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THE LOCAL INDEX THEOREM AND THE TANGENT GROUPOID

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Abstract

The primary goal of this thesis is to study the Getzler calculus of differential operators acting on Clifford modules using the tangent groupoid. In this view, the family of heat kernels associated with a Dirac operator extends to a smooth section of a smooth vector bundle over the tangent groupoid. Over the zero fiber of the tangent groupoid this section is the heat operator of a family of geometric harmonic oscillators on the fibers of the tangent bundle. This leads quickly to a geometric proof of the "local" Atiyah-Singer index theorem.

First we shall study in detail the construction of the tangent groupoid and, more generally, the construction of the deformation to the normal cone. Then we shall show that the heat kernel of any Dirac operator gives a Schwartz-class section of the rescaled spinor bundle of Higson and Yi over the tangent groupoid.

Finally we shall show that we may apply a generalized Getzler's method to Dirac operators corresponding to connections that have non-zero torsion. Bismut's Dirac operator is one such operator. It corresponds to any three-form over the manifold. When the three-form is closed there is a local formula for the index of Bismut Dirac operator. We will show, using residue cocycles, that we may obtain similar results in low dimensions even when the three-form is not closed.

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Chapter 1 Introduction

In this thesis, we explore several topics in local index theory that are tied together by the deformation to the normal cone construction in geometry. For a smooth embedding of manifolds, the deformation to the normal cone is a manifold formed by copies of the ambient manifold smoothly deforming to a copy of the normal bundle of the embedding. The deformation to the normal cone plays the role of a magnifying glass that amplifies geometrical structure in a neighborhood of an embedding in the normal direction. It provides us with a rich algebraic geometric context for the study of global properties of embeddings which has been mostly overlooked in differential geometry.

Associated with an embedding of manifolds, we may also define the concept Euler-like vector field. An Euler-like vector field vanishes on the submanifold and resembles the Euler vector field on the normal bundle. In [BLM19], Bursztyn, Lima, and Meinrenken showed that there is a one-to-one correspondence between Euler-like vector fields and tubular neighborhoods. In Chapter 3, using the deformation to the normal cone, we will provide a new proof and generalize this result to manifolds with filtered structure.

The tangent groupoid is an example of a deformation to the normal cone. The importance of tangent groupoid in index theory was highlighted in the work of Alain Connes [Con94], who used it to give a new proof of the Atiyah-Singer index theorem. Recently, Nigel Higson and Zelin Yi [HY19] underscored the significance of tangent groupoid in Getzler's calculus of differential operators. They showed that the Getzler's symbol calculus can be geometrically explained as a smooth deformation of operators, after introducing a rescaled version of the spinor bundle.

In Chapter 6, we will show that the family of heat kernels for the Dirac operator gives a smooth section of the rescaled bundle over the tangent groupoid that was constructed by Higson and Yi. More precisely, this section is a "Schwartz" section. We shall explain the relationship between this and the local proof of Atiyah-Singer index theorem.

In [Bis89], Bismut proved a version of the local index theorem for the Dirac operators arising from certain connections with torsion. These Dirac operators are essentially the Levi-Civita Dirac operators deformed by a closed 3-form. Bismut's proof uses stochastic methods. In Chapter 7, we use the method of rescaled bundle, and a modification of Getzler's rescaling technique, to provide a new proof of the local index theorem for such deformed operators.

For non-closed 3-forms, the Getzler calculus fails. However, we may study the Connes-Moscovici residue cocycle in cyclic cohomology corresponding to these Dirac operators, and in that context we may still apply the rescaling method to obtain a formula for the residue cocycles. In Chapter 8, we provide our results in dimension $n = 4, 6$. These are the first computations of the Connes-Moscovici residue cocycles beyond the case of the Levi-Civita Dirac operator, and the first where non-zero higher terms are calculated.

In the following four sections, we give more detailed introductions to these topics.

1.1 Euler-Like Vector Fields, and Deformation to the Normal Cone

This section is mostly a verbatim copy of the introduction to the paper [SH18].

The *Euler vector field* on a finite-dimensional real vector space V is the infinitesimal generator of the scalar multiplication flow. Thus if f is a smooth function on V, then its derivative in the direction of the Euler vector field is

$$E(f)(v) = \frac{d}{dt}\bigg|_{t=0} f(e^t v). \tag{1.1}$$

If $x_1, \ldots, x_q$ is any linear coordinate system on V (in other words, a basis for the dual vector space V^*), then

$$E = x_1 \frac{\partial}{\partial x_1} + \cdots + x_n \frac{\partial}{\partial x_n}. \tag{1.2}$$

The Euler vector field is also characterized by the property that if f is a smooth homogeneous function on V of degree q, then

$$E(f) = q \cdot f. \tag{1.3}$$

We have the following extension of the concept of Euler-vector field to the non-linear context:

Definition (See Definition 3.4.1, and [BLM19, Definition 2.5][1]). If M is a smooth embedded submanifold of a smooth manifold V, then an *Euler-like vector field* for the embedding of M into V is a vector field E on V with the property that if f is a smooth function on V that vanishes on M to order $q \geq 1$, then

$$E(f) = q \cdot f + r,$$

where the remainder r is a smooth function that vanishes to order $q+1$ or higher (a smooth function f on V *vanishes to order* $q \geq 1$ *on* M if Df vanishes on M for every linear differential operator D on V of order $q-1$ or less).

If V is the total space of a vector bundle over M, then the Euler vector field on V is Euler-like for the embedding of M into V as the zero section. More generally, recall that a *tubular neighborhood* of M in V is a diffeomorphism from an open neighborhood of the zero section in the total space of the normal bundle

$$N_V M = TV\big|_M \Big/ TM$$

to an open neighborhood of M in V such that:

1. the diffeomorphism is the identity on M (where M is embedded in the normal bundle as the zero section), and
2. the differential of the diffeomorphism, restricted to vertical tangent vectors, induces the identity map from $N_V M$ to itself.

If E is the Euler vector field on the normal bundle, then any tubular neighborhood embedding carries E to an Euler-like vector field defined in a neighborhood of M in V. Let us call this the Euler-like vector field *associated* to the tubular neighborhood embedding.

Our first purpose in chapter 3 is to comment on the following attractive result:

Theorem (See Theorem 3.4.2, and [BLM19, Proposition 2.6]). *The correspondence that associates to a tubular neighborhood embedding its associated Euler-like vector field determines a* bijection *from germs of tubular neighborhoods to germs of Euler vector fields.*

[1] In [BLM19] it is required that Euler-like vector fields be complete. That is not necessary for our purposes and does not affect the results below, which concern germs of Euler-like vector fields near M.

The theorem has a number of applications, and reader is referred to [BLM19] for full details, but here is a simple example. Let (M, ω) be a symplectic manifold and let m be a point in M. By the Poincaré lemma there is a 1-form α on M such that $d\alpha = \omega$ near m. In addition, α can be chosen so that the vector field X defined by $\iota_X \omega = 2\alpha$ is Euler-like for the embedding of $\{m\}$ into M (note that this is a first-order condition on the coefficients of α at m; simple linear algebra shows it can be satisfied). The corresponding tubular neighborhood identifies ω with a 2-form on $T_m M$ having constant coefficients in any linear coordinate system. This proves the Darboux theorem.

We shall examine Theorem 1.1 from the perspective of the *deformation to the normal cone* associated to the embedding of M into V, which in this paper we shall simply call the *deformation space* associated to the embedding. Among other things, the deformation space $\mathbb{N}_V M$ is a smooth manifold that is equipped with a submersion onto $\mathbb{R}$. The fibers of this submersion over all nonzero $x \in \mathbb{R}$ are copies of V, while the fiber over $x = 0$ is the normal bundle for the embedding of M into V. So the deformation space may be described, as a set, as a disjoint union

$$\mathbb{N}_V M = N_V M \times \{0\} \sqcup \bigsqcup_{\lambda \in \mathbb{R}^\times} V \times \{\lambda\}.$$

See Section 3.2 for further details, including, most importantly, a review of the smooth manifold structure on $\mathbb{N}_V M$.

If E is an Euler-like vector field on M, then there is an associated vector field $\mathbf{E}$ on $\mathbb{N}_V M$ that is vertical for the submersion to $\mathbb{R}$, restricts to a copy of E on each fiber $V \times \{\lambda\}$, and restricts to the Euler vector field on the zero fiber $N_V M \times \{0\}$. See Lemma 3.4.3 (this property characterizes Euler-like vector fields).

There is also a canonical vector field $\mathbf{A}$ on the deformation space that restricts to $\lambda \cdot \partial/\partial\lambda$ on the open set

$$V \times \mathbb{R}^\times = \mathbb{N}_V M\Big|_{\mathbb{R}^\times}$$

(moreover $\mathbf{A}$ is vertical on the fiber over $0 \in \mathbb{R}$, and is the negative of the Euler vector field there). The formula

$$\lambda \cdot \mathbf{T} = \mathbf{A} + \mathbf{E}$$

defines a third "translation" vector field $\mathbf{T}$ on the deformation space.

Theorem (See Section 3.4)**.** *The vector fields $\mathbf{E}$, $\mathbf{A}$ and $\mathbf{T}$ are all smooth. The time $t = 1$ flow map associated to the vector field $\mathbf{T}$ sends a neighborhood of the fiber of the deformation space over $\lambda{=}0$ to a neighborhood of the fiber over $\lambda{=}1$ and is a tubular* *neighborhood embedding. This associates a tubular neighborhood embedding to the Euler vector field* E.

Our approach throughout will be algebraic, treating vector fields very explicitly as derivations of algebras of smooth functions, and so on. Indeed we shall follow the algebraic-geometric approach and *define* the deformation space $\mathbb{N}_V M$ as the spectrum of the Rees algebra associated to the filtration of smooth functions on V by order of vanishing on M. This point of view because it fits very well with our second purpose, which is to study deformation spaces in the context of *filtered manifolds*.

A filtered manifold is a smooth manifold that is equipped with an increasing filtration on its tangent bundle which is compatible with Lie brackets of vector fields; see Definition 3.5.1 for details. This concept has arisen in a number of interrelated areas, including partial differential equations and sub-Riemannian geometry. More recently, filtered manifolds have received attention in noncommutative geometry thanks to work in index theory by Connes and Moscovici [CM95], Ponge [Pon01, Pon06] and Van Erp [vE05, vE10a, vE10b].

An r-step filtered manifold V is a manifold with sequence of subbundle of the tangent bundle

$$H^1 \subset H^1 \subset \cdots \subset H^r = TV$$

such that $[H^i, H^j] \subset H^{i+j}$, denoted by (V, H). We may define the bundle of Lie algebras over V

$$\mathfrak{h} := \bigoplus_i H^{i+1}/H^i$$

and by exponentiating $\mathcal{H} := \exp(\mathfrak{h})$, we obtain a bundle of Lie groups, the osculating groups, over V.

For a submanifold of $M \hookrightarrow V$ such that $G^i := H^i \cap TM$ is subbundle of TM, we define a version of the normal bundle $N_V^H M \to M$. Note that we have a bundle osculating groups $\mathcal{G} \to M$. The fiber of the normal bundle over a point $m \in M$, naturally identifies with the homogeneous space

$$\mathcal{H}_m/\mathcal{G}_m.$$

We then define a version of deformation to the normal cone foe the embedding of filtered manifolds, which as a set is given by

$$\mathbb{N}_V^H M = N_V^H M \times \{0\} \sqcup \bigsqcup_{\lambda \in \mathbb{R}^\times} V \times \{\lambda\}.$$

An example of this deformation space is tangent groupoid. Indeed for a filtered manifold (M, G) we may define the tangent groupoid given set-theoretically by

$$\mathbb{T}^G M := T^G M \times \{0\} \bigcup M \times M \times \mathbb{R},$$

where $T^G M$, the 0-fiber, canonically identifies with the bundle of osculating groups $\mathcal{G} \to M$. We obtain that the osculating groups are consistent with the groupoid structure:

Theorem (See Theorem 3.8.2)**.** *The multiplication on the fiber of* $\mathbb{T}^G M$ *over* $(m, 0)$ *that is induced from the groupoid structure on* $\mathbb{T}^G M$ *is the same as the group multiplication operation in the osculating groups.*

1.2 Heat Kernel and the Rescaled Bundle

The rescaled bundle is a smooth vector bundle over the tangent groupoid associated to a spin manifold. For a closed, even-dimensional spin manifold M, with bundle of spinors $S \to M$, the rescaled bundle $\mathbb{S} \to \mathbb{T}M$ as a set is given by the disjoint union

$$\begin{array}{ccc} S \boxtimes S^* & \bigsqcup & \Lambda^* T^* M \\ \downarrow & & \downarrow \\ M \times M \times \mathbb{R}^\times & \bigsqcup & TM \times \{0\}. \end{array} \tag{1.4}$$

This vector bundle was introduced in [HY19] to serve as a suitable geometric space to visualize Getzler's rescaling method. The most notable result about the rescaled bundle is the geometric formulation of the local proof of the Atiyah-Singer index theorem.

Proposition ([HY19])**.** *Let* D^2 *be the square of the Dirac operator on* M*. There is a smooth family of differential operators* $\mathcal{D}$ *acting on sections of* $\mathbb{S}$ *over the source fibers of the tangent groupoid, given by*

$$\mathcal{D} := \begin{cases} \mathcal{D}_{(m,t)} := t^2 D \curvearrowright C^\infty(S \boxtimes S^*; M \times M \times \{t\}) \\ \mathcal{D}_{(m,0)} := \sigma_2(D) \curvearrowright C^\infty(\Lambda^* T^* M \curvearrowright TM \times \{0\}). \end{cases}$$

Here $\sigma_2(D)$ *is the Getzler symbol of the operator* D*, which is a harmonic oscillator operator (see below.)*

In chapter 6 we introduce the space of Schwartz sections $\mathcal{S}(\mathbb{S})$ of the rescaled bundle of smooth sections that are rapidly decaying "along the zero fiber" with all their derivatives.

This space includes the compactly supported sections of the rescaled bundle, but is more useful for heat kernel purposes:

Theorem. (See Theorem 6.2.6) *Let* D *be the Dirac operator acting on the spinor bundle* $S \to M$*. If* $K_t^{D^2} \in C^\infty(S \boxtimes S^*; M \times M)$ *is the heat kernel associated to the Laplacian* D^2*, then we obtain a Schwartz section of the rescaled bundle:*

$$\mathbb{K} := \begin{cases} t^n K_{t^2}^{D^2} & \in C^\infty(S \boxtimes S^*; M \times M \times \{t\}) \\ K_{\tau=1}^{H} & \in C^\infty(\Lambda^* T^* M \curvearrowright TM \times \{0\}). \end{cases}$$

Here K_τ^H *is the heat kernel associated to the harmonic oscillator operator*

$$\sigma_2(D) = H = -\sum_i (\partial_i - \frac{1}{4} R_{ij} x^j)^2 + F$$

acting on $\Lambda^* T^* M \to TM \times \{0\}$.

The space of Schwartz sections forms a convolution algebra, with a family of supertraces. Applying these supertraces to Schwartz section above we obtain the Atiyah-Singer index theorem (See Chapter 6).

1.3 Connes-Moscovici Residue Cocycle for Bismut's Dirac Operators

This section is taken mostly verbatim from the introduction in [SLSJ21].

An even spectral triple consists of an algebra A acting on a $\mathbb{Z}_2$-graded Hilbert space H and an odd (unbounded) self-adjoint operator D on H such that $[D; a]$ is bounded and $a(1 + D^2)^{-1}$ is compact for $a \in A$. Under suitable analytic hypotheses, Connes and Moscovici [CM95] introduced an even cocycle $\Phi = (\varphi_0, \varphi_2, ..., \varphi_{2\lfloor \frac{n}{2} \rfloor})$ in the $(b; B)$-bicomplex computing the periodic cyclic cohomology of A, involving a complicated combination of residues of spectral ζ-functions for operators built from D, $\Delta = D^2$ and A:

$$\varphi_p(a_0, ..., a_p) = \sum_{|k| \le n-p} c_{pk} \operatorname*{Res}_{z=\frac{p}{2}+|k|} \mathrm{Tr}_s(a_0 [D, a_1]^{(k_1)} \cdots [D, a_p]^{(k_p)} \Delta^{-z})$$

The cocycle can be used to compute index pairings with elements of the K-theory

$K_0(A)$, and the resulting formula is referred to as the local index theorem in noncommutative geometry.

The first example of a spectral triple is the case $A = Cl(M)$ for a closed even-dimensional Riemannian spin manifold $(M^n; S \circlearrowleft Cl(T^*M))$, and $D = D^{LC}$ is the spin Dirac operator acting on the space of L^2 spinors $L^2(M; S)$. In this case the Getzler calculus [Get83] can be used to show that all but the lowest terms in the formula for the residue cocycle vanish identically, and to simplify the remaining terms. In [CM95, Remark II.1] it is very briefly remarked that this approach leads to the formula (for the $p+1$-multilinear component):

$$\varphi_p(a_0, \cdots, a_p) = \frac{(2\pi i)^{-n/2}}{p!} \int_M a_0 da_1 \cdots da_p . \det{}^{1/2}\Big(\frac{R^{LC}/2}{\sinh(R^{LC}/2)}\Big)_{[n-p]} \tag{1.5}$$

where $0 \le p \le n$ is even, $a_0, \cdots, a_p \in Cl(M)$ and $R^{LC} \in \Omega^2(M; \mathfrak{o}(TM))$ is the Riemannian curvature. More detailed discussions were given by Ponge [Pon04], Chern-Hu [CH97], and Lescure [Les98].

Bismut [Bis89] studied the local index problem for operators of the form

$$D = D^{LC} + c(B)$$

where D^{LC} is the spin Dirac operator as above and $B \in \Omega^3(M)$ is a closed 3-form (it is alsopossible to work with more general Clifford module bundles). Bismut showed that there is still a local index theorem in this case, where the local supertrace converges:

Theorem (See Theorem 7.2.5, [Bis89, Theorem 1.7])**.**

$$\lim_{t \to 0} k_t^B(x, x) dx = (2\pi i)^{-n/2} \det{}^{1/2}\Big(\frac{R_-/2}{\sinh(R_-/2)}\Big)$$

with k_t^B *is the heat kernel associated to* ${D^B}^2$*, and* R_- *the curvature of a certain metric connection* ∇_- *(depending on* B*).*

In [Bis89] probabilistic methods were used to prove this result, but it was also remarked that other methods, including Getzler's method, could be adapted to this situation. In Chapter 7 we spell out the details of this modification of Getzler's method, and use it to compute the residue cocycle for D. The main result is as follows:

Theorem (Theorem 7.3.5).

$$\varphi_p(a_0, \cdots, a_p) = \frac{(2\pi i)^{-n/2}}{p!} \int_M a_0 da_1 \cdots da_p . \det{}^{1/2}\Big(\frac{R_-/2}{\sinh(R_-/2)}\Big)_{[n-p]} \tag{1.6}$$

1.4 Bismut's Dirac Operator for Connections with Torsion

Part of the motivation for the work described in Section 1.3 was to gain insight into the complicated higher terms in the Connes-Moscovici cocycle in situations where the Getzler calculus is not available and the higher terms do not vanish identically. Toward this goal, in the second part of the chapter 7 we study the residue cocycle for $D = D^{LC} + c(B)$ in low dimensions ($n = 4, 6$) when B is not closed. In this case the Getzler calculus is not available and some of the higher terms do appear.

Theorem (See Section 8.1). *For* $n = 4$ *we have*

$$\varphi_0(a_0) = (2\pi i)^{-2} \int_M a_0 \left(\det{}^{1/2}\Big(\frac{R_-/2}{\sinh(R_-/2)}\Big) + \frac{1}{6} dd^* dB \right)$$

$$\varphi_2(a_0, a_1, a_2) = \frac{(2\pi i)^{-2}}{6} \int_M a_0 g(da_1, da_2) dB$$

$$\varphi_4(a_0, a_1, a_2, a_3, a_4) = \frac{(2\pi i)^{-2}}{6} \int_M a_0 da_1 da_2 da_3 da_4,$$

where κ *is the scalar curvature.*

In a sense φ_0 is the most involved to compute, because it depends on the greatest number of terms in the asymptotic expansion of the heat kernel. For $n = 6$ we obtain the following explicit formula:

Theorem (Theorem 8.2.1).

$$\varphi_0(a_0) = \frac{(2\pi i)^{-3}}{18} \int_M a_0 g^{ab} g^{cd} \big(\frac{1}{2} \nabla_a R^{\top}_{bc} + g^{ef} B_{ace} R^{\top}_{bf}\big) \wedge \nabla_d dB.$$

Chapter 2
Preliminaries

In this chapter we will present several background topics. We will highlight important topics that will be used in later chapters.

2.1 Deformation to the Normal Cone

The deformation to the normal cone originates in algebraic geometry. It is the analogue of the tubular neighborhood embedding in differential geometry. In this section, we discuss the local smooth structure of the deformation space. See [CR08] for more details.

Let $M \hookrightarrow V$ be a smooth embedding of smooth manifolds. We define the deformation to the normal cone $\mathbb{N}_V M$ as a set by the disjoint union

$$V \times \mathbb{R}^{\times} \bigcup N_V M \times \{0\},$$

where $N_V M$ is the normal bundle. In the next subsection, we describe the smooth structure on this set.

2.1.1 C^{∞}-structure on the Deformation Space

Let $U \subset \mathbb{R}^m \times \mathbb{R}^l$ and denote by U_0 the intersection $U \cap \left(\mathbb{R}^m \times \{0\}\right)$. We define the set $\widetilde{U} \subset \mathbb{R}^m \times \mathbb{R}^l \times \mathbb{R}$ to be

$$\widetilde{U} = \{(x, \xi, t) : (x, t\xi) \in U\}.$$

There is a natural bijection

$$\varphi_U : \mathbb{N}_U U_0 \to \widetilde{U}$$

given by the formula

$$(x, y, t) \mapsto (x, \frac{1}{t}y, t)$$

$$(x, y, 0) \mapsto (x, y, 0).$$

Let $U, U' \subset \mathbb{R}^m \times \mathbb{R}^l$ and let $F = (F_1, F_2) : U \to U'$ be a diffeomorphism such that $F_2(x, 0) = 0$ for all x. This will give a diffeomorphism

$$\widetilde{F} : \widetilde{U} \to \widetilde{U}'$$

by means of the formula

$$\begin{cases} (x, \xi, t) \mapsto (F_1(x, t\xi), \frac{1}{t}F_2(x, t\xi), t) & \text{if} \quad t \neq 0 \\ (x, \xi, 0) \mapsto (F_1(x, 0), \frac{\partial F_2}{\partial \xi}(x, 0).\xi, 0) & \text{if} \quad t = 0. \end{cases}$$

Lemma 2.1.1 ([CR08, Lemma 3.2]). *The map* $\widetilde{F} : \widetilde{U} \to \widetilde{U}'$ *is smooth.*

Proof. Since $F_2 : U \to \mathbb{R}^l$ is smooth and $F_2(x, 0) = 0$ for all x, we have

$$F_2(x, t\xi) = t\frac{\partial F_2}{\partial \xi}(x, 0).\xi + t^2 G(x, t\xi)$$

for some smooth function $G : U \to \mathbb{R}^l$. Hence $\widetilde{F}(x, y, t) = (F_1(x, t\xi), \frac{\partial F_2}{\partial \xi}(x, 0).\xi + tG(x, t\xi))$ for all t. Hence $\widetilde{F}$ is smooth. □

A diffeomorphism $F = (F_1, F_2) : U \to U'$, where $F_2(x, 0) = 0$for all x, induces a diffeomorphism of deformation spaces

$$\widehat{F} : \mathbb{N}_U U_0 \to \mathbb{N}_{U'} U'_0,$$

$$(x, \xi, t) \mapsto (F_1(x, t\xi), F_2(x, 0), t),$$

$$(x, \xi, 0) \mapsto (F_1(x, 0), \frac{\partial F_2}{\partial \xi}(x, 0).\xi, t).$$

The diffeomorphism $\widetilde{F}$ is a change of coordinates over deformation to the normal cone:

Lemma 2.1.2 ([CR08, Proposition 3.6]). *We have*

$$\begin{array}{ccc} \mathbb{N}_U U_0 & \xrightarrow{\widehat{F}} & \mathbb{N}_{U'} U_0 \\ \downarrow{\scriptstyle \varphi_U} & & \downarrow{\scriptstyle \varphi_{U'}} \\ \widetilde{U}_1 & \xrightarrow{\widetilde{F}} & \widetilde{U}' \end{array} \quad . \tag{2.1}$$

Let $\{(\mathcal{U}_\alpha, \psi_\alpha)\}$ be a C^∞-atlas on V such that $\psi_\alpha : \mathcal{U}_\alpha \to U_\alpha \subset \mathbb{R}^m \times \mathbb{R}^l$ maps $\mathcal{U}_\alpha \cap M$ to $\mathbb{R}^m \times \{0\}$. If

$$\widehat{\psi}_\alpha : \mathbb{N}_{\mathcal{U}_\alpha}\Big(\mathcal{U}_\alpha \cap M\Big) \to \mathbb{N}_{U_\alpha} U_{\alpha,0}$$

is the induced map on the deformation spaces, then it follows from Lemma 2.1.2 that the set

$$\{\mathbb{N}_{\mathcal{U}_\alpha}\Big(\mathcal{U}_\alpha \cap M\Big), \widetilde{\psi}_\alpha = \varphi_{U_{\alpha\alpha}} \circ \widehat{\psi}_\alpha\}$$

is a C^∞-atlas for the deformation to the normal cone $\mathbb{N}_V M$.

2.1.2 Differential Operators on the Deformation Space

Continuing with the smooth embedding $M \hookrightarrow V$, we clearly have:

Lemma 2.1.3. *If D is a differential operator on V of order k, and $f \in C^\infty(V)$ vanishes to order at least p on M, then Df vanishes to order $p - k$ or more along M.* □

Define the *principal cosymbol* of D, $\sigma_n(D) : C^\infty(N_V M \times \{0\}) \to C^\infty(N_V M \times \{0\})$ as follows. In a local coordinate $\psi = (x, \xi) : \mathcal{U} \to U \subset \mathbb{R}^m \times \mathbb{R}^l$ where $\psi(\mathcal{U}) \cap M \subset \mathbb{R}^m \times \{0\}$, assume D is given by

$$D = \sum_{\alpha,\beta} P_{\alpha,\beta} \frac{\partial^\alpha}{\partial \xi^\alpha} \frac{\partial^\beta}{\partial x^\beta},$$

where $\frac{\partial^\beta}{\partial x^\beta}$ are the derivations along M and $\frac{\partial^\alpha}{\partial \xi^\alpha}$ are derivations in the normal direction. Then the principal cosymbol $\sigma_n(D)$ acts vertically on the normal bundle and is given by the formula

$$\sigma_n(D) := \sum_{|\alpha|=k, \beta=0} P_{\alpha,0} \frac{\partial^\alpha}{\partial \xi^\alpha}.$$

Theorem 2.1.4. *Let D be a differential operator of order k on V. Then the family of operators*

$$\mathcal{D} := \begin{cases} t^k D : C^\infty(V \times \{t\}) \to C^\infty(V \times \{t\}) \\ \sigma_n(D) : C^\infty(N_V M \times \{0\}) \to C^\infty(N_V M \times \{0\}) \end{cases}$$

where $\sigma_n(D)$ is the principal cosymbol gives smooth differential operators on $\mathbb{N}_V M$.

Proof. Let's denote the coordinate components by $\psi = (x, \xi) : \mathcal{U} \to U$. Considering the local coordinate on the deformation space

$$\widehat{\psi} : \mathbb{N}_{\mathcal{U}} \mathcal{U} \cap M \to \widehat{U}$$

given by the formula

$$(v, t) \mapsto (x, \frac{1}{t}\xi, t)$$

$$(X_m, 0) \mapsto (d\psi(X_m), 0).$$

Under this local coordinate the differential operator $\mathcal{D}$ transfer on $\widehat{U}$ as

$$D = \sum_{\alpha,\beta} t^{k-|\alpha|} P_{\alpha,\beta} \frac{\partial^\alpha}{\partial \xi^\alpha} \frac{\partial^\beta}{\partial x^\beta}$$

which is clearly a smooth differential operator. □

2.2 Groupoids

A groupoid is a pair of of sets, the morphism space $\mathbb{G}$ and object space M, with two maps called the source and target maps $s, t : \mathbb{G} \to M$, together with

1. a partial multiplication $\mathbb{G}^{(2)} := \mathbb{G} \ {}_s\times_t \ \mathbb{G} \to \mathbb{G}$, $(\alpha, \beta) \mapsto \alpha \circ \beta$
2. a unit map $u : M \to \mathbb{G}$, $m \mapsto \mathrm{id}_m$
3. an inverse map $i : \mathbb{G} \to \mathbb{G}$, $i(\alpha) = \alpha^{-1}$

such that for every $\alpha, \beta, \gamma \in \mathbb{G}$, and $m \in M$ we have

- $\alpha \circ (\beta \circ \gamma) = (\alpha \circ \beta) \circ \gamma$ as long as the multiplications make sense.
- $s(\alpha \circ \beta) = s(\beta)$ and $t(\alpha \circ \beta) = t(\alpha)$
- $u(t(\alpha))\alpha = \alpha \circ u(s(\alpha)) = \alpha$
- $\alpha^{-1} \circ \alpha = \mathrm{id}_{s(\alpha)}$ and $\alpha \circ \alpha^{-1} = \mathrm{id}_{t(\alpha)}$

We use the notation $s, t : \mathbb{G} \rightrightarrows M$ for the groupoid.

2.2.1 Lie Groupoids

A Lie groupoid $s, t : \mathbb{G} \rightrightarrows M$ is a groupoid where the sets of objects M and morphisms $\mathbb{G}$ are smooth manifolds, the source, target, composition, unit and inversion maps are all smooth maps and additionally the source and target maps are submersions.

Example 2.2.1.

- *For every manifold* M, $M \rightrightarrows M$ *with source and target map the identity map is a Lie groupoid called the* unit groupoid*.*

- *For a smooth manifold* M, $\pi_1, \pi_2 : M \times M \rightrightarrows M$ *with projection to first and second components, as target and source maps is known as* pair groupoid*. The partial multiplication is given by*

$$(m_1, m_2) \circ (m_2, m_3) \mapsto (m_1, m_3).$$

- *If* $\pi : E \to M$ *is a smooth vector bundle, then* $\pi, \pi : E \rightrightarrows M$ *is a Lie groupoid. The partial multiplication is given by the formula:*

$$e_m \circ f_m \mapsto e_m + f_m$$

  for $e_m, f_m \in E_m$*.*

An important example of a Lie groupoid is the tangent groupoid. This groupoid is defined as the deformation to the normal cone corresponding to the diagonal embedding $M \mapsto M \times M$ denoted by $\mathbb{T}M$. Since the deformation to the normal cone is a covariant functor from the category of embedding of smooth manifolds to category of smooth manifolds (see [CR08, Remark 3.7]), the embedding of the Lie groupoid

$$M \rightrightarrows M \hookrightarrow M \times M \rightrightarrows M$$

induces a groupoid structure on the deformation space $\mathbb{T}M$.

Definition 2.2.1. The tangent groupoid, $\mathbb{T}M \rightrightarrows M \times M$, as a set, is given by the disjoint union

$$\begin{array}{ccc} M \times M \times \mathbb{R}^{\times} & \bigsqcup & TM \\ \downarrow & & \downarrow \\ M \times \mathbb{R}^{\times} & \bigsqcup & M \times \{0\} \end{array} \tag{2.2}$$

Note that on zero-fiber of the deformation to the normal cone $\mathbb{N}_{M \times M}M$ we have the normal bundle $N_{M \times M}M$, where we can identify with the tangent bundle:

$$N_{M \times M}M \to TM$$

given by the formula

$$[X_m, Y_m] \mapsto X_m - Y_m.$$

In Definition 4.1, we used this identification. The composition law of the tangent groupoid is given by the second and third bullet points of Example 2.2.1 for the nonzero-fibers and the zero-fiber, respectively.

2.2.2 Groupoid Convolution Algebra

Associated to the tangent groupoid is the convolution algebra of compactly supported functions.

Definition 2.2.2. A left Haar system on a Lie groupoid $s, t : \mathbb{G} \to M$ is a family of smooth measures μ_m on the source fibers $\mathbb{G}_m = s^{-1}(m)$, such that

1. For every compactly supported smooth function f on $\mathbb{G}$, the following map from M to $\mathbb{C}$ is smooth:
$$m \mapsto \int_{\mathbb{G}_m} f(\gamma) d\mu_m(\gamma).$$
2. If $\gamma_1 : m \to n$ and if $f \in C_c^\infty(\mathbb{G})$, then we have:
$$\int_{\mathbb{G}_n} f(\gamma_1 \circ \gamma) d\mu_m(\gamma) = \int_{\mathbb{G}_m} f(\gamma) d\mu_m(\gamma).$$

Let μ be a smooth measure on the smooth manifold M^n, which is considered as a smooth section of the density bundle $|\Lambda| \to M$. We may choose an atlas $\{(\mathcal{W}_\alpha, \psi_\alpha)\}_\alpha$ of coordinate charts, such that for every $f \in C^\infty(M)$ we have

$$\int_{\mathcal{W}_\alpha} f(m) d\mu(m) = \int_{\psi_\alpha(\mathcal{W}_\alpha)} t_\alpha(x) f \circ \psi_\alpha^{-1}(x) dx$$

where dx is the Haar measure on $\mathbb{R}^n$, and $t_\alpha : \psi_\alpha(\mathcal{W}_\alpha) \to \mathbb{R}_+$ are functions such that

$$t_\alpha t_\beta^{-1} = \det(\psi_\alpha \circ \psi_\beta^{-1})$$

on the intersection $\psi_\alpha(\mathcal{W}_\alpha) \cap \psi_\beta(\mathcal{W}_\beta)$.

Denote by $\mathcal{U}_\alpha := \mathcal{W}_\alpha \times \mathcal{W}_\alpha \subset M \times M$, and define

$$\varphi_\alpha : \mathcal{W}_\alpha \times \mathcal{W}_\alpha \to \mathbb{R}^n \times \mathbb{R}^n,$$

$$(x,y) \mapsto (\psi_\alpha(x), \psi_\alpha(x) - \psi_\alpha(y)).$$

Define

$$\widetilde{U}_\alpha := \{(x,y,t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} : \quad (x, ty) \in \varphi_\alpha(W_\alpha \times W_\alpha)\}.$$

We then obtain induced coordinate charts for the tangent groupoid that cover the zero-fiber:

$$\widetilde{\varphi}_\alpha : \widetilde{U}_\alpha := \mathbb{N}_{\mathcal{U}_\alpha} \mathcal{W}_\alpha \to \widetilde{U}_\alpha \tag{2.3}$$

$$\begin{cases} (x,y,t) & \mapsto (\psi_\alpha(x), \frac{\psi_\alpha(x) - \psi_\alpha(y)}{t}, t) \\ (x, X_x, 0) & \mapsto (x, X_x, 0). \end{cases}$$

Theorem 2.2.3. *The family of source-wise measures*

$$\mu := \begin{cases} \mu_{(m,t)} = |t|^{-\dim(M)} \mu & \text{on} \quad \mathbb{T}M_{(m,t)} = \{m\} \times M \times \{t\} \\ \mu_{(X_m,0)} := \mu_m & \text{on} \quad \mathbb{T}M_{(X_m,0)} = T_m M \times \{0\} \end{cases}$$

where the μ_m *is the induced smooth translation invariant measure on the tangent space* $T_m M$*, defines a Haar system on the tangent groupoid.*

Proof. The family of measures is clearly smooth on the open subset $M \times M \times \mathbb{R}^\times \subset \mathbb{T}M$. So we need to focus on open subset intersecting the zero-fiber. Without loss of generality, we can consider the case $M = \widetilde{\mathcal{U}}_\alpha$. Under the isomorphism $\widetilde{\varphi}_\alpha$, (2.3), we obtain a family of measures on $\widetilde{U}_\alpha$:

$$\widetilde{\varphi}_\alpha^{-1*} \mu_{(x,t)}(y) = \mu_{\mathbb{R}^n}, \quad t \neq 0$$

and

$$\widetilde{\varphi}_\alpha^{-1*} \mu_{(X_x,0)}(y) = \mu_{\mathbb{R}^n}, \quad t = 0$$

So we have $\widetilde{\varphi}_\alpha^{-1*} \mu_{(x,t)} = \mu_{\mathbb{R}^n}$ for all $t \in \mathbb{R}$, which is smooth. □

Proposition 2.2.1 ([CR08, Section 3.1])**.** *For two compactly supported function* $f_1, f_2 \in C_c^\infty(\mathbb{T}M)$ *the* convolution *product*

$$f_1 * f_2 = \begin{cases} (x,y,t) \mapsto t^{-n} \int_M f_1(x,z,t) f_2(z,y,t) d\mu(z), & M \times M \times \{t\} \\ (X_m, 0) \mapsto \int_{T_m M} f_1(X_m - Y_m, 0) f_2(Y_m, 0) d\mu_m(Y_m), & TM \times \{0\} \end{cases}$$

gives a compactly supported smooth function on $\mathbb{T}M$*. This product is associative.* □

2.3 Clifford Algebra and Rescaling

In this section we discuss Clifford algebras and spin groups. For more details see [LM89]. Let V be a real vector space with an inner product. We define the Clifford algebra $\mathrm{Cl}(V)$ defined by the property that there exists an injective linear map $c : V \to \mathrm{Cl}(V)$ where its image generates the algebra $\mathrm{Cl}(V)$ subject to the relation

$$c(v)^2 = -||v||^2 1$$

for all $v \in V$. The Clifford algebra is naturally filtered:

$$\mathbb{R} = \mathrm{Cl}_0(V) \subset \mathrm{Cl}_1(V) \subset \cdots \subset \mathrm{Cl}_n(V) = \mathrm{Cl}(V),$$

where

$$\mathrm{Cl}_k(V) = \mathrm{span}\{v_1 \cdots v_k : \ v_i \in V \sqcup \mathbb{R}\}.$$

We denote the complexified Clifford algebra $\mathrm{Cl}(V) \otimes \mathbb{C}$ by $\mathbb{C}\mathrm{l}(V)$.

