The Pennsylvania State University
The Graduate School
Department of Curriculum and Instruction

CHARACTERIZING THE DEVELOPMENT OF A SCHEMA FOR REPRESENTING
AND SOLVING ALGEBRA WORD PROBLEMS BY PRE-ALGEBRAIC STUDENTS
ENGAGED IN A STRUCTURED DIAGRAMMATIC ENVIRONMENT

A Dissertation in
Curriculum and Instruction

by

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Submitted in Partial Fulfillment
of the Requirements
for the Degree of

Doctor of Philosophy

December 2009
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ABSTRACT

In recent years, the learning of algebra by all students has become a significant national priority (Moses & Cobb, 2001; National Council of Teachers of Mathematics, 2000). Algebra is considered to be a foundational topic in mathematics (Usiskin, 1988) and some have argued that an understanding of algebra is fundamental to success in today’s technological society (Moses & Cobb 2001; Nathan & Koellner, 2007).

Whereas the National Council of Teachers of Mathematics [NCTM] (2000) advocates that students should “develop their skill in…solving linear equations in the middle grades” (p. 39), many middle school and high school algebra students have been reported to demonstrate a lack of understanding when solving equations. Students have also been found to experience difficulty when attempting to represent and solve algebra word problems. The purpose of this study is to explore the nature of the algebraic understandings developed by students who learned to solve algebra word problems within the context of a structured diagrammatic environment. Three sixth grade students who had never before taken algebra took part in a twelve-week teaching experiment in which they learned to represent and solve algebra word problems.

Students were given tasks which were similar in structure to those utilized by the Singaporean model method. However, the sequencing of the tasks used in this study differed from that of the Singaporean curriculum in that, immediately following their use of diagrams to model and solve algebra word problems, students in this study were introduced to the use of letters and numbers to accomplish the same.

The tenets of Realistic Mathematics Education’s theory of lesson design were employed in choosing the scope and sequencing of the tasks. Using APOS theory (Dubinsky, 1991), Simon
and colleagues’ (2004) activity-effect theory, and Gray and Tall’s (2007) procept theory, the nature and development of students’ algebraic understandings were examined. It was found that students developed four overarching understandings. Three of these understandings are prominent in the literature as being prerequisite to a students’ algebraic development. Underlying and supporting students’ development of these understandings were the following conceptions: an additive quantity can be treated as a process and an object (a procept), a quantitative whole is decomposable, and the parts of a quantitative whole are commutative.
TABLE OF CONTENTS

LIST OF FIGURES .......................................................................................................................................................... vii

LIST OF TABLES ............................................................................................................................................................. xiv

ACKNOWLEDGEMENTS .................................................................................................................................................... xv

Chapter 1  Statement of the Problem ............................................................................................................................ 1

Importance of the study of algebra ............................................................................................................................. 1
Importance of this study .................................................................................................................................................. 2

Chapter 2  Review of the Literature ............................................................................................................................. 5

Theoretical Framework .................................................................................................................................................. 5
The nature of understanding ....................................................................................................................................... 5
Reflective abstraction ..................................................................................................................................................... 7
Lesson Design ............................................................................................................................................................... 15
Ways of thinking and knowing ................................................................................................................................... 21
Student’s cognitive difficulties with algebra .............................................................................................................. 25
Students’ encountered difficulties with the meaning and use of letters in algebra .................................................. 25
Students’ encountered difficulties with the construction and interpretation of algebraic expressions ........... 33
Students’ encountered difficulties with the interpretation and solution of algebraic equations ......................... 36
Usefulness of Informal Problem Conditions ........................................................................................................... 42
Students’ use of informal problem-solving methods ............................................................................................. 42
Success of informal problem-solving environments ............................................................................................... 46
Diagram drawing as a promising heuristic in the development of algebraic reasoning ........................................... 51
Students’ successful use of diagrams to model the mathematical structure of an algebraic problem-solving situation ........................................................................................................................................ 53
Students’ reading of diagrams as a part of the problem-solving process ................................................................ 61
Usefulness of diagrams in enabling students to perceive implicit quantitative relationships ................................ 63

Chapter 3  Methodology ............................................................................................................................................... 66

Teaching Experiment .................................................................................................................................................... 66
Setting and Participants ............................................................................................................................................... 68
Setting ......................................................................................................................................................................... 69
Participants ................................................................................................................................................................. 70
Pre-requisite understandings ..................................................................................................................................... 71
Pre-Assessment ............................................................................................................................................................ 71
Assessment of prerequisite understandings ............................................................................................................ 71
Assessment of potential understandings ................................................................................................................ 80
Selection criteria for the teaching experiment .......................................................................................................... 88
Teaching Sessions ....................................................................................................................................................... 89
LIST OF FIGURES

Figure 2-1: Figures from two problems used to evaluate students’ conceptualization of letters in algebra in Küchemann’s study (1981, p. 102) .......................................................................................... 31

Figure 2-2: Comparing and exchanging task given to sixth grade students in Comparing Quantities unit (van Reeuwijk, 2001, p. 614). ........................................................................................................... 45

Figure 2-3: A diagram given to students representing the amount of candy in a candy shop at the beginning and end of the day (Gregg & Yackel, 2002, p. 494) ......................................................................................... 47

Figure 2-4: Students’ actions upon a diagram to determine how much candy had been made during the day (Gregg & Yackel, 2002, p. 494) ................................................................................................. 48

Figure 2-5: Screen capture of pictorial phase of EQUATION program (Aczel, 1998b, p. 143)...49

Figure 2-6: Screen capture of EQUATION program (Aczel, 1998b, p. 144) .................................................................................................................. 49

Figure 2-7: Three students’ representations of a situation in which “grain a” was poured into three containers (Mikulina, 1991/1969, p. 197)................................................................................................................... 54

Figure 2-8: A student’s diagram of a situation given in a verbal part-whole problem (Mikulina, 1991/1969, p. 200) .................................................................................................................. 54

Figure 2-9: A diagram given to students from which they generated the equations below it (Mikulina, 1991/1969, p. 209) .................................................................................................................. 55

Figure 2-10: A diagram similar to that which was drawn by a student to model an algebra word problem (Bodanskii, 1991/1969, p. 302) ........................................................................................................ 56

Figure 2-11: “Multiplication and division models for arithmetic word problems (at left) and algebraic word problems (at right)” (Ng & Lee, 2009, p. 289) ................................................................. 57

Figure 2-12: “Comparison models: arithmetic model (at left) and algebraic model (at right)” (Ng & Lee, 2009, p. 287) .................................................................................................................. 58

Figure 2-13 “Solutions to the Animal Problem: Use of the mass of the dog as generator (at left) and use of the mass of the cow as generator (at right)” (Ng & Lee, 2009, p. 301).............................. 59

Figure 2-14: A diagram given to students from which they generated the equations below it (Mikulina, 1991/1969, p. 209) .................................................................................................................. 61

Figure 2-15: The type of diagram drawn by 2nd and 3rd grades student to represent an algebra word problem (Bodanskii, 1991/1969, p. 305) ...................................................................................... 62

Figure 2-16: Diagram from which a student wrote a part-whole word problem (Mikulina (1991/1969, p. 216) ...................................................................................................................... 64
Figure 3-1: List of prerequisite understandings needed for participation in the teaching experiment ............................................................................................................................................................................................ 71

Figure 3-2: Possible representation of the Quantitative Relationships in Pre-Assessment Item 14 ......................................................................................................................................................................................................................................................... 87

Figure 4-1: Kelly’s Response to Item 4 of the Pre-Assessment................................................. 103

Figure 4-2: Kelly’s response to Harper Middle School Task #1 on 4/15/2008 ......................... 106

Figure 4-3: Kelly’s Response to Harper Middle School Task #3 on 4/15/2008 ....................... 109

Figure 4-4: Kelly’s Response to Harper Middle School Task #4 on 4/15/2008 ....................... 110

Figure 4-5: Kelly’s Response to the Beginning of Harper Middle School Task #1 on 4/17/2008 ......................................................................................................................................................................................... 114

Figure 4-6: Kelly’s Response to Harper Middle School Task #2a on 4/17/2008 .................... 115

Figure 4-7: Kelly’s Response to Beginning of Harper Middle School Task #3 on 4/17/2008 .... 117

Figure 4-8: Denise’s Response to Item 7 on the Pre-Assessment............................................ 118

Figure 4-9: Denise’s representation of six times as long.......................................................... 119

Figure 4-10: Harper Middle School Task #3 on 4/15/2008 .................................................. 120

Figure 4-11: Denise’s Response to Harper Middle School Task #3 on 4/15/2008 ................. 121

Figure 4-12: Denise’s Response to Harper Middle School Task #5 on 4/15/2008................. 123

Figure 4-13: Denise’s Response to Harper Middle School Task #6 on 4/15/2008............... 125

Figure 4-14: Valerie’s Response to Item 6 on the Pre-Assessment........................................ 127

Figure 4-15: Paper Strip Task #1 on 1/10/2008......................................................................... 128

Figure 4-16: Valerie’s Response to Paper Strip Task #1 on 1/10/2008................................... 128

Figure 4-17: Judith-Jamie Pennies Task on 2/19/2008............................................................ 130

Figure 4-18: Valerie’s and Alexis’s Response to the Judith-Jamie Pennies Task on 2/19/2008. 131

Figure 4-19: Valerie’s and Alexis’s Response to the Tripling of Judith’s Pennies Task on 2/19/2008 ............................................................................................................................................................................................ 133

Figure 4-20: Valerie’s Response to the Doubling of Frank’s Nickels Task on 2/19/2008 ....... 134

Figure 4-21: Valerie’s Response to the Tripling of Frank’s Pizzas Task on 2/19/2008 .......... 135
Figure 4-22: Valerie’s Response to the Harper Middle School Task #1 on 4/4/2008 .......... 137
Figure 4-23: Valerie’s Response to the Harper Middle School Task #2 on 4/4/2008 .......... 138
Figure 4-24: First Grade Students’ Diagrammatic Representation of the Amount of Grain Poured Evenly into Three Jars (Mikulina, 1991/1969, p. 197) ................................................................. 142
Figure 4-25: Kelly’s Response to Paper Strip Task #1 .............................................. 144
Figure 4-26: Kelly’s answer to pre-assessment Item 3 .............................................. 145
Figure 4-27: Kelly’s Response to Item 6 on the Pre-Assessment ............................. 146
Figure 4-28: Kelly’s Notational Response to Item 10 on the Pre-Assessment .......... 146
Figure 4-29: Kelly’s Response to Paper Strip Task 2 on 4/28/2008 ......................... 148
Figure 4-30: Kelly’s Response to Paper Strip Task 3 on 4/28/2008 ......................... 149
Figure 4-31: Kelly’s answer to Task 2a on 5/1/2008 .............................................. 151
Figure 4-32: Kelly’s answer to Task 3 on 5/1/2008 .............................................. 153
Figure 4-33: Kelly’s answer to Task 5 on 5/1/2008 .............................................. 155
Figure 4-34: Denise’s Response to Item 3 on the Pre-Assessment ......................... 160
Figure 4-35: Denise’s Response to Item 4 on the Pre-Assessment ......................... 160
Figure 4-36: Denise’s Response to Item 6 on the Pre-Assessment ......................... 161
Figure 4-37: Denise’s Response to Item 10 on the Pre-Assessment ....................... 162
Figure 4-38: Denise’s Example of an Equation ..................................................... 164
Figure 4-39: Denise’s Response to Paper Strip Task #3 (excluding the words in italics) ........ 167
Figure 4-40: Denise’s Response to Paper Strip Task #4 on 4/28/2008 ...................... 169
Figure 4-41: Denise’s Responses to Paper Strip Tasks 5c and 5d ......................... 169
Figure 4-42: Denise’s Responses to Paper Strip Tasks 7a and 7b ......................... 170
Figure 4-43: Valerie’s Response to Item 3 on the Pre-Assessment ......................... 172
Figure 4-44: Valerie’s Response to Item 4 on the Pre-Assessment ......................... 172
Figure 4-45: Valerie’s Response to Item 5 on the Pre-Assessment ......................... 172
Figure 4-46: Valerie’s Response to Item 9 on the Pre-Assessment ......................... 173
Figure 4-47: Valerie’s Response to Pre-Assessment Item 12

Figure 4-48: Valerie’s Response (bottom right) to the Candy Bar Task on 1/22/2008

Figure 4-49: Calculator with $a$ and $b$ “buttons”

Figure 4-50: Alexis and Valerie’s Responses to Candy Bar Task #3—near middle of teaching episode

Figure 4-51: Alexis and Valerie’s Responses to Candy Bar Task #3—end of teaching episode

Figure 4-52: Example of a Harper Middle School Type of Task

Figure 4-53: Sample representation of the relationship among quantities in the Harper Middle School sample task

Figure 4-54: Example of a Double Sided Unknown Quantity Task

Figure 4-55: Sample representation of the relationship among quantities in the Double Sided Unknown Quantity sample task

Figure 4-56: Kelly’s Response to Harper Middle School Task #5 on 4/15/2008

Figure 4-57: Kelly’s Response to Double Sided Unknown Quantity Task #2 on 4/17/2008

Figure 4-58: Kelly’s Response to Double Sided Unknown Quantity Task #2a on 4/17/2008

Figure 4-59: Kelly’s Response to Paper Strip Logic Puzzle #2 on 5/1/2008

Figure 4-60: Kelly’s Response to Paper Strip Puzzle #5 on 5/2/2008

Figure 4-61: Kelly’s Initial Response to Double Sided Unknown Quantity Task #1 on 5/5/2008

Figure 4-62: Kelly’s Response to Paper Strip Puzzle #4 on 5/5/2008

Figure 4-63: Kelly’s Final Response to Double Sided Unknown Quantity Task #1 on 5/5/2008

Figure 4-64: Denise’s Response to Harper Middle School Task #5 on 4/15/2008

Figure 4-65: Denise’s Response to Double Sided Unknown Quantity Task #1 on 4/18/2008

Figure 4-66: Denise’s Response to Double Sided Unknown Quantity Task #2 on 4/18/2008

Figure 4-67: Denise’s Response to Double Sided Unknown Quantity Task #2 on 5/5/2008

Figure 4-68: Valerie’s Response to Paper Strip Task #1 on 2/12/2008
Figure 4-69: Valerie’s Response to Harper Middle School Task #5 on 4/4/2008 .......................... 231
Figure 4-70: Valerie’s Response to Double Sided Unknown Quantity Task #1 on 5/6/2008 ...... 233
Figure 4-71: Harold-Jenna Task (adapted from Bodanskii, 1991/1969) ................................. 239
Figure 4-72: Kelly’s response to item 16 on the pre-assessment ........................................... 240
Figure 4-73: Kelly’s response to the Jenna-Harold Task on 5/6/2008 ................................... 241
Figure 4-74: Kelly’s response to paper Strip Task #2 on 5/6/2008 ...................................... 243
Figure 4-75: Kelly’s response to paper Strip Task #3 on 5/8/2008 ....................................... 247
Figure 4-76: Kelly’s response to paper Strip Task #3 on 5/13/2008 ..................................... 251
Figure 4-77: Kelly’s response to paper Strip Task #1 on 5/14/2008 ...................................... 254
Figure 4-78: Moveable Pieces Task #1—5/29/2008 ................................................................. 257
Figure 4-79: Kelly’s Response to Paper Strip Task #1a—5/29/2008 ....................................... 259
Figure 4-80: Kelly’s response to Hypothetical Equivalence Task #2—5/30/2008............... 263
Figure 4-81: Kelly’s response to Hypothetical Equivalence Task #3—5/30/2008............... 265
Figure 4-82: Denise’s Response to Item 16 on the Pre-Assessment ....................................... 268
Figure 4-83: Part 1 of Denise’s response to the Jenna-Harold Task on 4/21/2008 ................. 269
Figure 4-84: Part 3 of Denise’s response to the Jenna-Harold Task on 4/21/2008 .......... 270
Figure 4-85: Denise’s response to Paper Strip Task #7 on 5/2/2008 .................................... 280
Figure 4-86: Denise’s response to Paper Strip Task #1 on 5/6/2008 .................................. 281
Figure 4-87: Denise’s response to Paper Strip Task #2 on 5/6/2008 .................................. 286
Figure 4-88: Denise’s response to Identical Parts Task #1 on 5/8/2008 ............................ 288
Figure 4-89: Part 1 of Denise’s response to the Jenna-Craig Task on 5/12/2008 ............... 291
Figure 4-90: Part 2 of Denise’s response to Jenna-Craig Task on 5/12/2008 ...................... 294
Figure 4-91: Part 3 of Denise’s response to the Jenna-Craig Task on 5/12/2008 ............... 297
Figure 4-92: Part 1 of Denise’s response to Jennifer-Jose Task on 5/15/2008 ................. 300
Figure 4-93: Part 2 of Denise’s response to Jennifer-Jose Task on 5/15/2008 ................. 303
Figure 4-94: Part 1 of Denise’s response to the Jenna-Harold Task on 5/15/2008 .................. 306
Figure 4-95: Part 2 of Denise’s response to the Jenna-Harold Task on 5/15/2008 .................. 307
Figure 4-96: Valerie’s Response to Item 10 on the Pre-Assessment ........................................ 310
Figure 4-97: Valerie’s Response to Additive Deficit Task #1 on 5/19/2008 ............................ 311
Figure 4-98: Valerie’s Response to Pre-Drawn Additive Deficit Task #1 on 5/27/2008 ........ 316
Figure 4-99: Valerie’s Response to Pre-Drawn Additive Deficit Task #3 on 5/27/2008 .......... 319
Figure 4-100: Valerie’s Response to Additive Deficit Task #1 on 5/29/2008 .......................... 320
Figure 4-101: Valerie’s Response to Additive Deficit Task #1 on 5/27/2008 .......................... 323
Figure 4-102: Valerie’s Response to Double Sided Unknown Quantity Task #1 on 6/2/2008 .. 326
Figure 4-103: Valerie’s Response to Additive Deficit Task #1 on 5/27/2008 .......................... 328
Figure A-1: Tasks 1 through 3 from the Harper Middle School Task set ............................... 359
Figure A-2: Tasks 4 and 5 from the Harper Middle School Task set ...................................... 360
Figure A-3: Tasks 6 and 7 from the Harper Middle School set .............................................. 362
Figure A-4: Tasks 8 through 10 from the Additive Deficit Task set ...................................... 364
Figure A-5: Tasks 11 and 12 from the Additive Combination Task set .................................. 365
Figure A-6: Tasks 13 and 14 from the Double Sided Unknown Quantity Tasks ................... 366
Figure A-7: Tasks 15 through 22 from the Paper Strip Task set ............................................ 368
Figure A-8: Tasks 23 through 34 from the Paper Strip Puzzles set ........................................ 376
Figure A-9: Tasks 35 through 37 from the Double Sided Unknown Quantity set.................... 378
Figure A-10: Tasks 38 through 43 from the Paper Strip Puzzles set ...................................... 381
Figure A-11: Tasks 44 through 45 from the Non-Adjacent Parts set and 46-48 from the Remaining Parts set ................................................................. 384
Figure A-12: Tasks 49 through 50 from the Moveable Pieces set ........................................... 385
Figure A-13: Tasks 50 and 51 from the Structured Hypothetical Equivalence set ................... 387
Figure A-14: Tasks 52 through 55 from the Additive Quantity as Input set ............................ 388
Figure A-15: Tasks 56-58 from the Hypothetical Equivalence set (partial scaffolding) ........... 390
Figure A-16 Tasks 59 and 60 Equation only tasks ................................................................. 391
Figure A-17 Tasks 61 and 62 from the Additive Deficit Task set ........................................... 391
Figure A-18 Tasks 63 and 64 Non-scaffolding, equation only tasks ....................................... 391
LIST OF TABLES

Table 3-1: Pre-assessment items with prerequisite understandings being assessed.......................... 72

Table 3-2: List of understandings intended for students to develop during the teaching experiment .............................................................................................................................. 80

Table 3-3: Pre-assessment items with the potential understandings being assessed....................... 81
ACKNOWLEDGEMENTS

I would like to thank my committee for their continued support throughout the entire dissertation process. Thank you, also, for pushing me to think about “the hard questions.”

I would like to thank my dissertation advisor, Dr. Rose Mary Zbiek, for her ongoing guidance and tireless efforts on my behalf. Thank you so much for the countless hours which you have spent reading my work, providing vital editorial feedback, and helping me to develop as a scholar.

I wish to thank my committee chair, Dr. Kathleen Heid, for her ongoing support of me as a graduate student and for her unwavering belief in my ability to make a contribution within the field of urban education as a mathematics educator.

I wish to thank my father, William Green, for passing to me a love of knowledge and of scholarship. Thank you, Dad, for continually expressing your belief in my ability to achieve this goal.

I wish to thank my mother, Liller Green, for being my mentor and my friend. Thank you, Mom, for providing food, laughter, encouragement, and a listening ear throughout this entire process.

I wish to thank my sister, Pamela Coleman, for desperately needing me to finish this process so that she could “have closure.” Thank you, Pam, for continually cheering me on.

Most of all, I wish to thank my Lord and Savior Jesus Christ for giving me the courage and the strength that I needed to accomplish this goal. I love you, Lord.
Chapter 1

Statement of the Problem

This study is a teaching experiment that explores the algebraic understandings constructed by pre-algebraic sixth grade students who learned to model and solve algebra word problems through the use of diagrams having a particular structure. The transition of these students to the use of algebraic symbols in order to model and solve algebra word problems is also explored.

This study is motivated by the findings of multiple researchers that both pre-algebra and algebra students possess misconceptions related to the meaning of algebraic symbols, objects, and processes (Clement, 1982; Filloy & Rojano, 1989; Gray & Tall, 2007; Küchemann, 1981; Thompson, 1995). In light of the fact that an understanding of algebra is considered critical to students’ future success in higher mathematics (Usiskin, 1988) and that the learning of algebra by all students has become a high national priority (Moses & Cobb, 2001; NCTM, 2000), it seems imperative that an understanding of how students come to construct meaningful algebraic understandings be explored.

Importance of the study of algebra

The National Council of Teachers of Mathematics [NCTM] (2000) advocates that students should “develop their skill in…solving linear equations in the middle grades” (p. 39). However, some middle school and high school algebra students experience difficulty relating to
reasoning algebraically (Clement, 1982; Herscovics & Linchevski, 1994). Some students who have taken algebra misconstrue the meaning of algebraic symbols and expressions (Davis, 1984; Küchemann, 1981) while others have failed to demonstrate an understanding of the invariance of a solution across equivalent equations (Steinberg, Sleeman, & Ktorza, 1990). Still others have been found to lack an understanding of the underlying quantitative structure of an equation (Thompson, 1994). Given these results, it seems that students who have had some algebra instruction do not necessarily perform equation solving with understanding.

NCTM (2000) also recommended:

Students’ facility with symbol manipulation can be enhanced if it is based on extensive experience with quantities in contexts through which students develop an initial understanding of the meanings and uses of variables and an ability to associate symbolic expressions with problem contexts. (p. 227)

Since students have been found to encounter difficulties with regards to misinterpreting the meaning of algebraic symbols, it seems that a form of instruction which enables students to engage in algebraic reasoning without the initial use of such symbols could be advantageous. Several researchers (Aczel, 1998a; Bodanskii, 1991/1969; Gregg & Yackel, 2002) have found that teaching students to represent and solve realistic problem situations which involve an unknown quantity using pictorial representations has been helpful in enabling these students to represent and solve such problems using algebraic symbols.

**Importance of this study**

Whereas students’ actions upon objects within a structured diagrammatic environment seems to hold great potential for supporting students’ construction of algebraic understandings, very few studies have detailed the means by which such understandings in these environments may be constructed. It became my goal, then, to characterize such a development.
Based upon the successful results of the aforementioned diagram studies and upon the learning theories of, amongst others, Piaget (1971), Simon and colleague’s (Simon, Tzur, Heinz, & Kinzel, 2004) and the Realistic Mathematics Education framework (Gravemeijer, 1997; Streefland, 1993; Treffers, 1991), I began this study with the following conjectures. A student’s construction of algebraic understandings within particular diagrammatic environments seems to be supported by the fact that he or she is able to utilize particular icons that are cognitively accessible to him or her (Piaget, 1971) in order to create a model of a particular algebraic situation (van Reeuwijk, 1995a). The student can then act upon those objects in such a manner as to achieve his or her goal of finding the value of the unknown quantity. As a result, the diagram has the potential to become, for the student, a site for algebraic reasoning (van Reeuwijk, 1995a). That is, that student could notice those mathematically based actions which enabled him or her to achieve that goal and those actions which hindered him or her from that goal (Simon et al., 2004). By reflecting on the relationship between his or her successful actions and the effect of those actions, such a person might abstract a particular understanding (Simon et al., 2004) which he or she might generalize to new, but similarly structured algebraic situations.

Based upon these conjectures and based upon existent diagrammatic tools (Maier, 1997; Singapore Ministry of Education, 2006), students in this study were immersed in an environment in which they were asked to model particular realistic situations using rectangles to represent the unknown quantity. Students were encouraged to draw connecting rectangles of the same length to represent iterations of the unknown quantity and to norm the size of the originally generated rectangle in order to create representations of new quantities that were related to the original unknown quantity. The goal of this study, then, is to characterize the way in which students’ use of this type of diagram to model and solve algebra word problems afforded and informed their construction of particular algebraic understandings.
This study differs from many of the studies that have been previously conducted in this area in that it closely examines the mechanics of students’ constructions of algebraic understandings within a structured diagrammatic environment and it does so through the lens of students’ part-whole understandings. It is hoped that this study contributes to the field’s knowledge of the means by which students might develop algebraic understandings and the means by which students’ pre-instructional understandings might be harnessed in that effort. The research questions for this study are:

a) What is the nature of the algebraic understandings developed by students as a result of their engagement in algebra problem solving via the use of diagrams consisting of connected rectangles?

b) By what means did students seem to develop these understandings? What aspects of the environment seemed to constrain or afford this development?
Chapter 2

Review of the Literature

Theoretical Framework

Throughout the course of this study of students’ development of algebraic understandings, certain assumptions about teaching, learning, and mathematical understanding are made. Below is an explication of those assumptions.

The nature of understanding

The National Council for Teachers of Mathematics (2000) advocates mathematics teaching that is based upon students’ existing knowledge and which promotes the development of conceptual understanding. The meaning of the word “understanding” in mathematics has been highly debated (Hiebert & Carpenter, 1992; Simon, 2006). Hiebert and Carpenter (1992) stated, “…it is useful to think of students’ knowledge of mathematics as internal networks of representations…. [These networks] are built gradually as new information is connected to existing networks or as new relationships are constructed between previously disconnected information” (p. 69). According to Hiebert and Carpenter, it is during this process of reconfiguration that a learner comes to understand a concept.

Whereas Hiebert and Carpenter (1992) provided a model of the cognitive restructuring that occurs as a result of a learner’s knowledge construction, Simon (2006) described the change
that occurs in the way that the learner thinks mathematically. Simon stated that there is a “change in the learner’s ability to think about and/or perceive particular mathematical relationships” (p. 3). According to Simon, this change, or “conceptual advance” (p. 3), is indicative of a learner’s construction of what Simon calls a “key developmental understanding” (p. 2). Simon described a key developmental understanding as an anticipatory notion which, when held by a learner, enables the learner to determine that which must be mathematically true with regards to particular mathematical relationships. That is, a learner with a particular key developmental understanding is able to pre-determine the affects of varying a particular mathematical condition upon a mathematical object or quantity.

I would like to propose the following as an example of a key developmental understanding related to this study. Suppose that there were two children, only one of whom possessed an understanding of the inverse relationship between the size of a unit of measurement and the number of iterations of that unit required to measure a particular whole. Suppose both students were given the following problem: “Jill pretended to make a train by lining up four rectangular blocks of the same length. These blocks came from pile A. She pretended to make another train by lining up eight rectangular blocks from pile B. These blocks were the same length as one another. The length of the blocks in pile A was different from the length of the blocks in pile B. The two trains were the same length. Which train had longer blocks?”

A student who understood the inverse relationship between size of unit and number of units required to measure a particular whole would be able to read the problem and answer “Train A is longer” without having to mentally or physically run through an experiment using rectangular blocks (Simon, 2006). To this student, this answer is necessarily true based upon the key developmental understanding held by that student. However, a student who did not possess this understanding would, presumably, need to experiment, mentally or physically, with rectangular blocks in order to solve the above problem. It is this noted difference in mathematical
activity that enables an observer to differentiate between a student who possesses a particular key
developmental understanding and one who does not (Simon, 2006). An observer’s description of
a learner’s key developmental understanding is, necessarily, a construct of the observer (Steffe &
Thompson, 2000), but it is based upon that which is consistently observed in a learner’s actions.
Throughout this paper, the word “understanding” is used to signify that which Simon (2006) calls
a key developmental understanding.

Reflective abstraction

Piaget (1971) stated, “logico-mathematical construction…proceed[s] by means of
reflective abstraction” (p. 321-322). Logico-mathematical knowledge enables an individual to
draw particular conclusions about relationships between (conceptual or actual) objects (Kamii,
2000). Piaget (1972) contended that the construction of logico-mathematical knowledge by a
learner “consists in [the learner’s] action on objects, but with abstraction of knowledge based on
action and no longer on objects themselves” (p. 71). Piaget (1971) distinguished between two of
the ways by which an individual may act upon an object. One way involves acting in such a
manner as to ascertain the physical features of an object or set of objects. Such actions upon
objects support the construction of empirical knowledge, according to Piaget (1971). The other
means of interacting with an object involves the utilization of actions—such as ordering or
sorting—that are based in an individual’s “logico-mathematical structures” (Piaget, 1971, p. 342).
According to Piaget, when a learner utilizes such actions upon an object, he or she endows the
object with qualities that are not inherent in that object. As a result, that which the learner
abstracts is not due to the qualities of the object, but to the learner’s logico-mathematically-based
actions upon that object (Piaget, 1972). More specifically, Piaget (1971) defined “reflective
abstraction” as the process by which the “coordination of actions” (p. 181) used by a learner to
solve a task are displaced to a higher level of cognition and, there, is reconfigured in such a way as to create a new logico-mathematical structure.

In an effort to build upon Piaget’s (1971) description of the reflective abstraction process and to theorize the way in which a learner engaged in this process comes to construct a particular understanding, Simon, Tzur, Heinz, and Kinzel (2004) proposed their “reflection on activity-effect relationships” theory (p. 33).

Simon and his colleagues (2004) postulated that an individual’s learning process is facilitated by the efforts which he or she undertakes in order to accomplish a goal which he or she has set. Simon and colleagues labeled the “set of actions used in an attempt to meet [this] goal” the learner’s “activity sequence.” When using the word “activity,” however, Simon and colleagues are referring to the learner’s “mental activity” (p. 33) since it is the learner’s ongoing acts of cognizing and his or her present assimilatory structures that regulate and facilitate the learner’s observable actions. What are meant by assimilatory structures are those cognitive mechanisms by which a learner incorporates new conceptions into his or her network of already existing conceptions together with that extant network of conceptions (Piaget, 1971). In addition, it is those assimilatory structures which are being transformed during the reflective abstraction process.

In describing a learner’s activity sequence, at times, Simon and colleagues delineated the observable activity in which the learner engages while maintaining the underlying assumption that there are cognitive factors which are both shaping and being shaped by the learner’s external activity. In order to avoid the need for ongoing clarification, when describing a learner’s activity sequence here, “activity” may refer to any one or more of the actions undertaken by a learner during that sequence which may be observed or inferred by observation.

According to Simon and colleagues (2004), during learners’ enactment of an activity sequence, learners “attend to the results of their…activity, distinguishing between positive results
of their activity (closer to their goal) and negative results (farther from their goal)” (p. 29). As a result, learners will make a decision (consciously or unconsciously) about the way in which they may adapt their activity in order to better approximate the goal (Simon et al., 2004). Simon and colleagues called “these adjustments…the effects of the activities” (p. 30). In particular, as the learner reflects on “[his or] her activity (sequence) and its effects across a number of tasks of…[the same] type, [he or] she distinguishes a regularity” (p. 32). The regularity to which the learner attends (consciously or unconsciously) is that invariant property across effects which results in a better approximation of the goal (Simon et al., 2004; Thompson, 1985).

Like Simon and colleagues (2004), Dubinsky (1991) based his theory of mathematical learning on Piaget’s reflective abstraction. Whereas Simon and colleagues (2004) theorized about the mechanics of reflective abstraction, Dubinsky theorized about the cognitive results of that process. Dubinsky proposed a theory of learning which articulates the stages of a learner’s construction of a new (to the learner) mathematical concept. In Dubinsky’s theory, these stages are varying levels of a learner’s conception of that concept. According to Dubinsky, a learner moves from one level of conception to the next through the process of reflective abstraction. The stages within this theory—action, process, object, schema (APOS)—and the reflective abstraction processes which, according to Dubinsky, facilitate these transitions—are discussed below.

Although Dubinsky did not specify in detail the mechanics of a learner’s movement from one level of conception to the next, Simon and colleagues offered that their Reflection on Activity-Effect Relationship (RAER) theory is an effective tool in this effort. In this study, Simon and colleagues’ RAER theory is that which is assumed to provide an expanded explanation of the learner’s cognitive activities occurring during Dubinsky’s named reflective abstraction processes.

Like Piaget (1970), Dubinsky (1991) placed the ontogenesis of conceptual learning with a learner’s action upon that which is a real, comprehensible (abstract or concrete) object to the
The learner’s actions upon that object, according to Dubinsky, are tantamount to “calculating with [that] object” (Dubinsky, 1991, p. 106). Initially, the learner’s ability to anticipate the contextual significance of the result of those actions is limited. At this point, the learner would be considered to have an action conception. Asiala and colleagues (1996) stated, “when an action is repeated, and the individual reflects upon it, it may be interiorized into a process” (p. 10).

A learner with a process conception is able to anticipate the steps and the nature of the results of his or her actions without having to mentally or physically run through those actions (Asiala et al., 1996). Additionally, the learner is able to construct new processes through the coordination of several processes or through constructing the reversal of a process according to Dubinsky (1991). A process understanding with reversibility correlates with Piaget’s (1973) notion of operations which he called “interiorized…, reversible…actions” (p. 76).

The next stage in Dubinsky’s APOS theory is an object conception. Asiala and colleagues (1996) described the quality of an object conception.

When an individual reflects on operations applied to a particular process, becomes aware of the process as a totality, realizes that transformations (whether they be actions or processes) can act on it, and is able to actually construct such transformations, then he or she is thinking of this process as an object (p. 11).

According to Dubinsky (1991), a learner’s transition from a process to an object conception is accomplished through encapsulation, another form of reflective abstraction. The last stage in Dubinsky’s (1991) APOS theory is schema, “a more or less coherent collection of objects and processes” (p. 102) which, when constructed by a learner, is able to be generalized and used by the learner in other mathematical situations which the learner perceives to be applicable. Dubinsky stated that a person’s understanding of a particular mathematical concept is what drives which schema he or she utilizes to “make sense out of a perceived problem situation” (p. 102).
In order to illustrate the tenets of Simon and colleagues’ (2004) theory and the stages of Dubinsky’s (1991) APOS theory, the following example is given. Some of the results of Olive and Caglayan’s (2008) study of eighth grade algebra students are used as a part of the examples.

Olive and Caglayan (2008) conducted a study in which they analyzed the algebraic understandings of a classroom of eighth-grade students who were learning to solve algebra word problems. Included in the study’s data were interviews with the classroom teacher, interviews with particular students, and videotapes of the classes. Two examples of student interview data from this study were given in Olive and Caglayan (2008). Two different pairs of students were asked to solve the following problem.

Mrs. Speedy keeps coins for paying the toll crossing on her commute to and from work. She presently has three more dimes than nickels and two fewer quarters than nickels. The total value of the coins is $5.40. Find the number of each type of coin that she has (p. 273).

Olive and Caglayan (2008) drew from Steffe’s (1994) identification of “children’s construction of composite units at different levels of composition (a singleton unit, a unit of units, and a unit of units of units)” (p. 271) in order to make sense of the data that they analyzed in this study. Using the same levels as Steffe, Olive and Caglayan (2008) stated that, in order to solve the coin problem, a learner would have to construct the number of coins as a unit, the value of that set of coins as a unit of units and the total value of all of the coins in the coin problem as a unit of unit of units. Olive and Caglayan (2008) found that in order to be able to construct an equation to solve the coin problem in one variable, a learner had to have been able to construct and coordinate units on the three levels mentioned.

For example, a student whom they interviewed, Maria, was able to solve the coin problem correctly. Maria utilized the letters \( n \), \( d \), and \( q \) to represent the number of nickels, dimes, and quarters, respectively, in the coin problem. Maria represented the total value of each type of coin as \( 0.05n \), \( 0.1d \), and \( 0.25q \). According to Olive and Caglayan (2008), in order for a learner to
be able to create a representation such as $0.05n$, he or she would need to have coordinated units, in this case, number of nickels ($n$) and the value of one nickel ($0.05$), to create a new unit, total value of the nickels. Maria represented the total value of the coins in this problem as $0.05n + 0.1d + 0.25q$. In order to have accomplished this, according to Olive and Caglayan (2008), she would have had to combine the representations of the total value of each coin, thus creating a unit of units of units. Olive and Caglayan (2008) found that “the ability to coordinate different units in a quantitative situation is an important skill for students to develop in order to be successful in both representing and solving algebraic word problems” (p. 273).

Below is a thought experiment applying the APOS theory to the problem situation in Olive and Caglayan’s (2008) study. Suppose that a student, I will call him Dave, was given a simpler task such as: “Mr. Brown has three more dimes than nickels. If Mr. Brown has $n$ nickels, write how many dimes he has using $n$.” Let’s suppose that Dave had the understanding that a letter can represent a specific, unknown quantity but that he did not understand that one could operate upon an unknown quantity as if it were known. In order to consider how he might represent the number of dimes that Mr. Brown has, Dave might choose a number of nickels and add three to that number, thus giving him the number of dimes. At this point, Dave has an action conception of $n+3$. That is, he does not anticipate that he is engaging in particular instances of adding three to a generalized number $n$.

Dave’s activity sequence is as follows: 1) Choose a number for $n$, 2) add three to that number, and 3) write the sum. The effect of Dave’s activity is that he notices that, for each value of $n$, the sum is three greater than the original number of nickels. By reflecting on the relationship between his activity and its effect, Dave will abstract the notion that, because the sum of a certain number of nickels and three is three greater than $n$ for all values of $n$, the number of dimes must be $n+3$. Dave now has a process understanding of $n+3$. 
Finally, suppose that Dave was given the following problem: “The following describes Mr. Brown’s amount of money where $n$ stands for number of nickels and $d$ for number of dimes: $0.05n + 0.10d$ (because each nickel is worth $0.05$ and each dime is worth $0.10$). Write this expression in terms of $n$ only.

Dave might then engage in the activity of operating upon the expression $0.05n + 0.10d$ by replacing $n$ with a number and $d$ with a number that is three greater than $n$ and then finding the sum. The effect of Dave’s activity is that he would notice that the number that he uses for $d$ is always the number he uses for $n$ plus three. By reflecting on his activity and its effect, Dave would abstract the notion that he is multiplying 0.1 by $n + 3$. Therefore, to solve the problem, Dave would rewrite the expression so that it reads: $0.05n + 0.10(n+3)$. Dave now has an object conception of the expression $n + 3$ in that he is able to use this expression as an input into the larger expression.

In Olive and Cagalayan’s (2008) study, Maria was able to solve the coin problem by first creating the equation $0.05n + 0.1d + 0.25q = 5.40$ in order to represent the problem situation given. Using the additional information given in the problem, Maria then created the equation $0.05n + 0.1(n+3) + 0.25(n -2) = 5.40$ in order to find out how many nickels, dimes, and quarters Mrs. Speedy had. Maria was considered by the researchers as having an understanding of quantitative structure (Thompson, 1993). That is, Maria possessed the ability to reason about the quantities in the problem (number of nickels, number of dimes, number of quarters) and the relationship between those quantities without needing to evaluate those quantities (Thompson, 1993). According to Olive and Cagalayan (2008), Maria’s quantitative reasoning ability enabled her to construct and coordinate units on the three levels of units, units of units, and units of units of units.

According to APOS theory, it seems that Maria had a schema for situations containing quantities that were related additively or multiplicatively in that she held the relationships
between those quantities (in this case number of nickels, dimes, and quarters) as objects. Ahead of time, she anticipated that she would need to replace two of the letters in the expression $0.05n + 0.1d + 0.25q$ with the one letter, $n$, so that that expression was written in one variable. And she viewed $0.05n + 0.1d + 0.25q$ as an object in that she was able to act upon it by replacing $d$ and $q$ with the expressions that related those quantities to $n$. This collection of processes and objects formed a schema for Maria that, it seems, she would be able to utilize when encountering new relational situations such as this one.

Since the process and object understandings that comprise a learner’s conceptual schema enable him or her to anticipate the result of his or her acting in particular situations prior to acting, it seems that Dubinsky’s schema intersects with that which is defined by von Glasersfeld (1995) as a scheme. Von Glasersfeld stated:

I have come to specify the three parts of schemes as follows:

1. Recognition of a certain situation;
2. a specific activity associated with that situation; and
3. the expectation that the activity produces a certain previously experienced result.

(p. 65)

Seemingly, a learner’s scheme (as defined by von Glasersfeld) is regulated by that learner’s schema (using Dubinsky’s term) of the concept in question. It is the learner’s process and object understandings that enable him or her to “recognize” a situation as being an instance like one that was previously experienced and which calls for a particular set of actions. Further, it is those understandings that enable the learner to anticipate the results of his or her actions. Thus, it seems that an event which precedes the activation of a particular scheme is only a trigger for a learner with particular understandings (and experiences) who, based on those understandings (and experiences), summons particular actions to solve the problem in a particular way.
In this study, the word schema is used in Dubinsky’s sense of the word and refers to the comprehensive set of understandings that a learner seems to have in place related to a particular concept. The word scheme is used in the sense of von Glasersfeld’s (1995) definition of the term. It is used, in part, to refer to a particular set of actions repeatedly used by a learner in particular problem-solving situations. Whenever a learner’s scheme is discussed, it is understood that this characterization is an observer-constructed model of the learner’s understandings (Steffe & Wiegel, 1996) which is based upon the mathematical behavior of the learner.

The theories of Dubinsky (1991) and Simon and colleagues (2004) suggest that reflection is a key aspect of the process of logico-mathematical construction. Given this, it seems reasonable that mathematics lessons that are designed to give learners the opportunity to act in particular ways and to reflect on the results of those actions, have the potential of enabling students to construct an intended mathematical abstraction (Simon et al., 2004). The way in which this opportunity was intended to be realized in this study follows.

**Lesson Design**

Both Simon and colleagues (2004) and Realistic Mathematics Education researchers (Gravemeijer, 1997; Streefland, 1993; Treffers, 1991) have proposed theoretical frameworks for the process of designing lessons. According to them, lessons should build from students’ current mental operations. Simultaneously, lessons should engage students in experiences that have the potential to enable them to construct mathematical concepts that are more sophisticated than those which they currently hold (Dubinsky, 1991; Simon et al., 2004; Thompson, 1985).

The didactical frameworks of both Simon and colleagues and the Realistic Mathematics Education researchers are those which instructed the lesson design process in this study. Both frameworks are based in or are consistent with the theory of learning to which this paper
subscribes. And both didactical theories examine lesson design with a grain size and from a perspective that is different from, but complementary to, the way the other does. The theory of lesson design espoused by each group of researchers, as well as the way in which each theory integrates with the other, is examined below.

Simon and colleagues’ (2004) Reflection on Activity-Effect Relationship

Based upon the premise that the abstraction of a particular activity-effect relationship promotes the learner’s construction of a particular conception, Simon and colleagues (2004) proposed that lessons be designed that would give learners an opportunity to reflect upon and potentially abstract such a relationship. In order to achieve this goal, the lesson designer must plan the lesson in such a manner that students, given their current conceptions, will likely establish a particular goal for themselves and a particular activity sequence in order to accomplish that goal. It is anticipated that, after several iterations of engagement in that activity sequence, students reflect “on [their] activity and its effects” (Simon et al., 2004, p. 31) and “distinguish a regularity” (p. 31) in the nature of those effects (the adjustments needed to be made in order to reach their goal). When a student has developed an anticipatory notion of the type of adjustment that he or she will need to make to a particular set of actions in order to accomplish a particular goal, it is stated that that student has abstracted the relationship between an activity and its effect (Simon et. al, 2004). This abstraction, according to Simon and colleagues, is the “first stage in the development of a new conception” (pp. 30-31).
Researchers within Realistic Mathematics Education (RME) view mathematics learning as the result of “mathematizing” (Streefland, 1993, p. 111). According to Treffers (1987), mathematizing is the mental activity engaged in by a learner as he or she attempts to solve a situation-based mathematics problem that is realistic to him or her. By realistic, it is meant that the mathematical and situational settings of the problem are within the realm of the student’s comprehension (van Reeuwijk, 1995a).

RME identifies two types of mathematizing—horizontal and vertical—that occur during the solving of situation-based problems. According to Gravemeijer (citing Treffers, 1997), “horizontal mathematization…is mathematizing contextual problems” (p. 333). This type of mathematization is the mental process by which a learner constructs an external representation of the mathematical structure of a problem. This process utilizes the learner’s current conceptions and does not entail a shift to a more sophisticated conception. On the other hand, vertical mathematizing is “the mathematization of mathematical matter” (Gravemeijer, 1997, p. 332). During this process, the object of the learner’s reflection is his or her own mathematical activity. During the latter process, according to RME, a learner’s mathematical understanding about a particular concept becomes more sophisticated than that which he or she held prior to that reflection. That is, the learner’s conception shifts vertically to a more mature one during this process.

Gravemeijer (1997) provided an example and explanation of an RME-based problem: “1296 supporters want to visit the away soccer game of Feijenoord. The treasurer learns that one bus can carry 38 passengers and that a reduction will be given for every ten buses. How many buses are needed?” (p. 325). In order to find out how many buses will be needed, a problem solver who has never before learned long division could repeatedly subtract the number 380 from
1296 (since 10 buses of 38 provide a discount). According to Gravemeijer, the focus of the problem solver who acts in this manner is on the remaining number of people needing to be placed on a bus. Thus, he or she may continue to subtract 380 or 38 from the remaining number of people until there are no more sets of 38 left.

In this example, horizontal mathematization would be stated to have taken place in that the learner represented the conditions of the problem and acted upon that representation. However, during this activity, if the learner noticed that he or she could subtract larger multiples of 38 from 1296 than 380 because it would yield an answer more quickly, he or she might decide to find out the largest multiple of 38 that could be subtracted from 1296. Hence, the learner would be stated to have engaged in vertical mathematization in that the learner was reflecting on his or her own mathematical activity of subtracting rather than on the conditions of the problem itself. Further, that reflection could be stated to have led to a more sophisticated conception than the learner held previously.

In order to foster students’ development of a particular conception, researchers within RME seek to engage students in lessons where “it is attempted to teach the standard procedure by letting it evolve from informal ones in a learning process” (Gravemeijer, 1997, p. 324). For example, in the bus problem previously discussed, the problem was worded in such a manner as to possibly invoke repeated subtraction of sets of 380 and 38 from 1296 (Gravemeijer, 1997). Whereas this informal activity would not be considered as long division in its most condensed form, the learner’s activity is very similar to the long division process. Over time, as the learner notices particular patterns, he or she has the potential of developing a more succinct and formal procedure which looks very similar to long division. The creation of such a procedure is called curtailment and it is one of the intended goals of RME instruction.

Gravemeijer and Doorman (1999) identified particular attributes which such problems must have. They stated,
Well-chosen context problems offer opportunities for the students to develop informal highly context-specific solution strategies. These informal solution procedures then, may function as catalysts for curtailment, formalization, or generalization. (p. 117)

According to RME, mathematics lessons are designed with the anticipation that students will utilize their own informal actions and notations in order to represent the relationships in the context problem and to solve the problem. Collectively, these actions and notations are called a student’s “mathematical model” (Gravemeijer, 1997, p. 339) of the problem. When interpreting a situation mathematically, students are stated to be making a “model of [the] situation” (Gravemeijer, 1997, p. 339). In addition the lesson designer may present a model upon which students may act in order solve a problem. This “didactic model” (Klein, Beishuizen, & Treffers, 1998, p. 445) is presented in diagrammatic or concrete form and is that which the lesson designer anticipates will help to “facilitate” (Klein et al., p. 447) students’ reinvention of a particular mathematics concept based upon his or her current conceptions (Gravemeijer & Doorman, 1999).

As students solve a series of similar problems for which they have developed their own heuristic, it is anticipated that they will (consciously or unconsciously) seek to operate more efficiently. That is, they will search for a means by which they may abbreviate those solution methods which they have found to be efficacious. As a result, the model itself will become the object of students’ reflective activity (Gravemeijer, 1997). As such, the model, for the students, is stated to become a “model for mathematical reasoning” (Gravemeijer, 1997, p. 339). The result of such reasoning is the creation of “condensed and formalized” (p. 332) methods which more closely approximate the formalized algorithms used by the mathematical community. In lesson planning, it is the goal of RME that students realize this cognitive shift in their model use. Seemingly, students who have succeeded in formalizing earlier informal notations and strategies have encapsulated and generalized the essence of particular quantitative relationships and the effect of transformative actions upon those relationships.
Given Piaget (1971) and Dubinsky’s (1991) articulation of the primary role of a learner’s action upon objects in the reflective abstraction process, Simon and colleagues’ reflection on activity-effect theory, and RME’s theory of vertical mathematizing, the following assumption about teaching and learning is made in this study. The type of experience which potentially optimizes the construction of logico-mathematical knowledge by a learner is one in which the learner, having set or adopted a goal that is cognitively accessible to him or her, has the opportunity to create and act upon, what is to him or her, an object. Further, this experience should give the learner the opportunity to reflect upon the records of his or her actions upon that object and the (mental or physical) adjustments made which resulted in movement towards his or her established goal (Piaget, 1971; Simon et al., 2004). When the learner reflects upon and, subsequently, abstracts, the relationship between his or her action upon the object and the invariant aspect of the successful adjustments (effects) to that action, the learner is then able to anticipate the results of his or her actions without having to (mentally or physically) perform that action (Simon et al., 2004; Thompson, 1985). Taken together, these theories of learning and lesson design drove this study’s task design, the mathematical interactions which occurred during the teaching sessions, and the analysis of the study data.

**Algebra story problems**

The type of problems utilized in this study is an algebra number story problem according to Mayer’s (1981) characterization. Mayer (1981) developed a system by which he categorized over 1000 algebra story problems into “eight families based on the nature of the source formula involved” (p. 135, italics removed). An algebra “number story” (p. 142) problem was chosen because it has a realistic setting, but it is “not based on any source formula” (p. 142). That is, an algebra number story problem does not require students’ use of such geometric or relational
formulas as $A = \pi r^2$ and $d=rt$. Algebra word problems requiring the use of a source formula seem, minimally, to call upon schemas which invoke students’ rational number, geometric, and statistical (Mayer, 1981) thinking in addition to their algebraic thinking.

Because the focus of this study is on examining the way in which students develop their ability to reason algebraically as they work within a diagrammatic environment, it seems necessary to utilize problems that do not require additional understandings outside of that which is required for algebraic thinking. In that way, students’ encountered cognitive obstacles do not have the potential of being the result of a confluence of factors including students’ rational number knowledge and their understanding of particular geometric or statistical formulas. For this reason, the tasks that are given in this study and the correct answers to those tasks contain only whole numbers.

An example of an algebra number story problem follows: Chris has four times as many marbles as Curtis does. If Curtis doubled the number of marbles that he had and bought four more marbles, he and Chris would have the same number of marbles (Maier, 1997). Create a diagram to represent this situation. Use your diagram to find out how many marbles did each person start out with. Here, Chris’s number of marbles is $n$ times as great as Curtis’s number of marbles where $n$ is a whole number. Further, the answer to this question is a whole number.

Throughout the rest of this paper, algebra number story problems are referred to only as “algebra word problems.”

Ways of thinking and knowing

Quantitative reasoning
According to Thompson (1988), quantitative reasoning is “reason[ing] about quantities, their magnitudes, and their relationships with other quantities” (p. 164). An individual with this ability is able to conceive of a known or unknown quantity as signifying a property (such as an object’s weight or number of items in a group). Without needing to ascertain the actual value of that quantity, an individual who has quantitative reasoning ability is able to think about the way in which this quantity stands multiplicatively or additively in relation to other quantities (Thompson, 1994). And, that individual is able to think of the result of that operation between known or unknown quantities as a quantity itself (Thompson, 1993).

Quantitative reasoning seems to be essential for the construction of algebraic statements based upon problem information. This is because, in order construct such statements, students need to be able to “name [the] quantities” (Thompson, 1988, p. 168) in the situation, hold those quantities in mind without having the need to assign a value to any of them, and, where necessary, create new quantities which are multiplicatively or additively dependent upon other quantities. Further, as articulated by Clement (1982), in order to establish equivalence between two quantities that are not equal, a “hypothetical operation” needs to be constructed. The student-professor problem will be examined in light of the quantitative reasoning construct.

This problem states: “Write an equation using the variables S and P to represent the following statement: ‘There are six times as many students as professors at this university.’ Use S for the number of students and P for the number of professors” (Clement, 1982, p. 17). In order to solve this problem, students need to identify and name the unknown quantities as “number of students” and “number of professors.” They need to hold these unevaluated quantities in mind and “compare [them] multiplicatively” (Thompson, 1994, p. 185). Having determined that the number of students is six times as much as the number of professors in the situation described, the individual who is able to reason quantitatively must then recognize that he or she needs to (hypothetically) increase the number of professors by a factor of six as prescribed by Clement.
Thus, the individual must construct a new quantity, “six times the number of professors”, in order to create a hypothetical relationship of equality between number of students and number of professors.

The interaction between students’ ability to think and reason algebraically and their development of algebraic objects as quantities will be examined in the analysis section of this paper.

**Internal and external representations**

According to Goldin and Kaput (1996), a learner’s construction of internal representations related to a particular concept and his or her construction of contextual meaning for external representations are afforded and constrained by that learner’s current mathematical conceptions. Citing Palmer (1977), Goldin and Kaput (1996) define “a representation [as] a configuration of some kind that, as a whole or part by part, corresponds to, is referentially associated with, stands for, symbolizes,…or otherwise represents something else ” (p. 398). This configuration may be external and thus visible to an observer. Or the configuration may be that which is imputed to a learner by an observer who wishes to construct a model of that learner’s cognitive structure (Goldin & Kaput, 1996).

Notable amidst Goldin and Kaput’s (1996) discussion of internal and external representations is the notion that an external representation which is meant to “represent” an idea or concept does not necessarily evoke the same internal image within each learner. For example, suppose that two learners see the graph of a straight line. One learner may immediately think an equation of the form $y=mx+b$ while the other may think of the infinitely many ordered pairs whose coordinates comprise the line. In either case, a straight line on a graph carries with it particular connotation(s) to the learner (Goldin & Kaput, 1996). These connotations are
influenced by a learner’s internal representations of particular concepts according to Goldin and Kaput. But, it also seems that they may be influenced by the learner’s “concept image” (Tall & Vinner, 1981)—that is, the learner’s overall notion of a particular concept based upon his or her conceptions, misconceptions, and experiences related to a particular mathematical construct. A learner’s internal representations related to a particular concept, which are a part of his or her concept image related to the same, greatly influence the way in which a learner perceives and interacts with an external representation related to that concept.

One of the implications of the internal–external representational construct is the following. That which an external representation is meant to signify by its designer is not necessarily interpreted as such by another individual. Kinzel (2001) stated,

Notations are ‘potential representations’; they become representations when someone interprets them, that is, when someone establishes a signifying connection between symbol and referent. In exploring this connection from the student’s perspective, the emphasis is on how the student is perceiving the notation, what the student sees as being represented, and what role that representation plays within the mathematical activity.

An example of this may be found in reports that many pre-algebra and algebra students interpret algebraic symbols as referents to words or objects, rather than as referents to quantities (Stacey & MacGregor, 1997). A further implication of the representational construct is that a learner’s symbolizations cannot be presumed to signify a particular concept or idea usually associated with that representation. Since this cannot be presumed, the learner’s verbal and nonverbal mathematical behavior as a whole must be consulted in order to interpret the meaning of a learner’s external artifacts.

What follows is a discussion of students’ pre-instructional and post-instructional understandings related to algebraic symbols. Following this, students’ symbolic manipulation and algebraic problem-solving will be examined. In light of the NCTM’s (2000) recommendation that students should learn equation solving in the context of solving word problems, it is important to examine those understandings held by students. In that way, the point
of departure for the types of lessons that the NCTM recommends may be determined. Further, the cognitive difficulties that persist following traditional algebra instruction may be identified and pro-actively addressed.

Student’s cognitive difficulties with algebra

Students’ demonstrated misconceptions with regards to algebraic symbols, symbolic manipulation, and translation of verbal statements to algebraic symbols have been well documented (Clement, 1982; Filloy & Rojano, 1989; Gray & Tall, 2007; Küchemann, 1981). Amongst the most prevalent of the reported cognitive difficulties of both algebra and pre-algebra students is that of students’ lack of understanding of letters in algebra as will be discussed in the following section.

Students’ encountered difficulties with the meaning and use of letters in algebra

Both pre-algebra and algebra students have been reported to misinterpret the meaning of letters in algebra (Booth, 1984; Küchemann, 1981; MacGregor & Stacey, 1997; NCTM, 2000). Küchemann (1981) tested 13-15 year old children who had had some instruction in algebra, making note of the differences in students’ responses to questions whose answers were similar in structure, but whose solutions required different minimal levels of understanding concerning letters. Through his assessment, Küchemann (1981) identified six increasingly complex conceptualizations of letters in algebra. They are, in order of complexity:

Letter EVALUATED

Letter IGNORED
In this paper, the term “letter” (as opposed to variable) is used when analyzing students’ symbolic understandings because such usage allows for the varying levels of students’ conceptualizations of algebraic symbols. In the section that follows, the first four conceptions of letter are discussed as a means of defining the cognitive obstacles that the study seeks to help students to circumvent and as a means of defining the conception which this study hopes to build upon.

**Letter Evaluated**

According to Küchemann (1981), the first three conceptualizations of letters in algebra—letter evaluated, letter ignored, and letter as object—prevented students with those conceptions from successfully engaging in algebraic problem-solving with understanding. In fact, Küchemann stated that a learner holding any of these three conceptions of letter, when in an algebraic problem-solving situation, acts in such a way that allows him or her to “avoid having to operate on a specific unknown” (p. 105).

For example, Küchemann (1981) reported that almost all of the 14-year-old students whom he tested were successful when solving an equation having one unknown quantity: \(a+5=8\). However, when asked to solve the next problem, some students resorted to assigning to a letter a number contrived by them in order to solve the problem (see also Booth, 1984, for similarly
reported results). When asked “What can you say about \( r \) if \( r = s + t \) and \( r + s + t = 30 \)” (p. 105), 41\% of the 14-year-old students answered correctly while 21\% of those students answered “10.” Küchemann interpreted the answer generated by the latter set of students as resulting from their attribution of the number 10 to each of the letters \( r \), \( s \), and \( t \). Küchemann (1981) explained that this solution method enabled these students to circumvent “operat[ing] on the unknown” (p. 105).

This finding by Küchemann is in keeping with what was found by Stacey and MacGregor (1997). According to these researchers, the majority of a class of 11-year-old pre-algebra students who were asked to solve the following problem attempted to ascribe a number to the letter \( h \). “David is 10 cm taller than Con. Con is \( h \) cm tall. What can you write for David’s height?” (p. 110). Some students simply guessed what Con’s height might be and added 10 cm to that guess while other students added 8 (the position of \( h \) in the alphabet) to 10 to obtain their answer (Stacey & MacGregor, 1997).

Thompson (1994) theorized that an individual who is able to conceive of a known or unknown quantity as signifying an attribute (such as an object’s weight or number of items in a group) without needing to ascertain the actual value of that quantity possesses quantitative reasoning ability. Seemingly, the students in Stacey and MacGregor’s (1997) study had not yet constructed that understanding.

**Letter Ignored**

According to Küchemann (1981), Booth (1984), and MacGregor and Stacey (1997), some students tend to either completely circumvent or avoid the use of letters to solve algebraic problems in which such use is required. Küchemann (1981) contended that these students, like the students who have the need to evaluate a letter, do not view a letter as a specific unknown. Unlike those students who tend to evaluate a letter, these students ignore the letter.
Another of Küchemann’s (1981) findings seems to corroborate this result. Küchemann reported that 68% of the 14 year old students whom he tested correctly solved the problem, “Add 4 onto \( n + 5 \)’’ (p. 108) while only 20 % gave an answer of “9.” However, when solving the problem, “Add 4 onto \( 3n \)” (p. 108), only 36% of the students answered correctly while 31% of these students answered “\( 7n \)” and 16% answered “7.” Küchemann explained these seemingly disparate results by stating that in order to solve the former problem, the letter \( n \) could essentially be ignored. Students could simply add four to the five contained in the expression \( n + 5 \) and rewrite the \( n \) term without having to understand what \( n \) represents. However, in order to correctly solve the latter problem, students needed, among other things, to conceive of a letter as a specific unknown. Further, students needed to have an “acceptance of lack of closure” (Collis, 1974, p. 5) in order to successfully solve this problem. This understanding is discussed further in a later section.

These results seem to indicate that some students conceive of letters in algebra as nonspecific, unknown quantities upon which no action may occur or which may be assigned arbitrary values (Küchemann, 1981). However, in order to act upon a letter as a specific unknown, it seems that, an object conception of letter as a specific unknown is necessary. It was my intent, during this study, to enable study participants to construct an object understanding of letter so that they could avoid the misconceptions of letter reported here and in the sections that follow.

**Letter as Object**

This conception is not to be confused with an object conception of letter on the APOS continuum. In that case, it seems, a letter is, to the learner, a conceptual object that may be operated with and upon “as if it were known” (Stacey & MacGregor, 1999, p. 30). Students
whom Küchemann (1981) characterized as perceiving a letter as an object were stated to have the conception that letters are labels or collectible entities. Both of these conceptions seem to fall under the category of letter as object because, in each case, a letter represents, to the learner, a non-quantitative entity. Below is a reporting of studies which illustrate both of these misconceptions.

**Letter as label**

Küchemann (1981) and Booth (1984) studied the symbolic understandings of 13-15 year old students in England and MacGregor and Stacey (1997) studied the understandings of 11-15 year old students in Australia, all of whose mathematics courses were interspersed with algebraic concepts. Both pre-algebra and algebra students in the studies were found to interpret a letter as a referent to a word or object that may or may not begin with that particular letter (Küchemann, 1981; MacGregor & Stacey, 1997). In response to Booth’s (1984) question, “What do the $q$ and $w$ mean” (p. 13) in the expression “$5q – 3w$”, a 15 year old student responded, “… ‘it could be various things….It could be…5 bananas minus 3 apples, something like that’” (p. 13). According to Küchemann, a student could successfully solve problems such as those requiring them to simplify the expression “$3a + 5a$” while this misconception is still in place in that the student obtaining the answer of “$8a$” does so on the premise that he or she is combining a specified number of apples. In situations like this, the letter as label misconception might go unnoticed. Alternatively, perhaps students’ symbolic assumptions are not challenged in the midst of routine equation solving and simplification of expressions, thus the root of syntactic errors may go undetected.
Rosnick’s (1982) interviews of ten college students, the majority of whom had taken at least one semester of calculus, provides evidence of the durability of the letter as label conception. Rosnick found that, at some point in the problem-solving process, each student conflated a variable with the name of an object or word. The fact that each student committed this error is an indication that these students’ procedural knowledge was not entirely anchored in an accurate conception of letters in algebra. Rosnick uncovered a further consequence of the misconception of algebraic letters in problem-solving during an interview with a student. This student, who was enrolled in calculus, was given the following problem:

‘I went to the store and bought the same number of books as records. Books cost two dollars each and records cost six dollars each. I spent $40 altogether. Assuming that the equation $2B + 6R = 40$ is correct, what is wrong, if anything, with the following reasoning. Be as detailed as possible.’

$$2B + 6R = 40 \text{ Since } B = R, \text{ I can write}$$

$$2B + 6B = 40$$

$$8B = 40$$

This last equation says 8 books is equal to $40$. So one book costs $5$. (Rosnick, 1982, p. 17)

The student shared with Rosnick (1982) that the statement “$B=R$” seemed incorrect because “…$B$ is a book and $R$ is a record so a book doesn’t equal a record” (p. 18). Rosnick stated that, because of the letter as label misconception, the student “is unable to replace one letter with its numeric equivalent which would lead to a solution of the system of equations” (p. 19).

MacGregor and Stacey (1997) offered that students may commit the letter as label error based upon their prior experiences where they may have used a letter as an abbreviation for measurement units or to represent a variable measurement in a formula, thus equating a letter with the word that it represents. Students were also found to interpret a letter as representing the numerical equivalent of the ordinal position of that letter in the alphabet (Booth, 1984;
Küchemann, 1981; Stacey & MacGregor, 1997). MacGregor and Stacey postulated that students’ previous experience with puzzles which involve matching a number with a letter in order to decode a hidden word is a possible reason for students’ matching of letters with numbers.

*Letter as collectible entity*

Whereas some students who conceptualize a letter as an object might view a letter as representing a word so that $5a$ means “5 apples”, Küchemann (1981) also found that some students may conceptualize a letter as being a collectible item so that $5a$ means “5 $a$’s.” Some students were able to find the perimeter of a triangle, all of whose sides were labeled as $e$ (see Figure 2-1a), but were not able to find the perimeter of an object that had $n$ sides of length 2 (see Figure 2-1b). In order to be able to solve the latter, according to Küchemann, students needed to be able to conceptualize the side length $n$ “as a number…[whose] value is not known” (p. 103). However, in order to solve the former, students could surmise that because there were 3 $e$’s, the answer is $3e$ (a form of representation which they had likely seen and used before).

