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GROUPOIDS AND ALGEBRAS OF CERTAIN SINGULAR FOLIATIONS WITH A FINITE NUMBER OF LEAVES

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Abstract

Given any singular foliation, Androulidakis and Skandalis showed how to construct a holonomy groupoid, a smooth convolution algebra and a C*-algebra. In this thesis, we study certain classes of singular foliations which have only finitely many leaves. The main results take the form of classifications of these foliations and descriptions of their groupoids and algebras. Along the way, we prove a Lie groupoid analog of a theorem of Dixmier and Malliavin.

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Chapter 1

Introduction

In this dissertation we analyse groupoids and operator algebras associated to certain singular foliations with finitely many leaves, building on work of Androulidakis and Skandalis [4]. In this introductory chapter we first provide background and motivation and then summarize our main results. Note that, except for minor changes, Chapters 3, 4 and 5, and the corresponding portions of this introduction repeat content from [28], [27] and [26], respectively.

1.1 Background and motivation

Informally speaking, a *foliation* of a manifold M consists of a partition of M into immersed submanifolds called *leaves*. Besides being a classical topic in geometry, foliations play an important role in classical mechanics (see Chapter 3 of [39]) and optimal control theory (see Chapter 5 of [1]). Furthermore, the interplay between foliation theory and operator theory is a significant aspect of Connes' noncommutative geometry program [17]. A key construction in this area is the C*-algebra $C^*(\mathcal{F})$ of a regular foliation $\mathcal{F}$. This begins with the construction, due to Winkelnkemper [54], of a (possibly non-Hausdorff) Lie groupoid $G(\mathcal{F})$ called the *holonomy groupoid* or *graph* of $\mathcal{F}$. See [16] for a fuller exposition of this topic; we shall only sketch some of the main ideas below.

In order to motivate the construction of $G(\mathcal{F})$, let us first recall that solution operators of certain PDEs can be usefully represented by smooth integral kernels. For example, this is the case for the heat equation on a Riemannian manifold M. The value of the kernel at $(y, x) \in M \times M$ may be understood as a measure of how much the operator propagates from x to y. Whereas heat flow propagates in all directions, in other important situations one would like to consider operators which only propagate along the leaves of some foliation. A kernel representing such an operator should then be a smooth function on those pairs of points (y, x)

that belong to the same leaf. One is therefore led to the following problem:

Problem. *What is a smooth function on the space of all such pairs?*

The answer is not straightforward because the leaf equivalence relation need not be a submanifold (nor even an immersed submanifold) of $M \times M$. The reason (and also the remedy) for this failure is an interesting and important phenomenon known as *holonomy*. In rough terms, the holonomy groupoid $G(\mathcal{F})$ tries to desingularize the leaf equivalence relation, somewhat in the spirit of blowups in algebraic geometry, by accounting for holonomy. In the case of regular foliations, $G(\mathcal{F})$ is a Lie groupoid and the natural solution to the above problem is "a smooth function on $G(\mathcal{F})$".

Since many natural examples of foliations are not regular, but instead present singularities of some type or another, it is desirable to extend these constructions so that they also apply in singular cases. A number of authors have done work on this topic, see [40], [41], [42], [20], [19], [4], [6]. The most broadly applicable construction is the one given by Androulidakis and Skandalis in [4] and it is their approach that we are concerned with here.

We also follow [4] in understanding a foliation to be any locally finitely-generated $C^\infty(M)$-module $\mathcal{F}$ of compactly-supported vector fields on M that is closed under Lie bracket. A leaf of $\mathcal{F}$ is then the set of points accessible from a given point by composing flows of vector fields in $\mathcal{F}$. By work of Stefan and Sussmann ([48], [49]), the leaves of $\mathcal{F}$ constitute a partition of M into immersed submanifolds (generalizing the Frobenius theorem). A foliation is *regular* if all its leaves have the same dimension and *singular* otherwise. In the regular setting, the module of vector fields can be recovered from the partition, but this fails in the singular setting. In fact, varying the module $\mathcal{F}$ while keeping the partition the same will be a prominent theme in this work.

In [4], given any singular foliation $\mathcal{F}$, Androulidakis and Skandalis constructed the following objects:

1. A holonomy groupoid $G(\mathcal{F})$. In general, this is only a topological groupoid, and its topology can be very wild. It is a Lie groupoid if and only if $\mathcal{F}$ is *almost regular*, a hypothesis satisfied by all foliations studied in this thesis.

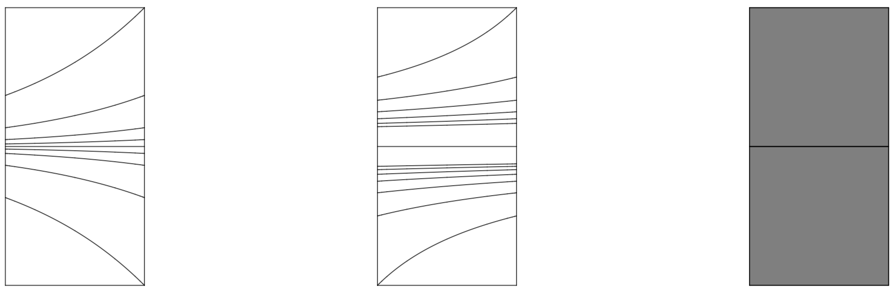
2. A smooth convolution algebra $\mathcal{A}(\mathcal{F})$. In the almost regular case, one has $\mathcal{A}(\mathcal{F}) \cong C_c^\infty(G(\mathcal{F}))$, after choosing a smooth Haar system in order to make sense of convolution.

3. A C*-algebra $C^*(\mathcal{F})$, obtained by completing $\mathcal{A}(\mathcal{F})$. In the almost regular case, this is the usual groupoid C*-algebra in the sense of [43].

We now give an informal introduction to the idea of holonomy. Figure 1.1 depicts the leaves of the following three foliations of the cylinder $S^1 \times \mathbb{R}$, which we regard as having coordinates (x, y) where x is $\mathbb{Z}$-periodic:

$$\mathcal{F}\{\tfrac{d}{dx} + y\tfrac{d}{dy}\} \qquad \mathcal{F}\{\tfrac{d}{dx} + y^2\tfrac{d}{dy}\} \qquad \mathcal{F}\{\tfrac{d}{dx}, y\tfrac{d}{dy}\}.$$

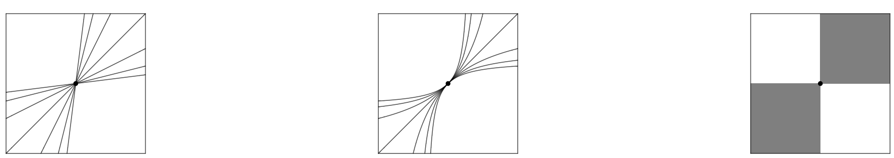
Here, the notation $\mathcal{F}\{X_1, \ldots, X_n\}$ refers to the foliation generated by a finite set of vector fields $X_1, \ldots, X_n$. The first two of these are regular foliations with 1-dimensional leaves while the third is a singular foliation whose leaves are $S^1 \times \mathbb{R}_+$, $S^1 \times \{0\}$ and $S^1 \times \mathbb{R}_-$, where $\mathbb{R}_+ := (0, \infty)$, $\mathbb{R}_- := (-\infty, 0)$.



$\mathcal{F}\{\tfrac{d}{dx} + y\tfrac{d}{dy}\}$ $\qquad$ $\mathcal{F}\{\tfrac{d}{dx} + y^2\tfrac{d}{dy}\}$ $\qquad$ $\mathcal{F}\{\tfrac{d}{dx}, y\tfrac{d}{dy}\}$.

Figure 1.1: Leaves of some foliations of $S^1 \times \mathbb{R}$.

All three of these foliations determine nonsmooth equivalence relations. The issue becomes apparent when we restrict the leaf equivalence relations to the submanifold $T = \{0\} \times \mathbb{R}$ passing through $p = (0, 0)$. The resulting subsets of $T \times T$ are depicted in Figure 1.2.



$\mathcal{F}\{\tfrac{d}{dx} + y\tfrac{d}{dy}\}$ $\qquad$ $\mathcal{F}\{\tfrac{d}{dx} + y^2\tfrac{d}{dy}\}$ $\qquad$ $\mathcal{F}\{\tfrac{d}{dx}, y\tfrac{d}{dy}\}$.

Figure 1.2: Equivalence relations of some foliations of $S^1 \times \mathbb{R}$, restricted to $T = \{0\} \times \mathbb{R}$.

Despite the singularities apparent in Figure 1.2, the holonomy groupoid of each of these foliations is smooth. In terms of these pictures, what occurs is that the problematic point (p, p) at the origin gets blown up and replaced with the *holonomy group* at p. For a regular foliation, given a point p and a transversal submanifold T through the leaf of p, the holonomy group at p may be viewed as the discrete group consisting of all (germs of) diffeomorphisms of T fixing p which can be obtained using flows of vector fields in $\mathcal{F}$. For both of the regular foliations in Figure 1.1, this holonomy group is infinite cyclic, and it is easy to imagine how such a replacement can resolve the singularity. To put it more plainly, for each of the two regular foliations, the equivalence relation in Figure 1.2 is the union of the graphs of a countable family of graphs of diffeomorphisms which intersect only at the origin. After performing the blowup, one is left instead with the *disjoint union* of these graphs.

A striking difference between the regular and singular settings is that, whereas for regular foliations holonomy is purely a discrete phenomenon, for singular foliations one can also have *continuous holonomy*. For the singular foliation $\mathcal{F}\{\frac{d}{dx}, y\frac{d}{dy}\}$ shown above, the group of diffeomorphism germs of T which can be obtained using compositions of flows is infinite-dimensional. One perspective is that the work of Androulidakis and Skandalis identifies the correct way to take a quotient of this infinite-dimensional group and obtain a finite-dimensional Lie group which serves as the natural generalization of the usual holonomy group.

For the foliation $\mathcal{F}\{\frac{d}{dx}, y\frac{d}{dy}\}$ above, the holonomy group at $p = (0,0)$ may be identified with the Lie group of linear, orientation-preserving diffeomorphisms of T and, in particular, it is isomorphic to $\mathbb{R}$. However, this is just one the many foliations of $S^1 \times \mathbb{R}$ whose leaves are $S^1 \times \mathbb{R}_+$, $S^1 \times \{0\}$ and $S^1 \times \mathbb{R}_-$. With the exception of some pathological examples, the holonomy group at p of any such foliation is naturally realized, for some positive integer k that encodes the "transverse order" of the foliation, as a one-dimensional subgroup of the group J^k of k-jets of diffeomorphisms of $\mathbb{R}$ which fix 0. Explicitly,

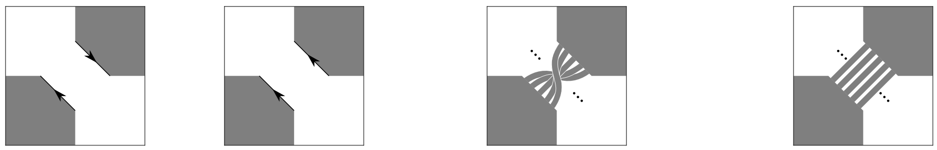
$$J^k = \{a_1 y + a_2 y^2 + \ldots a_\ell y^k : a_i \in \mathbb{R}, a_1 \neq 0\}$$

under the operation "compose and truncate". Some examples of holonomy groups which can occur are tabulated below:

Foliation	Holonomy group	Ambient group
$\mathcal{F}\{\frac{d}{dx}, y\frac{d}{dy}\}$	$\{e^t y : t \in \mathbb{R}\}$	J^1
$\mathcal{F}\{\frac{d}{dx}, y^2\frac{d}{dy}\}$	$\{y + ty^2 : t \in \mathbb{R}\}$	J^2
$\mathcal{F}\{\frac{d}{dx} + y\frac{d}{dy}, y^2\frac{d}{dy}\}$	$\{e^n y + ty^2 : n \in \mathbb{Z}, t \in \mathbb{R}\}$	J^2
$\mathcal{F}\{\frac{d}{dx} + y^2\frac{d}{dy}, y^4\frac{d}{dy}\}$	$\{y + ny^2 + n^2y^3 + ty^4 : n \in \mathbb{Z}, t \in \mathbb{R}\}$	J^4

Table 1.1: Holonomy groups at $p = (0,0)$ of several different foliations of $S^1 \times \mathbb{R}$, all of which have leaves $S^1 \times \mathbb{R}_+$, $S^1 \times \{0\}$ and $S^1 \times \mathbb{R}_-$.

The precise details of how (p,p) is blown up into a copy of the holonomy group at p also depends on two natural orderings of the group J^k, associated to the positive and negative half lines. As Figure 1.3 shows, the topological possibilities for the blowup space are actually quite rich, especially given how simple the leaf space of these foliations is. The last two surfaces, for example, are not homeomorphic, as can be seen by counting the number of topological ends.



$\mathcal{F}\{\frac{d}{dx}, y\frac{d}{dy}\}$ $\mathcal{F}\{\frac{d}{dx}, y^2\frac{d}{dy}\}$ $\mathcal{F}\{\frac{d}{dx} + y\frac{d}{dy}, y^2\frac{d}{dy}\}$ $\mathcal{F}\{\frac{d}{dx} + y^2\frac{d}{dy}, y^4\frac{d}{dy}\}$

Figure 1.3: Holonomy groupoids of several foliations of $S^1 \times \mathbb{R}$ whose leaves are $S^1 \times \mathbb{R}_+$, $S^1 \times \{0\}$ and $S^1 \times \mathbb{R}_-$, restricted to $T = \{0\} \times \mathbb{R}$.

We shall now summarize the contents of this dissertation.

1.2 Preliminaries

In Chapter 2, we gather various definitions and results which are be needed in the subsequent chapters. We review the work [4], on which this dissertation is based. We define precisely what is meant by a (singular) foliation $(M, \mathcal{F})$ and give the constructions of the holonomy groupoid $G(\mathcal{F})$ as well as the *full holonomy groupoid* $G_{\text{full}}(\mathcal{F})$, a larger groupoid containing $G(\mathcal{F})$. We also give a picture of the full holonomy groupoid as a groupoid of (equivalence classes of germs of) holonomy transformations:

Theorem (Theorem 2.3.18)**.** *As an abstract groupoid, the full holonomy groupoid of $(M, \mathcal{F})$ admits a description as a quotient of the groupoid of germs of local diffeomorphisms of M which preserve $\mathcal{F}$.*

This is similar in spirit to the picture obtained in [6], but slightly different because we do not make use of slices.

1.3 The smooth algebra of a one-dimensional singular foliation

In Chapter 3, we analyse in detail the groupoids and algebras of a family of singular foliations of the real line. For each positive integer k, denote by $\mathcal{F}^k_{\mathbb{R}}$ the singular foliation of $\mathbb{R}$ consisting of compactly-supported vector fields that vanish to kth order (or more) at 0. For every k, the leaves of $\mathcal{F}^k_{\mathbb{R}}$ are $\mathbb{R}_-, \mathbb{R}_+$ and $\{0\}$. The holonomy groupoid of $\mathcal{F}^k_{\mathbb{R}}$ is very simple; $G(\mathcal{F}^k_{\mathbb{R}})$ is canonically isomorphic to the transformation groupoid $\mathbb{R} \rtimes_\phi \mathbb{R}$ associated to the flow of any complete vector field generating $\mathcal{F}^k_{\mathbb{R}}$. Accordingly, the smooth algebra $\mathcal{A}(\mathcal{F}^k_{\mathbb{R}})$ is isomorphic to $C^\infty_c(\mathbb{R} \rtimes_\phi \mathbb{R})$ and the C*-algebra $C^*(\mathcal{F}^k_{\mathbb{R}})$ is isomorphic to $C_0(\mathbb{R}) \rtimes_\phi \mathbb{R}$, the crossed product of the associated action of $\mathbb{R}$ on $C_0(\mathbb{R})$.

We precisely determine the extent to which the holonomy groupoids, smooth algebras and C*-algebras of the foliations $\mathcal{F}^k_{\mathbb{R}}$ remember the positive integer k. The most substantial part of this story is the analysis of the smooth algebras and the main result of Chapter 3 is the following:

Theorem (Theorem 3.4.10). *The smooth convolution algebras of the foliations $\mathcal{F}^k_{\mathbb{R}}$ are pairwise nonisomorphic.*

For the C*-algebras, however, we have the following:

Theorem (Theorem 3.3.5). *The C*-algebras of the foliations $\mathcal{F}^k_{\mathbb{R}}$ are of two isomorphism types that are determined by the parity of k.*

This demonstrates the principle that, even for singular foliations as simple as these ones, there can be information stored in the smooth algebra which is washed away when one passes to the C*-algebra.

There is a natural faithful representation of $C^*(\mathcal{F}^k_{\mathbb{R}})$ on $L^2(\mathbb{R}_-) \oplus L^2(\mathbb{R}_+)$. Indeed, the C*-algebra of a foliation can always be naturally represented onto the L^2-space of each of its leaves. In the case at hand, the open, nonsingular leaves $\mathbb{R}_-$ and $\mathbb{R}_+$ suffice to give us a faithful representation. For these concrete C*-algebras, we obtain:

Theorem (Theorem 3.3.6). *The C*-algebras of the foliations $\mathcal{F}^k_{\mathbb{R}}$, represented as C*-algebras on $L^2(\mathbb{R}_-) \oplus L^2(\mathbb{R}_+)$, are pairwise distinct.*

We now outline the methods by which the above results on the algebras of $\mathcal{F}^k_{\mathbb{R}}$ are obtained. The flow of a smooth vector field on $\mathbb{R}$ with a finite order critical point at 0 and no other critical points is determined up to topological conjugacy and time-reversal symmetry by the parity of its vanishing order at 0. This remark already explains why there are at most two possibilities for the isomorphism type of the C*-algebra $C_0(\mathbb{R}) \rtimes_\phi \mathbb{R}$. The fact that two different C*-algebras do indeed occur follows from an index calculation.

The more substantial issue of showing that the smooth algebras $\mathcal{A}(\mathcal{F}^k_{\mathbb{R}}) \cong C^\infty_c(\mathbb{R} \rtimes_\phi \mathbb{R})$ are pairwise nonisomorphic requires different methods. The intuition here is as follows: if the flow ϕ fixes the origin to kth order in x, where x denotes the coordinate of the manifold $\mathbb{R}$, then the convolution product on $C^\infty_c(\mathbb{R} \rtimes_\phi \mathbb{R})$ is "commutative to kth order in x". In more precise terms, our argument follows the following outline:

1. In each of the convolution algebras $\mathcal{A}(\mathcal{F}^k_{\mathbb{R}})$, there is a nested sequence of ideals $x^p \cdot \mathcal{A}(\mathcal{F}^k_{\mathbb{R}})$, $p = 1, 2, 3, \ldots$, consisting of functions vanishing to order p on the isotropy group of 0.

2. The quotient $\frac{\mathcal{A}(\mathcal{F}^k_{\mathbb{R}})}{x^p \cdot \mathcal{A}(\mathcal{F}^k_{\mathbb{R}})}$ is commutative if and only if $p \leq k$.

3. Any isomorphism between $\mathcal{A}(\mathcal{F}^k_{\mathbb{R}})$ and $\mathcal{A}(\mathcal{F}^\ell_{\mathbb{R}})$ necessarily maps the ideal sequence of the first algebra onto the ideal sequence of the second.

Our approach to the third step of this program requires us to solve a problem concerning the existence of factorizations in the smooth convolution algebra $C^\infty_c(\mathbb{R} \rtimes_\phi \mathbb{R})$. A general form of this factorization problem is considered later in Chapter 4.

Lastly, we explicitly calculate the quotient of $\mathcal{A}(\mathcal{F}^k_{\mathbb{R}})$ by the ideal

$$x^\infty \cdot \mathcal{A}(\mathcal{F}^k_{\mathbb{R}}) := \bigcap_{p \geq 1} x^p \cdot \mathcal{A}(\mathcal{F}^k_{\mathbb{R}}).$$

This results in an interesting algebra of formal series whose indeterminate does not commute with its coefficients.

Theorem (Theorem 3.4.12). *The quotient algebra* $\frac{\mathcal{A}(\mathcal{F}^k_{\mathbb{R}})}{x^\infty \cdot \mathcal{A}(\mathcal{F}^k_{\mathbb{R}})}$ *is isomorphic to* $C^\infty_c(\mathbb{R})[[x]]$ *with a certain noncommutative product* $*_k$.

The formulas for the product $*_k$ are somewhat messy, but let us show the first two cases. Suppose $f = \sum_{p \geq 0} f_p x^p, g = \sum_{p \geq 0} g_p x^p \in C^\infty_c(\mathbb{R})[[x]]$ and write $f *_k g = \sum_{p \geq 0} h_p x^p$.

- If $k = 1$, then $h_p = \sum_{0 \leq n \leq p} f_n * (e^{nt} g_{p-m})$.
- If $k = 2$, then $h_p = \sum_{0 \leq n \leq m \leq p} \binom{m-1}{n-m} f_n * (t^{m-n} g_{p-m})$.

Here, $*$ denotes the usual (commutative) convolution product on the coefficient ring $C^\infty_c(\mathbb{R})$ and t is the standard coordinate multiplier for this ring.

1.4 A Dixmier-Malliavin theorem for Lie groupoids

In Chapter 4, we generalize a famous theorem of Dixmier-Malliavin from the context of Lie groups to the context of Lie groupoids.

In a 1960 paper, Ehrenpreis [25] posed a number of related questions including whether every smooth, compactly-supported function $\varphi \in C^\infty_c(\mathbb{R}^n)$ can be "deconvolved" as $f = g * h$ where $g, h \in C^\infty_c(\mathbb{R}^n)$. The latter question became known as the *Ehrenpreis factorization problem.* In 1978, Rubel-Squires-Taylor [45] showed that the answer is "no" if $n \geq 3$. In the same year, Dixmier-Malliavin [23] showed that the answer is still "no" if $n = 2$. The remaining case $n = 1$ was eventually settled in 1999 by Yulmukhametov [55] who showed the answer is "yes" for the real line. Also in the positive direction, [23] gives the answer to a weaker form of the factorization question to be "yes" for any Lie group whatsoever. This is the celebrated *Dixmier-Malliavin theorem.*[1]

Theorem ([23], 3.1 Théorème)**.** *Let G be a Lie group and form the smooth convolution algebra $C_c^\infty(G)$. Every $f \in C_c^\infty(G)$ can be expressed as*

$$f = g_1 * h_1 + \ldots + g_n * h_n,$$

for some positive integer n and $g_1, h_1, \ldots, g_n, h_n \in C_c^\infty(G)$.

This result has applications to the representation theory of real reductive groups (see [12], [8]).

The main result of Chapter 4 is the analogous result for Lie groupoids:

Theorem (Theorem 4.4.2)**.** *Let G be a Lie groupoid and form the smooth convolution algebra $C_c^\infty(G)$. Every $f \in C_c^\infty(G)$ can be decomposed as*

$$f = g_1 * h_1 + \ldots + g_n * h_n,$$

for some positive integer n and $g_1, h_1, \ldots, g_n, h_n \in C_c^\infty(G)$.

Note that defining a convolution product on $C_c^\infty(G)$ requires a choice of smooth Haar system but, as different Haar systems lead to canonically isomorphic algebras, issues relating to factorization do not depend on this choice. One could also avoid making a choice entirely by working with appropriate densities in place of functions.

We then apply our Dixmier-Malliavin theorem to obtain results on the arithmetic of certain ideals in the smooth convolution algebra of a Lie groupoid. Any closed, invariant submanifold X of the unit space of a Lie groupoid G determines a nested sequence of ideals $J_1 \supseteq J_2 \supseteq \ldots \supseteq J_\infty$ in $C_c^\infty(G)$, where J_p consists of the functions which vanish to order p on the restricted groupoid $G_X \subseteq G$. Our main findings here are:

Theorem (Theorem 4.6.1)**.** *$J_\infty * J_\infty = J_\infty$ and $(J_1)^{*p} = J_p$ for all $p > 0$.*

[1] More accurately, one of several closely-related Dixmier-Malliavin theorems.

In practical terms, this means that a function vanishing to infinite order on G_X can be written as a finite sum in which each term is a convolution of two functions vanishing to infinite order on G_X, and that a function vanishing to pth order on G_X can be written as a finite sum in which each term is a p-fold convolution of functions which vanish on G_X. Note these results on ideals are only interesting after one has generalized to the groupoid setting. In the group case, the unit space consists of a single point and these ideals do not arise at all.

1.5 Singular foliations tangent to a given hypersurface

In Chapter 5, we define a class of singular foliations which we call *transverse order k foliations* (Definition 5.3.1). We classify these foliations by a holonomy invariant and obtain descriptions of their groupoids and algebras.

Suppose $\mathcal{F}$ is a foliation of a connected manifold M whose leaves consist of a single codimension-1 submanifold $L \subseteq M$ together with the components of $M \setminus L$. The total number of leaves is therefore either two or three. Thanks to a splitting principle for singular foliations ([4] Proposition 1.12, [6] Proposition 1.2), the local structure of such a foliation is completely determined by a foliation of $\mathbb{R}$ modelling the transverse structure of the foliation near the leaf L. If this transverse foliation is $\mathcal{F}^k_{\mathbb{R}}$, we say that $(M, \mathcal{F})$ is a foliation of *transverse order* k (Definition 5.3.1). The foliations of $S^1 \times \mathbb{R}$ with singular leaf $S^1 \times \{0\}$ that appeared earlier in Table 1.1 were examples of transverse order k foliations. The purpose of Chapter 5 is to classify transverse order k foliations and provide explicit descriptions their groupoids and algebras.

If M and L are given, there is a unique foliation $\mathcal{F}$ of transverse order $k = 1$ whose singular leaf is L, namely the collection of all compactly-supported vector fields which are tangent along L. However, when $k \geq 2$, the structure of transverse order k foliations becomes much more interesting. For this reason, we will always assume k is an integer ≥ 2 in this chapter.

If $X_1, \ldots, X_n$ are smooth vector fields on a manifold M such that $[X_i, X_j]$ is a $C^\infty(M)$-linear combination of $X_1, \ldots, X_n$ for all i, j, we use the notation

$$\mathcal{F}\{X_1, \ldots, X_n\} := \mathrm{span}_{C^\infty_c(M)}\{X_1, \ldots, X_n\}$$

for the foliation they generate.

Example 1.5.1. Consider $\mathbb{R}^2$ with usual coordinates (x, y). Then

$$\mathcal{F}\left\{y^2\frac{\partial}{\partial y}, \frac{\partial}{\partial x}\right\} \qquad\qquad \mathcal{F}\left\{y^2\frac{\partial}{\partial y}, \frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right\}$$

are distinct transverse order 2 foliations with singular leaf the x-axis. Although these foliations are distinct, they are still isomorphic to each other; the pushforward of the first foliation by the diffeomorphism $(x, y) \mapsto (x, e^x y)$ is the second foliation.

The above example exposes the basic point that the module of vector fields on M which are "tangent to L to order 2" is not in fact well-defined when working up to smooth coordinate changes. To get nonisomorphic examples of transverse order 2 foliations with the same leaves, one needs the singular leaf to not be simply-connected.

Example 1.5.2. In the previous example, consider x as a $\mathbb{Z}$-periodic coordinate so that the space is the cylinder $S^1 \times \mathbb{R}$. Since the generating vector fields are invariant under horizontal translation, they also define transverse order 2 foliations on $S^1 \times \mathbb{R}$ with singular leaf the equator $S^1 \times \{0\}$. These foliations are not isomorphic to each other and this chapter will provide a framework explaining this nonisomorphism. Briefly, whereas for the first foliation the holonomy around the equator is trivial, for the second, the holonomy, suitably interpreted, is multiplication by e.

Suppose $(M, \mathcal{F})$ is a foliation of transverse order $k \geq 2$ with singular leaf L. We give a brief overview of the notion holonomy to be introduced. Let c be a path in L from a point x to a point x' and fix small, one-dimensional transversals T and T' at the endpoints. If $\mathcal{F}$ were a regular foliation of codimension-one, the classical notion of holonomy would assign to c a diffeomorphism germ $T \to T'$ sending $x \mapsto x'$. This does not occur in the case at hand, but it turns out one does have a well-defined holonomy mapping at the level of $(k-1)$-jets (this is related to the picture of the holonomy groupoid obtained in [6]). In particular, taking $x = x'$, $T = T'$ and choosing an identification of T with $\mathbb{R}$, one obtains from this assignment a homomorphism

$$\gamma : \pi_1(L, x) \to J^{k-1}, \tag{1.1}$$

where J^r denotes the group of r-jets at 0 of diffeomorphisms of $\mathbb{R}$ fixing 0. Concretely, J^r is the group of polynomials of the form $a_1 y + \ldots + a_r y^r$ with $a_i \in \mathbb{R}$, $a_1 \neq 0$ under the operation "compose and truncate". There is a canonical quotient map $J^r \to J^{r-1}$ for all $r \geq 2$ whose kernel is the group $\mathbb{R}$, embedded in J^r by way of $t \mapsto y + ty^r$. We obtain:

Theorem (Definition 5.5.6)**.** *The homomorphism* (1.1) *is well-defined up to inner automorphisms of* J^{k-1} *and gives rise to a holonomy invariant*

$$h(\mathcal{F}) \in [\pi_1(L), J^{k-1}] \tag{1.2}$$

for the foliation.

Here, if A and B are groups, we use $[A, B]$ to denote the quotient set of $\mathrm{Hom}(A, B)$ by the conjugation action of B, in the spirit of the similar notation frequently employed for homotopy classes of maps. Note it is not necessary to specify a basepoint for the fundamental group in (1.2) because L is connected, so its different fundamental groups are canonically isomorphic when working up to to inner automorphisms.

The holonomy invariant (1.2) is "L-local", in the sense that it only depends on the restriction of the foliation to a neighbourhood of the singular leaf, and natural with respect to isomorphisms of transverse order k foliations, in an appropriate way. In fact, it is a complete invariant for the structure of the foliation nearby to the singular leaf

Theorem (Theorem 5.9.2)**.** *If $(M_i, \mathcal{F}_i)$ is a foliation of transverse order k with singular leaf L_i for $i = 1, 2$ and there is a diffeomorphism $\theta_0 : L_1 \to L_2$ carrying $h(\mathcal{F}_1)$ to $h(\mathcal{F}_2)$, then θ_0 can be extended to a foliation-preserving diffeomorphism $\theta : U_1 \to U_2$, where U_i is neighbourhood of L_i in M_i.*

Furthermore, the possible values of this holonomy invariant are exhausted.

Theorem (Theorem 5.10.1)**.** *Given a connected manifold L and a homomorphism $\pi_1(L) \to J^{k-1}$, there exists a transverse order k singular foliation with singular leaf L whose holonomy invariant is (the class of) the given homomorphism.*

We formalize the above ideas about holonomy using principal bundles[2]. Given a transverse order $k \geq 2$ singular foliation $(M, \mathcal{F})$ with singular leaf L, we construct a sequence of principal bundles $P^r(\mathcal{F}) \to L$, $r = 1, 2, \ldots, k$. The elements of $P^r(\mathcal{F})$ are r-jets of certain submersions $M \to \mathbb{R}$ and may be thought of as dual versions of transversals. When $r \leq k - 1$, the structure group of $P^r(\mathcal{F})$ is J^r_d, the underlying discrete group of J^r. This captures the idea that the $(k-1)$-jets of a transversals can be parallel transported along paths in L. This rigidity breaks down at $r = k$; the structure group of $P^k(\mathcal{F})$ is $J^k_{\mathbb{R}}$, the one-dimensional Lie group structure on J^k obtained by decomposing it into the fibers of the natural projection $J^k \to J^{k-1}$. The main applications of these principal bundles $P^r(\mathcal{F})$ are as follows:

Theorem (Sections 5.5, 5.6, 5.7)**.**

1. *The monodromy of the principal J^{k-1}_d-bundle $P^{k-1}(\mathcal{F})$ is exactly the holonomy invariant $h(\mathcal{F})$ of* (1.2)*.*
2. *The gauge groupoid of the principal $J^k_{\mathbb{R}}$-bundle $P^k(\mathcal{F})$ reconstructs the holonomy groupoid of $\mathcal{F}$, restricted to L.*

[2]Our principal bundles will always be smooth, with structure group acting from the left.

3. *The monodromy of the principal J^1_d-bundle $P^1(\mathcal{F})$ determines a flat connection on the conormal bundle of L in M (note that $J^1 = \mathrm{GL}(1, \mathbb{R})$) which it is appropriate to call the* Bott connection*.*

In [4], in addition to the (singular analog of the) usual holonomy groupoid $G(\mathcal{F})$, the authors construct a *full holonomy groupoid* $G_{\text{full}}(\mathcal{F})$. This is a "big groupoid" containing the usual (i.e. minimal) holonomy groupoid, as well as various intermediate groupoids. It is helpful to work inside this big groupoid initially and later extract the usual one as its s-connected component.

The full holonomy groupoid of $\mathcal{F}^k_{\mathbb{R}}$ can be thought of as a smooth blowup of the singular equivalence relation $(\mathbb{R} \setminus \{0\})^2 \cup \{(0,0)\} \subseteq \mathbb{R}^2$ wherein the singular point $(0,0)$ is replaced by a copy of $J^k_{\mathbb{R}}$:

$$G_{\text{full}}(\mathcal{F}^k_{\mathbb{R}}) \cong (\mathbb{R} \setminus \{0\})^2 \cup J^k_{\mathbb{R}}.$$

More generally, the full holonomy groupoid of any transversely order k foliation $(M, \mathcal{F})$ with singular leaf L can be thought of as a smooth blowup of the singular equivalence relation $(M \setminus L)^2 \cup L^2 \subseteq M^2$ wherein the singular locus L^2 is replaced by a copy of the gauge groupoid of the principal $J^k_{\mathbb{R}}$-bundle $P^k(\mathcal{F}) \to L$:

$$G_{\text{full}}(\mathcal{F}) \cong (M \setminus L)^2 \cup \mathrm{Gauge}(P^k(\mathcal{F})).$$

The full holonomy groupoid of a transverse order k foliation provides an interesting example of a topological space equipped with a smooth atlas that is nearly, but not quite, a smooth manifold:

Theorem (Theorem 5.2.14)**.** *The topology of $G_{\text{full}}(\mathcal{F}^k_{\mathbb{R}})$ is Hausdorff, regular and separable, but not normal.*

There are known constructions for "manifolds" with these properties (see [35], Chapter 14), but it is interesting to encounter such a beast "in the wild".

The holonomy groupoid of a transverse order $k \geq 2$ foliation $(M, \mathcal{F})$ is the s-connected component of the full holonomy groupoid. We can give more concrete information: let $\gamma : \pi_1(L, x_0) \to J^{k-1}$ be any homomorphism representing the holonomy invariant $h(\mathcal{F}) \in [\pi_1(L), J^{k-1}]$. Let Γ be the range of γ, a countable subgroup of J^{k-1}, and let $\Gamma_{\mathbb{R}}$ be the preimage of Γ by the natural projection $J^k \to J^{k-1}$. The relationships between these various groups are shown in the following diagram:

$$\begin{array}{ccccc}
\mathbb{R} & \longrightarrow & J^k_{\mathbb{R}} & \longrightarrow & J^{k-1}_d \\
\Vert & & \uparrow & & \uparrow \\
\mathbb{R} & \longrightarrow & \Gamma_{\mathbb{R}} & \longrightarrow & \Gamma \\
 & & & & \uparrow{\scriptstyle \gamma} \\
 & & & & \pi_1(L, x_0)
\end{array}$$

Theorem (Theorem 5.7.5)**.** *The isotropy groups of $G(\mathcal{F})$ at points on the singular leaf are isomorphic to $\Gamma_{\mathbb{R}}$.*

Meanwhile, the restriction of $G(\mathcal{F})$ to one of the open leaves (there are at most two) is a pair groupoid so, applying standard results on groupoid C*-algebras, we obtain information about the structure of the foliation C*-algebra.

Theorem (Theorem 5.7.2, Corollary 5.7.6)**.** *The foliation C*-algebra $C^*(\mathcal{F})$ fits into an extension*

$$0 \to I \to C^*(G(\mathcal{F})) \to C^*(\Gamma_{\mathbb{R}}) \otimes \mathbb{K} \to 0. \tag{1.3}$$

where $\mathbb{K}$ is the C-algebra of compact operators on a separable Hilbert and I denotes either $\mathbb{K}$ or $\mathbb{K} \oplus \mathbb{K}$, according to whether $\mathcal{F}$ has one open leaf or two.*

It would be interesting to analyse the extension (1.3). The one-dimensional Lie group $\Gamma_{\mathbb{R}}$ is solvable, so this problem is likely to be tractable.

Since we are mainly concerned with what is happening near the singular leaf L, it is often sufficient to consider the case where M is the total space of a line bundle $\pi : E \to L$, with L embedded in E as the zero section. In this case, we already have a natural principal J^r-bundle $J^r(E, \mathbb{R}) \to L$, even without specifying a transverse order k-foliation.

Theorem. *There is a one-to-one correspondence between:*

1. *Flat connections on the principal J^{k-1}-bundle $J^{k-1}(E, \mathbb{R})$*
2. *Singular foliations of transverse order k on E whose singular leaf is L (embedded as the zero section)*

Once a flat connection on $J^{k-1}(\mathcal{F})$ has been fixed, the resulting J^{k-1}_d-bundle structure on $J^{k-1}(E, \mathbb{R})$ is canonically isomorphic to the principal J^{k-1}_d-bundle $P^{k-1}(\mathcal{F})$.

Chapter 2

Preliminaries

The present chapter establishes certain conventions to be used throughout and summarizes relevant aspects of Androulidakis and Skandalis's work on singular foliations and their groupoids and algebras. Proofs are omitted when they can be found in [4].

2.1 Conventions

2.1.1 Manifolds

In this dissertation, a **smooth manifold** refers to a topological space equipped with a smooth atlas of some constant, finite dimension that is furthermore metrizable (or, equivalently, paracompact and Hausdorff). Many authors require their manifolds to be second-countable, but that assumption is not convenient here. In any event, the distinction is only relevant for highly disconnected manifolds; every smooth manifold in our sense is a (possibly uncountable) disjoint union of second-countable smooth manifolds.

We sometimes employ the term **smooth space** to refer to a topological space that is equipped with a smooth atlas. Note that the same disclaimers which apply when one speaks of two manifolds being "equal" apply to smooth spaces also. This is to say, an atlas for a smooth space is not really an innate part of its structure. Rather, one introduces the usual notion of equivalence of two atlases and works either with equivalence classes of atlases, or with the unique maximal atlas in each class. Note the topology of a smooth space is uniquely determined by any atlas, so the former need not be specified in advance.

2.1.2 Groupoids

We shall tend to use calligraphic characters such as $\mathcal{G}$ to denote abstract groupoids (no topology) and reserve roman characters such as G for topological groupoids. For brevity, we often write $\mathcal{G} \rightrightarrows X$ to indicate that $\mathcal{G}$ is an (abstract) groupoid with unit space X. We typically denote the source and target projections of $\mathcal{G}$ by s and t, respectively. Multiplication is performed from right to left so that, given $a, b \in \mathcal{G}$, the product ab is defined if and only if $s(a) = t(b)$. The inversion map is denoted $\imath : \mathcal{G} \to \mathcal{G}$ or, frequently, just $a \mapsto a^{-1}$. We use (standard) notations such as $\mathcal{G}_x := s^{-1}(x)$ and $\mathcal{G}^x := t^{-1}(x)$ for the source and target fibers. We only speak of morphisms between groupoids that have the same unit space and always require that the underlying map on the unit space is the identity. Accordingly, the way in which we understand quotients of groupoids is constrained to the following:

Definition 2.1.1. A **normal subgroupoid** of a groupoid $\mathcal{G} \rightrightarrows X$ is a union $\mathcal{N} = \bigcup_{x \in M} \mathcal{N}_x$, where $\mathcal{N}_x$ is a subgroup of the isotropy group $\mathcal{G}_x^x$ such that, if $x, y \in X$, $a \in \mathcal{G}_x^y$, $b \in \mathcal{N}_x$, then $aba^{-1} \in \mathcal{N}_y$.