2.3.1 Rescaling and Exterior Algebra

Theorem 2.3.1. *The associated graded algebra of the Clifford algebra is naturally isomorphic to the exterior algebra via the homomorphism*

$$\frac{\mathrm{Cl}^k(V)}{\mathrm{Cl}^{k-1}(V)} \to \Lambda^k V \simeq \Lambda^k V^*,$$

$$[v_1 \cdots v_k] \mapsto v_1 \wedge \cdots \wedge v_k.$$

The inverse of this vector space isomorphism

$$q : \Lambda^* V \to \mathrm{Cl}(V),$$

is called the quantization map*. For an orthonormal basis* $\{e^1, \cdots, e^n\}$ *for* V *we have*

$$q(e^{i_1} \wedge \cdots \wedge e^{i_k}) = e^{i_1} \cdots e^{i_k}.$$

Under the quantization map, the Clifford algebra can also be considered a graded space, and hence it comes with a *number operator*

$$N : \mathrm{Cl}(V) \to \mathrm{Cl}(V)$$

that on a Clifford element a of pure degree k acting according to the formula

$$N(a) := ka.$$

We also define a *rescaling map*, for $t \neq 0$,

$$\delta_t : \mathrm{Cl}(V) \to \mathrm{Cl}(V) \tag{2.4}$$

where for a pure degree k element a,

$$\delta_t(a) = t^{-k}a.$$

Definition 2.3.2. We define the rescaled Clifford multiplication of two Clifford elements by

$$a._t b = \delta_{t^{-1}}(\delta_t(a).\delta_t(b))$$

The Clifford product and the exterior product are related through the rescaling map:

Lemma 2.3.3. $\lim_{t\to 0} a._t b = a \wedge b.$ □

2.3.2 Spin Group and Spin Representation

Denote the subspace $q^{-1}(\Lambda^i V) \subset \mathrm{Cl}_i(V)$ by $\mathrm{Cl}^i(V)$.

Theorem 2.3.4 ([BGV04, Proposition 3.7])**.** *The subspace* $\mathrm{Cl}^2(V)$ *is a Lie algebra with respect to the commutator in the Clifford algebra. This Lie algbera is naturally isomorphic to Lie algebra* $\mathfrak{so}(V)$ *via the morphism*

$$\tau : \mathrm{Cl}^2(V) \to \mathfrak{so}(V)$$

$$a \mapsto \left[v \mapsto av - va\right].$$

For every $a \in \mathrm{Cl}(V)$ we may define the algebraic exponential

$$\exp_C(a) = \sum_i \frac{a^i}{i!}.$$

Definition 2.3.5. By exponentiating the Lie algebra $\mathrm{Cl}^2(V)$ inside the Clifford algebra we obtain a Lie group, the *spin group* $\mathrm{Spin}(V) \subset \mathrm{Cl}_{ev}(V)$. The Lie algebra isomorphism $\tau : \mathrm{Cl}^2(V) \to \mathfrak{so}(V)$ exponentiates to homomorphism of Lie groups

$$\tau : \mathrm{Spin}(V) \to \mathrm{SO}(V),$$

and when $\dim(V) > 1$, this map is a double cover.

We finish this subsection with definition of the spinors.

Proposition 2.3.1 ([BGV04, Proposition 3.19]). *Let V be an even-dimensional vector space. Up to isomorphism, there exists a unique graded complex vector space $\triangle = \triangle^+ \oplus \triangle^-$ that is a $\mathrm{Cl}(V)$-module such that*

$$\mathrm{End}(\triangle) \xrightarrow{\simeq} \mathbb{C}\mathrm{l}(V).$$

We have $\dim(\triangle^+) = \dim(\triangle^-) = 2^{\frac{n}{2}-1}$.

Since $\mathrm{Spin}(V)$ consists only of even elements, the subspace $\triangle^\pm$ are representations of the spin group, called the half-spin representations.

2.4 Clifford Module and Curvature

In this section, we will give a quick overview of the Clifford modules. For more detail see [BGV04, Roe99].

An oriented Riemannian manifold M^n is called *spin* if there exists a principal $\mathrm{Spin}(n)$-bundle $P^{\mathrm{Spin}} \to M$ that is a double cover over the principal $\mathrm{SO}(n)$-bundle $P^{\mathrm{SO}} \to M$, compatible with the double cover $\mathrm{Spin}(n) \to \mathrm{SO}(n)$. Therefore P^{Spin} inherits a connection from the natural Levi-Civita connection.

We will assume M is even-dimensional, so we have a spinor bundle

$$\Delta := P^{\mathrm{Spin}} \times_{\mathrm{Spin}(n)} \triangle,$$

where $\triangle$ is the spinor representation of the Clifford algebra $\mathbb{C}\mathrm{l}(n)$. Note that $\Delta \to M$ has a natural connection induced from the Levi-Civita connection. Also keep in mind that we have a canonical identification

$$\mathrm{End}(\Delta) \simeq \mathbb{C}\mathrm{l}(TM).$$

2.4.1 Clifford Module

A $\mathbb{Z}_2$-graded Hermitian vector bundle over an even-dimensional manifold $S \to M$ is called a *Clifford module* if there is a graded action of the Clifford bundle $\mathrm{Cl}(TM)$. We may assume this bundle comes with a *Clifford connection* ∇ that is a Hermitian and compatible with the Clifford action

$$\nabla(a.s) = \nabla^{\mathrm{LC}} a.s + a.\nabla s$$

for $s \in \Gamma(S)$ and $a \in \Gamma(\mathrm{Cl}(TM))$.

We have the following decomposition

$$\mathrm{End}(S) \simeq \mathbb{C}\mathrm{l}(TM) \otimes \mathrm{End}_{\mathrm{Cl}}(S).$$

With respect to this decompostion, the curvature of the Clifford connection decomposes as

$$\nabla^2 = R^S + F^{S/\Delta}$$

where $R^S \in \Omega^2(M; \mathrm{End}(S)) \simeq \Omega^2(M; \mathbb{C}\mathrm{l}(TM))$ is the Clifford action of the Riemannian curvature,

$$R^S(e^i, e^j) = \frac{1}{4} \sum_{kl} (R(e^i, e^j)e^k, e^l)c(e^k)c(e^l),$$

and $F^{S/\Delta} \in \Omega(M; \mathrm{End}_{\mathrm{Cl}}(S))$ is called *twisting curvature* that commutes with Clifford action.

Remark 2.4.1. If M is a spin manifold, then every Clifford module $S \to M$ is a twisted spinor bundle

$$S = \Delta \otimes W,$$

where $W \to M$ is a Hermitian vector bunlde with a Hermitian connection. In this case, the twisting curvature identifies with the curvature of the twisting bundle

$$F^{S/\Delta} = F^W.$$

2.5 Periodic Cyclic Cohomology

In this section, we introduce the (b, B)-bicomplex for a unital algebra A over $\mathbb{C}$; the cohomology of its totalization gives the periodic cyclic cohomology. We then introduce the pairing of this cohomology theory with K-theory. For more detail on this topic see [Hig06, Lod13].

2.5.1 (b, B)-Bicomplex

Definition 2.5.1. For $p \in \mathbb{Z}_{\geq 0}$, denote by $C^p(A)$ the space of multi-linear functionals $\varphi : A^{p+1} \to \mathbb{C}$ such that $\varphi(a_0, \cdots, a_p) = 0$ if for some $i > 0, a_i = 1$. We define linear maps $b : C^p(A) \to C^{p+1}(A)$, and $B : C^{p+1}(A) \to C^p(A)$ given by following formulas

$$b\varphi(a_0, \cdots, \varphi_{p+1}) = \sum_{i=0}^{p} (-1)^i \varphi(a_0, \cdots, a_i a_{i+1}, \cdots, a_{p+1}) + (-1)^{p+1} \varphi(a_{p+1} a_0, a_1, \cdots, a_p)$$

and

$$B\varphi(a_0, \cdots, a_p) = \sum_i (-1)^{ip} \varphi(1, a^i, a^{i+1}, \cdots, a^{i-1}).$$

So we obtain a double complex, the (b, B)-*bicomplex*:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots \\
 & & \uparrow{\scriptstyle b} & & \uparrow{\scriptstyle b} & & \uparrow{\scriptstyle b} & & \uparrow{\scriptstyle b} \\
\cdots & \xrightarrow{B} & C^3(A) & \xrightarrow{B} & C^2(A) & \xrightarrow{B} & C^1(A) & \xrightarrow{B} & C^0(A) \\
 & & \uparrow{\scriptstyle b} & & \uparrow{\scriptstyle b} & & \uparrow{\scriptstyle b} & & \\
\cdots & \xrightarrow{B} & C^2(A) & \xrightarrow{B} & C^1(A) & \xrightarrow{B} & C^0(A) & & \\
 & & \uparrow{\scriptstyle b} & & \uparrow{\scriptstyle b} & & & & \\
\cdots & \xrightarrow{B} & C^1(A) & \xrightarrow{B} & C^0(A) & & & & \\
 & & \uparrow{\scriptstyle b} & & & & & & \\
\cdots & \xrightarrow{B} & C^0(A) & & & & & &
\end{array} \tag{2.5}$$

Lemma 2.5.2 ([Con94, Section 3.1]). *We have* $b^2 = B^2 = 0$ *and* $bB + Bb = 0$.

The totalization of (b, B)-bicomplex gives a periodic chain complex

$$\cdots \xrightarrow{b+B} C^{ev}(A) \xrightarrow{b+B} C^{odd}(A) \xrightarrow{b+B} C^{ev}(A) \xrightarrow{b+B} \ldots \tag{2.6}$$

where $C^{ev}(A) := \bigoplus_i C^{2i}(A)$ and $C^{odd}(A) := \bigoplus_i C^{2i+1}(A)$.

Definition 2.5.3. The cohomology of the periodic chain (2.6) gives the *periodic cyclic cohomology groups* $HP^{ev}(A)$ and $HP^{odd}(A)$.

2.5.2 Pairing with K-theory

Let A be a unital associative algebra over $\mathbb{C}$. If $\varphi : A^{p+1} \to \mathbb{C}$ is a multi-linear functional, for every $n > 0$, we obtain a multilinear functional then

$$\varphi : M_n(A)^{p+1} \to \mathbb{C}$$

$$\varphi(m_0 \otimes a_0, \cdots, m_p \otimes a_p) := \mathrm{tr}(m_0 \cdots m_p)\varphi(a_0, \cdots, a_p).$$

Definition 2.5.4.

- Let $\varphi = (\varphi_1, \varphi_3, \cdots)$ be an odd (b, B)-cocycle and let $u \in GL_n(A)$ be an invertible element. We have the following pairing

$$\langle \varphi, u \rangle := \frac{1}{\Gamma(1/2)} \sum_{i=0}^{\infty} (-1)^{i+1} i! \varphi_{2i+1}(u^{-1}, u, \cdots, u^{-1}, u).$$

- Let $\varphi = (\varphi_0, \varphi_2, \cdots)$ be an even (b, B)-cocycle and let $e \in M_n(A)$ be an idempotent element. We have the following pairing

$$\langle \varphi, u \rangle := \varphi_0(e) + \sum_{i=1}^{\infty} (-1)^k \frac{(2k)!}{k!} \varphi_{2i}(e - \frac{1}{2}, e, \cdots, e).$$

Theorem 2.5.5 ([Con94, Section 3.3])**.** *The pairing above induces pairing between periodic cyclic cohomology and K-theory:*

$$HP^{ev}(A) \times K_0(A) \to \mathbb{C}$$

and

$$HP^{odd}(A) \times K_1(A) \to \mathbb{C}.$$

Chapter 3
Deformation to the Normal Cone and Filtered Manifolds

This chapter is almost copied from [SH18], verbatim.

3.1 Smooth Manifolds from Algebras

In this section we shall give some elementary algebraic definitions that we shall use throughout the chapter, give criteria guaranteeing that the spectrum of an algebra carries a smooth manifold structure, and compare derivations on algebras to vector fields on spectra in the manifold case.

Definition 3.1.1. Let A be an associative and commutative[1] algebra (with a multiplicative identity) over the field of real numbers. A *character* of A is a nonzero algebra homomorphism

$$\varphi\colon A \longrightarrow \mathbb{R}.$$

The *spectrum* of A is the set of all characters. We equip it with the topology of pointwise convergence, that is, the topology having the fewest open sets so that the evaluation maps

$$\widehat{a}\colon \varphi \longmapsto \varphi(a)$$

are continuous functions on the spectrum, for every $a \in A$.

Definition

of continuous real-valued functions on the spectrum of A that includes all global sections

[1] It is not necessary to assume commutativity, but the definitions that follow are not very interesting in the noncommutative case.

of the form

$$f = g(\widehat{a_1}, \dots, \widehat{a_k}),$$

where $k \in \mathbb{N}$, where $a_1, \dots, a_k \in A$, and where g is a smooth, real-valued function on $\mathbb{R}^k$.

Definition 3.1.3. Let $\mathcal{S}$ be a sheaf of real-valued functions on a topological space X. Let $\Omega \subseteq X$ be an open subset. We shall say that functions $h_1, \dots, h_n \in \mathcal{S}(\Omega)$ *smoothly generate* $\mathcal{S}(\Omega)$ if for every $f \in \mathcal{S}(\Omega)$ there is a smooth function g on $\mathbb{R}^n$ such that

$$f = g(h_1, \dots, h_n).$$

In the case where $X = \text{Spectrum}(A)$ and $\mathcal{S} = \mathcal{S}_A$, we shall also say that elements $a_1, \dots, a_n \in A$ smoothly generate $\mathcal{S}_A(\Omega)$ if the functions $h_j = \widehat{a}_j|_\Omega$ satisfy the above condition.

Lemma 3.1.4. *Let A be a commutative algebra over the real numbers. The spectrum of A is a smooth manifold of dimension n, with $\mathcal{S}_A$ equal to its sheaf of smooth functions, if and only if for every point in the spectrum there is an open neighborhood Ω of that point, and there exist $a_1, \dots, a_n$ in A, such that*

1. *the elements $a_1, \dots, a_n$ smoothly generate $\mathcal{S}_A(\Omega)$, and*
2. *the map*
$$(\widehat{a}_1, \dots, \widehat{a}_n) \colon \text{Spectrum}(A) \longrightarrow \mathbb{R}^n$$
is a homeomorphism from Ω to an open set in $\mathbb{R}^n$.

Proof. If $a_1, \dots, a_n$ exist for every point in the spectrum, as in the statement of the lemma, then the spectrum is certainly a smooth n-manifold with local coordinates $\widehat{a}_1, \dots, \widehat{a}_n$. Conversely, suppose that the spectrum is a smooth n-manifold in such a way that $\mathcal{S}_A$ is the sheaf of smooth functions. Let φ be a point in the spectrum and let $x_1, \dots, x_n$ be local coordinates at φ. By definition of $\mathcal{S}_A$, there are elements $a_1, \dots, a_N \in A$ and smooth functions g_i on $\mathbb{R}^N$ for $i = 1, \dots, n$ such that

$$x_i = g_i(\widehat{a}_1, \dots, \widehat{a}_N)$$

near φ. We can assume that $\widehat{a}_j(\varphi) = 0$, for all j. Since the x_i are local coordinates, there are smooth functions h_j on $\mathbb{R}^n$ for $j = 1, \dots, N$ such that

$$\widehat{a}_j = h_j(x_1, \dots, x_n)$$

near φ. If $g\colon \mathbb{R}^N \to \mathbb{R}^n$ and $h\colon \mathbb{R}^n \to \mathbb{R}^N$ are the smooth functions whose components are g_i and h_j, then $g \circ h = \mathrm{id}_{\mathbb{R}^n}$ near 0. So g is a submersion at 0. By linear algebra and the inverse function theorem, there is an inclusion k of $\mathbb{R}^n$ into $\mathbb{R}^N$ as a coordinate subspace so that the composition

$$\mathbb{R}^n \xrightarrow{k} \mathbb{R}^N \xrightarrow{g} \mathbb{R}^n$$

is a local diffeomorphism at 0. If k maps the i'th standard basis vector of $\mathbb{R}^n$ to the k_i'th standard basis vector of $\mathbb{R}^N$, then the elements $a_{k_1}, \ldots, a_{k_n}$ have the properties (i) and (ii) in the statement of the lemma. □

For the rest of this section we shall assume that the spectrum of A is indeed a smooth manifold, with $\mathcal{S}_A$ the sheaf of smooth functions.

Definition 3.1.5. Let X be a derivation of the algebra A, and let Ω be an open subset of the spectrum of A. We shall say that X *is compatible with a vector field* $\widehat{X}$ *on* Ω if the diagram

$$\begin{array}{ccc} A & \xrightarrow{a \mapsto \widehat{a}} & C^\infty(\Omega) \\ {\scriptstyle X}\downarrow & & \downarrow{\scriptstyle \widehat{X}} \\ A & \xrightarrow[a \mapsto \widehat{a}]{} & C^\infty(\Omega) \end{array}$$

commutes.

An obvious necessary condition for X to be compatible with a vector field on Ω is that if $\Lambda \subseteq \Omega$ is any open subset, if

$$a, a_1, \ldots, a_k \in A,$$

if g is a smooth function of k variables, and if

$$\widehat{a}|_\Lambda = g(\widehat{a_1}, \ldots, \widehat{a_k})|_\Lambda, \tag{3.1}$$

then

$$\widehat{X(a)}|_\Lambda = \sum_{i=1}^{k} \widehat{X(a_i)}|_\Lambda \cdot g_i\Big(\widehat{a_1}, \ldots, \widehat{a_k}\Big)|_\Lambda \tag{3.2}$$

where g_i denotes the i'th partial derivative of g.

Definition 3.1.6. If Ω is an open subset of the spectrum, then we shall say that a derivation X of A is *smooth over* Ω if (3.2) holds for every open subset $\Lambda \subseteq \Omega$, all $a, a_1, \ldots, a_k \in A$, and all g as in (3.1).

Lemma 3.1.7. *If the spectrum of* A *is a smooth manifold, then every derivation of* A *that is smooth over an open subset* Ω *of the spectrum of* A *is compatible with a unique vector field on* Ω.

Proof. By Lemma 3.1.4, around every point of the spectrum there is a neighborhood Λ, and elements $a_1, \ldots, a_n$ of A, so that $\widehat{a_1}, \ldots, \widehat{a_n}$ are coordinate functions on Λ. Since any vector field on Λ is completely determined by its action on a system of coordinate functions, we see that there is at most one vector field on Ω that is compatible with any given derivation X.

As for existence, given local coordinates of the type $\widehat{a_1}, \ldots, \widehat{a_n}$ on some open subset Λ of Ω, we can define $\widehat{X}$ on Λ by

$$\widehat{X}\Big(g(\widehat{a_1}, \ldots, \widehat{a_n})\Big) = \sum_{i=1}^{n} \widehat{X(a_i)} \cdot g_i\Big(\widehat{a_1}, \ldots, \widehat{a_n}\Big).$$

This is a vector field, it is compatible with X on Λ by (3.2), and it is independent of the choice of local coordinates, again by (3.2). So we obtain a vector field defined on all of Ω, as required. □

In our calculations it will be helpful to observe the following fact:

Lemma 3.1.8. *Let* Ω *be an open subset of the spectrum of* A *whose complement has empty interior. Every derivation of* A *that is smooth over* Ω *is smooth over the full spectrum.*

Proof. Let Λ be an open subset of the spectrum. By hypothesis, any identity of smooth functions (3.1) over Λ leads to an identity of the type (3.2) over $\Lambda \cap \Omega$. But $\Lambda \cap \Omega$ is dense in Λ, so the identity (3.2) holds over Λ. □

3.2 The Deformation Space for Smooth Manifolds

Let V be a smooth manifold and let M be a smooth, embedded submanifold (both of them without boundary, as will always be the case in this chapter). In this section we shall review the construction of the *deformation to the normal cone*, or *deformation* *space*, associated to the inclusion of M into V. See [Ful98, Chapter 5] for the standard treatment in algebraic geometry and see for example [Hig10] for the C^∞-version.

We shall emphasize the algebraic aspects of the construction. These play only a modest role for ordinary manifolds, but they will be helpful when we consider filtered manifolds later on.

Here is a summary of what we shall do. The *deformation space* associated to the embedding of M into V may be described, as a set, as a disjoint union

$$\mathbb{N}_V M = N_V M \times \{0\} \sqcup \bigsqcup_{\lambda \in \mathbb{R}^\times} V \times \{\lambda\}, \tag{3.3}$$

as we noted in the introduction. It is given the weakest topology so that the obvious maps to $\mathbb{R}$ and to V are continuous, and so that, in addition, for every smooth function a on V that vanishes on M, the function

$$\begin{aligned} (X_m, 0) &\longmapsto X_m(a) \\ (v, \lambda) &\longmapsto \lambda^{-1} a(v) \end{aligned}$$

is also continuous. Here X_m is a normal vector at $m \in M$, that is, a vector in the quotient space $T_m V / T_m M$. The value $X_m(a)$ is well defined because a vanishes on M. We shall prove that the deformation space carries a smooth manifold structure so that all the functions above are smooth (in fact they smoothly generate the sheaf of all smooth functions on the deformation space in the sense of Definition 3.1.3).

Now we proceed with the details.

Definition 3.2.1. Denote by $A(V, M)$ the $\mathbb{R}$-algebra of all Laurent polynomials

$$f(t) = \sum_{q \in \mathbb{Z}} a_q t^{-q}$$

whose coefficients a_q are smooth, real-valued functions on V that satisfy the condition

$$q > 0 \quad \Rightarrow \quad a_q \text{ vanishes to order } q \text{ on } M$$

(we emphasize that by definition only finitely a_q are nonzero). The space $A(V, M)$ is indeed an algebra, because if a_p vanishes to order p on M, and a_q vanishes to order q on M, then the pointwise product $a_p a_q$ vanishes to order $p + q$. The *deformation space* $\mathbb{N}_V M$ is the spectrum of $A(V, M)$.

Our first objective is to identify $\mathbb{N}_V M$, defined as a spectrum, with (3.3). Associated to $t \in A(V, M)$ is the continuous map

$$\widehat{t}\colon \mathbb{N}_V M \longrightarrow \mathbb{R} \tag{3.4}$$

as in Definition 3.1.1, and we shall compute the fibers over each $\lambda \in \mathbb{R}$. These are the spectra of the following algebras:

Definition 3.2.2. For $\lambda \in \mathbb{R}$ denote by $A_\lambda(V, M)$ the quotient of $A(V, M)$ by the ideal generated by $t-\lambda$.

Lemma 3.2.3. *If $\lambda \in \mathbb{R}$ is nonzero, then $A_\lambda(V, M)$ is isomorphic to $C^\infty(V)$ via evaluation of Laurent polynomials at $t = \lambda$.*

Proof. If the element $\sum a_q t^{-q}$ lies in the kernel of evaluation at λ, then

$$\sum a_q t^{-q} = (t - \lambda) \cdot \sum_q \Big(\sum_{j \geq 0} a_{q-j} \lambda^j \Big) t^{-q-1},$$

and the right-most Laurent polynomial lies in $A(V, M)$, as required. □

To handle the case where $\lambda = 0$ we need some notation.

Definition 3.2.4. For each integer $q > 0$ denote by $I_q(V, M)$ the ideal of smooth functions on V that vanish to order q on M. Set $I_0(V, M) = C^\infty(V)$.

The spaces $I_q(V, M)$ form a decreasing filtration of the algebra of smooth functions on V, and we can form the associated graded algebra

$$\bigoplus_{q \geq 0} I_q(V, M)/I_{q+1}(V, M). \tag{3.5}$$

If $a \in I_q(V, M)$, then we shall write

$$\langle a \rangle_q \in I_q(V, M)/I_{q+1}(V, M) \tag{3.6}$$

for the coset of $a \in I_q(V, M)$ in the degree q component of (3.5).

Lemma 3.2.5. *The algebra $A_0(V, M)$ is isomorphic to the associated graded algebra (3.5) via the map*

$$\sum_{q \in \mathbb{Z}} a_q t^{-q} \longmapsto \sum_{q \geq 0} \langle a_q \rangle_q. \qquad \square$$

It is now easy to compute the spectrum of $A_0(V, M)$. The degree zero part of $A_0(V, M)$ is $C^\infty(M)$, and each character of $A_0(V, M)$ restricts to evaluation at some point $m \in M$ on the degree zero part. The character therefore factors through the quotient algebra $A_{0,m}(V, M)$ by the ideal in $A_0(V, M)$ generated by the vanishing ideal of m in $C^\infty(M)$.

Lemma 3.2.6. *There is a unique isomorphism from $A_{0,m}(V, M)$ to the algebra of real-valued polynomial functions on the normal vector space T_mV/T_mM for which*

$$\langle a \rangle_1 \longmapsto \Big[X_m \mapsto X_m(a) \Big].$$

for every normal vector X_m and every smooth function a on V vanishing on M. The spectrum of $A_{0,m}(V, M)$ identifies in this way with T_mV/T_mM. □

We may summarize the evalutaion formulas for the functions in the Rees algebra as follows:

Definition 3.2.7. For a function $f = \sum_{p \in \mathbb{Z}} f_p t^{-p}$ we denote its values on a point $(v, t) \in V \times \mathbb{R}$ in the nonzero fibers of the deformation space by $\varepsilon_{(v,t)}(f)$ that is given by

$$\varepsilon_{(v,t)}(f) = \sum_{p \in \mathbb{Z}} f_p(v) t^{-p} \tag{3.7}$$

For evaluation on the zero fiber we denote by $\varepsilon_{X_m}(f)$ the value at $[X_m] \in N_VM \times \{0\}$ that is given by the formula

$$\varepsilon_{X_m}(f) = \sum_{p \in \mathbb{Z}} \frac{1}{p!} X^p . f_p(m) \tag{3.8}$$

in which X is an arbitrary smooth extension of the X_m to a local vector field.

Remark 3.2.1. [HY19, Remark 2.3] If V is Riemannian manifold we may obtain the evaluation map ε_{X_m} alternatively by the formula

$$([X_m], 0) \mapsto \lim_{t \to 0} \sum_{p \in \mathbb{Z}} f_p(\exp_m(\lambda X_m)) \lambda^{-p}$$

Remark 3.2.2. We shall prove a more general result in Theorem 3.6.8.

Returning to the deformation space, the above considerations identify the fibers of (3.4) with V when $\lambda \neq 0$, and with the normal bundle N_VM when $\lambda = 0$. We obtain the description (3.3), as required. As for the topology on $\mathbb{N}_VM$, since $A(V, M)$ is generated by:

1. the element $t \in A(V, M)$,
2. the functions $a \cdot t^0 \in A(V, M)$, where $a \in C^\infty(V)$, and
3. monomials $a \cdot t^{-1} \in A(V, M)$, where a vanishes on M.

we find that the topology on $\mathbb{N}_V M$, viewed as a spectrum, agrees with the topology we described earlier.

Theorem 3.2.8. *The deformation space* $\mathbb{N}_V M$ *is a smooth manifold.*

Proof. We shall use Lemma 3.1.4. The only nontrivial case is that of a character φ in the fiber over $\lambda=0$, corresponding to a normal vector X_m. Introduce smooth functions $x_1, \ldots, x_n$ on V that are local coordinates in a neighborhood U of m in V, for which

$$M \cap U = \{ u \in U : x_{k+1}(u) = \cdots = x_n(u) = 0 \}.$$

Now define $\Lambda \subseteq \mathbb{N}_V M$ to be the open set consisting of those elements of the deformation space of the form (u, λ) for $u \in U$ and $\lambda \neq 0$, or $(X_u, 0)$ for $u \in M \cap U$. The elements

$$t, x_1, \ldots, x_k, x_{k+1}t^{-1}, \ldots, x_n t^{-1} \in A(V, M) \tag{3.9}$$

satisfy the hypotheses of Lemma 3.1.4; if $W \subseteq \mathbb{R}^n$ is the image of U under the coordinates $\{x_j\}$ on V, then the homeomorphic image of Λ under the functions (3.9) is the open set

$$\left\{ (\lambda, x_1, \ldots, x_n) : (x_1, \ldots, x_k, \lambda x_{k+1}, \ldots, \lambda x_n) \in W \right\}$$

in $\mathbb{R}^{n+1}$; and the smooth generation statement in the lemma follows from the Taylor expansion for smooth functions on V. □

3.3 The Tangent Groupoid

In this section we shall briefly describe the special features of the deformation space associated to the diagonal embedding of a smooth manifold into its square. This is in preparation for Section 3.8 where a more complicated version of the same thing will be considered.

Definition 3.3.1. Let M be a smooth manifold. The *tangent groupoid* $\mathbb{T}M$ is the deformation space associated to the diagonal embedding of M into $M \times M$.

The name *tangent groupoid* is due to Connes, who explained the importance of the tangent groupoid in index theory. See [Con94, Chapter 2, Section 5], and see [?] for more details concerning the construction of the tangent groupoid using smooth manifold techniques.

As the name promises, $\mathbb{T}M$ is not only a smooth manifold but a Lie groupoid (see [MM03, Chapter 5] for background information on Lie groupoids). The source, target and other structure maps are all obtained from the following functoriality property of the deformation space construction: from a commutative diagram of smooth manifolds and submanifolds

$$\begin{array}{ccc} M_1 & \longrightarrow & M_2 \\ \downarrow & & \downarrow \\ V_1 & \longrightarrow & V_2 \end{array}$$

(where the horizontal arrows are any smooth maps) we obtain a smooth map $\mathbb{N}_{V_1}M_1 \to \mathbb{N}_{V_2}M_2$. Moreover if the horizontal arrows are submersions, then so is the map of deformation spaces.

In the case at hand, think of M as diagonally embedded in $M\times M$ and $M\times M\times M$, and note that the deformation space for the identity embedding of M in itself is simply $M\times\mathbb{R}$. The first and second coordinate projections

$$\begin{array}{ccc} M & = & M \\ \downarrow & & \downarrow \\ M\times M & \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} & M \end{array}$$

determine target and source maps

$$\mathbb{T}M \overset{s}{\underset{t}{\rightrightarrows}} M\times\mathbb{R}.$$

The unit map is determined by the diagonal inclusion of M into $M\times M$, and the inverse map is determined by the flip map on $M\times M$. Finally the space of composable elements in $\mathbb{T}M$,

$$\mathbb{T}M^{(2)} = \{ (\gamma_1,\gamma_2) \in \mathbb{T}M \times \mathbb{T}M : s(\gamma_1) = t(\gamma_2) \}$$

is the deformation space for the diagonal embedding of M into $M\times M\times M$, while the projection

$$M\times M\times M \longrightarrow M\times M$$

onto the first and third factors gives the composition law for $\mathbb{T}M$.

All these maps are easy to compute explicitly in terms of the description (3.3) of the deformation space. The part of $\mathbb{T}M$ over each $\lambda \in \mathbb{R}$ is a subgroupoid, and when $\lambda \neq 0$ we obviously obtain a copy of the pair groupoid of M. When $\lambda = 0$ we obtain the tangent bundle TM, viewed as a bundle of abelian Lie groups over M; this computation will be carried out in a more general context in Section 3.8.

3.4 Vector Fields on the Deformation Space

In this section we shall give a proof of Theorem 1.1 (the theorem of Bursztyn, Lima and Meinrenken) using vector fields on the deformation space.

Definition 3.4.1 ([BLM19, Definition 2.5])**.** If M is a smooth embedded submanifold of a smooth manifold V, then an *Euler-like vector field* for the embedding of M into V is a vector field E on V with the property that if f is a smooth function on V that vanishes on M to order $q \geq 1$, then

$$E(f) = q \cdot f + r,$$

where the remainder r is a smooth function that vanishes to order $q{+}1$ or higher (a smooth function f on V *vanishes to order* $q \geq 1$ *on* M if Df vanishes on M for every linear differential operator D on V of order $q{-}1$ or less).

First we shall prove the *existence* of compatible tubular neighborhood embeddings:

Theorem 3.4.2. *If* E *is an Euler-like vector field for the inclusion of* M *into* V*, then there is a tubular neighborhood diffeomorphism*

$$\Phi\colon N_V M \longrightarrow V$$

(*defined on a neighborhood of the zero section*) *that carries the Euler vector field on the normal bundle to the germ of* E *near* M*. This determines a* bijection *from germs of tubular neighborhoods to germs of Euler vector fields.*

The first step in our proof is to construct from E a vector field on the deformation space. To start, let us denote by $\mathbf{E}$ the vector field on $V{\times}\mathbb{R}^\times$ that is tangent to the fibers of the projection map $V{\times}\mathbb{R}^\times \to \mathbb{R}^\times$, and that is a copy of E on each fiber $V{\times}\{\lambda\}$.

Lemma 3.4.3. *If* E *is Euler-like, then the vector field* $\mathbf{E}$ *above extends uniquely to a vector field on* $\mathbb{N}_V M$*. The extension is tangent to the fibers of the projection* $\mathbb{N}_V M \to \mathbb{R}$*,* *and the restriction to the fiber over* $0 \in \mathbb{R}$ *is the Euler vector field on the normal vector bundle* $N_V M$.

Proof. The extension is unique, if it exists, because $V \times \mathbb{R}^\times$ is dense in $\mathbb{N}_V M$. The existence of the extension, and its properties, are easy to check in local coordinates. Alternatively, if E is Euler-like, then, since E preserves the order of vanishing of functions on M, the formula

$$\sum a_q t^{-q} \longmapsto \sum E(a_q) t^{-q}$$

defines a derivation of $A(V, M)$ that is compatible, in the sense of Definition 3.1.5, with the vector field $\mathbf{E}$ on $V \times \mathbb{R}^\times$. We therefore obtain a smooth extension of $\mathbf{E}$ from Lemmas 3.1.7 and 3.1.8. It follows from the definition of an Euler-like vector field that the restriction of this smooth extension to $N_V M$ is the Euler vector field. □

Remark 3.4.1. The lemma actually *characterizes* Euler-like vector fields: if X is a vector field on V, and if the extension to a vector field $\mathbf{X}$ to $V \times \mathbb{R}^\times$, as above, further extends a vector field on $\mathbb{N}_V M$ that then restricts to the Euler vector field on the vector bundle $N_V M$, then X is Euler-like.

Next we shall introduce a canonical vector field on the deformation space.

Lemma 3.4.4. *The formula*

$$\gamma_s \colon \begin{cases} (v, \lambda) \longmapsto (v, e^s \lambda) \\ (X, 0) \longmapsto (e^{-s} X, 0) \end{cases}$$

defines a smooth action of the Lie group $\mathbb{R}$ *on the deformation space* $\mathbb{N}_V M$.

Proof. This is again easy to check directly in the local coordinates of Theorem 3.2.8. From the algebraic point of view, it suffices to note that the geometric flow is associated to the morphism

$$\gamma \colon A(V, M) \longrightarrow A(V, M) \otimes_{\mathbb{R}} C^\infty(\mathbb{R})$$

defined by the formula

$$\gamma \colon \sum a_q t^{-q} \longmapsto \sum a_q t^{-q} \otimes e^{-tq}$$

(the tensor product here is the ordinary algebraic tensor product). □

Definition 3.4.5. We shall denote by $\mathbf{C}$ the vector field on $\mathbb{N}_V M$ that generates the flow $\{\gamma_s\}$ above. Note that $\mathbf{C}$ restricts to the vector field $\lambda \cdot \partial / \partial \lambda$, on $V \times \mathbb{R}^\times$, while on the zero fiber of $\mathbb{N}_V M$ it agrees with the negative of the Euler vector field on the normal bundle.

Now we shall combine $\mathbf{E}$ with $\mathbf{C}$ to obtain a new vector field $\mathbf{T}$ on $\mathbb{N}_V M$.

Lemma 3.4.6. *Let* E *be an Euler-like vector field for the inclusion of* M *into* V*, and let* $\mathbf{E}$ *be the associated vector field on* $\mathbb{N}_V M$*, as in Lemma 3.4.3. The vector field*

$$\mathbf{T} = \lambda^{-1}\mathbf{E} + \frac{\partial}{\partial\lambda}$$

on the open subset $V\times\mathbb{R}^\times \subseteq \mathbb{N}_V M$ *extends to a (smooth) vector field on* $\mathbb{N}_V M$ *with*

$$\lambda \cdot \mathbf{T} = \mathbf{C} + \mathbf{E}.$$

Proof. If E is Euler-like, then the formula

$$\sum a_q t^{-q} \longmapsto \sum (E(a_q) - q a_q) t^{-(q+1)},$$

defines a derivation of $A(V, M)$. The derivation is compatible with the vector field $\mathbf{T}$ over the open set $V\times\mathbb{R}^\times \subseteq \mathbb{N}_V M$, and since the complement of this open set has empty interior it follows from Lemma 3.1.8 that the derivation is compatible (in the sense of Definition 3.1.5) with a unique vector field on all of $\mathbb{N}_V M$. □

It is clear from its definition that the vector field $\mathbf{T}$ on $\mathbb{N}_V M$ is $\widehat{t}$-related to the vector field $d/d\lambda$ on $\mathbb{R}$ (recall from (3.4) that $\widehat{t}$ is the natural projection from $\mathbb{N}_V M$ to $\mathbb{R}$). As a result, the time $t{=}1$ flow map for the vector field $\mathbf{T}$ maps the $\lambda{=}0$ fiber $N_V M \subseteq \mathbb{N}_V M$ to the $\lambda{=}1$ fiber $M \subseteq V$ (although we need to be a bit careful about the domain of definition of the flow map). We shall show that this fiber mapping is a tubular neighborhood, and that it carries the Euler vector field on the normal bundle to the Euler-like vector field E.

Definition 3.4.7. Denote by $\{\tau_s\}$ the local flow on $\mathbb{N}_V M$ associated to the vector field $\mathbf{T}$ in Lemma 3.4.6.

Recall that the maps τ_s assemble into a smooth map

$$\tau\colon \mathbb{R} \times \mathbb{N}_V M \longrightarrow \mathbb{N}_V M$$

that is defined on some neighborhood of $\{0\}\times\mathbb{N}_V M$ in $\mathbb{R}\times\mathbb{N}_V M$, such that

$$\mathbf{T}(f)(w) = \frac{d}{ds}\bigg|_{s=0} f(\tau_s(w))$$

for all smooth functions f on $\mathbb{N}_V M$ and all $w \in \mathbb{N}_V M$, and

$$\tau_{s+t}(w) = \tau_s(\tau_t(w))$$

in a neighborhood of $\{0\}\times\{0\}\times\mathbb{N}_V M$ in $\mathbb{R}\times\mathbb{R}\times\mathbb{N}_V M$. In all these formulas, we are writing $\tau_s(w) = \tau(s, w)$. For $s \neq 0$, then we shall write the restriction of the flow τ_s to the fiber of $N_V M$ over $\lambda=0$ in the form

$$\mathbb{N}_V M \ni (X, 0) \stackrel{\tau_s}{\longmapsto} (\varphi_s(X), s) \in \mathbb{N}_V M.$$

For any open subset $U \subseteq N_V M$ with compact closure, and all sufficiently small $|s|$, the map φ_s is a diffeomorphism from U to an open subset of V.

Lemma 3.4.8. *Let f be a smooth function on V that vanishes on M. There is a smooth function $h\colon V \to \mathbb{R}$ that vanishes to order 2 such that*

$$\frac{d}{ds} f(\varphi_s(X_m)) = s^{-1} f(\varphi_s(X_m)) + s^{-1} h(\varphi_s(X_m))$$

for every $X_m \in N_V M$ and all sufficiently small $|s|$.

