynomial for radiation impedance calculation, p. 21
\( \nu \) Non-dimensional, frequency dependent variable, p. 21
\( f \) Frequency, p. 21
\( R \) Distance from a radiating face along the central axis, p. 34
$P_n^m$ Legendre polynomial, p. 34

$u_0$ Acoustical particle velocity, p. 34

$h_n^m$ Spherical Hankel function, p. 34

$n$ Mode number for Legendre polynomials, p. 34

$Q_s$ Acoustical source strength, p. 35

$m$ Flare coefficient for Salmon horns, p. 41

$S_T$ Salmon horn throat cross-sectional area, p. 41
Acknowledgments

Firstly I’d like to thank my advisor, Dr. Stephen Thompson, for giving me the opportunity to work on this project and for his guidance throughout. His experience, breadth of knowledge, and wealth of anecdotes allowed me to learn far beyond the scope of my work, and more about the people behind the papers, something I value tremendously.

Thank you also to the Penn State Acoustics program for extending a hand during some unprecedented times and for their continued efforts to see all their students succeed. The professors I was able to connect with and learn from were instrumental in pulling back the curtain to all of what acoustics can be - far more than I believed possible before my arrival. Thank you in particular to Dr. Dan Russell for being a kind, supportive, and engaging supervisor and mentor, giving me support and flexibility to work and succeed over the past two years.

Finally, I want to express my infinite gratitude and thanks to my family and friends (dogs included). The love and support I received and continue to feel to this day is humbling - I wouldn’t be who or where I am without it.
Acoustical horns can be considered among the first and most common acoustical devices used in the world today. From speakers to musical instruments, horns play a large role in the tone, level, and efficiency of all sorts of audio devices. However, the way in which a horn’s radius changes with length makes it difficult to create a precise and robust model. Transmission lines and lumped elements are a way to break down a complex, distributed system, like the air inside a horn, into the building blocks of its medium to estimate it’s response to any input.

Transmission lines built as a cascade of 'T-model' segments is commonplace in fundamentals of acoustics courses and can be used for any acoustical waveguide shape. Often, cylinders are the first waveguide shaped used, but quickly the bore geometry can become much more complex while still using the identical transmission line formula. This simple model can help show the relation between number of elements used/cascaded with the reliability and frequency bandwidth of accuracy for the results. The answer can approach that of experimental analysis, but this accuracy often comes at a cost of increased computing requirements.

However, a piece of recently uncovered literature could hold the key to add a new, more accurate, and computationally equivalent (or lower) method of modeling these systems as acoustical domain circuits. Specifically, this paper was found in a review of my advisor Dr. Stephen Thompson’s advisor, Arthur Benade’s published work. It was released after his passing, and has largely been overlooked by relevant literature - the updated model presented in this paper aims to prove the legitimacy of Benade’s work, and the potential for future development.
1.1 Background

A complete model of a horn requires two parts: a transmission line to simulate wave propagation and a radiation impedance to calculate the sound propagation at the radiating end. The following techniques can be considered the most common teachings and practices to compute a fairly reliable estimate of a horn, or horn-like system, using numerical or equivalent circuit modeling. The focus of this thesis will be to build upon and improve these models.

1.2 Lumped Elements and Equivalent Circuits

To give context for the rest of this thesis, it’s important to understand the concept of equivalent circuits and how they can be used to accurately model acoustical systems, among those of many other domains. While the specific model derived later is entirely in the acoustical domain, the intent is for its use in larger systems that bridge between acoustical, mechanical, electrical, magnetic, and potentially any other domain.

The basic elements of these systems can be represented by their analog electrical domain components, specifically inductors, capacitors, and resistors. In the mechanical domain, the three building block components are masses, springs, and dampers which represent the inertial, compliant, and resistive properties of a system. In an impedance circuit diagram, they’re represented by inductors, capacitors, and resistors, respectively. The acoustical domain uses similar nomenclature with components of acoustical mass (inertia of the medium) represented by an inductor, acoustical compliance (springiness of the medium) represented by a capacitor, and an acoustical resistance (flow resistance/boundary interaction) represented by a resistor, as seen in Figure 1.1. For all applications in this paper, the medium will be air. Together, these three domains can model most common audio systems - for example, an amplifier (electrical) connected to a loudspeaker (mechanical) that vibrates in air and produces sound (acoustical).

These domains also contain domain-specific potential and flow variables. In the electrical impedance circuit, voltage serves as the potential variable and current the flow. The mechanical domain uses force and velocity, while the acoustical domain uses pressure and volume velocity. Transformers can be used to translate between domains and connect them together in multi-domain models.
1.3 Historical Modeling Techniques

1.3.1 Uniform Cylinder Transmission Line

The simplest way to build a lumped-element acoustical transmission line is also the technique that has withstood the test of time. This is to divide a waveguide into many uniform length cylinders where the radius is equal to the cavity’s radius at the leading edge. These cylinders, and their equivalent circuit counterparts, are cascaded to create an estimate of the shape of the waveguide’s bore. Gabrielson [1] calls these sub-elements of the transmission line “Basic Acoustic Elements”, but they are more commonly known as the T-model components. They include both mass and compliance components to represent the inertia and springiness of a small, cylindrical tube of air, such as the short cylindrical segment shown in Figure 1.2.

![Figure 1.2. Uniform cylindrical segment of length ℓ, radius a, and cross-sectional area A.](image)
Figure 1.3. Acoustical equivalent circuit representing a cylindrical element of air, as seen in Figure 1.2. Model includes two inductors, each for half of the air mass, and a capacitor for the compliance of the air cavity. The shape is that of a "T-model" circuit, commonly cascaded in transmission lines. Also referred to as a Basic Acoustic Element (BAE) by Gabrielson [1].

Summing the lengths $\ell$ of these evenly distributed cylinders lengths will equal the total length $L$ of the waveguide or cavity to be modeled. The technique can be visualized in the same way that a Riemann sum estimates the area under a curve in the derivation of an integral – the smaller $\ell$ is, the better the approximation. This is the same for our acoustical model. Whatever the radii might be on the edge of a circular waveguide, the combination of acoustical cylinders can, with variable accuracy depending on the length of $\ell$, model the system. For most applications in speaker design or for musical instruments, the cross-section at any single point along the central axis is circular - but this is not required.

The name T-model comes from how the mass and complaint components are distributed into the components and the resulting shape of the circuit. Instead of having one inductance, the cylinder’s mass is divided into two identical components, each contributing half of the acoustical mass. These connect in parallel to an acoustical compliance, as seen in Figure 1.3. The combined total of the two mass elements, as well as the value of the acoustical compliance, are used to translate the values to the acoustic inductance and compliance, frequency-dependent impedances. For the entire waveguide of length $\ell$, the acoustic mass and compliance are calculated

$$M_{acs} = \frac{\rho L}{A} \quad \text{and} \quad C_{acs} = \frac{Al}{\rho c^2}. \quad (1.1)$$
When the waveguide is broken into $N$ smaller cylindrical sections, like those in Figure 1.2, the length $L$ is replaced by $l = \frac{L}{N}$ and the mass is split evenly between the two inductances as shown by Gabrielson [1]. The impedances for both, then, are

$$Z_{M_{acs}} = j\omega M_{acs} = \frac{j\omega \rho l}{A}, \quad (1.2)$$

$$Z_{C_{acs}} = \frac{1}{j\omega C_{acs}} = \frac{\rho c^2}{j\omega Al}. \quad (1.3)$$

This circuit orientation can also be represented using transformation matrices to relate the input potential and flow to the output,

$$\begin{bmatrix} P_1 \\ U_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot \begin{bmatrix} P_2 \\ U_2 \end{bmatrix}. \quad (1.4)$$

Series and parallel impedances, $Z_s$ and $Z_p$, respectively, have unique 'ABCD' matrices, $T_s$ and $T_p$, which are multiplied together in the same order that the corresponding elements appear in the circuit. In this case the circuit is symmetrical, but the elements are being multiplied from left to right, or from the input to output of the cylinder.

$$T_s = \begin{bmatrix} 1 & Z_s \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad T_p = \begin{bmatrix} 1 & 0 \\ \frac{1}{Z_p} & 1 \end{bmatrix}. \quad (1.5)$$

The resulting transfer matrix for each T-model, cylindrical acoustic element can be computed with a matrix of the form

$$T = T_s \cdot T_p \cdot T_s = \begin{bmatrix} 1 & Z_s \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ \frac{1}{Z_p} & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & Z_s \\ 0 & 1 \end{bmatrix},$$

$$T = \begin{bmatrix} 1 + \frac{Z_s}{Z_p} & 2Z_s + \frac{Z_s^2}{Z_p} \\ \frac{1}{Z_p} & 1 + \frac{Z_s^2}{Z_p} \end{bmatrix}. \quad (1.6)$$

The drawbacks of this model for a flaring horn are two-fold. First is the requirement for a lot of elements to accurately estimate accurate bore dimensions. Whether you use the equivalent circuit or matrix version, every additional element increases the computational cost of the model. The only shape of waveguide where this is less of a problem is a cylinder because the radius of every element is identical, so also are the mass and spring components as well. The only consideration necessary is the number of degrees of freedom to give the system which will change the frequency bandwidth. The
geometry, though, will match exactly, so fewer components will be necessary especially for lower-frequency modeling.

Any deviation from a uniform radius will start to introduce inaccuracies, with a flaring horn introducing the greatest error as the radius has the possibility to change significantly from one end to another. The only solution, without changing the shape of the cascaded elements, is to make \( \ell \) smaller and/or increase the number of elements in the system. Figure 1.4 shows this effect.

Secondly, a sharp acoustics mind will also realize that the wave shape of propagation inside a cylinder and a horn are not the same. Cylinders largely support plane-wave propagation down their length where all energy moves parallel to the central axis. A horn, though, (flaring, linear, or otherwise) supports wave shapes much closer to that of spherical waves where energy propagates parallel to the central axis as well as at an angle away from the central axis as you get farther from the center. Effects of these differences are seen in the physical properties of the equivalent system, as well as how the final element in the cascade interacts with the load/radiation impedance at the far end.

A proper model will address both insufficiencies while maintaining the ease with which the models are built, calculated, and able to be visually understood.

Figure 1.4. A exponential curve \( y = e^{3x} \) (red), representing the bore geometry of a flaring horn, estimated using \( N \) equally spaced cylindrical elements. As \( N \) increases from 5 to 50, the accuracy greatly increases as indicated by the black curve largely covering the red.
1.3.2 Radiation Impedance of a Piston

The radiating face of any acoustical waveguide, including but not limited to cylinders, cones, and flaring horns, are most simply and commonly modeled as a piston source at the end of a long tube or in an infinite baffle. Commonly cited examples come from Levine and Schwinger [5] as well as Beranek [2] who provides both an analytical solution and an acoustical equivalent circuit representation for the radiation impedance from a pipe in an infinite baffle. The analytical solution is calculated as

\[ Z_{acs} = \pi a^2 \rho c \left[ 1 - \frac{J_1(2ka)}{ka} \right] + j\pi \rho c \frac{K_1(2ka)}{2k^2}, \]

(1.7)

where \( J_1 \) is a first-order Bessel function and \( K_1 \) is a first-order modified Bessel function of the second kind.

The acoustical equivalent circuit model that approximates that same radiation impedance contains four components: two resistors, one capacitor, and one inductor, as shown in Figure 1.5. In the calculation of impedance, the inductor \( M_{a1} \) accounts for the effective mass at low frequencies. \( R_{a2} \) represents the high frequency radiation resistance that, when combined in series with \( R_{a1} \) and in parallel with \( M_{a1} \), provides a good approximation of the radiation resistance at low frequencies. Lastly, \( C_{a1} \) helps to create a smooth transition from the low to high frequency regions.

![Figure 1.5. Beranek [2] circuit structure for the acoustic analogy of the radiation impedance of a piston in an infinite baffle or for a piston in the end of a long tube.](image-url)
The results from Eq. 1.7 and the circuit in Figure 1.5 are compared in Figure 1.6 as normalized, complex curves for an arbitrary piston size. Clearly the circuit approximation is similar to that of the analytical for the entire frequency bandwidth of interest.

More specifically, the keys to look for to make sure that these models are similar and useful are firstly that at lower frequencies the reactance dominates. This indicates that most energy is stored within the system in an inductance-like fashion. Secondly, as the frequency increases, and therefore the acoustical size $ka$ becomes larger, the resistive component equals and then surpasses the reactance until finally approaching a steady state value. Now much more energy is being radiated away from the system than is stored. Models of this type will all be able to semi-accurately represent many radiation types like a piston in a tube.

![Graph comparing analytical and circuit models](image.png)

**Figure 1.6.** Comparison of an analytical (Eq. 1.7) and equivalent circuit model (Figure 1.5) for the radiation impedance of a baffled piston in a tube, from Beranek [2].
In the acoustical equivalent circuit shown in Figure 1.5, at smaller $ka$ values the combination of $C_{a1}$ and $R_{a2}$ appears very large, and nearly all the load is represented by the inductance $M_{a1}$. At higher frequencies, where $ka \gg 1$, the resistance appears to shunt the inductance and account for most of the impedance of the system. Along with the circuit itself, Beranek [2] provides equations for all four components given the density and sound speed of the medium, $\rho$ and $c$, and the radius $a$ of the radiating face. The four components are calculated

\[ R_{a1} = \frac{0.1404\rho c}{a^2}, \quad R_{a2} = \frac{0.318\rho c}{a^2}, \quad M_{a1} = \frac{0.27\rho}{a}, \quad \text{and} \quad C_{a1} = \frac{5.94a^3}{\rho c^2}, \quad (1.8) \]

where the subscript notation identifies them as acoustical domain quantities and the component number by type.