Lemma 2.1.2. *Let $\mathcal{N}$ be normal subgroupoid of a groupoid $\mathcal{G} \rightrightarrows X$. Given $x, y \in X$ and $a, b \in \mathcal{G}_x^y$, put $a \approx b$ if and only if $a^{-1}b \in \mathcal{N}$. Then, $\approx$ is an equivalence relation and the groupoid operations of $\mathcal{G}$ descend in a well-defined way to give the quotient set $\mathcal{G}/\approx$ the structure of a groupoid on X.* □

Definition 2.1.3. Given a normal subgroupoid $\mathcal{N}$ of a groupoid $\mathcal{G}$, the **quotient groupoid** $\mathcal{G}/\mathcal{N}$ is the groupoid $\mathcal{G}/\approx$ of the above lemma.

2.1.3 Lie groupoids

A **Lie groupoid** is a groupoid $G \rightrightarrows B$ where G and B are smooth manifolds, the source and target maps $s, t : G \to B$ are submersions and all structure maps are smooth. If G is only a smooth space (see Section 2.1.1), we call $G \rightrightarrows B$ a **smooth groupoid**. One may refer to [37] for a detailed treatment of Lie groupoids.

Example 2.1.4. For any manifold M, the **pair groupoid** refers to the Lie groupoid structure on cartesian product $M \times M$ with source projection pr_2, target projection pr_1 and multiplication defined by $(x_3, x_2)(x_2, x_1) = (x_3, x_1)$ for all $x_1, x_2, x_3 \in M$.

Example 2.1.5. Given a smooth action of a Lie group H on a smooth manifold M, the **transformation groupoid** $H \ltimes M$ is the Lie groupoid whose underlying manifold is $H \times M$ with groupoid operations defined as follows:

Source projection:	$(h, x) \mapsto x$
Target projection:	$(h, x) \mapsto hx$
Multiplication:	$(h_2, h_1 x)(h_1, x) = (h_2 h_1, x)$

We sometimes find it convenient to reverse the order of the factors; the transformation groupoid $M \rtimes H$ has underlying manifold $M \times H$ and its Lie groupoid structure is such that $(h, x) \mapsto (x, h) : H \ltimes M \to M \rtimes H$ is a Lie groupoid isomorphism.

In this dissertation, all principal bundles are assumed to be smooth, with structure group acting on the left. Every principal bundle determines a so-called gauge groupoid (also known as the Atiyah groupoid) as described below. This construction will play an important role in Chapter 5.

Lemma 2.1.6. *Let $\pi : P \to B$ be a (smooth, left) principal H-bundle, where H is a Lie group. Then, there is a unique Lie groupoid structure on the quotient manifold $(P \times P)/H$, where H acts diagonally, whose operations are determined as follows:*

$$\begin{aligned} &\textit{source projection:} && [q,p] \mapsto \pi(p) \\ &\textit{target projection:} && [q,p] \mapsto \pi(q) \\ &\textit{multiplication:} && [r,q][q,p] = [r,p]. \end{aligned}$$

for all $p, q, r \in P$. Here $[q, p]$ denotes the class of (q, p) in $(P \times P)/H$. □

Definition 2.1.7. The **gauge groupoid** $\mathrm{Gauge}(P) \rightrightarrows B$ of a (smooth, left) principal bundle $P \to B$ is the Lie groupoid constructed in the above lemma.

Example 2.1.8. In the case of a trivial left H-bundle $B \times H$, it is easy to see that $(y, x, h) \mapsto [(y, 1), (x, h)]$ defines an isomorphism $B \times B \times H \to \mathrm{Gauge}(B \times H)$.

Gauge groupoids are always transitive and, in fact, every transitive Lie groupoid $G \rightrightarrows B$ is isomorphic to a gauge groupoid; for any choice of $x \in B$, one has that G^x is a principal G^x_x-bundle and $G \cong \mathrm{Gauge}(G^x)$. In [3], the correspondence between transitive Lie groupoids and principal bundles is extended in a way that takes extensions into account.

2.1.4 Monodromy of flat bundles

Given two groups A and B, we denote by $[A, B]$ the quotient of the set $\mathrm{Hom}(A, B)$ by the conjugation action of B.

Let M be a connected, smooth manifold and let Γ be a (discrete) group. Given basepoints $x, y \in M$, there is a canonical bijection between the quotient sets $[\pi_1(M, x), \Gamma]$ and $[\pi_1(M, y), \Gamma]$. This is so because the isomorphisms $\pi_1(M, x) \to \pi_1(M, y)$ determined by two different choices of paths from x to y only differ by an inner automorphism of $\pi_1(M, x)$, and it follows that their induced bijections

$\mathrm{Hom}(\pi_1(M,x),\Gamma) \to \mathrm{Hom}(\pi_1(M,y),\Gamma)$ differ by an inner automorphism of Γ. It therefore makes sense to speak of the set $[\pi_1(M),G)]$ without specifying a choice of basepoint. In a similar vein, we have the following

Proposition 2.1.9. *If $\theta : M_1 \to M_2$ is a diffeomorphism of connected manifolds, then pushing forward loops by θ determines a well-defined bijection*

$$\theta_* : [\pi_1(M_1),\Gamma] \to [\pi_1(M_2),\Gamma].$$

In fact, the procedure of the above proposition makes

$$M \mapsto [\pi_1(M),\Gamma]$$

into a functor from the category of connected, smooth manifolds and diffeomorphisms to the category of sets and bijections.

Let $\pi : Q \to M$ be a smooth principal bundle with connected base manifold M and discrete structure group Γ.[1] Fix a point $q_0 \in Q$ and put $x_0 := \pi(q_0)$. One may then define a group homomorphism

$$\gamma : \pi_1(M,x_0) \to \Gamma$$

in the following way: given any loop $c : [0,1] \to M$ based at M, let $\widetilde{c} : [0,1] \to Q$ be the unique lift of c with $c(0) = q_0$ and define $\gamma([c])$ by:

$$\widetilde{c}(1) = \gamma([c]) \cdot q_0.$$

One may check this gives a well-defined homomorphism γ, moreover, that the class in $[\pi_1(M),\Gamma]$ of this homomorphism does not depend on the chosen point $q_0 \in M$. For example, if $h \in \Gamma$ and $q_0' = h \cdot q_0$, then the monodromy homomorphism $\gamma' : \pi_1(M,x_0) \to \Gamma$ determined by q_0' satisfies $\gamma' = \mathrm{Ad}_h \circ \gamma$. It therefore makes sense to define:

Definition 2.1.10. Let Q be a smooth principal bundle with connected base manifold M and discrete structure group Γ. The **monodromy invariant** of Q is the element $h(Q) \in [\pi_1(M),\Gamma]$ represented by the homomorphism $\pi_1(M,x_0) \to \Gamma$ constructed above.

It is a standard result that principal bundles with discrete structure group (or equivalently flat bundles) are completely classified by their monodromy invariants. To be more precise, the following result holds:

[1] This is the same thing as as a normal covering space whose group of deck transformations has been identified with Γ.

Theorem 2.1.11. *For $i = 1, 2$, let M_i be a connected, smooth manifold and let $Q_i \to M_i$ be a smooth left principal bundle with discrete structure group Γ.*

1. *If $\theta : Q_1 \to Q_2$ is a Γ-bundle isomorphism and $\theta_0 : M_1 \to M_2$ is the underlying map of the base, then $(\theta_0)_*(h(Q_1)) = h(Q_2)$.*
2. *Conversely, if there exists a diffeomorphism $\theta_0 : M_1 \to M_2$ such that $(\theta_0)_*(h(Q_1)) = h(Q_2)$, then there exists a Γ-bundle isomorphism $\theta : Q_1 \to Q_2$ covering θ_0.* □

2.2 Singular foliations

2.2.1 Modules of smooth sections

Throughout the following, E is a smooth vector bundle over a smooth manifold M and $\mathcal{F}$ is a $C^\infty(M)$-submodule of $C_c^\infty(M; E)$, the smooth, compactly-supported sections of E.

Definition 2.2.1. The **fiber** of a submodule $\mathcal{F} \subseteq C_c^\infty(M; E)$ at $x \in M$ is the vector space $A_x\mathcal{F} := \mathcal{F}/I_x\mathcal{F}$, where $I_x \subseteq C^\infty(M)$ denotes the ideal of functions vanishing at the point x. We denote the quotient map $\mathcal{F} \to A_x\mathcal{F}$ by $X \mapsto [X]_x$.

From this definition, we have the relation

$$[fX]_x = f(x)[X]_x \qquad f \in C^\infty(M), X \in \mathcal{F}.$$

If $X \in \mathcal{F}$ has $X \in I_x\mathcal{F}$ for all $x \in M$, then $X = 0$, so one may legitimately regard $\mathcal{F}$ as a module of sections of the bundle

$$A\mathcal{F} := \bigsqcup_{x \in M} A_x\mathcal{F}.$$

However, it should be noted that $A\mathcal{F}$ need not have the structure of a smooth vector bundle and, indeed, the dimensions of its fibers may vary from point to point. Note also that, for each $x \in M$, the evaluation map $\mathcal{F} \to E_x$ contains $I_x\mathcal{F}$ in its kernel and therefore descends to a well-defined map $A_x\mathcal{F} \to E_x$ on the fiber. Thus, the "singular bundle" $A\mathcal{F}$ is equipped with a "singular bundle map" $A\mathcal{F} \to E$. Furthermore, this bundle map sends $\mathcal{F}$, viewed as a module of sections of $A\mathcal{F}$, identically onto $\mathcal{F}$, viewed as a module of sections of E.

Definition 2.2.2. A submodule $\mathcal{F} \subseteq C_c^\infty(M; E)$ is **finitely-generated** if there exist $X_1, \ldots, X_n \in C^\infty(M; E)$ such that every $X \in \mathcal{F}$ has a representation $X = f_1X_1 + \ldots + f_nX_n$ with $f_i \in C_c^\infty(M)$ and **free of rank n** if the latter representations are also unique.

Remark 2.2.3. Note that, in the above definition, generators are not required to be compactly-supported. So, for example, we consider $\mathfrak{X}_c(\mathbb{R}^n) = C_c^\infty(\mathbb{R}^n; T\mathbb{R}^n)$, the $C^\infty(\mathbb{R}^n)$-module of compactly-supported vector fields on $\mathbb{R}^n$, to be freely-generated by $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$, even though $\frac{\partial}{\partial x_i} \notin \mathfrak{X}_c(\mathbb{R}^n)$.

Definition 2.2.4. Given an open set $U \subseteq M$, the **restriction** of a submodule $\mathcal{F} \subseteq C_c^\infty(M; E)$ to U is defined to be the $C^\infty(U)$-module $\mathcal{F}_U \subseteq C_c^\infty(U; E_U)$ given by $\mathcal{F}_U := C_c^\infty(U)\mathcal{F}$. We say that $\mathcal{F}$ is **locally finitely-generated** (resp. **locally free of rank n**) if each point of M belongs to some open set U such that $\mathcal{F}_U$ is finitely-generated (resp. free of rank n).

If $\mathcal{F}$ is locally finitely-generated, then each fiber $A_x\mathcal{F}$ is a finite-dimensional vector space. Furthermore, the dimension of $A_x\mathcal{F}$ equals the minimum number of generators required for $\mathcal{F}_U$, when U is any sufficiently small neighbourhood of x ([4], Proposition 1.5).

Let us now say a bit more about the locally free case.

Proposition 2.2.5. *Let $\mathcal{F}$ be a locally finitely-generated $C^\infty(M)$-submodule of $C_c^\infty(M; E)$. Then, the following are equivalent:*

1. $\dim(A_x\mathcal{F}) = k$ *for all* $x \in M$.
2. $\mathcal{F}$ *is locally free of rank* k.
3. *There exists a k-dimensional smooth vector bundle $A \to M$ and a vector bundle map $A \to E$ which is injective over a dense subset of M such that the image of the induced map $C_c^\infty(M; A) \to C_c^\infty(M; E)$ is $\mathcal{F}$.*

Moreover, when these equivalent conditions hold, we may take $A = A\mathcal{F}$, equipped with the unique smooth structure for which $\mathcal{F}$, realized as a module of sections of $A\mathcal{F}$, coincides with $C_c^\infty(M; A\mathcal{F})$.

Proof. If (2) holds, then (a version of) the Serre-Swan theorem gives that $A\mathcal{F}$ is a vector bundle with respect to the unique smooth structure for which $\mathcal{F}$, realized as a module of sections of $A\mathcal{F}$, coincides with $C_c^\infty(M; A\mathcal{F})$. From this, it is simple to deduce (1), (3) and the "moreover" statement.

Suppose (3) holds. Note the "almost injectivity" assumption on the bundle map $A \to E$ is equivalent to injectivity of the induced map $C_c^\infty(M; A) \to C_c^\infty(M; E)$. Therefore, $\mathcal{F}$ is isomorphic to $C_c^\infty(M; A)$ as a $C^\infty(M)$-module and (2) follows.

Finally, assume (1) holds. Let $U \subseteq M$ be open and $X_1, \ldots, X_k \in C_c^\infty(U; E_U)$ be generators for $\mathcal{F}_U$. Suppose $f_1, \ldots, f_k \in C_c^\infty(U)$ have $f_1X_1 + \ldots + f_kX_k = 0$. Then, for any $x \in U$, we get $f_1(x)[X_1]_x + \ldots f_k(x)[X_k]_x$. Since the k vectors $[X_i]_x$ span the k-dimensional vector space $A_x\mathcal{F}$, they are a basis, and we obtain $f_1(x) = \ldots = f_k(x) = 0$. This shows that (2) holds. □

2.2.2 Foliations

In this section, we precisely define what will be meant by the word "foliation" and discuss related notions and constructions.

Definition 2.2.6 ([4], Definition 1.1)**.** A **foliation** $\mathcal{F}$ of a smooth manifold M is a locally finitely-generated $C^\infty(M)$-module of compactly-supported vector fields on M that is furthermore stable under taking Lie brackets.

The choice to work with compactly-supported vector fields, though not totally essential, is convenient in several ways. Flows of compactly-supported vector fields are automatically complete. Furthermore, a compactly-supported vector field defined on an open subset can always be extended by zero to the whole space. Alternative approaches include dropping the compact-support assumption altogether or working with the sheaf of locally-defined vector fields. The approach via compactly-supported vector fields is something of a compromise between a fully global and fully local approach.

Definition 2.2.7 ([4], Definition 1.7)**.** A **leaf** of a foliation $(M, \mathcal{F})$ is an orbit of $\exp(\mathcal{F})$, the group of diffeomorphisms of M generated by $\exp(X)$, $X \in \mathcal{F}$.

By work of Stefan and Sussmann ([48], [49]), the leaves of a foliation $(M, \mathcal{F})$ constitute a partition of M into immersed submanifolds. See Section 1.3 of [4] for more detailed information on the leafwise smooth structure.[2]

Example 2.2.8. If A is any Lie algebroid over a smooth manifold M, then the image of the map $C_c^\infty(M; A) \to \mathfrak{X}_c(M)$ induced by the anchor map is a foliation of M. Presently, it seems not to be known whether in fact *all* foliations can (perhaps only locally) be obtained in this way. It was noted in [5], Proposition 1.3 that one can construct foliations $(M, \mathcal{F})$, with M noncompact, such that $\sup_{x \in M} \dim(A_x \mathcal{F}) = \infty$. Obviously such a foliation cannot be induced by a single Lie algebroid on M. However, it appears to not be known whether a foliation whose fibers are bounded in dimension (which always happens if M is compact) must be induced by a Lie algebroid, nor does the local version of this question seem to be settled. The article [36] contains some partial work on this problem; see Proposition 4.33 therein.

Example 2.2.9. Specializing the above example, any Lie groupoid $G \rightrightarrows M$ induces a foliation by way of its Lie algebroid. If G is s-connected, the leaves of the foliation induced by G are exactly the orbits of G.

[2] There is an unimportant error in Remark 1.15 (1) of [4]. Let $N = \mathbb{R}$ and let $M = \mathbb{R}$ with the singular foliation singly-generated by $x\frac{d}{dx}$. Define $f : N \to M$ by $f(x) = x^3$. Then $(df_x)(T_xN) \subseteq F_{f(x)}$ is satisfied, but f is not leafwise in the sense of [4].

Definition 2.2.10. Let $(M, \mathcal{F})$ be a foliation and let $x \in M$.

- The **fiber** of $\mathcal{F}$ at x is the finite-dimensional vector space $A_x\mathcal{F} := \mathcal{F}/I_x\mathcal{F}$ (this is a particular case of Definition 2.2.1).
- The **tangent space** of $\mathcal{F}$ at x is the finite dimensional vector space $T_x\mathcal{F} := \{X(x) : X \in \mathcal{F}\}$.
- The **isotropy Lie algebra** of $\mathcal{F}$ at x is the kernel $\mathfrak{g}_x\mathcal{F}$ of the surjective linear map $A_x\mathcal{F} \to T_x\mathcal{F}$ that descends from the evaluation map $\mathcal{F} \to T_x\mathcal{F}$.

Consequent to these definitions, for each point $x \in M$, there is an exact sequence:

$$0 \to \mathfrak{g}_x\mathcal{F} \to A_x\mathcal{F} \to T_x\mathcal{F} \to 0.$$

As the notation and terminology would suggest, $\mathfrak{g}_x$ is a Lie algebra. The bracket on $\mathcal{F}$ descends to a well-defined bracket on $\mathfrak{g}_x\mathcal{F}$.

2.2.3 Regular and almost regular foliations

Definition 2.2.11. A foliation $(M, \mathcal{F})$ is called **regular** if the dimensions of its tangents spaces $T_x\mathcal{F}$, $x \in M$ are constant. A foliation which is not regular is said to be **singular**. A foliation is called **almost regular** if the dimensions of its fibers $A_x\mathcal{F}$, $x \in M$ are constant.

A foliation is regular if and only if its leaves all have the same dimension. For a regular foliation, the tangent spaces $T_x\mathcal{F}$, $x \in M$ form a subbundle of TM and $\mathcal{F}$ is equal to the compactly-supported sections of this subbundle. See [4], Example 1.3 (2) for further details.

Every regular foliation is almost regular. As explained in Proposition 2.2.5, if $\mathcal{F}$ almost regular, $A\mathcal{F} = \bigsqcup_{x \in M} A_x\mathcal{F}$ is a vector bundle. Indeed, transferring the bracket of $\mathcal{F}$ to $C_c^\infty(M; A\mathcal{F})$ makes $A\mathcal{F}$ into Lie algebroid whose anchor map is moreover injective on a dense open subset of M. In fact, one may equivalently define almost regular foliations as precisely the ones arising from a Lie algebroid with an almost injective anchor map. See also the discussion in [4], Section 3.2.

2.2.4 Pullbacks and automorphisms

Foliations can be pulled back by submersions (or, more generally, by maps satisfying an appropriate transversality assumption; see [4], Definition 1.9).

Definition 2.2.12. If $(N, \mathcal{E})$ is a foliation and $p : M \to N$ is a submersion, the **pullback foliation** $p^{-1}(\mathcal{E})$ is the foliation of M consisting of $C_c^\infty(M)$-linear combinations of vector fields on M which are p-projectable and project to elements of $\mathcal{E}$. In particular, if ι is the inclusion of an open set U into M, we write $\iota^{-1}(\mathcal{F}) = \mathcal{F}_U$ and call $\mathcal{F}_U$ the **restriction** of $\mathcal{F}$ to U.

It is clear that a foliation can be pushed forward or pulled back by a diffeomorphism, simply by pushing forward or pulling back its constituent vector fields. Indeed, this may be considered a special case of pullback by a submersion. We use the following terminology and notations.

Definition 2.2.13. Let $(M, \mathcal{F})$ be a foliation.

- An $\mathcal{F}$**-automorphism** is a diffeomorphism $\theta : M \to M$ satisfying $\theta_*(\mathcal{F}) = \mathcal{F}$. We denote the group of $\mathcal{F}$-automorphisms by $\mathrm{Aut}(\mathcal{F})$.
- A **local** $\mathcal{F}$**-automorphism** is a diffeomorphism $\theta : U \to V$, where U and V are open subsets of M, satisfying $\theta_*(\mathcal{F}_U) = \mathcal{F}_V$. The collection of all local $\mathcal{F}$-automorphisms is a pseudogroup.
- We write $\mathcal{G}(\mathcal{F})$ for the groupoid over M consisting of germs of local $\mathcal{F}$-automorphisms.

One has that $\exp(\mathcal{F})$ (Definition 2.2.7) is a normal subgroup of $\mathrm{Aut}(\mathcal{F})$ (see [4], Proposition 1.6).

2.2.5 Gluing foliations

Even though we never view *individual* foliations as sheaves, in Section 5.8 it will be useful for us to know that one can compare or construct foliations on the same manifold using a sheaf property. The proof, which we omit, is a routine verification using partitions of unity.

Proposition 2.2.14. *Let M be a smooth manifold and let $(U_i)_{i \in I}$ be an open cover of M.*

1. *If $\mathcal{E}$ and $\mathcal{F}$ are foliations of M and $\mathcal{E}_{U_i} = \mathcal{F}_{U_i}$ for all $i \in I$, then $\mathcal{E} = \mathcal{F}$.*
2. *Suppose $\mathcal{F}_i$ is a foliation of U_i for each $i \in I$. If $(\mathcal{F}_i)_{U_i \cap U_j} = (\mathcal{F}_j)_{U_i \cap U_j}$ is satisfied for all $i, j \in I$, then there exists a (unique by (1)) foliation $\mathcal{F}$ of M such that $\mathcal{F}_{U_i} = \mathcal{F}_i$ for all $i \in I$.*

2.3 The holonomy groupoid of a singular foliation

2.3.1 Bisubmersions

The following definition is the basic ingredient in Androulidakis and Skandalis's construction of the holonomy groupoid.

Definition 2.3.1 ([4], Definition 2.1)**.** An $\mathcal{F}$**-bisubmersion** of a foliation $(M, \mathcal{F})$ is a triple (W, t, s) where W is a smooth manifold and $s, t : W \to M$ are submersions satisfying $s^{-1}(\mathcal{F}) = t^{-1}(\mathcal{F}) = C_c^\infty(W; \ker(ds)) + C_c^\infty(W; \ker(dt))$.

We sometimes abuse notation and denote an $\mathcal{F}$-bisubmersion (W, t, s) simply by W. It is easy to see that, if $U \subseteq W$ is open, then $(U, t|_U, s|_U)$ is also an $\mathcal{F}$-submersion.

Definition 2.3.2. A **morphism** of $\mathcal{F}$-bisubmersions (W_1, t_1, s_1), (W_2, t_2, s_2) is a smooth map $f : W_1 \to W_2$ such that $s_1 = s_2 \circ f$ and $t_1 = t_2 \circ f$. A **local morphism** from W_1 to W_2 is a morphism from an open subset of W_1 to W_2.

It is clear that morphisms of bisubmersions can be composed and that the identity is always a morphism of bisubmersions. In [4], Corollary 2.11(c), it is shown that, if there is a local morphism of bisubmersions $W_1 \to W_2$ sending $w_1 \mapsto w_2$, then there is also a morphism of bisubmersions $W_2 \to W_1$ sending $w_2 \mapsto w_1$. The following definition is therefore justified.

Definition 2.3.3. Let $(M, \mathcal{F})$ be a foliation, and $(W_i)_{i \in I}$ be a collection of $\mathcal{F}$-bisubmerions. Then, we denote by $\sim$ the equivalence relation on $\bigsqcup_{i \in I} W_i$ given by $W_i \ni w_i \sim w_j \in W_j$ if and only if there exists a local morphism from W_i to W_j sending w_i to w_j. We denote the quotient map $\bigsqcup_{i \in I} W_i \to \left(\bigsqcup_{i \in I} W_i\right) / \sim$ by $Q = (Q_i)_{i \in I}$.

Definition 2.3.4. Let $(M, \mathcal{F})$ be a foliation and let $\mathcal{U} = (U_i)_{i \in I}$ and $\mathcal{V} = (V_j)_{j \in J}$ be collections of $\mathcal{F}$-bisubmersions. Put $U = \bigsqcup_{i \in I} U_i$ and $V = \bigsqcup_{j \in J} V_j$. We say that $\mathcal{U}$ is **adapted** to $\mathcal{V}$ if every $u \in U$ is $\sim$ to some $v \in V$. We say that $\mathcal{U}$ and $\mathcal{V}$ are **equivalent** if they are adapted to each other.

The following simple proposition is helpful in clarifying certain issues relating to forming the quotient by $\sim$.

Proposition 2.3.5. *The quotient map Q of the above definition is an open map.*

Proof. Let $W := \bigsqcup_{i \in I} W_i$. We need to prove that that the $\sim$-saturation of any open set $U \subseteq W$ is open. It suffices to consider the case where $U \subseteq W_i$ for some $i \in I$. To this end, suppose $w \in U$, $w' \in W_j$ for some $j \in I$ and $w \sim w'$. Therefore, there exists a local morphism f from W_j to W_i with $f(w') = w$. Then, $f^{-1}(U)$ is a neighbourhood of w' with the property that each point in $f^{-1}(U)$ is $\sim$ (by way of f) to a point in U. □

Corollary 2.3.6. *Let $\mathcal{U} = (U_i)_{i \in I}$ and $\mathcal{V} = (V_j)_{j \in J}$ be collections of bisubmersions of $(M, \mathcal{F})$. Put $U = \bigsqcup_{i \in I} U_i$ and $V = \bigsqcup_{j \in J} V_j$. If $\mathcal{U}$ is adapted to $\mathcal{V}$, then the map $U/\sim \to V/\sim$ sending $[u] \mapsto [v]$ whenever $u \sim v$ is an open embedding. If $\mathcal{U}$ and $\mathcal{V}$ are equivalent, this map $U/\sim \to V/\sim$ is a homeomorphism.*

Proof. Note that the restriction of an open mapping to an open set is an open mapping. In particular, this can be applied to U or V sitting in $U \sqcup V$. It follows that both $U/\sim$ and $V/\sim$ sit as open subsets in $(U \sqcup V)/\sim$. If $\mathcal{U}$ is adapted to $\mathcal{V}$, then V meets every equivalence class in $U \sqcup V$, so that $V/\sim$ is homeomorphic to $(U \sqcup V)/\sim$. If $\mathcal{U}$ and $\mathcal{V}$ are equivalent, then $U/\sim$ is homeomorphic to $(U \sqcup V)/\sim$ as well. □

Remark 2.3.7. Note that, in general, the restriction of a quotient map to an open set which meets every equivalence class is not a quotient map. For example, the map $[0, 1] \to S^1 : t \mapsto e^{2\pi it}$ is a quotient map, but its restriction to $[0, 1)$ is not.

2.3.2 Construction of the holonomy groupoid

There are natural notions of inverse and composition for bisubmersions. See [4], Proposition 2.4.

Definition 2.3.8. Let $(M, \mathcal{F})$ be a foliation.

- The **composition** of two $\mathcal{F}$-bisubmersions W_1, W_2 is the $\mathcal{F}$-bisubmersion $W_2 \circ W_1$ whose underlying manifold is the fiber product $W_2 \ {}_{s_2}\!\times_{t_1} W_1$ and whose source and target maps are given by $s(w_2, w_1) = s_1(w_1)$ and $t(w_2, w_1) = t_2(w_2)$, in an obvious notation.
- The **inverse** of an $\mathcal{F}$-bisubmersion W is the $\mathcal{F}$-bisubmersion W^{-1} obtained by keeping the same underlying manifold, but interchanging the source and target maps.

Definition 2.3.9 ([4], Definition 3.1)**.** Let $(M, \mathcal{F})$ be a foliation. A collection $\mathcal{W} = (W_i)_{i \in I}$ of $\mathcal{F}$-bisubmersions is called an **holonomy atlas** provided that:

(i) $\bigcup_{i \in I} s_i(W_i) = M$.

(ii) If $W \in \mathcal{W}$ and $w \in W$, then there exists $W' \in \mathcal{W}$ and $w' \in W'$ such that $w \in W^{-1}$ is $\sim$ to $w' \in W'$. .

(iii) If $W_1, W_2 \in \mathcal{W}$, $w_1 \in W_1$, $w_2 \in W_2$, and $s_1(w_1) = t_2(w_2)$, then exists $W' \in \mathcal{W}$ and $w' \in W'$ such that $(w_1 \circ w_2) \in W_1 \circ W_2$ is $\sim$ to $w' \in W'$.

Items (ii) and (iii) amount to saying $\mathcal{W}$ is closed under inverse and composition, if we work up to $\sim$.

Theorem 2.3.10 ([4], Proposition 3.2)**.** *Suppose $\mathcal{W} = (W_i)_{i\in I}$ is a holonomy atlas for a foliation $(M, \mathcal{F})$. Let $G(\mathcal{W}) := \left(\bigsqcup_{i\in I} W_i\right) / \sim$. Then there is a groupoid structure on $G(\mathcal{W})$ such that*

$$Q_{W_2}(w_2)Q_{W_1}(w_1) = Q_{W_2 \circ W_1}(w_2, w_1)$$

whenever W_1 and W_2 are bisubmersions adapted to $\mathcal{W}$ and $w_1 \in W_1$, $w_2 \in W_2$ are such that $s_2(w_2) = s_1(w_1)$. □

Every foliation $(M, \mathcal{F})$ admits a path holonomy atlas $\mathcal{W}_{\text{path}}$ and a full holonomy atlas $\mathcal{W}_{\text{full}}$ such that, if $\mathcal{W}$ is any holonomy atlas for $(M, \mathcal{F})$, then $\mathcal{W}_{\text{path}}$ is adapted to $\mathcal{W}$ and $\mathcal{W}$ is adapted to $\mathcal{W}_{\text{full}}$.

Definition 2.3.11 ([4], Definition 3.5, Example 3.4 (1))**.** Let $(M, \mathcal{F})$ be a foliation with path holonomy atlas $\mathcal{W}_{\text{path}}$) and full holonomy atlas $\mathcal{W}_{\text{full}}$. Then, the **path holonomy groupoid**, or simply the **holonomy groupoid**, of $\mathcal{F}$ is $G(\mathcal{F}) := G(\mathcal{W}_{\text{path}}$. Similarly, the **full holonomy groupoid** of $\mathcal{F}$ is $G_{\text{full}}(\mathcal{F}) := G(\mathcal{W}_{\text{path}}$.

By definition, given any holonomy atlas $\mathcal{W}$ for $\mathcal{F}$, there are canonical open inclusions $G(\mathcal{F}) \subseteq G(\mathcal{W}) \subseteq G_{\text{full}}(\mathcal{F})$.

2.3.3 Bisections

Definition 2.3.12. A **bisection** of a bisubmersion (W, t, s) of a foliation $(M, \mathcal{F})$ is a locally closed submanifold $N \subseteq W$ such that the restrictions of s and t to N are diffeomorphisms onto open subsets of M.

Proposition 2.3.13. *Suppose N is a bisection of a bisubmersion (W, t, s) of a foliation $(M, \mathcal{F})$. Then, $t|_N \circ (s|_N)^{-1}$ is a local $\mathcal{F}$-automorphism*

Proof. See Proposition 2.9 of [4]. □

Definition 2.3.14. Suppose that (W, t, s) is a bisubmersion of a foliation $(M, \mathcal{F})$. We say that a local $\mathcal{F}$-automorphism θ is **carried** by (W, t, s) at a point $w \in W$ if there is a bisection N of (W, t, s) with $w \in N$ such that θ has the same germ as $t|_N \circ (s|_N)^{-1}$ at $s(w)$.

2.3.4 Holonomy transformation picture of the full holonomy groupoid

In [6], it is shown that the holonomy groupoid of [4] can be realized as a groupoid of (equivalence classes of) holonomy transformations of a family of transversal slices. It is pointed out in [6], Remark 2.10(c) that the use of slices is essential because of an issue which can arise from nonorientable leaves. In this section, we lay out a rather cheap way to realize the groupoid of [4] without introducing slices which, though not very different from the original description in terms of bisubmersions, still has some of the flavour of a description by holonomy transformations.

Proposition 2.3.15. *Let W_1, W_2 be bisubmersions of $(M, \mathcal{F})$ and fix $w_i \in W_i$. If $w_1 \sim w_2$, then the set of local automorphisms carried by W_1 at w_1 is exactly the equal to the set of local automorphisms carried by W_2 at w_2. Conversely, if there exists a local automorphism carried at both w_1 and w_2, then $w_1 \sim w_2$.*

Proof. Suppose $w_1 \sim w_2$ and let f be a local morphism with $f(w_1) = w_2$. Let $N_1 \subseteq W_1$ be a bisection with $w_1 \in N_1$. Since N_1 is a section of s_1, we have $T_wW_1 = T_wN_1 \oplus \ker(ds_1)_w$ for all $w \in S_1$. From $s_1 = s_2 \circ f$, one may deduce that $T_wS_1 \cap \ker(df)_w = \{0\}$, and that $df_w(T_wS) \cap \ker(ds_2) = \{0\}$. Thus, $f|_{N_1}$ is an immersion which is transverse to s_2. In the same way, $f|_{N_1}$ is transverse to r_2. It follows that the image of a small neighbourhood of w_1 in N_1 is a bisection $N_2 \subseteq W_2$ with $w_2 \in N_2$. Clearly the local diffeomorphisms induced by N_1 and N_2 have the same germ at $s_1(w_1) = s_2(w_2)$, and we obtain that every local automorphism carried at w_1 is also carried at w_2. Interchanging the roles of w_1 and w_2, we get that same local automorphisms are carried at the two points. The converse statement is exactly [4], Corollary 2.11(b). □

By the above proposition, the following defines an equivalence relation.

Definition 2.3.16. Let $(M, \mathcal{F})$ be a foliation. Let θ_1 and θ_2 be germs at $x \in M$ of local $\mathcal{F}$-automorphisms with $\theta_1(x) = \theta_2(x)$. We write $\theta_1 \approx \theta_2$ if there exists and $\mathcal{F}$-bisubmersions W and a point $w \in W$ such that both θ_1 and θ_2 are carried by W at w.

Definition 2.3.17. Let $(M, \mathcal{F})$ be a foliation. A local $\mathcal{F}$-automorphism θ_0 is **null** at $x \in M$ if there exists an $\mathcal{F}$-bisubmersion W and a point $w \in W$ with $s(w) = x$ such that both θ_0 and id_M are carried by W at w. We denote the group of germs at x of local $\mathcal{F}$-automorphisms which are null at x by $\mathcal{N}(\mathcal{F})_x$ and put $\mathcal{N}(\mathcal{F}) := \bigsqcup_{x \in M} \mathcal{N}(\mathcal{F})_x$.

Theorem 2.3.18. *Let $(M, \mathcal{F})$ be a foliation. Then $\mathcal{N}(\mathcal{F})$ is a normal subgroupoid of $\mathcal{G}(\mathcal{F})$ and there is an abstract groupoid isomorphism*

$$G_{\text{full}}(\mathcal{F}) \to \mathcal{G}(\mathcal{F})/\mathcal{N}(\mathcal{F})$$

such that, if $W = (W, t, s)$ is an $\mathcal{F}$-bisubmersion and $w \in W$, then $Q_W(w) \in G_{\text{full}}(\mathcal{F})$ is mapped to the germ at $s(w)$ of any local $\mathcal{F}$-automorphism that is carried by W at w.

Proof. That this map is a bijection follows from Proposition 2.3.15. Multiplication and inversion are preserved by [4], Proposition 2.8. □

2.3.5 Smoothness of the holonomy groupoid of an almost regular foliations

The holonomy groupoid constructed in [4] can be quite poorly-behaved for general foliations. Almost regular foliations are precisely the foliations whose holonomy groupoids are Lie groupoids. This case was previously treated by Debord in [19], with a different approach than that of [4]. The following result follows from [4], Section 3.2 and can also be obtained in a more direct manner by adapting the arguments in [21].

Proposition 2.3.19. *Let M be an n-dimensional smooth manifold, and $\mathcal{F}$ an almost regular singular foliation of M with constant fiber dimension k. Then there is a unique smooth structure on $G_{\text{full}}(\mathcal{F})$ such that, if (W, t, s) is any $\mathcal{F}$-bisubmersion, the map $Q_W : W \to G_{\text{full}}(\mathcal{F})$ is smooth. The groupoid operations are smooth with respect to this smooth structure.*

When $\mathcal{F}$ is almost regular, $\mathcal{A}(\mathcal{F})$ is isomorphic to $C_c^\infty(G(\mathcal{F}))$, the smooth convolution algebra of the groupoid, and $C^*(\mathcal{F})$ is isomorphic to $C^*(G(\mathcal{F}))$, the C*-algebra of the groupoid. Here we are implicitly fixing a smooth Haar system on $G(\mathcal{F})$ in order to make sense of convolution and bypass any discussion of densities.

Chapter 3

The Smooth Algebra of a One-Dimensional Singular Foliation

Except for minor changes, this chapter reproduces the content of [28]. We consider the foliations $\mathcal{F}_{\mathbb{R}}^k$ of $\mathbb{R}$, k a positive integer, given by all compactly-supported, smooth vector fields which vanish to order k (or more) at 0. We obtain a complete classification of the smooth convolution algebras and C*-algebras of these foliations. Our main findings may be summarized as follows:

Theorem (Theorems 3.4.10, 3.3.5, 3.3.6)**.**

1. *The smooth convolution algebras of the foliations $\mathcal{F}_{\mathbb{R}}^k$ are pairwise nonisomorphic.*
2. *The C*-algebras of the foliations $\mathcal{F}_{\mathbb{R}}^k$ fall into two isomorphism classes determined by the parity of k.*
3. *The C*-algebra of each $\mathcal{F}_{\mathbb{R}}^k$ is faithfully represented in a natural way on $L^2(\mathbb{R}_-) \oplus L^2(\mathbb{R}_+)$ and the images of these representations are pairwise distinct.*

Let us now describe the organization of this chapter. In Section 3.1, we summarize relevant aspects of Androulidakis-Skandalis's work and justify our use of the models $G(\mathcal{F}_{\mathbb{R}}^k) \cong \mathbb{R} \rtimes_\phi \mathbb{R}$, $C_c^\infty(\mathbb{R} \rtimes_\phi \mathbb{R})$ and $C_0(\mathbb{R}) \rtimes_\phi \mathbb{R}$ for the groupoids, smooth convolution algebras and C*-algebras of the foliations $\mathcal{F}_{\mathbb{R}}^k$. In Section 3.2 we review the known fact that the so-called *Wiener-Hopf extension*

$$0 \to \mathbb{K} \to C_0(\mathbb{R} \cup \{+\infty\}) \rtimes_\tau \mathbb{R} \to C^*(\mathbb{R}) \to 0$$

arising from the translation action on a one-sided extension of the line is isomorphic to the (nonunital) Toeplitz extension. One standard approach is to use Laguerre functions to produce an operator of index 1 and appeal to Brown-Douglas-Fillmore theory (see [30] and [44]). We will instead use a suitably unitarized form of the Cayley transform to give an explicit conjugacy relating the Wiener-Hopf extension to (an uncompressed form of) the Toeplitz extension. This approach appears to harken back to [22]. In Section 3.3, we use the results of Section 3.2 to analyse the C*-algebras $C^*(\mathcal{F}^k_{\mathbb{R}})$ and obtain the above mentioned results on their isomorphism types and representations. Finally, in Section 3.4, we prove pairwise nonisomorphism of the smooth convolution algebras $\mathcal{A}(\mathcal{F}^k_{\mathbb{R}})$ by examining their quotients by the sequence of ideals $x^p \cdot \mathcal{A}(\mathcal{F}^k_{\mathbb{R}})$, as discussed above. We furthermore describe the quotient by the ideal $x^\infty \cdot \mathcal{A}(\mathcal{F}^k_{\mathbb{R}}) := \bigcap_{p\geq 1} x^p \cdot \mathcal{A}(\mathcal{F}^k_{\mathbb{R}})$.