![Figure 2-1: Figures from two problems used to evaluate students’ conceptualization of letters in algebra in Küchemann’s study (1981, p. 102)](a) (b)
context of algebraic situations. Students appear to begin their algebra instruction with pre-
conceived notions about the meaning of letter and, within the context of that instruction, may also
build misconceptions about letter. Further, it appears that such naïve conceptions of letter are not
necessarily detected or addressed during algebraic instruction (Stacey & MacGregor, 1997).

One of the implications of this result is that instructional interventions intended to enable
students to construct meaning for the referential nature of letters in algebra must begin with an
inquiry into students’ understanding of letters (Stacey & MacGregor, 1997) and continue with the
same. In this study, students’ understanding of a letter as a specific unknown quantity was pre-
assessed and built upon throughout the study.

**Letter as Specific Unknown**

Numerous examples have been offered which distinguish the thinking of students who
hold the first three of Küchemann’s (1981) letter characterizations—letter evaluated, letter
ignored, and letter as object—from those who conceive of a letter as a specific unknown.
Students with the latter conception are able to view a letter as both the representation of a
particular number and as an entity that is able to be operated upon without having to first assign a
value to that letter (Küchemann, 1981). The previously reported results of Küchemann’s studies
seem to indicate that the conception of a letter as a specific unknown and the ability to “operate
on a specific unknown” (p. 105) are not necessarily intuitive (Filloy & Rojano, 1989; Herscovics
& Linchevski, 1994). In the next section of this chapter, students’ understandings of expressions
and equations in algebra will be examined.
Students’ encountered difficulties with the construction and interpretation of algebraic expressions

Both pre-algebra and algebra students have been found to misinterpret the meaning of letters in algebra, sometimes thinking of them as objects or labels (Booth, 1984). Not surprisingly, many students have also been found to misinterpret algebraic expressions. Some of the common difficulties encountered are pre-algebra students’ inability to “operate with the unknown as if it were known” (Stacey & MacGregor, 1999, p. 30) and pre-algebra and algebra students’ misinterpretation of multiplicative quantities as representing a number of objects (Booth, 1984). In the section that follows, the range of students’ misconceptions with respect to algebraic expressions will be discussed.

Difficulty with “acceptance of lack of closure”

Both Booth (1984) and Küchemann (1981) discussed students whose responses to questions assessing their understanding of algebraic symbols demonstrated that they had not yet obtained “acceptance of lack of closure” (Collis, 1974, p. 5, capital letters removed) with regard to algebraic expressions. Collis (1974) stated that, as an individual develops intellectually, his or her ability to accept the lack of closure of increasingly more complex mathematical expressions occurs. Collis defined this acceptance as the “ability to refrain from” the necessity of performing the operation(s) indicated within an expression while still being able to “regard the outcome…as unique and ‘real’” (Collis, 1974, p. 5). Collis gave as an example the fact that, by ten years of age, a child may possess the ability to perceive of the expression 273 + 472 as “replaceable by a third [number] from the same set” (p. 5) without having to evaluate that expression. It is enough, to the child, that the potential for replacement exists. According to Collis, around the age of 12, children may accept the lack of closure of a non-numerical expression. However, at this age, the
child may not yet conceive of a letter as a variable, but, at most, as a generalized number (Collis, 1974). In other words, the child may be able to conceptualize the numerical uniqueness of each letter (Collis, 1974), but he or she may not necessarily be able to conceptualize the attributes of a variable as an entity in itself.

With regards to Küchemann’s (1981) task in which students were asked to add 4 to $3n$, learners who answered “$7n$” or “7” demonstrated that they were unable to accept the lack of closure of the answer. That is, these students may have felt the need to unite all of the terms into one because they did not feel comfortable having an “answer” that contained an unperformed operation as is the case with $3n + 4$ (Küchemann, 1981). In contrast, Küchemann found that the learners holding the three most sophisticated conceptions of letter—letter as specific unknown, letter as generalized number, and letter as variable—were able to solve these and other types of algebra problems with understanding. However, the degree to which these learners were able to solve more complex algebra problems depended on the level of sophistication of their concept of letters in algebra (Küchemann, 1981).

The “proceptual divide”

Just as Küchemann (1981) examined students’ understanding of letters in algebra, Gray and Tall (2007) addressed learners’ conceptualizations of expressions in algebra. These researchers stated that when a learner characterizes an algebraic expression as a “procept” he or she conceptualizes that expression as a mathematical entity that is both a process and a concept. In other words, to that learner, an expression is a process that could be evaluated if the letters within that expression had a particular value. At the same time, that learner views an expression as an object that can be manipulated and operated upon without the need for evaluation. In
contrast, Tall and Thomas (1991) described the type of learner who does not have this conception of an algebraic expression.

We have evidence that the lack of formation of the procept for an algebraic expression causes difficulties for pupils who see the symbolism representing only a general procedure for computation: an expression such as $2+3x$ may be conceived as a process that cannot be carried out because the value of $x$ is not known (p. 137).

According to this data, it seems that a learner who can accept the lack of closure in an expression like $2+3x$ may still view this expression as prescribing a series of steps to be carried out rather than as an entity in itself. Data from deLima and Tall’s (2008) study of 68 Brazilian high school students indicate that students who do not possess a proceptual understanding of algebraic expressions carry out equation solving without understanding.

The students in deLima and Tall’s (2008) study were asked to solve linear equations and then explain their reasoning. These students often cited procedural rules as their reason for solving the equations in the way that they did. DeLima and Tall stated that these students viewed equation solving as “picking up [symbols] and moving them around” (p. 3) in a fashion that was in accordance with external rules that they did not understand. In order for students to engage in algebraic problem-solving that is meaningful and able to be built upon, students must construct algebraic expressions as procepts according to deLima and Tall (2008). Gray and Tall (1994) called the “proceptual divide” (p. 132) one that separates procedural thinking from proceptual thinking. Gray and Tall’s (1994) characterization of the robustness of proceptual thinking can inform this study by serving as an explanatory mechanism for students’ mathematical behavior and hypothesized understandings. The misconception spoken of in the next section seems that it could have its roots in the very procedural conceptions spoken of by Gray and Tall (1994).
**Concatenation**

Booth (1984) cited students’ inability to accept the lack of closure as one of the reasons that students create a “conjoined answer” (p. 19) when faced with an addition of two letters. For example, when asked to tell how many goals were scored at a soccer game where “West Ham scores $x$ goals, M.U. scores $y$” (p. 17), eight out of the 28 students given this question answered “$xy$” (p. 19). Further seven of the 28 students who attempted to find the area of a pictorial rectangle whose height was labeled as $p$ and whose base was separated into two sections and labeled as $a$ and $m$, utilized the term $am$ in their solution process. Booth also stated that, “in some cases the conjoined term is seen as necessarily equivalent to the unclosed sum” (p. 18). Students have not only been found to experience difficulty with the concept of an algebraic expression, but with the “concept of equation” (Herscovics & Kieran, 1980, p. 574) as well. In the next section, students’ varied challenges with particular attributes of equations and equation solving will be discussed.

**Students’ encountered difficulties with the interpretation and solution of algebraic equations**

Even after students have taken algebra, they have been found to face tremendous difficulties in understanding the concept of equation (Baroody & Ginsburg, 1983; Behr, Erlwanger, & Nichols, 1980). They have also been found to engage in manipulation of symbols during equation solving without understanding (deLima & Tall, 2008). These cognitive obstacles underscore the necessity for the type of instruction that would lead to algebraic problem-solving with understanding. In the following section, students’ experienced difficulties with the concept of equation will be discussed.
Meaning of the equals symbol

Because of their prior school experiences, students do not readily consider the equal symbol as an indication of the equivalence of the two expressions on either side of it (Baroody & Ginsburg, 1983). Instead, the equal symbol is often misinterpreted as a “do something signal” (Behr, Erlwanger, & Nichols, 1980) which separates a problem to the left of it from a result to the right of it (Baroody & Ginsburg, 1983; Filloy & Rojano, 1989; Kieran, 1981). This limited view of the equal symbol persists even into middle school (Behr, Erlwanger, & Nichols, 1980; Herscovics & Kieran, 1980).

In Warren and Cooper’s (2005) study, third grade students were introduced to equation solving principles via the use of a pictorial balance scale. Circles were used to represent known quantities and a bag was used as a symbol to represent the unknown quantity. Students also worked with physical scales and modified equations (where a question mark was used instead of x). The students were said to have experienced a “shift…towards viewing addition and subtraction equations in terms of equivalence, where the situation is viewed in a multi-directional way (i.e., balance)” (p. 58). These studies inform the current study in that lessons for this study were designed which first enabled students to establish an understanding of the equivalence between quantities. Then, the equal sign was introduced as a symbol of “quantitative sameness” (Warren & Cooper, 2005, p. 59) which separates two algebraic expressions that represent the same quantity. As will be indicated in the next section, students’ construction of what is, to them, a paradigmatic equation must shift in order for them to successfully learn to solve equations with understanding.
Pre-algebra students’ spontaneous equation-solving activities

Filloy and Rojano (1989) and Herscovics and Linchevski (1994) sought to ascertain the equation-solving strategies employed by students who had not yet been taught algebraic equation solving. The work of these researchers is critical to the present study in that it allows for a distinction to be made between pre-algebraic thinking and algebraic thinking. Further, this knowledge of students’ pre-algebraic understandings provides information that was necessary to design lessons that were meant to cultivate particular algebraic understandings and that are based upon students’ spontaneous problem-solving activity.

Both Filloy and Rojano (1989) and Herscovics and Linchevski (1994) found that pre-algebraic students, when solving equations of the form $ax + b = c$, primarily utilized inverse operations. During the use of inverse operations, students undo “one by one, the operations given in the left hand sequence starting with the number $c$” (Filloy & Rojano, 1989, p. 19). What these authors seem to mean is that, given an equation like $ax + b = c$, in order to obtain an answer of $c$, first $a$ had to be multiplied by $x$, then $b$ had to be added to that product using the order of operations. In order to find out what $x$ is equal to, a person could perform the inverse of the last operation performed which would mean subtracting $b$ from $c$ and then, using the inverse operation to multiplication, divide that answer by $a$.

However when presented with equations that had “at least two appearances of the unknown” (Filloy & Rojano, 1989, p. 19), both researcher teams found that the students in their respective studies did not show the tendency to “operate spontaneously with or on the unknown” (Herscovics & Linchevski, 1994, p. 63). That is, students did not tend to add, subtract, multiply, or divide one unknown quantity to, from, or by another unknown quantity as a part of their equation solving process. Instead, these students tended to use some type of guess and test method in order to ascertain the solution to the equation.
Filloy and Rojano (1989) concluded that the insufficiency of students’ arithmetic knowledge to inform actions upon the unknown signifies a “didactic cut” (p. 20) which defines the boundary between arithmetic and algebraic thinking. Similarly, Herscovics and Linchevski (1994) stated that, within the context of attempting to solve equations, students’ “inability to operate spontaneously with or on the unknown indicated the existence of a cognitive gap that can be considered a demarcation between arithmetic and algebra” (p. 63). The type of thinking that is required in order to solve equations of the form $ax + b = cx$ and of the form $ax+b=cx+d$ without the use of guess and test is algebraic in nature (Filloy & Rojano, 1989) in that students must “operate with the unknown as if it were known” (Stacey & MacGregor, 1999, p. 30).

Both Filloy and Rojano’s (1989) and Herscovics and Linchevski’s (1994) theories contribute to the understanding that pre-algebraic students do not intuitively consider operating on the unknown when initially faced with a situation that requires this type of mathematical thinking. The implication of this result is that students may need experiences that enable them to construct an understanding of the unknown quantity as an entity which represents a specific unknown number and which is able to be acted with and upon “as if it were known” (Stacey & MacGregor, 1999, p. 30).

It seems that van Oers’ (2000) discussion on mathematical symbolization might provide a clue as to why some students experience difficulty in treating a letter as if it were a specific unknown. Van Oers states, “a sign (or symbol) always involves a form of predication, suggesting actions by which the referred-to object obtains its meaning” (p. 149, italics removed). Van Oers suggests that a numerical symbol exemplifies this theory. According to van Oers, the numeral 5 is a socially accepted symbol which, amongst other situation-based interpretations, may be considered as “the result of [‘a counting’] action” (p. 149). Learners who (consciously or unconsciously) associate this counting action with the symbol five had to first construct an understanding of the concept of number through their actions upon objects (counting, sorting,
ordering, etc.) and their reflection on the relationship between those actions and their effects. These learners then had to reify this action and accept the agreed upon symbol of that reification (van Oers, 2000).

However, with regard to a letter that is meant to specify a particular number that is not known, for example, $x$, students are often asked to accept a symbol of a concept for which they may not have constructed meaning—that is the idea of an operable unknown quantity (Filloy & Rojano, 1989). Unlike the conditions under which they developed the concept of number, students may not have had the opportunity to engage in actions on objects, interiorize those actions into processes, and then reify that process to an object known as an unknown quantity. Only then, it seems, would a student be able to perceive of $x$ as a meaningful symbol of both the result of a process (that is, the result of some specific counting action where the number of counts is unknown) and an object (a specific, unknown number) (Thompson, 1994). This is consistent with Von Glasersfeld (cited in Steffe & Olive, 1996) who stated, “In my terminology, a word is used as a symbol, only when it brings forth in the user an abstracted generalized re-presentation, not merely a response to a particular situation” (p. 126).

The implication of the above is that, before letters are introduced as representations for an unknown quantity, they need to be constructed as a meaningful referent by a learner who has never before encountered letters in algebra. Mikulina’s (1991/1969) successful introduction of letters to students as a means of representing the quantities in a concrete situation serves as an example of the way in which the use of letters can be introduced within the context of the diagrams used in this study. This usage of letters to represent quantities, combined with students’ use of diagram drawing in Mikulina’s study, is discussed later in the paper.
Equality of equations

Even after algebra instruction, some students do not appear to fully comprehend the principles that govern algebraic equation solving (Bodanski, 1969/1991; Steinberg, Sleeman, & Ktorza, 1990). That is, some students are unable to identify that the process of operating identically upon both sides of an equation leaves the solution to the “new” equation unchanged from that of the former (Steinberg, Sleeman, & Ktorza, 1990). Steingberg, Sleeman, and Ktorza showed 8th – 9th grade algebra students pairs of equations and, for each pair, asked students if the equations would yield the same solution. In some cases, one of the equations was as an altered version of the other in that the altered equation contained an instance of identical operations by the same quantity on both sides of the equal sign. In other cases, the equations were not equivalent to one another. In order to solve this problem, about 13% of the eighth graders and 33% of the ninth graders chose to solve both equations separately and compare the solutions. These students did not appear to have an understanding of the invariance of an equation solution under transformation (Steinberg, Sleeman, & Ktorza, 1990). Further, about 48% of the eighth graders and 5% of the ninth graders determined that an equation and its transformation were not equivalent.

Two of the reasons provided by students who fell into the last category were: 1) one equation contained more terms than the other, thus making the two equations non-equivalent and 2) both sides of one equation were reduced by a certain quantity, thus making the transformed equation “‘less’” (p. 117) than the source equation (Steinberg, Sleeman, & Ktorza, 1990). Some of these students’ understanding of equations seems to be limited to that of a collection of terms, the nature of which may or may not be relevant to these students, especially in the process of equation solving. Based upon the results of this study, it seems likely that students who were not able to recognize the equivalence between an equation and its transformation perform equation
solving by rote and without semantic understanding of their actions (Kaput, 1987; Steinberg, Sleeman, & Ktorza, 1990).

**Usefulness of Informal Problem Conditions**

Whereas much has been reported on both pre-algebra and algebra students’ cognitive difficulties with algebra, a growing number of studies have examined the means by which students may utilize informal problem-solving methods in order to solve algebra word problems (Hall, Kibler, Wenger, & Truxaw, 1989; Ng & Lee, 2009; van Reeuwijk, 1995b). It has been reported that students’ utilization of informal reasoning enables students to access the structure of a problem (Hall, Kibler, Wenger, & Truxaw, 1989). However, there is a limit to the degree to which utilizing certain types of informal reasoning will enable a student to abstract the particular algebraic understandings (van Reeuwijk, 1995). The following section discusses some of the findings related to students’ use of informal strategies within an algebraic problem-solving environment.

**Students’ use of informal problem-solving methods**

The use of informal reasoning in order to solve algebra word problems has been advocated by some researchers who argue that this approach enables a person who is not able to identify the structure of the problem to get at that structure (Hall, Kibler, Wenger, & Truxaw, 1989). The findings of these researchers support the choice made by the researcher of this study to build upon students’ informal problem-solving strategies in an effort to enable students to construct particular algebraic understandings.
According to Hall and colleagues (1989), a "situation model" is a representation that numerically captures the essence of the relationships between quantities in a problem situation and whose use does not rely on algebraic thinking or strategies. According to them, it is used when a person seems unable to discern the structure of the relationship between quantities in a problem.

Hall and colleagues posed the following “motion” problem to undergraduate computer science majors: “Two trains leave the same station at the same time. They travel in opposite directions. One train travels 60 km/h and the other 100 km/h. In how many hours will they be 880 km apart?” (Hall et al., 1989, p. 225).

One student utilized a situation model, in the midst of other solution attempts, to solve the problem. The student used a “building-up strategy”, in Lamon’s (1994) term, to determine how far each of the two trains would be from the starting point and from one another cumulatively after a certain number of hours. This student later wrote and utilized time–distance formulas to determine the answer. It seems that the final stage of this student’s solution process was informed by his initial informal reasoning. That is, by engaging in the building up activity, the student seemed to abstract the situational structure of that which this problem was an example and to finish the problem formally (Hall et al., 1989).

Hall and colleagues’ (1989) work implies that when the structure of a problem is recognized, a formal representation of this relationship may be constructed, as exemplified by the student in the episode above. Similar to this student, it seems likely that pre-algebra students would not possess a formal solution method with which they might solve algebra word problems. However, unlike the student in the above episode who was able to later draw upon a formal problem-solving method, pre-algebra students who are experiencing algebra word problems as novel would not likely have an extensive set of related formal structures upon which to draw. But, given a series of similar problems over time, these students may generalize and abstract their
own solution methods which they may, over time, curtail and, later, formalize. Thus, more formalized solution methods may be developed from students’ informal activity within problem-solving environments that support such activity.

The results of McFadden’s (1998) study corroborated the results of that of Hall and colleagues (1989) with regards to the effectiveness of informal problem-solving strategies in helping students to identify the structure of a problem. McFadden (1998) studied sixth grade pre-algebra and seventh and eighth grade algebra students’ ability to solve algebra word problems and sort those problems according to their structure. McFadden found that those students who used non-algebraic solution strategies were more likely to correctly identify problems having the same structure than were students who formulated algebraic equations. According to McFadden, “with informal strategies, such as guess-and-check, problem solvers do not distance themselves from the relationships of the problem” (p. 169).

Similarly, Koedinger and Nathan (2004) found that the high school algebra students they studied were more successful at solving word problems containing one instance of an unknown quantity than they were at solving an equation that modeled the underlying structure of that problem. To solve the realistic word problems, students utilized either guess and test or inverse operations. Koedinger and Nathan speculated that students were better able to solve these problems using informal methods because the students encountered situations and language that were familiar to them. On the other hand, for novice algebra students, solving structurally similar equations (absent the word problem context) may have been a more novel experience due to students understanding of the meaning of algebraic symbols.

Whereas the use of strategies such as guess and check seems to allow problem solvers to identify the structure of a word problem as reported by McFadden and to utilize students’ current conceptions as indicated by Koedinger and Nathan (2008), the use of such a strategy may not lead to a more formal solution method according to van Reeuwijk (2001).
reported on the algebraic development of 12 sixth grade students who engaged in lessons from a pilot version of the *Comparing Quantities* unit (Kindt, Abels, Meyer, & Pligge, 1998) of the Mathematics in Context curriculum. These lessons were designed to enable pre-algebraic students to use informal strategies to solve pictorial and word problems which, potentially, could be modeled using systems of equations.

According to van Reeuwijk (2001), students were given the following tools or strategies with which they could reason in order to solve the problems within the unit: “compare and exchange” (p. 614), combination chart, and notebook. The students were also given the option of using a guess and test strategy to solve problems. The strategy of exchanging was encouraged for use on pictorial word problems such as that given in Figure 2-2. It seems that in order to solve this problem, a student could replace the pineapple on the left side of the second scale with five bananas (given the relationship from the first scale). Then, a person could remove two bananas from each side of the second scale in order to find that one apple is the same weight as three bananas. The other two tools, combination chart and notebook, were means by which students could establish and use patterns to list the prices of various combinations of each of the two items in order to find the price of each item (van Reeuwijk, 1995b).

![Figure 2-2: Comparing and exchanging task given to sixth grade students in *Comparing Quantities* unit (van Reeuwijk, 2001, p. 614).](image)

Van Reeuwijk (1995b) stated that students could use any strategy of their choice to solve the problems in the unit. On the unit’s post-assessment, van Reeuwijk found that some students’
problem-solving methods had not developed beyond their pre-instructional strategies. Twelve percent of the students used guess and test. Van Reeuwijk found that approximately one-third of the students who used the exchange strategy on the post-assessment (39%) did not demonstrate a qualitative shift from their pre-instructional understandings. The assessment data in general was corroborated by “interviews with teachers and students” (p. 8) and classroom observations. One of van Reeuwijk’s conclusions concerning this result was that students’ unconstrained use of guess and test throughout the unit did not support their vertical mathematization of their informal strategies. Thus, problems in the next iteration of the unit were constructed in such a manner as to limit students’ use of guess and test and to encourage the use of informal or systematized exchanging strategies.

Success of informal problem-solving environments

The implication of Hall and colleagues' (1989), McFadden's (1998), and Koedinger and Nathan’s (2008) findings is that informal reasoning may serve as a powerful problem-solving tool in that it enables problem solvers to utilize their current conceptions to (consciously or unconsciously) to unpack the structure of an algebra word problem. However, van Reeuwijk’s (1995b) work shows that the type of informal reasoning that is supported has the potential to constrain or afford the abstraction of algebraic strategies for solving algebra word problems. Based on these observations, it seems that instruction to pre-algebra students in an algebra problem-solving environment should both build upon and evoke those informal strategies that have the potential to be vertically mathematized into more formal, sophisticated algebraic strategies (van Reeuwijk, 1995b).

Particular researchers have succeeded in providing students with environments which would enable them to construct the understandings necessary to solve equations and which started
from their use of informal actions. These environments had in common the use of representations upon which students could act informally to find the value of an unknown quantity. Further, these environments held the potential of supporting students’ generalization and abstraction of such actions. Finally, each setting afforded students the opportunity to observe or verify the results of their actions. Below are descriptions of three experiments which share the above characteristics and within which students were successful in learning how to simplify expressions or solve equations of the type $ax + c = bx + d$.

Gregg and Yackel (2002) designed a series of lessons with the intent of helping pre-college students in a “developmental-level beginning algebra course” (p. 493) to construct a means of adding and subtracting algebraic expressions with understanding. Students were told that, in a particular candy shop, the number of pieces of candy that were packed in a roll of candy differed according to flavor of the candy. Students were given diagrams like the one shown in Figure 2-3 and asked to determine how much candy was packed during the day.

According to Gregg and Yackel (2002), students operated upon the above diagram in a manner similar to that which is shown in Figure 2-4.
Students utilized the diagram given in Figure 2-3 as a means by which they could reason mathematically. Students matched those elements of the diagram that were identical at the beginning and end of the day. Using an arrow and a circle around an individual piece of candy, students then represented the fact that, at the end of the day, one of the candy rolls was missing a piece of candy. (The notion of a roll that is missing a piece of candy and a representation for this concept was introduced by the researchers in a previous pictorial problem). It seems that this activity enabled students to visualize the fact that, since there were four individual pieces of candy at the beginning of the day and only three individual pieces at the end of the day, one piece of candy must have been removed from one of the candy rolls in order to account for the missing fourth piece.

Gregg and Yackel stated that tasks such as those in Figure 2-3 “helped [students] begin to develop a notion of composite units” (p. 494). Thus, it seems that the opportunity afforded to students to act upon a diagram of the type in Figure 2-3 enabled them to reason algebraically about known and unknown quantities in a manner that did not require prior algebraic understandings. They seemed to reason that \(4x + 3 - (2x + 4) = 2x - 1\) based upon the two packages (with one piece of candy symbolized as being removed) that are not crossed out on the right side of the diagram in Figure 2-4.

As with Gregg and Yackel’s (2002) study, Aczel (1998a), created an environment in which students were able to act informally in order to represent algebraic situations and manipulate representations of the unknown quantity. Aczel (1998a) studied 14-15 year old students who had already been taught equation solving. Prior to the experiment, Aczel’s test results showed that many of these students were not proficient in solving algebraic equations containing negative quantities. Further, these students’ ability to construct algebraic equations for
realistic scenarios involving one or more unknown quantities was very weak. Aczel engaged 22 of these students in a two-hour experiment during which they solved “computerized balance” (p. 2) problems, equations, and algebra word problems using the EQUATION program.

Created by Aczel for his study, the EQUATION program began with a pictorial phase (see Figure 2-5) in which students could guess an answer, remove objects from the balance scale, or give up.

Figure 2-5: Screen capture of pictorial phase of EQUATION program (Aczel, 1998b, p. 143)

Over time, different changes were phased into the onscreen representations. For example, each unknown weight was later labeled with a letter and the two buttons for removing objects from the scale were replaced by “a single [-] button” (Aczel, 1998a, p. 3). Eventually, all pictures and words were replaced by algebraic notation. The program later asked students to solve equations that were independent of the balance model context, some of which contained negative quantities (see Figure 2-6). Finally, the program gave students word problems which they could choose to model and solve algebraically or whose answer they could guess.

Figure 2-6: Screen capture of EQUATION program (Aczel, 1998b, p. 144)
Following the EQUATION experience, students who engaged in the EQUATION program showed an improvement in their ability to solve an equation of the form \( ax + b = cx + d \) as well as in their ability to represent and interpret realistic situations using algebraic notation. Further, more students in this group utilized algebraic notation in order to model and solve an algebraic word problem.

The success of the EQUATION experience seems to lie in the fact that students were able to operate first upon the objects that were comprehensible to them (weights on a balance scale) before having to operate on letters in an equation.

Aczel (1998a) noted the interactions of two students in particular as they worked within the EQUATION environment. The transition of these two students, Rebecca and Nicola, to solving equations that symbolically represented the relationships that were embodied on the computer balance scale was very smooth according to Aczel (1998a). In fact, the data show that these students’ initially solved the equations presented to them even faster than they did the pictorial puzzles. According to Aczel, the students were able to associate objects in the balance environment with the “objects” in the equation (where the letter b was used in place of a barrel to represent the unknown quantity). Further, students appeared to be able to apply their generalized and abstracted solution method from the balance environment to the structurally similar equation environment.

Pirie and Martin’s (1997) case study of Alwyn, who taught pre-algebra students how to solve algebraic equations, reveals another instance in which students’ informal activity was used as a building block from which more formalized equation solving emerged. Alwyn’s goal was to enable students to construct an understanding of the attributes of an equation and the principles of equation solving.

In his lesson, Alwyn asked students to solve the following problem: \( \Box + \Box + 18 = \Box + 53 \). Students were invited to try to guess the number that should be placed in the boxes in order
to make the equality statement true. They were then invited to adjust their guess if it was incorrect.

It seems reasonable to assume that, when the problem was initially posed, the students in the class (having never seen work of this type before) perceived the equation with boxes as a model of a particular arithmetic situation. That is, the students may have perceived of the equation: \( \Box + \Box + 18 = \Box + 53 \) as a case where the sum of the double of a particular number and 18 is equal to the sum of that number and 53. However, the work of one of the students, Joan, seems to indicate that this student began to use the equation with boxes as a model for reasoning algebraically. Joan acted upon the boxes in the equation in such a manner as to abstract a method for finding the value of the unknown quantity.

One student, Joan, verbalized her means of solving a problem on the worksheet:

(Working with \( \Box + \Box + \Box + \Box + \Box - 10 = \Box + \Box + \Box + 4 \)) These three boxes (on the right hand side) cover up these three boxes (three of the five on the left hand side) like that (putting her fingers over the two sets of three drawn boxes) and then that will leave you two (boxes), So it’ll be two boxes take away ten and four, and then it’ll be two boxes equals fourteen. (This covering up action was used by several of the pupils). (Pirie & Martin, 1997, p. 170)

It seems that Alwyn’s students’ success in learning to solve equations of the form \( ax+c = bx+d \) was based in their formalization of their earlier informal solution processes when solving the equations with boxes.

In the section that follows, students’ use of diagram drawing as a means of solving algebra word problems is explored.

**Diagram drawing as a promising heuristic in the development of algebraic reasoning**

In the above discussion, it has been argued that a problem-solving environment which invites the use of informal strategies is a potentially advantageous setting for the development of
algebraic understandings by pre-algebra students. Research showing that pre-algebra students often possess misconceptions about algebraic symbols and that they are unable to “operate spontaneously with and on the unknown” (Herscovics & Linchevski, 1994, p. 63) has been discussed. Further, it has been pointed out that, even after instruction, students possess misunderstandings about the composition and transformation of an equation. Thus, it seems that a problem-solving environment that does not rely solely on the use of algebraic symbolism, as Simon (1988) observed, yet which affords the opportunity for the development of algebraic thinking, is needed. A setting in which students can draw a diagram in order to solve algebra word problems seems to be a feasible environment for the following reasons.

Diagrams have the potential to be created in such a manner that they serve as a model of the mathematical structure of an algebra word problem. That is, students may create diagrams that model the relationships between known and unknown quantities given in the problem statement. Representations of this nature are said to be advantageous in that they allow the problem solver to visualize quantitative relationships that are both implicit and explicit in the word problem (Bodanskii, 1991/1969; Paige & Simon, 1966). Such diagrams also have the potential to be acted upon by students in such a manner that the diagram may become, to those students, a model for their mathematical reasoning.

Through a learner’s reflection on the relationship between his or her actions upon a diagram and the effects of those actions (Simon et al., 2004), he or she may generalize and abstract those actions that he or she used to successfully solve such problems. As a result, students have the potential to vertically mathematize their actions upon diagrams in such a manner that they are able to construct an equation based upon a problem situation and operate analogously upon the letters and numbers within that equation (Simon, 1988).
Students’ successful use of diagrams to model the mathematical structure of an algebraic problem-solving situation


To this end, the teachers in each of the aforementioned Soviet researchers’ studies utilized a curriculum with their students (beginning when these students were in the first grade) which enabled them to represent concrete situations and realistic word problems using diagrams. Students, initially, engaged in problem-solving experiences in which an unspecified amount of liquid or grain was poured into two or more containers. Students were taught how to create a model of the part-whole relationship between measures using diagrams. Letters were used to represent the amount of the substance in each container. For example, in the diagram in Figure 2-7, two students in Mikulina’s study represented the volume of grain (called “grain $a$” by the teacher) that was poured from a jar into three glasses. In the first and third diagrams, the students seemed to have created models of the situation in that each diagram shows the quantity $a$ as a whole that is divided equally into three parts. Whereas the second diagram does not accurately portray the situation in that only one of three boxes is labeled with an $a$, this diagram shows a student’s attempt to model the situation in that three boxes are drawn.
Figure 2-7: Three students’ representations of a situation in which “grain a” was poured into three containers (Mikulina, 1991/1969, p. 197).

After the lessons described above, students in Mikulina’s (1991/1969) study were introduced to more abstract part-whole situations in the form of verbal word problems. These situations, too, were represented by students diagrammatically. For example, students were given the task to “draw the segment $f$ and then make $f$ become a part of a whole” (p. 200). The diagram in Figure 2-8 was created by one of the students in answer to this question. Mikulina stated that the majority of the students answered this type of question correctly.

![Diagram](image)

Figure 2-8: A student’s diagram of a situation given in a verbal part-whole problem (Mikulina, 1991/1969, p. 200)

It seems that, to make this diagram from the situation given, a student would have had to have an understanding of the relationship between parts of a whole and the whole itself. Based upon the fact that students had previously created diagrams to model realistic situations, it seems possible that students drew upon these diagram drawing experiences in order to conceptualize and represent the part-whole relationships in the verbal statements.

Following students’ experiences in modeling real life and abstract situations using diagrams, students in each study were given diagrams which represented part-whole relationships and asked to create as many equations as they could from that diagram. For example, in Mikulina’s (1991/1969) study, students were given a diagram like the one in Figure 2-9. Below the diagram is an example of the equations created by students for that situation.
It seems that, in order to create the equations like the ones shown above, a student would have to reason from the diagram by making explicit to him or herself the ways in which the parts and whole are related to one another. That is, a student would have to think of quantity $c$ as a part of quantity $a$, for example, or that quantity $a$ was a whole composed of parts $k$, $f$, and $c$. Initially used by students in Mikulina’s study as a means of modeling concrete and abstract situations, diagrams themselves eventually became students’ object of reasoning (Mikulina, 1991/1969). In other words, students used the same type of diagram that they had previously created to model the distribution of grain or liquid into containers as the source for their equation writing.

The current study drew heavily upon the part-whole emphasis utilized within Mikulina’s (1991/1969) study as a means of enabling students to construct an understanding of the relationship between known and unknown quantities in an equation. However, multiplicative part-whole reasoning combined with additive part-whole reasoning were employed more frequently in this study than was additive part-whole reasoning alone as in Mikulina’s study. The way in which this occurred is discussed in more detail in the task analysis and data analysis sections of this paper.

Like the students in Mikulina’s study, students in Bodanskii’s (1991/1969) study learned to symbolically represent simple realistic situations in which quantities were additively related. Unlike the students in Mikulina’s (1991/1969) study, students in Bodanskii’s study went on to learn to represent and solve complex algebra word problems including those of the form $ax + b =$
Students were taught how to represent the structure of an algebraic word problem utilizing diagrams. For example, the following problem was given to students: “In the kindergarten, there were 17 more hard chairs than soft ones. When 43 more hard chairs were added, there were 5 times more hard chairs than soft. How many hard and soft chairs were there?” (Bodanskii, 1991/1969, p. 302).

The diagram in Figure 2-10 is a recreation of that which was drawn by a student to solve the above problem. In this replica of a student’s diagram, the heavy lines represent the quantities that are explicitly given in the problem while the dotted line represents quantities that were not given in the problem. The diagram for the number of soft chairs (labeled M) has been extended, to become equal in length to the diagram for the number of hard chairs. According to Bodanskii, “there are now two equal quantities, I_b—the total number of hard chairs, and II_b the supplemented number of soft chairs” (p. 302).

Figure 2-10: A diagram similar to that which was drawn by a student to model an algebra word problem (Bodanskii, 1991/1969, p. 302)

With his or her depiction of a relationship of hypothetical equivalence between the number of soft chairs and the number of hard chairs, the problem solver has brought something to the diagram that was not present within the problem. In this way, it could be stated that the problem solver is now reasoning from the very diagram that was created as a model of the situation given. According to Bodanskii (1991/1969), problem solvers answering this type of question went on to successfully create an equation of the situation. Thus, it may be stated, again,
that students’ ability to represent and solve algebra word problems was facilitated by their creation of a diagrammatic model of the situation that they then utilized as a model for reasoning. The tasks which Bodanskii (1991/1969) utilized in his study greatly influenced the task structure and sequencing in the current study since it was my goal to enable students to develop similar understandings as those constructed by Bodanskii’s students. In particular, some of the questions used in Bodanskii’s (1991/1969) that required the use of hypothetical equivalence in order to solve were used, simultaneously, as assessment tasks and starting points for instruction.

As with students who participated in the Soviet studies, Singaporean students are taught to use diagrams to model the relationship between quantities as a means of solving word problems that are algebraic in nature (Ng & Lee, 2009). “The model method” has been used in Singaporean classrooms since 1983 and entails drawing the structure of algebraic and arithmetic word problems via the use of boxes to represent quantities (see Figure 2-11 and Figure 2-12). The representation of an unknown quantity (be it a rectangle or a brace outlining a figure whose measure is unknown) is labeled with a question mark when students are in the primary grades and letters when students are older. In Figure 2-11 and Figure 2-12, the letters $a$, $b$, $c$, and $d$ represent quantities whose value are given in a problem situation while the question mark in the diagram and the letter $x$ in an equation denote the unknown quantity for a given problem.

![Diagram](image)

Figure 2-11: “Multiplication and division models for arithmetic word problems (at left) and algebraic word problems (at right)” (Ng & Lee, 2009, p. 289)
Ng and Lee (2009) studied the algebraic understandings of 151 fifth grade children in five Singaporean schools who had been taught to use the model method to solve problems that were algebraic in nature. Students were already grouped into two levels of mathematics classes, EM1 and EM2, depending on their English and math proficiency. EM1 was the group with an overall higher proficiency in English and math. These primary school students were given a number of algebra word problems to solve. The problems ranged in difficulty from problems that had only one unknown quantity and contained only whole number known quantities in the question to problems that had two unknown quantities or that contained some fractional known quantities.

All students were asked to use the model method to solve these problems; however, they were allowed to use other methods if they needed to do so. Ng and Lee (2009) found that approximately 80% of the students in the EM1 group and approximately 30% of the students in the EM2 group were able to correctly answer the two problems containing only whole number quantities and one unknown quantity. One of these problems is as follows: “A cow weighs 150kg more than a dog. A goat weighs 130 kg less than the cow. Altogether the three animals weight 410 kg. What is the mass of the cow?” (Ng & Lee, 2009, p. 295).

Approximately 38% of the students who answered this question correctly utilized the model method to answer the question. According to Ng and Lee (2009), these students drew the

Figure 2-12: “Comparison models: arithmetic model (at left) and algebraic model (at right)” (Ng & Lee, 2009, p. 287)
lengths of the rectangles representing each animal’s weight relative to the lengths of the other rectangles, depending upon the information given in the problem. The known difference between quantities was represented by a labeled brace that bridged the gap between the right edges a given pair of rectangles. (In the type of diagram most frequently used, the implicit difference between the mass of the dog and the mass of the goat was represented using a dotted line). And they utilized a particular “generator” (p. 301), either the mass of the cow or the mass of the dog, as a means of solving the problem. That is, the length of the rectangle representing the mass of the cow or the mass of the dog was used as a norming unit. Students utilized the given differences between the generator and the mass of the other two animals to adjust the total given weight up or down, creating a quantity that was three times the amount of the generator. That result was divided by three in order to attain the value of the generator.

Figure 2-13 “Solutions to the Animal Problem: Use of the mass of the dog as generator (at left) and use of the mass of the cow as generator (at right).” (Ng & Lee, 2009, p. 301)

Overall, Ng and Lee (2009) found that many students gave answers that were only partially correct. They identified a number of students in this category whose diagrams, though seemingly accurate in its portrayal of the problem situation, may have contained one small error that resulted in an erroneous or partial solution. The errors that Ng and Lee (2009) stated were responsible for some of these faulty conclusions were: “misinterpretation of a piece of
information correctly captured in the model, misrepresentation of a piece of information, or changing the generator midway through the solution” (p. 310).

The results of Ng and Lee’s (2009) study inform this study in the sense that the types of errors committed by students that Ng and Lee found to lead towards incorrect or partial solutions could be used as a means of examining or explaining the difficulty that students in this study may have experienced. Further, the factors that these researchers found to lead to successful use of diagrams to solve algebra word problems also have explanatory value in analyzing students’ successful efforts in representing and solving algebra word problems in this study.

In addition to the Soviet studies, the Singapore model method has informed the structure of this study’s diagrammatic environment as well as this study’s emphasis on part-whole understandings. Many of the tasks in the current study mirror that which is utilized within the model method because of the potential that these tasks held for the mathematization and formalization of students’ informal, yet algebraically based solution processes.

The commonality between each of the studies in this section is that students’ creation of diagrams that enabled them to represent the structure of a problem situation was useful in enabling them to solve algebra word problems. The diagrams gave the problem solvers the opportunity to operate upon a representation of the unknown quantity in such a manner that the problem solvers avoided the cognitive obstacles usually associated with operating upon such a quantity. Further, the work of these researchers shows that diagrams that are correctly drawn are the most useful in enabling the problem-solving process. Finally, this collective body of research shows that diagrams may be operated upon in a manner that is synonymous with algebraic problem-solving using actions that may letter be represented symbolically (Simon, 1986). The next two sections discuss other benefits of diagrams to the algebraic problem solver.
Students’ reading of diagrams as a part of the problem-solving process

In Mikulina’s (1991/1969) study, students who correctly solved the task of writing equations based upon a given diagram (as shown in Figure 2-14) seem to have benefited from the opportunity to “read off” (Paige & Simon, 1966, p. 107) the diagram based upon their part-whole understandings (Mikulina, 1991/1969). That is, students had the opportunity to visualize the parts and whole in relation to one another on the diagram and to determine the quantitative relationships that existed.

Figure 2-14: A diagram given to students from which they generated the equations below it (Mikulina, 1991/1969, p. 209)

Bodanskii (1991/1969) stated that, when diagrams are used to represent the given relationship between quantities presented in a problem situation, students have the opportunity to visualize this relationship between the quantities from the situation. For example, the type of diagram in Figure 2-15 was drawn by second and third grade students in Bodanskii’s study to represent the following problem: “One piece of wire was 54 m longer than another. When 12 m was cut from each of them, the first turned out to be 4 times longer than the second. Find the length of each piece of wire” (p. 305).

Figure 2-15: A diagram from Bodanskii’s (1991/1969) study to represent the problem situation
Figure 2-15: The type of diagram drawn by 2nd and 3rd grades student to represent an algebra word problem (Bodanskii, 1991/1969, p. 305)

Bodanskii stated that the diagram “led to the choice of the remainder of the smaller segment as the unknown quantity and, correspondingly, to the composition of the elementary equation $3x = 54$” (p. 305). It seems that a student who chose to identify and name segment x had to have “read” from the diagram certain relationships. That is, this student, having identified the parts of both line segments that represented a length of 12, was able to reason that the length of the remainder of the bottom line segment had to be that which was multiplied by four to get the length of the remaining top segment.

The students in the aforementioned studies were learners who had not yet been exposed to formal instruction in algebra. However, through the creation and use of diagrams that modeled the structure of a problem situation, these students were able to correctly determine the relationships between quantities and accurately solve algebra word problems. Just as well-structured diagrams can be useful in interpreting and solving algebraic problem situations, Bodner and Goldin (1991) found that poorly or inaccurately structured diagrams can serve as a hindrance to students’ problem-solving activities, even amongst those who have taken formal algebra. College algebra students in Bodner and Goldin’s (1991) study were given the following word problem: “The length of a rectangle is one inch greater than twice its width. The perimeter is 26 inches. What are the dimensions of the rectangle?” (p. 160)

Twenty of the 22 students interviewed either labeled their diagram partially or incorrectly, did not label their diagram, or drew no diagram. At least 13 of these students created inaccurate equations to represent the problem information. Bodner and Goldin speculated that many of the errors were due to the way in which students, if they had used Paige and Simon's (1966) words, “read off” their incorrect diagrams since all of the students demonstrated
understanding of ‘perimeter’ earlier in the interview” (p. 162). Bodner and Goldin’s speculation suggests that correct diagram drawing can facilitate the writing of an equation.

However, the fact that nine students were able to create correct equations even though only two students created correct diagrams suggests that perhaps not all of the writers of correct equations needed an externally represented diagram in order to carry out their task. The implications of this possibility is that older learners may not need to use a diagram in order to solve algebra word problems, whereas younger, pre-algebraic students (as in Mikulina and Bodanskii’s studies) seem to be helped by their use.

Usefulness of diagrams in enabling students to perceive implicit quantitative relationships

Bodanskii (1991/1969) stated, “It is…necessary that in the first stages of instruction, the children be given a special means for the discernment and expression of the implicit relationships between quantities” (p. 298). Bodanskii (1991/1969) stated that diagram drawing was the means by which such implicit relationships may be revealed. For example, when solving the chair problem (mentioned previously), by creating a diagram of the situation, students in Bodanskii’s study had the opportunity to visualize the relationships between the quantities given in the problem. This made it possible for students to potentially determine the relations between “quantities [that] must…[be] equalized” (Bodanskii, 1991/1969, p. 302) in order to solve the problem.

Students in Mikulina’s study also seemingly benefitted from their use of diagrams as a tool in representing the quantitative relationships in word problems. Based upon the foundation laid in the first teaching phase of Mikulina’s study, students were asked to represent a given realistic situation or “text” (Mikulina, 1991/1969, p. 215) in as many ways as they could. A sample text follows: “There were \(a\) red pencils and \(b\) blue ones in a box, and the total number of
pencils was $c$” (p. 215). Students represented these texts both diagrammatically and formulaically. Students were then asked to create their own realistic situation based upon the original text that was given to them.

Mikulina (1991/1969) stated, “Here the drawing served as a major aid to the students. It helped them, first list the necessary number of quantities in the content of the text and, second, to place these quantities in a definite connection” (p. 215). In other words, the diagrams created by students seemed to help them to externalize and visualize the parts and whole implicit in the realistic situation and to conceptualize the way in which each part or whole related to the other. For example, based upon the diagram shown in Figure 2-16, a student wrote: “There were $d$ ships in a harbor, $f$ ships departed and $m$ ships remained” (p. 215).

\[\text{Figure 2-16: Diagram from which a student wrote a part-whole word problem (Mikulina (1991/1969, p. 216)}\]

In this instance, it seems that the diagrams helped to make obvious to the students those quantitative relationships embedded in the original text (Mikulina, 1991/1969). Based upon this observation, students were able to create new texts containing quantitative relationships that were isomorphic to those that were in the original text.

In conclusion, it seems that students’ use of diagrams in order to be able to model the structure of an algebra word problem seems to be helpful to them in the following ways: (a) In accordance with Piaget’s (1971) theory, students may act upon objects that are cognitively available to them and in such a manner that they meaningfully reflect upon their actions and the results of those actions (Simon et al., 2004); (b) students may, in the language of RME, vertically mathematize their mathematics activity so that their informal activity may be formalized and
symbolized; and (c) students may circumvent the misconceptions associated with symbols in algebra while still developing and utilizing algebraic thinking skills.

Given the wide-ranging misconceptions that permeate the mathematical thinking of pre-algebra and algebra students alike with regards to the symbols and language of algebra, it seems that there is an urgency to utilize alternative methods to traditional algebra teaching. The aforementioned Soviet studies which build from students’ current conceptions and provide students with an informal problem-solving environment in order to solve algebra word problems provide a rich foundation from which to build.
Chapter 3

Methodology

This study focuses on the means by which students might develop algebraic understandings and the means by which students’ pre-instructional understandings might be harnessed in that effort. To answer the research questions requires establishing: a) the nature of the algebraic understandings developed by students as a result of their engagement in algebra problem solving via the use of diagrams consisting of connected rectangles, and b) the means by which students seemed to develop these understandings, including the aspects of the environment that seemed to constrain or afford this development.

Teaching Experiment

The design of this study is a teaching experiment. A teaching experiment is one in which an investigator acts as both a researcher and teacher of one or more students in an effort to build a model of the process by which students construct particular mathematical understandings (Steffe & Thompson, 2000).

At the start of a teaching experiment, the researcher–teacher (heretofore, referred to as the researcher) begins with a conceptual goal for the students. After assessing the students’ current mathematical understandings with respect to that desired conception, the researcher builds a model of those understandings (Steffe & Wiegel, 1996). This model is, necessarily, from an observer’s perspective and is based upon students’ mathematical communication (nonverbal and verbal) across a number of conceptually related problem-solving tasks (Steffe & Thompson,
The researcher plans lessons based upon his or her conjecture of a likely path of development of students’ mathematical constructions—a “hypothetical learning trajectory” (Simon, 1995)—beginning from the students’ current conceptions to the intended conceptual advance (Simon, 1995).

After each lesson, the researcher views a video of that lesson. One of the purposes of this viewing is to provide the researcher with the opportunity to construct models of students’ mathematical understandings (Steffe & Thompson, 2000). The construction of such models enables the researcher to adjust his or her hypothetical learning trajectories, as needed, and to plan (or alter the plans for) the next day’s lesson (Simon, 1995). Further, the ongoing practice of specifying students’ understandings informs the researcher’s current and subsequent analysis of the way in which students’ understandings developed over time. The iterative process of assessment–analysis, model building, planning, and teaching continues throughout the life of the experiment after which “retroactive analysis” (Simon, 2000, p. 341) of the data occurs.

There are several reasons for the choice of a teaching experiment methodology for use in this study. First, the teaching practices recommended by this methodology are consistent with the researcher’s goal-directed stance concerning teaching and learning. The researcher reviewed the learner’s understandings on an ongoing basis based on the interactions that occurred during a teaching session. The researcher created or adjusted tasks based upon the learner’s current understandings. By engaging this process, the researcher made pedagogical decisions prior to the lesson based upon her perception of students’ evolving understandings and the schemes that seemed to be available to them (Steffe & Thompson, 2000).

Further, during teaching, the researcher inquired into and built upon students’ knowledge by observing students’ “ways and means of operating” (Steffe & Thompson, 2000) when they were placed in situations having particular mathematical constraints (Simon, 2000; Steffe & Thompson, 2000).
Secondly, the teaching experiment methodology was appropriate for use in this study because the researcher sought to specify a means by which formal algebraic thinking may emerge from students’ pre-formal activity in a diagrammatic environment. Implicit within the methodology is a model-building framework which afforded the researcher the opportunity to specify the development of particular mathematical understandings over time. As a result, the construction of a model for the development of algebraic thinking arose organically out of the research activities that were recommended by the teaching experiment methodology.

A theoretical assumption of the teaching experiment methodology is that there is a reflexive relationship between the construction of knowledge by an individual and the social context in which that individual interacted mathematically (Cobb, 2000). With regards to the occurrence of mathematics learning within a social context, Cobb and Whitenack (1996) stated,

> In particular, the notion of a learning opportunity indicates that mathematical learning is viewed as a process of conceptual self-organization as well as of enculturation. Individual students’ constructive activities are therefore considered to be socially situated in that they occur as they participate in classroom social processes. **However, the proposed linkage between psychological and social processes is indirect in that, in the last analysis, it is the students who interpret others’ actions and reorganize their mathematical activity.”** (p. 218, bold added).

The researcher in this study operated from the assumption that the social setting in which a learner interacts mathematically affords and constrains his or her mathematical constructions in particular ways. This assumption had implications for teaching, data collection and data analysis and those implications are addressed in those sections, respectively.

### Setting and Participants

A pilot study and a main study were conducted as a part of this research. Each of the pilot and main studies contained two phases: a pre-assessment and a teaching experiment. The pilot study was used as a means of assessing the potential efficacy of the tasks, the structure of
the teaching sessions, and the data collection methods in yielding the type of data that would allow the researcher to answer the research questions. What follows below is a discussion of the main study.

Setting

The first phase of the main study was a pre-assessment. The participants in this phase were sixth graders who attended a private, predominantly African American middle school in Philadelphia. This school was chosen for several reasons. Due to my employment at this school, I had the opportunity to conduct the study during school hours and immediately after school. As a math specialist and 7th and 8th grade math teacher at the school, I had never taught any of these students as a classroom teacher. However, as a means of professional development, I did conduct some demonstration lessons within these students’ classrooms whose mathematical content was non-algebraic in nature. Due to my classroom visits, I was familiar with the students and had established an informal rapport with them. Therefore, my conversations with the students were a natural extension of my earlier interactions with them.

Further, I was very familiar with the school’s math curriculum. Therefore, I was aware that the first through sixth grade students at this school, at the time, learned from a very traditional math curriculum and would not be introduced formally to algebra until the seventh grade.

I did not conduct this study, however, with an assumption that African American students learned mathematics differently from non-African American students. Quite the opposite, I made the assumption that the construction of logico-mathematical knowledge by every student takes place in the manner outlined in the theoretical framework section of this paper.
Participants

The two sixth grade classrooms at this school were visited and a letter of recruitment was given to each student to take home to their parents. Those students who were willing and able to participate in the study during the day or immediately after school and who received permission from their parents to participate were included in the first phase of the study.

Three students were issued a pre-assessment which was intended to determine whether the students demonstrated particular prerequisite understandings. This pre-assessment and these understandings are discussed in detail in the next section. Three students demonstrated the prerequisite understandings and thus participated in the second phase of this study.

The second phase of the study was the teaching experiment. Based upon their demonstration of the prerequisite understandings, all three students who took the pre-assessment were included in the second phase of the main study. However, one of those students will not be included in this paper’s analysis. Due to this student’s frequent unavailability, he was not able to finish a substantial portion of the task sequence.

Additionally, a fourth student who had participated in the pilot study was included in the second phase of the main study. This student, Valerie (a pseudonym), had been given the same pre-assessment as the main study participants at the beginning of the pilot study. During the pilot study, Valerie demonstrated a keen understanding of part-whole relationships. Therefore, I wished to see what type of understandings she would construct and by what means as a result of her engagement in the tasks given during the main study. Valerie was given the same tasks during her participation in the main study as the other participants with the exception of when she had already constructed the conception that those tasks were meant to engender. The other two students are Kelly and Denise.
Students were chosen for the main part of the study based upon their demonstration of the understandings that are listed in Figure 3-1. These understandings are necessary for successful engagement in the initial tasks of the teaching experiment.

1. Multiplication can be thought of as iterating a particular number of sets of a given size.
2. Division can be thought of as evenly distributing a specified total among a specified number of recipients.
3. A rectangle or letter can represent a specific, yet unknown quantity.
4. The solution to an equation is that number which, when substituted for the letter representing the unknown quantity, makes the equation true (Pirie & Martin, 1997).
5. In a balance situation, both sides of the scale must be operated upon identically in order to maintain that scale’s balance.

Figure 3-1: List of prerequisite understandings needed for participation in the teaching experiment

Pre-Assessment

Assessment of prerequisite understandings
A pre-assessment was issued in order to assess the understandings listed in Figure 3-1 and to ascertain students’ pre-experimental methods for solving problems that are algebraic in nature. Table 3-1 contains the pre-assessment items. The prerequisite understanding that is being assessed appears next to the assessment question. A discussion of the way in which each of the pre-assessment items in Table 3-1 has the potential to assess the prerequisite understanding that is listed next to it follows. Some pre-assessment items have no understandings listed next to them. These items assessed understandings that would be the target of teaching sessions during the study; pre-assessment of these understandings was essential to identify students who had not yet developed the intended goals of the lessons. Any students who already possessed such understandings would not have been included in the study. These items and the understandings that they assess are discussed in the section following the discussion of Table 3-1.

Table 3-1: Pre-assessment items with prerequisite understandings being assessed

<table>
<thead>
<tr>
<th>Pre-Assessment Item</th>
<th>Pre-requisite understanding being assessed</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jackie shared 48 pieces of cake with a certain number of people. Each person got the same number of pieces of cake and there were no leftovers. How many people could Jackie have shared her cake with and how many pieces would each person have received? List as many ways as possible that this could have occurred.</td>
<td>1, 2</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td><strong>Byron had a part-time job.</strong> On his last four days on the job, Byron earned $23 each day. Suppose that you were going to use a calculator to determine how much money Byron earned over that four day period. List the sequence of buttons that you would push. Explain why you gave this answer.</td>
<td>1</td>
</tr>
</tbody>
</table>
| **If** $u = v + 3$, **give a possible value for** $u$ **and** $v$.  
(Modified from Küchemann, 1981, p. 105) | 3, 4 |
| **If** $n+3=9$, **what is** $n+8$? **Explain how you got your answer.** (Modified from Küchemann, 1981, p. 105) | 3, 4 |
| **John has 8 pennies. Micah has three times as many pennies as John. How many pennies does Micah have?** | 1 |
| **Marcus worked for** $n$ **hours. Felicia worked 5 more hours more than Marcus. Express how many hours Felicia worked in terms of** $n$. (Modified from Küchemann, 1981, p. 105) |   |
Draw a board that is five times as long as the board below. Explain how you got your answer.

Chris has four times as many marbles as Curtis does. If Curtis doubled the number of marbles that he had and bought four more marbles, he and Chris would have the same number of marbles. Create a diagram to represent this situation. Use your diagram to find out how many marbles did each person start out with. (Modified from Maeir, 1997)

5p + 2 = 2p + 8. Find the value of p. Show all of your work.

Board B is four times as long as Board A. Write an equation to represent the relationship between the lengths of Boards A and B where a represents the length of Board A and b represents the length of Board B.
The circles below represent the amount of candy that Jeremiah has. Phillip bought boxes of candy the same size as the one below. He now has five times as much candy as Jeremiah. Draw a diagram of the amount of candy that Phillip has.

<table>
<thead>
<tr>
<th>Jeremiah’s Candy:</th>
<th>Phillip’s Candy:</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="null" alt="Candy Circles" /></td>
<td></td>
</tr>
</tbody>
</table>

“The [school auditorium] had 17 more chairs than the [band room]. When 43 chairs were added to the school auditorium, there were 5 times as many chairs in the [school auditorium] than in the [band room].” Draw a diagram to represent this situation. Use your diagram to find out how many chairs there were originally in each room. (Bodanskii, 1991/1969)

<table>
<thead>
<tr>
<th>“The [school auditorium] had 17 more chairs than the [band room]. When 43 chairs were added to the school auditorium, there were 5 times as many chairs in the [school auditorium] than in the [band room].”</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="null" alt="Diagram" /></td>
<td></td>
</tr>
</tbody>
</table>

“[Jenna] had 5 times as much money as [Harold] did. When [Jenna] and [Harold] both earned $45 shoveling snow, [Jenna] had 3 times as much money as [Harold].” (Bodanskii, 1991/1969). Write an equation or draw a diagram to represent this situation. Use your equation or diagram to find out how much money each person started out with.

<table>
<thead>
<tr>
<th>“[Jenna] had 5 times as much money as [Harold] did. When [Jenna] and [Harold] both earned $45 shoveling snow, [Jenna] had 3 times as much money as [Harold].”</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="null" alt="Equation" /></td>
<td></td>
</tr>
</tbody>
</table>
“[Claire] spent \(\frac{3}{4}\) of her money on a dictionary. She spent \(\frac{1}{2}\) of the remainder on a calculator. The dictionary cost $30 more than the calculator” (Singapore Ministry of Education, 2006, p. 21). Draw a diagram to represent this situation. Use your diagram to figure out how much the dictionary cost.

Wire A is seven times longer than Wire B. If the length of Wire B was quadrupled and 14 inches were added to it and if 10 inches of Wire A were removed, the two lines would be the same length. Write an equation to represent this situation. Find the original length of each line by using equations only. Show all work. (Modified from Maier, 1997; see also Bodanskii, 1991/1969)
The objects on the scale above make it balance exactly. According to this scale, if △ balances ○○○○, then □ balances which of the following?

a) ○

b) ○○

c) ○○○

d) ○○○○

(NCES, 2001, NAEP NQT v2.0-Questions)
There are six times as many students at the school as there are teachers. Suppose \( S \) represents the number of students and \( T \) represents the number of teachers. Which equation correctly expresses the relationship between the number of students at the school and the number of teachers at the school?

\[
\begin{align*}
a) & \quad T = 6xS \\
b) & \quad S = 6xT
\end{align*}
\]

(Modified from Clement, Lochhead, & Monk, 1981)

As a result of having to list as many ways as possible that 48 pieces of cake could be distributed evenly in response to assessment item 1, an individual could find the answer using division or factor pairs. If an individual was not able to provide a solution, then it seems likely that he or she did not understand the multiplicative nature of the situation; therefore, he or she did not possess Understandings 1 or 2. By having to provide the calculator keys that might be used to find the answer to pre-assessment item 2, the problem solver might give an answer of either 23+23+23+23 or 4x23. If the former answer had been given, then the participant would be asked if there was another method that he or she could use. If the participant were not able to provide the alternative answer of 4x23, then he or she would not have demonstrated that he or she did possess Understanding 1.

A person who successfully solves Item 3 would have chosen values for \( u \) and \( v \) such that \( u = v + 3 \). To do that, this person would have had to understand that he or she was only looking
for values to satisfy this equation. Further, if this person chose a particular value for \( u \) or \( v \), he or she would have had to understand that the value represented by the other letter is constrained to be three more or three less than the value chosen, respectively. Thus, the successful problem solver would have had to understand that each letter represents a specific, but unknown quantity.

Item 4 assesses Understandings 3 and 4 in the sense that, in order to solve the problem successfully, a problem solver would find the value of \( n \) such that the equation \( n+3=9 \). To do this, that person could understand that \( n \) stands for a specific number and that there is only one value of \( n \) that would make the equation true. Although a person solving this task correctly may have Understandings 3 and 4, it is possible that some may solve this problem correctly without it. A combination of Understandings 1 and 3 are needed in order to be able to correctly answer item 7 in the sense that the problem solver would need to understand that he or she is representing five times a certain length that is not known. Additionally, the problem solver would need to have Understanding 1 in order to know that he or she may draw five rectangles (or a figure that is the length of five rectangles) in order to represent five times the length of the original board.

Item 9 requires that a person possesses Understanding 3 in order that he or she might know that he or she may substitute a particular number for \( p \). Second, item 9 requires that a person possesses Understanding 4 in the sense that he or she must anticipate that he or she is looking for a value for which two sides of the equation equal one another. Item 11 assesses Understanding 1 in the sense that a problem solver must draw 15 circles to represent five times the amount of Philip’s candy to answer the problem correctly. In order to be able to do that, the person must know that five times a given amount can be represented by five sets of three circles.

Item 16 assesses Understanding 5 in the following way. In order to be able to ascertain how many circles balance one square, a person would need to act upon the diagram by mentally or physically removing or ignoring the triangle on the left side of the scale and three circles on the right side of the scale. Having done that, there would be two squares remaining on the left side of
the scale and four circles on the right side of the scale. The problem solver would then need to divide the number of circles remaining by the number of squares remaining in order to ascertain the number of circles that balance a square. In order to perform these operations, the problem solver would first need to know that one must remove the same quantity from each side of a balance scale in order to maintain the scale’s balance; this is Understanding 5.

**Assessment of potential understandings**

Some items on the pre-assessment were also intended to assess whether the study participants possessed the understandings that they were intended to learn during the teaching experiment. *Error! Reference source not found.* lists each of these understandings. The understandings are categorized according to whether they are necessary for the construction of an algebraic representation or for action upon that representation. Following this,

Table 3-3 lists the pre-assessment items that assess these understandings.

Table 3-2: List of understandings intended for students to develop during the teaching experiment

<table>
<thead>
<tr>
<th>Cluster</th>
<th>Potential Understanding</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Construction of</strong></td>
<td>a) A relationship of dependence may be constructed between two quantities that are additively or multiplicatively related, whether the values of those quantities are known or unknown (Thompson, 1988; Cortes, 1995).</td>
</tr>
<tr>
<td><strong>Representation</strong></td>
<td></td>
</tr>
<tr>
<td></td>
<td>b) It is necessary to create a (real or hypothetical) equivalence between two sets of quantities which contains only one of the problem’s unknown quantities and which is based upon the quantitative relationships embedded in the problem. This must be done in order to place an explicit constraint on the unknown quantity in</td>
</tr>
</tbody>
</table>
order to determine its value (Bodanskii, 1969/1991; Cortes, 1995).

<table>
<thead>
<tr>
<th>Action Upon Representation</th>
</tr>
</thead>
<tbody>
<tr>
<td>c) The enactment of an operation upon a quantity on one side of an equality statement must be similarly undertaken upon an identical quantity on the other side of the equality in order to maintain the “equality of...[the] expressions” (Vlassis, 2002, p. 344) occupying either side of the statement.</td>
</tr>
<tr>
<td>d) The unknown quantity may be “operate[d] with... as if it were known” (Stacey &amp; MacGregor, 1999, p. 30).</td>
</tr>
<tr>
<td>e) When an unknown quantity is isolated on one side of an equality statement and a known quantity is isolated on the other, the unknown quantity is multiplicatively related to the known quantity to which it is equal. That is, the known quantity is ( n ) times the unknown quantity where ( n ) is some real number.</td>
</tr>
</tbody>
</table>

Table 3-3: Pre-assessment items with the potential understandings being assessed.

<table>
<thead>
<tr>
<th>Pre-Assessment Item</th>
<th>Potential understandings being assessed</th>
</tr>
</thead>
<tbody>
<tr>
<td>6. Marcus worked for ( n ) hours. Felicia worked 5 more hours more than Marcus. Express how many hours Felicia worked in terms of ( n ). (Modified from Küchemann, 1981, p. 105)</td>
<td>a, d</td>
</tr>
</tbody>
</table>
8. Chris has four times as many marbles as Curtis does. If Curtis doubled the number of marbles that he had and bought four more marbles, he and Chris would have the same number of marbles. Create a diagram to represent this situation. Use your diagram to find out how many marbles did each person start out with. (Modified from Maier, 1997)

9. \(5p + 2 = 2p + 8\). Find the value of \(p\). Show all of your work.

10. Board B is four times as long as Board A. Write an equation to represent the relationship between the lengths of Boards A and B where \(a\) represents the length of Board A and \(b\) represents the length of Board B.

12. “The [school auditorium] had 17 more chairs than the [band room]. When 43 chairs were added to the school auditorium, there were 5 times as many chairs in the [school auditorium] than in the [band room]” (Bodanskii, 1991/1969). Draw a diagram to represent this situation. Use your diagram to find out how many chairs there were originally in each room.

13. “[Jenna] had 5 times as much money as [Harold] did. When [Jenna] and [Harold] both earned $45 shoveling snow, [Jenna] had 3 times as much money as [Harold].” (Bodanskii, 1991/1969). Write an equation or draw a diagram to represent this situation. Use your equation or diagram to find out how much money each person started out with.
14. “[Claire] spent \( \frac{3}{4} \) of her money on a dictionary. She spent \( \frac{1}{2} \) of the remainder on a calculator. The dictionary cost $30 more than the calculator” (Singapore Ministry of Education, 2006, p. 21). Draw a diagram to represent this situation. Use your diagram to figure out how much the dictionary cost.

