The Pennsylvania State University
The Graduate School

DYNAMIC SUPPLY CHAIN MODEL AND DISRUPTION ANALYSIS

A Dissertation in
Industrial Engeering
by
Ilsoo Lee

Submitted in Partial Fulfillment
of the Requirements
for the Degree of

Doctor of Philosophy

May 2011

The dissertation of Ilsoo Lee was reviewed and approved* by the following:

Terry L. Friesz
Harold and Inge Marcus Chaired Professor of Industrial Engineering
Dissertation Advisor, Chair of Committee

Vittal Prabhu
Professor of Industrial Engineering

Tao Yao
Assistant Professor of Industrial Engineering

Robert D. Weaver
Professor of Agricultural Economics

Paul Griffin
Head Chair of Peter and Angela Dal Pezzo Department of Industrial Engineering

*Signatures are on file in the Graduate School.

Abstract

Rapid changes and complexities in business environments have stressed the interactions between partners and competitors, leading supply chains to become the most important element of contemporary business environments. This dissertation shows how to structuralize complex supply chain to create integrated dynamic models of supply chains using differential variational inequalities under the assumption that the output market for the firms is organized as a non-cooperative spatial oligopoly. Production planning and distribution plans for finished goods are determined by dynamic oligopolistic competition among their producers. In addition to these assumptions, integrated supply chain model assumes that input factor prices are set by contracts extending over the planning horizon and that just-in-time use of factor inputs is pervasive. Furthermore, with the potential of causing financial, reputational, and operational loss, supply chain disruption must be managed through the optimized planning and design of supply chain networks. Finally, this dissertation presents a mathematically dynamic supply chain network model threatened by disruptions and formulates a mathematical model as differential variational inequalities(DVI). The DVI is solved using a fixed-point algorithm, and finally a simple numerical example, introduced to compare supply chain optimization with and without disruptions, is presented.

Table of Contents

List of Figures

List of Tables

Acknowledgments

Most of all I would like to express sincere gratitude to my thesis advisor, Professor Terry L. Friesz for the kind assistance and guidance throughout the research process. He always encouraged me during my graduate student life, giving of his valuable time and providing new ideas and comments to make this dissertation. I would like to thank to Dr. Prabhu for being on my committee and for his insightful comments. To Dr. Weaver, I appreciate his guidance for research and ideas he told me. I appreciate Dr. Yao's valuable comments to my work during my comprehensive exam as well as accepting to join my committee. All my appreciation goes to my colleagues at the Department of Industrial Engineering: Taeil Kim and Sunghoon Chung. Additionally, I appreciate all the Korean IE students who have always been with me. Finally, I would like to thank my parents: Kunil Lee and Gayong Lee for their unconditional love, care, support and guidance as well as Hosoon Lee with all my heart.

Dedication

To my wife, Seoyoung Kim and my lovely daughter, Chloe Jiyoon Lee...

Chapter 1

Introduction

1.1 Motivation

Due to rapid changes in business environments and increasingly complex relationships among diverse business partners and competitors, supply chains have become essential factors in the quest to improve profits and efficiency Kogan and Tapiero (2007). A supply chain can be defined as a network of autonomous or semi-autonomous agents collectively responsible for procurement, manufacturing and distribution activities associated with one or more sets of related outputs. In other words, a supply chain network is the actual system charged with transforming natural resources, raw materials, and components into finished products, and eventually moving a product or service from supplier to customer Swaminathan et al. (1998). The various entities comprising a supply chain, typically consisting of interdependent organizations, people, technology, activities, information and resources, operate according to different sets of constraints and objectives, which may also link and coordinate them. As such, these entities are generally interdependent from the point of view of improving on-time delivery, quality assurance and controlling costs. As a consequence, performance of any aspect of the supply chain depends on the performance of the rest of the supply chain. Because of this organic relationship among agents in the supply chain, a part of the supply chain or one of the agents in the supply chain may damage the overall performance of the supply chain. Like supply chain disruption, this breed of supply chain risk, known as supply chain uncertainty, yields a degree of vulnerability. As mentioned above, supply chains may be organically related with the production processes, transportation of the goods, and demand markets. Thus, supply chain disruptions and the related risks are major topics of interest in theoretical and applied research, as well as in industry.

Supply chain disruptions such as weather conditions, terrorist attacks and diseases, are unintended, unwanted situations, which resulted in a downward variation and negative supply chain performance. The disruptions are associated with a probability of occurrence with direct and indirect negative impacts and can be classified into five sources: (1) demand-side, (2) supply-side, (3) regulatory, legal and bureaucratic, (4) infrastructure, and (5) catastrophic Wagner and Bode (2008). Supply chain vulnerability is an exposure to serious disturbance Christopher and Peck (2004). It is the capacity of supply chain to anticipate, cope with, resist and recover from the disruptions. The supply chain vulnerability is a function of supply chain characteristics, which include its density, complexity (base reduction and global sourcing) and network (node and link) connectivity criticality Craighead et al. (2007). Thus, the goal of supply chain risk management is to design and implement a supply chain that it is capable of anticipating, coping with and rapidly recovering from disruptions. To date several major disruptions have demonstrated the need to address supply-side disruption. For instance, in March 2000 a fire at the Phillips Semiconductor plant in Albuquerque, New Mexico, caused a major customer, Ericsson, to lose $400 million in potential revenue. Contrarily, another major customer, Nokia, managed to allocate alternative supplies and, therefore, mitigated the impact of the disruption Latour (2001). As a consequence, the goal of supply chain risk management is to alleviate the consequences of disruptions and the related risks or, simply put, to increase the robustness of the supply chain. In particular, robust supply chain network configuration is critical since it is costly to construct, difficult to reverse, and impacts long time horizon.Snyder (2003).

To study supply chain disruption, supply chain network flow optimization and supply chain network design and redesign must be simultaneously considered. Supply chain network flows optimization refers to the study of existing multi-level distribution networks in hopes to optimize product, information and financial flows. In practice, supply chain network flow optimization problems involve tactical and operational decisions which fundamentally are transportation decisions. On the other hand, supply chain network design and redesign aims to select the optimal facility locations and size within the network for a given objective, which is designated by the networks' authorities. Beyond location and size, the design and redesign of large-scale networks includes strategic decisions which involve the capacities required to fulfil these activities, their allocation to specific product groups, and the control system incorporated to manage all activities.

This dissertation integrates strategic, tactical and operational levels in proposing a dynamic supply chain network design mathematical model, which manages disruption by formulating the maximization of each agent's revenue and minimization of supply chain disruption over finite time horizons based on differential variational inequality (DVI). As supply chain uncertainty must be considered during all phase of supply chain planning, this dissertation also examines supply chain network design for lessening supply disruption based on strategic facility location, capacity, tactical inventory and operational transportation decisions. Thus, we study the supply chain network design for lessening supply disruption based on strategic facility location and capacity decision, tactical inventory and operational transportation decisions. The model discussed is based on continuous time and dynamic mathematical programming. Simply put, a dynamic supply chain is a supply chain that is not in steady state; rather it is either in disequilibrium and moving toward an equilibrium or it has attained a moving equilibrium, a sequence of equilibrium states visited in succession without transit through any disequilibrium. In dynamic supply chains, inventories of essential inputs as well as inventories of products made from those inputs are fundamental and critical state variables that devolve from the construction of competitive business strategies. As such, the technical literature on production planning and multiechelon inventory problems is directly relevant to modeling, planning and operating dynamic supply chains.

Basically, it is possible to modify and extend the spatial oligopolistic competition model. The setting will be one for which a single homogenous good is manufactured by producers for sale by retailers; there will be no direct local consumption of production; shipments from producers to retailers are free onboard; there is a multi-echelon supply chain that prepares the input flows to producers and may create and exploit inventories at each level. Additionally, producers may also build up inventories and spatially reconfigure those inventories to their advantage. In the discussion that follows, it will simplify our notation to imagine all producers and all retailers may potentially exist at every node of a fully connected graph. That is, we allow transport between any origin-destination pair (i, j), where $i, j \in \mathcal{N}$ and $\mathcal{N}$ is the complete set of network nodes. This generality is merely a notational convenience; a model in which individual producers and retailers are restricted to subsets of nodes can be created.

1.2 Literature Review

We discuss the literature directly related to general supply chain network and supply chain disruption in this section. All companies that aim to be competitive on the market need to pay attention to the supply chain. In particular, they should analyze and optimize the supply chain in order to improve customer service quality without an unexpected cost increase. To this end, various studies pertaining to supply chain has been conducted.

Let me begin with the literature review regarding supply chain network and flow optimization. Research for supply chain modeling and optimization started very early on, with facility location and resource allocation problems formulated by Geoffrion and Graves (1974) who consider the problem of distribution system layout and sizing. Cohen and Lee (1989) consider a supply chain network that connects suppliers of raw materials, manufacturers of both intermediate and finished products, warehouses, distribution channels and customers who are geographically spread. The resource allocation decisions involve the network design which includes both the location and capacity of all the sites/ nodes and the material flow management within the network.

In this supply chain network, two types of entities can be existed such as competitors and cooperators. They can interact in two ways: horizontal and vertical. Christopher (1992) argues that the real competition does not see company against company, but rather supply chain against supply chain. Vertical as well as horizontal integrations are required for the flow optimization in the supply chain, and for the optimization of all related activities. That implies agreements among subjects that operate at different levels of the supply chain, so called vertical integration, and among entities of the same level, so called horizontal integration. For the flow optimization in supply chain network, Qu et al. (1999) provide an integrated approach where decisions for both inventory and transportation are made simultaneously. The supply chain considered for this research work has a centralized warehouse and several geographically dispersed suppliers with multiple items and stochastic demand.

Consequently, we can distinguish two kinds of general supply chain network analysis: 1) Network flows optimization: we consider a pre-designed or existing distribution network, and we want to optimize the flows of materials through the network. 2) Network design or re-design: we choose the best configuration of the facilities within the network in order to satisfy customers' needs more efficiently. This network design problems involve strategic decisions which influence tactical and operational decisions Crainic and Laporte (1997), such as facility location, transportation flow and inventory level decision, which affect the cost of the distribution system, and the quality of the service level Ambrosino and Scutellà (2005).

The problem of deciding optimal (shipments) flow in a pre-defined supply chain equilibrium network is firstly noted by Nagurney et al. (2002). Dong et al. (2004) develop a supply chain network model where a finite-dimensional variational inequality is formulated for the behavior of various decision makers. Zhang (2006) propose a supply chain model that comprises heterogeneous supply chains involving multiple products and competing for multiple markets.

In real business circumstances, most industrial companies have a distribution system (supply chain network) with many local warehouses geographically close to the customers. One or more warehouses in each country in Europe are not unusual Abrahamsson (1993). Production oriented companies usually have a long distribution channel to break down large production batches, step by step, warehouse by warehouse, to a product-mix demanded by the customers. A sales-oriented company uses a wide distribution channel, with many sales offices and warehouses, to be geographically close to the customers. In this kind of decentralized and traditional distribution structures, each link in the distribution chain usually manages both sales and warehousing with a great increase of the network complexity level. Abrahamsson (1993) stresses that by using modern information systems and by implementing a more effective distribution strategy, the goods can be delivered directly to the customer from a central warehouse or a production site, where the main focus is on centralization of the distribution structure.

Based on the previous studies, Abrahamsson (1995) defines the impacts of the centralized structure on the manufacturer and retailer in terms of logistics costs and value added activities:

1. Lower fixed distribution costs: decreased costs for staff, warehousing space and administration.

2. Lower variable distribution costs: reduced inventory costs and transport costs can be kept at a defined constant level (according to the traditional logistics theories the transportation costs were expected to increase considerably, but they didn't increase in any of the Swedish industrial cases. The reason was a complete assortment in the central warehouse stock in combination with a smooth flow of small deliveries out from the warehouse with a reduction in shortage and in express freights and an increment in turnover).

3. Integration feedback: centralized control of the material flow leads to the recourse decrease in sales department.

4. Shorter lead-time for all the markets and the entire assortment

5. Increased on-time deliveries

Since the cost of transshipment in practice is generally lower than both the shortage cost and the cost of an emergency delivery from the designated warehouse and the transshipment time is shorter than the regular replenishment lead time, most of research on supply chain assume transshipment time is negligible. However, Chiou (2008) mainly concentrates on an analysis of the operation of transshipment in a group consisting of multiple retailing outlets and a single/or multiple upper sources in a two-echelon supply chain system. In the future research, we might consider this transshipment operation and transportation concern.

Please turn your attention to supply chain uncertainty (risk) literature. Supply uncertainty issue has gained increased attention in recent years. A recent study Kembel (2000) estimates the cost of downtime by lost revenue for several on-line industries that cannot function if their computers are down. For example, Patterson (2002) shows the cost of one hour of downtime for Ebay is estimated at $225,000, for Amazon.com, $180,000, and for brokerage companies $6,450,000. Note that these numbers do not include the cost of paying employees who cannot work because of an outage or the cost of losing customers' goodwill. Moreover, a company that experiences a supply chain disruption can expect to face significant declines in sales growth, stock returns, and shareholder wealth for two years or more following the incident (Hendricks and Singhal, 2003, 2005). Moreover, since most supply chain design decisions (for example, facility location) are very expensive to change, it is not reasonable to use deterministic parameters to describe these uncertainties when designing a supply chain Qi and Shen (2007).

Generally, supply chain disruptions can influence on various factors of each entities in the network such as production cost, inventory level, transshipment cost, finished goods' price, and so on. Parlar and Berkin (1991) introduce disruptions in the EOQ model. Babich et al. (2007) discuss pricing competition between multiple suppliers, all of whom have a disruption risk. They focus on a single uncertain demand and model the retailer's profits based on the number of suppliers they choose to order from. Sheffi (2001, 2005) and Simchi-Levi et al. (2002) stress the importance of sharing risk throughout the supply chain and the dangers of disruptions to just-in-time (JIT) systems. They indicate that JIT systems can lack buffers for supply uncertainty and can be at high risk for interruption of material flow. There are different ways to categorize uncertainty in supply chains. Johnson (2001) suggested that when viewed as a whole, uncertainty falls into two major categories: supply uncertainty including capacity limitations, currency fluctuations, and supply disruptions and demand uncertainty including seasonal imbalances, volatility of demands, and new products.

We need to identify the differences between supply-side uncertainty and demand-side uncertainty in multiechelon supply chains. The primary focus of research to date has been on demand-side uncertainty, which is related to fluctuations in the demand for products, as opposed to the supply-side risk, which deals with uncertain conditions that affect the production and transportation processes of the supply chainQiang et al. (2009). However, a growing body of literature recently considers supply-side uncertainty like disruptions in the context of facility location (Church and Scaparra 2007, Snyder and Daskin 2005, Snyder et al. 2006) or location-inventory models Qi and Shen (2007). In general, these studies find that the optimal number of facilities increases when the disruption risk increase.

When we study supply chain uncertainty as well as supply chain network optimization, time horizon is very important factor in dynamic modeling. Lessening the time horizon to consider only a single period makes the study easier and may enable us to obtain closed-form results; several papers employ this approach (Chopra et al. 2007; Dada et al. 2007; Tomlin and Wang 2005). Dada et al. model a retailer with multiple unreliable suppliers in a newsboy setting, where inventory (from one or multiple suppliers) is used to mitigate generally distributed supply uncertainty, and they include disruptions as a possible realization. However, as we demonstrate, truncating also underestimates the disruption risk Schmitt and Snyder (2009).

In addition, to date, very little research has considered disruptions in a dynamic network with a multiechelon setting. Natarajan (2007) considers a three echelon supply chain with a single manufacturer, single warehouse and multiple retailers with one product flowing through the decentralized supply chain. The manufacturer is assumed to have infinite capacity and the objective is to obtain the ordering policies for all the retailers and the warehouse under different scenarios. In this case, if inventory sites are subject to disruptions, it may be preferable to hold inventory at the retailers rather than at the warehouse. In general, it is considered that under this decentralized supply chain, a disruption would affect the agents in the final echelon like retailers mostly; under a centralized, a disruption would affect the whole supply chain. In fact, the mean costs in the both supply chain strategies are the same, but the decentralized strategy results in a smaller variance of cost. This is referred to as the risk-diversification effect, which says that disruptions are equally frequent in either system, but they are less severe in the decentralized system Snyder and Shen (2006).

Consequently, in my dissertation, studies will be conducted under multiechelon/ multi-stages supply chain network and these studies are assumed to be under continuous time-based modeling. My dissertation will provide a supply chain network optimization in the deterministic condition as well as with uncertainty like supply chain disruptions. Based on dynamic Nash game theoretic analysis, the decentralized supply chain optimization problem will be reformulated to a Differential Variational Inequalities (DVI) and solved as well as integrated supply chain network will be re-designed in infinite dimensional setting, which is different from Nagurney et al (2002). Especially, supply-side uncertainty like disruptions will be addressed under the dynamic supply chain model based on differential Nash game. That is, time horizon is multi-period and continuous and multiechelon structure is assumed. Additionally, various numerical examples will be addressed.

Chapter 2

Production Planning

Considered is the problem of simultaneously making price and production decisions for a single product with a known deterministic demand function. The objective is to maximize profit. Results reducing the computations necessary for the pricing decision are given, and planning horizons are identified as in previous inventory research. This dissertation treats the continuous-time convex problem with variable price and time-dependent demand curve, and leads to an algorithm that assures an optimal and unique solution. In this chapter, this dissertation develops models that describe how prices, production rates and distribution activities evolve over time and influence one another for three output market structures: perfect competition, monopoly, and oligopoly. This dissertation deals with oligopolistic market and its characteristics in chapter 3.

2.1 The Perfect Competition

Firstly, let me consider the circumstance of perfect competition. The price of the single homogeneous output of each firm is a known function of continuous time

$$p(t) \in L^2[t_0, t_f]$$

since we assume every firm conducts its business in a perfectly competitive market and is, as a consequence, a price-taker. The time interval considered is $[t_0, t_f] \subseteq \Re^1_+$ where $t_0 \in \Re^1_+$ is the fixed initial time, $t_f \in \Re^1_{++}$ is the fixed terminal time. There is a constant nominal discount rate ρ, and continuous compounding is assumed. The firm's output rate is $q(t)$ with associated production cost $V(q)$; the rate of allocation of output to consumption is $c(t)$; and the firm's inventory is $I(t)$, a quantity of undelivered stock or a quantity of backorders according to its sign. Moreover the controls

$$q \in L^2[t_0, t_f]$$
$$c \in L^2[t_0, t_f]$$

completely determine the inventory level which may be viewed as the operator

$$I(q, c) : L^2[t_0, t_f] \times L^2[t_0, t_f] \longrightarrow \mathcal{H}^1[t_0, t_f]$$

where $L^2[t_0, t_f]$ is the space of square-integrable functions and $\mathcal{H}^1[t_0, t_f]$ is a Sobolev space for the real interval $[t_0, t_f] \in \Re^1_+$. The upper bounds on output, consumption and inventory are, respectively

$$Q \in \Re^1_{++}$$

$$C \in \Re^1_{++}$$

$$K \in \Re^1_{++}$$

In addition we use the notation

$$h_{inv}(I) : \mathcal{H}^1[t_0, t_f] \longrightarrow \mathcal{H}^1[t_0, t_f]$$

for the inventory holding cost functional.

2.1.1 Optimal Control Problem for Perfect Competition

A consequence of the above notation and assumptions is the firm's extremal problem:

$$\max\, p_{Inv} I(t_f) e^{-\rho t_f} + \int_{t_0}^{t_f} e^{-\rho t} \{p(t) c - V(q) - h_{inv}(I)\} dt \tag{2.1}$$

subject to

$$\frac{dI}{dt} = q - c \tag{2.2}$$

$$I(0) = I_0 \tag{2.3}$$

$$0 \le I \le K \tag{2.4}$$

$$0 \le q \le Q \tag{2.5}$$

$$0 \le c \le C \tag{2.6}$$

where p_{Inv} is the price per unit of inventory liquidated at the terminal time t_f. That is, in the objective function, $p_{Inv} I(t_f) e^{-\rho t_f}$ is the present value of terminal time inventory/backorders. Under the integral is the instantaneous net present value of profit for the firm, consisting of the price multiplied by the amount consumed minus the variable cost of production for each instant in time. This is integrated to give the net present value of the time stream of profits.

2.2 The Monopoly

Please note that aspatial monopoly in a dynamic setting. The firm of interest is now without competition and exploits the demand curve for its product in order to maximize profit. We employ the notion of an inverse demand curve; that is

$$p(t) = \Psi(c, t)$$

denotes the price consumers pay for the firm's product as a function of consumption rate c and time t. The explicit dependence on time arises from changes in consumer preferences over time and/or seasonal effects. Consequently, we consider the following model:

$$\max J = \eta I(t_f) e^{-\rho t_f} + \int_{t_0}^{t_f} e^{-\rho t} \{\Psi(c, t) c - V(q) - h_{inv}(I)\} dt$$

subject to

$$\frac{dI}{dt} = q - c$$
$$I(0) = I_0$$
$$I(t_f) = 0$$
$$0 \leq q \leq Q$$
$$0 \leq c \leq C$$

where once again η is the dual variable for the terminal condition $I(t_f) = 0$.

2.2.1 Necessary Conditions for the Monopoly

The relevant Hamiltonian

$$H = e^{-\rho t}\left\{\Psi(c,t)\,c - V(q) - h_{inv}(I)\right\} + \lambda(q - c) \tag{2.7}$$

The minimum principal requires

$$c^* = \left[\arg\left(\frac{\partial H}{\partial c} = 0\right)\right]_0^C$$
$$q^* = \left[\arg\left(\frac{\partial H}{\partial q} = 0\right)\right]_0^Q$$

The adjoint equation and transversality condition are

$$\frac{d\lambda}{dt} = (-1)\frac{\partial H}{\partial I} = e^{-\rho t}\frac{dh_{inv}(I)}{dI} \tag{2.8}$$
$$\lambda(t_f) = \frac{\partial \eta I(t_f)\, e^{-\rho t_f}}{\partial I(t_f)} = \eta e^{-\rho t_f} \tag{2.9}$$

Assuming none of the control constraints are binding for the optimal trajectory, we have

$$\frac{\partial H}{\partial c} = e^{-\rho t}\frac{\partial}{\partial c}\left[\Psi\left(c,t\right)c\right] - \lambda = 0 \tag{2.10}$$

$$\frac{\partial H}{\partial q} = -e^{-\rho t}\frac{\partial V(q)}{\partial q} + \lambda = 0 \tag{2.11}$$

which tells us

$$\frac{\partial}{\partial c}\left[\theta\left(c,t\right)c\right] = \frac{\partial V(q)}{\partial q} = \lambda e^{\rho t} \tag{2.12}$$

along the optimal trajectory. This last expression is a statement that marginal revenue with respect to consumption equals marginal variable cost with respect to output. Furthermore, the adjoint variable is the present value of marginal variable cost with respect to output:

$$\lambda = e^{-\rho t}\frac{\partial V(q)}{\partial q}$$

2.2.2 The Monopolistic Firm in a Network

Let me consider the firm of interest to be located at one of the nodes of a distribution network for which flows over paths generate flows over arcs controlled by shipping agents who set freight tariffs for the firm's output at the origin-destination (OD) pair level. There are three sets important to articulating a model of production and distribution on a network; these are as follow: $\mathcal{A}$ for directed arcs, $\mathcal{N}$ for nodes and $\mathcal{W}$ for origin-destination (OD) pairs. Subsets of these sets are formed as is meaningful by using the subscript i for a specific node and ij for a specific OD pair (i, j).

The firm controls production output rates expressed as a vector q, allocations of output to meet demand expressed as a vector c and shipping patterns expressed as a vector s. Inventories I are a vector of state variables determined by the controls. That is:

$$c \in \left(L^2\left[t_0, t_f\right]\right)^{|\mathcal{N}|}$$
$$q \in \left(L^2\left[t_0, t_f\right]\right)^{|\mathcal{N}|}$$
$$s \in \left(L^2\left[t_0, t_f\right]\right)^{|\mathcal{W}|}$$

$$I\left(c, q, s\right) : \left(L^2\left[t_0, t_f\right]\right)^{|\mathcal{N}|} \times \left(L^2\left[t_0, t_f\right]\right)^{|\mathcal{N}|} \times \left(L^2\left[t_0, t_f\right]\right)^{|\mathcal{W}|} \longrightarrow \left(\mathcal{H}^1\left[t_0, t_f\right]\right)^{|\mathcal{N}|}$$

where again $L^2\left[t_0, t_f\right]$ is the space of square-integrable functions and $\mathcal{H}^1\left[t_0, t_f\right]$ is a Sobolev space for the real interval $\left[t_0, t_f\right] \in \Re^1_+$.

Chapter 3

Integrated Supply Chain Model

Some sort of models of dynamic oligopolistic network competition including the one aforementioned have been addressed in spatial economics literature. Thus, this dissertation shows how such models of interest may be extended to create integrated dynamic models of supply chains using differential variational inequalities. Here we take the point of view that input factors for supply chains and freight network services for output distribution selected by firms in their business relationships may only be accurately modelled by considering the combined supply-production-distribution policies of those firms, as they compete with one another via a generalized multi-layer transport network. We assume that the output market for the firms is organized as a non-cooperative spatial oligopoly. Production planning and distribution plans for finished goods are determined by dynamic oligopolistic competition among their producers. In addition to these assumptions, integrated supply chain model assumes that input factor prices are set by contracts extending over the planning horizon and that just-in-time use of factor inputs is pervasive.

In detail, it is possible to modify and extend the spatial oligopolistic competition model presented in the previous chapter to include consideration of supply chains. The setting will be one for which a single homogenous good is manufactured by producers for sale by retailers; there will be no direct local consumption of production; shipments from producers to retailers are free onboard; there is a multi-echelon supply chain that prepares the input flows to producers and may create and exploit inventories at each level. Additionally, producers may also build up inventories and spatially reconfigure those inventories to their advantage. In the discussion that follows, it will simplify our notation to imagine all producers and all retailers may potentially exist at every node of a fully connected graph. That is, we allow transport between any origin-destination pair (i, j), where $i, j \in \mathcal{N}$ and $\mathcal{N}$ is the complete set of network nodes. This generality is merely a notational convenience; a model in which individual producers and retailers are restricted to subsets of nodes can be created.

3.1 Differential Variational Inequalities

To articulate an adequately general differential variational inequality with controls, we must specify the function spaces associated with the key mappings that arise in such a problem formulation.

3.1.1 Problem Definition

Let me start with considering the control vector

$$u \in \left(L^2\left[t_0, t_f\right]\right)^m$$

and associated operator

$$x(u,t) = \arg\left\{ \frac{dy}{dt} = f\left(y,u,t\right),\ y\left(t_0\right) = y_0,\ \Gamma\left[y\left(t_f\right),t_f\right] = 0 \right\} \in \left(\mathcal{H}^1\left[t_0,t_f\right]\right)^n \tag{3.1}$$

where

$$x_0 \in \Re^n \tag{3.2}$$

$$f : \left(\mathcal{H}^1\left[t_0,t_f\right]\right)^n \times \left(L^2\left[t_0,t_f\right]\right)^m \times \Re^1_+ \longrightarrow \left(L^2\left[t_0,t_f\right]\right)^n \tag{3.3}$$

$$\Gamma : \left(\mathcal{H}^1\left[t_0,t_f\right]\right)^n \times \Re^1_+ \longrightarrow \left(\mathcal{H}^1\left[t_0,t_f\right]\right)^r \tag{3.4}$$

and $\left(L^2\left[t_0,t_f\right]\right)^m$ is the m-fold product of the space of square-integrable functions $L^2\left[t_0,t_f\right]$ with inner product defined by

$$\langle v,u\rangle = \int_{t_0}^{t_f} v^T u dt, \tag{3.5}$$

while $\left(\mathcal{H}^1\left[t_0,t_f\right]\right)^n$is the n-fold product of the Sobolev space $\mathcal{H}^1\left[t_0,t_f\right]$. The entity $x(u,t)$ is to be interpreted as an operator that tells us the state vector x for each control vector u and each time $t \in [t_0,t_f] \subset \Re^1_+$ when there are end point conditions which the state variables must satisfy. Working with this operator is, in effect, a supposition that a two point boundary value problem involving the state variables has a unique solution for each control vector considered. Note that constraints on u are enforced separately, so in working with $x(u,t)$ we are not presuming existence of a solution of the variational inequality to be articulated below. Note that we do not actually have to explicitly solve for $x(u,t)$, as is made clear in our subsequent analysis.

Furthermore, we assume that every control vector is constrained to lie in a set

$$U \subseteq \left(L^2\left[t_0, t_f\right]\right)^m,$$

where U is defined to ensure the terminal conditions imposed on the state variables may be reached from the initial conditions intrinsic to (3.1). Given the operator (3.1), the variational inequality of interest to us takes the following form:

$$\text{find } u^* \in U \text{ such that}$$

$$\langle F\left(x(u^*, t), u^*, t\right), u - u^* \rangle \geq 0 \text{ for all } u \in U \tag{3.6}$$

where

$$F : \left(\mathcal{H}^1\left[t_0, t_f\right]\right)^n \times \left(L^2\left[t_0, t_f\right]\right)^m \times \Re^1_+ \longrightarrow \left(L^2\left[t_0, t_f\right]\right)^m$$

Note that, by virtue of the inner product (3.5), we may state the variational inequality (3.6) as

$$\langle F\left(x(u^*, t), u^*, t\right), u - u^* \rangle \equiv \int_{t_0}^{t_f} \left[F\left(x(u^*), u^*, t\right)\right]^T \left(u - u^*\right) \geq 0$$

We refer to (3.6) as a *differential variational inequality* (with explicit state equations and controls) and give it the symbolic name $DVI(F, f, U, \Gamma, x_0)$.

3.1.1.1 Regularity Conditions for $DVI(F, f, U, \Gamma, x_0)$

To analyze (3.6) we will rely on the following notion of regularity:

Definition 1. *Regularity of $DVI(F, f, U, \Gamma, x_0)$. We call $DVI(F, f, U, \Gamma, x_0)$ regular if:*

1. $x(u,t) : (L^2[t_0,t_f])^m \longrightarrow (\mathcal{H}^1[t_0,t_f])^n$ *exists and is unique for each* $u \in U$*;*

2. $x(u,t)$ *is continuous and G-differentiable with respect to* u*;*

3. $\Gamma(x,t)$: *is continuously differentiable with respect to* x;

4. $F(x,u,t)$ *is continuous with respect to* x *and* u*;*

5. $f(x,u,t)$ *is continuously differentiable with respect to* x *and* u*;*

6. U *is convex; and*

7. $x_0 \in \Re^n$ *is known and fixed.*

The motivation for this definition of regularity is to parallel as closely as possible those assumptions needed to analyze traditional optimal control problems from the point of view of infinite dimensional mathematical programming.

3.1.2 Existence of the State Operator

Before we provide an existence result for $DVI(F,f,U,\Gamma,x_0)$, the existence of the state operator is an essential issue we must explore. In particular, we wish to state and prove the following:

Theorem 2. *Existence and continuity of state operator. Take* $x \in (\mathcal{H}^1[t_0,t_f])^n$ *and* $u \in (L^2[t_0,t_f])^m$*. Let regularity conditions 3, 4, 5, 6 and 7 of Definition 1 hold. In addition, let us assume*

$$\begin{aligned} |f(x,u,t)| &\leq C \\ \left\| \frac{\partial}{\partial x} f(x,u,t) \right\| &\leq K \end{aligned}$$

for some constants C and K and for all t, x, u. Then for every $u \in U$, the problem

$$\frac{dx}{dt} = f\left(x, u, t\right) \qquad x\left(t_0\right) = x_0 \tag{3.7}$$

has a unique solution; hence, the state operator $x\left(u, t\right)$ exists and is unique for all $t \in \left[t_0, t_f\right]$, provided $x(t_f)$ is reachable. Furthermore the mapping $u\left(\cdot\right) \to x\left(u, \cdot\right)$ is continuous from $\left(L^2\left[t_0, t_f\right]\right)^m$ into $\left(\mathcal{H}^1\left[t_0, t_f\right]\right)^n$.

Proof: We follow the proof by Bressan and Piccoli (2007). We note every element in $\left(\mathcal{H}^1\left[t_0, t_f\right]\right)^n$ is continuous, because its first-order derivative is already well-defined and bounded. Hence, it suffices to prove the existence and continuity of the state operator in $\left(C^0\left[t_0, t_f\right]\right)^n$. The problem (3.7) is equivalent to the fixed point problem

$$x\left(t\right) = \Phi\left(u, x\right)\left(t\right) \equiv x_0 + \int_{t_0}^{t} f\left(x\left(s\right), u\left(s\right), s\right) ds \tag{3.8}$$

If $\left\{u_v\right\}_{v \geq 1}$ is a sequence of admissible controls approaching u, for any subsequence $u_{v'}$ we can extract a further subsequence $u_{v''}$ such that $u_{v''}\left(t\right) \to u\left(t\right)$ for almost every t. By the Lebesgue dominated convergence theorem, it follows

$$\lim_{v \to \infty} \int_{t_0}^{t_f} f\left(y\left(s\right), u_v\left(s\right), s\right) ds = \int_{t_0}^{t_f} f\left(y\left(s\right), u\left(s\right), s\right) ds$$

This limit holds for the entire sequence u_v since the subsequence $u_{v'}$ was arbitrary. Therefore for each continuous function y, the image $\Phi\left(u_v, y\right)$ converges to $\Phi\left(u, y\right)$ uniformly on $\left[t_0, t_f\right]$. This proves the continuity of Φ with respect to u.