Proof. Since E is an Euler vector field, we can write

$$E(f) = f + h,$$

where h vanishes on M to order 2. Now define $\mathbf{f}$ to be the composition

$$\mathbb{N}_V M \longrightarrow V \times \mathbb{R} \longrightarrow V \stackrel{f}{\longrightarrow} \mathbb{R}.$$

Then by definition of φ_s and the flow τ_s,

$$\frac{d}{ds} f(\varphi_s(X_m)) = \frac{d}{ds}\mathbf{f}(\tau_s(X_m, 0)) = \mathbf{T}_{\tau_s(X_m,0)}(\mathbf{f})$$

But it follows from the definition of $\mathbf{T}$ that

$$\begin{aligned}\mathbf{T}_{\tau_s(X_m,0)}(\mathbf{f}) &= s^{-1}\mathbf{E}_{(\varphi_s(X_m),s)}(\mathbf{f}) \\ &= s^{-1}E(f)(\varphi_s(X_m)) = s^{-1} f(\varphi_s(X_m)) + s^{-1} h(\varphi_s(X_m)),\end{aligned}$$

as required. □

The map φ_s takes the zero section $M \subseteq N_V M$ identically to $M \subseteq V$, because $\mathbf{T}$ restricts to $\partial/\partial\lambda$ on the submanifold $M \times \mathbb{R} \subseteq \mathbb{N}_V M$. So for every $m \in M$ and all sufficiently small $|s|$ the derivative of φ_s induces a map

$$\varphi_{s,*} \colon T_m V / T_m M \longrightarrow T_m V / T_m M \tag{3.10}$$

Lemma 3.4.9. *The mapping* (3.10) *is* $s \cdot \mathrm{id}$*.*

Proof. We shall calculate the linear algebraic adjoint φ_s^* of the linear transformation (3.10). The vector space of smooth functions on V that vanish to first order on M surjects onto the vector space dual of $T_m V/T_m M$ via the usual pairing of functions and tangent vectors, and functions that vanish to second order are in the kernel of the surjection. Applying Lemma 3.4.8 we find that

$$\frac{d}{ds}\varphi_s^* = s^{-1}\varphi_s^* \colon (T_m V/T_m M)^* \longrightarrow (T_m V/T_m M)^*,$$

and so by calculus $s^{-1}\varphi_s^*$ is a constant family of linear maps. To evaluate the constant we shall compute the limit of $s^{-1}\varphi_s^*$ as $s \to 0$. The function $s \mapsto \tau_s(X_m, 0)$ is a smooth curve in $\mathbb{N}_V M$, with value $(X_m, 0)$ at $s = 0$, and the function $(\nu, s) \mapsto s^{-1} f(\nu)$ is smooth on $\mathbb{N}_V M$, with values $(X_m, 0) \mapsto X_m(f)$ when $s = 0$. So

$$\lim_{s \to 0} s^{-1} f(\varphi_s(X_m)) = X_m(f).$$

As a result, if $[f]$ denotes the class in $(T_m V/T_m M)^*$ determined by f, namely

$$[f] \colon X_m \longmapsto X_m(f)$$

then

$$\varphi_s^*([f]) = [f \circ \varphi_s] \colon X_m \longmapsto s \cdot X_m(f)$$

and so $\varphi_s^* = s \cdot \mathrm{id}$, as required. □

Lemma 3.4.9 tells us that for any s the map $X_m \mapsto \varphi_s(s^{-1} X_m)$ is a tubular neighborhood mapping on the domain where it is defined, but this may not be a neighborhood of the full zero section of $N_V M$. To remedy this problem, we shall use the Lie bracket relations among $\mathbf{E}$, $\mathbf{C}$ and $\mathbf{T}$, which are as follows:

$$[\mathbf{T}, \mathbf{C}] = \mathbf{T}, \quad [\mathbf{T}, \mathbf{E}] = 0, \quad \text{and} \quad [\mathbf{C}, \mathbf{E}] = 0 \tag{3.11}$$

(note that it suffices to verify these relations on the dense set $V \times \mathbb{R}^\times \subseteq \mathbb{N}_V M$).

Lemma 3.4.10. *If* K *is a compact subset of* $N_V M$ *and* $k > 0$*, then there exists* $\varepsilon > 0$ *so that*

$$\varphi_{e^t s}(X) = \varphi_s(e^t X)$$

for all $X \in K$*, all* $|t| < k$*, and all* $s \in (-\varepsilon, \varepsilon)$.

Proof. It follows from the first relation in (3.11) that

$$\tau_{e^t s} = \gamma_t \circ \tau_s \circ \gamma_{-t} \tag{3.12}$$

(to be precise, the identity is well-defined and correct on any given compact set K, and for $|t|$ bounded by any given k, as long as $|s|$ is sufficiently small). The formula in the lemma follows by evaluating both sides on $(X, 0)$. □

Proof of Theorem 3.4.2. Choose a neighborhood of the zero section in $N_V M$ and a smooth positive function $s(m)$ so that $\varphi_s(X_m)$ is defined for all $X_m \in U$ and all $|s| < 2s(m)$. Using Lemma 3.4.10, we find that the germ of the map

$$\Phi(X_m) = \varphi_{s(m)}(s(m)^{-1} X_m)$$

near the zero section of $N_V M$ is independent of the map $m \mapsto s(m)$ and is a tubular neighborhood. The second relation in (3.11) implies that Φ carries the Euler vector field on the normal bundle to E. □

Theorem 3.4.2 also asserts that there is a *unique* (germ of a) tubular neighborhood embedding that carries the Euler vector field to any given Euler-like vector field. We have nothing really new to say about this uniqueness statement, but for completeness here is a proof.

Lemma 3.4.11. *Let* V *be a finite-dimensional vector space and let* $\Psi \colon U \to W$ *be a diffeomorphism from one open neighborhood of* $0 \in V$ *to another, with* $\Psi(0) = 0$*. If* Ψ *carries the Euler vector field to itself, near* 0*, and if the derivative of* Ψ *at* 0 *is the identity, then* Ψ *is the identity near* 0.

Proof. Let v be an element in a ball around 0 (with respect to some norm) that is contained in $U \subseteq V$. Both of the curves $\Psi(e^{-t} v)$ and $e^{-t}\Psi(v)$ $(t \geq 0)$ have the same derivatives for all t, given by the negative of the Euler vector field, and the same initial point at $t = 0$. Hence

$$\Psi(e^{-t}v) = e^{-t}\Psi(v) \qquad \forall t \geq 0. \tag{3.13}$$

Now by calculus, if Ψ_* is the derivative of Ψ at 0, then there is a positive constant so that

$$\|\Psi(u) - \Psi_* u\| \leq \text{constant} \cdot \|u\|^2 \tag{3.14}$$

for all $u \in U$ sufficiently close to 0. Writing $u = e^{-t}v$, multiplying (3.14) by e^t, and using (3.13), we obtain

$$\|\Psi(v) - \Psi_* v\| \leq e^{-t} \cdot \text{constant} \cdot \|v\|^2,$$

and so $\Psi(v) = \Psi_* v = v$. □

Proof of the uniqueness statement in Theorem 1.1. If two tubular neighborhood embeddings are given, under both of which E identifies with the Euler vector field, then the composition the first with the inverse of the second is a diffeomorphism Ψ from one neighborhood of the zero section in the normal bundle $N_V M$ to another that fixes the zero section, and carries the Euler vector field to itself. By repeating the argument in Lemma 3.4.11 we find that if $X_m \in N_V M$ is contained in a ball around 0 that is contained in the domain of definition of Ψ, then

$$\Psi(e^{-t}X_m) = e^{-t}\Psi(X_m) \qquad \forall t \geq 0.$$

Applying the projection $N_V M \to M$ to this equation and taking the limit as $t \to -\infty$, we find that $\Psi\colon N_V M \to N_V M$ is fiber-preserving near the zero section. Now apply the previous lemma fiberwise, using the condition (1.2) in the definition of tubular neighborhood embedding to verify that lemma's derivative hypothesis. □

3.5 Lie Filtrations and Unipotent Groups

In this section we shall review the definition of a *Lie filtration* on the tangent bundle of a smooth manifold, due to Tanaka [Tan70] (although the name for the concept that we use here was chosen by Melin [Mel82]) and give an algebraic description of the unipotent *osculating groups* that are attached to the points of a filtered manifold.

Definition 3.5.1. Let V be a smooth manifold. A *Lie filtration* on the tangent bundle TV is an increasing sequence of smooth vector subbundles

$$H^1 \subseteq H^2 \subseteq \cdots \subseteq H^r = TV$$

with the property that if X and Y are vector fields on V, and also sections of H^p and H^q, respectively, then the Lie bracket $[X, Y]$ is a section of H^{p+q} (we set $H^{p+q} = TV$ if $p+q \geq r$). An r-*step filtered manifold* is a smooth manifold whose tangent bundle is equipped with a Lie filtration of length r, as above.

Remark 3.5.1. The concept of filtered manifold arises in a number of places. Apart from [Tan70] and [Mel82], see also [Mor93] and [ČS09], for instance. Some of the treatments of filtered manifolds in sub-Riemannian geometry are particularly close to the perspective of this chapter; see for example [Bel97, Secs. 4,5] and [ABB16, Ch. 10].

We shall usually write (V, H) to make explicit reference to the Lie filtration. For simplicity we shall assume throughout that the bundles H^q in Definition 3.5.1 have constant rank, which of course they must have if V is connected.

Example 3.5.1. *An ordinary smooth manifold is obviously a* 1*-step filtered manifold. In the* 1*-step case the constructions in this and the next two sections will be identical with the constructions in Section 3.2.*

Example 3.5.2. *In the* 2*-step case the Lie bracket condition in Definition 3.5.1 is vacuous, so a 2-step filtered manifold is simply a smooth manifold together with a smooth vector subbundle of the tangent bundle (Beals and Greiner [BG88] coined the term* Heisenberg manifold *for the special case in which this bundle has codimension one in the tangent bundle). The calculations in this and the following sections are very easy in the* 2*-step case.*

For our purposes, the significant features of a filtered manifold (V, H) will be accessed through the algebra of linear partial differential operators on V, and in particular through an increasing filtration on differential operators that is determined by the Lie filtration on TV.

We begin with some generalities on differential operators, unrelated to Lie filtrations. If $X_1, \ldots, X_n$ is any local frame for the tangent bundle of a smooth manifold, then any linear partial differential operator D can be expressed in a unique way as a linear combination

$$D = \sum_{\alpha} f_\alpha X^\alpha, \tag{3.15}$$

where

1. the sum is over all multi-indices $\alpha = (\alpha_1, \dots, \alpha_n)$ with nonnegative integer entries,
2. $X^\alpha = X_1^{\alpha_1} \cdots X_n^{\alpha_n}$ (note that the order of $X_1, \dots, X_n$ is fixed), and
3. the coefficients f_α are smooth functions, all but finitely many of them zero.

Lemma 3.5.2. *Let v be a point in a smooth manifold V, let $\{X_1, \dots, X_n\}$ be a local frame for TV, defined near v. If a linear differential operator D is expressed in terms of the frame as in (3.15), and if D vanishes at v in the sense that $(Df)(v) = 0$ for every smooth function f on V, then all the functions f_α vanish at v.* □

The following two definitions are taken from the work of Choi and Ponge [CP19, Section 2] (which in turn adapts terminology from [Bel97, Section 4]).

Definition 3.5.3. Let (V, H) be an r-step filtered manifold. A *local* H-*frame* for V is a local frame $X_1, \dots, X_n$ for the tangent bundle such that for every $q = 1, \dots, r$, the vector fields

$$X_1, \dots, X_{\mathrm{rank}(H^q)}$$

are sections of H^q, and so constitute a local frame for H^q.

Definition 3.5.4. The *weight sequence* of V is the sequence

$$(q_1, \dots, q_n) = (1, \dots, 1, 2, \dots, 2, \dots, r, \dots, r)$$

in which each integer q is repeated $\mathrm{rank}(H^q) - \mathrm{rank}(H^{q-1})$ times.

Remark 3.5.2. With this terminology, if $\{X_a\}$ is a local H-frame, then X_a is a section of the vector bundle H^{q_a}.

Definition 3.5.5 ([Mel82, Section 3])**.** Let (V, H) be an r-step filtered manifold. Let D be a linear differential operator and let s be a nonnegative integer. We shall write

$$\mathrm{order}_H(D) \leq s,$$

and say that the H-order of D is no more than s, at a point $v \in V$, if for some (or equivalently every) local H-frame $X_1, \dots, X_n$ defined near v, the operator D can be expressed as a sum

$$D = \sum_\alpha f_\alpha X_1^{\alpha_1} \cdots X_n^{\alpha_n}, \tag{3.16}$$

in such a way that

$$q_1\alpha_1 + \cdots + q_n\alpha_n > s \quad \Rightarrow \quad f_\alpha = 0,$$

where $\{q_a\}$ is the weight sequence for (V, H).

Example 3.5.3. *In the* 1*-step case (see Example 3.5.1) this is of course the usual notion of order of a differential operator.*

Definition 3.5.6. Let (V, H) be a filtered manifold and denote by $\mathcal{D}(V)$ the algebra of linear partial differential operators on V. We shall denote by

$$\mathcal{D}^s(V) \subseteq \mathcal{D}(V)$$

the linear space of all operators that are of H-order no more than s at every point of V.

It is evident that if p and q are any nonnegative integers, then

$$\mathcal{D}^p(V) \cdot \mathcal{D}^q(V) \subseteq \mathcal{D}^{p+q}(V),$$

so the concept of H-order defines an increasing filtration on the algebra $\mathcal{D}(V)$. If X is a vector field on V, then X has H-order no more than q as a differential operator if and only if it is a section of H^q.

The notion of H-order on differential operators leads to the following notion of order of vanishing of a function at a point in a filtered manifold:

Definition 3.5.7. Let V be a filtered manifold and let v be a point in V. Let q be a positive integer. A smooth function f on V *vanishes to* H*-order* q *at* v if the function Df vanishes at v for every differential operator D of H-order $q-1$ or less. We shall denote by

$$I_q(V, v) \subseteq C^\infty(V)$$

the ideal of smooth, real-valued functions on V that vanish to H-order q. For convenience we shall also write $I_0(V, v) = C^\infty(V)$.

Of course, even though the notation does not indicate it, the ideals $I_q(V, v)$ depend on the filtration H. The spaces $I_q(V, v)$ decrease as q increases, and in addition

$$I_p(V, v) \cdot I_q(V, v) \subseteq I_{p+q}(V, v)$$

for all $p, q \geq 0$. So we obtain a decreasing filtration of the algebra $C^\infty(V)$ by ideals.

Definition 3.5.8. Let v be a point in a filtered manifold (V, H). Denote by $A_0(V, v)$ the associated graded algebra

$$A_0(V, v) = \bigoplus_{q \geq 0} I_q(V, v)/I_{q+1}(V, v).$$

In the context of ordinary manifolds this naturally identifies with the algebra of polynomial functions on the tangent space $T_v V$. Our objective in the remainder of this section is to show that $A_0(V, v)$ naturally identifies with the algebra of polynomial functions on a real unipotent group $\mathcal{H}_v$ attached to the Lie filtration and the point $v \in V$.

Definition 3.5.9. Let (V, H) be a filtered manifold and let $v \in V$. Denote by $\mathfrak{h}_v$ the direct sum

$$\mathfrak{h}_v = \bigoplus_{q=1}^{r} H^q_v / H^{q-1}_v.$$

Equip $\mathfrak{h}_v$ with a graded Lie algebra structure, as follows. Given elements $\langle X_v \rangle_p$ and $\langle Y_v \rangle_q$ in degrees p and q, represented by tangent vectors $X_v \in H^p_v$ and $Y_v \in H^q_v$, extend both to sections of H^p and H^q and define

$$\Big[\langle X_v \rangle_p, \langle Y_v \rangle_q \Big] = \langle [X, Y]_v \rangle_{p+q}.$$

For further details, and examples, see [Mel82], [CP19] or [vEY19].

Lemma 3.5.10. *The graded Lie algebra $\mathfrak{h}_v$ acts as derivations on the graded algebra $A_0(V, v)$ via the formula*

$$\delta_{\langle X_v \rangle_p} \colon \sum_{q \geq 0} \langle a_q \rangle_q \longmapsto \sum_{q \geq p} \langle X(a_q) \rangle_{q-p},$$

where X_v is extended to a section X of H^p, as in Definition 3.5.9 (and where the angle-bracket notation $\langle a \rangle_q$ is as in (3.6)). □

Definition 3.5.11. We shall denote by $\mathcal{H}_v$ the unipotent group with Lie algebra $\mathfrak{h}_v$. This is the *osculating group* attached to the point v. Denote by $A(\mathcal{H}_v)$ the algebra of real-valued polynomial functions on $\mathcal{H}_v$.

Remark 3.5.3. In the present context, *unipotent group* means the same thing as *simply connected nilpotent Lie group*, while $A(\mathcal{H}_v)$ is the algebra of functions on the group that correspond to polynomial functions on the Lie algebra $\mathfrak{h}_v$ under the exponential map

$$\exp \colon \mathfrak{h}_v \longrightarrow \mathcal{H}_v,$$

which, we recall, is a diffeomorphism. See for example [Hoc81, Chapter XVI, Section 4] for a more algebraic construction of $\mathcal{H}_v$.

Now if A is an algebra that is equipped with a locally finite-dimensional and locally nilpotent action of a finite-dimensional real nilpotent Lie algebra $\mathfrak{h}$ by derivations, then the action of $\mathfrak{h}$ exponentiates to an action of the associated unipotent group $\mathcal{H}$ by algebra automorphisms. And if ε is any character of A, then there is an *orbit homomorphism*[2]

$$A \longrightarrow A(\mathcal{H}) \tag{3.17}$$

into the algebra of real-valued polynomial functions on the associated unipotent group that is defined by the formula

$$a \longmapsto \Big[h \mapsto \varepsilon(h^{-1}(a))\Big] \qquad (a \in A, \quad h \in \mathcal{H}). \tag{3.18}$$

It is an $\mathcal{H}$-equivariant algebra homomorphism if we let $\mathcal{H}$ act on $A(\mathcal{H})$ by the left regular representation.

Definition 3.5.12. We shall call the character

$$A_0(V, v) \ni \sum \langle a_q \rangle_q \stackrel{\varepsilon}{\longmapsto} a_0(v) \in \mathbb{R}$$

the *counit* of $A_0(V, v)$.

We shall prove the following result.

Theorem 3.5.13. *Let* (V, H) *be a filtered manifold, and let* v *be a point in* V*. The orbit homomorphism*

$$A_0(V, v) \longrightarrow A(\mathcal{H}_v)$$

associated to the counit of $A_0(V, v)$ *is an* $\mathcal{H}_v$*-equivariant algebra isomorphism.*

Remark 3.5.4. The orbit homomorphism in the theorem is the *unique* $\mathcal{H}_v$-equivariant homomorphism for which the composition

$$A_0(V, v) \longrightarrow A(\mathcal{H}_v) \xrightarrow{\text{eval. at } e} \mathbb{R}$$

is the counit of $A_0(V, v)$.

[2]It is dual to the orbit map $\mathcal{H} \to \text{Spectrum}(A)$ given by $h \mapsto h(\varepsilon)$.

Lemma 3.5.14. *Let V be a filtered manifold of rank r, and let v be a point in V. Let $\{q_1, \dots, q_n\}$ be the weight sequence for (V, H) and let $\{X_a\}$ be a local H-frame, defined near v. There are local coordinates $\{x_a\}$ defined near v such that*

1. *each x_a vanishes at v to H-order q_a, and*
2. *$X_a(x_b) = \delta_{ab}$ at the point v, for all $a, b = 1, \dots, n$.*

Proof. Define a linear transformation from $\mathcal{D}(V)$ into the vector space dual of $C^\infty(V)$ by the formula

$$D \longmapsto \Big[f \mapsto (Df)(v)\Big].$$

It induces a linear map

$$\mathcal{D}^r(V) \longrightarrow \Big(C^\infty(V)\Big/I_{r+1}(V, v)\Big)^*. \tag{3.19}$$

Note that the quotient $C^\infty(V)\Big/I_{r+1}(V, v)$ is a *finite-dimensional* vector space.

It follows from Lemma 3.5.2 that the images under (3.19) of the monomial differential operators X^α of H-order no more than r are linearly independent. So by linear algebra there are functions $f_\beta \in C^\infty(V)$ with

$$(X^\alpha f_\beta)(v) = \delta_{\alpha\beta}$$

The members $\{x_a\}$ of this list of functions that correspond to the vector fields $\{X_a\}$ form a local coordinate system of the required type. □

Remark 3.5.5. The coordinates provided by the lemma above are called *privileged coordinates* in [CP19, Definition 4.9] and [Bel97], and their existence is proved in [CP19, Proposition 4.13] and in [Bel97, Theorem 4.15]. Our argument is only slightly different.

Proof of Theorem 3.5.13. Equip the algebra $A(\mathcal{H}_v)$ with the decreasing filtration given by order of vanishing, in the ordinary sense unrelated to Lie filtrations, at $e \in \mathcal{H}_v$. The associated graded algebra is the symmetric algebra on its degree one part, which identifies with $\mathfrak{h}_v^*$.

The algebra $A_0(V, v)$ also carries a decreasing filtration, in which an element has order j or more if it can be represented as a sum $\sum \langle a_q \rangle_q$, with each a_q vanishing, also in the ordinary sense, to order j or more. The associated graded algebra is a symmetric algebra on the degree-one classes determined by the elements $\langle x_a \rangle_{q_a}$, where $\{x_a\}$ is any coordinate system as in Lemma 3.5.14.

The filtrations of $A_0(V, v)$ and $A(\mathcal{H}_v)$ are compatible with one another under the map (3.17), and the generators $\langle x_a \rangle_{q_a}$ map to the dual basis elements

$$\langle X_{a,v} \rangle^*_{q_a} \in \mathfrak{h}^*_v,$$

with $\{X_a\}$ the local H-frame in Lemma 3.5.14. This proves the theorem. □

Remark 3.5.6. Let $\{X_a\}$ be a local H-frame near $v \in V$, and let $\{x_a\}$ be an associated system of privileged coordinates, as in Lemma 3.5.14. The frame determines a basis $\{\langle X_{a,v} \rangle_{q_a}\}$ for the Lie algebra $\mathfrak{h}_v$ and the local coordinates determine a local diffeomorphism

$$w \longmapsto \sum_a x_a(w) \langle X_{a,v} \rangle_{q_a}$$

from V to $\mathfrak{h}_v$, and hence, by exponentiation, a local diffeomorphism

$$V \xrightarrow{\cong} \mathcal{H}_v.$$

This in turn induces an isomorphism of algebras

$$A(\mathcal{H}_v) \xrightarrow{\cong} A_0(V, v).$$

The algebra isomorphism depends on the choice of coordinate systems $\{x_a\}$, in general, and is *not* in general inverse to the canonical isomorphism of Theorem 3.5.13. Those coordinates for which the two isomorphisms *are* inverse to one another are called *Carnot coordinates* in [CP19].

3.6 Normal Spaces for Filtered Manifolds

In this section we shall construct the filtered manifold analogue of the normal bundle. Its fibers will be most naturally viewed as unipotent homogeneous spaces rather than as quotients of tangent vector spaces.

Definition 3.6.1. Let (V, H) be an r-step filtered manifold. An embedded submanifold $M \subseteq V$ is a *filtered submanifold* if the intersections

$$G^q = H^q|_M \cap TM \qquad (q = 1, \dots, r)$$

are smooth vector subbundles of TM.

If M is a filtered submanifold of (V, H), then the bundles G^q form a Lie filtration of TM, so that (M, G) is a filtered manifold in its own right.

Definition 3.6.2. Let (M, G) be a filtered submanifold of a filtered manifold (V, H), and denote by $I_q(V, M)$ the ideal of smooth functions on V that vanish to H-order at least q on M. We shall denote by $A_0(V, M)$ the associated graded algebra

$$A_0(V, M) = \bigoplus_{q \geq 0} I_q(V, M)/I_{q+1}(V, M)$$

The *normal space* $N^H_V M$ is the spectrum of $A_0(V, M)$.

Theorem 3.6.3. *Let (M, G) be a filtered submanifold of a filtered manifold (V, H). The normal space $N^H_V M$ is a smooth manifold in such a way that the sheaf of smooth functions is the sheaf from Definition 3.1.2.*

The proof is not difficult, but it requires some information about vector fields and local coordinates adapted to the inclusion of M into V.

Definition 3.6.4. Let (M, G) be a filtered submanifold of a filtered manifold (V, H). A *local* (G, H)*-frame* for TV at a point of M is a local H-frame for V with the additional property that the vector fields in the frame that are tangent to M (upon restriction to M) form a local G-frame for M.

The vector fields in the local frame divide into two sets:

1. vector fields tangent to M upon restriction to M, which restrict to give a G-local frame for M, and

2. vector fields not tangent to M.

We shall call the latter the *normal* vector fields in the local frame. The normal vector fields X_a for which $a \leq \text{rank}(H^p)$ restrict to give a local frame for the quotient bundle $H^p\big|_M \big/ G^p$.

Lemma 3.6.5. *Let (V, H) be an r-step filtered manifold with order sequence $\{q_a\}$, and let (M, G) be a filtered submanifold of V. Let $\{X_a\}$ be a local (G, H)-frame defined near a point $m \in M$. There are smooth functions z_c defined near m, one for each normal vector field X_c in the frame, such that*

1. z_c vanishes on M to H-order q_c.

2. $X_c(z_d) = \delta_{cd}$ *on* M.

To prove this generalization of Lemma 3.5.14 we shall use the following generalization of Lemma 3.5.2.

Lemma 3.6.6. *Let* M *be an embedded submanifold of a smooth manifold* V*, and let* m *be a point in* M*. Let* $\{Z_1, \ldots, Z_k\}$ *be vector fields on* V*, defined in some neighborhood of* $m \in V$*, and assume that their values at* m *project to linearly independent vectors in the normal space* $TV|_M/TM$*. If a linear differential operator of the form*

$$D = \sum f_\alpha Z^\alpha$$

has the property that $(Df)(m) = 0$ *for every smooth function* f *on* V *that vanishes on* M*, then all the coefficient functions* f_α *vanish at* m*.* □

Proof of Lemma 3.6.5. Acccording to Lemma 3.6.6 the monomial operators X^α that use only normal vector fields in the local (G, H)-frame map by evaluation at m to a linearly independent set in $\mathrm{Hom}(I_1(V, M), \mathbb{R})$. If we consider only monomial operators of H-order r or less, then this linearly independent set lies in the finite-dimensional vector space

$$\mathrm{Hom}(I_1(V, M)/I_{r+1}(V, M), \mathbb{R}) \subseteq \mathrm{Hom}(I_1(V, M), \mathbb{R})$$

and so, by linear algebra, associated to this finite linearly independent set in a finite-dimensional vector space there are functions $g_\beta \in I_1(V, M)$ with $X^\alpha(g_\beta) = \delta^\beta_\alpha$ at the point m.

We want to adjust the functions g_β so that this relation holds near m in M, not only at the single point m. Let $h_{\alpha\beta} = X^\alpha(g_\beta)$. This matrix of functions is the identity at m, and so is invertible near m. Let $h^{\alpha\beta}$ be the entries of the inverse matrix and define

$$f_\beta = \sum_\gamma h^{\beta\gamma} g_\gamma.$$

Then $X^\alpha(f_\beta) = \delta_{\alpha\beta}$ on M, near m. Now, if we define z_c to be the function f_β associated to the vector field $X_c \in \{X^\beta\}$, then the functions $\{z_c\}$ have the required properties. □

Proof of Theorem 3.6.3. We shall use the vector fields and functions obtained above to show that the criteria in Lemma 3.1.4 are satisfied for every character φ of $A_0(V, M)$.

The degree zero part of $A_0(V, M)$ is $C^\infty(M)$, and φ restricts there to evaluation at some $m \in M$. Let $\{X_a\}$ be a local (G, H)-frame near m. Choose smooth functions $\{z_c\}$

on V as in Lemma 3.6.5. In addition, choose smooth functions $\{y_a\}$ on V, indexed by the members Y_a of the local (G, H)-frame that are tangent to M, so that

$$Y_a(y_b) = \delta_{ab} \quad \text{at } m \in V.$$

There is a neighborhood U of $m \in V$ such that functions $\{y_a, z_c\}$ are coordinates for U, while the functions $\{y_a\}$ restrict to coordinates for $M \cap U$.

Now let Λ be the open set in $N_V^H M$ consisting of all those characters whose restriction to the degree zero part of $A_0(V, M)$ is evaluation at some point of $M \cap U$. It follows from Taylor's theorem that the elements

$$\langle y_a \rangle_0 \quad \text{and} \quad \langle z_c \rangle_{q_c} \tag{3.20}$$

smoothly generate $A_0(V, M)$ over Λ.

Moreover $A_0(U, M \cap U)$ is freely generated as an algebra over its degree zero part $C^\infty(M \cap U)$ by the classes $\langle z_c \rangle_{q_c}$. So if dim($M$)=$k$ and dim(V)=n, then the map

$$\text{Spectrum}(A_0(V, M)) \longrightarrow \mathbb{R}^k \times \mathbb{R}^{n-k}$$

given by evaluation on the generators (3.20) sends Λ homeomorphically to the open set $W \times \mathbb{R}^{n-k}$, where $W \subseteq \mathbb{R}^k$ is the range of the coordinates $\{y_c\}$ on $M \cap U$. □

We shall now calculate the normal space $N_V^H M$ in terms of the osculating groups introduced in the last section. There is a natural map

$$N_V^H M \longrightarrow M \tag{3.21}$$

corresponding to the inclusion of $C^\infty(M)$ as the degree zero subalgebra of $A_0(V, M)$, and fiber of $N_V^H M$ over $m \in M$ identifies with the spectrum of the following algebra.

Definition 3.6.7. If $m \in M$, then we shall denote by $A_{0,m}(V, M)$ the quotient of $A_0(V, M)$ by the ideal in $A_0(V, M)$ generated by the vanishing ideal of m in $C^\infty(M)$. The formula

$$\varepsilon_m \colon \sum \langle a_q \rangle_q \longmapsto a_0(m)$$

defines a character of $A_{0,m}(V, M)$ that we shall call the *counit*.

Theorem 3.6.8. *Let* (M, G) *be a filtered submanifold of a filtered manifold* (V, H) *and let* m *be a point in* M. *Let* $\mathcal{H}_m$ *and* $\mathcal{G}_m$ *be the osculating groups for* $m \in V$ *and* $m \in M$, *respectively. There is a unique* H_m*-equivariant algebra isomorphism*

$$A_{0,m}(V, M) \longrightarrow A(\mathcal{H}_m/\mathcal{G}_m)$$

whose composition with evaluation at the identity coset in $\mathcal{H}_m/\mathcal{G}_m$ *is the counit* ε_m *of* $A_{0,m}(V, M)$.

Remark 3.6.1. Here $A(\mathcal{H}_m/\mathcal{G}_m)$ is the algebra of polynomial functions on the unipotent homogenous space $\mathcal{H}_m/\mathcal{G}_m$, or equivalently the algebra of polynomial functions on $\mathcal{H}_v$ that are invariant under right translations by elements of $\mathcal{G}_m$.

Proof. The Lie algebra $\mathfrak{h}_m$ acts on $A_{0,m}(V, M)$ by derivations according to the formula in Lemma 3.5.10, and this action exponentiates to a locally finite-dimensional action of H_v by automorphisms. The image of the orbit map

$$A_{0,m}(V, M) \longrightarrow A(\mathcal{H}_m)$$

associated to the counit ε_m is included in the right $\mathcal{G}_m$-invariant functions on $A(\mathcal{H}_m)$; this is a consequence of the fact that if $X \in \mathfrak{g}_m$, then

$$\varepsilon_m(\delta_X(a)) = 0$$

for every $a \in A_{0,m}(V, M)$. So we obtain an orbit homomorphism

$$A_{0,m}(V, M) \longrightarrow A(\mathcal{H}_m/\mathcal{G}_m),$$

and it remains to show that it is an isomorphism. We shall use a variation on the argument used to prove Theorem 3.5.13.

Filter $A_{0,m}(V, M)$ by order of vanishing of functions in the ordinary sense at m. Using the coordinates of the previous lemma, the associated graded algebra is freely generated by the classes $\langle z_c \rangle_{q_c}$.

Filter $A(\mathcal{H}_m/\mathcal{G}_m)$ by order of vanishing in the ordinary sense at the basepoint in $\mathcal{H}_m/\mathcal{G}_m$. The associated graded algebra is freely generated by the normal dual vectors $\langle Z_c \rangle^* \in (\mathfrak{h}_m/\mathfrak{g}_m)^*$.

Our orbit map is filtration preserving, we find that it induces an isomorphism on associated graded algebras; indeed it maps $\langle z_c \rangle_{q_c}$ to $\langle Z_c \rangle^*$. $\square$

Remark 3.6.2. The algebra $A_0(V, M)$ consists of those smooth functions on the normal space $N_V^H M$ whose restrictions to all of the fibers of (3.21) are polynomial functions.

3.7 Deformation Spaces for Filtered Manifolds

In this section we shall construct the deformation space associated to a filtered submanifold of a filtered manifold. We shall copy Section 3.2 almost verbatim.

Definition 3.7.1. Let (M, G) be a filtered submanifold of a filtered manifold (V, H). Denote by $A(V, M)$ the algebra of Laurent polynomials

$$\sum_{n\in\mathbb{Z}} a_q t^{-q}$$

whose coefficients are smooth, real-valued functions on V that satisfy the condition

$$q > 0 \quad \Rightarrow \quad a_q \text{ vanishes to } H\text{-order } q \text{ on } M.$$

The *deformation space* $\mathbb{N}_V^H M$ is the spectrum of $A(V, M)$.

As is the case for ordinary manifolds, the deformation space is a union

$$\mathbb{N}_V^H M = N_V^H M \times \{0\} \sqcup \bigsqcup_{\lambda\in\mathbb{R}^\times} V \times \{\lambda\},$$

(but of course with the normal space from the previous section).

Theorem 3.7.2. *The deformation space $\mathbb{N}_V^H M$ is a smooth manifold in such a way that the sheaf of smooth functions is the sheaf from Definition 3.1.2.*

Proof. We shall follow the proof of Theorem 3.2.8, and we shall use the same coordinate functions $\{y_a\}$ and $\{z_c\}$ as in the proof of Theorem 3.6.3, defined in a neighborhood U of $m \in V$. Let $\Lambda \subseteq \mathbb{N}_V M$ be the open subset consisting of all (u, λ) with $u \in U$ and $\lambda \neq 0$, together with all the elements $(X_m, 0)$, with $X_m \in \mathcal{H}_m/G_m$. The elements

$$t, \quad y_a, \quad \text{and} \quad z_c t^{-q_c} \tag{3.22}$$

of $A(V, M)$ satisfy the conditions of Lemma 3.1.4. If $W \subseteq \mathbb{R}^n$ is the image of the coordinates $\{y_a, z_c\}$, then the functions (3.22) map Λ homeomorphically to the open subset

$$\Big\{ \big(\lambda, \{y_a\}, \{z_c\}\big) : \big(\{y_a\}, \{\lambda^{q_c} z_c\}\big) \in W \Big\}$$

of $\mathbb{R}^{n+1}$. □

3.8 The Tangent Groupoid for Filtered Manifolds

In this section we shall briefly discuss the diagonal embedding of a filtered manifold into its square, where the deformation space carries a Lie groupoid structure. We shall describe this groupoid structure in terms of the osculating groups in Definition 3.5.11.

Definition 3.8.1. Let (M, G) be a filtered manifold, and define a Lie filtration of $M \times M$ by defining $H^p \subseteq TM \times TM$ to be $G^p \times G^p$. The *tangent groupoid* of (M, G) is the deformation space

$$\mathbb{T}^G M := \mathbb{N}_M^H M \times M.$$

associated to the diagonal embedding of M in $M \times M$.

The tangent groupoid for filtered manifolds was previously constructed by Van Erp [vE05] and Ponge [Pon06] in the 2-step case, and then by Choi and Ponge [CP19], and also by Van Erp and Yuncken [vEY17], in the general case. Connes gave a proof of the Atiyah-Singer theorem using the standard tangent groupoid considered in Section 3.3 [Con94, Chapter 2, Section 5]. See [vE10a] for a proof of an index theorem for contact manifolds using a similar approach.

As in Section 3.3, the tangent groupoid has a natural Lie groupoid structure with object space $M \times \mathbb{R}$. The part of $\mathbb{T}M$ over each $\lambda \neq 0$ is a copy of the pair groupoid of M, as before, and it remains to describe the groupoid structure over $\lambda = 0$.

If $\mathcal{G}_m$ is the osculating group at $m \in M$, as in Definition 3.5.11, then the isomorphism of Theorem 3.6.8 gives an identification

$$\mathbb{T}^G M|_{(m,0)} \cong (\mathcal{G}_m \times \mathcal{G}_m) / \mathcal{G}_m \cong \mathcal{G}_m. \tag{3.23}$$

Here $\mathcal{G}_m$ is embedded diagonally as a subgroup of $\mathcal{G}_m \times \mathcal{G}_m$, and the second isomorphism is induced from $(g_1, g_2) \mapsto g_1 {g_2}^{-1}$.

Theorem 3.8.2. *The multiplication on the fiber of $\mathbb{T}^G M$ over $(m, 0)$ that is induced from the groupoid structure on $\mathbb{T}^G M$ is the same as the group multiplication operation that is induced from the identification (3.23).*

To prove the proposition, let us return to the functoriality of the deformation space that was mentioned (for ordinary manifolds) in Section 3.3. Suppose given a commutative diagram

$$\begin{CD} M_1 @>>> M_2 \\ @VVV @VVV \\ V_1 @>>{\varphi}> V_2 \end{CD}$$

in which the columns are inclusions of filtered manifolds, as in Definition 3.6.1, and the differentials of the horizontal maps are filtration-preserving on tangent spaces. There is an induced map on deformation spaces, and in particular on normal spaces. Indeed if $\varphi(m_1) = m_2$ then composition with φ induces a morphism of algebras

$$\varphi^*\colon \mathcal{A}_{0,m_2}(V_2, M_2) \longrightarrow \mathcal{A}_{0,m_1}(V_1, M_1). \tag{3.24}$$

In addition, the differential of φ induces a Lie algebra homomorphism

$$\varphi_*\colon \mathfrak{h}_{1,m_1} \longrightarrow \mathfrak{h}_{2,m_2} \tag{3.25}$$

and so a group morphism

$$\varphi_*\colon \mathcal{H}_{1,m_1} \longrightarrow \mathcal{H}_{2,m_2}. \tag{3.26}$$

The morphisms (3.24) and (3.25) are related as follows: if $f \in \mathcal{A}_{0,m_2}(V_2, M_2)$, then

$$\delta_{\xi_1}\varphi^* f = \varphi^*\delta_{\varphi_*\xi_1} f \qquad \forall \xi_1 \in \mathfrak{h}_{m_1} \tag{3.27}$$

(for ordinary manifolds this is simply the definition of the differential φ_*).

Consider now the induced map on normal spaces

$$\varphi_*\colon N^{H_1}_{V_1} M_1\Big|_{m_1} \longrightarrow N^{H_2}_{V_2} M_2\Big|_{m_2}$$

(recall that the normal spaces are the spectra of the algebras in (3.24)). Identify the normal spaces with unipotent homogeneous spaces, as in Theorem 3.6.8, to obtain a map

$$\varphi_*\colon \mathcal{H}_{1,m_1}/\mathcal{G}_{1,m_1} \longrightarrow \mathcal{H}_{2,m_2}/\mathcal{G}_{2,m_2}. \tag{3.28}$$

We find from (3.27) that (3.28) is induced from (3.26).