15. Wire A is seven times longer than Wire B. If the length of Wire B was quadrupled and 14 inches were added to it and if 10 inches of Wire A were removed, the two lines would be the same length. Write an equation to represent this situation. Find the original length of each line by using equations only. Show all work. (Modified from Maier, 1997; see also Bodanskii, 1991/1969)
16. The objects on the scale above make it balance exactly. According to this scale, if \( \triangle \) balances \( \bigcirc \bigcirc \bigcirc \bigcirc \), then \( \square \) balances which of the following?

- a) \( \bigcirc \)
- b) \( \bigcirc \bigcirc \)
- c) \( \bigcirc \bigcirc \bigcirc \)
- d) \( \bigcirc \bigcirc \bigcirc \bigcirc \)

(c, d, e)

(NCES, 2001, NAEP NQT v2.0-Questions)
17. There are six times as many students at the school as there are teachers. Suppose S represents the number of students and T represents the number of teachers. Which equation correctly expresses the relationship between the number of students at the school and the number of teachers at the school?

a) $T = 6xS$

b) $S = 6xT$

(Modified from Clement, Lochhead, & Monk, 1981)

In the section that follows, the manner in which each pre-assessment item in Table 3-3 assesses the corresponding potential understandings is discussed. A letter in parentheses will accompany the discussion to indicate which understanding is being discussed at a given time.

In order to successfully solve Item 6, a person must establish a relationship of dependence between Marcus’s number of hours worked and Felicia’s number of hours worked (a). He or she must then operate on the letter $n$ in order to represent Felicia’s number of hours worked (d). Thus both understandings (a) and (d) are needed to solve Item 6. In order to solve item 8 algebraically, a person must create a relationship of dependence between Chris’s number of marbles and Craig’s number of marbles (a). The representation of Craig’s number of marbles would have to be operated upon in order that Chris’s number of marbles might be represented (d). It must then be supposed that Chris and Craig both acquired the number of marbles stated in the second part of the problem condition in order to establish a relationship of equivalence (b). Once
established, the problem solver must operate identically upon the newly established equivalent quantities in order that the unknown quantity might be set equal to a known quantity (c). Once that is accomplished, the problem solver could use the multiplicative relationship between the two quantities in order to find the value of the unknown quantity (e). Item 15 may be solved in the same manner using the same understandings.

In order to solve item 9 algebraically, a person would need to subtract or otherwise remove two times the unknown quantity from both sides of the equation (c, d). He or she would then need to understand that there is a multiplicative relationship between the known and unknown quantity in order that he or she could divide the known quantity by the number of instances of the unknown quantity to which the known quantity is equal (e).

In order to solve Item 10, a problem solver must establish that there is a multiplicative relationship between the size of Boards A and B (a). He or she must then represent the length of either board using a letter and operate upon that letter according to the multiplicative relationship given in the problem statement (a, d).

Items 12 and 13 require that a relationship of hypothetical equivalence be established in order that each of these problems can be solved algebraically (b). That is, once the additive and multiplicative relationships between quantities have been determined and established (a, d), one or both of the quantities must be operated upon in such a manner as to create two equivalent quantities based upon the relationships given in the problem statement. For example, for item 12, suppose that \( x \) represented the original number of chairs in the band room. Then \( x + 17 \) would represent the original number of chairs in the auditorium. Once additional chairs were added to the auditorium, there would be \( x + 60 \) chairs in the auditorium. In order to solve the problem, a hypothetical relationship of equivalence must be created between the number of chairs in the auditorium and the number of chairs in the band room (b). Based upon the relationships given in the problem, the number of chairs in the band room must be multiplied by five—even though
there are not that many chairs in the band room—in order to equal the number of chairs in the auditorium (Clement, 1982). Therefore, the following equation may be used to represent the problem situation: $5x = x + 60$. Understandings c, d, and e are needed to solve the equation algebraically.

The understandings necessary to solve item 14 include (a) through (e). However, two other understandings are needed to solve this problem algebraically is: the additive difference between two quantities is a procept and an unknown quantity can be partitioned. One of the ways in which this problem could be solved is by creating a diagram (see Figure 3-2). Using this diagram, I will explain how the solution to this problem requires the understandings given.

![Figure 3-2: Possible representation of the Quantitative Relationships in Pre-Assessment Item 14](image)

The cost of the dictionary is $\frac{3}{4}$ of Claire’s total amount of money. The cost of the calculator is $\frac{1}{8}$ of Claire’s total amount of money ($\frac{1}{2}$ of the remaining $\frac{3}{4}$ of Claire’s total amount of money). The cost of the dictionary is shown as the group of shaded rectangles in Figure 3-2. The representation of the cost of the dictionary may be partitioned so that it is represented in terms of the cost of the calculator (a). Then, in order to solve the problem algebraically, a relationship of equivalence must be created (b) utilizing the implicit relationships embedded within the problem statement (but made explicit by the diagram in Figure 3-2) (Bodanskii, 1991/1969). Since it is known that the cost of the dictionary is six times the cost of the
calculator (a) as shown in Figure 3-2 and since it is known that the additive difference between the cost of the dictionary and the cost of the calculator is $30, it may be stated that $30 + x = 6x$ where $x$ is the cost of the calculator. From there, understandings c, d, and e must be utilized in order to solve the equation.

Item 16 utilizes understandings c, d, and e in the following way. Both sides of the scale must be operated upon identically in order to maintain the balance of the scale (c). The triangle on the left side and three circles on the right side must be removed in order that the unknown quantity is isolated on one side of the scale and the known quantity on the other (d, e). Although the weight of a square is not known, the number of squares may divided into the number of remaining circles in order to find out how many circles balance a square (d, e).

Item 17 requires understanding d in that, to create an equation relating the number of students and the number of teachers (e), an individual would have to multiply the number of teachers by 12, even though the number of teachers is not known and there are not that many teachers at the school (Clement, 1982).

**Selection criteria for the teaching experiment**

Only those students who answered items 1, 2, 3, 5, and 7 correctly on the pre-assessment were invited to participate in the teaching experiment since the understandings required to solve these items were, minimally, the understandings needed to solve the initial tasks of the study. It was strongly preferred that students be able to solve item 16 accurately as well, although one student, Kelly, did not do so. Kelly was included in the teaching experiment because she demonstrated that she possessed the prerequisite understandings which were vital for participation in the study.
Students who represented and solved items 8, 9, 12, 13, 14, or 15 algebraically would not have been asked to participate in the teaching experiment since the understandings required to solve these items in that manner were those which were to be taught during the study. No students fit into this category.

Teaching Sessions

Teaching sessions were held during the school day or after school approximately two days a week for twelve weeks. The sessions typically lasted from 20 to 30 minutes. The tasks given and the reason for their inclusion in this study are given in Appendix A. In this paper, the word “task” is used to designate an oral or written problem that is given to a student by the researcher for the student to solve. Included in this designation will be all accompanying diagrams. The phrase “task sequence” is being used to refer to the group of tasks, sequentially ordered, that were given the students during the study. Most of the task sequence was created prior to the beginning of this study. However, there were times when unexpected cognitive obstacles were encountered or when learning took place in an unexpected fashion. A cognitive model was built of a student’s current conceptions by the researcher. In keeping with the teaching experiment methodology, tasks that were created that would help students to build from the understandings which they seemed to demonstrate at a particular time to the desired conception.

Throughout the experiment, I “initiate[d] and guide[d] the development of social norms” (Yackel & Cobb, 1996, p. 460) during the teaching sessions. At the beginning of the study, I explained to the students that I would be asking them to give the reason for their responses to certain tasks. I assured them that, if I asked them for that reasoning, it was not an indication of whether they answered a question correctly or incorrectly. Rather, I just wanted to know what they were thinking. One student, Denise, began to anticipate these questions and would state the
rationale for her mathematical actions immediately after they occurred. The other two students, Valerie and Kelly, would state their rationale when asked.

Whenever a student received a new task, I would ask that student to read the problem statement aloud. This was done in order that the text of the task or segment of the task being worked on might be captured on the audio and video recordings for future identification purposes. If the problem statement was divided into sections on the page, the student was asked to read one section, represent the relationships in that part of the problem, and then read the next section. This segmented approach was not done for data collection purposes. Rather, it was done to provide scaffolding for the students since they were just learning how to represent algebraic situations and I wanted them to be able to attend to the quantitative relationships within that segment of the task only. Eventually, the tasks were written so that the problem statement was in one paragraph.

After students represented the relationships in the problem, there was sometimes a hesitation on the part of the students. I would often ask students questions like, “What happens next [in the problem]?” or “What do we know now?” in order to enable the students to contextualize the information that they have represented and to ascertain whether or not they have represented all information completely and accurately. I also asked these questions so that students might consider whether or not they were misconstruing any information. Further, I asked students questions such as “What are you trying to find? [and] What did this calculation give you?” (Thompson, 1993, p. 204) in order to promote students’ reflection on the goal or result of their actions.

Additionally, there were certain mathematical behaviors that I tried to reinforce. These behaviors were those which had the potential of supporting students’ problem-solving efforts and conceptualizing the implicit relationships between quantities (Larkin & Simon, 1987). Namely, I asked students to always label their diagrams whenever they represented a given quantity or when
they found out the value of an unknown quantity (Maier, 1997; Singapore Ministry of Education, 2006). I did this so that students might be able to perceive the relationship between known and unknown quantities that they may not have abstracted from the problem statement (Larkin & Simon, 1987). I also asked that they always connect the boxes that belonged to the same representation and that they create boxes of the same size that represented the same quantity. I did this because it seemed that students might be more likely to notice relationships of equivalence when boxes or groups of boxes that were equal in length were properly aligned with one another.

Further, I asked students to refrain from guess and testing on any task. In itself, this method is a viable means of obtaining answers to the problems given. However, if students predominantly utilized the guess and test method, it did not seem as likely that they would engage in acting upon their diagrams to solve the problems. As a result, the understandings that students would have built by consistently utilizing the guess and test method, would, presumably, have been different from those constructed by students who acted upon diagrams to solve the problems. If this had been the case, I would not have been able to answer this study’s primary research question. Therefore, I discouraged students’ use of guess and test.

Data Collection

Each teaching session was videotaped. Most sessions were also audiotaped. According to Powell and colleagues (2003), videotapes of students’ mathematical activity offer an unparalleled access to real-time data. However, Powell and colleagues sound Hall’s (2000) caution that a researcher’s decision concerning which images to record is necessarily shaped by his or her theoretical lens. Thus, the means by which this source of data is captured, and other data sources as well, is value-laden. With this in mind, the following is a description of and justification for the way in which video data was recorded in this experiment.
One camera was used to record the teaching sessions. This camera was unmanned and focused upon the paper where the student worked. In this manner, all of the students’ written and pointing actions could be captured. I sat directly across from the student with the camera to my back. So that students’ writings could be clearly recorded, students were given black markers with which to write. All of the students’ writings were numbered, dated, and kept as data. All students’ written work, videotapes, audio tapes, transcripts with annotations, reflective notes by the researcher, and any artifacts collected during the teaching sessions are considered as data from this study.

Data Analysis

During the study

I conducted at least two teaching sessions each day in addition to carrying additional job responsibilities that began at the same time of the start of the main study. This made it very difficult to view the videotape of each teaching session on a daily basis. Ideally, viewing each session’s videotape would have provided me with the most accurate means to assess and analyze students’ conceptual understandings at a given point in time. However, I did make note of what appeared to be changes in students’ understandings as they occurred and characterized students’ misconceptions in order to determine the manner in which tasks needed to be altered or created to accommodate those conceptions. I also created summary reports of students’ progress which were sent regularly to my thesis advisor who corresponded with me on an ongoing basis about the nature and progress of the study.
After the study

According to Powell, Francisco, and Maher (2003), “an event is called critical when it demonstrates a significant or contrasting change from previous understanding, a conceptual leap from earlier understanding” (p. 416). Events may also be deemed as critical, according to Powell and colleagues, if they reveal evidence that is contrary to the researcher’s hypotheses about learners’ understandings or if they reveal evidence about learners’ misconceptions. Particular events which were deemed as critical were noted during the study. An event of this nature may have been characterized by evidence of a students’ misconception or learning. Another type of event that was deemed as critical is that which motivated the need for the design of a new task. These critical events were noted during the teaching experiment and served as the one of the points of focus when analyzing the study’s data after its completion.

Some data that was collected during the study will receive minimal discussion while other data collected will receive more emphasis as a result of the focus on those tasks and events that seemed to contribute towards the most salient understandings that were constructed in this study.

In order to analyze the data, all videotape of each participant’s teaching sessions was viewed at least once. Those places during the teaching session where a critical event or series of events seemed to occur were noted. When a student seemed to have constructed a particular understanding during a given session or over a period of time, the student’s work from immediately before, during, and after that session was closely scrutinized. If the written work showed any similarities or differences that were notable, I would view that portion of the videotape dealing with that particular task or tasks in order to determine the way in which the student solved it. When a student seemed to treat a particular mathematical entity in a way that seemed to be conceptually different or seemed to display a qualitative change in his or her
mathematical behavior or speech during a given teaching session, this behavior was characterized as evidence that a conceptual change had occurred or was occurring. I then looked for further evidence to either dispute or buttress that claim.

Once the identification of evidence was accomplished, I would send a “Claims and Evidences” document to my dissertation advisor. In this document, I would make a claim about what I believed a student did or did not understand and I provided evidence from various parts of the data. I would also state the reason why I felt that the claim that I made was supported by the evidence that I provided. I received feedback from my advisor on these claims and evidences as well. I then revised the claims and evidences according to that feedback. This process was carried out for all three of the study participants. These “Claims and Evidences” documents are what formed the primary foundation for my analyses in Chapter 4.

Following the combing of the data, I sought to create a story of the development of each students’ understandings over time. Four understandings emerged as the main understandings developed by each participant. These are discussed in the next chapter. Finally, based upon my interpretation of the videotape data, analysis of students’ work, and feedback from my advisor with regards to the claims and evidences documents, I created a storyline for each student which attempted to account for the change in each student’s conceptions over time.

In conclusion, the use of a teaching experiment methodology was valuable in this study in that the understandings that students developed help to drive the creation of new tasks and an adjustment to students’ hypothetical learning trajectory. The data collected came from multiple sources and served as a rich resource for analysis. The means of analysis allowed for patterns of students’ understandings and misconceptions to emerge.
Chapter 4

Analysis of Students’ Understandings

During this study students were taught how to use diagrams as a means of representing and solving algebra word problems. Students were then taught how to model their diagrammatic problem-solving activity using algebraic symbols. With that scenario in mind, the research questions posed for this study poses are:

a) What is the nature of the algebraic understandings developed by students as a result of their engagement in algebra problem-solving via the use of diagrams consisting of connected rectangles?

b) By what means did students seem to develop these understandings? What aspects of the environment seemed to constrain or afford this development?

In this chapter, I provide answers to these questions by articulating the understandings developed by the students, individually and collectively. I include cognitive obstacles reported in the literature related to representing and solving algebra word problems that seemingly were circumvented or experienced differently as a result of the diagrammatic instruction. I also interpret these results in terms of what they tell us about the development of algebraic reasoning.

Theoretical Framework

The theoretical framework for this study is based in Piagetian learning theory. Piaget (1971) proposed that students learn logico-mathematical concepts through the process of
reflective abstraction. Logico-mathematical knowledge is a network of “mental relationships between and among objects” (Kamii & Joseph, 2004, p. 16), constructed by an individual, that enables him or her to impose characteristics on objects that are not inherent to those objects (such as relative size, categorization, and the like). According to Piaget (1971), this knowledge is constructed when an individual’s logico-mathematically facilitated actions upon objects are coordinated in such a manner as to engender a new understanding that is more sophisticated than the understanding that facilitated the actions themselves.

Basing his theory upon that of Piaget, Dubinsky (1991) proposed that learners transition from one level of understanding of a mathematics concept to the next through the process of reflective abstraction. Each level of understanding, or, conception, is more sophisticated than the one before it. According to Dubinsky (1991) when a learner initially acts upon an object, he or she does so according to a prescribed procedure and in a way that is external to him or her. When the individual acts with regularity upon this object and reflects upon his or her actions, he or she internalizes that set of actions in such a manner that he or she is able to anticipate the results of those actions. The individual at this stage is stated to understand that concept as a process (Asiala et al., 1996). With that conception, the individual may operate with that process by intentionally combining it with other processes or reversing the process. When the process becomes, to the learner, an entity unto itself that can be operated upon, the learner is stated to have “encapsulated the process into a cognitive object” (Dubinsky, Weller, McDonald, & Brown, 2005, p. 339).

According to Dubinsky and colleagues (2005), a schema is a cognitive structure composed of “actions, processes, and objects” (p. 339) constructed by and available to an individual which enables him or her to appropriate those conceptions needed to solve a particular type of math problem.

Like Piaget, Tall (2008) stated that an individual’s mathematical learning begins with that person’s action upon objects. When the actions occur upon objects that are perceptible in the real
world (such as graphs and geometrical figures), Tall (2008) stated that the person is acting within the “conceptual-embodied world [which is] based on perception of and reflection on properties of objects” (p. 7). This acting may occur physically or mentally. When a person symbolizes those actions, he or she is stated to be acting within the “proceptual-symbolic world” (p. 7).

According to Tall, these worlds are parallel. Therefore, an individual who is thinking mathematically within the realm of school mathematics has the potential of transitioning between both worlds at any given time. Tall gave the example of a person who translates a figure on a graph. That person was stated to be acting within the embodied world. On the other hand, when that person symbolizes that translation using function notation, he or she is acting within the “proceptual-symbolic world” (p. 7). The stages of a person’s development within the realm of the symbolic world are those that Tall likens to Dubinsky’s APOS theory. Tall’s (2008) stages within this realm are: procedures, processes, and procept and they are similar to Dubinsky’s action, process, and object, according to Tall (2008).

The nature of a procept, according to Tall (2008) is that it is simultaneously, to the learner, a process and a concept. On the one hand, an individual that has a proceptual understanding of a particular mathematical symbol (such as 5+8) views that symbol as being a process that is being carried out. At the same time, that person is able to view that symbol as an entity that can be operated upon (Tall, 2008). Thomas and Tall (2001) call a symbol such as 2+3x a “potential process” since the operations of multiplication and addition cannot actually be carried out unless the letter x is replaced by a particular value.

According to Tall (2008), the means by which a person moves from one level of abstraction to the next is through compression, which involves a compacting of the understandings in the previous phase so that they become an object of thought.

The word scheme will be utilized in the sense proposed by von Glasersfeld (1995) who outlined “the three parts of schemes as follows: 1) Recognition of a certain situation; 2) a specific
activity associated with that situation; and 3) the expectation that the activity produces a certain previously experienced result (p. 65).” In other words, when a person is observed to act in a particular manner mathematically in a particular type of situation with, seemingly, a particular expected outcome, that person will be characterized as using a particular scheme.

APOS theory (Dubinsky et al, 2005) and Tall’s (2008) theory of cognitive development figured prominently in the analysis of this study’s data. These theories were used to distinguish between levels of students’ conceptual understandings as it relates to algebraic quantities.

**Understandings of Interest**

As a result of their involvement in this study, the participants constructed particular understandings that seem to be critical to algebraic reasoning. Two of these understandings (Understandings 2 and 4 from the list below) are widely discussed in the literature. Understanding 1 (listed below) was proposed in part as a result of Gregg and Yackel’s (2002) and Steffe’s (1994) work with composite units. I hypothesized that students might benefit from unitizing first the known and then the unknown quantity as a means of building up their algebraic understandings (Gregg & Yackel, 2002). Understanding 3 was based upon both Thompson’s (1995) quantitative reasoning framework and a decomposition of the skills required to solve an algebraic equation. The four understandings listed are those that were intentionally taught within the context of this experiment. Within each section of this chapter, the nature of and the means by which the study participants developed a particular understanding is explored and characterized. Within this same section will be a comparison of the manner in which the participants seemed to construct that understanding. Throughout the chapter and at the end of the chapter, those sub-understandings that students seemed to have developed as a result of or en
route to the development of the understandings listed here are discussed. The understandings that are explored in this chapter are:

1) An unknown quantity, $u$, is an iterable, composite unit. If $k$ is a known quantity and $u$ is an unknown quantity, then $ku$ consists of $k$ units of $u$.

2) A letter representing an unknown quantity is able to be operated upon “as if it were known” (Stacey & MacGregor, 1999, p. 30)

3) Given two equivalent wholes, either of which consists of one or more occurrences of a particular unknown quantity, the additive difference between the unknown parts of the two wholes is multiplicatively related to the additive difference between the known parts of the two wholes

4) Establishing a relationship of equivalence between two quantitative wholes is useful in finding the value of the unknown quantity contained within one of those wholes (Clement, 1982)

Understanding 1—An unknown quantity is an iterable, composite unit. If $n$ is a known quantity and $u$ is an unknown quantity, then $nu$ consists of $n$ units of $u$.

Steffe and colleagues (1983) characterized the development of children’s “number sequences” as one in which children develop an increasingly sophisticated concept of number through their counting actions and their reflection on those actions. The objects, or “elements”, of those counting actions are, at first, real external items. Then the “individual counting acts” (Steffe, 1994a, p. 15) themselves become the elements of counting. At that stage, number, for the child, is a “figurative composite unit” that “refers to a sequence of counting acts” (Steffe, 1994a, p. 15).
According to Steffe and colleagues, through progressive unitization of the results of their counting actions and coordination of those units, children, may develop a conception of a number as being an “iterable composite unit” (Steffe, 1994b, p.51). A composite unit, according to Steffe (1994), is “a unit that itself is composed of units” (p. 15). For example, the number three is a composite unit for a child who conceptualizes three as being composed of three units of one.

According to Steffe (1994), a child who holds a number as an iterable unit is able to anticipate, before counting, that he or she will create units of a particular size by which he or she will count. Further, a child who holds a composite unit as countable is able to anticipate counting the number of composite units that comprise a certain sized whole in order to ascertain the size of the whole. For example, suppose that a child who has a conception of number as an iterable countable unit plans to divide 27 by 3. According to Steffe (1994), this child would be likely to create and count the number of groups of three that are present in 27.

The choice and sequencing of the tasks for the initial teaching sessions were based on my intent to help students develop an understanding of the unknown quantity as an iterable, composite unit (Olive & Caglayan, 2008). Gregg and Yackel (2002) succeeded in helping pre-college students in a developmental algebra course to develop the unknown quantity as a composite unit through the use of a realistic scenario called “Candy Shop.” In the Candy Shop scenario, students were shown before and after scenarios in which “candies” were purported to have been packaged during the day. Students had to figure out how much candy constituted either the starting amount, the amount packed, or the ending amount.

I designed the Harper Middle School tasks using a similar motif. Study participants were told of students who attended a middle school where a school fundraiser was occurring in the form of a candy sale. Students at that school were stated to be able to purchase candy according to the number of pieces in a box. The study participants were told that the length of a box of candy depended on the number of pieces of candy inside of it.
I chose the candy sale scenario because, in keeping with the RME approach for lesson design, this scenario seemed to lend itself towards supporting students’ unitization of the known (and later) unknown quantities. Similar to Gregg and Yackel’s (2002) Candy Shop scenario where elongated ovals were used to represent “rolls” of candy, boxes of candy could be thought of as a packaged unit.

Further, utilizing a scenario employing boxes of candy afforded me the opportunity to introduce the use of rectangles to represent the known quantity (number of pieces of candy). This usage was of import because, if students drew the lengths of the rectangles according to size of the quantities they represented, side by side with no gaps, their relative lengths could be visually compared by students and utilized to find the value of the unknown quantity. By first introducing rectangles as icons containing the circles representing pieces of candy, I anticipated that a rectangle could become, to the student, a legitimate representation of the known (and later unknown) quantity itself.

By starting with known values, it was my intent that, when representing more than one box of candy, students might conceptualize the multiplicative relationship between the number of boxes that they drew, the number of circles in each boxes, and the total number of circles that were in the diagram. That is, it was my goal for students to develop the anticipation that, when drawing a representation of \( n \) times a given amount (which was represented by one box), they would have to draw \( n \) boxes. In this manner, students would be able to construct the unknown quantity as an iterable, composite unit and a multiple of the unknown quantity as a unit of units (Olive & Caglayan, 2008).
Understandings Developed by Individual Students

In the following sections, I describe the path of understanding taken by each student. I begin with Kelly’s story because her algebraic conceptions seem to be the least robust compared to that of Denise and Valerie. I end with Valerie’s story, which is the most complicated due to the fact that she participated in both the pilot study and the main study. Of the three students, Valerie also seemed to have developed the most sophisticated algebraic understandings.

Kelly’s Story

At the beginning of the study, Kelly demonstrated that she conceptualized $n$ times the length of a figure as the length of a concatenation of $n$ iterations of that figure. Kelly demonstrated this during my initial interview with her on 4/11/2008, during which she justified her response to Item 7 of the Pre-Assessment (see Figure 4-1).

Jan: Okay. So I see you drew a board here. Can you explain that to me, how you got that one [points to Kelly’s response to Item 7 on the sheet of paper]? [Kelly turns the paper so that it is sideways with the right side of the paper facing her.]

Kelly: Okay. Well I kinda measured it with my fingers, like this [Kelly frames the originally given rectangle with her thumb and index finger of her right hand]. …Okay, so I measured it with my fingers like this, and then I carried it here [Kelly places her fingers (still framed a certain distance apart) so that the index finger of her finger frame is resting on the leftmost side of the rectangle that she drew originally]. Marked where I got, marked where my finger was [Kelly puts the index finger of her left hand next to the thumb of her finger frame].

Jan: Okay.

Kelly [Keeping the same distance between her fingers, Kelly moves her finger frame to the right of the finger that was keeping place. She moves the place-keeping finger to the right and places the finger frame to the right of it. As a result, she reaches the rightmost edge of the diagram; she puts the place-keeping finger on the right edge of the rectangle that she had drawn]: But I can see that I messed up, because it only has four of ‘em.
By stating that “it only has four of ‘em”, it seems that Kelly was saying that the entire diagram, including the originally drawn rectangle, was the width of four unit rectangles. Because Kelly stated that she “messed up”, I invited her to add to her diagram. In hindsight, this seems to have been a leading suggestion as I am not sure that she would have added to her diagram otherwise. A better question to ask would have been, “Are you satisfied with your diagram as it is right now?” If Kelly answered in the negative, I could have asked her if she wished to draw a new diagram to reflect what she was thinking. The excerpt that follows begins with my suggestion.

7. Draw a board that is five times as long as the board below. Explain how you got your answer.

Figure 4-1: Kelly’s Response to Item 4 of the Pre-Assessment

Jan: Okay. Feel free to add something more, like [inaudible]

Kelly: Marker [picks up a black marker from the table].

Jan: Yeah.

Kelly: Can I outline it first?

Jan: Absolutely. [Using the marker, Kelly outlines the long rectangle that she had already drawn in pencil; attached to the right of this rectangle, Kelly draws a shorter rectangle] Okay, now, what makes you decide …. what made you decide that this [points to the rectangle that Kelly just drew] is five times longer than the original?

Kelly: Because, my fingers measured it like this, this long [Kelly frames the borders of the originally given rectangle].

Jan: Okay.

Kelly: And it’s kinda, it kinda shows me…. [Starting from the leftmost side of the long rectangle that she drew, Kelly places her finger frame on this rectangle. Kelly marks the rightmost edge of her finger frame with the index finger of her other
hand. She then places the leftmost side of her finger frame to the right of the “marking” finger. Kelly does this four times. She then frames the short rectangle with the same finger frame] It kinda showed me that I only had four of them.

Jan: Okay. Now why have, why would having five rectangles mean that it’s five times longer than that.

Kelly: Because it’s, it’s, because it is five, five of these is how long a rectangle [with two fingers, frames the length of the entire diagram that she drew].

In the beginning of the episode, Kelly framed the originally drawn rectangle (which I will now call the unit rectangle) with her fingers creating a “finger frame” that was the approximate width of this given rectangle. Then, throughout this episode, Kelly appeared to use this finger frame in order to find out the length of the rectangle that she drew on the pre-assessment relative to the length of the unit rectangle. This seems to indicate that Kelly saw the unit rectangle as the standard of measure.

Towards the end of the episode, Kelly demonstrated this understanding again when she then “measured” the length of her entire diagram using a finger frame based on the unit rectangle and is able to iterate this finger frame five times. Interestingly enough, the first time that Kelly measured the diagram that she drew (before she added one box) using a finger frame, she only had to iterate this finger frame three times to reach the end of the rectangle. She later only drew one more rectangle. Thus, it seems, Kelly desired that her entire diagram be the length of five of the unit rectangles. Kelly’s measuring activity is consistent with Kamii and Clark’s (1997) finding that, by the sixth grade, students have developed an understanding of the iterability of a nonstandard unit of measure.

At the end of this episode, when asked why “having five rectangles” would mean that her diagram is five times longer than the unit rectangle, Kelly stated, “because it’s, it’s, because it is five, five of these is how long a rectangle” and, with two fingers, she frames the length of the
entire diagram that she drew. Kelly’s answer seems to suggest that, once again, her expectation is that her diagram should be the length of five of the unit rectangles.

That Kelly feels comfortable in utilizing the unit rectangle as a unit seems to indicate that she conceptualized the unit rectangle as an object according to Dubinsky’s APOS theory. I say this because Kelly uses the unit rectangle as an input to measure and create a new entity. Further, Kelly’s decision to draw one long rectangle rather than five individual rectangles seems to indicate that Kelly conceptualizes the long rectangle as a composite unit made up of five unit rectangles.

In the first teaching session, I assigned Kelly the first of the Harper Middle School tasks (see Figure 4-2). During the episode below, Kelly demonstrated that she conceived of the originally given amount of candy in one box as a unit. After reading the task aloud (students were asked to do this in order that the text of the task that was being solved would be included as a part of the recording), Kelly began to represent the problem situation.

Kelly: [Next to “Phillip’s Candy”, draws a rectangle and then draws three circles inside; Kelly starts to draw another rectangle.] Is it okay if I just draw the rectangles first?

Jan: Sure.

[Kelly draws four rectangles next to the first one that she drew. Starting from the rightmost rectangle, she draws three circles in each rectangle.]

It seems that Kelly asked to draw the rectangles first in anticipation of later drawing three circles inside of each rectangle. This seems to indicate that, before drawing the circles, she thought of each rectangle as representing a composite unit of three.
After reading the problem statement, Kelly drew a box next to the originally given rectangle on Jeremiah’s diagram. When asked why she did so, Kelly replied, “Because he has one, and one plus one is two…this would have to be two.” Kelly seemed to think of doubling the amount of Jeremiah’s candy as being tantamount to doubling the number of boxes holding the candy. This seems to indicate that Kelly thought of the amount of candy represented by one box as a unit of measure. Kelly further demonstrated her thinking by units in the next excerpt where she calculates how many pieces of candy each person in the story had.

Kelly: So, Phillip has three, six, nine, twelve, fifteen, fifteen boxes, fifteen pieces of candy, and five boxes with three each. Jeremiah has one box, well two boxes, with three in, three each.
Jan: Okay.

Kelly: So that is, um, six. So he would have to have a, maybe, a longer box, a longer and bigger size box. [Kelly draws a large rectangle on the same line as Jeremiah’s candy.]

Jan: Okay.

Kelly: Then he’d have to have nine more pieces of candy. Do you want me to fill this in here?

Jan: Sure.

Kelly: One, two, three, [draws three circles vertically inside of the box that she just drew]… Kelly: One, two, three, one two three [draws two more vertical sets of three circles inside of the last box that she drew]. He’d have to have nine more, cause nine plus six equals fifteen.

Kelly’s counting by threes activity in the middle and at the end of the episode seems to be further evidence that Kelly was thinking in units of three. Kelly seemed to conceptualize the box containing three pieces of candy as both a three (as evidenced by her counting by threes activity at the end of the episode) and unit whole (as evidenced by her earlier statement that doubling Jeremiah’s amount of candy would amount to having two boxes “because he has one, and one plus one is two.”

As an intermediary step to transitioning Kelly from the use of boxes with circles to represent known quantities to the use of empty boxes to represent the unknown quantity, I gave Kelly tasks which asked her to represent the known number of pieces of candy inside each box as a number. In the episode that follows, Kelly showed that she continued to view each rectangle as representing units of the known quantity (see Figure 4-3).

After reading the problem statement, Kelly drew six connected rectangles next to “Denise’s Candy.” When asked why she had drawn six boxes, Kelly gave the following response.

Because um, [tapping the given diagram of Ivan’s Candy] there are six times and one [tapping the given diagram of Ivan’s Candy] plus five, one plus—one [taps first box that she drew] plus five more [taps each of next five boxes that she drew] would be six.
Similar to the way that Kelly reasoned about representing doubling in the previous episode, in this episode, Kelly stated that six times as much would be six rectangles because “one plus five more would be six.” It seems that Kelly predetermined that since one box represented a certain amount of candy, she had to multiply six times one in order to calculate the total number of boxes needed in the diagram. After Kelly read the next section of the problem, the following activity ensued.

Kelly: [draws a rectangle that she connects to the originally drawn diagram of Ivan’s candy. She draws and connects another rectangle to the one she just drew. She counts the circles in the original diagram] So, this is fifteen. Okay, five, five, [writes the number five in the last two rectangles on Ivan’s diagram; to herself, counts by 5’s up to 30] there would have to be fifteen pieces in this next box.

After drawing a long rectangle and writing the number 15 inside of it, Kelly then wrote the number 5 inside of each of the boxes in Denise’s diagram. In this episode, Kelly seemed to have unitized the given quantity as evidenced by her counting by fives before she determined the number of pieces of candy that she would need to put in Ivan’s new box of candy.
In order to introduce Kelly to the use of empty rectangles as a means of representing the unknown quantity, Kelly was given Harper Middle School Task #4 (see Figure 4-4). After reading the first paragraph of the scenario, Kelly indicated that she did not understand what was being asked. The following exchange then occurred.

Jan: Okay, now before I told you, okay, that’s five pieces, whatever, whatever [referring to instructions given on previous tasks]. Now, you don’t know how many pieces are in that box. But, you do know that Paulette bought four times as many, as much candy as Darrell did. Is there any way that you can show that on the diagram?

Kelly: Well, um, I could put four in here [points to the originally drawn box representing Darrell’s candy], and then four times would be sixteen pieces of candy.

Jan: Okay. Now, let’s suppose that we don’t know the number of pieces of candy in that first box. Is there any way you can represent the amount of candy Paula has, Paulette has without drawing any circles in it but using this picture right up here.

Kelly: Um, saying, this box and pretend there’s three more.
6) Darrell bought a box of candy with a certain number of pieces in it (please see diagram below). Paula bought four times the amount of candy that Darrell bought. Show this on the diagram below.

![Diagram of candy](image)

If Darrell doubled the amount of candy that he has and then bought a box of candy containing 10 pieces in it, he and Paula would have the same number of pieces of candy. Alter your diagram to show this situation.

Use your diagram to figure out how much candy each person started out with.

Darrell’s original # of pieces of candy: 5
Paula’s original # of pieces of candy: 20

Figure 4-4: Kelly’s Response to Harper Middle School Task #4 on 4/15/2008

When the number of pieces of candy and pennies were known in the first two episodes, Kelly was able to draw the representation of \( n \) times each of those amounts. Further, in both instances, counted by the given quantity and used that given quantity as a basis to find the unknown quantity. For this reason, it seems as though Kelly had unitized the known quantity.

However, in the third episode, when Darrell started off with just a “certain” number of pieces, Kelly did not spontaneously engage in the activity of representing the multiple of the unknown quantity. Further, towards the beginning of the third episode, she tried to ascribe a certain number to the unknown in order to solve the problem. It may have been the case that Kelly tried to assign a number to the unknown quantity in the Darrell-Paula task since a number had been assigned to the given rectangle in each of the previous problems. However, it seems
more likely that Kelly’s mathematical behavior in the Darrell-Paula task is an indication that she had an initial inability to operate on the unknown quantity (Küchemann, 1981) as evidenced by the fact that she did not choose to represent the iteration of the unknown quantity. Kelly may not have been able to operate with or upon the representation of a quantity unless its value was known. Thus, up to this point, it seems that Kelly’s thinking may have been arithmetic in nature (Filloy & Rojano, 1989).

I believe that Kelly did not spontaneously conceptualize a rectangle as representing a composite unit of unknown measure for the following reason. In previous instances during which the quantity was given pictorially, Kelly had material upon which she could operate. In other words, Kelly could form a mental image of the circles or the number represented by those circles and iterate that number. Even when only a number was given without a picture, Kelly was able to operate on the known quantity. However, when there was no number “out there” to assign to the rectangle, Kelly had no mental material upon which to operate. That is, there was no number that she could unitize and iterate. Therefore, unlike the instances during which the quantity had been given, Kelly was not able to conceptualize a rectangle as a figurative or abstract composite unit according to Steffe and colleagues’ (1983) theory.

When Kelly drew four rectangles next to “Paula’s Candy,” she did not connect all of the boxes on each diagram as she had been asked to previously, until she was reminded. This may mean that Kelly thought of Paula’s candy as being composed of four boxes of candy of a certain size, but not as a unified collection. In the excerpt below, Kelly experienced difficulty determining how to draw Darrell’s ten additional pieces of candy.

It is possible that Kelly was eventually able to represent four times Darrell’s amount of candy as “three more rectangles” because she had supposed that one of the rectangles contained four pieces of candy. Thus, when asked to think of Darrell’s number of pieces of candy as being unknown, Kelly was able to surmise that four times Darrell’s amount of candy could be
represented by four rectangles. In order to arrive at this conclusion, it is possible that Kelly multiplied four times four in order to find out the number of pieces of candy she would obtain. She may have then divided that total by four, her proposed number in each box. This explanation seems likely based upon her performance on the next part of the task.

Kelly: If Darrell doubled the amount of candy he has and then bought a box of candy containing ten pieces in it, he and Paula would have the same number of pieces of candy. Alter your diagram to show this situation. [Pause]

Jan: So what are you thinkin’ at this moment?

Kelly: Um, I can’t really think.

Jan: Well, even though we don’t know the number of pieces of candy in a box [points towards originally drawn box on Darrell’s diagram], we do know that he doubled the amount of candy he has. So how would, how could you show that, on the diagram?

Kelly: Um [long pause]. There could be one [points towards originally drawn box on Darrell’s diagram] No. [Long pause]

Jan: Well, let’s try this….Let’s suppose. Let’s suppose that—

Kelly: Well, if, if I’m thinking four [pointing to the originally drawn rectangle on Darrell’s diagram], um, I’m thinking four, and, hm. Hold on, I have to rethink it. [Long pause]

Here, Kelly seemed to be thinking of assigning the number four to Darrell’s amount of candy as before. She seemed to want to work through that possibility before I changed the task for her.

Jan: Okay.

Kelly: I can’t get the answer.

Jan: Okay, that’s okay. Now what were you thinking about the number four?

Kelly: I um. I kinda put, um, different numbers of boxes, different numbers of candy in there and got it wrong.

Here, Kelly seemed to be trying to guess the number rather than recreate the problem conditions on paper. It is possible that this means that she did not anticipate that representing the
structure of the problem externally can help her to ascertain the correct answer. Kelly’s actions in this episode seem to point towards an action conception of the multiple of the unknown quantity in that, in order to be able to represent this multiple, she had to first ascribe a number to the unknown quantity, determine its multiple, and then determine the number of necessary boxes needed to accommodate that multiple.

Jan: Oh, I see. Gotcha. What if, however, if there had been a box with four in it, how would you show doubling that?

Kelly: Um, four times two which equals eight.

Jan: So how many boxes would there be?

Kelly: Eight.

Jan: I mean, for Darrell. Like, let’s say there was, there’s not necessarily four in here, but let’s say there was four in here, right [taps originally drawn box on Darrell’s diagram]. And, but you wanted to show eight. And if that box represented four, then what picture would represent eight?

Kelly: Two, one other box.

Jan: Okay.

Having Kelly think about a particular number in each box seemed to help her to envision how many boxes she would need in order to represent a multiple of the unknown quantity. Using Dubinsky’s (1991) perspective, Kelly was in the action stage of understanding the multiple of an unknown quantity in that she had to engage in a step-by-step procedure (Tall, 2008) in order to ascertain the correct number of boxes she would need to represent double the amount of Darrell’s amount of candy.

Because of the difficulty that Kelly experienced in representing the multiple of the unknown quantity in the previous session, I posed additional tasks to Kelly which I hoped would enable her to construct the understanding that a number that is $k$ times the value of an unknown quantity consists of $k$ units of that quantity.
I assigned Kelly a task, part of which is shown in Figure 4-5. After Kelly read the beginning of the problem statement, I told her that it was all right if she included the circles in her diagram. I stated this because I felt that I may have had Kelly transition from circles to numbers too soon in the previous teaching session. Allowing Kelly to draw the number of circles in the task in Figure 4-5 gave her, it seems, the opportunity to act upon the representation, thus allowing for her construction of a process understanding of the multiple of the unknown quantity.

Jan:  [after Kelly reads the problem statement aloud.] And it’s okay if you include the circles.

Kelly:  Okay.  [Kelly draws a rectangle on the line next to “Frederick’s marbles.” Kelly counts the number of circles inside of the originally drawn box representing John’s marbles. Kelly draws three more rectangles connected to the first one that she drew]

John had a certain number of marbles as shown in the diagram below. Frederick had four times as many marbles as John. Draw a diagram of this situation.

John’s marbles:  

Frederick’s marbles:  

Figure 4-5: Kelly’s Response to the Beginning of Harper Middle School Task #1 on 4/17/2008

Kelly, it seems, may have used the number of circles in John’s original diagram in order to determine the number of boxes to draw in Frederick’s diagram. I say this because Kelly first drew a rectangle next to “Frederick’s marbles.” Before representing the last three boxes of Frederick’s diagram, Kelly first counted the number of circles in John’s original diagram. Kelly then drew the last three boxes of Frederick’s diagram connected to the first. This leads me to believe that Kelly may have noted the number of circles in John’s diagram, multiplied that
number by either three or four, and, based upon the total, determined that four boxes needed to be drawn to represent four times as many marbles.

I proceeded to give Kelly two more tasks that would enable her to build her understanding of the multiple of an unknown quantity. Her response to the task shown in Figure 4-6 is given below.

Jan: Okay. I have a question for you. I’m gonna draw this box, and this represents someone’s penny collection. Say it’s Jada’s penny collection. And suppose that Thomas had a penny collection that was three times bigger. [Jan puts paper in front of Kelly.] What would that look like?

Kelly [next to Thomas’s name, draws one rectangle. She begins to draw a second rectangle connected to this one]: Oh wait, do you want three more or three times more?

Jan: Well, let’s suppose Jada had five pennies in that first box, and Thomas had three times as many pennies as Jada.

Kelly: Oh, okay. I get it. [Kelly completes drawing the second rectangle on Thomas’s diagram and then draws a third one connected to this one].

Jan: Okay. Can you tell me why you drew that?

Kelly: Um, well you said, she has, suppose she has five, and three times more. First of all, three times more means three times including the box, so [Kelly taps each rectangle that she drew as she counts] one, two, three more.

Figure 4-6: Kelly’s Response to Harper Middle School Task #2a on 4/17/2008

When asked to represent a penny collection that was three times bigger than the original one given, Kelly began to draw two rectangles, then asked, “Oh wait, do you want three more or
three times more?” I did not realize it at the time, but my introduction of the term “$n$ times bigger” was probably a confounding factor in Kelly’s initial understanding of the task. I should have stated that Thomas’s penny collection was “$n$ times as big” as Jada’s penny collection. Because I was not aware of a language error committed at the time, I did not call attention to it. Instead, I asked Kelly to show how many boxes Thomas would have if Jada had a given number of pennies and Thomas had “three times as many pennies as Jada.” This intervention seemed to help Kelly to represent $n$ times Jada’s number of pennies.

Prior to this study, Kelly had at her disposal the conceptualization that $n$ times the length of a figure would result in $n$ iterations of that figure. She also had available to her an understanding of number as an iterable, countable unit. In the above excerpt, Kelly seemed to demonstrate the synthesis of those understandings when she was asked to draw a representation of three times a given amount. When asked why she drew three rectangles (to represent three times bigger), Kelly explained, “Um, well you said, she has, suppose she has five, and three times more. First of all, three times more means three times including the box.” Although syntactic complications were present, it seems that Kelly’s goal was to represent three times Jada’s amount of pennies using three times the number of boxes that were previously there.

With each of the Harper Middle School tasks given before Task 2a on 4/17/2008, Kelly showed that she was able to represent $n$ times a given amount only in the context in which one of the boxes contained a written or unwritten given amount assigned to the unit box. Then, to justify her answer, Kelly either counted by the known quantity or she added $n-1$ to 1 to state why she drew $n$ boxes. In this instance, however, Kelly explicitly stated that she multiplied the original number of boxes (one) by the multiple given (three) in order to determine the number of boxes to draw. Thus, Kelly’s counting activity within each of these tasks and a similar one that followed it differs from her counting activity at the beginning of the Harper Middle School tasks.
On the tasks that followed, Kelly drew \( n \) rectangles in order to represent \( n \) times the value represented by a particular rectangle. Kelly’s response to Harper Middle School Task #3 (see Figure 4-7) illustrates this. After reading the first paragraph in this task, Kelly drew five connected rectangles in the space next to “Jarrod’s Candy” as shown in the diagram in Figure 4-7.

Kelly performed this same action in subsequent tasks.

![Diagram](https://via.placeholder.com/150)

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Figure 4-7: Kelly’s Response to Beginning of Harper Middle School Task #3 on 4/17/2008

Based upon her mathematical actions and her justifications throughout the episodes given, it seems that Kelly demonstrated that she understood that, given unknown quantity \( k \) and known quantity, \( u \), \( ku \) was comprised of \( k \) units of \( u \). Kelly’s development of this understanding seemed to take place as a result of the following. When asked to represent \( n \) times an unknown quantity, Kelly participated in an activity sequence of ascribing a certain number to the unknown quantity (which was represented by a rectangle) and multiplying that number by \( n \). Kelly, then seemed to have observed the effect that \( n \) times the ascribed number would occupy \( n \) rectangles (Simon et al., 2004; Tall, 2008). By reflecting on the relationship between her activity and its effect, Kelly was able to abstract the notion that \( n \) times any quantity will occupy \( n \) rectangles. Thus, in Task 3 of 4/17 (and all subsequent tasks), Kelly was able to anticipate that the number of rectangles that she would need to draw in order to represent the multiple of an unknown quantity.
**Denise’s Story**

Denise began the study with a tentative conception that \( n \) times an unknown length can be represented by the concatenation of \( n \) boxes as evidenced by her response to Item 7 on the pre-assessment (see Figure 4-8). During the initial interview, I asked Denise to explain her response.

Jan: Okay, so I see where you wrote “I drew that box 5 more times.” Could you explain that a little bit?

Denise: You see, I counted this box as being drawn [taps the box that was pre-drawn on paper], I’m not sure if you wanted us to do that, and then I drew the box four more times [moving her pen from left to right, taps the inside of the box that she drew in four places]. It’s the same equal size.

Jan: Why would drawing that box five times mean that the box was five times as long?

Denise: Because if you count it as long, then that’s how long it would be, but I actually should have added another box I think.

7. Draw a board that is five times as long as the board below. Explain how you got your answer

![Diagram of a box with the caption: I drew that box 5 more times.]

Figure 4-8: Denise’s Response to Item 7 on the Pre-Assessment

Denise pointed out that she counted the unit box as a part of her drawing. She stated that she then drew the same box four more times. It seems that Denise was saying that adding four boxes would make the total length of the box five times the length of the original box which was her goal. At the end of the excerpt, however, Denise indicated that she was not sure whether she should have added another box. Therefore, as a follow-up question, I asked Denise to draw a box that was six times the length of a box that I drew on a separate sheet of paper. Denise drew six
connected rectangles underneath the rectangle that I drew (see Figure 4-9). When I asked Denise whether she was satisfied whether that was the right answer, she stated that she was not sure and then indicated that she thought that her response might be the right answer. When I asked why, Denise replied, “Because it’s six times the same size as this one [points at original box that I had drawn].” Denise’s tentativeness in expressing the correctness of her response seems to indicate that she needed further experience with the known quantity before she was asked to represent the unknown quantity using an empty box.

![Figure 4-9: Denise’s representation of six times as long](image)

Denise’s work on the Harper Middle School tasks seemed to help her in the sense that she was able to unitize the value of the known quantity while being able to view a box as representing a specific, iterable known quantity. The task in Figure 4-10 was given to Denise following the first two Harper Middle School Tasks. Even though it does not pertain to candy, the context of the problem is still such that the quantity being measured was discrete, countable, and in a collection. Therefore, this task is being grouped with the Harper Middle School Tasks. Denise’s response to the task in Figure 4-10 and the conversation that follows is an indication of her unitization of the unknown quantity.
4) Marcus started a penny collection. The circles below represent the number of pennies that Marcus has saved. Jada has a penny collection that is five times as large as Marcus’s. Draw a diagram to represent Jada’s penny collection.

Marcus’s penny collection: 

Jada’s penny collection:

Suppose that Marcus triples the amount of pennies that he has and then saves 10 more pennies. Alter your diagram to represent this situation. Use your diagram to determine how many more pennies Jada would need to add to her collection in order to have the same number of pennies as Marcus.

Number of pennies that Jada needs to add to her collection: 

Figure 4-10: Harper Middle School Task #3 on 4/15/2008

Denise: [after reading the task aloud] Okay. If there’s five times as much. Four times as much. Five times. [Next to the words, “Jada’s penny collection”, writes the number 4 and draws a box around it; to the right of the first box, writes the number 4 and draws a box around it; repeats this action three more times until there are five connected boxes.]

Denise seemed to be wavering about whether she was supposed to represent four times as much or five times as much. Perhaps, because there were to be four pennies in each rectangle, Denise may have become temporarily confused. The order in which Denise drew the representation of each penny collection seems to be of import. Unlike Kelly who drew all of the boxes first and wrote the number inside afterwards, Denise chose to first write the number four each time and then draw a box around it. It is possible that Denise was counting by fours as she drew in order to reach a certain total. Or, perhaps she was thinking, primarily, in units of four (as
opposed to units of one box). The latter explanation seems more likely given Denise’s actions subsequent to creating the diagram in Figure 4-11

![Diagram of Marcus and Jada's penny collections. Marcus has 4, 4, and 14, and Jada has 4, 4, 4, and 4. Denise has written 2 next to the diagram.](image)

Figure 4-11: Denise’s Response to Harper Middle School Task #3 on 4/15/2008

Denise: [after reading the second half of the problem statement aloud] If he triples it [next to the first box on Marcus’s diagram, Denise writes the number 4 and then draws a box around the 4; she does this once more] to make it 12 altogether [waves pen back and forth above the first three boxes of Marcus’s diagram].

Jan: Um hm.

Denise: And then he adds 10 more [next to the third box on Marcus’s diagram, Denise writes “4 4”]. Can I just put 5 or 4, 4 and 2, a half a box?

Jan: It’s up to you how you want to do it.

Denise: Okay [writes the number 2 to the right of where she wrote “4 4”].

Jan: In fact, let’s just make it, let’s see what happens if you add 10. See how you might draw that.

Denise: Do I just draw one big box?

Jan: Sure. [Denise crosses out the two 4’s and the 2 that she just wrote; she writes the number 10 and draws a box around the number. This box extends slightly past the last box on Jada’s diagram.]

When Denise began to write 4, 4, and 2 in order to represent Marcus’s 10 additional pennies, I should not have interfered. I had not anticipated that a student might utilize a form
other than one long box to represent the added known quantity. My thinking at the time was that, in future tasks, I wanted students to be able to visually compare the equal-sized representations of the added known quantity and the iterations of the unknown quantity as a means of “discovering” the multiplicative relationship between the two. However, I realize now that Denise was thinking in a very powerful fashion in that she exhibited an inclination to represent a new quantity in terms of the given quantity that she had, apparently, unitized. This type of thinking could have been built upon as we began to work with the unknown quantity.

Jan: Okay, now please tell me about how you figured out what length to draw this last box [points to the last box on Marcus’s diagram]?

Denise: I put the length as half of 4 because 2’s half of 4.

Jan: Got you, got you. Okay.

Denise: So, I think she would need 2 more pennies to add.

Jan: Okay, all right.

Denise: So that they would have the same.

In this episode, Denise demonstrated that she unitized the known quantity in such a fashion that she was able to imagine a new quantity in terms of the given quantity. Denise was able to anticipate, before counting, she would be creating 2 ½ units of four in order to represent the 10 additional pennies. Thus, Denise held the known quantity as an iterable composite unit according to Steffe (1994). Denise’s work on tasks like these seemed to help her in her construction of the multiple of an unknown quantity as a composite unit. When solving the first of the Harper Middle School Tasks in which Denise had to represent a multiple of the unknown quantity (see Figure 4-12), Denise did not show the tentativeness that she demonstrated when having to justify that \( n \) times the length of the representation of the unknown quantity was equal in length to \( n \) boxes.
After reading the problem statement at the top of the page, Denise drew four rectangles connected horizontally next to the words “Paula’s candy.” After she read the problem statement at the bottom of the page, Denise stated the following.

Denise: So, if you doubled it, which means it would be like that [draws a rectangle connected to the given rectangle on Darrell’s diagram], then bought ten more boxes of candy, he and Paula would have the same number of pieces. [Denise draws a rectangle that is connected to the second rectangle on Darrell’s diagram. The right side of this rectangle is aligned with the right side of the last rectangle on Paula’s diagram.] So the number of pieces in a box could possibly be five.

Jan: And why is that?

Denise: Because if he was to double it. If he started out with five [Denise points to first box on Darrell’s diagram] and Paula had four times as much which would make this 20 [points to Paula’s diagram] and if he doubled five, them that would give him ten and then ten more would be twenty that Paula had.

Figure 4-12: Denise’s Response to Harper Middle School Task #5 on 4/15/2008

By drawing four boxes to represent Paula’s amount of candy and adding one box to represent the doubling of Darrell’s amount candy, Denise seemed to demonstrate that she
understood that $n$ times the amount of the unknown quantity can be represented using $n$ boxes.

Although, earlier, Denise seemed to waver in her confidence about the correctness of her representation of $n$ times the length of a figure during her initial interview, in this episode, Denise did not seem to question that representation. It is possible that Denise’s work with the Harper Middle School tasks helped her in this effort.

During those tasks, Denise had the opportunity to abstract that the total length of the diagram representing the multiple of the known quantity, $c$, contained $n \times c = n$ boxes. For example, when the known quantity was four and Denise had to represent six times that amount, Denise drew six boxes, counting by fours. It is possible that, because the quantity to be iterated was known, Denise may have known to anticipate that she would have to represent a total of 24. Thus, she may have either counted by fours until she reached 24. Or, she may have anticipated ahead of time that she would need to represent six fours. In either case, Denise seemed to develop the anticipation that she would have to draw $n$ boxes in order to represent $n$ times an amount.

Denise’s performance on the next task (see Figure 4-13) underscores the conjecture that she developed an understanding that $n$ times an unknown quantity can be represented by the concatenation of $n$ boxes.

Denise: [After reading the problem statement at the top of the page, she draws a rectangle next to “Jarrod’s Candy.”] Five times [draws four more rectangles connected to the first one on Jarrod’s diagram; punctuates the air above each rectangle on Jarrod’s diagram with pen cap, starting from right to left]. Read this part?

Jan: Yes please.

Denise: If Crystal tripled the amount of candy she had and then bought a box of candy that had 12 pieces in it, she and Jerald would have the same number of pieces of candy. Show this situation on your diagram. So, if she tripled it, which would mean that up here [draws a rectangle connected to the first one on Crystal’s diagram. The right side of this rectangle is aligned with that of the third rectangle on Jarrod’s diagram; Denise draws a vertical line down the middle of the newly-drawn rectangle] and she bought 12 more pieces. [Denise
draws a rectangle connected to the third rectangle in Crystal’s diagram. The right side of this rectangle is aligned with the right side of the last rectangle on Jarrod’s diagram. Denise writes the numeral 12 inside of the last rectangle on Crystal’s diagram; she writes the number six on the first three boxes on Crystal’s diagram.] These could be six. [Denise writes the numeral 6 in the first rectangle on Jarrod’s diagram] I’m guessing because [she writes the numeral 6 in each of the remaining rectangles on Jarrod’s diagram] if she started off with six and six times five is thirty and six times three is 18 plus 12 is 30.

Jan: Okay—and how did you figure out the number six?

Denise: Because if this had to be twelve [points to 12-box on Crystal’s diagram] and these were two boxes [points to the last two boxes in Jarrod’s diagram] then it had to be something that could go into 12 divided by two.

Jan: Great. Okay.

Figure 4-13: Denise’s Response to Harper Middle School Task #6 on 4/15/2008

In both of these episodes, Denise demonstrated that she did not have the need to ascribe a value to the unknown quantity in order to accurately represent a multiple of the unknown quantity. This seems to be evidence that Denise understood that the original quantity given is a specific yet unknown quantity (Küchemann, 1981) that can be operated with and upon (Stacey & MacGregor, 1999).
Further evidence that Denise seemed to unitize the unknown quantity occurred when Denise explained the value of each box was six in the Crystal-Jarrod problem. She stated, “because if this had to be twelve [points to 12-box on Crystal’s diagram] and these were two boxes [points to the last two boxes in Jarrod’s diagram] then it had to be something that could go into 12 divided by two.”

Here, Denise seemed to be saying that she divided the number 12 by 2, the number of boxes that were equal to 12. In order to be able to act in this manner, it seems that Denise had to conceptualize each of the two boxes as representing an unknown, but specific quantity. Further, it seems that she had to conceptualize that the two boxes, together, as a new unit that represented the known value. As a result, Denise was able to operate with the unknown quantity by dividing the known quantity by the number of occurrences of the unknown quantity.

**Valerie’s story**

Valerie participated in both the pilot study and the main study. During the pilot study, Valerie worked with another pilot study participant, Alexis. When excerpts from the pilot study are discussed, because Valerie is the focus of this analysis, I will primarily use Valerie’s name when discussing my teaching decisions if those decisions relate things that are true for both Valerie and Alexis. In addition, in telling Valerie’s story, I primarily describe Valerie’s actions and words except when Alexis’s comments or actions seem to be central to the development of the mathematical conversation and activity.

Valerie’s path of development of Understanding 1 was very similar to that of Kelly. Similar to Kelly, Valerie demonstrated on the pre-assessment that she understood that \( n \) times the length of a certain figure can be represented by the concatenation of \( n \) iterations of that figure.
Below is an excerpt from the interview where Valerie’s explained to her response to Item 6 on the pre-assessment (see Figure 4-14)

Jan: Draw a board that is five times as long as the board below. Explain how you got your answer. So you wrote, “I drew five boards that are the same size as this board.” Okay, so, what makes you sure that this [frames fingers around the five boxes that Valerie drew] is five times longer, the one you drew is five times longer than this board [points at the pre-drawn board].

Valerie: Because they asked for five more of these and I just drew five more.

Jan: So does five times as long mean five more?

Valerie: Yes.

Jan: Why is that?

Valerie: Because if they’re saying five more than they’re just asking for the same size multiplied by five.

Figure 4-14: Valerie’s Response to Item 6 on the Pre-Assessment

In justifying her response to Item 6 on the pre-assessment, Valerie had stated, “Because if they’re saying five more than they’re just asking for the same size multiplied by five.” In this statement, it seems that Valerie was not just thinking about multiplying the number of boxes in the original quantity by five in order to obtain create a representation of the new quantity. Rather, Valerie seemed to be stating that the length, itself, of each box is being multiplied by five.

Similar to that which occurred during my interactions with Kelly, I erroneously utilized the terms “n times longer” and “n times as long as” interchangeably. Though this language use
did not affect my interaction with Valerie during the previous excerpt, it did have some affect upon future problem-solving interactions. During the pilot study, I assigned Valerie a task in which she was asked to represent a paper strip that was “three times longer” than another paper strip (see Figure 4-15 and Figure 4-16).

**Figure 4-15:** Paper Strip Task #1 on 1/10/2008

**Figure 4-16:** Valerie’s Response to Paper Strip Task #1 on 1/10/2008

Valerie, at first, drew one box to represent the length of Strip D and three boxes to represent the length of Strip C. Then, when Valerie had to consider doubling the length of Strip C and adding six inches to it, Valerie changed her mind and stated “maybe strip C was a little bit longer” and she created a fourth box for the length of Strip C. Again, the use of the language “three times longer” may have served as a source of confusion for Valerie. When asked why she
drew four boxes, Valerie stated that she was “thinking like three more.” It seems that Valerie correctly understood that “three times more” should be represented by maintaining the original box and then adding three times the original amount.

In order to represent doubling the length of Strip D, Valerie then drew two additional boxes which she connected to the first box in the Strip D diagram. She then added a fourth box to the Strip D diagram to represent the six additional inches of paper added to Strip D. Valerie then indicated the quantities represented in her diagram as described below.

Jan: Okay, now will you show me the original length of Strip D please? [Valerie draws a bracket above the first box in the Strip D diagram.] Okay, will you show me where you doubled it? [Valerie draws a bracket above the second and third boxes in the Strip D diagram. Okay, and would you label the six inches please? [Valerie writes “6 inches” underneath the last box on the Strip D diagram.]

In representing the doubling of Paper Strip D, it seems as though Valerie had generalized her use of \( n + 1 \) boxes which she used to represent \( n \) times more. Because of the language error mentioned earlier, it is seems likely that Valerie built this misconception in the midst of this study (given her prior performance on the pre-assessment). Valerie demonstrated this same misconception during a later teaching session in which she was asked to solve the task in Figure 4-17. The following passage illustrates her thinking:

Valerie: Jamie had five times as many pennies as Judith. If Judith tripled [her] amount of pennies and then was given 36 more pennies, they would have the same amount…. Maybe Judith had one box. [Valerie draws a box (see Figure 4-18).]

Jan: Okay, let’s label that with a letter. That way we’ll know—we’ll call it her original. At the bottom, can you write what A stands for please? What are we measuring? [Valerie writes “A=Judith’s pennies”.] Judith’s pennies. Okay. All right. And could you now represent, please, Jamie’s pennies? [Underneath what she just wrote, Valerie writes “B=Jamie’s pennies”.] Okay, and can you draw a diagram of that please?

Valerie: This? [She points at sheet where diagram is drawn; the camera view is blocked.]
Jan: Um, yeah, Of Jamie’s [taps place on paper where problem statement is written] pennies based on what you see here. [Underneath Judith’s diagram, Valerie writes the letter B then draws five connected boxes underneath that letter.] ... All right, now, Judith tripled her amount of pennies. Can you show me that? [Alexis draws three boxes connected to the first box on Judith’s diagram]. Okay, now I see you added three boxes. Why is that?

Alexis: Because if she only had one box, she tripled it [inaudible].

Jan: [to Valerie] What do you think?

Valerie: Yeah, and she tripled it, so she had three more [taps each box on Judith’s diagram]. Then, she was given 36 more pennies. So maybe, this box right here [draws a box to the right of the last box on Judith’s diagram] would be the 36. Then, they would have the same amount.

Jan: How convinced are you that this [traces an invisible circle around first four boxes on Judith’s diagram] represents Judith’s original amount [points at first box on Judith’s diagram], including the amount that was tripled.

Valerie: Because this is her original amount [pointing to first box on Judith’s diagram]—this one box. Then you added three more. Then you added 36 more pennies and they both ended up with five boxes.

### Jamie had 5x as many pennies as Judith. If Judith tripled her amount of pennies and then was given 36 more pennies, they would have the same amount

Figure 4-17: Judith-Jamie Pennies Task on 2/19/2008
In this scenario, Valerie correctly represented Jamie’s number of pennies as five boxes. However, after Alexis added three boxes to Judith’s diagram to represent the tripling of Judith’s pennies, Valerie agreed that Alexis’s representation was correct by stating, “Yeah, and she tripled it, so she had three more.” A little later, Valerie reaffirmed that she thought that Alexis’s representation was correct when she stated, “Because this is her original amount [pointing to first box on Judith’s diagram]—this one box. Then you added three more.”

Although Valerie seemed to demonstrate that she understood that the multiple of an unknown quantity is $n$ iterations of that quantity, Valerie’s misconception concerning the doubling or tripling of an unknown quantity seemed to persist. Perhaps Valerie’s conception was
based in her understanding of a multiple of the unknown quantity. In previous episodes, when asked why she drew \( n \) boxes to represent a quantity \( (b) \) that is \( n \) times a particular quantity \( (a) \), Valerie would indicate that quantity \( b \) had “nothing” and then \( n \) boxes were added to it. Therefore, when having to alter the diagram of an already existing quantity to represent the doubling or tripling of that quantity, it is possible that Valerie reasoned that one box already existed and \( n \) boxes were added to it.

In order to assist Valerie and Alexis in their understanding of doubling and tripling an amount, I drew a box with four circles in it and asked them the following question: “What if I said to you—okay, this is how many pennies Judith had, and she tripled this amount. Would you please draw for me what Judith’s new amount of pennies would be?” Alexis began to draw the rectangle containing 16 circles shown at the bottom Figure 4-19. The following conversation then occurred.

Jan: [to Valerie] First of all, what do you think? Do you agree?

Valerie: No, ‘cause, hold on. If she tripled it, it’s supposed to be twelve.

Alexis: Oh. [inaudible]

Valerie: Yeah, just take it. Take that [Alexis crosses out the last four circles inside the box].

Jan: And what made you decide to draw sixteen?

Alexis: Because I was thinking like if that if that was like this one [points from box drawn by Jan to the circles on the left side of the box drawn by Alexis]. And you tripled that one [covers both sides of Alexis’s box with her hands].

Jan: So how many boxes would this be [taps three sections of Alexis’s box]?

Alexis/Valerie: Four. Three. Three

Jan: Can you make boxes inside—draw lines inside so that we can see all those boxes [Alexis draws a vertical line after each four circles inside of the box that she drew].
Valerie correctly stated that there should be 12 circles inside of the box that Alexis drew in order to reflect three times as much. Therefore, even though I tapped the box that Alexis had drawn in three places, it is very possible that Valerie would have stated that three boxes should be drawn instead of four without my inadvertent help. The next task given to Valerie and Alexis was similar in nature due to the fact that some uncertainty was expressed during the previous task. This time, I drew a box with five circles in it (Figure 4-20) and asked the students to represent double that amount. Valerie responded, “so that would be ten” and she drew two boxes underneath the box that I drew.
After this episode, I returned back to the Judith-Jamie problem with the students. When I asked them if they still thought that they should represent triple the amount of Judith’s pennies using three additional boxes, they both stated that they should. I proceeded to work on one more example with the students (see Figure 4-21).