Now we show the existence of a solution to $x = \Phi\left(u, x\right)$. For any fixed u, the assumption

$$\left\| \frac{\partial}{\partial x} f(x,u,t) \right\| \leq K$$

implies

$$|f(x,u,t) - f(y,u,t)| \leq K|x-y|$$

Let us define an equivalent norm in $(C^0[t_0,t_f])^n$, namely

$$\|y\|_* \equiv \max_{t_0 \leq t \leq t_f} e^{-2K(t-t_0)} |y(t)| \tag{3.9}$$

and assume $\|y - y'\|_* = \delta$, which leads to

$$|y(s) - y'(s)| \leq \delta e^{2L(s-t_0)} \quad \forall s \in [t_0, t_f]$$

Therefore, for every $t > t_0$ we obtain

$$\begin{aligned}
& e^{-2Kt} |\Phi(u,y)(t) - \Phi(u,y')(t)| \\
& \qquad = e^{-2K(t-t_0)} \left| \int_{t_0}^{t} \{ f(y(s), u(s), s) - f(y'(s), u(s), s) \} ds \right| \\
& \qquad \leq e^{-2K(t-t_0)} \left| \int_{t_0}^{t} K |y(s) - y'(s)| ds \right| \\
& \qquad \leq e^{-2K(t-t_0)} \int_{t_0}^{t} K\delta e^{2K(s-t_0)} ds \\
& \qquad < \frac{\delta}{2}
\end{aligned}$$

which implies

$$\|\Phi(u,y) - \Phi(u,y')\|_* \leq \frac{1}{2} \|y - y'\|_*$$

By the contraction mapping theorem, there exists a unique fixed point $x = \Phi(u,x)$ for every given control u. By virtue of the assumed reachability and the definition of Φ, existence, uniqueness and continuity of a solution to problem (3.7) is assured. ■

Now, let us now establish that the state operator is differentiable in the Gateaux sense:

Theorem 3. *Differentiability of the state operator with respect to the control. In addition to regularity conditions 3, 4, 5, 6 and 7 of Definition 1, assume that f is defined for controls in V, an open neighborhood of U. Let $u(\cdot) \in U$ be a control whose corresponding solution $x(u, \cdot)$ is defined on $[t_0, t_f]$. Then, for every bounded measurable $\Delta u(\cdot)$ and every $t \in [t_0, t_f]$, it must be that $x(u, \cdot)$ is G-differentiable with respect to u. That is, the derivative*

$$\delta x(u, \Delta u) \equiv \lim_{\varepsilon \to 0} \frac{x(t, u + \varepsilon \Delta u) - x(t, u)}{\varepsilon}$$

exists for every Δu. In particular

$$\delta x(u, \Delta u) = \int_0^t M(t, s) D_u f(x(u, s), u(s), s) \cdot \Delta u(s) \, ds \tag{3.10}$$

where $D_u f$ denotes the Jacobian matrix of partial derivatives $\frac{\partial f_i}{\partial u_j}$, and M is the matrix fundamental solution for the linearized problem

$$\dot{v}(t) = D_x f(x(u, t), u(t), t) \cdot v(t)$$

Proof: We follow the proof by Bressan and Piccoli (2007). Let $z(t)$ be the right-hand side of (3.10). By Theorem 3.7 in Bressan and Piccoli (2007), z is a solution to

$$\dot{z}(t) = A(t) z(t) + D_u f(x(u, t), u(t), t) \cdot \Delta u(t), \quad z(t_0) = 0 \tag{3.11}$$

where $A(t) = D_x f(x(u,t), u(t), t)$. If we define

$$\begin{aligned} x_\varepsilon(t) &= x(u+\varepsilon\Delta u, t) \\ y_\varepsilon(t) &= x(u,t) + \varepsilon z(t) \end{aligned}$$

we need to show, in order to prove the theorem, that

$$\lim_{\varepsilon\to 0}\left|\frac{x_\varepsilon(t) - y(t)}{\varepsilon}\right| = 0 \tag{3.12}$$

Note that $x_\varepsilon(t)$ is the fixed point of the map $w \to \Phi(u+\varepsilon\Delta u, w)$ defined in (3.8), which is contractive with the norm $\|\cdot\|_*$ defined in (3.9). By the contraction mapping theorem, we have an estimate

$$\frac{1}{\varepsilon}\|x_\varepsilon - y_\varepsilon\|_* \leq \frac{2}{\varepsilon}\|\Phi(u+\varepsilon\Delta u, y_\varepsilon) - y_\varepsilon\|_*$$

Then, to prove (3.12), it suffices to show that

$$\lim_{\varepsilon\to 0}\sup_{t\in[t_0,t_f]}\frac{1}{\varepsilon}\left|x_0 + \int_{t_0}^{t} f(y_\varepsilon(s), u+\varepsilon\Delta u, s)\,ds - y_\varepsilon(t)\right| = 0 \tag{3.13}$$

Let us next define

$$\digamma \equiv \frac{1}{\varepsilon}\left|x_0 + \int_{t_0}^{t} f(y_\varepsilon(s), u+\varepsilon\Delta u, s)\,ds - y_\varepsilon(t)\right| \geq 0$$

We note that

$$\digamma = \frac{1}{\varepsilon}\left|x_0 + \int_{t_0}^{t} f(x(u,s)+\varepsilon z(s), u+\varepsilon\Delta u, s)\,ds - x(u,t) - \varepsilon z(t)\right| \tag{3.14}$$

Hence

$$\mathcal{F} = \frac{1}{\varepsilon} \left| \mathcal{F}_1 + \mathcal{F}_2 + \mathcal{F}_3 + \mathcal{F}_4 \right|$$

where

$$\mathcal{F}_1 = x_0 + \int_{t_0}^{t} f\left(x\left(u,s\right), u\left(s\right), s\right) ds + \int_{t_0}^{t} D_x f\left(x\left(u,s\right), u\left(s\right), s\right) \cdot \varepsilon z\left(s\right) ds$$

$$\mathcal{F}_2 = \int_{t_0}^{t} D_u f\left(x\left(u,s\right), u\left(s\right), s\right) \cdot \varepsilon \Delta u\left(s\right) ds - x\left(t\right) - \varepsilon z\left(t\right)$$

$$\begin{aligned} \mathcal{F}_3 &= \int_{t_0}^{t} \int_{0}^{1} \left[D_x f\left(x\left(u,s\right) + \sigma \varepsilon z\left(s\right), u\left(s\right) + \sigma \varepsilon \Delta u\left(s\right), s\right) \right. \\ &\qquad \left. - D_x f\left(x\left(u,s\right), u\left(s\right), s\right) \right] \cdot \varepsilon z\left(s\right) d\sigma ds \end{aligned}$$

$$\begin{aligned} \mathcal{F}_4 &= \int_{t_0}^{t} \int_{0}^{1} \left[D_u f\left(x\left(u,s\right) + \sigma \varepsilon z\left(s\right), u\left(s\right) + \sigma \varepsilon \Delta u\left(s\right), s\right) \right. \\ &\qquad \left. - D_u f\left(x\left(u,s\right), u\left(s\right), s\right) \right] \cdot \varepsilon z\left(s\right) d\sigma ds \end{aligned}$$

We have

$$x\left(t\right) = x_0 + \int_{t_0}^{t} f\left(x\left(u,s\right), u\left(s\right), s\right) ds$$

and from (3.11)

$$z\left(t\right) = \int_{t_0}^{t} \left[D_x f\left(x\left(u,s\right), u\left(s\right), s\right) z\left(s\right) + D_u f\left(x\left(u,s\right), u\left(s\right), s\right) \right] \cdot \Delta u\left(s\right) ds$$

Hence,

$$\mathcal{F}_1 + \mathcal{F}_2 = 0$$

By inspection

$$F \leq F_5 + F_6 \tag{3.15}$$

where

$$\begin{aligned} F_5 \quad = \quad & \int_{t_0}^{t_f} \int_0^1 \| D_x f \left(x \left(u, s \right) + \sigma \varepsilon z \left(s \right), u \left(s \right) + \sigma \varepsilon \Delta u \left(s \right), s \right) \\ & \qquad - D_x f \left(x \left(u, s \right), u \left(s \right), s \right) \| \cdot \left| z \left(s \right) \right| d\sigma ds \end{aligned}$$

$$\begin{aligned} F_6 = \int_{t_0}^{t_f} \int_0^1 & \| D_u f \left(x \left(u, s \right) + \sigma \varepsilon z \left(s \right), u \left(s \right) + \sigma \varepsilon \Delta u \left(s \right), s \right) \\ & \qquad - D_u f \left(x \left(u, s \right), u \left(s \right), s \right) \| \cdot \left| \Delta u \left(s \right) \right| d\sigma ds \end{aligned}$$

By the Lebesgue dominated convergence theorem, the right-hand side of (3.15) converges to zero, so that

$$0 \leq \lim_{\varepsilon \to 0} \sup_{t \in [t_0, t_f]} \frac{1}{\varepsilon} \leq 0$$

thereby enforcing (3.13). This completes the proof.

We have proved that the state operator $x(u, t)$ exists, is unique and is G-differentiable on $[t_0, t_f]$ when 3, 4, 5, 6 and 7 of Definition 1 are combined with additional mild regularity conditions.

3.1.3 Optimality Conditions

To develop necessary conditions for solutions of $DVI(F, f, U, \Gamma, x_0)$, we note that (3.6) may be restated as the following optimal control problem

$$\min \gamma^T \Gamma \left[x\left(t_f\right), t_f\right] + \int_{t_0}^{t_f} \left[F\left(x^*, u^*, t\right)\right]^T u dt \tag{3.16}$$

subject to

$$\frac{dx}{dt} = f\left(x, u, t\right) \tag{3.17}$$

$$u \in U \tag{3.18}$$

$$x\left(t_0\right) = x_0 \tag{3.19}$$

where $x^* = x(u^*, t)$ is the optimal state vector and $\gamma \in \Re^r$ is the vector of dual variables for the terminal constraints $\Gamma\left[x\left(t_f\right), t_f\right] = 0$. Care must be taken to correctly understand the optimal control problem (3.16), (3.17), (3.18), and (3.19). In particular, this optimal control problem is a mathematical abstraction and of no use for computation, since its criterion depends on knowledge of the variational inequality solution u^*. Nonetheless, it is valuable for deriving necessary conditions for $DVI(F, f, U, \Gamma, x_0)$. In particular, the necessary conditions for $DVI(F, f, U, \Gamma, x_0)$ follow directly from the minimum principle and related necessary conditions for (3.16), (3.17), (3.18), and (3.19).

In what follows we will need the Hamiltonian for (3.16), (3.17), (3.18), and (3.19), namely

$$H\left(x, u, \lambda, t\right) = \left[F\left(x^*, u^*, t\right)\right]^T u + \lambda^T f\left(x, u, t\right) \tag{3.20}$$

where $\lambda(t)$ is the adjoint vector that solves the adjoint equations and satisfies the transversality conditions for the given state variables and controls. Note that for a given state vector and a given instant in time (3.20) is convex in u when $DVI(F, f, U, \Gamma, x_0)$ is regular in the sense of Definition 1. It is now a relatively easy matter to derive the necessary conditions stated in the following theorem:

Theorem 4. *Necessary conditions for $DVI(F, f, U, \Gamma, x_0)$. When regularity in the sense of Definition 1 holds, necessary conditions for $u^* \in U$ to be a solution of $DVI(F, f, U, \Gamma, x_0)$ are:*

1. *the finite dimensional variational inequality principle:*

$$\left[F(x^*, u^*, t) + \nabla_u (\lambda^*)^T f(x^*, u^*, t)\right]^T (u - u^*) \geq 0 \quad \forall u \in U; \tag{3.21}$$

2. *the state dynamics*

$$\frac{dx^*}{dt} = f(x^*, u^*, t) \tag{3.22}$$

$$x^*(t_0) = x_0; \tag{3.23}$$

3. *the adjoint dynamics*

$$(-1)\frac{d\lambda^*}{dt} = \nabla_x (\lambda^*)^T f(x^*, u^*, t) \tag{3.24}$$

$$\lambda^*(t_f) = \gamma^T \frac{\partial \Gamma[x^*(t_f), t_f]}{\partial x^*(t_f)} \tag{3.25}$$

Proof: The Pontryagin minimum principle is a necessary condition for optimal control problem (3.16) through (3.19). Hence

$$u^* = \arg\left\{\min_{u\in U} H\left(x^*, u, \lambda^*, t\right)\right\} \tag{3.26}$$

for each $t \in [t_0, t_f]$, which in turn, by virtue of regularity, is equivalent to

$$\left[\nabla_u H\left(x^*, u^*, \lambda^*, t\right)\right]^T \left(u - u^*\right) \geq 0 \quad u, u^* \in U$$

Note that

$$\nabla_u H\left(x, u, \lambda, t\right) = F\left(x^*, u^*, t\right) + \nabla_u \lambda^T f\left(x, u, t\right)$$

where for given u

$$\begin{aligned}
\lambda\left(u, t\right) &= \arg\left\{(-1)\frac{d\lambda}{dt} = \nabla_x H\left(x, u, \lambda, t\right),\ \lambda\left(t_f\right) = \gamma^T \frac{\partial\Gamma\left[x\left(t_f\right), t_f\right]}{\partial x\left(t_f\right)}\right\} \\
&= \arg\left\{(-1)\frac{d\lambda}{dt} = \nabla_x \left[F\left(x^*, u^*, t\right)\right]^T u + \nabla_x \lambda^T f\left(x, u, t\right),\right. \\
&\qquad\qquad \left.\lambda\left(t_f\right) = \gamma^T \frac{\partial\Gamma\left[x\left(t_f\right), t_f\right]}{\partial x\left(t_f\right)}\right\} \\
&= \arg\left\{(-1)\frac{d\lambda}{dt} = \nabla_x \lambda^T f\left(x, u, t\right),\ \lambda\left(t_f\right) = \gamma^T \frac{\partial\Gamma\left[x\left(t_f\right), t_f\right]}{\partial x\left(t_f\right)}\right\}
\end{aligned} \tag{3.27}$$

since $x(u, t)$ is completely determined by knowledge of the controls u. The theorem follows immediately.

Note that item 1 of this theorem refers to a *finite* dimensional variational inequality because the Pontryagin minimum principle (3.26) from which it is derived minimizes the associated Hamiltonian for each instant of time and corresponding state and adjoint variables.

3.1.4 DVI Existence

We are ready to state and prove the following existence result:

Theorem 5. *Existence of a solution to $DVI(F, f, U, \Gamma, x_0)$. When regularity in the sense of Definition 1 holds and additionally U is compact, $DVI(F, f, U, \Gamma, x_0)$ has a solution.*

Proof: By the assumption of regularity $x(u, t)$ is well defined and continuous. So $F(x(u, t), u, t)$ is continuous in u. Also, we know U is convex by regularity and compact by the given. Consequently, by Theorem 2 in Browder (1968), $DVI(F, f, U, \Gamma, x_0)$ has a solution.

3.2 Modeling Competitive Supply Chains

In a competitive supply chain, there might be horizontal competition among producers or retailers and vertical competition between producer and retailer. We assume that a single homogenous good is manufactured by producers for sale by retailers, there will be no direct local consumption of production and this model is based on dynamic oligopolistic spatial competition. Furthermore, this dissertation considers supply disruption as a sort of supply uncertainty. To this end, supply uncertainty can be represented in a supply chain model by using probability parameters reflected on primary cost functions of each agent involved in the supply chain. That is, suppliers' transportation costs and manufacturers production functions are affected by supply-side disruption respectively. Consequently these have an overall influence on supply chain agents such as suppliers, producers, and retailers. In making a model of this condition, dynamic oligopolistic spatial competition is basically assumed.

3.2.1 Some Background and Notation

The oligopolistic firms of interest, embedded in a network economy, are in oligopolistic competition according to dynamics that describe the trajectories of inventories/backorders and correspond to flow conservation for each firm at each node of the network of interest. The oligopolistic firms, acting as shippers, compete as price takers in the market for physical distribution services which is perfectly competitive due to its involvement in numerous other markets of the network economy. The time scale we consider is neither short nor long, but rather of sufficient length to allow output and shipping pattern adjustments but not long enough for firms to re-locate or enter or leave the network economy.

We employ much of the notation introduced in previous sections. Because there are some key differences between the dynamic oligopolistic competition to now be studied and other type of competition like perfect competition and monopoly explored previously, we choose to give an exhaustive list of the notation to be employed below, even though that will involve a bit of duplication. In particular, we again let continuous time be denoted by the scalar $t \in \Re^1_+$, initial time by $t_0 \in \Re^1_+$, and final time by $t_f \in \Re^1_{++}$, with $t_0 < t_f$ so that $t \in [t_0, t_f] \subset \Re^1_+$. There are several sets important to articulating a model of oligopolistic competition on a network; these are as follow: $\mathcal{F}$ for firms, $\mathcal{A}$ for directed arcs, $\mathcal{N}$ for nodes and $\mathcal{W}$ for origin-destination (OD) pairs. Subsets of these sets are formed as is meaningful by using the subscript f for a specific firm, i for a specific node, and ij for a specific OD pair (i, j).

Each firm $f \in \mathcal{F}$ controls production output rates q^f, allocation of output to meet demand c^f and shipping pattern s^f. Inventories I^f are state variables determined by the controls. In particular, c^f, q^f and s^f constitute concatenations of the following vectors:

$$c \in \left(L^2\left[t_0, t_f\right]\right)^{|\mathcal{N}| \times |\mathcal{F}|}$$
$$q \in \left(L^2\left[t_0, t_f\right]\right)^{|\mathcal{N}| \times |\mathcal{F}|}$$
$$s \in \left(L^2\left[t_0, t_f\right]\right)^{|\mathcal{W}| \times |\mathcal{F}|}$$

$$I\left(c, q, s\right) : \left(L^2\left[t_0, t_f\right]\right)^{|\mathcal{N}| \times |\mathcal{F}|} \times \left(L^2\left[t_0, t_f\right]\right)^{|\mathcal{N}| \times |\mathcal{F}|} \times \left(L^2\left[t_0, t_f\right]\right)^{|\mathcal{W}| \times |\mathcal{F}|} \longrightarrow \left(\mathcal{H}^1\left[t_0, t_f\right]\right)^{|\mathcal{N}| \times |\mathcal{F}|}$$

where $L^2\left[t_0, t_f\right]$ is the space of square-integrable functions and $\mathcal{H}^1\left[t_0, t_f\right]$ is a Sobolev space for the real interval $\left[t_0, t_f\right] \in \Re^1_+$.

3.2.2 The General Firm's Objective and Constraints in optimization problems

Each firm has the objective of maximizing net profit expressed as revenue less cost and can be thought of as an operator acting on allocations of output to meet demands, production rates and shipment patterns. For each firm $f \in \mathcal{F}_P$, net profit can be represented as follows:

$$\Phi_f(c^f, q^f, s^f; c^{-f}, q^{-f}) = \int_{t_0}^{t_f} e^{-\rho t} \left\{ \sum_{i \in \mathcal{N}} \pi_i \left(\sum_{g \in \mathcal{F}_P} c_i^g, t \right) c_i^f - \sum_{i \in \mathcal{N}_f} V_i^f(q^f, t) - \sum_{w \in \mathcal{W}_f} r_w\left(t\right) s_w^f - \sum_{i \in \mathcal{N}} \psi_i^f(I_i^f, t) \right\} dt \tag{3.28}$$

where $\rho \in \Re^1_{++}$ is a constant nominal rate of discount, $r_w \in \Re^1_{++}$ is the freight rate (tariff) charged per unit of flow s_w for OD pair $w \in \mathcal{W}_f$, ψ_i^f is firm f's inventory cost at node i, and I_i^f is the inventory/backorder of firm f at node i. In (3.28), c_i^f is the allocation of the output of firm $f \in \mathcal{F}_P$ at node $i \in \mathcal{N}$ to consumption at that node. Our formulation is in terms of flows so we employ the inverse demand functions $\pi_i\left(c_i, t\right)$ where

$$c_i = \sum_{g \in \mathcal{F}} c_i^g$$

is the total allocation of output to consumption for node i. Furthermore q_i^f is the output of firm $f \in \mathcal{F}_P$ at node $i \in \mathcal{N}$. Also $V_i^f(q,t)$ is the variable cost of production for firm $f \in \mathcal{F}_P$ at node $i \in \mathcal{N}$. Note that $\theta_f(c^f, q^f, s^f; c^{-f}, q^{-f})$ is a functional that is completely determined by the controls c^f, q^f and s^f when non-own allocations to consumption and non-own production rates

$$\begin{aligned} c^{-f} &\equiv \left(c^{f'} \,:\, f' \neq f \right) \\ q^{-f} &\equiv \left(q^{f'} \,:\, f' \neq f \right) \end{aligned}$$

are taken as exogenous data by firm f. The first term of the functional $\theta_f(c^f, q^f, s^f; c^{-f}, q^{-f})$ in expression (3.28) is the firm's revenue; the second term is the firm's cost of production; the third term is the firm's shipping costs; and the last term is the firms's inventory or holding cost.

We also impose the terminal time inventory constraints

$$I_i^f\left(t_f\right) \geq \tilde{K}_i^f \quad \forall\, f \in \mathcal{F}_P, i \in \mathcal{N}_f \tag{3.29}$$

where the $\tilde{K}_i^f \in \Re^1_{++}$ are exogenous. All consumption, production and shipping variables are non-negative and bounded from above; that is

$$C^f \geq c^f \geq 0 \tag{3.30}$$

$$Q^f \geq q^f \geq 0 \tag{3.31}$$

$$S^f \geq s^f \geq 0 \tag{3.32}$$

where

$$C^f \in \Re_{++}^{|\mathcal{F}|}$$

$$Q^f \in \Re_{++}^{|\mathcal{F}|}$$

$$S^f \in \Re_{++}^{|\mathcal{W}_f|}$$

Constraints (3.30), (3.31) and (3.32) are recognized as pure control constraints, while (3.29) are terminal conditions for the state space variables. Naturally,

$$\Omega_f = \left\{ \left(c^f, q^f, s^f \right) : (3.30), (3.31), (3.32) \right\}$$

is the set of feasible controls.

Firm f solves an optimal control problem to determine its production q^f, allocation of production to meet demand c^f, and shipping pattern s^f – thereby also determining inventory I^f via dynamics we articulate momentarily – by maximizing its profit functional $\Phi_f(c^f, q^f, s^f; c^{-f}, q^{-f})$ subject to inventory dynamics expressed as flow balance equations and pertinent production and inventory constraints. The inventory dynamics for firm $f \in \mathcal{F}_P$, expressing simple flow conservation, obey

$$\frac{dI_i^f}{dt} = q_i^f + \sum_{w \in \mathcal{W}_i^d} s_w^f - \sum_{w \in \mathcal{W}_i^o} s_w^f - c_i^f \quad \forall\, i \in \mathcal{N}_f \tag{3.33}$$

$$I_i^f(t_0) = K_i^f \quad \forall\, i \in \mathcal{N}_f \tag{3.34}$$

$$I_i^f(t_f) \geq \tilde{K}_i^f \quad \forall\, i \in \mathcal{N}_f \tag{3.35}$$

where $K_i^f \in \Re^1_{++}$ and $\tilde{K}_i^f \in \Re^1_{+}$ are exogenous, while $\mathcal{W}_i^d$ is the set of OD pairs with destination node i and $\mathcal{W}_i^o$ is the set of OD pairs with origin node i. Note that the transportation time for the flow of finished goods is not captured explicitly in the inventory dynamics, however it is accounted for implicitly in the freight rate (tariff) charged per unit of flow. Further, in addition to the terminal time inventory (state) constraints (3.35), the model is general enough to handle inventory constraints over the entire planning horizon $[t_0, t_f]$. For instance, non-negativity of the inventory (state) variables could be imposed to restrict firms from taking backorders.

Consequently,

$$I(c, q, s) = \arg\left\{ \frac{dI_i^f}{dt} = q_i^f + \sum_{w \in \mathcal{W}_i^d} s_w^f - \sum_{w \in \mathcal{W}_i^o} s_w^f - c_i^f, \right.$$
$$\left. I_i^f(t_0) = K_i^f, I_i^f(t_f) \geq \tilde{K}_i^f \quad \forall\, f \in \mathcal{F}_P, i \in \mathcal{N}_f \right\}$$

where we implicitly assume that the dynamics have solutions for all feasible controls.

With the preceding development, we note that firm f's problem is: with the c^{-f} and q^{-f} as exogenous inputs, compute c^f, q^f and s^f (thereby finding I^f) in order to solve the following extremal problem:

$$\left.\begin{array}{ll} \max & \Phi_f(c^f, q^f, s^f; c^{-f}, q^{-f}) \\ \text{subject to} & \left(c^f, q^f, s^f\right) \in \Omega_f \end{array}\right\} \forall f \in \mathcal{F}_P \tag{3.36}$$

where

$$\Omega_f = \left\{ \left(c^f, q^f, s^f \right) : (3.29), (3.30), (3.31), (3.32) \text{ hold} \right\}$$

also for all $f \in \mathcal{F}_P$. That is, each firm is a Nash agent that knows and employs the current instantaneous values of the decision variables of other firms to make its own non-cooperative decisions. As such, (3.36) is a differential Nash game.

3.2.2.1 Inverse Demands

Agreements are in place that prevent producers from selling directly to consumers, so retailers and consumers will face subtly distinct inverse demand functions for the finished good of interest. To understand this, let $\mathcal{F}_P$ be the set of firm with producers at node $i \in \mathcal{N}$ and $\mathcal{F}_R$ the set of retailers, potentially occupying every network node. Also let w_j refer to the wholesale price paid by retailers $r \in \mathcal{F}_R$ for the producers' output in market $j \in \mathcal{N}$. We note that the following identity holds:

$$D_j(w_j + \alpha_j w_j) = \sum_{r \in \mathcal{F}_R} c_j^r \tag{3.37}$$

where $D_j(.)$ is the market demand function at node $j \in \mathcal{N}$ for the finished good and c_j^r is the consumption at node $j \in \mathcal{N}$ of the goods flow from retailer $r \in \mathcal{F}_R$; furthermore, $\alpha_j \in \Re^1_{++}$ is the retailers' margin at node $j \in \mathcal{N}$. It can be assumed that each such demand function has an inverse denoted by $\Theta_j(.,.)$ such that

$$w_j = \Theta_j(\sum_{r \in \mathcal{F}_R} c_j^r, \alpha_j) \quad \forall j \in \mathcal{N} \tag{3.38}$$

which is called the wholesalers' inverse demand. Next, the consumers' inverse demand for the finished good at node $j \in \mathcal{N}$ is denoted by $\Psi_j(.)$. While the inverse is obtained from (3.37), it is not identical to the wholesalers' inverse demand (3.38).

In particular, the consumers' inverse demand takes the form

$$w_j + \alpha_j w_j = v_j = \Psi_j(\sum_{r \in \mathcal{F}_R} c_j^r) \quad \forall j \in \mathcal{N} \tag{3.39}$$

and $w_j + \alpha_j w_j$ is the retail price paid by consumers for the finished good at node $j \in \mathcal{N}$. We assume that the inverse demand $\Psi_j(.)$ exists for every retail market $j \in \mathcal{N}$. Clearly an alternative form of (3.39) is

$$w_j = \frac{1}{1+\alpha_j}\Psi_j(\sum_{r \in \mathcal{F}_R} c_j^r) \quad \forall r \in \mathcal{F}_R,\ j \in \mathcal{N} \tag{3.40}$$

Expressions (3.38) and (3.40) make very clear that

$$\Theta_j(\sum_{r \in \mathcal{F}_R} c_j^r, \alpha_j) = \frac{1}{1+\alpha_j}\Psi_j(\sum_{r \in \mathcal{F}_R} c_j^r) \quad \forall j \in \mathcal{N} \tag{3.41}$$

3.2.3 Reflecting supply chain uncertainty

Suppliers' transportation costs and manufacturers' production functions are negatively affected due to supply uncertainty. Please note it is outside the scope of this dissertation to consider retailers' uncertainty because this type of uncertainty has been referred to as demand uncertainty. For each producer of firm $f \in \mathcal{F}_P$ at node i, probability parameters ξ_i and η_i are applied to the production and shipping cost functions with the corresponding cumulative distribution function denoted by $\digamma_i(\xi_i)$ and $\digamma_i(\eta_i)$, respectively. Thus, the expected production, input flow cost and variable transportation cost functions for a producer of firm $f \in \mathcal{F}_P$ at node

i are as follows:

$$E\left[\widetilde{F}_i^f(h_i^f)\right] = \int_{\xi_i} \widetilde{F}_i^f(h_i, \xi_i) d\digamma_i(\xi_i)$$
$$E\left[\widetilde{C}_i^f(u_i^f)\right] = \int_{\eta_i} \widetilde{C}_i^f(u_i, \eta_i) d\digamma_i(\eta_i)$$
$$E\left[\widetilde{r}_{ij}\right] = \int_{\eta_i} \widetilde{r}_{ij}(q_{ij}, \eta_i) d\digamma_i(\eta_i)$$

where $\widetilde{F}, \widetilde{C}, \widetilde{r}$ denote the random production function, random input flow cost function, and random transportation rate, respectively. For supplier s at node h, we apply probability parameter δ_h, reflecting the disruption effect on the variable transportation cost with the corresponding cumulative distribution function given by $\digamma_h(\delta_h)$. Thus, the expected transportation cost can be represented as follows:

$$E\left[\widetilde{V}_h^s(u_h)\right] = \int_{\delta_i} \widetilde{V}(u_h, \delta_h) d\digamma_h(\delta_h)$$

In addition, supply disruption risk minimization can be considered with a multi-objective optimization problem produced. Using variance to measure risk (Tomlin 2006, Silberberg and Suen 2000), a nonnegative weight, β_i, is applied to the variance of the cost functions for each producer to account for their attitudes toward disruption risks. Smaller and larger weights correspond to risk-taking and risk-averse producers, respectively.

3.2.4 Producers' Extremal Problem

3.2.4.1 Deterministic

To facilitate the story begun above, we employ the following state dynamics for producers:

$$\frac{dI_i^f}{dt} = F_i^f(h_i^f) + \sum_{j\in\mathcal{N}} s_{ji}^f - \sum_{j\in\mathcal{N}} s_{ij}^f - \sum_{r\in\mathcal{F}_R}\sum_{j\in\mathcal{N}} q_{ij}^{fr} \quad \forall f \in \mathcal{F}_P, \forall r \in \mathcal{F}_R, i \in \mathcal{N} \tag{3.42}$$

where $F_i^f(.)$ is the single factor production function for a producer of the firm $f \in \mathcal{F}_P$ at node i and $\mathcal{F}_P$ is the set of firms producing the homogeneous product of interest. In addition, N is the final echelon (stage) of supplying the single factor to producers at nodes $i \in \mathcal{N}$, while h_i^f is the flow of the single factor to producer of the firm $f \in \mathcal{F}_P$ at node $i \in \mathcal{N}$. The set of retailers we consider is $\mathcal{F}_R$. Additionally, q_{ij}^{fr} denotes the sales by producer of the firm $f \in \mathcal{F}_P$ from inventory or new production at node $i \in \mathcal{N}$ to retailer $r \in \mathcal{F}_R$ at node $j \in \mathcal{N}$. Although we refer to the input to production as a single factor, it is in fact a precisely constituted aggregate of several inputs, constructed from individual factors added at each level of the supply chain through which it passes. Moreover, the aggregate input factor flow u_N is disaggregated into individual flows h_i^f used by each producer of the firm $f \in \mathcal{F}_P$ at each node $i \in \mathcal{N}$ where

$$u_N = \sum_{f\in\mathcal{F}_P}\sum_{i\in\mathcal{N}} h_i^f \tag{3.43}$$

Naturally we employ the notation

$$h^f = (h_i^f : i \in \mathcal{N}) \tag{3.44}$$

to describe the vector of factor allocations controlled by producer of the firm $f \in \mathcal{F}_P$ at node i. We assume there are supply contracts in place between firm $f \in \mathcal{F}_P$ and the supply chain agent that have the effect of establishing a cost function $C_i^f(h_i^f, t)$ for the instantaneous cost to acquire the input flow h_i^f at each node $i \in \mathcal{N}$. Additionally upper and lower bounds are established by those contracts for factor flow to each producer:

$$A_f \leq \sum_{i \in \mathcal{N}} h_i^f \leq B_f \quad \forall f \in \mathcal{F}_P \tag{3.45}$$

where

$$[A_f, B_f] \subset \Re^1_{++} \quad \forall f \in \mathcal{F}_P \tag{3.46}$$

In light of this development, we may express the extremal problem for each firm $f \in \mathcal{F}_P$ as follows:

$$\begin{aligned}\max J_f^P(q^f, s^f, h^f; c) = \int_{t_0}^{t_f} e^{-\rho t} \Bigg\{ & \sum_{r \in \mathcal{F}_R} \sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{N}} \frac{1}{1+\alpha_j^r} \Psi_j (\sum_{g \in \mathcal{F}_P} c_j^g) q_{ij}^{fr} \\ & - \sum_{i \in \mathcal{N}} C_i^f(h_{iN}^f, t) - \sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{N}} r_{ij} \left(s_{ij}^f + \sum_{r \in \mathcal{F}_R} q_{ij}^{fr} \right) \\ & - \sum_{i \in \mathcal{N}} \psi_i^f (I_i^f, t) \Bigg\} dt \end{aligned} \tag{3.47}$$

subject to

$$\frac{dI_i^f}{dt} = F_i^f(h_i^f) + \sum_{j \in \mathcal{N}} s_{ji}^f - \sum_{j \in \mathcal{N}} s_{ij}^f - \sum_{r \in \mathcal{F}_R} \sum_{j \in \mathcal{N}} q_{ij}^{fr} \quad \forall f \in \mathcal{F}_P, \forall r \in \mathcal{F}_R, i \in \mathcal{N} \tag{3.48}$$

$$I_i^f\,(t_0) = K_i^f \quad \forall i \in \mathcal{N} \tag{3.49}$$

$$I_i^f\,(t_f) \geq \tilde{K}_i^f \quad \forall i \in \mathcal{N} \tag{3.50}$$

$$0 \leq q_{ij}^{fr} \leq \widetilde{q}_{ij}^{fr} \quad i\,,j \in \mathcal{N},\, r \in \mathcal{F}_r \tag{3.51}$$

$$0 \leq s_{ij}^{f} \leq \widetilde{s}_{ij}^{f} \quad i\,,j \in \mathcal{N} \tag{3.52}$$

$$A_f \leq \sum_{i \in \mathcal{N}} h_i^f \leq B_f \quad \forall f \in \mathcal{F}_P \tag{3.53}$$

$$u_N = \sum_{g \in \mathcal{F}_P} \sum_{i \in \mathcal{N}} h_{Ni}^g \tag{3.54}$$

where q^f, s^f, and h^f are vectors, that are respectively concatenations of the full vectors of output, shipping and input factor flows and are under the control of firm $f \in \mathcal{F}_P$. We recall that shipments to retailers are free onboard and thus paid by producers. Furthermore, r_{ij} is the freight rate for origin-destination (OD) pair (i, j) while inventory cost for producer of the firm $f \in \mathcal{F}_P$ at node $i \in \mathcal{N}$ is $\psi_i^f(.,.)$; also K_i^f is the initial inventory for producer of the firm $f \in \mathcal{F}_P$ at node $i \in \mathcal{N}$. Moreover, (3.50) is the terminal time inventory constraint; (3.51) and (3.52) are constraints expressing upper bounds on outputs and shipments, where $\tilde{K}_i^f$, $\widetilde{q}_{ij}^{fr}$ and $\widetilde{s}_{ij}^{f}$ are exogenous fixed parameters for all $f \in \mathcal{F}_P$ and $i, j \in \mathcal{N}$. Constraints (3.53) form the aforementioned upper and lower bounds on aggregate factor flows to producers. Note that

$$q_{ij}^{f} = \left(q_{ij}^{fr} : r \in \mathcal{F}_R\right) \tag{3.55}$$

$$q^{f} = (q_{ij}^{f} : i, j \in \mathcal{N}) \tag{3.56}$$

$$q^{-f} = (q^{g} : g \in \mathcal{F}_P \backslash f) \tag{3.57}$$

$$s^{f} = \left(s_{ij}^{f} : i, j \in \mathcal{N}\right) \tag{3.58}$$

$$h^{f} = (h_{i}^{f} : i \in \mathcal{N}) \tag{3.59}$$

$$h = \left(h^{f} : f \in \mathcal{F}_P\right) \tag{3.60}$$

$$h^{-f} = (h^{g} : g \in \mathcal{F}_P \backslash f) \tag{3.61}$$

$$c^{r} = \left(c_{j}^{r} : j \in \mathcal{N}\right) \tag{3.62}$$

$$c = (c^{r} : r \in \mathcal{F}_R) \tag{3.63}$$

Note that the constraints of this extremal problem depend on the vector h^{-f} and on the scalar unknown u_N, both of which are determined exogenously by the producers and the supply chain manager, respectively.