3.1 Preliminaries

The main objects of study in this chapter are the following foliations:

Definition 3.1.1 ([4], Example 1.3 (3))**.** For each positive integer k, we denote by $\mathcal{F}^k_{\mathbb{R}}$ the almost regular foliation of $\mathbb{R}$ singly-generated by $x^k \frac{d}{dx}$.

For every k, the leaves of $\mathcal{F}^k_{\mathbb{R}}$ are $\mathbb{R}_-$, $\{0\}$ and $\mathbb{R}_+$.

In this section we describe the models we will be using for the holonomy groupoids $G(\mathcal{F}^k_{\mathbb{R}})$, smooth algebras $\mathcal{A}(\mathcal{F}^k_{\mathbb{R}})$ and C*-algebras $C^*(\mathcal{F}^k_{\mathbb{R}})$ of the foliations $\mathcal{F}^k_{\mathbb{R}}$, and briefly justify their usage.

Lemma 3.1.2. *Let H be a connected Lie group acting on a smooth manifold M and let $\mathcal{F}$ be the singular foliation of M determined by this action. If the set of $x \in M$ with trivial isotropy is dense in M, then the canonical map (see Example 3.4 in [4]) from the transformation groupoid $M \rtimes H$ to the holonomy groupoid $G(\mathcal{F})$ is an isomorphism.*

Proof. According to Proposition 3.9 in [4], we just need to check that $M \rtimes H$ is an s-connected *quasigraphoid*, as defined at the reference. Since H is connected, the transformation groupoid is s-connected. Suppose that $x \in M$ and f is a smooth map from an open neighbourhood U of x to H which satisfies $f(y)y = y$ for all $y \in U$. Then, $f(y) = 1$ for a dense set of y in U by hypothesis, and so f is identically equal to 1. This shows that the only local bisections which induce the identity mapping are the trivial ones, i.e. $M \rtimes H$ is a quasigraphoid. □

Corollary 3.1.3. *Let X be a complete vector field on a smooth manifold M. Let $\mathcal{F}$ be the singular foliation singly-generated by X. If X has finitely many critical points and periodic orbits, then the transformation groupoid $M \rtimes_X \mathbb{R}$ determined by the flow of X is canonically isomorphic to the holonomy groupoid $G(\mathcal{F})$.*

By the corollary above one has a canonical isomorphism

$$G(\mathcal{F}^k_{\mathbb{R}}) \cong \mathbb{R} \rtimes_\phi \mathbb{R} \tag{3.1}$$

where ϕ is the flow of any complete vector field generating $\mathcal{F}^k_{\mathbb{R}}$. The most obvious generator for the singular foliation $\mathcal{F}^k_{\mathbb{R}}$ is $x^k \frac{d}{dx}$, but note that this vector field is not complete when $k \geq 2$. One can, however, rescale it in order to get a complete vector field X_k generating the same foliation. For example, one could use $X_k = (1+x^2)^{-\frac{k-1}{2}} x^k \frac{d}{dx}$ which resembles $x^k \frac{d}{dx}$ near the origin, but has sublinear growth.

Remark 3.1.4. In fact, it still makes perfect sense to talk about the transformation groupoid $M \rtimes_X \mathbb{R}$ associated to smooth vector field even when X is not complete. One just takes the underlying manifold of the groupoid to instead be the domain of the flow, an open subset of $\mathbb{R} \times M$, and defines the groupoid operations exactly as in the complete case. We could therefore get away with using the vector fields $x^k \frac{d}{dx}$ themselves rather than rescalings thereof.

The transformation groupoid $\mathbb{R} \rtimes_\phi \mathbb{R}$ has a natural right Haar system given by copying the usual Lebesgue measure on each source fiber. The convolution product determined on $C^\infty_c(\mathbb{R} \rtimes_\phi \mathbb{R})$ by this Haar system is as follows:

$$(f * g)(x,t) = \int_{\mathbb{R}} f(\phi_s(x), t-s) g(x,s)\ ds. \tag{3.2}$$

Note that changing the smooth Haar system results in a canonically isomorphic product. We also have an adjoint operation on $C^\infty_c(\mathbb{R} \rtimes_\phi \mathbb{R})$ given by

$$f^*(x,t) = \overline{f(\phi_t(x), -t)}.$$

The $*$-algebra structure on $C^\infty_c(\mathbb{R} \rtimes_\phi \mathbb{R})$ defined by these operations is canonically isomorphic to the smooth convolution algebra $\mathcal{A}(\mathcal{F}^k_{\mathbb{R}})$ defined in Section 4.3 of [4] (one needs to translate from the language of densities to the language of functions).

One also has that the C*-algebra $C^*(\mathcal{F}^k_{\mathbb{R}})$ is canonically isomorphic to the crossed product C*-algebra $C_0(\mathbb{R}) \rtimes_\phi \mathbb{R}$, where ϕ denotes the flow of any complete vector field generating $\mathcal{F}^k_{\mathbb{R}}$ as well as the corresponding action of $\mathbb{R}$ on $C_0(\mathbb{R})$ defined by $(\phi_t f)(x) = f(\phi_{-t}(x))$. Let us briefly explain why this is so. Firstly, it is well-known that there is a canonical isomorphism $\Psi : C^*(\mathbb{R} \rtimes_\phi \mathbb{R}) \to C_0(\mathbb{R}) \rtimes_\phi \mathbb{R}$ from the usual groupoid C*-algebra, in the sense of [43], to the crossed product C*-algebra. The groupoid C*-algebra is the completion of $C^\infty_c(\mathbb{R} \rtimes_\phi \mathbb{R})$ with respect to the largest C*-norm dominated by the L^1-norm

$$\|f\|_{1,\text{Groupoid}} = \sup_{x\in\mathbb{R}} \left\{ \int_{\mathbb{R}} |f(x,t)|\ dt, \int_{\mathbb{R}} |f^*(x,t)|\ dt \right\}. \tag{3.3}$$

The isomorphism Ψ is determined by its restriction to $C_c^\infty(\mathbb{R} \rtimes_\phi \mathbb{R})$ which has image contained in the dense subalgebra $C_c(\mathbb{R}, C_0(\mathbb{R})) \subseteq C_0(\mathbb{R}) \rtimes_\phi \mathbb{R}$ and is given by

$$((\Psi f)(t))(x) = f(\phi_{-t}(x), t).$$

In Section 4.4 of [4], the C*-algebra $C^*(\mathcal{F}_\mathbb{R}^k)$ is defined as the completion of $\mathcal{A}(\mathcal{F}_\mathbb{R}^k)$ with respect to the largest C*-norm dominated by an L^1-norm $\|\cdot\|_{1,AS}$ defined by a choice of Riemannian metric[1] on the base manifold ([4], Definition 4.8). Choosing the standard Riemann metric on $\mathbb{R}$, one may check that Androulidakis-Skandalis's L^1-norm becomes the norm on $C_c^\infty(\mathbb{R} \rtimes_\phi \mathbb{R})$ defined by

$$\|f\|_{1,\mathrm{AS}} = \sup_{x\in\mathbb{R}} \left\{ \int_\mathbb{R} |f(x,t)|\beta(x,t)\ dt, \int_\mathbb{R} |f^*(x,t)|\beta(x,t)\ dt \right\} \tag{3.4}$$

where

$$\beta(x,t) = \begin{cases} (\phi_t'(x))^{1/2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0. \end{cases}$$

One may wonder whether these (different) L^1-norms (3.3) and (3.4) really determine the same C*-completions. This is so, and follows from the fact that the map $\beta : \mathbb{R} \rtimes_\phi \mathbb{R} \to \mathbb{R}_+$ appearing in (3.4) is a 1-cocycle, i.e. satisfies $\beta(\gamma_1\gamma_2) = \beta(\gamma_1)\beta(\gamma_2)$, where $\gamma_1, \gamma_2 \in \mathbb{R} \rtimes_\phi \mathbb{R}$. One has that the Hilbert space representations dominated by $\|\cdot\|_{1,\mathrm{Groupoid}}$ are the same as the ones obtained by integrating representations of the underlying groupoid. This is shown in [43], 1.7 Proposition. Following the same argument, one may check that the same remains true when $\|\cdot\|_{1,\mathrm{Groupoid}}$ is adjusted by a 1-cocycle; the 1-cocycle drops out of the main estimate given on pp. 50 of [43].

3.2 The Wiener-Hopf extension

In this section we explain how a suitably unitarized form of the Cayley transform can be used to relate the Wiener-Hopf extension (defined below) to the more familiar Toeplitz extension. The precise statements are given in Theorem 3.2.5 and Corollary 3.2.6.

Form a one-sided extension $\mathbb{R} \cup \{+\infty\}$ of the real line by adjoining a positive infinity, but not a negative infinity. Let M denote the multiplication representation of $C_0(\mathbb{R}\cup\{+\infty\})$ on $L^2(\mathbb{R})$. Let λ denote the regular representation of $\mathbb{R}$ on $L^2(\mathbb{R})$ as well as its integrated form, a representation of $C^*(\mathbb{R})$. On $L^1(\mathbb{R}) \subseteq C^*(\mathbb{R})$, this is the usual convolution representation $\lambda(f)\xi = f * \xi$.

[1] The choice of metric does not affect the completion.

Definition 3.2.1. The **Wiener-Hopf** algebra is the C*-algebra $\mathcal{T}_{\mathbb{R}}$ on $L^2(\mathbb{R})$ generated by products $M(f)\lambda(g)$ where $f \in C_0(\mathbb{R} \cup \{+\infty\})$ and $g \in C^*(\mathbb{R})$. There is a $*$-homomorphism $\sigma_{\mathbb{R}} : \mathcal{T}_{\mathbb{R}} \to C^*(\mathbb{R})$ called the **symbol map**, determined by $\sigma_{\mathbb{R}}(M(f)\lambda(g)) = f(+\infty)g$ whose kernel is $\mathbb{K}(L^2(\mathbb{R}))$, the ideal of compact operators. The resulting extension

$$0 \longrightarrow \mathbb{K}(L^2(\mathbb{R})) \longrightarrow \mathcal{T}_{\mathbb{R}} \xrightarrow{\sigma_{\mathbb{R}}} C^*(\mathbb{R}) \longrightarrow 0$$

will be referred to as the **Wiener-Hopf extension**.

That the symbol map $\sigma_{\mathbb{R}}$ is defined at all will be a consequence of the discussion to follow, in which the Wiener-Hopf algebra will be related to the Toeplitz algebra. For the moment, let us assume $\sigma_{\mathbb{R}}$ exists and obtain a crossed product description of the Wiener-Hopf algebra.

Let $\mathbb{R}$ act on the extended line $\mathbb{R} \cup \{+\infty\}$ by translation, fixing the point $+\infty$. Let τ be the corresponding action of $\mathbb{R}$ on $C_0(\mathbb{R} \cup \{+\infty\})$ defined by

$$(\tau_t f)(x) = f(x - t).$$

Evaluation at $+\infty$ yields an $\mathbb{R}$-equivariant exact sequence of commutative C*-algebras:

$$0 \longrightarrow C_0(\mathbb{R}) \longrightarrow C_0(\mathbb{R} \cup \{+\infty\}) \longrightarrow \mathbb{C} \longrightarrow 0.$$

Taking the crossed product of the above sequence by $\mathbb{R}$, we obtain another exact sequence of C*-algebras.

$$0 \longrightarrow C_0(\mathbb{R}) \rtimes_\tau \mathbb{R} \longrightarrow C_0(\mathbb{R} \cup \{+\infty\}) \rtimes_\tau \mathbb{R} \longrightarrow C^*(\mathbb{R}) \longrightarrow 0.$$

This sequence is of some importance in noncommutative geometry. For example, in [44], it is used to realize the Connes-Thom isomorphism of [15] as a boundary map in K-theory.

Recall that a (maximal) crossed product C*-algebra $A \rtimes_\alpha G$ contains canonical copies of A and $C^*(G)$ in its multiplier algebra and that $A \rtimes_\alpha G$ is generated by elementary products $a \times b$, where $a \in A$, $b \in C^*(G)$. Each nondegenerate representation π of $A \rtimes_\alpha G$ arises uniquely from a *covariant pair* (π_A, π_G) where π_A is a representation of A, π_G is a unitary representation of G and $\pi_G(t)\pi_A(a)\pi_G(t)^{-1} = \pi_A(\alpha_t(a))$ holds for all $a \in A$, $t \in G$. The corresponding representation π of $A \rtimes_\alpha G$ is determined by $\pi(a \times b) = \pi_A(a)\pi_G(b)$ for all $a \in A$, $b \in C^*(G)$. Here we abuse notation, denoting the integrated form of π_G, a representation of $C^*(G)$, by the same symbol as its inducing unitary representation.

For more information on crossed products, one may refer to the extensive survey [53].

There is a natural representation $\pi : C_0(\mathbb{R} \cup \{+\infty\}) \rtimes_\tau \mathbb{R} \to \mathbb{B}(L^2(\mathbb{R}))$ coming from the covariant pair (M, λ) where, as above, M denotes the multiplication representation and λ denotes the regular representation of $\mathbb{R}$. The representation π is thus determined on elementary products by

$$\pi(f \times g) = M(f)\lambda(g) \qquad f \in C_0(\mathbb{R} \cup \{+\infty\}), g \in C^*(\mathbb{R}). \tag{3.5}$$

By construction, the image of the representation π is the Wiener-Hopf algebra $\mathcal{T}_\mathbb{R}$. It is well-known that the restriction of π to the ideal $C_0(\mathbb{R}) \rtimes_\tau \mathbb{R}$ is a faithful representation onto the C*-algebra $\mathbb{K}(L^2(\mathbb{R}))$ of compact operators. This is a C*-algebraic formulation of the *Stone-von Neumann theorem*, describing the representation theory of the canonical commutation relations. Actually, one has more generally that $C_0(G) \rtimes G \cong \mathbb{K}(L^2(G))$ for any locally compact group G. One can then see that π is also faithful on $C_0(\mathbb{R} \cup \{+\infty\}) \rtimes_\tau \mathbb{R}$ by consideration of the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_0(\mathbb{R}) \rtimes_\tau \mathbb{R} & \longrightarrow & C_0(\mathbb{R} \cup \{+\infty\}) \rtimes_\tau \mathbb{R} & \longrightarrow & C^*(\mathbb{R}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & & \| & & \\
0 & \longrightarrow & \mathbb{K}(L^2(\mathbb{R})) & \longrightarrow & \mathcal{T}_\mathbb{R} & \xrightarrow{\sigma_\mathbb{R}} & C^*(\mathbb{R}) & \longrightarrow & 0.
\end{array}$$

The vertical maps on the right and the left are isomorphisms, whence the one in the middle is as well.

We shall review the known fact that the Wiener-Hopf extension is isomorphic to the (nonunital form of the) Toeplitz extension. In [44] as well as [30] (Lemma 6), this is shown by using Laguerre functions to produce an operator of index 1 and appealing to Brown-Douglas-Fillmore theory. However, we feel that a clearer understanding of the Wiener-Hopf extension is reached by using the Cayley and Fourier transform in concert to explicitly relate it to the Toeplitz extension. This observation is not new, but deserves to be better known. The approach by way of the Cayley transform seems to originate from [22] (see pp. 82-83 therein). See also the historical remark on page 2 of [24] as well as Proposition 3.7.1 of [51].

Let $H^2(S^1)$ denote the *Hardy space* of the circle, the closed subspace of $L^2(S^1)$ spanned by the basis vectors $\frac{1}{\sqrt{2\pi}} z^n$ where $n \geq 0$. One may also characterize the elements of $H^2(S^1)$ as the boundary values of a corresponding space of holomorphic functions on the open disk. Recall that the multiplication representation $M : C^1(S^1) \to \mathbb{B}(L^2(S^1))$ commutes modulo compact operators with $P_{H^2(S^1)}$, the projection onto the Hardy space. The *Toeplitz operator* with symbol $f \in C(S^1)$ is defined as $T_f := P_{H^2(S^1)} M(f) P_{H^2(S^1)}$, the compression of $M(f)$ to the Hardy space. The *Toeplitz algebra* is the C*-algebra $\mathcal{T}$ on $H^2(S^1)$ generated by the Toeplitz operators. In fact, $\mathcal{T}$ is singly-generated by the unilateral shift operator T_z. The Toeplitz algebra sits in a canonical extension

$$0 \longrightarrow \mathbb{K}(H^2(S^1)) \longrightarrow \mathcal{T} \xrightarrow{\sigma} C(S^1) \longrightarrow 0$$

called the *Toeplitz extension*. The homomorphism σ is called the *symbol map* and satisfies $\sigma(T_f) = f$ for all $f \in C(S^1)$. Indeed, the assignment $f \mapsto T_f$ is a completely positive splitting for σ. One may also consider the *nonunital Toeplitz algebra* $\mathcal{T}_0$ defined as the preimage by σ of the codimension one ideal $C_0(S^1) \subseteq C(S^1)$ consisting of functions vanishing at $1 \in S^1$. It is singly-generated by T_{1-z} and fits into the *nonunital Toeplitz extension*:

$$0 \longrightarrow \mathbb{K}(H^2(S^1)) \longrightarrow \mathcal{T}_0 \xrightarrow{\sigma} C_0(S^1) \longrightarrow 0.$$

It is sometimes desirable to not compress to the Hardy space and work instead with

$$\overline{\mathcal{T}}_0 := \mathcal{T}_0 + \mathbb{K}(L^2(S^1)). \tag{3.6}$$

There is a unique extension $\overline{\sigma}$ of the symbol map to $\overline{\mathcal{T}}$ which has $\ker(\overline{\sigma}) = \mathbb{K}(L^2(S^1))$. From these definitions, we have the following commutative diagram in which the vertical maps are inclusions (or, in the case of the rightmost map, an equality).

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{K}(H^2(S^1)) & \longrightarrow & \mathcal{T}_0 & \xrightarrow{\sigma} & C_0(S^1) & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\| & & \\
0 & \longrightarrow & \mathbb{K}(L^2(S^1)) & \longrightarrow & \overline{\mathcal{T}_0} & \xrightarrow{\overline{\sigma}} & C_0(S^1) & \longrightarrow & 0
\end{array}$$

A convenient feature of this uncompressed form of the (nonunital) Toeplitz extension is that all three terms in the exact sequence on the bottom are represented on the same Hilbert space $L^2(S^1)$. Brown-Douglas-Fillmore theory gives, however, that the extensions on the top and bottom of the above diagram are in fact the same.

Lemma 3.2.2. *There exists an isometric isomorphism $V : H^2(S^1) \to L^2(S^1)$ such that Ad_V carries $\mathcal{T}_0$ onto $\overline{\mathcal{T}}_0$ and the following diagram is commutative:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{K}(H^2(S^1)) & \longrightarrow & \mathcal{T}_0 & \xrightarrow{\sigma} & C_0(S^1) & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle \mathrm{Ad}_V} & & \big\downarrow{\scriptstyle \mathrm{Ad}_V} & & \big\| & & \\
0 & \longrightarrow & \mathbb{K}(L^2(S^1)) & \longrightarrow & \overline{\mathcal{T}_0} & \xrightarrow{\overline{\sigma}} & C_0(S^1) & \longrightarrow & 0.
\end{array}$$

Proof. Let $S = T_z$, the unilateral shift operator on $H^2(S^1)$, and let $\overline{S}$ be the direct sum of S with the identity operator on the orthogonal complement of $H^2(S^1)$ in $L^2(S^1)$. Since S and $\overline{S}$ are both isometries of index -1, the classification of essentially normal operators implies the existence of an isometric isomorphism $V : H^2(S^1) \to L^2(S^1)$ such that VSV^{-1} equals $\overline{S}$ modulo compact operators[2]. Thus, $VT_{1-z}V^{-1}$ equals T_{1-z}, modulo compact operators. Recalling that $\mathcal{T}_0$ is singly-generated by T_{1-z} and contains $\mathbb{K}(H^2(S^1))$, the result follows. □

Recall that one similarly has a *Hardy space* $H^2(\mathbb{R}) \subseteq L^2(\mathbb{R})$ for the real line. As in the case of the circle, $H^2(\mathbb{R})$ admits two descriptions: a harmonic analysis description as the space of $\xi \in L^2(\mathbb{R})$ whose Fourier transform has support contained in $\mathbb{R}_+$, and a complex function theory description as the space of boundary values of a corresponding space of holomorphic functions on the upper half-plane. Recall the *Cayley transform*

$$w : \mathbb{R} \to S^1 \setminus \{1\} \qquad\qquad w(t) = \frac{t-i}{t+i}. \tag{3.7}$$

Like any other diffeomorphism $\mathbb{R} \to S^1 \setminus \{1\}$, the Cayley transform determines an isometric isomorphism $L^2(S^1) \to L^2(\mathbb{R})$ given by $\xi \mapsto |w'|^{1/2}(\xi \circ w)$. Recall however that, viewed as a Mobïus transformation, w carries the upper half plane to the unit disk. Furthermore, its derivative $w'(t) = \frac{2i}{(t+i)^2}$ is the square of a function holomorphic in the upper half-plane. One is therefore encouraged to adjust the phase of the naïve identification $L^2(S^1) \to L^2(\mathbb{R})$ given above and instead use the isometric isomorphism W defined by the following meromorphic formula:

$$W : L^2(S^1) \to L^2(\mathbb{R}) \qquad\qquad (W\xi)(t) = \frac{\sqrt{2}}{t+i}\xi(w(t)). \tag{3.8}$$

Although this adjustment to the phase might seem an inconsequential matter, it in fact leads to W having the highly useful property of mapping $H^2(S^1)$ onto $H^2(\mathbb{R})$. Indeed,

$$\left\{\tfrac{1}{\sqrt{2\pi}}z^n : n \geq 0\right\} \xrightarrow{\ W\ } \left\{\tfrac{1}{\sqrt{\pi}}\left(\tfrac{t-i}{t+i}\right)^n \tfrac{1}{t+i} : n \geq 0\right\}$$

$$\left\{\tfrac{1}{\sqrt{2\pi}}z^n : n < 0\right\} \xrightarrow{\ W\ } \left\{\tfrac{1}{\sqrt{\pi}}\left(\tfrac{t+i}{t-i}\right)^n \tfrac{1}{t-i} : n \geq 0\right\}.$$

[2]Obviously S and $\overline{S}$ are not actually conjugate; S has no eigenvectors, whereas $\overline{S}$ acts as the identity on an infinite-dimensional subspace.

We also have that W intertwines the multiplication representation of $C_0(\mathbb{R})$ on $L^2(\mathbb{R})$ with the multiplication representation of $C_0(S^1)$ on $L^2(S^1)$ in the expected way, because the phase adjustment is itself implemented by a (circle-valued) multiplication operator. We summarize the properties of W just discussed in the following lemma.

Lemma 3.2.3. *Define $W : L^2(S^1) \to L^2(\mathbb{R})$ by $(W\xi)(t) = \frac{\sqrt{2}}{t+i}\xi\left(\frac{t-i}{t+i}\right)$. Then:*

1. *W is an isometric isomorphism $L^2(S^1) \to L^2(\mathbb{R})$,*
2. *W maps $H^2(S^1)$ onto $H^2(\mathbb{R})$,*
3. *W satisfies $WM(f)W^{-1} = M(f \circ w)$ for all $f \in C_0(S^1)$, where $w(t) = \frac{t-i}{t+i}$ is the Cayley transform.*

We use the following convention for the Fourier transform:

$$\widehat{f}(s) = \int_{\mathbb{R}} f(t)e^{-2\pi ist}\, dt \qquad\qquad f \in C_c(\mathbb{R}). \tag{3.9}$$

The Fourier transform extends uniquely to a C*-algebra isomorphism, written simply as

$$f \mapsto \widehat{f} : C_0(\mathbb{R}) \to C^*(\mathbb{R}).$$

The Fourier transform also extends uniquely to an isometric isomorphism of Hilbert spaces, which we write as

$$U : L^2(\mathbb{R}) \to L^2(\mathbb{R})$$

in order to notationally distinguish these slightly different concepts. The key properties of the Fourier transform needed are gathered below.

Lemma 3.2.4. *Let $U : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the Fourier transform defined by $(U\xi)(s) = \int_{\mathbb{R}} \xi(t)e^{-2\pi ist}\, dt$ for $\xi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and extended by continuity. Then:*

1. *U is an isometric isomorphism $L^2(\mathbb{R}) \to L^2(\mathbb{R})$,*
2. *U maps $H^2(\mathbb{R})$ onto $L^2(\mathbb{R}_+)$,*
3. *U satisfies $UM(f)U^{-1} = \lambda(\widehat{f})$ for all $f \in C_0(\mathbb{R})$, where $f \mapsto \widehat{f}$ is the Fourier isomorphism $C_0(\mathbb{R}) \to C^*(\mathbb{R})$ and λ is the regular (i.e. convolution) representation of $C^*(\mathbb{R})$ on $L^2(\mathbb{R})$.*

We now present the conjugacy between the Wiener-Hopf and Toeplitz extensions.

Theorem 3.2.5. *Let $W : L^2(S^1) \to L^2(\mathbb{R})$ and $U : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the Hilbert space isomorphisms associated to the Cayley and Fourier transforms Then,* $\mathrm{Ad}_U \circ \mathrm{Ad}_W$ *carries $\overline{\mathcal{T}}_0$, the Toeplitz algebra in its uncompressed, nonunital form onto the Wiener-Hopf algebra $\mathcal{T}_{\mathbb{R}}$. Furthermore, the map $\sigma_{\mathbb{R}}$ defined by commutativity of the following diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{K}(L^2(S^1)) & \longrightarrow & \overline{\mathcal{T}}_0 & \xrightarrow{\overline{\sigma}} & C_0(S^1) & \longrightarrow & 0 \\ & & \Big\downarrow \mathrm{Ad}_U\circ\mathrm{Ad}_W & & \Big\downarrow \mathrm{Ad}_U\circ\mathrm{Ad}_W & & \Big\downarrow f\mapsto\widehat{f\circ w} & & \\ 0 & \longrightarrow & \mathbb{K}(L^2(\mathbb{R})) & \longrightarrow & \mathcal{T}_{\mathbb{R}} & \xrightarrow{\sigma_{\mathbb{R}}} & C^*(\mathbb{R}) & \longrightarrow & 0 \end{array}$$

satisfies $\sigma_{\mathbb{R}}(M(f)\lambda(g)) = f(+\infty)g$ for all $f \in C_0(\mathbb{R} \cup \{+\infty\})$, $g \in C^*(\mathbb{R})$.

Proof. By definition, $\mathcal{T}_{\mathbb{R}}$ is generated by products of the form $M(f)\lambda(g)$ where $f \in C_0(\mathbb{R} \cup \{+\infty\})$ and $g \in C^*(\mathbb{R})$. By definition, $\overline{\mathcal{T}}_0$ is generated by products $P_{H^2(S^1)}M_fP_{H^2(S^1)}$, $f \in C_0(S^1)$, together with $\mathbb{K}(L^2(S^1))$. Let χ denote the characteristic function of $\mathbb{R}_+$, so that $M(\chi)$ is the orthogonal projection onto $L^2(\mathbb{R}_+) \subseteq L^2(\mathbb{R})$. The image of $\overline{\mathcal{T}}_0$ by $\mathrm{Ad}_U \circ \mathrm{Ad}_W$ is then generated by products $M(\chi)\lambda(g)M(\chi)$, $g \in C^*(\mathbb{R})$ together with $\mathbb{K}(L^2(\mathbb{R}))$ (see Lemmas 3.2.3 and 3.2.4). Since $\mathcal{T}_{\mathbb{R}}$ contains the compacts, the claim that $\mathrm{Ad}_U \circ \mathrm{Ad}_W$ maps $\overline{\mathcal{T}}_0$ onto $\mathcal{T}_{\mathbb{R}}$ will be proven once we establish the following:

Claim 1. If $f \in C_0(\mathbb{R} \cup \{+\infty\})$, $g \in C^*(\mathbb{R})$ and $f(+\infty) = 1$, then $M(f)\lambda(g)$ is equal to $M(\chi)\lambda(g)M(\chi)$, modulo compact operators.

Firstly, since $M(C_0(\mathbb{R}))\lambda(C^*(\mathbb{R})) \subseteq \mathbb{K}(L^2(\mathbb{R}))$, it does no harm to assume that f is identically equal to 1 on $\mathbb{R}_+$. We can then write $f - \chi = (1 - \chi)h$ where $h \in C_0(\mathbb{R})$ and therefore, using that $M(h)\lambda(g)$ is compact and that compact operators form an ideal, conclude that $M(f)\lambda(g)$ is equal to $M(\chi)\lambda(g)$ modulo compact operators. Secondly, recall that the Hardy projection $P_{H^2(S^1)}$ commutes, modulo compact operators, with the multiplication representation of $C(S^1)$ on $L^2(S^1)$. Conjugating through $U \circ W : L^2(S^1) \to L^2(\mathbb{R})$, this says that the projection $M(\chi)$ commutes modulo compact operators with the regular representation λ of $C^*(\mathbb{R})$ on $L^2(\mathbb{R})$. Therefore, $M(\chi)\lambda(g) = M(\chi)\lambda(g)M(\chi)$, modulo compact operators, establishing the claim.

The assertion about $\sigma_{\mathbb{R}}$ also follows from the above claim. □

Since the uncompressed form of the nonunital Toeplitz extension is isomorphic to the usual nonunital Toeplitz extension, we therefore have

Corollary 3.2.6. *The Wiener-Hopf algebra $\mathcal{T}_{\mathbb{R}}$ is isomorphic to the nonunital Toeplitz algebra $\mathcal{T}_0$.*

Proof. Follows from Theorem 3.2.5 and Lemma 3.2.2. □

Corollary 3.2.7. *The K-theory boundary map $K_1(C^*(\mathbb{R})) \to K_0(\mathbb{K}(L^2(\mathbb{R}))) \cong \mathbb{Z}$ induced by the Wiener-Hopf extension $0 \to \mathbb{K}(L^2(\mathbb{R})) \to \mathcal{T}_{\mathbb{R}} \overset{\sigma_{\mathbb{R}}}{\to} C^*(\mathbb{R}) \to 0$ sends $[1-b] \mapsto -1$, where $b \in L^1(\mathbb{R}) \subseteq C^*(\mathbb{R})$ is given by*

$$b(t) = \begin{cases} e^{-t/2} & t \geq 0 \\ 0 & t < 0. \end{cases}$$

Proof. This follows from the corresponding calculation for the Toeplitz extension, and naturality of the K-theory boundary map. Note that $[1-b]$ is chosen to be the image under the Fourier isomorphism $C_0(\mathbb{R}) \to C^*(\mathbb{R})$ of the usual generator of $K_1(C_0(\mathbb{R}))$ given as the class of (any) loop of winding number 1. □

Remark 3.2.8. Obviously one can combine the Fourier transform of the line, the Cayley transform and the Fourier transform of the circle to realize $C^*(\mathbb{R})$ as a codimension-1 ideal in $C^*(\mathbb{Z})$. One may understand this section as explaining how, by appropriately unitarizing the Cayley transform, one can promote this observation to the level of extensions.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{K}(L^2(\mathbb{R})) & \longrightarrow & C_0(\mathbb{R} \cup \{+\infty\}) \rtimes \mathbb{R} & \longrightarrow & C^*(\mathbb{R}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{K}(L^2(\mathbb{Z})) & \longrightarrow & C_0(\mathbb{Z} \cup \{+\infty\}) \rtimes \mathbb{Z} & \longrightarrow & C^*(\mathbb{Z}) & \longrightarrow & 0 \end{array}$$

3.3 The C*-algebras of the foliations $\mathcal{F}_{\mathbb{R}}^k$

In this section, we classify the C*-algebras of the foliations $\mathcal{F}_{\mathbb{R}}^k$ up to isomorphism. It is quite straightforward to see that all the $C^*(\mathcal{F}_{\mathbb{R}}^k)$ with k even are isomorphic, and that all the $C^*(\mathcal{F}_{\mathbb{R}}^k)$ with k odd are isomorphic. The main task is therefore to distinguish these two cases by some invariant. For this purpose, we use that $C^*(\mathcal{F}_{\mathbb{R}}^k)$ contains a unique essential ideal I isomorphic to $\mathbb{K} \oplus \mathbb{K}$. The ideal I determines a map $K_1(C^*(\mathcal{F}_{\mathbb{R}}^k)/I) \to \mathbb{Z}$ given as the composition of the K-theory boundary map $K_1(C^*(\mathcal{F}_{\mathbb{R}}^k)/I) \to K_0(I)$ with the addition map $K_0(I) \cong \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$. We show this map $K_1(C^*(\mathcal{F}_{\mathbb{R}}^k)/I) \to \mathbb{Z}$ is zero when k is even and an isomorphism onto $2\mathbb{Z}$ when k is odd.

As in the case of regular foliations, the C*-algebra $C^*(\mathcal{F})$ of a singular foliation $\mathcal{F}$ is a naturally represented on the L^2 space of any of its leaves (or the holonomy cover of that leaf, as appropriate). See Section 4.5 in [4]. In the case of $\mathcal{F}^k_{\mathbb{R}}$, the leaves with no holonomy consist of the two half lines $\mathbb{R}_- := \mathbb{R}_-$ and $\mathbb{R}_+ := \mathbb{R}_+$. We show that the natural representation of $C^*(\mathcal{F}^k_{\mathbb{R}})$ on $L^2(\mathbb{R}_-) \oplus L^2(\mathbb{R}_+)$ is faithful and that the images of these representations as k varies are distinct C*-algebras (even though there are only of two isomorphism types). This also shows that the reduced and maximal C*-algebras are the same for these foliations.

As discussed in Section 3.1, the holonomy groupoid of $\mathcal{F}^k_{\mathbb{R}}$ is isomorphic to the transformation groupoid $\mathbb{R} \rtimes_\phi \mathbb{R}$ associated to the flow of any complete vector field generating $\mathcal{F}^k_{\mathbb{R}}$. The $C*$-algebra of $\mathcal{F}^k_{\mathbb{R}}$ is then isomorphic to the crossed product $C_0(\mathbb{R}) \rtimes_\phi \mathbb{R}$ of the associated action on $\mathbb{R}$. Throughout this section, ϕ will denote a smooth action of $\mathbb{R}$ on $\mathbb{R}$ which fixes the origin and is transitive on each of the open half lines $\mathbb{R}_-$ and $\mathbb{R}_+ := \mathbb{R}_+$. We use the same symbol ϕ to denote the corresponding action of $\mathbb{R}$ on $C_0(\mathbb{R})$ by $*$-automorphisms:

$$(\phi_t f)(x) = f(\phi_{-t}(x)). \tag{3.10}$$

We want to understand the structure of the crossed product C*-algebra $C_0(\mathbb{R}) \rtimes_\phi \mathbb{R}$. Working up to orientation-preserving topological conjugacy, there are four types of flows to consider. We assign each case a "bi-index", as tabulated below. The first component indicates whether the origin is a source or a sink for the left half line and the second component indicates whether the origin is a source or a sink for the right half line.

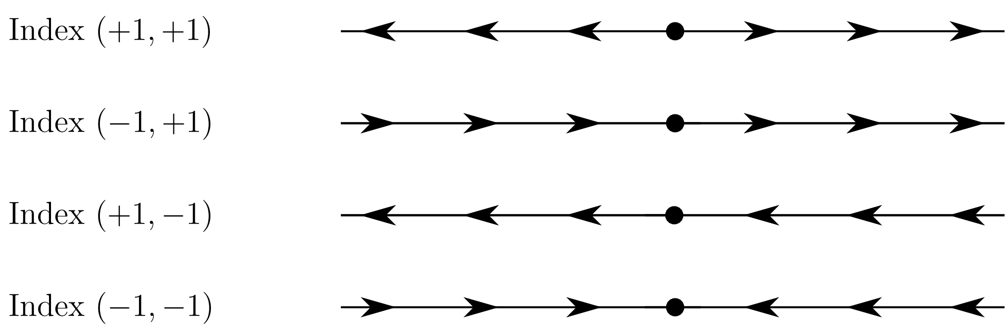


Figure 3.1: Indexing convention for flows on $\mathbb{R}$ with unique fixed point 0.

Remark 3.3.1. If orientation-reversing topological conjugacies are also allowed, we are reduced to three cases; the index $(-1, 1)$ case becomes equivalent to the index $(1, -1)$ case. If time-reversal symmetry is allowed, we are reduced to two cases; the index $(-1, 1)$ and $(1, -1)$ cases become equivalent and the index $(1, 1)$ and $(-1, -1)$ cases become equivalent. Such equivalences do not change the C*-algebra, so it follows that there are at most two possibilities for the isomorphism type of $C_0(\mathbb{R}) \rtimes_\phi \mathbb{R}$. That two types do in fact occur will follow from an index calculation.

Evaluation at 0 yields an $\mathbb{R}$-equivariant exact sequence of commutative C*-algebras

$$0 \longrightarrow C_0(\mathbb{R}_-) \oplus C_0(\mathbb{R}_+) \longrightarrow C_0(\mathbb{R}) \longrightarrow \mathbb{C} \longrightarrow 0,$$

where $\mathbb{R}_- := \mathbb{R}_-$, $\mathbb{R}_+ := \mathbb{R}_+$. The restriction of the action to either of the half lines is conjugate to the translation action of $\mathbb{R}$ on itself. Thus, taking the crossed product by $\mathbb{R}$ yields that $C_0(\mathbb{R}) \rtimes_\phi \mathbb{R}$ sits in an exact sequence of the form

$$0 \longrightarrow \mathbb{K} \oplus \mathbb{K} \longrightarrow C_0(\mathbb{R}) \rtimes_\phi \mathbb{R} \longrightarrow C^*(\mathbb{R}) \longrightarrow 0. \tag{3.11}$$

First we show that there are four possibilities for the isomorphism type of this natural extension if we slightly alter the usual definition of isomorphism of extensions by insisting that the order of the factors $\mathbb{K} \oplus \mathbb{K}$ is preserved.

Let M denote the multiplication representation of $C_0(\mathbb{R})$ on $L^2(\mathbb{R}_-) \oplus L^2(\mathbb{R}_+)$. Let U denote the unitary representation of $\mathbb{R}$ on $L^2(\mathbb{R}_-) \oplus L^2(\mathbb{R}_+)$ given by $(U_t\xi)(x) = (\phi'_{-t}(x))^{1/2}\xi(\phi_t(x))$. Then (M, U) is a covariant pair and determines a representation π of $C_0(\mathbb{R}) \rtimes_\phi \mathbb{R}$ on $L^2(\mathbb{R}_-) \oplus L^2(\mathbb{R}_+)$. Observe that the restriction of π to $C_0(\mathbb{R}_- \cup \mathbb{R}_+) \rtimes_\phi \mathbb{R}$ is an isomorphism onto $\mathbb{K}(L^2(\mathbb{R}_-)) \oplus \mathbb{K}(L^2(\mathbb{R}_+))$.