For modeling horns, the primary downfall of this model arises from the shape of the radiating face. Horns, linear and flaring alike, support wave shapes much closer to spherical than planar. Any point in a conical waveguide has a semi-spherical wave shape, but the effect is especially true as you move down the length of a horn with any flare and the slope of the outer dimension increases. At the mouth of horns, then, is a radiating wave that has a curve, not a flat surface. Without any spherical-like behavior allowed in the piston model, the radiating face shape cannot really represent the wave as it exists in reality. A better model of radiation impedance for a flaring horn will be able to match the shape of the spherical wave produced at the mouth.

### 1.4 Objectives

The goal of this project is to implement an alternative to the transmission line and radiation impedance models listed above by addressing the pitfalls identified. For the transmission line, the intent is to replace equivalent circuit model of cascaded cylindrical components with conical segments. An equivalent circuit model of a cone was published by Benade [3] over 35 years ago, and yet it has been passed over by almost every publication on the topic since. From a geometric perspective, the conical segments should be able to match the a flaring horns radius much more accurately. As a circuit, the conical segments are only slightly more complicated than the T-model for a cylinder, so implementation appears straight forward and hopefully will yield improved results.

This transmission line will be paired with a radiation impedance model of a radiating spherical cap on a spherical shell from Hélie and Rodet [9]. A simple optimization
will allow this numerical model to be translated to acoustical domain components in Simscape using the same radiation impedance structure as for a piston in a tube from Beranek [2]. These models will be built in MATLAB’s Simscape, where they can be included in frequency and time domain simulations. The ability to use the model in the time domain will be a big step in opening the door for more exploration, and to get the Benade model used in many more contexts, for example Thompson’s [10] clarinet model.

Included in this paper is both the background and derivation of each component, and their numerical implementation as proof of concept. Once validated, the transition into Simscape components will be outlined. These components will make use of a Simscape acoustical library, available for use with any MATLAB license. An estimation of the far-field radiated pressure will be discussed along with the results of the model for increasingly complex acoustical waveguide shapes.

Legitimacy of this new model will be based partially on how accurately simulation results compare to experimental data as compared to the historical methods. Resonance frequencies of the input impedance will be used to quantify this accuracy. In addition, the efficiency, or how many transmission line components are necessary to achieve a low percent error, will also be discussed based on the resonance calculations. Finally, suggestion will be made about how to include thermoviscous losses into the model which will increase computational requirement, but increase accuracy of the results further and set-up potential future work.

1.5 Literature Review

The historical model for radiation impedance, as described earlier, comes from Beranek’s well known text, Acoustics [2]. The same model will be used as a structure for the updated model as well. While not pioneered by him, computation of the cylindrical elements included in the transmission line comes from Gabrielson [1]; this will be combined with the rest of the cone equivalent circuit from Benade [3]. The radiation impedance that will be used was derived by Hélie and Rodet [9] to model a rigid, spherical cap pulsating on a spherical baffle. The complex expression used is their second order high pass model. For use with a given horn geometry, Eveno et. al. [6] created a helpful summary which alters the second-order high pass model to include a power series expansion based on the geometry of the horn opening. Both return complex impedance curves of impedance for any frequency range and spacing.

To test the model, computational results of input impedance resonance frequencies of
a Strasser oboe are used from Plitnik and Strong [7]. They used a model very similar to that of the historical model presented previously where the transmission line is made of cylindrical elements and the radiation impedance is a rigid piston at the end of a tube. Also, experimental data for two musical instrument bores and their resulting resonance frequencies is provided in the same paper from Eveno et. al [6] that gave us the polynomial expansion version of the radiation impedance. They captured the data in an anechoic chamber using a chirp pressure input perturbation and an impedance sensor at the bell. Unfortunately the numerical data was corrupted since publication, but the resonance peaks below the cutoff frequency of each were provided and will be compared to the model’s results. Finally, a reliably accurate numerical reference for impedance calculation of a cone is used in some instances, which comes from Stewart and Linsday [4].

As described earlier, the conical equivalent circuit has been cited by a few authors, only one of which used it in a similar context. Gustafsson [11] uses it to help with comparison of his model for radiation and spherical modes of antennas. Instead of looking at acoustically-small elements, this paper considers electrically-small elements, and the conical model in the acoustical domain provides an alternative derivation of the same result he found in the electrical domain. Smith [12] briefly notes it’s use in modeling the end of single reed instruments, assuming that the bell is conical, without any flare. Instead of cascading elements, the paper instead uses a single cone, and the conical models are never considered without the rest of the instrument included - no details or results are provided for comparison either. Scavone [13] is a third to refer to the Benade paper, and actually uses it as a transmission line for musical instrument bores. However, the transmission line isn’t of conical elements, but rather he uses a single conical elements with multiple cylindrical elements inside, so only models of conical bores with a linear taper can be used. His analysis is also translated to discrete-time filters for synthesizing musical instrument sounds in the time domain. Of all these, none apply it to a shape other than a cone (linear change in radius along the length) nor do they append multiple conical elements together in a transmission line. So, they clearly don’t address the issue this paper intends to solve.

The paper has also been referenced in a comprehensive text on horn loudspeaker systems by Kolbrek and Dunker [14]. In this text, though, it isn’t used as an alternative element for a transmission line, while the cylindrical approach is shown in detail. Instead, their analysis is much closer to that of Scavone [13] where only conical horns are considered with no results provided on experimental analysis of the horn apart from the rest of a
In addition, and although it doesn’t make use of the conical model, Leach [15] provides a thorough background on horn behavior and modeling. Significant examples within LTSpice for many horn geometries are provided, including linear and varying flares within the Salmon family of horns [16].

The final reference from Mapes-Riordan [17], like this project, creates a transmission line of conical elements using Benade’s work. However, these simulations are done purely in the frequency domain using the analytical transfer matrix version of the model rather than building an equivalent circuit of the components themselves. It makes sense, however, because this paper was published far before computers and their available software packages would be able to comprehend and handle such a model. Reassuringly, it concludes by indicating that the conical model appears to achieve more accurate results than the cylindrical approach. Lastly, it also created a dissipative conical element model, as is provided for conjecture in this work. Comparable results were not provided, but this reference will serve beneficial if future work is to be done with an adapted Simscape model including thermoviscous losses. A paper from Thompson et. al. [18] indicates how to include thermoviscous damping to a cylinder as an acoustical domain ladder network building off of the T-model circuit.
Chapter 2 | Numerical Validation

Both the transmission line and the radiation impedance of the updated model can be tested numerically before translating into Simscape. Showing this approach will elucidate the upgrades from the historical version previously described. Once validated, the transmission line and radiation impedance will be translated to acoustical domain equivalent circuit components.

As stated in Chapter 1, the first goal should be the ability of a model to efficiently match the shape of a flaring horn via the acoustical elements used in the transmission line. The other goals are to address and achieve accurate depiction of the propagating wave shape through the waveguide, as well as the shape of the radiating element and propagating waves into free-space.

2.1 Cone Model

Because flaring horns have an inner diameter that changes with length, the solution to estimate the shape more accurately is to replace the cascade of cylindrical elements with a cascade of conical elements. While these elements still have a linear taper, as opposed to the non-linear shape of a flare, the slope of each component will significantly reduce the discontinuity at the end of each cascaded segment. Where normally the cylinder will have the same radius, the conical segments will increase their end radius to be closer to the horn’s radius at that same point. This should allow the model to get a more accurate physical representation of the horn with fewer elements in the transmission line, as seen in Figure 2.1 where the black and red curves are almost indistinguishable.
Figure 2.1. A exponential curve $y = e^{3x}$ (red), representing the bore geometry of a flaring horn, estimated using equally spaced cylindrical elements (left) and equally spaced conical elements (right). Both black curves use 10 elements along the length and are superimposed on top of a red curve.

As previously stated, the use of cones in the transmission line is additionally beneficial because the propagating wave shape in a cone is much closer to what you’d see in a flaring horn. While neither are truly spherical, the semi-spherical shape does a significantly better job than plane waves in a cylindrical element, and for this model that improvement is enough.

While there are various acoustical cone models available, the one selected from Benade [3] presents a method of modeling a conical section of an acoustical horn, like that in Figure 2.2, with an acoustical domain equivalent circuit as seen in Figure 2.3. This model is an extension of the BAE/T-model approach described in Chapter 1 for cylindrical elements.
Figure 2.2. Conical waveguide element of length $\ell$ measured on the outer edge, radii $a_0$ and $a_e$ at the throat and mouth, respectively, and the distance to an imaginary apex on the throat side, $x_0$. A coordinate system along the center line is provided as well. Figure recreated from Benade [3].

Figure 2.3. Acoustical equivalent circuit structure recreated from Benade [3]. The diagram appends three components to a uniform line (T-model): inertances $M_0$ and $M_e$, as well as an acoustical-to-acoustical domain transformer with a ratio $r = a_e/a_0$. The uniform line can include one or more cylindrical elements, like the one seen in Figure 2.3.

The model works by appending two dynamic inertances $M_e$ and $M_0$ in parallel with the uniform cylindrical element. $M_0$ is added on the left side of the cone, or at the cone’s
throat. \( M_e \) is added on the right at the cone’s mouth. Also, an acoustical-to-acoustical transformer is inserted in parallel between the right side of the uniform line and \( M_e \). The transformer has a ratio

\[
    r = \frac{a_e}{a_0},
\]

where \( a_0 \) is the radius at the throat and \( a_e \) is the radius at the mouth, as seen in Figure 2.2.

Just as \( T_s \) and \( T_p \) were transfer matrices for series and parallel components, a transformer also has an equivalent transfer matrix

\[
    T_r = \begin{bmatrix} r & 0 \\ 0 & \frac{1}{r} \end{bmatrix},
\]

which accounts for the changing radius of the cone and therefore the characteristic impedances on either end of the element. It’s important to note that the uniform line can include more than one cylindrical element. Increasing element number can be used if any part of the waveguide has a linear taper, or to simply increase the number of elements in the model to improve accuracy. All elements should remain acoustically much smaller than a wavelength, and this can be a way to confirm that. Due to the way that the data for this model will be supplied, even the longest conical segment of a horn will remain much smaller in length than the wavelength of the highest frequencies. If this was not the case, or if we were modeling a uniform horn, the entire transmission line could be made of cylinders and would be combined with a single transformer and two inertances to model the linearly tapered cone.

The conical circuit has a unique way to account for the change in mass at either end, which also helps its integration into a transmission line. Compared to a cylinder of equal length and with a radius of the average of the cone’s throat and mouth radius, at one end there will be more acoustical mass, and at the other there will be less. The two inertances, which account for mass in the acoustical domain, help to re-distribute the overall mass of the cylinder by making \( M_0 \) positive (left) and \( M_e \) negative (right). The inertances are calculated

\[
    M_0 = \frac{x_0 \rho}{\pi a_0^2 F} \quad \text{and} \quad M_e = -\frac{x_e \rho}{\pi a_e^2 F}.
\]

Again, \( a_e \) and \( a_0 \) are the radius at the mouth and throat of the conical segment, respectively, and \( x_0 \) is the distance from the throat to an apex point along the central axis. \( F \)
is a scalar to convert from planar to spherical wave front area, calculated

$$F = \frac{2 \cdot (1 - \cos(\frac{a_0}{x_0}))}{\sin^2(\frac{a_0}{x_0})}. \quad (2.4)$$

When cascading multiple conical sections, the radius at the mouth of one and the throat of the next will be identical. Yet, the magnitude of the two inertances will not be identical so they won’t entirely cancel at the junction of two cones. The cascade of many of these conical elements will leave small positive inertances in the circuit at each junction in addition to the inerance of the uniform line T-model. It’s these small additions that help to more accurately model the increased surface area of a spherical-like wave, and the redistributed acoustical mass throughout the element.

### 2.1.1 MATLAB Implementation

For use in an analytical model in MATLAB, the acoustical domain circuit of a cone, and more specifically the circuit’s corresponding transformation matrices, are multiplied to create a single expression for the entire element. $M_0$ and $M_e$ are each an additional parallel component which get added to the left and right of the equation along with $T_r$ in its place to the left of $M_e$.

$$T = T_p(M_0) \cdot T_s\left(\frac{M_{acs}}{2}\right) \cdot T_p(C_{acs}) \cdot T_s\left(\frac{M_{acs}}{2}\right) \cdot T_r \cdot T_p(M_e). \quad (2.5)$$

These equations can be implemented in MATLAB for any horn shape. To validate the results, a linearly tapered horn (or cone) was built for comparison across multiple existing equation sets and models. This cone is depicted in Figure 2.4.

Comparison with an analytical solution as well as the results using the cylindrical transmission line shows the legitimacy of the conical transmission line model. The analytical solution for the input impedance of a cone comes from Stewart and Lindsay [4] and is calculated where

$$Z_{in} = \frac{\rho c}{S_1}\left[\frac{jZ_{rad}\frac{\sin(k(L-\theta_2))}{\sin(\theta_2)}}{Z_{rad}\frac{\sin(k(L+\theta_1-\theta_2))}{\sin(\theta_1)}} + \frac{\rho c}{S_2}\frac{\sin(kL)}{\sin(k\theta_2)} - \frac{j\rho c}{S_2}\frac{\sin(k(L+\theta_1))}{\sin(k\theta_1)}\right], \quad (2.6)$$

$$\theta_1 = \tan^{-1}(kx_0), \quad (2.7)$$

$$\theta_2 = \tan^{-1}(kx_e), \quad (2.8)$$
and $S_1$ and $S_2$ are the cross-sectional areas of the throat and mouth, respectively.