3.2.4.2 With disruptions

To facilitate the situation described above, the following state dynamics for producers are employed:

$$\frac{dI_i^f}{dt} = E\left[\widetilde{F}_i^f\left(\sum_{s\in\mathcal{F}_S}\sum_{h=1}^{g} u_{hi}^{sf}\right)\right] + \sum_{j\in\mathcal{N}} s_{ji}^{f} - \sum_{j\in\mathcal{N}} s_{ij}^{f} - \sum_{r\in\mathcal{F}_R}\sum_{j\in\mathcal{N}} q_{ij}^{fr} \quad \forall\, f \in \mathcal{F}_P, \forall\, r \in \mathcal{F}_R, i \in \mathcal{N} \tag{3.64}$$

where $E\left[\widetilde{F}_i^f(.)\right]$ is the expected single factor production function for producer of firm $f \in \mathcal{F}_P$ at node i and $\mathcal{F}_P$ is the set of firms producing the homogeneous product of interest. Note that a producer is a firm's production agent locates in a specific node i. Furthermore, N represents the final echelon (stage) of supplying the single factor to producers at node $i \in \mathcal{N}$, while h_i^f is the flow of the single factor to producer of firm $f \in \mathcal{F}_P$ at node $i \in \mathcal{N}$. The set of retailers we consider is $\mathcal{F}_R$. Additionally, q_{ij}^{fr} denotes the sales by producer of firm $f \in \mathcal{F}_P$ at node i from inventory or new production at node $i \in \mathcal{N}$ to retailer $r \in \mathcal{F}_R$ at node $j \in \mathcal{N}$. Moreover, the input factor flow u_{hi}^{sf} is aggregated into the input flow u_i^f used by each producer at each node $i \in \mathcal{N}$ of firm $f \in \mathcal{F}_P$. This instance is represented by

$$u_i^f = \sum_{s \in \mathcal{F}_S} \sum_{h=1}^{g} u_{hi}^{sf} \tag{3.65}$$

$$u^f = (u_i^f : i \in \mathcal{N}) \tag{3.66}$$

to describe the vector of factor allocations controlled by firm $f \in \mathcal{F}_P$ with producers at node $i \in \mathcal{N}$. Supply contracts between the producer at node i of firm $f \in \mathcal{F}_P$ and the supply chain agent are assumed, having the effect of establishing an expected cost function $E\left[\widetilde{C}_i^f(u_i^f, t)\right]$ for the instantaneous cost to acquire the input flow u_i^f at each node $i \in \mathcal{N}$. Additionally, upper and lower bounds are established by those contracts for factor flow to each producer at node i of firm $f \in \mathcal{F}_P$:

$$A_f \leq \sum_{i \in \mathcal{N}} u_i^f \leq B_f \quad \forall f \in \mathcal{F}_P \tag{3.67}$$

where

$$[A_f, B_f] \subset \Re^1_{++} \quad \forall f \in \mathcal{F}_P \tag{3.68}$$

In light of this development, the extremal problem for each firm $f \in \mathcal{F}_P$ with producers at node $i \in \mathcal{N}$ can be expressed as follows:

$$\max J_f^P(q^f, s^f, h^f; c) = \int_{t_0}^{t_f} e^{-\rho t} \quad \Bigg\{ \sum_{r\in\mathcal{F}_R} \sum_{i\in\mathcal{N}} \sum_{j\in\mathcal{N}} \frac{1}{1+\alpha_j^r} \Psi_j(\sum_{g\in\mathcal{F}_P} c_j^g) q_{ij}^{fr}$$
$$- \sum_{i\in\mathcal{N}} E\left[\widetilde{C}_i^f(u_i^f)\right] - \sum_{i\in\mathcal{N}} \sum_{j\in\mathcal{N}} E\left[\widetilde{r}_{ij}\right] \left(s_{ij}^f + \sum_{r\in\mathcal{F}_R} q_{ij}^{fr} \right)$$
$$- \sum_{i\in\mathcal{N}} \psi_i^f(I_i^f, t) - \sum_{i\in\mathcal{N}} \beta_i^f \Bigg[Var(\widetilde{F}_i^f(\sum_{s\in\mathcal{F}_S} \sum_{h=1}^{g} u_{hi}^{sf})) \tag{3.69}$$
$$+ Var(\widetilde{C}_i^f(u_i^f) + \sum_{j\in\mathcal{N}} Var(\widetilde{r}_{ij}) \left(s_{ij}^f + \sum_{r\in\mathcal{F}_R} q_{ij}^{fr} \right) \Bigg] \Bigg\} dt \tag{3.70}$$

subject to

$$\frac{dI_i^f}{dt} = E\left[\widetilde{F}_i^f(\sum_{s\in\mathcal{F}_S} \sum_{h=1}^{g} u_{hi}^{sf}) \right] + \sum_{j\in\mathcal{N}} s_{ji}^f - \sum_{j\in\mathcal{N}} s_{ij}^f - \sum_{r\in\mathcal{F}_R} \sum_{j\in\mathcal{N}} q_{ij}^{fr} \tag{3.71}$$
$$\forall\, f \in \mathcal{F}_P, \forall\, r \in \mathcal{F}_R, i \in \mathcal{N}$$

$$I_i^f(t_0) = K_i^f \quad \forall i \in \mathcal{N} \tag{3.72}$$

$$I_i^f(t_f) \geq \tilde{K}_i^f \quad \forall i \in \mathcal{N} \tag{3.73}$$

$$0 \leq q_{ij}^{fr} \leq \widetilde{q}_{ij}^{fr} \quad i, j \in \mathcal{N},\, r \in \mathcal{F}_r \tag{3.74}$$

$$0 \leq s_{ij}^f \leq \widetilde{s}_{ij}^f \quad i, j \in \mathcal{N} \tag{3.75}$$

$$A_f \leq \sum_{i\in\mathcal{N}} u_i^f \leq B_f \quad \forall f \in \mathcal{F}_P \tag{3.76}$$

where q^f, s^f, and u^f are vectors, that are respectively concatenations of the full vectors of output, shipping and input factor flows and are under the control of firm $f \in \mathcal{F}_P$ with producers at node $i \in \mathcal{N}$. As discussed above, shipments to retailers are free onboard and thus paid by producers of firm $f \in \mathcal{F}_P$. Furthermore, r_{ij} is the freight rate for the origin-destination (OD) pair (i, j), while inventory costs for producer at node $i \in \mathcal{N}$ of firm $f \in \mathcal{F}_P$ is $\psi_i^f(.,.)$ and K_i^f is the initial inventory for producer at node $i \in \mathcal{N}$ of firm $f \in \mathcal{F}_P$. Please note that (3.50) describes the terminal time inventory constraint while (3.51) and (3.52) describe constraints expressing upper bounds on outputs and shipments, where $\tilde{K}_i^f$, $\widetilde{q}_{ij}^{fr}$ and $\widetilde{s}_{ij}^f$ are exogenous fixed parameters for all firms $f \in \mathcal{F}_P$ and $i, j \in \mathcal{N}$. Please note that

$$q_{ij}^f = \left(q_{ij}^{fr} : r \in \mathcal{F}_R\right) \tag{3.77}$$

$$q^f = \left(q_{ij}^f : i, j \in \mathcal{N}\right) \tag{3.78}$$

$$q^{-f} = \left(q^g : g \in \mathcal{F}_P \backslash f\right) \tag{3.79}$$

$$s^f = \left(s_{ij}^f : i, j \in \mathcal{N}\right) \tag{3.80}$$

$$u^f = \left(u_i^f : i \in \mathcal{N}\right) \tag{3.81}$$

$$u = \left(u^f : f \in \mathcal{F}_P\right) \tag{3.82}$$

$$u^{-f} = \left(u^g : g \in \mathcal{F}_P \backslash f\right) \tag{3.83}$$

$$c^r = \left(c_j^r : j \in \mathcal{N}\right) \tag{3.84}$$

$$c = \left(c^r : r \in \mathcal{F}_R\right) \tag{3.85}$$

3.2.5 Retailers' Extremal Problem

Turning our attention to retailers, we stipulate that only retailers may sell finished goods. Since the single homogeneous finished good must be obtained from producers, the pertinent dynamics for retailers are

$$\frac{dR_j^r}{dt} = \sum_{f\in\mathcal{F}_P}\sum_{i\in\mathcal{N}} q_{ij}^{fr} - c_j^r \quad \forall r \in \mathcal{F}_R,\ j \in \mathcal{N} \tag{3.86}$$

where R_j^r denotes the inventory of retailer $r \in \mathcal{F}_R$ at node $j \in \mathcal{N}$, while $\mathcal{F}_R$ is the set of retailers and $\mathcal{N}$ is the set of nodes at which retailer r is located. Note also that c_j^r denotes the consumption activity served by retailer $r \in \mathcal{F}_R$ at node $j \in \mathcal{N}$. Therefore, the extremal problem faced by each retailer $r \in \mathcal{F}_R$ is the following:

$$\max J_r^R(c^r; c^{-r}, q^r)$$

$$= \int_{t_0}^{t_f} e^{-\rho t} \left[\sum_{i\in\mathcal{N}}\sum_{j\in\mathcal{N}} \left(c_j^r - \frac{1}{1+\alpha_j^r}\sum_{i\in\mathcal{N}} q_{ij}^{fr}\right) \Psi_j(\sum_{g\in\mathcal{F}_P} c_j^g) - \sum_{j\in\mathcal{N}} \phi_j^r(R_j^r, t)\right] dt \tag{3.87}$$

subject to

$$\frac{dR_j^r}{dt} = \sum_{f\in\mathcal{F}_P}\sum_{i\in\mathcal{N}} q_{ij}^{fr} - c_j^r \quad \forall r \in \mathcal{F}_R,\ j \in \mathcal{N} \tag{3.88}$$

$$0 \le c_j^r \le \widetilde{c}_j^r \quad \forall\, r \in \mathcal{F}_R, j \in \mathcal{N} \tag{3.89}$$

$$R_j^r(t_0) = Q_j^r \quad \forall\, r \in \mathcal{F}_R, j \in \mathcal{N} \tag{3.90}$$

$$R_j^r(t_f) = \widetilde{Q}_j^r \quad \forall\, r \in \mathcal{F}_R, j \in \mathcal{N} \tag{3.91}$$

In (3.87) $\phi_i^r\left(R_j^r, t\right)$ denotes the inventory costs at node $i \in \mathcal{N}$ for retailer $r \in \mathcal{F}_R$. Additionally Q_j^r is the initial inventory and $\widetilde{Q}_j^r$ is the terminal time inventory, while $\widetilde{c}_j^r$ is the upper bound on consumption, for retailer $r \in \mathcal{F}_R$ at node $i \in \mathcal{N}$. Note that

$$c^r = \left(c_j^r : j \in \mathcal{N}\right) \tag{3.92}$$

$$c^{-r} = (c^g : g \in \mathcal{F}_P - \{r\}) \tag{3.93}$$

$$q_{ij}^r = \left(q_{ij}^{fr} : f \in \mathcal{F}_P\right) \tag{3.94}$$

$$q^r = \left(q_{ij}^r : i, j \in \mathcal{N}\right) \tag{3.95}$$

Note that the constraints of this extremal problem depend on the vector q^r, which is exogenous, since output allocations are determined by the producers.

3.2.6 Suppliers' Extremal Problem

3.2.6.1 Deterministic

Now let us consider a multi-echelon supply chain stretching from unrefined raw materials to factor flows ready for use by producers. We use u_k to denote the flow of the input factor exiting stage k (that is, the flow from stage k to stage $k+1$). If we use S_k to denote the inventory at stage k of the supply chain, we may write

$$\frac{dS_k}{dt} = u_{k-1} - u_k \qquad k = 1, ..., N$$

where it is understood that only the terminal flow u_N is ready for use in producing the homogeneous finished good of present interest to us. Recall that we have already assumed there are contracts in place specifying a fee schedule $C_i^f\left(h_i^f, t\right)$ and guarantied upper and lower bounds for factor flow to each producer $f \in \mathcal{F}_p$ at node $i \in \mathcal{N}$ at time $t \in [t_0, t_f]$, where the allocations u_N^f are controlled by producer $f \in \mathcal{F}_P$. The controls available to the supply chain agent are captured by the vector

$$u = (u_k : k = 1, ..., N) \tag{3.96}$$

As a consequence the single manager who operates all supply chain stages $k \in [1, N]$ seeks to minimize his/her total cost; that is, he/she seeks to solve the following optimal control problem:

$$\min J^S(u) = \int_{t_0}^{t_f} e^{-\rho t} \sum_{k=1}^{N} [V_k(u_k, t) + \varphi_k (S_k, t)] \, dt \tag{3.97}$$

subject to

$$\frac{dS_k}{dt} = u_{k-1} - u_k \quad k = 1, ..., N \tag{3.98}$$

$$S_k(0) = S_k^0 \qquad k = 1, ..., N \tag{3.99}$$

$$u_N = \sum_{f \in \mathcal{F}_P} \sum_{i \in \mathcal{N}} h_i^f \tag{3.100}$$

$$0 \leq u_k \leq U_k \quad k = 0, ..., N \tag{3.101}$$

where $V_k(.,.)$ denotes the variable costs of preparing the stage k flow, $\varphi_k (.,.)$ is the inventory cost function, S_k^0 is the initial inventory, and U_k is the technological upper bound for stage k flow of the supply chain. Note that constraint (3.100) was introduced previously as (3.43). Note that the constraints of this extremal problem depend on the vector h which is determined exogenously since the producers decide factor flows to their production facilities within the bounds set by the contracts they hold with the supply chain manager.

3.2.6.2 With disruptions

In considering a multi-assigned suppliers model, producers must be able to contract with multiple suppliers for input factor materials to lessen supply disruption risk. In that, input factor materials are assumed to be ready for use by producers. Let u_{hi} denote the flow of the input factor from a supplier at node h to a producer at node i, S_h denotes the inventory at node h of the supplier and $S_h(0)$ denotes the initial inventory such that

$$\frac{dS_h^s}{dt} = S_h^s(0) - \sum_{i=1}^{m} u_{hi}^s \qquad h = 1, ..., g$$

To account for supply chain disruption, each supplier is assumed to not make any material; rather, each supplier has an initial inventory and does not possess a production function. Please recall that there are contracts in place specifying a random fee schedule $C_i^f\left(u_i^f, t\right)$ and guarantied upper and lower bounds for the factor flow to each producer of firm $f \in \mathcal{F}_p$ at node $i \in \mathcal{N}$ at time $t \in [t_0, t_f]$, where the allocations u_{hi}^{sf} are controlled by supplier $s \in \mathcal{F}_S$. The controls available to the supply chain agent are captured by the vector

$$u_h^s = (u_{hi} : i = 1, ..., m) \tag{3.102}$$

As a consequence, the supplier s seeks to minimize total cost by solving the following optimal control problem:

$$\min J^S(u) = \int_{t_0}^{t_f} e^{-\rho t} \left[\sum_{h=1}^{g} E\left[\widetilde{V}_h^s(u_h^s, t)\right] + \varphi_h^s\left(S_h^s, t\right) + \beta_h^s \cdot Var\left(\widetilde{V}_h^s(u_h^s, t)\right)\right] dt \tag{3.103}$$

subject to

$$\frac{dS_h^s}{dt} = S_h^s(0) - \sum_{i=1}^{m} u_{hi}^s \qquad h = 1, ..., g \tag{3.104}$$

$$0 \leq u_h \leq U_h \qquad h = 1, ..., g \tag{3.105}$$

where $E\left[\widetilde{V}_h(.,.)\right]$ is the expected variable costs at the node h, $\varphi_h(.,.)$ is the inventory cost function, and U_h is the technological upper bound of input factor flow at node h of the supply chain. Please note that the constraints of this extremal problem depend on the vector u which is determined exogenously since the producers decide factor flows to their production facilities within the bounds set by the contracts they hold with the supply chain manager.

3.3 The Differential Variational Inequality for Competitive Supply Chains

In this section we give an overview of how the relevant differential variational inequality for our combined producer-retailer-supply chain game may be formed.

3.3.1 Maximum Principle for the Producers in the deterministic case

With

$$\begin{pmatrix} c \\ u_N \\ q^{-f} \end{pmatrix} \tag{3.106}$$

as exogenous, each firm $f \in \mathcal{F}_P$ solves

$$\max J_f^P(q^f, s^f, h^f; c) \quad \text{s.t.} \quad \left(q^f, s^f, h^f\right) \in \Lambda_P^f(h^{-f}, u_N) \tag{3.107}$$

where

$$\Lambda_P^f(h^{-f}, u_N) \equiv \left\{ \begin{pmatrix} q^f \\ s^f \\ h^f \end{pmatrix} : (3.48), (3.49), (3.50), (3.51), (3.52), (3.53), (3.54) \text{ hold} \right\} \tag{3.108}$$

The corresponding Hamiltonian is

$$\begin{aligned} &H_P^f(q^f, s^f, h^f, I^f, \lambda^f; c) \\ &= e^{-\rho t} \Bigg\{ \sum_{r \in \mathcal{F}_R} \sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{N}} \frac{1}{1 + \alpha_j^r} \Psi_j \Big(\sum_{g \in \mathcal{F}_P} c_j^g \Big) q_{ij}^{fr} \\ &\qquad - \sum_{i \in \mathcal{N}} C_i^f(h_{iN}^f, t) - \sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{N}} r_{ij} \left(s_{ij}^f + \sum_{r \in \mathcal{F}_R} q_{ij}^{fr} \right) \\ &\qquad - \sum_{i \in \mathcal{N}} \psi_i^f(I_i^f, t) \Bigg\} \\ &\qquad + \sum_{i \in \mathcal{N}} \lambda_i^f \left[F_i^f(h_i^f) + \sum_{j \in \mathcal{N}} s_{ji}^f - \sum_{j \in \mathcal{N}} s_{ij}^f - \sum_{r \in \mathcal{F}_R} \sum_{j \in \mathcal{N}} q_{ij}^{fr} \right] \end{aligned} \tag{3.109}$$

where $I^f = \left(I_i^f : i \in \mathcal{N}\right)$ and $\lambda^f = \left(\lambda_i^f : i \in \mathcal{N}\right)$ is a vector of adjoint variables. We will use the notation

$$\nabla_z H_P^{f*} = \nabla_z H_P^f(q^{f*}, s^{f*}, h^{f*}, I^{f*}, \lambda^{f*}; c^*) \tag{3.110}$$

to denote the gradient of the Hamiltonian with respect to the control vector

$$z = \begin{pmatrix} q^f \\ s^f \\ h^f \end{pmatrix}$$

of producer f evaluated at a Nash equilibrium. The maximum principle for producer $f \in \mathcal{F}_P$ leads to:

$$\left[\nabla_{q^f} H_P^{f*}\right]^T (q^f - q^{f*}) + \left[\nabla_{s^f} H_P^{f*}\right]^T (s^f - s^{f*}) + \left[\nabla_{h^f} H_P^{f*}\right]^T (h^f - h^{f*}) \leq 0 \quad (3.111)$$

$$\begin{pmatrix} q^f \\ s^f \\ h^f \end{pmatrix}, \begin{pmatrix} q^{f*} \\ s^{f*} \\ h^{f*} \end{pmatrix} \in \Lambda_P^f(h^{-f*}, u_N^*) \quad (3.112)$$

3.3.2 Maximum Principle for the Retailers in the deterministic case

With

$$\begin{pmatrix} c^{-r} \\ q^r \end{pmatrix} \quad (3.113)$$

as exogenous, each retailer $r \in \mathcal{F}_R$ solves

$$J_r^R(c^r; c^{-r}, q^r) \quad \text{s.t.} \quad c^r \in \Lambda_R^r(q^r) \quad (3.114)$$

where

$$\Lambda_R^r(q^r) \equiv \{c^r : (3.88), (3.89), (3.90), (3.91) \text{ hold}\} \quad (3.115)$$

The corresponding Hamiltonian is

$$H_R^r(c^r, R^r, \gamma^r; c^{-r}, q^r) = \quad e^{-\rho t} \quad \Bigg[\sum_{i\in\mathcal{N}} \sum_{j\in\mathcal{N}} \left(c_j^r - \frac{1}{1+\alpha_j^r} \sum_{i\in\mathcal{N}} q_{ij}^{fr} \right) \Psi_j(\sum_{g\in\mathcal{F}_R} c_j^g) - \sum_{j\in\mathcal{N}} \phi_j^r(R_j^r, t) \Bigg] + \sum_{j\in\mathcal{N}} \gamma_j^r \left(\sum_{f\in\mathcal{F}_P} \sum_{i\in\mathcal{N}} q_{ij}^{fr} - c_j^r \right) \tag{3.116}$$

where $R^r = \left(R_j^r : j \in \mathcal{N}\right)$ and $\gamma^r = \left(\gamma_j^r : j \in \mathcal{N}\right)$ is a vector of adjoint variables. We will use the notation

$$\nabla_{c^r} H_R^{r*} = \nabla_{c^r} H_R^r \left(c^{r*}, R^{r*}, \gamma^{r*}; c^{-r*}, q^{r*}\right) \tag{3.117}$$

to denote the gradient of the Hamiltonian with respect to the controls of retailer r evaluated at a Nash equilibrium. The maximum principle for retailer $r \in \mathcal{F}_R$ leads to:

$$\left[\nabla_{c^r} H_R^{r*}\right]^T \left(c^r - c^{r*}\right) \leq 0 \qquad c^r, c^{r*} \in \Lambda_R^r\left(q^{r*}\right) \tag{3.118}$$

3.3.3 Minimum Principle for the Suppliers in the deterministic case

With h as exogenous, the supply chain manager solves

$$\min J^S(u) \quad \text{s.t.} \quad u \in \Lambda_S(h) \tag{3.119}$$

where

$$h = \left(h^f : f \in \mathcal{F}_P\right) \tag{3.120}$$

and

$$\Lambda_S(h) \equiv \{u : (3.98), (3.99), (3.100), (3.101) \text{ hold}\} \tag{3.121}$$

The corresponding Hamiltonian is

$$H_S(u, S, \zeta; h) = \ \ e^{-\rho t} \ \ \sum_{k=1}^{N} [V_k(u_k, t) + \varphi_k(S_k, t)] + \sum_{k=1}^{N} \zeta_k (u_{k-1} - u_k)$$

where $S = (S_k : k \in [i, N])$ and $\zeta = (\zeta_k : k \in [1, N])$ is a vector of adjoint variables. We will use the notation

$$\nabla_u H_S^* = \nabla_u H_S \ (u^*, S^*, \zeta^*; h^*) \tag{3.122}$$

to denote the gradient of the Hamiltonian with respect to supply chain controls evaluated at a Nash equilibrium. The minimum principle for the single supply chain manager leads to:

$$[\nabla_u H_S^*]^T (u - u^*) \geq 0 \quad\quad u, u^* \in \Lambda_S(h^*) \tag{3.123}$$

3.3.4 The DVI in the deterministic case

We note that the finite dimensional variational inequalities derived above hold for each instant of continuous time. So we may integrate the individual variational inequalities over time and sum them over discrete agent indices to obtain a single necessary condition: the solution

$$\begin{pmatrix} q^* \\ s^* \\ h^* \\ c^* \\ u^* \end{pmatrix} \in \Omega \tag{3.124}$$

must satisfy

$$\sum_{f\in\mathcal{F}_P} \int_{t_0}^{t_f} \left\{ \left[-\nabla_{q^f} H_P^{f*}\right]^T (q^f - q^{f*}) + \left[-\nabla_{s^f} H_P^{f*}\right]^T (s^f - s^{f*}) + \left[-\nabla_{h^f} H_P^{f*}\right]^T (h^f - h^{f*}) \right\} dt + \int_{t_0}^{t_f} \sum_{r\mathcal{F}_R} \left[-\nabla_{c^r} H_R^{r*}\right]^T (c^r - c^{r*}) dt + \int_{t_0}^{t_f} \left[\nabla_u H_S^*\right]^T (u - u^*) dt \geq 0 \tag{3.125}$$

for all

$$\begin{pmatrix} q \\ s \\ h \\ c \\ u \end{pmatrix} \in \Omega \tag{3.126}$$

where

$$\Omega = \Lambda \times \Gamma \tag{3.127}$$

$$\Lambda = \Lambda_S(h^*) \times \prod_{f\in\mathcal{F}_P} \Lambda_P^f(h^{-f}, u_N) \times \prod_{r\in\mathcal{F}_R} \Lambda_R^r(q^{r*}) \tag{3.128}$$

and Γ is the set of adjoint variables determined by the adjoint equations and the transversality conditions. The articulation of Γ is left as an exercise for the reader.

3.3.5 Existence of a solution

We assume the competitive supply chain model is regular in the sense of the following definition:

Definition 6. *The competitive supply chains problem described above will be considered regular if: (1) the state operators $I(h,s,q)$, $R(c,q)$, and $S(u)$ exist and are unique, while each of those components is continuous and G-differentiable; (2) the inverse demand, production, input flow cost, transportation cost, inventory cost functions are continuously differentiable with respect to controls and states.*

Theorem 7. *Differential variational inequality formulation of competitive supply chains problem. Any solution of (3.125) is a solution of the dynamic oligopolistic network competition problem when regularity in the sense of Definition 6 holds.*

Proof: We begin by noting that (3.125) is equivalent to the following optimal control problem

$$\max G(q,s,h,c,u) = \sum_{f\in\mathcal{F}_P}\int_{t_0}^{t_f}\left[\sum_{i\in\mathcal{N}_f}\frac{\partial H_f^*}{\partial q_i^f}q_i^f + \sum_{i\in\mathcal{N}_f}\frac{\partial H_f^*}{\partial s_i^f}s_i^f + \sum_{i\in\mathcal{N}_f}\frac{\partial H_f^*}{\partial h_i^f}h_i^f\right]dt$$

$$+\sum_{f\in\mathcal{F}_R}\int_{t_0}^{t_f}\left[\sum_{i\in\mathcal{N}_r}\frac{\partial H_r^*}{\partial c_i^f}c_i^f\right]dt - \sum_{f\in\mathcal{F}_S}\int_{t_0}^{t_f}\left[\sum_{k\in\mathcal{N}_s}\frac{\partial H_s^*}{\partial u_k^f}u_k^f\right]dt$$

$$\text{s.t.} \qquad (3.49), (3.50), (3.51), (3.52), (3.53), (3.54),$$

$$(3.89), (3.90), (3.91), (3.99), (3.100), (3.101)$$

where it is essential to recognize that $G(q,s,h,c,u)$ is a linear functional that assumes knowledge of the solution to our oligopolistic game; as such $G(q,s,h,c,u)$

is a mathematical construct for use in analysis and has no meaning as a computational device. The augmented Hamiltonian for this artificial optimal control problem is

$$\begin{aligned} H_0 \;=\; & \sum_{f\in\mathcal{F}_P}\left[\sum_{i\in\mathcal{N}_f}\frac{\partial H_f^*}{\partial q_i^f}q_i^f+\sum_{i\in\mathcal{N}_f}\frac{\partial H_f^*}{\partial s_i^f}s_i^f+\sum_{i\in\mathcal{N}_f}\frac{\partial H_f^*}{\partial h_i^f}h_i^f\right] \\ & +\sum_{f\in\mathcal{F}_R}\left[\sum_{i\in\mathcal{N}_r}\frac{\partial H_r^*}{\partial c_i^f}c_i^f\right]-\sum_{f\in\mathcal{F}_S}\left[\sum_{k\in\mathcal{N}_s}\frac{\partial H_s^*}{\partial u_k^f}u_k^f\right] \\ & +\sum_{f\in\mathcal{F}_P}\Psi_f^P+\sum_{f\in\mathcal{F}_R}\Psi_f^R+\sum_{f\in\mathcal{F}_S}\Psi_f^S \end{aligned}$$

where

$$\begin{aligned} \Psi_f^P &= \sum_{i\in\mathcal{N}_f}\lambda_i^f\left[F_i^f(h_i^f)+\sum_{j\in\mathcal{N}}s_{ji}^f-\sum_{j\in\mathcal{N}}s_{ij}^f-\sum_{r\in\mathcal{F}_R}\sum_{j\in\mathcal{N}}q_{ij}^{fr}\right] && f\in\mathcal{F}_P,\forall r\in\mathcal{F}_R, i\in\mathcal{N} \\ \Psi_f^R &= \sum_{i\in\mathcal{N}_r}\lambda_i^r\left[\sum_{f\in\mathcal{F}_P}\sum_{i\in\mathcal{N}}q_{ij}^{fr}-c_j^r\right] && \forall r\in\mathcal{F}_R,\ j\in\mathcal{N} \\ \Psi_f^S &= \sum_{k\in\mathcal{N}_s}\lambda_k^s\left[u_{k-1}-u_k\right] && k=1,...,N \end{aligned}$$

The associated maximum principal requires

$$\begin{aligned} &\max H_0 \quad \text{s.t.} \quad \{(3.49),(3.50),(3.51),(3.52),(3.53),(3.54), \\ &(3.89),(3.90),(3.91),(3.99),(3.100),(3.101)\ \} \end{aligned}$$

The corresponding necessary conditions for this mathematical program are identical to (3.111), 3.118), and (3.123) since

$$\frac{\partial H_0^*}{\partial q_i^f}=\frac{\partial H_f^*}{\partial q_i^f}+\frac{\partial \Psi_f^{P*}}{\partial q_i^f}=\frac{\partial H_f^*}{\partial q_i^f}$$

$$\frac{\partial H_0^*}{\partial s_i^f} = \frac{\partial H_f^*}{\partial s_i^f} + \frac{\partial \Psi_f^{P*}}{\partial s_i^f} = \frac{\partial H_f^*}{\partial s_i^f}$$

$$\frac{\partial H_0^*}{\partial h_i^f} = \frac{\partial H_f^*}{\partial h_i^f} + \frac{\partial \Psi_f^{P*}}{\partial h_i^f} = \frac{\partial H_f^*}{\partial h_i^f}$$

$$\frac{\partial H_0^*}{\partial c_i^f} = \frac{\partial H_r^*}{\partial c_i^f} + \frac{\partial \Psi_f^{R*}}{\partial c_i^f} = \frac{\partial H_r^*}{\partial h_i^f}$$

$$\frac{\partial H_0^*}{\partial u_k^f} = \frac{\partial H_s^*}{\partial u_k^f} + \frac{\partial \Psi_f^{S*}}{\partial u_k^f} = \frac{\partial H_s^*}{\partial u_k^f}$$

where

$$\begin{aligned} H_0^* \;=\; & \sum_{f\in\mathcal{F}_P}\left[\sum_{i\in\mathcal{N}_f}\frac{\partial H_f^*}{\partial q_i^f}q_i^{f*} + \sum_{i\in\mathcal{N}_f}\frac{\partial H_f^*}{\partial s_i^f}s_i^{f*} + \sum_{i\in\mathcal{N}_f}\frac{\partial H_f^*}{\partial h_i^f}h_i^{f*}\right] \\ & + \sum_{f\in\mathcal{F}_R}\left[\sum_{i\in\mathcal{N}_r}\frac{\partial H_r^*}{\partial c_i^f}c_i^{f*}\right] - \sum_{f\in\mathcal{F}_S}\left[\sum_{k\in\mathcal{N}_s}\frac{\partial H_s^*}{\partial u_k^f}u_k^{f*}\right] \\ & + \sum_{f\in\mathcal{F}_P}\Psi_f^{P*} + \sum_{f\in\mathcal{F}_R}\Psi_f^{R*} + \sum_{f\in\mathcal{F}_S}\Psi_f^{S*} \end{aligned}$$

and

$$\begin{aligned} \Psi_f^{P*} &= \Psi_f^P\left(q^{f^*}, s^{f*}, h^{f*}, I^{f*}, \lambda^{f*}\right) \\ \Psi_f^{R*} &= \Psi_f^R\left(c^{f^*}, R^{f*}, \lambda^{r*}\right) \\ \Psi_f^{S*} &= \Psi_f^S\left(u^{f^*}, S^{f*}, \lambda^{s*}\right) \end{aligned}$$

We next note that the following existence result holds:

Theorem 8. *Existence of competitive supply chains problem solution. When the variational inequality of Theorem 7 is regular in the sense of Definition 6, there exists a solution of the dynamic oligopolistic network competition problem.*

Proof: Note that the feasible region of controls Ω is convex and compact by the virtue of the given and the explicit lower and upper bounds of the formulation. Note also that continuity is assured by regularity. Existence is then immediate from the results of Section 3.1.