Theorem 3.3.2. *Let ϕ be a smooth action of $\mathbb{R}$ on $\mathbb{R}$ with 0 as its unique fixed point. Then:*

1. *The representation π of $C_0(\mathbb{R}) \rtimes_\phi \mathbb{R}$ on $L^2(\mathbb{R}_-) \oplus L^2(\mathbb{R}_+)$ defined above is faithful.*
2. *The K-theory boundary map $K_1(C^*(\mathbb{R})) \to \mathbb{Z} \oplus \mathbb{Z}$ arising from the exact sequence* (3.11) *sends the generator $[1-b]$ from Corollary 3.2.7 to the index of the flow ϕ, as tabulated in Figure 3.1.*

Proof. We define $\mathcal{T}_\mathbb{R}^{-1}$ to be $\mathcal{T}_\mathbb{R}$ equipped with its usual symbol map $\sigma_\mathbb{R}^{-1} := \sigma_\mathbb{R}$. We define $\mathcal{T}_\mathbb{R}^{+1}$ to be $\mathcal{T}_\mathbb{R}$ equipped with the time-reversed symbol map $\sigma_\mathbb{R}^{+1} := r \circ \sigma_\mathbb{R}$, where $r : C^*(\mathbb{R}) \to C^*(\mathbb{R})$ is determined on $L^1(\mathbb{R})$ by $(rf)(t) = f(-t)$. Given $\varepsilon = (\varepsilon_1, \varepsilon_2) \in \{\pm 1\}^2$, we define $\mathcal{T}_\mathbb{R} \oplus_\varepsilon \mathcal{T}_\mathbb{R}$ to be the C*-algebra on $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ given as the pullback algebra

$$\{(T_1, T_2) \in \mathcal{T}_\mathbb{R} \oplus \mathcal{T}_\mathbb{R} : \sigma_\mathbb{R}^{\varepsilon_1}(T_1) = \sigma_\mathbb{R}^{\varepsilon_2}(T_2)\}$$

equipped with the map $\sigma_\mathbb{R}^\varepsilon : \mathcal{T}_\mathbb{R} \oplus_\varepsilon \mathcal{T}_\mathbb{R} \to C^*(\mathbb{R})$ defined by

$$\sigma_\mathbb{R}^\varepsilon(T_1, T_2) := \sigma_\mathbb{R}^{\varepsilon_1}(T_1) = \sigma_\mathbb{R}^{\varepsilon_2}(T_2).$$

By definition, $\mathcal{T}_{\mathbb{R}} \oplus_\varepsilon \mathcal{T}_{\mathbb{R}}$ sits in an exact sequence

$$0 \longrightarrow \mathbb{K}(L^2(\mathbb{R})) \oplus \mathbb{K}(L^2(\mathbb{R})) \longrightarrow \mathcal{T}_{\mathbb{R}} \oplus_\varepsilon \mathcal{T}_{\mathbb{R}} \xrightarrow{\sigma^\varepsilon_{\mathbb{R}}} C^*(\mathbb{R}) \longrightarrow 0.$$

One may check, using Corollary 3.2.7, that the K-theory boundary map

$$K_1(C^*(\mathbb{R})) \to K_0(\mathbb{K}(L^2(\mathbb{R})) \oplus \mathbb{K}(L^2(\mathbb{R}))) \cong \mathbb{Z} \oplus \mathbb{Z}$$

determined by this extension sends $[1-b] \mapsto \varepsilon$.

Identify each of $\mathbb{R}_- \cup \{0\}$ and $\{0\} \cup \mathbb{R}_+$ with $\mathbb{R} \cup \{+\infty\}$ in such a way that the flow ϕ is translated to either the usual translation flow (in the case of a sink) or the time-reversed translation flow (in the case of a source). The identifications chosen lead to corresponding isometric isomorphisms $V_- : L^2(\mathbb{R}_-) \oplus L^2(\mathbb{R})$ and $V_+ : L^2(\mathbb{R}_+) \to L^2(\mathbb{R})$. These choices are such that $V := V_- \oplus V_+$ carries the image of π onto $\mathcal{T}_{\mathbb{R}} \oplus_\varepsilon \mathcal{T}_{\mathbb{R}}$. One may check that the following diagram is commutative:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_0(\mathbb{R}_- \cup \mathbb{R}_+) \rtimes_\phi \mathbb{R} & \longrightarrow & C_0(\mathbb{R}) \rtimes_\phi \mathbb{R} & \longrightarrow & C^*(\mathbb{R}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{Ad}_V \circ \pi} & & \downarrow{\scriptstyle \mathrm{Ad}_V \circ \pi} & & \| & & \\
0 & \longrightarrow & \mathbb{K}(L^2(\mathbb{R})) \oplus \mathbb{K}(L^2(\mathbb{R})) & \longrightarrow & \mathcal{T}_{\mathbb{R}} \oplus_\varepsilon \mathcal{T}_{\mathbb{R}} & \xrightarrow{\sigma^\varepsilon_{\mathbb{R}}} & C^*(\mathbb{R}) & \longrightarrow & 0.
\end{array}$$

The vertical maps on the left and right are isomorphisms, whence so is the map in the center, proving (1). This diagram also establishes (2), by the above calculation and naturality of the K-theory boundary map. □

Corollary 3.3.3. *Let ϕ and ψ be smooth actions of $\mathbb{R}$ on $\mathbb{R}$ with 0 as their unique fixed point. Then the following are equivalent:*

1. *There exists an isomorphism $C_0(\mathbb{R}) \rtimes_\phi \mathbb{R} \to C_0(\mathbb{R}) \rtimes_\psi \mathbb{R}$ and an isomorphism $\mathbb{K} \oplus \mathbb{K} \to \mathbb{K} \oplus \mathbb{K}$ preserving the order of the factors making the following diagram commutative*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{K} \oplus \mathbb{K} & \longrightarrow & C_0(\mathbb{R}) \rtimes_\phi \mathbb{R} & \longrightarrow & C^*(\mathbb{R}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathbb{K} \oplus \mathbb{K} & \longrightarrow & C_0(\mathbb{R}) \rtimes_\psi \mathbb{R} & \longrightarrow & C^*(\mathbb{R}) & \longrightarrow & 0
\end{array} .$$

   Here, the top and bottom rows are the exact sequences coming from evaluation at 0 *as in* (3.11)*.*

2. *ϕ and ψ have the same index, as tabulated in Figure 3.1.*

Proof. Follows from the naturality of the K-theory long exact sequence. □

We know from faithfulness of the representation of $C_0(\mathbb{R}) \rtimes_\phi \mathbb{R}$ on $L^2(\mathbb{R}_-) \oplus L^2(\mathbb{R}_+)$ that the ideal $I \cong \mathbb{K} \oplus \mathbb{K}$ given as the kernel of the map $C_0(\mathbb{R}) \rtimes_\phi \mathbb{R} \to C^*(\mathbb{R})$ is an essential ideal. It is also, as we now show, an *intrinsic* ideal in the sense that it is preserved by any $*$-automorphism of $C_0(\mathbb{R}) \rtimes_\phi \mathbb{R}$.

Lemma 3.3.4. *Let A be a C*-algebra. Then A can contain at most one one closed, $*$-invariant ideal which is an essential ideal and is isomorphic to a finite direct sum of simple C*-algebras. In particular, if I is such an ideal in A, then $\theta(I) = I$ for every $*$-automorphism $\theta : A \to A$.*

Proof. Clearly an essential ideal must contain any ideal which is simple as a C*-algebra. Recall that, in the C*-algebra context, ideals of ideals are again ideals in the ambient C*-algebra. Therefore an essential ideal must in fact contain any ideal that is isomorphic to a finite direct sum of simple C*-algebras. The conclusion follows. □

Let B be a C*-algebra containing an ideal I which is isomorphic to the direct sum of finitely-many copies of $\mathbb{K}$, the algebra of compact operators on a separable Hilbert space. Then we have an map

$$\text{Index} : K_1(B/I) \to \mathbb{Z}$$

given as the composition of the boundary map $K_1(B/I) \to K_0(I)$ associated to the short exact sequence $0 \to I \to B \to B/I \to 0$ and the canonical map $K_0(I) \to \mathbb{Z}$ sending the class of any minimal projection to 1. If I is also an essential ideal in B, then it is intrinsic by the above lemma. Therefore, any automorphism $\theta : B \to B$ descends to an automorphism of $\overline{\theta} : B/I \to B/I$. Naturality of the K-theory boundary map as well as invariance of the addition map $\mathbb{Z}^n \to \mathbb{Z}$ under permuting the factors, then gives that $\overline{\theta}_* : K_1(B/I) \to K_1(B/I)$ commutes with $\text{Index} : K_1(B/I) \to \mathbb{Z}$.

If ϕ is the flow of a complete vector field generating $\mathcal{F}^k_{\mathbb{R}}$ for k odd, then ϕ has index $(1,1)$ or $(-1,-1)$. If ϕ is the flow of a complete vector field generating $\mathcal{F}^k_{\mathbb{R}}$ for k even, then ϕ has index $(-1,1)$ or $(1,-1)$. Recalling that $C^*(\mathcal{F}^k_{\mathbb{R}}) \cong C_0(\mathbb{R}) \rtimes_\phi \mathbb{R}$, the above discussion finally gives us:

Theorem 3.3.5. *Let k and ℓ be a positive integers. Then, $C^*(\mathcal{F}^k_{\mathbb{R}})$ is isomorphic to $C^*(\mathcal{F}^\ell_{\mathbb{R}})$ if and only if k and ℓ have the same parity.*

Lastly, since all the C*-algebras $C^*(\mathcal{F}^k_{\mathbb{R}})$ have natural faithful representations on $L^2(\mathbb{R}_-) \oplus L^2(\mathbb{R}_+)$, it makes sense to compare the images of these representations. Here we obtain the following:

Theorem 3.3.6. *Let π_k denote the canonical representation of $C^*(\mathcal{F}^k_{\mathbb{R}})$ on the Hilbert space $L^2(\mathbb{R}_-) \oplus L^2(\mathbb{R}_+)$ and let A_k denote the image of π_k. If k and ℓ are distinct positive integers, then $A_k \neq A_\ell$. Indeed,*

$$A_k \cap A_\ell = \mathbb{K}(L^2(\mathbb{R}_-)) \oplus \mathbb{K}(L^2(\mathbb{R}_+)).$$

Proof. Recall that $C^*(\mathcal{F}^k_{\mathbb{R}}) \cong C_0(\mathbb{R}) \rtimes_\phi \mathbb{R}$ where ϕ is the flow of a complete vector field agreeing with $x^k \frac{d}{dx}$ near $x = 0$. Similarly, $C^*(\mathcal{F}^\ell_{\mathbb{R}}) \cong C_0(\mathbb{R}) \rtimes_\psi \mathbb{R}$ where ψ is the flow of a complete vector field agreeing with $x^\ell \frac{d}{dx}$ near $x = 0$.

Recall that, by choosing a diffeomorphism $u : \mathbb{R}_- \sqcup \mathbb{R}_+ \to \mathbb{R} \sqcup \mathbb{R}$ and conjugating the flow ϕ to a pair of translation flows, we can conjugate A_k to a pushout of Wiener-Hopf algebras. We may do the same for A_ℓ, using a diffeomorphism $v : \mathbb{R}_- \sqcup \mathbb{R}_+ \to \mathbb{R} \sqcup \mathbb{R}$. We may reformulate our problem to be about comparing these pushout algebras under the unitary induced by the diffeomorphism $w = u \circ v^{-1}$. The key thing to notice is that, if $k > \ell$, then this diffeomorphism must be such that $\lim_{x\to\infty} w'(x) = +\infty$ (on each copy of $\mathbb{R}$), due to the different asymptotics of ϕ and ψ close to 0. In this way, one can reduce the problem to the proposition below. ◻

Proposition 3.3.7. *Let u be an orientation-preserving diffeomorphism of $\mathbb{R}$. Let $U : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the corresponding Hilbert space isomorphism defined by $(U\xi)(x) = (u'(x))^{1/2}\xi(u(x))$. If $\lim_{x\to\infty} u'(x) = \infty$, then the Wiener-Hopf algebra $\mathcal{T}_{\mathbb{R}}$ is not preserved by U. Indeed, $\mathcal{T}_{\mathbb{R}} \cap \mathrm{Ad}_U(\mathcal{T}_{\mathbb{R}}) = \mathbb{K}(L^2(\mathbb{R}))$.*

Proof. Let P denote orthogonal projection onto $L^2(\mathbb{R}_+) \subseteq L^2(\mathbb{R})$. Let $f_1, f_2 \in C_0(\mathbb{R})$ so that $\widehat{f_1}$ and $\widehat{f_2}$ are arbitrary elements of $C^*(\mathbb{R})$. Let $T_i = \lambda(\widehat{f_i})P$, so that $T_1, T_2 \in \mathcal{T}_{\mathbb{R}}$ and $\sigma_{\mathbb{R}}(T_i) = \widehat{f_i}$. We need to show that, if f_1 and f_2 are not both zero, then $T_1 - U^{-1}T_2U$ is not compact. We consider only the case where $f_1 \neq 0$ (the case where f_2 is not zero and f_1 is possibly zero is similar).

Choose a continuous compactly-supported function ξ on $\mathbb{R}$ with $\|\xi\|_2 \neq 0$ such that $\|\lambda(\widehat{f_1})\xi\|_2 = a > 0$. Since $\lambda(f_1)$ commutes with the translation, we may assume without loss of generality that ξ is supported in $\mathbb{R}_+$. Let $b > 0$ be larger than the diameter of the support of ξ so that $\xi_n := \tau_{nb}(\xi)$, $n = 0, 1, 2, \ldots$ defines an orthonormal sequence in $L^2(\mathbb{R}_+)$. For each n, we have $\|T_1\xi_n\|_2 = a$.

We claim that $U^{-1}T_2U\xi_n \to 0$. Once this is proved, we will have that the image by $T_1 - U^{-1}T_2U$ of an orthonormal sequence does not converge to zero, which shows $T_1 - U^{-1}T_2U$ is not a compact operator. For n sufficiently large, $U\xi_n$ is supported inside $\mathbb{R}_+$, and so $\|U^{-1}T_2U\xi_n\|_2 = \|\lambda(\widehat{f_2})U\xi_n\|_2 = \|f_2\eta_n\|_2$ where $\eta_n \in L^2(\mathbb{R})$ is defined by $\widehat{\eta_n} = U\xi_n$. Since $\|U\xi_n\|_2 = 1$, but the diameter of the support of $U\xi_n$ goes to 0 as $n \to \infty$, an application of the Cauchy-Schwartz inequality, gives that the L^1-norm of $U\xi_n$ vanishes as $n \to \infty$. On the other side of the Fourier transform, we then have that $\eta_n \in C_0(\mathbb{R}) \cap L^2(\mathbb{R})$ and $\|\eta_n\|_\infty \to 0$. The result will follow if we show that $\|f_2\eta_n\| \to 0$. This follows from the following simple estimate, valid for any $R > 0$:

$$\|f_2\eta_n\|_2^2 \leq \|\eta_n\|_\infty^2 \left(\int_{|x|<R} |f_2(x)|^2 \, dx \right) + \sup_{|x|>R} |f_2(x)|^2.$$

□

3.4 The smooth algebras of the foliations $\mathcal{F}_{\mathbb{R}}^k$

In this section we come to central result in this chapter, that the smooth convolution algebras $\mathcal{A}(\mathcal{F}_{\mathbb{R}}^k)$, as k ranges over the positive integers, are pairwise nonisomorphic. Note that the notion of isomorphism being considered here is slightly stronger simple algebraic isomorphism.

Definition 3.4.1. Let k and ℓ be positive integers. An **isomorphism** $\mathcal{A}(\mathcal{F}_{\mathbb{R}}^k) \to \mathcal{A}(\mathcal{F}_{\mathbb{R}}^\ell)$ will refer to a $*$-algebra isomorphism which can furthermore be (uniquely) extended to a C*-algebra isomorphism $C^*(\mathcal{A}_{\mathbb{R}}^k) \to C^*(\mathcal{F}_{\mathbb{R}}^\ell)$.

By the results of Section 3.3, we therefore already have that $\mathcal{A}(\mathcal{F}_{\mathbb{R}}^k)$ cannot be isomorphic to $\mathcal{A}(\mathcal{F}_{\mathbb{R}}^\ell)$ if k and ℓ do not have the same parity.

Remark 3.4.2. Nearly any continuity hypothesis one might think to impose on a map at the level of smooth algebras guarantees the existence of an extension to a map of C*-algebras. It is entirely plausible that every (algebraic) $*$-automorphism of $\mathcal{A}(\mathcal{F}_{\mathbb{R}}^k)$ is necessarily continuous, and therefore of the above form. However, we prefer not to get bogged down with delicate questions about automatic continuity.

Recall that the smooth convolution algebra $\mathcal{A}(\mathcal{F}_{\mathbb{R}}^k)$ is canonically isomorphic to $C_c^\infty(\mathbb{R} \rtimes_\phi \mathbb{R})$, where ϕ denotes the flow of any complete vector field generating $\mathcal{F}_{\mathbb{R}}^k$. The convolution product and adjoint operation on $C_c^\infty(\mathbb{R} \rtimes_\phi \mathbb{R})$ are defined by as follows:

$$\begin{aligned} (f * g)(x,t) &= \int_{\mathbb{R}} f(\phi_u(x), t-u) g(x,u) \, du \\ f^*(x,t) &= \overline{f(\phi_t(x), -t)}. \end{aligned} \tag{3.12}$$

Evaluation at $x = 0$ leads to an exact sequence

$$0 \longrightarrow C_c^\infty(\mathbb{R} \rtimes_\phi \mathbb{R}) \cdot x \longrightarrow C_c^\infty(\mathbb{R} \rtimes_\phi \mathbb{R}) \longrightarrow C_c^\infty(\mathbb{R}) \longrightarrow 0$$

The algebra on the left is the ideal in $C_c^\infty(\mathbb{R} \rtimes_\phi \mathbb{R})$ of functions which vanish to first order on $\{0\} \times \mathbb{R}$, the isotropy group at 0. The logic behind the notation for this ideal is as follows: one understands the symbol x as referring to the coordinate function of $\mathbb{R}$ and then uses the fact that the algebra $C^\infty(\mathbb{R})$ of smooth functions on the base manifold multiplies $C_c^\infty(\mathbb{R} \rtimes_\phi \mathbb{R})$ from the left and the right according to the formulas below.

$$\begin{aligned} &(f \cdot g)(x,t) = f(\phi_t(x))g(x,t) \\ &(g \cdot f)(x,t) = f(x)g(x,t) \qquad f \in C^\infty(\mathbb{R}) \qquad g \in C_c^\infty(\mathbb{R} \rtimes_\phi \mathbb{R}). \end{aligned}$$

Note that this product satisfies the expected associative laws:

$$\begin{aligned} &f \cdot (g * h) = (f \cdot g) * h \\ &(g * h) \cdot f = g * (h \cdot f) \qquad f \in C^\infty(\mathbb{R}) \qquad g, h \in C_c^\infty(\mathbb{R} \rtimes_\phi \mathbb{R}). \end{aligned}$$

We shall need to know the following:

Proposition 3.4.3. *The ideal* $C_c^\infty(\mathbb{R} \rtimes_\phi \mathbb{R}) \cdot x \subseteq C_c^\infty(\mathbb{R} \rtimes_\phi \mathbb{R})$ *is "intrinsic" in the sense that every automorphism of* $C_c^\infty(\mathbb{R} \rtimes_\phi \mathbb{R})$ *maps this ideal onto itself.*

Proof. Working inside the C*-completion $C_0(\mathbb{R}) \rtimes_\phi \mathbb{R}$, we have that $x \cdot C_c^\infty(\mathbb{R} \rtimes_\phi \mathbb{R})$ is the intersection with $C_c^\infty(\mathbb{R} \rtimes_\phi \mathbb{R})$ of the ideal $I \cong \mathbb{K} \oplus \mathbb{K}$ of Section 3.3 given as the kernel of the map $C_0(\mathbb{R}) \rtimes_\phi \mathbb{R} \to C^*(\mathbb{R})$. The conclusion follows from Proposition 3.3.4 and Definition 3.4.1. □

More generally, we can consider nested sequence of $*$-invariant ideals

$$C_c^\infty(\mathbb{R} \rtimes_\phi \mathbb{R}) \cdot x^p \subseteq C_c^\infty(\mathbb{R} \rtimes_\phi \mathbb{R})$$

corresponding to the functions which vanish to pth order on $\{0\} \times \mathbb{R}$ as well as the infinite vanishing order ideal

$$C_c^\infty(\mathbb{R} \rtimes_\phi \mathbb{R}) \cdot x^\infty := \bigcap_{p=1}^{\infty} C_c^\infty(\mathbb{R} \rtimes_\phi \mathbb{R}) \cdot x^p.$$

That these spaces of functions are ideals with respect to convolution follows from Corollary 3.4.6 below. One may see Section 4.6 of Chapter 4 for treatment of a more general situation. Let us first recall the following definition.

Definition 3.4.4. Let G be a Lie groupoid. A smooth function $\Delta : G \to \mathbb{R}_+$ is called if a 1-cocycle if $\Delta(gh) = \Delta(g)\Delta(h)$ is satisfied for all composable $g, h \in G$.

If Δ is a smooth 1-cocycle of a Lie groupoid G, then the pointwise product $f \mapsto \Delta f$ is an algebra automorphism for the convolution product on $C_c^\infty(G)$ determined by (any) smooth Haar system.

Proposition 3.4.5. *Let ϕ be the flow of a complete vector field generating $\mathcal{F}^k_{\mathbb{R}}$. Then, there exists a smooth 1-cocycle $\Delta : C_c^\infty(\mathbb{R} \rtimes \phi\mathbb{R}) \to \mathbb{R}_+$ such that $\Delta(x,t) = \frac{\phi_t(x)}{x}$ is satisfied whenever $(x,t) \in \mathbb{R} \rtimes_\phi \mathbb{R}$ and $x \neq 0$.*

Proof. The vector field X whose flow is ϕ has the form $X = h(x)x^k \frac{d}{dx}$, where h is a smooth, nowhere vanishing function. We have $\phi_t'(x) = \frac{h(\phi_t(x))(\phi_t(x))^k}{h(x)x^k}$ whenever $x \neq 0$, and so

$$\frac{\phi_t(x)}{x} = \left(\frac{h(x)}{h(\phi_t(x))}\phi_t'(x)\right)^{1/k}$$

for $x \neq 0$. The expression on the right defines a smooth, positive-valued function $(x,t) \mapsto \Delta(x,t)$ on all of $\mathbb{R} \rtimes_\phi \mathbb{R}$. The expression on the left gives us that Δ is multiplicative on the dense subgroupoid of $\mathbb{R} \rtimes_\phi \mathbb{R}$ consisting of (x,t) with $x \neq 0$. By continuity, Δ is multiplicative everywhere. □

Corollary 3.4.6. *In the context of the proposition above, for every positive integer p, we have that $C_c^\infty(\mathbb{R} \rtimes_\phi \mathbb{R}) \cdot x^p = x^p \cdot C_c^\infty(\mathbb{R} \rtimes_\phi \mathbb{R})$ is a $*$-invariant ideal in $C_c^\infty(\mathbb{R} \rtimes_\phi \mathbb{R})$.*

Proof. This follows from the relation $x \cdot f = (\Delta f) \cdot x$ for every $f \in C_c^\infty(\mathbb{R} \rtimes_\phi \mathbb{R})$ given in the above proposition. □

In fact the sequence of ideals $x^p \cdot C_c^\infty(\mathbb{R} \rtimes_\phi \mathbb{R})$ is also intrinsic. To see this, we need the following Dixmier-Malliavin theorem.

Proposition 3.4.7. *The smooth convolution algebra $C_c^\infty(\mathbb{R} \rtimes_\phi \mathbb{R})$ satisfies*

$$C_c^\infty(\mathbb{R} \rtimes_\phi \mathbb{R}) * C_c^\infty(\mathbb{R} \rtimes_\phi \mathbb{R}) = C_c^\infty(\mathbb{R} \rtimes_\phi \mathbb{R}).$$

Proof. This is a special case of Theorem 4.4.2, the main theorem in Chapter 4 to follow. In fact, rather than use the full force of Theorem 4.4.2, one can instead use Theorem 4.1.1 to write any $f \in C_c^\infty(\mathbb{R} \rtimes_\phi \mathbb{R})$ as $f = g_0 * h_0 + g_1 * h_1$ where $g_0, g_1 \in C_c^\infty(\mathbb{R})$ and $h_0, h_1 \in C_c^\infty(\mathbb{R} \rtimes_\phi \mathbb{R})$. Left multiplying by a suitable cutoff function then gives a representation of f as a two-term sum in which each term is the convolution of functions in $C_c^\infty(\mathbb{R} \rtimes_\phi \mathbb{R})$. In this way, one avoids many of the most technical parts of Chapter 4. □

Corollary 3.4.8. *If θ is an automorphism of $C_c^\infty(\mathbb{R} \rtimes_\phi \mathbb{R})$ and $p \in \{1, 2, \dots, \infty\}$, then $\theta(x^p \cdot C_c^\infty(\mathbb{R} \rtimes_\phi \mathbb{R})) = x^p \cdot C_c^\infty(\mathbb{R} \rtimes_\phi \mathbb{R})$.*

Proof. The case $p = 1$ is Proposition 3.4.3. For p a positive integer, it follows from writing $(x \cdot C_c^\infty(\mathbb{R} \rtimes_\phi \mathbb{R}))^{*p} = x^p * (C_c^\infty(\mathbb{R} \rtimes_\phi \mathbb{R}))^{*p} = x^p \cdot C_c^\infty(\mathbb{R} \rtimes_\phi \mathbb{R})$ where the last equality uses Proposition 3.4.7. Taking the intersection over all positive integers p gives the case $p = \infty$. □

The following result is the means by which we distinguish the different $\mathcal{A}(\mathcal{F}^k_{\mathbb{R}})$.

Theorem 3.4.9. *Let k and p be positive integers. Then, the quotient of $\mathcal{A}(\mathcal{F}^k_{\mathbb{R}})$ by the ideal $x^p\mathcal{A}(\mathcal{F}^k_{\mathbb{R}})$ is commutative for $p \leq k$ and noncommutative for $p > k$.*

Taken together, Theorem 3.4.9 and Corollary 3.4.8 imply

Theorem 3.4.10. *The smooth convolution algebras $\mathcal{A}(\mathcal{F}^k_{\mathbb{R}})$, k a positive integer, are pairwise nonisomorphic.*

It remains to establish Theorem 3.4.9. We prove the following more precise result which has Theorem 3.4.9 as a corollary.

Theorem 3.4.11. *Fix positive integers p and k. Let X be a complete vector field generating $\mathcal{F}^k_{\mathbb{R}}$ and assume for simplicity that X coincides with $x^k\frac{d}{dx}$ on a neighbourhood of 0. Let ϕ be the flow of X and form the transformation groupoid $\mathbb{R} \rtimes_\phi \mathbb{R}$ with usual coordinates (x, t). Let*

$$T : C^\infty_c(\mathbb{R} \rtimes_\phi \mathbb{R}) \to \frac{C^\infty_c(\mathbb{R})[x]}{(x^{p+1})}$$

be the surjective linear map which assigns to a function $f \in C^\infty_c(\mathbb{R} \rtimes_\phi \mathbb{R})$ its pth order Taylor expansion $T(f) = f_0 + f_1x + \ldots f_px^p$, where $f_n(t) = \frac{1}{n!}\frac{\partial^n f}{\partial x^n}(0, t)$. Since the kernel of T is $x^{p+1} \cdot \mathcal{A}(\mathcal{F}^k_{\mathbb{R}})$, we may use the map T to transfer the ring structure of $\frac{C^\infty_c(\mathbb{R}\rtimes_\phi\mathbb{R})}{x^{p+1}\cdot C^\infty_c(\mathbb{R}\rtimes_\phi\mathbb{R})}$ to $C^\infty_c(\mathbb{R})[x]/(x^{p+1})$.

1. *If $p < k$, then the product on $C^\infty_c(\mathbb{R})[x]/(x^{p+1})$ induced by T is the usual one given by multiplication followed by truncation (the product on the coefficient ring $C^\infty_c(\mathbb{R})$ is convolution).*

2. *If $p = k \geq 2$, then the product on $C^\infty_c(\mathbb{R})[x]/(x^{k+1})$ induced by T is determined by the single relation $xf = fx + \delta(f)x^k$, where $\delta : C^\infty_c(\mathbb{R}) \to C^\infty_c(\mathbb{R})$ is the derivation defined by $\delta(f)(t) = tf(t)$.*

3. *If $p = k = 1$, then the product on $C^\infty_c(\mathbb{R})[x]/(x^2)$ induced by T is determined by the single relation $xf = \Delta(f)x$, where $\Delta : C^\infty_c(\mathbb{R}) \to C^\infty_c(\mathbb{R})$ is the algebra automorphism defined by $\Delta(f)(t) = e^tf(t)$.*

Proof. We shall only consider the (most complicated) case $p = k \geq 2$. It is convenient to enlarge $C^\infty_c(\mathbb{R} \rtimes_\phi \mathbb{R})$ slightly and work instead in the *smooth translation algebra* $C^\infty_u(\mathbb{R} \rtimes_\phi \mathbb{R})$ consisting of smooth functions f on $\mathbb{R} \rtimes_\phi \mathbb{R}$ supported inside $M \times [-r, r]$ for some $r > 0$ (depending on f). The convolution product and adjoint operation on $C^\infty_u(\mathbb{R} \rtimes_\phi M)$ are defined in the same way, using (3.12). The translation algebra has the convenient property of containing $C_c^\infty(\mathbb{R})$ as a subalgebra. Specifically, given $f \in C_c^\infty(\mathbb{R})$, define $\widetilde{f} \in C_u^\infty(\mathbb{R} \rtimes_\phi \mathbb{R})$ by $\widetilde{f}(x,t) = f(t)$ for all (x,t). The assignment

$$f \mapsto \widetilde{f} : C_c^\infty(\mathbb{R}) \to C_u^\infty(\mathbb{R} \rtimes_\phi \mathbb{R})$$

is an injective $*$-homomorphism satisfying $T(\widetilde{f}) = f$ for $f \in C_c^\infty(\mathbb{R})$. We also have the obvious relation $T(f \cdot x) = T(f)x$, $f \in C_u^\infty(\mathbb{R} \rtimes_\phi \mathbb{R})$, whence we may extend $f \mapsto \widetilde{f}$ to a linear splitting $S : C_c^\infty(\mathbb{R})[x]/(x^{k+1}) \to C_u^\infty(\mathbb{R} \rtimes_\phi \mathbb{R})$ of T defined as follows:

$$S(f_0 + f_1 x + \ldots + f_k x^k) = \widetilde{f}_0 + \widetilde{f}_1 \cdot x + \ldots + \widetilde{f}_k \cdot x^k$$

It therefore remains to understand the product on elements of the form $\widetilde{f}_0 + \widetilde{f}_1 \cdot x + \ldots + \widetilde{f}_k \cdot x^k$, modulo the ideal $x^{k+1} \cdot C_u^\infty(\mathbb{R} \rtimes_\phi \mathbb{R})$. Let $\widetilde{\delta}$ be the derivation of $C_u^\infty(\mathbb{R} \rtimes_\phi \mathbb{R})$ given by $(\widetilde{\delta} f)(x,t) = tf(x,t)$. Then we have

$$\widetilde{\delta f} = \widetilde{\delta} \widetilde{f} \qquad\qquad f \in C_c^\infty(\mathbb{R}).$$

Recall that $\mathcal{F}_{\mathbb{R}}^k$ is singly-generated by $x^k \frac{d}{dx}$. When $k = 1$, this is just the Euler vector field; its flow is complete and given by the scaling action $\phi_t(x) = e^t x$. When $k \geq 2$, the flow is not complete. The flow is given by

$$\phi_t(x) = \frac{x}{\sqrt[k-1]{1-(k-1)tx^{k-1}}}.$$

and is defined on the domain $\{(t,x) : tx^{k-1} < \frac{1}{k-1}\}$. More important for us than this specific formula is the way that the Taylor series of the flow starts:

$$\phi_t(x) \sim x + tx^k + \ldots \tag{3.13}$$

Since X is a complete vector field generating $\mathcal{F}_{\mathbb{R}}^k$, $k \geq 2$ which furthermore coincides with $x^k \frac{d}{dx}$ on a neighbourhood of 0, it follows that the flow of ϕ can be expressed as

$$\phi_t(x) = x + tx^k + h(x,t)$$

where h is a smooth function vanishing to order $k+1$ on $\{0\} \times \mathbb{R}$. We therefore have, for any $f \in C_u^\infty(\mathbb{R} \rtimes_\phi \mathbb{R})$,

$$x \cdot f = f \cdot x + (\widetilde{\delta} f) \cdot x^k + r$$

where $r \in x^{k+1} \cdot C_c^\infty(\mathbb{R} \rtimes_\phi \mathbb{R})$. Replacing f above with $\widetilde{f}$ for $f \in C_c^\infty(\mathbb{R})$, we obtain

$$x \cdot \widetilde{f} = \widetilde{f} \cdot x + \widetilde{\delta(f)} \cdot x^k \pmod{x^{k+1} \cdot C_u^\infty(\mathbb{R} \rtimes_\phi \mathbb{R})}.$$

Together with the fact that $\widetilde{f} * \widetilde{g} = \widetilde{f * g}$, for $f, g \in C_c^\infty(\mathbb{R})$, the above determines the product on $C_c^\infty(\mathbb{R})[x]/(x^{k+1})$ to be as claimed. □

Finally, we describe the product structure on $\frac{C_c^\infty(\mathbb{R}\rtimes_\phi\mathbb{R})}{x^\infty\cdot C_c^\infty(\mathbb{R}\rtimes_\phi\mathbb{R})}$. According to Borel's theorem ([32], pp 16), the mapping $T : C_c^\infty(\mathbb{R}\rtimes_\phi\mathbb{R}) \to C_c^\infty(\mathbb{R})[[x]]$ to formal series with coefficients in $C_c^\infty(\mathbb{R})$ given by forming the Taylor expansion at $x = 0$ is surjective. Clearly its kernel is equal to $x^\infty \cdot C_c^\infty(\mathbb{R}\rtimes_\phi\mathbb{R})$, so we may understand the product on $\frac{C_c^\infty(\mathbb{R}\rtimes_\phi\mathbb{R})}{x^\infty\cdot C_c^\infty(\mathbb{R}\rtimes_\phi\mathbb{R})}$ by transferring it to $C_c^\infty(\mathbb{R})[[x]]$.

Some special notations will be needed. Given a smooth action ϕ of $\mathbb{R}$ on $\mathbb{R}$ fixing the origin, let

$$\phi_t(x) \sim \sum_{m=1}^{\infty} \phi_m(t) x^m$$

be the Taylor expansion of the flow in x, so that ϕ_m is a smooth function on $\mathbb{R}$. More generally, we consider the Taylor expansion of any power of the flow:

$$(\phi_t(x))^n \sim \sum_{m=n}^{\infty} \phi_m^n(t) x^m.$$

The smooth functions ϕ_m^n will be used to describe the product on $C_c^\infty(\mathbb{R})[[x]]$ induced by T.

Theorem 3.4.12. *Let ϕ be a smooth action of $\mathbb{R}$ on $\mathbb{R}$ which fixes the origin. The product $*$ on $C_c^\infty(\mathbb{R})[[x]]$ determined by the Taylor expansion map $T : C_c^\infty(\mathbb{R}\rtimes_\phi\mathbb{R}) \to C_c^\infty(\mathbb{R})[[x]]$ is such that, if $f = \sum_{n\geq 0} f_n x^n$ and $g = \sum_{n\geq 0} g_n x^n$, then $f * g = h$ where $h = \sum_{n\geq 0} h_n x^n$ is given by*

$$h_p = \sum_{n\leq m\leq p} f_n * (\phi_m^n g_{p-m}).$$

Notice that if $\phi_t(x) = x$, or even if ϕ simply fixes the origin to infinite order, then $\phi_n^m(t) = 1$ if $m = n$ and 0 otherwise, restoring the usual (commutative) product structure on $C_c^\infty(\mathbb{R})[[x]]$.

Proof. Throughout, $[m]$ denotes the operation of extracting the coefficient of x^m in a formal series. Let $F, G \in C_c^\infty(\mathbb{R}\rtimes_\phi\mathbb{R})$ be lifts of f, g through the map $T : C_c^\infty(\mathbb{R}\rtimes_\phi\mathbb{R}) \to \mathbb{C}[[x]]$. Let $H = F * G$. Then, we have

$$\begin{aligned}
[p]F(t-s,\phi_s(x))G(s,x) &= \sum_{m=0}^{p} \left([m]F(t-s,\phi_s(x))\right)\left([p-m]G(s,x)\right)\\
&= \sum_{m=0}^{p}\sum_{n=0}^{m} \left([m]\phi_s(x)^n\right)\left([n]F(t-s,x)\right) g_{p-m}(s)\\
&= \sum_{m=0}^{p}\sum_{n=0}^{m} \phi_m^n(s) f_n(t-s) g_{p-m}(s)
\end{aligned}$$

Integrating with respect to s then gives

$$[p]H(x,t) = \sum_{m=0}^{p} \sum_{n=0}^{m} (f_n * (\phi_m^n g_{p-m})(t)$$

□

Corollary 3.4.13. *The quotient* $\frac{\mathcal{A}(\mathcal{F}_{\mathbb{R}}^1)}{x^\infty \cdot \mathcal{A}(\mathcal{F}_{\mathbb{R}}^1)}$ *is isomorphic to* $C_c^\infty(\mathbb{R})[[x]]$ *with product* $*_1$ *defined as follows: if* $f = \sum_{n\geq 0} f_n x^n$ *and* $g = \sum_{n\geq 0} g_n x^n$*, then* $f *_1 g = \sum_{n\geq 0} h_n x^n$*, where*

$$h_p = \sum_{0\leq n\leq p} f_n * (e^{nt} g_{p-m}).$$

Proof. We may take $\mathcal{F}_{\mathbb{R}}^1$ to be generated by $x\frac{d}{dx}$ with flow $\phi(t) = e^t x$. □

Corollary 3.4.14. *Fix a positive integer* $k \geq 2$*. Then, the quotient* $\frac{\mathcal{A}(\mathcal{F}_{\mathbb{R}}^k)}{x^\infty \cdot \mathcal{A}(\mathcal{F}_{\mathbb{R}}^k)}$ *is isomorphic to* $C_c^\infty(\mathbb{R})[[x]]$ *with product* $*_k$ *defined as follows: if* $f = \sum_{n\geq 0} f_n x^n$ *and* $g = \sum_{n\geq 0} g_n x^n$*, then* $f *_k g = \sum_{n\geq 0} h_n x^n$*, where:*

$$h_p = \sum_{0\leq n\leq m\leq p} f_n * (\phi_m^n g_{p-m})$$

and the functions $\phi_m^n \in C^\infty(\mathbb{R})$ *are defined by* $\phi_m^n \equiv 0$ *when* $m-n$ *is not divisible by* $k-1$ *and*

$$\phi_m^n(t) = (-1)^r (k-1)^r \binom{-\frac{n}{k-1}}{r} t^r = (k-1)^r \binom{r + \frac{n}{k-1} - 1}{r} t^r$$

when $n + (k-1)r$ *for* r *a nonnegative integer.*

Proof. Let X be a complete vector field on $\mathbb{R}$ that generates $\mathcal{F}_{\mathbb{R}}^k$ and coincides with $x^k \frac{d}{dx}$ on a neighbourhood of 0. Then, the flow ϕ of X satisfies

$$\phi_t(x) = \frac{x}{\sqrt[k-1]{1-(k-1)tx^{k-1}}}$$

for all (t,x) on a neighbourhood of $\mathbb{R} \times \{0\} \subseteq \mathbb{R}^2$. It is a simple matter to calculate the Taylor expansion of $x \mapsto (\phi_t(x))^n$ as a binomial series. □

We conclude by pointing out the following relation among the functions ϕ_m^n which were used to define the product on $C_c^\infty(\mathbb{R})[[x]]$ in Theorem 3.4.12

Proposition 3.4.15. *Let ϕ be a smooth action of $\mathbb{R}$ on $\mathbb{R}$ fixing the origin. As above, define smooth functions ϕ_m^n, for $m, n \geq 0$, by $\phi_m^n(t) = [m](\phi_t(x))^n$, where $[m]$ denotes the operation of extracting the coefficient of x^m in a formal series. Then, the following identities are satisfied.*

$$\phi_m^n(t) = \sum_{i=n}^{m} \phi_m^i(t-s)\phi_i^n(s)$$

Proof. For any smooth functions f and g on $\mathbb{R}$ which vanish at 0, one has

$$[m](f \circ g)^n = \sum_{i=n}^{m}([m]g^i)([i]f^n).$$

Taking $f = \phi_{t-s}$ and $g = \phi_s$ so that $f \circ g = \phi_t$ we get the desired identity. □

The above identity suggests the presence of a Hopf algebra behind the scenes. It would be interesting to look for a connection between the algebras $\frac{C_c^\infty(\mathbb{R} \rtimes_\phi \mathbb{R})}{x^\infty \cdot C_c^\infty(\mathbb{R} \rtimes_\phi \mathbb{R})}$ and the work of Connes and Kreimer [18]. See also [9], [13].