An analytical approximation of complex radiation impedance for an unflanged cylindrical pipe from Levine and Schwinger [5] is used with all three models. For openings much smaller than a wavelength, or $ka \ll 1$ the radiation impedance at the open end is

$$Z_{\text{rad}} = \frac{\rho c}{A} \left( \frac{1}{4} (ka)^2 + 0.61 jka \right).$$

(2.9)

Both the cylindrical and conical transmission lines were modeled with $N = 100$ segments. The input impedance for the transformation matrices is calculated

$$Z_{\text{in}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot T^N \cdot \begin{bmatrix} Z_{\text{rad}} \\ 1 \end{bmatrix},$$

(2.10)

where $T^N$ represents the cascade of $N$ cones from Eq. 2.5 with radial dimensions taken from their respective portion of the larger waveguide. For visual comparison, the input
impedance calculated by each of the three models are superimposed in Figure 2.5. The frequency of each of the first eight resonance frequencies, or peaks of the curves, are displayed in Table 2.1.

It’s clear to the eye that these results are not identical, but that’s expected. What’s most important is that the results were close, within only a few Hz in most cases, which should bode well moving to Simscape. Also, both transmission lines took largely the same amount of time to compute. This isn’t a concrete indication that the few extra components necessary to model conical sections won’t prove detrimental to speed in future implementations. But, if the solvers operate similarly, the difference should be on the same, negligible order of magnitude and won’t need to be commented on further.

![Figure 2.5. Input impedance of the cone in Figure 5.1 using three models for validation: a transmission line of cylindrical elements, a transmission line of conical elements, and an analytical solution from Stewart and Lindsay [4] (S&L). All three models use the radiation impedance of an unflanged cylindrical pipe from Levine and Schwinger [5].](image)

$$Z_0 = \rho c \left[ \frac{jZ_{rad}(kL - a_1)}{S_1} \sin(kL) + \frac{\rho c}{S_2} \sin(kL) \right]$$
Table 2.1. Resonance frequencies of a conical horn modeled both as a conical and cylindrical transmission line with N=100, as well as Stewart and Lindsay’s analytical solution [4].

<table>
<thead>
<tr>
<th>Model</th>
<th>$f_1$</th>
<th>$f_2$</th>
<th>$f_3$</th>
<th>$f_4$</th>
<th>$f_5$</th>
<th>$f_6$</th>
<th>$f_7$</th>
<th>$f_8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conical</td>
<td>235.9</td>
<td>472.4</td>
<td>714.9</td>
<td>967.0</td>
<td>1229.7</td>
<td>1500.6</td>
<td>1776.8</td>
<td>2055.9</td>
</tr>
<tr>
<td>Cylindrical</td>
<td>233.6</td>
<td>469.3</td>
<td>710.9</td>
<td>1222.5</td>
<td>1491.7</td>
<td>1766.2</td>
<td>2043.9</td>
<td>2323.7</td>
</tr>
<tr>
<td>Analytical</td>
<td>233.6</td>
<td>467.5</td>
<td>708.6</td>
<td>958.9</td>
<td>1219.4</td>
<td>1487.9</td>
<td>1761.3</td>
<td>2037.5</td>
</tr>
</tbody>
</table>

2.2 Radiation Impedance Model

To improve upon the piston source in the end of a tube, the radiating face used in this work comes from Hélie and Rodet [9] which models the end of a horn as a radiating spherical cap on a sphere. The size and shape of the sphere is determined by the flare angle $\theta_0$ at the edge of the horn. The radiating face is made up of every point along the invisible sphere that fits inside the horn, as seen in Figure 2.6. The derivation and implementation presented is purely numerical in both the original work by Hélie and Rodet [9], as well as a useful follow-up paper from Eveno et. al. [6].

The major benefit of this model is that the radiating face is curved, as are waves propagating from the mouth of a horn. Not only is the surface area better estimated, but the spreading of energy away from the central axis is also accounted for. Again, exact wave shapes in a cone or flaring horn aren’t identical to the shape of the spherical cap, but the estimate is much improved on a flat piston.

![Figure 2.6. Radiation impedance of a horn where the bell is approximated as a sphere, $S$, where only the portion $S_0$ radiates. So, the radiating face is a spherical cap set in a spherical baffle. The sphere has a radius $r_0$ and the half angle of the cap is denoted by $\theta_0$.](image)
2.2.1 MATLAB Implementation

Once the radii of the horn is measured at many points along the length, the last two of those points and the distance between them can be used to calculate the slope and therefore angle of the flared opening. This radiation model can be created for any horn shape. Slight errors in calculating the radius of the sphere, the slope of the line to the cap’s edge, and the radiating face size will have significant impacts on the reliability of the model. So, great care was taken to validate the accuracy of this model. A MATLAB script computes the 'physical' shape and size of $S$, $S_0$, $r_0$, and $\theta_0$ and plots it for visual inspection before calculating the resulting radiation impedance. This is shown in Figure 2.7.

$$r_0 = 0.27006 \, [\text{m}], \quad a = 0.15 \, [\text{m}], \quad \Theta_0 = 0.507 \, [\text{rad}]$$

**Figure 2.7.** Visualization of $S$, $S_0$, $r_0$, and $\theta_0$ for manual inspection to confirm model is set up according to Figure 2.6 for the given horn geometry. In this case, the horn, shown in red, is a Salmon horn with a flare rate of $m = 4.5$.

The actual computation of complex impedance is implemented using a reorganization of what Hélie and Rodet [9] derived. The second-order high-pass model is provided as a frequency dependent function of $\nu = r_0 f / c$ and includes $P_\alpha$, $P_\xi$, and $P_\nu$ which are sixth-order polynomial expansions of $\theta_0$. Thus, the computation is directly dependent on
the flare angle at the mouth of the horn. This form comes from Eveno et. al. [6] where

\[ Z_{\text{rad}} = \frac{\rho c}{A} \cdot \frac{j\nu P_\alpha - (\nu P_\nu)^2}{1 + 2j\nu P_\xi P_\nu - (\nu P_\nu)^2}, \]  

(2.11)

and the \( P \) coefficients are the polynomial expansions with coefficients as seen in Figure 2.8.

<table>
<thead>
<tr>
<th>( P_a )</th>
<th>( P_\xi )</th>
<th>( P_\nu )</th>
<th>( \theta_0^0 )</th>
<th>( \theta_0^1 )</th>
<th>( \theta_0^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.8788</td>
<td>0.7200</td>
<td>-0.0220</td>
<td>-1.24200</td>
<td>0.22100</td>
<td>-0.07946</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( P_a )</th>
<th>( P_\xi )</th>
<th>( P_\nu )</th>
<th>( \theta_0^3 )</th>
<th>( \theta_0^4 )</th>
<th>( \theta_0^5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1620</td>
<td>-0.6360</td>
<td>0.1113</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-0.1440</td>
<td>0.0207</td>
<td>0.0000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-0.4240</td>
<td>0.2607</td>
<td>-0.1980</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Figure 2.8.** Coefficients of the \( \theta_0 \)-dependent polynomial expansions used in Eq. 2.11. Graphic borrowed from Eveno et. al. [6].

This expression can be calculated in MATLAB, but it can’t be used directly in any existing Simscape component. However, the real and imaginary curves can be used as a reference to match with the performance of an existing model.

### 2.2.2 Comparison to Spherical and Planar Radiation

Comparing the results of this model to that of plane and spherical radiation share some insight on how differently it works. To compare it to spherical radiator where an entire sphere is pulsating equally in all directions, a large-angle approximation was used where \( \theta_0 = \pi/2 \) so the cap was equal to half of the sphere. Both spheres were also given identical radii. In this case, the spherical cap only accounts for half of the radiating surface area of the full radiating sphere, and in the real part of the impedance, as plotted in Figure 2.9, this relationship holds as the spherical cap radiation is just about half the magnitude of the spherical, especially at lower frequencies.
Figure 2.9. Comparison of complex radiation impedance of a spherical radiator and the spherical cap model from Hélie and Rodet using equal radii for the spheres and $\theta_0 = \pi/2$.

To compare it to a piston source, a small-angle approximation was made where $\theta_0 = \pi/6$. With a smaller angle, there’s less curvature in the cap so it more closely resembles the rigid piston. The area of both of the radiating faces was equated for the two, and the results are plotted in Figure 2.10. There isn’t as direct a relationship for the differences seen like with the spherical radiator, but both figures show that there are clearly differences in the models, so we should expect a difference in our modeled results. Their similarity also proves the legitimacy of this model, which can now be implemented in Simscape.
2.2.3 Optimization to Acoustical Domain Circuit

Using the Beranek structure described in Chapter 1, it’s possible to overcome the fact that this model is purely analytical. Inspection of the real and imaginary curves show a strong similarity in behavior to the impedance of a piston source, as is seen in Figure 2.10. At low frequencies the reactance dominates, they cross in the mid-frequencies, and at higher frequencies the resistance dominates and approaches a steady-state value. So, it should be possible to change the four component values in the Beranek’s [2] acoustical domain circuit structure to return complex impedance curves identical to that of the Hélie and Rodet [9] model from Eq. 2.11.

To do so, a simple cost function was built to optimize the parameters of the four components of Figure 1.5 to accurately reproduce the complex impedance of the spherical
cap. Parameter values for $R_{a1}$, $R_{a2}$, $M_{a1}$, and $C_{a1}$ as calculated in Eq. 1.8 for a piston source can serve as a good starting point for the optimization, which, using MATLAB’s fminsearch, can manipulate the real and imaginary curves until they match across the entire frequency range of interest. This operation takes less than 20 seconds for a 3 kHz range at 0.1 Hz spacing.

The result of this optimization is shown in Figure 2.11. It highlights that the $Z_{rad}$ calculated using the acoustical domain circuit in Figure 1.5 with optimized values for $R_{a1}$, $R_{a2}$, $M_{a1}$ and $C_{a1}$ matches the complex curves from Eq. 2.11 with high accuracy for any frequency range. Although physical meaning is stripped when doing it in this way, the numerical result is all that’s necessary to implement the structure in an acoustical domain. Now, the resulting component can be used in Simscape to compute time and frequency domain simulations.

![Figure 2.11. Real and imaginary components of radiation impedance using the results of Eq. 2.9 as well as the Beranek structure with optimized values. After the optimization, the real and imaginary curves match with high accuracy.](image)

25
Chapter 3  Methodology

The software selection for this work was influenced primarily by the need for equivalent circuit modeling, ideally in the acoustical domain. Both the conical transmission line and radiation impedance used have analytical solutions, but can be easier used in other applications if the equivalent circuit model is used and accessible for anyone. Programs like LTSpice have historically been used for reasons including price, convenience, and the inability of other programs to match their level of accuracy for frequency domain simulations. However, MathWorks has, in recent years, made improvements to its multi-domain modeling software called Simscape which is a sub-category of the more commonly known Simulink.

Simulink, unlike more traditional MATLAB computations, are primarily a graphical coding software. Instead of a project that looks like many lines of text, a Simulink project will show a diagram of components and wires connected together, all which are interactive that a user can add, remove, or alter as they please. Simscape is a sub-directory within Simulink that enables users to create models of physical systems across many domains. Each domain contains a broad library of pre-made components all which are available with a license for the software. If the component library doesn’t have exactly what you need, text-based code can be used to create a custom components for local use or that can be shared with others as long as they have the relevant domain.

3.1 Simscape Multi-Domain Modeling

The benefits of Simulink are, firstly, the ease in which multiple domains can be used together in the same model without having to retro-fit the electrical-only components as previously done with LTSpice. The foundational library, included with a full license of the software and highly documented for usability, includes Electrical, Mechanical,
Magnetic, and many more domains commonly used in audio systems. Additionally, an Acoustical Domain has been developed by Stephen Thompson and is slated for release to the public soon. The building blocks in this library provide a basis to build the more complicated networks described in the early part of this paper. Access to this resource makes Simscape a clear choice in itself.

A second benefit of this program is its ability to compute both frequency and time domain simulations. This allows, for the purposes of a horn model, to see input impedance plots as well as to integrate the components with an input pressure source that’s time varying. Once built, the components of this model can easily be integrated into systems like a clarinet, as created by Thompson [10], that returns time-domain waveforms that can be played for comparison to the real instrument.

So, with Simscape selected and the additional libraries at hand, all the circuits were built in the acoustical domain. Although the specific acoustic domain used isn’t included for public use now, it will hopefully be in the future.

3.1.1 Custom Components

To simplify circuit diagrams while expanding the capabilities of any system, Simscape allows users to build sub-systems and custom components that can be shared amongst anyone with the libraries installed. These custom components are built in a straightforward fashion by creating fixed or variable sub-systems from the acoustical library foundational library components. While the option exists to build these first in block-diagram modeling and assign them to a sub-system afterwards, the more robust option is to build them with text-based code. A user can create nodes to connect the block to the rest of the circuit, and then internally define, connect, and set parameter values for each component used within. Each component must include foundational material and potential and flow variables from a single domain, so the custom components built for this model will be part of the acoustical domain only.