Proof of Proposition 3.8.2. It follows from (3.27) that the groupoid operation

$$\mathbb{T}^G M|_{(m,0)} \times \mathbb{T}^G M|_{(m,0)} \longrightarrow \mathbb{T}^G M|_{(m,0)},$$

when viewed as a map

$$\mathcal{G}_m \times \mathcal{G}_m \longrightarrow \mathcal{G}_m$$

using (3.23), is equivariant for the left and right multiplication actions of $\mathcal{G}_m$ (on the left and right factors, respectively, in the case of the left-hand side). In addition, the groupoid operation maps (e, e) to e. So it must be group multiplication. □

3.9 Euler-Like Vector Fields on Filtered Manifolds

Definition 3.9.1. Let (M, G) be a filtered submanifold of a filtered manifold (V, H). An *Euler-like vector field* for the embedding of M into V is a vector field E with the property that if f is a smooth function on V that vanishes on M to H-order q, then

$$E(f) = q \cdot f + r$$

where r is a smooth function that vanishes on M to H-order $q+1$ or higher.

Example 3.9.1. *If $m \in M$ and if $\{y_a, z_c\}$ is the local coordinate system defined near $m \in V$, that was used in the proofs of Theorems 3.6.3 and 3.7.2, then formula*

$$E = \sum_c q_c \cdot z_c \cdot \frac{\partial}{\partial z_c}$$

defines an Euler-like vector field near m. A global Euler-like vector field can be assembled from locally defined Euler-like vector fields of this type using a partition of unity.

Our aim is to relate Euler-like vector fields to tubular neighborhood embeddings, as in Theorem 1.1. An interesting feature of the filtered manifold case that we are now considering is that it is not immediately clear what the appropriate notion of tubular neighborhood embedding should be (for instance, the normal space $N^H_V M$ is not itself a filtered manifold, so we cannot insist that tubular neighborhood embeddings be isomorphisms of filtered manifolds). So we shall let the analogue of Theorem 1.1 determine the definition of a tubular neighborhood embedding.

To define the appropriate notion of a tubular neigborhood embedding we shall need to define a "zero section" of the normal space, and then examine the vertical tangent bundle for the submersion

$$N^H_V M \longrightarrow M$$

at the zero section. First, the homomorphism

$$\begin{aligned} A_0(V, M) &\longrightarrow C^\infty(M) \\ \sum \langle a_q \rangle_q &\longmapsto \langle a_0 \rangle_0 \end{aligned}$$

defines an inclusion of M into $N_V^H M$ that will be our zero section. Next, the vertical tangent space at a point m in the zero section identifies with the quotient of Lie algebras $\mathfrak{h}_m/\mathfrak{g}_m$. Each of $\mathfrak{h}_m$ and $\mathfrak{g}_m$ is a graded Lie algebra, and we shall write

$$\mathfrak{h}_m^q = H_m^q / H_m^{q-1} \quad \text{and} \quad \mathfrak{g}_m^q = G_m^q / G_m^{q-1}.$$

Definition 3.9.2. Let (M, G) be a filtered submanifold of a filtered manifold (V, H). A *tubular neighborhood embedding* of $N_V^H M$ into V is a diffeomorphism from a neighborhood of $M \subseteq N_V^H M$ to a neighborhood of $M \subseteq V$ with the following properties:

1. The diffeomorphism is the identity on M
2. At each point of M the differential maps the vertical space $\mathfrak{h}_m^q/\mathfrak{g}_m^q$ into H_m^q, and the composition
$$\mathfrak{h}_m^q/\mathfrak{g}_m^q \longrightarrow H_m^q \longrightarrow \mathfrak{h}_m^q/\mathfrak{g}_m^q$$
with the natural projection is the identity.

The normal space $N_V^H M$ carries a natural vector field, which we shall call the Euler vector field, as follows:

Definition 3.9.3. The *Euler vector field* on $N_V^H M$ is the vector field associated to the smooth derivation of $A_0(V, M)$ given by

$$\sum_q \langle a_q \rangle_q \longmapsto \sum_q q \cdot \langle a_q \rangle_q.$$

Remark 3.9.1. The normal space $N_V^H M$ is not naturally a filtered manifold, in general. But if M is a point, then $N_V^H M$ is simply the unipotent group $\mathcal{H}_v$ and this is a filtered manifold. In this case, the Euler vector field is Euler-like in the sense of Definition 3.9.1.

The Euler vector field generates a flow $\{\rho_s\}$ on $N_V^H M$ that is easy to describe in group-theoretic terms. First, there is a one-parameter group of Lie algebra automorphisms of the graded Lie algebra

$$\mathfrak{h}_m = \bigoplus_{q=1}^{r} H^q_m / H^{q-1}_m$$

that multiplies the degree q summand by e^{tq}. This one-parameter group exponentiates to a one-parameter group of automorphisms of the unipotent group $\mathcal{H}_m$ that maps the subgroup $\mathcal{G}_m$ to itself, and therefore induces a flow $\{\rho_s\}$ on the homogeneous space $\mathcal{H}_m/\mathcal{G}_m$, as required.

Definition 3.9.4. Denote by $\mathbf{C}$ the vector field on $\mathbb{N}^H_V M$ that generates the flow

$$\gamma_s \colon \begin{cases} (\nu, \lambda) \longmapsto (\nu, e^s \lambda) \\ (X, 0) \longmapsto (\rho_{-s} X, 0) \end{cases}$$

Lemma 3.9.5. *If E is an Euler-like vector field for the inclusion of M into V, then the vector field*

$$\mathbf{T} = \lambda^{-1} E + \frac{\partial}{\partial \lambda}$$

on the open subset $V \times \mathbb{R}^\times \subseteq \mathbb{N}^H_V M$ extends to a vector field on $\mathbb{N}^H_V M$ with

$$\lambda \cdot \mathbf{T} = \mathbf{C} + \mathbf{E},$$

where $\mathbf{E}$ smoothly extends the λ-independent vector field on $V \times \mathbb{R}^\times$ that is defined by E. □

Repeating the argument from Section 3.4 we find that:

Theorem 3.9.6. *Let (M, G) be a filtered submanifold of a filtered manifold (V, H). The correspondence that associates to each tubular neighborhood embedding the associated Euler-like vector field on V is bijection from germs of tubular neighborhood embeddings to germs of Euler-like vector fields.* □

Remark 3.9.2. In the case where M is a point, the inverse

$$V \longrightarrow \mathcal{H}_m$$

of the tubular neighborhood embedding corresponds to a system of Carnot coordinates, as in [CP19, Section 7] and Remark 3.5.6.

Chapter 4
Overview of the Rescaled Bundle

Let M be an oriented even-dimensional Riemannian manifold with a Clifford module $S \to M$ (see subsection 2.4.1). This vector bundle comes with a Hermitian metric and a connection. When M is spin and when $S \to M$ is the spinor bundle, Yi and Higson, [HY19], introduced the "rescaled bundle" $\mathbb{S} \to \mathbb{T}M$, which is given set-theoretically by the disjoint union

$$\begin{array}{ccc} S \boxtimes S^* & \bigsqcup & \Lambda^* T^* M \\ \downarrow & & \downarrow \\ M \times M \times \mathbb{R}^\times & \bigsqcup & TM \times \{0\}. \end{array} \tag{4.1}$$

In this chapter, we give an overview of the construction and geometric structure of this vector bundle.

4.1 Getzler's Order of Differential Operators

Let D be a differential operator acting on $S \to M$. In [HY19] the Getzler order of D is defined to be at most m if locally D can be written as sum of terms

$$c(Y_1) \cdots c(Y_r) \nabla_{X_1} \cdots \nabla_{X_s}$$

where X_i, Y_j are vector fields over M, and $s + r \leq m$.

Definition 4.1.1. Let P be a differential operator acting on $\Gamma(S \boxtimes S^*)$ that is a family of "source-wise" differential operators $P_m \otimes 1$ acting on sections of the tensor bundle

$$S \otimes S^*_m \to M \times \{m\}$$

over source-fibers of the $M \times M$ at m. We call such differential operator a *source-wise* differential operator and define its Getzler order to be the maximum of the Getzler order of P_m for all m.

Example 4.1.1.

- *Let* D *be a differential operator on* $S \to M$*. By differentiating along the first component,* D *may be considered as a source-wise differential operator on* $S \boxtimes S^*$*.*

- *The covariant derivation* ∇_X *and Clifford multiplication* $c(X)$*, for a nonzero vector field* X*, have Gezler order 1.*

- *The Dirac operator* D *and its square* D^2 *both have Getzler order* 2*. This follows for* D^2 *from Lichnerowicz formula:*

$$D^2 = -\sum_i (\nabla^2_{e^i} - \nabla_{\nabla_{e^i} e^i}) + \frac{sc}{4} + \mathcal{F}^{S/\Delta}$$

Remark 4.1.1. The definition of Getzler order given here is based on [Roe99], which is slightly different from the original definition of differential order given by [BGV04]. For source-wise operators induced from operators acting on $S \to M$, both definition accord. However, for general source-wise operators acting on $S \boxtimes S^*$, such as the Riemmanian Euler vector field to be introduced in section 4.3.1, these two definitions give different orders. We will discuss this in more detail later, in Remark 4.6.1.

4.2 Scaling Order

Now let's discuss the bundle $S \boxtimes S^* \to M \times M$. This bundle restricted to the diagonal $M \overset{\Delta}{\hookrightarrow} M \times M$ is canonically isomorphic to the Clifford bundle with values in Clifford endomorphisms of the Clifford bundle:

$$S \otimes S^* \simeq \mathrm{End}(S) \simeq \mathbb{C}\mathrm{l}(TM) \otimes \mathrm{End}_{\mathrm{Cl}}(S).$$

Definition 4.2.1. We define the Clifford order of a nonzero section $s \in \Gamma(S \boxtimes S^*)$, denoted by Clifford-order(s), to be the smallest k such that $s|_{\Delta M} \in \mathbb{C}\mathrm{l}^k(TM)$. If $s = 0$, we define its Clifford order to be $-\infty$.

Note that

- the Clifford order is bounded above by $\dim(M)$.
- The Clifford order may still be defined for a locally-defined sections of $S \boxtimes S^*$ over any neighborhood of a point on the diagonal by restricting to the diagonal.

By combining the vanishing order and the Clifford order, we obtain the *scaling order*:

Definition 4.2.2. We say a section $s \in \Gamma(S \boxtimes S^*)$ has scaling order order at least $p \in \mathbb{Z}$ if for every source-wise differential operator D with Getzler order q

$$p \leq q - \text{Clifford-order}(Ds)$$

We denote the scaling order of s by Scaling-order(s) and we have

$$\text{Scaling-order}(s) = \min_{D \text{ source-wise}} (\text{Getzler-order}(D) - \text{Clifford-order}(Ds))$$

Remark 4.2.1. The definition of the scaling order is a variation of that of [HY19, Definition 3.6]. This definition will lead to the same *rescaled module* and *rescaled bundle*.

Corollary 4.2.3. *For every section* $s \in \Gamma(S \boxtimes S^*)$

$$\textit{Scaling-order}(s) \leq -\textit{Clifford-order}(s).$$

4.3 The Taylor Order

In this section, we describe how to build a local section of any scaling order greater than or equal to $-n$.

Definition 4.3.1.

- We define the "Riemannian Euler vector field" or "radial vector field", $\mathcal{R}$, on a neighborhood of the diagonal of $M \times M$ as follows

$$\mathcal{R}(\exp X_m, m) = \frac{d}{dt}|_{t=1}(\exp_m tX_m, m) \tag{4.2}$$

  This vector field is an Euler-like vector field for the diagonal embedding.

- A section $s \in \Gamma(S \boxtimes S^*)$ is "synchronous" if $\nabla_{\mathcal{R}} s = 0$ in a neighborhood of the diagonal.

Remark 4.3.1. Let $\tau(x, y) : S_y \otimes S_y^* \to S_x \otimes S_y^*$ denote the parallel transport operator along the unique (short) geodesic from y to x. For every section $u \in \Gamma(S \otimes S^* \to M)$ considered as a section of $S \boxtimes S^*$ over the diagonal $M \hookrightarrow M \times M$, we may obtain a local synchronous section of $s \in \Gamma(S \boxtimes S^*)$ defined over $(x, y) \in M \times M$ by the formula

$$s(x, y) := \tau(x, y)u(x) \in S_x \otimes S_y^*.$$

Every local synchronous frame may be obtained this way.

1. Let $(x^1, \cdots, x^n)$ be a geodesic local cordinate system on M centered at $m \in M$. If $u \in \Gamma(S)$ is a section of the spinor bundle. We associate to u a Taylor series

$$u \sim \sum_{\alpha \geq 0} x^\alpha u_\alpha$$

   where u_α's are synchronous and

$$s - \sum_{|\alpha| \leq N} x^\alpha u_\alpha$$

   vanishes to order at least N on the diagonal.

2. Let $\{x^1, \cdots, x^n\}$ be a family of smooth functions in neighborhood of a diagonal point (m, m), on $M \times M$. Assume for every m' close to m, restricting these functions to $M \times \{m'\}$ gives geodesic local coordinates on M. So in other words, these functions give a family of geodesic coordinates in the source-wise direction. Now we may associate to every section $s \in \Gamma(S \boxtimes S^*)$ a Taylor expansion:

$$s \sim \sum_{\alpha \geq 0} x^\alpha s_\alpha$$

   where s_α's are synchronous and

$$s - \sum_{|\alpha| \leq N} x^\alpha s_\alpha$$

   vanishes to order at least N on the diagonal.

Definition 4.3.2. We may define the taylor order of s to be

$$\text{Taylor-order}(s) := \min_\alpha \{|\alpha| - \text{Clifford-order}(s_\alpha)\} \tag{4.3}$$

We will show that the Taylor order and scaling order are indeed equal:

Proposition 4.3.1. Scaling-order(s) = Taylor-order(s).

Proof. In [HY19, Proposition 6.3], it is shown that the scaling order is bounded below by the Taylor order:

$$\text{Scaling-order}(s) \geq \text{Taylor-order}(s).$$

Hence we need to show the opposite inequality.

Let $s \sim \sum_{\alpha \geq 0} x^\alpha s_\alpha$ be a Taylor expansion of s in the sense of Definition 4.3.2. By choosing $N > 2\dim(M)$, the partial sum $s_N = \sum_{|\alpha| \leq N} x^\alpha s_\alpha$ will have the same scaling order and Taylor order as s. So we may assume for some multi-index $\beta = (\alpha_1, \cdots, \alpha_n)$, $|\beta| \leq N$,

$$\text{Taylor-order}(s) = |\beta| - \text{Clifford-order}(s_\beta).$$

If $D = \nabla^{\alpha_1}_{\frac{\partial}{\partial x_1}} \cdots \nabla^{\alpha_n}_{\frac{\partial}{\partial x_k}}$, for $\alpha \neq \beta$, then

$$D(x^\alpha s_\alpha)_{x=0} = 0.$$

Hence

$$\begin{aligned}
\text{Scaling-order}(s) &= \text{Scaling-order}(s_N) \\
&\leq \text{Getzler-order}(D) - \text{Clifford-order}(s_\beta) \\
&= \text{Taylor-order}(s).
\end{aligned}$$

□

Corollary 4.3.3. *If* $s \in \Gamma(S \boxtimes S^*)$ *is synchronous, then*

$$\text{Scaling-order}(s) = -\text{Clifford-order}(s).$$

Corollary 4.3.4. *Let* $f \in C^\infty(M \times M)$ *be function that vanishes to order at least* l *along the diagonal, and let* $s \in \Gamma(S \boxtimes S^*)$ *be a synchronous section of scaling order at most* k. *Then*

$$\text{Scaling-order}(fs) \geq l - k.$$

4.4 Rescaled Module

To define a vector bundle $\mathbb{S}$ over the tangent groupoid $\mathbb{T}M$, Higson and Yi [HY19, Definition 3.12] define a module $\mathcal{S}(\mathbb{T}M)$ over the Rees algebra $\mathcal{A}(\mathbb{T}M)$ which plays the the role of the set of "regular" sections of $\mathbb{S}$.

Definition 4.4.1. Let $\mathcal{S}(\mathbb{T}M) \subset \Gamma(S \boxtimes S^*)[t, t^{-1}]$ be the set of all elements

$$\sum_{p\in\mathbb{Z}} s_p t^{-p}$$

where $s_p \in \Gamma(S \boxtimes S^*)$ has scaling order at least p.

This vector space is clearly a module over the Rees algebra $\mathcal{A}(\mathbb{T}M)$, and is filtered. Dividing by "the space of sections vanishing on the zero fiber" we obtain the quotient space

$$\mathcal{S}_0(\mathbb{T}M) := \frac{\mathcal{S}(\mathbb{T}M)}{t.\mathcal{S}(\mathbb{T}M)}$$

that is a graded module over $\mathcal{A}_0(\mathbb{T}M)$ with degrees starting from $-\dim(M)$ up.

Note that for $(m_1, m_2, t) \in M \times M \times \mathbb{R}^\times$, there is an obvious evaluation map

$$\varepsilon_{(m_1,m_2,t)} : \mathcal{S}(\mathbb{T}M) \to S_{m_1} \otimes S^*_{m_2}$$
$$s = \sum_{p\in\mathbb{Z}} s_p t^{-p} \mapsto \sum_{p\in\mathbb{Z}} s_p(m_1, m_2)t^{-p}.$$

Note that

Lemma 4.4.2. *For $f \in \mathcal{A}(\mathbb{T}M)$ we have*

$$\varepsilon_{(m_1,m_2,t)}(f.s) = \varepsilon_{(m_1,m_2,t)}(f).\varepsilon_{(m_1,m_2,t)}(s)$$

where the $\varepsilon_{(m_1,m_2,t)}(f)$ is the value of f at (m_1, m_2, t). □

Definition 4.4.3. For every $m \in M$, define the evaluation map ε_m on the quotient module $\mathcal{S}_0(\mathbb{T}M)$ as follows:

$$\varepsilon_m : \mathcal{S}_0(\mathbb{T}M) \to \Lambda^* T_m M \otimes \mathrm{End}_{\mathrm{Cl}}(S_m)$$
$$[\sum_{p\in\mathbb{Z}} s_p t^{-p}] \mapsto \sum_{p\leq 0} [s_p]_{-p},$$

where for $k \geq 0$

$$[.]_k : \mathbb{C}l^k(T_mM) \to \frac{\mathbb{C}l^k(T_mM)}{\mathbb{C}l^{k-1}(T_mM)} \simeq \Lambda^k T_m M \simeq \Lambda^k T_m^* M.$$

For $f \in \mathcal{A}_0(\mathbb{T}M)$ and $s \in \mathcal{S}_0(\mathbb{T}M)$ we have

$$\varepsilon_m(f.s) = \varepsilon_m(f).\varepsilon_m(s).$$

For a section $s \in \Gamma(S \boxtimes S^*)$, we denote by $\nabla_X s$ the covariant derivative along the first component. This gives a derivation on $\mathcal{S}(\mathbb{T}M)$:

$$\boldsymbol{\nabla}_X := t\nabla_X : \mathcal{S}(\mathbb{T}M) \to \mathcal{S}(\mathbb{T}M)$$
$$s = \sum_{p\in\mathbb{Z}} s_p t^{-p} \mapsto \sum_{p\in\mathbb{Z}} \nabla_X s_p t^{-p+1}$$

For $f \in \mathcal{A}(\mathbb{T}M)$ we have

$$\boldsymbol{\nabla}_X(f.s) = (\mathbf{X}f).s + f.\boldsymbol{\nabla}_X s$$

Lemma 4.4.4. *The derivation $\boldsymbol{\nabla}_X$ descends to a nilpotent derivation of degree -1 on the quotient module*

$$\boldsymbol{\nabla}_X : \mathcal{S}_0(\mathbb{T}M) \to \mathcal{S}_0(\mathbb{T}M),$$

and hence we may define the algeraic exponential map

$$\exp(\boldsymbol{\nabla}_X) : \mathcal{S}_0(\mathbb{T}M) \to \mathcal{S}_0(\mathbb{T}M)$$

by the power series of the exponential. If $f \in \mathcal{A}_0(\mathbb{T}M$ then

$$\exp(\boldsymbol{\nabla}_X)([f.s]) = \exp(\mathbf{X})(f). \exp(\boldsymbol{\nabla}_X)([s])$$

Definition 4.4.5. The evaluation map at $X_m \in T_mM$ is defined by

$$\varepsilon_{X_m} : \mathcal{S}_0(\mathbb{T}M) \to \Lambda^* T_m M \otimes \mathrm{End}_{\mathrm{Cl}}(S_m)$$
$$\varepsilon_{X_m}([s]) = \varepsilon_m(\exp(\boldsymbol{\nabla}_X)([s])),$$

where X is an arbitrary extension of X_m to a local vector field.

Here we provide a geometric formula for this evaluation map:

Proposition 4.4.1. *For* $s \in \mathcal{S}(\mathbb{T}M)$

$$\varepsilon_{X_m}[s] = \lim_{t \to 0} \delta_t \tau(m, \exp(tX_m)) s(\exp(tX_m), m, t)$$

Here δ_t *is the Clifford algebra rescaling map 2.4.*

Proof. Without loss of generality we may assume $s = s_p t^{-p}$, and that $s_p = f.\sigma$, where $\sigma \in \Gamma(S \boxtimes S^*)$ is a synchronous section of Clifford order l and f has vanishing of order at least k, $k - l \geq p$. We consider a local extension X of X_m by parallel transport along geodesic rays centered at m. The evaluation $\nabla_X^i \sigma(m)$ depends just on the values of σ along the geodesic ray $\exp(tX_m)$. Since σ is a synchronous section $\nabla_X^i \sigma(m) = 0$ for every $i > 0$. Now we may write

$$\begin{aligned}
\varepsilon_{X_m}\sigma &= \varepsilon_m\Big(\sum_i \frac{1}{i!}\nabla_X^i (f\sigma) t^{i-p}\Big) \\
&= \sum_i \frac{1}{i!} X^i.f(m,m)[\sigma(m,m)]_{i-p} \\
&= \frac{1}{k!} X^k.f(m,m)[\sigma(m,m)]_{k-p} \\
&= \lim_{t\to 0} f(\exp(tX_m), m) t^{-k} [\tau(m, \exp(tX_m))\sigma(\exp(tX_m), m)]_{k-p} \\
&= \lim_{t\to 0} f(\exp(tX_m), m) t^{-p}. \lim_{t\to 0} t^{p-k} \{\tau(m, \exp(tX_m))\sigma(\exp(tX_m), m)\} \\
&= \lim_{t\to 0} \delta_t \tau(m, \exp(tX_m))\sigma(\exp(tX_m), m, t),
\end{aligned}$$

where in the third equality we used Remark 3.2.1. □

4.5 The Rescaled Bundle

Consider the following family of vector spaces

$$\mathbb{S}_\gamma = \begin{cases} S_{m_1} \otimes S^*_{m_2} & \gamma = (m_1, m_2, t) \\ \Lambda^* T_m M \otimes \mathrm{End}_{Cl}(S_m) & \gamma = (X_m, 0). \end{cases}$$

parametrized by $\gamma \in \mathbb{T}M$. We have a natural injection

$$\mathcal{S}(\mathbb{T}M) \to \Pi_\gamma \mathbb{S}_\gamma$$

$$\sigma \mapsto \widehat{\sigma}$$

where $\widehat{\sigma}(\gamma) = \varepsilon_\gamma \sigma$.

Let $\mathbf{S}_{\mathbb{T}M}$ be the sheaf over the $\mathbb{T}M$ that consist of sections of the bundle

$$\Pi_\gamma \mathbb{S}_\gamma \to \mathbb{T}$$

that are locally of the form $\sum_i f_i \widehat{\sigma}_i$ where $f_i \in C^\infty(\mathbb{T}M)$ and $\sigma_i \in \mathcal{S}(\mathbb{T}M)$.

Proposition 4.5.1 ([HY19]). $\mathbf{S}_{\mathbb{T}M}$ *is a locally free and finitely generated sheaf of* $\mathcal{A}(\mathbb{T}M)$*-modules over the tangent groupoid. So we obtain a smooth vector bundle* $\mathbb{S} \to \mathbb{T}M$ *with fibers* $\mathbb{S}_\gamma$*.*

4.6 Getzler Symbol

Let Y be a vector field over M. We may define two derivations on the rescaled module acting along the first component:

$$\mathbf{c}(\mathbf{Y}) := tc(Y) : \mathcal{S}(\mathbb{T}M) \to \mathcal{S}(\mathbb{T}M)$$

$$\boldsymbol{\nabla}_{\mathbf{Y}} := t\nabla_Y : \mathcal{S}(\mathbb{T}M) \to \mathcal{S}(\mathbb{T}M)$$

These extend to operators on the space of smooth sections of the rescaled bundle.

Theorem 4.6.1. *For* $s \in \Gamma(\mathbb{S})$, $Y \in \Gamma(TM)$, *and* $X_m \in T_m M$ *we have following formula:*

- [HY19, Lemma 3.29]

  $$\varepsilon_{X_m}[\mathbf{c}(\mathbf{Y})s] = Y_m \wedge \varepsilon_{X_m} s$$

  In other word, along the zero fiber $\mathbf{c}(\mathbf{Y})$ *restricts to wedge product by* Y*.*

- [HY19, Lemma 3.30]

  $$\varepsilon_{X_m}[\boldsymbol{\nabla}_{\mathbf{Y}} s] = \partial_{Y_m} \varepsilon_{X_m} s + \frac{1}{2}\kappa(X_m, Y_m) \wedge \varepsilon_{X_m} s$$

  In other words, along the zero fiber $\boldsymbol{\nabla}_{\mathbf{Y}}$ *restricts to partial derivative with respect to* Y*, plus wedge product with the curvature two-form. So in particular, for* $X_m = \sum_i x_j \partial_j$ *and* $Y_m = \partial_{e_i}$ *we have*

  $$\boldsymbol{\nabla}_{\mathbf{Y}} s|_{X_m} = (\partial_i - \frac{1}{4} R_{ij} x_j \wedge) s|_{X_m}.$$

More generally, let P be a source-wise differential operator of Getzler order k defined on $S \boxtimes S^* \to M \times M$. Then the family of operators

$$\mathbf{P} := t^k P : \mathcal{S}(\mathbb{T}M) \to \mathcal{S}(\mathbb{T}M)$$

extends a smooth differential operator acting on the smooth sections of the rescaled bundle $\Gamma(\mathbb{S})$.

Definition 4.6.2. Let P be a source-wise differential operator of Getzler order k. We define the Getzler symbol of P to be the restriction to zero-fiber of the tangent groupoid of $\mathbf{P} = tP$:

$$\sigma_G(P) := \mathbf{P}|_{t=0} : \Gamma(\pi^*\Lambda^*T^*M \otimes \mathrm{End}_{\mathrm{Cl}}(S)) \to \Gamma(\pi^*\Lambda^*T^*M \otimes \mathrm{End}_{\mathrm{Cl}}(S)).$$

If D is a smooth differential operator on $S \to M$, considered as an operator acting on $S \boxtimes S^* \to M \times M$, and if D is given locally as the sum

$$\sum c(Y_1) \cdots c(Y_r) \nabla_{X_1} \cdots \nabla_{X_s}$$

then the Getzler symbol is given by

$$\sigma_G(D) := \sum_{r+s=k} Y_1 \wedge \cdots \wedge Y_r \wedge (\partial_{X_1} - \frac{1}{4} R_{ij} x_j \wedge) \cdots (\partial_{X_s} - \frac{1}{4} R_{ij} x_j \wedge).$$

Example 4.6.1. *Let $\mathcal{R}$ be the Riemannian Euler vector field defined on a neighborhood of the diagonal of $M \times M$. Clearly, $\mathcal{R}$ is a source-wise operator of Getzler order 1. So $t\nabla_{\mathcal{R}}$ extends to smooth differential operator on the rescaled bundle. However, its Getzler's symbol is quite boring:*

$$\sigma_G(\nabla_{\mathcal{R}}) = 0.$$

To see this, consider $\nabla_{\mathcal{R}}$ acting on the fibers $t \neq 0$, without multiplying with t which gives the derivation

$$\nabla_{\mathcal{R}} : \mathcal{S}(\mathbb{T}M) \to \mathcal{S}(\mathbb{T}M)$$
$$\sum_{p \in \mathbb{Z}} s_p t^{-p} \mapsto \sum_{p \in \mathbb{Z}} \nabla_{\mathcal{R}} s_p t^{-p}.$$

On the zero fiber, this gives the derivation by Euler vector field, $\mathcal{E}$, on the vector bundle $\pi^*\Lambda^*T^*M$. Therefore by multiplying with t, the derivation on the zero fiber vanishes.

Remark 4.6.1. The original definition of the Getzler order of a source-wise differential operator P, given in [BGV04], is the smallest $k \in \mathbb{Z}$ such that $t^k P$ extends to an operator on the zero section. Hence, in the original definition, $\nabla_{\mathcal{R}}$ is of order 0 with Getzler symbol $\partial_{\mathcal{E}}$. Roughly speaking, the original definition of the Getzler's order takes into account vanishing order of the coefficients of the differential operators, which current definition does not.

4.7 The Convolution Algebra and the Supertraces

Let $\mathbb{T}^{(2)}M$ be the space of composable pairs of the tangent groupoid. We have the obvious projection maps

$$p_1, p_2 : \mathbb{T}^{(2)}M \to \mathbb{T}M$$

and the composition map

$$c : \mathbb{T}^{(2)}M \to \mathbb{T}M.$$

Theorem 4.7.1 ([HY19, Theorem 4.5])**.** *The rescaled bundle has a "multiplicative structure", i.e. there exists a smooth vector bundle morphism*

$$p_1^*\mathbb{S} \otimes p_2^*\mathbb{S} \xrightarrow{\circ} c^*\mathbb{S}$$

which on the zero fibers is given by

$$\mathbb{S}_{(X_m,0)} \otimes \mathbb{S}_{(Y_m,0)} \to \mathbb{S}_{(X_m+Y_m)}$$
$$\alpha \otimes \beta \mapsto \alpha \wedge \beta \wedge \exp(-\frac{1}{2}\kappa(X_m, Y_m)),$$

and which on nonzero fibers is given by the canonical composition map:

$$\mathbb{S}_{(m_1,m_2,\lambda)} \otimes \mathbb{S}_{(m_2,m_3,\lambda)} \to \mathbb{S}_{(m_1,m_2,\lambda)}$$

$$\alpha \otimes \beta \mapsto \alpha \circ \beta.$$

For the proof, in [HY19], a version of rescaled bundle $\mathbb{S}^{(2)} \to \mathbb{T}^{(2)}M$ over the space of composable arrows is introduced. We have a canonical external product

$$\otimes : \Gamma(\mathbb{S}) \otimes \Gamma(\mathbb{S}) \to \Gamma(\mathbb{S}^{(2)}).$$

We give more detail on this in Appendix.

By fixing a Haar system μ on the tangent groupoid, we obtain a convolution product on the space of smooth compactly supported sections of rescaled bundle:

Proposition 4.7.1 ([HY19, Lemma 4.4])**.** *There following formula defines a associative product*

$$* : \Gamma_c(\mathbb{S}) \otimes \Gamma_c(\mathbb{S}) \to \Gamma_c(\mathbb{S})$$
$$f_1 * f_2(\eta) = \int_{\mathbb{T}M^{s(\eta)}} f_1(\gamma) \circ f_2(\eta \circ \gamma^{-1}) d\mu^{s(\eta)}(\gamma).$$

Remark 4.7.1. In next chapter, we will extend the convolution product to a bigger space, the Schwartz sections.

The convolution algebra $\Gamma_c(\mathbb{S})$ has smooth family of supertraces parametrized by $t \in \mathbb{R}$:

Proposition 4.7.2 ([HY19, Theorem 5.6])**.** *We have the following map of supertraces*

$$\mathrm{Str} : \Gamma(\mathbb{S}) \to C^\infty(\mathbb{R})$$

given by the following formula

$$\mathrm{Str}_t(s) = \begin{cases} t^{-\dim(M)} \int_M \mathrm{str}(\varepsilon_{(m,m,t)} s) d\mu(m) & t \neq 0 \\ \int_M \mathrm{str}(\varepsilon_{(0_m,0)} s) d\mu(m) & t = 0. \end{cases}$$

Chapter 5 | The Schwartz Algebra of the Rescaled Bundle

In this chapter we will develop some machinery for index theory on the rescaled bundle. More or less following [CR08], we define the algebra of "Schwartz" functions over the tangent groupoid. This algebra contains the smooth functions with compact support, and all the functions in this algebra restrict to standard Schwartz functions over the zero fiber of the tangent groupoid. The convolution product over the compactly supported functions extends to the Schwartz space and makes the Schwartz space into an associative algebra.

We then define the space of Schwartz sections of the rescaled bundle, and extend the convolution product over the compactly supported sections to the Schwartz space. We show that the resulting algebra is a Fréchet algebra.

5.1 Schwartz Functions on the Deformation Space

Our definition of Schwartz functions is a variation of the one given in [CR08]. The benefit of the new approach is that we drop the "conic compact support" condition in [CR08, Definition 4.1].

Definition 5.1.1. Let $D \subset \mathbb{R}^m \times \mathbb{R}^l$ be any set. For every $\varepsilon > 0$, we define the set

$$\widetilde{\mathcal{D}}_{[-\varepsilon,\varepsilon]} \subset \mathbb{R}^m \times \mathbb{R}^l \times [-\varepsilon, \varepsilon]$$

as

$$\widetilde{D}_{[-\varepsilon,\varepsilon]} := \{(x, \xi, t) \in \mathbb{R}^m \times \mathbb{R}^l \times [-\varepsilon, \varepsilon] : \ (x, t\xi) \in \mathcal{S}\}. \tag{5.1}$$

We allow $\varepsilon = \infty$, in which case we denote $\widetilde{\mathcal{D}}_{[-\varepsilon,\varepsilon]}$ simply by $\widetilde{\mathcal{D}}$.

Assume V is an open subset of $\mathbb{R}^m \times \mathbb{R}^l$ and denote $M = V \cap \left(\mathbb{R}^m \cap \{0\}\right)$.

Lemma 5.1.2. *We have an isomorphism* $\mathbb{N}_V M \simeq \widetilde{V}$ *via*

$$\begin{aligned} \mathbb{N}_V M &\to \widetilde{V} \\ (x, y, t) &\mapsto (x, \frac{1}{t}y, t). \\ (x, y, 0) &\mapsto (x, y, 0) \end{aligned}$$

Let U_0 denote the intersection $U \cap \left(\mathbb{R}^m \times \{0\}\right)$. For $\varepsilon > 0$, we denote by $\mathbb{N}_V M_{[-\varepsilon,\varepsilon]}$ the deformation to the normal cone with fibers over $[-\varepsilon, \varepsilon]$, as a set given by

$$V \times [-\varepsilon, \varepsilon] \bigcup \mathbb{N}_V M.$$

Using the natural identification

$$\begin{aligned} \varphi_U : \mathbb{N}_U U_{0[-\varepsilon,\varepsilon]} &\to \widetilde{U}_{[-\varepsilon,\varepsilon]} \\ (x, \xi, t) &\to (x, \frac{1}{t}\xi, t). \end{aligned}$$

we define the space of Schwartz functions $\mathcal{S}(\mathbb{N}_U U_{0[-\varepsilon,\varepsilon]})$ as the pullback $\varphi_U^* \mathcal{S}(\widetilde{U}_{[-\varepsilon,\varepsilon]})$.

Definition 5.1.3. Let $U \subset \mathbb{R}^m \times \mathbb{R}^l$ be an open set.

- The space of r-differentiable rapidly decaying functions, $\mathcal{S}_r(\widetilde{U}_{[-\varepsilon,\varepsilon]})$, consists of functions $f \in C^r(\widetilde{U}_{[-\varepsilon,\varepsilon]})$, such that for every compact set $K \subset U$, all $k \geq 0$ and all $|\alpha| + |\beta| + l \leq r$

$$||f||^{[-\varepsilon,\varepsilon]}_{K,k,l,\alpha,\beta} := \sup_{(x,t\xi)\in K} (1 + ||\xi||^2)^{k/2} |\partial_t^l \partial_x^\beta \partial_\xi^\alpha f(x, \xi, t)| < \infty. \tag{5.2}$$

- The space of Schwartz functions $\mathcal{S}(\widetilde{U}_{[-\varepsilon,\varepsilon]})$ is defined to be the inverse limit

$$\mathcal{S}(\widetilde{U}_{[-\varepsilon,\varepsilon]}) := \varprojlim_r \mathcal{S}_r(\widetilde{U}_{[-\varepsilon,\varepsilon]}).$$

We will show that this definition is invariant under change of coordinates. Let $U, U' \subset \mathbb{R}^m \times \mathbb{R}^l$ and $F = (F_1, F_2) : U \to U'$ be a diffeomorphism that restricts to a diffeomorphism from $U \cap \left(\mathbb{R}^m \times \{0\}\right)$ to $U' \cap \left(\mathbb{R}^m \times \{0\}\right)$. We then obtain a change of coordinates for the deformation to the normal cone $\widetilde{F} : \widetilde{U}_{[-\varepsilon,\varepsilon]} \to \widetilde{U}'_{[-\varepsilon,\varepsilon]}$ given by the formula

$$\begin{cases} (x, \xi, t) \mapsto (F_1(x, t\xi), \frac{1}{t}F_2(x, t\xi), t) & \text{if} \quad t \neq 0 \\ (x, \xi, 0) \mapsto (F_1(x, 0), \frac{\partial F_2}{\partial \xi}(x, 0).\xi, 0) & \text{if} \quad t = 0. \end{cases} \tag{5.3}$$

Lemma 5.1.4. *The induced diffeomorphism* $\widetilde{F} : \widetilde{U}_{[-\varepsilon,\varepsilon]} \to \widetilde{U}'_{[-\varepsilon,\varepsilon]}$ *gives a continuous map on the Schwartz spaces:*

$$\mathcal{S}_r(\widetilde{U}'_{[-\varepsilon,\varepsilon]}) \to \mathcal{S}_r(\widetilde{U}_{[-\varepsilon,\varepsilon]}),$$

and

$$\mathcal{S}(\widetilde{U}'_{[-\varepsilon,\varepsilon]}) \to \mathcal{S}(\widetilde{U}_{[-\varepsilon,\varepsilon]})$$

$$f \mapsto f \circ \widetilde{F}.$$

Proof. We want to show Schwartz estimates for $f \circ F$. We use the notations $\widetilde{F}(x, \xi, t) = (\widetilde{F}_1(x, \xi, t), \widetilde{F}_2(x, \xi, t), t)$. For multi-indices α, β and integer l, by the chain rule, we may write

$$\partial_t^l \partial_x^\beta \partial_\xi^\alpha (f \circ F) = \sum_{\alpha', \alpha'', \alpha''', \beta', \beta'', \beta''', l', l'', l'''} (\partial_t^{l'} \partial_x^{\alpha'} \partial_\xi^{\beta'} \widetilde{F}_1).(\partial_t^{l''} \partial_x^{\alpha''} \partial_\xi^{\beta''} \widetilde{F}_2).(\partial_t^{l'''} \partial_{x'}^{\alpha'''} \partial_{\xi'}^{\beta'''} f) \tag{5.4}$$

There exists a smooth function $G : \mathcal{U} \to \mathbb{R}^l$ where $F_2(x, t\xi) = t\frac{\partial F_2}{\partial \xi}(x, 0).\xi + t^2 G(x, t\xi)$. So we may write

$$\widetilde{F}_1(x, \xi, t) = F_1(x, t\xi)$$

$$\widetilde{F}_2(x, \xi, t)(x, \xi, t) = \frac{\partial F_2}{\partial \xi}(x, 0).\xi + tG(x, t\xi).$$

So as long as $(x, t\xi) \in \mathcal{K}$, the coefficients $\partial_t^{l'} \partial_x^{\alpha'} \partial_\xi^{\beta'} \widetilde{F}_1(x, \xi, t)$ and $\partial_t^{l''} \partial_x^{\alpha''} \partial_\xi^{\beta''} \widetilde{F}_2(x, \xi, t)$ in formula 5.4 are bounded. Also note that $(x, t\xi) \in \mathcal{K}$ if and only if $(\widetilde{F}_1(x, \xi, t), t\widetilde{F}_2(x, \xi, t)) \in \mathcal{K}'$. Hence the Schwartz estimates for g follows from Schwartz estimates for f.