Chapter 4

A Dixmier-Malliavin Theorem for Lie Groupoids

Except for minor changes, this chapter reproduces the content of [27]. A famous theorem of Dixmier and Malliavin ([23], 3.1 Théorème) states that every smooth, compactly-supported function on a Lie group can be expressed as a finite sum in which each term is the convolution (with respect to Haar measure) of two such functions. In this chapter we prove the following Lie groupoid generalization.

Theorem (Theorem 4.4.2)**.** *Let G be a Lie groupoid with a smooth Haar system and form the smooth convolution algebra $C_c^\infty(G)$. Every $f \in C_c^\infty(G)$ can be written as $g_1 * h_1 + \ldots + g_n * h_n$ for some positive integer n and $g_1, h_1, \ldots, g_n, h_n \in C_c^\infty(G)$.*

The above theorem is then used to obtain results on the multiplication structure of certain ideals in $C_c^\infty(G)$ arising from functions vanishing to given order along a given invariant submanifold of the unit space. Such results are needed to complete the proof of main result Theorem 3.4.10 of Chapter 3, where it was necessary to know that every element of the smooth convolution algebra of the foliation $\mathcal{F}_{\mathbb{R}}^k$ could be expressed as a finite sum of convolution products.

We now explain the organization of the present chapter. In Section 4.1, we use a key lemma of Dixmier-Malliavin to obtain, as a corollary, the preliminary factorization result Theorem 4.1.1. This allows us, given a smooth $\mathbb{R}$-action on a manifold M, to express functions on M as two-term sums in which both terms are the convolution of a function on $\mathbb{R}$ with a function on M. Section 4.2 largely serves a notational purpose. In it, we lay out our conventions for the Lie algebroid of a Lie groupoid, for Haar measures, and for Lie groupoid actions. Section 4.3 is quite technical and culminates with Lemma 4.3.3, giving a framework in which functions on $\mathbb{R}$ can be convolved to yield a function on a Lie groupoid with appropriate properties. In Section 5, we derive the main result, Theorem 4.4.2. Section 4.5 prepares the ground for the final section by analysing the product structure of ideals in the (commutative) algebra of smooth functions on a manifold under pointwise multiplication. This is again somewhat technical, though we do make use of an interesting inversion principle which connects Schwartz functions to bump functions (Lemma 4.5.6). Finally, in Section 4.6, we obtain our results on the product structure of ideals in the smooth convolution algebra of a Lie groupoid. The main finding here is Theorem 4.6.1.

4.1 Dixmier-Malliavin for $\mathbb{R}$ actions

In this section, we use a key lemma from [23] to obtain a preliminary factorization result which will be essential to proving our Lie groupoid generalization of Dixmier-Malliavin.

Let X be a complete vector field on a smooth manifold M. We denote by $t \mapsto e^{tX}$ the corresponding 1-parameter group of diffeomorphisms (in other words the flow) that is related to X by

$$\frac{d}{dt}\Big|_{t=0} f(e^{tX}m) = (Xf)(m).$$

This action of $\mathbb{R}$ on M can alternatively be encoded by a representation π of the smooth convolution algebra $C_c^\infty(\mathbb{R})$ on $C_c^\infty(M)$ which we call the **integrated form** of the action. This representation π is defined by the following formula:

$$(\pi(f)\psi)(m) = \int_{\mathbb{R}} f(-t)\psi(e^{tX}m)\, dt. \tag{4.1}$$

Theorem 4.1.1. *Let $\mathbb{R}$ act smoothly on a manifold M via a complete vector field X and let π be the representation of $C_c^\infty(\mathbb{R})$ on $C_c^\infty(M)$ defined by (4.1). Then, for any $\varphi \in C_c^\infty(M)$, there exists $f_0, f_1 \in C_c^\infty(\mathbb{R})$ and $\psi_0, \psi_1 \in C_c^\infty(M)$ such that*

$$\varphi = \pi(f_0)\psi_0 + \pi(f_1)\psi_1. \tag{4.2}$$

Moreover, this factorization can be taken such that, for $i = 0, 1$, $\operatorname{supp}(\psi_i) \subseteq \operatorname{supp}(\varphi)$ and $\operatorname{supp}(f_i) \subseteq (-\epsilon, \epsilon)$, where $\epsilon > 0$ is fixed in advance.

Although Theorem 4.1.1 is not recorded in [23], it is a straightforward consequence of the results and techniques therein. The main challenge is to achieve the factorization (4.2) with smooth f_i. It is far easier to achieve the factorization if the f_i are only required to be differentiable of a large, but finite, order (see [10], pp. 23). The means by which this simpler result is achieved suggest a strategy for the more difficult result, so it seems worthwhile to provide a short outline here.

Lemma 4.1.2. *For every nonnegative integer k, there exists $f \in C_c^k(\mathbb{R})$, $g \in C_c^\infty(\mathbb{R})$ such that*

$$\delta = f^{(k+2)} + g$$

where δ denotes the delta distribution at 0.

Proof. Antidifferentiate the delta distribution $k+2$ times, always picking the antiderivative that vanishes on the negative half line. The result is the C^k function F vanishing on the negative-half line and satisfying $F(t) = \frac{1}{(k+1)!}t^{k+1}$ for $t \geq 0$. Of course, $F^{(k+2)} = \delta$, but F is not compactly-supported. Let G be a C^∞ function that agrees with F outside of a bounded interval. Then, $\delta = F^{(k+2)} = (F-G)^{(k+2)} + G^{(k+2)} = f^{(k+2)} + g$, where $f = F - G \in C_c^k(\mathbb{R})$ and $g = G^{(k+2)} \in C_c^\infty(\mathbb{R})$. □

This elementary lemma has a weak version of Theorem 4.1.1 as a corollary. Note the representation π defined by (4.1) still makes sense for functions that aren't smooth, but only, say, continuous. Moreover, $\pi(f)\psi$ still belongs to $C_c^\infty(M)$ when $f \in C_c(\mathbb{R})$, provided $\psi \in C_c^\infty(M)$. We can even extend the representation π to compactly-supported distributions on $\mathbb{R}$, for instance $\pi(\delta) = \mathrm{id}_{C_c^\infty(M)}$. This is mainly a notational point—it is entirely possible to eliminate distributions from the discussion. However, because the corollary below is only being included for motivational reasons, it hardly seems worth it to do so.

Corollary 4.1.3. *Let X be a complete vector field on a manifold M and let π be the representation of $C_c(\mathbb{R})$ on $C_c^\infty(M)$ defined by* (4.1)*. Then, for any $\varphi \in C_c^\infty(M)$ and any integer $k \geq 0$, one can write*

$$\varphi = \pi(f_0)\psi_0 + \pi(f_1)\psi_1$$

where $f_0 \in C_c^k(\mathbb{R})$, $f_1 \in C_c^\infty(\mathbb{R})$ and $\psi_0, \psi_1 \in C_c^\infty(M)$.

Proof sketch. Applying the above lemma, we can write $\delta = f^{(k+2)} + g$ where $f \in C^k(\mathbb{R})$, $g \in C_c^\infty(\mathbb{R})$. Noting the relation $\pi(h')\psi = \pi(h)X\psi$ (an instance of the integration by parts formula), we get $\varphi = \pi(\delta)\varphi = \pi(f^{(k+2)})\varphi + \pi(g)\varphi = \pi(f)X^{k+2}\varphi + \pi(g)\varphi$ so, setting $f_0 = f$, $f_1 = g$, $\psi_0 = X^{k+2}\varphi$, $\psi_1 = \varphi$, we are finished. □

A major innovation of [23] is to achieve approximate representations of δ, analogous to that of Lemma 4.1.2, but in terms of C^∞ functions. Precisely, they prove the following.

Theorem 4.1.4 ([23], 2.5 Lemme)**.** *Given any positive sequence $c_k > 0$, there exist functions $f, g \in C_c^\infty(\mathbb{R})$ and scalars a_k with $|a_k| \le c_k$ such that*

$$\delta = g + \sum_{k=1}^{\infty} a_k f^{(k)},$$

where the convergence is in the sense of compactly-supported distributions.[1]

Note that an obvious rescaling argument implies that the functions f, g of the above theorem can be taken to have support contained in $(-\epsilon, \epsilon)$, for any $\epsilon > 0$.

In spite of its simplicity, the methods by which the above statement about 1-dimensional distributions is proved are highly nontrivial. As Casselman put it in his exposition [11], "Its proof is very intricate, a real tour de force." The crucial point turns out to be to find conditions on the growth of a sequence $\lambda = \{0 < \lambda_1 < \lambda_2 < \ldots\}$ that will guarantee that the function χ_λ defined as the restriction to $\mathbb{R}$ of the reciprocal of the entire function represented by the infinite product $\prod(1 + \frac{z^2}{\lambda_i^2})$ defines a function in the Schwartz space $\mathcal{S}(\mathbb{R})$. Functions of this type are used to prove a Schwartz function analog of Theorem 4.1.4 which, in turn, is used to prove Theorem 4.1.4.

Theorem 4.1.4 is exactly the tool needed to give the

Proof of Theorem 4.1.1. Fix $\varphi \in C_c^\infty(M)$. Note that $X\varphi, X^2\varphi, \ldots$ have supports contained in the support of φ. We can choose a sequence of positive constants c_k that decays rapidly enough that, for any sequence of scalars a_k with $|a_k| < c_k$, the sum $\sum_k a_k X^k \varphi$ is uniformly convergent to a C^∞ function (this requires a little diagonal selection trick, but is elementary). Obviously, the support of the sum is contained in that of φ. From Theorem 4.1.4, we can choose scalars a_k, $|a_k| < c_k$ and $f, g \in C_c^\infty(\mathbb{R})$ with supports contained in $(-\epsilon, \epsilon)$ such that $g + \sum_k a_k f^{(k)} \to \delta$, in the distributional sense. Thus, defining $h_n = g + \sum^n a_k f^{(k)}$, we have that $\pi(h_n)\varphi \to \varphi$ pointwise over M. On the other hand,

$$\pi(h_n)\varphi = \pi(g)\varphi + \sum^n a_k \pi(f^{(k)})\varphi = \pi(g)\varphi + \pi(f) \sum^n a_k X^k \varphi.$$

By the choice of constants c_k, the series $\sum^n a_k X^k \varphi$ converges uniformly to a function $\psi \in C_c^\infty(\mathbb{R})$ which furthermore satisfies $\mathrm{supp}(\psi) \subseteq \mathrm{supp}(\varphi)$. Thus, we arrive at the weak factorization $\varphi = \pi(f)\psi + \pi(g)\varphi$, which has all the features advertised in the statement. □

[1]Explicitly, $h_n \in C_c^\infty(\mathbb{R})$ converge to δ in the distributional sense if their supports are uniformly bounded and $\int h_n(x) f(x)\, dx \to f(0)$ for every $f \in C_c^\infty(\mathbb{R})$.

4.2 Lie groupoid preliminaries

To a large extent, the purpose of the present section is to fix our notation. Throughout, $G \rightrightarrows B$ denotes a Lie groupoid with source map s and target map t. The inversion map is denoted $\imath : G \to G$, or just $\gamma \mapsto \gamma^{-1}$. We assume the submersions s and t have k-dimensional fibers and that the unit space B is realized as a closed submanifold of G.

With some superficial differences, our conventions for Lie algebroids and Haar measures accord with those found in [37] and [43]. We follow [37] in viewing sections of the Lie algebroid of a Lie groupoid as right-invariant vector fields, but depart from [43] by working with right Haar measures in place of left Haar measures. This is done so that our measures and our vector fields will both live along the same fibers (namely the source fibers), but is of no particular importance; left and right and can always be exchanged using the inversion map.

The Lie algebroid of a Lie groupoid

Definition 4.2.1. A vector field on G is said to be **right-invariant** if it is tangent to the source fibers (i.e. is a section of the distribution $\ker(ds) \subseteq TG$) and satisfies

$$(R_\gamma)_* X = X \qquad\qquad \gamma \in G,$$

where R_γ denotes right-multiplication by γ.

The equation above is a bit imprecise because R_γ is not defined on all of G, but instead is a diffeomorphism $G_{t(\gamma)} \to G_{s(\gamma)}$. A more accurate formulation would be

$$(R_\gamma)_* \left(X\big|_{G_y} \right) = X\big|_{G_x} \qquad\qquad \gamma \in G_x \cap G^y,$$

where the restricted vector fields make sense because X was assumed tangent to the source fibers. Because we can write any $\gamma \in G$ as $R_\gamma(t(\gamma))$, it follows that a right-invariant vector field is completely determined by its restriction to B.

Definition 4.2.2 ([37], Definition 3.1)**.** As a vector bundle over the base manifold B, the **Lie algebroid** AG of G is the restriction of the source fiber tangent bundle $\ker(ds) \subseteq TG$ to B.

Every section of AG extends uniquely to a right-invariant vector field on G. Thus, right-invariant vector fields on G are in 1-1 correspondence with sections of the Lie algebroid AG. This is the Lie groupoid counterpart of the analogous pair of descriptions for the Lie algebra of a Lie group. We shall tend to abuse notation, denoting a right-invariant vector and its restriction to B (a section of AG) by the same symbol.

Example 4.2.3. If G is the pair groupoid $B \times B$, then the source fibers are just the slices $B \times \{b\}$, $b \in B$. A right-invariant vector field on G amounts to a single vector field X on B copied on each slice. Meanwhile, the Lie algebroid obviously identifies with TB, so sections of the Lie algebroid also correspond in an obvious way to vector fields on B.

There is a bundle map $AG \to TB$ called the **anchor map** defined simply by restricting the differential $dt : TG \to TB$ of the target submersion to AG. It is common practice to denote the anchor map by $\# : AG \to TB$, but we shall avoid this notation because it overlaps with common notation for fundamental vector fields in the context of a Lie group actions, and we will be considering groupoids acting on manifolds. Instead, given $X \in C^\infty(B, AG) = \{\text{right-invariant vector fields on G}\}$, we write X^B for the corresponding vector field on B. Thus,

$$X^B(b) = \tfrac{d}{dt} t(e^{tX} b)\Big|_{t=0}. \tag{4.3}$$

The vector fields X and X^B are t-related; if X is a complete, right-invariant vector field on G, then X^B is a complete vector field on B and the target submersion $t : G \to B$ is equivariant for the $\mathbb{R}$-actions on G and B.

An irritation that does not arise in the Lie group context, but does for Lie groupoids, is the potential for right-invariant vector fields to have incomplete flows. For example, in the case of the pair groupoid $G = B \times B$, the flow of a right-invariant vector field is just the flow of an arbitrary vector field on B, copied on each slice $B \times \{b\}$. The following proposition, however, at least shows that complete, right-invariant vector fields are in plentiful supply.

Proposition 4.2.4. *Let $G \rightrightarrows B$ be a Lie groupoid, X a compactly-supported section of the Lie algebroid AG. Then, X, considered as a right-invariant vector field*[2] *on G, is complete.*

Proof. Let $\phi : W \to G$ be the (maximal) flow of X. So, W is an open subset of $\mathbb{R} \times G$ containing $\{0\} \times G$. To see ϕ is complete, it suffices to show $(-\epsilon, \epsilon) \times G \subseteq W$ for some $\epsilon > 0$. Let $K \subseteq B$ be a compact set containing $\{b \in B : X(b) \neq 0\}$. By an easy compactness argument, there exists an $\epsilon > 0$ such that $(-\epsilon, \epsilon) \times K \subseteq W$. In fact, since X vanishes on $B \setminus K$, we actually have $(-\epsilon, \epsilon) \times B \subseteq W$. But then, for any $\gamma \in G$, we have, using the right-invariance, the integral curve $t \mapsto \phi_t(t(\gamma))\gamma : (-\epsilon, \epsilon) \to G$ passing through γ at $t = 0$, and so $(-\epsilon, \epsilon) \times G \subseteq W$, as was to be proven. □

[2] Possibly no longer compactly-supported; consider what happens if G is a noncompact Lie group.

Note that right-invariance of a complete vector field X on G can also be formulated as a property of its flow; each diffeomorphism e^{tX} should be right-invariant in the sense that it preserves the s-fibers and satisfies $e^{tX}(\gamma_1\gamma_2) = (e^{tX}\gamma_1)\gamma_2$ whenever $\gamma_1\gamma_2$ is defined.

Haar systems and convolution

Defining a convolution product on $C_c^\infty(G)$ for a Lie groupoid $G \rightrightarrows B$ requires a choice of (smooth) Haar system. This choice is ultimately unimportant; the algebras associated to different Haar measures are canonically isomorphic. Indeed, by replacing functions on G with sections of an appropriate density bundle, it is possible to obtain the convolution algebra in a fully intrinsic way which does not require a choice of Haar system. See the discussion following Definition 2 in [17], Section 2.5. In this chapter, we will stick with functions, however, in large part to better resemble the classical Dixmier-Malliavan theorem.

For a given a submersion $p : W \to B$, say with k-dimensional fibers, a (smooth) **fiberwise measure** is a collection $\lambda = (\lambda_b)$ of smooth measures on the fibers $p^{-1}(b)$, $b \in B$ such that, in any open subset of W small enough to be identified with $\mathbb{R}^k \times U$ for U an open set in B in such a way that p is identified with the factor projection $\mathbb{R}^k \times U \to U$, the measures take the form $\lambda_b = \rho(\cdot, b)dt_1 \cdots dt_k$, $b \in U$, where ρ is a smooth, positive-valued function on $\mathbb{R}^k \times U$ and $dt_1 \cdots dt_k$ is the standard volume measure copied on each fiber $\{b\} \times \mathbb{R}^k$. Equivalently, one can think of λ as a globally positive section of the density bundle of the p-vertical subbundle $\ker(dp) \subseteq TW$. We list a few basic properties:

1. Fiberwise measures are unique up to rescaling (corresponding to the fact that the density bundle is trivial and oriented); if λ and ν are fiberwise measures for $p : W \to B$, then there is a unique positive-valued, smooth function ρ on W such that $\nu = \rho\lambda$.

2. A fiberwise measure λ for $p : W \to B$ determines a corresponding "integration along fibers" map $p_! : C_c^\infty(W) \to C_c^\infty(B)$.

3. Fiberwise measures can be pulled back. Suppose, $p : W \to B$ is a submersion and $\mu : M \to B$ is a smooth map so that $\mathrm{pr}_2 : W \times_{p,\mu} M \to M$ is a

submersion.[3]

$$\begin{array}{ccc} W \times_{p,\mu} M & \xrightarrow{\mathrm{pr}_1} & W \\ \Big\downarrow{\scriptstyle \mathrm{pr}_2} & & \Big\downarrow{\scriptstyle p} \\ M & \xrightarrow{\quad \mu \quad} & B \end{array}$$

Then, a fiberwise measure λ for p determines a fiberwise measure λ_M for pr_2 by identifying $\mathrm{pr}_2^{-1}(m)$ with $p^{-1}(\mu(m))$ in the obvious way. If preferable, one may obtain this pullback construction in two stages, expressing it in terms of appropriate product and restriction constructions.

A (smooth, right) **Haar system** λ for a Lie groupoid $G \rightrightarrows B$ is a fiberwise measure $\lambda = (\lambda_b)_{b \in B}$ for the source submersion $s : G \to B$ that is right-invariant in the sense that, for any $\gamma \in G$, the right-multiplication R_γ is a measure isomorphism from $(G_{t(\gamma)}, \lambda_{t(\gamma)})$ to $(G_{s(\gamma)}, \lambda_{s(\gamma)})$.

Recall the Haar measure of a Lie group can be constructed by "bare hands" by choosing a positive density on the tangent space of the identity and then using translation operations to trivialize the tangent bundle and obtain a corresponding globally-positive density on the whole group, which is translation-invariant by construction. The construction of a Haar system for a Lie groupoid $G \rightrightarrows B$ proceeds along analogous lines: for any $\gamma \in G$, the right-translation $R_\gamma : G_{t(\gamma)} \to G_{s(\gamma)}$ sends $t(\gamma) \mapsto \gamma$. Thus, any globally-positive section of the density bundle of $AG \to B$ can be canonically extended to a globally-positive section of the density bundle of $\ker(ds) \subseteq TG$, the subbundle of tangent spaces to the source fibers. By construction, this extension is right-invariant in an obvious sense. Along the same lines, one sees that the (smooth) Haar measure of a Lie groupoid is unique up to multiplication (appropriately defined) by a smooth, positive-valued function on the base manifold.

Once a Haar measure λ has been fixed for G, we obtain a convolution operation $*$ with respect to which $C_c^\infty(G)$ becomes a (generally noncommutative) algebra.

$$(f * g)(\gamma_0) = \int_{G_{t(\gamma_0)}} f(\gamma^{-1}) g(\gamma\gamma_0)\ d\lambda_{t(\gamma_0)} \tag{4.4}$$

Lie groupoid actions

Definition 4.2.5. Let $G \rightrightarrows B$ be a Lie groupoid. Let M be a manifold with a given smooth map $\mu : M \to B$ called the **momentum map**. By definition, a

[3] The fiber product $W \times_{p,\mu} M = \{(\gamma, m) : p(\gamma) = \mu(m)\}$ is a closed submanifold of $W \times M$. This follows from writing it as the preimage of the diagonal $\Delta \subseteq M \times M$ under the map $p \times \mu : G \times M \to M \times M$ and noting that, because p is a submersion, the latter map is transverse to Δ.

left action of G on M is a smooth product $G \times_{s,\mu} M \ni (\gamma, m) \mapsto \gamma \cdot m \in M$, such that $\mu(\gamma \cdot m) = t(\gamma)$ for all $(\gamma, m) \in G \times_{s,\mu} M$, $\mu(m) \cdot m = m$ for all $m \in M$ and $(\gamma_1\gamma_2) \cdot m = \gamma_1 \cdot (\gamma_2 \cdot m)$ for all $\gamma_1, \gamma_2 \in G$, $m \in M$ with $s(\gamma_1) = t(\gamma_2)$, $s(\gamma_2) = \mu(m)$.

Example 4.2.6. A Lie group H acting smoothly on the left of a manifold M can be considered as a Lie groupoid action by taking $G = H$, $B = \{1_G\}$ and s, t, μ to be the collapsing maps onto the one-point space B.

Example 4.2.7. Every Lie groupoid $G \rightrightarrows B$ acts on its own arrow space G by taking the momentum map to be the target submersion $t : G \to B$ and the action map to be the groupoid multiplication.

Example 4.2.8. Every Lie groupoid $G \rightrightarrows B$ acts on its own unit space B by taking the momentum map to be the identity map on B. The fiber product $G \times_{s,\mathrm{id}} B$ canonically identifies with G via, $\gamma \mapsto (\gamma, s(\gamma))$ and, under this identification, the action map is just the target submersion $t : G \to B$.

Recall that, when a Lie group H acts on a manifold M, each Lie algebra element $X \in \mathfrak{h}$ determines a corresponding *fundamental vector field* $X^\#$ on M. Similarly, when a Lie groupoid G acts on a manifold M with momentum map μ, each Lie algebroid section $X \in C_c^\infty(B, AG)$ determines a vector field X^M on a M by the formula:

$$X^M(m) = \tfrac{d}{dt}\Big[e^{tX} \cdot m\Big]_{t=0}. \tag{4.5}$$

The vector field X^M is complete if X is complete and satisfies the following right-invariance condition: $e^{tX^M}(\gamma \cdot m) = (e^{tX}\gamma) \cdot m$ whenever $\gamma \cdot m$ is defined.

If, additionally, a Haar system λ has been specified for G, then an action of G on M determines a representation π of the convolution algebra $C_c^\infty(G) = C_c^\infty(G, \lambda)$ on $C_c^\infty(M)$ called the **integrated form of the action** according to the following formula:

$$(\pi(f)\psi)(m) = \int_{G_{\mu(m)}} f(\gamma^{-1})\psi(\gamma \cdot m)\; d\lambda_{\mu(m)}. \tag{4.6}$$

Take note that, in the special case when G is acting on itself from the left, we recover (4.4), the convolution product on $C_c^\infty(G)$.

The action of G on M can be packaged as a Lie groupoid $G \ltimes M \rightrightarrows M$ called the **transformation groupoid** of the action. This is done by taking

$G \ltimes M = G \times_{s,\mu} M$ with structure maps defined as follows:

$$\begin{aligned} \text{source } \sigma : \quad & (\gamma, m) \mapsto m \\ \text{target } \tau : \quad & (\gamma, m) \mapsto \gamma \cdot m \\ \text{inversion } \jmath : \quad & (\gamma, m) \mapsto (\gamma^{-1}, \gamma \cdot m) \\ \text{multiplication} : \quad & (\gamma_2, \gamma_1 \cdot m)(\gamma_1, m) = (\gamma_1 \gamma_2, m) \end{aligned}$$

Note that the relation $\tau = \sigma \circ \jmath$ shows that the action map is in fact a submersion. The Haar measure λ on G determines a corresponding Haar system λ_M on $G \ltimes M$, using the obvious identification of each $(G \ltimes M)_m = G_{\mu(m)} \times \{m\}$ with $G_{\mu(m)}$.

4.3 Relating $\mathbb{R}$ actions to groupoid actions

This section is devoted to proving the somewhat technical Lemma 4.3.3 below. Accordingly, most of the notations set down below can safely be forgotten once this end has been achieved. As always, $G \rightrightarrows B$ is a Lie groupoid with source s, target t and inversion map $\imath$ and Haar system λ. Let M be a G-space with momentum map $\mu : M \to B$. Let π be the corresponding representation of $C_c^\infty(G)$ on $C_c^\infty(M)$ defined by (4.6).

$$(\pi(f)\psi)(m) = \int_{G_{\mu(m)}} f(\gamma^{-1})\psi(\gamma \cdot m)\ d\lambda_{\mu(m)}$$

Let $X_1, \ldots, X_k \in C_c^\infty(B, AG)$, thought of as complete, right-invariant vector fields on G. Correspondingly, we have complete vector fields $X_1^B, \ldots, X_k^B$ on B and and $X_1^M, \ldots, X_k^M$ on M defined by (4.3) and (4.5), respectively. Let $\pi_1, \ldots, \pi_k$ be the representations of $C_c^\infty(\mathbb{R})$ on $C_c^\infty(M)$ associated to the complete vector fields $X_1^M, \ldots, X_k^M$ in accordance with (4.1).

$$(\pi_i(f)\psi)(m) = \int_{\mathbb{R}} f(-t)\psi(e^{tX_i^M} m)\ dt$$

Our basic goal is to work out the relationship between π and $\pi_1, \ldots, \pi_k$ in neighbourhoods of G that are parametrized by the map $u : \mathbb{R}^k \times B \to G$ defined by

$$u(t_1, \ldots, t_k, b) = e^{t_1 X_1} \cdots e^{t_k X_k} b.$$

We find it helpful to introduce the following operation, as an intermediary between π and the π_i. Given $f \in C_c^\infty(\mathbb{R}^k \times B)$, we define a convolution operation $\widetilde{\pi}(f)$ on

$C_c^\infty(M)$ by

$$(\widetilde{\pi}(f)\psi)(m) = \int \cdots \int_{\mathbb{R}^k} f(-t_k, \cdots, -t_1, \mu(e^{t_1 X_1^M} \cdots e^{t_k X_k^M} m)) \\ \psi(e^{t_1 X_1^M} \cdots e^{t_k X_k^M} m)\, dt_1 \cdots dt_k.$$

The reason for using precisely the above expression will hopefully be made clear shortly. For now, let us note that the following relationship between $\pi_1, \ldots, \pi_k$ and $\widetilde{\pi}$ is a trivial consequence of the definitions.

Lemma 4.3.1. *Suppose $f_1, \ldots, f_k \in C_c^\infty(\mathbb{R})$, $\chi \in C_c^\infty(B)$ and define $f \in C_c^\infty(\mathbb{R}^k \times B)$ by $f = f_k \otimes \cdots \otimes f_1 \otimes \chi$. Then,*

$$\widetilde{\pi}(f)\psi = \pi_k(f_k) \cdots \pi_1(f_1)\psi$$

holds whenever $\psi \in C_c^\infty(M)$ has $\chi \equiv 1$ on $\mu(\mathrm{supp}(\psi))$.

Next, we relate $\widetilde{\pi}$ to π.

Lemma 4.3.2. *Suppose that W is an open subset of $\mathbb{R}^k \times B$ that is mapped diffeomorphically by u onto an open subset of G. Then, there exists a linear bijection θ_W from $C_c^\infty(W) \subseteq C_c^\infty(\mathbb{R}^k \times B)$ to $C_c^\infty(u(W)) \subseteq C_c^\infty(G)$ such that*

$$\widetilde{\pi}(f)\psi = \pi(\theta_W(f))\psi$$

holds for all $f \in C_c^\infty(W)$ and $\psi \in C_c^\infty(M)$.

We remark that the bijection θ_W above is independent of the manifold M and the given action of G; the same θ_W works for all G-sets. The basic idea is to define $\theta_W(f)$ as the pushforward of f by u, followed by multiplication by a suitable Jacobian factor.

Before proceeding to the proof of Lemma 4.3.2 we find it useful to re-express the representations π and $\widetilde{\pi}$ in a somewhat more abstract form. Form the transformation groupoid $G \ltimes M \rightrightarrows M$ with source σ, target τ, inversion map $\jmath$ and induced Haar system λ_M. Given $f \in C_c^\infty(G)$ and $\psi \in C_c^\infty(M)$, we define $f \times \psi \in C_c^\infty(G \ltimes M)$ by restricting $f \otimes \psi$ to $G \ltimes M \subseteq G \times M$

$$(f \times \psi)(\gamma, m) = f(\gamma)\psi(m).$$

We can express π in terms of the above operations as follows:

$$\pi(f)\psi = \sigma_!((f \times \psi) \circ \jmath) \qquad f \in C_c^\infty(G), \psi \in C_c^\infty(M), \tag{4.7}$$

where $\sigma_! : C_c^\infty(G \ltimes M) \to C_c^\infty(M)$ is the integration along fibers map associated to λ_M.

Next, we find an analogous expression for $\widetilde{\pi}$. First, define the following maps:

$$\begin{aligned}
&u : \mathbb{R}^k \times B \to G && (t_1, \dots, t_k, b) \mapsto e^{t_1 X_1} \cdots e^{t_k X_k} b \\
&\widetilde{s} : \mathbb{R}^k \times B \to B && (t_1, \dots, t_k, b) \mapsto b \\
&\widetilde{\imath} : \mathbb{R}^k \times B \to \mathbb{R}^k \times B && (t_1, \dots, t_k, b) \mapsto (-t_k, \dots, -t_1, e^{t_1 X_1^B} \cdots e^{t_k X_k^B} b)
\end{aligned}$$

$$\begin{aligned}
&v : \mathbb{R}^k \times M \to G \ltimes M && (t_1, \dots, t_k, m) \mapsto (e^{t_1 X_1} \cdots e^{t_k X_k} \mu(m), m) \\
&\widetilde{\sigma} : \mathbb{R}^k \times M \to M && (t_1, \dots, t_k, m) \mapsto m \\
&\widetilde{\jmath} : \mathbb{R}^k \times M \to \mathbb{R}^k \times M && (t_1, \dots, t_k, m) \mapsto (-t_k, \dots, -t_1, e^{t_1 X_1^M} \cdots e^{t_k X_k^M} m).
\end{aligned}$$

We give $\widetilde{s}$ and $\widetilde{\sigma}$ the obvious fiberwise measures, copying the standard volume measure of $\mathbb{R}^k$ on each fiber, and denote the associated integration along fiber maps by $\widetilde{s}_!$ and $\widetilde{\sigma}_!$. Notice that $\widetilde{\imath}$ and $\widetilde{\jmath}$ are order-2 diffeomorphisms and that the following intertwining relations are satisfied.

$$u \circ \widetilde{\imath} = \imath \circ u \qquad \widetilde{s} = s \circ u \qquad v \circ \widetilde{\jmath} = \jmath \circ v \qquad \widetilde{\sigma} = \sigma \circ v$$

Given $f \in C_c^\infty(\mathbb{R}^k \times B)$ and $\psi \in C_c^\infty(M)$, we define $f \times \psi \in C_c^\infty(\mathbb{R}^k \times M)$ by

$$(f \times \psi)(t_1, \dots, t_k, m) = f(t_1, \dots, t_k, \mu(m))\psi(m).$$

By analogy with (4.7), we give the following expression for $\widetilde{\pi}$ in terms of the above operations.

$$\widetilde{\pi}(f)\psi = \widetilde{\sigma}_!((f \times \psi) \circ \widetilde{\jmath}) \qquad f \in C_c^\infty(\mathbb{R}^k \times B), \psi \in C_c^\infty(M) \tag{4.8}$$

With these preparations and notations, we proceed to the

Proof of Lemma 4.3.2. The relation $u\widetilde{\imath} = \imath u$ implies that $W' := \widetilde{\imath}(W)$ is also mapped diffeomorphically by u onto an open subset of G. Thus, there exists a smooth, positive-valued function $\rho_{W'}$ on W' such that the pullback of the Haar measure along u to $W' \subseteq \mathbb{R}^k \times B$ equals $\rho_{W'}\, dt_1 \cdots dt_k$. Thus, for all $f \in C_c^\infty(u(W'))$, we have

$$\widetilde{s}_!((f \circ u|_{W'})\rho_{W'}) = s_!(f).$$

Let $\Omega = (\mathrm{id} \times \mu)^{-1}(W)$, an open subset of $\mathbb{R}^k \times M$, and note that v maps Ω diffeomorphically onto an open subset of $G \ltimes M$. The relation $v\widetilde{\jmath} = \jmath v$ implies that $\Omega' := \widetilde{\jmath}(\Omega)$ is also mapped diffeomorphically by v onto an open subset of $G \ltimes M$.

The pullback of the Haar measure λ_M of $G \rtimes M$ along $v|_{\Omega'}$ is $\rho_{\Omega'}\, dt_1 \dots dt_k$, where $\rho_{\Omega'} = \rho_{W'}(\mathrm{id} \times \mu)$. Thus, for all $f \in C_c^\infty(v(\Omega'))$, we have

$$\widetilde{\sigma}_!((f \circ v|_{\Omega'})\rho_{\Omega'}) = \sigma_!(f). \tag{4.9}$$

Take θ_W to be the bijection $C_c^\infty(W) \to C_c(u(W))$ determined by

$$(\theta_W(f) \circ u|_W)(\rho_{W'} \circ \widetilde{\imath}) = f \qquad\qquad f \in C_c^\infty(W).$$

Then, for any $f \in C_c^\infty(W)$ and $\psi \in C_c^\infty(M)$, a simple calculation shows that

$$f \times \psi = ((\theta_W(f) \circ u|_W)(\rho_{W'} \circ \widetilde{\imath})) \times \psi = ((\theta_W(f) \times \psi) \circ v|_\Omega)(\rho_{\Omega'} \circ \widetilde{\jmath})$$

and so

$$(f \times \psi) \circ \widetilde{\jmath} = ((\theta_W(f) \times \psi) \circ \jmath \circ v|_{\Omega'})\rho_{\Omega'}.$$

Thus, applying (4.7), (4.8) and (4.9), we find that

$$\widetilde{\pi}(f)\psi = \widetilde{\sigma}_!((f \times \psi) \circ \widetilde{\jmath}) = \sigma_!((\theta_W(f) \times \psi) \circ \jmath) = \pi(f)\psi,$$

as was to be proven. □

Combining Lemma 4.3.2 and Lemma 4.3.1, we obtain the following technical lemma below, which has been our goal throughout the section.

Lemma 4.3.3. *Let $G \rightrightarrows B$ be a Lie groupoid with a given Haar system. Let K be a compact subset of B and let W be an open subset of G containing K. Suppose $X_1, \dots, X_k \in C_c^\infty(B, AG)$ constitute a base for AG over each point of K. Then, there exists an $\epsilon > 0$ such that, for any $f_1, \dots, f_k \in C_c^\infty(\mathbb{R})$ having $\mathrm{supp}(f_i) \subseteq (-\epsilon, \epsilon)$, there exists an $f \in C_c^\infty(G)$ with $\mathrm{supp}(f) \subseteq W$ with the property that, for any G-space M with momentum map μ and any $\psi \in C_c^\infty(M)$ with $\mu(\mathrm{supp}(\psi)) \subseteq K$, one has*

$$\pi(f)\psi = \pi_1(f_1) \cdots \pi_k(f_k)\psi$$

where $\pi_1, \dots, \pi_k$ are the representations of $C_c^\infty(\mathbb{R})$ on $C_c^\infty(M)$ associated by (4.1) to the corresponding complete vector fields $X_1^M, \cdots, X_k^M$ on M and π denotes the representation of $C_c^\infty(G)$ on $C_c^\infty(M)$ given by (4.6).

Proof. Let $u : \mathbb{R}^k \times M \to G$ be defined by $u(t_1, \dots, t_k, b) = e^{t_1 X_1} \dots e^{t_k X_k} b$. By an inverse function theorem/compactness argument, there exists an open subset U of B containing K and an $\epsilon > 0$ such that u maps $(-\epsilon, \epsilon)^k \times U$ diffeomorphically onto an open subset of G. By possibly shrinking ϵ and U, we may assume that $u((-\epsilon, \epsilon)^k \times U) \subseteq W$. Let $\chi \in C_c^\infty(B)$ satisfy $\mathrm{supp}(\chi) \subseteq U$

and $\chi \equiv 1$ on K. Let $f_1, \ldots, f_k \in C_c^\infty(R)$ have $\operatorname{supp}(f_i) \subseteq (-\epsilon, \epsilon)$ so that $f = f_k \otimes \cdots \otimes f_1 \otimes \chi \in C_c^\infty(\mathbb{R}^k \times B)$ has $\operatorname{supp}(f) \subseteq (-\epsilon, \epsilon)^k \times U$. Then, for any $\psi \in C_c^\infty(M)$ with $\operatorname{supp}(\psi) \subseteq K$, we have

$$\pi_k(f_k) \cdots \pi_1(f_1)\psi = \widetilde{\pi}(f)\psi = \pi(\theta_W(f))\psi$$

where the first equality comes from Lemma 4.3.1 and the second from Lemma 4.3.2. Since $\theta_W(f)$ is supported in $u((-\epsilon, \epsilon)^k \times U) \subseteq W$, we are finished. □

4.4 Proof of main theorem

In this section, having made the necessary preparations, we prove the main result Theorem 4.4.2, which was stated in the introduction. In fact, we can prove the following more general result.