A simple example of a sub-system is that of a single T-model element, like in Figure 1.3. This circuit requires a two-port network, meaning there are four connections from this circuit element to the rest of the system that must be made: positive in, positive out, negative in, negative out. The four ports are created and then each is assigned to be attached to the positive or negative terminal of the component it’s connected to. The unconnected side of the two inductances and capacitance are then attached together to complete the circuit. If desired, the potential and flow variables can also be equated and manipulated so the input and output will pass information through the sub-system to
the rest of the component. Reference code for context is provided in Figure 3.1.

```matlab
nodes
    port1 = acoustical.acoustical; % + in
    port2 = acoustical.acoustical; % - in
    port3 = acoustical.acoustical; % + out
    port4 = acoustical.acoustical; % - out
end
components(ExternalAccess=none)
    Macs1 = acoustical.elements.inertance(m = Macs);
    Macs2 = acoustical.elements.inertance(m = Macs);
    Cacs = acoustical.elements.compliance(c = Cacs);
end
connections
    connect(port1, Macs1.pp);
    connect(port3, Macs2.nn);
    connect(port2, Cacs.nn, port4);
    connect(Macs1.nn, Cacs.pp, Macs2.pp);
end
```

**Figure 3.1.** Sub-circuit built in Simscape code for the T-model structure. The ports of a two-port network are created, three foundational components are called, and connections are made attaching the circuit together and with the external ports.

Once created, the text-code file can be referenced from within the acoustical library and the sub-system, including all three acoustical components, will appear as a single component in the block diagram. Multiple examples of this will be shown in Chapter 4 where components like an "acoustical_converter" are user-created blocks integrated in more complicated systems.

Dynamic structures can also be created using loops; this allows a single block to represent a much wider variety of systems without being stuck with a single number of components. For example, the amount of cylindrical or conical components in a transmission line is determined by the length $\ell$. The smaller $\ell$ gets, the larger $N$ is - this shouldn’t force the creation of a new component entirely. Instead, the code uses the user-defined inputs to determine how many elements need to be made, and they’ll be created and attached in the correct order with each mass and spring element getting their own unique parameter values. If $\ell$ or $N$ changes again, the sub-system will re-assess and to match whatever the physical dimensions of the waveguide might be.

The ability to do this makes the block diagram much simpler, much more universally applicable, and, once the code in the component is verified, it can be shared to anyone with the acoustical domain Simscape library. A transmission line of 200 conical elements
won’t require a user to select and attach each and every one - this component will automate the process saving time and reducing error during such a monotonous task.

3.2 Integration with MATLAB

The time and frequency domain simulations in Simscape, and most other software packages within the MathWorks suite, can not only be run from the editor themselves but also from a MATLAB script. The compatibility of a more traditional script and the Simulink libraries allows access in both directions for parameters and results from the models as they’re running. Variables in the MATLAB workspace can be used to set/adjust parameter values within a component, or Simscape can run a linear frequency domain analysis (linearization) of a model and send the necessary information to replicate the results into the MATLAB workspace. Additionally, this allows for many instances of a model to be run without needing to manually change parameters every time.

While Simscape and Simulink are very powerful in some ways, they lack in others. The text-based code spans the majority of basic MATLAB functionality but it is limited within that small scope and many packages aren’t yet available for use within the components. What this means is that a calculation or function used in a MATLAB *.m-file won’t necessarily be able to be replicated within a Simscape or Simulink script. Ideally, future versions of the software will continue to include more of the MATLAB library within it’s sub-programs, which will allow for much easier future use of this model.

Pertinent to this work is the inability to use MATLAB’s optimization toolbox which, as described earlier, is critical to defining the parameters for the radiation impedance structure. Instead of trying to use a non-built-in optimization routine, which may or may not have worked due to these limitations anyway, the choice was made to use a MATLAB script to calculate the parameters for the radiation impedance which, along with the horn’s physical dimensions, can be accessed by the Simscape model through their overlapping work spaces. The script then returns the linearized results which can be analyzed within the MATLAB environment, something that would have been done either way. This limitation highlights yet another benefit of the dynamic components created for the transmission line and radiation impedance. Without having to edit a circuit or open Simscape the parameters can be changed for each simulation.
3.3 Input Impedance Calculation

While there exists no explicit input impedance measurement device in Simscape, there is a method to return it using the model linearizer. To do so, the acoustical model needs to be driven with a volume velocity (flow) source with magnitude $1 \, \text{m}^3/\text{s}$. By measuring the pressure (potential) across the terminals of the input source, the result is proportional to the impedance across the input. Still some manipulation needs to be done to be able to compare these to experimental data, but it saves a big step in getting there. The linearization information, Bode plots and impedance-equivalent curves, are returned through an ABCD matrix. For any input, the response can be calculated to match what one would see in the model linearizer built-in window.
Chapter 4  
Simscape Modeling

The finalized model, as it is translated into Simscape, includes custom components for modeling the acoustical transmission line and radiation impedance. Within the transmission line, components were built to model an acoustical cone as well as the acoustical-to-acoustical converter, both of which are cascaded in the transmission line. Within the radiation impedance block, an estimate of the radiated pressure is made based on the geometry of the radiating surface modeled. This model is run, as described in Chapter 4, using a MATLAB script that specifies the horn’s physical dimensions and returns the frequency domain input impedance curves and relevant resonance peaks.

4.1 Acoustical Cone Sub-Circuit

In the circuit model for a cone, Benade introduces a transformer, but one that doesn’t change domains [3]. In Simscape, a component was created that maintains the pressure difference, or the potential, across the input and output terminals, but it scales the particle velocity, the flow, by the ratio in Eq. 2.2. This component can be used in the acoustical domain in Simscape, as seen in Figure 4.1.

Building the remainder of the cone requires a few other components, all of which appear once and are given default parameter values but accurate units. Eq. 2.3 provides the accurate values for the positive and negative inertances attached on either side of the circuit, and the uniform line/T-model is built using two inertances and a capacitance which were defined earlier in Eq. 1.1 for the cylindrical transmission line. The code in Appendix B shows how exactly this component is created with the text-based approach, and how the custom converter is referenced. If built graphically using the custom components, a single cone takes the form seen in Figure 4.1 which bears almost identical resemblance to Figure 2.3.
Figure 4.1. Re-creation of Figure 2.3 in Simscape’s acoustical domain. Also this model includes the two inertances and compliance for the cylindrical uniform line as well as a custom transformer component.

In Chapter 5, numerical results are shared to confirm the model’s legitimacy as compared to previous literature of cone input impedance calculations.

4.2 Conical Element Transmission Line

For any acoustical horn or circular waveguide of known length and with adequate radius measurements, a transmission line can be created to model the system. The necessary inputs for an accurate model are those two requirements: the total length of the waveguide \( L \) and a vector of radii that includes both \( a_c \) and \( a_0 \) for each segment \( N \). The block parameter page is shown in Figure 4.2. The length of the vector of radii, therefore, is \( N + 1 \). For example, a cone of length \( L = 0.5 \) meters divided into \( N = 10 \) segments would have the values

\[
a_{\text{vec}} = [a_1, a_2, a_3, ..., a_{10}, a_{11}],
\]

where \( a_1 \) is the radius at the throat, and \( a_{11} \) is the radius of the mouth at \( x = L \).

Each conical element’s circuit components are created, the parameter values are calculated and assigned, and the input and output ports are attached in series with one another. For only a few conical elements, this circuit can be recreated by hand with multiple sub-circuits built and attached one after another. However, each component requires its own set of parameters, which would need to be manually assigned. For any more than three, this requires significant time and allows for incorrect connections or misaligned parameters. For an accurate model with upwards of a two hundred or more conical sections, this is all but impossible. This problem is alleviated by the creation of the dynamic Simscape component of the conical transmission line. Not only is it
more time-efficient, but it can also be used bi-directionally with no extra work. So, the throat and mouth, with small and large radius, respectively, could easily be reversed in the acoustical system. All that would need to happen is to flip the component and/or reconnect the input and output going to the the set of nodes on the other side.

**4.3 Radiation Impedance Sub-Circuit**

The complex radiation impedance for a spherical cap on a sphere is calculated easily in MATLAB using the second-order high-pass model in Eq. 2.11. Code in Appendix C shows exactly how the optimization is done to return component values for \( R_{a1} \), \( R_{a2} \), \( M_{a1} \), and \( C_{a1} \).

In Simscape, this structure only requires four components be attached, and they’re done so identically to Figure 1.5. This operates largely in the same way that the acoustical domain piston radiation impedance block works; however, the input doesn’t require physical dimensions. Instead, because the values are already calculated, the four component values are input into the component as seen in Figure 4.3. This method, therefore, is no more advanced that adding the four components themselves into the model and assigning values from the MATLAB script. Until the optimization can be done in the Simscape code, this is the best option and helps to keep the model as simple as possible.
4.4 Far-Field Pressure Estimation

An additional benefit of having a radiation impedance block is that it provides access to time-varying potential and flow variables through the input and output ports. This allows for the calculation of far-field pressure at a distance $R$ away from the radiating surface. In a time-domain simulation, this pressure output will simulate the response of the horn to whatever input signal is provided. For a baffled piston, the pressure is calculated

$$P = \frac{j\omega \rho U}{2\pi R},$$

where $U$ is the acoustic volume velocity. This equation can serve as a rough estimate for many radiating sources including the cap on a sphere of which the radiation impedance block is meant to model. That said, the results are based on the pressure and flow equations only, so the acoustical circuit modeling the spherical cap, and its unique component values, won’t have an impact on the output.

In an attempt at improving this calculation, the derivation for pressure radiated from a spherical cap on a sphere was sourced from a derivation by Morse and Ingard [19]. The pressure, in terms of distance $R$ and angle $\theta$, is

$$P(R, \theta) = \frac{-j\rho u_0 c}{2} \sum_{n=0} \left[ P_{n-1}(\cos(\theta)) - P_{n+1}(\cos(\theta)) \right] \frac{h^{(2)}_n(kR)}{h^{(2)}_{n-1}(kR_0)} P_n(\cos(\theta)).$$

(4.3)
Unfortunately, for trying to create any time-domain estimations, the equation relies on frequency dependent Bessel and spherical Hankel functions which don’t have an approachable analog for the time domain.

Using MATLAB and the output of a time domain simulation to attain the acoustical particle velocity, this frequency-dependent pressure can be calculated. An example of this is shown in Figure 4.4 which was calculated for an arbitrary cap shape and size. Due to frequency dependence as well as the computational requirements of the more accurate equation the result cannot yet be replicated as a time domain estimation in Simscape as it stands.

Figure 4.4. Frequency dependent pressure radiation in dB from a spherical cap on a sphere as calculated by Eq. 4.3 for a sinusoidal input.

However, the limiting case of this equation where \( \theta \ll 1 \) acts like a point-source on a spherical baffle. Clearly this is not an accurate representation of the radiating cap on a sphere, but the simplifications allow a decent estimation to be made where the far field pressure is

\[
P(R, \theta) \approx \frac{\rho c Q_s}{4\pi r_0^2} \left[ j(kr_0)^2 + 3j \cos(\theta) \left( \frac{j(kr_0)^3}{2} \right) \right] \frac{e^{jkR}}{kR},
\]

and, using the assumption also that \( kr_0 \ll 1 \), can be simplified to

\[
P(R) \approx \frac{j\rho c kQ_s}{4\pi} \frac{e^{jkR}}{R},
\]

(4.4)
Comparing to Eq. 4.3, the difference in amplitude amounts simply to a factor of two between the piston and point source pressure radiation. Neither serves as an exact result, so the point source on a sphere was selected. For acoustically small openings, which many of the applications of this model are, the point source approximation can do well enough for a quick, first approximation by the model.

4.5 Finalized Simscape Model

Putting all these components together, Figure 4.5 shows what the final Simscape model looks like including an acoustical flow source and pressure measurement device to calculate the input impedance. The circuit also includes acoustical "ground" references, a medium-parameters block for air, as well as scopes to identify where the software should tap for pressure and input impedance. This circuit is what is referenced at "HornModel" in the MATLAB script to run a simulation in Appendix C.

Figure 4.5. Completed horn model in Simscape. Model includes an acoustical source and pressure measurement device to drive the conical transmission line and spherical cap radiation impedance.
Chapter 5  |  Results

To quantify accuracy of the model, resonance frequencies of the input impedance will be compared between models using identical horn geometries. Resonance frequencies are defined in this thesis as the locations where a maximum value of input impedance occurs. The frequency range of the peaks considered will extend from 1 to 3000 Hz, or as high as experimental data was provided. The frequency range was selected with musical instruments in mind as it spans well above the cutoff frequency of the horns with available data that will be used for comparison later. Acoustical horns in other applications may include many resonance peaks within that range, and others only a few.

For comprehensive observation, increasingly complex horn bores were simulated with variable values of $N$ in the transmission line. These were simulated with 0.1 Hz spacing, and were compared with similar models as well as experimental results from the published literature. Additionally, MATLAB numerical calculations and their equivalent Simscape model are compared to show consistency within their respective solvers.

Each waveguide was modeled first using the traditional T-model element transmission line with cascaded cylindrical sections and a radiation impedance of a piston at the end of a tube. These results were compared with the model of a conical transmission line and spherical cap radiation impedance. Validation of the proposed model is shown first using a conical waveguide and then an exponentially flaring horn, both shapes which are used often as part of acoustical systems. These results will directly compare the historical and proposed models for convergence.