□

Let $M \hookrightarrow V$ be embedding of smooth manifolds and let $U \subset V$ be an open set.

Definition 5.1.5. Let $M \hookrightarrow V$ be an embedding of smooth manifolds. We call a local coordinate

$$\psi : U \to \mathcal{U} \subset \mathbb{R}^m \times \mathbb{R}^l \tag{5.5}$$

an *embedding coordinate* for the embedding $M \hookrightarrow V$ if

$$\psi(U \cap M) \subset \mathbb{R}^m \times \{0\}. \tag{5.6}$$

From an embedding coordinate we obtain a coordinate chart for the deformation to the normal cone

$$\widetilde{\psi} : \mathbb{N}_U(U \cap M)_{[-\varepsilon,\varepsilon]} \to \mathbb{N}_{\mathcal{U}}\mathcal{U}_{0[-\varepsilon,\varepsilon]} \xrightarrow{\varphi_{\mathcal{U}}} \widetilde{\mathcal{U}}_{[-\varepsilon,\varepsilon]}.$$

Now we define the Schwartz functions over general deformation spaces.

Definition 5.1.6. A function $f \in C^\infty(\mathbb{N}_V M_{[-\varepsilon,\varepsilon]})$ belongs to $\mathcal{S}_r(\mathbb{N}_V M_{[-\varepsilon,\varepsilon]})$ and $\mathcal{S}(\mathbb{N}_V M_{[-\varepsilon,\varepsilon]})$ if for every local coordinate $\psi : U \to \mathcal{U}$, the function $f \circ \widetilde{\psi}^{-1}$ belongs to $\mathcal{S}_r(\widetilde{\mathcal{U}}_{[-\varepsilon,\varepsilon]})$ and $\mathcal{S}(\widetilde{\mathcal{U}}_{[-\varepsilon,\varepsilon]})$, respectively. We call spaces $\mathcal{S}_r(\widetilde{\mathcal{U}}_{[-\varepsilon,\varepsilon]})$ and $\mathcal{S}(\widetilde{\mathcal{U}}_{[-\varepsilon,\varepsilon]})$ the space of r-differentaible rapidly decaying and the space Schwartz functions, respectively.

From here on we assume V is a smooth closed manifold. Consider a finite open covering $\{U_p\}_p$ of V with embedding coordinate charts $U_p \to \mathcal{U}_p$. Also consider a family of compact subsets $\{K_s\}_s$ such that

- their interiors cover V and
- for every s there exist an index p_s such that

$$K_s \subset U_{p_s}. \tag{5.7}$$

We may obtain a family of seminorms for functions on $\mathbb{N}_V^\varepsilon M$.

$$||f||^{[-\varepsilon,\varepsilon]}_{s,k,l,\alpha,\beta} := ||f|_{\widetilde{\mathcal{U}}_{p_s}}||^{[-\varepsilon,\varepsilon]}_{K_s,k,l,\alpha,\beta}$$

where $||.||_{K_s,k,l,\alpha,\beta}$ is the seminorm 5.2 on functions over $\widetilde{\mathcal{U}}_{s,[-\varepsilon,\varepsilon]}$. We then obtain the norms

$$||f||^{[-\varepsilon,\varepsilon]}_{r,k} = \sum_{|\alpha|+\beta+l\le r, i\le k} ||f||^{[-\varepsilon,\varepsilon]}_{s,i,l,\alpha,\beta} \tag{5.8}$$

on functions over $\mathbb{N}_V M_{[-\varepsilon,\varepsilon]}$.

Lemma 5.1.7. *Different choices of an open covering result in an equivalent family of seminorms.* □

We have the following Fréchet spaces:

Theorem 5.1.8.

- $\mathcal{S}_r(\mathbb{N}_V M_{[-\varepsilon,\varepsilon]})$ *is complete with respect to following family of norms*

$$\{||.||_{r,k}^{[-\varepsilon,\varepsilon]}\}_{k\geq 0}.$$

- $\mathcal{S}(\mathbb{N}_V M_{[-\varepsilon,\varepsilon]})$ *is complete with respect to following family of norms*

$$\{||.||_{r,k}^{[-\varepsilon,\varepsilon]}\}_{r\geq 0,k\geq 0}.$$

We will prove the first bullet point. The second bullet point follows from the first as the inverse limit of Fréchet spaces is a Fréchet space. Let $\mathcal{O} \subseteq \mathbb{R}^n$ be any open set.

Lemma 5.1.9. *If* $\{g_n\} \in C^1(\mathcal{O})$ *converges uniformly to* g *and* $\partial_i g$ *converges uniformly to* f_i *for* $1 \leq i \leq n$*, then* $g \in C^1(\mathcal{O})$ *and* $\partial_i g = f_i$*.*

Proof of Theorem 5.1.8. Since we only need norm estimates the norms on local neighborhoods of the zero fiber of the deformation space, we may just check the statements for a neighboorhood $\mathbb{N}_{U_{p_s}}\left(U_{p_s[-\varepsilon,\varepsilon]} \cap M\right) \subset \mathbb{N}_V M_{[-\varepsilon,\varepsilon]}$. We use the identification $\mathbb{N}_{U_{p_s}}\left(U_{p_s} \cap M_{[-\varepsilon,\varepsilon]}\right) \simeq \widetilde{\mathcal{U}}_{[-\varepsilon,\varepsilon]}$. Let $\{f_n\} \in \mathcal{S}_r(\widetilde{\mathcal{U}}_{[-\varepsilon,\varepsilon]})$ be a Cauchy sequance. For every compact $\mathcal{K}$, and k, l, α, β with $|\alpha| + \beta| + l \leq r$ we have a uniform convergence over $\widetilde{\mathcal{K}}_{[-\varepsilon,\varepsilon]} = \{(x, \xi, t) \in \mathbb{R}^m \times \mathbb{R}^l \times [-\varepsilon, \varepsilon] : (x, t\xi) \in \mathcal{K}\}$

$$\lim_n (1 + ||\xi||^2)^{k/2} \partial_t^l \partial_x^\beta \partial_\xi^\alpha f_n(x, \xi, t) = f_{K,k,l,\alpha,\beta}$$

for some smooth function $f_{K,k,l,\alpha,\beta}$. In particular, we know the sequence $\{f_n\}_n$ converges uniformly to some continuous function f. Using Lemma 5.1.9 and induction we deduce f is smooth and we have uniform convergences

$$\partial_t^l \partial_x^\beta \partial_\xi^\alpha f_n(x, \xi, t) \to \partial_t^l \partial_x^\beta \partial_\xi^\alpha f(x, \xi, t).$$

So we have the following uniform convergences over $\widetilde{\mathcal{K}}_s^\varepsilon$

$$(1 + ||\xi||^2)^{k/2} \partial_t^l \partial_x^\beta \partial_\xi^\alpha f_n(x, \xi, t) \to (1 + ||\xi||^2)^{k/2} \partial_t^l \partial_x^\beta \partial_\xi^\alpha f(x, \xi, t).$$

□

Example 5.1.1. *Let* ρ *be the injectivity radius of the manifold* M*, and* $U_\rho \subset M \times M$ *is the* ρ*-neighborhood of the diagonal. The distance function* $d(x, y) : U_\rho \to \mathbb{R}^+$ *induces a smooth function* $f = \frac{d(x,y)}{t}$ *over the tangent groupoid* $\mathbb{T}M$*. We also obtain a Schwartz* *function over a neighborhood of the zero fiber of the tangent groupoid:*

$$e^{-d^2(x,y)/t^2}.$$

Lemma 5.1.10. *Let* $\varepsilon > 0$ *be any positive number.*

- *The compactly supported functions form a dense subspace of the Schwartz space:*

$$C_c^\infty(\mathbb{N}_V M_{[-\varepsilon,\varepsilon]}) \subset \mathcal{S}(\mathbb{N}_V M_{[-\varepsilon,\varepsilon]}).$$

- *A function* $f \in C_c^\infty(V \times [-\varepsilon, \varepsilon])$ *that vanishes in a neighborhood of* $V \times \{0\}$*, may be regarded as a Schwartz function*

$$f \in \mathcal{S}(\mathbb{N}_V M_{[-\varepsilon,\varepsilon]}).$$

- *For every* $f \in \mathcal{A}(\mathbb{N}_V M_{[-\varepsilon,\varepsilon]})$ *and* $g \in \mathcal{S}_r(\mathbb{N}_V M_{[-\varepsilon,\varepsilon]})$

$$fg \in \mathcal{S}_r(\mathbb{N}_V M_{[-\varepsilon,\varepsilon]}).$$

- *For every* $f \in \mathcal{A}(\mathbb{N}_V M_{[-\varepsilon,\varepsilon]})$ *and* $g \in \mathcal{S}(\mathbb{N}_V M_{[-\varepsilon,\varepsilon]})$

$$fg \in \mathcal{S}(\mathbb{N}_V M_{[-\varepsilon,\varepsilon]}).$$

Proof. For the first bullet point, without loss of generality we may assume $\mathbb{N}_V M_{[-\varepsilon,\varepsilon]} \simeq \widetilde{\mathcal{U}}_{[-\varepsilon,\varepsilon]}$ for an open set $\mathcal{U} \subset \mathbb{R}^m \times \mathbb{R}^l$. Choose $\varphi(x, \xi, t) \in C_c^\infty(\widetilde{\mathcal{U}}_{[-\varepsilon,\varepsilon]})$, where $\varphi(x, \xi, t) = 1$ for $|\xi| \leq 1$ and $\varphi(x, \xi, t) = 0$ for $|\xi| \geq 2$. For $g \in \mathcal{S}(\mathbb{N}_V M_{[-\varepsilon,\varepsilon]})$ we define compactly supported functions $g_k(x, \xi, t) = \varphi(x, \xi/k, t)g(x, \xi, t)$ for $k \in \mathbb{N}$. Then $g_k \to g$ in the Fréchet topology.

The fourth bullet point follows from third bullet point since the space of smooth Schwartz functions is the inverse limit of the spaces of r-differentiable rapidly decaying functions, $\varprojlim_r \mathcal{S}_r(\mathbb{N}_V M_{[-\varepsilon,\varepsilon]})$. So we only prove the third bullet point. Continuing with the case of $\mathbb{N}_V M_{[-\varepsilon,\varepsilon]} \simeq \widetilde{\mathcal{U}}_{[-\varepsilon,\varepsilon]}$. A function $f = ht^{-1} \in \mathcal{A}(\mathbb{N}_V M_{[-\varepsilon,\varepsilon]})$ may be interpreted as a function $f \in C^\infty(\widetilde{\mathcal{U}}_{[-\varepsilon,\varepsilon]})$ of the form

$$f(x, y, t) = \begin{cases} \frac{1}{t} h(x, t\xi) & \text{if} \quad t \neq 0 \\ \frac{\partial}{\partial \xi} h(x, 0).\xi & \text{if} \quad t = 0 \end{cases}$$

where $h \in C^\infty(\mathcal{U})$ such that $h|_{\mathcal{U}\cap\left(\mathbb{R}^m\times\{0\}\right)} = 0$. There exists a smooth function $k \in C^\infty(\mathcal{U})$, such that $h(x, ty) = t\frac{\partial}{\partial\xi}h(x, 0).\xi + t^2k(x, t\xi)$. Considering

$$f(x, \xi, t) = \frac{\partial}{\partial\xi}h(x, 0).\xi + tk(x, t\xi),$$

and $g \in \mathcal{S}(\widetilde{\mathcal{U}}_{[-\varepsilon,\varepsilon]})$, we see that the function $g.f$ satisfies the Schwartz estimates. Since functions of the form $ht^{-1}, h', h''t$ generate the Rees algebra, the proof is complete. □

Finally we define the space of Schwartz functions on whole deformation to the normal cone.

Definition 5.1.11. We define Schwartz functions over the deformation to the normal cone, $\mathcal{S}(\mathbb{N}_V M)$, to be functions $f \in C^\infty(\mathbb{N}_V M)$ such that for every $\varepsilon > 0$ they belong to $\mathcal{S}(\mathbb{N}_V M_{[-\varepsilon,\varepsilon]})$.

5.2 Tangent Groupoid and the Schwartz Algebra

The tangent groupoid is a deformation to the normal cone, so it has a Schwartz space. In this section, we will show that the convolution product on compactly supported functions extends to an associative product on the Schwartz space.

The space of composable arrows of the tangent groupoid $\mathbb{T}^{(2)}M$ is a deformation space

$$\mathbb{T}M^{(2)} := \mathbb{N}_{M\times M\times M}M$$

with respect to the diagonal embedding $M \hookrightarrow M \times M \times M$.

Note that we have a natural map, the external product

$$\mathcal{A}(\mathbb{T}M) \otimes \mathcal{A}(\mathbb{T}M) \to \mathcal{A}(\mathbb{T}^{(2)}M)$$

$$(\sum_p f_p t^{-p}, \sum_q g_q t^{-q}) \mapsto \sum_{p,q} f_p(m_1, m)g_q(m, m_2)t^{-(p+q)}$$

that extends to

$$\otimes : C^\infty(\mathbb{T}M) \otimes C^\infty(\mathbb{T}M) \to C^\infty(\mathbb{T}^{(2)}M)$$

$$(f, g) \mapsto f \otimes g = \begin{cases} (m_1, m, m_2, t) \mapsto f(m_1, m, t)g(m, m_2, t)] \\ (X_m, Y_m, 0) \mapsto f(X_m, 0)g(Y_m, 0) \end{cases} \tag{5.9}$$

Using the Leibniz rule, we see that this external product restricts to Schwartz space:

Lemma 5.2.1. *The external product restricts to Schwartz functions:*

$$\otimes : \mathcal{S}(\mathbb{T}M) \otimes \mathcal{S}(\mathbb{T}M) \to \mathcal{S}(\mathbb{T}^{(2)}M).$$

□

Proposition 5.2.1. *The integration map*

$$C_c^\infty(\mathbb{T}^{(2)}M) \xrightarrow{\mathcal{P}} C_c^\infty(\mathbb{T}M)$$

defined by the formula

$$F \mapsto \begin{cases} (m_1, m_2, t) \mapsto \int_M F(m_1, m, m_2, t) t^{-\dim(M)} d\mu(m) \\ (X_m, 0) \mapsto \int_{T_m M} F(X_m - Y_m, Y_m, 0) d\mu_m(Y_m) \end{cases}$$

extends continuously to a map from $\mathcal{S}(\mathbb{T}^{(2)}M)$ *to* $\mathcal{S}(\mathbb{T}M)$.

To prove this proposition we need a local version of the statement. Assume $\mathcal{V} \subset \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ is an open set and we define the open set $\widetilde{\mathcal{V}} \subset \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ as

$$\widetilde{\mathcal{V}} := \{(x, y, z, t) : (x, ty, tz) \in \mathcal{V})\}.$$

This open set is of the form (5.1) and the standard coordinates are those that we use for the deformation space $\mathbb{T}^{(2)}M$. So we have space of Schwartz functions $\mathcal{S}(\widetilde{\mathcal{V}})$. Consider the open set $\mathcal{U} := \pi_{1,2}(\mathcal{V})$ where $\pi_{1,2} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ is the projection to the first two components.

Lemma 5.2.2. *We have a well-define map*

$$\mathcal{P} : \mathcal{S}(\widetilde{\mathcal{V}}) \to \mathcal{S}(\widetilde{\mathcal{U}})$$

$$\mathcal{P}(F)(x, y, t) := \int_{\{z \in \mathbb{R}^n : (x,y,z,t) \in \widetilde{\mathcal{V}}\}} F(x, y, z, t) dz$$

Proof. This proof is copied from [CR08, Lemma 4.11] almost verbatim.

There is a constant for every compact $\Omega \subset \mathcal{V}$, and α, β, l there exists a constant $C > 0$ such that

$$\partial_t^l \partial_x^\beta \partial_y^\alpha F(x, y, z, t) \leq C \frac{1}{(1 + ||y||^2 + ||z||^2)^{k/2}}$$

for all $(x, ty, tz) \in \mathcal{Q}$. Therefore

$$\begin{aligned}
\partial_t^l \partial_x^\beta \partial_y^\alpha \mathcal{P}(F)(x, y, t) &\leq \int_{\mathbb{R}^n} C \frac{1}{(1 + ||y||^2 + ||z||^2)^{k/2}} dz \\
&\leq \frac{C}{(1 + ||y||^2)^{k/2}} \int_{\mathbb{R}^n} \frac{1}{(1 + ||z||^2)^{k/2}} dz \\
&\leq C' \frac{1}{(1 + ||y||^2)^{k/2}}
\end{aligned}$$

for some constant $C' > 0$ for all $(x, ty) \in \mathcal{K} := \pi_{1,2}(\mathcal{Q})$ where $\mathcal{K}$ is considered as a compact set inside $\mathcal{U}$. $\square$

Using this lemma we now prove the Proposition 5.2.1.

Proof of Proposition 5.2.1. This proof is copied from [CR08, Proposition 4.8] almost verbatim.

Let $V \subset M \times M \times M$ be an open subset and $\Phi : V \to \mathcal{V} \subset \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ be an embedding coordinate chart for the diagonal embedding $M \hookrightarrow M \times M \times M$. Let $\mathcal{U} := \pi_{1,2}(\mathcal{V})$, $U := \pi_{1,2}(V)$ and $\varphi : U \to \mathcal{U}$ be the induced embedding coordinate chart for the diagonal embedding $M \hookrightarrow M \times M$:

$$\begin{array}{ccc}
V & \xrightarrow{\Phi} & \mathcal{V} \\
{\scriptstyle \pi_{1,2}} \downarrow & & \downarrow {\scriptstyle \pi_{1,2}} \\
U & \xrightarrow[\varphi]{} & \mathcal{U}.
\end{array}$$

Denote by $\widetilde{V} := \mathbb{N}_V\big(V \cap M\big)$ and $\widetilde{U} := \mathbb{N}_U\big(U \cap M\big)$ the standard open subsets of the deformation spaces $\mathbb{T}^{(2)}M$ and $\mathbb{T}M$, respectively. We also have the induced coordinates charts

$$\widetilde{\Phi} : \widetilde{V} \to \widetilde{\mathcal{V}}$$

and

$$\widetilde{\varphi} : \widetilde{U} \to \widetilde{\mathcal{U}}.$$

Now since we have the following commuting diagram

$$\begin{array}{ccc} \mathcal{S}(\widetilde{V}) & \xrightarrow{\Phi} & \mathcal{S}(\widetilde{\mathcal{V}}) \\ \downarrow{\scriptstyle \mathcal{P}} & & \downarrow{\scriptstyle \mathcal{P}} \\ \mathcal{S}(\widetilde{U}) & \xrightarrow[\varphi]{} & \mathcal{S}(\widetilde{\mathcal{U}}), \end{array}$$

the statement follows from the Lemma 5.2.2. □

Definition 5.2.3. Composing the exterior product and integral map we obtain the convolution product over the Schwartz space:

$$\mathcal{S}(\mathbb{T}M) \otimes \mathcal{S}(\mathbb{T}M) \xrightarrow{\otimes} \mathcal{S}(\mathbb{T}^{(2)}M) \xrightarrow{\mathcal{P}} \mathcal{S}(\mathbb{T}M),$$

$$f \otimes g \mapsto f * g.$$

This convolution product of the Schwartz functions clearly is an extension of the convolution product on the space of compactly supported functions. Since the convolution product over the compactly supported functions is associative, it follows from Lemma 5.1.10 that

Lemma 5.2.4. *The Schwartz space* $\mathcal{S}(\mathbb{T}M)$ *is an associative convolution algebra.*

5.3 Space of Schwartz Sections

In this section, we combine ideas from [LY20] and [CR08] to define the space of "Schwartz" sections of the rescaled bundle. In Section 4.7, we introduced a rescaled bundle over the space of composable arrows of the tangent groupoid

$$\mathbb{S}^{(2)} \to \mathbb{T}^{(2)}M.$$

along with an external product

$$\otimes : \Gamma(\mathbb{S}) \otimes \Gamma(\mathbb{S}) \to \Gamma(\mathbb{S}^{(2)}).$$

We know a smooth section $s \in \Gamma(\mathbb{S})$ maybe written as a finite sum

$$s = f_1 s_1 + \cdots + f_k s_k$$

where $f_i \in C^\infty(\mathbb{T}M)$, and $s_i \in \mathcal{A}(\mathbb{S})$.

Definition 5.3.1.

- We define the space of Schwartz sections $\mathcal{S}(\mathbb{S})$ to be space of sections $s \in \Gamma(\mathbb{S})$ that can be written as a finite sum

$$s = f_1 s_1 + \cdots + f_k s_k$$

  where $f_i \in \mathcal{S}(\mathbb{T}M)$, and $s_i \in \mathcal{A}(\mathbb{S})$.

- We define the space $\mathcal{S}_r(\mathbb{S})$ in similar way, but this time coefficients $f_i \in \mathcal{S}_r(\mathbb{T}M)$.

- We also define the spaces $\mathcal{S}_r(\mathbb{S}^{(2)})$, and $\mathcal{S}(\mathbb{S}^{(2)})$, of sections of the form

$$s = f_1 s_1 + \cdots + f_k s_k$$

  where $s_i \in \mathcal{A}(\mathbb{S}^{(2)})$, and f_i belongs to $\mathcal{S}_r(\mathbb{T}^{(2)}M)$, and $\mathcal{S}(\mathbb{T}^{(2)}M)$, respectively.

Lemma 5.3.2. *The external product restricts to the Schwartz spaces:*

$$\otimes : \mathcal{S}_r(\mathbb{S}) \otimes \mathcal{S}_r(\mathbb{S}) \to \mathcal{S}_r(\mathbb{S}^{(2)}), \tag{5.10}$$

$$\otimes : \mathcal{S}(\mathbb{S}) \otimes \mathcal{S}(\mathbb{S}) \to \mathcal{S}(\mathbb{S}^{(2)}). \tag{5.11}$$

Proof. See Appendix. □

Lemma 5.3.3. *Every $s \in \Gamma(\mathbb{S}^{(2)})$ can be written as a finite sum*

$$s = f_1 s_1 + \cdots f_k s_k$$

where $f_i \in C^\infty(\mathbb{T}^{(2)}M)$ and $s_i = \sum_p \sigma_{i,p} t^{-p} \in \mathcal{A}(\mathbb{S}^{(2)})$ such that the sections

$$\sigma_{i,p}(m_1, m, m_2) \in \Gamma(S \boxtimes 1 \boxtimes S^*)$$

are independent of the component m in the middle. □

Lemma 5.3.4. *The formula*

$$F \mapsto \begin{cases} (m_1, m_2, t) \mapsto \int_M F(m_1, m, m_2, t) t^{-\dim(M)} d\mu(m)] \\ (X_m, 0) \mapsto \int_{T_m M} F(X_m - Y_m, Y_m, 0) d\mu_m(Y_m) \end{cases}$$

defines a continuous map

$$\mathcal{P} : \mathcal{S}(\mathbb{S}^{(2)}) \to \mathcal{S}(\mathbb{S}).$$

Proof of Theorem 5.3.4. Using Lemma 5.3.3, we may prove the proposition just for $s \in \mathcal{S}(\mathbb{S})$ of the form

$$s = f.\sigma t^{-p}$$

where $f \in \mathcal{S}(\mathbb{T}^{(2)}M)$, and where $\sigma(m_1, m, m_2) \in \Gamma(S \boxtimes 1 \boxtimes S^*)$ is independent of the middle term m.

We then have

$$\mathcal{P}s = \begin{cases} (m_1, m_2, t) \mapsto [\varepsilon_{(m_1,m,m_2)}\sigma t^{-p}] \int_M f(m_1, m, m_2, t) t^{-\dim(M)} d\mu(m)] \\ (X_m, 0) \mapsto \varepsilon_{(X_m,0,-Y_m)}[\sigma t^{-p}] \int_{T_m M} f(X_m - Y_m, Y_m, 0) d\mu_m(Y_m) \end{cases}$$

and so $\mathcal{P}s = [\mathcal{P}f].\sigma t^{-p}$, where $\mathcal{P}f \in \mathcal{S}(\mathbb{S})$ is defined in Lemma 5.2.1. □

Composing the external product and the integral map we obtain the convolution product

$$* : \mathcal{S}(\mathbb{S}) \otimes \mathcal{S}(\mathbb{S}) \xrightarrow{\otimes} \mathcal{S}(\mathbb{S}^{(2)}) \xrightarrow{\mathcal{P}} \mathcal{S}(\mathbb{S})$$

$$F * G := \mathcal{P}(F \circ G)$$

Clearly this convolution product is an extension of the convolution product over the space of compactly supported sections.

5.4 Completeness of the Convolution Algebra

We will introduce a family of seminorms on the convlution algebra $\mathcal{S}(\mathbb{S})$. This will make the convolution algebra into a Frechet space. We will only consider sections defined on certain neighborhoods of the zero fiber of the tangent groupoid. This will suffice for the heat kernel purposes we are considering in Chapter 6.

In this section $\mathbb{N}_V M_{[-\varepsilon,\varepsilon]}$ and $\mathbb{T}M_{[-\varepsilon,\varepsilon]}$ denote the inverse image of the interval $[-\varepsilon, \varepsilon]$ under the natural projection these deformation spaces to $\mathbb{R}$. We denote by $\mathbb{S}_{[-\varepsilon,\varepsilon]}$ the restricted bundle over the tangent groupoid $\mathbb{T}M_{[-\varepsilon,\varepsilon]}$.

Let ρ be the injectivity radius of the closed Riemannian manifold M^n, and let $\mathcal{W} \subset M$ be an open set with diamater less than ρ. Consider a coordinate chart $\psi : \mathcal{W} \to W \subset \mathbb{R}^n$.

Define open sets

$$\mathcal{U} := \mathcal{W} \times \mathcal{W} \tag{5.12}$$
$$U := \{(x, y) \in W \times \mathbb{R}^n : x + y \in W\}, \tag{5.13}$$

We have the induced embedding coordinate chart

$$\mathcal{U} \to U$$

$$(x, y) \mapsto (\psi(x), \psi(y) - \psi(x))$$

for the diagonal embedding $M \hookrightarrow M \times M$ (see Definition5.1.3). So we obtain a local chart for the tangent groupoid:

$$\widetilde{\mathcal{U}} := \mathbb{N}_{\mathcal{U}}\Big(\mathcal{U} \cap M\Big) = \mathbb{T}\mathcal{W} \to \widetilde{U}.$$

The space of restricted algebraic sections $\mathcal{A}(\mathbb{S}_{\widetilde{\mathcal{U}}})$ is a free module over $\mathcal{A}(\mathbb{T}\mathcal{W})$. A basis for this module is

$$\{s_\gamma\} \subset \mathcal{A}(\mathbb{S}_{\widetilde{\mathcal{U}}}) \tag{5.14}$$

of the form $s_\gamma = e^\gamma t^{|\gamma|}$, indexed by $\gamma = (\gamma_1, \cdots, \gamma_n)$ with $\gamma_i = \pm 1$, and e^γ is a synchronous local frame obtained by parallel transport in direction of the first component, of the Clifford section $e_1^{\gamma_1} \cdots e_n^{\gamma_n}$ over the diagonal, for some local orthonormal frame $(e_1, \cdots, e_n)$ (See 4.)

So every $s \in \Gamma(\mathbb{S}|_{\widetilde{\mathcal{U}}})$, may be written uniquely as a sum

$$s = \sum_\gamma f_\gamma s_\gamma$$

where $f_\gamma \in C^\infty(\widetilde{\mathcal{U}})$. Using such a representaion, for every $\varepsilon > 0$, we define the following norms

$$||s||_{r,k}^{\widetilde{\mathcal{U}}_{[-\varepsilon,\varepsilon]}} := \sum_{l \le k, r} ||f_\gamma||_{r,l}^{[-\varepsilon,\varepsilon]}. \tag{5.15}$$

Lemma 5.4.1. *Different choices of the orthonormal frame gives an equivalent family of norms on* $\mathcal{S}(\widetilde{\mathcal{U}}_{[-\varepsilon,\varepsilon]})$. □

Assume $\{\mathcal{W}_i\}_i$ is a finite open cover of M with coordinate charts $\mathcal{W}_i \to W_i \subset \mathbb{R}^n$, such that the diameter of each $\mathcal{W}_i$ is less than the injectivity radius of M. Define the open subsets $\mathcal{U}_i := \mathcal{W}_i \times \mathcal{W}_i \subset M \times M$ and $U_i := \{(x,y) \in W_i \times \mathbb{R}^n : x + y \in W_i\}$. Then we have the induced embedding coordinates $\mathcal{U}_i \to U_i$ for the diagonal embedding $M \hookrightarrow M \times M$. We obtain a family of coordinate charts $\widetilde{\mathcal{U}}_i \to \widetilde{U}_i$ for the tangent groupoid $\mathbb{T}M$. Define the open sets $\mathcal{U} := \bigcup_i \mathcal{U}_i \subset M \times M$, and $\widetilde{\mathcal{U}} := \bigcup_i \widetilde{\mathcal{U}}_i \subset \mathbb{T}M$ and define the following norms on sections of $\mathbb{S}|_{\widetilde{\mathcal{U}}_{[-\varepsilon,\varepsilon]}}$:

$$||s||_{r,k}^{[-\varepsilon,\varepsilon]} := \sum_i ||s|_{\mathcal{U}_i}||_{r,k}^{\widetilde{\mathcal{U}}_{i,[-\varepsilon,\varepsilon]}}. \tag{5.16}$$

Starting with a different finite open cover $\{\mathcal{W}'_i\}_i$ and coordinate charts $\mathcal{W}'_i \to W'_i$, and hence $\{\mathcal{U}'_i := \mathcal{W}'_i \times \mathcal{W}'_i\}_i$, and different induced embedding coordinate charts $\mathcal{U}'_i \to U'_i$, we then obtain different norms. Since the open subsets of the tangent groupoid $\widetilde{\mathcal{U}}_{[-\varepsilon,\varepsilon]}, \widetilde{\mathcal{U}}'_{[-\varepsilon,\varepsilon]} \subset \mathbb{T}M_{[-\varepsilon,\varepsilon]}$,

$$\widetilde{\mathcal{U}}_{[-\varepsilon,\varepsilon]} := \bigcup_i \widetilde{\mathcal{U}}_{i,[-\varepsilon,\varepsilon]}$$

and

$$\widetilde{\mathcal{U}'}_{[-\varepsilon,\varepsilon]} := \bigcup_i \widetilde{\mathcal{U}'}_{i,[-\varepsilon,\varepsilon]}$$

differ by a relatively compact sets, it follows

Lemma 5.4.2. *For different choices of open sets $\{\mathcal{U}_i\}_i$, local orthonormal frames $\{e^i\}_i$ and embedding coordinate charts $\mathcal{U}_i \to U_i$, we obtain equivalent families of norms.*

From Theorem 5.1.8, it follows that

Theorem 5.4.3.

- *The space $\mathcal{S}_r(\mathbb{S}|_{\widetilde{\mathcal{U}}_{[-\varepsilon,\varepsilon]}})$ is complete with following seminorms:*

$$\{||.||_{r,k}^{[-\varepsilon,\varepsilon]}\}_r.$$

- *The space $\mathcal{S}(\mathbb{S}|_{\widetilde{\mathcal{U}}_{[-\varepsilon,\varepsilon]}})$ is complete with following seminorms:*

$$\{||.||_{r,k}^{[-\varepsilon,\varepsilon]}\}_{r,k}.$$

Similar to functions over tangent groupoid we may define the space of Schwartz sections of the rescaled bundle as follows

Definition 5.4.4. We define Schwartz sections to be sections $s \in \mathcal{S}(\mathbb{S})$ such that $s\mathcal{S}(\mathbb{S}|_{\widetilde{\mathcal{U}}_{[-\varepsilon,\varepsilon]}})$ for some $\varepsilon > 0$. Similarly we define the space $\mathcal{S}_r(\mathbb{S})$.

Since Schwartz functions can be approximated by compactly supported functions (Lemma 5.1.10), the Schwartz sections can be approximated by compactly supported sections in the Fréchet topology. Therefore we conclude

Proposition 5.4.1. *The convolution product gives an associative algebra structure on the space of Schwartz sections*

$$*: \mathcal{S}(\mathbb{S}) \otimes \mathcal{S}(\mathbb{S}) \to \mathcal{S}(\mathbb{S}).$$

Lemma 5.4.5. *For every* r, k*,and* ε*, there exist a* $C > 0$ *such that for every* $s_1, s_2 \in \mathcal{S}(\mathbb{S}_{\widetilde{\mathcal{U}}_{[-\varepsilon,\varepsilon]}})$ *we have*

$$||s_1 * s_2||_{r,k}^{[-\varepsilon,\varepsilon]} \leq C||s_1||_{r,k}^{[-\varepsilon,\varepsilon]}.||s_2||_{r,k}^{[-\varepsilon,\varepsilon]}.$$

Chapter 6
Heat Kernels and the Rescaled Bundle

The theme of this chapter is inspired by the Atiyah-Singer index theorem for the Dirac operators. We know that the supertrace $\mathrm{Str}(e^{-tD^2}) = \int_M \mathrm{str}(K_t^{D^2}(x,x))dx$ equals the analytic index of the Dirac operator D, and is independent of t. Using Getzler's rescaling methid and by taking the limit $t \to 0^+$, we obtain the topological formula for the index:

$$\lim_{t\to 0} \mathrm{Str}_t(K_\tau) = \lim_{t\to 0} \int_M \mathrm{str}(K_{\tau t^2}^{D^2}(m,m))dm$$
$$\int_M \widehat{\mathcal{A}}(\frac{\tau R}{2\pi}) \wedge \mathrm{tr}\exp(-\frac{\tau F}{2\pi i}).$$

A relevant question would be that whether the family of heat kernels lives as a continuous section of a bundle over the tangern groupoid such that its value at $t = 0$ is the density of the topological index. The answer is yes, the rescaled bundle. In this chapter, we show that the family of heat kernels corresponding to the square of Dirac operator gives a family of Schwartz sections of the rescaled bundle. Our approach is inspired by the Volterra series method in local index theory.

6.1 Asymptotic Expansion of the Heat Kernel

Let M^n be a closed Riemannian manifold and let D be the Dirac operator associated with a Clifford module $S \to M$. From Section 4.6, we know the following family of differential operators on the rescaled bundle forms a smooth differential operator on the rescaled bundle:

$$\boldsymbol{\Delta} := \begin{cases} t^2 D^2 :\circlearrowleft \Gamma(S \otimes S_m^*; M \times m \times \{t\}) & m \in M, \quad t \neq 0 \\ H \circlearrowleft \Gamma(\Lambda^* T_m^* M; T_m M \times \{0\}) & m \in M, \quad t = 0. \end{cases}$$

Here, H is the *harmonic oscillator operator* and symbol of the square of the Dirac operator. In local coordinates H is given by

$$\sigma_G(D^2) = H = -\sum_i (\partial_i - \frac{1}{4} R_{ij} x^j)^2 + F.$$

Let $K_t^{D^2}$ denote the kernel of the heat operator e^{-tD^2} and let K_τ^H denote Mehler's kernel for the heat operator $e^{-\tau H}$. We have a formula for the latter:

$$K_\tau^H(X_m) = (4\pi\tau)^{-\frac{1}{2}} \det{}^{1/2}\Big(\frac{\tau R/2}{\sinh \tau R/2}\Big) \exp\Big(-\frac{1}{4}\langle X_m | \frac{\tau R}{2} \coth(\frac{\tau R}{2}) | X_m \rangle\Big) \exp(-\tau F).$$

See [BGV04, Roe99].

For the kernel $K_t^{D^2}$ we do not have an explicit formula. However, from [BGV04, Roe99], we have an asymptotic expansion of the heat kernel $K_t^{D^2}$:

$$K_t^{D^2}(x, y) \sim q_t(x, y) \sum_{i=0}^{\infty} t^i \Phi_i(x, y), \tag{6.1}$$

where $q_t(x, y) = (4\pi t)^{-\frac{n}{2}} \exp(-\frac{d(x,y)^2}{4t})$, and where the coefficients $\Phi_i(x, y)$ belong to the space $\Gamma(S \boxtimes S^*; M \times M)$. The heat kernel satisfies the heat equation

$$(\partial_t + D^2) K_t^{D^2}(x, y) = 0$$

where D differentiates with respect to the x-component. To obtain the coefficients of the asymptotic exapnsion of the heat kernel, the right-hand side of (6.1) can be treated as a formal solution for the heat equation (see [BGV04, Section 2.5] or [Roe99, Theorem 7.15]). This way we obtain recurrence relations between the coefficients Φ_i's. Let $\mathcal{W} \subset M$ be an open subset with diameter less than the injectivity radius of M, and define the open subset $\mathcal{U} := \mathcal{W} \times \mathcal{W} \subset M \times M$. Let $\mathcal{R}$ be the Riemannian Euler vector field on $\mathcal{U}$, as given in (4.2).