3.4 The Differential Variational Inequality for Supply Chains under uncertainty

In this section we give an overview of how the relevant differential variational inequality for our combined producer-retailer-supply chain game may be formed.

3.4.1 Maximum Principle for the Producers with disruptions

With

$$\begin{pmatrix} c \\ q^{-f} \end{pmatrix} \tag{3.129}$$

as exogenous, each firm $f \in F_P$ solves

$$\max J_f^P(q^f, s^f, u^f; c) \quad \text{s.t.} \quad \left(q^f, s^f, u^f\right) \in \Lambda_P^f(u^{-f}) \tag{3.130}$$

where

$$\Lambda_P^f(u^{-f}) \equiv \left\{ \begin{pmatrix} q^f \\ s^f \\ u^f \end{pmatrix} : (3.48), (3.49), (3.50), (3.51), (3.52), (3.54) \text{ hold} \right\} \tag{3.131}$$

The corresponding Hamiltonian is

$$
\begin{aligned}
&H_P^f(q^f, s^f, u^f, I^f, \lambda^f; c) \\
&= e^{-\rho t}\Bigg\{\sum_{r\in\mathcal{F}_R}\sum_{i\in\mathcal{N}}\sum_{j\in\mathcal{N}}\frac{1}{1+\alpha_j^r}\Psi_j(\sum_{g\in\mathcal{F}_P} c_j^g)q_{ij}^{fr} \\
&\qquad - \sum_{i\in\mathcal{N}}\sum_{h\in\mathcal{N}} E\left[\widetilde{C}_i^f(u_{hi}^{sf}, t)\right] - \sum_{i\in\mathcal{N}}\sum_{j\in\mathcal{N}} E\left[\widetilde{r}_{ij}\right]\left(s_{ij}^f + \sum_{r\in\mathcal{F}_R} q_{ij}^{fr}\right) - \sum_{i\in\mathcal{N}}\psi_i^f(I_i^f, t) \\
&\qquad -\beta_i^f\left[Var(\widetilde{F}_i^f(\sum_{s\in\mathcal{F}_S}\sum_{h=1}^{g} u_{hi}^{sf})) + Var(\widetilde{C}_i^f(u_i^f) + Var(\widetilde{r}_{ij})\left(s_{ij}^f + \sum_{r\in\mathcal{F}_R} q_{ij}^{fr}\right)\right]\Bigg\} \\
&\qquad + \sum_{i\in\mathcal{N}}\lambda_i^f\left[E\left[\widetilde{F}_i^f(u_i^f)\right] + \sum_{j\in\mathcal{N}} s_{ji}^f - \sum_{j\in\mathcal{N}} s_{ij}^f - \sum_{r\in\mathcal{F}_R}\sum_{j\in\mathcal{N}} q_{ij}^{fr}\right]
\end{aligned}
\tag{3.132}
$$

where $I^f = \left(I_i^f : i \in \mathcal{N}\right)$ and $\lambda^f = \left(\lambda_i^f : i \in \mathcal{N}\right)$ is a vector of adjoint variables. We will use the notation

$$\nabla_z H_P^{f*} = \nabla_z H_P^f(q^{f*}, s^{f*}, u^{f*}, I^{f*}, \lambda^{f*}; c^*) \tag{3.133}$$

to denote the gradient of the Hamiltonian with respect to the control vector

$$z = \begin{pmatrix} q^f \\ s^f \\ u^f \end{pmatrix}$$

of producer f evaluated at a Nash equilibrium. The maximum principle for producer $f \in \mathcal{F}_P$ leads to:

$$\left[\nabla_{q^f} H_P^{f*}\right]^T (q^f - q^{f*}) + \left[\nabla_{s^f} H_P^{f*}\right]^T (s^f - s^{f*}) + \left[\nabla_{h^f} H_P^{f*}\right]^T (u^f - u^{f*}) \leq 0 \quad (3.134)$$

$$\begin{pmatrix} q^f \\ s^f \\ u^f \end{pmatrix}, \begin{pmatrix} q^{f*} \\ s^{f*} \\ u^{f*} \end{pmatrix} \in \Lambda_P^f(u^{-f*}) \quad (3.135)$$

3.4.2 Maximum Principle for the Retailers with disruptions

With

$$\begin{pmatrix} c^{-r} \\ q^r \end{pmatrix} \quad (3.136)$$

as exogenous, each retailer $r \in \mathcal{F}_R$ solves

$$J_r^R(c^r; c^{-r}, q^r) \quad \text{s.t.} \quad c^r \in \Lambda_R^r(q^r) \quad (3.137)$$

where

$$\Lambda_R^r(q^r) \equiv \{c^r : (3.88), (3.89), (3.90), (3.91) \text{ hold}\} \quad (3.138)$$

The corresponding Hamiltonian is

$$
\begin{aligned}
H_R^r(c^r, R^r, \gamma^r; c^{-r}, q^r) = \quad e^{-\rho t} \quad & \left[\sum_{i\in\mathcal{N}} \sum_{j\in\mathcal{N}} \left(c_j^r - \frac{1}{1+\alpha_j^r} \sum_{i\in\mathcal{N}} q_{ij}^{fr} \right) \Psi_j (\sum_{g\in\mathcal{F}_R} c_j^g) \right. \\
& \left. - \sum_{j\in\mathcal{N}} \phi_j^r (R_j^r, t) \right] \\
& + \sum_{j\in\mathcal{N}} \gamma_j^r \left(\sum_{f\in\mathcal{F}_P} \sum_{i\in\mathcal{N}} q_{ij}^{fr} - c_j^r \right)
\end{aligned}
\tag{3.139}
$$

where $R^r = \left(R_j^r : j \in \mathcal{N}\right)$ and $\gamma^r = \left(\gamma_j^r : j \in \mathcal{N}\right)$ is a vector of adjoint variables. We will use the notation

$$
\nabla_{c^r} H_R^{r*} = \nabla_{c^r} H_R^r \left(c^{r*}, R^{r*}, \gamma^{r*}; c^{-r*}, q^{r*}\right) \tag{3.140}
$$

to denote the gradient of the Hamiltonian with respect to the controls of retailer r evaluated at a Nash equilibrium. The maximum principle for retailer $r \in \mathcal{F}_R$ leads to:

$$
\left[\nabla_{c^r} H_R^{r*}\right]^T \left(c^r - c^{r*}\right) \leq 0 \qquad c^r, c^{r*} \in \Lambda_R^r \left(q^{r*}\right) \tag{3.141}
$$

3.4.3 Minimum Principle for the Suppliers with disruptions

With h as exogenous, the supply chain manager solves

$$
\min J^S(u^s) \quad \text{s.t.} \quad u^s \in \Lambda_S(u^f) \tag{3.142}
$$

and

$$\Lambda_S(u^f) \equiv \{u : (3.100), (3.101) \text{ hold}\} \tag{3.143}$$

The corresponding Hamiltonian is

$$\begin{aligned} H_S(u, S, \zeta) &= e^{-\rho t} \sum_{h=1}^{g} \left[E\left[\widetilde{V}_h^s(u_h^s, t) \right] + \varphi_h^s \left(S_h^s, t \right) + \beta_h^s \cdot Var\left(\widetilde{V}_h^s(u_h^s, t) \right) \right] \\ &+ \sum_{h=1}^{g} \zeta_h \left[S_h^s(0) - \sum_{i=1}^{m} u_{hi}^s \right] \end{aligned} \tag{3.144}$$

where $S = (S_k : k \in [i, N])$ and $\zeta = (\zeta_k : k \in [1, N])$ is a vector of adjoint variables. We will use the notation

$$\nabla_u H_S^* = \nabla_u H_S \left(u^*, S^*, \zeta^* \right) \tag{3.145}$$

to denote the gradient of the Hamiltonian with respect to supply chain controls evaluated at a Nash equilibrium. The minimum principle for the single supply chain manager leads to:

$$\left[\nabla_u H_S^* \right]^T \left(u^s - u^{s*} \right) \geq 0 \qquad u^s, u^{s*} \in \Lambda_S(u^{f*}) \tag{3.146}$$

3.4.4 The DVI with disruptions

We note that the finite dimensional variational inequalities derived above hold for each instant of continuous time. So we may integrate the individual variational inequalities over time and sum them over discrete agent indices to obtain a single necessary condition: the solution

$$\begin{pmatrix} q^* \\ s^* \\ u^{f*} \\ c^* \\ u^{s*} \end{pmatrix} \in \Omega \tag{3.147}$$

must satisfy

$$\begin{aligned} \sum_{f\in\mathcal{F}_P} \int_{t_0}^{t_f} \Big\{ \Big[-\nabla_{q^f} H_P^{f*}\Big]^T (q^f - q^{f*}) + \Big[-\nabla_{s^f} H_P^{f*}\Big]^T (s^f - s^{f*}) \\ + \Big[-\nabla_{u^f} H_P^{f*}\Big]^T (u^f - u^{f*}) \Big\} dt \\ + \int_{t_0}^{t_f} \sum_{r\mathcal{F}_R} [-\nabla_{c^r} H_R^{r*}]^T (c^r - c^{r*}) dt \\ + \int_{t_0}^{t_f} [\nabla_{u^s} H_S^*]^T (u^s - u^{s*}) dt \geq 0 \end{aligned} \tag{3.148}$$

for all

$$\begin{pmatrix} q \\ s \\ u^f \\ c \\ u^s \end{pmatrix} \in \Omega \tag{3.149}$$

where

$$\Omega = \Lambda \times \Gamma \tag{3.150}$$

$$\Lambda = \Lambda_S(u^*) \times \prod_{f\in\mathcal{F}_P} \Lambda_P^f(u^{-f}) \times \prod_{r\in\mathcal{F}_R} \Lambda_R^r(q^{r*}) \tag{3.151}$$

and Γ is the set of adjoint variables determined by the adjoint equations and the transversality conditions.

3.4.5 Existence of a solution for the case with disruptions

We assume the competitive supply chain model with disruptions is regular in the sense of the Definition 6 as well:

Theorem 9. *Differential variational inequality formulation of competitive supply chains with disruptions problem. Any solution of (3.148) is a solution of the dynamic oligopolistic network competition with disruptions problem when regularity in the sense of Definition 6 holds.*

Proof: In the same manner, (3.148) is equivalent to the following optimal control problem

$$\begin{aligned}
\max G(q,s,h,c,u) = & \sum_{f\in\mathcal{F}_P}\int_{t_0}^{t_f}\left[\sum_{i\in\mathcal{N}_f}\frac{\partial H_f^*}{\partial q_i^f}q_i^f+\sum_{i\in\mathcal{N}_f}\frac{\partial H_f^*}{\partial s_i^f}s_i^f+\sum_{i\in\mathcal{N}_f}\frac{\partial H_f^*}{\partial h_i^f}h_i^f\right]dt \\
& +\sum_{f\in\mathcal{F}_R}\int_{t_0}^{t_f}\left[\sum_{i\in\mathcal{N}_r}\frac{\partial H_r^*}{\partial c_i^f}c_i^f\right]dt-\sum_{f\in\mathcal{F}_S}\int_{t_0}^{t_f}\left[\sum_{k\in\mathcal{N}_s}\frac{\partial H_s^*}{\partial u_k^f}u_k^f\right]dt \\
& \text{s.t.} \qquad (3.49),(3.50),(3.51),(3.52),(3.53),(3.54), \\
& (3.89),(3.90),(3.91),(3.99),(3.100),(3.101)
\end{aligned}$$

where it is essential to recognize that $G(q,s,h,c,u)$ is a linear functional that assumes knowledge of the solution to our oligopolistic game; as such $G(q,s,h,c,u)$ is a mathematical construct for use in analysis and has no meaning as a computational device. The augmented Hamiltonian for this artificial optimal control problem is

$$
\begin{aligned}
H_0 \;=\; & \sum_{f\in\mathcal{F}_P}\left[\sum_{i\in\mathcal{N}_f}\frac{\partial H_P^{f*}}{\partial q_i^f}q_i^f+\sum_{i\in\mathcal{N}_f}\frac{\partial H_P^{*}}{\partial s_i^f}s_i^f+\sum_{i\in\mathcal{N}_f}\frac{\partial H_P^{*}}{\partial h_i^f}h_i^f\right]\\
&+\sum_{f\in\mathcal{F}_R}\left[\sum_{i\in\mathcal{N}_r}\frac{\partial H_R^{r*}}{\partial c_i^f}c_i^f\right]-\sum_{f\in\mathcal{F}_S}\left[\sum_{k\in\mathcal{N}_s}\frac{\partial H_S^{s*}}{\partial u_k^f}u_k^f\right]\\
&+\sum_{f\in\mathcal{F}_P}\Psi_P^f+\sum_{f\in\mathcal{F}_R}\Psi_R^r+\sum_{f\in\mathcal{F}_S}\Psi_S^s
\end{aligned}
$$

where

$$
\begin{aligned}
\Psi_P^f &= \sum_{i\in\mathcal{N}}\lambda_i^f\left[E\left[\widetilde{F}_i^f(u_i^f)\right]+\sum_{j\in\mathcal{N}}s_{ji}^f-\sum_{j\in\mathcal{N}}s_{ij}^f-\sum_{r\in\mathcal{F}_R}\sum_{j\in\mathcal{N}}q_{ij}^{fr}\right] && \forall f\in\mathcal{F}_P,\forall r\in\mathcal{F}_R, i\in\mathcal{N}\\
\Psi_R^r &= \sum_{j\in\mathcal{N}}\gamma_j^r\left[\sum_{f\in\mathcal{F}_P}\sum_{i\in\mathcal{N}}q_{ij}^{fr}-c_j^r\right] && \forall r\in\mathcal{F}_R,\; j\in\mathcal{N}\\
\Psi_S^s &= \sum_{k=1}^{g}\zeta_k\left[S_k^s(0)-\sum_{i=1}^{N}u_{ki}^s\right] && k=1,...,g
\end{aligned}
$$

The associated maximum principal requires

$$
\begin{aligned}
&\max H_0 \quad \text{s.t.} \quad \{(3.49),(3.50),(3.51),(3.52),(3.53),(3.54),\\
&(3.89),(3.90),(3.91),(3.99),(3.100),(3.101)\;\}
\end{aligned}
$$

The corresponding necessary conditions for this mathematical program are identical to (3.111), 3.118), and (3.123) since

$$
\frac{\partial H_0^*}{\partial q_i^f}=\frac{\partial H_P^{f*}}{\partial q_i^f}+\frac{\partial \Psi_P^{f*}}{\partial q_i^f}=\frac{\partial H_P^{f*}}{\partial q_i^f}
$$

$$
\frac{\partial H_0^*}{\partial s_i^f}=\frac{\partial H_P^{f*}}{\partial s_i^f}+\frac{\partial \Psi_P^{f*}}{\partial s_i^f}=\frac{\partial H_P^{f*}}{\partial s_i^f}
$$

$$\frac{\partial H_0^*}{\partial h_i^f} = \frac{\partial H_P^{f*}}{\partial h_i^f} + \frac{\partial \Psi_P^{f*}}{\partial h_i^f} = \frac{\partial H_P^{f*}}{\partial h_i^f}$$

$$\frac{\partial H_0^*}{\partial c_i^f} = \frac{\partial H_R^{r*}}{\partial c_i^f} + \frac{\partial \Psi_R^{r*}}{\partial c_i^f} = \frac{\partial H_R^{r*}}{\partial h_i^f}$$

$$\frac{\partial H_0^*}{\partial u_k^f} = \frac{\partial H_S^{s*}}{\partial u_k^f} + \frac{\partial \Psi_S^{s*}}{\partial u_k^f} = \frac{\partial H_S^{s*}}{\partial u_k^f}$$

where

$$\begin{aligned} H_0^* \;=\; & \sum_{f \in \mathcal{F}_P} \left[\sum_{i \in \mathcal{N}_f} \frac{\partial H_P^{f*}}{\partial q_i^f} q_i^{f*} + \sum_{i \in \mathcal{N}_f} \frac{\partial H_P^{f*}}{\partial s_i^f} s_i^{f*} + \sum_{i \in \mathcal{N}_f} \frac{\partial H_P^{f*}}{\partial h_i^f} h_i^{f*} \right] \\ & + \sum_{f \in \mathcal{F}_R} \left[\sum_{i \in \mathcal{N}_r} \frac{\partial H_R^{r*}}{\partial c_i^f} c_i^{f*} \right] - \sum_{f \in \mathcal{F}_S} \left[\sum_{k \in \mathcal{N}_s} \frac{\partial H_S^{s*}}{\partial u_k^f} u_k^{f*} \right] \\ & + \sum_{f \in \mathcal{F}_P} \Psi_f^{P*} + \sum_{f \in \mathcal{F}_R} \Psi_f^{R*} + \sum_{f \in \mathcal{F}_S} \Psi_f^{S*} \end{aligned}$$

and

$$\begin{aligned} \Psi_P^{f*} &= \Psi_P^f \left(q^{f^*}, s^{f*}, h^{f*}, I^{f*}, \lambda^{f*} \right) \\ \Psi_R^{r*} &= \Psi_R^r \left(c^{f^*}, R^{f*}, \lambda^{r*} \right) \\ \Psi_S^{s*} &= \Psi_S^s \left(u^{f^*}, S^{f*}, \lambda^{s*} \right) \end{aligned}$$

We next note that the following existence result holds:

Theorem 10. *Existence of competitive supply chains with disruptions problem solution. When the variational inequality of Theorem 9 is regular in the sense of Definition 6, there exists a solution of the dynamic oligopolistic network competition with disruptions problem.*

Proof: Note that the feasible region of controls Ω is convex and compact by the virtue of the given and the explicit lower and upper bounds of the formulation.

Note also that continuity is assured by regularity. Existence is then immediate from the results of Section 3.1.

3.5 Price of Anarchy

The *price of anarchy* is the name given by Roughgarden to the ratio of total congestion arising from user equilibrium traffic assignment to minimum total congestion, where the latter is achieved by system optimal traffic assignment Roughgarden (2002). As such the price of anarchy captures the inefficiency associated with a Nash equilibrium among network users/agents like firms, retailers, and suppliers. As a consequence of the above presentation we may offer the following formal definition of the price of anarchy:

Definition 11. *Price of anarchy The price of anarchy* ρ *is*

$$\rho = \frac{\sum_{a \in \mathcal{A}} u_a \left(s, c, u\right)}{w(s, c, u)} \tag{3.152}$$

where u_a *and* w *are, respectively, the user equilibrium revenue/utility and the system optimal revenue/utility..*

3.5.1 Vertically integrated model for overall supply chain

Considering all the parties as one firm in the deterministic case, we can make maximizing model of firm's revenue by totaling three parties' objective equations as follows.

$$\max J(s,c,u) = \int_{t_0}^{t_f} e^{-\rho t} \left\{ \sum_{i\in\mathcal{N}} (c_i\Psi_i(c_i) - \phi_i(I_i,t) - C_i(h_i,t)\ ) - \sum_{i\in\mathcal{N}}\sum_{j\in\mathcal{N}} r_{ij}\, s_{ij} - \sum_{k=1}^{N} [V_k(u_k,t) + \varphi_k\,(S_k,t)] \right\} dt \tag{3.153}$$

subject to

$$\frac{dI_i}{dt} = F_i(h_i) + \sum_{j\in\mathcal{N}} s_{ji} - \sum_{j\in\mathcal{N}} s_{ij} - \sum_{i\in\mathcal{N}} c_i \quad i,j \in \mathcal{N} \tag{3.154}$$

$$\frac{dS_k}{dt} = u_{k-1} - u_k \quad k = 1,...,N \tag{3.155}$$

$$I_i\,(t_0) = K_i \quad \forall i \in \mathcal{N} \tag{3.156}$$

$$I_i\,(t_f) \geq \tilde{K}_i \quad \forall i \in \mathcal{N} \tag{3.157}$$

$$S_k(0) = S_k^0 \qquad k = 1,...,N \tag{3.158}$$

$$0 \leq s_{ij} \leq \widetilde{s}_{ij} \quad i\,,j \in \mathcal{N} \tag{3.159}$$

$$0 \leq c_i \leq \widetilde{c}_i \quad i \in \mathcal{N} \tag{3.160}$$

$$A_f \leq \sum_{i\in\mathcal{N}} h_i \leq B_f \tag{3.161}$$

$$u_N = \sum_{i\in\mathcal{N}} h_i \tag{3.162}$$

$$0 \leq u_k \leq U_k \quad k = 0,...,N \tag{3.163}$$

3.5.2 Comparison between Nash game theoretic and vertically integrated

So far, we have dealt with the supply chain based on non-cooperative Nash game theoretic model. However, we can consider one firm which controls all the agents in the supply chain network, so called vertically integrated supply chain. The performance or revenue difference between two models is displayed by the result of numerical example provided in chapter 6. As denoted in below table, the total revenue of one firm in the vertically integrated supply chain network is larger than one of all the agents in the non-cooperative network environment.

	Non-Cooperative Supply Chain	Vertically Integrated Supply Chain
Characteristics	User Optimization	System Optimization
	Nash Equilibrium	Generally, total benefit is bigger than the sum of user optimized benefit
Numerical Example Results displayed in chapter 6	Firm1(Producer1,2,3): 364	
	Firm2(Producer 4,5): 167	
	Retailer 6: 12	
	Retailer 7: 35	
	Suppliers: -1,524	
	Total Revenue: -946	Total Revenue: 2,121

Table 3.1. Comparison between Non-Cooperative and Vertically integrated SC

3.6 Network design for reliable supply chain

Supply chain design for minimizing supply chain disruption risk in terms of a centralized and integrated supply chain is discussed. In essence, one centralized decision-maker/firm chooses producer facility locations to satisfy retailers' demand and to minimize total cost. We do not consider suppliers in this model because it is believed that this model can represent the relation between suppliers and producers in a similar manner.

$$\min \sum_{i\in\mathcal{N}} f_i X_i + \sum_{s\in\mathcal{S}} p_s \int_{t_0}^{t_f} e^{-\rho t} \left\{ \sum_{i\in\mathcal{N}} \sum_{j\in\mathcal{N}} r_{ij} q_{ijs} + \sum_{i\in\mathcal{N}} \psi_{is}(I_{is}, t) \right\} dt$$

subject to

$$\frac{dI_{is}}{dt} = F_i(\sum_{h=1}^{g} u_{hi}) - \sum_{j\in\mathcal{N}} q_{ijs} \quad i \in \mathcal{N} \tag{3.164}$$

$$I_i\,(t_0) = K_i \quad \forall i \in \mathcal{N} \tag{3.165}$$

$$I_i\,(t_f) \geq \tilde{K}_i \quad \forall i \in \mathcal{N} \tag{3.166}$$

$$0 \leq q_{ij} \leq \widetilde{q}_{ij} \quad i\,, j \in \mathcal{N} \tag{3.167}$$

$$\int_{t_0}^{t_f} \sum_{i\in\mathcal{N}} q_{ijs}(1 - a_{is}) = \int_{t_0}^{t_f} D_j(t) \quad i, j \in \mathcal{N} \tag{3.168}$$

$$0 \leq q_{ij} \leq \widetilde{q}_{ij} \quad i\,, j \in \mathcal{N} \tag{3.169}$$

$$X_i \in \{0, 1\} \quad \forall i \in \mathcal{N} \tag{3.170}$$

where f_i is the annual fixed cost, p_s is the probability that scenario s occurs and $\mathcal{S}$ is the set of scenarios. Let $a_{is} = 1$ if node i fails in scenario s. In addition, $D_j(t)$ denotes the demand function at node j in the instant time t and q_{ijs} denotes the flows from node i to node j in the scenario s. This dissertation does not focus on the facility location problem as it will be determined and optimized through one time decision-making. Therefore, the algorithm for this mixed integer mathematical program is not considered; rather product's flows among suppliers, producers, and retailers are optimized under the assumption that manufacturing facilities of firms have already been selected.

3.7 Risk Management in Supply Chains

Supply chain uncertainty causes risk of agents belonging to the supply chain network. Risk management is a methodological approach to the management of outcome uncertainty. The concept of risk is subject to various definitions. Generally, risk can be regarded as the measurable uncertainty. From the financial perspective, risk is the deviation of return. From the project management perspective, risk is a measure of the probability and consequence of not achieving a defined project goal. Supply chain risk can be defined variously as well. According to Zsidisin and Ritchie (2008), supply chain risk is defined as the potential occurrence of an incident or failure to seize opportunities with inbound supply in which its outcomes result in a financial loss for the firm. In the same manner, supply chain risk in this dissertation is defined as a product of the probability of occurrence of a negative event and the resulting amount of damage because in this dissertation, a disruption is measured by a probability of occurrence and impact on each agent's cost function. In the literature on supply chain management, the term risk is also replaced with vulnerability. Risk management usually comprises the stages of identifying possible risks, an analysis of these, elaborating control actions and risk controlling. In addition, a systematic risk management process usually comprises the stages of (1) risk identification, (2) risk analysis including risk assessment and classification, (3) risk management in the narrow sense, e.g., risk treatment, and (4) risk monitoring.

Supply chains are vulnerable to risky events. A critical subject of demand side disruptions in supply chain network is the bullwhip effect, which is characterized by an amplification of demand volatility in the upstream direction of the supply chain. Lee et al.(1997) analyzed this phenomenon and identified delayed and distorted information, sales promotions, order batching, price fluctuations and rationing or shortage gaming as major causes. Demand volatility still presents a major risk source for many firms. However this dissertation considers supply side disruption as above mentioned because many researchers have dealt with demand side uncertainty. Supply side risks are as follows.

- Price fluctuations on the supply markets
- Supplier quality problems
- Capacity fluctuations or shortages on the supply markets
- Poor logistics per performance of suppliers
- Poor logistics performance of logistics service providers
- Sudden supplier defaults

Supply chain design is critical to reducing vulnerability to supply side disruptions. The same disruption can have very different implications depending on how organizations have designed their supply chain and planned for such an event. In other words, recovery planning and systems are very important. Those could enable supply chain partners to gain visibility of inventories, sales, shipments, etc. throughout the entire supply chain. More importantly, supply chain visibility would enhance the capability of the supply chain partners to coordinate their operations in an efficient manner. However, it is also important to establish a recovery planning system so that a supply chain can recover quickly from a major disruption.

For example, even before the September 11 terrorist attacks, Continental Airlines worked with Caleb Technologies to develop the CrewSolver decision support system to generate globally optimal recovery solutions. The optimal recovery solution enables Continental Airlines to reassign crews quickly to cover open flights and to return them to their original schedules in a cost-effective manner while honouring government regulations, contractual obligations and customer expectations. Tang and Tomlin (2008)

For supply chain risk management, there can be existed a lot of strategies and tactics. However, considered are the strategies relevant to or utilized by the integrated supply chain model mentioned above. Firstly, we can consider a strategic stock. In the past where product design and production lines hold unchanged, one may consider carrying additional safety stock inventories of certain critical components to ensure that the supply chain can continue to function smoothly when facing a disruption in supply chain. However, as product life cycle shortens and as product variety increases, the inventory holding and obsolescence costs of these additional safety stock inventories could be exorbitant. Instead of carrying more safety stocks, a firm may consider storing some inventories at certain locations/warehouse/distribution centres to be shared by multiple supply chain agents. For example, Toyota and Sears keep certain inventories of cars and appliances at certain locations so that all retailers in the nearby region share these inventories. By doing so,Toyota and Sears can achieve a higher customer service level without incurring high inventory cost when dealing with regular demand fluctuations. When a disruption occurs, these shared inventories at strategic locations will allow a firm to deploy these strategic stocks quickly to the affected area as well. Secondly, we can consider make-and-buy strategy. When facing potential supply disruptions, a supply chain is more resilient if certain products are produced in-house while other products are outsourced to other suppliers. For instance, HP used to make a fraction of their DeskJet printers at their Singapore factory and outsourced the remaining portion of their production to a contract manufacturer in Malaysia Lee and Tang (1998). In addition, both Brooks Brothers and Zara produce their fashion items at their in-house factories and outsource other basic items to their suppliers in China Ghemawat (2003). This make-and-buy strategy offers flexibility that allow firms to shift production quickly should a supply disruption occur. Additionally flexible transportation strategy is regarded as an effective one. In supply chain network, transportation is the most critical factor that makes a supply chain network disrupted. Therefore, flexible transportation operation should be very proactive. Proactive tactics are as follows.

- Multi-modal transportation. To prevent the supply chain operations from coming to a halt when disruptions occur in the ocean, in the air, on the road, etc. some companies utilize a flexible logistics strategy that relies on multiple modes of transportation.

- Multi-carrier transportation. To ensure continuous flow of materials in the case of political disruptions (landing rights, labour strikes, etc.), various air cargo companies have formed an alliance that will enable them to switch carriers quickly in the event of political disruptions.

- Multiple routes. To avoid a complete shutdown, various companies are considering alternative routes so as to ensure smooth material flows along the supply chains. For example, due to long delays at the west coast ports and heavy traffic jams along various west coast freeways, some east coast companies are encouraging shippers to develop new routes in addition to the traditional route. Specifically, after the west coast ports were shut down for

2 weeks in 2002, some shippers considered shipping various manufacturing goods from Asia to east coast ports via Panama Canal.

Finally, we can utilize revenue management via dynamic pricing. Dynamic pricing is a common mechanism for selling perishable products/services. For instance, when selling limited seats on an airplane with uncertain demand, airlines have always adjusted their ticket price dynamically to meet uncertain demand with limited supply. Revenue management via dynamic pricing can also be an effective way to manage demand when the supply of a particular product is disrupted. Specifically, a retailer can use pricing mechanism to entice customers to choose products that are widely available. For example, when Dell was facing supply disruptions from their Taiwanese suppliers after an earthquake in 1999, Dell immediately deployed a contingency plan by offering special "low-cost upgrade" options to customers if they chose similar computers with components from other suppliers. This dynamic pricing and promotion strategy enabled Dell to satisfy its customers during a supply crisis Martha and Subbakrishna (2002).

Consequently, accepting the fact that uncertainty cannot be completely eliminated and given that there are several possible failure modes that can affect a supply chain network; there are two choices for building resilient supply chains, which means supply chains with ability to maintain, resume and restore operations after a disruption. The first approach involves the time tested "just in case" way of maintaining inventories all along the chain, employing dual or multisourcing and manufacturing at multiple sites. This is a highly inefficient option. A better option would be to first design a sourcing strategy taking into account the disruption costs for the most relevant failure modes and then putting in place contingency plans for each disruption that include both description of the procedures to follow and a definition of roles and responsibilities. In both cases it is first necessary to identify the exceptions that can occur in the chain, estimate the probabilities of their occurrence, map out the chain of immediate and delayed consequential events that propagate through the chain and quantify their impact.

In the aspect of demand side uncertainty, we may also consider some strategies, which are out of scope of my research. However, the strategies relevant to inventory of firms in supply chain network may need to be simply looked into. While every firm/agent in supply chain bears supply risk, some firms may avoid inventory risk with wholesale price contracts by lessening the cost of unsold inventory. Generally, in order to mitigate inventory risk like shortage and oversupply, firms can have two sort of strategy: Push and Pull contract. In case of push contract there is a single wholesale price and the retailer, by ordering his entire supply before the selling season, bears all of the supply chain's inventory risk. A pull contract also has a single wholesale price, but the supplier bears the supply chain's inventory risk because only the supplier holds inventory while the retailer replenishes as needed during the season. Additionally, firms can consider advance-purchase discounts contract, which allows for intermediate allocations of inventory risk: The retailer bears the risk on inventory ordered before the season while the supplier bears the risk on any production in excess of that amount.