Lastly, real-world conical and flaring horn shapes are loaded into the model. Instead of comparing with a cylindrical element transmission line for validation, as this was accomplished with the previous two test cases, the results are compared with experimental data available of input impedance and resonance frequencies. Results here will show the model’s accuracy and reliability over a variety of shapes.
The first experimental comparison uses a computer program to calculate the input impedance of a Strasser oboe excluding tone holes and without a reed from Plitnik and Strong [7]. Those familiar with oboe dimensions will realize that the model presents the resonances of a horn with a linear taper, or a conical waveguide. The model used is a cylindrical element transmission line with the radiation impedance of a rigid circular piston mounted in an unflanged pipe - similar to the historical model presented in Chapter 1 with slight variation in the computation and software used. The computer program was able to run for high values of \( N \), so the results should be reliable but still have the same pitfalls as the historical model.

For flaring horn shapes, experimental data is provided by Eveno et. al. [6]. Plots are shown of input impedance for two musical instrument horns. The first two resonances of input impedance for a trombone, and first five for a trumpet, excluding holes and mouthpieces, are provided numerically alongside the plots. Unfortunately, since publication the data files have been corrupted, so we could only access what was shown in the paper.

Unfortunately, most other available experimental data is unusable in this application because the horn is modeled and measured as part of a larger system. The model from this paper intends to isolate the flaring horn from the rest of an instrument or loudspeaker. While a Simscape model could be created to include these other changes, the scope and timeline of this work didn’t allow for that inclusion. Additionally, this could introduce excess error from simplifications of a very complicated system that could mislead any takeaways from the results.

Accuracy is measured by percent error to the experimental/alternative data from these sources listed. Individual resonances are compared, as well as is an average from those within the 1-3000 Hz spectrum.

An important note is that the comparison data used is assumed to be accurate and without numerical error or inaccuracies with the capture method. Small imperfections (i.e. dents in the horns or different mic placement) can have a significant impact on the acoustical response. Because there was not time enough for an experimental set-up to be tested, but experimental data was desired for comparison, these possible imperfections are assumed to be negligible and the data is taken at face value.
5.1 Acoustical Waveguide Test-Cases

Results are shown in Tables 5.1 through 5.8, and are compared to experimental data from previous literature, when available. Each shape and its results are discussed in some detail below. In addition to the numerical results, the calculated impedance spectrum is also provided for every test case. These aim to provide some reference of how the magnitudes of the horns compare, how the peaks decay at higher frequencies, and most importantly to show where the numerical data came from.

5.1.1 Conical

The first test shape, a cone (linear horn) is plotted in Figure 5.1. Figure 5.2 shows the calculated impedance spectrum. Creating a conical transmission line of conical segments is akin to making a cylindrical transmission line to model a cylinder. The outer dimensions of the differential elements will match the radius exactly at all points, so \( N \) will simply change the degrees of freedom, the accuracy of the model, and will extend the accuracy to higher frequencies.

The resonance frequencies for the cone are listed in Table 5.1. Comparing a conical transmission line to a cylindrical, however, the improvements in efficiency aren’t clear when modeling a cone. In fact, the cylindrical transmission line achieves convergence with far fewer elements than the conical, as seen in Table 5.1. Especially at lower frequencies, while the conical transmission line requires around \( N = 50 \), the cylindrical needs closer to \( N = 30 \) and then the frequencies remain within a single Hz of each other. Above \( f_3 \), the results are very similar between the two models though, where up to \( N = 150 \) both are still reaching their steady-state approximate value.

An important note of these, and all results shown for comparison between the two models, is that their steady-state values are not expected to be identical. This is because even though they’re intended to model the same physical waveguide, the radiation impedance treats the end like a different physical shape - in one case a flat disk and the other a curved, spherical cap. So, the solutions, and resulting resonance frequencies, are actually for two different, although very similar, problems.
Figure 5.1. Geometry of a circular horn with a linear change in radius, or cone.

Figure 5.2. Input impedance of the waveguide in Figure 5.1 modeled in Simscape with a conical transmission line and spherical cap radiation impedance. Resonance peaks found in Table 5.1.
Table 5.1. Resonance frequencies of a cone using with a conical/spherical model (top) vs a cylindrical/piston model (bottom) for increasing $N$.

<table>
<thead>
<tr>
<th>N</th>
<th>$f_1$</th>
<th>$f_2$</th>
<th>$f_3$</th>
<th>$f_4$</th>
<th>$f_5$</th>
<th>$f_6$</th>
<th>$f_7$</th>
<th>$f_8$</th>
<th>$f_9$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>241.4</td>
<td>455.3</td>
<td>685.3</td>
<td>926.9</td>
<td>1153.9</td>
<td>1350.3</td>
<td>1509.5</td>
<td>1637.9</td>
<td>1738.6</td>
</tr>
<tr>
<td>30</td>
<td>244.6</td>
<td>484.7</td>
<td>736.5</td>
<td>1006.3</td>
<td>1282.2</td>
<td>1556.5</td>
<td>1826.9</td>
<td>2092.5</td>
<td>2352.4</td>
</tr>
<tr>
<td>50</td>
<td>244.7</td>
<td>487.3</td>
<td>743.1</td>
<td>1016.4</td>
<td>1296.5</td>
<td>1577.0</td>
<td>1856.3</td>
<td>2133.8</td>
<td>2409.5</td>
</tr>
<tr>
<td>75</td>
<td>244.7</td>
<td>488.6</td>
<td>745.8</td>
<td>1020.4</td>
<td>1301.9</td>
<td>1584.4</td>
<td>1866.4</td>
<td>2147.7</td>
<td>2428.3</td>
</tr>
<tr>
<td>100</td>
<td>244.6</td>
<td>489.3</td>
<td>747.1</td>
<td>1022.1</td>
<td>1304.2</td>
<td>1587.3</td>
<td>1870.3</td>
<td>2152.9</td>
<td>2435.2</td>
</tr>
<tr>
<td>150</td>
<td>244.6</td>
<td>489.8</td>
<td>748.1</td>
<td>1023.7</td>
<td>1306.1</td>
<td>1589.7</td>
<td>1873.3</td>
<td>2156.9</td>
<td>2440.4</td>
</tr>
</tbody>
</table>

10 250.8 496.2 737.5 965.7 1171.4 1502.1 1621.8 1715.8
30 245.3 494.4 755.7 1027.2 1300.7 1571.8 1839.2 2101.9 2359.3
50 244.4 492.7 755.2 1030.3 1308.9 1587.7 1865.5 2141.7 2416.1
75 243.3 491.7 754.5 1030.3 1310.9 1592.2 1873.2 2153.7 2433.5
100 243.3 491.2 754.0 1030.2 1311.4 1593.5 1875.7 2157.8 2439.5
150 242.7 490.7 753.5 1029.9 1311.5 1594.3 1877.4 2160.5 2443.6

5.1.2 Exponential

The radii along the length of an exponentially flaring horn can be computed easily if the flaring horn is part of the Salmon family of horns [16] where $m$ determines the exact flare shape. The cross-sectional area at each point $x$ along the length of the horn is calculated

$$S(x) = S_T e^{2mx},$$

where $S_T$ is the cross-sectional area of the horn at the throat, or when $x = 0$.

Figure 5.3 shows an exponential horn with $m = 4.5$, and Figure 5.4 shows the calculated impedance spectrum of that horn. Table 5.2 compares the resonance frequencies from the model using a conical transmission line and spherical cap radiation impedance to the cylindrical/piston model. As with the cone, the conical transmission line doesn’t immediately appear to be any more efficient than a cylindrical waveguide. In this case though, across all frequencies the results are much more similar, needing approximately the same $N$ to get within a few percent of their steady-state estimate.
Figure 5.3. Geometry of an exponentially flaring horn as calculated by Eq. 5.1 with $m = 4.5$.

Figure 5.4. Input impedance of the waveguide in Figure 5.3 modeled in Simscape with a conical transmission line and spherical cap radiation impedance. Resonance peaks found in Table 5.2.
Table 5.2. Resonance frequencies of an exponentially flaring horn using with a conical/spherical model (top) vs a cylindrical/piston model (bottom) for increasing $N$.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$f_1$</th>
<th>$f_2$</th>
<th>$f_3$</th>
<th>$f_4$</th>
<th>$f_5$</th>
<th>$f_6$</th>
<th>$f_7$</th>
<th>$f_8$</th>
<th>$f_9$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>282.6</td>
<td>465.3</td>
<td>698.1</td>
<td>937.4</td>
<td>1157.9</td>
<td>1347.3</td>
<td>1503.7</td>
<td>1634.6</td>
<td>1735.5</td>
</tr>
<tr>
<td>30</td>
<td>307.3</td>
<td>494.8</td>
<td>742.5</td>
<td>1012.8</td>
<td>1287</td>
<td>1559.8</td>
<td>1829.3</td>
<td>2094.2</td>
<td>2353.7</td>
</tr>
<tr>
<td>50</td>
<td>312</td>
<td>499.8</td>
<td>748.4</td>
<td>1020.5</td>
<td>1298.5</td>
<td>1577.5</td>
<td>1856</td>
<td>2133.1</td>
<td>2408.6</td>
</tr>
<tr>
<td>75</td>
<td>314.4</td>
<td>502.2</td>
<td>750.9</td>
<td>1023.5</td>
<td>1302.6</td>
<td>1583.4</td>
<td>1864.6</td>
<td>2145.6</td>
<td>2426.1</td>
</tr>
<tr>
<td>100</td>
<td>315.5</td>
<td>503.4</td>
<td>752</td>
<td>1024.7</td>
<td>1304.1</td>
<td>1585.6</td>
<td>1867.8</td>
<td>2150.1</td>
<td>2432.3</td>
</tr>
<tr>
<td>150</td>
<td>316.7</td>
<td>504.5</td>
<td>753.1</td>
<td>1025.8</td>
<td>1305.4</td>
<td>1587.3</td>
<td>1870.1</td>
<td>2153.4</td>
<td>2436.8</td>
</tr>
</tbody>
</table>

While these results don’t clearly show any superiority of the conical over cylindrical, this goal of this work isn’t simply that. It’s also to validate the conical and spherical cap approach as another of multiple legitimate strategies to solve the problem. So, data and experimental results, then, were sourced to see how accurately and efficiently the input impedance and resonance frequencies can be calculated on a real-world system. Standing alone, the conical and exponential results only prove, again, that it’s a viable estimate and that it can accurately reach convergence on par with alternative models.

### 5.2 Musical Instrument Bores

To quantify the results, all instruments discussed below required resonance frequency data along with their physical dimensions. With that information, simulations were run using the conical/spherical model for varying $N$ and the resonance frequencies calculated are compared to what is presented in their respective papers.

43
5.2.1 Oboe

Data for the oboe was taken from Plitnik and Strong [7]. Figure 5.5 shows the dimensions and Figure 5.6 shows the calculated impedance. The results of the oboe are the least surprising of the three instruments because the accuracy of a linear horn, which is all an oboe bore is, was previously confirmed with the cone shown before in Figure 5.1. The "experimental" values in Table 5.3 comes from simulations using a cylindrical transmission line. In that model a differential length $\ell = 0.25 \text{ cm}$ is used. Table 5.3 and Table 5.4 show that the conical transmission line approach achieved accurate results (only 2.7% error on average) with a length of as long as $\ell = 3 \text{ cm}$. At the lower frequencies primarily, the results start to converge very quickly, and adding extra elements continues to improve accuracy within the frequency range.

![Figure 5.5. Geometry of an oboe as defined by Plitnik and Strong [7].](image)
Figure 5.6. Input impedance of the oboe shown in Figure 5.5 modeled in Simscape with a conical transmission line and spherical cap radiation impedance. Peak (resonance) frequencies found in Table 5.3.

Table 5.3. Resonance frequencies of an oboe bore modeled with a conical transmission line and spherical cap radiation impedance for increasing values of $N$. Experimental results from Plitnik [7].

<table>
<thead>
<tr>
<th>$N$</th>
<th>$f_1$</th>
<th>$f_2$</th>
<th>$f_3$</th>
<th>$f_4$</th>
<th>$f_5$</th>
<th>$f_6$</th>
<th>$f_7$</th>
<th>$f_8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>257.4</td>
<td>468</td>
<td>685.6</td>
<td>914</td>
<td>1130.8</td>
<td>1323</td>
<td>1483</td>
<td>1606.1</td>
</tr>
<tr>
<td>30</td>
<td>257.4</td>
<td>505.2</td>
<td>752.9</td>
<td>1003.3</td>
<td>1255.9</td>
<td>1508.9</td>
<td>1760.3</td>
<td>2008.9</td>
</tr>
<tr>
<td>50</td>
<td>256.4</td>
<td>508.1</td>
<td>760.7</td>
<td>1015.9</td>
<td>1273.5</td>
<td>1532.6</td>
<td>1792.4</td>
<td>2052</td>
</tr>
<tr>
<td>75</td>
<td>255.9</td>
<td>509.1</td>
<td>763.8</td>
<td>1020.9</td>
<td>1280.3</td>
<td>1541.6</td>
<td>1804.2</td>
<td>2067.3</td>
</tr>
<tr>
<td>100</td>
<td>255.6</td>
<td>509.6</td>
<td>765.1</td>
<td>1023</td>
<td>1283.2</td>
<td>1545.4</td>
<td>1808.9</td>
<td>2073.4</td>
</tr>
<tr>
<td>150</td>
<td>255.3</td>
<td>510</td>
<td>766.3</td>
<td>1024.9</td>
<td>1285.8</td>
<td>1548.6</td>
<td>1812.9</td>
<td>2078.3</td>
</tr>
<tr>
<td>Experimental</td>
<td>256.0</td>
<td>513.0</td>
<td>772.0</td>
<td>1033.0</td>
<td>1296.0</td>
<td>1561.0</td>
<td>1827.5</td>
<td>2095.0</td>
</tr>
</tbody>
</table>
Table 5.4. Percent error between calculated frequencies and experimental results found in Table 5.3 for the oboe. Average percent error across all resonances for each value of N is provided.