Proposition 6.1.1 ([BGV04, Theorem 2.26], [Roe99, Theorem 7.15])**.** *Over the open set $\mathcal{U}$, we have following recurrence relations*

$$\Phi_0 = \tau(x, y) \tag{6.2}$$
$$(\nabla_{\mathcal{R}} + i + \nabla_{\mathcal{R}} \log g^{1/4})\Phi_i(x, y) = -D^2\Phi_{i-1}(x, y) \qquad i > 0 \tag{6.3}$$

where $g(x, y) = \det(g_{ij})$ *is the determinant of the Riemannian metric in normal coordinates centered at* y *and* $\tau(x, y) : S_y \to S_x$ *is the parallel transport.*

The coefficients $\Phi_i(m, m) \in S_m \otimes S_m^* \simeq \mathbb{C}l^{2i}(T_mM)$ restricted to the diagonal have Clifford order at most $2i$. Moreover these coefficients are related to Mehler's kernel:

Proposition 6.1.2 ([BGV04, Theorem 4.12])**.**

$$K_\tau^H(X_m) = \sum_{i=1}^{n/2} \tau^i [\Phi_i]_{2i}(m, m),$$

where $[.]_{2i} : \mathbb{C}l^{2i}(T_mM) \to \Lambda^{2i}T_mM$ *denotes the symbol map.*

In [LY20, Proposition 2.9], Yi and Ludewig obtained lower bounds for the scaling order of the coefficients Φ_i. We sketch their argument here:

Lemma 6.1.1. *If* $i > 0$*, then for every section* $s \in \Gamma(S \boxtimes S^*)$ *we have*

$$\text{Scaling-order}((\nabla_{\mathcal{R}} + i + \nabla_{\mathcal{R}} \log g)s) = \text{Scaling-order}(s).$$

Proof. Since scaling and Taylor order are the same (See Proposition 4.3.1), we may assume $s = x^\alpha u$ where u is a parallel section. Then

$$(\nabla_{\mathcal{R}} + i + \nabla_{\mathcal{R}} \log g)s = (|\alpha| + i + \nabla_{\mathcal{R}} \log g)s.$$

Note that the function $\nabla_{\mathcal{R}} \log g$ vanishes along the diagonal. Hence the right-hand side of the formula above has the same Taylor order as s and the statement follows. □

Now we are ready to give an estimate for the scaling order of the coefficients Φ_i:

Theorem 6.1.2. *For every* $i \geq 0$*,* $\text{Scaling-order}(\Phi_i) \geq -2i$.

Proof. We will use induction. Since $\Phi_0(x, y) = \tau(x, y)$, by Corollary 4.3.3 we have

$$\text{Scaling-order}(\Phi_0) = 0.$$

So assume we know

$$\text{Scaling-order}(\Phi_{i-1}) \geq -2(i-1).$$

Since D^2 has Getzler order 2,

$$\begin{aligned}
\text{Scaling-order}(\Phi_i) &= \text{Scaling-order}((\nabla_{\mathcal{R}} + i + \nabla_{\mathcal{R}} \log g)\Phi_i) \\
&= \text{Scaling-order}(-D^2\Phi_{i-1}) \\
&\geq \text{Scaling-order}(\Phi_{i-1}) - \text{Getzler-order}(D^2) \\
&\geq -2i.
\end{aligned}$$

□

Corollary 6.1.3. *For every* $i \geq 0$, $\Phi_i t^{2i} \in \mathcal{A}(\mathbb{S})$. □

Definition 6.1.4. For every $\tau > 0$, and $N \geq 0$, denote by Φ_τ^N the following element of the resclaed module:

$$\sum_{i=0}^{N} \tau^i t^{2i} \Phi_i(x, y) \in \mathcal{A}(\mathbb{S})$$

.

Remark 6.1.1. For $i > n/2$, the section $\Phi_i t^{2i}$ vanishes on the zero fiber of the tangent groupoid. Hence, the values of the sections Φ_τ^N over the zero fiber are independent of $N \geq n/2$.

6.2 Volterra Series and the Schwartz property for the Heat Kernel

Let $\tau > 0$ be a positive parameter and let $N \in \mathbb{Z}_{\geq 0}$. From Example 5.1.1, we know $t^n q_{\tau t^2}$ extends to a Schwartz function on $\mathbb{T}M$. Since $\Phi_\tau^N = \sum_{i=0}^N \tau^i t^{2i} \Phi_i(x, y)$ is an element in the rescaled module (Remark 6.1.1), we obtain a family of Schwartz sections

$$K_\tau^N := t^n q_{t^2} \sum_{i=0}^{N} \Phi_i t^{2i} \tau^i \in \mathcal{S}(\mathbb{S}).$$

parametrized by τ and N.

Remark 6.2.1. The values of K_τ^N at $t = 0$ are independent of $N \geq n/2$.

Definition 6.2.1. For $\tau > 0$ and $N \geq 0$, we define the *remainder kernel*

$$r^N_\tau = (\partial_\tau + t^2 D^2) K^N_\tau$$

which is an element in the Schwartz space $\mathcal{S}(\mathbb{S})$.

The section r^N_τ can be written in terms of the heat coefficient Φ_N in a very simple form. Indeed, applying the heat operator $\partial_\tau + t^2 D^2$ to $t^n q_{t^2} \sum_{i=0}^N \Phi_i t^{2i} \tau^i$, by the recurrence relations (6.2), all terms involving Φ_i with $i < N$ cancel and we obtain (see proof of [BGV04, Theorem 2.29]):

Proposition 6.2.1.

$$r^N_\tau = t^n q_{t^2}.(D^2 \Phi_N) t^{2N} \tau^N.$$

□

In particular, for $N > n/2$, the remainder kernel r^N_τ is a Schwartz section that vanishes along the zero fiber to order at least $2N - n$. This has consequences for the Schwartz norms of r^N_τ. We recall how the Schwartz norms are defined on the tangent groupoid, as done in Chapter 5. Assume $\{\mathcal{W}_i\}_i$ is a finite open cover of M with coordinate charts $\psi_i : \mathcal{W}_i \to W_i \subset \mathbb{R}^n$, such that diameter of each $\mathcal{W}_i$ is less than the injectivity radius of M. Define open subsets $\mathcal{U}_i := \mathcal{W}_i \times \mathcal{W}_i \subset M \times M$ and $U_i := \{(x, y) \in W_i \times \mathbb{R}^n : x + y \in W_i\}$. We have induced embedding coordinates

$$\mathcal{U}_i \to U_i$$

$$(x, y) \mapsto (\psi_i(x), \psi_i(y) - \psi_i(x))$$

for the diagonal embedding $M \hookrightarrow M \times M$. We then obtain local coordinates for the tangent groupoid $\mathbb{T}M_{[-\varepsilon,\varepsilon]}$ around the zero fiber,

$$\widetilde{\mathcal{U}}_{i,[-\varepsilon,\varepsilon]} \to \widetilde{U}_{i,[-\varepsilon,\varepsilon]},$$

and denote $\widetilde{\mathcal{U}}_{[-\varepsilon,\varepsilon]} := \bigcup_i \widetilde{\mathcal{U}}_{i,[-\varepsilon,\varepsilon]}$. The Schwartz norms are given by the formula

$$||s||^{[-\varepsilon,\varepsilon]}_{r,k} := \sum_i ||s||^{\widetilde{\mathcal{U}}_{i,[-\varepsilon,\varepsilon]}}_{r,k},$$

where $||.||^{\widetilde{\mathcal{U}}_{i,[-\varepsilon,\varepsilon]}}_{r,k}$ is given by (5.15).

Lemma 6.2.2. *Assume $\tau > 0$, $N > n/2$ and $k \in \mathbb{Z}_{\geq 0}$. For small enough $\varepsilon > 0$, the restriction of $r_\tau^N \in \mathcal{S}(\mathbb{S}_{\widetilde{\mathcal{U}}_{[-\varepsilon,\varepsilon]}})$ has*

$$||r_\tau^N||_{r,k}^{[-\varepsilon,\varepsilon]} \leq \frac{1}{2}.$$

Proof. Without loss of generality we may assume $r_\tau^N \in \mathcal{S}(\mathbb{S}_{\widetilde{\mathcal{U}}_{i,[-\varepsilon,\varepsilon]}})$ and we will show its Schwartz norms are small enough over the open set $\widetilde{\mathcal{U}}_{i,[-\varepsilon,\varepsilon]}$. As in 5.14. over $\widetilde{\mathcal{U}}$ we have sections

$$\{s_\gamma\}_\gamma \subset \mathcal{A}(\mathbb{S}) \tag{6.4}$$

of the form $s_\gamma = e^\gamma t^{|\gamma|}$ where e^γ is a basis Clifford section obtained from a local orthonormal frame $\{e^i\}_i$. We may write

$$r_\tau^N = \sum_\gamma f_\gamma s_\gamma$$

where $f_\gamma \in C^\infty(\widetilde{\mathcal{U}})$ and

$$||r_\tau^N||_{r,k}^{[-\varepsilon,\varepsilon]} = \sum_\gamma ||f_\gamma||_{r,k}^{[-\varepsilon,\varepsilon]}.$$

So we need to show that for small enough $\varepsilon > 0$, $||f_\gamma||_{r,k}^{[-\varepsilon,\varepsilon]}$ is small. Considering f_γ as functions over U_i, these norms are given by the sum $||f_\gamma||_{r,k}^{[-\varepsilon,\varepsilon]} = \sum_{|\alpha|+\beta+l\leq r, i\leq k} ||f|_{\widetilde{\mathcal{U}}_{p_s}}||_{K_s,k,l,\alpha,\beta}^{[-\varepsilon,\varepsilon]}$ (see (5.8)) where

$$||f_\gamma||_{K_s,k,l,\alpha,\beta}^{[-\varepsilon,\varepsilon]} := \sup_{(x,t\xi)\in K} (1+||\xi||^2)^{k/2} |\partial_t^l \partial_x^\beta \partial_\xi^\alpha f_\gamma(x,\xi,t)| < \infty \tag{6.5}$$

for some compact sets $K_s \subset U_i$. Note that right-hand side of (6.5) is bounded for every k and $1+||\xi||^2$ approaches infinity as $\xi \to \infty$. Since for $|\alpha| + \beta + l \leq r$ the the function $\partial_t^l \partial_x^\beta \partial_\xi^\alpha f_\gamma(x,\xi,t)$ vanishes for $t = 0$ we deduce that as $\varepsilon \to 0$ the norm $||f_\gamma||_{K_s,k,l,\alpha,\beta}^{[-\varepsilon,\varepsilon]}$ approaches to zero and this is enough to prove the lemma. □

Let $\tau\Delta_k$ be the rescaled k-simplex in $\mathbb{R}^k$:

$$\{(\tau_1, \cdots, \tau_k) : 0 \leq \tau_1 \leq \cdots \leq \tau_k \leq \tau\}.$$

Equip with the Lebesgue measure $d\tau_1 \cdots d\tau_k$. So in particular, we have the volume $\mathrm{Vol}(\tau\Delta_k) = \tau^k/k!$.

Definition 6.2.3. We define the following Schwartz sections using the convolution product on the Schwartz algebra $\mathcal{S}(\mathbb{S})$:

$$K_\tau^{N,m} := \int_{\tau\Delta_m} K^N_{\tau-\tau_m} * r^N_{\tau_m-\tau_{m-1}} * \cdots * r^N_{\tau_1} \, d\tau_1 \cdots d\tau_m,$$

$$r_\tau^{N,m+1} := \int_{\tau\Delta_m} r^N_{\tau-\tau_m} * r^N_{\tau_k-\tau_{m-1}} * \cdots * r^N_{\tau_1} \, d\tau_1 \cdots d\tau_m.$$

We need these sections for the Volterra series technique for the construction of the heat kernel. We have

Lemma 6.2.4.

$$(\partial_\tau + t^2 D^2) K_\tau^{N,m}(x, y) = r_\tau^{N,m+1} + r_\tau^{N,m}.$$

Proof. This follows verbatim from the proof of [BGV04, Lemma 2.22]. □

Lemma 6.2.5. *For fixed $\tau > 0$ and $N > n/2$ and fixed k, there exist small enough $\varepsilon > 0$ and a constant $C > 0$ such that*

$$||K_\tau^{N,m}||_{r,k}^{[-\varepsilon,\varepsilon]} \leq C(\frac{1}{2})^m.$$

Proof. It follows immediately from Lemmas 5.4.5 and 6.2.2, and the fact volume of the rescaled k-simplex $\tau\Delta_m$ is $\tau^m/m!$. □

Fix an integer $r > 0$.

Theorem 6.2.6. *For large enough* N

- *There exists $\varepsilon > 0$ such that the alternating series*

$$p_\tau^N := \sum_{k=0}^{\infty} (-1)^m K_\tau^{N,m}$$

converges in the space $\mathcal{S}_r(\mathbb{S}_{\widetilde{\mathcal{U}}_{[-\varepsilon,\varepsilon]}})$.

- *We have*

$$(\partial_\tau + t^2 D^2) p_\tau^N = 0.$$

- *The section p_τ^N is independent of N and equals*

$$K_\tau := \begin{cases} t^n K_{t^2\tau}^{D^2} & t \neq 0 \\ K_\tau^H & t = 0. \end{cases}$$

Proof. The first bullet point follows immediately from Lemma 6.2.5.

For the second bullet point we may use Lemma 6.2.4: Applying the heat operator $\partial_\tau + t^2 D$ to the section p_τ^N gives a telescoping series that where all terms cancel out.

To see the third bullet point, note that for every $t \neq 0$, the solution to the heat eqatuion $(\partial_\tau + t^2 D^2)s = 0$ is unique, subject to an initial condition. So the values of the section p_τ^N over fibers $t \neq 0$ is independent of N

$$p_\tau^N|_{t\neq 0} = t^n K_{t^2\tau}^{D^2}.$$

Then by continuity, the value of p_τ^N over the zero fiber is also unique and is given by

$$p_\tau^N|_{t=0} = K_\tau^H.$$

□

Chapter 7
The Residue Cocycle and Bismut's Dirac Operators

This chapter is copied verbatim from [SLSJ21].

7.1 The Connes-Moscovici Residue Cocycle

In [CM95] Connes and Moscovici introduced a powerful new (b, B)-cocycle into noncommutative geometry. The cocycle is associated to a spectral triple (A, H, D) satisfying suitable technical hypotheses, and involves residues of spectral zeta functions for $\Delta = D^2$. It is cohomologous (in the (b, B)-bicomplex) to Connes' Chern character for the K-homology class $[F] \in K^0(A)$ defined by the operator $F = D(1 + D^2)^{-1/2}$, and hence in particular may be used to compute index pairings (see for example [Con94, GS89, Hig06]). In this section we give a short introduction to the Connes-Moscovici cocycle following Higson's notes [Hig06], and then briefly discuss the relationship to heat kernel asymptotics. One simplifying assumption we shall make is to take the operator D to be Fredholm. The non-Fredholm case is important in certain applications, but introduces minor technical complications, and we refer the interested reader to [Hig06] for discussion of how to reduce to the Fredholm (even the invertible) case.

7.1.1 The Connes-Moscovici Theorem

Recall (cf. [Con94]) that an *even spectral triple* (A, H, D) for an (ungraded) algebra A consists of a $\mathbb{Z}_2$-graded Hilbert space H, a representation of A on H by operators of even degree, and an unbounded odd self-adjoint operator D such that for all $a \in A$,

1. $a(1 + D^2)^{-1}$ is a compact operator, and

2. $a \cdot \mathrm{dom}(D) \subset \mathrm{dom}(D)$ and the commutator $[D, a]$ extends to a bounded operator on H.

The standard example is $A = C_c^\infty(M)$, M a complete even-dimensional Riemannian manifold, $H = L^2(M, S)$ and D a Dirac operator acting on sections of a $\mathbb{Z}_2$-graded Clifford module S.

The Connes-Moscovici result applies to spectral triples satisfying some technical hypotheses that we now describe. Following [Hig06], we introduce an algebra of 'generalized differential operators' for (A, H, D). Let $\Delta = D^2$ and define

$$H^\infty = \bigcap_{s=1}^{\infty} \mathrm{dom}(\Delta^s).$$

Suppose $a \cdot H^\infty \subset H^\infty$ for all $a \in A$. Let $\mathcal{D}(A, D)$ be the smallest algebra of operators on H^∞ that contains A, $[D, A]$ and such that if $X \in \mathcal{D}(A, D)$ then $[\Delta, X] \in \mathcal{D}(A, D)$.

We suppose the algebra $\mathcal{D}(A, D)$ is equipped with an increasing filtration $\mathcal{D}(A, D) = \cup_{q \geq 0} \mathcal{D}_q(A, D)$, where the operators X in $\mathcal{D}_q(A, D)$ are said to have '(generalized) order q' (notation: $o(X) = q$), having the following properties:

1. There is an even integer $r \geq 2$ such that

$$[\Delta, \mathcal{D}_q(A, D)] \subset \mathcal{D}_{q+r-1}(A, D), \qquad [D, a] \in \mathcal{D}_{\frac{r}{2}-1}(A, D), \quad \forall a \in A. \tag{7.1}$$

2. If $X \in \mathcal{D}_q(A, D)$ then there is a constant $\varepsilon > 0$ such that

$$\varepsilon \|Xv\| \leq \|v\| + \|\Delta^{\frac{q}{r}} v\|. \tag{7.2}$$

The integer r should be thought of as the generalized order of Δ, even though Δ need not itself lie in the algebra $\mathcal{D}(A, D)$. The correct choice of r depends on the details of the application; in the standard example mentioned above $r = 2$, but for example Connes and Moscovici study an interesting example with $r = 4$.

We now make the simplifying assumption that D is a Fredholm operator (cf. [Hig06, Section 6.1] for a discussion of how to reduce to this situation). For $\mathrm{Re}(z) > 0$ define

$$\Delta^{-z} = \int_{\mathbb{C}} \lambda^{-z} (\lambda - \Delta)^{-1} d\lambda,$$

where the contour $\mathbb{C}$ is a downward-oriented vertical line in the complex plane which separates 0 from the strictly positive part of the spectrum of Δ.

Definition 7.1.1. The algebra $\mathcal{D}(A, D)$ has *finite analytic dimension* if there is a real number $n \geq 0$ such that if $X \in \mathcal{D}_q(A, D)$ then for all $z \in \mathbb{C}$ with real part $\Re(z) > \frac{q+n}{r}$, the operator $X\Delta^{-z}$ extends by continuity to a trace-class operator on H. The minimal $n \geq 0$ with this property is called the *analytic dimension* of $\mathcal{D}(A, D)$. The function $z \mapsto \mathrm{Tr}(X\Delta^{-z})$ is then well-defined and holomorphic in the right half-plane $\Re(z) > \frac{q+n}{r}$. We say that $\mathcal{D}(A, D)$ has the *analytic continuation property* if this function extends to a meromorphic function on $\mathbb{C}$.

If $\Delta = D^2$ for a Dirac operator D on a compact Riemannian manifold M, the above analytic continuation property is well-known (originally due to Minakshisundaram-Pleijel) and the analytic dimension coincides with the dimension of M. For further context see for example the account in [Hig04], which describes a fairly general analytic continuation result. Note in particular that if $\mathcal{D}(A, D)$ has finite analytic dimension n then for $q > \frac{r}{n}$ and for all $a \in A \subset \mathcal{D}_0(A, D)$, $a(1 + D^2)^{-q}$ is trace class, hence (A, H, D) is a *finitely summable* spectral triple and Connes' Chern character is defined (cf. [Con94]).

The following is a case of the Connes-Moscovici local index formula [CM95], although we have formulated the hypotheses in the language of [Hig06].

Theorem 7.1.2 (Connes-Moscovici [CM95])**.** *Let (A, H, D) be an even spectral triple with D Fredholm, and let $\Delta = D^2$. Suppose A preserves $H^\infty = \cap_{s\geq 1}\mathrm{dom}(\Delta^s)$, and define the filtered algebra $\mathcal{D}(A, D)$ as above. Assume $\mathcal{D}(A, D)$ satisfies (7.1), (7.2), has finite analytic dimension n, and satisfies the analytic continuation property. Then there is an even (b, B)-cocycle $\Phi = (\varphi_0, \varphi_2, ..., \varphi_{2\lfloor \frac{n}{2} \rfloor})$ cohomologous to Connes' Chern character, which is given, for all $a_0, ..., a_p \in A$, by the following expressions:*

$$\varphi_0(a_0) = \operatorname*{Res}_{z=0}(\Gamma(z)\mathrm{Tr}_s(a_0\Delta^{-z})) + \mathrm{Tr}_s(a_0\Pi)$$

where Π is orthogonal projection onto the kernel of D and Tr_s denotes the supertrace, and if $p > 0$,

$$\varphi_p(a_0, ..., a_p) = \sum_{|k|\leq n-p} c_{pk} \operatorname*{Res}_{z=\frac{p}{2}+|k|} \mathrm{Tr}_s(P_k(a_0, ..., a_p)\Delta^{-z}),$$
$$P_k(a_0, ..., a_p) = a_0[D, a_1]^{(k_1)} \cdots [D, a_p]^{(k_p)}$$

where $k = (k_1, ..., k_p) \in (\mathbb{Z}_{\geq 0})^p$ is a multi-index, $X^{(\ell)} = \mathrm{ad}_\Delta^\ell(X)$, and

$$c_{pk} = \frac{(-1)^k}{k!} \frac{\Gamma(\frac{p}{2} + |k|)}{(k_1 + 1)(k_1 + k_2 + 2) \cdots (k_1 + \cdots + k_p + p)}, \qquad |k| = k_1 + \cdots + k_p.$$

We will refer to Φ as the *Connes-Moscovici cocycle* or the *residue cocycle*.

Remark 7.1.1. The bounds $p \leq 2\lfloor \frac{n}{2} \rfloor$ and $|k| \leq n - p$ come from generalized order considerations: the operator $P_k(a_0, ..., a_p) \in \mathcal{D}(A, D)$ has generalized order at most $|k|(r-1) + p(\frac{r}{2} - 1)$, hence by the analytic continuation assumption, $\frac{p}{2} + |k|$ will lie in the half plane where the supertrace is holomorphic provided that $n < p + |k|$.

Being cohomologous to Connes' Chern character, the residue cocycle yields a formula for the index of the operator in the spectral triple, and more generally for the index pairing with an element in the K-theory group $K_0(A)$. A multi-linear functional $\varphi_p \colon A^{\otimes(p+1)} \to \mathbb{C}$ may be extended to $M_k(A)^{\otimes(p+1)}$ by defining

$$\varphi_p(m_0 \otimes a_0, ..., m_p \otimes a_p) = \mathrm{tr}(m_0 \cdots m_p)\varphi_p(a_0, ..., a_p).$$

For the following see for example [GS89, Proposition 1.1, Theorem D] and [Hig06, Theorem 2.27].

Theorem 7.1.3. *Let* (A, H, D), Φ *be as in Theorem 7.1.2 and assume in addition that* A *is unital. Let* $e \in M_k(A)$ *be an idempotent. Then*

$$\textit{index}(e(D \otimes 1_k)e) = \varphi_0(e) + \sum_{p>0,\ \textit{even}} (-1)^{p/2} \frac{p!}{(p/2)!} \varphi_p(e - \tfrac{1}{2}, e, ..., e).$$

7.1.2 Residues and Heat Kernel Asymptotics

Throughout this subsection we assume (A, H, D) are as in Theorem 7.1.2. The residue cocycle is closely related to the small time behavior of the heat kernel $e^{-t\Delta}$. Let $\Pi^\perp = 1 - \Pi$ be projection onto the orthogonal complement of $\ker(D) = \ker(\Delta)$. The following is well-known, but we include it for completeness.

Proposition 7.1.1. *For all* $t > 0$ *the operator* $P_k(a_0, ..., a_p)e^{-t\Delta}$ *is trace class. As* $t \to 0^+$ *the trace norm is* $O(t^{-s})$ *for any* $s > \frac{n}{r} + |k|(1 - \frac{1}{r}) + p(\frac{1}{2} - \frac{1}{r})$. *As* $t \to \infty$ *the trace norm of* $P_k(a_0, ..., a_p)e^{-t\Delta}\Pi^\perp$ *decays exponentially.*

Proof. Let $s \in \mathbb{R}$ be as in the statement. By order considerations (see Remark 7.1.1), the operator $P_k(a_0, ..., a_p)(1 + \Delta)^{-s}$ is trace class. On the other hand by functional calculus $(1 + \Delta)^s e^{-t\Delta}$ is a bounded operator with norm at most $C_s t^{-s} e^t$, where C_s is a constant. It follows that $P_k(a_0, ..., a_p)e^{-t\Delta} = P_k(a_0, ..., a_p)(1 + \Delta)^{-s}(1 + \Delta)^s e^{-t\Delta}$ is trace class for all $t > 0$, and that its trace norm has the claimed asymptotic behavior as $t \to 0^+$. For $t > 1$,

$$P_k(a_0, ..., a_p)e^{-t\Delta}\Pi^\perp = P_k(a_0, ..., a_p)e^{-\Delta}e^{-(t-1)\Delta}\Pi^\perp$$

thus the trace norm is bounded by the trace norm of $P_k(a_0, ..., a_p)e^{-\Delta}$ times the operator norm $\|\Pi^\perp e^{-(t-1)\Delta}\Pi^\perp\| = e^{-(t-1)b}$ where $b > 0$ is the lower bound of $\Pi^\perp\Delta\Pi^\perp$ on $\Pi^\perp H$. $\square$

Proposition 7.1.2. *Suppose the supertrace admits an asymptotic expansion as $t \to 0^+$ of the form*

$$\mathrm{Tr}_s(P_k(a_0, ..., a_p)e^{-t\Delta}) \sim t^{-N_k} \sum_{s \geq 0} t^s \psi_{k,s}(a_0, ..., a_p).$$

Then the components of the residue cocycle are given by

$$\varphi_p(a_0, ..., a_p) = \sum_{|k| \leq n-p} c'_{pk}\psi_{k,s_{p,k}}(a_0, ..., a_p), \qquad s_{p,k} = N_k - \frac{p}{2} - |k|,$$

where

$$c'_{pk} = \frac{(-1)^k}{k! \cdot (k_1+1)(k_1+k_2+2)\cdots(k_1+\cdots+k_p+p)}.$$

Proof. For brevity let $P_k = P_k(a_0, ..., a_p)$. Since Δ^{-z} was defined to vanish on $\ker(\Delta)$, one has $\Delta^{-z} = \Delta^{-z}\Pi^\perp$. By the Mellin transform,

$$\Gamma(z)P_k\Delta^{-z} = \int_0^\infty t^{z-1}P_k e^{-t\Delta}\Pi^\perp dt$$

where the integral converges in the trace norm when $\Re(z) \gg 0$ by Proposition 7.1.1. Therefore taking the trace and splitting the range of integration, we have for $\Re(z) \gg 0$,

$$\Gamma(z)\mathrm{Tr}_s(P_k\Delta^{-z}) = \int_0^1 t^{z-1}\mathrm{Tr}_s(P_k e^{-t\Delta}\Pi^\perp)dt + \int_1^\infty t^{z-1}\mathrm{Tr}_s(P_k e^{-t\Delta}\Pi^\perp)dt \qquad (7.3)$$

Proposition 7.1.1 implies further that the integral over $(1, \infty)$ is a holomorphic function $f_k(z)$. Substituting $\Pi^\perp = 1 - \Pi$, $e^{-t\Delta}\Pi = \Pi$ in (7.3) yields

$$\Gamma(z)\mathrm{Tr}_s(P_k\Delta^{-z}) = \int_0^1 t^{z-1}\mathrm{Tr}_s(P_k e^{-t\Delta})dt - \frac{1}{z}\mathrm{Tr}_s(P_k\Pi) + f_k(z). \qquad (7.4)$$

Using the asymptotic expansion of $\mathrm{Tr}_s(P_k e^{-t\Delta})$ as $t \to 0^+$, the integral on the right hand side has an analytic continuation (with simple poles) to a neighborhood of $\frac{p}{2}+|k|$, namely

$$\sum_{s=0}^{M}\frac{1}{s-N_k+z}\psi_{k,s}+\int_0^1 t^{z-1}R_M(t)dt$$

for any $M \geq \max(0, N_k - \frac{p}{2} - |k| + 1)$ where $R_M(t) = o(t^{M-N_k})$ is the remainder. The analytic continuations of the two sides of (7.4) must agree near $z = \frac{p}{2} + |k|$ and consequently we may take residues of both sides, which gives the result. Note in particular that when $p > 0$ the second term on the right hand side of (7.4) is holomorphic near $z = \frac{p}{2} + |k|$ so does not contribute to the residue. When $p = 0$ (so $k = \emptyset$), $P_\emptyset = a_0$ and the residue of the second term is $-\mathrm{Tr}_s(a_0\Pi)$. □

7.2 Dirac-type Operators and Getzler Order

This section is mostly expository. We describe Bismut's generalization [Bis89] of the Lichnerowicz formula to Dirac-type operators $D = D^{LC} + c(B)$. We then specialize to the case where B is a 3-form and briefly introduce a Getzler symbol calculus 'adapted to B', which is a slight variation of the usual Getzler calculus. The observation that a variation of the usual Getzler calculus is appropriate for studying the heat operator e^{-tD^2} is due to Bismut [Bis89]. Our discussion of Getzler calculus draws from the approaches in [BGV04, Roe99], and as in these references, we will only need a less elaborate version of Getzler's original calculus [Get83], sufficient for handling compositions of the form $P \circ Q$ where P is a differential operator and Q is a smoothing operator. Throughout we will work on a closed Riemannian spin manifold, the extension to operators acting on general Clifford modules being well understood (cf. [BGV04]).

7.2.1 Dirac-type Operators

Let (M^n, g) be a closed Riemannian spin manifold with $\mathbb{Z}_2$-graded spinor bundle S, and let $c\colon \mathbb{C}l(T^*M) \xrightarrow{\sim} \mathrm{End}(S)$ denote the Clifford action. Equip S with a Hermitian structure such that for $v \in T^*M$, $c(v)$ is skew-Hermitian. There is a canonical isomorphism (the *Clifford symbol map*) of $\mathbb{Z}_2$-graded complex vector spaces $\mathbb{C}l(T^*_xM) \simeq \wedge T^*_xM_{\mathbb{C}}$ that sends $e_{i_1}\cdots e_{i_k} \in \mathrm{Cl}(T^*_xM)$ to $e_{i_1}\wedge\cdots\wedge e_{i_k} \in \wedge T^*_xM$, where $e_1,\ldots,e_n$ is any orthonormal frame of T^*_xM (cf. [BGV04]), and we will use this isomorphism to identify $\mathbb{C}l(T^*M)$ and $\wedge T^*M_{\mathbb{C}}$.

The Levi-Civita connection ∇^{LC} on $TM \simeq T^*M$ determines a canonical connection (the spin connection) on S that we also denote by ∇^{LC}. *The spin Dirac operator* D^{LC} acting on smooth sections of S is the odd, essentially self-adjoint, first-order differential operator given by the composition

$$\Gamma(S) \xrightarrow{\nabla^{LC}} \Gamma(T^*M \otimes S) \xrightarrow{c} \Gamma(S). \tag{7.5}$$

More generally a *Dirac operator* acting on smooth sections of S is an operator given by a composition similar to (7.5), but allowing the spin connection ∇^{LC} to be replaced with a connection of the form $\nabla^{LC} + \sqrt{-1}\alpha$ where α is a $\mathbb{R}$-valued 1-form (equivalently, twist S by a trivial Hermitian line bundle and couple D^{LC} to it using a possibly non-trivial Hermitian connection). In this article by a *Dirac-type operator* we shall mean an odd, essentially self-adjoint first order differential operator D acting on smooth sections of S that differs from D^{LC} by a smooth bundle endomorphism. The most general such operator is of the form

$$D = D^{LC} + c(B)$$

where $B \in \Gamma(\wedge T^*M_{\mathbb{C}})$ is a differential form of odd (possibly mixed) degree satisfying $c(B)^* = c(B)$. When B is a 1-form, D is a Dirac operator in the above sense.

For any B as above, there is a spectral triple $(C_c^\infty(M), L^2(M, S), D)$ satisfying all the hypotheses of the Connes-Moscovici theorem. All of these spectral triples represent the same element in the K-homology group of the closed manifold M, and hence although their residue cocycles will differ, they are guaranteed to be (b, B)-cohomologous. The operator $\Delta = D^2$ is a generalized Laplacian in the sense of [BGV04, Chapter 2], hence the heat kernel $e^{-t\Delta}$ has an asymptotic expansion as $t \to 0^+$ to which Proposition 7.1.2 applies.

7.2.2 Bismut's Lichnerowicz Formula

Let $B \in \Gamma(\wedge T^*M_{\mathbb{C}})$ be a differential form of odd degree such that $c(B)^* = c(B)$. Define a new connection $\nabla \colon \Gamma(S) \to \Gamma(T^*M \otimes S)$ on S given by

$$\nabla_X = \nabla_X^{LC} + c(\iota_X B), \qquad X \in \mathfrak{X}(M) \tag{7.6}$$

where ι_X denotes contraction with the vector field X. (To avoid a possible misconception, we mention that if $e_1, \ldots, e_n$ is a local orthonormal frame, then $\sum_i c(e_i)\nabla_{e_i} = D^{LC} + k\, c(B)$ if $B \in \Gamma(\wedge^k T^*M_{\mathbb{C}})$, which is *not* the operator D unless $k = 1$. The operator D is however the Dirac operator associated to the Clifford superconnection $\mathbb{A} = \nabla^{LC} + B$ in the sense of [BGV04, p.116].) The formal adjoint of ∇ with respect to the Riemannian L^2 inner products is denoted $\nabla^*\colon \Gamma(T^*M \otimes S) \to \Gamma(S)$.

The odd differential form B gives rise to a collection B_j, $j = 1, 2, 3, \ldots$ of even degree differential forms (with degrees $(k-2j-1)^2$ if B has odd degree k), given in terms of a local orthonormal frame $e_1, \ldots, e_n$ by

$$B_j = \sum_{i_1 < \cdots < i_{2j+1}} \Big(\iota(e_{i_1}) \cdots \iota(e_{i_{2j+1}})B\Big)^2.$$

(The result is independent of the choice of local orthonormal frame.) Bismut proved the following formula for the square of the Dirac-type operator $D = D^{LC} + c(B)$.

Proposition 7.2.1 ([Bis89], Theorem 1.1)**.** *Let $D = D^{LC} + c(B)$ where D^{LC} is the spin Dirac operator and B is an odd differential form. Let $\Delta = D^2$. Then*

$$\Delta = \nabla^*\nabla + \frac{\kappa}{4} + c(dB) + 2\sum_{j \geq 1} (-1)^j j c(B_j),$$

where κ is the scalar curvature. When $B \in \Gamma(\wedge^3 T^*M)$ the formula simplifies to

$$\Delta = \nabla^*\nabla + \frac{\kappa}{4} + c(dB) - 2|B|^2.$$

In terms of a local orthonormal frame $e = \{e_1, \ldots, e_n\}$,

$$\nabla^*\nabla = -\Big(\sum_{i=1}^n \nabla^2_{e_i}\Big) - \nabla_{\nu_e}, \qquad \nu_e = \sum_{i=1}^n \nabla^{LC}_{e_i} e_i = -\sum_{i=1}^n \mathrm{div}_g(e_i) e_i.$$

The formula in Proposition 7.2.1 appears slightly simpler than [Bis89, Theorem 1.1] because we have omitted twists by an auxiliary bundle.

Throughout the rest of the article, we restrict to the case where B is a 3-form. The condition $c(B)^* = c(B)$ implies $B \in \Gamma(\wedge^3 T^*M)$ is a real 3-form. A special feature of the 3-form case is that the connection ∇ is the lift (via the isomorphism $\mathfrak{o}_n = \mathfrak{so}_n \simeq \mathfrak{spin}_n$) of a metric connection on the tangent bundle—also denoted ∇ when there is no risk of confusion—given by the formula

$$\nabla = \nabla^{LC} + B_{\mathfrak{o}}, \qquad B_{\mathfrak{o}} \in \Omega^1(M, \mathfrak{o}(TM)), \tag{7.7}$$

where for any $A \in \Omega^k(M)$, $k \geq 2$ we define $A_\mathfrak{o} \in \Omega^{k-2}(M, \mathfrak{o}(TM))$ using the metric:

$$g(A_\mathfrak{o}(X_1, ..., X_{k-2})X_{k-1}, X_k) = 2A(X_1, ..., X_k).$$

The lift property ensures that

$$[\nabla_X, c(Y)] = c(\nabla_X Y).$$

The extra skew-symmetry of (7.7) implies that the torsion T_∇ of ∇ is skew-symmetric,

$$g(T_\nabla(X, Y), Z) = 4B(X, Y, Z),$$

and $\nabla_X X = \nabla^{LC}_X X$ for all $X \in \mathfrak{X}(M)$, hence ∇, ∇^{LC} have the same geodesics.

Remark 7.2.1. When $dB = 0$ there is an interesting perspective on ∇ coming from generalized geometry in the sense of Hitchin, where ∇ may be thought of as the analogue of the Levi-Civita connection when doing geometry with a closed 3-form 'background', see for example [Hit11, ?].

7.2.3 Getzler Calculus

Let $B \in \Gamma(\wedge^3 T^*M)$ be a 3-form on M, and let ∇ denote the corresponding connection on TM (equation (7.7)) or its lift to S (equation (7.6)). A key feature of the Getzler symbol calculus applied to the spin Dirac operator D^{LC} is that, by the Lichnerowicz formula, the square $(D^{LC})^2$ has order 2. The appearance of the connection ∇ in Proposition 7.2.1 suggests that in the study of heat kernel asymptotics for the operator D, a variation of the Getzler calculus that replaces the Levi-Civita connection ∇^{LC} with ∇ should be used.

Let

$$\mathrm{pr}\colon TM \to M, \qquad s\colon M \times M \to M,$$

be the bundle projection, resp. projection map to the second factor (i.e. the source map of the pair groupoid $M \times M$). The Riemannian exponential map identifies a tubular neighborhood $\mathbf{U}$ of the 0-section in TM with a neighborhood U of the diagonal in $M \times M$:

$$\exp\colon v \in \mathbf{U} \subset TM \mapsto (\exp_y(v), y) \in U \subset M \times M \tag{7.8}$$

where $v \in T_yM$. The inverse of the diffeomorphism (7.8) is denoted

$$\mathbf{rx}\colon U \to \mathbf{U} \subset TM. \tag{7.9}$$

The maps (7.8), (7.9) intertwine $s|_{\mathcal{U}}$, $\mathrm{pr}|_{\mathbf{r}\mathcal{U}}$. For $y \in M$ let $\mathcal{U}_y = \mathcal{U} \cap (M \times \{y\})$, a geodesic ball around y, and let $\mathbf{r}x_y \colon \mathcal{U}_y \to T_y M$ be the restriction of $\mathbf{r}x$ to $\mathcal{U}_y$.

Using ∇-parallel translation along radial geodesics, followed by the Clifford symbol map, we obtain isomorphisms

$$S \boxtimes S^*|_{\mathcal{U}} \simeq s^*(S \otimes S^*) = s^* \operatorname{End}(S) = s^* \mathbb{C}l(T^*M) \simeq s^* \wedge T^* M_{\mathbb{C}}. \tag{7.10}$$

Using the inverse of the exponential map (7.9) on the base combined with the isomorphism (7.10) on the fibres, we obtain an isomorphism

$$\Gamma(S \boxtimes S^*|_{\mathcal{U}}) \simeq \Gamma(\mathrm{pr}^* \wedge T^* M_{\mathbb{C}}|_{\mathbf{U}}), \tag{7.11}$$

that will be used frequently below.