Chapter 4

Robust Version of Integrated Supply Chain model

4.1 Introduction

Most research dealing with uncertainty can be divided into two approaches: the probabilistic one and the scenario planning. Probabilistic models consider uncertainty through addressing probability parameters as random variables with probability distributions. Scenario planning assumes that it needs no probability distribution and only requires distinct scenarios including the most events having the possibility that happen. There are several methods to deal with uncertainty when it cannot be represented by using probability; in other words when the uncertainty associated to a factor is not characterized by probability distributions. Unfortunately, dynamic programming assumes complete knowledge of the probability distributions and suffers from the curse of dimensionality.

Hence, the need arises to develop a new optimization approach that incorporates the stochastic character in the supply chain without making any assumptions on its distribution, is applicable to a wide range of network topologies, is easy to understand intuitively, and combines computational tractability with the structural properties of the optimal policy Bertsimas and Sim (2004). Robust optimization addresses the problem of data uncertainty by guaranteeing the feasibility and optimality of the solution for the worst instances of the parameters. However, because it is intrinsically a worst-case approach, feasibility often comes at the cost of performance and generally leads to overconservative solutions Bertsimas and Thiele (2006). The problem of demand uncertainty has motivated a significant amount of literature in the field of revenue management and pricing Adida and Perakis (2006). Raman and Chatterjee model the stochasticity of the demand by introducing an additive model where the random noise is a continuous time Wiener process Raman and Chatterjee (1995).

Robust optimization seeks an optimal solution of a problem when its data is uncertain. A robust optimization formulation was first considered by Soyster (1973) in the case of a linear optimization problem where the data were uncertain within a convex set. He addresses uncertainty by taking a worst-case approach. Nevertheless, such an approach decreases the performance of the solution significantly. Ben-Tal and Nemirovski (1998) studied robust convex optimization . They show that some polynomial-time algorithms allow to efficiently solve exactly or approximately some of these problems. Bertsimas and Sim (2004) studied the trade-off between robustness of a solution to a linear programming problem and the suboptimality of the solution. Bertsimas and Thiele (2006) apply robust optimization principles to supply chain management.

In this dissertation, supply-side uncertainty is addressed through robust optimization and ideas from robust optimization is applied to deduce the robust counterpart of the nominal integrated supply chain optimization problem presented above.

4.2 The Robust Optimization Approach

Let me consider again the nominal form of the firm (producers) as above mentioned

$$\max J_f^P(q^f, s^f, h^f; c) = \int_{t_0}^{t_f} e^{-\rho t} \left\{ \sum_{r\in\mathcal{F}_R} \sum_{i\in\mathcal{N}} \sum_{j\in\mathcal{N}} \frac{1}{1+\alpha_j^r} \Psi_j(\sum_{g\in\mathcal{F}_P} c_j^g) q_{ij}^{fr} - \sum_{i\in\mathcal{N}} C_i^f(h_i^f, t) - \sum_{i\in\mathcal{N}} \sum_{j\in\mathcal{N}} r_{ij} \left(s_{ij}^f + \sum_{r\in\mathcal{F}_R} q_{ij}^{fr} \right) - \sum_{i\in\mathcal{N}} \psi_i^f(I_i^f, t) \right\} dt \tag{NP}$$

subject to

$$\frac{dI_i^f}{dt} = F_i^f(h_i^f) + \sum_{j\in\mathcal{N}} s_{ji}^f - \sum_{j\in\mathcal{N}} s_{ij}^f - \sum_{r\in\mathcal{F}_R} \sum_{j\in\mathcal{N}} q_{ij}^{fr} \quad \forall f \in \mathcal{F}_P, \forall r \in \mathcal{F}_R, i \in \mathcal{N}$$

$$I_i^f(t_0) = K_i^f \quad \forall i \in \mathcal{N}$$

$$I_i^f(t_f) \geq \tilde{K}_i^f \quad \forall i \in \mathcal{N}$$

$$0 \leq q_{ij}^{fr} \leq \widetilde{q}_{ij}^{fr} \quad i, j \in \mathcal{N}, r \in \mathcal{F}_r$$

$$0 \leq s_{ij}^f \leq \widetilde{s}_{ij}^f \quad i, j \in \mathcal{N}$$

$$A_f \leq \sum_{i\in\mathcal{N}} h_i^f \leq B_f \quad \forall f \in \mathcal{F}_P$$

$$u_N = \sum_{g\in\mathcal{F}_P} \sum_{i\in\mathcal{N}} h_i^g$$

Through robust optimization approach, we can reformulate the above mathematical programming NP by adding uncertainty terms to the production function F^f, and transportation cost C^f without any consideration of probabilistic distribution assumption. This approach does not make any assumption on the probabilistic distribution of the producers' production and transportation and the suppliers' transportation cost. In addition, we can reformulate similarly the suppliers' extremal problem by adding uncertainty term to the variable cost V^S. However, the firms' reformulation to robust optimization is dealt with in this chapter.

4.2.1 Modeling Supply-side Uncertainty

Let me introduce an additive model of production function as follows:

$$F_i^f(h_i^f, t) = a_i^f(t) + b_i^f(t). \cdot h_i^f + \xi_i^f(t)$$

where $\xi_i^f(t)$ is the uncertain parameter.

In this model of production function, the parameter $a_i^f(t)$ can be regarded as having uncertainty. Therefore I denote $a_i^f(t)$ the nominal function, $\widetilde{a}_i^f(t)$ the realization, and I suppose that the realization belongs in an interval centered around the nominal function with half-length $\widehat{a}_i^f(t)$. In addition, I introduce a budget of uncertainty function $\Theta_i^f(t)$ taking values in $[0, T]$. Function $\Theta_i^f(t)$ is assumed to be an increasing function of time. This function allows us to adjust the trade-off between the level of conservatism sought for the robust solution and its performance. It is assumed to be non-decreasing in order to allow the aggregate error over time to increase. Furthermore, $\dot{\Theta}_i^f(t) \leq 1 \; \forall i, t$ is assumed in order to ensure that the function does not grow faster than new variables are added. In here, the constraints related to these assumptions are as follows.

$$\left|\widetilde{a}_i^f(t) - a_i^f(t)\right| \leq \widehat{a}_i^f(t)$$
$$\int_{t_0}^{t_f} \frac{\left|\widetilde{a}_i^f(t) - a_i^f(t)\right|}{\widehat{a}_i^f(t)} ds \leq \Theta_i^f(t)$$

Let

$$z_i(t) \equiv \frac{\widetilde{a}_i^f(t) - a_i^f(t)}{\widehat{a}_i^f(t)}$$

be the scaled variation of uncertainty $\widetilde{a}_i^f(t)$,then

$$z_i(t) \in [-1, 1]$$
$$\int_{t_0}^{t_f} |z_i(s)|\, ds \leq \Theta_i^f(t)$$

Similarly, an additive model of transportation cost function is addressed as follows:

$$C_i^f(h_i^f, t) = \alpha_i^f(t) + \beta_i^f(t). \cdot h_i^f + \epsilon_i^f(t)$$

where $\epsilon_i^f(t)$ is the uncertain parameter.

In the same as above mentioned, $\alpha_i^f(t)$ is denoted as the nominal function, $\widetilde{\alpha}_i^f(t)$ the realization, which is assumed to belong in an interval centered around the nominal function with half-length $\widehat{\alpha}_i^f(t)$. In addition, a budget of uncertainty function $\Gamma_i^f(t)$ taking values in $[0, T]$. Function $\Gamma_i^f(t)$ is assumed to be an increasing function of time. $\dot{\Gamma}_i^f(t) \leq 1\ \forall i, t$ is also assumed in order to ensure that the function does not grow faster than new variables are added. The constraints related to these assumptions are as follows.

$$\left|\widetilde{\alpha}_i^f(t) - \alpha_i^f(t)\right| \le \widehat{\alpha}_i^f(t)$$
$$\int_{t_0}^{t_f} \frac{\left|\widetilde{\alpha}_i^f(t) - \alpha_i^f(t)\right|}{\widehat{\alpha}_i^f(t)} ds \le \Gamma_i^f(t)$$

Let

$$w_i(t) \equiv \frac{\widetilde{\alpha}_i^f(t) - \alpha_i^f(t)}{\widehat{\alpha}_i^f(t)}$$

be the scaled variation of uncertainty $\widetilde{\alpha}_i^f(t)$,then

$$w_i(t) \in [-1, 1]$$
$$\int_{t_0}^{t_f} |w_i(s)|\, ds \le \Gamma_i^f(t)$$

To simplify the analysis of this robust optimization model, the shipping rate r_{ij} is assumed to be deterministic differently from the model considering supply disruptions in the above.

4.2.2 Robust Optimization Model

Since information about what values the uncertain variable will take is unknown to each agent such as producer and supplier, what objective function should the firm and supplier maximize? As is denoted in the literature, if an a priori distribution is assumed for the uncertain variable, the producer and supplier could adopt a policy that maximizes the expected revenue as formulated above chapter 3. In the case of nominal problem NP,the robust counterpart can be formulated as

$$\max U$$

subject to

$$U \leq \int_{t_0}^{t_f} e^{-\rho t} \quad \left\{ \sum_{r\in\mathcal{F}_R} \sum_{i\in\mathcal{N}} \sum_{j\in\mathcal{N}} \frac{1}{1+\alpha_j^r} \Psi_j(\sum_{g\in\mathcal{F}_P} c_j^g) q_{ij}^{fr} \right. \tag{4.1}$$

$$\left. - \sum_{i\in\mathcal{N}} \left(\widetilde{\alpha}_i^f(t) + \beta_i^f(t).\cdot h_i^f \right) - \sum_{i\in\mathcal{N}} \sum_{j\in\mathcal{N}} r_{ij} \left(s_{ij}^f + \sum_{r\in\mathcal{F}_R} q_{ij}^{fr} \right) - \sum_{i\in\mathcal{N}} \psi_i^f(\widetilde{I}_i^f, t) \right\} dt$$

$$\frac{d\widetilde{I}_i^f}{dt} = \widetilde{a}_i^f(t) + b_i^f(t).\cdot h_i^f + \sum_{j\in\mathcal{N}} s_{ji}^f - \sum_{j\in\mathcal{N}} s_{ij}^f - \sum_{r\in\mathcal{F}_R} \sum_{j\in\mathcal{N}} q_{ij}^{fr} \quad \forall f \in \mathcal{F}_P, \forall r \in \mathcal{F}_R, i \in \mathcal{N}$$

$$\widetilde{I}_i^f(t_0) = K_i^f \quad \forall i \in \mathcal{N}$$

$$\widetilde{I}_i^f(t_f) \geq \tilde{K}_i^f \quad \forall i \in \mathcal{N}$$

$$0 \leq q_{ij}^{fr} \leq \widetilde{q}_{ij}^{fr} \quad i, j \in \mathcal{N}, r \in \mathcal{F}_r$$

$$0 \leq s_{ij}^f \leq \widetilde{s}_{ij}^f \quad i, j \in \mathcal{N}$$

$$A_f \leq \sum_{i\in\mathcal{N}} h_i^f \leq B_f \quad \forall f \in \mathcal{F}_P$$

$$u_N = \sum_{g\in\mathcal{F}_P} \sum_{i\in\mathcal{N}} h_i^g$$

Here, the inventory level at time t can be written as follows

$$\begin{aligned} \widetilde{I}_i^f(t) &= K_i^f + \int_{t_0}^{t} \left(a_i^f(s) + z_i(s)\widehat{\alpha}_i^f(s) + b_i^f(s).\cdot h_i^f + \sum_{j\in\mathcal{N}} s_{ji}^f(s) - \sum_{j\in\mathcal{N}} s_{ij}^f(s) - \sum_{r\in\mathcal{F}_R} \sum_{j\in\mathcal{N}} q_{ij}^{fr}(s) \right) \\ &= I_i^f(t) + \int_{t_0}^{t} z_i(s)\widehat{\alpha}_i^f(s) ds \end{aligned}$$

where

$$I_i^f(t) = K_i^f + \int_{t_0}^{t} \left(a_i^f(s) + b_i^f(s).\cdot h_i^f + \sum_{j\in\mathcal{N}} s_{ji}^f(s) - \sum_{j\in\mathcal{N}} s_{ij}^f(s) - \sum_{r\in\mathcal{F}_R} \sum_{j\in\mathcal{N}} q_{ij}^{fr}(s) \right) ds$$

is the nominal inventory level. In order to reformulate this problem, we need to determine the realization of $\widetilde{\alpha}_i^f$, $\widetilde{a}_i^f$, that is, the worst-case realization. Thus we can re-state constraint (4.1) as follows

$$
\begin{aligned}
U \leq \int_{t_0}^{t_f} e^{-\rho t} & \Bigg\{ \sum_{r \in \mathcal{F}_R} \sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{N}} \frac{1}{1+\alpha_j^r} \Psi_j(\sum_{g \in \mathcal{F}_P} c_j^g) q_{ij}^{fr} \\
& - \sum_{i \in \mathcal{N}} \left(\alpha_i^f(t) + \beta_i^f(t). \cdot h_i^f \right) - \sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{N}} r_{ij} \left(s_{ij}^f + \sum_{r \in \mathcal{F}_R} q_{ij}^{fr} \right) \\
& - \sum_{i \in \mathcal{N}} \psi_i^f(I_i^f, t) - \sum_{i \in \mathcal{N}} \left(\widehat{\alpha}_i^f(t) w_i(t) \right) - \sum_{i \in \mathcal{N}} \psi_i^f(\int_{t_0}^{t} z_i(s) \widehat{\alpha}_i^f(s) ds) \Bigg\} dt
\end{aligned}
\tag{4.2}
$$

$$
U \leq G - \int_{t_0}^{t_f} e^{-\rho t} \left\{ \sum_{i \in \mathcal{N}} \left(\widehat{\alpha}_i^f(t) w_i(t) \right) + \sum_{i \in \mathcal{N}} \psi_i^f(\int_{t_0}^{t} z_i(s) \widehat{\alpha}_i^f(s) ds) \right\} dt \tag{4.3}
$$

for all $w(t)$ and $z(t)$ such that

$$
\begin{aligned}
& w_i(t) \in [-1, 1] \\
& \int_{t_0}^{t_f} |w_i(s)| \, ds \leq \Gamma_i^f(t)
\end{aligned}
$$

$$
\begin{aligned}
& z_i(t) \in [-1, 1] \\
& \int_{t_0}^{t_f} |z_i(s)| \, ds \leq \Theta_i^f(t)
\end{aligned}
$$

where

$$
\begin{aligned}
G = \int_{t_0}^{t_f} e^{-\rho t} & \Bigg\{ \sum_{r \in \mathcal{F}_R} \sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{N}} \frac{1}{1+\alpha_j^r} \Psi_j(\sum_{g \in \mathcal{F}_P} c_j^g) q_{ij}^{fr} \\
& - \sum_{i \in \mathcal{N}} \left(\alpha_i^f(t) + \beta_i^f(t). \cdot h_i^f \right) - \sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{N}} r_{ij} \left(s_{ij}^f + \sum_{r \in \mathcal{F}_R} q_{ij}^{fr} \right) - \sum_{i \in \mathcal{N}} \psi_i^f(I_i^f, t) \Bigg\} dt
\end{aligned}
\tag{4.4}
$$

this equation is same as the nominal form's objective function so that it is independent of w and z.

To guarantee the feasibility of above inequality (4.3), we need to find w and z that minimize the right-hand side. In addition, we can separate this problem across nodes and adjust the interval of z_i and w_i because $\widehat{\alpha}_i^f(t)$ and $\widehat{\alpha}_i^f(s)$ are positive assuming inventory holding cost rate $h_{i,int}^f(t)$ and linear function. Consequently, we need to solve

$$\max_{w,z} \int_{t_0}^{t_f} e^{-\rho t} \left(\widehat{\alpha}_i^f(t) w_i(t) + H_{i,int}^f(t) z_i(t) \widehat{\alpha}_i^f(t)\right) dt$$

subject to

$$0 \leq w_i(t) \leq 1$$
$$\int_{t_0}^{t_f} w_i(s) ds \leq \Gamma_i^f(t)$$

$$0 \leq z_i(t) \leq 1$$
$$\int_{t_0}^{t_f} z_i(s) ds \leq \Theta_i^f(t)$$

where $H_{i,int}^f(t) \equiv \int_t^{t_f} h_{i,int}^f(s) ds$. This is a particular instance of a continuous linear program. Therefore, we can write the robust optimization formulation for problem

NP described above as follows:

$$\max J_f^P(q^f, s^f, h^f, w, z; c) = \int_{t_0}^{t_f} e^{-\rho t} \Bigg\{ \sum_{r\in\mathcal{F}_R} \sum_{i\in\mathcal{N}} \sum_{j\in\mathcal{N}} \frac{1}{1+\alpha_j^r} \Psi_j(\sum_{g\in\mathcal{F}_P} c_j^g) q_{ij}^{fr}$$

$$- \sum_{i\in\mathcal{N}} C_i^f(h_i^f, t) - \sum_{i\in\mathcal{N}} \sum_{j\in\mathcal{N}} r_{ij} \left(s_{ij}^f + \sum_{r\in\mathcal{F}_R} q_{ij}^{fr} \right) - \sum_{i\in\mathcal{N}} \psi_i^f(I_i^f, t)$$

$$+ \widehat{\alpha}_i^f(t) w_i(t) + H_{i,int}^f(t) z_i(t) \widehat{\alpha}_i^f(t) \Bigg\} dt$$

subject to

$$\frac{dI_i^f}{dt} = a_i^f(t) + b_i^f(t) . \cdot h_i^f + \sum_{j\in\mathcal{N}} s_{ji}^f - \sum_{j\in\mathcal{N}} s_{ij}^f - \sum_{r\in\mathcal{F}_R} \sum_{j\in\mathcal{N}} q_{ij}^{fr} \quad \forall f \in \mathcal{F}_P, \forall r \in \mathcal{F}_R, i \in \mathcal{N}$$

$$I_i^f(t_0) = K_i^f \quad \forall i \in \mathcal{N}$$

$$I_i^f(t_f) \geq \tilde{K}_i^f \quad \forall i \in \mathcal{N}$$

$$0 \leq q_{ij}^{fr} \leq \widetilde{q}_{ij}^{fr} \quad i, j \in \mathcal{N}, r \in \mathcal{F}_r$$

$$0 \leq s_{ij}^f \leq \widetilde{s}_{ij}^f \quad i, j \in \mathcal{N}$$

$$A_f \leq \sum_{i\in\mathcal{N}} h_i^f \leq B_f \quad \forall f \in \mathcal{F}_P$$

$$u_N = \sum_{g\in\mathcal{F}_P} \sum_{i\in\mathcal{N}} h_i^g$$

$$0 \leq w_i(t) \leq 1 \tag{4.5}$$

$$\int_{t_0}^{t_f} w_i(s) ds \leq \Gamma_i^f(t) \tag{4.6}$$

$$0 \leq z_i(t) \leq 1 \tag{4.7}$$

$$\int_{t_0}^{t_f} z_i(s)ds \leq \Theta_i^f(t) \tag{4.8}$$

$$H_{i,int}^f(t) \equiv \int_t^{t_f} h_{i,int}^f(s)ds$$

Through this formulation, we can remove uncertain variables because all the parameters are known and pre-determined.

Additionally, we can consider the robust optimization model of the supplier. To remind us of the supplier's nominal problem, let me describe again

$$\min J^S(u) = \int_{t_0}^{t_f} e^{-\rho t} \sum_{k=1}^{N} \left[V_k(u_k, t) + \varphi_k\left(S_k, t\right)\right] dt$$

subject to

$$\frac{dS_k}{dt} = u_{k-1} - u_k \quad k = 1, ..., N$$

$$S_k(0) = S_k^0 \qquad k = 1, ..., N$$

$$u_N = \sum_{f \in \mathcal{F}_P} \sum_{i \in \mathcal{N}} h_i^f$$

$$0 \leq u_k \leq U_k \quad k = 0, ..., N$$

To begin with, let me assume an additive model of supplier's variable cost as follows:

$$V_k(u_k, t) = v_k(t) + g_k(t) \cdot u_k(t) + \zeta_k(t)$$

where $\zeta_k(t)$ is the uncertain parameter. In the same manner,

$$|\widetilde{v}_k(t) - v_k(t)| \leq \widehat{v}_k(t)$$
$$\int_{t_0}^{t_f} \frac{|\widetilde{v}_k(t) - v_k(t)|}{\widehat{v}_k(t)} ds \leq \Xi_k(t)$$

Let

$$y_k(t) \equiv \frac{\widetilde{v}_k(t) - v_k(t)}{\widehat{v}_k(t)}$$

be the scaled variation of uncertainty $\widetilde{v}_k(t)$,then

$$y_k(t) \in [-1, 1]$$
$$\int_{t_0}^{t_f} |y_k(t)|\, ds \leq \Xi_k(t)$$

Therefore, robust counterpart of supplier can be as

$$Min\ Y$$

subject to

$$Y \geq \int_{t_0}^{t_f} e^{-\rho t} \sum_{k=1}^{N} \left[\widetilde{V}_k(u_k, t) + \varphi_k\left(S_k, t\right)\right] dt \tag{4.9}$$

$$\frac{dS_k}{dt} = u_{k-1} - u_k \quad k = 1, ..., N$$
$$S_k(0) = S_k^0 \qquad k = 1, ..., N$$
$$u_N = \sum_{f \in \mathcal{F}_P} \sum_{i \in \mathcal{N}} h_i^f$$
$$0 \leq u_k \leq U_k \quad k = 0, ..., N$$

We can restate constraint (4.9) as follows

$$Y \geq R + \int_{t_0}^{t_f} e^{-\rho t} \left\{ \sum_{k=1}^{N} y_k(t) \widehat{v}_k(t) \right\} dt$$

for all $y_k(t)$ such that

$$y_k(t) \in [-1, 1]$$
$$\int_{t_0}^{t_f} |y_k(t)| \, ds \leq \Xi_k(t)$$

where

$$R = \int_{t_0}^{t_f} e^{-\rho t} \sum_{k=1}^{N} [v_k(t) + g_k(t) \cdot u_k(t) + \varphi_k (S_k, t)] \, dt$$

Consequently, we need to solve

$$\max \int_{t_0}^{t_f} e^{-\rho t} \left\{ \sum_{k=1}^{N} y_k(t) \widehat{v}_k(t) \right\} dt$$

subject to

$$y_k(t) \in [0, 1]$$
$$\int_{t_0}^{t_f} y_k(t) ds \leq \Xi_k(t)$$

To consider this subproblem with supplier's extremal problem, we need to consider dual formulation. A dual formulation for this class of problems was analyzed by Tyndall (1965) and Pullan (1997). Based on these research results, strong duality can be written as follows:

$$\min_{w,r} \omega_k(t)\Xi_k(t) + \int_{t_0}^{t} \sum_{k=1}^{N} r_k(s,t)ds$$

$$s.t \;\; \omega_k(t) + r_k(s,t) \geq \widehat{v}_k(t) \;\; k = 1, ..., N \quad \forall s \in [t_0.t]$$

$$\omega_k(t) \geq 0 \;\; \forall t$$

$$r_k(s,t) \geq 0 \;\; \forall t \quad \forall s \in [t_0.t]$$

As a result, the robust optimization formulation for supplier's extremal problem is:

$$\min J^S(u) = \int_{t_0}^{t_f} e^{-\rho t} \quad \sum_{k=1}^{N} [V_k(u_k,t) + \varphi_k (S_k,t)]\, dt + \omega_k(t)\Xi_k(t) + \int_{t_0}^{t} \sum_{k=1}^{N} r_k(s,t)ds$$

subject to

$$\frac{dS_k}{dt} = u_{k-1} - u_k \quad k = 1, ..., N$$

$$S_k(0) = S_k^0 \qquad\qquad k = 1, ..., N$$

$$u_N = \sum_{f \in \mathcal{F}_P} \sum_{i \in \mathcal{N}} h_i^f$$

$$0 \leq u_k \leq U_k \qquad k = 0, ..., N$$

$$s.t \;\; \omega_k(t) + r_k(s,t) \geq \widehat{v}_k(t) \;\; k = 1, ..., N \quad \forall s \in [t_0.t]$$

$$\omega_k(t) \geq 0 \;\; \forall t$$

$$r_k(s,t) \geq 0 \;\; \forall t \quad \forall s \in [t_0.t]$$

4.3 DVI Formulation

By assuming the firm(producer) has uncertainty in supply chain network, we can derive and use the DVI form for the robust formulation.

4.3.1 Maximum Principle for the firm

With

$$\begin{pmatrix} c \\ u_N \\ q^{-f} \end{pmatrix}$$

as exogenous, each firm $f \in \mathcal{F}_P$ solves

$$\max J_f^P(q^f, s^f, h^f, w, z; c) \quad \text{s.t.} \quad \left(q^f, s^f, h^f, w, z\right) \in \Lambda_P^f(h^{-f}, u_N) \times \mathcal{U}_P^f(w, z)$$

where

$$\Lambda_P^f(h^{-f}, u_N) \equiv \left\{ \begin{pmatrix} q^f \\ s^f \\ h^f \end{pmatrix} : (3.48), (3.49), (3.50), (3.51), (3.52), (3.53), (3.54) \text{ hold} \right\}$$

$$U_P^f(w, z) \equiv \left\{ \begin{pmatrix} w \\ z \end{pmatrix} : (4.5), (4.6), (4.7), (4.8) \text{ hold} \right\}$$

The corresponding Hamiltonian is

$$
\begin{aligned}
&H_P^f(q^f, s^f, h^f, w, z, I^f, \lambda^f; c) \\
&= e^{-\rho t}\Bigg\{\sum_{r\in\mathcal{F}_R}\sum_{i\in\mathcal{N}}\sum_{j\in\mathcal{N}} \frac{1}{1+\alpha_j^r}\Psi_j(\sum_{g\in\mathcal{F}_P} c_j^g) q_{ij}^{fr} \\
&\qquad - \sum_{i\in\mathcal{N}} C_i^f(h_{iN}^f, t) - \sum_{i\in\mathcal{N}}\sum_{j\in\mathcal{N}} r_{ij}\left(s_{ij}^f + \sum_{r\in\mathcal{F}_R} q_{ij}^{fr}\right) \\
&\qquad - \sum_{i\in\mathcal{N}}\left(\psi_i^f(I_i^f, t) + \widehat{\alpha}_i^f(t) w_i(t) + H_{i,int}^f(t) z_i(t) \widehat{\alpha}_i^f(t)\right)\Bigg\} \\
&\qquad + \sum_{i\in\mathcal{N}} \lambda_i^f \left[F_i^f(h_i^f) + \sum_{j\in\mathcal{N}} s_{ji}^f - \sum_{j\in\mathcal{N}} s_{ij}^f - \sum_{r\in\mathcal{F}_R}\sum_{j\in\mathcal{N}} q_{ij}^{fr}\right]
\end{aligned}
$$

where $I^f = \left(I_i^f : i \in \mathcal{N}\right)$ and $\lambda^f = \left(\lambda_i^f : i \in \mathcal{N}\right)$ is a vector of adjoint variables. The maximum principle for producer $f \in \mathcal{F}_P$ leads to:

$$
\begin{aligned}
&\left[\nabla_{q^f} H_P^{f*}\right]^T (q^f - q^{f*}) + \left[\nabla_{s^f} H_P^{f*}\right]^T (s^f - s^{f*}) + \left[\nabla_{h^f} H_P^{f*}\right]^T (h^f - h^{f*}) \\
&\quad + \left[\nabla_w H_P^{f*}\right]^T (w - w^*) + \left[\nabla_z H_P^{f*}\right]^T (z - z^*) \leq 0
\end{aligned}
$$

$$
\begin{pmatrix} q^f \\ s^f \\ h^f \end{pmatrix}, \begin{pmatrix} q^{f*} \\ s^{f*} \\ h^{f*} \end{pmatrix} \in \Lambda_P^f(h^{-f*}, u_N^*)
$$

$$
\begin{pmatrix} w \\ z \end{pmatrix}, \begin{pmatrix} w^* \\ z^* \end{pmatrix} \in U_P^f(w^*, z^*)
$$

4.3.2 The DVI for Robust optimization formulations

We note that the finite dimensional variational inequalities derived above hold for each instant of continuous time. So we may integrate the individual variational inequalities over time and sum them over discrete agent indices to obtain a single necessary condition: the solution

$$\begin{pmatrix} q^* \\ s^* \\ u^{f*} \\ w^* \\ z^* \\ c^* \\ u^{s*} \end{pmatrix} \in \Omega \tag{4.10}$$

must satisfy

$$\begin{aligned} \sum_{f \in \mathcal{F}_P} \int_{t_0}^{t_f} \Bigg\{ & \left[-\nabla_{q^f} H_P^{f*}\right]^T (q^f - q^{f*}) + \left[-\nabla_{s^f} H_P^{f*}\right]^T (s^f - s^{f*}) \\ & + \left[-\nabla_{u^f} H_P^{f*}\right]^T (u^f - u^{f*}) + \left[-\nabla_{w} H_P^{f*}\right]^T (w - w^*) + \left[-\nabla_{z} H_P^{f*}\right]^T (z - z^*) \Bigg\} dt \\ & + \int_{t_0}^{t_f} \sum_{r\mathcal{F}_R} \left[-\nabla_{c^r} H_R^{r*}\right]^T (c^r - c^{r*}) dt \\ & + \int_{t_0}^{t_f} \left[\nabla_{u^s} H_S^{*}\right]^T (u^s - u^{s*}) dt \geq 0 \end{aligned} \tag{4.11}$$

for all

$$\begin{pmatrix} q \\ s \\ u^f \\ w \\ z \\ c \\ u^s \end{pmatrix} \in \Omega$$

where

$$\Omega = \Lambda \times \Gamma \times \mathcal{U}$$

$$\Lambda = \Lambda_S(u^*) \times \prod_{f \in \mathcal{F}_P} \Lambda_P^f(u^{-f}) \times \prod_{r \in \mathcal{F}_R} \Lambda_R^r(q^{r*})$$

and Γ is the set of adjoint variables determined by the adjoint equations and the transversality conditions as same as the preceding.

Chapter 5

Algorithms

5.1 Fixed Point Algorithm

In order to apply the results developed above regarding the relationship of dynamic Nash games to differential variational inequalities, we must be able to compute the solutions to differential variational inequalities. It should come as no surprise that there is often an equivalent functional fixed point problem corresponding to a given differential variational inequality. This formulation provides an immediate, simple and often quite effective algorithm for solving $DVI(F, f, U, \Gamma, x_0)$.

5.1.1 Formulation

In particular, we are now ready to state and prove the following result:

Theorem 12. *Fixed point formulation of $DVI(F, f, U, \Gamma, x_0)$. When regularity in the sense of Definition 1 holds, $DVI(F, f, U, \Gamma, x_0)$ is equivalent to the following fixed point problem:*

$$u = P_U \left[u - \alpha F \left(x(u,t), u, t \right) \right]$$

where $P_U[.]$ *is the minimum norm projection onto* $U \subseteq \left(L^2[t_0,\tau]\right)^m$ *and* $\alpha \in \Re^1_{++}$ *is an arbitrary positive constant.*

Proof: The fixed point problem under consideration requires that

$$u = \arg\min_v \left\{ \frac{1}{2} \left\| u - \alpha F\left(x(u,t), u, t\right) - v \right\|^2 : v \in U \right\} \tag{5.1}$$

where $\alpha \in \Re^1_{++}$ is any positive real scalar. That is, we seek the solution of the optimal control problem

$$\min_v \gamma^T \Gamma\left[x\left(t_f\right), t_f\right] + \int_{t_0}^{t_f} \frac{1}{2}\left[u - \alpha F\left(x, u, t\right) - v\right]^2 dt$$

subject to

$$\begin{aligned} \frac{dx}{dt} &= f\left(x, v, t\right);\ x\left(t_0\right) = x_0 \\ v &\in U \end{aligned}$$

where u is treated as fixed for the purpose of projection. Because of regularity and the assumed convexity of $f(x,v,t)$, a necessary and sufficient condition for a solution $v^* \in U$ of this optimal control problem is

$$\left[\nabla_v H_1\left(x^*, v^*, \eta^*, t\right)\right]^T \left(v - v^*\right) \geq 0 \qquad \forall v \in U \tag{5.2}$$

where

$$H_1\left(x, v, \eta, t\right) = \frac{1}{2}\left[u - \alpha F\left(x, u, t\right) - v\right]^2 + \eta^T f\left(x, v, t\right)$$

and for given x and v

$$\eta = \arg \left\{ (-1) \frac{d\eta}{dt} = \nabla_x H_1 (x, v, \eta, t) \, , \;\; \eta (t_f) = \gamma^T \frac{\partial \Gamma [x (t_f) \, , t_f]}{\partial x (t_f)} \right\}$$

Note that

$$\nabla_v H_1 (x, v, \eta, t) = -u + \alpha F (x, u, t) + v + \nabla_v \eta^T f (x, v, t)$$

Because $u = v$ by virtue of (5.1) we have

$$\nabla_u H_1 (x, v, \eta, t) = \alpha F (x, u, t) + \nabla_u \eta^T f (x, u, t) \tag{5.3}$$

Now if we set $\lambda = \frac{\eta}{\alpha}$; we have

$$\left[F (x^*, u^*, t) + \nabla_u (\lambda^*)^T f (x^*, u^*, t) \right]^T (u - u^*) \geq 0 \qquad \forall v \in U \tag{5.4}$$

which is identical to the finite dimensional variational inequality principle of Theorem 4. The other optimality conditions are also identical. This completes the proof, since (5.4) is both a necessary and sufficient condition.