<table>
<thead>
<tr>
<th>N</th>
<th>$f_1$</th>
<th>$f_2$</th>
<th>$f_3$</th>
<th>$f_4$</th>
<th>$f_5$</th>
<th>$f_6$</th>
<th>$f_7$</th>
<th>$f_8$</th>
<th>Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.5%</td>
<td>8.8%</td>
<td>11.2%</td>
<td>11.5%</td>
<td>12.7%</td>
<td>15.2%</td>
<td>18.9%</td>
<td>23.3%</td>
<td>12.8%</td>
</tr>
<tr>
<td>30</td>
<td>0.5%</td>
<td>1.5%</td>
<td>2.5%</td>
<td>2.9%</td>
<td>3.1%</td>
<td>3.3%</td>
<td>3.7%</td>
<td>4.1%</td>
<td>2.7%</td>
</tr>
<tr>
<td>50</td>
<td>0.2%</td>
<td>1.0%</td>
<td>1.5%</td>
<td>1.7%</td>
<td>1.7%</td>
<td>1.8%</td>
<td>1.9%</td>
<td>2.1%</td>
<td>1.5%</td>
</tr>
<tr>
<td>75</td>
<td>0.0%</td>
<td>0.8%</td>
<td>1.1%</td>
<td>1.2%</td>
<td>1.2%</td>
<td>1.2%</td>
<td>1.3%</td>
<td>1.3%</td>
<td>1.0%</td>
</tr>
<tr>
<td>100</td>
<td>0.2%</td>
<td>0.7%</td>
<td>0.9%</td>
<td>1.0%</td>
<td>1.0%</td>
<td>1.0%</td>
<td>1.0%</td>
<td>1.0%</td>
<td>0.8%</td>
</tr>
<tr>
<td>150</td>
<td>0.3%</td>
<td>0.6%</td>
<td>0.7%</td>
<td>0.8%</td>
<td>0.8%</td>
<td>0.8%</td>
<td>0.8%</td>
<td>0.8%</td>
<td>0.7%</td>
</tr>
</tbody>
</table>

5.2.2 Trumpet

The trumpet data comes from Eveno et. al. [6]. The dimensions are plotted in Figure 5.7, and Figure 5.8 shows the calculated impedance spectrum. The paper includes values for the first 5 resonance frequencies, all of which were also modeled in Simscape with results provided in Table 5.5. These are, unlike the computer-modeled oboe results, true experimental data taken from continuous input impedance measurements. As mentioned previously, the data were corrupted since publication, so a software program, DataThief [8], was used to extract the data from figures in the paper, and the curve-fit toolbox in MATLAB helped to align it closely as possible to the correct geometry.

The percent error in Table 5.6 shows that an average of less than 2% accuracy can be achieved with as less than $N = 30$, or $\ell = 2.2$ cm. The trumpet results also have an anomaly at the fourth resonance where the percent error is much higher than the rest, especially the first, second, and fifth. Without having taken the data, the exact source of this divergence is unable to be reliably determined. However, the modeled resonance frequencies are consistent in their divergence, and the error reduces as more transmission line components are added, as is expected.
Figure 5.7. Geometry of a straight trumpet used for experimental analysis by Eveno et. al. [6]. Radii taken using the smallest sample spacing available in DataThief [8].

Figure 5.8. Input impedance of the straight trumpet shown in Figure 5.7 modeled in Simscape with a conical transmission line and spherical cap radiation impedance. Peak (resonance) frequencies found in Table 5.5.
Table 5.5. Resonance frequencies of a straight trumpet bell modeled with a conical transmission line and spherical cap radiation impedance for increasing values of N. Experimental results from an identical, straight trumpet bell from Eveno et. al. [6] are provided for comparison.

<table>
<thead>
<tr>
<th>N</th>
<th>$f_1$</th>
<th>$f_2$</th>
<th>$f_3$</th>
<th>$f_4$</th>
<th>$f_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>205.6</td>
<td>439.1</td>
<td>664.6</td>
<td>984.1</td>
<td>1178.9</td>
</tr>
<tr>
<td>30</td>
<td>202.3</td>
<td>439.8</td>
<td>690.9</td>
<td>942.1</td>
<td>1188.4</td>
</tr>
<tr>
<td>50</td>
<td>202.1</td>
<td>437.3</td>
<td>694.4</td>
<td>946.3</td>
<td>1195.6</td>
</tr>
<tr>
<td>75</td>
<td>201.7</td>
<td>439</td>
<td>694.1</td>
<td>949.1</td>
<td>1196.6</td>
</tr>
<tr>
<td>100</td>
<td>202.1</td>
<td>439</td>
<td>694.8</td>
<td>947.7</td>
<td>1197</td>
</tr>
<tr>
<td>150</td>
<td>203</td>
<td>440.4</td>
<td>694.4</td>
<td>949.5</td>
<td>1197.7</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Trumpet Resonance Frequencies, Hz</th>
</tr>
</thead>
<tbody>
<tr>
<td>Experimental</td>
<td>202.87  443.09  676.53  913.41  1177.20</td>
</tr>
</tbody>
</table>

Table 5.6. Percent error between calculated frequencies and experimental results found in Table 5.5 for the straight trumpet bell. Average percent error across all resonances for each value of N is provided.

<table>
<thead>
<tr>
<th>N</th>
<th>$f_1$</th>
<th>$f_2$</th>
<th>$f_3$</th>
<th>$f_4$</th>
<th>$f_5$</th>
<th>Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>1.3%</td>
<td>0.9%</td>
<td>1.8%</td>
<td>7.7%</td>
<td>0.1%</td>
<td><strong>2.4%</strong></td>
</tr>
<tr>
<td>30</td>
<td>0.3%</td>
<td>0.7%</td>
<td>2.1%</td>
<td>3.1%</td>
<td>1.0%</td>
<td><strong>1.4%</strong></td>
</tr>
<tr>
<td>50</td>
<td>0.4%</td>
<td>1.3%</td>
<td>2.6%</td>
<td>3.6%</td>
<td>1.6%</td>
<td><strong>1.9%</strong></td>
</tr>
<tr>
<td>75</td>
<td>0.6%</td>
<td>0.9%</td>
<td>2.6%</td>
<td>3.9%</td>
<td>1.6%</td>
<td><strong>1.9%</strong></td>
</tr>
<tr>
<td>100</td>
<td>0.4%</td>
<td>0.9%</td>
<td>2.7%</td>
<td>3.8%</td>
<td>1.7%</td>
<td><strong>1.9%</strong></td>
</tr>
<tr>
<td>150</td>
<td>0.1%</td>
<td>0.6%</td>
<td>2.6%</td>
<td>3.4%</td>
<td>0.7%</td>
<td><strong>1.8%</strong></td>
</tr>
</tbody>
</table>
5.2.3 Trombone

The final musical instrument with data available data was a trombone, the data of which came from Eveno et. al. [6], along with the trumpet. Dimensions, also extracted using DataThief, can be seen in Figure 5.9. Figure 5.10 shows the calculated impedance spectrum.

With only the first two resonance frequencies provided for comparison, as shown in Table 5.7 the model and experimental results show a slightly different picture than that of the oboe and trumpet. Compared to the experimental data as percent error, shown in Table 5.8, the results are only just above 1% accurate for $N = 10$. However, as $N$ increases, the accuracy actually reduces slightly and the model overestimates the resonance frequencies. This could be due to imperfect data for frequencies or horn shape, the fact that the model doesn’t include losses, or a similar case as the cone previously (Figure 5.1) where lower frequencies weren’t as accurate as the higher frequencies were.

![Figure 5.9. Geometry of a tenor trombone used for experimental analysis by Eveno et. al [6]. Radii taken using the smallest sample spacing available in DataThief [8].](image-url)
**Figure 5.10.** Input impedance of the oboe shown in Figure 5.9 modeled in Simscape with a conical transmission line and spherical cap radiation impedance. Peak (resonance) frequencies found in Table 5.7.

**Table 5.7.** Resonance frequencies of a tenor trombone bell modeled with a conical transmission line and spherical cap radiation impedance for increasing values of N. Experimental results from an identical, tenor trombone bell from Eveno et. al. [6] are provided for comparison.

<table>
<thead>
<tr>
<th>N</th>
<th>Trombone Resonance Frequencies, Hz</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>$f_1$</td>
<td>243.9</td>
<td>523.1</td>
</tr>
<tr>
<td>30</td>
<td>$f_2$</td>
<td>243.8</td>
<td>531.1</td>
</tr>
<tr>
<td>50</td>
<td></td>
<td>244.6</td>
<td>531.5</td>
</tr>
<tr>
<td>75</td>
<td></td>
<td>246.2</td>
<td>531.4</td>
</tr>
<tr>
<td>100</td>
<td></td>
<td>247.4</td>
<td>531.4</td>
</tr>
<tr>
<td>150</td>
<td></td>
<td>248.9</td>
<td>531.7</td>
</tr>
<tr>
<td></td>
<td>Experimental</td>
<td>241.43</td>
<td>517.22</td>
</tr>
</tbody>
</table>
Table 5.8. Percent error between calculated frequencies and experimental results found in Table 5.7 for the tenor trombone bell. Average percent error across all resonances for each value of N is provided.

<table>
<thead>
<tr>
<th>N</th>
<th>f_1</th>
<th>f_2</th>
<th>Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>1.0%</td>
<td>1.1%</td>
<td>1.1%</td>
</tr>
<tr>
<td>30</td>
<td>1.0%</td>
<td>2.7%</td>
<td>1.8%</td>
</tr>
<tr>
<td>50</td>
<td>1.3%</td>
<td>2.8%</td>
<td>2.0%</td>
</tr>
<tr>
<td>75</td>
<td>2.0%</td>
<td>2.7%</td>
<td>2.4%</td>
</tr>
<tr>
<td>100</td>
<td>2.5%</td>
<td>2.7%</td>
<td>2.6%</td>
</tr>
<tr>
<td>150</td>
<td>3.1%</td>
<td>2.8%</td>
<td>2.9%</td>
</tr>
</tbody>
</table>

5.3 Analytical and Simscape Comparison

To see if there could be any error due to solver limitations or rounding in the Simscape environment, the trumpet bore, as seen in Figure 5.7, was modeled using MATLAB matrix multiplication with the transfer matrices from Chapter 1. The trumpet was selected arbitrarily, and any of the test cases could have been used as the comparison is between the solvers, and has nothing to do with the actual data or results themselves. The results in Table 5.9 show the first 5 resonance frequencies for the same values of N as see in Table 5.5. Assuming that the experimental values are accurate, the percent error in Table 5.10 shows not only that these return nearly identical results, but for lower values of N the percent error is smaller from Simscape than the matrix version from MATLAB. Still as N increases, both solvers converge to very similar values. This same trend can be seen using other horn shapes as well; however, because the results are not the same for otherwise identical components and damping, there are clearly different processes that could lead to larger issues in other simulations. In this case the differences caused the percent error to be smaller, but just as easily the solver could cause an issue with the data and the frequencies could be farther from the experimental data, and one another. Within the context of this work, though, the Simscape solver is up to the task to at least be able to return estimates on par with that of MATLAB and within a few percent error of experimental results.
Table 5.9. Resonance frequencies of a straight trumpet bell modeled numerically with ABDC matrices from Equation 1.6. Model still uses a conical transmission line and spherical cap radiation impedance, but without the Simscape solver.

<table>
<thead>
<tr>
<th>N</th>
<th>f₁</th>
<th>f₂</th>
<th>f₃</th>
<th>f₄</th>
<th>f₅</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>208</td>
<td>453.6</td>
<td>717.6</td>
<td>951.9</td>
<td>1164.6</td>
</tr>
<tr>
<td>30</td>
<td>202.8</td>
<td>443</td>
<td>698.2</td>
<td>957.5</td>
<td>1208.4</td>
</tr>
<tr>
<td>50</td>
<td>202.4</td>
<td>438.9</td>
<td>698</td>
<td>953.4</td>
<td>1205.5</td>
</tr>
<tr>
<td>75</td>
<td>201.8</td>
<td>439.7</td>
<td>695.6</td>
<td>952.4</td>
<td>1201.4</td>
</tr>
<tr>
<td>100</td>
<td>202.1</td>
<td>439.4</td>
<td>695.7</td>
<td>949.5</td>
<td>1199.5</td>
</tr>
<tr>
<td>150</td>
<td>203</td>
<td>440.2</td>
<td>694.1</td>
<td>949.1</td>
<td>1197.5</td>
</tr>
<tr>
<td>Experimental</td>
<td>202.87</td>
<td>443.09</td>
<td>676.53</td>
<td>913.41</td>
<td>1177.2</td>
</tr>
</tbody>
</table>

Table 5.10. Percent error between calculated frequencies and experimental results found in Table 5.9 for the straight trumpet bell using the MATLAB solver, compared to Simscape. Average percent error across all resonances for each value of N is provided.