Jan: All right. Let’s do one more. Frank [writes the name “Frank” on a sheet of paper] had a certain number of pizzas [draws a box under the name Frank; inaudible]. Then he tripled the amount of pizzas he had. Now, when I say tripled, let’s make sure, let’s see if we’re talking about the same thing. For example, when London [points at the box on the first line of London’s first diagram] doubled his nickels [points at the second line of London’s diagram], are you thinking of all three of these boxes as his nickel collection [traces a circle around the first and second lines of London’s diagram] or just these two [indicates the two boxes on the second line of London’s diagram]?

Valerie: Just [inaudible; points at the two boxes on the second line of London’s diagram].

Jan: So, Frank tripled the number of pizzas he had. Can you show me the total number of pizzas Frank now has?

Alexis: He had four.

Valerie: [inaudible; draws three connected boxes underneath the first box on Frank’s diagram.] This is tripled. Three [taps diagram that she just drew].

Figure 4-20: Valerie’s Response to the Doubling of Frank’s Nickels Task on 2/19/2008
Jan: So this is gone [covers the first line of Frank’s diagram]; this is just this [scans pen along second line of Frank’s diagram]. Is that correct?

Alexis & Valerie: Yes.

Unfortunately, I did not ask Valerie the reason why she thought that three boxes should represent the tripling of Frank’s number of pizzas. Valerie may have used the same reasoning that she did when I had drawn, on a separate sheet of paper, an example in which Judith started out with four pennies. After Alexis drew 16 circles to represent triple the amount of Judith’s pennies, Valerie had stated, “No, ’cause, hold on. If she tripled it, it’s supposed to be twelve.”

![Frank's Diagram]

Figure 4-21: Valerie’s Response to the Tripling of Frank’s Pizzas Task on 2/19/2008

When we returned to the Judith and Jamie Pennies Task after I had supposed that Judith had three pennies originally in her box, Valerie affirmed that after tripling, Judith would have nine pennies, and that only two more boxes needed to be added to Judith’s diagram.

In this instance, it seems that having to consider the number of pieces of candy that Judith would have after tripling helped Valerie to anticipate the additional number of boxes that would be needed to represent Judith’s new amount of candy.
Valerie seemed to have an action understanding that the tripling of a given amount could be represented using three rectangles. That is, in the midst of the situation, Valerie was able to determine the number of rectangles needed if she were provided an original amount. However, Valerie had not yet demonstrated that she able to anticipate that three rectangles would be needed to represent tripling without first acting upon a given quantity.

I assigned Valerie the Harper Middle School Tasks as a means of buttressing what seemed to be her developing understanding of the doubling or tripling of a known quantity. It was my goal for Valerie to understand that generating the representation of a multiple, \( n \), of a given quantity when the representation of the original quantity was still present was akin to iterating that quantity \( n-1 \) times. It was then my hope that Valerie would be able to generalize this concept to the unknown quantity as well. On the first Harper Middle School Task, seen in Figure 4-22, Valerie represented tripling by adding two boxes to the original diagram given.

Jan: Okay. So, first of all, what is this—what do you—what is this problem saying to you?

Valerie: Like this is an example of a box candy, and it has two pieces of candy in it. And it says that he tripled the amount of candy that he started with.

Jan: Okay. Now, could you show me that, on that diagram please? [Valerie draws a rectangle connected to the pre-drawn rectangle. She draws two circles inside of the second rectangle. She attaches a third rectangle to the second one and draws two circles inside]. Okay. Can you tell me why you drew what you drew?

Valerie: Because he tripled it, so I added two more [moves pen across the last two boxes in the diagram] for it to be like the three of them. And now he has six pieces of candy.
Harper Middle School was selling candy as a fund-raiser. The candy was packed into boxes by the students. Each box differed in size according to the number of pieces of candy that it could hold. At lunchtime, the boxes of candy were sold.

Julius bought a box of candy that had two pieces in it (as shown in the diagram below). Suppose that after buying more boxes of the same size, Julius ended up with triple the amount of candy that he started out with. Alter the diagram below to show the amount of candy that Julius has now.

Figure 4-22: Valerie’s Response to the Harper Middle School Task #1 on 4/4/2008

In the last line of this excerpt, Valerie referred to the unit box instead of referring to the number of pieces of candy inside when justifying her diagram. This was unlike Valerie’s justifications previously when giving her reasoning for creating a representation that represented doubling or tripling. Although Valerie then stated, “he now has six pieces of candy”, this seems to be a report of the result of the tripling rather than the reason for the addition of two extra boxes.

When justifying her representation on the next Harper Middle School Task (shown in Figure 4-23), Valerie primary made reference to the circles inside of the box, rather than to the unit box itself:

[Attached to the pre-drawn box on the page, Valerie draws a box and then draws four circles inside of it.]

Jan: Okay. Again, will you tell me why you drew what you drew?

Valerie: The same way I doubled it. Like there’s one here [points at pre-drawn box], and then I added another one [points at the box that she just drew]. Now she has double the amount, so now she has eight [scans pen left to right across diagram].

Jan: Okay. Were you thinking doubling in terms of doubling the number of boxes? Or were you counting the pieces and making it.

Valerie: Like four times two is eight, so I just [pointing at pre-drawn box] did that.
Figure 4-23: Valerie’s Response to the Harper Middle School Task #2 on 4/4/2008

When justifying why she drew one additional box to represent the doubling of Ashai’s candy, Valerie twice referred to doubling the number of circles inside of the given box as when she said “four times two is eight, so I just [pointing at pre-drawn box] did that.” For this reason, it seems that Valerie may have utilized the given quantity to determine the number of boxes.

It is interesting to note, however, that when solving both of the Harper Middle School tasks, Valerie drew each box before writing the given number of circles inside of it. Thus, while Valerie justified her doubling answer using multiplication, it seems that Valerie was thinking in terms of units of boxes and not only in terms of the number of items that should go inside of each box. By drawing each box before drawing the pieces of candy inside, Valerie may have even been anticipating and counting by a particular unit.

To summarize, on 2/19/2008, Valerie engaged in a series of tasks that involved her acting upon visual objects that were in the form of circles. Similar to Kelly, when asked to represent the doubling or tripling of a number, Valerie engaged in an activity sequence in which she would multiply the number of circles shown in a box by the factor that she was supposed to represent. Valerie had the opportunity to observe the effect that the product would need to be distributed into $n$ boxes, each of which contained the given number of items. It seems that Valerie interiorized this process so that, by the time she solved the Harper Middle School Tasks, Valerie anticipated that she would be drawing the number of boxes that it would take to hold double or
triple the given amount. Valerie exhibited the same mathematical behavior whether the original box was included in the representation (as in the Judith-Jamie task) or whether she was asked to represent the result of doubling or tripling of a quantity on a separate line from the individual diagram.

Although Valerie worked only with doubling and tripling in these instances, Valerie later demonstrated the ability to represent the quadrupling of an unknown quantity once she was told that quadrupling meant four times as much.

*Comparison of Students’ Development of Understanding 1*

Each participant started out with the understanding that $n$ times the length of a rectangle was the length of a concatenation of $n$ iterations of that figure (though Denise’s understandings were somewhat tentative). And, they each constructed the understanding of the unknown quantity as a composite unit and the multiple of the unknown quantity as a unit of units in different ways.

All three participants conceived of a rectangle representing the known quantity as an iterable composite unit as evidenced by their success in representing the multiple of a known quantity within the diagrammatic situation. However, when Kelly encountered the an empty rectangle for the first time, she felt the need to ascribe a value to the unknown quantity. And Valerie, though she was able to accurately represent $n$ times longer and, sometimes, $n$ times as long, she would also sometimes utilize the former representation for the latter. It seems possible that Valerie later experienced problems with the representation of doubling and tripling because proper clarification was not provided to her. Finally, when Denise encountered the empty box problems for the first time, she did not seem to experience any difficulty.
According to Tall’s (2008) theory, the students were acting within the embodied-symbolic world in that their representations were box iconic (boxes and circles) and, later, symbolic (with numbers replacing the circles). They at first acted procedurally in order to represent the multiple of a known quantity. They first had to count the number of circles inside of the given box. Then the students either counted by the number of circles inside of the box or counted by the number of boxes in order to produce a representation of the multiple of the known quantity. Because each of the Harper Middle School Tasks at this stage required the students to write the known quantity inside of each box, it is difficult to say at what point each of them internalized this procedure. However, Denise was the only student who wrote the number first and then drew a box around it when representing a multiple of the known quantity. Valerie and Kelly both drew the box first and then wrote a number inside.

Since Denise was the only participant who did not appear to experience any difficulty once she transitioned to the use of empty boxes, the question remains whether or not there is a relationship between her attention to unitizing and counting by the known number and her preparedness to represent the unknown quantity. Perhaps it was the case that as a result of her deliberate focus upon representing the unit of counting, Denise noticed the effect of her actions (according to Tall’s theory) whereby she “saw” that she had to represent that particular unit \( n \) times. On the other hand, by representing the box first and then drawing the circles or writing the number inside, perhaps Valerie and Kelly were not anticipating that they were trying to reach a particular number of circles, only a particular number of boxes. It is only when the students’ attention was brought to the circles themselves that Valerie and Kelly were able to represent the multiple of an unknown quantity accurately. Thus, students’ construction of the understanding that a known quantity, \( k \), multiplied by an unknown quantity, \( u \), is equal to \( k \) units of \( u \) seemed to have occurred as a result of their reflection on the relationship between their action upon the box representing the known quantity and the effect of that action.
Students’ difficulty with their interpretation of letters in algebra has been widely reported (Booth, 1984; Küchemann, 1981; Stacey & MacGregor, 1997). According to these authors, students often confuse a letter in algebra with an abbreviation or a label. It was my goal in this study to introduce letters as a means of representing the unknown quantity after the students had already built an understanding of the unknown quantity using an icon that held less ambiguity to them (i.e., a box). Then, the use of letters was contextualized to represent the length of the box representing the unknown quantity.

The use of letters in the main study was introduced to Kelly and Denise during the paper strip tasks. Valerie engaged in these tasks during the pilot study. These tasks were designed based upon Mikulina’s (1991/1969) work with elementary school students and had three intended purposes: (a) To enable the students to build a part-whole understanding of the relationship between unknown and known quantities within the same or equivalent wholes, (b) to enable students to construct the understanding that \( n \) times a certain quantity was \( n \) iterations of that quantity, and (c) to enable students to express a relationship between unknown quantities. These goal understandings differed slightly from the goal understandings that had been established for students in Mikulina’s (1991/1969) study.

According to Mikulina (1991/1969), it was intended that the students in his study develop “a general concept of the relationship between the whole and its parts” (p. 192) and that they be able to represent this relationship diagrammatically and symbolically. Further, it was his intent that students would be able to solve a variety of word problems (including problems in which the initial quantity was unknown) in which either the whole or one of the parts was not given.
Students in Mikulina’s (1991/1969) study engaged in problem-solving experiences in which they first represented the relationship between measurable attributes of objects in a realistic situation (such as the volume of containers of grain or the lengths of paper strips) using diagrams. Students then represented these same relationships in diagram form. The students in his study did not know the numerical value of each measure, but each quantity was referred to by the substance being measured and a letter name, such as “grain $a$.” Students observed actions where a substance was poured evenly into several containers or like substances in several containers were poured into one container; they were asked whether the quantity in question was “a whole or a part” (Mikulina, 1991/1969, p. 203).

Students in that study were then asked to represent the relationship between the parts and the whole using diagrams. For example, when the grain from one container was poured evenly into three other containers, students were asked to draw this situation using “paper strips”, which meant that they should use rectangular boxes to represent the situation (please see three students’ work in Figure 4-24).

Figure 4-24: First Grade Students’ Diagrammatic Representation of the Amount of Grain Poured Evenly into Three Jars (Mikulina, 1991/1969, p. 197)
The part-whole understandings that were constructed by students in Mikulina’s study and those meant to be cultivated in this study differed from one another in the nature of the part-whole understandings that were developed. Students in Mikulina’s study solved word problems that were, primarily, of the form $x + a = c$ or $a + x = c$ where the first quantity of each equation represented a starting amount, and $c$ represented an ending amount. Further, $a$ and $c$ were known quantities and $x$ was an unknown quantity. Therefore, it might be stated that the students in Mikulina’s study constructed part-whole understandings that were additive in nature.

Students in this study, primarily, solved problems which were of the form, $ax = b$, $ax + b = c$, or $ax + b = cx + d$. Thus, the part-whole understandings that students constructed were additive and multiplicative in nature. In the initial paper strip tasks, students were asked to draw paper strips that were a certain number times as long as a paper strip given to them. During the paper strip tasks of the main study, Denise and Kelly were given pen, paper, scissors, a pre-cut paper strip, and an unmarked straight edge\(^1\) in order to accomplish this task. The students were then asked to create a diagram of the paper strip and its multiple on a piece of paper. This was done in order to enable the students to conceptualize the original paper strip as an iterable unit. An example of Kelly’s work on this is shown in Figure 4-25. Next, Kelly and Denise were asked to create number sentences based upon given information in the problem statement. Following this portion of the tasks, Kelly and Denise were each asked to complete a series of tasks in which they had to construct equations relating the lengths of paper strips that they cut out.

\(^1\) During the pilot study, Valerie had been given these same tools, except she was given a ruler instead of an unmarked straight edge. I decided to give Kelly and Denise an unmarked straight edge. I had found that, during the pilot study, in order to create the new paper strip from the old one, the students focused on the measurements of the paper strips rather than the relationship between the lengths of the paper strips.
Kelly began this study with a “letter evaluated” conception of letter, using Küchemann’s (1981) categories. Students possessing this conception have the ability to reason about letters contained within an expression or equation as long as the letter has been given a particular value. This is demonstrated by Kelly’s response to the pre-assessment items in Figure 4-26. Kelly was able to provide values for \( u \) and \( v \) that would satisfy the conditions of the equation. When asked about how she solved the problem in Figure 4-26, Kelly responded in the following way.

Kelly: \( u \), I think …would be 5 and \( v \), I think…would be two.
Jan: And why is that?

Kelly: Because $u$ could be five, because two plus three equals five, and the $v$ could be two because two plus three equals five.

Jan: Gotcha. Okay. Thank you very much.

In order to be able to correctly answer the aforementioned question, it seems that Kelly may have first assigned a value to either $u$ or $v$ as evidenced by her statement that “$u$ could be five, because two plus three equals five” or “the $v$ could be two because two plus three equals five.” Or, Kelly may have named a pair of numbers which, added together, were equal to five. Both Denise and Valerie provided similar answers to the task.

![Figure 4-26: Kelly’s answer to pre-assessment Item 3](image)

At the beginning of the study, Kelly only demonstrated a partial understanding of a letter as a representation of a specific, unknown quantity (Küchemann, 1981). She did not fully demonstrate that a letter can be operated upon as if it were known as evidenced by her response to Item 6 from the pre-assessment (shown in Figure 4-27). Both Booth (1984) and Küchemann (1981) found that many of the 13-15 year old algebra and pre-algebra students whom they studied did not demonstrate the ability to operate upon an unknown quantity “as if it were known” (Stacey & MacGregor, 1999, p. 30). Given a problem such as that which was found on pre-assessment item #6, those students might have given an answer of “$5n$” because of their inability to accept a lack of closure (Booth, 1984).

In her response to Item 6, Kelly represented Felicia’s number of hours worked as $n^5$. Although this answer does not reflect the additive relationship between each person’s number of hours worked, it does show that Kelly attempted to combine the important quantitative
information in the problem, and possibly operate upon the given number of hours \((n)\) in order to show Felicia’s number of hours worked. Unfortunately, I did not ask Kelly about her response to this task during the interview following the pre-assessment.

Figure 4-27: Kelly’s Response to Item 6 on the Pre-Assessment

Kelly drew a question mark across the statement of the task in Figure 4-28. It is possible that Kelly did not know how to operate upon a letter in such a manner as to represent the relationship between two quantities using letters. It is also possible that Kelly did not know how to construct an equation that would relate two letters. Kelly’s actions during a later teaching episode proved the latter speculation, at least, to be true.

Figure 4-28: Kelly’s Notational Response to Item 10 on the Pre-Assessment

A letter within an equation is an operable composite part of a whole that is in relation to other parts within the same or an equivalent whole

Kelly began the study with a partial understanding that a letter could be operated upon. However, she did not have the understanding of the way in which the process of adding a known quantity onto an unknown quantity could be represented.
After participating in the creation and diagramming of paper strips and their multiples, students were given problem-solving scenarios in which they had to create equations relating the known lengths of paper strips. Then students had to create equations using letters. Before using letters in these equations, however, students were asked to utilize the terms “Length of Strip 1” and “Length of Strip 2” in order to represent the unknown quantity and its multiple. The intent of the tasks was to enable students to construct “Length of Strip 1” and “Length of Strip 2” as quantitative entities that could be operated upon as if they were known. By first having students find the lengths of Strips 1 and 2 when the length of the other was known in Task 2, it was my anticipation that students would be able to generalize their activity in Task 3. This task was designed based upon Thompson’s (1988) work.

Thompson (1988) found that the 8th grade students with whom he worked experienced difficulty in creating relationships of dependence between quantities that were given in algebra word problems. This difficulty was due, in part, to students’ inability to properly “name [the] quantities” (p. 168) in the problem. Instead of naming a quantity according to what it was measuring, students would attempt to name a quantity according to its measure. Thompson gave the example that if, in a word problem, one of the known quantities was $250 (the amount of money in a bank account belong to a person named Sam), some students might name the box referring to this quantity as “‘250 dollars’ instead of ‘Sam’s Savings’” (p. 168). As a result, students were not able to readily identify and construct relationship between unknown quantities in the problem statement (Thompson, 1988).

Kelly’s answers to Paper Strip Tasks 2 and 3a provide a means of benchmarking where Kelly’s quantitative understandings were prior to this instructional intervention. To solve Paper Strip Tasks 2a and 2b in Figure 4-29, it seems that Kelly utilized the given relationship between the lengths of Strips 1 and 2. In order to find the length of Strip 2, Kelly stated that she divided
51 by 3 because “this [points at the words “Length of Strip 2” on the sheet in Figure 4-29] is three
times the length of this [points at the words “Length of Strip 1” on the same sheet].

<table>
<thead>
<tr>
<th>Figure 4-29: Kelly’s Response to Paper Strip Task 2 on 4/28/2008</th>
</tr>
</thead>
<tbody>
<tr>
<td>Suppose that the length of Strip 1 was 30 inches.</td>
</tr>
<tr>
<td>a) Write a number sentence to show how you would find out the length of Strip 2:</td>
</tr>
<tr>
<td>Length of Strip 2 = (30 \times 3) = 90</td>
</tr>
<tr>
<td>Suppose that the length of Strip 2 was 51 inches.</td>
</tr>
<tr>
<td>b) Write a number sentence to show how you would find out the length of Strip 1:</td>
</tr>
<tr>
<td>Length of Strip 1 = (\frac{51}{3}) = 17</td>
</tr>
</tbody>
</table>

In spite of her success with solving Paper Strip Tasks 2a and 2b in Figure 4-29, Kelly
experienced difficulty in representing the act of operating upon the length of Strip 2 when
attempting to solve the task in Figure 4-30.

Jan: Suppose you knew the length of Strip 2 [holds up Strip 2], but you didn’t know
the length of Strip 1 [holds up Strip 1]. How would you find the length of Strip 1
if you only knew the length of Strip 2?

Kelly: Divide by three.

Jan: Okay. Can you make a kind of number sentence or letter and number sentence
that would, that would explain that?

Kelly: Okay. L of S2 is 51 inches [writes “L of S2 is 51 inches” (see Figure 4-30)] L of
S1 is [writes “L of S1=?”] Fifty-one [begins to write something and then
scratches it out]—Can I write the facts down first though?

Jan: Sure. [Kelly begins to write something on the second line in the space between
the question mark and the mark that she just crossed out; she then scratches out
the new mark as well.]
Kelly: Fifty one inches divided by 3 equals seventeen inches [writes 51 inches ÷3 = 17 inches]. L of S1 is now seventeen inches. [Writes “L of S1 is now 17 inches.”]

Figure 4-30: Kelly’s Response to Paper Strip Task 3 on 4/28/2008

It seems likely that Kelly’s decision to write a specific number sentence rather than a generalized one in this episode occurred because she may have misunderstood my instructions when I stated “Suppose you knew the length of strip two.” Kelly may have interpreted those instructions to mean that she should start with the length of Strip 2 from the previous problem. After I rephrased the question, Kelly stated that she did not know what to do.

Jan: What if I said to you, okay, L of S1 equals [at bottom of page, writes “L of S1=”] and then you had to show me how you would find L of S2. I mean, how you would, you knew L of S2, you don’t know L of S1. It’s like a general formula.

Kelly: What do you mean?

Jan: So, let’s pretend we didn’t know this was fifty-one [points to Strip 2]. We just know this is L of S2. How would you find L of S1?

Kelly: If we didn’t know the length [points to where she wrote “L of S1” on the problem sheet] of the first, of number one, then -
Jan: Yeah. We didn’t know fifty-one.

Kelly: Oh. I don’t know how we would get that.

Jan: Okay, okay. Um, so let’s pretend, right? You’re telling somebody. Okay. You’re describing it over the phone, you’re saying “Cut out a paper length of a certain size”, and they cut it out. And then you would say to them, “Okay, now in order to make this first strip you need to do blank.” What would you tell them?

Kelly: Cut it into thirds.

Jan: Okay. Would you write that maybe in sentence form? [Points at where she wrote “L of S1 =” at the bottom of the page; Kelly writes “1st, cut long strip into thirds”]

It seems that in this task, Kelly’s problem was not in naming the lengths of Strips 1 and 2 as quantities as Thompson (1988) experienced with the 8th grade students in his study, but in operating upon her symbolization of that quantity.

When Kelly did succeed in writing a sentence to describe the way to find the length of Strip 1 given the length of Strip 2, it was only after she was asked to describe how to operate upon the strip itself rather than upon the length of the strip. Thus, it seems like, for Kelly, “L of S2” was not an operable entity in that she was able to describe the action that could be taken upon Strip 2 to create Strip 1, but she was not able to operate directly upon a representation of the length of Strip 2 (L of S2) in order to obtain the length of Strip 1.

Before solving the problem shown in Figure 4-31, Kelly was given the diagram shown in Figure 4-31 on a separate sheet of paper. No values were assigned to any of the strip lengths. Kelly was asked to reason about how paper strips 1, 2, and 3 could have been created if only one of those strip lengths was known. The purpose of engaging Kelly in this exercise was to enable her to focus her attention on the part-whole relationships between each of the paper strips. Kelly was able to accurately state the relationships between each pair of paper strips as exemplified in the dialogue below (due to technical difficulties, only the audio recording of this teaching session was available for transcription and annotation).
Jan: Okay, okay. So how about Strip 2, how would you create Strip 2 if you just had Strip 1?

Kelly: If I only had Strip 1? Um, I’d have to get another strip. I’d have to get four more squares, put them together that would be strip \(b\) [\(b\) was equal to the length of Strip 2].

![Diagram of strips](image)

Suppose that \(a = 15\) inches. Show this on the diagram

\[
\begin{align*}
\text{Strip 4:} & \quad \begin{array}{c}
15" \quad 15"
\end{array} \\
\text{Strip 5:} & \quad \begin{array}{c}
c \\
c \\
c
\end{array}
\end{align*}
\]

\(a = \text{length of Strip 1}\quad b = \text{length of Strip 2}\quad c = \text{length of Strip 3}\)

a) Write a number sentence to show how to find out what \(b\) is equal to:

\[b = \frac{4 \times a}{2} = 60\text{ inches}\]

b) Write a number sentence to show how to find out what \(c\) is equal to:

\[c = 2 \times a = 30"\]

Figure 4-31: Kelly’s answer to Task 2a on 5/1/2008

In doing exercises like the one that Kelly worked on in the episode above, Kelly showed her ability to unitize each strip depending upon what strip needed to be created. That is, Kelly could state the number of strips or the portion of one strip that was needed in order to create another strip. Kelly seemed to use her understanding of identifying the length of a particular paper strip as a unit in her work on the problem in Figure 4-31.

Kelly: Write a number sentence to show how you will find what, what \(b\) is equal to. Okay. \(b\) is equal to…four strip \(a\)’s and that is 60 inches

Jan: Okay.
Kelly: Okay. Four times $a$ equals 60 inches. [Kelly writes “4 x $a$” to the left of the equal sign and “60” to the right of the equals sign.]

Jan: Okay, All righty. And how did you get 60?

Kelly: Um, because you know how this—two strip $a$’s are equal to one strip $c$ and two strips $c$’s are equal to um one strip $b$.

Jan: Okay.

During this problem-solving episode, Kelly seemed to experience a cognitive shift to operating upon a letter that represents an unknown quantity. Kelly used the fact that “$b$ is equal to...four strip $a$’s” in order to determine that the value of $b$ must be four times the value of $a$. It seems that Kelly was able to generalize the relationship between $a$ and $b$ within the context of a known whole. It seems possible that, since Kelly operated only in this manner when the whole was known, that she understood creating an equation relating the two quantities, $a$ and $b$, as an action according to APOS theory. That is, Kelly may not have been able to anticipate, without the use of numbers, that $b$ was four iterations of $a$.

To solve the task in Figure 4-32, finding the value of $a$ when the value of $b$ was given, Kelly utilized the same thinking process as she did on the task in Figure 4-31, as can be seen in the teaching episode below.

[Kelly changes the 2 in “20” to a 1 inside of each $a$ box on the Strip 4 diagram (see Figure 4-32)]

Jan: Okay can you tell me why you put ten there?

Kelly: Because four, four strip $a$’s fit into one strip $b$ and ten—and four divided by four equals ten.

At this point, Kelly demonstrated that she had the understanding that because four $a$ boxes “fit into” one $b$ box, the value of $a$ is 40 (the value of $b$) divided by four.

Jan: Okay so [reading from the calculator as Kelly pushes the buttons] Four divided by—clear—forty divided by four is ten. Okay so that's super. Now would you please read [the next question]?

Kelly: Write a number sentence to show to, find out what $a$ is equal to. What do they mean?
Jan: So here just like here where you put four times $a$ is sixty inches, what did you do to find $a$ this time?

Kelly: Um, to find $a$? Um, I, well there’s four parts of this and the first number of forty is four and ten times four equals forty so four tens is forty.

Jan: Okay.

---

Figure 4-32: Kelly’s answer to Task 3 on 5/1/2008

Although she showed the ability to reason about the relationships between parts of the whole in the episodes above, Kelly was not able to generalize her problem-solving process to write a “letter and number sentence” that would show how to find $b$ given $a$ when neither value was known. Below is an episode from the teaching session when Kelly was solving the task in Figure 4-33.
Kelly: Write a letter and number sentence to show how to find, how you would find $a$ if you only knew $b$.

Jan: Okay. Now what do you think that question means?

Kelly: Um, I think it means that you have to use $b$ to figure out what length $a$ is.

Jan: Okay. And this time we don’t know what $b$ is but you can just call it $b$.

Kelly: Um, so I don’t, so I don’t know $b$ but I have to imagine what $b$ would be.

Jan: Well instead of that thinking um, we don’t know $b$. We know it’s a certain length. So instead of saying twenty or sixty let’s just say $b$. What would you do to $b$ to find $a$?

Kelly: $b$ divided by $a$, $b$ divided by $c$ equals $a$. [writes “$b \div c = a$”]

Jan: Okay why don’t you tell me why you wrote that?

Kelly: Because I thought the answer would have to be $a$ since we have to see what $a$ equals.

Jan: Okay so why do you think that $b$ divided by $c$ is $a$?

Kelly: Um, don’t really know. I just thought that $a$ had to be here, I didn’t think I should put $d$, $b$ divided by $d$. If that makes sense.

Jan: Okay. If you knew this was a certain number like let’s say you knew that was sixteen. What would $a$ be?

Kelly: Well $a$ would be eight inches.

Jan: Eight. Um, why do you say eight?

Kelly: Because um, eight times two equals, eight times two equals sixteen.
Figure 4-33: Kelly’s answer to Task 5 on 5/1/2008

Here, Kelly seemed to be assuming that $b$ is two times $a$ (since we were supposing that $b$ was equal to 16). Therefore, I had to clarify what her assumptions were.

Jan: So you’re saying that $b$ is twice $a$?
Kelly: No four times. It would be four inches.
Jan: And how do you get that?
Kelly: Because four times four equals sixteen.
Jan: Okay. If you knew $b$ was, let’s say, twenty inches right then what would $a$ be?
Kelly: Five inches.
Jan: Okay.
Kelly: Cause five times four equals twenty.
Jan: All right. And finally if you knew $b$ was thirty-six inches what would $a$ be?
Kelly: Um, nine inches?

Jan: … So now, so those are just some specific numbers. What if I said to you, you know what, you know $b$ right, use that to make a number sentence, a letter and number sentence to find $a$. So in other words let’s see, looking on the calculator, let’s pretend that I had an $a$ button on the calculator and I had a $b$ button on the calculator, so what button would you push to find $a$? You don’t know what $a$ is, but you do know what $b$ is.

Kelly: What’s $b$? [inaudible]

Jan: Let’s just pretend $b$ is an unknown number but it’s on the calculator and when you press it, it will come up.

Kelly: $b$?

Jan: Uh huh.

Kelly: Divided by $a$ equals $e$ [pushes calculator buttons off camera]

Jan: Equals “$e$”? [It appears that Kelly may have pushed the $b$ “button” written at the top of the calculator, the division symbol on the calculator, the $a$ “button” written at the top of the calculator, and then the equals symbol on the calculator. This would have given Kelly the error message that she saw.]


Jan: Now I see you wanting to divide; is there a reason why?

Kelly: I don’t think we could get $b$ [inaudible]. $b$ times $a$ equals

Jan: How about this, for each one of these, whenever I gave you a number for $b$, what did you do to that number to get $a$?

Kelly: Ah I don’t remember.

Kelly did not seem to be able to generalize the process by which she found $a$ when I gave her a value for $b$. However, perhaps Kelly had thought of the buttons that she pushed on the calculator as her generalization. At this point, I gave Kelly specific values of $b$ to think about.

Jan: Okay. For example $b$ was twenty.

Kelly: Oh I divided it by four.

Jan: Okay. So instead of me giving you a number like twenty or sixteen the number is $b$, what do you do to $b$ to get $a$?
Kelly: Um, I divided by four.

Jan: Okay, can you write me a number sentence?

Kelly: [writes “b ÷ 4 = a”; rewrites the b.] I don’t want people to think it’s a six; I have bad handwriting.

At this point, Kelly was able to state and write a number and letter sentence relating $a$ and $b$. I pursued her understanding of operating on the letters as if they were known by asking her to explain.

Jan: Okay. Why don’t you explain that to me?

Kelly: $b$ divided, $b$ divided by four equals $a$. Let’s say if it’s forty. $b$, so it would be forty divided by four equals $a$. $a$ equals 10

Although Kelly was successful in constructing letter and number sentences relating $a$ and $b$, it seems that, at this moment, Kelly’s thinking was still tied to her process of dividing specific values of $b$ by 4 in order to obtain the value of $a$. At this point, Kelly seemed to have a process conception of a letter in that she viewed it as representing a specific, unknown quantity that, when substituted, can be used to evaluate a letter and number sentence. However, Kelly seemed to have an action concept of an equation in that, in order be able to relate two quantities in an equation, Kelly first needed to substitute the letters with numbers. Kelly was not able to anticipate that she could create a relationship of equivalence given the relationship between two quantities. The following conversation occurred after I asked Kelly to read the second task on the page in Figure 4-33.

Kelly: Okay. Write a number sentence to show how you find $b$ with only $a$. Okay. [Kelly writes “$a \times 4 = b$”]

Jan: Interesting. Okay tell me please why you wrote that.

Kelly: Um, um, because I divided $b$, I divided $b$ by four and I would, I would multiply $a$ by four.

Jan: Okay cool…. Now would you please read that last one?
Kelly: Write a letter and number sentence to show how you will find $c$ with only $a$.
   [Kelly writes “$a \times 2 = c$”]

Jan: Okay and again why did you do it that way?

Kelly: Because the two $a$’s go into $c$ twice.

By the end of this teaching episode, Kelly seemed to transition to reasoning more abstractly about the relationship between quantities $a$ and $c$ in that she seemed to utilize the fact that two $a$ boxes would fit inside of a $c$ box, therefore, the value of $c$ must be twice the value of $a$. In that sense, it seems that the diagram was helpful in enabling Kelly to think about the part-whole relationships between quantities $a$, $b$, and $c$.

The development of Kelly’s understanding of a letter as an operable composite unit seemed to have been facilitated by her engagement in tasks that built upon her part-whole understanding and utilized her ability to reason visually. Kelly had the opportunity to reason from a diagram that represented a realistic situation in which different parts of equivalent wholes were multiplicatively related. It seems possible that being able to see how many boxes of one size “fit into” a box of another size enabled Kelly to abstract the relationship between the box lengths.

Within the context where one of the box lengths was given, Kelly was able to state the relationship between the lengths of two boxes using letters. However, when no lengths were given, Kelly had to rely upon numerical examples in order to reason about the relationship between box lengths. It was not until the end of the teaching episode involving the task in Figure 4-33 that Kelly seemed to generalize her mathematical activity. She seemed to reason strictly from the diagram (without the need for a numerical context) when she stated that “$a \times 2 = c$” because “the two $a$’s go into $c$ twice.”

Kelly’s development of a letter representing an unknown quantity as an operable composite unit seemed to follow the path as her construction of a rectangle representing an
unknown quantity as an operable composite unit. Kelly first needed exemplars upon which to operate in order to be able to conceptualize a letter as a representation of an unknown quantity that could be operated with and upon before she could generalize her activity. However, unlike during her development of the rectangle as a composite unit, in the more recent episodes, Kelly’s object of focus was the length of the rectangles rather than the discrete measure of the number of circles that each rectangle might hold. Further, Kelly was being asked to make explicit the relationship between the lengths of the two rectangles in the form of an equation. In this effort, Kelly’s part-whole understandings seemed to play a more prominent role. This is discussed next.

To develop the understanding that \( na=b \) (where \( a \) is the labeled length of a drawn rectangle, \( A \); \( b \) is the labeled length of another drawn rectangle, \( B \); and \( n \) is the number of iterations of Rectangle A which, together, equal the length of Rectangle B), Kelly seemed to engage in the following activity sequence. She determined the number of iterations of Rectangle A that needed to be combined to equal the length of Rectangle B, she assigned a numerical value to the length of Rectangle A and multiplied that numerical value by the number of iterations of Rectangle A needed. Kelly may have then observed that the effect of her activity was that the multiple of \( a, na \), was the length of \( n \) rectangles of length \( a \). Therefore, Kelly seems to have abstracted the notion that \( na=b \). According to Tall’s (2008) procept theory, during the Paper Strip Puzzle tasks, Kelly traversed back-and-forth between the proceptual-symbolic and the conceptual-embodied worlds in that she acted upon the diagrams by visually comparing them to determine the relationship between their lengths and she utilized the letters labeling the lengths of the diagrams to generalize and represent her actions. In this way, Kelly’s use of the diagrams and the structure of the diagrams seemed to facilitate Kelly’s transition to algebraic thinking.

Denise’s Story
A letter can represent a specific, unknown quantity that can be operated upon “as if it were known” (Stacey & MacGregor, 1999, p. 30)

Like Kelly, at the beginning of the study, Denise demonstrated that she understood that letters within an equation represent particular quantities which make the equation true. In her response to Item 3 (see Figure 4-34), Denise suggested “if $u$ was 5, then $v$ could be 2.” This seems to be an indication that Denise was able to assign a value to $u$, and based upon that value, determine that $v$ was 2. Additionally, when solving item 4 (see Figure 4-35), Denise seemed to have first determined that the value of $n$ was six given the fact that $n$ plus three is equal to 9. Denise then determined that $n$, which was 6, plus 8 equals 14.

![Figure 4-34: Denise’s Response to Item 3 on the Pre-Assessment](image)

Unlike Kelly, however, Denise also demonstrated at the beginning of the study the understanding that a letter can represent a specific, unknown quantity (Küchemann, 1981) that can be operated upon as if that quantity were known. This conclusion is based upon her responses to pre-assessment Item 6 in Figure 4-36. Denise’s initial response to Item 6 on the pre-assessment was, “$n \times 5 = ?$” The following discussion took place concerning Denise’s response.

Jan: Would you please tell me why you chose that answer?

![Figure 4-35: Denise’s Response to Item 4 on the Pre-Assessment](image)
Denise: I chose that answer because Marcus worked $n$ hours and Felicia worked five more hours.

Jan: Okay, okay.

Denise: So, really, I should have put $n$ plus.

Jan: You can put that in there if you’d like. Do you want to write that in?

Denise: Sure. [Denise writes a plus sign over the multiplication sign in the expression “$n+5$.”]

Figure 4-36: Denise’s Response to Item 6 on the Pre-Assessment

Unfortunately, I asked Denise a question that might have led her to change her written response from multiplication to addition. In spite of this, however, it seems that Denise demonstrated that she viewed the letter $n$ as representing a specific unknown quantity that can be operated upon as if it were known. This is evidenced by her initial symbolization of multiplying $n$ by five and then adding 5 to $n$. Thompson (1988) would characterize Denise as having the ability to “reason quantitatively” (p. 164) in that, she was able to reason about the way in which one quantity was additively related to another quantity without having the need to ascertain the value of the first quantity. Whereas Denise was able to represent Felicia’s number of hours worked in terms of Marcus’s number of hours worked, when asked to write an equation to express the relationship between the lengths of two boards, Denise chose to use a different kind of representation (see Figure 4-37).
Denise’s response to Item 10 on the pre-assessment (shown in Figure 4-37) provides some insight into her thinking about the relationship between an unknown quantity and its multiple. In particular, Denise seemed to have conceptualized the quantity $b$, from Item 10, as being a number that is four times the value of $a$.

Jan: So let’s talk about this [pointing to the task] and--what does all this in your answer mean?

Denise: $a$ [points at the letter $a$ on her diagram] is how big $a$ is. Two [points to “2x” in her diagram]—that’s two times as big [points to the words “2x as big” in the diagram]; three times as big [points to the words “3x as big” in the diagram]; and $b$ [points to the letter $b$ in the diagram] is four times as big [inaudible.]

Jan: So it’s almost like—

Denise: A scale.

Jan: Oh, all right. Thank you.

In this episode, it seems that Denise conceptualized the letters $a$ and $b$ as representing specific unknown lengths where $b$ is four times as large as $a$. This is indicated by her creation of a diagram where the letter $a$, the words “2x as big” and “3x as big”, and the letter $b$ were written in such a manner that they were evenly spaced from one another. Denise considered this representation as a type of “scale.” Given the seemingly equal spacing between the four elements
of her diagram, it seems that Denise thought of her diagram as a model of the situation where a particular unit is being iterated.

She identified \( b \) as representing a number that is 4 times as big as \( a \). However, Denise’s diagram did not seem to indicate that she thought of \( a \) as the iterable unit, though it is not possible to rule out that she had this conception. It would have been very helpful if I had followed up Denise’s comment about her representation being like “a scale” with a question about why she stated that and how that related to the original task. It is possible to state, however, that Denise did not spontaneously choose to represent “2x as big” and “3x as big” in terms of \( a \).

It is interesting that Denise was able to represent five more than \( n \) in task 6 of the pre-assessment, while she did not spontaneously choose to represent “2x as big” and “3x as big” in terms of \( a \) on Item 10 of the pre-assessment. It is possible that Denise did not feel the need to write \( 2xa \) or \( 3xa \) since the letter \( a \) was the first term of her representation. It also might be the case that Denise’s response to Item 10 was her best approximation as to what an equation should look like in this situation.

When I asked Denise what her understanding of the word “equation” was immediately following her justification of her work on Item 10, Denise replied in the following manner.

Denise: It’s like an example or when you can know something

Jan: Can you give me an example of an equation? Just anything you can think of would be fine [writes the word “Equation” on a separate piece of paper (see Figure 4-38)].

Denise: Um, have you ever heard of H two O [writes H\(_2\)O on the paper]. It’s two parts hydrogen [writes “2 H”] and one part oxygen [writes “1 O”]. That’s an equation. Or \( e \) equals \( mc \) squared or five times two, ten—something like that.
Based upon the examples that she gave, it seems that Denise thought of an equation as a structure that combined letters or numbers and which represented a certain process (as in 5x2=10) or concept (as in H₂O). Perhaps, at this time in the study, the quantity 3xa did not represent a conceptual object to Denise in the same sense that H₂O may have. Further, since one of the ways Denise stated that she viewed an equation as “an example” and based upon the fact that she described her representation as being like a “scale”, it is possible that Denise concerned herself with representing the length of Board B, using “2x as big” and “3x as big” as locations on the metaphorical scale.

I speculate that if the equation task had been structured differently, Denise would have answered the question using one term only. For example, suppose the task had been worded as follows: “Board A is a inches long. Board B is 4 times as long. Write how long Board B was in terms of a.” It seems that in that instance, Denise would have written 4a as her response based upon her initial answer to the Felicia-Marcus question. This would have been a good follow up question to have asked Denise at the time.
A letter within an equation is an operable composite part of a whole that is in relation to other parts within the same or an equivalent whole

Denise was given the same paper strip tasks as Kelly in which she had to create two paper strips based upon the length of a given paper strip. The length of Strip 2 was supposed to be three times as long as the original strip, Strip 1, and the length of Strip 3 was supposed to be a combination of the lengths of Strips 1 and 2. After representing these strips in a diagram, Denise had to answer questions about their length, given the length of one of those strips. For example, “Suppose that the length of Strip 2 was 51 inches, write a number sentence to show how you would find the length of Strip 1.” Although Denise was able to readily identify the process that she used to obtain the lengths of Strip 1 or Strip 2 in these types of tasks when only the length of the other strip was known, I found that Denise was not able to generalize her process in Task 3, which is shown in Figure 4-39. It seemed that Denise was unable to operate upon the term “Length of Strip 1” as if it were an object when asked to do so. This can be seen in the following discussion.

Denise: Write a letter and number sentence that shows how you can find the length of Strip 1 if you only knew that the length of Strip 2. So write—

Jan: So what I mean by that, instead of me giving you a number, like this is 80 or this is 40 [pointing to Strip 2] or this is 20 [pointing to Strip 1], let’s say I just said, [begins to pick up Strip 1] Length of Strip 1 equals—I just said Length of Strip 2 equals, so what’s the length of Strip 1? So here [picks up paper with problem statement on it], Length of Strip 1 equals, how could you write something that—let’s say you were giving someone instructions. You were saying, “I want you to create this strip [picks up Strip 1, puts down and picks up Strip 2], I’m not telling you the length of this strip, but this is what I want you to do to this strip. Do you know what I mean? [Denise writes “3÷3.”] Okay, now what made you decide to say 3 divided by 3?

Denise: Because if Strip 2 is three times as much to get, so if it was Strip 1, then you would do 3 divided by 3 or 4 divided by 4 if it was 4 times as much.
Jan: Okay, I got you. I got you. What if I said, again, you’re giving someone instructions—Length of strip 1 equals length of strip 2 [writes “Length of Strip 2”], and then what should go next?

Denise: Are you saying that this is the length of Strip 2? [Points at where Jan wrote “Length of Strip 2.”]

Jan: Yes, so I’m saying that I’m trying to find out this [picks up Strip 1], and we’re given this one [picks up Strip 2]. That’s what this is [creates a finger frame around where she wrote “Length of Strip 2”]. It’s not a number, but it’s—you know it’s that length [holds up Strip 2]. What would have to do to this [circles where she wrote “Length of Strip 2” on the paper] quantity to get this length [points to Strip 1]?

Denise: Divide.

Jan: Okay, do you want to show that in that sentence, in terms of what you would divide by or how you would do that.

Denise: As much as the number was. I think [inaudible].

Jan: Okay. Can you figure out how you can show that on this particular, using those words and then where, what would you divide into or you can even say it in sentence form like take the length of Strip 2 and blah, blah, blah.

Denise: Oh, so write it as a sentence.

Jan: Sure, you can write it as a sentence.

3) Write a letter & number sentence that shows how you could find the length of Strip 1 if you only knew the length of Strip 2.

Length of Strip 1 = \( \frac{3}{3} = 1 \)  

*Use the length of strip 2 and divide it by the number it's bigger than.*

Write a letter & number sentence that shows how you could find the length of Strip 2 if you only knew the length of Strip 1.

Length of Strip 2 = \( 1 \times 3 = 3 \)

*Length of Strip 2 = Length of Strip 1 \times 3*
Figure 4-39: Denise’s Response to Paper Strip Task #3 (excluding the words in italics)

Although we have yet to see what she writes, in the above dialogue, it seems that Denise was unable to act on the instructions that I gave to her concerning writing a sentence that shows what should be done to the length of Strip 2 in order to obtain the length of Strip 1. It seems that either Denise did not understand the instructions, or like Thompson’s (1988) students, she had not yet encapsulated “Length of Strip 2” as an object, thus preventing her from operating upon this term using operation symbols and numbers. In the following excerpt, Denise attempted to write a sentence relating the lengths of the two paper strips. This excerpt immediately follows the one preceding this paragraph.

Denise:  [writes “Use the length of Strip 2 and divide it by a numer [sic] it’s bigger than”] Okay. I put use the length of Strip 2 and divide it by the number it’s bigger than, so if you had one, if it was Strip 1 and that’s just a number by itself, but we don’t know what that number is—it’s just like a mystery number—and Strip 2 is three times as much, that’s what I meant by the bigger number.

Jan:  Okay, so how much it’s bigger than, let’s say, Strip 1 for example?

Denise:  Uh huh.

Jan:  I got you, okay. What if you had to write a sentence that shows how you would find the length of Strip 2 if you only knew Strip 1? So now, we’re going to start with Strip 1. [Denise writes 1x3=3.] Okay, now where did the 1 come from?

Denise:  The three is how much that Strip 2 is bigger than, and sometimes like, it’s three times more, then the one came from when you divide it by 3 and you have 1, and that’s how I did it.

Denise’s decision to write a sentence to describe the process by which Strip 2 might be attained supports the conclusion that she thought of multiplying Length of Strip 1 by 3 as a process, but that she may not have thought of three times length of Strip 1 as a quantitative object.

The task in Figure 4-40 was given to Denise in order that she might create two equivalent wholes of which three unknown quantities were parts. After taping together Strips 1 and 2 and
creating Strip 3, Denise was asked to create a diagram of Strips 1, 2, and 3. In order to create each diagram, Denise chose to use Strip 1 as the unit strip, as can be seen in the excerpt below.

Jan: Now I’m going to ask, would you please create a diagram of the strips as they appear now? Since they’re a little long, you may have to draw a miniature version, but if you can just keep the relationships intact.

Denise: Can I like draw a line to represent where Strip 2 is and Strip 1?

Jan: Sure.

Denise: [Underneath the words, “Strips 1 & 2”, draws four connected rectangles; writes the numeral 2 in each of the 2nd to 4th boxes in the diagram; writes the numeral 1 inside of the first box.; underneath the words “Strip 3”, draws a long rectangle which she subdivides into four rectangles; she writes the numeral 3 inside of each of these rectangles.] So you can tell that Strip 1 is 1 times as much, Strip 2 is 3 times as much and Strip 3 is 4 times as much.

Jan: Okay, that’s why you did the boxes. Okay, you anticipated my question. That’s good.

In this excerpt, it seems that Denise chose to use the length of Strip 1 as the iterative unit length as evidenced by her decision to draw Strips 1 and 2 using individual boxes. She still appears to have used the length of Strip 1 as the unit of length when drawing Strip 3 as evidenced by her subdivision of the diagram of Strip 3 into four equal sized parts. Denise gave indication that she thought of the length of each strip with relation to the length of Strip 1 when, after drawing, she volunteered, “So you can tell that Strip 1 is 1 times as much, Strip 2 is 3 times as much and Strip 3 is 4 times as much”
In the excerpt below, Denise explains her solution process for solving Paper Strip Task #5 in Figure 4-41.

Since it was four times as much, I did, and 20 was how much Strip 3 was, I did 20 divided by 4 and I got 5, and I multiplied it by how much this one is. That was 3, so 5 times 3 equals 15 inches and 20 divided by 4 equals 5 times 1.
I sought to build upon Denise’s work of finding the length of a paper strip when all quantities were known by having her to generalize her process in Task 7. Denise’s work on this task is seen in Figure 4-42. Denise, on her own, ascribed a letter to each of the lengths of Strips 2 and 3 (although she stated that the letters b and c, respectively, were equal to Strip 2 and Strip 3). The fact that Denise chose to spontaneously give each length a letter seems to indicate that she had encapsulated “Length of Strip 2” and “Length of Strip 3” as quantitative entities which could be operated upon. Further, Denise’s action seems to indicate that, to her, a letter was a suitable representation for an unknown length.

![Figure 4-42: Denise’s Responses to Paper Strip Tasks 7a and 7b](image)

Similar to the students in Mikulina’s (1991/1969) study, Denise seemed to have benefited from having a diagrammatic representation from which to operate. The diagrams that Denise created in Figure 4-40 seemed to provide a basis from which Denise could observe, mentally operate upon, and abstract part-whole relationships embedded in the task. Available to Denise in this task were her understandings about the multiplicative relationship between a whole that contains only the iterated units of an unknown quantity and the whole itself. Denise also had the understanding that a letter can be operated upon as if it were known. Based upon these, Denise
was able to symbolize her numerical activity at a process level of abstraction. At this point, Denise seemed to have a proceptual understanding of a letter as a representation of the unknown quantity in that she was able to operate upon a letter as if it were known, use a letter as an input and conceptualize the quantitative nature of the output.

Denise began the study with an understanding that a letter could represent an unknown quantity and was able to be operated upon. However, it seems as though these tasks enabled Denise to express a more complex relationship between unknown quantities. In particular, Denise was able to create equations that expressed the relationship between unknown parts of equivalent wholes. The implication of this is that Denise understood that there is a multiplicative relationship between the lengths of unknown parts of equivalently-sized wholes that are comprised of the same number of same sized parts.

Denise’s unitization of Strip 1 in the task in may have aided her in solving the task in that Denise was readily able to identify the relationship between the lengths of Strips 1, 2, and 3. That Denise already had the understanding that she could operate upon an unknown quantity as if it were known seems to have enabled her to generalize her actions in the task in Figure 4-42.

Valerie’s Story

*A letter can represent a specific, unknown quantity that can be operated upon “as if it were known” (Stacey & MacGregor, 1999, p. 30)*

Valerie began this study with an understanding that letters within an equation represent particular quantities that make the equation true. On Item 3 of the pre-assessment (see Figure 4-43), Valerie first provided a value for $u$. She then supplied a value for $v$, seemingly based upon
the value that she chose for $u$. When asked to justify her answer on item 4 of the pre-assessment (see Figure 4-44), Valerie stated:

“Well, six equals $n$ because if you say ‘blank plus three equals nine’, you could just do nine subtracted by three which is six. And then, if you do for $n$ here [pointing at where she wrote “$n=6$”] equals six, then you could do six plus eight which equals 14.”

<table>
<thead>
<tr>
<th>3. If $u = v + 3$, give a possible value for $u$ and $v$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u = \Box \hspace{1cm} v = 6$</td>
</tr>
</tbody>
</table>

Figure 4-43: Valerie’s Response to Item 3 on the Pre-Assessment

<table>
<thead>
<tr>
<th>4. If $n+3=9$, what is $n+8$? Explain how you got your answer.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$14 \hspace{1cm} n=6$</td>
</tr>
</tbody>
</table>

Figure 4-44: Valerie’s Response to Item 4 on the Pre-Assessment

By stating that $n$ equals six because $n+3=9$, Valerie showed that she understood that $n$ had to be a unique value to make the equation true. Valerie was then able to substitute that value into the expression, $n+8$ in order to obtain an answer of 14.

At the beginning of the study, Valerie did not demonstrate that she had the ability to operate upon the unknown quantity based upon her responses to pre-assessment items five and nine (see Figure 4-45 and Figure 4-46).

| 5. Marcus worked for $n$ hours. Felicia worked 5 more hours more than Marcus. |
| Express how many hours Felicia worked in terms of $n$. $14$ hours |

Figure 4-45: Valerie’s Response to Item 5 on the Pre-Assessment
Figure 4-46: Valerie’s Response to Item 9 on the Pre-Assessment

Valerie’s explanation of her response to pre-assessment problem item 5 is below.

Jan: I see you wrote 14 hours. What made you decide to write that?

Valerie: I was just guessing on this one also because I didn’t understand what \( n \) was.

Jan: So, if I told you, for example, what \( n \) was, what would you then be able to write?

Valerie: Well, say if you said that \( n \) was nine, so that would say that Marcus worked for nine hours and Felicia worked five more hours than Marcus. Then, the answer would be 14.

By assigning a value to \( n \) and then evaluating \( n+5 \) for that value of \( n \), it seems that Valerie may have had an inability to accept a lack of closure (Collis, 1974). This is underscored by the fact that Valerie stated, “I was just guessing on this one…because I didn’t understand what \( n \) was.” Evidence of an inability to accept a lack of closure in Valerie’s responses suggests she would not be able to operate on a letter as an unknown.

When asked to explain her response to pre-assessment item 9, Valerie provided the following explanation:

“It says that board B was four times as long as Board A so I just drew one board of A and then plus four, as in like times four. Then it equals four boards like this [points to the rectangle drawn underneath the letter A].”

In light of the fact that Valerie corrected herself when saying, “I just drew one board of A and then plus four, as in like times four,” I asked her whether the sign should have been a plus
sign or a multiplication sign. This question seemed to lead Valerie to change her answer as she then indicated that the sign should have been a multiplication sign.

In her response, Valerie represented Boards A and B using boxes. She labeled each representation with the letters A and B. These letters could have represented, to Valerie, either the name of each board or the length of each board. Valerie’s response seems to suggest that she viewed the number of boxes representing Board B as being four times the number of boxes representing Board A. The fact that Valerie chose to draw and operate upon the representation of Board A, rather than operating on the letter representing the length of Board A may be indicative of the fact that she was not able to conceive of the letter $a$ as being able to be operated upon without being evaluated. Or, it could be the case that Valerie drew what was familiar to her, especially since, in a previous problem on the pre-assessment, she had been asked to create the representation of a board that was five times as long as a given picture.

Although Valerie did not demonstrate the ability to operate upon a letter or to create an equation that related two quantities, Valerie did demonstrate that she could reason about and even manipulate an equation. In the following discussion, Valerie justifies her response to a modified version of the classic “Student-Professor” problem, as seen in Figure 4-47.

Jan: Now what made you choose equation b?

Valerie: Because it said that there was [points at the equation “$S=6xT$” on the pre-assessment] six as many times teachers as there were students so if this was backwards [moves finger from right to left along equation] and you said $T$ [taps blank space at bottom of paper] which is teachers times six [moving finger right, taps paper twice] equals $S$ [slides finger to the right] which is students and that would be the correct answer.
12. There are six times as many students at the school as there are teachers. Suppose \( S \) represents the number of students and \( T \) represents the number of teachers. Which equation correctly expresses the relationship between the number of students at the school and the number of teachers at the school?

\[
\begin{align*}
\text{a. } & T = 6S \\
\text{b. } & S = 6T
\end{align*}
\]

Figure 4-47: Valerie’s Response to Pre-Assessment Item 12

Because Valerie misstated the problem statement when she said, “six as many times teachers as there were students”, I repeated the question so that there would be no ambiguity.

Jan: Now, okay, so it’s saying there are six times as many students as there are teachers and so what is your reasoning again [points at the equation “\( S=6XT \)”]?

Valerie: Well, if, I like rephrased it like and said it was backwards, [taps finger along blank space along the bottom of the page as she states each word] \( T \) times 6 equals \( S \). So that, say if there was five teachers, and then say this was five [pointing to the letter \( T \) in the equation “\( S=6XT \)” in the problem] so it would be five times six [moves finger as she taps finger on the paper] which … equals 30.

Jan: ….You said this is backwards. What do you mean by backwards?

Valerie: I just like flipped it backwards. Instead of this, instead of the \( S \) equal going first, I did \( T \) times six.

Valerie “flipped [the equation \( S=6xT \)] backwards” in order to be able to reason about it. Valerie’s actions seem to indicate that she viewed the equals sign as a “‘do something signal’” (Behr, Erlwanger, & Nichols, 1976) in that she preferred to have the “problem”, \( 6xT \), on the left side of the equation and the “answer”, \( S \), on the right side of the equation. After having done so, Valerie then ascribed a value to \( T \), seemingly, in order to make sense of the equation. This seems to indicate that Valerie had an inability to accept a lack of closure in that she needed to evaluate the expression \( 6xT \), in order to be able to reason about the quantities involved. However, this
episode also shows that Valerie has an understanding of letter such that she was able to reason about the relationship quantities that were embedded in a complex situation (Clement, 1982). Because Valerie seemed to feel the need to substitute a number for a letter in order to be able to reason about quantities, it seems that, at this point in the study, Valerie had a process concept of letter and an action conception of an equation.

_A letter within an equation is an operable composite part of a whole that is in relation to other parts within the same or an equivalent whole_

The full sequence of the paper strip tasks was not developed until after the pilot study ended. However, I will examine the way in which Valerie seemed to develop the understanding of a letter as being an operable composite unit within the context of the tasks that she was given both during and after the pilot study.

In order to ascertain whether Valerie and Alexis (the student who was Valerie’s partner in the pilot study) understood the multiplicative relationship between two unknown quantities, I posed a task that is captured in the beginning in the excerpt below. In this task, I was, essentially, asking the students how they would operate on the representation of one unknown quantity in order to attain the other unknown quantity.

Jan: I’m gonna draw a box [draws a box on a new sheet of paper (this box and subsequent material appear in Figure 4-48)]. Now we’re gonna suppose that this is, I’m gonna call this a, all right. And a is the length of Mike’s candy bar [writes \( a = \text{Length of Mike’s candy bar} \)]. Everybody with me that far? [Both students: Um-hm.] Now, this is a [points at the letter a above the a box] then [draws three connected boxes underneath the a box]. Can anyone tell me, if you had to push buttons on the calculator—I’m going to call this length of Judy’s candy bar [draws a brace underneath the three connected boxes and writes “Length of Judy’s candy bar” underneath that]. If you had to tell me the length of Judy’s candy bar, what would you wind up writing — what buttons would you push on the calculator?
Valerie: Well it depends on what length this is [creates a finger frame around the box].

Alexis: What’s the length of \( a \)?

Valerie: \( a \). I mean, say if it was like 6 inches or so.

Jan: Then what would you wind up having to push?

Alexis & Valerie: Six times three equals 18.

Alexis: Write it down?

Jan: Yes, please. [Alexis writes “6x3=18” at the bottom of the page.] Now, why six times three?

Valerie: Because six is the length of the original length and then it’s three times longer so it would be six times three.

Here, Valerie tries to assign a number to \( a \). I next sought to redirect her in order to see the manner in which she would choose to operate upon \( a \) in the absence of numerical examples.
Jan: Okay. Now, could you write an expression for me if we didn’t know it was three—I mean, six—if we just knew it was called $a$.

Valerie: Well then you could say that Judy’s is three times longer than $a$.

Because I wanted to know if Valerie would be able to express the relationship between the quantities $a$ and $b$ symbolically, I decided to build upon her numerical activity by creating the idea of an $a$ button on the calculator. This decision, which arose spontaneously during this session, was meant to enable Valerie to picture the action which she took when multiplying the numerical value ascribed to $a$ by three and generalizing that action. By having Valerie use press the letter $a$ on the calculator, it seems that she had the opportunity to conceptualize $a$ as an input, thus moving her towards encapsulating $a$ as a procept.

Jan: So let’s write. I want you to pretend that there was a $a$ button on the calculator. There’s one, two, three, four, five. Just write [points to bottom of Page 3 (Figure 4-48)] what buttons you would push in order to find the length of Judy’s candy bar...Pretend that there’s an $a$ button on the calculator. You know how you have all those numbers? Pretend there was an $a$ button.

...Valerie: $a$ would be six.

Jan: Okay, now, pretend all these were letters instead of numbers. Okay? Let’s just call this the $a$ button here. If you wanted to find out—let me put it like this—I’m going to call this $b$ [writes “b=” next to the line that says “Length of Judy’s candy bar”]. Length of Judy’s candy bar. So I want to say $b = \text{ }$ [writes “b=” at the top of the page] and I want you to tell me what buttons you would push on some imaginary calculator to get $a$.

Valerie: To get the length of $b$?

Jan: Yes.

Valerie: Would press $a$ is three, right?

Because Valerie did not seem to be able to visualize the $a$ “button” on the calculator, I decided to draw one so that there would be a “button”, in fact, to push (see Figure 4-49). In so doing, I thought that Valerie might generalize the action that she would take if she were to multiply a given value of $a$ by three.
Jan: Well $a$ is—I’m gonna—In fact, I’m gonna draw $a$ right here [writes the letter $a$ near the top of the calculator; taps the calculator]. That’s $a$. So if you wanted to find out the length of $b$, what would you push?

...  

Valerie: [pushing buttons on the calculator off-camera] I would say $a$ times three.

Jan: Okay…Why would you say $a$ times 3. Oh, can you write that down first?

Valerie: What, $a$ times 3? [Jan: Um hm; Valerie writes “$a \times 3 = b$”.

Jan: [points at the letter $a$ that Valerie wrote] Can you make that— it almost looks like a nine. [Valerie changes $a$ to $A$].….Now, tell me why you said $b$ is $a$ times three.

Valerie: Because $b$, okay, you do $a$ [Alexis: $a$ and then it’s three more]. More. ‘Cause $a$’s the original and then you times it by three which equals $b$.

Figure 4-49: Calculator with $a$ and $b$ “buttons”

Working through the “calculator” example seemed to help Valerie to consider the way to operate upon an unknown quantity. Valerie began the study with the understanding that a letter
can represent a specific unknown quantity. She did not have the understanding that a letter can be operated upon as if it were known, perhaps because she did not have the ability to accept a lack of closure. Thus, the teaching intervention that occurred in the above excerpt appeared to serve as a bridge from her previous understandings to the understanding that a letter is an operable composite unit.

By assigning the number six to $a$, Valerie was able to operate upon that six and determine that the answer was 18. Then, by having to consider how she would operate on a “generalized number”, $a$, on the calculator, Valerie was forced to consider the process of operating on a quantity in order to attain a quantity that was three times as big. Like Kelly, Valerie’s work seemed to be tied to a numerical exemplar (“6x3”). This similarity in their responses might follow from the observation that Kelly began the study with similar understandings that Valerie had with the exception of Kelly’s quasi-understanding of being able to operate upon a letter representing an unknown quantity.

Although both pre-algebra and algebra students have been widely purported to have a misconception of letters in algebra and what they represent (Booth, 1984; Stacey & MacGregor, 1997), Kelly and Valerie seemed to have circumvented those issues in that the following were true: (a) the unknown quantity was first represented by a box, as opposed to a letter, (b) they were given a meaningful part-whole environment in which to operate upon the diagrammatic representation of the unknown quantity according to the rules of arithmetic (Larkin & Simon, 1987), and (c) they were given the opportunity to mathematize their own mathematics by generalizing the process that they undertook to find the value of a known quantity.

In order to build upon the work done in the task in Figure 4-48, I gave Valerie and Alexis a similar, but slightly more complex task. In this task, they were still being asked to write an equation which related the values of $a$ and $b$. Because my instructions to them were not clear enough, both Valerie and Alexis confused the value of $a$ and $b$ with the number of boxes
representing each of those quantities. In the episode that follows, I presented Valerie and Alexis with two diagrams (see Figure 4-50). I asked Valerie and Alexis to write equations relating quantities $a$ and $b$.

[Jan draws two connected boxes at the top of a new sheet of paper (Figure 4-50). She draws three connected boxes underneath those two. Above the two connected boxes, Jan draws a brace above which she writes the letter $a$; beneath the three connected boxes, Jan draws a brace beneath which she writes the letter $b$.]

Figure 4-50: Alexis and Valerie’s Responses to Candy Bar Task #3—near middle of teaching episode
Jan: These are two candy bars [taps each diagram that she just drew]. Now before you write anything, I just want you to talk. If you had to push buttons on a calculator to get...a, what would you push?

In my instructions to Valerie and Alexis, I should have stated that a represented the length of the candy bar beneath it and b represented the length of the candy bar beneath it. Because I did not anticipate that we would be operating under different assumptions, I did not make this clarification. The following excerpt details the conversation that took place after the task was given.

Valerie: Including this a button?

Jan: Yes.

Valerie: To get b?

Jan: Oh I’m sorry, now we gotta have a b button [writes the letter b at the top of the calculator next to where the letter a is already written]. That’s a, that’s b. [draws a circle around each of the letters on the calculator]

Valerie: So we have to press a, b, and the numbers in order to get a and b?

Jan: If you want to find a [points to the a diagram] what do you need to do with b [points to the b diagram]?

Alexis: If you want to find a, what do you have to do with b?

Jan: Um hm.

Alexis: Um [points at the last box on the b diagram], b minus one.

Jan: b minus one

Valerie: Yeah, cause you could do b minus one because if you took away this one [points to the third box on the b diagram], they’d be the same length [points at both diagrams].

In this excerpt, Valerie and Alexis seemed to be thinking of the value of b as being the number of boxes used to represent b. At the time, I did not realize that this may have been due to the fact that I did not explicitly state what the letters a and b represented. In order to help the students to rethink subtracting one from b, I followed through on their recommendation. I offered a
numerical example for one of the boxes so that the students could compare the actual result of subtracting one from the value of \( b \) with their expectation of the same.

Jan: Okay, so let’s do that—let’s take away one of these [draws an X through the third box on the \( b \) diagram]. \( b \) minus one.