5.1.2 The Algorithm

Naturally there is an associated fixed point algorithm based on the iterative scheme

$$u^{k+1} = P_U \left[u^k - \alpha F \left(x \left(u^k, t \right) , u^k, t \right) \right]$$

The positive scalar may be chosen empirically to assist convergence and may even be changed during as the algorithm progresses. The detailed structure of the fixed point algorithm is given below:

Fixed Point Algorithm

Step 0. Initialization. Identify an initial feasible solution $u^0 \in U$ and set $k = 0$.

Step 1. Solve optimal control problem. Solve the following optimal control problem:

$$\begin{aligned} \min_v J^k(v) &= \gamma^T \Gamma[x(t_f), t_f] + \int_{t_0}^{t_f} \frac{1}{2} \left[u^k - \alpha F\left(x^k, u^k, t\right) - v\right]^2 dt \quad (5.5) \\ \text{subject to } \frac{dx}{dt} &= f(x, v, t);\ x(t_0) = x_0 \quad (5.6) \\ v &\in U \quad (5.7) \end{aligned}$$

Call the solution u^{k+1}.

Step 2. Stopping test. If $\left\|u^{k+1} - u^k\right\| \leq \varepsilon_1$ where $\varepsilon_1 \in \Re^1_{++}$ is a preset tolerance, stop and declare $u^* \approx u^{k+1}$. Otherwise set $k = k + 1$ and and go to Step 1.

The convergence of this algorithm is guaranteed by the following result:

Theorem 13. *When $DVI(F, f, U, \Gamma, x_0)$ is regular in the sense of Definition 1, while additionally $F(x, u, t)$ is strongly monotonic for $u \in U$, the fixed point algorithm presented above converges.*

Proof: Consider

$$u^{k+1} - u^* = P_U\left[u^k - \alpha F\left(x\left(u^k, t\right), u^k, t\right)\right] - P_U\left[u^* - \alpha F\left(x(u^*), u^*, t\right)\right]$$

and note that P_U is a contraction; that is, the projection of a vector is never greater in length than the length of the vector itself. Thus

$$\|P_U(v)\| \leq \|v\|$$

for any $v \in U \subseteq (L^2[t_0, \tau])^m$. Define

$$F^k = F\left(x\left(u^k, t\right), u^k, t\right); \; F^* = F\left(x(u^*), u^*, t\right)$$

Because F obeys a strong monotonicity condition, we have

$$\langle F^k - F^*, u^k - u^* \rangle \geq \varepsilon \left\| u^k - u^* \right\|$$

where $\varepsilon \in \Re^1_{++}$. We also know that both $\left\| F^k - F^* \right\|$ and $\left\| u^k - u^* \right\|$ are bounded, by virtue of the boundedness of U and the continuity of F. Consequently, there must exist $\beta \in \Re^1_{++}$ such that

$$\left\| F^k - F^* \right\|^2 \leq \beta \left\| u^k - u^* \right\|^2 \tag{5.8}$$

The contractive property of P_U and the strong monotonicity of F together with property (5.8) mean

$$\begin{aligned}
\left\| u^{k+1} - u^* \right\|^2 &\leq \left\| \left(u^k - u^*\right) - \alpha\left(F^k - F^*\right) \right\|^2 \\
&= \left\| u^k - u^* \right\|^2 + \alpha^2 \left\| F^k - F^* \right\|^2 - 2\alpha \langle F^k - F^*, u^k - u^* \rangle \\
&\leq (1 + \beta - 2\alpha\varepsilon) \left\| u^k - u^* \right\|^2
\end{aligned}$$

Note that we may chose $\alpha > 0$ such that $1 + \beta - 2\alpha\varepsilon < 1$ which is equivalent to $\alpha > \frac{\beta}{2\varepsilon}$ a condition ensuring

$$\left\|u^{k+1} - u^*\right\|^2 < \left\|u^k - u^*\right\|^2$$

Consequently, the algorithm is a strict contraction mapping and convergence is assured.

5.1.3 Solving the Sub-Problems

It is important to realize that the fixed point algorithm of Section 5.1 can be carried out in continuous time provided we employ a continuous time representation of the solution of each subproblem (5.5)-(5.7) from Step 1 of the fixed point algorithm. This may be done using a continuous time gradient projection method. For our present circumstances, that algorithm may be stated as

Descent Algorithm in Hilbert Space for the Projection Sub-Problems

Step 0. Initialization. Pick $v^{k,0}(t) \in U$ and set $j = 0$.

Step 1. Finding State Variables. Solve the state dynamics

$$\frac{dx}{dt} = f\left(x, v^{k,j}, t\right) \tag{5.9}$$

$$x(t_0) = x_0 \tag{5.10}$$

Call the solution $x^{k,j}(t)$ .In the event a discrete time method is used to solve the state dynamics (5.9) and (5.10), curve fitting is used to obtain the continuous time state vector $x^{k,j}(t)$.

Step 2. Finding adjoint variables. Solve the adjoint dynamics

$$(-1)\frac{d\lambda}{dt} = \nabla_x H^k \Big|_{\substack{v=v^{k,j}\\ x=x^{k,j}}} \tag{5.11}$$

$$\lambda(t_f) = \frac{\partial \Gamma\left[x^{k,j}(t_f), t_f\right]}{\partial x(t_f)} \tag{5.12}$$

where

$$H^k = \frac{1}{2}\left[u^k - \alpha F\left(x^k, u^k, t\right) - v\right]^2 + \lambda^T f\left(x, v^{k,j}, t\right)$$

Call the solution $\lambda^{k,j}(t)$.In the event a discrete time method is used to solve the adjoint dynamics (5.11) and (5.12), curve fitting is used to obtain the continuous time adjoint vector $\lambda^{k,j}(t)$.

Step 3. Finding the gradient. Determine

$$\nabla_v J^{k,j}(t) = \nabla_v H^k$$

Step 4. Stopping test. For a fixed and suitably small fixed step size

$$\theta_k \in \Re^1_{++}$$

determine

$$v^{k,j+1}(t) = P_U\left[v^{k,j}(t) - \theta_k \nabla_v J^{k,j}\right] \tag{5.13}$$

In the event a discrete time method is used to solve the above projection subproblem, curve fitting is used to obtain the continuous time control vector (5.13).

Step 5. Stopping test. For $\varepsilon_2 \in \Re^1_{++}$, a pre-set tolerance, stop if $\left\|v^{k,j+1} - v^{k,j}\right\| < \varepsilon_2$and declare $v^{k*} \approx v^{k,j+1}$. Otherwise set $j = j + 1$ and go to Step 1.

5.2 Gap Function Method for *DVI(F, f, U,* Γ*,*x_0*)*

Additionally, we can consider another algorithm for solving differential variational inequality (DVI) problem. Using the notion of a gap function, a variational inequality problem can be converted to an equivalent optimization problem, whose objective function is always nonnegative and whose optimal objective function value is zero if and only if the optimal solution solves the original variational inequality problem.

5.2.1 Gap Functions for *DVI(F, f, U,* Γ*,*x_0*)*

When the regularity conditions given in Definition 1 hold, *DVI(F, f, U,* Γ*,*x_0*)* belongs to the class of infinite dimensional variational inequalities considered by Konnov, Kum, and Lee (2002), wherein U is a non-empty closed and convex subset and F is a continuously differentiable mapping of u. This allows us to analyze *DVI(F, f, U,* Γ*,*x_0*)* by considering gap functions, which we define as the following based on Friesz (2010):

Definition 14. *A function* $G : U \longrightarrow \Re_+$ *is called a gap function for DVI(F, f, U,* Γ*,*x_0*) when the following statements hold:*

1. $G(u) \geq 0$ *for all* $u \in U$
2. $G(u) = 0$ *if and only if* u *is the solution of DVI(F, f, U,* Γ*,*x_0*).*

In particular, we will consider gap functions of the form

$$G_\alpha(u) = \max_{v \in U} \Phi_\alpha(u, v) \tag{5.14}$$

where

$$\Phi_\alpha(u,v) = \langle F[x(u,t),u,t], u-v\rangle - \alpha\phi(u,v) \tag{5.15}$$

$$x(u,t) = \arg\left\{\frac{dy}{dt} = f(y,u,t),\ y(t_0)=y_0,\ \varPsi[y(t_f),t_f]=0\right\} \in \left(\mathcal{H}^1[t_0,t_f]\right)^n \tag{5.16}$$

$$U \subseteq \left(L^2[t_0,t_f]\right)^m \tag{5.17}$$

$$\alpha \in \Re^1_{++} \tag{5.18}$$

where the function ϕ appearing in (5.14) satisfies the following assumptions:

A1. ϕ is continuously differentiable on $\left(L^2[t_0,t_f]\right)^{2m}$;

A2. ϕ is non-negative on $\left(L^2[t_0,t_f]\right)^{2m}$;

A3. $\phi(u,v)=0$ if and only if $u=v$; and

A4. $\phi(u,v)$ is strongly convex in $v\in U$ with modulus $c>0$ for any $u \in \left(L^2[t_0,t_f]\right)^m$; that is

$$\phi(u,v) + \langle\nabla_v\phi(u,v), u-v\rangle + \frac{1}{2}c\|u-v\|^2 \le \phi(u,u) = 0$$

for all $u\in U$.

Yamashita, Taji, and Fukushima (1997) propose, for finite-dimensional spaces, the following ϕ-functions that satisfy the four assumptions listed above:

1. $\phi_1(u,v) = \tau_1(u-v)$, where τ_1 is non-negative, continuously differentiable, strongly convex, and $\tau_1(0)=0$;

2. $\phi_2(u,v) = \tau_2(v) - \tau_2(u) - \langle \nabla \tau_2(u), u - v \rangle$, where τ_2 is twice continuously differentiable, and strongly convex; and

3. $\phi_3(u,v) = \langle u - v, M(u)(u - v) \rangle$, where $M(u)$ is a continuously differentiable, symmetric, and uniformly positive-definite matrix.

Lemma 15. *Gap function for DVI(F, f, U, Γ,x_0). The function $G_\alpha(u)$ defined by (5.14) is a gap function for DVI(F, f, U, Γ,x_0). In particular, u is the solution to DVI(F, f, U, Γ,x_0), if and only if $u = v_\alpha(u)$.*

Proof: The proof is in two parts.

(i) $[u = v_\alpha(u) \Longrightarrow DVI(F,\ f,\ U,\ \Gamma, x_0)]$ The optimality condition for (5.14) is

$$\left\langle \frac{\partial \Phi_\alpha(u, v_\alpha)}{\partial v}, v - v_\alpha \right\rangle \leq 0 \qquad \forall v \in U$$

That is

$$\langle -F(x,u,t) - \alpha \nabla_v \phi(u, v_\alpha), v - v_\alpha \rangle \leq 0 \qquad \forall v \in U \tag{5.19}$$

Substituting u for v in (5.19), we obtain

$$\langle F(x,u,t), u - v_\alpha \rangle \geq -\alpha \langle \nabla_v \phi(u, v_\alpha), u - v_\alpha \rangle \qquad \forall u \in U \tag{5.20}$$

Note that strong convexity for the ϕ-function intrinsic to the gap function means the following extension of the tangent line property holds:

$$\phi(u, v_\alpha) + \langle \nabla_v \phi(u, v_\alpha), u - v_\alpha \rangle + \frac{1}{2} c \|u - v_\alpha\| \leq \phi(u,u) \tag{5.21}$$

By virtue of relationships (5.20) and (5.21), we have

$$\begin{aligned} G_\alpha(u) &= \Phi_\alpha(u, v_\alpha) \\ &= \langle F(x,u,t), u - v_\alpha \rangle - \alpha\phi(u, v_\alpha) \\ &\geq -\alpha \langle \nabla_v \phi(u, v_\alpha), u - v_\alpha \rangle - \alpha\phi(u, v_\alpha) \\ &= \alpha \left[\phi(u,u) - \phi(u, v_\alpha) - \langle \nabla_v \phi(u, v_\alpha), u - v_\alpha \rangle \right] \\ &\geq \frac{\alpha c}{2} \| v_\alpha - u \|^2 \geq 0 \quad , \end{aligned} \tag{5.22}$$

where the property $\phi(u,u) = 0$ is used. Therefore, $G_\alpha(u) \geq 0$ for all $u \in U$. Moreover, if $G_\alpha(u) = 0$, then by (5.22) we have $u = v_\alpha$. From (5.20), we see by inspection that $u = v_\alpha$ solves $DVI(F,\ f,\ U,\ \Psi,\ \ x_0,\ t_0,\ t_f)$.

(ii) $[DVI(F,\ f,\ U, \Gamma, x_0) \Longrightarrow u = v_\alpha(u)]$ Suppose now that u is a solution of $DVI(F,\ f,\ U,\ \Gamma, x_0)$. Then

$$\langle F(x,u,t), v - u \rangle \geq 0 \qquad \forall v \in U$$

and it follows that

$$\begin{aligned} \Phi_\alpha(u, v) &= \langle F(x,u,t), u - v \rangle - \alpha\phi(u, v) \\ &\leq -\alpha\phi(u, v) \end{aligned}$$

for all $v \in U$. It follows that

$$G_\alpha(u) = \max_{v \in U} \Phi_\alpha(u, v) \leq -\alpha\phi(u, v_\alpha)$$

which contradicts the nonnegativity property of $G_\alpha(u)$, unless $G_\alpha(u) = 0$ and

$u = v_\alpha(u)$.

5.2.2 D-gap Function

Note that the preceding definition of the gap function does not ensure that $G_\alpha(u)$ is in general differentiable, a limitation we would like to overcome. To that end, let us introduce the so-called a *D-gap function*, which is based on the primitive gap functions G_α and G_β introduced above and has the form

$$\psi_{\alpha\beta}(u) = G_\alpha(u) - G_\beta(u) \tag{5.23}$$

for $0 < \alpha < \beta$. While $G_\alpha(u)$ is not differentiable in general, $\psi_{\alpha\beta}(u)$ is G-differentiable. To show that $\psi_{\alpha\beta}(u)$ is a gap function, we only need to show the essential nonnegativity property holds. We continue to invoke assumptions A1, A2, A3 and A4; hence, by virtue of strong convexity of $\phi(u)$, we have

$$\begin{aligned}
\psi_{\alpha\beta}(u) &= G_\alpha(u) - G_\beta(u) \\
&= \Phi_\alpha(u, v_\alpha) - \Phi_\beta(u, v_\beta) \\
&\geq \Phi_\alpha(u, v_\beta) - \Phi_\beta(u, v_\beta) \\
&= \langle F(x, u, t), u - v_\beta \rangle - \alpha\phi(u, v_\beta) - \langle F(x, u, t), u - v_\beta \rangle + \beta\phi(u, v_\beta) \\
&= (\beta - \alpha)\phi(u, v_\beta)
\end{aligned}$$

This demonstrates $\psi_{\alpha\beta}(u) \geq 0$, and, of course, $G_\alpha(u) \geq G_\beta(u)$ for all $u \in U$. So (5.23) does in fact define a gap function.

We will employ, as our D-gap function, the gap function Fukushima (1992) has named the *regularized gap function* for finite dimensional spaces and Konnov, Kum, and Lee (2002) have extended to Hilbert spaces. In particular, we introduce as a generator of the regularized gap function the following

$$\phi(u,v) = \frac{1}{2}\|v-u\|^2 \tag{5.24}$$

which satisfies the relevant assumptions on $\phi(\cdot)$, especially its strong convexity with modulus $\alpha > 0$. From (5.15) and (5.24) we get

$$\Phi_\alpha(u,v) = \langle F(x,u,t), u-v\rangle - \frac{\alpha}{2}\|v-u\|^2$$

The corresponding D-gap function becomes

$$\psi_{\alpha\beta}(u) = G_\alpha(u) - G_\beta(u) = \max_{v\in U}\Phi_\alpha(u,v) - \max_{v\in U}\Phi_\beta(u,v)$$

or, alternatively

$$\psi_{\alpha\beta}(u) = \langle F(x,u,t), v_\beta(u) - v_\alpha(u)\rangle - \frac{\alpha}{2}\|v_\alpha(u)-u\|^2 + \frac{\beta}{2}\|v_\beta(u)-u\|^2 \tag{5.25}$$

where

$$v_\alpha(u) = \arg\max_{v\in U}\Phi_\alpha(u,v) \tag{5.26}$$

$$v_\beta(u) = \arg\max_{v\in U}\Phi_\beta(u,v) \tag{5.27}$$

Furthermore, it should be noted that, for a fixed $u \in U$, the maximization problem (5.26) is equivalent to the following:

$$v_\alpha(u) = \arg\min_{v\in U}\left\|v - \left(u - \frac{1}{\alpha}F(x,u,t)\right)\right\|^2$$

which may be rewritten in the form of a fixed point problem involving a projection operator, namely

$$v_{\alpha}(u) = P_U\left[u - \frac{1}{\alpha}F(x,u,t)\right] \tag{5.28}$$

as observed in Fukushima (1992) for finite dimensions and Konnov, Kum, and Lee (2002) for infinite dimensions.

Using a D-gap function (5.25), *DVI(F, f,* x_0*,* t_0*,* t_f*)* may be re-stated as the following equivalent optimal control problem *OCP(*F_0*, f,* x_0*)*:

$$\begin{aligned}
&\min \psi_{\alpha\beta}(u) \\
&\quad = \langle F(x,u,t), v_{\beta}(u) - v_{\alpha}(u)\rangle - \frac{\alpha}{2}\|v_{\alpha}(u) - u\|^2 + \frac{\beta}{2}\|v_{\beta}(u) - u\|^2 \\
&\quad = \int_{t_0}^{t_f}\left\{F(x,u,t)[v_{\beta}(u) - v_{\alpha}(u)] - \frac{\alpha}{2}[v_{\alpha}(u) - u]^2 + \frac{\beta}{2}[v_{\beta}(u) - u]^2\right\}dt \\
&\quad = \int_{t_0}^{t_f} F_0(x,u,t)\,dt
\end{aligned} \tag{5.29}$$

subject to

$$\frac{dx}{dt} = f(x,u,t) \tag{5.30}$$

$$x(t_0) = x_0 \tag{5.31}$$

where

$$F_0(x,u,t) \equiv F(x,u,t)[v_{\beta}(u) - v_{\alpha}(u)] - \frac{\alpha}{2}[v_{\alpha}(u) - u]^2 + \frac{\beta}{2}[v_{\beta}(u) - u]^2$$

For brevity, in the above, there are neither control constraints nor terminal constraints. A more general differential variational inequality and a correspondingly more general optimal control problem could be considered without complications.

Now we are interested in the gradient of the objective functional $\psi_{\alpha\beta}(u)$, which is equivalent to the gradient of the corresponding Hamiltonian in the theory of optimal control, owing to the particular function spaces we have elected in this chapter. In particular, the Hamiltonian for *OCP(*F_0*, f,* x_0*) is*

$$\begin{aligned} H(x,u,\lambda,t) &= F(x,u,t)\left[v_\beta(u) - v_\alpha(u)\right] \\ &\quad - \frac{\alpha}{2}\left[v_\alpha(u) - u\right]^2 + \frac{\beta}{2}\left[v_\beta(u) - u\right]^2 + \lambda f(x,u,t) \end{aligned} \tag{5.32}$$

To obtain the gradient of $\psi_{\alpha\beta}(u)$, we need to carefully consider the role of $v_\alpha(\cdot)$ and $v_\beta(\cdot)$ which are maximizers defined by (5.26) and (5.27). In particular, $v_\alpha(\cdot)$ and $v_\beta(\cdot)$ are unique by the strong concavity of $\Phi_\alpha(u,v)$ and $\Phi_\beta(u,v)$ in v and the convexity of the set U. To continue our analysis, we employ, without proof, the following lemma from Pshenichnyi (1971):

Lemma 16. *Gradient and G-derivatives. Let V be an abstract Hilbert space, $U \subseteq V$ and $h : V \times U \to \Re$ a mapping whose gradient $\nabla_u h(u,v)$ exists everywhere on U and is continuous on $V \times U$. Define two functions as follows:*

$$w(u) = \max_{v \in U} h(u,v)$$
$$z(u) = \{v \in U : w(u) = h(u,v)\}$$

Then the G-derivative of $w(u)$ and the G-derivative of $h(u,v)$ in the direction ρ are related according to

$$\delta w(u,\rho) = \max_{v \in z(u)} \delta h(u,\rho; v,\rho)$$

Furthermore, if $z(u)$ is a singleton for all $u \in V$, and z is a continuous function *on V, then w is continuously differentiable and its gradient is given by*

$$\nabla w(u) = \nabla \max_{v \in U} h(u,v) = \nabla_u h(u, z(u)).$$

Now we are in a position to articulate the gradient of the objective functional $\psi_{\alpha\beta}(u)$:

Proposition 17. *Gradient of D-gap function. Suppose $F(x,u,t)$ is Lipschitz continuous on every bounded subset of $\left(L^2[t_0,t_f]\right)^m$. Then $\psi_{\alpha\beta}(u)$ is continuously differentiable in the sense of Gateaux, and*

$$\begin{aligned}\nabla\psi_{\alpha\beta}(u) &= \frac{\partial}{\partial u} H(x,u,\lambda,t) \\ &= \frac{\partial F(x,u,t)}{\partial u}\left[v_\beta(u) - v_\alpha(u)\right] \\ &+ \alpha\left[v_\alpha(u) - u\right] - \beta\left[v_\beta(u) - u\right] + \lambda \frac{\partial f(x,u,t)}{\partial u}\end{aligned}$$

Proof: Rewrite the objective functional as

$$\begin{aligned}\psi_{\alpha\beta}(u) &= G_\alpha(u) - G_\beta(u) \\ &= \max_{v\in U}\left\{\int_{t_0}^{t_f} F(x,u,t)\left[u - v\right] - \frac{\alpha}{2}\left[v-u\right]^2 dt\right\} \\ &- \max_{v\in U}\left\{\int_{t_0}^{t_f} F(x,u,t)\left[u - v\right] - \frac{\beta}{2}\left[v-u\right]^2 dt\right\}\end{aligned}$$

Let us define

$$\begin{aligned}g_\alpha(u,v) &= F(x,u,t)\left[u-v\right] - \frac{\alpha}{2}\left[v-u\right]^2 \\ g_\beta(u,v) &= F(x,u,t)\left[u-v\right] - \frac{\beta}{2}\left[v-u\right]^2\end{aligned}$$

Then, by Lemma 16 we have

$$\delta G_\alpha(u,\rho)=\int_{t_0}^{t_f}\left\{\frac{\partial g_\alpha(u,v_\alpha)}{\partial x}y+\frac{\partial g_\alpha(u,v_\alpha)}{\partial u}\rho\right\}dt$$
$$\delta G_\beta(u,\rho)=\int_{t_0}^{t_f}\left\{\frac{\partial g_\beta(u,v_\beta)}{\partial x}y+\frac{\partial g_\beta(u,v_\beta)}{\partial u}\rho\right\}dt$$

so that

$$\delta\psi_{\alpha\beta}(u,\rho)=\int_{t_0}^{t_f}\left\{\frac{\partial g}{\partial x}y+\frac{\partial g}{\partial u}\rho\right\}dt \tag{5.33}$$

where we for simplicity of notation we write

$$g(u)=g_\alpha(u,v_\alpha)-g_\beta(u,v_\beta)$$

and $y=\delta x$ is a variation in x which implicitly depends on ρ. Furthermore, by definition

$$x(t)=x_0+\int_{t_0}^{t}f(x,u,y)\,dt$$

Also, we know from the analysis of continuous time optimal control problems in Friesz (2010) that

$$y=\int_{t_0}^{t}\left[\frac{\partial f}{\partial x}y+\frac{\partial f}{\partial u}\rho\right]dt$$

We introduce the adjoint vector defined by the final value problem

$$-\frac{d\lambda}{dt}=\left(\frac{\partial f}{\partial x}\right)^T\lambda+\left(\frac{\partial g}{\partial x}\right)^T \tag{5.34}$$

$$\lambda(t_f)=0 \tag{5.35}$$

so that (5.33) becomes

$$\delta\psi_{\alpha\beta}\left(u,\rho\right)=\int_{t_0}^{t_f}\left\{\left[-\left(\frac{d\lambda}{dt}\right)^T-\lambda^T\frac{\partial f}{\partial x}\right]y+\frac{\partial g}{\partial u}\rho\right\}dt$$

Noting that $y\left(t_0\right)=0$ and $\lambda\left(t_f\right)=0$, the by now familiar step of integration by parts yields

$$\begin{aligned}\int_{t_0}^{t_f}-\left(\frac{d\lambda}{dt}\right)^T y dt &= \int_{t_0}^{t_f}\lambda^T\frac{dy}{dt}dt\\ &= \int_{t_0}^{t_f}\lambda^T\left[\frac{\partial f}{\partial x}y+\frac{\partial f}{\partial u}\rho\right]dt\end{aligned}$$

It follows that

$$\begin{aligned}\delta\psi_{\alpha\beta}\left(u,\rho\right) &= \int_{t_0}^{t_f}\left\{\lambda^T\left[\frac{\partial f}{\partial x}y+\frac{\partial f}{\partial u}\rho\right]-\lambda^T\frac{\partial f}{\partial x}y+\frac{\partial g}{\partial u}\rho\right\}dt\\ &= \int_{t_0}^{t_f}\left\{\lambda^T\frac{\partial f}{\partial u}+\frac{\partial g}{\partial u}\right\}\rho dt\\ &= \left\langle\lambda^T\frac{\partial f}{\partial u}+\frac{\partial g}{\partial u},\rho\right\rangle\end{aligned}$$

Therefore, the gradient of $\psi_{\alpha\beta}\left(u\right)$ becomes

$$\begin{aligned}\nabla\psi_{\alpha\beta}\left(u\right) &= \lambda^T\frac{\partial f}{\partial u}+\frac{\partial g}{\partial u}\\ &= \nabla_u H\left(x,u,\lambda,t\right)\end{aligned}$$

Furthermore, we note that

$$\begin{aligned}\nabla \psi_{\alpha\beta}(u) &= \nabla_u H(x,u,\lambda,t) \\ &= \lambda \frac{\partial f(x,u,t)}{\partial u} + \frac{\partial F(x,u,t)}{\partial u}\left[v_\beta(u) - v_\alpha(u)\right] \\ &\qquad + \alpha\left[v_\alpha(u) - u\right] - \beta\left[v_\beta(u) - u\right]\end{aligned}$$

Also

$$-\frac{d\lambda}{dt} = \nabla_x H(x,u,\lambda,t)$$

To solve an extremal problem with fixed initial time, fixed terminal time and free terminal state using the D-gap function in Hilbert space, one needs certain additional information. In particular, the terminal time constraint

$$\Psi\left[x(t_f), t_f\right] = 0 \tag{5.36}$$

and the final value of the adjoint vector

$$\lambda(t_f) = \mu^T \frac{\partial \Psi\left[x(t_f), t_f\right]}{\partial x}$$

where μ is the Lagrange multiplier for (5.36) are needed.

5.2.3 D-gap function Numerical Application

Consider the following simple example of three firms with fixed inventories and both upper and low bounds on their prices and the demands they fulfill. We elect to solve this problem using time discretization and a finite dimensional gap function. The three sellers' initial endowments are $K_s = \{3000, 2000, 2500\}$ $s = 1, 2, 3$. Time horizon $t_0 = 0$ and $t_f = 5$. Observed demand for the output of each seller $s = 1, 2, 3$ is

$$h_s\left[\pi(t)\right] = a_s(t) - b_s(t)\pi_s(t) + \sum_{i \neq s} c_i(t)\pi_i(t)$$

$$\begin{aligned} a_s(t) &= \{300, 200, 300\} \\ b_s(t) &= 2 - 0.2t \qquad s = 1, 2, 3 \\ c_s(t) &= 0.1t + 0.5 \qquad s = 1, 2, 3 \end{aligned}$$

$$\begin{aligned} h_1(t) &= 300 - (2 - 0.2t)\pi_1(t) + (0.1t + 0.5)\pi_2(t) + (0.1t + 0.5)\pi_3(t) \\ h_2(t) &= 200 - (2 - 0.2t)\pi_2(t) + (0.1t + 0.5)\pi_1(t) + (0.1t + 0.5)\pi_3(t) \\ h_3(t) &= 300 - (2 - 0.2t)\pi_3(t) + (0.1t + 0.5)\pi_1(t) + (0.1t + 0.5)\pi_2(t) \end{aligned}$$

Both upper and low bounds on their prices and demands are as follows

$$D_{\min} = 0$$

$$\pi_{\min} = \begin{pmatrix} 50 \\ 35 \\ 45 \end{pmatrix}$$

$$\pi_{\max} = \begin{pmatrix} 300 \\ 300 \\ 300 \end{pmatrix}$$

This example is solved using D-gap function method. Once a differential gap function has been formed for DVI, it is used to create a nonlinear program. This nonlinear program is solved by Decent Method. In decent method, $\alpha = 10$ and $\beta = 10.2$ $(0 < \alpha < \beta)$ is used. This example forms $DVI(F_0,\, f,\, U,\, \Gamma, x_0)$ as follows:

$$F_0 = \begin{pmatrix} D \\ \pi \end{pmatrix} \qquad u = \begin{pmatrix} \pi \\ D \end{pmatrix}$$

$$\Psi = \begin{matrix} y_1(10) = 3000 \\ y_2(10) = 2000 \\ y_3(10) = 2500 \end{matrix} \qquad f = \frac{dy}{dt} = D$$

$$U = \left\{ u : \pi_{\min} - \pi_s \leq 0,\, \pi_s - \pi_{\max} \leq 0,\, D_s \geq 0,\, D_s - h_s\left(\pi_s, \pi^{-s}\right) \quad s = 1, 2, 3 \right\}$$

In this example, we employ a D-gap function of the form

$$\psi_{\alpha\beta} = \varphi_\alpha(u) - \varphi_\beta(u)$$

where

$$\varphi_\alpha(u) = \max < F(y,u,t), v-u > -\frac{\alpha}{2}\|v-u\|^2$$

$$\varphi_\beta(u) = \max < F(y,u,t), v-u > -\frac{\beta}{2}\|v-u\|^2$$

Therefore

$$\psi_{\alpha\beta} =< F(y,u,t), v_\beta(u) - v_\alpha(u) > -\frac{\alpha}{2}\|v_\alpha(u)-u\|^2 + \frac{\beta}{2}\|v_\beta(u)-u\|^2$$

where

$$v_\alpha(u) = P_U\left[u - \frac{1}{\alpha}F(y,u,t)\right] = P_U\left[\begin{pmatrix}\pi\\ D\end{pmatrix} - \frac{1}{10}\begin{pmatrix}D\\ \pi\end{pmatrix}\right]$$

$$v_\beta(u) = P_U\left[u - \frac{1}{\beta}F(y,u,t)\right] = P_U\left[\begin{pmatrix}\pi\\ D\end{pmatrix} - \frac{1}{10.2}\begin{pmatrix}D\\ \pi\end{pmatrix}\right]$$

Then the gradient information we need is

$$\nabla\psi_{\alpha\beta} = \frac{\partial F(y,u,t)}{\partial u}\left[v_\beta(u) - v_\alpha(u)\right] + \alpha\left[v_\alpha(u) - u\right] - \beta\left[v_\beta(u) - u\right] + \lambda\frac{\partial f(y,u,t)}{\partial u}$$

We employ the constant step size $\theta_k = 0.5$. Consequently, we solve the above differential variational inequality using D-gap function. In our calculations GAMS/PATHNLP and Matlab are used to solve this problem with 10000 iterations and discretization by time approximation with $N = 1000$ equal time steps. The results are generated as following table and figures.

interation k	*gap* $\psi_{\alpha\beta}(u^k)$
1	16.1861
2	0.1335
3	0.1333
.	.
.	.
.	.
500	0.1132
1000	0.0922
2000	0.0412
5000	0.0368
8000	0.0312
10000	0.0283

Table 5.1. D-gap function example result

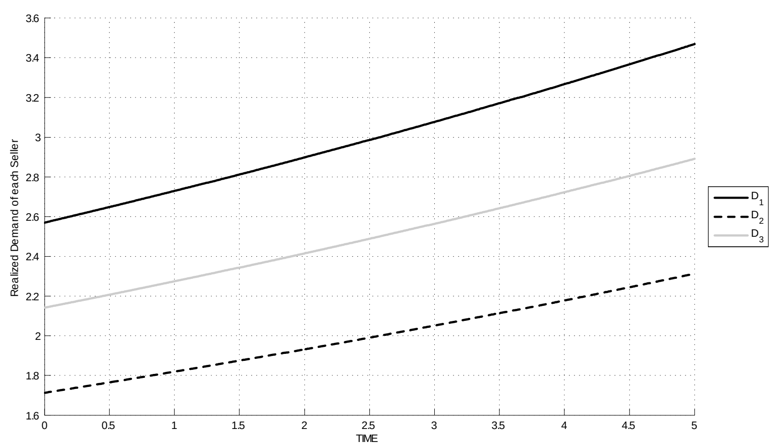


Figure 5.1. Realized demand trajectories of sellers

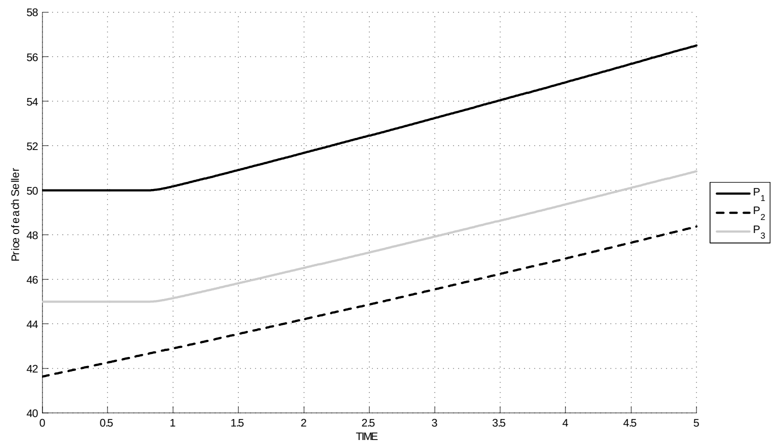


Figure 5.2. Price trajectories of sellers

Chapter 6

Numerical Example

This numerical example deals with both the deterministic case and the case considering uncertain production and transportation functions. While network topologies for suppliers, producers, and retailers are the same for both cases , suppliers' transportation costs and producers' production and transportation costs differ reflecting the network disruption risk. In this example, the network consists of 2 arcs and 3 nodes for suppliers and 9 arcs and 7 nodes for producers and retailers. In addition, there are 5 arcs between suppliers and producers. Producer firm 1 and producer firm 2 have activities located at nodes $i = 1, 2, 3, 4, 5$. Retailer 1 is located at node 6 and retailer 2 is located at node 7. The time interval of interest is $[0, 10]$, meaning $t_0 = 0$ and $t_f = 10$.