Theorem 4.4.1. *Let $G \rightrightarrows B$ be a Lie groupoid with a given Haar system, let M be a G-space with momentum map μ and let π be the corresponding representation of $C_c^\infty(G) = C_c^\infty(G, \lambda)$ on $C_c^\infty(M)$, i.e. the integrated form of the action defined by (4.6). Then, for every $\varphi \in C_c^\infty(M)$, there exist $f_1, \ldots, f_N \in C_c^\infty(G)$ and $\psi_1, \ldots, \psi_N \in C_c^\infty(M)$ such that*

$$\varphi = \pi(f_1)\psi_1 + \ldots + \pi(f_N)\psi_N.$$

Moreover, this factorization can be taken such that, for all i, $\operatorname{supp}(\psi_i) \subseteq \operatorname{supp}(\varphi)$ and $\operatorname{supp}(f_i) \subseteq W$, where $W \subseteq G$ is a prescribed open set containing $\mu(\operatorname{supp}(\varphi))$.

Proof. Let $K = \mu(\operatorname{supp}(\varphi))$ and fix an open set W in G containing K. It is enough to prove the theorem under the additional hypothesis that the Lie algebroid AG is trivial (as a bundle) over a neighbourhood of K. Indeed, we can use a partition of unity on B to write any φ as a finite sum of functions that satisfy this extra hypothesis and have support contained in that of the original φ. If each summand can be decomposed in the desired way, then so can the sum, simply by adding up the decompositions.

Under this extra assumption, there exist sections $X_1, \ldots, X_k \in C_c^\infty(B, AG)$ that constitute a frame of AG over each point in K. Let $X_1^M, \ldots, X_k^M$ denote the corresponding complete vector fields on M and let $\pi_1, \ldots, \pi_k$ denote the corresponding representations of $C_c^\infty(\mathbb{R})$. Let $\epsilon > 0$ come from Lemma 4.3.3.

Applying Theorem 4.1.1 with $X = X_k^M$ and $\psi = \varphi$, we can write

$$\varphi = \pi_k(f_0)\psi_0 + \pi_k(f_1)\psi_1$$

where $f_0, f_1 \in C_c^\infty(\mathbb{R})$ with supports contained in $(-\epsilon, \epsilon)$ and $\psi_0, \psi_1 \in C_c^\infty(M)$ with supports contained in $\mathrm{supp}(\varphi)$. Applying Theorem 4.1.1 with $X = X_{k-1}^M$ for $\psi = \psi_0$ and $\psi = \psi_1$ then gives

$$\varphi = \pi_1(f_0)\pi_2(f_{00})\psi_{00} + \pi_1(f_0)\pi_2(f_{01})\psi_{01} + \pi_1(f_1)\pi_2(f_{10})\psi_{10} + \pi_1(f_1)\pi_2(f_{11})\psi_{11}$$

where the fs are in $C_c^\infty(\mathbb{R})$ with supports contained in $(-\epsilon, \epsilon)$ and ψs are in $C_c^\infty(M)$ with supports contained in $\mathrm{supp}(\varphi)$. Continuing in this manner, we eventually get φ as the sum of 2^k terms of the form

$$\pi_1(f_1) \cdots \pi_k(f_k)\psi$$

where $f_1, \ldots, f_k \in C_c^\infty(\mathbb{R})$ have supports contained in $(-\epsilon, \epsilon)$ and $\psi \in C_c^\infty(M)$ has support contained in $\mathrm{supp}(\varphi)$. If ϵ is sufficiently small, then Lemma 4.3.3 guarantees that each of these terms can be written as $\pi(f)\psi$ where $f \in C_c^\infty(G)$ has $\mathrm{supp}(f) \subseteq W$. □

Specializing to the case where G is acting on itself, we obtain the desired generalized Dixmier-Malliavin theorem as a corollary.

Theorem 4.4.2. *Let $G \rightrightarrows B$ be a Lie groupoid with a given Haar system. Then, for any $\varphi \in C_c^\infty(G)$, there exist $f_1, \ldots, f_N, \psi_1, \ldots, \psi_N \in C_c^\infty(G)$ such that*

$$\varphi = f_1 * \psi_1 + \ldots + f_N * \psi_N.$$

Moreover, this factorization can be taken such that, for all i, $\mathrm{supp}(\psi_i) \subseteq \mathrm{supp}(\varphi)$ and $\mathrm{supp}(f_i) \subseteq W$, where W is a prescribed open subset of G containing $t(\mathrm{supp}(\varphi))$.

4.5 Product structure of ideals of smooth functions under multiplication

In this section, we study ideals of smooth functions vanishing to given order along a submanifold when the operation is pointwise multiplication. This will lay the groundwork for the subsequent section in which the operation is convolution. The result we wish to generalize to the convolution setting is the following.

Theorem 4.5.1. *Let X be a closed submanifold of a smooth manifold M. Let $I_p \subseteq C_c^\infty(M)$ denote the ideal of functions vanishing to pth order on X. Then,*

1. $(I_\infty)^2 = I_\infty$
2. $(I_1)^p = I_p$ *for every positive integer p.*

The relation $(I_\infty)^2 = I_\infty$ actually remains true even when X is any closed subset of M, and not necessarily a submanifold. This stronger result is due to Tougeron. See [52], Proposition V.2.3 as well as [50], Section 4. Note that, although the results in these references are stated in terms of germs of functions, it is a simple matter to use partitions of unity to convert them into statements about compactly-supported functions. The second relation $(I_1)^p = I_p$ is much more elementary than the first and can be established by applying Taylor's theorem locally.

Theorem 4.5.1 is not quite sufficient for our purposes, however. We need to consider a submersion $\pi : N \to M$ (later taken to be the source or target projection of Lie groupoid) and the resulting $C_c^\infty(N)$-module structure on $C_c^\infty(M)$. It will be convenient for us to combine the cases of infinite and finite vanishing order into a single statement, but we hasten to point out that the case of infinite vanishing order is by far the more substantive one. Note also that the MathOverflow question [7] (still not fully resolved at time of writing) centers around quite similar issues.

Theorem 4.5.2. *Let $\pi : N \to M$ be a submersion. View $C_c^\infty(N)$ as a $C_c^\infty(M)$-module with product $f \cdot g = (f \circ \pi)g$, where $f \in C_c^\infty(M)$, $g \in C_c^\infty(N)$. Let X be a closed submanifold of M and set $Y := \pi^{-1}(Y)$. For $p \in \mathbb{N} \cup \{\infty\}$, write $I_p \subseteq C_c^\infty(M)$ and $J_p \subseteq C_c^\infty(M)$ for the ideals of functions that vanish to pth order on X and Y respectively. Then, the relation*

$$J_{p+q} = I_p \cdot J_q$$

is satisfied for all $p, q \in \mathbb{N} \cup \{\infty\}$, where $I_p \cdot J_q$ means the set of all sums of products $g \cdot h$, where $g \in I_p$, $h \in J_q$.

The problem is obviously local in nature; one can use a partition of unity to chop up a function f on N into smaller functions all of which vanish to the same order as f on Y. In fact, it is enough to consider the case where $N = \mathbb{R}^k \times \mathbb{R}^\ell$, $M = \mathbb{R}^k$, $X = \{0\}$ and π is the standard projection, so that $Y = \{0\} \times \mathbb{R}^\ell$. Throughout this section, $n = k+\ell$ and $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^\ell$ has coordinates $(x, y) = (x_1, \dots, x_k, y_1, \dots, y_\ell)$. We use the usual multi-index notation for partial derivatives: given $\gamma = (\alpha, \beta) \in \mathbb{N}^n = \mathbb{N}^k \times \mathbb{N}^\ell$ we write $\partial^\gamma := \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_k}}{\partial x_k^{\alpha_k}} \frac{\partial^{\beta_1}}{\partial y_1^{\beta_1}} \cdots \frac{\partial^{\beta_\ell}}{\partial y_\ell^{\beta_\ell}}$.

We treat the $p < \infty$ and $p = q = \infty$ cases of the local problem separately in the following two lemmas. The bulk of our effort will go towards establishing the second of these.

Lemma 4.5.3. *If $f \in C^\infty(\mathbb{R}^n)$ vanishes to order $p+q$ on $\{0\} \times \mathbb{R}^\ell$, where $p \in \mathbb{N}$, $q \in \mathbb{N} \cup \{\infty\}$, then one can write*

$$f(x, y) = \sum_{|\alpha|=p} x^\alpha f_\alpha(x, y),$$

where each f_α belongs to $C^\infty(\mathbb{R}^n)$ and vanishes to order q on $\{0\} \times \mathbb{R}^\ell$.

Lemma 4.5.4. *If $f \in C_c^\infty(\mathbb{R}^n)$ vanishes to order ∞ on $\{0\} \times \mathbb{R}^\ell$, then one can write*

$$f(x, y) = \rho(x)h(x, y),$$

where $\rho \in C^\infty(\mathbb{R}^k)$ has a zero of order ∞ at 0 and is strictly positive on $\mathbb{R}^k \setminus \{0\}$, and $h \in C^\infty(\mathbb{R}^n)$ vanishes to order ∞ on $\{0\} \times \mathbb{R}^\ell$.

It is a simple matter to derive Theorem 4.5.2 from these lemmas.

Proof of Theorem 4.5.2. We just need to show $J_{p+q} \subseteq I_p \cdot J_q$, the reverse containment being obvious. Suppose, therefore, that $f \in J_{p+q}$. Using a partition of unity argument and standard facts about the local structure of submersions and submanifolds, we may assume one of the following two alternatives holds:

(i) The support of f is disjoint from Y.

(ii) The support of f is contained in an open set U such that:

  (a) U is diffeomorphic to $\mathbb{R}^k \times \mathbb{R}^{\ell_1} \times \mathbb{R}^{\ell_2}$ and $\pi(U)$ is diffeomorphic to $\mathbb{R}^k \times \mathbb{R}^{\ell_1}$,

  (b) under these diffeomorphisms, $\pi : U \to \pi(U)$ identifies with the standard projection $\mathbb{R}^k \times \mathbb{R}^{\ell_1} \times \mathbb{R}^{\ell_2} \to \mathbb{R}^k \times \mathbb{R}^{\ell_1}$, and

  (c) under these diffeomorphisms, $X \cap \pi(U)$ identifies $\{0\} \times \mathbb{R}^{\ell_1}$, so that $Y \cap U$ identifies with $\{0\} \times \mathbb{R}^{\ell_1} \times \mathbb{R}^{\ell_2}$.

If (i) is satisfied, take any $g \in C_c^\infty(M)$ that is equal to 1 on $\pi(\mathrm{supp}(f))$ and equal to 0 outside of some open set not intersecting X. Then, $f = (g \circ \pi)f$, where g vanishes to order $\infty \geq p$ on X and f vanishes to order $p + q \geq q$ on Y.

If (ii) is satisfied and $p < \infty$, then, applying Lemma 4.5.3 in the given chart with $\ell = \ell_1 + \ell_2$, we may assume that $f = x^\alpha h$, where $|\alpha| = p$ and $h \in C_c^\infty(\mathbb{R}^k \times \mathbb{R}^{\ell_1} \times \mathbb{R}^{\ell_2})$ vanishes to order q on $\{0\} \times \mathbb{R}^{\ell_1} \times \mathbb{R}^{\ell_2}$. Then, $f = (g \circ \pi)h$, where $g \in C_c^\infty(\mathbb{R}^k \times \mathbb{R}^{\ell_1})$ is given by $g = x^\alpha \varphi$ for an appropriate cutoff function $\varphi \in C_c^\infty(\mathbb{R}^k \times \mathbb{R}^{\ell_1})$. Obviously, g vanishes to order p on $\{0\} \times \mathbb{R}^{\ell_1}$. The expression $f = (g \circ \pi)h$ can be made global simply by extending g and h to be identically 0 outside of $\pi(U)$ and U, respectively.

If (ii) is satisfied and $q = \infty$, we proceed in the same way using Lemma 4.5.4. First, in the given chart, write $f(x, y) = \rho(x)h(x, y)$ where $\rho \in C_c^\infty(\mathbb{R}^k)$ has a zero of infinite order at 0 and is strictly positive on $\mathbb{R}^k \setminus \{0\}$, and $h \in C_c^\infty(\mathbb{R}^k \times \mathbb{R}^{\ell_1} \times \mathbb{R}^{\ell_2})$ vanishes to order ∞ on $\{0\} \times \mathbb{R}^{\ell_1} \times \mathbb{R}^{\ell_2}$. Then, $f = (g \circ \pi)h$, where $g \in C_c^\infty(\mathbb{R}^k \times \mathbb{R}^{\ell_1})$ is given by $g = \rho \cdot \varphi$ for an appropriate cutoff function $\varphi \in C_c^\infty(\mathbb{R}^k \times \mathbb{R}^{\ell_1})$. Obviously g vanishes to order $\infty \geq p$ on $\{0\} \times \mathbb{R}^{\ell_1}$. □

It remains to prove the Lemmas 4.5.3 and 4.5.4.

Proof of Lemma 4.5.3. If $p = 0$, there is nothing to prove, so assume $p \geq 1$. It clearly suffices to prove that we can write

$$f(x, y) = x_1 f_1(x, y) + \dots x_k f_k(x, y)$$

where $f_i \in C^\infty(\mathbb{R}^n)$ vanish to order $p+q-1$ and proceed recursively. The functions $f_1, \dots, f_k$ defined by

$$\begin{aligned} f_1(x, y) &= \int_0^1 \tfrac{\partial f}{\partial x_1}(tx_1, x_2, \dots, x_k, y_1, y_2, \dots, y_\ell)\ dt \\ f_2(x, y) &= \int_0^1 \tfrac{\partial f}{\partial x_2}(0, tx_2, x_3, \dots, x_k, y_1, y_2, \dots, y_\ell)\ dt \\ &\vdots \\ f_k(x, y) &= \int_0^1 \tfrac{\partial f}{\partial x_k}(0, \dots, 0, tx_k, y_1, y_2, \dots, y_\ell)\ dt \end{aligned}$$

serve this purpose. □

The proof of Lemma 4.5.4 will rely on several further lemmas, specifically Lemmas 4.5.8, 4.5.11 and 4.5.12. Recall that a function f on $[1, \infty)$ is **rapidly decaying** if $\lim_{t\to\infty} t^m f(t) = 0$ for every nonnegative integer m. We say f is a **Schwartz function** if it is C^∞ and it and all its derivatives are rapidly decaying. Since there are many more functions of rapid decay than there are Schwartz functions, it seems plausible that there could exist a function of rapid decay that vanishes more slowly than any Schwartz function. The following lemma shows this does not occur by providing a "Schwartz envelope" for any rapidly decaying function. The original reference for this fact may be [14], Lemma 3.6, pp. 127. One can also find it in the expository note [29].

Lemma 4.5.5.

1. *If f is a bounded, rapidly decaying function on $[1, \infty)$, then there exists a positive-valued, monotone decreasing Schwartz function g on $[1, \infty)$ such that $|f| \leq g$.*

2. *If (f_k) is a sequence of rapidly decaying functions on $[1, \infty)$, then there exists a positive-valued Schwartz function g on $[1, \infty)$ such that $\lim_{t\to\infty} \frac{f_k(t)}{g(t)} = 0$ for all k.*

Proof. For the first part, assume without loss of generality that f is monotone decreasing, or else replace it by $t \mapsto \sup_{s \geq t} |f(s)|$. Let $\varphi \in C^\infty(\mathbb{R})$ be a nonnegative-valued function with support contained in $[0, 1]$ satisfying $\int_{\mathbb{R}} \varphi(t)\, dt = 1$. Define g to be the convolution $\varphi * f$, that is, $g(t) = \int_0^1 \varphi(s) f(t - s)\, ds$. We remark that there is a small issue with this definition of g near $t = 1$, but this is easily fixed by enlarging the domain of f, say by defining $f(t) = \sup_{s \geq 1} f(s)$ for $t \leq 1$. It is easy to see that g is monotone decreasing and that $f \leq g$. One can check that the convolution of two rapidly decaying functions is rapidly decaying (imposing sufficient regularity properties so that convolution makes sense), and it follows that the convolution of a rapidly decaying function with a Schwartz function is Schwartz (since the derivatives can be put on the Schwartz function).

For the second part, assume without loss of generality that each f_k is bounded and use the first part to produce, for each k, a positive-valued Schwartz function g_k such that $\lim_{t \to \infty} \frac{f_k(t)}{g_k(t)} = 0$ (if $f_k \leq g_k$ holds, but $\frac{f_k}{g_k}$ does not vanish at infinity, replace g_k with $t \mapsto t g_k(t)$). An easy diagonal selection argument guarantees the existence of constants $c_k > 0$ such that $g = \sum c_k g_k$ is a Schwartz function. Since $g > g_k$, it is clear that $\frac{f_k}{g}$ vanishes at infinity for every k. □

The next step is to convert Lemma 4.5.5 into a statement about smooth functions with an infinite order zero at 0 by performing an inversion in the variable. Much more sophisticated accounts of the connection between Schwartz functions and functions that remain smooth after being extended by zero can be found in the literature, see [2], Theorem 5.4.1. For present purposes, the simple-minded lemma below is enough.

Lemma 4.5.6. *The inversion map $t \mapsto 1/t : (0, 1] \to [1, \infty)$ puts functions f on $(0, 1]$ with $\lim_{t \to 0^+} f(t) t^{-m} = 0$ for all positive integers m into bijection with the rapidly decaying functions on $[1, \infty)$, and also puts the smooth functions f on $(0, 1]$ for which putting $f(t) = 0$ for $t \leq 0$ yields a smooth extension into bijection with the Schwartz functions on $[1, \infty)$.*

To prove Lemma 4.5.6, we need the following simple fact.

Lemma 4.5.7. *Suppose f is a smooth function on $(0, \infty)$ with $\lim_{t \to 0^+} f^{(m)}(t) = 0$ for every nonnegative integer m. Then, in setting $f(t) = 0$ for $t \leq 0$, one obtains a C^∞ extension of f to all of $\mathbb{R}$.*

Proof. An application of the mean value theorem shows the extension is differentiable with derivative 0 at the origin. The statement follows by induction. □

Proof of Lemma 4.5.6. The first correspondence is obvious. Towards the second, suppose f is a Schwartz function on $[1, \infty)$ and define g on $(0, 1]$ by $g(t) = f(1/t)$.

Then, $g'(t) = f_1(1/t)$, where $f_1(t) = -t^2 f'(t)$ is yet another Schwartz function. By induction, each derivative of g has the form $f_k(1/t)$ for some Schwartz function f_k on $[1, \infty)$. In particular, $\lim_{t\to 0^+} g^{(k)}(t) = 0$ for all k so that, by Lemma 4.5.7, setting $g(t) = 0$ for $t \le 0$ effects a smooth extension of g. The converse direction, that $f(t) = g(1/t)$ is a Schwartz function on $[1, \infty)$ when g is a smooth function with $g(t) = 0$ for $t \le 0$, proceeds similarly. □

Applying the correspondence of Lemma 4.5.6, the second part of Lemma 4.5.5 translates to the following.

Lemma 4.5.8. *Let f_k be a sequence of functions on $[0, \infty)$ that vanish to infinite order at 0, i.e. $f_k(0) = 0$ and $\lim_{t\to 0^+} f_k(t)t^{-m} = 0$ for all m. Then, there exists a C^∞ function g on $\mathbb{R}$ with $g(t) = 0$ for $t \le 0$, and $g(t) > 0$ for $t > 0$ such that $\lim_{t\to 0^+} \frac{f_k(t)}{g(t)} = 0$ for all k.*

Remark 4.5.9. A direct proof of Lemma 4.5.8 was given by George Lowther at [7] (Lemma 2). Nonetheless, as the Schwartz function formulation of this result appears to be better known, it seemed worthwhile to draw out this connection here.

Lemmas 4.5.11 and 4.5.12, stated and proved below, will rely on the following mild generalization of Lemma 4.5.7 whose proof we omit. Recall that $n = k + \ell$ and $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^\ell$ has coordinates $(x, y) = (x_1, \ldots, x_k, y_1, \ldots, y_\ell)$.

Lemma 4.5.10. *Let f be a smooth function on $\mathbb{R}^n \setminus (\{0\} \times \mathbb{R}^\ell)$ such that, for every $\gamma \in \mathbb{N}^n$, the partial derivative $\partial^\gamma f$ has limit zero at every point of $\{0\} \times \mathbb{R}^\ell$. Then, f extends to a C^∞ function on all of $\mathbb{R}^n$ vanishing to infinite order on $\{0\} \times \mathbb{R}^\ell$.*

In particular, when $\ell = 0$, the above says that a smooth function on $\mathbb{R}^k \setminus \{0\}$ with all higher partials vanishing at the origin extends smoothly to all of $\mathbb{R}^k$. This is helpful in checking the following.

Lemma 4.5.11. *Let φ be a smooth function on $\mathbb{R}$ with a zero of infinite order at 0. Then, the function f on $\mathbb{R}^k$ defined by $f(x) = \varphi(|x|)$, where $|x| = \sqrt{x_1^2 + \ldots + x_k^2}$, is a C^∞ function on $\mathbb{R}^k$ with a zero of infinite order at 0.*

Proof. Obviously f is smooth on $\mathbb{R}^k \setminus \{0\}$. On the latter domain, $\frac{\partial f}{\partial x_i}(x) = \psi(|x|)$ where $\psi(t) = \begin{cases} -\frac{\varphi(t)}{t} & t \neq 0 \\ 0 & t = 0 \end{cases}$ is another C^∞ function on $\mathbb{R}$ with a zero of infinite order at 0. By induction, f satisfies the conditions of Lemma 4.5.10 (with $\ell = 0$), whence is smooth as claimed. □

The next lemma gives sufficient conditions under which the quotient of two smooth functions on $\mathbb{R}^n$ that vanish to infinite order on $\{0\} \times \mathbb{R}^\ell$ is another such function.

Lemma 4.5.12. *Let f and g be C^∞ functions on $\mathbb{R}^n$ that vanish to infinite order on $\{0\} \times \mathbb{R}^\ell$ and assume $g > 0$ on $\mathbb{R}^n \setminus (\{0\} \times \mathbb{R}^\ell)$. If, for every $\gamma \in \mathbb{N}^n$ and $m \in \mathbb{N}$, the function $\frac{\partial^\gamma f}{g^m}$ has limit 0 at each point of $\{0\} \times \mathbb{R}^\ell$, then $\frac{f}{g}$ extends to a C^∞ function on all of $\mathbb{R}^n$ vanishing to infinite order on $\{0\} \times \mathbb{R}^\ell$.*

Proof. Let $\mathscr{F}$ denote the collection of all smooth functions on $\mathbb{R}^n \setminus (\{0\} \times \mathbb{R}^\ell)$ obtained as $C^\infty(\mathbb{R}^n)$-linear combinations of the functions $\frac{\partial^\gamma f}{g^m}$. By assumption, the functions in $\mathscr{F}$ all have limit 0 at each point $\{0\} \times \mathbb{R}^\ell$. Observe that $\mathscr{F}$ is closed under taking partial derivatives. Indeed, if γ is a multi-index, m is a positive integer, $h \in C^\infty(\mathbb{R}^n)$ and ∂ is one of the first-order partials $\frac{\partial}{\partial x_i}$ or $\frac{\partial}{\partial y_j}$, then $\partial(h\frac{\partial^\gamma f}{g^m}) = (\partial h)\frac{\partial^\gamma f}{g^m} + h\frac{\partial \circ \partial^\gamma f}{g^m} - m(\partial g)h\frac{\partial^\gamma f}{g^{m+1}}$. Thus, thinking of $\frac{f}{g}$ as a smooth function on $\mathbb{R}^n \setminus (\{0\} \times \mathbb{R}^\ell)$, we have by induction that all of its higher order partial derivatives have limit 0 at every point of $\{0\} \times \mathbb{R}^\ell$ and so, by Lemma 4.5.10, $\frac{f}{g}$ extends to a smooth function on $\mathbb{R}^n$ that vanishes to infinite order on $\{0\} \times \mathbb{R}^\ell$. $\square$

We are now in a position to give the

Proof of Lemma 4.5.4. Suppose that $f \in C_c^\infty(\mathbb{R}^n)$ vanishes to order ∞ on $\{0\} \times \mathbb{R}^\ell$. Given $\gamma \in \mathbb{N}^n$ and $m, r \in \mathbb{N}$, define a continuous function $f_{\gamma,m,r}$ on $[0, \infty)$ by

$$f_{\gamma,m,r}(t) = \sup_{\substack{x \in \mathbb{R}^k \\ |x| \leq t}} \sup_{\substack{y \in \mathbb{R}^\ell \\ |y| \leq r}} |\partial^\gamma f(x, y)|^{1/m}.$$

The assumption that f vanishes to infinite order on $\{0\} \times \mathbb{R}^\ell$ implies that each $f_{\gamma,m,r}$ vanishes to infinite order at $t = 0$, i.e. $\lim_{t \to 0^+} f_{\gamma,m,r}(t)t^{-s} = 0$ for any positive integer s. It therefore follows from Lemma 4.5.8 that there exists a C^∞ function φ on $\mathbb{R}$ vanishing to infinite order at $t = 0$ with $\varphi(t) > 0$ for $t > 0$ such that $\lim_{t \to 0^+} \frac{f_{\gamma,m,r}(t)}{\varphi(t)} = 0$ for all γ, m, r. By Lemma 4.5.11, the function ρ on $\mathbb{R}^k$ defined by $\rho(x) = \varphi(|x|)$ is a C^∞ function, positive on $\mathbb{R}^k \setminus \{0\}$ and vanishing to infinite order at 0. By design, for any $\gamma \in \mathbb{N}^n$ and $m, r \in \mathbb{N}$, one has the bound

$$\left| \frac{\partial^\gamma f(x, y)}{\rho(x)^m} \right| \leq \left(\frac{f_{\gamma,m,r}(|x|)}{\varphi(|x|)} \right)^m$$

for $x \neq 0$ and $|y| < r$, which shows that the left hand side vanishes as $x \to 0$. Thus, applying Lemma 4.5.12, one has that $h(x, y) = \frac{f(x,y)}{\rho(x)}$ extends smoothly to a function on all of $\mathbb{R}^n$ that vanishes to infinite order on $\{0\} \times \mathbb{R}^\ell$, completing the proof. $\square$

4.6 Product structure of ideals in the smooth convolution algebra of a Lie groupoid

In this final section, we apply our generalization of the Dixmier-Malliavin theorem to obtain the Lie groupoid analog of Theorem 4.5.1 by reducing it to its commutative counterpart. Throughout, G denotes a Lie groupoid over the manifold M with fixed Haar system λ and we assume that $X \subseteq M$ is an **invariant** closed submanifold in the sense that $s^{-1}(X) = t^{-1}(X)$. The restriction $G_X := s^{-1}(X) = t^{-1}(X)$ of G to X is, in its own right, a Lie groupoid $G_X \rightrightarrows X$. The Haar system λ on G can be restricted to a Haar system λ_X on G_X and doing so makes the restriction map $C_c^\infty(G) \to C_c^\infty(G_X)$ into a homomorphism of the smooth convolution algebras. The kernel of this homomorphism is the ideal $J_1 \subseteq C_c^\infty(G)$ of functions that vanish on G_X. More generally, one can consider $J_p \subseteq C_c^\infty(G)$, the functions which vanish to pth order on G_X. It is simple to confirm that each J_p is an ideal with respect to the convolution product (either by arguing directly, or by applying Proposition 4.6.3 below). The quotients $C_c^\infty(G)/J_p$ for $p > 1$ can be thought of as extensions of the convolution algebra $C_c^\infty(G_X)$, fitting as they do into exact sequences of the form

$$0 \to J_1/J_p \to C_c^\infty(G)/J_p \to C_c^\infty(G_X) \to 0.$$

Roughly speaking, the kernel J_1/J_p contains Taylor series information up to order $p-1$ in directions transverse to G_X.

The Lie groupoid algebra analog of Theorem 4.5.1 is the following:

Theorem 4.6.1. *Let $G \rightrightarrows M$ be a Lie groupoid with given Haar system. Let X be an invariant, closed submanifold of M and let $G_X := s^{-1}(X) = t^{-1}(X)$. Let $J_p \subseteq C_c^\infty(G)$ denote the ideal, with respect to convolution, of functions that vanish to order p on G_X. Then,*

1. $J_\infty * J_\infty = J_\infty$
2. $(J_1)^{*p} = J_p$ *for every positive integer p.*

As in the preceding section, for the sake of efficiency, we shall in fact prove a more general result which treats the cases of finite and infinite vanishing order on equal footing.

Theorem 4.6.2. *Let $G \rightrightarrows M$ be a Lie groupoid with a given Haar system. Let X be an invariant, closed submanifold of M and let $G_X := s^{-1}(X) = t^{-1}(X)$. For $p \in \mathbb{N} \cup \{\infty\}$, let $J_p \subseteq C_c^\infty(G)$ denote the ideal of functions that vanish to order p on G_X. Then,*

$$J_{p+q} = J_p * J_q$$

holds for all $p, q \in \mathbb{N} \cup \{\infty\}$.

It is easy to see that $J_p * J_q \subseteq J_{p+q}$ is satisfied (again, either by arguing directly or by applying Proposition 4.6.3 below). The goal is therefore to sharpen these containments to equalities. Note that, whereas in the commutative setting the $p = q = 0$ is trivial, in Theorem 4.6.2 above the $p = q = 0$ case is exactly Theorem 4.4.2, our extension of the Dixmier-Malliavin theorem . Conversely, Theorem 4.4.2, in tandem with Proposition 4.6.3 below, reduces the proof of Theorem 4.6.2 to a formal manipulation.

Recall that $C_c^\infty(G)$ is a $C_c^\infty(M)$-bimodule with respect to the products defined by

$$f \cdot \varphi = (f \circ t)\varphi \qquad\qquad \varphi \cdot f = \varphi(f \circ s)$$

and, moreover, that these products satisfy the expected associativity identities

$$f \cdot (\varphi * \psi) = (f \cdot \varphi) * \psi \qquad\qquad (\varphi * \psi) \cdot f = \varphi * (\psi \cdot f),$$

where $f \in C_c^\infty(M)$ and $\varphi, \psi \in C_c^\infty(G)$.

The following proposition shows that the ideals $I_p \subseteq C_c^\infty(M)$ of functions vanishing to pth order on X determine the ideals $J_p \subseteq C_c^\infty(G)$ of functions vanishing to pth order on G_X by way of this module structure; one may write $J_p = I_p \cdot C_c^\infty(G) = C_c^\infty(G) \cdot I_p$. It is a quick corollary of the results in the preceding section.

Proposition 4.6.3. *Let* $G \rightrightarrows M$ *be Lie groupoid with a given Haar system. Let* X *be an invariant closed submanifold of* M*. For each* $p \in \mathbb{N} \cup \{\infty\}$*, let* $I_p \subseteq C_c^\infty(M)$ *and* $J_p \subseteq C_c^\infty(G)$ *denote the collection of functions vanishing to pth order on* X *and* G_X *respectively. Then,*

$$J_{p+q} = I_p \cdot J_q = J_q \cdot I_p$$

holds for all $p, q \in \mathbb{N} \cup \{\infty\}$.

Proof. Apply Theorem 4.5.2 with $N = G$ and $\pi = s$, respectively $\pi = t$. □

Theorem 4.6.2 is now a trivial consequence of Theorem 4.4.2 and Proposition 4.6.3.

Proof of Theorem 4.6.2. We have

$$J_p * J_q = I_p \cdot C_c^\infty(G) * C_c^\infty(G) \cdot I_q = I_p \cdot C_c^\infty(G) \cdot I_q = J_p \cdot I_q = J_{p+q},$$

where the second equality holds by Theorem 4.4.2 and the rest hold by Proposition 4.6.3. □

Chapter 5

Singular Foliations Tangent to a Given Hypersurface

Except for minor changes, this chapter reproduces the content of [26]. We are interested a specific family of almost regular foliations which we call *transverse order k* foliations (Definition 5.3.1). Roughly speaking, a transverse order k foliation is a foliation $(M, \mathcal{F})$ of a connected manifold M that has exactly one codimension-1 singular leaf L around which the transverse structure of the foliation is modeled on the foliation $\mathcal{F}^k_{\mathbb{R}}$ analysed in Chapter 3. Such a foliation has either two or three leaves in total, according as $M \setminus L$ is connected or not.

Unlike what happens in the context of regular foliations, a loop in L does not determine a holonomy transformation in the usual sense of a diffeomorphism germ on a given transversal. We show, however, that one does have a well-defined holonomy mapping at the level of $(k-1)$-jets. In this way, we are able to assign to each transverse order k foliation $\mathcal{F}$ a *holonomy invariant* (Definition 5.5.6):

$$h(\mathcal{F}) \in [\pi_1(L), J^{k-1}].$$

Here, J^{k-1} denotes the group of $(k-1)$-jets of diffeomorphisms of $\mathbb{R}$ fixing the origin and $[\pi_1(L), J^{k-1}]$ denotes the quotient set of $\mathrm{Hom}(\pi_1(L), J^{k-1})$ by the conjugation action of J^{k-1}. We prove that $h(\mathcal{F})$ is a complete invariant for the structure of $\mathcal{F}$ around its singular leaf (Theorem 5.9.2).

Featuring prominently in our approach are certain natural principal bundles $P^r(\mathcal{F}) \to L$, $r = 1, \ldots, k$, constructed in Section 5.5. The holonomy invariant $h(\mathcal{F})$ comes to us as the monodromy representation of $P^{k-1}(\mathcal{F})$. We also use the *gauge groupoid* of $P^k(\mathcal{F})$ to characterize the holonomy groupoid of $G(\mathcal{F})$ by first obtaining (Theorem 5.6.6) a description of the form

$$G_{\mathrm{full}}(\mathcal{F}) = (M \setminus L)^2 \cup \mathrm{Gauge}(P^k(\mathcal{F})).$$

for the *full holonomy groupoid* (Section 2.3.2). The usual holonomy groupoid $G(\mathcal{F})$ may then be extracted as the s-connected component of the full holonomy groupoid. In this way (Theorem 5.7.5), we are able to deduce that the holonomy group of $\mathcal{F}$ at any point of the singular leaf L is isomorphic to a one-dimensional, solvable Lie group $\Gamma_{\mathbb{R}} \subseteq J^k$. This group $\Gamma_{\mathbb{R}}$ is easily read off from our holonomy invariant $h(\mathcal{F})$.

Finally, using known facts about groupoid C*-algebras, one can conclude that the foliation C*-algebra $C^*(\mathcal{F})$ sits in an extension of one of the two forms

$$0 \longrightarrow \mathbb{K} \longrightarrow C^*(\mathcal{F}) \longrightarrow C^*(\Gamma_{\mathbb{R}}) \otimes \mathbb{K} \longrightarrow 0$$

$$0 \longrightarrow \mathbb{K} \oplus \mathbb{K} \longrightarrow C^*(\mathcal{F}) \longrightarrow C^*(\Gamma_{\mathbb{R}}) \otimes \mathbb{K} \longrightarrow 0$$

according as $\mathcal{F}$ has one open leaf or two.

We now explain the organization of the present chapter. In Section 5.1, we discuss the group J^k of jets diffeomorphisms of $\mathbb{R}$ fixing 0, and a nonstandard topology we will be using. In Section 5.2, we calculate the full holonomy groupoid of $J^k_{\mathbb{R}}$ and discuss its point set topological properties. In Section 5.3, we define transverse order k foliations. In Section 5.4, we prove that the foliation-preserving diffeomorphisms and submersions between transverse order k foliations have certain rigidity properties. In Section 5.5, use the aforementioned rigidity properties to construct the principal bundles $P^r(\mathcal{F}) \to L$ discussed above. We also assign a holonomy invariant $h(\mathcal{F})$ to every transverse order k foliation. In Section 5.6, we show that the restriction of the full holonomy groupoid of a transverse order k foliation to the singular leaf of a transverse order k foliation $\mathcal{F}$ is isomorphic to the gauge groupoid of $P^k(\mathcal{F})$. In Section 5.7, we explain how to extract the usual holonomy groupoid of a transverse order k foliation from its full holonomy groupoid and derive some results about the foliation C*-algebra. In Section 5.8, we consider transverse order k foliations on line bundles whose singular leaf is the zero section. We show that such foliations are in one-to-one correspondence with flat connections of on the natural J^{k-1}-bundle made up of $(k-1)$-jets from the fibers of the line bundle to $\mathbb{R}$. In Section 5.9, we show that the holonomy invariant $h(\mathcal{F})$ is a complete invariant of $\mathcal{F}$.

5.1 Groups of jets on the line

In this section we introduce certain groups of jets of diffeomorphisms of the real line which will play an important role. Let us briefly recall the concept of the jet of a smooth mapping. For more information, one may refer the exposition in [34], Section 12.

Definition 5.1.1. Let M be a smooth manifold and f a smooth real-valued function on M. Given $x_0 \in M$ and r a positive integer, we say that f **vanishes to order r** at x_0 if $f \in (I_{x_0})^r$, where $I_{x_0} \subseteq C^\infty(M, \mathbb{R})$ denotes the ideal of functions which vanish at x_0.

Lemma 5.1.2. *Let M, N be smooth manifolds, let r be a nonnegative integer and let $x_0 \in M$, $y_0 \in N$. Choose a diffeomorphism $\phi = (\phi_1, \ldots, \phi_n)$ from an open neighbourhood $V \subseteq N$ of y_0 onto an open set in $\mathbb{R}^n$ and define an equivalence relation $\sim_{r,x_0}$ on the set of smooth functions $M \to N$ that send $x_0 \mapsto y_0$ by $f \sim_{r,x_0} g$ if and only if $\phi_i \circ f - \phi_i \circ g$ vanishes to order $r+1$ at x_0 for $i = 1, \ldots, n$. Then, the equivalence relation $\sim_{r,x_0}$ does not depend on the choice of chart ϕ.* $\square$

Definition 5.1.3. Let M, N be smooth manifolds and $f : M \to N$ a smooth function. Given $x_0 \in M$ and r a nonnegative integer, the **r-jet** of f at x_0 is the equivalence class $j^r_{x_0}(f)$ of f under the relation $\sim_{r,x_0}$ of above lemma.

If $M = \mathbb{R}^n$ and $N = \mathbb{R}$ in the above definition, then $f \sim_{r,x_0} g$ if and only if the rth order Taylor polynomials of f and g at x_0 are the same. For this reason, it makes sense to identify $j^r_{x_0}(f)$ with the rth order Taylor polynomial of f. We shall make such identifications without comment.

There is a well-defined composition operation on jets. If $f, f' : M_1 \to M_2$ have the same r-jet at $x_0 \in M$ and $g, g' : M_2 \to M_3$ have the same r-jet at $y_0 := f(x_0) = f'(x_0) \in M_2$, then $g \circ f$ and $g' \circ f'$ have the same r-jet at x_0. It therefore makes sense to define $j^r_{y_0}(g) \circ j^r_{x_0}(f) := j^r_{x_0}(g \circ f)$. When working in coordinates, this operation on jets is the usual "compose and truncate" on rth order Taylor polynomials.

The following groups of jets will play an important role for us.

Definition 5.1.4. For r a positive integer, J^r denotes the group of r-jets at 0 of diffeomorphisms of $\mathbb{R}$ fixing 0.