Definition 7.2.1. An s-*fibred differential operator* on $\Gamma(S \boxtimes S^*|_{\mathcal{U}})$ is differential operator T with smooth coefficients on $\Gamma(S \boxtimes S^*|_{\mathcal{U}})$ given by a family $\{T_y\}_{y \in M}$, where T_y is a differential operator acting on $\Gamma(S|_{\mathcal{U}_y} \boxtimes S^*_y)$. The space of s-fibred differential operators forms an algebra under composition. An s-fibred differential operator T is said to *vanish on the diagonal* if $T(\Gamma(S \boxtimes S^*|_{\mathcal{U}}))$ is contained in the subspace of $\Gamma(S \boxtimes S^*|_{\mathcal{U}})$ consisting of sections that vanish on the diagonal. Using the identification (7.11), T yields a differential operator

$$\mathbf{T} \colon \Gamma(\mathrm{pr}^* \wedge T^* M_{\mathbb{C}}|_{\mathbf{U}}) \to \Gamma(\mathrm{pr}^* \wedge T^* M_{\mathbb{C}}|_{\mathbf{U}})$$

given by a family $\{\mathbf{T}_y\}_{y \in M}$ of differential operators along the fibres of $\mathbf{U} \to M$.

We mention several examples that will appear frequently below.

Example 7.2.1. *Any differential operator* P on M *acting on sections of* S *determines an* s*-fibred differential operator by 'copying'* P *on each* $M \times \{y\} \subset M \times M$*, and using the fact that* $s^* S^*|_{M \times \{y\}} \simeq M \times S^*_y$ *is canonically trivial to extend* P *to act on sections of* $S \boxtimes S^*|_{M \times \{y\}} = S \boxtimes S^*_y$.

Example 7.2.2. *Smooth functions on* $\mathcal{U}$ *act by multiplication on* $\Gamma(S \boxtimes S^*|_{\mathcal{U}})$ *and hence determine* s*-fibred differential operators. One example that appears frequently below is the function* $\mathbf{r}x^2 = g(\mathbf{r}x, \mathbf{r}x)$ *giving the squared distance to the diagonal. Further examples are the powers* ρ^r *for* $r \in \mathbb{R}$*, where by definition the restriction* ρ^r_y *of* ρ^r *to the normal coordinate chart* $\mathcal{U}_y = \mathcal{U} \cap (M \times \{y\})$ *is*

$$\rho^r_y(\mathbf{r}x_y) = \det(g_{ab})^{\frac{r}{4}}$$

where $g_{ab} = g(\partial_a, \partial_b)$ *are the components of the metric in the normal coordinate system on* U_y*.*

Example 7.2.3. *Generalizing the previous example, any smooth section* $Q \in \Gamma(S \boxtimes S^*|_U)$ *determines an* s*-fibred differential operator, via the identification* $\Gamma(S \boxtimes S^*|_U) \simeq \Gamma(s^* \operatorname{End}(S))$*, and letting* $Q|_U$ *act on sections of* $\Gamma(S \boxtimes S^*|_U) \simeq \Gamma(\mathrm{pr}_2^* \operatorname{End}(S))$ *by pointwise composition of endomorphisms of* S*.*

Example 7.2.4. *Let* $\mathcal{E}$ *be the vector field on* $U \simeq \mathbf{U}$ *corresponding to the Euler vector field on* $\mathbf{U} \subset TM$ *under the exponential map, whose integral curves are radial geodesics. Then* $\nabla_{\mathcal{E}}$ *is an* s*-fibred differential operator that vanishes on the diagonal. The identification* (7.10) *trivializes the bundle* S *along radial geodesics, and hence instead of* $\nabla_{\mathcal{E}}$ *we will often simply write* $\mathcal{E}$*.*

The inclusion $TM \times \mathbb{C} = \mathrm{pr}^* \wedge^0 T^*M_{\mathbb{C}} \hookrightarrow \mathrm{pr}^* \wedge T^*M_{\mathbb{C}}$ induces a map on sections $\eta\colon C^\infty(TM) \hookrightarrow \Gamma(\mathrm{pr}^* \wedge T^*M_{\mathbb{C}})$. If $\mathbf{T}\colon \Gamma(\mathrm{pr}^* \wedge T^*M_{\mathbb{C}}|_{\mathbf{V}}) \to \Gamma(\mathrm{pr}^* \wedge T^*M_{\mathbb{C}}|_{\mathbf{V}})$ is a differential operator defined on some open set $\mathbf{V} \subset TM$, then the composition

$$\mathbf{T} \circ \eta\colon C^\infty(\mathbf{V}) \to \Gamma(\mathrm{pr}^* \wedge T^*M_{\mathbb{C}}|_{\mathbf{V}})$$

is again a differential operator on $\mathbf{V}$.

Definition 7.2.2 (compare [Roe99] exercises 12.31, 12.32 and [BGV04] pp. 156–157)**.** For a section $\alpha \in \Gamma(S \boxtimes S^*|_U) \simeq \Gamma(\mathrm{pr}^* \wedge T^*M_{\mathbb{C}}|_{\mathbf{U}})$ and $u \in \mathbb{R}_{>0}$ define $\delta_u \alpha \in \Gamma(\mathrm{pr}^* \wedge T^*M_{\mathbb{C}}|_{u^{-1}\mathbf{U}})$ by

$$\delta_u \alpha(\mathbf{r}x) = \sum_{k=0}^{n} u^{-k} \alpha_{[k]}(u\mathbf{r}x), \tag{7.12}$$

where $\alpha_{[k]}$ is the component lying in the k-th exterior power. The operation δ_u is the *Getzler re-scaling for the connection* ∇. For an s-fibred differential operator T and $m \in \mathbb{Z}$ define

$$\sigma_m^G(T) = \lim_{u \to 0^+} u^m \delta_u \circ \mathbf{T} \circ \delta_u^{-1} \circ \eta \in \mathfrak{D}(TM) \otimes \wedge T^*M \tag{7.13}$$

when the limit exists, where $\mathfrak{D}(TM) \to M$ is the bundle of algebras whose fibre over $y \in M$ consists of differential operators on T_yM with polynomial coefficients. In (7.13), $\delta_u \circ \mathbf{T} \circ \delta_u^{-1}$ is to be viewed as a differential operator on $u^{-1} \cdot \mathbf{U} \subset TM$, which in the limit produces a differential operator on TM. If the limit exists for some $m \in \mathbb{Z}$ then T is said to have *Getzler order* m (notation: $o^G(T) = m$), and (7.13) is the m-th order *Getzler symbol* of T. The *constant term* of the Getzler symbol, denoted $\sigma_m^{G,0}(T)$ is the element of $\mathrm{Sym}(TM) \otimes \wedge T^*M$ (where $\mathrm{Sym}(TM)$ should be thought of as the bundle of constant coefficient differential operators along the fibres of TM) obtained by evaluating the polynomial coefficients of $\sigma^G_m(T)$ along the zero section. Getzler order determines a filtration of the algebra of s-fibred differential operators, and the Getzler symbol satisfies (cf. [Roe99, Proposition 12.22])

$$\sigma^G_{m_1+m_2}(T_1T_2) = \sigma^G_{m_1}(T_1)\sigma^G_{m_2}(T_2)$$

with T_1, T_2 being s-fibred differential operators with Getzler orders $o^G(T_i) = m_i$.

Remark 7.2.2. The small deviation from the usual setup was that ∇-parallel translation was used in (7.10), *not* ∇^{LC}-parallel translation.

To clarify the definition and for use in later calculations, we describe a number of examples.

Example 7.2.5. *Let $f \in C^\infty(U)$ be a smooth function that vanishes to order k on the diagonal. Then $o^G(f) = -k$ and*

$$\sigma^G_{-k}(f) = \lim_{u\to 0} u^{-k} f(urx)$$

is a smooth function on TM which is polynomial along the fibres of $TM \to M$ (it is the homogeneous k-th order Taylor polynomial of f in directions normal to the diagonal). In particular a function always possesses a 0-th order Getzler symbol $\sigma^G_0(f) = \mathrm{pr}^*(f|_{Diag_M})$.

Example 7.2.6. *Let $\alpha = \alpha_1 \cdots \alpha_k \in \Gamma(\wedge^k T^*M)$ be a decomposable k-form, and let $c(\alpha)$ be the 0-th order operator on S given by the Clifford action. We view $c(\alpha)$ as a s-fibred differential operator as in Example 7.2.1. Under the isomorphism $S \otimes S^* \simeq \wedge T^*M_{\mathbb{C}}$, Clifford multiplication by $\alpha_j \in \wedge^1 T^*M_{\mathbb{C}}$ becomes $\varepsilon(\alpha_j) - \iota(\alpha_j)$, where ε (resp. ι) denotes exterior multiplication (resp. contraction). It follows that $o^G(c(\alpha)) = k$ and*

$$\sigma^G_k(c(\alpha)) = 1 \otimes \alpha \in \mathfrak{D}(TM) \otimes \wedge T^*M.$$

Example 7.2.7. *Generalizing the previous two examples, any smooth section $Q \in \Gamma(S \boxtimes S^*|_U)$ has Getzler order (at most) $n = \dim(M)$. Its n-th order Getzler symbol is the n-form part of its restriction to the diagonal.*

Example 7.2.8. *The Getzler order $o^G(\mathcal{E}) = 0$ and*

$$\sigma^G_0(\mathcal{E}) = \mathcal{E} \otimes 1$$

is the Euler vector field on TM*.*

7.2.3.1 Getzler Symbol of Covariant Derivatives

Let $y \in M$, let $e_1, ..., e_n$ be an orthonormal basis of T_yM, and let $\mathbf{r}x_{y,a} = g(\mathbf{r}x_y, e_a)$ be the corresponding normal coordinates on U_y. By ∇-parallel translation along radial geodesics, extend $e_1, ..., e_n$ to an orthonormal frame of $\mathsf{TM}|_{U_y}$. Let $\omega = \sum_a \omega_a d\mathbf{r}x_{y,a} \in \Omega^1(U_y, \mathfrak{o}_n)$ be the connection 1-form for ∇ on U_y relative to the frame $e_1, ..., e_n$. By construction

$$\iota(\mathcal{E})\omega = 0, \tag{7.14}$$

and using the Cartan formula, the Lie derivative

$$Ł_{\mathcal{E}}\omega = \iota(\mathcal{E})R \tag{7.15}$$

where $R \in \Omega^2(U_y, \mathfrak{o}_n)$ is the curvature. Since ω vanishes at y (where $\mathbf{r}x_y = 0$), equation (7.15) implies (cf. [BGV04, Proposition 1.18]),

$$\omega_a(\mathbf{r}x_y) = -\frac{1}{2}\sum_b R(\partial_a, \partial_b)_y \mathbf{r}x_{y,b} + O(\mathbf{r}x_y^2), \tag{7.16}$$

where $O(\mathbf{r}x_y^2)$ denotes an $\mathfrak{o}_n$-valued smooth function vanishing to order 2 at the point y. Via the isomorphism $\mathfrak{o}_n = \mathfrak{so}_n \simeq \mathfrak{spin}_n$, the operator ∇_{∂_a} on S is

$$\nabla_{\partial_a} = \partial_a + \frac{1}{8}\sum_{b,c,d}(R_{cdab})_y \mathbf{r}x_{y,b} c(e_c)c(e_d) + O(\mathbf{r}x_y^2).$$

Therefore if $X \in \mathfrak{X}(M)$ is a vector field, $X_y = \sum_a X_a \partial_a$, then near $y \in M$,

$$\nabla_X = X + \frac{1}{8}\sum_{a,b,c,d}(R_{cdab})_y X_a \mathbf{r}x_{y,b} c(e_c)c(e_d) + O(\mathbf{r}x_y^2).$$

It follows that (compare [BGV04, Proposition 4.20]), at the point $y \in M$,

$$\sigma_1^G(\nabla_X)_y = X_y \otimes 1 + \frac{1}{8}\sum_{a,b,c,d}(R_{cdab})_y X_a \mathbf{r}x_{y,b} \otimes e_c e_d. \tag{7.17}$$

The sum over a, b, c, d in (7.17) is independent of the choice of orthonormal frame $e_1, ..., e_n$ at the point $y \in M$, and therefore adopting the abstract index notation convention, we may write

$$\sigma_1^G(\nabla_X) = X \otimes 1 + \frac{1}{8} \sum_{a,b,c,d} R_{cdab} X_a \mathbf{r} x_b \otimes e_c e_d. \tag{7.18}$$

To make the resulting expression more transparent, we introduce the following notation.

Definition 7.2.3. For any $A \in \Gamma(\wedge^2 T^*M \otimes \mathfrak{o}(TM))$ define $A^\top \in \Gamma(\wedge^2 T^*M \otimes \mathfrak{o}(TM))$ by

$$g(A^\top(X,Y)W,Z) = g(A(W,Z)X,Y).$$

When $B = 0$ one has $R^\top = R$ by a well-known property of the curvature of the Levi-Civita connection. We also define $R^\top_{ab} = g(R^\top e_b, e_a) \in \wedge^2 T^*M$, the matrix elements of $R^\top$ in the orthonormal frame.

In terms of $R^\top$, the Getzler symbol of ∇_X reads

$$\sigma_1^G(\nabla_X) = X \otimes 1 - \frac{1}{4} \sum_{a,b} X_a \mathbf{r} x_b \otimes R^\top_{ab}. \tag{7.19}$$

7.2.3.2 Getzler Symbol of Δ, $\Delta - c(dB)$

Definition 7.2.4. Let $D = D^{LC} + c(B)$ where $B \in \Gamma(\wedge^3 T^*M)$ is a 3-form, and recall that we defined Δ to be the square D^2. Let

$$\overline{\Delta} = \Delta - c(dB) = \overline{\Delta} = \nabla^*\nabla + \frac{\kappa}{4} - 2|B|^2,$$

where the second expression is Proposition 7.2.1.

If $dB \neq 0$ then by Example 7.2.6 $o^G(\Delta) = 4$ and

$$\sigma_4^G(\Delta) = 1 \otimes dB \in \mathfrak{D}(TM) \otimes \wedge T^*M.$$

More interesting is the operator $\overline{\Delta}$. By Proposition 7.2.1 and equation (7.19), $o^G(\overline{\Delta}) = 2$ and

$$\sigma_2^G(\overline{\Delta}) = -\sum_a \left(\partial_a \otimes 1 - \frac{1}{4} \sum_b \mathbf{r} x_b \otimes R^\top_{ab} \right)^2. \tag{7.20}$$

7.2.4 Heat Kernels

It is well-known (cf. [BGV04, Theorem 2.30]) that the heat operators $e^{-t\Delta}$, $e^{-t\overline{\Delta}}$ have integral kernels $\Theta_t, \overline{\Theta}_t \in C^\infty(M \times M \times (0,\infty), S \boxtimes S^*)$ depending smoothly on $(x,y,t) \in M \times M \times (0,\infty)$, with asymptotic expansions on U as $t \to 0^+$ (in the space $C^\ell(U, S \boxtimes S^*|_U)$ for any ℓ),

$$\Theta_t(\mathbf{r}x) \sim h_t(\mathbf{r}x) \sum_{j \geq 0} t^j \Theta_j(\mathbf{r}x), \tag{7.21}$$

$$\overline{\Theta}_t(\mathbf{r}x) \sim h_t(\mathbf{r}x) \sum_{j \geq 0} t^j \overline{\Theta}_j(\mathbf{r}x), \tag{7.22}$$

where

$$h_t(\mathbf{r}x) = (4\pi t)^{-n/2} e^{-\mathbf{r}x^2/4t}, \tag{7.23}$$

is the Euclidean approximation to the heat kernel. Moreover the expansions remain valid after differentiating both sides with respect to t any number of times. It is convenient to let $\Theta_j = 0$ when $j < 0$.

The Getzler orders and symbols of the heat kernel coefficients $\overline{\Theta}_j \in C^\infty(U, S \boxtimes S^*|_U)$, $j = 0, \ldots, \frac{n}{2}$ can be computed using (7.20) and Mehler's formula for the solution of the harmonic oscillator. The result is as follows.

Theorem 7.2.5 ([BGV04], Theorem 4.21)**.** *For $j = 0, \ldots, \frac{n}{2}$ the Getzler order $o^G(\overline{\Theta}_j) = 2j$, and the Getzler symbols are given by the generating function*

$$\sum_{j=0}^{n/2} t^j \sigma^G_{2j}(\overline{\Theta}_j) = \left(1 \otimes \mathit{det}^{1/2}\left(\frac{tR^\top/2}{\sinh(tR^\top/2)}\right)\right) \cdot \mathit{exp}\left(-\frac{1}{4t}\sum_{a,b} \mathbf{r}x_a \mathbf{r}x_b \otimes f(R^\top)_{ab}\right) \tag{7.24}$$

where $f(z) = \frac{z}{2}\coth(\frac{z}{2}) - 1$ and $R^\top \in \Gamma(\wedge^2 T^*M \otimes \mathfrak{o}(TM))$ is as in Definition 7.2.3.

The constant part of the Getzler symbol is obtained by setting $\mathbf{r}x = 0$ in (7.24), resulting in the differential form:

$$\sum_{j=0}^{n/2} t^j \sigma^{G,0}_{2j}(\overline{\Theta}_j) = \det{}^{1/2}\left(\frac{tR^\top/2}{\sinh(tR^\top/2)}\right). \tag{7.25}$$

As mentioned in Definition 7.2.3, when $B = 0$, $R^\top = R$ is the Riemann curvature tensor of the metric g, hence upon setting $t = 1$ the form (7.25) becomes the usual Chern-Weil representative of the $\widehat{A}$-class. In general we have the following formula for $R^\top$, which slightly generalizes [Bis89, Theorem 1.6].

Proposition 7.2.2. *Let* B *be a 3-form. Let* $\nabla = \nabla^{LC} + B_o$ *and* $\nabla_- = \nabla^{LC} - B_o$ *with curvature tensors* R, R_- *respectively. Then* $R^\top = R_- + (dB)_o$. *In particular when* $dB = 0$, $R^\top = R_-$, *and (7.25) is the Chern-Weil representative of the* $\widehat{A}$*-class (up to factors of* $2\pi\imath$*) constructed using the connection* ∇_-.

Proof. Let $d_{\nabla^{LC}}$ be the exterior covariant differential defined by the Levi-Civita connection. Then

$$R = (d_{\nabla^{LC}} + B_o)^2 = R^{LC} + d_{\nabla^{LC}} B_o + B_o^2$$

and thus

$$R^\top = R^{LC} + (d_{\nabla^{LC}} B_o)^\top + (B_o^2)^\top. \tag{7.26}$$

Let $e_1, ..., e_n$ be a local orthonormal frame. Then

$$\begin{aligned} g((B_o^2)^\top(W, X)Y, Z) &= g([B_o(Y), B_o(Z)]W, X) \\ &= 4\sum_i B(Y, e_i, X)B(Z, W, e_i) - B(Z, e_i, X)B(Y, W, e_i), \end{aligned}$$

and if one expands $g(B_o^2(W, X)Y, Z)$ in the same way one finds the same expression (after using the antisymmetry of B to permute the entries). Therefore $(B_o^2)^\top = B_o^2$. On the other hand

$$dB(W, X, Y, Z) = (\nabla^{LC}_W B)(X, Y, Z) - (\nabla^{LC}_X B)(W, Y, Z) + (\nabla^{LC}_Y B)(W, X, Z) - (\nabla^{LC}_Z B)(W, X, Y),$$

and since $(\nabla^{LC}_X B)_o = \nabla^{LC}_X B_o$, we find (grouping the four terms in the expression for dB in two groups of two):

$$(dB)_o = d_{\nabla^{LC}} B_o + (d_{\nabla^{LC}} B_o)^\top.$$

Thus equation (7.26) becomes

$$R^\top = R^{LC} - d_{\nabla^{LC}} B_o + B_o^2 + (dB)_o = R_- + (dB)_o.$$

□

7.3 The Residue Cocycle for $D = D^{LC} + c(B)$

In this section we study the residue cocycle for the spectral triple $(C^\infty(M), L^2(M, S), D)$, where the operator $D = D^{LC} + c(B)$, $B \in \Gamma(\wedge^3 T^* M)$. We describe some constraints on what contributions can occur in general, and show how to calculate the cocycle completely when $dB = 0$ using the Getzler calculus. We will use notation introduced in the previous section. In particular recall $\Delta = D^2$, $\overline{\Delta} = \Delta - c(dB)$, as well as the heat kernels Θ_t, $\overline{\Theta}_t$, the asymptotic expansion coefficients Θ_j, $\overline{\Theta}_j$, and the Euclidean approximation to the heat kernel $h_t(\mathbf{r}x) = (4\pi t)^{-n/2} e^{-\mathbf{r}x^2/4t}$.

According to Proposition 7.1.2 (and using $[D, a] = c(da)$ for $a \in C^\infty(M)$), for $0 \le p \le n$ the component $\varphi_p(a_0, ..., a_p)$, $a_0, ..., a_p \in C^\infty(M)$, of the residue cocycle can be determined from the asymptotic expansion as $t \to 0^+$ of

$$\mathrm{Tr}_s(P_k(a_0, ..., a_p)e^{-t\Delta}), \tag{7.27}$$

where $k = (k_1, ..., k_p) \in (\mathbb{Z}_{\ge 0})^p$, $|k| \le n - p$ and

$$P_k = P_k(a_0, ..., a_p) = a_0\Big(\mathrm{ad}_\Delta^{k_1} c(da_1)\Big) \cdots \Big(\mathrm{ad}_\Delta^{k_p} c(da_p)\Big). \tag{7.28}$$

As p and $a_0, ..., a_p$ will be fixed, to simplify notation we will write P_k instead of $P_k(a_0, ..., a_p)$ below. It is also convenient to introduce the operator

$$\overline{P}_k = a_0\Big(\mathrm{ad}_{\overline{\Delta}}^{k_1} c(da_1)\Big) \cdots \Big(\mathrm{ad}_{\overline{\Delta}}^{k_p} c(da_p)\Big), \tag{7.29}$$

obtained by replacing Δ with $\overline{\Delta} = \Delta - c(dB)$.

The integral kernel of $P_k e^{-t\Delta}$ is $P_k\Theta_t$, where the notation means that P_k, Θ_t are composed as s-fibred differential operators. (See Example 7.2.1 for the sense in which P_k is an s-fibred differential operator; another reasonable, though cumbersome, notation would be $P_k(x)\Theta_t(x, y)$ to emphasize that P_k acts along the first factor in the product $M \times M$.) The supertrace (7.27) is given by integration of the pointwise supertrace of the kernel along the diagonal:

$$\mathrm{Tr}_s(P_k e^{-t\Delta}) = \int_M \mathrm{tr}_s\Big(P_k\Theta_t|_{\mathbf{r}x=0}\Big)dV, \tag{7.30}$$

where dV is the Riemannian measure. Thus computing the asymptotic expansion of (7.27) amounts to studying the low-lying terms in the asymptotic expansion in t of the integrand (7.30), and in particular it is enough to work in an arbitrarily small neighborhood of the diagonal in $M \times M$. Using the asymptotic expansion (7.21) of Θ_t,

$$\mathrm{tr}_s(P_k\Theta_t|_{\mathbf{r}x=0}) \sim (4\pi t)^{-n/2} \sum_{j \ge 0} t^j \mathrm{tr}_s(h_t^{-1} P_k h_t \Theta_j|_{\mathbf{r}x=0}), \tag{7.31}$$

and likewise for $\mathrm{tr}_s(\overline{P}_k\overline{\Theta}_t|_{rx=0})$.

Lemma 7.3.1. *Let $\varnothing$ be an* s*-fibred differential operator of order* m*. Then*

$$h_t^{-1}\varnothing h_t = \sum_{\ell=0}^{m} t^{-m+\ell}\varnothing_\ell \tag{7.32}$$

where $\varnothing_\ell$ is an s*-fibred differential operator of order ℓ. For $\ell < \lceil m/2 \rceil$, $\varnothing_\ell$ vanishes on the diagonal (in the sense of Definition 7.2.1).*

Proof. Since $h_t(\mathbf{r}x) = (4\pi t)^{-n/2}e^{-\mathbf{r}x^2/4t}$, the coefficient of $t^{-m+\ell}$ comes from applying $m - \ell$ derivatives to h_t, leaving $m - (m - \ell) = \ell$ derivatives. This proves the claim regarding the order of $\varnothing_\ell$.

Let $\Psi \in \Gamma(S \boxtimes S^*|_U)$ and set $\varnothing(t) = h_t^{-1}\varnothing h_t$. Fix t and consider the re-scaling $\varphi_{0,u}\colon t \mapsto ut$. Along the diagonal, the re-scaled section $\varphi_{0,u}^*\varnothing(t)\Psi|_{\mathbf{r}x=0}$ has an asymptotic expansion in u as $u \to 0^+$, with some lowest power u^{-m+p_Ψ} where $p_\Psi \geq 0$. Taking the infimum of $-m + p_\Psi$ over all choices of Ψ yields the lowest power $-m + p \in \mathbb{Z}$ of t in (7.32) such that the s-fibred differential operator $\varnothing_p$ does not vanish on the diagonal.

On the other hand we may obtain a lower bound on $-m+p$ by considering the growth rate of the section $\varnothing(t)\Psi \in \Gamma(S \boxtimes S^*|_U)$ (no restriction to the diagonal $\mathbf{r}x = 0$) under the combined re-scaling $\varphi_u\colon (t, \mathbf{r}x) \mapsto (ut, u^{1/2}\mathbf{r}x)$. Clearly $\varphi_u^* e^{-\mathbf{r}x^2/4t} = e^{-\mathbf{r}x^2/4t}$, and as $\varnothing$ has order m, the section $\varnothing(t)\Psi$ grows at most at the rate $u^{-m/2}$ under the re-scaling. Hence $-m + p \geq -m/2$ and so $p \geq m/2$. Since p is an integer, $p \geq \lceil m/2 \rceil$. □

In particular the above Lemma applies to the order $|k|$ differential operators P_k, $\overline{P}_k$. For the operator $\varnothing = \overline{P}_k$ we will also need the Getzler orders of the operators $\overline{P}_{k,\ell} = \varnothing_\ell$.

Proposition 7.3.1. *In normal coordinates*

$$h_t^{-1}\Delta h_t = \Delta + \frac{1}{t}\nabla_{\mathcal{E}} + \frac{n}{2t} + \frac{1}{t}\mathcal{E}(\log\rho) - \frac{\mathbf{r}x^2}{4t^2}$$

where $\rho = |g|^{1/4}$ (see Example 7.2.2). The same formula holds with Δ replaced by $\overline{\Delta}$.

Proof. One has

$$h_t^{-1}\Delta h_t = \Delta + h_t^{-1}[\Delta, h_t] = \Delta - 2h_t^{-1}\nabla_{\nabla h_t} + h_t^{-1}\Delta h_t.$$

On the other hand

$$h_t^{-1}\nabla h_t = -\frac{1}{2t}\mathcal{E}$$

and a short calculation in normal coordinates (cf. [Roe99, p.100]) shows that

$$h_t^{-1}(\Delta h_t) = -\frac{\mathbf{r}x^2}{4t} + \frac{n}{2t} + \frac{1}{t}\mathcal{E}(\log\rho).$$

□

Lemma 7.3.2. *Let* $m > 0$ *and* $f \in C^\infty(M)$*. On* $U \subset M \times M$ *we have*

$$\mathrm{ad}^m_{h_t^{-1}\overline{\Delta}h_t}(c(df)) = t^{-m}\varnothing_0 + t^{-m+1}\varnothing_1 + \cdots + t^0\varnothing_m.$$

where $\varnothing_\ell$ *is an* s*-fibred differential operator with Getzler order* $o^G(\varnothing_\ell) \leq 2\ell$*.*

Proof. Proceed by induction on m. For the base case $m = 1$ we have

$$[h_t^{-1}\overline{\Delta}h_t, c(df)] = t^{-1}[\nabla_\mathcal{E}, c(df)] + [\overline{\Delta}, c(df)] = t^{-1}\varnothing_0 + \varnothing_1.$$

The operator $\varnothing_0 = c(\nabla_\mathcal{E} df)$ has Getzler order $o^G(\varnothing_0) = 1 - 1 = 0$, since $o^G(c(\alpha)) = 1$ for $\alpha \in \Omega^1(M)$ but $\mathcal{E}$ vanishes to order 1 on the diagonal (contributing -1, see Example 7.2.5). Since $o^G(\overline{\Delta}) = 2$, $o^G(c(df)) = 1$ we have $o^G(\varnothing_1) \leq 3$. But in fact $o^G(\varnothing_1) = 2$ because the Getzler symbols

$$\sigma_2^G(\overline{\Delta}) = -\sum_a \left(\partial_a \otimes 1 - \frac{1}{4}\sum_b \mathbf{r}x_b \otimes R_{ab}^\top\right)^2, \qquad \sigma_1^G(c(df)) = 1 \otimes df$$

commute. This establishes the base case. For the inductive step, suppose

$$\mathrm{ad}^{m-1}_{h_t^{-1}\overline{\Delta}h_t}(c(da)) = t^{-(m-1)}\varnothing_0 + t^{-(m-1)+1}\varnothing_1 + \cdots + t^0\varnothing_{m-1} = \sum_\ell t^{-(m-1)+\ell}\varnothing_\ell,$$

where $o^G(\varnothing_j) = 2j$. Then

$$\mathrm{ad}^m_{h_t^{-1}\overline{\Delta}h_t}(c(df)) = t^{-(m-1)}[h_t^{-1}\overline{\Delta}h_t, \varnothing_0] + t^{-(m-1)+1}[h_t^{-1}\overline{\Delta}h_t, \varnothing_1] + \cdots + t^0[h_t^{-1}\overline{\Delta}h_t, \varnothing_{m-1}].$$

By Proposition 7.3.1, $h_t^{-1}\overline{\Delta}h_t = \overline{\Delta} + t^{-1}T + t^{-2}F$ where $o^G(T) = 0$, $o^G(F) = -2$. The Getzler orders are $o^G([\overline{\Delta}, \varnothing_\ell]) = 2\ell + 2$, $o^G([T, \varnothing_\ell]) = 2\ell$, and $o^G([F, \varnothing_\ell]) = 2\ell - 2$. Hence a typical term

$$t^{-(m-1)+\ell}[h_t^{-1}\overline{\Delta}h_t, \varnothing_\ell] = t^{-m+(\ell+1)}\varnothing'_{\ell+1} + t^{-m+\ell}\varnothing'_\ell + t^{-m+(\ell-1)}\varnothing'_{\ell-1},$$

has Getzler orders as claimed, completing the inductive step. □

The following summarizes the result of applying the previous two lemmas to P_k, $\overline{P}_k$.

Corollary 7.3.3. *On $U \subset M \times M$ we have*

$$h_t^{-1} P_k h_t = \sum_{\ell=0}^{|k|} t^{-|k|+\ell} P_{k,\ell},$$

where each $P_{k,\ell}$ is an s-fibred differential operator of order $o(P_{k,\ell}) = \ell$. For $\ell < \lceil |k|/2 \rceil$, $P_{k,\ell}$ vanishes on the diagonal. There is a similar expansion for $h_t^{-1}\overline{P}_k h_t$, with the additional property the Getzler order $o^G(\overline{P}_{k,\ell}) \le 2\ell + N_{=0}(k)$, where $0 \le N_{=0}(k) \le p$ is the number of indices i such that $k_i = 0$.

Proof. The claims regarding the order of $P_{k,\ell}$ and its vanishing along the diagonal are immediate consequences of Lemma 7.3.1. By Lemma 7.3.2, the Getzler order $o^G(\overline{P}_{k,\ell})$ is 2ℓ if $k_i \neq 0$ for all $i = 1, \dots, p$. For each index i such that $k_i = 0$, the Getzler order count becomes 1 larger than this because $o^G(c(da_i)) = 1$ instead of 0. □

Corollary 7.3.4. *There is an asymptotic expansion*

$$\mathrm{Tr}_s(P_k e^{-t\Delta}) \sim (4\pi)^{-n/2} t^{-n/2-\lfloor |k|/2 \rfloor} \sum_{j \ge 0} \sum_{r=0}^{\lfloor |k|/2 \rfloor} t^{j+r} \int_M \mathrm{tr}_s(P_{k,r+\lceil |k|/2 \rceil} \Theta_j|_{rx=0}) dV.$$

The component φ_p of the residue cocycle is given by

$$\varphi_p(a_0, \dots, a_p) = \sum_{|k| \le n-p} \frac{c'_{pk}}{(4\pi)^{n/2}} \sum_{j=(n-p)/2-|k|}^{(n-p)/2-\lceil |k|/2 \rceil} \int_M \mathrm{tr}_s(P_{k,(n-p)/2-j} \Theta_j|_{rx=0}) dV.$$

Proof. The expansion for $\mathrm{Tr}_s(P_k e^{-t\Delta})$ follows from substituting the expansion from Corollary 7.3.3 into (7.31) and making the change of variables $r = \ell - \lceil |k|/2 \rceil$. The formula for φ_p is an immediate consequence of Proposition 7.1.2. □

Theorem 7.3.5. *If $dB = 0$, then*

$$\varphi_p(a_0, \dots, a_p) = \frac{(2\pi\imath)^{-n/2}}{p!} \int_M a_0 da_1 \cdots da_p \cdot det^{1/2}\Big(\frac{R_-/2}{\sinh(R_-/2)}\Big)_{[n-p]},$$

where R_- is the curvature of the connection $\nabla_- = \nabla^{LC} - B_\sigma$.

Proof. Let $\Psi_{j,k} = P_{k,(n-p)/2-j}\Theta_j|_U \in \Gamma(S \boxtimes S^*|_U)$. The $n = \dim(M)$ order Getzler symbol of $\Psi_{j,k}$ is the n-form part of the Clifford symbol of $\Psi_{j,k}|_{rx=0}$. By [BGV04, Proposition

3.21],

$$\mathrm{tr}_s(\Psi_{j,k}|_{rx=0}) = (-2\imath)^{n/2}\mathcal{B}(\sigma_n^{G,0}(\Psi_{j,k}))$$

where $\mathcal{B}\colon \wedge^n T^*M \to M \times \mathbb{R}$ is the Berezin integral ([BGV04, p.40]) determined by the orientation and the metric. Therefore by Corollary 7.3.4

$$\varphi_p(a_0, ..., a_p) = \sum_{|k| \le n-p} c'_{pk}(2\pi\imath)^{-n/2} \sum_{j=(n-p)/2-|k|}^{(n-p)/2-\lceil |k|/2 \rceil} \int_M \mathcal{B}(\sigma_n^{G,0}(\Psi_{j,k}))dV.$$

When $dB = 0$, $P_k = \overline{P}_k$, $\Theta_j = \overline{\Theta}_j$ hence by Theorem 7.2.5, $o^G(\Theta_j) = 2j$. By Corollary 7.3.3, $o^G(P_{k,\ell}) = 2\ell + N_{=0}(k)$ where $0 \le N_{=0}(k) \le p$ is the number of indices $i \in \{1, ..., p\}$ such that $k_i = 0$. Therefore

$$o^G(\Psi_{j,k}) \le n - p + N_{=0}(k).$$

When at least one $k_i \neq 0$, $N_{=0}(k) < p$ and $o^G(\Psi_{j,k}) < n$, and so $\sigma_n^G(\Psi_{j,k}) = 0$. Otherwise if $k = (0, ..., 0)$ then $c'_{pk} = 1/p!$ and by Example 7.2.6 and equation (7.25),

$$\sigma_n^{G,0}(\Psi_{j,k}) = a_0 da_1 \cdots da_p \cdot \det{}^{1/2}\left(\frac{R^\top/2}{\sinh(R^\top/2)}\right)_{[n-p]}.$$

The result follows from this and Proposition 7.2.2. □

When Theorem 7.1.3 is specialized to the triple $(C^\infty(M), L^2(M, S), D)$ and the idempotent $e = 1$, the result is the Atiyah-Singer formula:

$$\mathrm{index}(D) = \varphi_0(1) = (2\pi\imath)^{-n/2}\int_M \det{}^{1/2}\Big(\frac{R_-/2}{\sinh(R_-/2)}\Big).$$

Chapter 8 Bismut's Dirac Operators for Connections with Torsion

This chapter is copied verbatim from [SLSJ21].

In this chapter, we will consider the Bismut Dirac operator twisted with non-closed three form. Applying the Getzler's rescaling is more challenging in this case. However, in low dimensions we still can find local formulas. By dividing the calculation of the residue cocycle into pieces that on some pieces the rescaling method still applies, and for the rest we may take advantage of the low-dimensionality.

8.1 Residue Cocycle Calculations when $n = 4$

In this chapter, we compute the Connes-Moscovici cocycle completely for the operator $D = D^{LC} + c(B)$, $B \in \Gamma(\wedge^3 T^*M)$ when $dB \neq 0$ and the dimension $n = 4$. The case $p = 4$ may be disposed of immediately: by Corollary 7.3.4, only the $|k| = 0$, $j = (4-4)/2 = 0$ term contributes, thus

$$\varphi_4(a_0, a_1, a_2, a_3, a_4) = \frac{(2\pi i)^{-2}}{4!} \int_M a_0 da_1 da_2 da_3 da_4.$$

The remaining two components φ_0, φ_2 of the residue cocycle are computed in the sections below.