6.1 Deterministic case (Without Uncertainty)

The initial inventories at each node for producers $i = 1, 2, 3, 4, 5$ of firm $f \in \mathcal{F}_P$ $(f = 1, 2)$ and retailers $r \in \mathcal{F}_R$ $(r = 1, 2)$ at node $j = 6, 7$ are the following:

$$I_1^1(0) = 2 \quad I_2^2(0) = 3 \quad I_3^1(0) = 2 \quad I_4^2(0) = 3 \quad I_5^2(0) = 2 \quad R_6^1(0) = 1 \quad R_7^1(0) = 1$$

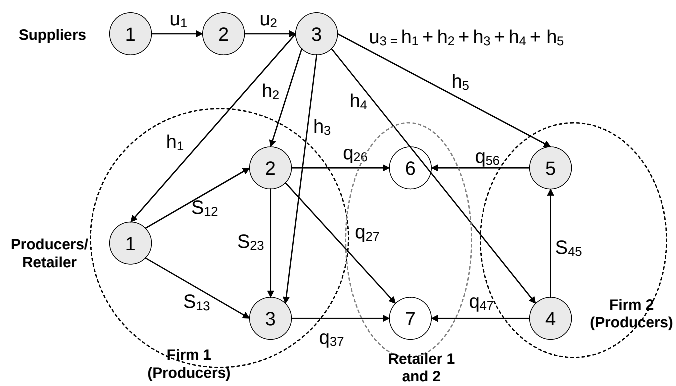


Figure 6.1. Supply Chain Network

The initial inventories for suppliers are the following:

$$S_0(0) = 10 \quad S_1(0) = 3 \quad S_2(0) = 1$$

The discount rate ρ is assumed to be 0.05 and the retailers' margins at nodes $j = 6$ and $j = 7$ are 0.1 and 0.08, respectively. Keeping in mind that the retailers occupy distinct nodes, the consumers' inverse demand functions for nodes $j = 6$ and $j = 7$ are assumed to be the following:

$$\Psi_6(c_6, c_7) = 11 - c_6 \qquad \Psi_7(c_6, c_7,) = 11 - 1.5 \cdot c_7$$

where c_j is the consumption at node j from the sole retailer located there. The production cost functions at nodes $i = 1, 2, 3, 4, 5$ are the following:

$$F_1^1(h_1^1) = 1. \cdot \frac{\left(h_1^1\right)^2}{2} \quad F_2^1(h_2^1) = 0.2 \cdot \frac{\left(h_2^1\right)^2}{2} \quad F_3^1(h_3^1) = 0.3 \cdot \frac{\left(h_3^1\right)^2}{2}$$

$$F_4^2(h_4^2) = 0.4 \cdot \frac{\left(h_4^2\right)^2}{2} \quad F_5^2(h_5^2) = 0.6 \cdot \frac{\left(h_5^2\right)^2}{2}$$

where h_i^f denotes the input factor flow from the final (3^{rd}) stage of the supply chain to producer i of firm $f \in \mathcal{F}_P$ ($f = 1, 2$). The inventory cost functions for producers $i = 1, 2, 3, 4, 5$ of firm $f \in \mathcal{F}_P$ ($f = 1, 2$) are as follows:

$$\psi_1^1(I_1^1, t) = 3 \cdot \frac{\left(I_1^1\right)^2}{2} \quad \psi_2^1(I_2^1, t) = 3 \cdot \frac{\left(I_2^1\right)^2}{2} \quad \psi_3^1(I_3^1, t) = 3 \cdot \frac{\left(I_3^1\right)^2}{2}$$

$$\psi_4^2(I_4^2, t) = 4 \cdot \frac{\left(I_4^2\right)^2}{2} \quad \psi_5^2(I_5^2, t) = 3 \cdot \frac{\left(I_5^2\right)^2}{2}$$

The inventory cost functions for retailers $r \in \mathcal{F}_R$ ($r = 1, 2$) at nodes $j = 6, 7$, respectively, are

$$\phi_6^1(R_6^1, t) = 6 \cdot \frac{\left(R_6^1\right)^2}{2} \qquad \phi_7^2(R_7^2, t) = 5 \cdot \frac{\left(R_7^2\right)^2}{2}$$

The inventory cost functions for stages $k = 1, 2, 3$ of the supply chain are as follows:

$$\varphi_1(S_1, t) = 5 \cdot (S_1)^2 \quad \varphi_2(S_2, t) = \frac{3}{8} \cdot (S_2)^2 \quad \varphi_3(S_3, t) = (S_3)^2$$

The freight rates r_{ij} from node i to node j are constant and are shown below:

$$r_{12} = 2 \quad r_{13} = 2 \quad r_{23} = 0.5 \quad r_{45} = 0.5 \quad r_{26} = 4 \quad r_{27} = 3 \quad r_{37} = 5 \quad r_{56} = 4 \quad r_{47} = 3$$

The costs to producers at nodes $i = 1, 2, 3, 4, 5$ of firm $f \in \mathcal{F}_P$ ($f = 1, 2$) for acquiring input flows are the following:

$$C_1^1(h_1^1, t) = 2 \cdot h_1^1 \quad C_2^1(h_2^1, t) = 2.5 \cdot h_2^1$$

$$C_3^1(h_3^1, t) = 2.1 \cdot h_3^1 \quad C_4^2(h_4^2, t) = 3 \cdot h_4^2 \quad C_5^2(h_5^2, t) = 2.1 \cdot h_5^2$$

Finally, the variable costs of preparing the stage $k = 1, 2, 3$ supply chain flows are presented below:

$$V_1(u_1, t) = 1.2 \cdot u_1 \quad V_2(u_2, t) = 1.3 \cdot u_2 \quad V_3(u_3, t) = 2 \cdot u_3$$

The following upper bounds on control variables are imposed:

$$\widetilde{q} = \begin{bmatrix} 10 \\ 10 \\ 10 \\ 10 \\ 10 \end{bmatrix} \quad \widetilde{s} = \begin{bmatrix} 10 \\ 10 \\ 10 \\ 10 \end{bmatrix} \quad \widetilde{h} = B_f = \begin{bmatrix} 8 \\ 8 \\ 8 \\ 8 \\ 8 \end{bmatrix} \quad \widetilde{c} = \begin{bmatrix} 10 \\ 10 \end{bmatrix} \quad \widetilde{u} = \begin{bmatrix} 15 \\ 15 \\ 15 \end{bmatrix}$$

Keeping in mind that the subnetworks for producers and retailers are disjoint, the initial value problems that constitute inventory dynamics for producers $i = 1, 2, 3, 4, 5$ of firm $f \in \mathcal{F}_P$ $(f = 1, 2)$ are the following flow balance equations:

$$\begin{aligned}
\tfrac{dI_1^1}{dt} &= F_1^1(h_1^1) - s_{12}^1 - s_{13}^1 & I_1^1(t_0) &= I_1^0 = 2 \\
\tfrac{dI_2^1}{dt} &= F_2^1(h_2^1) + s_{12}^1 - s_{23}^1 - q_{26}^{11} & I_2^1(t_0) &= I_2^0 = 3 \\
\tfrac{dI_3^1}{dt} &= F_3^1(h_3^1) + s_{13}^1 + s_{23}^1 - q_{37}^{12} & I_3^1(t_0) &= I_3^0 = 2 \\
\tfrac{dI_4^2}{dt} &= F_4^2(h_4^2) - s_{45}^2 - q_{47}^{22} & I_4^2(t_0) &= I_4^0 = 3 \\
\tfrac{dI_5^2}{dt} &= F_5^2(h_5^2) + s_{45}^2 - q_{56}^{21} & I_5^2(t_0) &= I_5^0 = 2
\end{aligned}$$

where q_{ij}^{fr} is the flow from a producer at node i of firm $f \in \mathcal{F}_P$ $(f = 1, 2)$ to a retailer $r \in \mathcal{F}_R$ $(r = 1, 2)$ at node j and s_{ij}^f is the shipment flow from a producer at node i to a producer at node j of firm $f \in \mathcal{F}_P$ $(f = 1, 2)$. Inventory dynamics for retailer $r \in \mathcal{F}_R$ $(r = 1, 2)$ at nodes $j = 6, 7$ are the following flow balance equations:

$$\frac{dR_6^1}{dt} = q_{26}^{11} + q_{56}^{21} - c_6 \qquad R_6^0 = 1$$
$$\frac{dR_7^2}{dt} = q_{27}^{12} + q_{37}^{12} + q_{47}^{22} - c_7 \quad R_7^0 = 1$$

Inventory dynamics for the supply chain stages $k = 1, 2, 3$ are the following flow balance equations:

$$\frac{dS_1}{dt} = u_0 - u_1 \qquad S_1^0 = 10$$
$$\frac{dS_2}{dt} = u_1 - u_2 \qquad S_2^0 = 3$$
$$\frac{dS_3}{dt} = u_2 - u_3 \qquad S_3^0 = 1$$
$$u_3 = h_1^1 + h_2^1 + h_3^1 + h_4^2 + h_5^2$$

The corresponding Hamiltonian for producer firm $f \in \mathcal{F}_P$ $(f = 1, 2)$ is

$$\begin{aligned} H_P^1(q^1, s^1, h^1, I^1, \lambda^1; c^r) \;=\; & e^{-\rho t} \Bigg\{ \frac{1}{1+\alpha_6^1} \Psi_6(c_6) q_{26}^{11} + \frac{1}{1+\alpha_7^2} \Psi_7(c_7)(q_{27}^{12} + q_{37}^{12}) \\ & - \; 2h_1^1 - 2.5h_2^1 - 2.1h_3^1 - 2s_{12}^1 - 2s_{13}^1 - 0.5s_{23}^1 \\ & - 4q_{26}^{11} - 3q_{27}^{12} - 5q_{37}^{12} - \sum_{i=1}^{3} \psi_i^1(I_i^1, t) \Bigg\} + \lambda_1 \left[F_1^1(h_1^1) - s_{12}^1 - s_{13}^1 \right] \\ & + \lambda_2 \left[F_2^1(h_2^1) + s_{12}^1 - s_{23}^1 - q_{26}^{11} - q_{27}^{12} \right] + \lambda_3 \left[F_3^1(h_3^1) + s_{13}^1 + s_{23}^1 - q_{37}^{12} \right] \end{aligned}$$

$$\begin{aligned} H_P^2(q^2, s^2, h^2, I^2, \lambda^2; c^r) = \; & e^{-\rho t} \Bigg\{ \frac{1}{1+\alpha_6^1} \Psi_6(c_6) q_{56}^{21} + \frac{1}{1+\alpha_7^2} \Psi_7(c_7) + q_{47}^{22} \\ & -3h_4^2 - 2.1h_5^2 - 0.5s_{45}^2 - 3q_{47}^{22} - 4q_{56}^{21} - \sum_{i=4}^{5} \psi_i^2(I_i^2, t) \Bigg\} \\ & + \lambda_4 \left[F_4^2(h_4^2) - s_{45}^2 - q_{47}^{22} \right] + \lambda_5 \left[F_5^2(h_5^2) + s_{45}^2 - q_{56}^{21} \right] \end{aligned}$$

We will use the notation

$$H_P^{f*} = H_P^f(q^{f*}, s^{f*}, h^{f*}, I^{f*}, \lambda^{f*}; c^*)$$

to denote the Hamiltonian evaluated at a Nash equilibrium where $I^f = \left(I_i^f : i = 1,2,3,4,5\right)$ and $\lambda^f = \left(\lambda_i^f : i = 1,2,3,4,5\right)$ are vectors of adjoint variables. The pertinent gradients of the Hamiltonians for producers are shown below:

$$\nabla_{q^1} H_P^{1*} = \begin{bmatrix} \nabla_{q_{26}^{11}} H_P^{1*} \\ \nabla_{q_{27}^{12}} H_P^{1*} \\ \nabla_{q_{37}^{12}} H_P^{1*} \end{bmatrix} = \begin{bmatrix} e^{-\rho t}(\frac{1}{1+\alpha_6^1}\Psi_6(c_6^*) - 4) - \lambda_2 \\ e^{-\rho t}(\frac{1}{1+\alpha_7^2}\Psi_7(c_7^*) - 3) - \lambda_2 \\ e^{-\rho t}(\frac{1}{1+\alpha_7^2}\Psi_7(c_7^*) - 5) - \lambda_3 \end{bmatrix}$$

$$\nabla_{s^1} H_P^{1*} = \begin{bmatrix} \nabla_{s_{12}^1} H_P^{1*} \\ \nabla_{s_{13}^1} H_P^{1*} \\ \nabla_{s_{23}^1} H_P^{1*} \end{bmatrix} = \begin{bmatrix} e^{-\rho t}(-2) - \lambda_1 + \lambda_2 \\ e^{-\rho t}(-2) - \lambda_1 + \lambda_3 \\ e^{-\rho t}(-0.5) - \lambda_2 + \lambda_3 \end{bmatrix}$$

$$\nabla_{h^1} H_P^{1*} = \begin{bmatrix} \nabla_{h_1^1} H_P^{1*} \\ \nabla_{h_2^1} H_P^{2*} \\ \nabla_{h_3^1} H_P^{3*} \end{bmatrix} = \begin{bmatrix} e^{-\rho t}(-2) + \lambda_1 h_1^{1*} \\ e^{-\rho t}(-2.5) + 0.4\lambda_2 h_2^{1*} \\ e^{-\rho t}(-2.1) + 0.6\lambda_3 h_3^{1*} \end{bmatrix}$$

$$\nabla_{q^2} H_P^{2*} = \begin{bmatrix} \nabla_{q_{56}^{21}} H_P^{f*} \\ \nabla_{q_{47}^{22}} H_P^{f*} \end{bmatrix} = \begin{bmatrix} e^{-\rho t}(\frac{1}{1+\alpha_6^1}\Psi_6(c_6^*) - 4) - \lambda_5 \\ e^{-\rho t}(\frac{1}{1+\alpha_7^2}\Psi_7(c_7^*) - 3) - \lambda_4 \end{bmatrix}$$

$$\nabla_{s^2} H_P^{2*} = \begin{bmatrix} \nabla_{s_{45}^2} H_P^{2*} \end{bmatrix} = \begin{bmatrix} e^{-\rho t}(-0.5) - \lambda_4 + \lambda_5 \end{bmatrix}$$

$$\nabla_{h^2} H_P^{2*} = \begin{bmatrix} \nabla_{h_4^2} H_P^{2*} \\ \nabla_{h_5^2} H_P^{2*} \end{bmatrix} = \begin{bmatrix} e^{-\rho t}(-3) + 0.4\lambda_4 h_4^{2*} \\ e^{-\rho t}(-2.1) + 0.6\lambda_5 h_5^{2*} \end{bmatrix}$$

The Hamiltonian for retailer $r \in \mathcal{F}_R$ $(r = 1, 2)$ is as follows:

$$\begin{aligned} H_R^1(c^1, R^1, \gamma^1; c^{-1}, q^f) = \; & e^{-\rho t} \left[\left(c_6 - \frac{1}{1+\alpha_6^1}(q_{26}^{11} + q_{56}^{21}) \right) \Psi_6(c_6) - \phi_6^1(R_6^1, t) \right] \\ & + \gamma_6 \; \left(q_{26}^{11} + q_{56}^{21} - c_6 \right) \end{aligned}$$

$$\begin{aligned} H_R^2(c^2, R^2, \gamma^2; c^{-2}, q^f) = \; & e^{-\rho t} \left[\left(c_7 - \frac{1}{1+\alpha_7^2}(q_{27}^{12} + q_{37}^{12} + q_{47}^{22}) \right) \Psi_7(c_7) - \phi_7^2(R_7^2, t) \right] \\ & + \gamma_7 \; \left(q_{27}^{12} + q_{37}^{12} + q_{47}^{22} - c_7 \right) \end{aligned}$$

We will use the notation

$$H_R^{r*} = (c^{r*}, R^{r*}, \gamma^{r*}; c^{-r*}, q^{r*})$$

to denote the Hamiltonian evaluated at a Nash equilibrium where $R^r = \left(R_j^r : j = 6, 7 \right)$ and $\gamma^r = \left(\gamma_j^r : j = 6, 7 \right)$ is are vectors of adjoint variables. The pertinent gradients of the Hamiltonians for retailers are shown below:

$$\nabla_{c^1} H_R^{1*} = [\nabla_{c_6} H_R^{r*} \;] = \left[e^{-\rho t} \; (\frac{1}{1+\alpha_6^1}(q_{26}^{11} + q_{56}^{21}) - 2c_6 + 11) - \gamma_6 \; \right]$$

$$\nabla_{c^2} H_R^{2*} = [\nabla_{c_7} H_R^{r*} \;] = \left[e^{-\rho t}(\frac{1.5}{1+\alpha_7^2}(q_{27}^{12} + q_{37}^{12} + q_{47}^{22}) - 3c_7 + 11) - \gamma_7 \; \right]$$

The Hamiltonian for the supply chain is as follows:

$$H_S(u,S,\zeta;h) = e^{-\rho t} \left(1.2u_1 + 1.3u_2 + 2u_3 + 5S_1^2 + \frac{3}{8}S_2^2 + S_3^2\right) + \zeta_1(u_0 - u_1) + \zeta_2(u_1 - u_2) + \zeta_3(u_2 - u_3)$$

where $S = (S_k : k = 1,2,3)$ and $\zeta = (\zeta_k : k = 1,2,3)$ are vectors of adjoint variables. The pertinent gradients of the Hamiltonians for supplier are shown below:

$$\nabla_u H_S^* = \begin{bmatrix} \nabla_{u_0} H_S^* \\ \nabla_{u_1} H_S^* \\ \nabla_{u_2} H_S^* \\ \nabla_{u_3} H_S^* \end{bmatrix} = \begin{bmatrix} \zeta_1 \\ e^{-\rho t}(1.2) - \zeta_1 + \zeta_2 \\ e^{-\rho t}(1.3) - \zeta_1 + \zeta_2 \\ e^{-\rho t}(1.2) - \zeta_3 \end{bmatrix}$$

As a result, the solution of this example is the following:

$$\begin{bmatrix} q^* \\ s^* \\ h^* \\ c^* \\ u^* \end{bmatrix} \in \Omega$$

satisfying

$$\sum_{f\in\mathcal{F}_P} \int_0^{10} \left\{ \left[-\nabla_{q^f} H_P^{f*}\right]^T (q^f - q^{f*}) + \left[-\nabla_{s^f} H_P^{f*}\right]^T (s^f - s^{f*}) + \left[-\nabla_{h^f} H_P^{f*}\right]^T (h^f - h^{f*}) \right\} dt + \int_0^{10} \sum_{r\in\mathcal{F}_R} \left[-\nabla_{c^r} H_R^{r*}\right]^T (c^r - c^{r*}) dt + \int_0^{10} \left[\nabla_u H_S^*\right]^T (u - u^*) dt \geq 0$$

for all

$$\begin{bmatrix} q \\ s \\ h \\ c \\ u \end{bmatrix} \in \Omega$$

where

$$\Omega = \Lambda \times \Gamma$$

$$\Lambda = \Lambda_S(h^*) \times \prod_{f \in \mathcal{F}_P} \Lambda_P^f(h^{-f}, u_N) \times \prod_{r \in \mathcal{F}_R} \Lambda_R^r(q^{r*})$$

6.2 With-Disruption Case (With Uncertainty)

In the case in which network disruptions occur, initial inventories at each node for producers $i = 1, 2, 3, 4, 5$ of firm $f \in \mathcal{F}_P$ $(f = 1, 2)$ and retailers $r \in \mathcal{F}_R$ $(r = 1, 2)$ at node $j = 6, 7$ are the same as the deterministic case:

$$I_1^1(0) = 2 \quad I_2^2(0) = 3 \quad I_3^1(0) = 2 \quad I_4^2(0) = 3 \quad I_5^2(0) = 2 \quad R_6^1(0) = 1 \quad R_7^1(0) = 1$$

The initial inventories for suppliers are the following:

$$S_0(0) = 10 \quad S_1(0) = 3 \quad S_2(0) = 1$$

The discount rate ρ is assumed to be 0.05 and the retailers' margins at nodes $j = 6$ and $j = 7$ are 0.1 and 0.08, respectively. Keeping in mind that the retailers occupy distinct nodes, the consumers' inverse demand functions for nodes $i = 6$ and $i = 7$ are the following:

$$\Psi_6(c_6, c_7) = 11 - c_6 \quad \Psi_7(c_6, c_7,) = 11 - 1.5 \cdot c_7$$

where c_j is the consumption at j from the sole retailer located there. The production functions at nodes $i = 1, 2, 3, 4, 5$ are the following:

$$\begin{aligned}
F_1^1(h_1^1) &= 1. \cdot \frac{\left(h_1^1\right)^2}{2} - 0.1\xi_1 h_1^1 \\
F_2^1(h_2^1) &= 0.2 \cdot \frac{\left(h_2^1\right)^2}{2} - 0.05\xi_2 h_2^1 \\
F_3^1(h_3^1) &= 0.3 \cdot \frac{\left(h_3^1\right)^2}{2} - 0.05\xi_3 h_3^1 \\
F_4^2(h_4^2) &= 0.4 \cdot \frac{\left(h_4^2\right)^2}{2} - 0.05\xi_4 h_4^2 \\
F_5^2(h_5^2) &= 0.6 \cdot \frac{\left(h_5^2\right)^2}{2} - 0.1\xi_5 h_5^2
\end{aligned}$$

where h_i^f denotes the input factor flow from the final (3^{rd}) stage of the supply chain to producer i of firm $f \in \mathcal{F}_P$ ($f = 1, 2$) and all ξ_i follow the uniform distribution $\xi_i\tilde{}[0, 2]$.

$$\begin{aligned}
E\left[F_1^1(h_1^1)\right] &= 1. \cdot \tfrac{\left(h_1^1\right)^2}{2} - 0.1h_1^1 & Var(F_1^1(h_1^1)) &= \tfrac{0.01}{3}(h_1^1)^2 \\
E\left[F_2^1(h_2^1)\right] &= 0.2 \cdot \tfrac{\left(h_2^1\right)^2}{2} - 0.05h_2^1 & Var(F_2^1(h_2^1)) &= \tfrac{0.0025}{3}(h_2^1)^2 \\
E\left[F_3^1(h_3^1)\right] &= 0.3 \cdot \tfrac{\left(h_3^1\right)^2}{2} - 0.05h_3^1 & Var(F_3^1(h_3^1)) &= \tfrac{0.0025}{3}(h_3^1)^2 \\
E\left[F_4^2(h_4^2)\right] &= 0.4 \cdot \tfrac{\left(h_4^2\right)^2}{2} - 0.05h_4^2 & Var(F_4^2(h_4^2)) &= \tfrac{0.0025}{3}(h_4^2)^2 \\
E\left[F_5^2(h_5^2)\right] &= 0.6 \cdot \tfrac{\left(h_5^2\right)^2}{2} - 0.1h_5^2 & Var(F_5^2(h_5^2)) &= \tfrac{0.01}{3}(h_5^2)^2
\end{aligned}$$

The inventory cost functions for producers $i = 1, 2, 3, 4, 5$ of firm $f \in \mathcal{F}_P$ $(f = 1, 2)$, retailers $r \in \mathcal{F}_R$ $(r = 1, 2)$ at nodes $j = 6, 7$, respectively and stages $k = 1, 2, 3$ of the supply chain are all the same as those employed in the deterministic case. The freight rates r_{ij} from node i to node j are assumed to be as follows:

$$r_{12} = 2 + \eta_1 \qquad r_{13} = 2 + 2\eta_1 \qquad r_{23} = 0.5 + 0.5\eta_2$$

$$r_{45} = 0.5 + 3\eta_4 \qquad r_{26} = 4 + \eta_2 \qquad r_{27} = 3 + 2\eta_2$$

$$r_{37} = 5 + \eta_3 \qquad r_{56} = 4 + 0.5\eta_5 \qquad r_{47} = 3 + 0.5\eta_4$$

where η_i follows normal distribution $N\tilde{}(0, 4)$.

$$E\left[r_{12}\right] = 2 \qquad Var(r_{12}) = 4 \qquad E\left[r_{13}\right] = 2 \qquad Var(r_{13}) = 16 \qquad E\left[r_{23}\right] = 0.5 \qquad Var(r_{23}) = 1$$

$$E\left[r_{45}\right] = 0.5 \qquad Var(r_{45}) = 36 \qquad E\left[r_{26}\right] = 4 \qquad Var(r_{26}) = 4 \qquad E\left[r_{27}\right] = 3 \qquad Var(r_{27}) = 16$$

$$E\left[r_{37}\right] = 5 \qquad Var(r_{37}) = 4 \qquad E\left[r_{56}\right] = 4 \qquad Var(r_{56}) = 1 \qquad E\left[r_{47}\right] = 3 \qquad Var(r_{47}) = 1$$

The costs to producers at nodes $i = 1,2,3,4,5$ of firm $f \in \mathcal{F}_P$ $(f = 1,2)$ for acquiring input flow are the following:

$$C_1^1(h_1^1,t) = (2+\eta_1)\cdot h_1^1 \qquad E\left[C_1^1(h_1^1,t)\right] = 2\cdot h_1^1 \quad Var(C_1^1) = 4\cdot(h_1^1)^2$$

$$C_2^1(h_2^1,t) = (2.5+\eta_2)\cdot h_2^1 \quad E\left[C_2^1(h_2^1,t)\right] = 2.5\cdot h_2^1 \quad Var(C_2^1) = 4\cdot(h_2^1)^2$$

$$C_3^1(h_3^1,t) = (2.1+\eta_3)\cdot h_3^1 \quad E\left[C_3^1(h_3^1,t)\right] = 2.1\cdot h_3^1 \quad Var(C_3^1) = 4\cdot(h_3^1)^2$$

$$C_4^2(h_4^2,t) = (3+\eta_4)\cdot h_4^2 \qquad E\left[C_4^2(h_4^2,t)\right] = 3\cdot h_4^2 \quad Var(C_4^2) = 4\cdot(h_4^2)^2$$

$$C_5^2(h_5^2,t) = (2.1+\eta_5)\cdot h_5^2 \quad E\left[C_5^2(h_5^2,t)\right] = 2.1\cdot h_5^2 \quad Var(C_5^2) = 4\cdot(h_5^2)^2$$

Finally, the variable costs of preparing stages $k = 1,2,3$ supply chain flows are the following

$$V_1(u_1,t) = 1.2\cdot u_1 + 3\delta_1 u_1 \quad E\left[V_1\right] = 4.2\cdot u_1 \quad Var(V_1) = 3(u_1)^2$$

$$V_2(u_2,t) = 1.3\cdot u_2 + 2\delta_2 u_2 \quad E\left[V_2\right] = 3.3\cdot u_2 \quad Var(V_2) = \tfrac{4}{3}(u_2)^2$$

$$V_3(u_3,t) = 2\cdot u_3 + \delta_3 u_3 \qquad E\left[V_3\right] = 3\cdot u_3 \quad Var(V_3) = \tfrac{1}{3}(u_3)^2$$

where the parameter δ_h is assumed to follow a uniform distribution $\tilde{}[0,2]$.Upper bounds imposed on control variables are the same as those employed in the deterministic case. Finally, risk aversion parameters are set to be: $\beta_i^1 = 0.6$ $(i = 1,2,3)$, $\beta_i^2 = 0.2$ $(i = 4,5)$,and supplier's $\beta_h = 1.2$ $(h = 1,2,3)$.

6.3 Numerical Example Result Summary

In both numerical cases, production functions of producers are increasing returns to scale while the demand function is linear. The differential variational inequalities described above are solved using a fixed point algorithm. GAMS/PATHNLP and Matlab are used to solve these programs with 1500 iterations and discretization by time approximation with $N = 1000$ equal time steps. The primary results are displayed below in Figure $2 - 5$.

Compared with the computational results of two numerical cases, our conclusions are as follows.

1. *The supply chain disruptions cause the production quantity to decrease.* This is caused by production function productivity decrease due to disruption effect. Compared with their counterparts in the deterministic case, the lower production under disruptions resulted in fewer shipments among producers as well as fewer shipments from producers to retailers as represented below in Figure 2.

2. *Less wholesale good flows with disruption.* Please observe Figure 2. The fewer in production, the fewer is the consumption goods movements between producers and retailers under disruption.

3. *A higher shipment rate under disruption.* The disruption cases the shipment rate to increase. This decreases the shipments between producers, please observe Figure 3. The higher shipment rate also affects on the input factor flows. This resulted in different input factor flow trajectory patterns as shown in Figure 4 when compared the deterministic and disruption cases.

4. *A higher input factor flow with a higher production variance.* If we observe

the input factor flow trajectories in the disruption case alone displayed below in Figure 4-(b), a higher input factor is associated with a higher production variance. Compared with nodes 2 and 3, production node 5 requires higher input factors. The increase is necessary to minimize the disruption risk/variance, since its production variance is larger than both nodes 2 and 3.

5. *The raw materials flows among suppliers display a similar trajectory pattern but different peaks.* Please observe Figure 5. Since the supply sub-network has merely a single path with no alternative paths available, the optimal trajectories of raw material flows have shown a similar pattern. However, trajectories have different peaks, at $t = 2.3$ and at $t = 3.5$ for respective deterministic and disruption cases. The reason is that the suppliers are sensitive to the variance of raw material flows. In order to minimize the supply cost, a high value of risk aversion factor required the suppliers to narrow the variance of raw material flows. This resulted in a gradually change in the raw material flows until to its peak. In addition, the raw material flow (u_3) from the supplier node $h = 2$ to the node $h = 3$ is increased to 15 in some time span in the disruption case. This is because the supplier needs to obtain additional raw material to supply more input factor material to producers since the production function's efficiency will lessen with disruptions.