Alexis: Which will make it two of them.

Jan: Now, here’s the thing [places a finger on each of the boxes in the \( a \) diagram], let’s say each of these were worth 8. If I took away one [points at the third box on the \( b \) diagram], that would make this value [pointing at the first and second boxes on the \( b \) diagram]—Let’s say, if each of these boxes were 8, then the value of \( b \) would be what?

Valerie: 24.

Jan: So if we took one away, it would be 23. In that case [Valerie: 24], I mean, yes. But if we took away one box, that would be taking away –

Alexis: It would make it 16.

Jan: Could you put \( b \) minus one for me there?

Alexis: Right here?

Jan: Yeah. So we’re testing out our theory. [Alexis writes “\( b-1=16 \)” next to where Jan wrote “\( a=\)” at the bottom of the page.] So, if \( b \) is 24 [points at the \( b \) diagram], right, that would make \( a \)

Alexis: 16

Jan: [writes “\( a = 24-1=23 \)”]. 23. But there’s a difference [points at first and second equations].

Valerie: 23?

Jan: Yeah, ‘cause we’re taking away one [points at the numeral 1 in the second equation], right?

Alexis: Oh, 24 minus one.

Jan: So, are we taking away one?

Valerie: We’re taking away one [taps bottom equation] from \( b \) [taps the last box on the \( b \) diagram].

Alexis: We’re taking away one from \( b \).
Jan: This whole box, right [indicating the third box on the b diagram]?
Valerie: Yes, from b.
Alexis: Yes.
Jan: So –
Valerie: We’re not taking away one from a.
Jan: Oh, so I’m thinking –
Valerie: To get b, you would add one, you would add a plus –
Alexis: You would make 16—you would add 8 to 16.
Jan: Okay [pointing at the third box in the b diagram].
Alexis: To get b. b to get a [points at the a diagram] you would take away 8 to make that 16.
Jan: Wow. If you take away eight [points at the third box in the b diagram]
Valerie: You get eight plus. All right, eight plus nine, I mean 8 plus a equals b.

Here, a shift in the conversation began to occur in that Alexis began to talk in terms of the number of units that one box was equal to instead of the number of boxes that a letter was equal to in width. This shift carried with it power in that it seemed to help Valerie to think about the operations that were occurring in terms of that same unit of 8. This same type of thinking is what supported the students in Gregg and Yackel’s study as they transitioned from using ovals to represent an unknown composite unit to letters.

Jan: Okay, I see. Will you write that down please? Eight plus a equals b. Okay, interesting, interesting. [Valerie writes “8 + a = b”.] Eight plus a equals b.
Valerie: Because if each box is eight [taps the a diagram] and then you add another eight [taps the space to the right of the a diagram] plus that’s, you know, a. That will give you b. This whole thing [taps the b diagram]. So a [taps the a diagram] plus another box [creates a finger frame in the space next to the a diagram which is about the width of one of the boxes] will give you this [creates a finger frame around the b diagram].
In this task, Valerie demonstrated that she had an understanding that a letter can be operated upon as if it were known when she wrote the equation “8 + a = b” as well as when she was agreeing with Alexis that “b-1=a”. The difficulty, then, that Valerie seemed to face is that she (and Alexis) were at first equating the values of a and b with the number of boxes representing those quantities due to a misunderstanding. Therefore, Valerie experienced no cognitive conflict when trying to justify that a was equal to b minus one. To her, this meant that two boxes were equal to three boxes minus one box.

However, when a numerical value for one box was introduced, Valerie had to rethink their equation. That is, Valerie realized that in order to subtract one box from b as she and Alexis suggested, they would have to subtract a unit of eight from b. Seemingly, the problem that both students were facing was the nature of the unit that was being added to or subtracted from a or b. When that unit became clearly established using a numerical example, Valerie was able to construct an equation in which she added a, which was represented by two boxes, to the number 8, which was the value of one box. Thus, numerically unitizing the value of one box seemed to help Valerie to operate correctly upon the quantity.

However, after this discussion, when asked to give more equations relating a, b, and later, c (represented by one box) both Valerie and Alexis continued to treat those quantities as if they were equal in value to the number of boxes to which they were equal in length writing the first through sixth equations appearing on Figure 4-51. For example, I wrote the letter b on the paper. When I asked Valerie to write an equation that showed the relationship between b and c, she wrote “c\times3” next to the letter b that I had written (see Equation 4 on Figure 4-51). The following conversation then ensued.

Jan: Okay, now you said c times 3? c times 3 [points at the c diagram]… And why did you say c times 3?

Valerie: Because one times three is three. b is three. Put that?
Jan: Yes please [Valerie writes “cx3” next to where Jan wrote “b=” (see 4th equation on Figure 4-51)].

![Figure 4-51: Alexis and Valerie’s Responses to Candy Bar Task #3—end of teaching episode](image)

However, Valerie’s reversion to equating each letter name with the number of boxes to which that letter was equal in length shows that Valerie did not develop the understanding that I initially thought that she had. She did not understand that each letter was equal to the combined value of the boxes that the letter represented rather than the number of boxes. I then relied on the fact that Valerie seemed to demonstrate that understanding while in the context of using...
numerical examples. Therefore, I challenged Valerie and Alexis’s 5th and 6th equations on the paper in Figure 4-51 of using numerical examples.

Jan: Now um so let’s give it a shot. If this was 24 [circling the b diagram with her pen], then 24 divided by 24 would be one. [Valerie (simultaneously): One.] But, each one of these are 8 right [tapping each of the boxes inside of the b diagram]?

Valerie & Alexis: Yes.

Jan: We need to come up with an answer of eight [pointing at the c diagram]. So, I’m not sure if that actually works.

Valerie: What? How?

Jan: Like, okay, if this is eight and this is eight [pointing at the two boxes in the a diagram], then, this is 16, right [pointing at the two boxes on the a diagram]?

Alexis: Yeah.

Jan: And then 16 divided by 16 is one. [Alexis: 16 divided by 16 equals one.] But we want c to be eight [pointing at the c diagram].

Valerie: So you do 24 divided by three [uses calculator] equals eight.

Jan: If you did 24 divided by three. Now, let’s think in terms of a, b, and, c. What would you have to divide by what to get….what divided by what would give you c?

Valerie: b

Alexis: What divided by what would give you c? b divided by b is c. a divided by a is c.

Jan: Okay, but I guess what I’m saying is—let’s pretend, this is eight, this is eight [writes the numeral 8 inside of each box in the a diagram.] The value of this, the length of this [indicating the a diagram] would be 16, let’s say. If you do 16 divided by 16, then that would be one. But the value of this is still eight, not one, right? [Valerie: Um hm; Jan writes the numeral 8 inside of the box in the c diagram.] So, I’m wondering if there’s something different we need to do?

Valerie: Hold on, so you’re saying a equals 16 [writes “=16” next to “a=” on the a diagram]

At this point in the teaching session, Valerie seems to realize that I had been speaking of a as if it were the length of the candy bar instead of the number of blocks that it included.
Alexis: We were saying one block.

Valerie: Equals eight [writes “=8” next to “c=” on the c diagram]. And this equals 24 [writes “=24” next to “b=” on the b diagram]

Jan: Yeah.

Valerie: But all these are eight [writes the numeral 8 inside each of the boxes on the b diagram.]

Jan: Correct. So with that in mind, I need to be able to divide something—b by something or a by something in to get c.

Valerie: Twenty [tapping the “b=24” on the b diagram]. b divided by [pushes buttons on the calculator]. Twenty four. b [tapping the “b=24” on the b diagram] divided by three will give you c.

Jan: Why is that?

Valerie: Because 24 divided by 3 is 8.

Jan: Oh, okay. That’s what you pushed on the calculator. [Valerie: Um hm.] Would you write that down?

Valerie: [writes as she talks] b divided by three gives you c [writes “b÷3=c”]

In this excerpt, Valerie redefined for herself what a, b, and c were equal to once she understood that those letters were supposed to be equal to the length of a strip of boxes rather than the number of boxes. That shift did not seem to occur until I wrote the number 8 inside each of the boxes on the a diagram. Having established this relationship, Valerie was quickly able to describe an equation stating the way in which b needed to be operated upon in order to obtain c.

Comparison of Students’ Development of Understanding 2

Valerie and Kelly engaged in a similar progression as it pertained to their development of the understanding that a letter representing an unknown quantity is able to be operated upon as if it were known. Both students seemed to have begun the study with a process conception of letter. And both students needed to work through numerical examples in order to generalize their
counting activity. Kelly needed numerical examples in order to be able to use letters in order to generalize the relationships given in a diagrammatic problem situation.

Working through the “calculator” example seemed to help Valerie to consider the way to operate upon an unknown quantity. By having to consider how she would operate on a “generalized number”, $a$, on the calculator, Valerie was forced to consider the process of operating on a quantity in order to attain a quantity that was three times as big. Like Kelly, Valerie’s work seemed to be tied to a numerical exemplar. This similarity in their responses might follow from the observation that Kelly began the study with similar understandings that Valerie had with the exception of Kelly’s quasi-understanding of being able to operate upon a letter representing an unknown quantity.

Denise seemed to have begun the study with a proceptual understanding of letter in that she was able, right away, to represent operations on the unknown quantity using a letter. Denise seemed unable, at first, to construct and operate upon the length of the paper strip as a quantity in sentence form. However, she was able to represent the relationships embedded in the diagram that included Paper Strips 1, 2, and 3 were able to using numbers. This may have assisted her when moving to represent these relationships using letters. Denise’s mathematical behavior seems to signify that she had encapsulated the lengths of the paper strips as quantitative objects in the sense that she was able to represent complex relationships using letter representations of those quantities.

**Understanding 3—Given two equivalent wholes, either of which consists of one or more occurrences of a particular unknown quantity, the additive difference between the unknown parts of the two wholes is multiplicatively related to the additive difference between the known parts of the two wholes.**
It has often been reported that many algebra students, when solving equations, do so in a rote manner and without an underlying understanding of what they are doing (Cortés & Pfaff, 2000; de Lima & Tall, 2008; Stacey & MacGregor, 1997; Steinberg, Sleeman, & Ktorza, 1990). For example, deLima and Tall (2008) examined the work of 68 Brazilian high school students who were asked to solve linear equations. When asked to justify their solution process, the students appealed to particular rules or procedures that they had learned such as “‘when [the term] changes its place, [it] changes sign as well’ [or] ‘as it was multiplying on one side, it passes to the other dividing’” (p. 6). In response to this, de Lima and Tall (2008) stated that for such students, symbol manipulation during equation solving was akin to moving around physical (non-conceptual) objects according to external rules. The effect that this “procedural embodiment” (de Lima & Tall, 2008, p. 11) has upon students is that their algebraic knowledge is fixed, non-robust, and unable to be built upon in a meaningful way. As a result, these students are susceptible to making errors in solving equations due to misremembered or misapplied “rules.”

Gray and Tall (2007) have postulated that the reason that some students are more successful in school mathematics than others is that, when learning a procedure (such as evaluating an expression), those in the latter group practice the procedure with the goal of memorization and accuracy. Students in the former group, on the other hand, draw meaning from their actions while operating with that procedure and observe a commonality in the “effect” that different procedures have upon the same object according to Gray and Tall (2007). As a result, these students then “compress” those procedures into a singular process, the nature of whose result is already known (Gray & Tall, 2007). When the learner views the process as an entity that can be acted upon and that retains its capability as an operator, then, that process is stated to become, to the learner, a procept.

Gray and Tall (2007) call the difference in thinking between a student who views a particular process as a fixed set of procedures and one who thinks of the same process as,
simultaneously, a process and a concept, “the proceptual divide.” According to these researchers, this divide is what inhibits many mathematics students from thinking and reasoning flexibly during problem-solving and what prevents them from developing a rich base of algebraic understandings upon which to build.

One of the goals of this study, then, was to ascertain the way in which students might build meaningful algebraic understandings based upon their pre-study understandings through the use of modeling and solving algebra words problems via diagram drawing. Certain widely reported cognitive obstacles had to be navigated. These include pre-algebraic students’ difficulty with conceptualizing that an algebraic expression is an entity to itself that does not need to be evaluated (Chalouh & Herscovics, 1988; Collis, 1974) and pre-algebraic students’ inability to operate spontaneously with or upon the unknown quantity (Filloy & Rojano, 1989; Herscovics & Linchevski, 1994).

In order to enable students to develop an understanding of an unknown quantity as an iterable composite unit, the Harper Middle School Tasks were designed. This was discussed under Understanding 1. Then, the goal became to enable students to build an understanding of the way in which the quantity $ku$ (where $k$ is an known quantity and $u$ is an unknown quantity) is a composite whole made up $k$ units of $u$. The Harper Middle School Tasks continued to be used in this effort (as is discussed below).

This was accomplished through posing tasks to the students in which they were required to represent the structure of a problem using different sized-rectangles to represent known and unknown quantities. The tasks were embedded within realistic situations and were designed in such a manner as to afford students the opportunity to create and informally act upon diagrams that were a model of the situation. I anticipated that, while solving these tasks, students would act upon the diagrams in such a manner that their mathematical activity itself could become the
object of reflection (Gravemeijer, 1997; Streefland, 1993; Treffers, 1987). When this occurs, a student is stated to have “mathematically” his or her mathematics.

The context of the Harper Middle School tasks was a candy sale that took place at the school. Students were stated to be able to buy candy according to the number of pieces that were in a box. The size of the box depended on the number of pieces inside of it. The unknown quantity was represented by an empty box while a known quantity was represented by a box with either the known value written inside or above the box.

For example, suppose a person were solving the task in Figure 4-52. The quantitative relationships in this task could be represented as shown in Figure 4-53.

Figure 4-52: Example of a Harper Middle School Type of Task

Maria had five times as much candy as Troy had. If Troy doubled his amount of candy, and bought a box containing 15 more pieces of candy, the two would have had the same amount of candy. Draw a diagram to represent this situation. Use your diagram to figure out how much candy each person started out with.

Figure 4-53: Sample representation of the relationship among quantities in the Harper Middle School sample task

In order to find the number of pieces of candy that one box represents, a person could use the fact that the last box in Troy’s diagram is equal in length to the last three boxes in Maria’s diagram. In other words, the problem solver would have to assume that Troy’s amount of candy, $x+x+15$, was decomposable into $x+x$ and 15. The same is true for Maria’s amount of candy,
193

\[x + x + x + x + x\], which the problem solver would have to assume is decomposable into \(x + x\) and \(x + x + x\). Then, the problem solver would be in the position to multiplicatively compare 15 and \(x + x + x\).

Filloy and Rojano (1989) and Herscovics and Linchevski (1994) have stated that while equations of the form \(ax + b = c\) can be solved by “undoing” the operations, in order to solve equations of the form \(ax + b = cx + d\), the unknown quantity has to be operated upon. Both groups of researchers have found this type of thinking to be problematic for pre-algebraic learners. Filloy and Rojano (1989) therefore called this conceptual boundary the “didactic cut” which separates algebraic from arithmetic thinking.

I wanted students to be able to abstract the understanding that there is a multiplicative relationship between the difference between the unknown quantities and the difference between the known quantities in situations having the following structure: \(ax + b = cx + d\). However, I also wanted students to be able to circumvent the cognitive obstacle that students ordinarily encounter when solving problems of this type. Therefore, the Double Sided Unknown Quantity Tasks were given. An example is given in Figure 4-54. These were tasks that were similar in format to the Harper Middle School Tasks. They were so named because of the occurrence of the unknown quantity on both sides of the equation. Students were asked to represent these situations using boxes and numbers. Below is a discussion of the way in which such a task might be solved, using the diagrams shown in Figure 4-55.

Marcus had six times as many pennies as Joy in his penny collection. If Joy tripled her number of pennies and collected 29 more pennies and if Marcus collected 5 more pennies, the two would have had the same number of pennies. Draw a diagram to represent this situation. Use your diagram to figure out how many pennies each person started out with.

Figure 4-54: Example of a Double Sided Unknown Quantity Task
One of the ways in which this task could be solved is to think of Joy’s number of pennies and Marcus’s number of pennies as two equivalent wholes. Since both of these wholes have the same value, if equivalent quantities were removed from each whole, the equivalence of the wholes would be maintained. A student utilizing the diagram might reason that, since three unit boxes and a 5-box is equal in length to the 29-box and since the length of the 5-box is equal to part of the length of the 29-box, then the remaining length of the 29-box, 24 inches, must be the total length of the three unit boxes.

The quantities in this diagram are represented in a fashion in which the length of a box represents the relative size of the quantity being represented. Further, a multiple, $n$, of the unknown quantity is represented as $n$ singletons. Therefore, it seems that this representation makes it more likely for the problem solver to notice the potential part-whole relationship between 29 and $x + 5$ than if the two quantities had been represented in equation form (Larkin & Simon, 1987). I say this because, unlike when working with numbers that are written solely in symbolic form, the problem solver has the opportunity to “see” that part of Marcus’s number of pennies which is equal to 29 and that part which is not. Further, within the part of Marcus’s diagram that is equal in length to the 29-box, the problem solver can make the distinction between that part of 29 that is “occupied” by five and that part which is not.

![Diagram of quantities](image-url)
This type of representation also seems like it would be helpful to a pre-algebraic problem solver because the terms $6x$ and $3x$ are units which would, likely, be very difficult for such a problem solver to “unpack.” Further, while, likely, being unable to spontaneously operate with or upon the unknown quantity when that quantity is represented in symbolic form (Filloy & Rojano, 1989), the problem solver can probably operate upon the unknown quantity in diagram form. This is due to the fact that the unknown quantity is represented by the length of a box, something that is familiar to the student. Finally, by having the multiple of the unknown quantity represented as the concatenation of $n$ unit boxes, the problem solver is able to match those parts of the representation of six times the unknown quantity that are equal in length to the representation of three times the unknown quantity and those that are not.

In speaking of children’s use of derived facts to solve addition problems, Gray and Tall (2001) stated, “When the symbols [for example 3+5] act freely as cues to switch between mental concepts to think about and processes to carry out operations, they are called procepts. These can be composed and decomposed at will to derive new facts” (p. 68). In other words, if a child has a proceptual understanding of the expression 3+5, he or she would be able to decompose and recompose each quantity within that expression in order to find the sum. For example, a child might reason that since three is two less than five and five and five is ten, then five plus three must be eight. Or, a child might decompose the five in his or her mind say that it is equal to three and two. A child who knows his or her doubles facts then might reorganize the expression 3+5 into the expression $3 + 3 + 2$ and, using his or her doubles facts, state that 3+5 equals eight because three plus three is six and six plus two equals eight.

As a result of their engagement in the diagrammatic environment described above, students had the opportunity to construct an understanding of an additive quantity as being a decomposable unit that is comprised of a known quantity and $n$ units of the unknown quantity.
According to Olive and Caglayan’s (2001) definition, this constructed entity could be described as a “unit of unit of units” (p. 280).

**Understandings Developed by Individual Students**

*Kelly’s Story*

If a known quantity, \( k \), is equivalent to \( n \) units of an unknown quantity, \( u \), the two quantities are multiplicatively related in the following way: \( k = nu \).

The first task of the Harper Middle School Tasks in which the unit quantity was not known was the Darrell-Paula task, shown in Figure 4-56. When solving this task, Kelly did not demonstrate that she had the understanding that, given two equivalent wholes where one whole is a known quantity and the other consist of one or more occurrences of the unknown quantity, the known quantity is multiplicatively related to the unknown quantity. Kelly’s first obstacle, as discussed in an earlier section, was to determine how she could represent four times Darrell’s amount of candy and double the amount of Darrell’s candy. The latter discussion is shown in the beginning of the excerpt below. Kelly determined that she should use two boxes to represent the doubling of Darrell’s candy based on the fact that I asked her to suppose that Darrell had four pieces of candy. However, Kelly seemed to be experiencing some tentativeness about the situation.
Darrell bought a box of candy with a certain number of pieces in it (please see diagram below). Paula bought four times the amount of candy that Darrell bought. Show this on the diagram below.

Jan: Let’s say there was four in here, right [taps originally drawn box on Darrell’s diagram]. And, but you wanted to show eight. And if that box represented four, then what picture would represent eight?

Kelly: Two, one other box.

Jan: Okay.

Kelly: Want me to draw that?

Jan: Yeah. [Kelly draws a rectangle that is connected to the originally drawn diagram of Darrell’s candy.] So it’s almost like we’re saying we’re doubling it, even though we don’t know how much is inside, we’re doubling it. And then it says, does that part, okay, so let me know what you think about that. Does that make sense to you?

Kelly: Um. Um, I’m trying to make sense, but then the way I’m doing it, it would be um, Paula would have to add two more pennies, two more pieces of candy to her boxes.

If Darrell doubled the amount of candy that he has and then bought a box of candy containing 10 pieces in it, he and Paula would have the same number of pieces of candy. Alter your diagram to show this situation.

Use your diagram to figure out how much candy each person started out with.

Darrell’s original # of pieces of candy: 5 Paula’s original # of pieces of candy: 20

Figure 4-56: Kelly’s Response to Harper Middle School Task #5 on 4/15/2008
Even though Kelly was able to think about how many boxes that it would take to double Darrell’s amount of candy, her thinking was centered in utilizing guess and test to determine the unknown amount.

Jan: Okay. How are you doing it?

Kelly: Um, I’m thinking four, four [taps the two boxes on Darrell’s diagram starting with the one on the left], but then he has to [Jan: Oh, I get you] add ten of them and that would be eighteen. And she has sixteen since there’s six more.

Jan: Okay, I got you. Well, let’s do this. There’s a way that you can find out the answer for sure. And you don’t have to guess and test a number. All you have to do, if you draw the diagram, right, if you finish up with this diagram, you’ll see. You want to try it?

Kelly: Um, sure.

Whenever a student became “stuck” in terms of guessing and testing, I returned to the rationale that, once the relationships in the problem were represented, they would be better able to ascertain the value of the unknown quantity without guessing. This rationale was definitely external to the students in that guess and test seemed to be their means of making sense of the structure of the problem (Hall, Kibler, Wenger, & Truxaw, 1989). In the next excerpt, I attempted to reason with Kelly about the quantitative relationships given in the problem statement.

Jan: Okay. Now Darrell has doubled, right? And then what does the next sentence say, or what is the next part of the sentence?

Kelly: If Darrell doubled the amount of candy he has and then bought a box of candy containing 10 pieces in it, he and Paula would have the same number of pieces of candy.

Jan: Okay.

Kelly: They could have, um, five each.

Jan: And why do you say five?
Kelly: Five, and then ten [taps each of the two boxes on Darrell’s diagram as she counts], and then five, ten, fifteen, twenty [taps each of the four boxes on Paula’s diagram as she counts]. Five, ten [taps each of the two boxes on Darrell’s diagram as she counts], and then he adds ten more pieces and that’s twenty.

Jan: Okay. Now, what made you decide on the number five?

Kelly: Huh?

Jan: What made you decide on the number five?

Kelly: I was thinking about different numbers, and I think five can go into certain numbers [Jan: Okay. Did--] evenly.

Jan: I’m sorry. Did the ten pieces of candy part help at all?

Kelly: Um hm.

Jan: Okay. Um, can you tell me how that would [inaudible]

Kelly: Okay. Five, then ten. [Kelly taps each of the boxes on Darrell’s diagram. She frames these two boxes with her fingers, then moves her fingers to the right of those boxes, maintaining the distance between them. While keeping her fingers framed on the paper, Kelly then draws a rectangle connected to the first two that is the length of her finger frame; she states something inaudibly.]

Jan: Okay. All right, and what is the size of this box [points to last box drawn] compared to these boxes [points to the last three boxes on Paula’s diagram]?

Kelly: It’s supposed to be doubled [inaudible].

Jan: Okay, okay. Would you write the number that represents inside please? [Kelly writes the number 10 inside of the last rectangle that she drew]. All right. Now based on that, how much, how many pieces of candy were originally in each box? [Kelly writes the number 5 on the line next to “Darrell’s original # of pieces of candy.”]

Kelly: Like, um, the one she already used?

Jan: Um hm. Um hm. [Kelly writes the number 20 on the line next to “Paula’s original number of pieces of candy.”] Okay. All right. Thank you.

In this task, Kelly did not demonstrate that she had the understanding that, given two equivalent wholes where one whole is a known quantity and the other consist of one or more
occurrences of the unknown quantity, the known quantity is multiplicatively related to the unknown quantity. Her default strategy was to use guess and test to solve the problem. This may have been an indication that Kelly was trying to understand the structure of the problem (Hall, Kibler, Wenger, & Truxaw, 1989) or that she did not perceive the usefulness of representing all quantitative relationships given in the problem situation (Larkin & Simon, 1987). Kelly demonstrated this same type of reasoning when solving the next problem of similar structure.

Unfortunately, after that, I did not give Kelly another problem of the form $ax + b = c$ in order to assist her in constructing an understanding of the multiplicative relationship that exists between $x$ and $(c-b)$. Instead, I gave Kelly a problem that had the underlying structure $ax+b=cx+d$. These will be called the “Double Sided Unknown Quantity Tasks.” After reading the initial paragraph in the Double Sided Unknown Quantity task in Figure 4-57, Kelly drew a rectangle on the line representing Ayanna’s number of CDs and she drew 6 connected rectangles on the line representing Avery’s number of CDs. After reading the second paragraph of this task, Kelly drew two rectangles connected to the first rectangle on Ayanna’s diagram. The following conversation then ensued.

Jan: So what’s happening in this situation here?

Kelly: They keep getting more CDs.

Jan: Okay. And after they get all the CDs they have, what happens?

Kelly [draws a long rectangle attached to the first three rectangles on Ayanna’s diagram]: They end up having the same number of CDs.

Jan: Okay, good. [Kelly writes the number 17 inside of the rectangle that she just drew.] And have you shown, can you show the five CDs as well? That Avery bought?

Kelly: Oh, yes. [Inaudible] fit this.
Jan: Now the five he bought is in addition to what he already has [taps the rightmost edge of Avery’s diagram].

Kelly: Okay. I can’t fit it over here [waves pen above the right side of Avery’s diagram].

Ayanna has a certain number of CDs in her collection. Avery has six times as many CDs in his collection as Ayanna. Draw a diagram to represent this situation.

Ayanna’s number of CDs:

Avery’s number of CDs:

Suppose that Ayanna received more CDs on her birthday. As a result, she has triple the number of CDs that she started with. Suppose that Ayanna then bought 17 more CDs and that Avery bought 5 more CDs. As a result of this, Ayanna and Avery now have the same number of CDs in their collection. Alter your diagram to show this.

Use your diagram to figure out how many CDs each person started out with.

Ayanna’s original number of CDs: 4  
Avery’s original number of CDs: 24

Figure 4-57: Kelly’s Response to Double Sided Unknown Quantity Task #2 on 4/17/2008

The fact that Kelly seemed to think that she needed more space for the five CDs is an indication that she was not considering that Avery and Ayanna’s two collections were supposed to be equal. It is possible that she thought that the 5-box should be longer given the length of the 17-box in order for her drawing to be accurate. I tried to then focus Kelly’s attention on representing the problem-solving situation.

Jan: I have a question for you. What does it mean that after Ayanna bought seventeen more, and Avery bought five more, they have the same amount? [Kelly: Hm?]

What would it look like if after Ayanna bought seventeen more CDs and Avery bought five more CDs, they would have the same amount. What would that look like in diagram form?

Kelly: Um, [pause] I don’t know. I’m just kinda stuck.
Jan: Okay. Now what I don’t want you to worry about is how to figure out the size of one box yet. Because once we get it all on paper, then we can work with that. Okay? So, what if I told you, okay, [with pen cap, traces right end of rightmost rectangle on Ayanna’s diagram] this is how much Ayanna’s CD is. Where should Avery’s line stop?

Kelly: Here. [Kelly draws a line extending from the top of the last rectangle on Avery’s diagram to the end of the page.]

Jan: And why is that?

Kelly: Because he only has five. He’s only getting five more. [Kelly draws a line extending from the bottom of the last rectangle on Avery’s diagram to the end of the page].

Jan: So, can you draw on the diagram what represents five? [Kelly draws a box underneath the first box on Avery’s diagram. It is slightly shorter than the box above it.] Okay. Now why don’t you tell me about this [points to the box that Kelly just drew], why you drew that?

Kelly: I think, I think Ayanna has maybe six or seven CDs.

Jan: Okay. Well, before we even talk about that, let’s um, explore why, why you drew the box down at the bottom.

Kelly: Cause it couldn’t fit; the rest of it couldn’t fit here [points to the rightmost box on Avery’s diagram].

Once again, Kelly demonstrated that she desired to extend the length of the 5-box without considering that doing so would make Avery’s diagram longer than that of Ayanna. Because of this, I affirmed to Kelly that the 5-box was in the right place. Then I stated the following.

Jan: Okay, let me show you this…. This diagram here [points to rightmost edge of Avery’s diagram], should actually end here [outlines rightmost box on Avery’s diagram in red]. Because you drew this correctly [points to Avery’s diagram], one, two, three, four, five, six [taps each of the first six boxes in Avery’s diagram as she counts]. He has six times as much as Ayanna, and then she adds on seventeen [traces over box labeled “17”], and that’s the end of hers [puts a red vertical line on the rightmost edge of Ayanna’s diagram; does the same to Avery’s diagram]. So the end of his would be the same [Kelly draws an X on the...
box that she drew underneath the first box on Avery’s diagram; Jan outlines the rightmost box on Avery’s diagram.

Even though I did not ordinarily alter students’ diagrams structurally, I created a new left side for the fourth box in Avery’s diagram by drawing a vertical line inside of that box and crossing out the original line that separated this box from the third box on that diagram. I did this because I wanted Kelly to be able to “see” any implicit relationships that existed within the diagram (Arcavi, 1999; Bodanskii, 1991/1969; Paige & Simon, 1966) and I was concerned that she would not be as able to reason from the diagram, at least initially, if equal quantities were not aligned.

Jan: Okay. Now let’s take a look at this. These are the same, right? [Taps each of the first, then second, then third boxes in Avery and Ayanna’s diagrams.]

Kelly: Um hm.

Jan: [traces a circle around the 17-box in Ayanna’s diagram and the last four boxes in Avery’s diagram] Let’s look at this part of the diagram. How much is this worth [last four boxes in Avery’s diagram]? If you were to match it up with this [points at the left side of the 17-box].

Kelly: Seventeen.

Jan: Okay. Can you write seventeen above that please [moves pen back and forth amongst last four boxes on Avery’s diagram]?

Kelly: Um hm. [Kelly writes the number 17 above the fifth box on Avery’s diagram.]

Jan: Okay, so you’re saying this whole this is worth seventeen, right? [Jan draws a bracket above the last four boxes in Avery’s diagram.] Is there any way, and we know that these [points to 4th, 5th, and 6th boxes on Avery’s diagram] are the same size, right? [Kelly: Um hm.] All right, is there any way you can use this information now to help you figure out the [taps second to last box on Avery’s diagram], what one box represents?

Kelly: Um hm. [6 second pause]

Jan: If you’re not sure, it’s okay. I can, I can pose it a different way.
Kelly: I need to [inaudible]

Jan: Okay, how about this.

Kelly: Cause that’s kind of hard.

In this excerpt, I imposed more constraints than usual on Kelly’s diagram. Even after showing Kelly where each box would line up and confirming with her what parts of the diagram were equal to each other, Kelly was still not able to figure out how to find the value of the unknown quantity. On the subject of diagrams, Petre and Green (1993) stated, “knowing what to expect, where to look, and what to look for—the cognitive components of an inspection—affects the strategies the individual employs in reading an information structure” (p. 56). Because Kelly did not necessarily have the anticipation that it would be helpful for her to utilize the multiplicative relationship between the quantities represented by the unmatched parts of both Avery and Ayanna’s diagrams, she did not search the diagram for such a relationship. Therefore, on a separate sheet of paper, shown in Figure 4-58, I drew a diagram for her consisting of just these unmatched parts.

Jan: Yeah, let’s look at it this way. [Jan draws two rectangles on a separate sheet of paper off-camera.] We got three [puts this sheet of paper in front of Kelly]—so, I’m just going to take this and put it here [moves original problem sheet to the side]. So we got three of those, right? [Jan draws a third rectangle on the paper] And we got a five [draws a shorter rectangle connected to the first three and draws five circles on the inside]. I’m gonna make it, one, two, three, four, five, and from here, there to there [taps the bottom left and right corners of the “17” box on Ayanna’s diagram on page 4] is seventeen. So I’m gonna draw seventeen circles. [Jan draws a long rectangle above the 4 rectangles just drawn. She begins to draw circles inside of this rectangle.] Four, five, six. [Jan continues drawing until she gets to the 10th circle.]

Kelly: Are each of them worth four by any chance?

Jan: How would you, why do you say that?
Kelly: If all this equals seventeen [moves her finger from left to right along top box on new page] and this is five [taps 5-box on new page]. Twelve plus five equals seventeen. So, four, eight, twelve [taps each of the first three rectangles drawn then points to last box containing five circles] seventeen.

Figure 4-58: Kelly’s Response to Double Sided Unknown Quantity Task #2a on 4/17/2008

This intervention seemed to help Kelly in several ways. First, by seeing the circles drawn, perhaps Kelly was better able to visualize that the five circles were part of a larger whole. That may be what prompted her to subtract five from 17. Further, it is possible that Kelly was better able to solve the problem because it was isolated from the rest of the larger diagram. Hahn and Kim (1999) stated that “different diagrammatic representations will provide different perceptual cues that affect the amount and effort of search that is required for problem-solving” (p. 185). Isolating the unmatched parts of the diagram may have reduced the “visual noise” for Kelly and enabled her to focus on the relationship between part and wholes in each diagram.

From this example, it appears that Kelly understood that when there are two equivalent wholes, $ax+b$ and $c$, the difference between the known quantities is multiplicatively related to one another.

During most of the teaching sessions between the 4/17/2008 and 5/5/2008 teaching sessions, Kelly worked on the paper strip tasks. These tasks included a series of about 12 logic puzzles which were spread out over two teaching sessions (5/1/2008 and 5/2/2008) in which Kelly had to explicate the relationships between quantities based on the diagrams given and
ascertain the value of the unknown quantities. The tasks on 5/1/2008 focused on the
multiplicative relationship between parts of equivalent wholes given a known part or whole while
the tasks on 5/2/2008 contained equivalent parts of wholes that had as their structure \( a + x = c \),
where \( x \) was known. Figure 4-59 and Figure 4-60 show two examples of Kelly’s work on these
puzzles.

![Figure 4-59: Kelly’s Response to Paper Strip Logic Puzzle #2 on 5/1/2008](image)

In order to solve Puzzle 2 on 5/1 in the manner in which she did, Kelly had to have
understood that quantity \( b \) was comprised of four iterations of quantity \( a \) and, simultaneously, two
iterations of quantity \( c \). Further, Kelly had to understand that quantities \( a \) and \( b \) were
multiplicatively related to one another.
In order to solve Puzzle 3 on 5/2 in the manner in which she did, Kelly had to have understood that quantity $b$ was comprised of three iterations of quantity $a$ and that quantity $a$ plus 10 equaled 25 in. Further, Kelly had to understand that quantities $a$ and $b$ were multiplicatively related to one another.

\[
a = 25'' - 10'' = 2\cdot 15''
\]

2) Write a number sentence to show how to find what $b$ is equal to:

\[
b = 15'' \times 3 = 45'' (6)
\]

If $a$, $b$, and $c$ are known quantities, $x$ is an unknown quantity, and $ax+b = c$, then $c$ is, simultaneously, a quantitative whole and a composition of the parts $ax$ and $b$. Further, the unknown quantity is multiplicatively related to the difference between the known quantities, $b$ and $c$.

When the Double Sided Unknown Quantity Tasks were re-introduced to Kelly on 5/5/2008, three weeks after her first introduction to them, Kelly did not demonstrate the same
understanding that she had three weeks earlier. When attempting to solve the task in Figure 4-61, Kelly experienced difficulty.

Kelly first represented the problem by drawing two unit sized boxes and a long box on the Strip A line and five unit-sized boxes and a long box on the Strip B line. At my request, Kelly wrote the known quantities inside of each box. Kelly then began to use the calculator.

Jan:  [reading the buttons that Kelly pushes on the calculator] 10 plus 41 [inaudible]. Okay, so you’re like testing out numbers? Is there any way that you can, without having to test numbers, just looking, use the diagram to help you?

Kelly:  Um [taps the first five boxes of Strip B diagram]. I can’t use it to help me.

Jan:  Okay, that’s all right. What I’m going to do then, is I’m gonna hold this, and we’ll come back to it. Is that okay? So, I’m gonna put a one on this. I wanna come back to [problem] one at the end.

Figure 4-61: Kelly’s Initial Response to Double Sided Unknown Quantity Task #1 on 5/5/2008
At the beginning of each of the previous Double Sided Unknown Quantity Tasks and the current task, once Kelly had correctly represented the structure of the problem, she did not know what to do. I surmised that Kelly did not understand that there was a multiplicative relationship between the unknown quantity and the difference between the known quantities of two equivalent wholes. Therefore, I assigned additional Kelly the Paper Strip Puzzle Tasks. The last two of these tasks that I assigned, one of which is shown in Figure 4-62, were different from the others in the sense that they had the following structure: \[2x + a = b\] (where \(a\) and \(b\) were known).

Kelly: Okay, so, one, two, three strip \(a\)’s equal one strip \(b\) and two strip \(a\)’s plus maybe a strip \(c\), I guess, equal one strip \(d\) [Kelly took to naming those strips that were unnamed with a letter]

Jan: Okay. [Reading the buttons that Kelly pushes on the calculator] 42 minus 18 equals 24. Divided by 2 equals 12.

Kelly: Okay, [inaudible] try this. Yay. These 12 inches apiece

Jan: Okay, you did 42 minus 18. [Kelly: Divided by 2.] Divided by 2. Why is that?

Kelly: Because there’s two \(a\) boxes and 42 minus 18 equals \([Kelly presses buttons on the calculator; Jan reads them off: 42 minus 18.]\) It equals 24. [Jan: Okay.] It equals 24 and 24 divided by 2 [inaudible] these is … 12. Do you want me to write that?

Jan: Yes, please… Um, why is it or what did you do—I noticed that after you got 12, you pushed some more buttons on the calculator, but I didn’t catch it. What did you do after that?

Kelly: Oh, um, I just wanted to check my answer. Oh yeah, I did 12 plus 12 plus 18 equals 42.
In this excerpt, Kelly stated that she subtracted 18 from 42 “because there’s two \(a\) boxes and 42 minus 18 equals… 24 and 24 divided by 2 is … 12.” It seems that in this statement, Kelly was saying that her goal was to find out what the 2 \(a\) boxes were worth and she could do that by subtracting 18 from 42. Further, when Kelly was checking her answer, she added together, 12, 12, 18, and 24. Kelly’s actions seem to indicate that she saw \(a + a + 18 + 24\) as equivalent to 42. Further, Kelly’s actions suggest that she viewed each original whole as decomposable (judging from her focus on \(a + a + 18 + 24\) from Strip A and 42 from Strip B) and that the parts created by this decomposition can be equated to one another. Kelly used this same type of reasoning when solving a similarly structured problem.

When we returned to Paper Strip Task #1 at the end of the 5/5/2008 session, Kelly seemed to be able to work with the task differently. Kelly’s additional work on this task is shown in Figure 4-63.

Jan: Now this is what you drew here. One thought I want to give you. These two should be [draws a line connecting the right side of the second boxes on the Strip A and B diagrams]—these are lined [inaudible]. Now, based on what you just
Kelly: Um, to find this? Well, um, all between here [uses hands to frame between 3rd box of each diagram and last box of each diagram] or is that just marked like that?

Jan: Well, um, I just did that to say that these–

Kelly: To get it to line up. Okay, um, 41 minus 17 and maybe divide by one, two, three, four, five [taps each of the five unit boxes on the Strip B diagram].

Jan: Okay, now why would you divide by 5 [moves pen along five unit boxes on the Strip B diagram]?

Kelly: Just because there’s five boxes here [taps each of the five unit boxes on the Strip B diagram].
equivalent wholes, once Kelly found the difference between the known quantities in each of the wholes, she divided that difference by the number of occurrences of the unknown quantity in one of the wholes. At the time, Kelly seemed to perform this operation with understanding as evidenced by the manner in which she checked her answer. However, it is possible that she memorized her activity as a faulty algorithm. It became my goal, then, to help Kelly to be able to identify, on the diagram, the parts that were equal in length to 41.

Jan: Okay. Um, let’s see now. Show me what 41 minus 17, [Kelly begins to use the calculator] where that would –oh, in fact, just do what you are thinking. [reading from the calculator] 41 minus 17 equals 24.

Kelly: It can’t be divided.

Jan: Where does the difference between 41 [points at the 41-box] and 17 [points at the 17-box]– can you put a bracket around it on the diagram?

Kelly: The difference between?

Jan: Yeah. [Kelly draws a bracket from the approximate middle of the 41-box upwards and to the right of the 41-box; she draws a bracket from underneath the left side of the 17-box, to the right and upwards to meet the other bracket; to the right of this long bracket, Kelly writes to the right of this bracket off-camera]

Jan: [reading from the calculator] 42 minus 17—was it 42?

Kelly: Wait, wait.

Jan: [reading from the calculator] 42 minus 17 equals—

Kelly: Yay.

Kelly probably proclaimed her delight because 42 minus 17 is equal to 25. Since she originally wanted to divide the difference between the two known quantities by five and since 25 is divisible by five, she probably thought that she would now be able to solve the problem in the way that she had intended.

Jan: Now remember this was 41 [points to the 41-box].

Kelly: Oh, that’s right.
At this point, I pointed out to Kelly that the left side of the 17-box and 41-boxes showed what the 17 and 41-box had in common. I then decided to create a bracket above the 41-box that was aligned with the borders of the 17-box so that I could illustrate to Kelly which part of the 41-box represented 17 inches. This decision was carried out in order to make more salient to Kelly those aspects of the diagram which had the potential of assisting her in perceiving of the part-whole relationship between 17 and 41 (Hahn & Kim, 1999; Ng & Lee, 2009).

Jan: …The difference, you’re saying, is this here [traces along bracket that Kelly drew connecting the 41 and 17-boxes]? So this is what the two have in common [uses her pen to trace upward from the left side of the 17-box to a space inside of the41-box; draws a dotted line upward through the 41-box; creates a bracket from the top of the dotted line to the right side of the 41-box.] These are both 17, right?

Kelly: Do you want me to write that?

Jan: Um, sure. [Kelly writes the number 17 to the right of the vertical line that I just created inside of the 41-box]

After Kelly writes the number 17, she subtracts 17 from 41 on the calculator.

Kelly: Okay, yeah, 24. This equals 17. [Jan: okay.] Um, okay, so 24. So these are 8 inches [moves pen left to right along third through fifth boxes of Strip B Diagram].

Jan: Why is that?

Kelly: Because, um, so, let’s, so minus 17…would be 24. 24 divided by these three boxes [moves pen left to right along third through fifth boxes of Strip B Diagram].equals eight , so—

Jan: Okay, would you write that inside please? So you’re saying that those three boxes together—these three equal 24? How do you know?

Kelly: Because 8 times 3 equals 24.

Jan: Okay. I mean, how do you know that, since that’s 17 [points at the 17-box], that’s 41[points at the 41-box], how do you know that should be 24 [points at the third through fifth boxes of Strip B Diagram]?

Kelly: Because 41 [points at the number 41 written inside of the 41-box] minus 17 [points at the number 17 written inside of the 41-box equals 24]. Since it’s
cutting off this [moves pen down line that Jan drew inside of the 41-box], this
[points at left side of 41-box] would automatically become 24 and you would
divide by 3.

The first thing that seemed to help Kelly to solve this problem was isolating those parts of
both diagrams that were not identical. Then, directing Kelly’s attention to that part of the 41-box
which matched the 17-box seemed to enable her to surmise that the remaining part of Diagram B
that was not matched with any other part must have been equal to 24. This is something that
Kelly was unable to do previously without my explicitly pointing out that part of the diagram that
represented the difference between the two known quantities. It seems that Kelly had a better
understanding of the relationship between the parts and whole of two equivalent quantities during
this episode than she did when she first began solving the Double Sided Unknown Quantity tasks
in the following sense. It was Kelly who chose to focus on the remaining unmatched boxes in the
Strip B diagram.

Based on her performance on the next task and on the next one like it in which she
required no assistance, Kelly seemed to demonstrate the understanding that the additive
difference between the unknown parts of two equivalent wholes is multiplicatively related to the
additive difference between the known parts of the two wholes.

It seems that the Paper Strip Puzzle tasks helped to fortify Kelly’s part-whole
understandings. It is possible that, by having to make explicit her problem-solving process when
solving tasks, that Kelly had to reflect upon her actions and the result of those actions. It is also
possible that labeling the unknown quantities with letters was helpful to Kelly in that she could
name and operate upon such letters as a part of her problem-solving process.

Denise’s Story
If a known quantity, $k$, is equivalent to $n$ units of an unknown quantity, $u$, the two quantities are multiplicatively related in the following way: $k = nu$.

While solving the Harper Middle School Tasks, Denise demonstrated that she understood the multiplicative relationship between the known and unknown quantities of two equivalent wholes as seen in Figure 4-64 and in the excerpt below.

Denise: Darrell bought a box of candy with a certain number of pieces in it. Please see diagram below. Paula bought four times the amount of candy that Darrell brought. Show this on the diagram below.

Jan: Okay. And you can draw that right on there please? [Denise draws four rectangles connected horizontally next to the words “Paula’s candy”]

Denise: If Darrell doubles the amount of candy that he has and then brought a box of candy containing ten pieces in it, he and Paula would have the same amount of pieces of candy. Alter your diagram to show this situation. So, if you doubled it, which means it would be like that [draws a rectangle connected to the given rectangle on Darrell’s diagram], then bought ten more boxes of candy, he and Paula would have the same number of pieces. [Denise draws a rectangle that is connected to the second rectangle on Darrell’s diagram. The right side of this rectangle is aligned with the right side of the last rectangle on Paula’s diagram.] So the number of pieces in a box could possibly be five.

Jan: And why is that?

Denise: Because if he was to double it. If he started out with five [Denise points to first box on Darrell’s diagram] and Paula had four times as much which would make this 20 [points to Paula’s diagram] and if he doubled five, then that would give him ten and then ten more would be twenty that Paula had.
When asked why she chose five as her answer to the problem, Denise justified her answer by substituting five for each box and verifying that Paula and Darrell would both have 20 pieces of candy. However, it seems that Denise may have obtained the number five through her interpretation of her diagram, which showed that the 10-box was the length of two boxes. Thus, Denise may have obtained the answer of five by dividing by two.

Denise performed similar operations when solving the next Harper Middle School Task, this time specifying that she divided the remaining total, the number of remaining occurrences of the unknown quantity.

If $a$, $b$, and $c$ are known quantities, $x$ is an unknown quantity, and $ax+b=c$, then $c$ is, simultaneously, a quantitative whole and a composition of the parts $ax$ and $b$. Further, the unknown quantity is multiplicatively related to the difference between the known quantities, $b$ and $c$. 
When solving Double Sided Unknown Quantity Task #1 on 4/18/2008 (see Figure 4-65), Denise showed that she did not understand that there is a constraint placed upon the additive difference between two quantities when it is equivalent to a multiple of the unknown quantity. After representing the relationships in the problem statement, Denise stated the following.

Denise: 8 minus 20 could have equaled 12 and each square could have been 4.

Jan: And why is that?

Denise: Because 4 can go into 12, 4 can go into 8 and if you had 8 [taps 8-box in Shelf B Diagram] plus 8 [taps 5th box in Shelf B diagram]; 8 times 1 [taps 5th box in Shelf B diagram], 1, 2, 3, 4, 5, 6. [taps each of the six boxes on the Shelf B diagram from right to left starting from the 8-box] I actually should have made that bigger [moves pen to the right of the 8-box] to show how it could have been more [traces down middle of 8-box with pen] to make that 7 [taps each of the boxes in the Shelf B diagram] and if it was 7, 4 times 7 is 28 and if that started out [taps first box in Shelf A diagram] with—actually, it could have been a different number, so it could have been 8.

At first, Denise seemed to choose an answer, 4, based on its divisibility into some of the key numbers in the problem. Denise quickly changed her approach and seemed to try to use a type of guess and test strategy where she assumed that each box represented the number eight (as indicated by her statement “8 times” and subsequent tapping of each box in the Shelf B diagram). Denise then stated that she should have made the 8-box bigger so that she could have created seven boxes (instead of six in the Shelf B diagram). The different approaches that Denise took seem to be indication that she did not have an understanding of the multiplicative relationship between the unknown quantity and the additive difference of 12.
At the school library, Shelf A had a certain number of books on it. Shelf B had five times as many books on it. Draw a diagram to represent this situation.

Number of books on Shelf A: 20
Number of books on Shelf B: 8

If 8 books were added to Shelf B and 20 books were added to Shelf A, the two shelves would have the same number of books. Alter your diagram to show this. Use your diagram to figure out how many books each bookshelf held initially.

Original number of books on Shelf A: 3
Original number of books on Shelf B: 15

Figure 4-65: Denise’s Response to Double Sided Unknown Quantity Task #1 on 4/18/2008

Jan: Okay, let me make sure I understand. So where did the 4 come from, you said 4 times 7?

Denise: If there were 7 blocks [taps each of the first 5 boxes in Shelf B diagram] and I cut this one in half [traces pen vertically through 8-box in Shelf B diagram].

Jan: Okay, there’s 1, 2, 3, 4, 5, 6, 7 [taps each of the six boxes in the Shelf B diagram], okay. Where did the number 7 come from?

Denise: If there was five times as many books and he added another, he added 8 more, but we didn’t know what the number was, and each book had a shelf.

Jan: Okay, so was 7 like a guess or –

Denise: It was kind of like a guess, but now I’m thinking if, what if it was a number and it’s like 2 and 2 can go into 20 ten times, so each one could have made 2, 2 times [points to first box on the Shelf A diagram], if this was 4 [points to 8-box in Shelf A diagram]—I should have made that longer, if it was 4 and this was 5 more [points to first five boxes on Shelf B diagram], then there would have been 9, so 2 times 9 equals 18, so that could have been an example. But if this one [points to 8-box on the Shelf B diagram], and I don’t think anything else could have been in except for 8, but 8 [traces pen along length of Shelf B diagram].
Diagram], it wouldn’t be that big if this was 20 [points to 20-box on Shelf A diagram].

At the end of this episode, Denise was still wrestling with the size of the 8-box. She took a different approach, however, by assigning one number (2) to the unknown and another (4) to the 8-box. Denise did not take into account the fact that, by doing this, the problem condition would have been violated. Although this seems to be further evidence of Denise’s lack of Understanding 3, Denise’s difficulty also reveals the shortcoming of the diagrammatic tool introduced to the students in the study. In order to draw the unknown and known quantities the correct size relative to one another, the unknown quantity would have to have been known. Therefore, the students had to take their best guess. It seems that, once students represented a situation, the information that they “read off” (Page & Simon, 1966) the diagram became more salient to them than the quantitative information provided to them in the problem statement.

As a result of Denise’s misconception, I sought to focus her attention on the length of the diagrammatic representation of the additive difference between 8 and 20.

Jan: I understand that these boxes [traces circles inside of 4th and 5th boxes on Shelf B diagram with pen cap], just we didn’t know how big to make 8 and 20. Maybe we can’t really judge by the size of the box what it should be. [Pointing to the 5th box on Shelf B diagram.] But we can judge by something else. For example, this length right here [creates a bracket underneath the 2nd through 5th boxes in Diagram B], is there anything you know about that length?

Denise: That they had 4 on it, 4 boxes.

Jan: Okay, it’s 4 boxes. Do you know the total number of books that this represents? [Jan sweeps pen cap back and forth amongst the first five boxes on the Shelf B diagram]

Denise: No, we don’t know that yet because we don’t know what one of these [taps first box on Shelf B diagram].
Jan: That’s true. We do know something else. What is this whole amount equal to? [Draws a bracket above the Shelf B diagram that extends from the second to the sixth boxes.]

Denise: If this right here was cut off [traces pen cap upward from left side of 8-box in Shelf B diagram through same location in 20-box in Shelf A diagram; 20 is like [creates imaginary lines inside of 20-box corresponding with left sides of the 4th and 3rd boxes in Shelf B diagram], about a fourth of it, and 8 was a fourth [traces upwards from left side of 8-box in Shelf B diagram to 20-box in Shelf A diagram]—no wait, 8 minus 20 would make it, um, 12 minus 8 would make it 4, so that couldn’t be the example.

Jan: Okay, so what are you, as you were thinking, what were you trying to figure out?

Denise: If this could be cut in half [pointing to the 8-box in the Shelf B diagram] and each one could equal 4, and 4 could go into 20 to make it 5, so 5 squares as there were originally before and maybe each square could be worth 5 or 4.

Jan: Okay, let’s try it out for example. Let’s say that was 5 [points to first box in Shelf A diagram] and these are all the same numbers [taps the first box in the Shelf B diagram], right? So that’s going to be 5 [points to the first box in the Shelf A diagram], to make this 25 [scans Shelf A diagram from right to left], and if this were 5 [points to the first box in the Shelf B diagram], 5, 5, 5, 5, 5 [taps each of the 1st through 5th boxes on the Shelf B diagram] 1, 2, 3, 4, 5 [taps each of the 1st through 5th boxes on the Shelf B diagram], that would be 25 right there, and so that[scans Shelf B diagram back and forth with her pen cap]—

Denise: That’s the complicated part. What happens to the 8 though?

Earlier in the teaching episode, Denise had subtracted eight from 20, but she did not seem to know what 12 represented. In this instance, I was trying to focus Denise’s attention on the additive difference between the two quantities. Thompson (1993), in his work with fifth graders on complex quantitative reasoning tasks, also found that students tended to subtract two numbers without having an understanding of what the result of that operation represented. As a result, Thompson recommended that teachers ask students questions such as, “What are you trying to find?” and “What did this calculation give you?” (p. 204) in order to promote reflection.

In this excerpt, even though I began the discussion trying to establish with Denise what lengths on the diagram we could determine for certain, Denise resumed to relying on the
appearance of the diagram. Denise seemed to use a combination of guess and test and visual estimation in order to try to arrive at a solution. Denise’s reliance on these informal problem-solving methods underscore what was postulated by Hall and colleagues (1999) when they stated that students who do not understand the structure of a problem will attempt to get at that structure through informal methods that are available to them. I tried to engender cognitive conflict within Denise by testing the number that she recommended. When that number failed to yield the same answer for the Shelf B diagram and the Shelf A diagram, I tried to then direct Denise’s focus towards the representation of the additive difference between the two known quantities.

Jan: Yeah. So here we go. I heard you say 20 minus 8 is 12. Why don’t we just go ahead and put the 12 here [writes the numeral 12 underneath the bracket on Shelf B diagram that frames 2nd through 5th boxes]?

Denise: Okay.

Jan: Now is that accurate to put that 12 there? What do you think?

Denise: Yes because it’s lined up with this [traces down from left edge of 20-box in Shelf A diagram to left edge of second box in Shelf B diagram] and it’s lined up with this [traces up from right edge of 8-box in Shelf B diagram to right side of 20-box in Shelf A diagram].

In this statement, Denise acknowledged that the 20-box was aligned with the last five boxes of Diagram B. Thus, it seems that Denise was saying that, since the 20-box was equal in length to the eight box and the four boxes of unknown length, then the combined length of the latter four boxes must have been equal to 12.

Jan: Okay, now is there anything that you know, [circling finger above 2nd through 5th boxes on Shelf B diagram] based on the fact that this whole thing is worth 12, does that help you find out the value of one box?

Denise: Yes because there’s 4 in here so each box could be worth 3.

It seems that the occupation of the four boxes of unknown length in the space between the left side of the 20-box and the left side of 8-box enabled Denise to visualize the four boxes’ equivalence to 12. Earlier in this teaching episode, Denise had subtracted 8 from 20 and gotten
an additive difference of 12. However, Denise did not know how to contextualize this difference. Thus, she offered that the value of each box could have been four. It was not until Denise’s attention was drawn to the location of the additive difference of 12 on the diagram that Denise was able to compare, multiplicatively, 12 and the value of the unknown quantity.

In order to help Denise to build Understanding 3, I gave her another Double Sided Unknown Quantity task, shown in Figure 4-66. After she represented all of the relationships given in the problem, Denise attempted to recreate the steps that I had taken with her to help her to solve the bookshelf problem.

Now I’m going to do the thing that you did…just now, which would be this [draws a bracket from underneath the second box to underneath the middle of the 13-box in the Fabric B diagram] and add 13 minus [inaudible]; that should equal [crosses off part of the bracket that she just drew]. If it was [pushing buttons on the calculator] 34 minus 13. This could be 21 [hovers pen above the 2nd to 5th boxes on Fabric B diagram], that’s five.

Cynthia was making headwraps for her and her cousin. She found two strips of fabric to work with. The length of Fabric Strip B was five times as long as the length of Fabric Strip A. Draw a diagram to represent this situation.

Length of Fabric Strip A: 34
Length of Fabric Strip B: 7

Cynthia added 13 inches of fabric to Fabric Strip B. She also doubled the length of Fabric Strip A and then added 34 more inches of fabric to it. When she did this, the two fabric strips became the same length.

Alter your diagram to show this. Use your diagram to figure out how long each fabric strip was initially.

Original length of Fabric Strip A: 7
Original length of Fabric Strip B: 35

Figure 4-66: Denise’s Response to Double Sided Unknown Quantity Task #2 on 4/18/2008
In this excerpt, Denise drew a bracket like I did in the previous problem. However, she did not start and stop the bracket where the 34-box began and ended. Instead, her bracket extended from underneath the second box in Fabric B diagram to underneath the middle of the 13-box in that same diagram. Thus, she seemed to be repeating my previous action of drawing a bracket but doing so without knowing why.

Next, Denise subtracted 13 from 34 and indicated that the 2\textsuperscript{nd} through 5\textsuperscript{th} boxes could be equal to 21, the additive difference between those two numbers. Denise seemed to still be unaware that the additive difference between 13 and 34 was a quantity in itself that was represented by the boxes occupying the space between the left edge of the 34-box and the left edge of the 13-box. Denise was engaging in an algorithm that was external to her. Because I wanted Denise’s actions to spring from her understandings, I sought to focus her attention back to the part whole relationships embodied within the diagram. In this way, Denise would have the opportunity to build meaningful connections between her current mathematical knowledge and the algebraic knowledge that was intended for her to learn rather have to resort to a series of memorized algorithms (Tall, 2008).

In order to help Denise to properly locate the additive difference on the diagram, I asked her the following question.

Jan: Where would the 21 start?

Denise: 21 would start [taps left side of second box on Fabric B diagram; taps each of the second through fifth boxes on the Fabric B diagram]

Jan: Well let’s put it differently. Can you show me on this diagram [moves finger from right edge of Fabric B diagram to left edge] where this 34 [taps 34-box] would show up?

Denise: Here to here [Denise draws an arc above Shelf B diagram extending from left side of 3rd box to right side of 13-box.]

Jan: Does that help you figure out where the 21 would be?

Denise: Yes.
Jan: Where?

Denise: From here [begins drawing a bracket from underneath the left edge of the third box on the Fabric B diagram] to here [ends the bracket underneath the right edge of the fifth third box on the Fabric B diagram; inaudible] 3, and there could be 7. [Pushes button on calculator] I multiplied it, sorry. [Pushes button on calculator] There could be seven in each square.

Jan: Now what made you say that this part right here is 21 [scans along bracket that Denise just drew]?

Denise: Because I was thinking from when we started on the last one [pointing to left edge of second box on Fabric B diagram] to where the 13 inch stopped [points to the left edge of the 13-box on the Fabric B diagram], so I should have went down on each one [points to left side of 34-box on Strip A diagram and pulls downward to corresponding location on Fabric B diagram; points to right side of 34-box on Strip A diagram and does the same] to see where it was [moves pen cap from left edge of third box on Strip A diagram to right side of Strip A diagram] and then test it to see if it was 7 plus 7. [Points to first two boxes on Strip A diagram; pushes buttons on the calculator.] 7 plus 7 plus 34 would equal 48. 7 times 5 plus 13 equals 48.

As happened during the previous Double Sided Unknown Quantity task, when Denise was asked to focus upon the representation of the remaining quantities—in this case, the known whole, 34 and the part, 21—she was able to discern the part-whole relationship embedded within the situation. The event that preceded Denise’s acknowledgement of where to locate the “21” on the diagram was when I asked Denise to identify the 34 inches on the Strip B diagram. After Denise drew a bracket over the boxes of the Strip B diagram, she figured out what part of the Strip B diagram was 21 inches in length. As occurred with Kelly, it seems that drawing Denise’s attention to those parts of the diagram that contained important quantitative information had the effect of enabling Denise to mentally isolate and act upon those parts. Perhaps this enabled Denise to be selective in the parts of the diagram upon which to place her focus since she probably did not know in advance what relationships within the diagram to search for or upon which parts of the diagram she should act (Petre & Green, 1993). By drawing a bracket over the last three boxes of Diagram B, Denise’s focus was brought to the part of the diagram with the
unmatched quantities within each whole. Perhaps that focus enabled her to unitize 34 as the new whole and to see 21 as part of that whole.

Throughout each of the episodes given, Denise consistently subtracted the two known wholes. It is unclear what this difference represented to her in each episode because Denise would often try to divide by a number of boxes that did not occupy the same amount of space as the difference (Denise also engaged in this set of actions in the task that immediately followed the one described above). Thus, it does not seem that the representation of the additive difference was an object to Denise. When looking at the diagrams of the two known quantities, Denise did not spontaneously conceptualize that the space occupying the left side of the longer box to the left side of the shorter box was the difference in question. Each time, Denise needed to be reminded of this fact.

During the Task 2 episode, Denise visually indicated where the representation of the larger known quantity began and ended. This seemed to help Denise to think about where, within that context, the difference between the two known quantities would be located. Ng and Lee (2009) found that Singaporean students who created a diagrammatic model of an algebra word problem and who correctly labeled all of the quantitative information that was given were more successful in solving the problem than those students who omitted just one piece of quantitative information from the diagram. It is possible that having the representation of the smaller known directly underneath the representation of the larger known quantity helped Denise to make this distinction.

The next time that Denise was given a Double Sided Unknown Quantity task was on 5/2/2008. When solving this task, Denise, again, did not immediately utilize the space representing the difference of the two known quantities in order to find the value of the unknown quantity. However, she was asked to attend to the part of the diagram where the additive difference was represented, Denise was able to solve the problem correctly. In the next episode,
we will see that Denise seems to have undergone a conceptual shift during which she became able to anticipate that she would be finding the distance of the space between the representations of the two known quantities when she subtracted those known quantities.

After representing the relationships given in the problem statement of the task, shown in Figure 4-67, Denise used the calculator to subtract 17 from 41 and divide that answer by three. She obtained an answer of eight. Denise then tapped each of the first five boxes in the Strip B diagram and used the calculator to multiply eight by five. Denise then performed the following actions.

[Denise traces upwards from the left side of the 3rd box in the Strip B diagram to the left side of the 41-box in the Strip A diagram. Denise then traces her pen from the left side of the 17-box in the Strip B diagram directly upward to the corresponding space inside of the 41-box in the Strip A diagram. Denise then taps the 17-box in the Strip B diagram. 1 second pause.]

Jan: Can I ask you a question? [Denise: Um hm.] Can you tell me why you subtracted 17 from 41?

Denise: To see how much distance was. [Denise frames the 3rd through 5th boxes in the Strip B diagram with her pen and index finger: Jan: Okay.] And each one, I found out, was worth eight. [Jan: Okay.] So, if that was true, then this would be 40 [traces first five boxes of the Strip B diagram with her pen] and this would be 8 [points to first box in the Strip A diagram].
Figure 4-67: Denise’s Response to Double Sided Unknown Quantity Task #2 on 5/5/2008

In this episode, Denise demonstrated that she had an understanding that there is a multiplicative constraint placed upon the additive difference between two quantities when that difference is equivalent to $n$ occurrences of the unknown quantity. When asked why she subtracted 17 from 41, Denise framed the three boxes that represented the additive difference between 17 and 41 and indicated that she was trying to find that difference. She then divided this difference by three which is the number of boxes that occupied the space representing that additive difference.

Denise’s solution process seems to indicate that she understood that since the additive difference was equal in length to three same sized boxes, then that difference must necessarily be three times the length of one box. This seems to be the reason why Denise divided the additive difference by three. Thus, Denise seemed to understand that the additive difference between the
unknown parts of two equivalent wholes is multiplicatively related to the additive difference between the known parts of the two wholes

Valerie’s Story

When a known quantity, $k$, is equivalent to $n$ units of an unknown quantity, $u$, the two quantities are multiplicatively related in the following way: $k = nu$.

When given a task of the form $ax + b = cx$, Valerie demonstrated that she understood that there was a multiplicative relationship between the unknown quantity and the difference between the known quantities. Valerie’s work on this task is shown in Figure 4-68.
Paper Strip A is five times as long as Paper Strip B. Write an equation relating the length of Paper Strip A and the length of Paper Strip B.

If 24 inches were added to Paper Strip A, Paper Strips A and B would be the same length. Alter your diagram to show this situation. Use your diagram to find the original length of each paper strip.

\[
\begin{align*}
\text{Paper Strip A} & = a + 24 \\
\text{Paper Strip B} & = b
\end{align*}
\]

\[
\begin{align*}
A & = \text{length of paper strip A} \\
b & = \text{length of paper strip B}
\end{align*}
\]

\[
\begin{align*}
A & = 60 \text{ inches} \\
b & = 30 \text{ inches}
\end{align*}
\]

Figure 4-68: Valerie’s Response to Paper Strip Task #1 on 2/12/2008

In order to represent the relationships in the first section of the problem statement, Valerie drew one box and wrote the letter A inside. Underneath this, she drew five connected boxes and wrote the letter B inside of the third box of that diagram. In response to the second problem statement, Valerie drew a long box attached to the first box in the Strip A diagram. She wrote “24 inches” above that box and then subdivided that box into four equal sections. Then, Valerie wrote “6 in” inside of each of the 2\textsuperscript{nd} through 5\textsuperscript{th} boxes in the A diagram.
After having drawn the boxes representing the original lengths of both paper strips, Valerie drew a long box to represent the 24 inches added on to Paper Strip A. Interestingly, Valerie chose to divide the 24-box into the number of boxes which occupied the same amount of space as the 24-box within the Strip B diagram. Valerie accomplished this without any input from me. Valerie appeared to then divided the number 24 by the total number of boxes that occupied the space of the 24-box.