The group J^r has a canonical r-dimensional Lie group structure coming from its identification with the group of real polynomials of the form $a_1 y + \ldots + a_r y^r$ where $a_1 \neq 0$ with respect to the "compose and truncate" operation.

For each $k \geq 2$, there is canonical exact sequence of Lie groups

$$0 \longrightarrow \mathbb{R} \longrightarrow J^k \longrightarrow J^{k-1} \longrightarrow 0$$

where the projection map $J^k \to J^{k-1}$ is given by deleting the order k term and the inclusion map $\mathbb{R} \to J^k$ is defined by $t \mapsto y + ty^k$.

We will frequently want to equip these jet groups with a nonstandard topology.

Definition 5.1.5. For $r \geq 1$, we write J_d^r for J^r considered as an (uncountable) discrete group. For $k \geq 2$ we write $J_{\mathbb{R}}^k$ for J^k equipped with the one-dimensional Lie group structure arising from its partition into the cosets of $\mathbb{R}$ in J^k (of which there are uncountably many).

We then have an extension of (non-second-countable) Lie groups

$$0 \longrightarrow \mathbb{R} \longrightarrow J_{\mathbb{R}}^k \longrightarrow J_d^{k-1} \longrightarrow 0,$$

where $\mathbb{R}$ has its standard smooth structure.

The following proposition is intended to show that, from the perspective of abstract group theory, these jet groups are quite tame.

Proposition 5.1.6. *For every $k \geq 2$, the group J^k is solvable. Indeed, we may express J^k as the semidirect product of a nilpotent group by an abelian group.*

Proof. For every $k \geq 2$, there is an evident exact sequence:

$$0 \to J^{2,k} \to J^k \to J^1 \to 0$$

where $J^{2,k} := \{y + a_2 y^2 + \ldots + a_k y^k : a_i \in \mathbb{R}\}$. This sequence splits on the right via $ay \mapsto ay$, so J^k is the semidirect product of $J^{2,k}$ by the abelian group J^1. For $k = 2$, we have $J^{2,k} \cong \mathbb{R}$. For $k \geq 3$, we have a central extension:

$$0 \to \mathbb{R} \to J^{2,k} \to J^{2,k-1} \to 0.$$

By induction, $J^{2,k}$ is nilpotent for all $k \geq 2$. □

The groups J^k for $k \geq 2$ are not themselves nilpotent. In fact, the center of J^k is trivial. By a curious coincidence, J^2 is isomorphic to the "ax+b group" of affine bijections of the real line. An example of an isomorphism is $ay+b \mapsto a^{-1}y + ba^{-2}y^2$.

Remark 5.1.7. It will later be relevant to take a countable subgroup $\Gamma \subseteq J_d^{k-1}$ and consider its preimage $\Gamma_{\mathbb{R}} \subseteq J_{\mathbb{R}}^k$ under the projection $J_{\mathbb{R}}^k \to J_d^{k-1}$. Because subgroups of solvable groups are solvable, the above proposition gives that the one-dimensional group $\Gamma_{\mathbb{R}}$ is second-countable and amenable. One then has that extensions of $C^*(\Gamma_{\mathbb{R}})$ can be described in terms of K-theoretic data (by the universal coefficient theorem) and that the K-theory of $C^*(\Gamma_{\mathbb{R}})$ can be geometrically computed (by the Baum-Connes conjecture).

5.2 The full holonomy groupoid of $\mathcal{F}_{\mathbb{R}}^k$

In this section, we describe the full holonomy groupoid (Definition 2.3.10) $G_{\text{full}}(\mathcal{F}_{\mathbb{R}}^k)$ of $\mathcal{F}_{\mathbb{R}}^k$ and discuss its point-set topological properties. When $k = 1$, it is simple to see that $G_{\text{full}}(\mathcal{F}_{\mathbb{R}}^1) \cong \mathrm{GL}(1, \mathbb{R}) \ltimes \mathbb{R}$, so we concentrate our discussion on the case $k \geq 2$ where the groupoid is larger. The minimal holonomy groupoid of $\mathcal{F}_{\mathbb{R}}^k$ was already discussed in Chapter 3; there is a unique Lie groupoid isomorphism

$$G(\mathcal{F}_{\mathbb{R}}^k) \cong \mathbb{R} \ltimes_\phi \mathbb{R},$$

where ϕ denotes the flow of any complete vector field X generating $\mathcal{F}_{\mathbb{R}}^k$. It will be more convenient, however, to replace $\mathbb{R} \ltimes_\phi \mathbb{R}$ by an isomorphic bisubmersion with polynomial structure maps.

Definition 5.2.1. Let $\Omega = \{(t, y) \in \mathbb{R}^2 : 1 + ty^{k-1} > 0\}$ and define $\sigma, \tau : \Omega \to \mathbb{R}$ by $\sigma(t, y) = y$ and $\tau(t, y) = y + ty^k$.

Note the inequality $1 + ty^{k-1} > 0$ is precisely the condition guaranteeing $\sigma(t, y)$ and $\tau(t, y) = y + ty^k$ have the same sign.

Lemma 5.2.2. *The triple $\Omega = (\Omega, \tau, \sigma)$ is an $\mathcal{F}_{\mathbb{R}}^k$-bisubmersion adapted to the path holonomy atlas and the natural map $Q_\Omega : \Omega \to G(\mathcal{F}_{\mathbb{R}}^k)$ is a diffeomorphism.*

Proof. Identify $G(\mathcal{F}_{\mathbb{R}}^k)$ with $\mathbb{R} \ltimes_\phi \mathbb{R}$, where ϕ is the flow of some complete vector field X on $\mathbb{R}$ generating $\mathcal{F}_{\mathbb{R}}^k$. We have $X = f(y)y^k \frac{d}{dy}$ where f is smooth and nonvanishing. From this, it follows that we can write

$$\phi_t(y) = y + h(t, y)y^k \qquad t, y \in \mathbb{R}, \tag{5.1}$$

where $h : \mathbb{R}^2 \to \mathbb{R}$ is smooth. We claim that

$$(t, y) \mapsto (h(t, y), y) : \mathbb{R} \ltimes_\phi \mathbb{R} \to \Omega$$

defines an isomorphism of bisubmersions. When $y \neq 0$, note that $\{t \in \mathbb{R} : 1 + ty^{k-1} > 0\}$ equals the set of all $t \in \mathbb{R}$ such that $y + ty^k$ has the same sign as y. Since ϕ is free and transitive on the positive and negative half lines, it follows that $(t, y) \mapsto (h(t, y), y)$ is a bijection from $\mathbb{R}^2 \setminus (\mathbb{R} \times \{0\})$ to $\Omega \setminus (\mathbb{R} \times \{0\})$.

Differentiating (5.1) with respect to t and rearranging gives

$$\frac{\partial h}{\partial t}(t, y) = \left(\frac{\phi_t(y)}{y}\right)^k > 0 \qquad t, y \in \mathbb{R}, y \neq 0.$$

from which one may deduce that $(t, y) \mapsto (h(t, y), y)$ is a diffeomorphism from $\mathbb{R}^2 \setminus (\mathbb{R} \times \{0\}) \to \Omega \setminus (\mathbb{R} \times \{0\})$.

Finally, for the sake of simplicity, choose X to coincide with $y^k \frac{d}{dy}$ on a neighbourhood of 0. Then, $\phi_t(y) = \frac{y}{\sqrt[k-1]{1-(k-1)ty^{k-1}}}$ holds on a neighbourhood of $\mathbb{R} \times \{0\}$. In particular, the Taylor series of ϕ_t begins $\phi_t(y) \sim y + ty^k + \frac{1}{2}t^2 y^{2k-1} + \ldots$ and we have $h(t,0) = t$ for all $t \in \mathbb{R}$. The rest follows. $\square$

By Theorem 2.3.18, the full holonomy groupoid of a foliation $(M, \mathcal{F})$ is isomorphic to the groupoid $\mathcal{G}(\mathcal{F})$ of germs of local $\mathcal{F}$-automorphisms modulo the normal subgroupoid $\mathcal{N}(\mathcal{F})$ of germs of null automorphisms (Definition 2.3.17). Therefore, to determine $G_{\text{full}}(\mathcal{F}^k_{\mathbb{R}})$ as an abstract groupoid, we need only determine the automorphisms and null automorphisms of $\mathcal{F}^k_{\mathbb{R}}$.

Lemma 5.2.3. *Let θ be a diffeomorphism of $\mathbb{R}$ defined on a neighbourhood of 0.*

1. *θ preserves $\mathcal{F}^k_{\mathbb{R}}$ if and only if $\theta(0) = 0$.*
2. *θ is null at 0 if and only if $j^k_0(\theta) = y$.*

Proof. If $f \in C^\infty_c(\mathbb{R})$, we have $\theta_*(f \frac{d}{dy}) = (f \circ \theta^{-1})\theta_*(\frac{d}{dy})$. Since $f \mapsto \theta^{-1}$ preserves the ideal of functions that vanish to order k at 0 and $\theta_*(\frac{d}{dy})$ is a positive, smooth function-multiple of $\frac{d}{dy}$, assertion (1) follows. If $j^k_0(\theta) = y$, we may write write $\theta(y) = y + f(y)y^k$ where f is a smooth function with $f(0) = 0$. Then, for $\epsilon > 0$ appropriately small, $\{(f(y), y) : y \in (-\epsilon, \epsilon)\}$ is a bisection of Ω containing $(0,0)$ which induces θ. Conversely, any bisection passing through $(0,0)$ is locally of the form $\{(f(y), y) : y \in (-\epsilon, \epsilon)\}$ for some smooth f, and (2) follows. $\square$

Proposition 5.2.4. *There is a unique isomorphism of abstract groupoids*

$$G_{\text{full}}(\mathcal{F}^k_{\mathbb{R}}) \to (\mathbb{R} \setminus \{0\})^2 \cup J^k$$

such that, if (W, t, s) is any $\mathcal{F}^k_{\mathbb{R}}$-bisubmersion and $w \in W$ has $s(w) = 0$, then $Q_W(w) \mapsto j^k_0(\theta)$ where θ is any diffeomorphism of $\mathbb{R}$ carried by W at w.

Proof. Clearly the restriction of $G_{\text{full}}(\mathcal{F}^k_{\mathbb{R}})$ to $\mathbb{R} \setminus \{0\}$ is uniquely isomorphic to the pair groupoid $(\mathbb{R} \setminus \{0\})^2$. By Lemma 5.2.3, the group $\mathcal{G}(\mathcal{F}^k_{\mathbb{R}})_0$ of germs at 0 of local $\mathcal{F}^k_{\mathbb{R}}$ automorphisms is the group of germs of diffeomorphisms θ of $\mathbb{R}$ with $\theta(0) = 0$ and the normal subgroup $\mathcal{N}(\mathcal{F}^k_{\mathbb{R}})_0$ of germs of null automorphisms at 0 is the group of germs of diffeomorphisms θ of $\mathbb{R}$ whose k-jet at 0 is y. Thus, by Theorem 2.3.18, $G_{\text{full}}(\mathcal{F}^k_{\mathbb{R}})_0 \cong \mathcal{G}(\mathcal{F}^k_{\mathbb{R}})_0 / \mathcal{N}(\mathcal{F}^k_{\mathbb{R}})_0 = J^k$. $\square$

Remark 5.2.5. The orbits of the minimal holonomy groupoid $G(\mathcal{F})$ of a foliation $(M, \mathcal{F})$ are exactly the leaves of $\mathcal{F}$. Similarly, if $\mathcal{W}$ is any holonomy atlas for $\mathcal{F}$, then the orbits of the holonomy groupoid $G(\mathcal{W})$ are unions of leaves related by $\mathcal{W}$. This explains why $G(\mathcal{F}^k_{\mathbb{R}})$ is a blow up of $(\mathbb{R}_-)^2 \cup (\mathbb{R}_+)^2 \cup \{(0,0)\}$ and $G_{\text{full}}(\mathcal{F}^k_{\mathbb{R}})$ is a blowup of $(\mathbb{R} \setminus \{0\})^2 \cup \{(0,0)\}$.

It is quite easy to see that the above identification of $G_{\text{full}}(\mathcal{F}^k_{\mathbb{R}})_{\mathbb{R}\setminus\{0\}}$ with $(\mathbb{R}\setminus\{0\})^2$ is also a diffeomorphism. However, we shall see that the isotropy group $G_{\text{full}}(\mathcal{F}^k_{\mathbb{R}})_0$ is in fact diffeomorphic to the one-dimensional Lie group $J^k_{\mathbb{R}}$, rather than the k-dimensional Lie group J^k. In order to illuminate the topological structure of $G_{\text{full}}(\mathcal{F}^k_{\mathbb{R}})$ near its isotropy group $G_{\text{full}}(\mathcal{F}^k_{\mathbb{R}})_0$, we need to introduce an explicit full holonomy atlas for $\mathcal{F}^k_{\mathbb{R}}$. We use freely the results and terminology of Section 2.3.2.

Definition 5.2.6. For each $\theta \in \text{Diff}_0(\mathbb{R})$, put $\Omega_\theta := (\Omega, \theta \circ \tau, \sigma)$.

By construction, Ω_θ is an $\mathcal{F}^k_{\mathbb{R}}$-bisubmersion carrying θ at the point $(0,0)$.

Proposition 5.2.7. *1. $\{\Omega_\theta : \theta \in \text{Diff}_0(\mathbb{R})\}$ is a full holonomy atlas for $\mathcal{F}^k_{\mathbb{R}}$. That is, any $\mathcal{F}^k_{\mathbb{R}}$-bisubmersion is adapted to this holonomy atlas (see Section 2.3.2).*

2. For each $\theta \in \text{Diff}_0(\mathbb{R})$, the canonical map $Q_{\Omega_\theta} : \Omega_\theta \to G_{\text{full}}(\mathcal{F}^k_{\mathbb{R}})$ is a diffeomorphism onto its image.

Proof. If θ is orientation-preserving, then Ω_θ restricted to $\mathbb{R}\setminus\{0\}$ is isomorphic to $(\mathbb{R}_-)^2 \cup (\mathbb{R}_+)^2$. If θ is orientation-reversing, then θ restricted to $\mathbb{R}\setminus\{0\}$ is isomorphic to $(\mathbb{R}_- \times \mathbb{R}_+) \cup (\mathbb{R}_+ \times \mathbb{R}_-)$. This implies that, for $y \in \mathbb{R}\setminus\{0\}$, every $\mathcal{F}^k_{\mathbb{R}}$-automorphism germ at y is carried by some Ω_θ. By construction, if $\theta \in \text{Diff}_0(\mathbb{R})$, then θ is carried by Ω_θ at $(0,0)$. We have shown that every local automorphism of $\mathcal{F}^k_{\mathbb{R}}$ is carried by some Ω_θ so, by Proposition 2.3.15, the Ω_θ form a full holonomy atlas.

For (2), note that the dimension of Ω_θ equals the dimension of $\mathbb{R}$ plus the fiber dimension of $\mathcal{F}^k_{\mathbb{R}}$, so the map of Ω_θ to $G_{\text{full}}(\mathcal{F}^k_{\mathbb{R}})$ is a local diffeomorphism (see Proposition 3.11(b) in [4]). Since the set of points in Ω_θ with trivial isotropy is dense, every local morphism $\Omega_\theta \to \Omega_\theta$ is the identity and the map $\Omega_\theta \to G_{\text{full}}(\mathcal{F}^k_{\mathbb{R}})$ is injective. □

One rudimentary property of the topology of $G_{\text{full}}(\mathcal{F}^k_{\mathbb{R}})$ is that it has two connected components. The basic idea of the proof, which we omit, already appeared in the first paragraph of the proof of Proposition 5.2.7.

Proposition 5.2.8. *The holonomy groupoid $G_{\text{full}}(\mathcal{F}^k_{\mathbb{R}})$ has two connected components*[1]*. Under the isomorphism of Proposition 5.2.4, the components map to $J^k_+ \cup (\mathbb{R}_-)^2 \cup (\mathbb{R}_+)^2$ and $J^k_- \cup (\mathbb{R}_- \times \mathbb{R}_+) \cup (\mathbb{R}_+ \times \mathbb{R}_-)$, where J^k_+ and J^k_- denote the k-jets of the orientation-preserving and orientation-reversing diffeomorphisms, respectively.*

[1] Or, equivalently, path components, since $G_{\text{full}}(\mathcal{F}^k_{\mathbb{R}})$ is locally Euclidean.

Next we describe the smooth structure of the isotropy group $G_{\text{full}}(\mathcal{F}^k_{\mathbb{R}})$.

Proposition 5.2.9. *Giving J^k its one-dimensional Lie group structure $J^k_{\mathbb{R}}$ (Definition 5.1.5) makes the group isomorphism $G(\mathcal{F}^k_{\mathbb{R}})_0 \to J^k_{\mathbb{R}}$ provided by Proposition 5.2.4 into a Lie group isomorphism.*

Proof. If $\theta \in \mathrm{Diff}_0(\mathbb{R})$ and $t \in \mathbb{R}$, then $(\{t\} \times \mathbb{R}) \cap \Omega_\theta$ is a bisection of Ω_θ carrying the local diffeomorphism $y \mapsto \theta(y + ty^k)$ at the point $(t, 0)$. The composition

$$\Omega_\theta \overset{\Omega_\theta}{\to} G_{\text{full}}(\mathcal{F}^k_{\mathbb{R}}) \to (\mathbb{R} \setminus \{0\})^2 \cup J^k$$

therefore sends

$$\Omega_\theta \ni (t, 0) \mapsto j^k_0(\theta) \circ (y + ty^k) \in J^k$$

so that $\Omega_\theta \cap (\mathbb{R} \times \{0\})$ is carried diffeomorphically onto the coset of $\mathbb{R} \subseteq J^k_{\mathbb{R}}$ containing $j^k_0(\theta)$. □

Proposition 5.2.9 shows that $G_{\text{full}}(\mathcal{F}^k_{\mathbb{R}})$ is necessarily a somewhat strange space; it is a blowup of the singular equivalence relation $(\mathbb{R} \setminus \{0\})^2 \cup \{(0,0)\} \subseteq \mathbb{R}^2$ for which the singular point at the origin is replaced by continuum-may copies of the the real line. This already shows it is not a manifold (even under in our relaxed sense of the word, see Subsection 2.1.1) because it has only two components (Proposition 5.2.8), but is not second countable.

The remainder of this section is devoted to investigating the separation properties of $G_{\text{full}}(\mathcal{F}^k_{\mathbb{R}})$. The basic observation is as follows: if $\theta_1, \theta_2 \in \mathrm{Diff}_0(\mathbb{R})$ have different Taylor series at 0, then the intersection of their graphs with some punctured neighbourhood of the origin are disjoint and, therefore, can be separated open subsets of the punctured plane. The following lemma shows that, by choosing neighbourhoods carefully, we can separate the different cosets of $\mathbb{R}$ in $J^k_{\mathbb{R}}$ from each other by open sets in $G_{\text{full}}(\mathcal{F}^k_{\mathbb{R}})$.

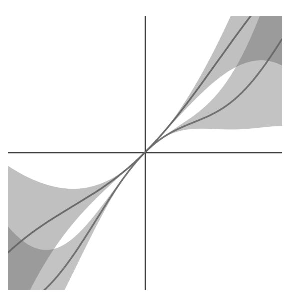


Figure 5.1: The geometry behind Lemma 5.2.10.

Lemma 5.2.10. *For any $\theta \in \mathrm{Diff}_0(\mathbb{R})$, there is a unique smooth function f_θ on $\mathbb{R}^2$ such that $\theta(y+ty^k) = \theta(y) + f_\theta(t,y)y^k$ for all $(t,y) \in \mathbb{R}^2$. Define*

$$U_\theta := \{(t,y) \in \Omega_\theta : |y|^{1/2}|f_\theta(t,y)| < 1\},$$

so that U_θ is an open subset of Ω_θ containing $\mathbb{R}\times\{0\}$. Then, given $\theta_1, \theta_2 \in \mathrm{Diff}_0(\mathbb{R})$ with different $(k-1)$-jets at 0, there exists $\epsilon > 0$ such that U_{θ_1} and $U_{\theta_2} \cap (\mathbb{R}\times(-\epsilon,\epsilon))$, viewed as $\mathcal{F}^k_{\mathbb{R}}$-bisubmersions, have disjoint images in $G_{\mathrm{full}}(\mathcal{F}^k_{\mathbb{R}})$.

Proof. We identify $G_{\mathrm{full}}(\mathcal{F}^k_{\mathbb{R}})$ with $(\mathbb{R}\setminus\{0\})^2 \cup J^k_{\mathbb{R}}$ without comment. It is straightforward to deduce the existence of f_θ from Taylor's theorem.

Suppose $\theta_1, \theta_2 \in \mathrm{Diff}_0(\mathbb{R})$ and $j_0^{k-1}(\theta_1) \neq j_0^{k-1}(\theta_2)$. The images of $U_{\theta_1} \cap (\mathbb{R}\times\{0\})$ and $U_{\theta_2} \cap (\mathbb{R}\times\{0\})$ are the (disjoint) cosets of $\mathbb{R}$ in $J^k_{\mathbb{R}}$ which contain $j_0^k(\theta_1)$ and $j_0^k(\theta_2)$, respectively. We need therefore only need to check that there is some $\epsilon > 0$ such that, whenever $(t_1,y) \in U_{\theta_1}$, $(t_2,y) \in U_{\theta_2}$ and $0 < |y| < \epsilon$, we have $\theta_1(y+t_1y^k) \neq \theta_2(y+t_2y^k)$. For any $(t_1,y) \in U_{\theta_1}$ and $(t_2,y) \in U_{\theta_2}$, we have

$$\begin{aligned}|\theta_1(y+t_1y^k) - \theta_2(y+t_2y^k)| &\geq |\theta_1(y) - \theta_2(y)| - |y^k f_{\theta_1}(t_1,y)| - |y^k f_{\theta_2}(t_2,y)\\ &\geq |\theta_1(y) - \theta_2(y)| - 2|y|^{k-\frac{1}{2}}\end{aligned}$$

Because the $(k-1)$-jet of $\theta_1 - \theta_2$ at 0 is nonzero, $2|y|^{k-\frac{1}{2}}$ vanishes more quickly than $|\theta_1(y) - \theta_2(y)|$ at $y = 0$. It follows that there exists an $\epsilon > 0$ such that, for $0 < |y| < \epsilon$, the right hand side of the above inequality is strictly positive. □

Corollary 5.2.11. *Let A be any coset of $\mathbb{R}$ in $J^k_{\mathbb{R}}$ and put $B = J^k_{\mathbb{R}} \setminus A$. Then, there are disjoint open sets $U, V \subseteq G_{\mathrm{full}}(\mathcal{F}^k_{\mathbb{R}}) \cong (\mathbb{R}\setminus\{0\})^2 \cup J^k_{\mathbb{R}}$ such that $A \subseteq U$ and $B \subseteq V$ (see Figure 5.2).*

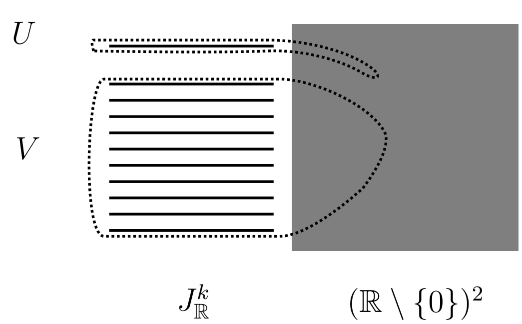


Figure 5.2: Separating one component of $J^k_{\mathbb{R}}$ from the rest inside $G_{\mathrm{full}}(\mathcal{F}^k_{\mathbb{R}})$.

Proof. Fix $\theta_0 \in \mathrm{Diff}_0(\mathbb{R})$. In the notation of Lemma 5.2.10 above, for each $\theta \in \mathrm{Diff}_0(\mathbb{R})$ with $j_0^{k-1}(\theta) \neq j_0^{k-1}(\theta_0)$, there exists an ϵ_θ such that U_{θ_0} and $U_\theta \cap (\mathbb{R} \times (-\epsilon_\theta, \epsilon_\theta))$ have disjoint images in $G_{\mathrm{full}}(\mathcal{F}_{\mathbb{R}}^k)$. Let U be the image of U_θ and let V be the union of the images of the $U_\theta \cap (\mathbb{R} \times (-\epsilon_\theta, \epsilon_\theta))$, ranging over $\theta \in \mathrm{Diff}_0(\mathbb{R})$ with $j_0^{k-1}(\theta) \neq j_0^{k-1}(\theta_0)$. □

Corollary 5.2.12. *The topology of $G_{\mathrm{full}}(\mathcal{F}_{\mathbb{R}}^k)$ is Hausdorff and regular.*

Proposition 5.2.13. *For every $k \geq 2$, the topology of $G_{\mathrm{full}}(\mathcal{F}_{\mathbb{R}}^k))$ is not normal.*

Proof. For notational convenience we take $k = 2$; the argument for $k > 2$ is essentially the same. Let $A \subseteq J_{\mathbb{R}}^2$ consist of all $a_1 y + a_2 y^2 \in J^2$ with a_1 rational. Put $B = J_{\mathbb{R}}^2 \setminus A$. Then A and B are disjoint closed sets partitioning $J_{\mathbb{R}}^2$ (each is a union of cosets). In a similar spirit to Niemytzki's Tangent Disc (see [46], pp. 100), one can use the Baire category theorem to show that A and B cannot be separated by disjoint open sets in $G_{\mathrm{full}}(\mathcal{F}_{\mathbb{R}}^2)$. □

We summarize various topological properties of $G_{\mathrm{full}}(\mathcal{F}_{\mathbb{R}}^k)$ discussed above in the following theorem.

Theorem 5.2.14. *For every $k \geq 2$, the full holonomy groupoid $G_{\mathrm{full}}(\mathcal{F}_{\mathbb{R}}^k)$:*

1. *is equipped with a smooth atlas (i.e. is a smooth space, in the sense of Section 2.1.1),*
2. *is Hausdorff and regular, but not normal,*
3. *is separable, but not second countable,*
4. *has two connected components.*

Proof. (1) holds for the full holonomy groupoid of any almost regular foliation (see Section 2.3.5). (2) is a repetition of Corollary 5.2.12 and Corollary 5.2.13. $G_{\mathrm{full}}(\mathcal{F}_{\mathbb{R}}^k)$ is separable because it contains $(\mathbb{R} \setminus \{0\})^2$ as a dense open subset. It is not second countable because it contains $J_{\mathbb{R}}^k$ and $J_{\mathbb{R}}^k$ is not second-countable. (4) is a repetition of Proposition 5.2.8. □

5.3 Transverse order k foliations

We now define this chapter's main objects of study and give some of their basic properties. Recall that $\mathcal{F}_{\mathbb{R}}^k := \mathcal{F}\{y^k \frac{d}{dy}\}$. This is the foliation of $\mathbb{R}$ consisting of all compactly-supported vector fields on $\mathbb{R}$ which vanish to order k or more at 0.

Definition 5.3.1. Let M be a connected, smooth manifold. A foliation $(M, \mathcal{F})$ is a **transverse order k** foliation if:

1. for each $x \in M$, there exists an open set $U \subseteq M$ with $x \in U$ and a local submersion $p : U \to \mathbb{R}$ such that $p^{-1}(\mathcal{F}^k_{\mathbb{R}}) = \mathcal{F}_U$.
2. $\mathcal{F}$ has exactly one singular leaf L.

Given a transverse order k foliation $(M, \mathcal{F})$, we call a submersion such as the one in (1) a **local $\mathcal{F}$-$\mathcal{F}^k_{\mathbb{R}}$-submersions**.

Remark 5.3.2. The important assumption above is (1). Assumption (2) is included mainly for convenience.

The prototypical example of a transverse order k foliation is the following.

Example 5.3.3. Let ℓ be a positive integer and $n := \ell + 1$. Equip $\mathbb{R}^n = \mathbb{R}^\ell \times \mathbb{R}$ with coordinates $(x, y) = (x_1, \ldots, x_\ell, y)$. Then

$$\mathcal{F}^k_{\mathbb{R}^n} := \mathcal{F}\left\{y^k \tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial x_1}, \ldots, \tfrac{\partial}{\partial x_\ell}\right\}$$

is a transverse order k foliation of $\mathbb{R}^n$. Indeed, $\mathcal{F}^k_{\mathbb{R}^n} = \mathrm{pr_2}^{-1}(\mathcal{F}^k_{\mathbb{R}})$, where pr_2 is the final coordinate projection $(x, y) \mapsto y$. The singular leaf of $\mathcal{F}^k_{\mathbb{R}^n}$ is the horizontal hyperplane $\mathbb{R}^\ell \times \{0\}$.

Every transverse order k foliation of an n-dimensional manifold is locally isomorphic to the foliation in Example 5.3.3.

Proposition 5.3.4. *Let $(M, \mathcal{F})$ be a transverse order k foliation with singular leaf L and $n := \dim(M) \geq 2$. Then, for any $x_0 \in L$, there exists a diffeomorphism $\theta : U \to V$, where $U \subseteq M$ is an open neighbourhood of x_0 and $V \subseteq \mathbb{R}^n = \mathbb{R}^\ell \times \mathbb{R}$ is an open neighbourhood of $(0, 0)$, such that $\theta(x_0) = (0, 0)$ and $\theta_*(\mathcal{F}_U) = (\mathcal{F}^k_{\mathbb{R}^n})_V$. Moreover, given any local $\mathcal{F}$-$\mathcal{F}^k_{\mathbb{R}}$-submersions p and any local retraction π onto L, both defined near x_0, there exists a diffeomorphism θ under which p becomes* pr_2 *and π becomes* pr_1.

Proof. Let $p : U_0 \to \mathbb{R}$ be a local $\mathcal{F}$-$\mathcal{F}^k_{\mathbb{R}}$-submersions with $x_0 \in U_0$. Any submersion is locally a Euclidean projection so, shrinking U_0 and choosing coordinates appropriately, we may identify U_0 with a ball in $\mathbb{R}^n$ centred at $x_0 = (0, 0)$ in such a way that $p = \mathrm{pr}_2|_{U_0}$. We then have $\mathcal{F}_{U_0} = (\mathrm{pr}_2|_{U_0})^{-1}(\mathcal{F}^k_{\mathbb{R}}) = (\mathcal{F}^k_{\mathbb{R}^n})_{U_0}$. In particular, $L \cap U_0 = (\mathbb{R}^\ell \times \{0\}) \cap U_0$. Now, possibly shrinking U_0 further, let $\pi : U_0 \to \mathbb{R}^\ell$ be a submersion satisfying $\pi(x, 0) = x$ for all $(x, 0) \in U_0 \cap (\mathbb{R}^\ell \times \{0\})$. By the inverse function theorem, there is a smaller ball $U \subseteq U_0$ centred on the origin such that $(x, y) \mapsto (\pi(x, y), y)$ defines a diffeomorphism $\theta : U \to V$, where $V = \theta(U) \subseteq U_0$ as well. Since, by construction, $\mathrm{pr}_2|_V \circ \theta = \mathrm{pr}_2|_U$, we have $\theta_*(\mathcal{F}_U) = \theta_*((\mathrm{pr}_2|_U)^{-1}(\mathcal{F}^k_{\mathbb{R}})) = (\mathrm{pr}_2|_V)^{-1}(\mathcal{F}^k_{\mathbb{R}}) = (\mathcal{F}^k_{\mathbb{R}^m})_V$. □

Corollary 5.3.5. *Let $(M, \mathcal{F})$ be a transverse order k foliation with singular leaf L. Then, L is a closed submanifold of codimension-*1

Proof. The above proposition shows that L is a codimension-1 embedded submanifold of M. Since the other leaves of $\mathcal{F}$ are open, it follows that L is closed. □

5.4 Local results on transverse order k foliations

In this section, we study the prototypical transverse order k foliation $\mathcal{F}^k_{\mathbb{R}^n}$ of Example 5.3.3. We assume throughout that $k \geq 2$. The main results for $\mathcal{F}^k_{\mathbb{R}^n}$ are Theorem 5.4.6, which characterizes $\mathcal{F}^k_{\mathbb{R}^n}$-$\mathcal{F}^k_{\mathbb{R}}$-submersions (Definition 5.3.1) in terms of their infinitesimal behaviour along the singular hyperplane, and Theorem 5.4.15, which computes the restriction of the full holonomy groupoid to the singular hyperplane. Also of importance is Theorem 5.4.12 which shows that, for any transverse order k foliation $(M, \mathcal{F})$ and x a point in the singular leaf, k-jet equivalence of $\mathcal{F}$-$\mathcal{F}^k_{\mathbb{R}}$-submersions at x is the same as orbit equivalence under the action of the group of null $\mathcal{F}$-automorphisms at x.

Let $n \geq 2$ be an integer and put $\ell := n - 1$. We equip $\mathbb{R}^n = \mathbb{R}^\ell \times \mathbb{R}$ with coordinates $(x, y) = (x_1, \ldots, x_\ell, y)$. The transverse order k foliation under discussion is:

$$\mathcal{F}^k_{\mathbb{R}^n} := \mathcal{F}\left\{y^k \tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial x_1}, \ldots, \tfrac{\partial}{\partial x_\ell}\right\} = \mathrm{pr}_2^{-1}(\mathcal{F}^k_{\mathbb{R}}).$$

The singular leaf of $\mathcal{F}^k_{\mathbb{R}^n}$ is the horizontal hyperplane:

$$L := \mathbb{R}^\ell \times \{0\}.$$

The following result shows that $\mathcal{F}^k_{\mathbb{R}^n}$-automorphisms and $\mathcal{F}^k_{\mathbb{R}^n}$-$\mathcal{F}^k_{\mathbb{R}}$-submersions are closely related.

Proposition 5.4.1. *Let $\theta : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism. Then, θ preserves $\mathcal{F}^k_{\mathbb{R}^n}$ if and only if $p^{-1}(\mathcal{F}^k_{\mathbb{R}}) = \mathcal{F}^k_{\mathbb{R}^n}$, where $p := \mathrm{pr}_2 \circ \theta : \mathbb{R}^n \to \mathbb{R}$.*

Proof. We have $\theta^*(\mathcal{F}^k_{\mathbb{R}^n}) = \theta^{-1}(\mathrm{pr}_2^{-1}(\mathcal{F}^k_{\mathbb{R}})) = p^{-1}(\mathcal{F}^k_{\mathbb{R}})$. □

The following terminology will be convenient.

Definition 5.4.2. Let θ be a diffeomorphism of $\mathbb{R}^n$. We say that θ is *vertical* if it has the form $\theta(x, y) = (x, \theta_x(y))$ where θ_x, $x \in \mathbb{R}^\ell$ is a smoothly-varying diffeomorphism of $\mathbb{R}$. Similarly, we say that θ is *horizontal* if it has the form $\theta(x, y) = (\theta_y(x), y)$ where θ_y, $y \in \mathbb{R}$ is a smoothly varying diffeomorphism of $\mathbb{R}^\ell$.

Since a horizontal diffeomorphism θ satisfies $\mathrm{pr}_2 \circ \theta = \theta$, the following is an immediate consequence of Proposition 5.4.1.

Lemma 5.4.3. *Any horizontal diffeomorphism preserves $\mathcal{F}^k_{\mathbb{R}^n}$.* □

On the other hand, an inverse function theorem argument gives the following.

Lemma 5.4.4. *Let θ be any diffeomorphism of $\mathbb{R}^n$ which preserves L. Then, locally near any point of L, we can write $\theta = \theta_h \circ \theta_v$ where θ_v is vertical and θ_h is horizontal.* □

The crux, therefore, is to understand which vertical diffeomorphisms θ preserve $\mathcal{F}^k_{\mathbb{R}^n}$. It is obviously necessary that θ preserve L, but this does not suffice. For example, $(x, y) \mapsto (x, e^x y)$ does not preserve $\mathcal{F}^2_{\mathbb{R}^2}$ (see Example 1.5.1).

Lemma 5.4.5. *Let $\theta(x, y) = (x, \theta_x(y)$ be a vertical diffeomorphism of $\mathbb{R}^n$ which preserves L. Then θ preserves $\mathcal{F}^k_{\mathbb{R}^n}$ if and only if $\theta'_x(0), \theta''_x(0), \ldots, \theta^{(k-1)}_x(0)$ are independent of x.*

Proof. Suppose that θ is vertical. Firstly, even without the assumptions on the derivatives, one has that θ preserves the foliation singly-generated by $y^k \frac{\partial}{\partial y}$. Indeed, for any $f \in C^\infty_c(\mathbb{R}^n)$, we have $\theta^*(f \frac{d}{dy}) = (f \circ \theta^{-1})\theta^*(\frac{\partial}{\partial y})$. Since $f \mapsto f \circ \theta^{-1}$ preserves the ideal $I^k_L \subseteq C^\infty_c(\mathbb{R}^n)$ of functions vanishing to order k on L, and since $\theta_*(\frac{\partial}{\partial y}) = \theta'_x(y) \frac{d}{dy}$ where $\theta'_x(y)$ is nowhere vanishing, the claim follows.

Next, $\theta_*(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial x_i} + (f_i \circ \theta^{-1}) \frac{\partial}{\partial y}$, where $f_i(x, y) = \frac{\partial}{\partial x_i} \theta_x(y)$. The foliation $\mathcal{F}^k_{\mathbb{R}^n}$ is preserved by θ if and only if $f_i \circ \theta^{-1}$, or equivalently f_i, belongs to the ideal I^k_L for $i = 1, \ldots, \ell$. In other words, we need $\frac{\partial}{\partial x_i} \theta^{(r)}_x(0) = 0$ for all $x \in \mathbb{R}^\ell$, $i = 1, \ldots, \ell$, $r = 1, \ldots, k-1$, which proves (2). □

We now translate Lemma 5.4.5 into the following result about submersions which will play an important role.

Theorem 5.4.6. *Let $U \subseteq \mathbb{R}^n$ be a convex open set containing $(0, 0)$. Let $p : U \to \mathbb{R}$ be a submersion with $p^{-1}(0) = L \cap U$. Then, the following are equivalent:*

1. *$p^{-1}(\mathcal{F}^k_{\mathbb{R}}) = (\mathcal{F}^k_{\mathbb{R}^n})_U$, i.e. p is a local $\mathcal{F}^k_{\mathbb{R}^n}$-$\mathcal{F}^k_{\mathbb{R}}$-submersion.*
2. *$\frac{\partial^r p}{\partial y^r}$ is constant on $L \cap U$ for $r = 1, \ldots, k-1$.*
3. *There exist constants $a_1, \ldots, a_{k-1} \in \mathbb{R}$ with $a_1 \neq 0$ and a smooth function f on $L \cap U$ such that $j^k_{(x_0,0)}(p) = a_1 y + \ldots a_{k-1} y^{k-1} + f(x_0) y^k$ for all $(x_0, 0) \in L \cap U$.*

4. There exist constants $a_1, \dots, a_{k-1} \in \mathbb{R}$ with $a_1 \neq 0$ and a smooth function f on U such that $p(x,y) = a_1 y + \dots a_{k-1} y^{k-1} + f(x,y) y^k$ for all $(x,y) \in U$.