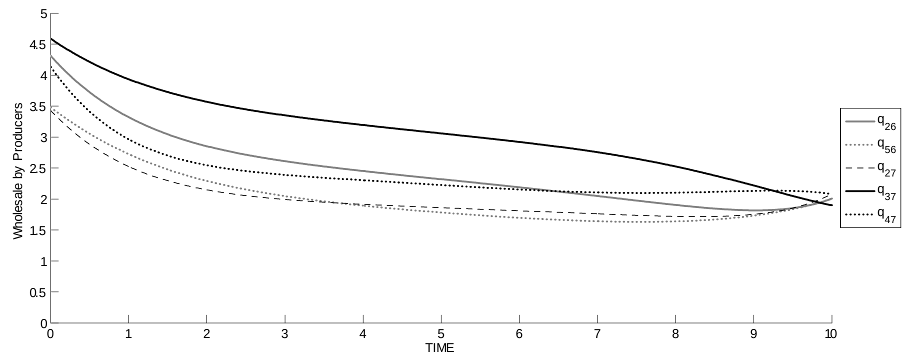


Figure 6.2. Wholesale goods flow from producers to retailers - (a) Deterministic

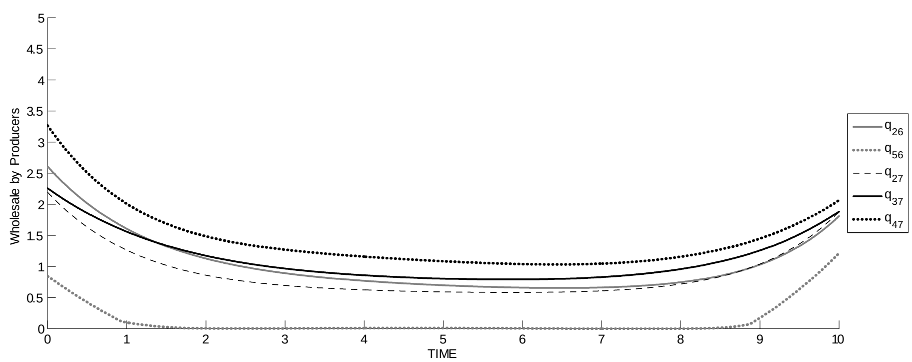


Figure 6.3. Wholesale goods flow from producers to retailers - (b) With disruptions

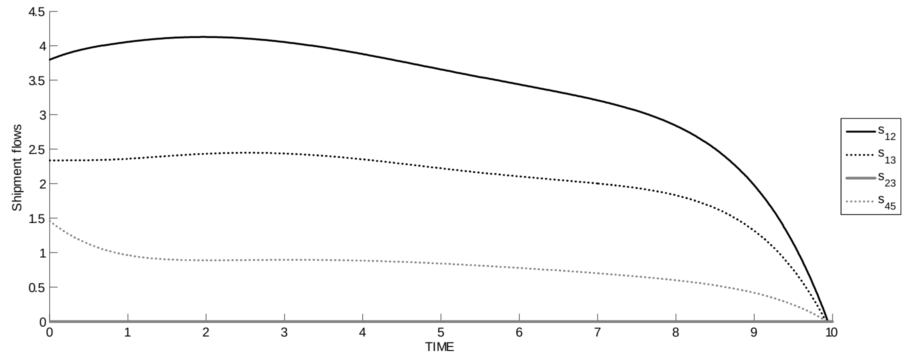


Figure 6.4. Shipment trajectories of producers at nodes1,2,3,4,5 - (a) Deterministic

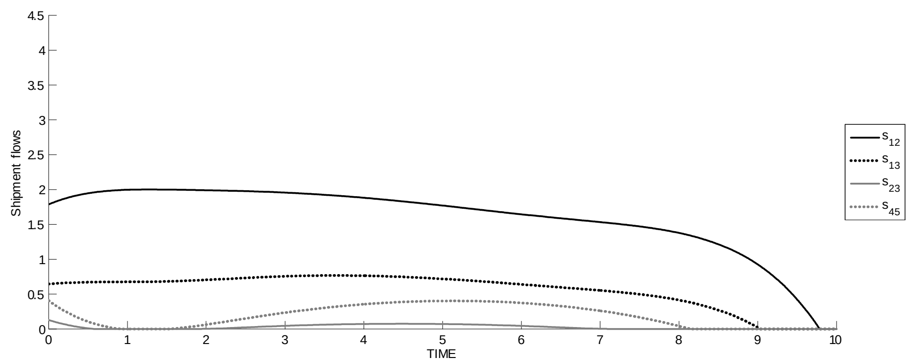


Figure 6.5. Shipment trajectories of producers at nodes1,2,3,4,5 - (b) With disruptions

6.4 Other case with recovery assumption

This case deals with the case assuming that any disaster occurs in 5th time which influences all the networks for a length of 2 time periods, till 7th time. In addition, production functions, transportation cost functions, and suppliers' variable cost functions assume to be influenced with the these functions change to the with-uncertainty functions form when disruption happens. Network topologies for suppliers, firms with their producers at node $i \in \mathcal{N}$, and retailers are the same for the above both cases. Other assumptions including the time interval of in-

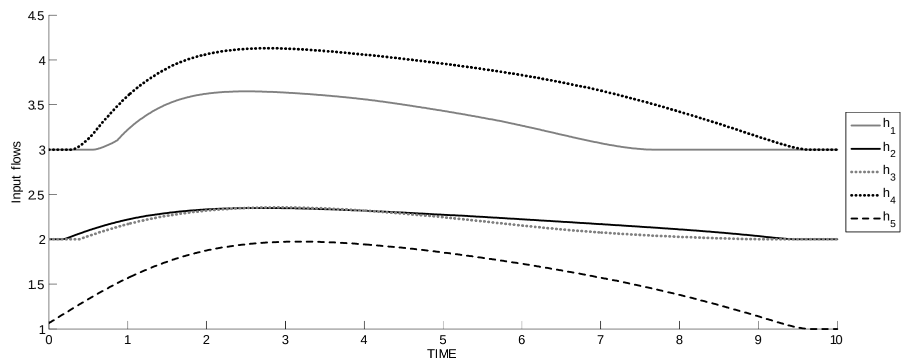


Figure 6.6. Input factor flows from supplier to producers - (a) Deterministic

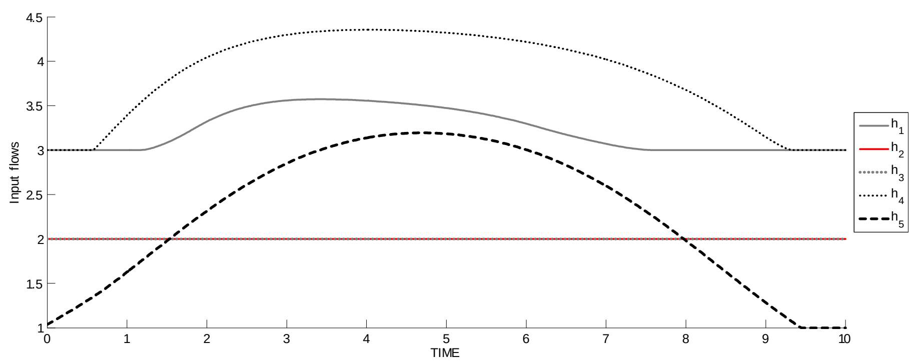


Figure 6.7. Input factor flows from supplier to producers - (b) With disruptions

terest is same as above cases. In other words, at 5th time, the related functions like each agent's maximization functions and constraints are changed into those under disruption case from those under deterministic case. Additionally, simple implementation is used just because the object is to know how optimal trajectories would change under recovery assumption case. Therefore, except for $t = 5, 6$, all the agents in the supply chain network are assumed to follow their deterministic model in the same way mentioned in the deterministic numerical example.

At 5th time, their model changes into disruption case. In other words, all the agents' objectives and constraints are addressed expections and variance terms as

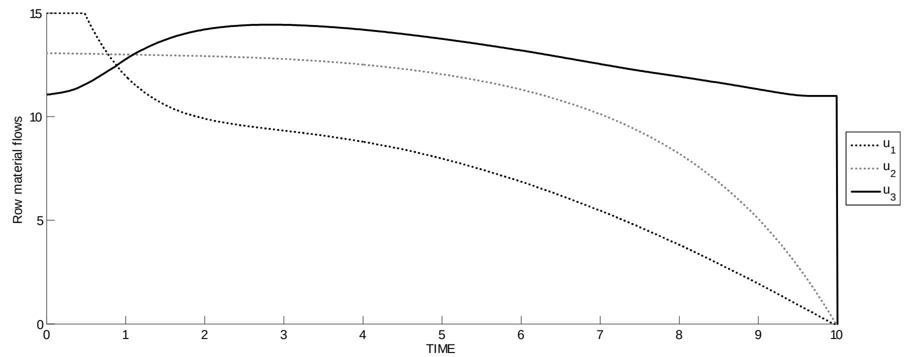


Figure 6.8. Raw input flow among suppliers - (a) Deterministic

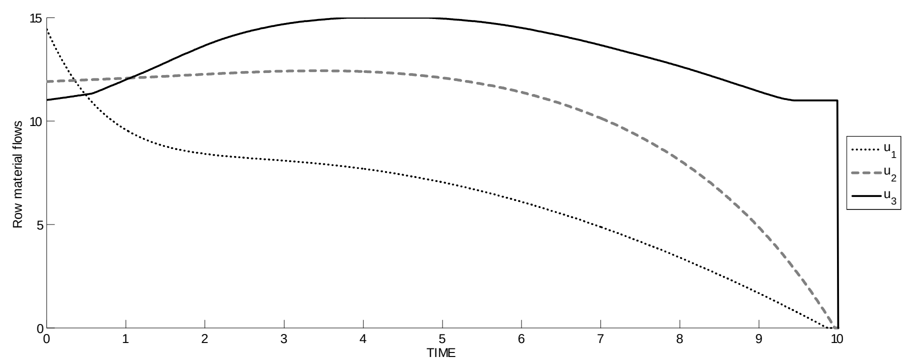


Figure 6.9. Raw input flow among suppliers - (b) With disruptions

same as those in the disruption case example.

In this case, we can derive the following results.

1. Wholesale quantity decreased and Inventory level fluctuated: From figure 7, we can see that wholesale quantity q_{56} under disruption in 5th period decreases and inventory level I_1fluctuates because the disruption affects most on producer 5 according to production function and transportation function assumption.

2. Shipment flow changed differently from deterministic case: From figure 9,

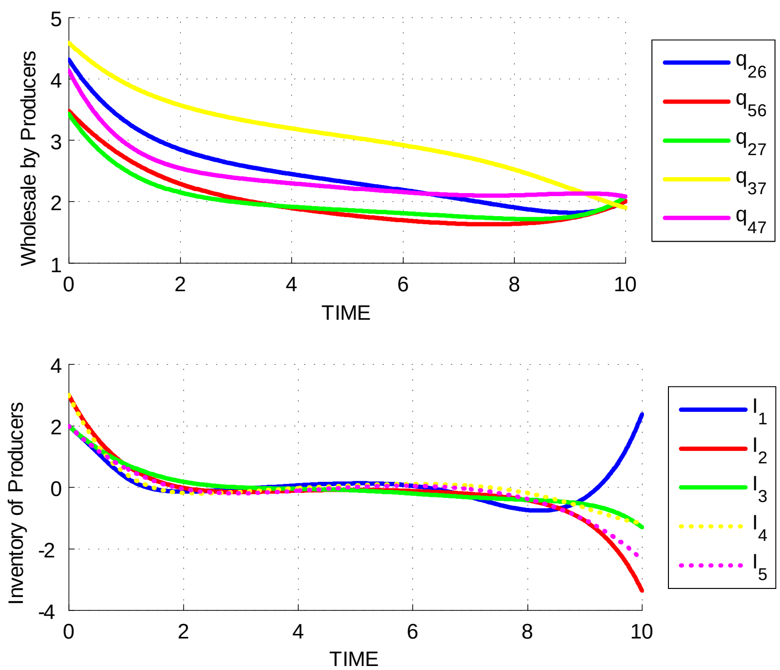
shipment flow s_{12} increases other shipment flows decrease during disruption happens because disruption also affects transportation cost so that s_{12} increases since the transportation cost from node 1 and 2 is affected (is smaller) less than from node 1 and 3.

3. Raw input flow less affected: Due to the assumption that raw material transportation cost changes a little, overall suppliers' input flows and inventory levels are little different from in the deterministic case.

4. On the whole, the supply chain network system composed of firms, retailers, and suppliers with some control variables bounds back to its deterministic solution after the damage caused by disruptions is recovered (a length of 2 time periods). It is caused by that we assume the production function, shipping cost function, and transportation cost function are recovered to the same as the deterministic ones to simplify the example. Thus, other assumptions on recovery are required in the future in order to investigate a recovery process or trajectory.

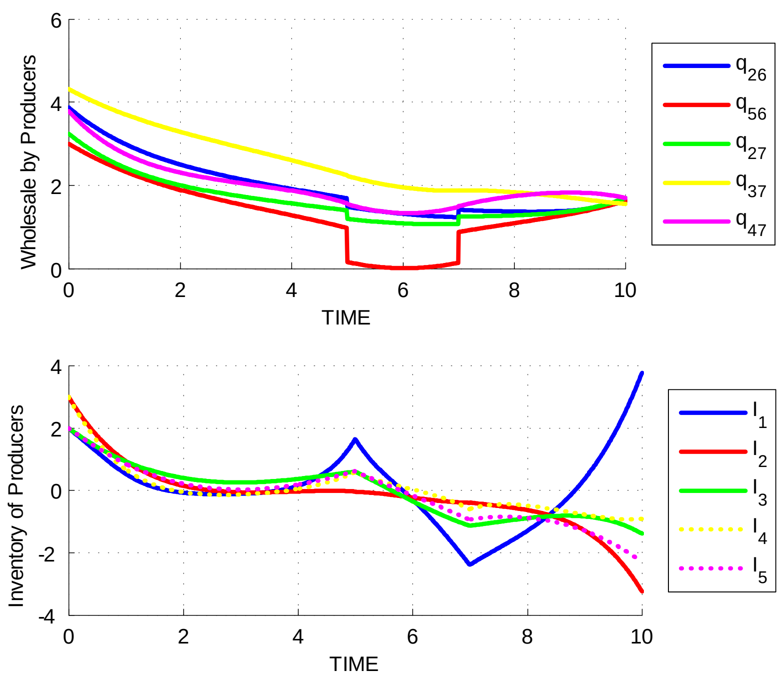
5. We can see from Figure 6 and 7 that by the disruption assumed by this example, the wholesale quantity of Firm 1 is decreased by 2.4 or 33% on the basis of time horizon 6 (from $q_{26} = 2.4, q_{27} = 1.8, q_{37} = 3$ to $q_{26} = 1.6, q_{27} = 1.2, q_{37} = 2$). In the same manner, Firm 2 has a wholesale loss of 2.5 or 61% because production function and shipping cost are assumed to be highly influenced at node 5. In this way, we can analyze the severity of disruption to each control variable as well as the total severity on each agent such as firm, retailer and supplier.

We can find out the production function and shipping cost function between producer and retailer affect on overall supply chain equilibrium in this example. In future study, we need to try various uncertainty parameters of each function such as production, shipping and raw material transportation cost function.



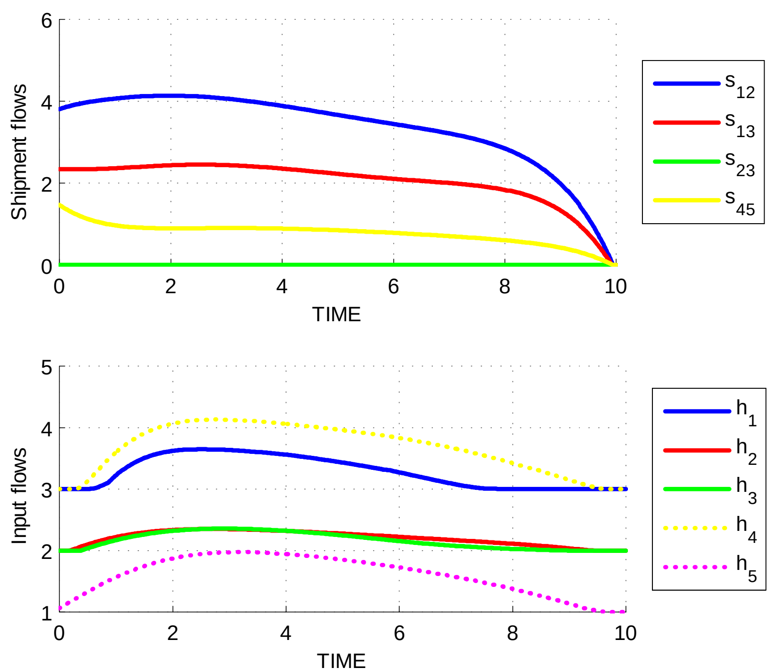
(**Above**: Wholesale goods flow from producers to retailers,
Below: Inventory of producers at node 1,2,3,4,5)

Figure 6.10. Without uncertainty, goods flow and Inventory



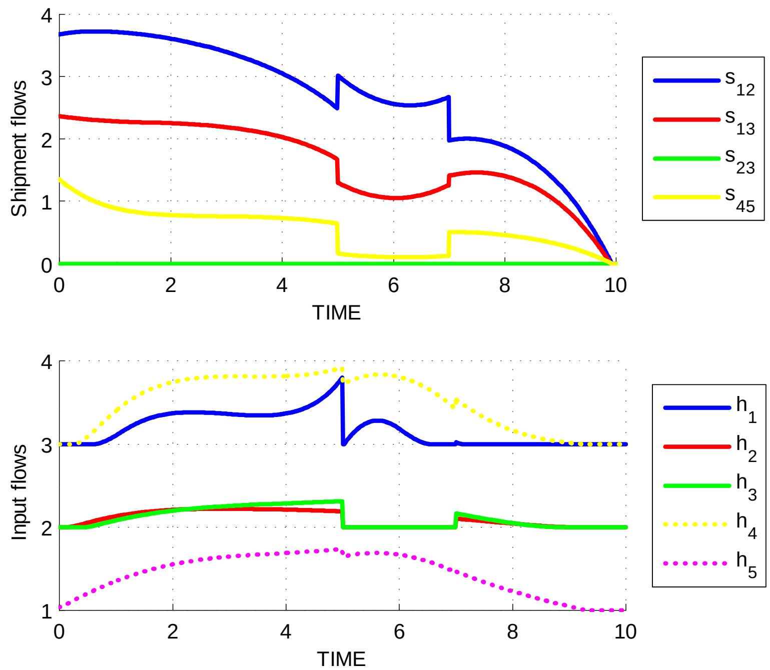
(**Above**: Wholesale goods flow from producers to retailers,
Below: Inventory of producers at node 1,2,3,4,5)

Figure 6.11. Under disruption, goods flow and Inventory



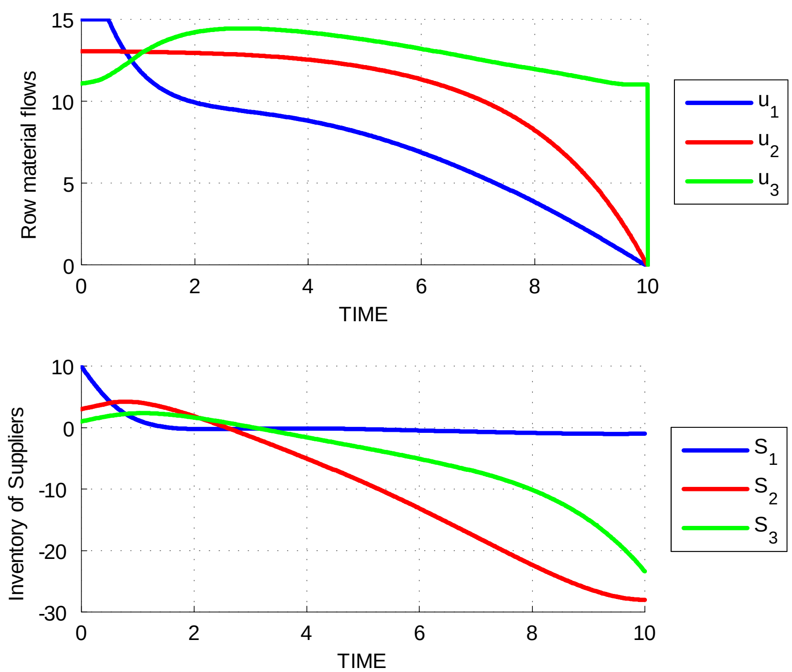
(**Above:** Shipment trajectories of producers at node 1,2,3,4,5,
Below: Input factor flows from supplier to producers)

Figure 6.12. Without uncertainty, shipment and Input flow



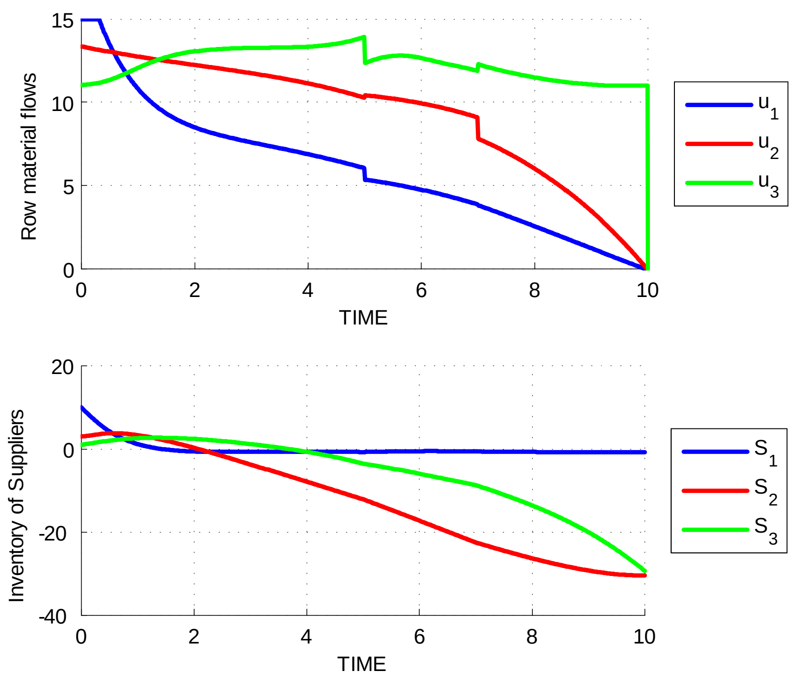
(**Above:** Shipment trajectories of producers at node 1,2,3,4,5, **Below:** Input factor flows from supplier to producers)

Figure 6.13. Under disruption, shipment and Input flow



(**Above:** Raw input flow among suppliers,
Below: Inventory of suppliers)

Figure 6.14. Without uncertainty, raw input flow



(**Above:** Raw input flow among suppliers,
Below: Inventory of suppliers)

Figure 6.15. Under disruption, raw input flow

Chapter 7

Concluding Remarks

This dissertation has explored a supply chain network model with supply chain uncertainty denoted by probability distribution and a differential variational inequality (DVI) formulation of the supply chain network model. In addition, this dissertation has mentioned the design of supply chain networks available to optimally deal with supply chain disruption risk without an algorithm for solving the mixed integer programming problem. Concretely, probabilistic parameters have reflected the threat of disruptions in supply chain networks. In the numerical examples, a uniform distribution and normal distribution were applied to the firm's production function, transportation cost function and supplier's raw material flow cost function. These are based on the condition that supply chain network disruptions affect the production function significantly, drastically increasing the transportation cost among suppliers, producers, and retailers. Risk aversion parameters, β_i and β_h, were incorporated to consider decision makers' risk allowances, but not to observe their affect on the optimal solution trajectories. This concern must be addressed in future studies.

Finally, mathematical modeling for optimizing supply chain networks with the threat of disruptions do not account for extreme disruption risk, such as unprecedentedly large earthquakes, 100-year flood, and lethal infectious diseases. Therefore, robust optimization and extreme value theory of statistics should be employed to study and effectively manage the huge risk based on the extreme events discussed above. Recovery time and recovery level of supply chain networks after disruptions must also be represented in future mathematical models. To conclude, this dissertation has demonstrated/accomplished the following:

1. A supply chain network optimization problem based on the dynamic oligopolistic model has been established using the differential variational inequality, which can be easily analyzed using the minimum principle from optimal control theory.

2. The threat of supply chain disruption has been addressed and reflected by a probability distribution.

3. Variance terms have been applied to establish each agent's risk attitude;

4. Numerical examples of the deterministic and disruption cases have been compared, demonstrating that supply chain disruption decreases production quantity and alters shipping and raw material flow trajectories.

5. Fixed point algorithm has been shown to be an effective means of solving the supply chain model with uncertainty.

Bibliography

Abrahamsson, M. (1993). Time-based distribution. *International Journal of Logistics Management 4*, 75–84.

Abrahamsson, M. and S. Brege (1995). Distribution channel reengineering: organisation and logistic. *Linköping Univ., Dept. Of Management and Economics*.

Adida, E. and G. Perakis (2006). A robust optimization approach to dynamic pricing and inventory control with no backorders. *Mathematical Programming 107*, 97–129.

Ambrosino, D. and M. Scutellà (2005). Distribution network design: new problems and related models. *European Journal of Operational Research 165*, 610–624.

Babich, V., P. Ritchken, and A. Burnetas (2007). Competition and diversication ed'ects in supply chains with supplier default risk. *Manufacturing & Service Operations Management 9*, 123–146.

Ben-Tal, A. and A. Nemirovski (1998). Robust convex optimization. *Mathematics of Operations Research 23*, 769–805.

Bertsimas, D. and M. Sim (2004). The price of robustness. *Operations Research 52*, 35–53.

Bertsimas, D. and A. Thiele (2006). A robust optimization approach to inventory theory. *Operations Research 54*(1), 150–168.

Bressan, A. and B. Piccoli (2007). *An Introduction to the Mathematical Theory of Control*. Springfield, MO: American Institute of Mathematical Sciences.

Browder, F. E. (1968). The fixed point theory of multivalued mappings in topological vector spaces. *Mathematische Annalen 177*, 283–301.

Chiou, C. (2008). Transshipment problems in supply chainsystems: Review and extensions. In V. Kordic (Ed.), *Supply Chain Theory and Applications*, Vienna, Austria, pp. 427–446. I-Tech Education and Publishing.

Chopra, S., G. Reinhardt, and U. Mohan (2007). The importance of decoupling recurrent and disruption risks in a supply chain. *Naval Research Logistics 54*, 544–555.

Christopher, M. (1992). *Logistics and Supply Chain Management: Strategies for Reducing Costs and Improving Services*. London: Financial Time Pitman Publishing.

Christopher, M. and H. Peck (2004). Building the resilient supply chain. *International Journal of Logistics Management 15*(2), 1–13.

Church, R. L. and M. P. Scaparra (2007). Protecting critical assets: the r-interdiction median problem with fortification. *Geographical Analysis 39*, 129–146.

Cohen, M. and H. Lee (1989). Resource deployment analysis of global manufacturing and distribution networks. *Journal of Manufacturing and Operations Management 2*, 81–104.

Craighead, C. W., J. Blackhurst, and M. J. Rungtusanatham (2007). The severity of supply chain disruptions: design characteristics and mitigation capabilities. *Decision Sciences 38*(1), 131–156.

Crainic, T. and G. Laporte (1997). Planning models for freight transportation. *European Journal of Operational Research 97*, 409–438.

Dada, M., N. Petruzzi, and L. Schwarz (2007). A newsvendor's procurement problem when suppliers are unreliable. *Manufacturing & Service Operations Management 9*(1), 9–32.

Dong, J., D. Zhang, and A. Nagurney (2004). A supply chain network equilibrium model with random demands. *European Journal of Operational Research 156*, 194–212.

Friesz, T. and I. Lee (2008). Dynamic oligopolistic competition in integrated supply, production and distribution chains. *Working Paper*.

Friesz, T. L. (2010). *Dynamic Optimization and Differential Games*. Spirnger-Verlag.

Friesz, T. L., D. Bernstein, T. Smith, R. Tobin, and B. Wie (1993). A variational inequality formulation of the dynamic network user equilibrium problem. *Operations Research 41*, 80–91.

Friesz, T. L., R. Mookherjee, and M. A. Rigdon (2005). An evolutionary game-theoretic model of network revenue management in oligopolistic competition. *Journal of Revenue and Pricing Management 4*, 156–173.

Friesz, T. L., M. A. Rigdon, and R. Mookherjee (2006). Differential variational inequalities and shipper dynamic oligopolistic network competition. *Transportation Research Part B 40*, 480–503.

Fukushima, M. (1992). Equivalent differentiable optimization problems and descent methods for asymmetric variational inequality problems. *Mathematical Programming 53*, 99–110.

Geod'rion, A. and G. Graves (1974). Multicommodity distribution system design by benders decomposition. *Management Science 29*, 822–844.

Ghemawat, P. (2003). Semiglobalization and international business strategy. *Journal of International Business Studies 34*(2), 138–152.

Hendricks, K. and V. Singhal (2003). The effect of supply chain glitches on shareholder wealth. *Journal of Operations Management 21*, 501–522.

Hendricks, K. and V. Singhal (2005). An empirical analysis of the ed'ect of supply chain disruptions on long-run stock price performance and equity risk of the firm. *Production and Operations Management 14*, 35–52.

Johnson, M. E. (2001). Learning from toys: lessons in managing supply chain risk from the toy industry. *California Management Review 43*, 106–124.

Kembel, R. (2000). The fibre channel consultant: a comprehensive introduction. Technical report, Northwest Learning Associates, Tucson, Ariz.

Kogan, K. and C. S. Tapiero (2007). *Supply Chain Games: Operations Management and Risk Valuation*. Spirnger.

Konnov, I. V. and S. Kum (2001). Descent methods for mixed variational inequalities in a Hilbert space. *Nonlinear Analysis: Theory, Methods and Applications 47*(1), 561–572.

Konnov, I. V., S. Kum, and G. M. Lee (2002). On convergence of descent methods for variational inequalities in a Hilbert space. *Mathematical Methods of Operations Research 55*, 371–382.

Latour, A. (2001, January 29). Trial by fire: A blaze in albuquerque sets off major crisis for cell-phone giants. *Wall Street journal*.

Lee, H. L., V. Padmanabhan, and S. Whang (1997). Information distortion in a supply chain: The bullwhip effect. *Management Science 43*(4), 546–558.

Lee, H. L. and Tang (1998). Variability reduction through operations reversal. *Management Science 44*(2), 162–172.

Martha, J. and S. Subbakrishna (2002). Targeting a just-in-case supply chain for the inevitable next disaster. *Supply Chain Management Review 6*(5), 18–24.

Nagurney, A., J. Dong, and D. Zhang (2002). A supply chain network equilibrium model. *Transportation Research Part E 38*(5), 281–303.

Natarajan, A. (2007). *Multi-criteria supply chain inventory models with transportation costs.* Ph. D. thesis, The Pennsylvania State University, University Park, PA.

Parlar, M. and D. Berkin (1991). Future supply uncertainty in eoq models. *Naval Research Logistics 38*, 107–121.

Pshenichnyi, B. N. (1971). *Necessary Conditions for an Extremum.* New York: Marcel Dekker.

Pullan, M. (1997). Existence and duality theory for separated continuous linear programm. *Mathematical and Computer Modelling of Dynamical Systems 3*, 219–245.

Qi, L. and Z. Shen (2007). A supply chain design model with unreliable supply. *Naval Research Logistics*.

Qiang, Q., A. Nagurney, and G. Dong (2009). Modeling of supply chain risk under disruptions with performance measurement and robustness analysis. In T. Wu and J. Blankhurst (Eds.), *Managing Supply Chain Risk and Vulnerability: Tools and Methods for Supply Chain Decision Makers* (in press ed.). Berlin, Germany: Springer.

Qu, W., J. B. and P. Iyogun (1999). An integrated inventorytransportation system with modified periodic policy for multiple products. *European Journal of Operational Research 115*, 254–269.

Raman, K. and R. Chatterjee (1995). Optimal monopolist pricing and ordering decisions by a monopolist. *Management Science 41*, 144–162.

Roughgarden, T. (2002). *Selfish Routing.* Ph. D. thesis, Cornell University, Ithaca, N.Y.

Schmitt, A. J. and L. V. Snyder (2009). Infinite-horizon models for inventory control under yield uncertainty and disruptions. *Working Paper*.

Sheffi, Y. (2001). Supply chain management under the threat of international terrorism. *The International Journal of Logistics Management 12*, 1–11.

Sheffi, Y. (2005). *The Resilient Enterprise: Overcoming Vulnerability for Competitive Advantage* (1st ed.). Cambridge, MA: MIT Press.

Silberberg, E. and W. Suen (2000). *The Structure of Economics: A Mathemathical Analysis* (3rd ed.). Irwin: McGraw-Hill.

Simchi-Levi, D., L. Snyder, and M. Watson (2002). Strategies for uncertain times. *Supply Chain Management Review 6*, 11–12.

Snyder, L. and Z. Shen (2006). Disruptions in multi-echelon supply chains: a simulation study. Submitted for publication.

Snyder, L. V. (2003). *Supply Chain Robustness and Reliability: Models and Algorithm.* Ph. D. thesis, Northwestern University, Evanston, Illinois.

Snyder, L. V. and M. S. Daskin (2005). Reliability models for facility location: the expected failure cost case. *Transportation Science 39*, 400–416.

Snyder, L. V., M. P. Scaparra, M. S. Daskin, and R. L. Church (2006). Planning for disruptions in supply chain networks. In *Tutorials in operations research.* Informs.

Soyster, A. (1973). Convex programming with set-inclusive constraints and applications to inexact linear programming. *Operations Research 21*, 1154–1157.

Swaminathan, J., S. Smith, and N. Sadeh (1998). Modeling supply chain dynamics: a multiagent approach. *Decision Sciences 29*(3), 607–632.

Tang, C. and B. Tomlin (2008). The power of flexibility for mitigating supply chain risks. *International Journal of Production Economics 116*(1), 12–27.

Tomlin, B. and Y. Wang (2005). On the value of mix exibility and dual sourcing in unreliable newsvendor networks. *Manufacturing & Service Operations Management 7*, 37–57.

Tomlin, B. T. (2006). On the value of mitigation and contingency strategies for managing supply chain disruption risks. *Management Science 52*.

Tyndall, W. (1965). A duality theorem for a class of continuous linear programming problems. *SIAM Journal of Applied Mathematics 13*, 644–666.

Wagner, S. M. and C. Bode (2008). An empirical examination of supply chain performance along several dimensions of risk. *Journal of Business Logistics 29 29 29*(1).

Yamashita, N., K. Taji, and M. Fukushima (1997). Unconstrained optimization reformulations of variational inequality problems. *Journal of Optimization Theory and Applications 92*(3), 439–456.

Zhang, D. (2006). A network economic model for supply chain versus supply chain competition. *Omega 34*, 283–295.

Zsidisin, G. and B. Ritchie (2008). *Supply chain risk: A handbook of assessment, management and performance*, Chapter Supply chain risk management-development, issues and challenges, pp. 1–11. New York: Spirnger.

Vita

Ilsoo Lee is a Ph.D. candidate in the Department of Industrial and Manufacturing Engineering at the Pennsylvania State University. He received his M.S. degree in Financial Engineering from the Seoul National University, Seoul, Republic of Korea in 2000, where he received his B.S. in Industrial Engineering. His research interest is supply chain network optimization under uncertainty, revenue management, and dynamic pricing. He is a member of INFORMS.