By subdividing the 24-box into four boxes, it seems that Valerie was equating the length of the 24-box with the length of the portion of the diagram of Strip A that was unmatched. In other words, Valerie appears to have conceptualized the additive difference between the lengths of the two paper strips as being equivalent to four instances of the unknown quantity. By dividing this difference by the number of instances of the unknown quantity, Valerie seems to be demonstrating her understanding of the multiplicative relationship between the two.

When solving the task in Figure 4-69, Valerie did not physically subdivide the boxes, however, she did still mentally compare the 10-box with the two boxes below it. Valerie began by creating diagrams for Darrell’s and Paula’s amounts of candy based on the information given in the problem statement. After doing so, Valerie pointed to the last box on Darrell’s diagram and asked, “So this one contains ten pieces?” In response to my answer in the affirmative, Valerie drew a bracket above that box and wrote “10 pieces”. The following conversation then ensued.

Valerie: And that means each box has five pieces of candy.

Jan: Okay. Why is that?

Valerie: Because, if you split this in half [draws an imaginary line down the center of the third box in Darrell’s diagram], it would be the same—do it individually, it would be five pieces. Five and five [points to the left and right sides of the 3rd box in Darrell’s diagram].

Jan: Okay. Gotcha. Could you write the fives where they should go?
Valerie: In each box?

Jan: Okay. Sure. [Valerie writes the numeral five in the first two boxes of Darrell’s diagram and each box of Paula’s diagram.] Okay. Super. Thank you

Valerie’s actions in this episode are very similar to her actions to the first one, however, Valerie did not subdivide the box representing the known quantity. It seems possible that Valerie did not subdivide these boxes on paper because she could imagine “splitting in half” the box representing the known quantity as she stated in the excerpt. Or, it is possible that Valerie may have anticipated that she would be dividing the known quantity by two since the box representing the known quantity was equal in length to two boxes representing the unknown quantity.

Through her actions in both episodes, Valerie demonstrated that she understood that when a known quantity, is equivalent to \( n \) units of an unknown quantity, the two quantities are multiplicatively related. Valerie was introduced to the Double Unknown Quantity Tasks during

Figure 4-69: Valerie’s Response to Harper Middle School Task #5 on 4/4/2008

Darrell bought a box of candy with a certain number of pieces in it (please see diagram below). Paula bought four times the amount of candy that Darrell bought. Show this on the diagram below.

Darrell’s Candy: 

Paula’s Candy: 

If Darrell doubled the amount of candy that he has and then bought a box of candy containing 10 pieces in it, he and Paula would have the same number of pieces of candy. Alter your diagram to show this situation.

Use your diagram to figure out how much candy each person started out with.

\[
\text{Darrell} = 5 \\
\text{Paula} = 20
\]
the main study. In the episode that follows, Valerie demonstrated a developing understanding of
the multiplicative relationship between the difference between the known quantities of two
equivalent wholes and the difference between the unknown quantities of those same two wholes.
Her work on this task is shown in Figure 4-70.

If \(a, b, \) and \(c\) are known quantities, \(x\) is an unknown quantity, and \(ax+b = c\), then \(c\) is,
simultaneously, a quantitative whole and a composition of the parts \(ax\) and \(b\). Further, the
unknown quantity is multiplicatively related to the difference between the known quantities, \(b\) and
\(c\).

Valerie began representing the relationships in the first section of the problem statement
by drawing one box on the Strip A line and drawing five connected boxes on the Strip B line.
After reading the second section of the problem statement, Valerie added one box to the Strip B
diagram and one small box and one long box to the Strip A diagram. At my request, Valerie
wrote the known quantities in their respective boxes. The following conversation ensued.

Jan: So what are you thinking? [Valerie begins to use the calculator.] Thirty-four divided by—I’m sorry, what did you divide by?

Valerie: Three.

In this excerpt, it seems that Valerie was attempting to enact the scheme that she developed
previously when solving problems that had the underlying structure \(ax+b= cx\). That is, after
ignoring the representations of the unknown quantity common to both diagrams, she would divide
the remaining known quantity by the number of instances of the remaining unknown quantity. In
this instance, however, Valerie seemed to have applied a “mal-rule” (Sleeman, 1984) to the
situation since she divided 34 by three without regarding the 13.

Jan: … Now why did you divide 34 by three?
Valerie: Because there is three boxes here [points at the 3rd through 5th boxes on the Strip B diagram], and then there’s a 13 here [points at the 13-box on the Strip B diagram]. I thought if I did—this plus this plus this [points at each of the 3rd through the 5th boxes on the Strip B diagram] plus three [points at the 13-box on the Strip B diagram] would give me this [points at the 34-box on the Strip A diagram]. So then I know [waves pen above both diagrams] how much they started off.

In this excerpt, Valerie showed that she was aware that the quantities represented by the 3rd through 5th boxes plus 13 was equal to 34 (Valerie seemed to have said “three” by accident because she was pointing at the 13-box).

Cynthia was making headwraps for her and her cousin. She found two strips of fabric to work with. The length of Fabric Strip B was five times as long as the length of Fabric Strip A. Draw a diagram to represent this situation.

Length of Fabric Strip A: 

Length of Fabric Strip B: 

Cynthia added 13 inches of fabric to Fabric Strip B. She also doubled the length of Fabric Strip A and then added 34 more inches of fabric to it. When she did this, the two fabric strips became the same length.

Alter your diagram to show this. Use your diagram to figure out how long each fabric strip was initially.

Original length of Fabric Strip A: \(7\) in \quad Original length of Fabric Strip B: \(35\) in

Figure 4-70: Valerie’s Response to Double Sided Unknown Quantity Task #1 on 5/6/2008

Jan: So you’re saying this plus this plus this [points at each of the 3rd through 5th boxes on the Strip B diagram] plus 13 [points at the 13-box on the Strip B diagram] is 34 [points at the 34-box on the Strip A diagram]?

Valerie: Um hm.

Jan: If you divided 34 [points at the 34-box] by three [circles the the 3rd through 5th boxes on the Strip B diagram with her pen cap]—

Valerie: You’d get 11.3.
Jan: Okay. Does that help you at all?

Valerie: No.

Jan: No? Okay.

Valerie: Because it’s not an even number.

Valerie seemed to think that the answer of 11.3 was incorrect because when she divided 34 by three, she did not obtain an “even number.” (By “even number”, I believe that Valerie may have meant a whole number). Valerie did not at all seem perturbed that her solution method did not take into consideration the part of the 34-box that was not occupied by the three unit boxes. Therefore, I invited Valerie to consider what would have happened if she had obtained a whole number. I did this in order to cause a disequilibrium in Valerie’s mind with regards to her solution method.

Jan: Okay. What if the answer had been 11, for example—cause this is kind of close to 11. If each of these boxes were 11 [points to each of the 3rd through the 5th boxes on the Strip B diagram], then the three together would have been what?

Valerie: The three would have been 33.

Jan: And so –

Valerie: It still would have been wrong because you would have to add 13 to it. It had to be the same.

In this excerpt, I asked Valerie to suppose that an approximation of the answer that she obtained by dividing 34 by 3 was correct. My purpose in doing so was to enable Valerie to experience a cognitive conflict when she multiplied that approximation, 11, by 3. It is interesting that, at first, Valerie did not perceive that 34 inches was distributed amongst three boxes and the 13-box. However, given the stipulation that each box’s length was equal to 11, it seems that Valerie was able to see that the total length of the three boxes in Diagram B would be too close to 34 and that she had not taken into account the 13-box. This became an opportune time to
encourage for Valerie to reconsider her solution method according to the relationships in the problem statement.

Jan: So how can we account for that 13 then? [Taps the 13-box; Valerie begins to use the calculator; Jan reads the buttons she pressed.] Thirty-four

Valerie: Minus 13 equals 21. [inaudible as she taps each of the 3rd through 5th boxes on the Strip B diagram.] So each box would be seven.

Jan: Okay. And how’d you get that?

Valerie: Because since I subtracted it [points to the 13-box], so I’m taking away the 13 and so like this [moves finger from 13-box to area on the 34-box that is right above the 13-box], and then you 21 left, and 21 divided by three [scans 3rd through 5th boxes on the Strip B diagram with her pen] is seven.

By asking Valerie, “So how can we account for that 13 then”, I hoped to direct Valerie’s attention to the fact that the length of the 13-box was part of the length of the 34-box. Having her attention directed to the 13-box on the diagram seemed to have helped Valerie to notice that the difference in length between the 34-box and the 13-box was that which was equal to the length of the three boxes in Diagram B.

Valerie’s final justification of her answer seems to indicate that the diagram transitioned, for her, from a model of the head wrap situation to a model for reasoning mathematically about the relationship between quantities (Gravemeijer, 1997; Treffers, 1991) when she stated: “So I’m taking away the 13 and so like this [moves finger from 13-box to area on the 34-box that is right above the 13-box], and then you have 21 left.” It seems that Valerie conceptualized the 34-box as being simultaneously a unit of 34 and as a composition of a unit of 13 (as indicated by her upward movement from the 13-box to the 34-box) and three units of the unknown quantity (3a). This is the understanding that, it seems, would be necessary to conceptualize the multiplicative relationship between the additive difference between the known quantities and the unknown quantity.
Comparison of Students’ Development of Understanding 3

Given similar patterns in their actions, it seems Kelly and Denise experienced similar learning trajectories on their path to constructing Understanding 3. They both attempted to use a guess and test strategy in order to find the value of the unknown quantity and they both sought to alter their diagrams in a manner that was inconsistent with the quantitative information given in the problem statement. However, whereas Kelly sought to utilize one approach or the other to solve the problem, Denise often combined that effort by creating visual estimates based on a number that she proposed. In these cases, she allowed perceptual cues to override the quantitative information.

Although Valerie did not choose a guess and test or visual estimation strategy, she did attempt to apply a strategy that had some semblance to the one that she utilized when solving equations of the form $ax+c=bx$. Denise did the same. In Valerie’s case, after having represented the Cynthia-headwrap problem, there were two unit boxes and a 34-box aligned with five unit boxes and a 13-box. Three unit boxes and the 13-box were aligned with the 34-box. Valerie divided the larger known quantity, 34, by three, the number of occurrences of the unknown quantity that were a part of the larger known quantity. When invited to test her solution, Valerie realized that she had not taken the 13 into account. Denise committed a similar error in that she attempted to apply a misremembered rule to the situation at hand. She subtracted 13 from 34 and then located the additive difference of 21 in the wrong place on the diagram.

In each situation, the intervention that consistently seemed to assist students in arriving at a solution to the problem was the one in which they were asked to focus on those parts of each diagram that were equal in length but unmatched. That is, students were asked to identify and label the additive difference. Once this occurred, the students “saw” that the difference between the known quantities was equal in length to the concatenation of unit boxes. Thus, students were
able to multiplicatively compare the two quantities in order to find the value of the unknown quantity.

The students in this study seemed to have avoided Filloy and Rojano’s (1989) didactic cut in the sense that they were able to operate with the unknown quantity, albeit in the form of a diagram, in order to find the value of the unknown quantity. Having to operate upon an object of unknown length, as opposed to a letter, seemed to make the unknown quantity more accessible to the students.

The fact that students attempted to apply algorithms constructed by them when solving problems of the form \( ax+b=cx \) to problems of the form \( ax+b=dx+cx \) seems to indicate that, the students did not recognize the difference in the structures of the problem. Or, perhaps they had procedurally embodied problems of the form \( ax+b=dx \) without fully understanding the nature of their activities (Tall, 2008).

The interventions used in the current study seemed to help the students focus their attention on the representation of the quantitative difference between the known quantities in problems with the structure \( ax+b=dx+cx \). This use of the diagram appeared to enable the students to conceptualize the multiplicative relationship between the additive difference between the known quantities and the unknown quantity, if only on a participatory level. The students did not seem to fully anticipate that they would need to search for or create a relationship of equivalence between two quantitative wholes, one or both of which consisted of an instance of the unknown quantity, in order to find the value of the unknown quantity. Therefore, their use of the diagram was not fully maximized (Petre & Green, 1993).

The manner in which students developed an understanding of the usefulness of creating a relationship of equivalence in order to find the value of the unknown quantity is explored in the next section.
Understanding 4—Establishing a relationship of equivalence between two quantitative wholes is useful in order to find the value of the unknown quantity contained within one of those wholes

One of the understandings that I had intended for students to develop as a result of their engagement in this study was that establishing a relationship of equivalence between two quantitative wholes is useful in finding the value of the unknown quantity contained within one of those wholes. To that end, I initially gave students tasks which enabled them to develop the following understanding: Given \( ax + b = cx + d \) where \( a \), \( b \), \( c \), \( d \), and \( x \) are known quantities and \( x \) is an unknown quantity, there is a multiplicative relationship between the additive difference between the known quantities and the additive difference between the unknown quantities contained within two equivalent wholes. This was Understanding 3 and was discussed in the previous section.

I also wanted to help to facilitate students’ development of “hypothetical equivalence” (Clement, 1982). According to Clement (1982), hypothetical equivalence enables a person to anticipate that, in order to represent a relationship of equivalence between two non-equivalent quantities, one of those quantities would have to be operated upon in a way that does not occur in the given situation. For example, to represent the fact that “there are six times as many students as professors at [a] university” (Clement, Lochhead, & Monk, 1981, p. 288) as an equation, the relationship between the number of professors and the number of students would have to be “equalized” by multiplying the number of professors by six. Thus, the equation \( S = 6P \) represents the relationship between the number of students and the number of professors even though there are not that many professors at the university (Clement, 1982).

The Harold-Jenna task (see ) was, structurally, one of the most complex of the tasks given during this study (Bednarz & Janvier, 1996) in that the expression of one quantity, \( A \), is dependent both on the present and future state of another quantity, \( B \). In order to establish a
relationship of equivalence, the problem solver would have to generate and compare two representations of \( A \) from both states of \( B \).

Jenna had 5 times as much money as Harold did. After Jenna and Harold both earned $45 shoveling snow, Jenna had 3 times as much money as Harold. Draw a diagram to represent this situation. Use your diagram to find out how much money each person started out with.

Figure 4-71: Harold-Jenna Task (adapted from Bodanskii, 1991/1969)

To solve this problem, a person could represent Jenna’s original amount, Harold’s original amount, and the $45 added to each quantity. A person could then create a third diagram that represented three times Harold’s new amount of money. Then, the relationship between three times Harold’s new amount and five times Harold’s original amount would have to be made explicit—that is, Jenna’s amount of money has not changed, therefore, these two quantities both represent Jenna’s current amount of money.

Understandings Developed by Individual Students

Kelly’s Story

Kelly began the study with the understanding that if two wholes are equal to one another, the equivalence of those wholes is maintained after the removal of equivalent quantities from each of those wholes. Kelly demonstrated this understanding in the latter part of the excerpt below during which Kelly and I discussed her response to item 19 on the pre-assessment (see Figure 4-72). Kelly’s original response to item 19 of the pre-assessment was letter d. She based her answer on the number of sides that each shape had. Since a triangle balanced 3 circles, Kelly reasoned that a square would balance four circles because it had more sides than a triangle. When
providing this response, it seems that Kelly was not taking into account the situation occurring on the scale.

Figure 4-72: Kelly’s response to item 16 on the pre-assessment

Once I asked Kelly to focus on the number of objects on each side of the scale and on the number of circles that were balance by a triangle, Kelly was able to articulate that there were “four [circles] left”, therefore, each square must balance two circles. Even though Kelly did not physically cross off any circles from the diagram, Kelly seemed to have mentally removed the triangle and three circles from the scale in order to ascertain that two squares were equal to two circles. It seems that Kelly understood that removing equivalent amounts from two equivalent wholes maintains the equality of those wholes.

As discussed in the previous section, Kelly demonstrated the understanding that unknown and known quantitative parts of equivalent wholes that remain once equivalent parts of both of those wholes are matched are multiplicatively related.
In order to assess whether Kelly had a notion of hypothetical equivalence, I gave her the Jenna and Harold task (see Figure 4-73). When she received this task, Kelly represented Jenna’s and Harold’s original amount of money using five rectangles and one rectangle, respectively. When Kelly drew the box for Harold’s additional $45, she extended that box until Harold’s diagram was the same length as Jenna’s diagram. Thus, Harold’s $45 box was much longer than Jenna’s $45 box. Kelly may have drawn the diagram this way because, in most of the previous tasks, the quantities being represented were already equivalent. I asked Kelly to make both $45 boxes the same size. I did this in order that Kelly would not “read off” (Mikulina, 1991/1969) any false cues (such as one that Jenna and Harold had the same amount of money once both of them earned $45) or make any false assumptions based upon the diagram. After making that correction, Kelly explained her diagram.

Kelly: She has [points at Jenna’s diagram] five times as much money as he does [points at Harold’s diagram]. Then she owns $45. And he only has one little bar [pointing to the first box on Harold’s diagram] and he owns $45 [points at last box on Harold’s diagram], then she only has three times as much as him. I have a way on the calculator to figure this out.
Jan: You do? Do me a favor though, before you do it…. I was wondering if you could show me what that would look like; if you could show on the paper what would three times as much as Harold’s money look like?

Kelly: [writes “J:”] J would be [inaudible; draws three rectangles next to the letter J]. She would have three bars and he would have one bar underneath the three rectangles [draws one rectangle].

Jan: Okay. Now, here’s the question. This [puts a finger frame around the group of three rectangles just drawn]. Does this represent [points at the last bar drawn (underneath the three rectangles)] this put together [frames Harold’s diagram] and this represents [frames the three boxes next to the letter J] this [taps Jenna’s original diagram]? Is there any way you can use that diagram to help you solve the problem?

Kelly: [writes “H:” next to the last rectangle drawn.] If I can do 45 divided by three, real quick in my head.

Jan: And why do you want to do that?

Kelly: She has three times as much [points at Jenna’s original diagram]. So it could be kind of like that [points at right and left sides of Jenna’s original diagram].

The manner in which Kelly began to solve the task was by assuming that three rectangles represented $45. As a result, it does not seem that Kelly sought to spontaneously create a relationship of equivalence between two quantities according to the problem condition in order to find the value of the unknown quantity. I then asked Kelly the following: “Can you show me on here (pointing at Diagram 3) where all of this (pointing at each box in Diagram 1) would show up?” In retrospect, the question that I asked Kelly was flawed. I should have asked her if she would redraw Diagram 1 on the same scale that she had draw Diagrams 3 and 4. As a result of my question, Kelly draw a line down the middle of the second box in Diagram 3. Because I had not helped Kelly develop tools to use to consider how to compare two unequal quantities, I decided not to engage in a further line of questioning. Instead, I began the next teaching session by giving Kelly a task designed to assess students’ understanding of the commutativity of the quantitative parts of equivalent wholes (see Figure 4-74).
Kelly: Suppose that the following diagram represents strips that someone cut out and taped together. Find the value of $c$ in this strip. Okay. [16 second pause] I'd say it's 8 inches.

Jan: Um, why eight?

Kelly: Because if you take out this [covers the leftmost box on Strip D with her hand] and move it over [points to the leftmost three boxes on Strip C], it equals 3 $c$'s and 24 divided by three equals eight.

Figure 4-74: Kelly’s response to paper Strip Task #2 on 5/6/2008

In response to Kelly’s answer, I pointed out to Kelly (by drawing lines from the Strip C to the Strip D diagram) that the 24-box would be skinnier than the combination of three of the boxes, each with a length of $c$. Then I asked Kelly if there was a different way that she could find the value of $c$. She recommended cutting the third box in the Strip C diagram in half “because you’d get more space” in there.

In attempting to solve Paper Strip Task #2, it seems that Kelly relied on the appearance of the boxes to determine their length. Kelly may have initially made a visual estimate of how many $c$ boxes could fit into a 24-box because many of the tasks that Kelly previously solved utilized this type of reasoning. However, once I told her that it was not the case that she could utilize that
strategy for this task, Kelly suggested chopping one of the c boxes in half in order to “create more space.”

Perhaps Kelly made that suggestion because I had stated that the 24-box was not long enough to contain three c boxes and that the proof of this was that the space between the two lines that I drew underneath the 24-box was a little skinnier than the length of one of the c boxes. Thus, maybe Kelly wanted to make one of the c boxes shorter so that three of them could fit into the 24-box. Or, perhaps she wanted to “lend” some space to the 24-box in order to make it longer. The latter alternative seems more likely to be the case given that Kelly started to use her pen to draw a line down the middle of the first c box in the strip C diagram. She then quickly changed her mind and created a line through the middle of the third c box. This c box was the one closest to the 24-box and, potentially, could have given the 24-box “more space” to the left.

Kelly’s inability to solve this problem without using visual estimation seems to be an indication that she may not have had an understanding of the logical necessity of the equivalence of two non-identical sets of parts of the same sized wholes that are not spatially aligned. Noss, Healy, and Hoyles (1997) stated, “We know that when powerful visual images are present, students tend to exhibit a preference for solving problems simply by perception rather than mobilizing any mathematical knowledge” (p. 209). This seems to have been the case with Kelly on a consistent basis. And, because Kelly did yet not have the prerequisite understanding to anticipate that the accuracy of the structure of the diagram in portraying the quantitative relationships in the problem is more important than the appearance of the diagram, Kelly tended to favor the visual cues offered by her diagram. This tendency is similar to that which was described by Hillel, Kieran, and Gurtner (1989) who engaged 12-year-old students in drawing tasks using Logo programming. These researchers found that the students tended to rely upon the onscreen visual feedback provided them by running their programs to determine the accuracy of
their program rather than monitoring the ongoing fidelity of their programs in maintaining the
given mathematical relationships.

Kelly was able to mentally shift the three c’s from Strip C until they were aligned with
the 24-box on Strip D. However, it does not seem that Kelly spontaneously thought of
rearranging and matching the boxes from Strips C and D so that she could find the length of box
C. It seems that this anticipatory set of actions must be preceded by the understanding that the
unmatched boxes from each strip are equal in length to each other.

The next teaching intervention that I attempted was in anticipation of a set of tasks that I
was about to give to Kelly. I wanted to see if Kelly, after identifying those parts of each diagram
that were identical, would spontaneously recognize and utilize the equivalence of the unmatched
parts within each whole in order to find the value of the unknown quantity. The following
conversation is a continuation from the above excerpt.

Jan: Okay, okay, what if I asked you okay, um, what in these two diagrams are the
same [points back and forth between the diagrams of Strip C and Strip D]—
represent the same value?

Kelly: These two [points to the first “24 ft.” box in each diagram].

Jan: Okay, um. Is there any way you can kind of mark those off since you know they
are the same?

Kelly: Uh, no.

Jan: Why not?

Kelly: Um, I just can't think of anything.

Jan: Okay. Well, what I'm going to do then—I have something to give you that’s
gonna help you – it’s just a problem just like it, but it’s set up a little different.

Because Kelly seemed to be having difficulty understanding how to utilize the
instructions that I was giving her in order to determine the value of the unknown quantity, I gave
her a series of tasks that had the same instructions, but that entailed more structure. In the
Identical Parts Tasks, Kelly was asked to find the “twins” contained within each whole and draw
those twins on a separate section of the paper. She was then asked to identify and draw the remaining parts of each whole separately. My goal was that, as a result of isolating those parts of each whole that were identical, Kelly would be able to isolate and focus on the equivalence of the remaining parts. The first Identical Parts Task solved by Kelly is shown in Figure 4-75.

Jan: Could you read these directions for me please?

Kelly: Draw as many boxes from each diagram that has an identical twin in the other diagram.

Jan: So, for example, these [checks the first box in each diagram at the top of the page] each have a g so that they’re the same. So once I drew them, I check them. If you see any other g’s, I’m going to ask you to—in both—I’m going to ask you to check it off and attach it to those g’s [points to pre-drawn boxes labeled g in the first identical parts section at the bottom of the paper].

Kelly: Okay. [In the first identical parts section of the paper, on the Strip G line, Kelly draws a rectangle next to the rectangle labeled g…..Kelly draws a rectangle next to the rectangle labeled g on the Strip H line of the first identical parts section of the paper.]

Jan: Now that you’ve included one from each, could you check it off please?
By having Kelly identify and isolate those parts of the diagram that were identical in size, it was my intent that Kelly would be able to isolate and focus on the equivalence of the remaining parts.

Kelly: [Checks leftmost box marked $g$ on each diagram at the top of the paper] All this stuff is just $g$. [Draws a vertical line in the “Remaining Parts” section of the paper.]

Jan: And, hold on one sec before you go there. Um [points to rectangles that Kelly drew in the first identical parts section of paper], can you label your $g$’s there please [Kelly writes $g$’s under both rectangles]? Now, before we go to remaining parts, I need you to look for any sets of pairs that are not $g$’s that are the same.

Kelly proceeded to identify, draw, and label the second set of identical parts within the Strip G and Strip H diagrams. She did the same for the remaining parts. When drawing the remaining part from Strip H, Kelly made the 80-box much shorter than the combined length of the parts that remained from Strip G. Kelly may have done this in order to maintain the size that she made the other 80-boxes. On the other hand, Kelly may not have drawn the 80-box as long as the combined length of the other boxes because she did not anticipate that the 80-box represented a quantity that was equivalent to the sum of the quantities represented by the 12, $g$, and 24-boxes.

Because I wanted Kelly to be able to visualize the relationships between quantities if she did not anticipate that the remaining parts in each section were equivalent, I asked Kelly to redraw the 80-box so that it was the correct size relative to the size of the other remaining parts.

After Kelly represented the remaining parts, I asked her about the relationship between the remaining parts of each diagram. I did this both to ascertain her understanding of the relationship between those parts and, once identified, to enable her to utilize the equivalence of the quantities represented by the remaining parts in order to find the value of the unknown quantity.

Jan: If you put these three together [traces pen along the 12”, $g$, and 44” boxes in the remaining parts section of the paper] and compared it to this [points to the 80-box
in the remaining parts section], would this be less than these together [scans pen across the 12-, g, and 44-boxes in the remaining parts section of the paper], greater than, or equal to these three together?

Kelly: Equal to.

Jan: How do you know?

Kelly: They look like it [indicates both sets of diagrams in the remaining parts section].

Jan: They look like it. Where—down here [points to the Strip G line of the remaining parts section] or up there [points to Strip G diagram at top of page]?

Kelly: Over here [indicates the remaining parts section].

Jan: Okay. All right. Um, is there any way you can look up here and say for sure—yes, this [points at unmarked 80-box in Strip H diagram at top of page] equals this, this, this [points at each of the 44-, g, and 12-boxes in the Strip G diagram at the top of the page]. Is there any way you can tell me for sure?

Kelly: It doesn't.

Jan: It doesn't look equal.

Kelly: It looks like half of a g.

Jan: Um, why do you say that?

Kelly: Starts at the 44 [traces her finger from left side of the 44-box in the Strip G diagram at the top of the page down to the left side of the first 80-box in the Strip H diagram; moves her finger right until it stops on the right edge of the 80-box then traces her finger directly upward through the middle of the third g box on the Strip G diagram] and it’s just half of it

Based upon Kelly’s actions, it seems that Kelly was saying that the 80-box was equal in length to the 44-box and one and one half g boxes. It is possible that Kelly did not think of the 80-box as being equal in length to the 44-, g, and 12-boxes in the Strip G diagram because the 12-box was not adjacent to the first two of those boxes (Hahn & Kim, 1999). The fact that Kelly could not affirm that the combination of the 44-, g, and 12-boxes were equal in length to the 80-box is indication that she did not have an understanding of the logical necessity of the equivalence of the quantities represented by the unmatched remaining parts. In the excerpt that
follows, I tried to draw Kelly’s attention to the steps that she had taken, thus enabling her, possibly, to deduce that the remaining parts were equivalent.

Jan: What if I were to tell you, okay, do you, would you agree that these two strips [taps both Strip G and Strip H diagrams], individually, like this strip [creates a finger frame around Strip H diagram] and this strip [creates a finger frame around Strip G diagram] equal each other? [Kelly responds nonverbally off camera.] Okay. Would you agree that you took out from these strips equal parts [points to both diagrams in first Identical Parts section at bottom of the page]? Parts that were equal to each other?

Kelly: Um, not really. [Taps Strip G and H diagrams] Because I couldn't draw exactly.

Jan: Okay. Saying, you're saying not really?

Kelly: No, I didn't draw it exactly.

Jan: That's okay. Um, but by removing these right [points at two boxes on the Strip H diagram], would you say that when you removed the g here [indicates the third box in the Strip G diagram] and the g here [indicates the third box in the Strip H diagram], you were removing the same sized part?

Kelly: Well, yes and no because this has actually more parts [points at the Strip G diagram], but its smaller parts.

Kelly’s continued emphasis on the appearance of the diagram is further indication of her lack of understanding of the equivalence of the length of the unmatched parts (Petre & Green, 1993). Kelly did not seem to have the understanding that the equivalence of two additive quantities is invariant when those quantities are decomposed into equivalent parts. This may be an indication that Kelly, at this point, did not have an understanding of an additive quantity as a procept. I tried to focus Kelly’s attention on the identicalness of the parts that she removed from each diagram.

Jan: What I mean is, you took a g out here [points at third box in the Strip G diagram]. You took a g out here [points at third box in the Strip H diagram]. Are both of those g’s the same?

Kelly: Uh, yes.

Jan: You took out—I took out two of the g’s, you took out two of the g’s, and then you took out two 80s. Everything that was removed before what was left, was equal—is that correct—to each other? Um, is that enough to convince you that
what’s left is equal? [Kelly responds nonverbally off camera.] No. What will convince you that what was left was equal?

By reviewing with Kelly each set of identical parts that were removed from the diagrams, I was trying to engage Kelly in a deductive reasoning process whereby she might conclude that the remaining parts must necessarily equal each other in length. However, I made a faulty assumption in thinking that Kelly understood the necessity of the equivalence of the lengths of the remaining parts.

Kelly: If this was bigger [points at 80-box in remaining parts section] and these were bigger [points at 12-, g, and 44-boxes in the Remaining Parts section] and they were attached. If they were all attached to each other [traces pen across 12-, g, and 44-boxes in the Remaining Parts section, they would [inaudible]

Jan: So the way you drew it makes a difference. All right. Do you want to redraw these [points at the 44-, g, and 12-boxes in the Strip G diagram] so that they are lined up with this [points at the unmarked 80-box in Strip H diagram]?

Kelly: There's still places where they might turn up the same.

Jan: Okay, what would help if you were allowed to cut them?

Kelly: It might.

Jan: Okay. I'm going to go get some scissors, I'll be right back.

I suggested having Kelly redraw the two groups of remaining parts so that she could perhaps “see” that they were equal in length. The fact that Kelly did not wish to do so because “there's still places where they might turn up the same” caused me to realize that Kelly might need to physically manipulate the parts of the diagram so that she could establish that the remaining parts did in fact equal one another in length. However, I was not able to engage Kelly in this activity because the teaching session had to end. In the next teaching session, I gave Kelly a similar task (see Figure 4-76) and came prepared with a pair of scissors. I left the option open to Kelly as to whether she would like to use scissors to cut the diagram into parts.

After Kelly read the task and identified and drew the two sets of identical parts and the remaining parts, the following conversation ensued.
Jan: Okay. Now, what do you know—is there anything you know about these three parts [points to the three rectangles on the Strip G line of the remaining parts section], as compared to this part [points to the 70-box on the Strip H line of the remaining parts section]?

Kelly: Um, well, g could be um—can I just? [Kelly frames her fingers around what appears to be the second and third g boxes on the Strip G diagram (diagram is partially blocked from view) and then frames them around the 34-box on the Strip G diagram.] Um, well, one g could equal [inaudible] 34 maybe.

Figure 4-76: Kelly’s response to paper Strip Task #3 on 5/13/2008

Kelly’s decision to try to estimate the length of the g box is an indication that she had not yet developed the understanding that she could utilize the equivalence of two differently configured quantitative wholes in order to find the value of the unknown quantity as she did not seek out such a relationship.

Jan: Okay. Well let me ask you a question. Let me ask you a question…. If I put this part, this part, and this part together [points at the g, 34-, and 12-boxes on the Strip G line of the Remaining Parts section]. So this—I’m just going to put a dot.
This part, this, and this [puts a dot on the 34-box, second g box, and 12-box on the Strip G diagram]. Do you think, that compared to the 70 [traces a circle around second 70-box in Strip H diagram], it would be greater, less than, or equal to this?

Kelly: I think it would look kind of be equal to.

Jan: Why do you think that?

Kelly: ‘Cause these [Kelly creates a finger frame around the 34-box and the second g box on the Strip G diagram; she pulls this finger frame down to the left side of the second 70- box on the Strip H diagram] and plus this [Kelly creates a finger frame around the 12-box on the Strip G diagram; she pulls down this finger frame to the right side of the second 70- box on the Strip H diagram] kind of equals.

Jan: Okay, if they were equal, then what—how much would the g [points at the g box on the Strip G line of the Remaining Parts section] be worth?

Kelly: The g be worth? [Jan: Um hm.] Oh, I don't know. Hm. [inaudible; 12 second pause.]

Jan: Do you mind if we even cut it?

Because Kelly was able to identify the two groups of remaining parts as equal in length but was not able to utilize that information, I suggested that she cut the pieces so that she had the opportunity to line them up. As a result, Kelly might recognize the situation as a part-whole one and utilized one of her prior schemes to find the value of g.

Kelly: Oh no, it's okay …can I just—? [Kelly reaches for the calculator.]

Jan: You don't need to cut it? Okay. [Kelly: Sorry.] No, that's fine. [Kelly begins to use the calculator; Jan reads the entries on the calculator.] 70 minus 56 equals 14….. Can you tell me why you were doing 70 minus 56?

Kelly: Um. Because 34 plus 12 equals [reading buttons on the calculator as she presses them; pause] Oh. That is very weird. I [thought] I pushed—I must have had it wrong. Wow. It's 46 then. [Jan: Okay.] Sorry. 20. 70 minus 46 [reading buttons on the calculator as she presses them]. Oops. Four.

Jan: It said 24 before. I think you might have accidentally pushed the equal button twice. [Kelly: Oh.] Cause that said 24. Um. Can you please tell me why you subtracted 46 from 70?

Kelly: Because, you add these two [taps the 34- and 12-boxes on the Strip G diagram] and, then you—with their sum, you subtract it from 70. From the whole thing.
Jan: Okay. So, what do you th-, do you think you know what \( g \) equals at this point?

Kelly: 24 inches.

By adding 34 and 12 and subtracting that answer from 70 in order to find out the value of \( g \), Kelly demonstrated that she understood that 34, 12, and \( g \) were quantitative parts of the whole, 70. Earlier in this episode, Kelly had an inclination towards relying on her visual estimation of the lengths of each set of parts to determine their equivalence. As evidenced by her measuring or comparing two sets of boxes with her fingers. This is an indication that Kelly was not convinced of the logical necessity of the two sets of boxes that comprise the remaining parts of each diagram, are equal. It is only after I asked Kelly to focus on the equality of the lengths of the Remaining Parts that Kelly was able to utilize a non-visual method to find the value of \( g \).

The understanding that Kelly used to solve this task, then, seems to be situated in the context in which Kelly’s attention was drawn to the equality of the length of two differently configured diagrams representing equivalent quantities.

In the next teaching session, I gave Kelly a similar task (see Figure 4-77), however, after representing the Identical and Remaining Parts, she resorted to trying visually comparing the lengths of the Remaining Parts as a means of asserting their lack of equality in length.

Jan: So these pieces [points to the \( h \), \( h \), and 14-boxes on the Strip L line of the Remaining Parts diagram] belong to Strip L right? And this is Strip M [points at the 40-box on the Strip M line of the Remaining Parts diagram.] What do you know about, if you put this \( h \), \( h \), and 14 together [points to the \( h \), \( h \), and 14-box on the Strip L line of the Remaining Parts diagram], how two would that compare with 40 inches [points at the 40-box on the Strip M line of the Remaining Parts diagram]?

Kelly: I don’t think the other \( h \) [taps the two \( h \) boxes on the Strip L line of the Remaining Parts diagram] would fit in here [taps the 40-box on the Strip M line of the Remaining Parts diagram] too well.

Jan: So are you thinking that these three put together [taps the \( h \), \( h \), and 14-boxes on the Strip L line of the Remaining Parts diagram] would be longer than 40 inches? [Kelly responds off camera.] And why do you think that?
Kelly: Well, I can kind of measure it from here [in the Remaining Parts section, frames the 14-box with her fingers and pulls that finger frame down to the left side of the 40-box in that section] and this [creates a finger frame around the second \( h \) box in the Remaining Parts section and pulls the finger frame down to the right side of the 40-box] will take up all the other space, so you would be left with no space.

Figure 4-77: Kelly’s response to paper Strip Task #1 on 5/14/2008

Since Kelly was not convinced that the two sets of remaining parts would necessarily equal one another in length, I gave her a pair of scissors and another sheet of paper containing the two diagrams from Paper Strip Task #1. I asked Kelly to cut the pieces from each diagram and arrange them according to the way they appeared in the original diagrams. After Kelly did this, I asked her to rearrange the pieces in such a manner that the identical parts from each diagram were opposite one another. I asked Kelly to do this so that the remaining parts from each diagram
would be lined up with each other as well. After Kelly performed this task, she proceeded to subtract 14 from 40.

Jan:  [reading calculator as Kelly presses buttons] Forty minus 14 equals 26

Kelly:  I don’t think I can figure it out.

Jan:  How come?

Kelly:  Cause I was trying to see if 26 – 26 divided by 4 is. That’s 6.5

Jan:  Okay… what made you decide to divide 26 by four?

Kelly:  Because there was four h’s [indicates four h’s along top row of pieces]

It was not until I asked Kelly to remove from the table all identical pairs from the top and bottom paper strips that she made the decision to divide the difference between the two known remaining quantities by two, the number of remaining occurrences of the unknown quantity in one of the paper strips.

It is likely that Kelly wanted to divide the difference between the remaining known quantities by the total number of occurrences of the unknown quantity in one of the paper strips because of her prior work with comparing quantities. In previous tasks, Kelly divided the difference between the known parts of two equivalent quantitative wholes by the difference between the unknown parts of the same two wholes in order to find the value of the unknown quantity. Kelly may have mis-generalized this solution method to include all instances of the unknown quantity. By having to isolate the remaining parts, Kelly was able to focus only on the number of instances of the unknown quantity that remained.

Kelly’s misconception that the known difference between equivalent wholes should be divided by the number of occurrences of the unknown quantity in order to find the value of the unknown quantity persisted throughout several problems that she solved. For example, Kelly had been given Moveable Pieces Task #1 (see Figure 4-78) and instructed to cut and arrange the pieces of each diagram on a large piece of orange paper exactly as they appeared before they
were cut. Kelly was then asked to find the value of the unknown quantity. Kelly proceeded to divide the difference between the known quantities in each whole by the total number occurrences of the unknown quantity.

Jan: Okay. Now based on what you see here, can you possibly tell me the value of d?

Kelly: Can I use the calculator?

Jan: Yup. [Reading the buttons that Kelly pressed on the calculator] 60.

Kelly: [inaudible; presses clear button; reads the buttons that she presses on the calculator] 15 plus 15 plus 6 equals.

Jan: [reading the buttons that Kelly presses on the calculator] Divided by four equals 9.

Kelly: Maybe it’s 9 feet?

Jan: Now, why did you divide by four?

Kelly: To get the um, to—I added these up [points at three different pieces on the bottom arrangement]—and that was 36. So [inaudible]. It’s wrong.

Jan: Why do you say it’s wrong?

Kelly: Um, because this represents five of these here [points at large sheet holding both arrangements.] It’s still wrong.

Jan: What’d you say?

Kelly: It’s five of these here [punctuates finger over both arrangements]. I thought it was four. [inaudible]
In the above excerpt, Kelly stated that she divided 36, which was the sum of the three known quantities in the bottom paper strip, by five, the number of occurrences of the unknown quantity in both paper strips. Following this, Kelly attempted to total all of the known quantities in both paper strips and divide that total by five. When I asked Kelly why she did this, she answered, “Because I wanted to see if it would equal a number that I could divide by 5.” She indicated that she wanted to be able to divide by five because there were five instances of the unknown quantity in the problem.

Kelly’s suspension of her previously demonstrated part-whole understandings is puzzling. On no other occasion but this one did Kelly proceed to add together all of the known quantities and divide by the total number of unknown quantities. In the next excerpt, I attempted to help Kelly to refocus on the information that she already knew based upon the problem statement.
Jan: I see, all right. So what else do you know that might be able to help you? [Kelly pulls two pieces from the bottom arrangement and places them on top of one of the $d$ boxes at the bottom of the arrangement.] Now what are you thinking of doing there?

Kelly: See if these two equal the same size. So 15 and 6. 21. It’s 21 inches.

Jan: Well, test it out. Why don’t you plug in 21 for $d$ and see if you get the same amount on top and bottom? [Reading buttons on the calculator as Kelly presses them] 21 plus 21 plus 21 plus 15 plus 15 plus 6.

Kelly: [Reading buttons on the calculator as she presses them.] 21 times 3 equals 63. Plus 15 plus 15 plus 6 equals. It’s 99. So 21 plus 21 plus 60 equals. [Kelly presses more buttons on the calculator.] Yeah, so, off by like 2.

I asked Kelly to test out her theory that $d$ might equal 21 so that she could see that that was not the solution. Because Kelly was starting to focus on the appearance of the pieces of the diagram, I decided to pose the same task to Kelly in letter and number form (see Figure 4-79).

Jan: [Moves large sheet of paper containing the puzzle pieces; hands Kelly the sheet of paper in Figure 4-79] I’m gonna ask you to think of these two [points to Diagrams 1 and 2 on Paper Strip Task #1a]. Thinkin’ of these, this top number [creates a finger frame around Diagram 1] should equal the bottom number because we know these two lengths [points at the top and bottom paper strips on the large sheet of paper] are the same, right? So, if that’s the case, is there any way you can use these [points to Diagrams 1 and 2 on Page 2], this kind of picture or diagram to help you solve the problem? [Kelly: Um (shakes her head).] Okay. Do me a favor—will you circle on this paper, on this diagram –

Kelly: The twins?
Figure 4-79: Kelly’s Response to Paper Strip Task #1a—5/29/2008

I asked Kelly to circle the identical “twins” in Diagrams 1 and 2 so that she would be able to isolate those parts that remained.

Jan: Yes. [Kelly circles the first two $d$’s on Diagram 2] And remember, one’s top, one’s bottom. [Kelly begins to circle the first $d$ on Diagram 1; she then crosses this out and uses a different colored marker to circle the two $d$’s on Diagram 1.] Okay. So what matches with what? [Kelly: Huh]? What matches with what?

Kelly: This [points at Diagram 1 and then pulls her finger down to Diagram 2.]

Jan: Like, what matches this one? [Points to second $d$ on Diagram 2; Kelly points at second $d$ on Diagram 1.] Okay, great. Now, we know then that that’s on the side [waves away circled letters]. What does that mean for the rest of the numbers? [Waves hand above the right side of both Diagrams 1 and 2.]

Kelly: They equal something.

Jan: They equal something. What do they equal, do you know?

Kelly: A number that can’t possibly be the answer to $d$. 
Jan: Okay. What I mean is, um, do these numbers all put together [circles 15, $d$, 6, and 15 on Diagram 2] equal anything that you know of?

Kelly: 36?

Jan: And why do you say 36?

Kelly: 15 plus 15 plus six equals 36.

In this instance, Kelly seemed to be focusing only on the remaining known quantities in Diagram 2 and not on the remaining $d$. I attempted to bring this fact to her attention.

Jan: So that all equals 36, plus $d$ [points at the last $d$ on Diagram 2].

Kelly: Can I see something? [Kelly presses buttons on the calculator.]

Jan: Yup. [Reading from the calculator.] 36 divided by 3 equals 12.

Kelly: Can I try it out to see if this [scans finger above Diagram 2] –

Jan: Um. Quick question, what made you divide by 3? [Kelly: Hm?] What made you divide by 3?

Kelly: Because there’s three of these down here [points to each of the $d$’s on Diagram 2].

It is very puzzling why Kelly sought to divide 36 by three when, in previous tasks, she had demonstrated the understanding that the difference between known quantities contained within equivalent wholes is multiplicatively related to the additive difference between the unknown quantities. Because I did not want Kelly to continue to pursue a direction that was not going to be profitable, I asked Kelly to focus her attention on the relationship between the parts of the two equivalent wholes.

Jan: Got you. Before you try that, I just want to ask you one other question. The top—this number 60 [points at the 60 on Diagram 1] – and these four numbers [points at the 15, $d$, 6, and 15 on Diagram 2]. This $d$ represents some number. Now I’m gonna make a statement. I want you to just say, please, true, false, or not sure. 60 [points at the 60 on Diagram 1] is the same number as 15, $d$, 6, and 15 [points at the 15, $d$, 6, and 15 on Diagram 2]. Would you say true, false, or not sure?

Kelly: True.
Jan: True. Why is it true? Because, um, these all equal each other [traces up and down between the circled d’s on Diagrams 1 and 2] so, there would only be these four [circles the 15, d, 6, and 15 on Diagram 2 with her pen] on the diagram [points at the second assembled paper strip on the orange paper].

Jan: So if that’s true, then, um, can you use that fact to help you? [Kelly: Um hm; presses buttons on the calculator.] 60 minus 36 equals—

Kelly: 24

Jan: Okay, so what do you think?

Kelly: I think that d is 24.

Jan: All right. How sure are you of that?

Kelly: Very sure.

Jan: Very. You want to test it out? [Reading the buttons that Kelly pressed on the calculator.] 24 plus 24 plus 60 equals

Kelly: 108. [Reading the buttons that she presses on the calculator.] 24 plus 24 plus 24 plus 36 equals 108.

Jan: Nice job.

The fact that Kelly was able to ascertain the value of d so quickly after I asked her about the equivalence of the remaining quantities leads me to believe that she was not as seriously focused on the material at hand as was possible. Kelly had clearly demonstrated in the past that she understood that, after isolating both sets of equivalent quantities within equivalent wholes, the remaining quantities within each whole were equivalent. Therefore, Kelly’s repeated demonstration of the misconception whereby she would divide the difference between the known quantities by the total number of occurrences of the unknown quantity seems to indicate that either her previously successful performances on the remaining parts tasks were done by rote, or that she was not taking the latter tasks seriously. It is my conjecture that the latter explanation is the case given the fact that in this, as well as other cases, Kelly was able to quickly affirm the equivalence of the remaining parts and to use that equivalence in order to find the value of the unknown quantity.
During the next teaching session, I gave Kelly three hypothetical equivalence tasks. After Kelly represented all of the relationships in the problem statement while solving each of the first two tasks, I asked Kelly which diagrams represented equivalent quantities to encourage her to utilize that relationship of equivalence in order to find the value of the unknown quantity.

For example, when solving Hypothetical Equivalence Task #2 (see Figure 4-80), the following conversation ensued after Kelly had represented all of the quantitative relationships in the problem.

Jan: Does this [draws parentheses around “a a a a” on Line 1] equal this [draws parentheses around “a a + 6” on Line 3]?

Kelly: Yes, this amount [hovering pen above “a a + 6” on Line 3] does not change and this does [points at “a + 3” on Line 4].

Jan: Why do you say that these two equal each other [points at “a a a a” on Line 1 and points at “a + 3” on Line 4]?

Kelly: This amount stays the same [points at “a a a a” on Line 1] but three more people come to play rope [points at “a+3” on Line 4]

Jan: Okay….

Kelly: Because in the paragraph [points at the problem statement at the middle of the page], nobody else comes to play kickball.

Jan: Oh, okay, okay.

Kelly: People just come to play rope.

Jan: Okay.
During the above excerpt, Kelly stated that the two quantities represented on Lines 1 and 3 were equal to one another. She stated that this was true because no more people were added to kickball while three more people began jumping rope. It seems that Kelly understood that the number of people playing kickball could be represented in two different ways according to the problem statements and, since that number of people playing had not changed, the two quantities represented in Lines 1 and 3 were equivalent. After this interaction, Kelly used her identical parts scheme and multiplicative-additive scheme in order to find the value of the unknown quantity. While solving the third hypothetical equivalent task (see Figure 4-81), Kelly required very little intervention from me.
After reading the problem statement at the top of the page, Kelly wrote “a a a a a” next to “# of miles ridden by Jacob” on Line 1 and “a” next to “# of miles ridden by Robert” on Line 2. After reading the problem statement in the middle of the page, Kelly wrote “a+3” next to “# of miles ridden by Jacob” on Line 3.

Jan: Oh, um, Jacob rode six more miles.

Kelly: Oh sorry, I said three [changes the numeral 3 in “a+3” to the numeral 6; writes “a a a a a + 18” next to “# of miles ridden by Robert” on Line 4].

Jan: Question. What made you decide to put five a’s?

Kelly: Oops [crosses out the five a’s on Line 4; writes “a a a a a’]. Sorry.

Jan: No, that’s fine.

Kelly: Do the a’s equal – I’m getting the impression that the a’s equal nine.

Jan: And why is that?

Kelly: Because it says [scanning over the problem statement in the middle of the page with her pen] “three times” and then here it says “five times” [points at the problem statement at the top of the page] and then I wrote the three plus the 18 [points at the a’s and the 18 in “a a a a a + 18” on Line 4] because three times six equals 18 and there’s the a [points at the a in “a+6”on Line 3]. So, and then there’s two left [taps twice in the area of the a’s on Line 4.] Divide this [points at the 18 on Line 4] by two a’s [taps paper twice] equals nine.
In the above excerpt, Kelly demonstrated that she understood that she could utilize the equivalence of two differently configured quantities in order to find the value of the unknown quantity. She showed this by equating \( a a a + 18 \) with \( a a a a a \) and then solving for \( a \). Kelly did not seem to set these quantities equivalent to one another by rote given her response while solving the previous task, Hypothetical Equivalence Task #2. During that episode, Kelly had stated that since the number of people playing kickball hadn’t changed, the two diagrams representing the number of people playing kickball represented the same amount. If asked, it seems that Kelly would have used the same reasoning to state that Robert had not ridden any additional miles while Jacob had, therefore, \( a a a + 18 = a a a a a \).

Kelly’s understanding that establishing a relationship of equivalence between two quantitative wholes is useful in finding the value of the unknown quantity contained within one of
those wholes seemed to be held on a participatory level. That is, by the end of the series of tasks described above, Kelly was able to create a relationship of equivalence between two non-equivalent quantities according to the given problem condition and to utilize that relationship in order to find the value of the unknown quantity. However, throughout her work on these tasks, Kelly did not seem to anticipate that the creation of such a relationship of equivalence would be beneficial. Instead, attention had to be drawn to the potential equivalence of two quantities in order for Kelly to act.

There were several factors that seemed to contribute to the growth of Kelly’s understanding about the usefulness of equivalence. First, the Identical Parts Tasks seemed to help Kelly by enabling her to act upon those parts of each diagram that were equal in length so that she could isolate those parts that had no “twin” within the other whole. Since Kelly had a pre-experimental understanding that the removal of equivalent quantities from equivalent wholes maintains the equivalence of those two wholes, Kelly was able to develop the understanding that the remaining parts from each diagram were, collectively, equal to one another in length.

Further, since Kelly did not know to anticipate that she could create a relationship of hypothetical equivalence in order to find the value of the unknown quantity, the tasks which provided two places on the worksheet to represent the same quantity in two different ways seemed to be helpful. First, it required Kelly to represent a quantity in a way that was not prescribed by the problem statement. For example, by having to represent the number of people playing kickball in terms of the new number of people jumping rope when solving Hypothetical Equivalence Task #2, Kelly had to represent a situation as it was not occurring in reality. That is, the configuration of the number of people playing kickball did not change, but the way to represent that number did. Second, the structure of having two places to represent the same quantity enabled Kelly to be able to simultaneously attend to two representations of the same quantity and to act upon them based upon the equivalence of the quantities that they represented.
Finally, it seems that the tasks given to Kelly enabled her to build the understanding of the decomposability and commutativity of the parts within an additive quantity. By being able to physically cut and move around parts of each quantitative whole, it seems that Kelly was able to better understand the relationship between the remaining parts—that is, that those parts are equivalent to one another.

Although Kelly did not seem to be consistent in the demonstration of her understandings, she did seem to know that once she was able to establish a relationship of equivalence, she could find the value of the unknown quantity.

**Denise’s Story**

At the beginning of the study, Denise showed that she understood that when two wholes are equal to one another, the equality of those wholes is maintained after the removal of equivalent quantities from each of those wholes. This understanding is demonstrated by Denise’s response to item 16 on the pre-assessment (see Figure 4-82).

Jan: Now I see you indicated two circles [points to Denise’s written response to the assessment item]. Can you tell me how you got that?

Denise: What I did was, since they’re balanced exactly and a triangle equals three of them [points to problem statement where it is shown that a triangle balances three circles], I subtracted three of these [points to the last three circles on the right side of the scale] which will, that will bring you four [traces across the first four circles on the right side of the scale]. And there were two of the squares [points to the two squares on the left side of the scale], so I divided four into two and I got two.

Denise equated the three circles with a triangle. After she “subtracted” the last three circles from the right side of the scale, she presumably, mentally removed the triangle from the left side of the scale. Following this mental removal, two squares would have remained on the left side of the scales and four circles on the right side. Denise stated that she divided “four into
two and …got two.” I take this statement to mean that Denise divided four by two and thus found out that one square was equal to two circles. Denise’s actions seem to indicate that she was operating under the assumption that the quantities that remained within each whole after the removal of equivalent quantities from both wholes were themselves equivalent.

Figure 4-82: Denise’s Response to Item 16 on the Pre-Assessment

As with Kelly, I gave Denise the Jenna-Harold task in order to assess her understanding of hypothetical equivalence (see Figure 4-83). Denise experienced a tremendous amount of difficulty when trying to solve this task. At first, Denise did not draw all of the boxes the same size when representing Jenna’s amount of money. I asked Denise to redraw Jenna’s diagram so that they would all be the same size. I did this so that Denise would have a better opportunity to visualize any implicit relationships contained within the problem statement as mentioned by Larkin & Simon (1987). After representing Jenna and Harold’s original amount of money, Denise guessed that the value of the unknown quantity could be 5. She then drew three boxes to
represent each person’s additional $45. When asked why she drew three boxes, Denise stated that she was wavering between thinking that one box could have represented five or it could have represented 15.

Figure 4-83: Part 1 of Denise’s response to the Jenna-Harold Task on 4/21/2008

Feeling that she made a mistake in her diagram, Denise attempted to draw another diagram that would accommodate her idea that the value of the unknown quantity could be five. After Denise expressed her confusion the second time, I asked her to redraw the problem conditions on the paper as shown in Figure 4-84. Denise drew a long rectangle next to “Jenna’s $”, subdivided this rectangle into four equal parts, and then added a fifth box to the right edge of the last box. She drew a rectangle that is equal in length to the first rectangle on Jenna’s diagram next to “Harold’s $.” At my request, Denise then drew representations for the additional $45 for each person.

Jan: Okay, now we have—let’s see if we have the situation here. Jenna had five times as much so you drew 1, 2, 3, 4, 5 [points to each of the first five boxes in Jenna’s diagram] and that was one [points to the first box in Harold’s diagram]. After Jenna and Harold both earned $45.00 shoveling snow, Jenna had three times as much money [points to Jenna’s diagram] as Harold [points to Harold’s diagram]. Now, how could you show the three times as much money as Harold?
Denise: I can get this is worth [points to fifth box in Jenna’s diagram] and what this is worth [points to the 45-box in Jenna’s diagram]. So 45, 3 goes into that, that goes into that. That would make it 15.

Jan: Well where does the number 3 come from?

Denise: Three times as much.

Jan: Oh, oh, oh—I got you. I got you. Now remember, this whole thing [frames Jenna’s diagram with her fingers] has to be three times this whole thing [frames Harold’s diagram with her fingers].

Denise: Oh.

Jan: Including the 45, yup.

Denise: [Inaudible; points to first box in Harold’s diagram.] I say this was worth five [moves pen along 2nd through 4th boxes in Jenna’s diagram] I can say it’s worth fifteen [points to first box in Harold’s diagram].

Jan: Okay.

Denise: So this [points to the 45-box in Harold’s diagram] will mean 60 and then three times 60 – can I work right here [points to the bottom of the paper]?

Jan: Sure, absolutely.

Figure 4-84: Part 3 of Denise’s response to the Jenna-Harold Task on 4/21/2008
For a moment, Denise ascribed the number five to the value of one unit box. She then appeared to change her mind and assumed that each box was worth 15 as evidenced by her statements “I can say it’s worth fifteen” when pointing at Harold’s first box and “this will mean 60” after pointing to the 45-box in Harold’s diagram.

Denise: I can use a calculator.
Jan: Yeah, that’s right.
Jan: So you did 60 times 3, uh-huh. [Denise pushes more buttons on the calculator]…Okay, you subtracted 180 minus 45. Okay.
Denise: [Inaudible] 27
Jan: Okay, so you divided 135 by 5 and got 27. What does that tell you?
Denise: That each box could be worth 27, so – [uses the calculator]
Jan: [Reading the buttons that Denise pushes and the calculator results.] Okay, so 27 plus 45 is 72, times 3 is 216, minus 45 is 171, divided by 5 is 34.2.
Denise: I’m kind of confused.

Denise appears to have multiplied 60 by three to find out how much money Jenna now had (perhaps because Jenna’s new amount was three times as much as Harold’s new amount). She then seemed to assume that Jenna had $180 altogether, thus, she subtracted 45 from 180 in order to find out the combined value of five unit boxes. Once Denise found this value, she divided the difference, 135, by five. Thus, Denise reached the conclusion that “each box could be worth 27.” Once she obtained that answer, Denise tried to verify it by comparing three times Harold’s amount with Jenna’s new amount. When these answers did not coincide, Denise became confused.

Denise’s work on the problem up to this point shows that she had a good understanding of the quantitative construction of each of Harold and Jenna’s amount of money. In particular, once she believed that she knew Jenna’s amount of money ($180), she engaged in the process of
“undoing” the operations used to construct Jenna’s amount of money to find the value of one box Herscovics & Linchevski, 1994). However, her work also shows that she did not anticipate the limitations of assigning a value to the unit box in an effort to find the value of the unit box. Once Denise “checked” her work, she saw that her solution method did not yield the correct answer.

Denise’s previous actions suggest that she did not spontaneously seek to find or create two quantitative wholes that were equivalent to each other in order to find the value of the unknown quantity. It was at this point that I introduced the idea of representing Jenna’s new amount of money by representing three times Harold’s new amount of money.

Jan: Okay, well actually, you’re doing a good job because you’re maintaining the relationships given here, so that’s good. Maybe we can talk about what three times [taps Harold’s diagram] Harold’s money would look like. Can you show me, if this is Harold’s money [frames Harold’s diagram with her fingers], can you show me what three times Harold’s – just underneath here – show me what three times Harold’s money would look like.

Denise: Do you want me to start right here [points to left side of paper]?

Jan: Sure. [Denise draws small rectangle, then attaches a larger one to its right. She writes the number 45 inside of the larger rectangle. Denise frames these two rectangles with her fingers. Denise then draws two long rectangles connected to the 45-box that are approximately equal in length the first two adjoined rectangles.]

Jan: Okay. Okay, great. Can you tell me about this drawing [taps Diagram 3] why you drew it that way?

Denise: Because I really don’t know what this number is [taps the first box in Diagram 3] and that’s the 45 [taps the 45-box in Diagram 3] and that altogether [frames the first two boxes of Diagram 3 with her fingers] makes one time as much and I did that two more times, so we get three altogether.

Denise correctly drew the representation of three times Harold’s new amount of money. Denise seemed to conceptualize Harold’s amount of money as an iterable unit as evidenced by the fact that, after drawing the first representation of Harold’s money, she drew two long rectangles which she did not subdivide until asked to do so.

Jan: Great. Got you. Now whenever we know a certain amount of what we’re drawing, I’d like us to label it. So on these two drawings [taps each of the
two long, unmarked boxes on Diagram 3], can you tell me what part of it we already know?

Denise: [draws a line down the left side of the first long, unmarked box in Diagram 3; inside of the right box created by this partition, Denise writes the number 45] The 45. [Denise repeats this action on the second long, unmarked box on Diagram 3.]

At this point in the teaching session, my goal was to bring to Denise’s attention the fact that three times Harold’s new amount of money was stated to equal Jenna’s new amount of money. As a result, I wanted to instruct Denise that she could compare the diagrams of those two quantities in order to find the value of the unknown quantity. Students in Ng and Lee’s (2009) study utilized this method of comparison as a part of their solution method when solving algebra word problems. They represented two unequal, additively related unknown quantities given in a problem situation with rectangles. A brace was drawn in the space where one rectangle extended past the other. This brace was utilized to indicate the difference between the two quantities.

Jan: Okay. Super. So this is three times what Harold has now. [Points to Diagram 3.] What are we told is three times what Harold has now?

Denise: What are we told?

Jan: Yeah. So after Jenna and Harold both earn $45.00 shoveling snow, Jenna has three times as much money as Harold [reading from problem statement]. So is there anywhere else on this paper [taps Page 2b] that – anything on this paper that tells us what this [taps the first box on Diagram 3] is equal to?

Although I was asking Denise to find a diagram on the page that represented a quantity that was equivalent to the one represented in Diagram 3, I only pointed to the first box of Diagram 3, and Denise seemingly mistook my question. She seemingly thought that I was asking if there was a quantity on the page that was equivalent to the unit box. Thus, Denise visually compared the unit box with the 45-box. I tried to then refocus her attention on the representations of the two quantitative wholes that were equal to one another.

Denise: 15?

Jan: 15. Why do you say 15?
Denise: Because I can cut this into thirds [points to the first 45-box on Diagram 3].

Jan: Okay, well let’s do this. Can we make a deal not to guess, even though I know the numbers are –?

Denise: It’s so difficult.

After this interchange, I asked Denise the following:

Jan: Okay. Is there any way you can use the diagram of Jenna’s money to help you figure out how much each box is worth? [Denise types in the number 45 on the calculator] Once again, without guessing. [Denise pushes the clear button on the calculator.] Um. All right, let me show you. If we put [frames Diagram 1 with her fingers] if we made [frames Diagram 1 with her fingers], okay, is this [points at the left edge of Diagram 1] equal to this [points at the left edge of Diagram 3]? Would you say yes or no?

Denise: Yes.

Jan: And why do you say yes?

Denise: Because this was the amount that Harold had [points at the first box on Diagram 2] and Jenna had five times as much [taps each of the first five boxes in Diagram 1], and that was one times as much [taps the first box in Diagram 1], so this and this [points at the first box on Diagram 3] are the same amount because this [taps the first box in Diagram 2] and that’s the 45, so this is the same amount [taps the first box in Diagram 3].

In Denise’s last statement in the above excerpt, it seems that she misunderstood my question to her about the equivalence of the quantities represented in Diagrams 1 and 3. Because I was pointing to the edge of both diagrams, Denise may have thought that I was asking her if the quantity represented by the unit box in Diagram 1 was equal to the quantity represented by the unit box in Diagram 3. However, it is not clear if Denise provided that explanation because she did not understand the question that I was asking. At this point in the teaching session, we had run out of time.

My work with Denise on this task led me to believe that she did not anticipate that she could utilize the equivalence of two quantitative wholes in order to find the value of the unknown quantity. I drew this conclusion from the fact that she did not spontaneously seem to search for
or try to establish relationships of equivalence between quantities. I also surmised this from the fact that she had a tendency towards using a numerical strategy—that of guessing and testing—to find the value of one box.

Because this task was meant to be an assessment of Denise’s understanding of hypothetical equivalence, I did not return to this problem with Denise until three teaching sessions later on 5/5/2008. In the excerpt below, I asked Denise to reason about the diagram that she had previously drawn.

Jan: Okay. So this was what you drew for Jenna [taps Diagram 1], Harold [taps Diagram 2] and then the three times as much as Harold [taps Diagram 3]. Can you interpret for me what this, what you might have meant when you drew it that way [moves finger along Diagram 3]?

Denise: This was the 45 [points to last box on Diagram 1], and [moves pen back and forth between fifth and sixth boxes on Diagram 1] it looks like the same size, and five times as much [points to first box on Diagram 1] and one times as much [points to first box on Diagram 2]. So that means that if you brought this down [traces down from right edge of first box in Diagram 1 to right edge of first box in Diagram 2] and put it in each place [taps the first 45-box on Diagram 3, then moves pen left to right along Diagram 3] and it would have been that, but I see you had me write it over somehow [circles Diagram 3 with her pen] and put it down here [taps first box of Diagram 3] as three of these [taps each of the 45-boxes in Diagram 3] with one little square and 45, one little square and 45, one little square and 45 [taps each box in Diagram 3 as she speaks].

Jan: Now do you think this [points at last box on Diagram 3] is an accurate portrayal of three times this [points at last box on Diagram 2]?

Denise: No.

Jan: I mean three times this [traces a circle around Diagram 2]. You say no. Why?

Denise: Because I don’t know what the 45 is and I don’t know how big to make it and it seems pretty difficult.

In this excerpt, Denise indicated she could not verify that Diagram 3 was an accurate portrayal of three times the quantity represented in Diagram 2. It does not seem that Denise’s concern was based on a lack of understanding of the concept of three times the amount of the
quantity represented in Diagram 2. Rather, Denise’s concern seemed to be that she had no way of knowing whether or not her original diagram even portrayed the relative sizes of the 45-box and the unit box.