<table>
<thead>
<tr>
<th>N</th>
<th>f₁</th>
<th>f₂</th>
<th>f₃</th>
<th>f₄</th>
<th>f₅</th>
<th>Mean</th>
<th>Simscape</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>2.5%</td>
<td>2.4%</td>
<td>6.1%</td>
<td>4.2%</td>
<td>1.1%</td>
<td>3.3%</td>
<td>2.4%</td>
</tr>
<tr>
<td>30</td>
<td>0.03%</td>
<td>0.02%</td>
<td>3.2%</td>
<td>4.8%</td>
<td>2.7%</td>
<td>2.1%</td>
<td>1.4%</td>
</tr>
<tr>
<td>50</td>
<td>0.2%</td>
<td>0.9%</td>
<td>3.2%</td>
<td>4.4%</td>
<td>2.4%</td>
<td>2.2%</td>
<td>1.9%</td>
</tr>
<tr>
<td>75</td>
<td>0.5%</td>
<td>0.8%</td>
<td>2.8%</td>
<td>4.3%</td>
<td>2.1%</td>
<td>2.1%</td>
<td>1.9%</td>
</tr>
<tr>
<td>100</td>
<td>0.4%</td>
<td>0.8%</td>
<td>2.8%</td>
<td>4.0%</td>
<td>1.9%</td>
<td>2.0%</td>
<td>1.9%</td>
</tr>
<tr>
<td>150</td>
<td>0.1%</td>
<td>0.7%</td>
<td>2.6%</td>
<td>3.9%</td>
<td>1.7%</td>
<td>1.8%</td>
<td>1.8%</td>
</tr>
</tbody>
</table>

5.4 Including Thermoviscous Damping

A final attempt was made to improve this model by introducing losses into the system. While there is still much unknown about how to model thermoviscous losses in conical waveguides, for cylindrical sections it is well understood and explained by Thompson et. al. [18]. Because the conical circuit builds off of cylindrical elements, a first approximation for a lossy conical transmission line model can be created by applying the work of Thompson et. al. into the Simscape component. Specifically, a lossy tube component is available in the acoustical library which adds additional elements as branches to each component in the T-line. This causes the computation time to increase significantly, so
to get a reasonable estimate, and to prove that insertion into the model works, enough branches were used to get accuracy just above the highest resonance frequency. Lastly, each cylindrical sections within the conical elements were sub-divided into 3 uniform cylinders per component requirements.

Simulation results provided are again for the trumpet shown in Figure 5.7, and the resonance frequencies in Table 5.11 show another legitimate estimate of the experimental results. The physics of this system requires a significant investigation before asserting the accuracy of any of these results, as well as if it’s worth it based on the results from the lossless conical transmission line elements. In addition, increased computational requirements from adding so many more components in the circuit limits how many simulations can be run. Whereas a lossless conical transmission line took minutes to run with $N = 100$, an equivalent model with lossy cones took almost three days.

**Table 5.11.** Resonance frequencies of a straight trumpet bell modeled with a conical transmission line that includes thermoviscous losses. Radiation impedance is that of a spherical cap. Experimental results from an identical, straight trumpet bell from Eveno et. al. [6] are provided for comparison.

<table>
<thead>
<tr>
<th>N</th>
<th>$f_1$</th>
<th>$f_2$</th>
<th>$f_3$</th>
<th>$f_4$</th>
<th>$f_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>205.3</td>
<td>442.5</td>
<td>685.7</td>
<td>940.5</td>
<td>1172.3</td>
</tr>
<tr>
<td>50</td>
<td>201.1</td>
<td>435.1</td>
<td>692.1</td>
<td>945.2</td>
<td>1196.3</td>
</tr>
<tr>
<td>100</td>
<td>200.5</td>
<td>436.5</td>
<td>691.8</td>
<td>945</td>
<td>1194.6</td>
</tr>
<tr>
<td>Experimental</td>
<td>202.87</td>
<td>443.09</td>
<td>676.53</td>
<td>913.41</td>
<td>1177.20</td>
</tr>
</tbody>
</table>
Chapter 6  |  Conclusion

6.1 Summary

Any model that uses lumped elements to represent a complex, physical system will need to balance computational efficiency with accuracy and reliability. On the outset of this thesis, there was optimism that using conical segments to model a flaring horn would significantly reduce computational requirements while maintaining or improving upon accuracy. This optimism wasn’t entirely unfounded, but there are still limitations that hold back any significant improvements on historical modeling techniques.

A transmission line of conical sections can, in fact, achieve a closer estimation of the physical system with fewer cones than the traditional cylinders. In addition, it correctly accounts for a closer estimation of the propagating wave shape and acoustical mass of each cascaded element. However, the lower value of $N$ is misleading. Yes, the transmission line has fewer cones, but to maintain the same level of accuracy as compared to using a transmission of cylindrical sections only, the conical circuit’s uniform line will require additional cylindrical elements.

This is seen in some of the inconsistency between $N$ and resonance frequency estimation across the waveguide shapes presented in Chapter 5. The conical section showed promise with the long trumpet and oboe, however for the cone and exponential horn, the fewer segments simply wasn’t enough to accurately estimate the lower resonance frequencies. However, once $N$ increased enough both models converged to very similar results. Conical segments can be used, but, at least in Simscape as it stands, don’t improve computational efficiency significantly enough to make them a direct replacement for a cylindrical transmission line.

That said, some of the issues could be due to numerical/computational error. No matter how close the solvers get to one another, their comparison to experimental results
will always fall prey to rounding. The models both require lots of individual calculations which allows the possibility for small errors to compound and significantly diverge the results. In general, the model acted consistently, but wasn’t without an anomaly that points to the possibility of a numerical error in MATLAB, Simscape, or the linear system analyzer.

The radiating element on the other hand, does serve as a marginal improvement over the piston in a baffle and incur no additional computation cost. Again, all credit goes to Hélie and Rodet [9] along with Eveno et. al. [6] for the derivation of this method. Once the underlying code created for this thesis to determine the parameter values can be ported to a Simscape component, it makes much more sense to use for a flaring horn than a flat piston.

A final limitation and source for confusion was in comparison between the MATLAB and Simscape solvers, which proved to be different. It was an assumption from the beginning that proof in MATLAB using text-based code would correlate with an improvement using an identical model in Simscape. In the early tests, this seemed to be the case and was continually checked moving forwards. It wasn’t until a model on the magnitude of hundreds of components was created that the inconsistencies shone through. Sometimes the Simscape solver and linearizer seemed to improve the results, as seen in Table 5.11, but other times the MATLAB solver proved superior. This didn’t prevent the previous conclusions from being drawn as both models could generate accurate results within a few percent error. However, it highlights that with increasingly complex systems there is still more than can be done to improve the workspace in the acoustical domain.

The benefits of these Simscape components, yet, are their ability to adapt to any horn shape and create an equivalent circuit for attachment to an entirely acoustical, or multi-domain model. With some polishing of the far-field pressure calculation, the dynamic transmission line and radiation impedance serve primarily as another approach to modeling acoustical horns, and a platform for further investigation if desired. The conical element pioneered by Benade hopefully will now get more recognition in the literature, and can be the topic of more advanced estimation of flaring horns in any acoustical context.

6.2 Future Work

This thesis came to a conclusion with a few potential leads to follow. The first next steps might be to dive deep into both MATLAB and Simscape solvers to try and understand
their limitations, how acoustical foundational library components interact with one another in the linearization toolbox, and to try and see if there are alternative transmission lines, still made up on conical elements, that might indeed improve computational efficiency.

Direct manipulation of the spherical cap radiation impedance component could see it become independent of an external MATLAB script to calculate and set component values. Unfortunately, and why it wasn’t done within this thesis, it mostly comes down to improvements from MathWorks and inclusion of more functions in the Simulink and Simscape code libraries. In addition, a more accurate estimation of the radiated pressure would improve the time-domain modeling capabilities. As with the optimization for parameters, the calculation is not currently possible within a Simscape block as the most accessible equation relies upon legendre polynomials as well as spherical Bessel and Hankel functions. These equations could hypothetically be re-written as time-domain algebraic equations, and, thanks to the optimized radiation impedance components, the volume velocity (flow variable) now is much more accurate than it was with the piston. Combination of the two, or alternatively if Simscape gains access to the "legendreP" and "besselj" commands in MATLAB, would create an improved time-domain pressure estimation for the component.

Finally, if any of the above steps were followed, the ability to measure the resonances in lab would significantly improve the scope to which the model could be validated. Not only could horns from musical instruments be used where the radius is always increasing, but horns could be measured in the opposite direction where the radius gets smaller along the length - or any other custom waveguide geometry. Doing this could also provide a plethora of data to start work understanding thermoviscous losses’ effect on propagation in cones. The work from Eveno et. al. [6] provides some insight about their set-up, as well as Kausel et. al. [20] lays the groundwork for a promising technique to accurately measure pressure, particle velocity, and the resulting impedance curves.
Appendix A
Numerical Simulation (MATLAB)

A.1 Transfer Matrix Script

```matlab
%% Linear Taper
clearvars; close all;

N = 150;  % number of segments
L = 0.6;  % total length
a1 = 0.01;  % start radius
a2 = 0.11;  % end radius
a_vec = a1:(a2-a1)/N:a2;  % radii for a linear taper
x = 0:(L)/N:L;  % radii for a linear taper

close all; figure(1);
plot(x,a_vec,'r','LineWidth',2);

%% Exponential Taper

clearvars; close all;

N = 30;  % number of segments
L = 0.6;  % total length of tube
dl = L/N;  % length of each sub-section
x = 0:dl:L;  % length of segments
m = 4.5;  % flare constant
a1 = 0.01;  % start radius
Sexp = pi.*a1.^2.*exp(2.*m.*x);  % Surface area of horn
```
a_vec = sqrt(Sexp./pi); %radii of horn

close all; figure(1);
plot(x,a_vec,'r','LineWidth',2);

%%% Oboe Dimensions
clc; clearvars; close all;

N = 150; %number of segments
L = 0.6253; %total length
a1 = 0.0023/2; %start radius
a2 = 0.0364/2; %end radius
a_vec = a1:(a2−a1)/N:a2; %radii for a linear taper
x = 0:(L)/N:L; %radii for a linear taper

close all; figure(1);
plot(x,a_vec,'r','LineWidth',2);

%%% Load Dimensions from a Text File
clc; clearvars; close all;

N = 100;

%Trumpet Data Options
ID1 = 'trumpet_data.txt'; %original data

%Trombone Data Options
% ID1 = 'trombone_data.txt'; %original data

% Take evenly spaced points from the data
data = load(ID1);
ids = ceil(linspace(1,length(data),N+1));
x_orig = data(:,1).';
a_orig = data(:,2).';
x = x_orig(ids);
a_vec = a_orig(ids);
L = x(end)−x(1);
%if using piston
S = pi*a_vec(end)^2;

plot(x_orig,a_orig,'k','LineWidth',1.5); hold on;
plot(x,a_vec,'r:','LineWidth',2);
legend('Raw Data','Simulation Data','FontSize',15);

%% Calculate Horn Impedance
clc; close all;

c = 343; rho = 1.21;

f_start = 1;
f_end = 3000;
df = 0.1;

f = f_start:df:f_end; %frequency vector
w = 2.*pi.*f; %angular frequency
k = w./c; %wave number for all frequencies

%calculate radiation impedance at the end of the mouth
slope = (a_vec(end)-a_vec(end-1))/(x(end)-x(end-1));
r0 = sqrt((a_vec(end)/slope)^2+(a_vec(end))^2); %radius of circle
th0 = atan(a_vec(end)/r0); %half angle of opening

Zrad = zhelieandrodet(f,rho,c,a_vec,r0,th0); %return Zrad
ZinBen = zeros(1,length(w)); %allocation for input impedance

for ii = 1:length(f)
    TT_prev = 1;
    for jj = N:-1:2
ae = a_vec(jj); a0 = a_vec(jj-1);
L_ax = x(jj) - x(jj-1);
L_co = sqrt(L_ax^2+(ae-a0)^2); %on-edge length of cone

x0_ax = a0*L_ax/(ae-a0); %on-axis length from 'origin'
x0 = sqrt(x0_ax^2+a0^2); %length from 'origin'
x0 = L_co + x0; %length from 'origin' to end
a = (ae+a0)/2; %average of the two radii

%uniform line transfer matrix
m_acs = 0.5*rho*L_ax/(pi*a^2); %acoustic mass of slice
c_acs = (pi*a^2)*L_ax/(rho*c^2); %acoustic compliance

Z1 = 1i*w(ii)*m_acs; %inertance term
Z2 = 1/(1i*w(ii)*c_acs); %compliance term

T_uni = [1+Z1/Z2,2*Z1+Z1^2/Z2;1/Z2,1+Z1/Z2];

%calculation of surface area of spherical wave
F = 2*(1-cos(a0/x0))/sin(a0/x0)^2;

%extra inductances and their impedances
M0 = x0*rho/(pi*a0^2*F);
Me = -xe*rho/(pi*ae^2*F);

Zm0 = 1i*w(ii)*M0;
Zme = 1i*w(ii)*Me;

%each inertance gets a transfer matrix is in parallel
T_m0 = [1,0;1/Zm0,1];
T_me = [1,0;1/Zme,1];

%transformer impedance
r = ae/a0;
T_tran = [r,0;0,1/r]; %gets a transformer impedance matrix

%create transfer matrix for the parts of the benade circuit
T_benade = T_m0 * T_uni * T_tran * T_me;

TT = T_benade * TT_prev; % multiply through for all sections
TT_prev = TT; % save for next iteration

end

% Calculate Zin with total T
ZinBen(ii) = ([1,0]*TT*[Zrad(ii);1]) / ([0,1]*TT*[Zrad(ii);1]);

end

% find resonance peaks of impedance
z_mag = 20*log10(abs(ZinBen));
[~, locs] = findpeaks(z_mag);
resonances = f(locs);
disp(['Conical Resonances (Hz): ',num2str(resonances)]);

figure(2);
set(gcf,'position', [300,400,700,600]);
semilogy(f, z_mag, 'r', 'LineWidth', 1.5); hold on;
ylabel('Magnitude [N*s/m]', 'fontsize', 13);
xlabel('Frequency [Hz]', 'fontsize', 13);
title('Input Impedance', 'fontsize', 15);
xlim([f(1), f(end)]); % ylim([120,175]);
legend('Conical TL', 'FontSize', 13, 'Location', 'southeast');