8.1.1 Recursion Relation for the Heat Kernel Coefficients

There are well-known recursion relations for the heat kernel asymptotic expansion coefficients Θ_j, $\overline{\Theta}_j$ that we briefly recall here. The setup of Section 7.2.3 will be used in the calculations. In particular we use the inverse of the Riemannian exponential map to identify $U \subset M \times M$ with a neighborhood of the 0-section $\mathbf{r}U \subset TM$, and ∇-parallel translation along radial geodesics is used to identify $S \boxtimes S^*|_U$ with $s^* \operatorname{End}(S)$. Thus for example Θ_j, $\overline{\Theta}_j$, $c(dB)$ are identified with $\operatorname{End}(\mathrm{pr}^* S)$-valued smooth functions on $\mathbf{r}U$. Under the identifications the operator $\nabla_{\mathcal{E}}$ becomes $\mathcal{E}$. The operators Δ, $\overline{\Delta}$, $c(dB)$ are viewed as s-fibred differential operators (see Example 7.2.1). With this understanding, the recurrence relation satisfied by the coefficients Θ_j is (cf. [BGV04, Theorem 2.26]):

$$\mathcal{E}\Theta_j + (j + \mathcal{E}(\log \rho))\Theta_j = -\Delta\Theta_{j-1}, \quad j \geq 1; \qquad \Theta_0 = \rho^{-1}$$

where recall $\rho = |g|^{1/4}$, with $|g|$ the determinant of the Riemannian metric in normal coordinates (see Example 7.2.2). The solutions are given recursively by the formula

$$\Theta_j(\mathbf{r}x) = -\rho^{-1}(\mathbf{r}x) \int_0^1 t^{j-1} \rho(t\mathbf{r}x) \Delta\Theta_{j-1}(t\mathbf{r}x) dt.$$

Of course the same equations hold with Θ_j, Δ replaced by $\overline{\Theta}_j$, $\overline{\Delta}$. Using $\Delta = \overline{\Delta} + c(dB)$, for the first few terms $j = 0, 1, 2$ we find

$$\Theta_0 = \overline{\Theta}_0 = \rho^{-1}, \qquad \Theta_1 = \overline{\Theta}_1 + \Theta_1^B \tag{8.1}$$

where

$$\Theta_1^B(\mathbf{r}x) = -\rho^{-1}(\mathbf{r}x) \int_0^1 c(dB)_{t\mathbf{r}x} dt, \tag{8.2}$$

and

$$\begin{aligned}\Theta_2(\mathbf{r}x) &= -\rho^{-1}(\mathbf{r}x) \int_0^1 t\rho(t\mathbf{r}x)(\Delta\Theta_1)(t\mathbf{r}x) dt \\ &= \overline{\Theta}_2(\mathbf{r}x) - \rho^{-1}(\mathbf{r}x) \int_0^1 t\rho(t\mathbf{r}x)\Big(c(dB)_{t\mathbf{r}x}\overline{\Theta}_1(t\mathbf{r}x) + (\Delta\Theta_1^B)(t\mathbf{r}x)\Big) dt.\end{aligned} \tag{8.3}$$

In a normal coordinate neighborhood $U_y = U \cap (M \times \{y\})$, the Laplacian Δ is given by

$$\Delta = -\sum_{i=1}^n \Big(\big(e_i + \omega(e_i)\big)^2 - \big(\nabla_{e_i} e_i + \omega(\nabla_{e_i} e_i)\big)\Big) + \frac{\kappa}{4} + c(dB) - 2|B|^2, \tag{8.4}$$

where $\omega \in \Omega^1(U_y, \operatorname{End}(S_y)) \simeq \Omega^1(U_y, \mathbb{C}l(T_y^* M))$ is the connection 1-form relative to the identification $S|_{U_y} \simeq U_y \times S_y$ given by ∇-parallel translation along radial geodesics, and $e_1, ..., e_n$ is a local orthonormal frame on U_y obtained from an orthonormal basis of $T_y M$ by ∇-parallel translation along radial geodesics. In particular, at the origin $\mathbf{r}x_y = 0$ of the chart, $\omega(e_i)|_{\mathbf{r}x_y=0} = 0$, $\nabla_{e_i} e_i|_{\mathbf{r}x_y=0} = 0$, $e_i\omega(e_i)|_{\mathbf{r}x_y=0} = 0$ (this last follows from (7.16) and skew-symmetry of R), and hence near $y \in M$,

$$\Delta = -\sum_{i=1}^{n} e_i^2 + \frac{\kappa_y}{4} + c(dB)_y - 2|B_y|^2 + O(|\mathbf{r}x_y|\partial), \tag{8.5}$$

where $O(|\mathbf{r}x_y|\partial)$ denotes a 1-st order differential operator with coefficients that vanish at y. The operator $\overline{\Delta}$ has a similar expression, leaving out the $c(dB)_y$ term.

8.1.2 Computation of φ_0 $(p = 0)$

When $p = 0$, $P_k = a_0$ and the commutators (indexed by $k = (k_1, ..., k_p)$) are absent. By Corollary 7.3.4,

$$\varphi_0(a_0) = (4\pi)^{-2} \int_M a_0 \mathrm{tr}_s\Big(\Theta_2|_{\mathbf{r}x=0}\Big) dV.$$

The pointwise supertrace $\mathrm{tr}_s(\Theta_2|_{\mathbf{r}x=0})$ can be computed explicitly in terms of the differential forms $R^\top$ (see Definition 7.2.3) and B, leading to the following.

Theorem 8.1.1. *Let* $\dim(M) = 4$ *and* $D = D^{LC} + c(B)$ *where* $B \in \Gamma(\wedge^3 TM)$*. The component* φ_0 *of the residue cocycle is*

$$\varphi_0(a_0) = (2\pi\imath)^{-2} \int_M a_0 \left(\mathit{det}^{1/2}\left(\frac{R^\top/2}{\sinh(R^\top/2)}\right) + \left(\frac{\kappa}{12} - 2|B|^2\right) dB + \frac{1}{6} dd^* dB \right),$$

where κ *is the scalar curvature and* $R^\top$ *is the differential form of Definition 7.2.3.*

Proof. The term involving $R^\top$ in the statement of Theorem 8.1.1 comes from $\mathrm{tr}_s(\overline{\Theta}_2|_{\mathbf{r}x=0})$; see the proof of Theorem 7.3.5. On the other hand $(\Theta_2 - \overline{\Theta}_2)(\mathbf{r}x)$ is given by (8.3):

$$-\rho^{-1}(\mathbf{r}x) \int_0^1 t\rho(t\mathbf{r}x)\Big(c(dB)_{t\mathbf{r}x}\overline{\Theta}_1(t\mathbf{r}x) + (\Delta\Theta_1^B)(t\mathbf{r}x)\Big) dt. \tag{8.6}$$

Evaluating at $\mathbf{r}x = 0$, using $\rho(0) = 1$, and performing the integral over t, (8.6) becomes

$$(\Theta_2 - \overline{\Theta}_2)|_{\mathbf{r}x=0} = -\frac{1}{2}\Big(c(dB)\overline{\Theta}_1|_{\mathbf{r}x=0} + (\Delta\Theta_1^B)|_{\mathbf{r}x=0}\Big). \tag{8.7}$$

We calculate the supertrace of the two terms I, II of (8.7) in turn.

Term I: we claim that

$$-\frac{(4\pi)^{-2}}{2}\mathrm{tr}_s(c(dB)\overline{\Theta}_1|_{rx=0}) = \frac{(2\pi\imath)^{-2}}{2}\Big(\frac{\kappa}{12} - 2|B|^2\Big)\mathcal{B}(dB),$$

where $\mathcal{B}\colon \bigwedge^n T^*M \to M\times\mathbb{R}$ is the Berezin integral. To show this we compute at $y\in M$ using equations (8.1), (8.5),

$$\overline{\Theta}_1|_{rx_y=0} = -\overline{\Delta}\rho_y^{-1}|_{rx_y=0} = \sum_{i=1}^{4}(e_i)^2\rho_y^{-1}\Big|_{rx_y=0} - \frac{\kappa_y}{4} + 2|B_y|^2. \tag{8.8}$$

Using the Taylor series of g_{ij} in normal coordinates (cf. [BGV04, Proposition 1.28]), one has

$$\sum_{i=1}^{4}(e_i)^2\rho_y^{-1}\Big|_{rx_y=0} = \frac{\kappa_y}{6}, \tag{8.9}$$

at any $y\in M$. Thus taking the supertrace (using [BGV04, Proposition 3.21]),

$$\mathrm{tr}_s(c(dB)\overline{\Theta}_1|_{rx=0}) = (-2\imath)^2\Big(\frac{\kappa}{6} - \frac{\kappa}{4} + 2|B|^2\Big)\mathcal{B}(dB) = -(-2\imath)^2\Big(\frac{\kappa}{12} - 2|B|^2\Big)\mathcal{B}(dB),$$

which gives the claim.

Term II: we claim that

$$-\frac{(4\pi)^{-2}}{2}\mathrm{tr}_s(\Delta\Theta_1^B)|_{rx=0} = \frac{(2\pi\imath)^{-2}}{2}\Big(\frac{\kappa}{12} - 2|B|^2\Big)\mathcal{B}(dB) + \frac{(2\pi\imath)^{-2}}{6}\mathcal{B}(dd^*dB).$$

To verify this we compute at $y\in M$ using (8.2), (8.5),

$$\begin{aligned}
\Delta\Theta_1^B|_{rx_y=0} &= \Big(-\sum_i e_i^2 + \frac{\kappa_y}{4} + c(dB)_y - 2|B_y|^2\Big)\Big(\rho_y^{-1}\int_0^1 c(dB)_{trx_y}dt\Big)\Big|_{rx_y=0}\\
&= -\sum_i e_i^2 c(dB)\Big|_{rx_y=0}\int_0^1 t^2dt + \Big(\frac{\kappa_y}{12} + c(dB)_y - 2|B_y|^2\Big)c(dB)_y\\
&= \frac{1}{3}c(dd^*dB)_y + \Big(\frac{\kappa_y}{12} + c(dB)_y - 2|B_y|^2\Big)c(dB)_y
\end{aligned} \tag{8.10}$$

where to obtain the second equality we used $\rho_y|_{rx_y=0} = 1$, $e_i\rho_y|_{rx_y=0} = 0$, (8.9), and for the third equality we used that $-\sum_i e_i^2\nu|_{rx_y=0} = (dd^*\nu)_y$ on 4-forms ν (since $\dim(M)=4$). Another consequence of $\dim(M)=4$ is that $c(dB)^2$ is scalar, hence $-\mathrm{tr}_s(c(dB)^2)=0$. Taking the supertrace of (8.10) as in the computation of Term I gives the claim. □

By expressing $R^\top$ in terms of the curvature R_- of $\nabla_- = \nabla^{LC} - B_o$, further simplification of the expression in Theorem 8.1.1 is possible.

Theorem 8.1.2. *Let* $\dim(M) = 4$ *and* $D = D^{LC} + c(B)$ *where* $B \in \Gamma(\wedge^3 TM)$. *The component* φ_0 *of the residue cocycle is*

$$\varphi_0(a_0) = (2\pi\imath)^{-2} \int_M a_0 \left(det^{1/2}\left(\frac{R_-/2}{\sinh(R_-/2)}\right) + \frac{1}{6} dd^* dB \right).$$

Proof. Using the identities

$$\det{}^{1/2}\left(\frac{R^\top/2}{\sinh(R^\top/2)}\right)_{[4]} = -\frac{1}{48}\mathrm{tr}(R^{\top,2})$$

and (see the proof of Proposition 7.2.2)

$$R^\top = R_- + (dB)_\mathfrak{o}, \qquad R_- = R^{LC} - d_{\nabla^{LC}} B_\mathfrak{o} + B_\mathfrak{o}^2,$$

one has

$$\det{}^{1/2}\left(\frac{R^\top/2}{\sinh(R^\top/2)}\right)_{[4]} = -\frac{1}{48}\mathrm{tr}\Big(R_-^2 + 2(R^{LC} - d_{\nabla^{LC}} B_\mathfrak{o} + B_\mathfrak{o}^2)(dB)_\mathfrak{o} + (dB)_\mathfrak{o}^2\Big).$$

By a short calculation one finds that $\mathrm{tr}((dB)_\mathfrak{o}^2) = 0$, $\mathrm{tr}((d_{\nabla^{LC}} B_\mathfrak{o})(dB)_\mathfrak{o}) = 0$ while

$$\mathrm{tr}(R^{LC}(dB)_\mathfrak{o}) = 2\kappa\, dB, \qquad \mathrm{tr}(B_\mathfrak{o}^2 (dB)_\mathfrak{o}) = -48|B|^2 dB.$$

□

8.1.3 Computation of φ_2 $(p = 2)$

By Corollary 7.3.4,

$$\varphi_2(a_0, a_1, a_2) = \sum_{|k| \le 2} \frac{c'_{2k}}{(4\pi)^2} \sum_{j=1-|k|}^{1-\lceil |k|/2 \rceil} \int_M \mathrm{tr}_s(P_{k,1-j} \Theta_j|_{rx=0}) dV, \tag{8.11}$$

and sum ranges over $k = (k_1, k_2)$ with $k_1, k_2 \ge 0$.

Lemma 8.1.3. *The* $k = (0, 0)$ *contribution in* (8.11) *is*

$$\frac{(2\pi\imath)^{-2}}{2} \int_M a_0 g(da_1, da_2) dB.$$

Proof. If $|k| = 0$ then we only have the $j = 1$ term in (8.11), which reads

$$\frac{(4\pi)^{-2}}{2}\int_M a_0 \mathrm{tr}_s(c(da_1)c(da_2)\Theta_1|_{rx=0})dV.$$

We may write $\Theta_1 = \overline{\Theta}_1 + (\Theta_1 - \overline{\Theta}_1)$. The contribution of $\overline{\Theta}_1$ was computed in Theorem 7.3.5, and is trivial in this case because the dimension $n = 4$: $da_1 da_2$ is a 2-form and $\det^{1/2}((R^\top/2)/\sinh(R^\top/2))$ has vanishing 2-form component. On the other hand by equation 8.2 and since $\rho|_{rx=0} = 1$,

$$(\Theta_1 - \overline{\Theta}_1)|_{rx=0} = c(dB).$$

Taking the supertrace yields

$$\mathrm{tr}_s(c(da_1)c(da_2)c(dB)) = (-2\imath)^2 g(da_1, da_2)\mathcal{B}(dB),$$

which gives the lemma. □

Lemma 8.1.4. *The* $|k| = 2$ *contribution in* (8.11) *vanishes.*

Proof. The contribution involves the supertrace $\mathrm{tr}_s(P_{k,1}\Theta_0|_{rx=0})$. The s-fibred differential operators P_k, $\overline{P}_k$ have order $|k|$ and the same symbol (in the classical sense), hence $P_k - \overline{P}_k$ is an s-fibred differential operator of order $|k| - 1$. By Lemma 7.3.1 applied to $P = P_k - \overline{P}_k$, it follows that the s-fibred differential operator $P_{k,1} - \overline{P}_{k,1}$ vanishes on the diagonal, and hence $\mathrm{tr}_s(P_{k,1}\Theta_0|_{rx=0}) = \mathrm{tr}_s(\overline{P}_{k,1}\Theta_0|_{rx=0})$. But $\Theta_0 = \overline{\Theta}_0$, and so this supertrace vanishes by the Getzler order calculations in Theorem 7.3.5. □

Theorem 8.1.5. *Let* $\dim(M) = 4$ *and* $D = D^{LC} + c(B)$ *where* $B \in \Gamma(\wedge^3 TM)$. *The component* φ_2 *of the residue cocycle is*

$$\varphi_2(a_0, a_1, a_2) = \frac{(2\pi\imath)^{-2}}{6}\int_M a_0 g(da_1, da_2)dB.$$

Proof. The $k = (0, 0)$ term of (8.11) was calculated in Lemma 8.1.3. By the lemmas above, the only remaining contributions in (8.11) come from $k = (1, 0)$, $(0, 1)$ and $j = 0$. We have

$$P_{(1,0),1} = \overline{P}_{(1,0),1} + a_0[c(dB), c(da_1)]c(da_2), \quad P_{(0,1),1} = \overline{P}_{(0,1),1} + a_0 c(da_1)[c(dB), c(da_2)].$$

Inserting these expressions in (8.11), the supertraces involving $\overline{P}_{k,1}$ vanish by the Getzler order considerations in the proof of Theorem 7.3.5. On the other hand $[c(dB), c(da)] = -2c(da)c(dB)$ since $\dim(M) = 4$ and dB is a 4-form. Consequently

$$(4\pi)^{-2}\mathrm{tr}_s(P_{(1,0),1}\Theta_0|_{rx=0}) = -2(2\pi\imath)^{-2}a_0 g(da_1, da_2)\mathcal{B}(dB)$$

$$(4\pi)^{-2}\mathrm{tr}_s(P_{(1,0),1}\Theta_0|_{rx=0}) = 2(2\pi\imath)^{-2}a_0 g(da_1, da_2)\mathcal{B}(dB).$$

Combining these terms with the combinatorial prefactors $c'_{1,(1,0)} = -\frac{1}{6}$, $c'_{1,(0,1)} = -\frac{1}{3}$ and adding to the $k = (0,0)$ term from Lemma 8.1.3 yields the result. □

8.1.4 An Index Formula

When Theorem 7.1.3 is specialized to the triple $(C^\infty(M), L^2(M,S), D)$ with $n = \dim(M) = 4$, and the idempotent $e = 1$, we recover (since dd^*dB is exact):

$$\mathrm{index}(D) = (2\pi\imath)^{-2}\int_M \det{}^{1/2}\Big(\frac{R_-/2}{\sinh(R_-/2)}\Big).$$

8.2 Residue Cocycle Calculations when $n = 6$

In this brief section we describe the outcome of our calculations of the φ_0 component of the residue cocycle for the operator $D = D^{LC} + c(B)$ (with $dB \neq 0$) in dimension $n = 6$. Already in this dimension the calculations are considerably more involved than in the previous section.

The φ_0 term is given in terms of the 3-rd term Θ_3 in the asymptotic expansion of the heat kernel $e^{-t\Delta}$:

$$\varphi_0(a_0) = (4\pi)^{-3}\int_M a_0 \mathrm{tr}_s\Big(\Theta_3|_{rx=0}\Big)dV.$$

As in Section 8.1, $\Theta_3 = \overline{\Theta}_3 + \Theta_3^B$ where $\overline{\Theta}_3$ is the asymptotic expansion coefficient for $\overline{\Delta} = \Delta - c(dB)$. In fact $\mathrm{tr}_s(\overline{\Theta}_3|_{rx=0}) = 0$ in this case as $(x/2)/\sinh(x/2)$ is an even function hence the top part of the Chern-Weil form representing the $\widehat{A}$-class vanishes for degree reasons. We computed Θ_3^B directly using the recursion relations in Section 8.1.1. A useful basic Clifford algebra fact that allows to eliminate several terms is that if $\alpha_1, \alpha_2 \in \Omega^4(M)$ (and $\dim(M) = 6$) then $c(\alpha_1)c(\alpha_2) = c(\alpha)$ where α is an inhomogeneous sum of forms with degrees $0, 2, 4$; moreover if $\alpha_1 = \alpha_2$ then only degrees $0, 4$ appear. Thus for example the operators $c(dB)^2$, $c(dB)^3$ (amongst others involving the curvature form, etc.) do not appear in the resulting formula as they have vanishing local supertrace. Another ingredient in the calculation is the method explained in [BGV04, Proposition 1.18, Proposition 1.28], [ABP75, Appendix II] for computing the Taylor expansion coefficients of the metric and connection in a synchronous frame over a geodesic coordinate patch. The result of the calculations is expressed here using abstract index notation (on $\mathfrak{o}(TM)$ indices) and the summation convention.

Theorem 8.2.1. *In dimension $n = 6$ the φ_0 term in the residue cocycle for the operator $D = D^{LC} + c(B)$ is*

$$\varphi_0(a_0) = \frac{(2\pi\imath)^{-3}}{18} \int_M a_0 g^{ab} g^{cd} \left(\frac{1}{2}\nabla_a R^{\top}_{bc} + g^{ef} B_{ace} R^{\top}_{bf}\right) \wedge \nabla_d dB.$$

Appendix
More on the Rescaled Bundle

In this supplemental chapter, we will give comments on some furthur topics concerning the rescaled bundle. Throughout the chapter, we will assume M is an oriented Riemannian manifold, with a Clifford module $S \to M$. We discuss the multiplicative structure in detail. We also will obtain covariant derivatives on the rescaled bundle which are the analogues of the vector fields over the deformation to the normal cone in Chapter 3.

1 Noncommutative Evaluations

We recall that we have the characters

$$\varepsilon_{X_m} : \mathcal{A}_0(\mathbb{T}M) \to \mathbb{C}$$

and evaluation maps

$$\varepsilon_{X_m} : \mathcal{S}_0(\mathbb{T}M) \to \Lambda^* T_m M$$

where both are given by the formula

$$\varepsilon_{X_m} = \varepsilon_m \exp(t\nabla_X).$$

See definiton 3.2.7 and Section 4.4. The derivation $\nabla_X = \nabla_{(X,0)}$ differentiates along the first component of $M \times M$. However, we may also consider the derivation on the second $\nabla_{(0,Y)}$ or even both components $\nabla_{(X,Y)}$. Note that

$$\nabla_{(X,0)} = \nabla_{(X,X)} - \nabla_{(0,X)}.$$

Since $\nabla_{(X,X)}$ keeps the Clifford order the same, it decreases the scaling order by at most one. Therefore, $\nabla_{(0,X)}$, and more generally $\nabla_{(X,Y)}$, decreases the scaling order by at most one. Hence we have induced derivations

$$(\mathbf{X}, \mathbf{Y}) : t(X, Y) : \mathcal{A}(\mathbb{T}M) \to \mathcal{A}(\mathbb{T}M)$$

$$\boldsymbol{\nabla}_{(X,Y)} := t\nabla_{(X,Y)} : \mathcal{S}(\mathbb{T}M) \to \mathcal{S}(\mathbb{T}M).$$

So we may define the characters

$$\varepsilon_{(X_m,Y_m)} := \varepsilon_m(\exp((\mathbf{X}, \mathbf{Y}))) : \mathcal{A}_0(\mathbb{T}M) \to \mathbb{C}$$

and *noncommutative* evaluation maps

$$\varepsilon_{(X_m,Y_m)} := \varepsilon_m(\exp(\boldsymbol{\nabla}_{(X,Y)})) : \mathcal{S}_0(\mathbb{T}M) \to \Lambda^* T_m M.$$

Over the Rees algebra $\mathcal{A}_0(\mathbb{T}M)$ we have:

$$\varepsilon_{(X_m,Y_m)} = \varepsilon_{X_m - Y_m}.$$

However, on $\mathcal{S}_0(\mathbb{T}M)$ we have the following formula

Theorem 1.1 ([HY19, Theorem 4.5]).

$$\varepsilon_{(X_m,Y_m)} = \exp(\frac{1}{2}\kappa(X_m, Y_m)) \wedge \varepsilon_{X_m - Y_m}.$$

We will obtain a geometric formula for the noncommutative character $\varepsilon_{(X_m,Y_m)}$, similar to Proposition 4.4.1:

Proposition 1.1. *For* $s \in \Gamma(\mathbb{S})$

$$\varepsilon_{(X_m,Y_m)} s = \lim_{t \to 0} \delta_t \tau(m, \exp(tX_m)) s(\exp(tX_m), \exp(tY_m)) \tau(\exp(tY_m), m).$$

Proof. Assume that s_1, s_2 are compactly supported section the rescaled bundle. We will use change of coordinate $\exp_m : T_m M \to M$ and denote the Jacobian determinant by $j_m(X) = |\det(d_X \exp_m)|$. By Proposition 4.4.1, we have

$$\varepsilon_{X_m} s_1 * s_2 = \lim_{t \to 0} \delta_t \tau(m, \exp(tX_m)) s_1 * s_2(\exp(tX_m), m, t)$$

$$= \lim_{t\to 0} \delta_t t^{-n} \int_M \tau(m, \exp(tX_m)) s_1(\exp(tX_m), z) s_2(z, m) dz.$$

By changing the coordinate, we obtain

$$\begin{aligned}
&= \lim_{t\to 0} \delta_t t^{-n} \int_{T_m M} \tau(m, \exp(tX_m)) s_1(\exp(tX_m), \exp(Y_m)) s_2(\exp(Y_m), m) j_m(Y_m) dY_m \\
&= \lim_{t\to 0} \delta_t \int_{T_m M} \tau(m, \exp(tX_m)) s_1(\exp(tX_m), \exp(tY_m)) s_2(\exp(tY_m), m) j_m(tY_m) dY_m \\
&= \lim_{t\to 0} \int_{T_m M} \Big[\delta_t \tau(m, \exp(tX_m)) s_1(\exp(tX_m), \exp(tY_m)) \tau(\exp(tY_m), m)\Big] \cdot_t \\
&\qquad \Big[\delta_t \tau(m, \exp(tY_m)) s_2(\exp(tY_m), m)] j_m(tY_m)\Big] dY_m,
\end{aligned}$$

where in the last integral, we have a scaled Clifford product (see Definition 2.3.2). Note that $j_m(tY_m) \to 1$ as $t \to 0$. Because of the existence of the limits involved, for every s_2, and the fact

$$\lim_{t\to 0} a(t) \cdot_t b(t) = a(0) \wedge b(0),$$

we deduce that the limit

$$\lim_{t\to 0} \delta_t \tau(m, \exp(tX_m)) s_1(\exp(tX_m), \exp(tY_m)) \tau(\exp(tY_m), m)$$

exists and hence

$$\varepsilon_{(X_m, Y_m)} s_1 = \lim_{t\to 0} \delta_t \tau(m, \exp(tX_m)) s_1(\exp(tX_m), \exp(tY_m)) \tau(\exp(tY_m), m).$$

□

2 Multiplicative Structure

Let $\mathbb{T}^{(2)}M$ denote the space of composable arrows of the tangent groupoid. In [HY19, Section 4.2], the construction of a rescaled module and rescaled vector bundle over $\mathbb{T}^2M$ was briefly mentioned. In this section, we will provide more details on the construction of this rescaled bundle.

We define a "source-wise" differential operator D on the vector bundle

$$S \boxtimes \mathbb{C} \boxtimes S^* \to M \times M \times M$$

to be an operator that differentiate along the first component, i.e. for every $x, y \in M$, there exists a differential operator $D_{x,y}$ acting on $\Gamma(S)$

$$(Ds)|_{M\times\{x\}\times\{y\}} = D_{x,y}(s|_{M\times\{x\}\times\{y\}})$$

for every $a \in \Gamma(S \otimes \mathbb{C} \otimes S_y^*)$. We may define its Getzler order of D to be the maximum of the Getzler orders operators $D_{x,y}$.

Consider a section $s \in \Gamma(S \boxtimes \mathbb{C} \boxtimes S^*; M \times M \times M)$. By restricting to the diagonal, we obtain a section of the Clifford bundle $\mathbb{Cl}(TM)$; so we define Clifford order of s to be Clifford order of this restriction to the diagonal. The scaling order of s is defined as

$$\text{Scaling-order}(s) = \min_{D \text{ source-wise}} (\text{Getzler-order}(D) - \text{Clifford-order}(Ds))$$

Remark 2.1. Note that the covariant derivatives $\nabla_{(X,X,X)}$ and $\nabla_{(X,0,X)}$ do not increase the Clifford order and they decrease the scaling order by at most one. Because

$$\nabla_{(X,0,0)} = \nabla_{(X,0,X)} - \nabla_{(0,0,X)}$$

and

$$\nabla_{(0,X,0)} = \nabla_{(X,X,X)} - \nabla_{(X,0,X)},$$

we deduce that in the definition of the scaling order we could have used derivations along the other components, not just source-wise directions.

We define the Riemannian Euler vector field $\mathcal{R}^{(2)}$ on $M \times M \times M$ in a neighborhood of the diagonal by:

$$\mathcal{R}^{(2)}(\exp(X_m), m, \exp(Y_m)) := \frac{d}{dt}\Big|_{t=1} (\exp(tX_m), m, \exp(tY_m))$$

Definition 2.1. A section $s \in \Gamma(S \boxtimes \mathbb{C} \boxtimes S^*)$ is *synchronous* if $\nabla_{\mathcal{R}^{(2)}} s = 0$.

Let $\{x^1, \cdots, x^n\}$ be a family of smooth functions in neighborhood of a diagonal point (x, x, x), on $M \times M \times M$. Assume for all y, z close to x, restricting these functions to $M \times \{y\} \times \{z\}$ gives a geodesic local coordinate system. So we have a family of source-wise geodesic coordinates $\{x^i\}_i$. In a neighborhood of the point (x, x, x), every section $s \in \Gamma(S \boxtimes \mathbb{C} \boxtimes S^*)$ has a Taylor expansion of the form

$$s \sim \sum_{\alpha \geq 0} x^\alpha s_\alpha$$

such that s_α's are synchronous.

Definition 2.2. We define the Taylor order of s to be

$$\text{Taylor-order}(s) := \min_\alpha\{|\alpha| - \text{c.o.}(s_\alpha)\}.$$

A similar argument as in Proposition 4.3.1 will give:

Theorem 2.3. *The scaling order is equal to the Taylor order.* □

Now we define the rescaled module:

Definition 2.4. The rescaled module $\mathcal{A}(\mathbb{S}^{(2)})$ as the space of Laurant polynomials

$$\sum_p s_p t^{-p}$$

where $s_p \in \Gamma(S \boxtimes \mathbb{C} \boxtimes S^*; M \times M \times M)$ is of scaling order at least p.

We have following evaluation maps

$$\varepsilon_{(m_1,m,m_2,t)} : \mathcal{A}(\mathbb{S}^{(2)}) \to S_{m_1} \otimes S^*_{m_2}$$
$$s = \sum_{p\in\mathbb{Z}} s_p t^{-p} \mapsto \sum_{p\in\mathbb{Z}} s_p(m_1, m, m_2) t^{-p}.$$

Similarly we may define the quotient space

$$\mathcal{A}_0(\mathbb{S}^{(2)}) := \frac{\mathcal{A}(\mathbb{S}^{(2)})}{t.\mathcal{A}(\mathbb{S}^{(2)})}$$

and the evaluation maps

$$\varepsilon_m : \mathcal{S}_0(\mathbb{T}^{(2)}M) \to \Lambda^* T_m M \otimes \text{End}(W_m)$$

$$[\sum_{p\in\mathbb{Z}} s_p t^{-p}] \mapsto \sum_{p\leq 0} [s_p]_{-p},$$

and

$$\varepsilon_{(X_m,,0,-Y_m)} : \mathcal{S}_0(\mathbb{T}^{(2)}M) \to \Lambda^* T_m M \otimes \text{End}_{\mathbb{C}l}(S_m)$$

$$\varepsilon_{(X_m,,0,-Y_m)}([s]) = \varepsilon_m(\exp(t\nabla_{(X,0,-Y)})([s])).$$

Using these evaluaation maps, a similar argument to the one given in Chapter 4 will give a smooth vector bundle

$$\mathbb{S}^{(2)} \to \mathbb{T}^{(2)}M$$

with fibers

$$\begin{cases} S_{m_1} \otimes \mathbb{C} \otimes S^*_{m_2} & (m_1, m, m_2, t) \\ \Lambda^* T_m M \otimes \mathrm{End}(W_m) & (X_m, Y_m, 0). \end{cases}$$

Similar to the exterior product on functions, (5.9), we have external product on the rescaled bundles:

Lemma 2.5 ([HY19]). *There is a well-defined* external product

$$\Gamma(S \boxtimes S^*) \otimes \Gamma(S \boxtimes S^*) \mapsto \Gamma(S \boxtimes 1 \boxtimes S^*)$$

$$(s_1, s_2) \mapsto \Big[s_1 \otimes s_2 : (m_1, m, m_2) \mapsto s_1(m_1, m) \circ s_2(m, m_2) \Big].$$

□

Under this map we have

Lemma 2.6.

$$\text{Scaling-order}(s_1 \otimes s_2) \leq \text{Scaling-order}(s_1) + \text{Scaling-order}(s_2).$$

□

The external product restricts to a morphism of spaces of algebraic sections

$$\mathcal{S}(\mathbb{T}M) \otimes \mathcal{S}(\mathbb{T}M) \to \mathcal{S}(\mathbb{T}^{(2)}M),$$

$$(\sum_p s_p t^{-p}, \sum_q s_q t^{-q}) \mapsto \sum_{p,q} s_p \otimes s_q t^{-(p+q)}$$

which extends to

$$\Gamma(\mathbb{S}) \otimes \Gamma(\mathbb{S}) \to \Gamma(\mathbb{S}^{(2)})$$

Proposition 2.1 ([HY19, Theorem 4.5]). *For* $s_1, s_2 \in \mathbb{S}$*, we have*

$$\varepsilon_{(X_m, 0, -Y_m)}(s_1 \otimes s_2) = \varepsilon_{X_m}(s_1)\varepsilon_{Y_m}(s_2) \exp(-\frac{1}{2}\kappa(X_m, Y_m)).$$

3 The Rescaled Module and Covariant Derivatives

As we saw for the Rees algebra $\mathcal{A}(\mathbb{T}M)$, the rescaled module $\mathcal{S}(\mathbb{T}M)$ inherits a canonical covariant derivative

$$\mathcal{C} := t\frac{\partial}{\partial t} : \mathcal{S}(\mathbb{T}M) \to \mathcal{S}(\mathbb{T}M)$$
$$\sum_p s_p t^{-p} \mapsto -\sum_p p s_p t^{-p}.$$

We will introduce another derivation using the Riemannian Euler-like vector field $\mathcal{R}$.

Lemma 3.1. *Take any section* $s \in \Gamma(S \boxtimes S^*)$. *We have*

$$\text{Scaling-order}(\nabla_{\mathcal{R}} s) \geq \text{Scaling-order}(s).$$

Proof. Assume the scaling order is at most k. Note that the scaling order is equal to the Taylor order (see Proposition 4.3.1). So consider a local Taylor expansion of the section s:

$$s \sim \sum_\alpha x^\alpha s_\alpha,$$

where s_α's are synchronous sections, and where $\text{Scaling-order}(s) = \min_\alpha |\alpha| - \text{Clifford-order}(s_\alpha)$. Assuming $N \gg 0$, we can write

$$s = \sum_{\alpha \leq N} x^\alpha s_\alpha + r$$

where r is a section with vanishing order at least $N + k + 1$. We may write

$$\nabla_{\mathcal{R}} s = \sum_\alpha (\mathcal{R}.x^\alpha) s_\alpha + \nabla_{\mathcal{R}} r.$$

Since the derivation by $\mathcal{R}$ does not reduce the vanishing order, the result follows. □

Hence we can define the following covariant derivative:

Definition 3.2.

$$\nabla_{\mathcal{R}} : \mathcal{S}(\mathbb{T}M) \to \mathcal{S}(\mathbb{T}M)$$
$$\sum_p s_p t^{-p} \mapsto \sum_p \nabla_{\mathcal{R}} s_p t^{-p}.$$

4 The Number Operator

Let $\{e^i\}_{i=1}^n$ be a local orthonormal frame for TM on M. For a multi-index $\gamma = (\gamma_1, \cdots, \gamma_n)$, $\gamma_i = 0, 1$, we denote by

$$e^\gamma := e_1^{\gamma_1} \cdots e_n^{\gamma_n} \in \mathbb{C}l(TM)$$

the basis Clifford element. View it as a local section of $S \otimes S^* \to M$. Taking M diagonally embedded in $M \times M$, using parallel transport in first coordinate, we obtain a synchronous section of

$$\Gamma(S \boxtimes S^*),$$

which we denote again by e^γ. Every section $s \in \Gamma(S \boxtimes S^*)$ may be written locally as

$$s = \sum_\gamma f_\gamma e^\gamma.$$

Definition 4.1. We define the *number operator* by the following formula:

$$\mathcal{N} : \Gamma(S \boxtimes S^*) \to \Gamma(S \boxtimes S^*)$$
$$s = \sum_\gamma f_\gamma e^\gamma \mapsto \sum_\gamma |\gamma| f_\gamma e^\gamma$$

where $|\gamma| = \gamma_1 + \cdots \gamma_n$.

Lemma 4.2. *This definition is independent of the choice of the local orthonormal frame.*

Proof. Let $\{e_1^i\}$ and $\{e_2^i\}$ be two local orthonormal frames with transition functions $e_1^i = \alpha_j^i e_2^j$. So we have $\sum_k \alpha_k^i \alpha_k^j = \delta^{ij}$. Let $\mathcal{N}_1$ and $\mathcal{N}_2$ denote the number operators corresponding to the bases $\{e_1^i\}$ and $\{e_2^i\}$, respectively. For every multi-index γ, e_1^γ can be written in terms of the basis $\{e_2^i\}$ as

$$e_1^\gamma = \sum_{|\alpha| \leq |\gamma|} m_\alpha e_2^\alpha.$$

We argue that all m_α for $|\alpha| < |\gamma|$ vanish. Without loss of generality we may assume $\gamma = (1, 1, \cdots, 1, 0, 0, \cdots, 0)$ where there are k many 1's. So

$$e_1^\gamma = \sum_{i_1, \ldots, i_k} \alpha_{i_1}^1 \cdots \alpha_{i_k}^k e_2^{i_1} \cdots e_2^{i_k}$$

$$= \sum_{i_1,\ldots,i_k \text{ distinct}} \alpha^1_{i_1} \cdots \alpha^k_{i_k} e_2^{i_1} \cdots e_2^{i_k} + I,$$

where the term I includes terms $\alpha^1_{i_1} \cdots \alpha^k_{i_k} e_2^{i_1} \cdots e_2^{i_k}$, suc that for $r \neq s$, there exists equal indices $i_r = i_s$. We may write $I = I_1 + \cdots + I_{k-1}$ where I_r includes only terms $\alpha^1_{i_1} \cdots \alpha^k_{i_k} e_2^{i_1} \cdots e_2^{i_k}$ where r is the smallest number that there exist $s > r$ such that $i_r = i_s$. We may further write

$$I_r = I_{r,r+1} + \cdots + I_{r,k}$$

where $I_{r,s}$ includes only terms $\alpha^1_{i_1} \cdots \alpha^k_{i_k} e_2^{i_1} \cdots e_2^{i_k}$, and s is the smallest $s > r$ such that $i_r = i_s$. So we need to show that $I_{r,s} = 0$ which we may write

$$I_{r,s} = \sum \alpha^1_{i_1} \cdots \alpha^k_{i_k} e_2^{i_1} \cdots e_2^{i_k} = (-1)^{s-r} \sum_j \alpha^r_j \alpha^s_j \sum \alpha^k_{i_k} e_2^{i_1} \cdots \widehat{e_2^{i_r}} \cdots \widehat{e_2^{i_s}} \cdots e_2^{i_k} = 0.$$

Hence $\mathcal{N}_2 e_1^\gamma = \mathcal{N}_1 e_1^\gamma$. □

Remark 4.1. Note that Scaling-order$(\mathcal{N}s) \geq$ Scaling-order(s).

Using the number operator, we obtain a third covariant derivative on the rescaled module:

$$\mathcal{N} : \mathcal{S}(\mathbb{T}M) \to \mathcal{S}(\mathbb{T}M)$$
$$\sum_p s_p t^{-p} \mapsto \sum_p \mathcal{N} s_p t^{-p}.$$

Now we may define the *horizontal* covariant derivative.

Definition 4.3. We define the horizontal covariant derivativie as follows

$$\mathcal{T} := \frac{1}{t}(\mathcal{C} + \nabla_{\mathcal{R}} - \mathcal{N}) : \mathcal{S}(\mathbb{T}M) \to \mathcal{S}(\mathbb{T}M)$$
$$\sum_p s_p t^{-p} \mapsto \sum_p (-p + \nabla_{\mathcal{R}} - \mathcal{N}) s_p t^{-p-1}.$$

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- (Invited Talk) Euler-like vector fields and deformation to the normal cone, Göttingen Mathematics Colloquium, 2021
- (Invited Talk) Euler-like vector fields and deformation to the normal cone, Cornell-PSU Joint Symplectic Seminar, 2016
- Deformation to the normal cone. What is it good for?, Noncommutative Geometry Seminars, Penn State, 2020
- Tangent groupoids and deformation quantization, Student-Directed NCG Seminars, 2020
- Spinor representation and Clifford algebras, Noncommutative Geometry Seminars, Penn State, 2020
- Euler-like vector fields and Moser-type proofs, Noncommutative Geometry Seminars, Penn State, 2018
- The local index formula in non-commutative geometry, Hypoelliptic Laplacian Seminar, Penn State, 2017