Jan: I got you, I got you. All right, let me say this to you then. This [taps Diagram 3] is definitely an accurate portrayal of this [taps Diagram 2] in the sense that, even though we don’t know how much this is [circles first box of Diagram 2 with her pen], this may be [points at the 45-box in Diagram 2], maybe the 45 should be smaller than this box [points at the first box of Diagram 2], it’s hard to say. But whatever this is [circles first box of Diagram 2 with her pen] and whatever this is [circles 45-box of Diagram 2 with her pen], if you wanted to triple this [circles diagram 2 with her pen], then you just have to triple these [taps the unmarked box in Diagram 2, then taps each of the unmarked boxes in Diagram 3] and triple these [taps the 45-box in Diagram 2, then taps each of the 45-boxes in Diagram 3]. Now what do you think about what I just said?

Denise: That this is triple [points at Diagram 3].

Jan: Okay, why?

Denise: Because you just said if it was three times as much as these [taps the unmarked box in Diagram 2, then taps each of the unmarked boxes in Diagram 3] and three times as much as these [taps the 45-box in Diagram 2, then taps each of the 45-boxes in Diagram 3].

Denise’s response to my question about whether she viewed the quantity represented in Diagram 3 as representing three times the quantity in Diagram 2 was nearly a repetition of my words therefore, her response could not be taken as a demonstration of her understanding. However, in a previous teaching session she did give indication of this understanding. On 4/21/2008, when justifying her original creation of this drawing, Denise had stated:

Because I really don’t know what this number is [taps the first box in Diagram 3] and that’s the 45 [taps the 45-box in Diagram 3] and that altogether [frames the first two boxes of Diagram 3 with her fingers] makes one time as much and I did that two more times, so we get three altogether.

In both teaching sessions, Denise seems to have alluded to the fact that she was uncertain of the accuracy of her diagram of the quantity with the structure $x+45$ because she did not know the size of the unknown quantity relative to 45. In the excerpt that follows, Denise further
demonstrated that her perception of the accuracy of the diagram was a hindrance to her in solving the problem.

Jan: Okay, great. Now there’s one more thing the problem tells us. Can you read this last part right here? This section right here?

Denise: [reading] “When Jenna and Harold both earn $45.00 shoveling snow, Jenna had three times as much money as Harold did. Draw a diagram to represent the situation. Use your diagram to find out how much money each person started out with.”

Jan: Okay. Are there any two diagrams on this page that are equivalent to each other in your opinion?

Denise: Yes.

Jan: Which two?

Denise: Actually no. If I would drag this [frames Diagram 1 with her fingers] over here [moves her fingers to the left], I wouldn’t be sure how that would be lined up.

Jan: Okay, and what would you want to line this up with [taps last box of Diagram 1]?

Denise: This [points at the right edge of Diagram 3].

In this excerpt, I asked Denise if any two diagrams on the page were equivalent. Denise seemed to take my question to mean, “Are there any diagrams on the page that are equal in length?” I should have asked, instead, whether she thought that any two diagrams on the page represented equivalent quantities. It is interesting to note that Denise concluded that Diagrams 1 and 3 were not “equivalent” based solely upon the lack of internal fidelity of the diagrams and not at all based upon the relationship implied by the problem statement about those two quantities. This seems to indicate that either Denise was not aware of the equivalence between the two quantities represented or she did not know how to use that relationship of equivalence to find the value of the unknown quantity. Denise seems to further show a lack of understanding of the
equivalence of the quantities represented by Diagrams 1 and 3 when she suggests, in the next excerpt, that she add to her diagram in a manner that is not consistent with the problem statement.

[Denise puts her pen at the right edge of Diagram 1 and drags it diagonally to the right edge of Diagram 3.]

Denise: You know what I see right now?
Jan: What?
Denise: That this [points at the fifth box in Diagram 1] is three times right here [moves pen across the second through fourth boxes on Diagram 1], that this [points at the last box in Diagram 1] is like one of these [points at the last box in Diagram 3].

Jan: Okay.

Denise: So if I was to drag some, if I would have put another one right here [outlines the space next to Diagram 1 that is above the first two boxes of Diagram 3], then it would be equivalent to this [points at Diagram 1].

In this excerpt, Denise seemed to be searching for a way to make Diagrams 1 and 3 equal in length. She implied that, by adding a unit box and a 45-box to the left side of Diagram 1, the lengths of the two diagrams would become equal. Denise seemed to be basing this conclusion upon the fact that, in her diagram, the 45-box was the same length as three unit boxes. Using Denise’s logic, by adding two boxes to Diagram 1, all of the boxes on one diagram would be evenly matched with a box or group of boxes of equal length on the other diagram.

Denise’s proposed solution for making the diagrams equal in length suggests a few things. First, Denise understood that, in order to be able to properly represent that equivalence of two quantities, the lengths of the diagrams representing those quantities should be equal to each other. Next, Denise understood that parts within the diagram that were aligned and equal in length represented equivalent quantities.

However, Denise did not find it problematic that she did not maintain the original problem conditions as she sought to alter Diagram 1. She seemingly did not understand that, in order to be able to reason about the quantities represented in a diagram, the problem conditions
needed to be met. The fact that Denise did not anticipate this seems to indicate that she did not have an understanding that finding a relationship of equivalence would enable her to ascertain the value of the unknown quantity. Had she anticipated this, it seems, Denise would have sought to create two diagrams representing equivalent quantities based upon the problem conditions. With the understanding that the problem conditions had been met and the knowledge that she had represented two quantities that were equivalent, Denise would have been able to reason from the diagrams, even if one diagram was longer or shorter than the other.

During the next teaching session, it became my goal to give Denise a task where she had to find the value of the unknown quantity given two diagrams that were aligned and equal in length, but whose identical parts were not aligned. I wanted to determine whether Denise needed for the equal-sized parts of each diagram to be aligned in order for her to find the value of the unknown quantity. In particular, I wanted to ascertain Denise’s understanding of the commutativity of quantitative parts within quantitative wholes. But, I wanted to create a problem-solving environment in which the possibility of confusion related to size differences in the diagrams was eliminated.

In most of the previous tasks that Denise encountered, the lengths of the diagrams that Denise drew or that were provided to her were equal, the diagrams were aligned with one another, and most of the parts of the diagram that were equal in size to one another were aligned with each other (please see Figure 4-85 for an example). The remaining, unequal-sized parts of each diagram were also in alignment. Denise would find the value of the unknown quantity by ignoring the aligned, identical parts (as in the first box of each diagram in Figure 4-85) and focusing on those parts of each diagram that were aligned yet non-identical.
Figure 4-85: Denise’s response to Paper Strip Task #7 on 5/2/2008

*Reasoning from a diagram in which multiplicatively related parts are not aligned*

I gave Paper Strip Task #1 (see Figure 4-86) to Denise in order to assess the way in which she would solve a problem where the identical parts in each diagram are next to one another and aligned as in previous problems. Denise solved the problem by multiplicatively comparing the lengths of the two unit boxes in the Strip A diagram with the length of the 18-box in the Strip B diagram.

Denise: [reading] “… Find out the value of a, of a in this diagram.”

Jan: Yeah. a. And you know what I mean by a right?

Denise: Um hm. This. [Moves her pen underneath the four a boxes in the Strip A diagram.] This whole thing or just one? [Rests pen on the first a box in the Strip A diagram.]

Jan: Just one. Yep. [Denise: Oh, Okay. So; writes “18÷2” and “9” on the blank lines on either side of the equal sign underneath the Strip B diagram.] And what made you decide to do 18 divided by two?

Denise: I saw that this line right here [draws an imaginary line down the middle of the last box on the Strip B diagram], if you cut this line in half [retraces the imaginary line], cause it’s like that [traces from the left and right sides of the brace underneath the last box on the Strip B diagram to the middle of the brace]? So, if you do that, you can kind of see that this line [draws a line from underneath the left side of the last box on the Strip B diagram to the
middle of that box] is the same as this line [draws a line underneath the second $a$ box on the Strip B diagram].

Jan: I see.

Denise: So, half of this [draws an imaginary line down the middle of the last box on the Strip B diagram] would mean, um, an $a$. And $a$ is obviously 9 inches now that we know.

Figure 4-86: Denise’s response to Paper Strip Task #1 on 5/6/2008

In this episode, Denise used the equivalence of the lengths of second and third unit boxes in Diagram A and the 18-box in Diagram B to find the value of $a$. This is consistent with Denise’s prior mathematical actions through which she demonstrated that she understood that if $n$ boxes of length $a$ were equal in length to a box of length $b$, then $na=b$. However, when attempting to solve Paper Strip Task #2 on 5/6/2008 (see Figure 4-87), Denise did not seem to make use of this knowledge. This may be due to the fact that, unlike in previous instances when there were two quantitative wholes drawn, the parts that were multiplicatively related were not
aligned. Therefore, the parts of each whole that were equivalent to one another were not as easily perceptible as when the parts of each diagram representing equivalent quantities were aligned.

Denise: Suppose that the following diagram represents strips that someone cut out and taped together. Find the value of \( c \) in this diagram. So \( c \) is each one right [inaudible; points to 4th \( c \) box in the Strip C diagram]?

Jan: Yes. Um hm. [Denise draws a line underneath the third \( c \) box on the Strip C diagram; she frames this line with her fingers; she then draws a line underneath the 24-box that extends from the left side to the middle of that box, hesitates, and then completes the line to the right side of the 24-box; Next to “\( c = \)” at the bottom of the page, Denise writes “24÷4 = 6”] And how do you get six? Or, why did you divide by four, I should say.

Denise: I did, I did the same thing here [points to Strip B diagram from previous problem]. [Jan: Okay.] And I saw what that was [frames the line under the 6-box], I saw what this was [frames small portion underneath 24-box. [Jan: Um hm.] Except I should have made it a little bit longer [makes the line under the \( c \) box slightly wider in both directions]. But, if you count [sic] it in half [underneath the first 24-box in the Strip C diagram, draws a horizontal line to the middle of that box], you can see that two of these [points to \( c \) box with line underneath it] fit into half of this [points to 24-box with line underneath it]. [Jan: Okay] So if you had a whole, you would do 2 times and you got 4. [Jan: Okay.] So I divided 24 by 4 and I got 6 feet.

As in previous tasks of this nature, Denise searched for the number of times that a box of unknown length could fit into a box of known length so that she could find the length of the box of unknown length. In this case, however, I had designed the task so that the length of the \( c \) box was not multiplicatively related to any of the known lengths. My intent was to observe whether Denise would spontaneously choose to match those boxes in one diagram with the boxes that were equal in length in the other diagram in order to find the length of one box.

Jan: I got you. Now, what if I said to you that I created these \( c \)’s so that they’re a little bit off. They’re not quite what they look like. Um, and so your answer is very close. And instead of us, maybe, trying another number up or down, let me ask you a question. Is there any way, by looking at this diagram [points back and forth between the third \( c \) box in the Strip C diagram and the first \( c \) box in the Strip D diagram] and matching the things that you know are already equal with each other, you can find the value of \( c \)?
Towards the end of the excerpt above, I should have asked Denise if there was another method that she could use to find the value of \( c \). Instead, I unfortunately prescribed a solution method for Denise by asking her to match the boxes from each diagram that were equivalent in length. I did this to enable Denise to operate identically upon both representations of each quantitative whole in order to maintain the equivalence of those wholes. However, I do not think that Denise had that conception herself; rather it was suggested to her.

Denise: Looking at this one? [Denise points to the second \( c \) box on the Strip D diagram.]

Jan: Um hm. Like, looking at these two there [simultaneously points at the first \( c \) box in both diagrams]. So, these two [points back and forth between the first \( c \) box in both diagrams] equal the same amount—they’re the same length. [Denise: Um hm.] Is there anything about, in each of them [points back and forth between both diagrams] that are the same?

Denise: They have the same length and they both have the same value?

Jan: … That’s true. How about any parts of them that are the same?

Denise: The length of how they’re spaced? [Denise frames the first \( c \) box on the Strip D diagram with her fingers.]

Jan: Okay. Can you put a check mark inside each pair of boxes that’s the same on the top and the bottom? [Denise puts a check mark inside of the first \( c \) box in both diagrams; she checks the second \( c \) box in the Strip C diagram then points to the first 24-box in the Strip D diagram.]

I gave Denise the directions to “put a check mark inside each pair of boxes that’s the same on the top and the bottom” in order to enable her to “see” the boxes that remained once she matched all of the identical boxes in the Strip C and D diagrams. Denise was able to carry out these instructions in part in that she put a check mark inside of the first \( c \) box in each diagram. However, several times after checking the second \( c \) box in the Strip C diagram, Denise moved her pen towards the first 24-box in the Strip D, seemingly in an attempt to check the first 24-box in the D diagram. The last time that this occurred, I then tried to give more specific instructions.

Jan: Is there another box down there that is the same, exactly the same size—that has the length of \( c \)? [Denise checks the second \( c \) box on the Strip D
Okay. Are there any other lengths that are exactly the same – top and bottom? [Denise checks the third c box on the Strip C diagram.] Okay, now, do you see another c down at the bottom that’s already written as a c? [Denise: No.] Okay, so maybe we can’t check that one off. But there is definitely something else you can check off. Do you see anything else?

By continuing to move her pen towards the first 24-box in the Strip D diagram in response to my request that she find a “twin” for the second c box in the Strip C diagram, it may be that Denise was thinking of the c box as a part of the 24-box in Strip D. Or, it may have been the case that Denise thought of herself as checking the same number of boxes in the top and bottom diagrams [since the 24-box in the Strip D diagram was the second box of that diagram]. In either case, it seems clear that Denise was not aware that an assumption of maintaining equivalence between quantities was being made.

Denise: This one. [Denise points to the third c box in the Strip C diagram.]

Jan: Let’s see. All right. So here we have c, c [points at the first c box in both diagrams], c, c [points at the second c box in both diagrams]. These, there are two c’s here [points back and forth between the third and fourth c boxes in the Strip C diagram] but I’m thinking one up top, one at the bottom [points to a location on the left side of both diagrams that is blocked from view] – you know what I mean? So I’m saying, is there another c down here [points at the second c box on the Strip D diagram] that’s not checked and there isn’t. Are there any more boxes—a pair of boxes – one on top, one on the bottom, that’s the same. None. What do you think about the 24?

Denise: Yes

Jan: Okay, can you check those off? [Denise checks the first 24-box in both diagrams; she then checks the second 24-boxes in both diagrams.] Now, because there was not another c here [points at the 4th c box in the Strip C diagram], down here [taps the Strip D diagram], could you cross out that check [points at the 4th c box in the Strip C diagram; Denise crosses of the check in that box] – would it be okay to do that? All right, what’s left?

Denise: Uh, two c’s and a 14.

Jan: Okay. Does that help?

Denise: Yes.

Jan: All right. What does that tell you that c equals?
Denise: 7.

Although Denise did not appear to understand the rationale behind the matching of identical boxes within the equivalent wholes, it does seem as though Denise understood that, once matching parts of each whole had been eliminated, the remaining parts of each whole were equal to one another.

Jan: Good. How do you know?

Denise: Because I saw that 14 [points to the 14-box] was almost the same size because this [underlines the third c box on Strip C diagram], which is like this [underlines the second c box on the Strip D diagram]. [Jan: Um hm.] So, if I wrote this [draws a line underneath part of the second 24-box on the Strip D diagram] and this [draws two lines, one right above the other, underneath the 14-box] which are apparently the same size.

Jan: Okay, so you did it by sight.

Denise: Yes and I also did it because if there are two of those [Jan: Um hm] left out of all the whole thing and there’s only one of these left [points at the 14-box], [Jan: Um hm] then there’s obviously going to be two of these that equal one of that. [Jan: Gotcha.] And if you divide 14 by 2 you’d get 7.
Because Denise had not seemed to fully understand the instructions that I had given her while solving Paper Strip Task #2 during the previous teaching session, I gave Denise the Identical Parts tasks, a series of tasks that had the same instructions, but that entailed more structure than the Paper Strip Tasks. Denise was asked to find the “twins” contained within each whole and draw those twins in a separate part of the paper. My goal was that, as a result of isolating those parts of each whole that were identical, Denise would be able to isolate and focus on the equivalence of the remaining parts.

Denise was given Paper Strip Task #1 (see Figure 4-88) during the next teaching session. She was asked to check off those parts of both diagrams that were identical, represent them in the “Identical Parts” sections at the bottom of the page, and then to find the remaining parts and draw
them at the bottom of the page. After Denise accomplished this, I drew Denise’s attention to the “Remaining Parts” section of the paper.

Jan:  Okay. Now I have a question for you. This is great [points to the Remaining Parts section]. What do you know about these three parts here [points to each of the three boxes that she drew on the Strip G line of the Remaining Parts section] in relation to this part here [taps the 80-box that she just drew on the Strip H line of the Remaining Parts section]?

Denise:  Um. That, if this was longer [points to the 80-box on the Strip H line of the Remaining Parts section], they would equal the same size. [Rests pen cap on the second 80-box of the Strip H diagram.] They could equal the same size.

Jan:  Why could they?

Denise:  Can I see the calculator?

Jan:  Sure.

Denise:  To see what would happen. [Denise begins to use the calculator.] 34 plus 24 [inaudible] Eighty minus 46. 34. [11 second pause]

Jan:  What are you thinking?

Denise:  That the g. That they’re the same size. If you add these two up [points to the 34- and 12-boxes in the Remaining Parts section] and you subtract it from this [points to the 80-box in the Remaining Parts section], then you could find out what this is [points to the g box on the Remaining Parts section].
In this excerpt, Denise was asked about the relationship between the $g$, 34, and 12-boxes with that of the 80-box. Even though Denise originally drew the 80-box so that it did not appear to be equal in length to the lengths of the $g$-, 34-, and 12-boxes combined, Denise treated the two sets of quantities as if they were equal in order to solve the problem. Thus, her mathematical reasoning was not affected by the perceptual cues given by the diagram (Arcavi, 1999). It may be the case that Denise thought that the two wholes in the Remaining Parts section were supposed to be equal in length given the fact that these were the only parts left from each whole and because each of the two “Identical Parts” columns had the "same thing." This represented progress for Denise in that she did not rely on the accuracy of her drawing in order to assume the equivalence of the quantities represented. Given that supposition, Denise was able to accurately find the value of the $g$ box by subtracting 34 and 12 from 80.

Jan: Okay. So what is $g$ equal to?
Denise: Um, I think it was equal to [pushes a button on the calculator] 80. It was like, I think it was 34, I’m not sure, but I’ll check. [Denise pushes buttons on the calculator.] Yeah.

Jan: [Reads the buttons that Denise just pushed on the calculator.] 80 minus 46 is 34. Okay. Thank you. Would you label where the 34 would go [Denise draws a bracket above the g box on the Remaining Parts section above which she writes “34”.] Okay, is this diagram accurate in relation to this one, do you think?

Denise: It’s not accurate in size, but it should be longer as opposed to this. [Draws a box underneath the 80-box in the Remaining Parts section. This box is longer than the 80-box.] Cause I saw what I did here [points to the second Identical Parts section] then I just [inaudible].

Jan: Oh, okay. Thank you….And one more question. Would you please label this length [points to the last box that Denise drew]? Or, does this [points to the words “80 in”] belong to both [points to both 80-boxes in the Remaining Parts section]?

Denise: This is 80 [writes “80 in” underneath the last box that she drew]. That’s [inaudible; crosses out the shorter 80-box in the Remaining Parts section].

Denise’s performance on this task was different from her performance on Paper Strip Task #2 on 5/2/2008. In the earlier task, the remaining parts from each equivalent whole were not visually isolated. Therefore, Denise had to, within the construct of the diagram, mentally isolate and compare the known and unknown quantities that were equal to one another. At first, the idea of making this comparison was not readily apparent to Denise. This is evidenced both by her initial attempt to estimate the length of the c box relative to the 24-box and her lack of surety in how to proceed when checking the boxes that were “twins.” After having checked the identical boxes in both diagrams when solving Paper Strip Task #1 on 5/6/2008 and drawing them at the bottom of the page, Denise identified the remaining parts that were equal to each other and drew them at the bottom of the page. Isolating the Remaining Parts within their own diagram seems to have been one of the main factors that enabled Denise to solve the task successfully.

One of the main differences between Paper Strip Task #2 on 5/6/2008 and Paper Strip Task #1 on 5/8/2008 is the fact that the latter task asked Denise to explicitly identify and isolate
the remaining parts. This seemed to enable Denise to build from her already extant knowledge
that when a known quantity is equivalent to multiple occurrences of the unknown quantity, there
is a multiplicative constraint.

In order to assist Denise in identifying unequal quantities in a problem situation that had
a potential of being equalized, I assigned her “hypothetical equivalence tasks” during the next
two teaching sessions. I structured the problem sheet in such a manner as to enable Denise to
represent the same quantity in two different ways. My assumption, based upon her prior
mathematical activity, was that Denise would not spontaneously look for and create two such
diagrams that represented an equalization of unequal quantities. I did this so that she could later
compare the two representations of that quantity in order to find the value of the unknown
quantity. Because of the complex nature of this task,

Denise’s work on this task is shown in Figure 4-89. After reading the question aloud,
Denise drew one box next to words “Craig’s amount of money” on Line 2. She drew five
connected rectangles next to the words “Jenna’s amount of money” on Line 1. She then drew a
second box connected to the first box on Craig’s diagram on Line 2 and labeled it $15. The
excerpt below shows the continuation of Denise’s work on the Jenna-Craig task.

Jan: Okay, super. Great, now, would you please read the next part?

Denise: [reading] “Jenna now has three times as much money as Craig. Draw a
diagram of this situation.” I know that she has three times as much more
than. [Jan: Um hm.] And, if this goes down [pulls pen down from right edge
of second box in Jenna’s diagram to left edge of second box in Craig’s
diagram; pulls pen down from right edge of fourth box in Jenna’s diagram to
space in Craig’s diagram] and that was longer—[pulls pen from left edge of
fifth box in Jenna’s diagram to the right, past the edge of Jenna’s diagram].

Jan: … Um, and I saw you just now doing something like this [mimics Denise’s
action of pulling her pen down from Diagram 1 to Craig’s]. What were you
thinking?

Denise: Um, that, when that comes down [pulls fingers down from right edge of the
first box in Diagram 1 and up from the right edge of the first box in Craig’s
diagram until they meet in a pinching position in between both diagrams] –
one times as much, three times as much [repeats same pinching action in space beneath right edge of third box Diagram 1] – cause it’s a whole [frames first two boxes in Craig’s diagram with her fingers]. That’s one that he got, unless you’re talking about that 15 [points to the 15-box in Diagram 2].

Jan: Okay. So, um, what would be three times what [points to entire diagram] – can you show me what you are indicating here?

Denise: If this was like this. [Denise draws a rectangle attached to the 5th box on Diagram 1; Jan: Um hm.] And this would come down. [Denise draws a line extending from the right edge of the second rectangle in Diagram 1 to the right edge of the second rectangle in Craig’s diagram; Jan: I see.] And this would come down. [Denise draws a line extending from the right edge of the fourth rectangle in Diagram 1 to the right edge of the fourth rectangle in Craig’s diagram; Jan: Um hm.] And this would come down [Denise draws a line extending from the right edge of the last rectangle that she drew to the space above Craig’s diagram].

Figure 4-89: Part 1 of Denise’s response to the Jenna-Craig Task on 5/12/2008

It seems that Denise was trying to establish the way in which she could alter Jenna’s diagram so that it appeared to be three times as long as Craig’s diagram in order to follow the
instructions to “draw a diagram of the situation in which Jenna now has three times as much money as Craig.” This probably occurred because I had not yet instructed Denise to create a separate diagram in order to show this relationship. But, the fact that Denise was willing to operate upon her original diagram of Jenna’s amount of money seems to indicate that she was not anticipating creating two different diagrams as a means of finding the unknown quantity. Alternatively, Denise did not anticipate that by altering the diagram which showed Jenna’s amount of money in relation to Craig’s original amount of money, she would lose the ability to compare this diagram with a different diagram showing Jenna’s amount of money. I then asked Denise to redraw the scenario with the new information at the bottom of the page. Denise drew two rectangles next to “Craig’s amount of money” on Line 4. She wrote $15 on the inside of the second rectangle.

Jan: And then based on that, please show Jenna’s money [Jan points to space next to “Jenna’s amount of money” at bottom of page] based on three times that. [Next to “Jenna’s amount of money” at the bottom of the page, Denise draws three connected rectangles, each of which is the approximate size of the diagram for Craig’s money that she just drew.]

Denise: One of the, it looks like one is a whole [traces second long box on Jenna’s diagram at the bottom of page]. And it’s still five times as much [waves pen above Jenna’s diagram at top of page], it could be one [draws a line inside of first box on Jenna’s diagram at bottom of page]. It could be one [draws a line inside of second box on Jenna’s diagram at bottom of page], two [draws a line inside of third box on Jenna’s diagram at bottom of page]. So, wait – one, two, three, four, five. [moving her pen across Jenna’s diagram at bottom of page; inaudible]. Do you have any scrap paper?

Jan: Yup [hands Denise a piece of paper].

It seems that in the above excerpt, Denise was looking for a way to equate the length of Jenna’s diagram at the bottom of the page with the length of Jenna’s diagram at the top of the page when she said, “It looks like one is a whole. And it’s still five times as much.” This represents growth on Denise’s part in the sense that, in a previous problem, she was not able to reason about both two diagrams representing the same quantity when one diagram was shorter
than the other. When Denise received the extra sheet of paper that she asked for (see Figure 4-90), she represented the quantities in Diagrams 1 and 4 again, but in such a way that their diagrams were equal to one another in length.

Denise: [working on Page 2] If this was Craig’s amount [writes “C:” and draws a rectangle next to it. Denise then draws a vertical line inside of this rectangle and writes the number 15 inside of the right side of this rectangle.]

Jan: Okay.

Denise: And this was three times as much [draws three rectangles connected together. Each rectangle is approximately the size of the rectangle that she drew for Craig’s amount (before it was subdivided)]. But, if it didn’t have the circles [Jan: Um hm], like, you know, it didn’t have the lines. [Jan: Yeah; Denise draws one long rectangle above the three rectangles (see Diagram 2)] You can put it in fives. So [Jan: Okay], I’ll just draw five lines. So, one, two, three, four [counting out loud as she draws each of four evenly spaced vertical lines inside of the long box that she just drew] – and then you have one, two, three, four, five [starting from right to left, puts a dot inside each of the five rectangles that were created inside of the long box]

Jan: Interesting, interesting.

Denise: It’s still three times as much [waves pen above Diagram 1].

Jan: I see. Um, and so, um, what made you think of drawing five boxes up here? [Jan taps the fifth box of the five connected boxes at the top of the page.]

Denise: So that I can—‘Cause, it’s five times as much [moves pen along five boxes at top of page] and I had to see, ‘cause it was only three times as much [moves pen along three connected boxes underneath the five boxes; Jan: Gotcha], to see how much that five would be worth—see, it’s not exactly on the line [points to three of the edges of the five boxes]
Denise’s work in the above excerpt shows that she understood that when two diagrams represent equivalent quantitative wholes, the representation of the parts within each of the wholes that are equal to each other, do not have to be aligned. At my request, Denise labeled the boxes in Diagram 1. She used the letter O (for “Original”) to mark the unit boxes. After Denise finished labeling her representations, I asked her to find the parts of each diagram that matched parts of the other diagram. After doing so, Denise represented the remaining parts.

Denise: The 15 [writes the number 15 in a space below and to the right of Diagram 1] which is the second line, so line b [writes the letter b to the left of the 15 that she just wrote] 15. It’s a 15 [draws a box around the number 15 that she just wrote]. Another 15 [draws a box connected to the first 15-box; writes the number 15 inside]. Another 15 [draws a box connected to the second 15-box; writes the number 15 inside]. And then it’s two more zeroes [draws two boxes and writes the numeral 0 on the inside of each].

Jan: Okay. Is there anything that we can say about these two zeroes [I used Denise’s language; points to the last two boxes just drawn]. Are these from right here? [Points to the two unchecked “O” boxes on first line of Sheet 2.]

Denise: Yes.
Jan: Is there anything you can say about the relationship between these three 15’s and these two zeroes?

Denise: These two zeroes [points to the zero boxes in the remaining parts diagram] look like the fifteens [points to the first two 15-boxes in the remaining parts diagram]. They equal, and it looks like it equals fifteen.

Jan: Okay, and how do you—? Okay—equals one fifteen or all of them together?

Denise: One, um—Well, it equals two [points back and forth between the two zero boxes; Jan: Okay], so one box could be worth 15.

Jan: Okay. Now let’s see – what’s left up here? This is the original amount of money, right? [Jan points to the last O box in Diagram 2 on Sheet 2; Denise: Um hm.] And there are two O’s left up there right? [Denise: Yes.] And down at the bottom, there are three fifteens left. Are those two amounts equal to each other?

Denise: Yes.

Jan: How do you know?

Denise: Well, actually, I don’t honestly know.

In this excerpt, it does not seem that Denise understood the logical necessity of the remaining parts having to equal one another. Therefore, I drew a diagram for Denise at the top of the page (see Diagram 4 in Figure 4-91) in order for her to figure out the way that having such a diagram would be helpful in finding the value of the unknown quantity.

Jan: Okay. And so, let’s say that you knew, this [places a check mark inside of the first box of four box diagram] equaled this [places a check mark inside of first box of Diagram 4b] and this [places a check mark inside of third box of four box diagram] equaled this [places a check mark inside of second box of Diagram 4b]. Like, they were all originally drawn. Whatever’s left – the two things have to equal each other. Um, do you agree, disagree with that statement, or not sure.

Denise: I agree and I think I see what you’re trying to say. [Jan: Okay.] I think two of these [points to the two 9-boxes in the four box diagram] equal this [points to the third box in Diagram 4b].

Jan: Okay. And why is that?

Denise: Because it’s this left and this left and this left and this left [points to each of the first checked boxes and the second checked boxes in both diagrams]. I see that if you scrunch this over a little bit [frames fingers around second box
of Diagram 4b and pulls them together; Jan: Oh, okay], those are the same [points to each of the second checked boxes in both diagrams] and those are the same [points to each of the first checked boxes in both diagrams; Jan: Um hm]. Put these together [frames the left side of the first 9-box and the right side of the second 9-boxes with her fingers and pulls her fingers together] and that equals that [frames the unmarked box on Diagram 4b].

In this excerpt, Denise demonstrated that she was able to think dynamically (Dreyfus, 1991) while reasoning about the parts in Diagrams 4a and 4b. In this episode, Denise mentally “scrunches” the second box in Diagram 4a, seemingly, so that it is directly above the second box in Diagram 4b. She also seems to mentally “put together” the two 9-boxes in Diagram 4a so that they could be aligned with the unmarked box in Diagram 4a. This mental movement of the parts of the diagram is evidence that Denise became aware of the commutativity of the quantitative parts of an additive whole. This type of reasoning seemed to assist her when I asked her to return to the original task.

Jan: Great, okay. So, based on that, what could happen here in this diagram [traces an imaginary circle around Diagrams 3a and 3b with her pen cap] – to show what up here [points to last O box on Diagram 2] equals what here [points to last 15-box on Diagram 1]?

Denise: So [on Page 3 (see Figure 4-91), Denise draws a rectangle which she subdivides into two unequal parts; she writes the number 15 inside of the part created on the right; Denise draws, subdivides, and labels two more sets of boxes like this which are connected to the original rectangle]. I’m doing one of these but [points to first subdivided rectangle (composed of an unmarked box and a 15-box)]—

Jan: Okay.

Denise: Up here [draws a long rectangle above the set of rectangles that she just created; this long rectangle is equal in length to the others combined; she subdivides this rectangle into five equal sized parts]. So if this [places a check mark inside of the first unmarked box of Diagram 1 on Sheet 3] is this [places a check mark inside of the first box of Diagram 2 on Sheet 3; Jan: Um hm]. This [places a check mark inside of the second unmarked box of Diagram 1], supposedly, equals this [places a check mark inside of the second box of Diagram 2].

Jan: I understand. Great, okay.
Denise: Um, and this [places a check mark inside of the third unmarked box of Diagram 1] equals this [places a check mark inside of the third box of Diagram 2; Jan: Okay]. You still have these left [moves pen along bottom of Diagram 1; Jan: Um hm]. These three [places fingers on left side of first 15-box and right side of last 15-box and pulls fingers together; taps each of the 15-boxes from right to left, then left to right; writes 15 x 3 vertically on the paper]. I think I know the answer in my head, I’m just checking it.

Jan: Okay sure.

Denise: I think it’s 45. [solves 15x3 problem] Okay, it’s 45. [Jan: Okay.] Um, you have one, two, three [taps each of the 15-boxes on Diagram 1] could equal 15 [on the first unchecked box of Diagram 2, writes the number 15], 15 [writes a 1 in the far right of this same box and a 5 in the far left side of the second unchecked box], 15 [writes the number 15 inside the second unchecked box] up here [circles the three 15s that she just wrote]. This would [inaudible] right here.

Figure 4-91: Part 3 of Denise’s response to the Jenna-Craig Task on 5/12/2008
In this excerpt, after identifying the identical boxes in Diagrams 1 and 2, Denise dynamically squeezed together the three 15-boxes together by pinching her fingers. She then wrote the number 15 in three locations across the last two unit boxes of Diagram 2. This shows that Denise conceptualized the combination of the three 15-boxes as being equivalent in length to two unmarked boxes. Interestingly, Denise did not hesitate to make this connection even though the combined length of the two unit boxes in Diagram 2 appears to be shorter than the combined length of the three 15-boxes in Diagram 1. At this point, Denise seemed to understand that the quantitative parts of additive quantities are commutative and that, once equivalent quantities have been matched, the additive combination of quantities that remain in each whole are equal to one another.

After the above excerpt, I asked Denise if she could use this information to help her to find the value of one box. Surprisingly, Denise then tried to figure out how to make the diagrams of the remaining parts equal to each other by cutting the size of certain boxes (as can be seen in the fourth box of Diagram 1). Although Denise understood that the group of remaining parts from each diagram should equal one another in length, she did not seem to feel comfortable reasoning about the quantities represented by those boxes unless the diagrams were equal in length. Therefore, I drew, for Denise, two diagrams which depicted the equality of the quantities represented by the remaining parts in each diagram (see Diagrams 3 and 4 in Figure 4-91).

Jan: Would you say that’s an accurate depiction of what’s happening up here? [Jan points back and forth between the last boxes of Diagram 2 and Diagram 1; Denise: Yes.] Okay. All right. So based on that, can you use that info to help you find out what one of these [points to the first O box in Diagram 3] original boxes is? And here’s a calculator if you need one.

Denise: That helps. So, [begins to use calculator].

Jan: 15 times three. [Denise: I got 45.] Okay. [Denise: But.] Divided by two equals [Denise says something unclear]…

Denise: 22.5…. 
While solving the Jenna-Craig task, Denise demonstrated that she understood that, after the equivalent parts of quantitative wholes have been matched, the remaining parts of those wholes are equivalent. More specifically, when Denise imagined moving the remaining pieces of each diagram dynamically, she could affirm the equivalence of the quantities represented by the remaining pieces. Although Denise understood this, she found it difficult to reason from the diagrams of those remaining quantities if the diagrams were not equal in length.

It seems possible that Denise needed a highly structured diagrammatic environment in order to successfully solve a remaining parts problem at this point in her trajectory. In the identical parts tasks, Denise was presented with pre-drawn diagrams. In those tasks where there was no pre-drawn diagram, Denise did not become successful in completing the task until I provided her with a diagram that showed the equal lengths of the remaining parts. Thus, it seems possible that, at this point in her trajectory, in order to find the value of the unknown quantity, Denise needed an environment in which there was an opportunity to operate with and upon a diagram that accurately showed the relative size of the parts. Thus, Denise’s understanding of the equality of the remaining parts seems to be contingent upon her ability to work from an accurately drawn diagram.

This need to have equal sized diagrams suggests that Denise’s understanding of the equivalence of the remaining parts was situational at this point (Noss, Healy, & Holyes, 1997). It may be the case that, although Denise knows that the two groups of remaining parts are, theoretically, equal in length, she does not know how to use that information to find the value of the unknown quantity. With that in mind, I gave Denise several hypothetical equivalence tasks during the next teaching session.

To solve the second hypothetical equivalence task on 5/15/2008 (see Figure 4-92), Denise first represented Jennifer’s and Jose’s distances walked at the top of the page. At the bottom of the page, Denise redrew the two diagrams that she had drawn at the top of the page.
Denise then altered the diagrams at the bottom of the page by drawing lines extending from Jose’s diagram to Jennifer’s diagram and by attaching a small box to the right of the 20-box in Jennifer’s diagram, drawing a bracket above the 20-box and the newly drawn box, and writing the number 20 above this bracket.

Figure 4-92: Part 1 of Denise’s response to Jennifer-Jose Task on 5/15/2008

By drawing lines vertically from parts of Jose’s diagram to Jennifer’s diagram, it seems that Denise was attempting to match the parts of Jennifer’s diagram that were identical in length to parts of Jose’s diagram. However, Denise created a new box in Jennifer’s diagram in violation of the original problem condition. She seemed to do this in order to match the length of this diagram with the first three boxes of Jose’s diagram. Denise may have done this because she thought that she was fulfilling the problem condition by making three of Jose’s boxes equal the
length of Jennifer’s diagram. Or, Denise may have thought that Jose’s diagram should have contained three sets of the boxes in Jennifer’s diagram. The latter explanation seems likely because Denise created an upwards line at the right edge of the sixth box in Jose’s diagram, thus establishing that there were two sets of three boxes in Jose’s diagram that were equal in length to Jennifer’s new diagram. It is possible that when Denise saw that there was only one box left in Jose’s diagram, she became confused. At this moment, I should have asked Denise why she chose to add another box to Jennifer’s diagram, but unfortunately, I did not do so. Alternatively, I directed Denise to represent the rest of the problem condition.

Jan: Okay. Well, why don’t we pull one more tool in to help? Would you please draw right underneath this [points to Jose’s diagram] the three times as much [points to Jennifer’s diagram] under here [points to space underneath Jose’s diagram]?

Denise: So [draws a box in space below Jose’s first diagram; draws a longer box connected to this first box] that was 20 [writes the number 20 inside of the box she just drew; draws a long box connected to the first 20-box]. Twenty [draws a vertical line inside of the long box just drawn and writes a 20 to the right of this line; draws a long box connected to the second 20-box]. Twenty [draws a vertical line inside of the long box just drawn. Writes a 20 to the right of this line].

Denise’s actions when drawing the last two sets of boxes seem to indicate that she thought of Jennifer’s distance walked as one unit (as indicated by her creation of two long boxes, each of which she then subdivided into an unlabeled box and a 20-box).

Jan: Okay. Thanks. What do we know about these two? These two distances [points back and forth between the two diagrams drawn next to the words “Jose’s distance walked” at the bottom of the page]?

Denise: That they’re supposed to be the same…in length – because if it’s three times as much, it’s supposed to be three times as this [points to Jennifer’s diagram].

Jan: Okay. All right. And why is three times as much as this [points to Jose’s second diagram at bottom of page] equal to this [points to Jose’s first diagram at bottom of page]?

Denise: Because it’s supposed to be the same length as the – because this is a bad example [points to Jose’s first diagram at bottom of page] because I didn’t [inaudible] out far enough to see what I was doing.
In this excerpt, Denise seemed to be stating that she thought that she did not draw her second diagram of Jose’s distance walked long enough because both of Jose’s diagrams should have been the same length. I then operated from the assumption that both diagrams should have been the same length in our conversation.

Jan: Okay. Based on the fact that those two are equal, does that help you any?

Denise: Yes.

Jan: Okay.

Denise: So, by what I know of this [traces pen along Diagram 3], what would you want me to do with that?

Jan: So would you please tell me how much one of these blocks is worth?

…

[Jan gives Denise a piece of paper (see Figure 4-93). Denise writes “10x3 = 30”; underneath this she writes “20 x 3 = 60”.]

Denise: Okay, so that together [draws an addition sign and a line underneath the numeral 60], 90 [writes the number 90 underneath the line drawn].

…

Jan: So what are you thinking right now?

Denise: That I think that one of these boxes could equal ten, but two of these boxes – because it looks like two I’m thinking two of the boxes equal – two boxes equal 20, and one box equals – and divided by two, that equals ten each.

…

Jan: Okay, and is there any way you can prove it [taps paper containing the calculations she just did] that that is true?

Denise: Um. To prove that one, each box equals ten?

Jan: Um-hm.

Denise: If I try doing ten times six plus [on Page 2a, writes “10 x 6 +”; crosses out the addition sign]. Might as well just do ten times six first. Ten times six equals 60 [writes “10 x 6 = 60”]. Plus 20 is 80 [writes an addition sign underneath the 60, draws a line underneath the 20, and then writes 80 underneath that line]. Hm [moves pen back and forth along Jennifer’s
diagram at bottom of Page 2a]. Twenty times three [vertically writes “20 x 3”; underneath draws a line and writes the numeral 60]. 60. Plus 30 [underneath 60, writes an addition sign and a 30 next to it] equals 90 [draws an underline and the numeral 90 underneath the line], so I think it was wrong, it’s wrong.

Figure 4-93: Part 2 of Denise’s response to Jennifer-Jose Task on 5/15/2008

In this excerpt, Denise acknowledged that the two distances represented by Jose’s two diagrams were supposed to be equal. Nevertheless, Denise resorted to guess and test in order to find the value of the unknown quantity. Although Denise had successfully solved problems of this nature on previous occasions, she did not seem to draw upon her understanding of the equivalence of the remaining parts in order to solve the problem. For that reason, I instructed Denise to identify the identical parts within each diagram as she had done in previous problems. Denise then checked the identical parts of the two diagrams at the bottom of the page in Figure 4-92. The following conversation ensued.

Jan: Okay. So what’s left?

Denise: Oh, I see what you’re trying to do. It’s two of these [points to the last two 20-boxes on Jose’s second diagram] and it’s two 20s and three of those [points to the three unmarked boxes in Jose’s first diagram].

Jan: Yeah.
Denise: So if I do [begins to use calculator] 20 plus 20. I could do times, but—divided by three… 40. I’ll just start from the 40 [pushes 40 ÷ 3 on the calculator].

Jan: Okay. Okay, now—

Denise: 13.3

Jan: Oh, I’m sorry. So why do you think that answer’s 13.3 repeating? Do you think it’s accurate?

Denise: Um. [6 second pause] Yes.

Jan: Why?

Denise: Because if there’s only two of these left [points at the two boxes in Diagram 3], um – one, two, three, four, five, six [counting the first six boxes in Jose’s first diagram]. Oh, I was just making sure I got the right amount. So three of those left, and then two of them left, then 20 more were added so – and that 20 [inaudible] there, so 20 plus 20, 40 divided by three. That equals—I’m going to try to do this on paper [solves 40 divided by 3 on paper; gets an answer of 13.333].

Jan: Okay.

In this episode, Denise demonstrated that she understood that since the quantities represented by the remaining parts of each diagram were equivalent, she could use her additive-multiplicative scheme to find out the value of the unknown quantity. Interestingly, Denise’s work on this hypothetical equivalence task represents the first time that she did not need to have a remaining parts diagram drawn for her in which each concatenation of boxes representing the remaining quantities had to be equal in length. This seems to indicate that Denise understood that when in a problem-solving situation in which two quantities are known to be equivalent, she could utilize that equivalence in order to find the value of the unknown quantity. However, the fact that Denise did not think of using her identical parts scheme on her own seems to be evidence that she was not able to anticipate that she should set two quantities equal to one another based upon the problem condition in order to find the value of the unknown quantity.
In the next episode, I gave Denise another hypothetical equivalence problem (see Figure 4-94). Denise’s performance during this task demonstrated that she anticipated that she should find and set quantities equal to one another in order to find the value of the unknown quantity. After Denise represented the quantities according to the problem condition, but before she represented the fact that Jenna had three times as much money as Harold, Denise began checking the identical parts in each diagram.

Denise: I’m going to need to find out how much each one originally—

Jan: Um hm.

Denise: So they have this [puts a check mark inside first box in Jenna’s diagram] and they had this [puts a check mark inside first box in Harold’s diagram].

Jan: Oh, before you do that though, have you shown the three times as much yet?

Denise: No [draws a line from the right edge of the third box in Jenna’s diagram to the right edge of the 45-box in Harold’s diagram]. I probably should do that. This [underneath Harold’s diagram, draws an unmarked box that is equal in length to the unmarked box in Harold’s diagram]. 45 [draws a box connected to the last drawn box; writes the number 45 inside of this box]. I’m sorry, I didn’t mean—do you have some more scrap paper?

Jan: Sure.
Figure 4-94: Part 1 of Denise’s response to the Jenna-Harold Task on 5/15/2008

On Page 3a (see Figure 4-95), Denise drew three sets of boxes in which the first box was short and the second box was longer. Denise wrote the number 45 inside each of the longer boxes. She then drew five connected boxes of the same size to which she attached a slightly longer box. After labeling the last box drawn with a 45, she stated, “It might be off in size.” Denise then proceeded to act upon the diagrams that she drew.

Denise: This one [checks the first box in the first diagram]. This one [checks the first box in the second diagram]. This one [checks the second box in the first diagram]. This one [checks the third box in the second diagram; checks the third box in the first diagram, the fifth box in the second diagram, the sixth box in the first diagram, and the second box in the second diagram]. I need this one again [adjusts calculator screen].

Jan: Okay.

Denise: This time I know what I’m doing. Do you know if this one’s gonna be a decimal?

Jan: Probably not. [Denise begins to use the calculator; Jan reads from the calculator] 45 plus 45 equals 90 divided by—
Denise: Divided by two. So what if we did 45 times five [using calculator; Jan: Okay]. And we got 225. [Continues to use the calculator.] Plus another 45 So it’s really 45 times six. [Jan: Okay.] And for him [points to Harold’s diagram on Page 3], well if he had three times as much, so it was [using calculator] 45 times two times three. Oh, will it do that problem [referring to the calculator’s inability to perform order of operations]?

Jan: Times two times three. It’s going to triple the double, so – oh, it’ll be okay, yeah.

Denise: 270 and 270. So each one is worth 45.

Jan: Okay.

Denise: So, if Harold had – I know Harold only had 45 because he only had one box which is right there [points to Harold’s diagram on Page 3]. Um, but Jenna had 45 times five [using calculator] and we gave her 225.

Figure 4-95: Part 2 of Denise’s response to the Jenna-Harold Task on 5/15/2008

In this episode, Denise encountered problems similar to those found in the other hypothetical equivalence tasks. In those tasks, Denise was not given a diagram from which to operate. However, unlike her problem-solving activity while solving those tasks, Denise operated with much more confidence when solving the Jenna-Harold problem. When she represented the problem in Figure 4-95, Denise acknowledged that the diagram “might be off in size.” However, this did not seem to be problematic to Denise as on previous occasions. In fact, Denise stated,
“this time I know what I’m doing.” She then proceeded to use her identical parts scheme and her additive-multiplicative scheme to successfully find the value of the unknown quantity.

Denise seemed to be operating from understanding and not by rote as evidenced by the manner in which she checked her answer. Denise multiplied by five and then added 45 in order to find the amount of money represented by the boxes in the top diagram on Page 3a. In order to ascertain the value represented by the boxes in the bottom diagram, Denise expressed a desire to multiply “45 times two times three.” Here, Denise seemed to want to double 45 (as a means of finding out the value for an unknown box and a 45-box since she has ascertained that the unknown quantity is 45). Denise then seemed to want to triple that result that amount (since there are three sets of unknown-45-boxes).

As illustrated in Denise’s performance on the above task, it seems that Denise learned that creating a relationship of equivalence between quantities would enable her to match the equivalent quantities so that she could find the value of the unknown quantity. The fact that Denise began to use her identical-parts scheme before she represented three times the amount of Harold’s new amount of money leads me to believe that this understanding is contextual. That is, it seems that Denise is aware of the benefit of setting two quantities equal to one another according to the problem condition. However, she seemed to understand this only within the context of having already represented both quantities.

The development of Denise’s understanding seemed to have been assisted by the scaffolding that was provided to her, which required that she represent the same quantity in two ways according to the problem condition. Further assisting her, it seems, was her ability to visualize the movement of pieces so that they could be aligned with other pieces. This ability to reason visually served Denise well in being able to state with assurance that two quantities were equal even though their representations were not adjacent to one another in a diagram.
Valerie’s Story

At the beginning of the study, Valerie showed that she understood that when two quantitative wholes are equivalent to one another, the equivalence of those wholes is maintained after the removal of equivalent quantities from each of those wholes. This understanding is demonstrated by Valerie’s response to Item 10 on the pre-assessment (see Figure 4-96) and in the excerpt below.

Valerie: If one triangle [points at the triangle in the initial problem condition] balances three of these [points at the three circles in the initial problem condition] so I sort of said well three of these [points at the three circles in the initial problem condition]. Then one circle [points at left side of scale] must count for two circles [points at answer choice b] and then there’s four left [points at right side of scale] so that two squares right here [points at left side of scale].

Jan: Okay.

Valerie: So that’s two and two [taps right side of scale twice] which is four so that’s why I circled b.

Jan: I see, okay. So you’re saying one – you said one circle might count for two circles?

Valerie: No, I said one triangle counts for three – equals three and one square counts for two circles

Jan: Gotcha. Okay, thank you.

In the above excerpt, Valerie stated that “one triangle balances three [circles].” She later stated, while pointing at the right side of the scale, “and then there’s four left.” Valerie’s statements here seem to indicate that she had mentally removed the triangle on the left side of the scale and the three circles on the right side of the scale leaving “four [circles] left.” Valerie then utilized the fact that there were four circles and two squares left to determine that each square balanced two circles when she said “so that’s two and two” while tapping the right side of the scale.
One of my goals for Valerie during this study was that she would develop the understanding that establishing a relationship of equivalence between two quantitative wholes is useful in finding the value of the unknown quantity contained within one of those wholes. To this end, one set of tasks that I gave to Valerie I have named the Additive Deficit tasks. These tasks presented realistic situations that had the underlying structure $ax+b=cx-d$. In order to solve these tasks algebraically, the problem solver must equalize the relationship between two non-equivalent quantities by creating a quantity that does not exist within the realistic situation. Thus, a person having an understanding of hypothetical equivalence (Clement, 1982) would be likely to successfully solve such a task.

The first Additive Deficit task that I gave Valerie can be seen in Figure 4-97. In order to solve this task, one must make the lengths of the paper strips equal to one another in a manner...
that does not exist within the realistic situation. For example, Additive Deficit Task #1 can be represented in the following way:

1) Length of Paper Strip A: \( a + 60 \)

2) Length of Paper Strip B: \( a + a + a - 16 \)

It is stated that the two quantities represented on Lines 1 and 2 would be equivalent if 60 inches of paper was added to Strip A and 16 inches of paper was cut from Strip B, which was three times as long as Strip A. In order to utilize a relationship of equivalence to solve the problem, one must suppose that 16 inches of paper were removed from Strip B and that the length of Paper Strip A was increased by 60 inches. Then, one could “restore” 16 inches to Paper Strip B, thus removing the 16 inch “deficit” from the original length of Strip B. At the same time, 16 inches need to be added to the new length of Paper Strip A in order to maintain the supposed equality of the two paper strip lengths. As a result, the length of Paper Strip A would be \( a + 76 \) which is an amount that exists nowhere in the realistic situation.

![Diagram](image)

**Figure 4-97: Valerie’s Response to Additive Deficit Task #1 on 5/19/2008**
When solving Additive Deficit Task #1, Valerie first represented the length of Paper Strip A as one short box and a longer box which she labeled with a 60. Valerie represented the length of Paper Strip B as three equal-sized, connected boxes (box 2a, boxes 2b and 2c combined, and box 2d and the shaded 16-box combined). The last of the three boxes, she partially shaded in. She labeled the shaded box with the number 16. After drawing the representations and being requested to find the value of the unknown quantity, Valerie paused for a moment. When asked what she was thinking, Valerie stated the following:

I don’t know how long 16 inches [points at the shaded 16-box in Diagram 2] would be. I wouldn’t know how long 60 inches [points at the 60-box in Diagram 1] would be, so I have to figure out, like, say if I divide some [points to the 60-box in Diagram 1] of this if it could come out to give me the amount that each originally was.

Valerie then provided an example of what she meant by supposing that the length of the 60-box in Diagram 1 was equal to the combined length of the second box [boxes 2b and 2c] and the unshaded portion of the third box [box 2d] in Diagram 2. She then drew a line down the middle of the second box in Diagram 2 saying, “I could divide that to three here.” Valerie then divided 60 by three, implying that she assumed that the newly created second through fourth boxes (boxes 2b, 2c, and 2d) in Diagram 2 were equal in length.

Valerie’s suggested strategy reveals something about her understanding of the usefulness of establishing a relationship of equivalence between two quantities in order to find the value of the unknown quantity. Valerie’s strategy was to divide the unknown part (boxes 2b and 2c combined) of one whole into smaller parts and combine that part with another unknown part (box 2d) of the same whole. She then utilized the equality between the length of that concatenation of parts with the length of the known part (60 inches) of an equivalent whole in order to find the value of the unknown quantity.

Valerie’s suggestion contained an unfounded assumption in that she assumed that box 2d would be equal in length to the two subdivisions contained within the second box of Diagram 2.
(boxes 2b and 2c). She stated that the appearance of the unshaded portion of the third box
appeared to be the same length as the two boxes to its left. Although Valerie only provided this
strategy as an example, her words and actions seem to imply that she was inadvertently
attempting to find and apply a relationship of equivalence as a means of finding the value of the
unknown quantity.

In order to help Valerie to establish what was necessarily true within the diagram based
upon the problem conditions, I asked her to draw lines extending from both sides of the 60-box in
Diagram 1 to the boxes in Diagram 2 that were aligned with this box. The following conversation
then ensued.

Jan: Okay, now based on that, you’re saying that this is 60 [points at the three
small boxes in the middle of Diagram 2], right?

Valerie: Um hm.

Jan: So, what does that tell you about this whole amount [moves finger from left
to right beginning with left side of second box in Diagram 1 and ending with
right side of shaded 16-box in Diagram 2]? So from [hovers pen above
Diagram 2]. I’ll tell you what. Could you bring this 16 up here? Would that
help at all? [Valerie draws a box to the right of the 60-box in Diagram 1.]

Valerie: Like that?

Jan: Yeah. So what does that tell you about this whole amount [draws a bracket
above the 60-box and the box to the right of it in Diagram 1]?

Valerie: That this is 60 [points at the 60-box] plus 16 [points at the box to the right of
the 60-box in Diagram 1].

Jan: Okay, how much would that be?

Valerie: Seventy-six.

By having Valerie draw a shaded 16-box on Diagram 1, I was trying to help her to make
explicit to herself the implicit relationships existent within the problem situation. That is, the sum
of the amount added to Paper Strip A and removed from Paper Strip B was equal to double the
original length of Paper Strip A. Because 16 inches were never added to Paper Strip A, this
alteration was not a part of the realistic situation. By representing all of the information that she knew to be true diagrammatically, Valerie had the opportunity to create a relationship of equivalence not contained in the problem situation. I then tried to direct Valerie’s attention to the representation in Diagram 2 that was equal in length to the 60- and 16-boxes in Diagram 1.

Jan: Originally, what did this quantity represent [traces pen around last two boxes of Diagram 1]?

Valerie: This? [Valerie points at 60-box in diagram 1.]

Jan: Um hm.

Valerie: Sixty.

Jan: Oh, I’m sorry. I mean, This amount [draws an outline around last two boxes of Diagram 1] was what in relation to the original problem [points at the problem statement at the top of the page]? Do you know what I mean?

Valerie: Three times. Like, oh this [points at 60-box]? This [points at 60-box] was just doubled of this [points at first box in Diagram 1; Jan: Okay]. Originally.

Jan: Double of that? Can you use that information to help you find out how much [taps first box of Diagram 1] one of these was? [Valerie begins to use calculator; Jan reads the buttons as presses them] 76 divided by two equals—

Valerie: Thirty-eight. So, that means if this [circles 60-box] equals two of those [points at the second small box in Diagram 2] originally. That means that this box [points at first box in Diagram 1] would equal 38 [writes a bracket underneath the first box in Diagram 1 under which she writes the number 38].

Although Valerie was able to obtain the correct answer of 38 for the value of the unknown quantity, it seems that she did so by making the wrong assumption. When I asked her what she knew about the last two boxes on Diagram 1, Valerie seemed to be speaking of the 60-box when she stated that something was double the value of the unknown quantity. I did not catch that error during the teaching session.

By having her attention diverted to those parts of each diagram that were equal in length, not based on appearance, but of logical necessity, Valerie became more able to use the visual information from the diagram to reason about the existent relationship between quantities. This
differed from the way that Valerie utilized visual information earlier in the episode. Then, she speculated that a part of a box might represent a certain quantity because of the size of that box relative to the size of the other boxes.

While solving this deficit task and the next one, Valerie seemed to experience difficulty because of her tendency to rely on the way that the diagram was drawn, even if the drawing was erroneously done. Therefore, I assigned Valerie the pre-drawn Additive Deficit tasks (as shown in Figure 4-98). These tasks contained situations in which two paper strips with a particular multiplicative relationship were pictured. Underneath these representations was a second set of diagrams showing the paper strips after their lengths were altered. One paper strip was added to in length while the other was shortened in length. As a result, the two paper strips became the same size.
Figure 4-98: Valerie’s Response to Pre-Drawn Additive Deficit Task #1 on 5/27/2008

My reasons for assigning these tasks were both to see how Valerie would solve the problem without the hindrances presented by incorrectly drawn diagrams. And, I wanted Valerie to have the opportunity to operate upon and reason from a diagram in which the size of the boxes relative to one another was correct. By having to solve this task, it was my intent that Valerie would develop the anticipation that she would need to equalize two unequal quantities in order to find the value of the unknown quantity.

During Pre-Drawn Additive Deficit Task #1, Valerie was asked to first describe what she thought was happening in the situation. She then was asked to utilize the information given to find the value of the unknown quantity.

Jan: So the question is what was the original length of each strip? [Reads buttons as Valerie uses calculator] Forty minus 22 equals 18. And can you tell me why you subtracted 22 from –?

Valerie: Because this portion [scans pen from left side of second box to right side of third box on Diagram 3] equals 40, and then like this [points to the 22-box on Diagram 3] piece of this is 22. So it’s 40 minus 22 will give me 18, saying maybe a certain part is 18 inches.

Jan: Um hm. Could you show me—do you know what part would be 18?

Valerie: I’m not sure yet. [Jan: Okay; 16 second pause.]

By asking Valerie to show me the part of the diagram that represented 18 inches, I was attempting to ground her in the situation so that she might experience a cognitive conflict with regards to trying to contextualize the result of subtracting 22 from 40. I then made the following request of Valerie.

One thing I am going to ask you to do—and whenever you have a diagram like this—is would you make sure that all the information you know down here [points at the 40-box on Diagram 4] is also written up here [points at the second box on Diagram 3]?

I made this request of Valerie so that, by having to show the known quantities all on one diagram (Mikulina, 1991/1969), Valerie might be more readily able to visualize quantitative
relationships that may not have been previously apparent. For example, by having to label “40 in” on Diagram 3, Valerie had the potential of noticing that the 40- and 22-boxes in Diagram 3 were equal in length to the two unit boxes in Diagram 1. Thus, she might be able to utilize that fact in order to find the value of the unknown quantity.

Jan: As you look at this, do you have any particular thoughts about it?

Valerie: Yeah, it looks like, you know, how these are the same [points back and forth between the second and third boxes of Diagram 3 and the 40-box of Diagram 4], and this is cut some [pulls pen down left side of third box on Diagram 3] because this was cut [scans pen along 22-box]?

Jan: Um hm.

Valerie: It looks like this [points at third box on Diagram 3] is one third.

Jan: It looks like that’s one third? [Valerie: Um hm.] Okay. Of what, now?

Valerie: This [circles pen in air above the second and third boxes on Diagram 3].

The fact that Valerie resorted to visually estimating the relative size of two boxes as a means of reasoning about the relationship between the quantities represented by those boxes seems to be an indication that she was not aware of the quantitative constraints imposed by the problem condition. In order to assist Valerie in focusing on those relationships that would enable her to find the value of the unknown quantity, I directed her attention to that part of Diagram 3 whose length was known.

Jan: What do we know about this piece [draws a bracket underneath the last three boxes on Diagram 3]?

Valerie: It equals this [traces underneath the last two boxes on Diagram 1].

Jan: Okay, does that help?

Valerie: Well, they’re saying that this part [traces pen down dotted line between Diagrams 3 and 4]—if 22 [points at 22-box on Diagram 3] plus 40 [points at 40-box on Diagram 4], so that would be 62, so this whole portion [traces underneath last three boxes on Diagram 3] equals 62.

Jan: Okay…..would you write that down please? [Underneath the bracket that is drawn under the last three boxes on Diagram 3, Valerie writes “62 in”]. Since
that’s the case, what can you say about one of these [points at the second box on Diagram 3]? 

Valerie: One? Well that would be [begins to use calculator]—

Jan [reading buttons from the calculator] 62 divided by two equals—

Valerie: That would equal 31 [points to first box on Diagram 3].

Two things may have helped Valerie in solving this problem. First, asking Valerie to represent that part of Diagram 3 which must be 40 inches enabled Valerie to represent all known information on one diagram. Secondly, directing Valerie’s attention to the last three boxes of Diagram 3 by drawing a bracket underneath them seemed to help Valerie to visually isolate and unitize those three boxes as a whole. Asking Valerie “what we know about them” prompted her to compare their combined length to that of the last two boxes in Diagram 1 and to create a 62-unit. Having indirectly equated the 62-unit with the last two boxes of Diagram 1, it seems that Valerie was able to identify the length of one box.

When Valerie solved the next two Pre-Drawn Additive Deficit Task, she was able to quickly utilize the relationships implicit within the diagram in order to find the value of the unknown quantity. Below is Valerie’s work on the third Pre-Drawn Additive Deficit Task (see Figure 4-99).

Valerie: So it looks like this portion [tracing last three boxes on Diagram 4] is three boxes.

Jan: Okay, how do you know?

Valerie: Because from here [points at left side of third box on Diagram 1] is three [traces pen to the right], and this is the same thing [scans pen across the last three boxes of Diagram 4], but this [moves pen across 15-box in Diagram 4] is cut off. This [draws a bracket above the third and fourth boxes in Diagram 4] still represents 12 [writes a 12 above the bracket that she just drew], so 12 plus 15 is 27. So three boxes—so [begins to use calculator]

Jan: [Reads the calculator as Valerie uses it] Twenty-seven divided by three equals—

Valerie: Nine. So each box is nine inches.
Although Valerie seemed to have abstracted the understanding that she could create a relationship of equivalence between non-equivalent quantities in order to find the value of the unknown quantity, Valerie did not seem to exhibit that understanding at the start of the next teaching session. At that time, I reissued to Valerie the first Additive Deficit Task (see Figure 4-100) that I had given her in hopes of seeing how she would operate differently with this task. Valerie properly represented the relationships between quantities, but when I asked her to attempt to use the diagram to find the value of the unknown quantity, Valerie experienced difficulty.

Jan: Now, can you please use the diagram and find the original length of each strip? [27 second pause] Can you tell me what you’re thinking right now?

Valerie: Like if the 16 [points to the 16-box] is kind of related to the 60 [points to the 60-box].
Jan: Okay, do you think it’s related at all?

Valerie: I’m not sure.

Jan: Okay [reading the calculator as Denise pushes the buttons]. Sixty divided by 16 equals 3.75. Okay, I have a question for you. What do we know from this situation? What do we know, besides the fact that that’s 60 [points at the 60-box] and that’s 16 [points at the shaded 16-box]? What else do we know about the two strips?

Valerie: That they’re the same length now.

Jan: Can you draw a line indicating where those two meet?

Valerie: Like, where they meet?

Jan: Yes. [Valerie draws a vertical line extending from the right side of the 60-box to the left side of the shaded 16-box.] Does that help any?

Valerie: Not really.

Figure 4-100: Valerie’s Response to Additive Deficit Task #1 on 5/29/2008

Valerie’s decision to divide 60 by 16 did not seem to be intentionally directed toward finding a relationship of equivalence between a quantity represented in Diagram 1 and a quantity
represented in Diagram 2 in that she did not utilize parts of each diagram that were equal in length. Because I wished to focus Valerie in that direction, I asked Valerie to draw a line where the “two [diagrams] meet.” In this awkwardly worded question, I was referring to the place on each diagram where one box in each diagram, if connected, would share a common border with a box in the other diagram. Valerie’s statement that she did not know what to do after drawing the line connecting Diagrams 1 and 2 was surprising given her performance on the Pre-Drawn Additive Deficit Tasks in the previous session.

I then asked Valerie to draw and label the 16-box in Diagram 1 so that she could see all of the known information on one diagram. When I asked Valerie if this helped her at all, she stated the following.

Valerie: Somewhat because it’s telling me that this much [draws a bracket from the left side of the second box in Diagram 2 to the right side of the third box in that diagram] is 60, and this much is 16 [points to the shaded 16-box on Diagram 2].

Jan: Okay. What could you do with that information? Oh, I’m sorry—would you label the 60 there also please [points at second and third boxes in Diagram 2]?

Valerie: Sure. [Writes 60 above second and third boxes in Diagram 2.]

It is interesting that Valerie chose, on her own, to identify the place on Diagram 2 that was equal in length to the 60-box that was in Diagram 1. This action did not seem to have been part of a global search for equivalence on her part in an effort to find the value of the unknown quantity. Rather, this seemed to be a more local search for equivalence prompted by the information provided by the newly added presence of the 16-box in Diagram 1.

Because Valerie had all of the known parts in place on each diagram, I next sought to bring to her attention the combined length of the 60-box and the shaded 16-box on Diagram 2.

Jan: If this hadn’t been cut off [points to 16-box on Diagram 2], what would be the width of this whole [traces finger along last three boxes of Diagram 2; reads screen as Valerie uses calculator]? 60 plus 16 equals 76 divided by two equals 38.
By asking Valerie, “if this hadn’t been cut off” while pointing at the last three boxes of Diagram 2, I brought the original condition to Valerie’s attention. It is possible that Valerie had not thought of adding 60 and 16 previously because that sum does not occur in the actual situation. Even though the number on the calculator screen read “38”, in explaining her solution process, Valerie refers to her answer as “35.”