A.2 Radiation Impedances and Cost Function

function Z_HR = zhelieandrodet(f, rho, c, a, r0, th0)
% Impedance Calculation from Helie and Rodet Impedance Model
%
% INPUTS:  f - frequency vector
%          rho - density of medium
%          c - sound speed in medium
%          a - radius of opening
%          r0 - radius of sphere
%          th0 - angle to r0 from horizontal
% OUTPUTS: Z_HR - complex impedance vector

h = r0 - sqrt(r0^2-(a(end))^2); % height of cap
S = 2*pi*r0*h; % surface area of spherical cap
z0_cap = rho*c/S;

nu = r0.*f./c; % unitless f-dependent term, polynomials for theta0
P = [0.8788, 1.0830, -1.2420, 1.1620, -0.6360, 0.1113; ...
0.7200, 0.0799, 0.2210, -0.1440, 0.0207, 0.0000; ...
-0.0220, 4.7040, -0.0794, -0.4240, 0.2607, -0.1980];

% P values are coefficients of a power series for theta0 (th0)
Pa = P(1,1) + P(1,2)*th0 + P(1,3)*th0^2 + ...
P(1,4)*th0^3 + P(1,5)*th0^4 + P(1,6)*th0^5;
Pz = P(2,1) + P(2,2)*th0 + P(2,3)*th0^2 + ...
P(2,4)*th0^3 + P(2,5)*th0^4 + P(2,6)*th0^5;
Pv = P(3,1) + P(3,2)*th0 + P(3,3)*th0^2 + ...
P(3,4)*th0^3 + P(3,5)*th0^4 + P(3,6)*th0^5;

Z_HR = z0_cap*((1i.*nu.*(Pv./Pa)-(nu.*Pv).^2)./...
(1+2.*1i.*nu.*Pz.*Pv-(nu.*Pv).^2));

function Z_B = zberanek(f,ra1,ra2,ca1,ma1)
% Beranek Impedance Structure from "Acoustics" Textbook
%
% INPUTS: f - frequency vector
% ra1 - resistor 1 value
% ra2 - resistor 2 value
% ca1 - capacitor value
% ma1 - inductor value
%
% OUTPUTS: Z_B - complex impedance vector

w = 2*pi*f;
Za = 1./(1./ra1 + 1i.*w.*ca1) + ra2; % impedance of the "branch"
Zb = 1i.*w.*ma1; % impedance of the inductor in parallel
function Cost = zcost (ra1 , ra2 , ca1 , ma1 , f , rho , c , a , r0 , th0)
% Cost Function for Helie and Rodet Parameter Optimization
% INPUTS:  ra1  -  resistor 1 value
%           ra2  -  resistor 2 value
%           ca1  -  capacitor value
%           ma1  -  inductor value
%           f    -  frequency vector
%           rho  -  density of medium
%           c    -  sound speed in medium
%           a    -  radius of opening
%           r0   -  radius of sphere
%           th0  -  angle to r0 from horizontal
% OUTPUTS:  Cost  -  numeric difference between the two curves
zhr = zhelieandrodet (f , rho , c , a , r0 , th0);
zb = zberanek (f , ra1 , ra2 , ca1 , ma1);
Cost = sum (abs ( real (zb) - real (zhr))) + sum (abs ( imag (zb) - imag (zhr))) ;
end
Appendix B
Acoustical Components (Simscape)

B.1 Acoustical Converter

```verbatim
component acoustical_converter < acoustical.four_port

parameters
  R = 1 % Area of rigid piston coupling the domains
end

equations
  p2 == p1/R;
  u2 == -u1*R;
end
```

B.2 Single Conical Element

```verbatim
component (Propagation = blocks) Benade_Cone

nodes
  pp1 = acoustical.acoustical; % + throat
  nn1 = acoustical.acoustical; % - throat
  pp2 = acoustical.acoustical; % + mouth
  nn2 = acoustical.acoustical; % - mouth
end
```
parameters

a0 = {0.01, 'm'}; %throat radius
ae = {0.02, 'm'}; %mouth radius
L = {1, 'm'}; %length of cone
end

parameters( Access=private )

L_co = sqrt(L^2 + (ae - a0)^2); %on-edge length of cone
x0_ax = a0*L/(ae - a0); %on-axis length from "origin"
x0 = sqrt(x0_ax^2 + a0^2); %length from "origin"
x_e = L_co + x0; %length from "origin" to end
a = (ae + a0)/2; %average of the two radii
Macs = 0.5*nn2.rho.*L/(pi*a^2); %inertance
Cacs = (pi*a^2)*L/(nn2.rho*nn2.c^2); %compliance
F = 2*(1 - cos(a0/x0))/sin(a0/x0)^2; %surface area ratio
M0 = x0*nn2.rho/(pi*a0^2+F); %throat inertance
Me = -xe*nn2.rho/(pi*ae^2+F); %mouth inertance
r = ae/a0; %transformer ratio
end

components ( ExternalAccess=none )

Za1 = acoustical.elements.inertance (m = Macs, ... 
  r_m = {0, 'Pa*s/m^3'}, g_m = {0, 'm^3/(Pa*s)'})
Za2 = acoustical.elements.inertance (m = Macs, ... 
  r_m = {0, 'Pa*s/m^3'}, g_m = {0, 'm^3/(Pa*s)'})
Zm0 = acoustical.elements.inertance (m = M0, ... 
  r_m = {0, 'Pa*s/m^3'}, g_m = {0, 'm^3/(Pa*s)'})
Zme = acoustical.elements.inertance (m = Me, ... 
  r_m = {0, 'Pa*s/m^3'}, g_m = {0, 'm^3/(Pa*s)'})
\[ Z_b = \text{acoustical.elements.compliance}(c = C_{acs}, \ldots) \]
\[ r_c = \{0, \ 'Pa\cdot s/m^3'\}, \ g_c = \{0, \ 'm^3/(Pa\cdot s)'\}; \]
\[ RR = \text{acoustical.elements.acoustical_converter}(R = r); \]
end

colors
connections
\[ \text{connect}(pp1,Zm0.pp,Za1.pp); \]
\[ \text{connect}(Za1.nn,Zb.pp,Za2.pp); \]
\[ \text{connect}(Za2.nn,RR.pp1); \]
\[ \text{connect}(nn1,Zm0.nn,Zb.nn,RR.nn1); \]
\[ \text{connect}(RR.pp2,Zme.pp,pp2); \]
\[ \text{connect}(RR.nn2,Zme.nn,nn2); \]
end
end

**B.3 Conical Transmission Line**

\[ \text{component (Propagation = blocks)} \text{ BenadeConeTL} \]

\[ \text{nodes} \]
\[ pp1 = \text{acoustical.acoustical}; \ % + \text{ in} \]
\[ nn1 = \text{acoustical.acoustical}; \ % - \text{ in} \]
\[ pp2 = \text{acoustical.acoustical}; \ % + \text{ out} \]
\[ nn2 = \text{acoustical.acoustical}; \ % - \text{ out} \]
end

\[ \text{annotations} \]
\[ pp1 : \text{ Side} = \text{ left}; \]
\[ nn1 : \text{ Side} = \text{ left}; \]
\[ pp2 : \text{ Side} = \text{ right}; \]
\[ nn2 : \text{ Side} = \text{ right}; \]
end

\[ \%\text{create and set the parameters that the user can define} \]

\[ \text{parameters} \]
\[ a = \{[0,1,2,3,4],\ 'm'\}; \ %\text{vector of radii} \]
L\_tube = \{0.5, 'm'\}; \%total length
end

%calculate the necessary values for the BAEs
parameters(Access=private)
n\_elem = length(a)-1; \%how many cones will need to be created
dl = L\_tube/n\_elem; \%how long the cone segments will be
end

%create n\_elem number of cones , acs reference in between
for ii=1:n\_elem
    components(ExternalAccess=none)
        cones(ii) = acoustical\_elements.Benade\_Cone(a0 = a(ii), ae = a(ii+1), L = dl);
    end
end
components(ExternalAccess=none)
    ref = acoustical\_elements.reference;
end

%connect the beginning and end of the first/last cone
connections
    connect(pp1, cones(1).pp1);
    connect(nn1, cones(1).nn1);
    connect(pp2, cones(end).pp2);
    connect(nn2, cones(end).nn2);
end

%connect the rest of the cones in the middle
for ii = 2:n\_elem \%only runs loop if n\_elem > 1, otherwise from prev
    connections
        connect(cones(ii-1).pp2, cones(ii).pp1);
        connect(cones(ii-1).nn2, cones(ii).nn1, ref.pp);
    end
end
end
Appendix C
Final Model (MATLAB/Simscape)

C.1 Simscape Companion Script

```matlab
%% DIMENSIONS LOADED THE SAME AS IN APPENDIX A

%% Run Horn Impedance Simulation
clc; close all; tic;

f_start = 1;
f_end = 3000;
df = 0.1;

% Lossless
[Z_horn, resonances, respeaks] = HornSim(x,a,f_start,f_end,df);

% Lossy
% [Z_horn, resonances, respeaks] = TVHornSim(x,a,f_start,f_end,df);

figure(2);
plot(f_start:df:f_end,Z_horn,'b','LineWidth',2);
xlabel('Frequency [Hz]', 'fontsize',13); ylabel('Magnitude [dB]');
title('Horn Input Impedance', 'fontsize',15); hold on;
xlim([f_start,f_end]); ylim([min(Z_horn)-10,max(Z_horn)+10]);
for ii = 1:length(resonances)
```

68
plot (resonances (ii), respeaks (ii), 'ro', 'LineWidth', 2);
end

title (titleID, 'FontSize', 15); ylim ([ymin, ymax]);
disp ([ 'Resonances (Hz): ', num2str (resonances)]); toc;

C.2 Simscape Referencing Function

function [Z_horn, fResLocs, fResPks] = HornSim(x, a, fstart, fend, df)
% INPUTS: x − horn section lengths for each radius
% a − radii corresponding to each distance
% fstart − lower bound for linearization
% fend − upper bound for linearization
% df − frequency resolution
%
% OUTPUTS: Z_horn − impedance of the horn within specified range
% fRes − resonance frequencies (peak impedances)
%
% USAGE: [Z_horn, f_resonances] = HornSim(x, a, 1, 5000, 0.1);

warning off;
c = 343; rho = 1.21;

%% Set Horn Dimensions

slope = (a(end) − a(end−1))/(x(end)−x(end−1));
r0 = sqrt ((a(end)/slope)²+(a(end))²);

th0 = atan (a(end)/r0);

%For pressure calculation
h = r0−sqrt (r0²−(a(end))²); %height of cap
S_cap = 2*pi*r0*h; %surface area of spherical cap
assignin ('base', 'S_cap', S_cap);
assignin ('base', 'r0', r0);
assignin ('base', 'th0', th0);
%Set Parameters for the Simscape Model (HRZradTest)
a_horn = a;
L_horn = x(end);

%% Optimization of Beranek Parameters
iter_num = 1E3;
f_opt = 0:0.1:5000; %arbitrary frequency range to optimize over
guess = [0.1404*rho*c/(end)^2,rho*c/(pi*a(end)^2),5.94*a(end)^3/(rho*c^2),8*rho/(3*pi^2*a(end))];
options = optimset( 'MaxFunEvals' ,iter_num) ; %number of iterations
%find the minimum value of the cost function
BeranekVals = fminsearch(@(v) zcost(v(1),v(2),v(3),v(4),f_opt,rho,c,a,r0,th0),guess,options);

Ra1 = BeranekVals(1);
Ra2 = BeranekVals(2);
Ca1 = BeranekVals(3);
Ma1 = BeranekVals(4);

%Need to assign all to base workspace so Simscape can access them
assignin( 'base' , 'a_horn' ,a_horn);
assignin( 'base' , 'L_horn' ,L_horn);
assignin( 'base' , 'Ra1' ,Ra1);
assignin( 'base' , 'Ra2' ,Ra2);
assignin( 'base' , 'Ca1' ,Ca1);
assignin( 'base' , 'Ma1' ,Ma1);

%% Specify the model name
model = 'HornModel';

%% Specify the analysis I/Os and get the analysis I/Os from the model
io = getlinio(model);
%% Specify the operating point
op = operpoint(model);

%% Linearize the model
sys = linearize(model, io, op);

%% Defining the ABCD Matrix
A = sys.A; B = sys.B; C = sys.C; D = sys.D;

%initialize the input sweep to use
ff = fstart : df : fend;
ww = 2.*pi.*ff;

%using a one input, one output system
u = 1; %input is a unit amplitude sine wave
[dim1, dim2] = size(B);
x = zeros(dim1, dim2, length(ww));
y = zeros(length(ww), 1); %output will get a single for every input w

for ii = 1:length(ww)
    I = eye(size(A,1)); %create identity matrix that matches A
    %state variables will match dimensions of A
    x(:,:,ii) = (1i*ww(ii)*I-A)
    y(ii) = C*x(:,:,ii) + D*u;
end

Z_horn = 10*log10(abs(y));
[pks, locs] = findpeaks(Z_horn);
fResLocs = ff(locs); fResPks = pks;
warning on;
end
Bibliography