Valerie: So before this cut off [points to Diagram 2], and this is three boxes [points at the first three boxes on Diagram 2], it looks like each box was 35 [points at first box in Diagram 1].

Jan: 35. And why 35?

Valerie: Because I added these two together [points to the numbers 60 and 16 on Diagram B], which makes the two boxes [moves pen from left side of second box in Diagram 2 to right side of shaded 16-box in Diagram 2] divided by two to get the equal of one box [points at first box in Diagram 1].

Jan: When you divide by two, I noticed you got 38. Where did 35 come from?

Valerie: Thirty-five? What do you mean?

Jan: Oh, okay. So what’s the length of each box?

Valerie: Thirty-eight.

Jan: Okay, all right.

At the end of this teaching episode, Valerie appeared to be able to visualize Diagram 2 before the 16 inches were cut off. As a result, Valerie was able to refer to the combination of 60 and 16 as the length of two boxes. The two boxes to which Valerie was referring seemed to be the two unit boxes that existed in Diagram 2 before 16 inches was cut as indicated by the sweep of her pen across the second box, the third box, and the shaded 16-box in Diagram 2. This seemed to help Valerie to decide to divide the total of 76 by the number of uncut unit boxes.

In a later teaching session, Valerie was asked to use letters and numbers to solve an Additive Deficit Task (see Figure 4-101). Valerie first represented the original height of each
tree. She then subtracted 13 from Line 1 and added 15 to Line 2 in order to fulfill the problem conditions. Valerie was then asked how she would find the value of B.

Jan: Now, is there any way you could use that information to help you find out the original height of each [tree]?

Valerie: I think so.

Jan: Okay. [29 second pause]

Valerie: Plus, plus, plus, plus [write a “+” sign between each B on Line 1]. Plus [writes a “+” sign after the B on Line 2; taps each term on Lines 1 and 2 with her pen; says something inaudible; 23 second pause]

Jan: What are you thinking as you’re looking at that?

Valerie: Well, since, like, these are the same [pointing at the first B on both lines], I’m kind of knocking that out [in the air, swipes pen to the left], so that means that these four Bs [moves pen left to right along Line 1] minus 13 equals 15.

Jan: Okay. Does that help you to figure out what B might be?

Valerie: Um hm.

Jan: Okay. What do you think?

Valerie: That somehow, I have to figure out like, how I would subtract the 13 from the four Bs [moves pen left to right along Line 1].

2) Tree A is five times taller than Tree B. If Tree B grew 15 more feet and if 13 feet were cut from Tree A, the two trees would be the same height. What was the original height of each tree?

\[ 35 = \text{Tree}_A B + B + B + B + B - 13 \]

\[ 7 = \text{Tree}_B B + 15 \]

Figure 4-101: Valerie’s Response to Additive Deficit Task #1 on 5/27/2008

Although Valerie demonstrated that removing B from both quantities would leave the remaining quantitative wholes equal to one another, she did not know what to do with those
remaining quantities. Unlike the Additive Deficit Tasks where the unknown and known quantities were represented by diagrams, the letter and number representation did not present a visual image of the relative lengths of the boxes representing those quantities. Without the presence of the boxes upon which to act, it was very difficult to motivate adding 13 inches back to Tree A’s new length and adding 13 inches to Tree B’s new length in order to remove the deficit of 13 inches in Tree A’s height (short of espousing a rule). Therefore, I made the decision to step outside of the problem context and appeal to Valerie’s understanding of whole number operations in order to solve the problem.

Jan: Can I ask you a question?
Valerie: Um hm.

Jan: What number would you need to have here [moves pen left to right along Line 1], in order to subtract 13 and get 15?
Valerie: Um, 28.

Jan: And how do you know that?
Valerie: Because this [points at the 13 on Line 1] plus this [points at the 15 on Line 2] equals 28, and 28 can be divided by four [moves pens across 2nd through 5th B’s on Line 2; inaudible] four B’s.

Jan: Does that help you figure out how much one B is?
Valerie: Yes.

Jan: What is it?
Valerie: Seven.

Valerie showed that she was able to use the equivalence of two quantities in order to find the value of the unknown quantity. However, it is not clear whether she was intentionally searching for such a relationship of equivalence (although she did reduce the problem to a simpler one by eliminating one B from both lines).
Although my question concerning what number the four B’s should be equal to prompted Valerie to find the solution to the problem, it did not enable her to operate algebraically upon the remaining quantities. DeLima and Tall (2008) spoke of this when they stated that embodied representations for algebraic situations (such as a balance model) have limitations once certain conditions are introduced (such as negative numbers). In hindsight, it may have been more beneficial to introduce a quasi-diagrammatic representation of the situation before the literal representation used in the task. This representation could have included letters and, instead of numbers, shaded and unshaded circles that represented one less or one additional foot of tree length, respectively. After operating upon those circles, Valerie could have then been prompted to generalize her process using positive and negative numbers.

Immediately after Valerie finished solving Additive Deficit Task #1 on 6/2/2008, I gave Valerie an equation of the form \( ax+b=.cx+d \) to solve. I did this so that I could pose an Additive Deficit Task to her in this form afterwards. After writing the equation “\( 3b+4=b+20 \)” (see Figure 4-102), I asked Valerie if she knew what \( 3b \) meant. Valerie stated, “another number.” The following conversation then ensued:

Jan: Yes. In fact, let me write it a different way because I forgot you haven’t gotten to that part yet—\( b \) plus \( b \) plus \( b \)—that’s the same thing—plus four equals \( b \) plus 20. I’m sorry. Would you please tell me, based on that, what \( b \) is? Yes, what \( b \) is? [20 second pause]

Valerie: I think \( b \) is eight.

Jan: And why do you think that?

Valerie: Because I kinda knocked out this \( b \) [points the first \( b \) on the left side of Equation 2] and this \( b \) [points the first \( b \) on the right side of Equation 2] since they were the same, and then I had these two [points to the 2\text{nd} and 3\text{rd} \( b \)’s on the left side of Equation 2]—so these two \( b \)’s [points to the 2\text{nd} and 3\text{rd} \( b \)’s on the left side of Equation 2] and this four [points to the number 4 on the left side of Equation 2] have to equal 20 [points to the number 20 on the right side of Equation 2]. So then—oh, 20 minus 14 is 16, and then—so these two [points at the 2\text{nd} and 3\text{rd} \( b \)’s on the left side of Equation 2] have to equal 16 to get to 20, so then each one will be eight.
Jan: Okay, good. Could you write equals eight down there, please? Why did you subtract four from 20?

Figure 4-102: Valerie’s Response to Double Sided Unknown Quantity Task #1 on 6/2/2008

Valerie stated that she had subtracted 14 from 20 to get 16. However, because she said and pointed to the number four on the left side of Equation 2 when referring to the fact that two $b$’s plus four was equal to 20, I assumed that she meant that she subtracted four from 20.

Valerie: Oh, to get this number because it had—for these—$b$ plus $b$ plus four had to equal 20.

Jan: Okay. I have a question for you. The idea of $b$ plus $b$ plus $b$ equaling three $b$—what do you think? Does that seem to make sense, or does that seem not to make sense?

Valerie: It seems to make sense.

Jan: How come?

Valerie: Because it’s just telling you a number by not telling you a number—like saying that three—hold on—I mean eight plus eight plus eight [points at each of the three $b$’s on the left side of the equation in Line 2] plus four equals eight plus 20 [points at the $b$ and 20 on the right side of the equation in Line 2], but it’s just not saying the number.

Jan: Okay, and here, it’s like we’re saying eight, plus eight, plus eight [points at each of the three $b$’s on the left side of the equation in Line 2] is the same as saying three times eight [points at the three and the $b$ in the $3b$ term on Line 1]. Is that okay?
Valerie: Um hm.

In the latter part of the above excerpt, when asked if it made sense that “\( b + b + b \)” was equal to \( 3b \), Valerie stated that it did and justified her answer by stating that “it’s just telling you a number by not telling you a number.” In doing this, Valerie seemed to give a rationale for why \( b \), and even \( b + b + b + 4 \) were legitimate quantities. However, Valerie did not seem to justify why it made sense that \( 3b \) was equal to \( b + b + b \). Therefore, I started from her line of reasoning and stated, “it’s like we’re saying eight, plus eight, plus eight [points at each of the three \( b \)’s on the left side of the equation in Line 2] is the same as saying three times eight.” Even though Valerie was probably at the point where she could understand that explanation, ideally, it would have been beneficial for Valerie to have received a far less superficial treatment of that concept.

Two tasks later, and during the same teaching session, I gave Valerie the task in Figure 4-103. This was the third Additive Deficit task of that day. This time, no situational context was given.

Valerie: [underneath “\( 8d-21 \)”, Valerie writes “\( d+d+d+d+d+d+d+d \)”; underneath “\( 4d+19 \), writes “\( d+d+d+d \)”; draws an X through the first four \( d \)’s on both sides of the equation; taps each of the unmarked \( d \)’s on the left side of the equation; moves pen left to right amongst these \( d \)’s; 24 second pause] \( d \) equals ten.

Jan: And how do you know?

Valerie: Because this 19 plus 21 equals 40, so then that’s four here [moves pen along the four \( d \)’s that remain on the left side of the equation] and 40 divided four is ten.

Jan: Okay. Would you write your answer, please? [Valerie writes “\( d=10. \)”] And, could you remind me, again, why you add 21 and 19?

Valerie: Because this [points at the four remaining \( d \)’s on the left side of the equation] and this [points at the number 21 on the left side of the equation] is what’s left, and then, this is just what’s left [points at the number 19 on the right side of the equation] from here, so I add these two [points at the numbers 19 and 21 in the equation] to get the whole of this [circles pen above the remaining “\( d+d+d+d \)” on the left side of the equation].
In her solution process, it seems that Valerie added 19 to 21 in order to find out the value of “the whole” $d+d+d+d$. Valerie may have chosen that strategy based on my conversation with her as she solved the Additive Deficit Task #1. In that conversation, Valerie had to solve the equation $B+B+B+B - 13 = 15$. I asked Valerie what number $B+B+B+B$ had to be in order that, once reduced by 13, the answer would be 15. Valerie then chose to add 13 and 15 to obtain that answer.

Valerie may have thought that, in order to ascertain the number from which she needed to subtract 21 in order to obtain a result of 19, she had to add 21 and 19. That seems to be what Valerie meant when she stated she had to “get the whole of this” while pointing to where she had written “$d+d+d+d$” on her paper.

Valerie’s solution processes throughout the Additive Deficit Tasks seemed to become increasingly more sophisticated and efficient. By the end of this series of tasks, Valerie seemed to demonstrate that she understood the usefulness of creating a relationship of equivalence in order to find the value of the unknown quantity. However, because, nearly until the very end, Valerie needed to be reminded to look for equivalent quantities, it seems possible that Valerie’s understanding might be task specific. It cannot be definitively stated that Valerie would search
for two equivalent quantities when solving any other type of algebraic word problem, but it does seem much more likely than when Valerie began these sets of tasks and others.

*Comparison of Students’ Development of Understanding 4*

By the end of the study, Kelly, Denise, and Valerie each demonstrated that they understood the usefulness of creating a relationship of equivalence in order to find the value of the unknown quantity. They demonstrated this understanding while in the context of a particular mathematical situation, therefore, it cannot be stated that they would necessarily utilize this knowledge in other situations. However, it does seem as though each of them has tools to negotiate algebraic situations that have been reported by Filloy & Rojano (1989) to pose difficulty for students to solve with understanding.

When given the Identical Parts tasks, Kelly and Denise began along similar paths in that both started out relying on the appearance of the diagram. Denise’s transition to understanding that the remaining parts of each equivalent whole were necessarily equal to one another seemed to have been facilitated by the constraint that she isolate and act upon those quantities from each equivalent whole that were non-identical. For Kelly, on the other hand, there remained a difficulty with a reliance on the appearance of the diagram. The use of moveable pieces seemed to help Kelly to conceptualize the commutativity of the quantitative parts of an additive quantity. However, the transition to the use of letters to represent the unknown quantity seemed to provide the greatest sense of cognitive relief to Kelly in that minimal spatial issues had to be negotiated when utilizing letters and numbers on a two line format.

The development of Kelly’s and Denise’s search for equivalence within the context of a complex hypothetical equivalence task grew differently. The scaffolding provided on the worksheets like the one containing the Jenna-Craig task seemed to help Denise to conceptualize
the same quantity in two different forms. The fact that the worksheet required Denise to represent the same quantity in two different ways in close proximity to one another seemed to enable her to compare the two quantities. Kelly’s work on the same type of task using only letters and numbers seemed to make such problems relatively easier for her in that, once represented, Kelly utilized her partitive-multiplicative scheme to find the value of the unknown quantity.

The development of Valerie’s understanding that it is useful to create a relationship of equivalence between two quantities in order to find the value of the unknown quantity seemed to be best illustrated through her experience negotiating the Additive Deficit Tasks. While in the diagrammatic environment, Valerie learned that, by removing an additive deficit, she could equate a known quantity with a multiple of the unknown quantity. Although Valerie seemed to demonstrate this understanding in the midst of several Additive Deficit puzzles, she needed to be reminded to represent all quantities on the diagram so that she might be able to compare the length of the concatenation of \( n \) unit boxes with the newly formed known quantity.

When Valerie transitioned to solving equations of the form \( ax-b=cx+d \), she was able to operate upon the unknown quantity in order to simplify the equation to \((a-c)x - b = d\). However, it was difficult to motivate the removal of the “deficit” from the left side of the equation without resorting to dispensing a rule, therefore, Valerie was encouraged to consider what number \((a-c)x\) needed to be in order that subtracting \(b\) from that quantity would yield a result of \(d\).

Both Kelly and Denise were able to solve equations of the form \(ax+c=bx+d\) by the end of the study with understanding. However, Kelly needed to be reminded that \(ax\) could be written as \(x+x+\ldots x\) with \(x\) an addend appearing \(a\) times. Kelly, Densie, and Valerie did not seem to experience the cognitive difficulties with the equal symbol that many students have been reported to have faced (Baroody & Ginsburg, 1983; Kieran, 1981). That is, at the end of the study, they did not seem to view the equal symbol in an equation as an indication that an operation must have
been enacted upon the left side of the equation in order to produce the expression on the right side of the equation. Instead, Kelly, Denise, and Valerie seemed to conceptualize an equal sign as an indication of “quantitative sameness” (Warren & Cooper, 2005, p. 59). This conception seems to have been facilitated by the fact that, throughout most of the study, the equal symbol was not used in order to relate two quantities in a word problem. Instead, the two line format was used. Then, when the use of a symbol to depict the equivalence of two quantities was necessary, the equal symbol arose organically (Janvier, Bednarz, & Belanger, 1987; Thompson, 1996)

Finally, Kelly, Denise, and Valerie successfully negotiated the “cognitive gap” spoken of by Stacey and MacGregor, (1999) in that they were able to operate upon the unknown quantity “as if it were known” (p. 30).

**Conclusion**

There were four understandings that students seemed to have constructed during this study as highlighted in this chapter. Students who participated in this study seemed to have constructed an additive quantity as a procept (Gray & Tall, 2007) in that they were able to view additive quantities as both a process of adding together known and unknown quantities and as an object. The evidence that each student viewed an additive quantity as an object is given as follows. Each student was able to act upon the additive quantity as if its value were known. Each student could decompose and recompose the parts within the additive quantity with the knowledge that the value of the quantity does not change. And each student was able to utilize an additive quantity as an input in an equivalence statement with another additive quantity and, using that relationship of equivalence, act upon each quantity independently in order to find the value of the unknown quantity.
There were several aspects of this study that seemed to contribute to this cognitive development by students. First, students were given experiences during which they were able constructed an understanding of the unknown quantity as a composite unit. Gregg and Yackel (2002) stated, “conceptualizing an unknown as a composite allows a student to think of \(x + x + 2x\), where \(2x\) is the result of counting \(x\) units, followed by counting \(x\) units again (p. 492). The early Harper Middle School tasks were designed to enable students to develop the unknown quantity as a composite unit. Students were able to count the number of pieces in each box of candy and could then iterate that number. At first, students were counting up by a particular known unit. Later, however, students were counting by \(x\) units as result of having generalized their counting process. In speaking of number, Slavit (1999) stated:

If counting is reified and number is understood as a mathematical object, then the actions involved in completing an addition task can be performed on object-oriented inputs, rather than inputs which are themselves viewed as actions. (p. 259)

In like manner, the students in this study seem to have reified the unknown quantity “as a mathematical object.” As a result, students were able to operate upon and manipulate this type of quantity with understanding.

Students constructed the notion that additive quantities were decomposable and recomposable through their isolation and rearrangement of identical parts and remaining parts of diagrams representing equivalent quantities. As a result of having to compare the remaining parts of both diagrams, students were able to abstract the notion that, although different in appearance, the remaining parts of each quantitative whole were equivalent because equivalent pieces of each quantitative whole had been removed. Thus, students seemed to have developed “structure sense” as defined by Linchevski and Linveh (1999) who stated, “this means that [students] will be able to use equivalent structures of an expression flexibly and creatively” (p. 291)

Two students in particular, Kelly and Denise, seemed to rely heavily upon visual input in order to interpret their diagrams. This finding is in keeping with that of Hillel, Kieran, and
Gurtner (1989) who taught children geometrical concepts through the use of LOGO programming. These researchers found that students “rel[ied] almost exclusively on visual cues from the output on the screen” (p. 1). As a result, they found, the students did not focus on those “features of a given task” (p. 2) that were mathematically salient. In order to circumvent this obstacle, these two students needed to be reminded to engage in a “dialectic” (Thompson, 1996) between their diagram and conditions of the problem statement.

Another behavior repeatedly engaged in by students was their tendency to guess and test when encountering new tasks. Students, at times, seemed to take their cues about the correctness of their answer based upon whether a number divided evenly into another. This was indicative that, at that moment, the student had not abstracted the desired understanding. An explanation for this mathematical behavior might lie, in part, in Linchevski and Linveh’s (1999) statement, “Certain number combinations, especially when combined with some specific mathematical operations, trigger deeply ingrained cognitive schema which, probably compete with the new structural perspective” (p. 192). Thus, even if a student demonstrated a particular algebraic understanding during a previous task, at times, students’ perception of “friendly numbers” which divided evenly into one another might have motivated their actions.

Students’ transition to symbols was afforded by the pairing of algebraic symbols with diagrams. Like Gregg and Yackel (2002), I embraced the goal to enable students to “use the standard notation in a way that mirrored the students’ reasoning when they used pictures in their solutions.” (p. 495). This was accomplished by having students model a particular word problem diagrammatically and then having them to model the same problem algebraically. Connections were explicitly made with students so that they were aware that they were dealing with the same type of situation.
In conclusion, students seemed to have abstracted algebraic notions based upon their actions upon objects that were available to them in this study and their reflection upon the results of the coordination of those actions (Olive & Caglayan, 2008; Piaget, 1971).
Chapter 5

Conclusion

Students within this study engaged in a twelve-week teaching experiment during which they learned to solve algebra word problems via the use of diagram drawing. The students seemed to have developed a rich algebraic foundation that enabled them to reason about relationships between quantities on a proceptual level (Gray & Tall, 2007). The participants in this study did seem to avoid some of the cognitive obstacles common to both algebra and pre-algebra students. In this chapter, the nature of these students’ algebraic understandings and the means by which that these understandings seemed to have developed is overviewed using the research questions as a framing vehicle. The implications of this study and future research are discussed in this chapter as well.

Research Questions

This study set out to answer the following research questions. The way in which these questions were answered by this study is addressed in the following section. Following this, I discuss cognitive obstacles reported in the literature related to representing and solving algebra word problems that seemingly were circumvented or experienced differently as a result of the diagrammatic instruction. I also interpret these results in terms of what they tell us about the development of algebraic reasoning.
a) What is the nature of the algebraic understandings developed by students as a result of their engagement in algebra problem solving via the use of diagrams consisting of connected rectangles?

b) By what means did students seem to develop these understandings? What aspects of the environment seemed to constrain or afford this development?

**Nature and means of understandings developed by the students**

Two students in this study developed an understanding of the unknown quantity as an object that can be operated with and upon as if it were known (the third student began this study with that understanding). For these students, this development seemed to occur due to students’ constant use of a numerical exemplar when attempting to understand a multiple of the unknown quantity. Even though it was widely reported in Filloy and Rojano (1989) and Herscovics and Linchevski (1994) that students do not spontaneously operate with or upon an unknown quantity, it seems that these students’ negotiated that obstacle by utilizing the relationships in the problem statement to create numerical relationships. Students then utilized the boxes to represent the numbers and then generalized their activity.

Students also developed a part-whole understanding of an additive quantity. Students learned that an additive quantity is decomposable and recomposable. The use of diagrams which were segmented according to the number of iterations of the unknown quantity that existing within each quantity seemed to help students to be able to match or operate upon the diagram (mentally or physically). The segmented appearance of the diagram also seemed to assist students in comparing lengths, matching identically-sized boxes and creating an additive-multiplicative scheme to find the value of the unknown quantity.
As a result of their interactions within the diagrammatic environment, students also developed an understanding of the multiplicative relationship between unknown and known quantities with an equation of the form $ax + b = cx$. Students’ use of their part-whole understandings seemed to enable them to develop an additive-multiplicative scheme in which students, when faced with two equivalent quantities, found the additive difference between the unknown quantities and divided that answer by the known quantity. The fact that alignment could be established between differently configured representations of equivalent quantities seemed to help in this effort. Also, when first working with problems with relationships of the form $ax + b = cx + d$, students had to become acclimated to focusing on the additive difference between known quantities as it appears in the diagram. Students did not spontaneously focus on this additive difference, but were able to utilize that difference to solve the problem when their attention was drawn to it. Thus, the use of the diagram was important in helping students to “see” the manner in which the additive difference between the known quantities was equal in length to the additive difference between the known quantities. As a result, students developed an understanding of the multiplicative relationship between those two sets of additive differences.

Students’ also built upon their part-whole understanding in such a manner as to enable them to view the parts of an additive quantity as commutative. The use of a diagram facilitated the development of this understanding in that students were asked to isolate those parts of each diagram that were identical and to work from the diagrams of the “remaining parts.” Further, students were also asked to rearrange parts of the diagram so that identical pieces were opposite one another. As a result of their engagement in this activity, the students were able to anticipate, within a certain type of situation, creating a relationship of equivalence is helpful in finding the value of the unknown quantity. When Valerie was given a task of the form $ax + b = cx - d$, she began to anticipate that she could establish a relationship of equivalence between the additive difference between $b$ and $d$ and the additive difference between $cx$ and $ax$. Further, when
encountering realistic situations in which the beginning state of quantity $a$ was a multiple of the beginning state of quantity $b$ and, after a change occurred to quantity $b$, quantity $a$ was also a multiple of the ending state of quantity $b$, Valerie, Kelly, and Denise learned to anticipate equating the two representations of quantity $a$.

One of the pitfalls of diagram use, however, was the manner in which two of the students tended to rely on the visual input provided by that diagram when the diagram was not drawn fully to scale. Denise and Kelly, especially, were conflicted at times because, after drawing, for example, a 20-box, if the 8-box drawn was not about half the length of the 20-box, the student wanted to utilize other tools such as a guess and test strategy to answer a problem. Because they had not yet developed certain algebraic understandings, each of these students was willing to suspend their knowledge of the problem conditions in order to either “fix” the diagram or alter the numbers within the diagram. This cognitive obstacle seems to be unique to a diagram drawing environment in that, when letters were used, there was not an emphasis on spatial arrangement.

Finally, students in the diagram scenario transitioned to the use of algebraic symbols occurred without the usual pitfalls experienced by students in other algebra learning settings. This seemed to occur because the use of letters accompanied the diagrams of the unknown quantity towards the end of the study when students represented algebra word problems of the following form: $ax + b = cx + d$. Students’ transition to the use of letters seemed to be almost a natural progression as symbol use arose out of a need to express quantitative relationships in a more succinct way. Finally, students’ equation solving activity seemed to be enacted with understanding because students understood that, when acting upon two equivalent quantities, one must maintain that equivalence by acting identically upon equivalent parts of those quantities.

In conclusion, the development of students’ algebraic understandings seemed to be afforded by their use of diagram drawing in that those diagrams enabled students to model the
structure of an algebra word problem, to act upon that diagram in a way that was synonymous with equation solving (Simon, 1988) and to formalize their activity using algebraic symbols.

Cognitive obstacles

Students seemed to have circumvented the misconception that the equals symbol is a “do something signal.” Because a two-line diagram was used throughout the study where the quantity represented on each line was either equivalent to the other quantity represented or related to that quantity in some way, students constructed an understanding of the equivalence of quantities before that was represented in equation form. Thus, the use of the equal sign arose organically out of a need to express the relationship between quantities.

The cognitive obstacles that students faced, while not the same as those reported in the literature, nevertheless, hindered students in their problem-solving. One of these obstacles surfaced when Kelly and Denise attempted to solve the hypothetical equivalence tasks. Both students exhibited, at one point, an over-reliance on visual input when solving these problems. That is, when having to compare the remaining parts of two equivalent wholes, each student would state that the remaining parts were not equal because they did not appear to be equal. It seems that the obstacle that these two students were facing was that the students did not understand the necessity of the equivalence of two groups of parts from equivalent wholes that remain after all other equivalent parts have been removed from each whole.

The intervention that seemed to help Denise and Kelly was their construction of the commutativity of the parts of two equivalent wholes. By moving around paper cut-outs of diagrams of paper strips, Kelly was able to isolate those parts of both equivalent wholes that were identical. She was then able to physically match those parts that were equivalent, yet differently configured.
Another cognitive obstacle encountered by students during this study occurred when the icon for the unknown quantity was introduced—the rectangle. Only one student was able to spontaneously act upon the representation of the unknown quantity right away, while the other two students had to work through exemplars. Similar to the cognitive obstacles reported in the literature, these students exhibited an inability to spontaneously operate with and upon the representation of the unknown. In contrast, students who were taught algebra in a traditional manner have been found to consider letters as objects (Booth, 1984) and to manipulate symbols during equation solving in a manner that does not reflect an underlying understanding of the situation (deLima & Tall, 2008).

Development of algebraic reasoning

The findings of this study may inform the field concerning the way that certain algebraic understandings develop. That is, there are certain algebraic understandings that developed within this study and certain problem conditions that seemed amenable to the development of those understandings that are worthy of note. First, students were given an environment in which they were able to informally operate with and upon a representation that, while being cognitively available to them, had the potential to be acted upon algebraically. Next, students were given a means by which they could transition from diagrams to letters while still building upon the understandings developed within the diagram environment. Third, students were given the tools to manipulate quantities in such a way that they both understood the change that they were making and the nature of the quantities that they were manipulating.

Having stated that, it seems that, in this study, the development of students’ ability to reason algebraically followed a certain path. First, by acting upon objects that represented known quantities, students abstracted the notion that the multiple, \( n \), of a known quantity can be
represented by \( n \)-iterations of that quantity. Having abstracted that understanding, students
generalized the results of those actions in order to act upon objects representing unknown
quantities and abstract that notion for unknown quantities. Students then developed an
understanding of additive quantities as decomposable and recomposable, having built an
understanding of the unknown quantity as operable and iterable. Finally, having developed an
understanding of the additive quantity as operable, students seemed to have developed the notion
of an additive quantity as a procept. Additionally, they developed an understanding of the
additive and multiplicative relationships between parts of equivalent quantities that are additive,
multiplicative, or a combination of the two, students were able to find the value of the unknown
quantity. It seems that this progression of constructing understandings of first the known quantity
to the unknown quantity to algebraic expressions to equations is critical in that each previous
concept was a building block of the next. Further, the development of students’ quantitative
reasoning ability was essential (Thompson, 1995) in that it is the adhesive, as it were, that holds
together expressions and equations is quantitative relationships. I concur with Olive and
Caglayan (2008) that students’ construction and coordination of composite units and students’
understanding of quantitative structure are two of the most essential understandings that form a
foundation for algebraic reasoning.

**Implications**

The implications of this study are wide reaching in that students, through their use of
diagram drawing were able to construct quantitative and algebraic understandings that informed
their algebraic problem-solving activity. Ideally, the findings of this study could inform
educational content and policy in the sense that, the manner in which algebra is has traditionally
been taught in many classrooms could be reexamined. Because of the oft-reported
misconceptions and “buggy algorithms” of algebra students, it seems that a systemic change is needed with regards to algebra curricula and teaching. Math students in Singapore have been found to excel beyond their peers in the United States in the area of mathematics (Mullis, Martin, Gonzalez, & Chrostowski, 2004). And, already, the model method utilized by Singapore Math has been adopted within many United States classrooms (Colvin, 2007; Hazelton & Brearly, 2008). However, the outcome of many of these implementations has not been rigorously studied (Ng & Lee, 2009).

Although many of the problem sets and diagrams utilized in this study are similar in form and structure to that which is used by Singapore’s model method, this study differs from Singapore Math for primary students (which includes sixth grade) in a significant way. That is, students utilizing the Singapore Math curriculum are not transitioned to the use of algebraic symbols in the primary grades (Ng & Lee, 2009). In contrast, the intended path of development for students in this study was to have students utilize diagrams to model and solve algebra word problems and then, based upon the understandings garnered during that phase of the study, have students transition to the use of symbolic representations. Although this difference exists, this study can help to contribute to the national conversation about the effectiveness of a diagrammatic approach and the means by which this approach enables students to construct critical algebraic understandings.

**Further Research**

There are further areas of research that could potentially be explored using the results of this study as a starting point. First, how is hypothetical equivalence constructed by students? Although the participants in this study learned how to anticipate searching for or creating a relationship of equivalence within a particular context, this did not seem to be a globalized
understanding. Therefore, future research could examine the factors that contribute to students’
development of hypothetical equivalence.

Further, the students within this study benefitted from the notion of actually physically or
mentally moving parts of diagrams around in order to solve equations. Coupling that result with
the fact that at least two of them were consistently hindered by the way that some parts of the
diagram were not accurately portrayed relative to other known parts, it seems that a computer
microworld would be an ideal environment to conduct this study. In this environment, relative
lengths could be drawn accurately and students could move parts of quantities dynamically while
preserving certain relationships. Different research questions might arise as a result of this
program use, but minimally, the same questions used in this study could be asked and answered
in perhaps a more robust way. As a result of garnering this type of robust data, the researcher
might gain a greater understanding about the way students develop algebraic understandings and
new contributions could be made with the field’s current understanding of this development.
However, some nuances of the data collected in the current study might be lost in a microworld
environment due to the fact that students do not have to consider how they might construct
representations for the known and unknown quantities. This activity in itself affords students the
opportunity to utilize their part-whole knowledge to compose quantities in a way that a pre-
constructed length might not.

Finally, this study could be built upon so that research on the construction of students’
proportional reasoning ability could be explored. At different junctures in this study, a diagram
that a student drew or that I drew would have provided a tremendous stepping off point to talk
about proportional relationships. The tools utilized in this study naturally lend themselves to such
reasoning because multiplicative relationships can be represented with them. Since proportional
reasoning is another mathematical ability that students have been widely reported to experience
cognitive difficulties (Lamon, 1994), this would be a rich area to explore.
References


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Appendix

Tasks

Within the following section, the tasks that were given to the participants of the main study are presented. In all cases, the choice and sequencing of the tasks for the initial teaching session were based on the desire to help students develop particular understandings and/or to assess particular student understandings. Preceding each task presentation is an explanation as to the reason why that task or set of tasks was utilized including a discussion of the way in which that task or set of tasks was intended to teach or assess the given understanding.

The Harper Middle School tasks (Figure A-1) were the starting point for the first session. Tasks one through three were given in order to enable the students to develop a conceptualization of a rectangle, first, as representing an iterable, known quantity. By starting with known values, it was my intent that students might conceptualize that there is a multiplicative relationship between the number of rectangles that they drew, the number of circles in each rectangle, and the total number of circles that were in the diagram. That is, it was my goal for students to develop the anticipation that, when drawing a representation of \( n \) times a given amount (which was represented by one box) they would have to draw \( n \) boxes. Then, when asked to draw a representation of \( n \) times an unknown amount, I anticipated that students would reflect on their earlier construction of the multiplicative relationship between the amount each box represents, the number of boxes, and the total quantity.
Harper Middle School was selling candy as a fund-raiser. The candy was packed into boxes by the students. Each box differed in size according to the number of pieces of candy that it could hold. At lunchtime, the boxes of candy were sold.

1) The circles below represent the amount of candy that Jeremiah has. Phillip bought boxes of candy the same size as the one below. He now has five times as much candy as Jeremiah. Draw a diagram of the amount of candy that Phillip has.

Jeremiah’s Candy: 🍱

Phillip’s Candy:

If Jeremiah doubles the amount of candy that he has and then buys a different-sized box of candy, he and Phillip will have the same amount of candy. Alter your diagram to represent this situation. Use your diagram to determine the amount of candy in the last box that Jeremiah bought.

(Modified from Gregg & Yackel, 2002; Maier, 1997)

2) The circles below represent the amount of candy that Paulette has. Joshua bought boxes the same size as the one below. He has seven times as much candy as Paulette. Draw a diagram of the amount of candy that Joshua has.

Paulette’s Candy: 🍱ål

Joshua’s Candy:

If Paulette triples the amount of candy that she has and then buys a different-sized box of candy, she and Joshua will have the same amount of candy. Alter your diagram to represent this situation. Use your diagram to determine the amount of candy in the last box that Paulette bought.

3) The circles below represent the amount of candy that Ivan has. Denise bought boxes the same size as the one below. She now has six times as much candy as Ivan. Draw a diagram of the amount of candy that Denise has. (Please do not draw the circles).

Ivan’s Candy: 🍱ålål

Denise’s Candy:

If Ivan triples the amount of candy that he has and then buys a different-sized box of
The purpose of tasks four and five was to enable students to develop an understanding of a quantitative difference as a process. That is, it was my goal that students build the understanding that the amount by which one quantity exceeds another is itself, a quantity, which is the result of an additive comparison (Thompson, 1995). Students are given two known iterations of the same quantity. The lesser quantity is then increased by a known amount. Students are asked to represent this situation and to determine the amount by which one quantity exceeds the other. By having to attend to determine this difference, it was my goal that students would build an anticipation that, in deficit situations, they were searching for a particular number that, when added to the lesser quantity, would become a part of a new quantity that is equal to the (originally) greater quantity. This task was given in anticipation of students’ work in future tasks in which the quantity to be iterated would not be known.

For example, in task five, students are asked to determine the number of marbles that Fred would have to acquire in order to have the same number as John (after John acquires more marbles). After representing the situation, a student could ascertain the number of marbles that each person had and then subtract the smaller from the larger. Or, he or she might observe that, within his last set of six, Fred is missing a certain number of marbles. In that case, the student might count up from the number of marbles are present, but that do not fit into a group of six. As a result of reflecting on his or her activity of drawing and counting and its effect—there are three less than six marbles in Fred’s collection, a student might construct the notion that he or she is
searching for a localized deficit which, when added to a group, creates an equivalence between two quantities.

4) Marcus started a penny collection. The circles below represent the number of pennies that Marcus has saved. Jada has a penny collection that is five times as large as Marcus’s. Draw a diagram to represent Jada’s penny collection.

Marcus’s penny collection: ⬤⬤⬤⬤

Jada’s penny collection:

Suppose that Marcus triples the amount of pennies that he has and then saves 10 more pennies. Alter you diagram to represent this situation. Use your diagram to determine how many more pennies Jada would need to add to her collection in order to have the same number of pennies as Marcus.

Number of pennies that Jada needs to add to her collection: ________

5) John had a certain number of marbles as shown in the diagram below. Frederick had four times as many marbles as John. Draw a diagram of this situation.

John’s marbles: ⬤⬤⬤⬤⬤

Frederick’s marbles:

Suppose that John doubled the number of marbles that he had and then bought 15 more marbles. Alter your diagram to represent this situation.

Use your diagram to determine how many more marbles Frederick would need to add to his collection in order to have the same number of marbles that John has.

Number of marbles that Frederick needs to add to his collection: ________

Figure A-2: Tasks 4 and 5 from the Harper Middle School Task set

Tasks six and seven were meant to serve as an informal assessment of students’ understanding of a rectangle as the representation of an unknown, yet specific quantity which is
able to be iterated. By correctly representing four times Darrell’s amount of candy, a student was demonstrating that he or she understood that there was a certain, unspecified number of pieces of candy in Darrell’s box and that that number was iterated four times in order to arrive at Paula’s amount of candy.

If a student was not able to spontaneously represent $n$ times an unknown quantity, I would give them a sub-task in which I drew a box with circles in it and asked them how many boxes would be needed to be drawn in order to have say, five times as many circles. After this experience, students would then be able to draw the correct number of rectangles.

Tasks six and seven were also intended to serve as a means of enabling students to represent a problem situation whose structure was algebraic in nature. The problem situation was written in such a manner that was cognitively accessible and able to be represented by students but whose representation had the potential to serve as a model about which students could reason algebraically.
6) Darrell bought a box of candy with a certain number of pieces in it (please see diagram below). Paula bought four times the amount of candy that Darrell bought. Show this on the diagram below.

Darrell’s Candy: 

Paula’s Candy:

If Darrell doubled the amount of candy that he has and then bought a box of candy containing 10 pieces in it, he and Paula would have the same number of pieces of candy. Alter your diagram to show this situation.

Use your diagram to figure out how much candy each person started out with.

7) Crystal bought a box of candy with a certain number of pieces in it (please see below). Jarrod bought five times the amount of candy that Crystal bought. Show this on the diagram below.

Crystal’s Candy: 

Jarrod’s Candy:

If Crystal tripled the amount of candy that she had and then bought a box of candy containing 12 pieces in it, she and Jarrod would have the same number of pieces of candy. Show this situation on your diagram.

Use your diagram to figure out how much candy each person started out with.

Figure A-3: Tasks 6 and 7 from the Harper Middle School set

Tasks eight through ten were also combined assessment and instructional tasks. These tasks presented realistic situations which had the underlying structure \( ax+b = cx-d \). In order to solve these tasks algebraically, the problem solver must equalize the relationship between two non-equivalent quantities by creating a quantity that does not exist within the realistic situation. Thus, this task is meant to assess students’ understanding of hypothetical equivalence (Clement, 1982).
A student who was not able to answer these tasks correctly would have been asked to draw as much as they knew (presumably, students at this point could draw a representation for the unknown quantity). Then, students would be asked, show where the diagram for each person’s amount of candy would end since they had the same amount of candy. Then, students could be asked the amount of candy that the remaining space in each diagram was worth.

8) Ayanna has a certain number of CDs in her collection. Avery has six times as many CDs in his collection as Ayanna. Draw a diagram to represent this situation.

Ayanna’s number of CDs:

Avery’s number of CDs:

Suppose that Ayanna received more CDs on her birthday. As a result, she has triple the number of CDs that she started with. Suppose that Ayanna then bought 17 more CDs and that Avery bought 5 more CDs. As a result of this, Ayanna and Avery now have the same number of CDs in their collection. Alter your diagram to show this.

Use your diagram to figure out how many CDs each person started out with.

Ayanna’s original number of CDs: ______

Avery’s original number of CDs: ______

(Modified from Bodanskii, 1991/1969; Maier, 1997)

9) At the school library, Shelf A had a certain number of books on it. Shelf B had five times as many books on it. Draw a diagram to represent this situation.

Number of books on Shelf A:

Number of books on Shelf B:

If 8 books were added to Shelf B and 20 books were added to Shelf A, the two shelves would have the same number of books. Alter your diagram to show this. Use your diagram to figure out how many books each bookshelf held initially.

Original number of books on Shelf A: ______
Original number of books on Shelf B: _______

(Modified from Maier, 1997)

10) Cynthia was making headwraps for her and her cousin. She found two strips of fabric to work with. The length of Fabric Strip B was five times as long as the length of Fabric Strip A. Draw a diagram to represent this situation.

Length of Fabric Strip A: 

Length of Fabric Strip B: 

Cynthia added 13 inches of fabric to Fabric Strip B. She also doubled the length of Fabric Strip A and then added 34 more inches of fabric to it. When she did this, the two fabric strips became the same length. Alter your diagram to show this. Use your diagram to figure out how long each fabric strip was initially.

Original length of Fabric Strip A: _______

Original length of Fabric Strip B: _______

Figure A-4: Tasks 8 through 10 from the Additive Deficit Task set

Tasks 11 and 12 were assessment tasks meant to determine students’ understanding of additive quantities. In other words, in order to represent James’ number of video games in question 11, students would need to represent David’s (unknown) number of video games along with an icon that represents 18 more video games. The manner in which students represent the additional video games demonstrates their understanding of the additive combination of the unknown quantity and known quantity. It also demonstrates their conception of the known quantity as being comprised of singletons or as being a unit in itself.

Further, students’ means of representing the fact that there are a total of 48 games allows the researcher to observe students’ conceptualization of the relationship between the unknown and known parts of a known whole. Question 12 allows for this same inquiry into students’ thinking about the relationship between the unknown parts of a known whole.
11) David had a certain number of video games in his collection. James had 18 more video games than David. Draw a diagram of this situation.

   David’s number of video games:

   James’s number of video games:

   Altogether, the boys had 48 video games. Show this information on your diagram. Use your diagram to figure out how many video games each person has.

   (Modified from Bodanskii, 1991/1969)

12) A movie theatre has three screening rooms. There are a certain number of people in Room A. Room B has four times as many people in it as Room A. Room C has three times as many people as Room A. Draw a diagram to represent this situation.

   Number of people in Room A:

   Number of people in Room B:

   Number of people in Room C:

   Altogether, there are 80 people in the movie theatre. Show this information on your diagram. Use your diagram to figure out how many people each screening room has.

Figure A-5: Tasks 11 and 12 from the Additive Combination Task set

Problems like tasks 13 through 14 were introduced with the intent of enabling students to construct an anticipation that it is necessary to equalize the known and unknown quantities (even if they are not originally equal to one another) in order to find the value of the unknown quantity. I hypothesized that, by having to represent the fact that the auditorium has five times as many chairs as the band room, students would need to reflect upon the relationship between the two quantities in such a manner as to create a diagram which displayed an equivalence that does not exist.
13) The school auditorium had 17 more chairs than the band room. When 43 chairs were added to the [school auditorium], there were 5 times [as many chairs in the school auditorium than in the band room].” (Modified from Bodanskii, 1991/1969, p. 302). Draw a diagram to represent this situation. Use your diagram to find out how many chairs there were originally in each room.

Number of chairs in auditorium:

Number of chairs in band room:

14) Jenna had 5 times as much money as Harold did. After Jenna and Harold both earned $45 shoveling snow, Jenna had 3 times as much money as Harold. Draw a diagram to represent this situation. Use your diagram to find out how much money each person started out with. (Modified from Bodanskii, 1991/1969)

Figure A-6: Tasks 13 and 14 from the Double Sided Unknown Quantity Tasks

Students demonstrated difficulty in answering tasks 13 and 14. One student’s encountered difficulties, for example, were in representing problem 14. The student was unable to spontaneously represent the fact that after $45 was added to both Harold and Jenna’s amounts, Jenna had 3 times as much money as Harold.

I determined that students needed to build a more abstract notion of the quantitative relationships that exist between differently-configured equivalent wholes, therefore I designed tasks 15 through 37. Tasks 15 through 19 were based, in large part, on Mikulina’s work with elementary school students. These tasks allowed students to have the experience of creating (and later representing) an object that is multiplicatively related to a given object. Students then had that experience upon which to draw when considering the multiplicative relationships between parts of two equivalent wholes.

Task 20 through 22 allow students to build a conceptualization of the relationship of dependency between parts of equivalent wholes and to build equations based in an experience
that is experientially and cognitively accessible to them (and that could be re-created at any time as necessary).

In task 15, a pre-cut strip of paper was given to students along with a large piece of paper, scissors, and a straight edge.

15) Create a paper strip that is three times as long as the strip given to you. Create a diagram of the two strips.

  Strip 1:

  Strip 2:

(Modified from Mikulina, 1991/1969)

16) Suppose that the length of Strip 1 was 30 inches. Write a number sentence to show how you would find out the length of Strip 2:

   Length of Strip 2 = _______________ = ________

17) Suppose that the length of Strip 2 was 50 inches. Write a number sentence to show how you would find out the length of Strip 1:

   Length of Strip 1 = _______________ = ________

18) Suppose that the lengths of Strips 1 & 2 were 60 inches altogether.

   a) Write a number sentence to show how you would find out the length of Strips 1& 2:

      Length of Strip 1 = _______________ = ________
      Length of Strip 2 = _______________ = ________

   b) Write a letter & number sentence that shows how you could find the length of Strip 1 if you only knew the length of Strip 2

      Length of Strip 1 =

   c) Write a letter & number sentence that shows how you could find the length of Strip 2 if you only knew the length of Strip 1.
Length of Strip 2 =

19) Tape Strips 1 and 2 together. Create a new strip (Strip 3) that is equal in length to this combination.

Create a diagram of the three strips as they appear now.

Strips 1 & 2:

Strip 3:

20) Suppose that the length of Strip 2 was 60 inches.

Write a number sentence to show how you would find out the length of Strip 3:

Length of Strip 3 = _______________ = ________

21) Suppose that the length of Strip 1 was 40 inches.

Write a number sentence to show how you would find out the length of Strip 3:

Length of Strip 1 = _______________ = ________

22) Suppose that the length of Strip 3 was 20 inches.

a) Write a number sentence to show how you would find out the length of Strip 1 and the length of Strip 2:

Length of Strip 1 = _______________ = ________

Length of Strip 2 = _______________ = ________

b) Write a letter & number sentence that shows how you could find the length of Strip 2 if you only knew the length of Strip 3

Length of Strip 2 =

c) Write a letter & number sentence that shows how you could find the length of Strip 3 if you only knew the length of Strip 2

Length of Strip 3 =
Tasks 23 through 34 were paper strip puzzles. The purpose of these tasks was to enable students to construct an understanding of the multiplicative relationship between parts of equivalent wholes given a known part or whole. Students had to act upon the diagram by counting the number of rectangles of one size that occupied the same amount of space as a rectangle belonging to a different whole and then utilize the given length and the relationship between the two rectangle lengths in order to find the missing length. Students then had to explicate the relationship between the rectangle lengths. By having them to engage in this activity, it was my goal that students might have the opportunity to reflect on the relationship between the number of identical parts comprising a given whole, the value of that part, and the value of the whole.

23) Suppose that the following diagram represents strips that someone cut out and taped together. No rulers were used in this process.

![Diagram of Strip 4 and Strip 5]

Strip 4: 

Strip 5: 

23) Suppose that the following diagram represents strips that someone cut out and taped together. No rulers were used in this process.

Strip 4: 

Strip 5: 

24) Suppose that $a = 15$ inches. Show this on the diagram
a = length of Strip 1         b = length of Strip 2           c = length of Strip 3

a) Write a number sentence to show how to find out what b is equal to:
   \[ b = \text{__________} = \text{______} \]

b) Write a number sentence to show how to find out what c is equal to:
   \[ c = \text{__________} = \text{______} \]

(Adapted from Mikulina, 1991/1969)

25) Suppose that b = 40 inches. Show this on the diagram

a = length of Strip 1         b = length of Strip 2           c = length of Strip 3

a) Write a number sentence to show how to find out what a is equal to:
   \[ a = \text{__________} = \text{______} \]
b) Write a number sentence to show how to find out what c is equal to:

\[ c = \underline{\quad} \quad = \quad \underline{\quad} \]

26) Suppose that \( c = 60 \) inches. Show this on the diagram

Strip 4:

\[ \text{Strip 4:} \]

\[ \text{a} \quad \text{a} \quad \text{b} \]

Strip 5:

\[ \text{Strip 5:} \]

\[ \text{c} \quad \text{c} \quad \text{c} \]

a) Write a number sentence to show how to find out what a is equal to:

\[ a = \underline{\quad} \quad = \quad \underline{\quad} \]

b) Write a number sentence to show how to find out what b is equal to:

\[ b = \underline{\quad} \quad = \quad \underline{\quad} \]

27)

Strip 4:

\[ \text{Strip 4:} \]

\[ \text{a} \quad \text{a} \quad \text{b} \]

Strip 5:

\[ \text{Strip 5:} \]

\[ \text{c} \quad \text{c} \quad \text{c} \]

a) Write a letter & number sentence to show how you would find a if you only knew b

\[ a = \quad \]

b) Write a letter & number sentence to show how you would find b if you only knew a

\[ b = \quad \]
c) Write a letter & number sentence to show how you would find c if you only knew a

\[ c = \]

28) Suppose that the following diagram represents strips that someone cut out and taped together. No rulers were used in this process.

\[ \text{Strip 4: } \]
\[ \text{Strip 5: } \]
\[ a = \text{length of Strip 1} \]
\[ b = \text{length of Strip 2} \]
\[ c = \text{length of Strip 3} \]

Explain how you think Strips 2 and 3 were created.

29) Suppose that \( a = 12 \text{ inches} \). Show this on the diagram

\[ \text{Strip 4: } \]
\[ \text{Strip 5: } \]
\[ a = \text{length of Strip 1} \]
\[ b = \text{length of Strip 2} \]
\[ c = \text{length of Strip 3} \]

a) Write a number sentence to show how to find out what \( b \) is equal to:
b = ____________  =  ______

b) Write a number sentence to show how to find out what c is equal to:

c = ____________  =  ______

30) Suppose that b = 30 inches. Show this on the diagram

Strip 4:

Strip 5:

a = length of Strip 1
b = length of Strip 2
c = length of Strip 3

a) Write a number sentence to show how to find out what a is equal to:

a = ____________  =  ______

b) Write a number sentence to show how to find out what c is equal to:

c = ____________  =  ______

31) Suppose that c = 25 inches. Show this on the diagram

Strip 4:

Strip 5:

a = length of Strip 1
b = length of Strip 2
c = length of Strip 3

a) Write a number sentence to show how to find out what a is equal to:
   
   a = ____________  =  ______

b) Write a number sentence to show how to find out what b is equal to:
   
   b = ____________  =  ______

32) Suppose that a + 10 = 40 inches. Show this on the diagram

\[ \text{Strip 4:} \]
\[ \text{Strip 5:} \]

\[ a = \text{length of Strip 1} \]
\[ b = \text{length of Strip 2} \]
\[ c = \text{length of Strip 3} \]

a) Write a number sentence to show how to find out what b is equal to:
   
   b = ____________  =  ______

b) Write a number sentence to show how to find out what c is equal to:
   
   c = ____________  =  ______

33) Suppose that the following diagram represents strips that someone cut out and taped together.
34) Suppose that the following diagram represents strips that someone cut out and taped together.

Strip 2:

| g | g |

Strip 3:

| g | g | q | q | q |

g = length of Strip 1

a) Suppose that Strip 3 is 51 inches longer than Strip 2. Show this information on your diagram.

b) Complete this equation: g + g + g = ______

c) How long are Strips 1, 2, and 3?

Length of Strip 1 = ____________
Figure A-8: Tasks 23 through 34 from the Paper Strip Puzzles set

Tasks 35 through 37 were intended to build upon the part-whole understandings that students developed during the paper strip puzzles tasks and, while in that familiar context, enable them to represent situations in which there is more than one instance of the unknown quantity.

35) Strip 2 is three times longer than Strip 1. Draw a diagram to represent this situation.

Length of Strip 1:

Length of Strip 2:

If 14 inches were added to Strip 2 and 36 inches were added to Strip 1 the two strips would be the same length. Alter your diagram to show this.

Use your diagram to find the original length of each strip.
Paper Strip B is five times longer than Strip A. Draw a diagram to represent this situation.

Length of Strip A:

Length of Strip B:

The length of Strip A was doubled and then 17 inches were added onto it. 31 inches of paper was added onto Strip B. The two strips are now the same length. Alter your diagram to show this.

Use your diagram to find the original length of each strip.

Original Length of Strip A: ___________ Original Length of Strip B: ___________

36) Paper Strip B is three times longer than Strip A. Draw a diagram to represent this situation.

Length of Strip A:

Length of Strip B:

The length of Strip A was doubled and then 25 inches was added onto it. 12 inches of paper was added onto Strip B. The two strips are now the same length. Alter your diagram to show this.

Use your diagram to find the original length of each strip.

Original Length of Strip A: ___________ Original Length of Strip B: ___________
Figure A-9: Tasks 35 through 37 from the Double Sided Unknown Quantity set

Tasks 38-43 were intended to assess what students understood about the additive and multiplicative relationships between different sized parts of equivalent wholes. The intent of these tasks was also to enable students to construct the understanding that a quantitative whole is decomposable into known and unknown parts. Further, students were intended to learn that when the known parts of equivalent wholes are matched, the unmatched remaining parts are equivalent and multiplicatively related. For example, in Task 41, the left side of the fourth $a$-box on the Strip A diagram and the right side of the 18-box on that same diagram are aligned with the right and left sides of the 42-box on the Strip B diagram. Students may utilize this visual information to determine that since the 18-box occupies part of the length of the 42-box, the remaining portion of the 42-box must be equivalent to the length of two $a$-boxes. Students could then utilize that information in order to find the length of one $a$-box.

Suppose that the following diagram represents strips that someone cut out and taped together. Solve the following puzzles:

38)
39)

Strip A: 

\[ b = \_\_\_\_\_\_\_\_\_\_\_\_ = \_\_\_\_\_\_\_\_\_\_\_\_ in. \]

\[ a = \_\_\_\_\_\_\_\_\_\_\_ = \_\_\_\_\_\_\_\_\_\_\_\_ ft. \]

Strip B:

\[ b = \_\_\_\_\_\_\_\_\_\_\_\_ = \_\_\_\_\_\_\_\_\_\_\_\_ ft. \]

40)

Strip A:

\[ a = \_\_\_\_\_\_\_\_\_\_\_\_ = \_\_\_\_\_\_\_\_\_\_\_\_ in. \]

\[ b = \_\_\_\_\_\_\_\_\_\_\_\_ = \_\_\_\_\_\_\_\_\_\_\_\_ ft. \]

Strip B:

\[ b = \_\_\_\_\_\_\_\_\_\_\_\_ = \_\_\_\_\_\_\_\_\_\_\_\_ ft. \]

41)
Strip A:  
\[ a = \_\_\_\_\_\_ = \_\_\_\_\_\_ \text{ in.} \]

Strip B:  
\[ b = \_\_\_\_\_\_ = \_\_\_\_\_\_ \text{ in.} \]

42)  
Strip A:  
\[ 55 \text{ in.} \]

Strip B:  
\[ 13 \text{ in.} \]

a = \_\_\_\_\_\_ = \_\_\_\_\_\_ \text{ in.} \]

b = \_\_\_\_\_\_ = \_\_\_\_\_\_ \text{ in.} \]

43)  
Strip A:  
\[ a = \_\_\_\_\_\_ = \_\_\_\_\_\_ \text{ in.} \]

b = \_\_\_\_\_\_ = \_\_\_\_\_\_ \text{ in.} \]

a = \_\_\_\_\_\_ = \_\_\_\_\_\_ \text{ in.} \]
Figure A-10: Tasks 38 through 43 from the Paper Strip Puzzles set

Tasks 44 through 48 were created in response to students’ difficulties in being able to solve problems in which the unknown quantities were not adjacent to one another on the diagram. Tasks 44 to 45 were meant to assess students’ ability to find the value of the unknown quantity when identical parts were not aligned. Tasks 46 to 48 were designed to enable students to conceptualize that, following the removal of identical parts from equivalent wholes, the remaining, un-identical parts are equivalent.

44) Suppose that the following diagram represents strips that someone cut out and taped together. Find the value of a in this diagram.
45) Suppose that the following diagram represents strips that someone cut out and taped together. Find the value of c in this diagram.

Strip C:

```
  c  c  c  24 ft.  c  24 ft.
```

Strip D:

```
  c  24 ft.  c  24 ft.  14 ft.
```

c = _____________ = _____ ft.

46)

Strip G:

```
  g  34  g  g  80 in.  12 in.
```

Strip H:

```
  g  800 in.  g  80 in.
```

Draw as many boxes from each diagram that has an identical “twin” in the other diagram (see example)
47)

Strip G:

Strip H:

Identical Parts

Identical Parts

Remaining Parts

Strip C:

Strip D:

Draw as many boxes from each diagram that has an identical “twin” in the other diagram (see example)

Draw as many boxes from each diagram that has an identical “twin” in the other diagram
In an effort to enable Kelly to further understand the relationship between non-identical, but equivalent parts. Strips of paper (as shown in Figure A-12) were glued to card stock and cut by Kelly and I. Kelly reassembled the pieces on a large sheet of paper as they were originally given. She was then requested to find the length of the unmarked rectangles. Kelly solved approximately four tasks of this nature and Valerie solved three of these types of tasks.

Figure A-11: Tasks 44 through 45 from the Non-Adjacent Parts set and 46 through 48 from the Remaining Parts set

<table>
<thead>
<tr>
<th>Strip C</th>
<th>Identical Parts</th>
<th>Identical Parts</th>
<th>Remaining Parts</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strip</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Figure A-12: Tasks 49 through 50 from the Moveable Pieces set

Tasks 51 and 52 were created to present hypothetical equivalence problems in a more structured way so that study participants might be more likely to perceive the implicit relationships between quantities. These tasks were used specifically by Denise who was experiencing difficulty with solving the hypothetical equivalence tasks. In task 51, study participants were asked to represent Dennis’s number of pennies (according to the manner in which it was originally described) in two places — at the top and bottom of the paper — and his number of pennies according to Jonathan’s number of pennies at the bottom of the paper. It was my intent that students’ attention be drawn to the fact that they are representing the same quantity (Dennis’s number of pennies) in two ways and that, based upon that, they might choose to operate upon the diagrams representing Dennis’s number of pennies accordingly.

51. Dennis had six times as many pennies as Jonathan. Jonathan then collected 60 more pennies. Draw a diagram of this situation.
Jonathan’s number of pennies: 

Dennis’s number of pennies: 

Dennis now has two times as many pennies as Jonathan. In the box below, draw Jonathan’s total number of pennies. Then show Dennis’s total number of pennies based on the new information.

Jonathan’s number of pennies: 

Dennis’s number of pennies: 

Dennis’s number of pennies: 

(From original diagram) 

52. During the walkathon, Jose walked six times as many blocks as Jennifer. Jose and Jennifer then each walked 20 more blocks. Draw a diagram of this situation.

Jennifer’s distance walked: 

Jose’s distance walked: 

Jose has now walked twice as far as Jennifer. In the box below, draw Jennifer’s distance walked. Then show Jose’s distance walked based on the new information.
Jennifer’s distance walked: 

Jose’s distance walked: 

Jose’s distance walked: 
(From original diagram)

Figure A-13: Tasks 51 and 52 from the Structured Hypothetical Equivalence set

Tasks 53-56 were used with the intent of enabling students to develop the conception that it is necessary for a statement of equality to be written in terms of one unknown quantity in order to be able to ascertain the value of that quantity. Another purpose of these tasks was to enable students to develop the understanding of additive and multiplicative quantities as objects. For example, in task 53, two wire lengths sum to 48 inches. A student who successfully solves this problem algebraically would have to represent one of the wire lengths in terms of the other wire length and, combine this representation with the representation of the generating wire length (Ng & Lee, 2009) to attain a particular total. Having to conceptualize one quantity in terms of another and then apply a quantitative operation to that newly represented quantity seems to be that which could enable a student to view an additive quantity as a legitimate input. Thus, a student might be likely to develop an understanding of and additive quantity as an object.

53. When added together, the lengths of Wire A and Wire B are 48 inches. Draw a diagram of this situation.

If Wire A was 5 inches longer and if Wire B was doubled in length, the sum of the lengths of the two wires would be 65 inches. Show this situation on a diagram.

Use your diagram to figure out the original length of each wire.
54. When added together, the lengths of Wire C and Wire D are 30 inches. Draw a diagram of this situation.

If Wire C was 10 inches longer and if Wire D was tripled in length, the sum of the lengths of the two wires would be 90 inches. Show this situation on a diagram.

Use your diagram to figure out the original length of each wire.

55. When added together, the lengths of String E and String F are 45 inches. Create a diagram using letters and numbers only to represent this situation.

If 15 inches was added to String E and if String F was quadrupled in length, the sum of the lengths of the two wires would be 120 inches. Show this situation on a diagram.

Use your diagram to figure out the original length of each wire.

Figure A-14: Tasks 54 through 55 from the Additive Quantity as Input set

Tasks 56-58 were designed as a tool to help the problem solver to develop the understanding that establishing the equivalence of two quantities that are written in two different ways is helpful in finding the value of the unknown quantity. This set of tasks was utilized specifically by Kelly. For example, in task 57, when three additional people are stated to have started jumping rope, it is stated that there is now two times as many people playing kickball as jumping rope.

Students were requested to represent the number of people now playing kickball in terms of the new number of people jumping rope. Students were then requested to compare the new kickball representation with the old one in order to find the value of the unknown quantity. By doing so, I anticipated that students would develop the notion that finding and establishing a
relationship of equality between two differently configured, but equivalent wholes was beneficial to finding the value of the unknown quantity.

Task 58 was also used to help to introduce numbers and letters as a means of representing the situation. By having a number and letter “diagram” accompany a rectangular diagram, it was my intent that students might draw an association between the two representations of the unknown quantity—letters and boxes—and draw the conclusion that a letter is a more succinct means of representing the same quantity.

56. At the mall, there were five times as many people in Store B as there were in Store A. Create a diagram of this situation in the space below.

# of people in Store A:
# of people in Store B:

Four people walked into Store A. Store B now has three times as many people as Store A. Draw a new diagram to represent this situation. Please use numbers instead of boxes to represent the four people who walked into the store.

# of people in Store A:
# of people in Store B:

Use your diagram to figure out how many people were originally in each store.

57. At recess, four times as many people were playing kickball as were jumping rope. Create a diagram of this situation in the space below.

# of people playing kickball
# of people jumping rope:

When three more people start jumping rope, there are then two times as many people playing kickball as there were jumping rope. Draw a new diagram to represent this situation. Please use numbers instead of boxes to represent the additional people who started jumping rope.

# of people playing kickball:
# of people jumping rope:
Use your diagram to figure out how many people were originally in each store.

58. Robert had ridden his bike five times farther than his brother Jacob. Create a diagram of this situation using letters only.

# of miles ridden by Jacob:

# of miles ridden by Robert:

Robert stopped to take a break. When Jacob rode his bike six more miles, Robert had then ridden three times farther than Jacob. Draw a new diagram to represent this situation using letters and numbers only. (Please use numbers to represent the additional miles that Jacob rode).

# of miles ridden by Jacob:

# of miles ridden by Robert:

Use your diagram to figure out how many people were originally in each store.

Figure A-15: Tasks 56 through 58 from the Hypothetical Equivalence set (partial scaffolding)

Tasks 59-60 were designed to enable students to transition from the use of diagrams to the use of letters and numbers. These tasks still contained content that focused upon linear measure, but students were required to use letters and numbers only to represent and solve the problem. At this point in the study, students were able to represent and solve most of the previous tasks using diagrams. Thus, these tasks were also given to assess students’ algebraic understandings at this point in the study.

59. Two pipes are sitting on the ground. Pipe A is three times longer than Pipe B. If the length of Pipe B was doubled and then extended 40 feet and if the length of Pipe A was extended 20 feet, they would have the same length. Create an equation to represent this situation. Use your equation to find out how long each pipe is right now.

(Modified from Bodanskii, 1991/1969)

60. Wire A is six times longer than Wire B. If Wire B was triple the size that it is now and then was extended 12 yards longer and if Wire A was extended 8 yards longer, the wires would be the same length. Create an equation to represent this situation. Use your equation to figure out the original length of each wire.
Tasks 61-62 were assessment items. I wanted to ascertain how students, using only letter and numbers, represented and solved a problem in which the unknown quantity was reduced in size.

61. At the sports tournament, there were four times as many people playing volleyball as were playing basketball. If 8 people stopped playing volleyball and 19 people joined basketball, there would be the same number of people playing each sport. Create an equation to represent this situation. Use your equation to figure out how many people are currently playing each sport.

(Modified from Maier, 1997)

62. Pipe A is four times as long as Pipe B. If 10 inches were cut from Pipe A and 23 inches of pipe were welded to Pipe B, the pipes would be the same length. Write an equation to represent this situation. Use this equation to figure out the original lengths of each pipe.

Tasks 63-64 were designed to assess students’ ability to solve algebra word problems without scaffolding. These tasks were only given once students had demonstrated the pre-requisite understandings needed to solve these tasks.

Solve the following problem using only letters and numbers.

63. Debra walked a certain distance down the street. Starting from the same point, Marsha walked four times as far as Debra. If Debra doubled the distance that she walked and then walked 10 more feet, she and Marsha would have walked the same distance.

64. Tree A is five times taller than Tree B. If Tree B grew 15 more feet and if 13 feet were cut from Tree B, the two trees would be the same height. What was the original height of each tree?
VITA

Jan Green

EDUCATION

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<tr>
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<th>Institution</th>
<th>Location</th>
<th>Degree</th>
<th>Field</th>
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<tr>
<td>1999 - 2009</td>
<td>Penn State University</td>
<td>University Park, PA</td>
<td>Ph.D., Curriculum and Instruction, Mathematics Education</td>
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<tr>
<td>1995 - 1999</td>
<td>Arcadia University</td>
<td>Glenside, PA</td>
<td>M. Ed., Mathematics Education</td>
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<tr>
<td>1984 - 1987</td>
<td>Hampton University</td>
<td>Hampton, VA</td>
<td>B.A., Psychology</td>
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CERTIFICATION

PA Mathematics Certification (7-12)

EXPERIENCE

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<th>Location</th>
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<tbody>
<tr>
<td>September, 2009 – Present</td>
<td>Algebra 2 Teacher</td>
<td>Mastery Charter School</td>
<td>Philadelphia, PA</td>
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<td>September, 2008 – June, 2009</td>
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<td>New Media Technology Charter High School</td>
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<td>Supervisor for Pre-service Elementary Teachers</td>
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