Proof. Define $\theta : U \to \mathbb{R}^n$ by $\theta(x,y) = (x, p(x,y))$. By the inverse function theorem, θ is a diffeomorphism in a neighbourhood of L. Shrinking U, we may assume that θ maps U diffeomorphically onto $\theta(U)$. By Proposition 5.4.1, θ preserves $\mathcal{F}^k_{\mathbb{R}^m}$ if and only if $p^{-1}(\mathcal{F}^k_{\mathbb{R}}) = (\mathcal{F}^k_{\mathbb{R}^n})_U$. The equivalence of statements (1) and (2) then follows from Lemma 5.4.5. Obviously (3) implies (2). Conversely, (2) and the fact that p itself vanishes on L imply that

$$\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_\ell}\right)^{\alpha_\ell} \left(\frac{\partial}{\partial y}\right)^{\beta} p(x,0) = 0$$

whenever $\beta \leq k-1$ and at least one of $\alpha_1, \dots, \alpha_\ell$ is nonzero. Statement (3) follows. Clearly (4) implies (3) and, by a Taylor series argument, (4) implies (3) as well. □

One may interpret the above theorem as saying that the infinitesimal structure of local $\mathcal{F}^k_{\mathbb{R}^n}$-$\mathcal{F}^k_{\mathbb{R}}$-submersions is very rigid along the singular leaf L: their kth order Taylor expansions involve only the variable y, and none of the variables $x_1, \dots, x_\ell$. Furthermore, the coefficients of $y, \dots, y^{k-1}$ remain constant as the basepoint of the Taylor expansion varies in L. As a further demonstration of this rigidity principle, the following corollary says that, working locally and up to order $k-1$, any two local $\mathcal{F}$-$\mathcal{F}^k_{\mathbb{R}}$-submersions are related by a unique polynomial.

Corollary 5.4.7. *Let $(M, \mathcal{F})$ be any transverse order k foliation with singular leaf L. Let p and q be local $\mathcal{F}$-submersions defined at a point $x_0 \in L$. Then, on some neighbourhood U of x_0, there exist unique constants $a_1, \dots, a_{k-1} \in \mathbb{R}$ and a unique smooth, real-valued function $f : U \to \mathbb{R}$ such that*

$$q = a_1 p + \dots a_{k-1} p^{k-1} + f p^k$$

holds on U. Necessarily, $a_1 \neq 0$.

Proof. By Proposition 5.3.4, we may assume $M = \mathbb{R}^n$, $\mathcal{F} = \mathcal{F}^k_{\mathbb{R}^n}$, $x_0 = (0,0)$ and $p = \mathrm{pr}_2$, whence the claim follows from Theorem 5.4.6 (4). □

By Proposition 5.4.1 and Theorem 5.4.6 together give a good understanding of the structure of $\mathcal{F}^k_{\mathbb{R}^n}$-automorphisms and $\mathcal{F}^k_{\mathbb{R}^n}$-$\mathcal{F}^k_{\mathbb{R}}$-submersions. Our next task is to mod out by null automorphisms (Definition 2.3.17). It is therefore necessary to involve some $\mathcal{F}^k_{\mathbb{R}^n}$-bisubmersions. We can easily convert an $\mathcal{F}^k_{\mathbb{R}}$-bisubmersion into an $\mathcal{F}^k_{\mathbb{R}^n}$-bisubmersion by taking the product with the pair groupoid L^2.

Lemma 5.4.8. *If (W, t, s) is an $\mathcal{F}^k_{\mathbb{R}}$-bisubmersion, then $(L^2 \times W, \mathrm{pr}_1 \times t, \mathrm{pr}_2 \times s)$ is an $\mathcal{F}^k_{\mathbb{R}^n}$-bisubmersion.*

Proof. This conclusion follows from consideration of the commutative diagrams:

$$\begin{array}{ccc} L^2 \times W & \longrightarrow & W \\ \Big\downarrow{\scriptstyle \mathrm{pr}_2 \times \sigma} & & \Big\downarrow{\scriptstyle \sigma} \\ \mathbb{R}^n & \longrightarrow & \mathbb{R} \end{array} \qquad \begin{array}{ccc} L^2 \times W & \longrightarrow & W \\ \Big\downarrow{\scriptstyle \mathrm{pr}_1 \times \tau} & & \Big\downarrow{\scriptstyle \tau} \\ \mathbb{R}^n & \longrightarrow & \mathbb{R}\,. \end{array}$$

□

In particular, recall (Definition 5.2.1) that

$$\Omega = \{(t, y) \in \mathbb{R}^2 : 1 + ty^{k-1} > 0\} \qquad \sigma(t, y) = y \qquad \tau(t, y) = y + ty^y$$

defines an $\mathcal{F}^k_{\mathbb{R}}$-bisubmersion $\Omega := (\Omega, \tau, \sigma)$ and that, more generally (Definition 5.2.6), $\Omega_\theta := (\Omega, \theta \circ \tau, \sigma)$ is an $\mathcal{F}^k_{\mathbb{R}}$-bisubmersions for any $\theta \in \mathrm{Diff}_0(\mathbb{R})$.

Definition 5.4.9. Let $\widetilde{\Omega}$ denote the $\mathcal{F}^k_{\mathbb{R}^n}$-bisubmersion $L^2 \times \Omega$. More generally, for any $\theta \in \mathrm{Diff}_0(\mathbb{R})$, let $\widetilde{\Omega}_\theta$ denote the $\mathcal{F}^k_{\mathbb{R}^n}$-bisubmersion $L^2 \times \Omega_\theta$.

Note that $N_0 := \{(x, x, 0, y) : (x, y) \in \mathbb{R}^n\}$ is a bisection of $\widetilde{\Omega}_\theta$ inducing the constant vertical diffeomorphism $(x, y) \mapsto (x, \theta(y))$. In particular, $\Omega = \Omega_{\mathrm{id}}$ carries the identity map and we may characterize the local $\mathcal{F}^k_{\mathbb{R}^n}$-automorphisms which are null at a point $x \in L$ as the ones which are carried by $\widetilde{\Omega}$ at at the point $(x, x, 0, 0)$. For example, we have the following:

Lemma 5.4.10. *Let θ be a local $\mathcal{F}^k_{\mathbb{R}^n}$-automorphism with $\theta(0, 0) = (0, 0)$. If θ is horizontal, then θ is null (Definition 2.3.17) at $(0, 0)$.*

Proof. Near $(0, 0)$, since θ is horizontal, we may write $\theta(x, y) = (\theta_y(x), y)$ where θ_y depends smoothly on y. Let $N = \{(\theta_y(x), x, 0, y) \in \widetilde{\Omega} : (x, y) \in U\}$, where $U \subseteq \mathbb{R}^n$ is an appropriately chosen neighbourhood of $(0, 0)$. Then, N is a bisection of $\widetilde{\Omega}$ through the point $(0, 0, 0, 0)$ inducing the germ of θ at $(0, 0)$. □

In a similar spirit, we have the following generalization of Lemma 5.2.3 (2).

Lemma 5.4.11. *Let $\theta_0 \in \mathrm{Diff}_0(\mathbb{R})$ be a diffeomorphism of $\mathbb{R}$ defined near 0 and define a constant, vertical diffeomorphism θ of $\mathbb{R}^n$ by $(x, y) \mapsto (x, \theta(y))$. Then, θ preserves $\mathcal{F}^k_{\mathbb{R}^n}$ and is null at $(0, 0)$ if and only if $j^k_0(\theta) = y$.* □

The next result has some importance in later chapters. We state it for general foliations of transverse order k, though we quickly reduce to coordinates in the proof.

Theorem 5.4.12. *Let $(M, \mathcal{F})$ be a transverse order $k \geq 2$ foliation with singular leaf L. Let p and q be local $\mathcal{F}$-$\mathcal{F}^k_{\mathbb{R}}$-submersions defined at $x_0 \in L$. Then, $j^k_{x_0}(p) = j^k_{x_0}(q)$ if and only if there exists a local $\mathcal{F}$-automorphism θ which is null at x_0 such that $q = p \circ \theta$ on a neighbourhood of x_0.*

Proof. Using Proposition 5.3.4, we may suppose without loss of generality that $(M, \mathcal{F}) = (\mathbb{R}^n, \mathcal{F}^k_{\mathbb{R}^n})$, $x_0 = (0,0)$ and $p = \mathrm{pr}_2$. By the inverse function theorem, $\eta(x,y) = (x, q(x,y))$ defines a diffeomorphism nearby to $(0,0)$. By definition, $q = p \circ \eta$ holds near $(0,0)$. It remains to confirm that η is null at $(0,0)$. Since $j^k_{(0,0)}(q) = y$, Theorem 5.4.6 (4) implies that we can write $q(x,y) = y + f(x,y)y^k$ near $(0,0)$ for a smooth function f satisfying $f(0,0) = 0$. Then, the bisection

$$N_f = \{(x, x, f(x,y), y) : (x,y) \in U\},$$

where U is an appropriate neighbourhood of $(0,0) \in \mathbb{R}^n$, induces η. Since $(0,0,0,0) \in N_f$ and the identity is also carried at $(0,0,0,0) \in \widetilde{\Omega}$, we have that η is null at $(0,0)$. □

Along similar lines, we now show that all local $\mathcal{F}^k_{\mathbb{R}^n}$-automorphisms can be brought, modulo a null automorphism, into a simple form.

Proposition 5.4.13. *Let $x_1, x_2 \in L$ and let θ be a local $\mathcal{F}^k_{\mathbb{R}^n}$-automorphism with $\theta(x_1, 0) = (x_2, 0)$. Then, there is a local $\mathcal{F}^k_{\mathbb{R}^n}$ automorphism η which is null at $(x_1, 0)$ and a diffeomorphism $\theta_0 \in \mathrm{Diff}_0(\mathbb{R})$ such that $(\theta \circ \eta)(x,y) = (x - x_1 + x_2, \theta_0(y))$ holds on a neighbourhood of $(x_1, 0)$.*

Proof. By translating, we may reduce to the case $x_1 = x_2 = 0$. Using Lemmas 5.4.4 and 5.4.10, we furthermore reduce to the case where θ is a vertical $\mathcal{F}^k_{\mathbb{R}^n}$-automorphism $(x,y) \mapsto (x, \theta_x(y))$. Composing with the constant vertical diffeomorphism $(x,y) \mapsto (x, \theta_0^{-1}(y))$, we may furthermore reduce to the case where $\theta_0(y) = y$. The result then follows from Theorem 5.4.12, taking $p(x,y) = y$ and $q(x,y) = \theta_x(y)$. □

Proposition 5.4.14.

1. $\left\{\widetilde{\Omega}_\theta : \theta \in \mathrm{Diff}_0(\mathbb{R})\right\}$ *is a full holonomy atlas for* $\mathcal{F}^k_{\mathbb{R}^n}$.
2. *For each* $\theta \in \mathrm{Diff}_0(\mathbb{R})$, *the canonical map* $Q_{\widetilde{\Omega}_\theta} : \widetilde{\Omega}_\theta \to G_{\mathrm{full}}(\mathcal{F}^k_{\mathbb{R}^n})$ *is a diffeomorphism onto its image.*

Proof. Denote the open upper and lower half spaces of $\mathbb{R}^n$ by $\mathbb{R}^n_+ := \mathbb{R}^\ell \times \mathbb{R}_+$ and $\mathbb{R}^n_- := \mathbb{R}^\ell \times \mathbb{R}_-$. It is easy to see that the restriction of $\widetilde{\Omega}_\theta$ to $\mathbb{R}^n \setminus L$ is isomorphic $(\mathbb{R}^n_-)^2 \cup (\mathbb{R}^n_+)^2$ if θ is orientation-preserving and $(\mathbb{R}^n_- \times \mathbb{R}^n_+) \cup (\mathbb{R}^n_+ \times \mathbb{R}^n_-)$ if θ is orientation-reversing. Thus, every local diffeomorphism of $\mathbb{R}^n \setminus L$ is already carried by $\widetilde{\Omega}_{\mathrm{id}}$ and $\widetilde{\Omega}_{-\mathrm{id}}$. It remains to show that, given any point $x_0 \in L$ and any local $\mathcal{F}^k_{\mathbb{R}^n}$-automorphism θ defined at x_0, the germ of θ at x_0 can by induced by a bisection of one of $\{\widetilde{\Omega}_\theta : \theta \in \mathrm{Diff}_0(\mathbb{R})\}$. After an easy reduction, we may take $x_0 = (0,0)$ and assume $\theta(0,0) = (0,0)$. By Proposition 5.4.13, after composing θ with a local $\mathcal{F}^k_{\mathbb{R}^n}$-automorphism that is null at $(0,0)$, we may assume that $\theta(x,y) = (x, \theta_0(y))$ holds on a neighbourhood U of $(0,0)$ for some $\theta_0 \in \mathrm{Diff}_0(\mathbb{R})$. Then, then $N = \{(x,x,0,y) : (x,y) \in U\}$ is a bisection of $\widetilde{\Omega}_\theta$ which induces $\theta|_U$, proving (1). The proof of (2) is the same as in Proposition 5.2.7. □

We can now show that the restriction of $G_{\mathrm{full}}(\mathcal{F}^k_{\mathbb{R}^n})$ is isomorphic, as a Lie groupoid, to $L^2 \times J^k_{\mathbb{R}}$, the product of the pair groupoid L^2 and the 1-dimensional Lie group $J^k_{\mathbb{R}}$ (see Definition 5.1.5).

Theorem 5.4.15. *There is a unique isomorphism of Lie groupoids*

$$G_{\mathrm{full}}(\mathcal{F}^k_{\mathbb{R}^n})_L \to L^2 \times J^k_{\mathbb{R}}$$

such that

$$Q_{\widetilde{\Omega}_\theta}(x_2, x_1, t, 0) \mapsto (x_2, x_1, j^k_0(\theta) \circ (y + ty^k))$$

for all $\theta \in \mathrm{Diff}_0(\mathbb{R})$, $x_1, x_2 \in L$ and $t \in \mathbb{R}$.

Proof. Theorem 2.3.18 gives an isomorphism $G_{\mathrm{full}}(\mathcal{F}^k_{\mathbb{R}})_L \to \mathcal{G}/\mathcal{N}$ where, for the sake of brevity, we write $\mathcal{G} := \mathcal{G}(\mathcal{F}^k_{\mathbb{R}^n})_L$ and $\mathcal{N} := \mathcal{N}(\mathcal{F}^k_{\mathbb{R}^n})_L$. Given $x_1, x_2 \in L$, $\theta \in \mathrm{Diff}_0(\mathbb{R})$, let $T_{x_2,x_1,\theta}$ denote the germ at x_1 of the diffeomorphism $(x,y) \mapsto (x + x_2 - x_1, \theta(x))$. The set of all $T_{x_2,x_1,\theta}$ constitute a subgroupoid $\mathcal{H} \subseteq \mathcal{G}$. It is easy to see that $T_{x_2,x_1,\theta} \to (x_2, x_1, j^k_0(\theta)) : \mathcal{H} \to L^2 \times J^k$ is a surjective groupoid homomorphism. By Lemma 5.4.11, the kernel of the latter homomorphism is exactly $\mathcal{H} \cap \mathcal{N}$, so that

$$\frac{\mathcal{H}\mathcal{N}}{\mathcal{N}} \cong \frac{\mathcal{H}}{\mathcal{H} \cap \mathcal{N}} \cong L^2 \times J^k.$$

By Proposition 5.4.13, $\mathcal{H}\mathcal{N} = \mathcal{G}$, so we have an isomorphism

$$\mathcal{G}/\mathcal{N} \ni [T_{x_2,x_1,\theta}] \mapsto (x_2, x_1, j^k_0(\theta) \to L^2 \times J^k.$$

Finally, note that the bisection $N_{x_1,x_2,t} := \{(x + x_2 - x_1, x, t, y) \in \widetilde{\Omega}_\theta : (x,y) \in \mathbb{R}^n\}$ induces the diffeomorphism $(x,y) \mapsto (x + x_2 - x_1, \theta(y + ty^k)$ so that the resulting isomorphism $G_{\mathrm{full}}(\mathcal{F}^k_{\mathbb{R}^n}) \to L^2 \times J^k$ indeed sends

$$Q_{\widetilde{\Omega}_\theta}(x_2, x_1, t, 0) \mapsto (x_2, x_1, j^k_0(\theta) \circ (y + ty^k)) \in L^2 \times J^k.$$

By Proposition 5.4.14, the $\widetilde{\Omega}_\theta$ constitute a full holonomy atlas for $\mathcal{F}^k_{\mathbb{R}^n}$ and each $Q_{\widetilde{\Omega}_\theta}$ is a diffeomorphism onto its image. It follows that this groupoid isomorphism becomes a Lie groupoid isomorphism when J^k is replaced by $J^k_{\mathbb{R}}$. □

5.5 Principal bundles of a transverse order k foliation

In what follows, $(M, \mathcal{F})$ denotes a transverse order k foliation with singular leaf L and $k \geq 2$. The purpose of this section is to leverage the rigidity phenomena encountered in the preceding section to construct certain natural principal bundles over L whose elements are jets of local $\mathcal{F}$-$\mathcal{F}^k_{\mathbb{R}}$-submersions. The main result is Theorem 5.5.3.

Because the group $\mathrm{Diff}_0(\mathbb{R})$ of diffeomorphisms of $\mathbb{R}$ fixing 0 preserves $\mathcal{F}^k_{\mathbb{R}}$, the group $\mathrm{Diff}_0(\mathbb{R})$ acts by composition from the left on the set of local $\mathcal{F}$-$\mathcal{F}^k_{\mathbb{R}}$-submersions defined at $x \in M$. Since composition of jets is well-defined, the following definition makes sense.

Definition 5.5.1. Given $x \in L$ and $r \in \{1, \ldots, k\}$, we write $P^r_x(\mathcal{F})$ for the set of r-jets at x of local $\mathcal{F}$-$\mathcal{F}^k_{\mathbb{R}}$-submersions endowed with the left action of J^r that descends from the left action of $\mathrm{Diff}_0(\mathbb{R})$. Put also $P^r(\mathcal{F}) := \bigsqcup_{x \in L} P^r_x(\mathcal{F})$, so that $P^r(\mathcal{F})$ is a bundle of sets over L with an action of J^r on each fiber.

Our goal is to equip $P^r(\mathcal{F})$ with the structure of a smooth principal bundle over L. Our constructions will rely on the following elementary lemma which provides a mechanism by which a smooth principal bundle structure may be induced from an appropriate family of sections. In essence, this is the construction of a principal bundle from a 1-cocycle.

Lemma 5.5.2. *Let H be a Lie group and M a smooth manifold. Suppose that P is a set equipped with a map $\pi : P \to M$ and an action of H that is free and transitive on each fiber of π. Let $\{U_i\}_{i \in I}$ be an open cover of M and let $\{s_i : U_i \to P\}_{i \in I}$ be a collection of local sections of π such that the transition maps $\{h_{ij} : U_i \cap U_j \to H\}_{i,j \in I}$, uniquely defined by $s_i(x) = h_{ij}(x)s_j(x)$ for all $x \in U_i \cap U_j$, are smooth. There is a unique smooth structure on P making it a smooth principal H-bundle with respect to which each s_i is a smooth section.* □

Recall (Definition 5.1.5) that J^r_d denotes J^r considered as a discrete group and $J^r_{\mathbb{R}}$ denotes J^r with a certain one-dimensional Lie group structure.

Theorem 5.5.3. *Let $(M, \mathcal{F})$ be a transverse order k foliation with singular leaf L and $k \geq 2$.*

1. *There is a unique principal $J^k_{\mathbb{R}}$-bundle structure on $P^k(\mathcal{F})$ such that, for any local $\mathcal{F}$-$\mathcal{F}^k_{\mathbb{R}}$-submersions $p : U \to \mathbb{R}$, the map $x \mapsto j^k_x(p) : U \cap L \to P^k(\mathcal{F})$ is a smooth section.*

2. *If $1 \leq r \leq k-1$, there is a unique principal J^r_d-bundle structure on $P^r(\mathcal{F})$ such that, for any local $\mathcal{F}$-$\mathcal{F}^k_{\mathbb{R}}$-submersions $p : U \to \mathbb{R}$, the map $x \mapsto j^r_x(p) : U \cap L \to P^r(\mathcal{F})$ is a smooth section.*

Proof. First we claim that, for any $r \in \{1, \ldots, k\}$, the action of J^r on $P^r(\mathcal{F})$ is free and transitive on fibers. By Proposition 5.3.4, it suffices to consider the case $(M, \mathcal{F}) = (\mathbb{R}^n, \mathcal{F}^k_{\mathbb{R}^n})$ and work near the point $(0,0) \in \mathbb{R}^n$. By Theorem 5.4.6, given a local $\mathcal{F}^k_{\mathbb{R}^n}$-$\mathcal{F}^k_{\mathbb{R}}$-submersion p defined at $(0,0)$, the jet $j^k_{(0,0)}(p)$ (a priori a polynomial of degree $\leq r$ in $x_1, \ldots, x_\ell, y$) has the form $a_1 y + \ldots + a_k y^k$ where $a_i \in \mathbb{R}$, $a_1 \neq 0$. The action of J^k, meanwhile, is the usual compose and truncate operation. It follows that J^r acts freely and transitively on $P^r_{(0,0)}(\mathcal{F})$ for $r = 1, \ldots, k$.

Next, suppose that $p, q : U \to \mathbb{R}$ are local $\mathcal{F}^k_{\mathbb{R}^n}$-$\mathcal{F}^k_{\mathbb{R}}$-submersions, where $U \subseteq \mathbb{R}^n$ is a convex, open neighbourhood of $(0,0)$. Without loss of generality (Proposition 5.3.4) we may take $p = \mathrm{pr}_2$. By Theorem 5.4.6, we have

$$j^k_{(0,x_0)}(q) = a_1 y + \ldots + a_{k-1} y^{k-1} + f(x_0) y^{k-1}$$

for $x_0 \in U \cap \mathbb{R}^\ell$, where $f : U \cap \mathbb{R}^\ell \to \mathbb{R}$ is smooth. The map $h : U \cap \mathbb{R}^\ell \to J^k_{\mathbb{R}}$ defined by $h(x_0) = a_1 y + \ldots + a_{k-1} y^{k-1} + f(x_0) y^k$ is smooth (the smooth structure on $J^{k-1}_{\mathbb{R}}$ only permits us to vary the coefficient of y^k). We have $j^k_{(x_0,0)}(q) = h \circ j^k_{(x_0,0)}(p)$ for all $x_0 \in U \cap \mathbb{R}^\ell$ and so, applying Lemma 5.5.2, we obtain (1). The proof of (2) is similar, keeping fewer terms of the Taylor expansions. □

By construction, these bundles are functorial for foliation-preserving diffeomorphisms defined near the singular leaf.

Proposition 5.5.4. *For $i = 1, 2$, let $(M_i, \mathcal{F}_i)$ be transverse order k singular foliations with singular leaves L_i. Suppose $U_i \subseteq M_i$ is an open set containing L_i and $\theta : U_1 \to U_2$ is a diffeomorphism with $\theta_*((\mathcal{F}_1)_{U_1}) = (\mathcal{F}_2)_{U_2}$. Then θ restricts to a diffeomorphism $\theta_0 : L_1 \to L_2$ and, for $r = 1, \ldots, n$, there is a unique isomorphism of principal bundles $P^r(\theta) : P^r(\mathcal{F}_1) \to P^r(\mathcal{F}_2)$ covering θ_0 sending $j^r_x(p) \mapsto j^r_{\theta(x)}(p \circ \theta^{-1})$, whenever p is a local $\mathcal{F}_1$-$\mathcal{F}^k_{\mathbb{R}}$-submersions defined at $x \in L$.*

$$\begin{array}{ccc} P^r(\mathcal{F}_1) & \xrightarrow{P^r(\theta)} & P^r(\mathcal{F}_2) \\ \downarrow & & \downarrow \\ L_1 & \xrightarrow{\theta|_L} & L_2 \end{array}$$

□

Proof. We may assume $U_1 = M_1$, $U_2 = M_2$. Then $\theta : M_1 \to M_2$ induces a bijection $p \mapsto p \circ \theta^{-1}$ from the set of local $\mathcal{F}_1$-$\mathcal{F}^k_{\mathbb{R}}$-submersions to the set of local

$\mathcal{F}_2$-$\mathcal{F}^k_{\mathbb{R}}$-submersions. This bijection commutes with the left action of $\mathrm{Diff}_0(\mathbb{R})$ by composition. Passing to r-jets gives the result. □

Since $P^{k-1}(\mathcal{F})$ has discrete structure group J^{k-1}_d, taking monodromy immediately gives the following. It is appropriate to think of an $\mathcal{F}$-$\mathcal{F}^k_{\mathbb{R}}$-submersion at a point $x \in L$ as a dual version of the transversal. Accordingly, the statement below may be interpreted as saying that a path in L induces a holonomy transformation between transversals *at the level of* $(k-1)$*-jets*, in alignment with the classical notion of holonomy for regular foliations.

Corollary 5.5.5. *Suppose* $(M, \mathcal{F})$ *is a transverse order* k *singular foliation with singular leaf* L*. Then a path* $c : [0,1] \to L$ *from a point* x *to a point* y *induces a* J^{k-1}_d*-equivariant map* $P^{k-1}_x(\mathcal{F}) \to P^{k-1}_y(\mathcal{F})$ *defined by sending* $p_0 \mapsto p(1)$ *where* $p : [0,1] \to P^{k-1}(\mathcal{F})$ *is the unique lift of* c *with* $p(0) = p_0$*.*

We furthermore use the monodromy of the J^{k-1}_d-bundle $P^{k-1}(\mathcal{F})$ to define the following invariant of a transverse order k foliation. See Section 2.1.4 for notation and terminology.

Definition 5.5.6. Let $(M, \mathcal{F})$ be a transverse order k foliation with singular leaf L. The **holonomy invariant** of $\mathcal{F}$ is

$$h(\mathcal{F}) := h(P^{k-1}(\mathcal{F})) \in [\pi_1(L), J^{k-1}].$$

That is, $h(\mathcal{F})$ is the monodromy invariant (Definition 2.1.10) of $P^{k-1}(\mathcal{F})$.

Here, $[\pi_1(L), J^{k-1}]$ denotes, the quotient of the set $\mathrm{Hom}(\pi_1(M, x_0), J^{k-1})$ by the conjugation action of J^{k-1}, where $x_0 \in L$ is any basepoint, as in Section 2.1.4.

The following proposition shows that $h(\mathcal{F})$ is indeed an *invariant* of $\mathcal{F}$. More precisely, $h(\mathcal{F})$ is an "L-local invariant" in the sense that it only depends on the restriction of $\mathcal{F}$ to any neighbourhood of its singular leaf.

Proposition 5.5.7. *For* $i = 1, 2$*, let* $(M_i, \mathcal{F}_i)$ *be transverse order* k *singular foliations with singular leaves* L_i*. Suppose* $U_i \subseteq M_i$ *is an open set containing* L_i *and* $\theta : U_1 \to U_2$ *is a diffeomorphism with* $\theta_*((\mathcal{F}_1)_{U_1}) = (\mathcal{F}_2)_{U_2}$*. Then* θ *restricts to a diffeomorphism* $\theta_0 : L_1 \to L_2$ *and* $(\theta_0)_*(h(\mathcal{F}_1)) = h(\mathcal{F}_2)$*. See Proposition 2.1.9 for the definition of the induced map* $(\theta_0)_* : [\pi_1(M_1), J^{k-1}] \to [\pi_1(M_2), J^{k-1}]$*.*

Proof. First apply Theorem 5.5.4 and then Theorem 2.1.11 (1). □

If we take the underlying J^1-bundle of the J^1_d-bundle $P^1(\mathcal{F}) \to L$ and identify J^1 with $\mathrm{GL}(1, \mathbb{R})$ in the obvious way, then we see that, through the usual correspondence between vector bundles and their frame bundles, $P^1(\mathcal{F})$ determines a flat line bundle over L. As one might guess (since a 1-jet $j^1_x(p)$ of a real-valued map is essentially the same thing as its differential dp_x), this line bundle is canonically isomorphic to $\nu^*_M(L)$, the conormal bundle of L in M. For convenience, we identify $\nu^*_M(L)$ with the subbundle of $T^*M|_L$ which annihilates $TL \subseteq TM|_L$. We omit the verification of the following.

Proposition 5.5.8. *Let $(M, \mathcal{F})$ be a transverse order $k \geq 2$ foliation with singular leaf L. View $P^1(\mathcal{F})$ as a $\mathrm{G}(1, \mathbb{R})$-bundle, as described above. Then*

$$\mathbb{R} \underset{\mathrm{GL}(1,\mathbb{R})}{\times} P^1(\mathcal{F}) \ni [\lambda, j^1_x(p)] \mapsto \lambda dp_x \in \nu^*_M(L)$$

is an isomorphism of line bundles.

Thinking of $P^1(\mathcal{F})$ as a flat $\mathrm{GL}(1, \mathbb{R})$-bundle, we may therefore make the following definition.

Definition 5.5.9. Let $(M, \mathcal{F})$ be a transverse order $k \geq 2$ foliation with singular leaf L. The **Bott connection** of $\mathcal{F}$ is the flat connection $b(\mathcal{F})$ on the conormal bundle $\nu^*_M(L)$ induced by Proposition 5.5.8 above.

We conclude this section with the observation that one really only needs to construct the bundle $P^k(\mathcal{F})$; the bundles $P^r(\mathcal{F})$ for $r < k$ can be recovered as quotients of the former. Recall that if P is a principal H-bundle and K is a closed normal subgroup of H, then P/K is naturally a principal H/K bundle over the same base.

Proposition 5.5.10. *Let r be an integer with $2 \leq r \leq k$ and identify J^{r-1} with $J^r/\mathbb{R}$. Then the map $P^r(\mathcal{F}) \to P^{r-1}(\mathcal{F})$ given by taking the underlying $(r-1)$-jet of an r-jet induces an isomorphism of J^{r-1}_d-bundles $P^r(\mathcal{F})/\mathbb{R} \to P^{r-1}(\mathcal{F})$.* □

5.6 Gauge groupoid description of the full holonomy groupoid restricted to a singular leaf

Suppose $(M, \mathcal{F})$ is a transversely order k singular foliation with singular leaf L. Since the restriction $G_{\mathrm{full}}(\mathcal{F})_L$ of the full holonomy groupoid to L is transitive, it must be a gauge groupoid (see Definition 2.1.7). In this section, we show that it is isomorphic to the gauge groupoid of the principal $J^k_{\mathbb{R}}$-bundle $P^k(\mathcal{F})$ constructed in Section 5.5. The main result is Theorem 5.6.6.

The following lemma indicates the method we use to produce elements of the gauge groupoid. The proof is a simple algebraic verification, which we omit.

Lemma 5.6.1. *Let $P \to B$ be a (smooth, left) principal H-bundle. Given $x, y \in B$, denote by Λ_x^y the set of all maps $\lambda : P_y \times P_x \to H$ that satisfy*

$$\lambda(kq, hp) = k\lambda(q,p)h^{-1}$$

for all $h, k \in H$, $p \in P_x$, $q \in P_y$.

1. *Given $\lambda \in \Lambda_x^y$, the element*

$$\overline{\lambda} := [q, \lambda(q,p)p] \in \mathrm{Gauge}(P)_x^y$$

   is independent of choice of $p \in P_x$, $q \in P_y$.

2. *$\lambda \mapsto \overline{\lambda}$ defines a bijection $\Lambda_x^y \to \mathrm{Gauge}(P)_x^y$.*

3. *Given $x, y, z \in B$; $\lambda_1 \in \Lambda_x^y$; $\lambda_2 \in \Lambda_y^z$; $\lambda \in \Lambda_x^z$, the following are equivalent:*
   (a) $\overline{\lambda}_2\overline{\lambda}_1 = \overline{\lambda}$
   (b) *$\lambda_2(r,q)\lambda_1(q,p) = \lambda(r,p)$ for all $p \in P_x$, $q \in P_y$, $r \in P_z$.* □

We need the following simple lemma concerning bisubmersions.

Lemma 5.6.2. *Let $(M, \mathcal{F})$ and $(N, \mathcal{F}_N)$ be singular foliations. Let $U, V \subseteq M$ be open and let $p : U \to N$ and $q : V \to N$ be submersions such that $p^{-1}(\mathcal{F}_N) = \mathcal{F}|_U$ and $q^{-1}(\mathcal{F}_N) = \mathcal{F}|_V$.*

1. *If (W, t, s) is an $\mathcal{F}$-bisubmersion, then $W_{q,p} := (s^{-1}(U) \cap t^{-1}(V), q \circ t, p \circ s)$ is an $\mathcal{F}_N$-bisubmersion.*

2. *Suppose W' is another $\mathcal{F}$-bisubmersion and let $w \in W_{q,p}$, $w' \in W'_{q,p}$. If there is a local morphism of $\mathcal{F}$-bisubmersions from W to W' sending $w_1 \mapsto w_2$, then there is also a local morphism of $\mathcal{F}_N$-bisubmersions from $W_{q,p}$ to $W'_{q,p}$ sending $w_1 \mapsto w_2$.*

Proof. For brevity, put $s_N := p \circ s$, $t_N := q \circ t$ and $\mathcal{F}_W := s_N^{-1}(\mathcal{F}_N) = t_N^{-1}(\mathcal{F}_N)$. Since $C_c^\infty(\ker(ds_N)) \subseteq s_N^{-1}(\mathcal{F}_N)$ and $C_c^\infty(\ker(dt_N)) \subseteq t_N^{-1}(\mathcal{F}_N)$, we have

$$C_c^\infty(\ker(ds_N)) + C_c^\infty(\ker(dt_N)) \subseteq \mathcal{F}_W.$$

On the other hand, $\ker(ds) \subseteq \ker(ds_N)$ and $\ker(dt) \subseteq \ker(dt_N)$, so

$$\mathcal{F}_W = C_c^\infty(\ker(ds)) + C_c^\infty(\ker(dt)) \subseteq C_c^\infty(\ker(ds_N)) + C_c^\infty(\ker(dt_N)),$$

establishing (1). For (2), one needs simply to restrict the domain of the given morphism appropriately. □

Remark 5.6.3. The above lemma shows that $p : U \to N$ and $q : V \to N$ determine a well-defined function (typically not a groupoid morphism) $G_{\text{full}}(\mathcal{F})_U^V \to G_{\text{full}}(\mathcal{F}_N)$ sending $Q_W(w) \mapsto Q_{W_{q,p}}(w)$. This function is smooth when $\mathcal{F}$ and $\mathcal{F}_N$ are almost regular.

The notation in the following definition will facilitate the statement of Theorem 5.6.6.

Definition 5.6.4. Let $(M, \mathcal{F})$ be a transverse order $k \geq 2$ foliation with singular leaf L. Let (W, t, s) be an $\mathcal{F}$-bisubmersion, let $w \in W$ and assume that $x_1 := s(w)$ and $x_2 := t(w)$ belong to L. Let p and q be $\mathcal{F}$-$\mathcal{F}^k_{\mathbb{R}}$-submersions defined at x_1 and x_2 respectively. Then, we put

$$\lambda_W(w, q, p) := j^k_0(\theta) \in J^k,$$

where θ is any diffeomorphism of $\mathbb{R}$ carried by the associated $\mathcal{F}^k_{\mathbb{R}}$-bisubmersion $W_{q,p}$ (Lemma 5.6.2, (1)) at w. Equivalently, $\lambda_W(w, q, p)$ is the image of $Q_{W_{q,p}}(w) \in G(\mathcal{F}^k_{\mathbb{R}})_0$ under the isomorphism $G(\mathcal{F}^k_{\mathbb{R}})_0 \to J^k$ of Proposition 5.2.9.

Lemma 5.6.5. *The k-jet $\lambda_W(w, q, p)$ in Definition 5.6.4 above only depends on:*

1. *the groupoid element $Q_W(w) \in G_{\text{full}}(\mathcal{F})$, and*
2. *the k-jets $j^k_{x_1}(p) \in P^k_{x_1}(\mathcal{F})$ and $j^k_{x_2}(q) \in P^k_{x_2}(\mathcal{F})$.*

Proof. Lemma 5.6.2, (2) gives immediately that $\lambda_W(w, q, p)$ only depends on $Q_W(w)$ (this is in the same vein as Remark 5.6.3). It remains to investigate the dependence of $\lambda_W(w, q, p)$ on p and q. To this end, suppose p' and q' are local $\mathcal{F}$-$\mathcal{F}^k_{\mathbb{R}}$-submersions with $j^k_{x_1}(p) = j^k_{x_1}(p')$ and $j^k_{x_2}(q) = j^k_{x_2}(q')$. Then, by Theorem 5.4.12, there exists, for $i = 1, 2$, a local $\mathcal{F}$-automorphism θ_i which is null at x_i, such that $p' = p \circ \theta_1$ near x_1 and $q' = q \circ \theta_2$ near x_2. Because θ_1, θ_2 are null at w, the $\mathcal{F}$-bisubmersion $W' := W_{\theta_2,\theta_2}$ carries the same local $\mathcal{F}$-automorphisms at w as does W. Equivalently, by Lemma 2.3.15, there exists a local morphism from W to W' sending $w \mapsto w$. Thus, applying Lemma 5.6.2, (2), there is a local morphism from $W_{q,p}$ to $W'_{q,p} = W_{q\circ\theta_2, p\circ\theta_1} = W_{q',p'}$ sending $w \mapsto w$ and it follows that $\lambda_W(w, q, p) = \lambda_W(w, q', p')$. □

We have come to the main result of this section.

Theorem 5.6.6. *Let $(M, \mathcal{F})$ be a transverse order $k \geq 2$ foliation with singular leaf L. There is a unique isomorphism of Lie groupoids*

$$G_{\text{full}}(\mathcal{F})_L \to \text{Gauge}(P^k(\mathcal{F}))$$

such that if (W, t, s) *is an* $\mathcal{F}$*-bisubmersion, and if* $w \in W$ *is such that* $x_1 := s(w)$ *and* $x_2 := t(w)$ *belong to* L*, then*

$$Q_W(w) \mapsto [j^k_{x_2}(q), \lambda_W(w, q, p) \cdot j^k_{x_1}(p)],$$

where p *and* q *are any local* $\mathcal{F}$*-*$\mathcal{F}^k_{\mathbb{R}}$*-submersions defined at* x_1 *and* x_2*, respectively.*

Proof. Suppose that W is a $\mathcal{F}$-bisubmersion and $w \in W$ has $s(w) \in L$. Let p be an $\mathcal{F}$-$\mathcal{F}^k_{\mathbb{R}}$-submersions defined at $s(w)$ and q and $\mathcal{F}$-$\mathcal{F}^k_{\mathbb{R}}$-submersions defined at $t(w)$. If $W_{q,p}$ carries θ at w and $\theta_1, \theta_2 \in \mathrm{Diff}_0(\mathbb{R})$ are given, then it is easy to see $W_{q\circ\theta_2, p\circ\theta_1}$ carries $\theta_2 \circ \theta \circ \theta_1^{-1}$ at w. Therefore, combining Lemma 5.6.5 and Lemma 5.6.1 (2), the procedure in the theorem statement determines a well-defined map $G_{\mathrm{full}}(\mathcal{F})_L \to \mathrm{Gauge}(P^k(\mathcal{F}))$.

Next we claim the map $G_{\mathrm{full}}(\mathcal{F})_L \to \mathrm{Gauge}(P^k(\mathcal{F}))$ under discussion preserves groupoid multiplication. Let (W, t, s) and (W', t', s') be $\mathcal{F}$-bisubmersions and suppose $w \in W$ and $w' \in W'$ have $s'(w') = t(w) \in L$. Put $x_1 := s(w)$, $x_2 := s'(w') = t(w)$, $x_3 := t'(w')$. Let p_i be an $\mathcal{F}$-$\mathcal{F}^k_{\mathbb{R}}$-submersions defined at x_i for $i = 1, 2, 3$. Observe that $W'_{p_3,p_2} \circ W_{p_2,p_1}$ is a closed submanifold of $(W' \circ W)_{p_3,p_1}$. The latter observation implies that, if θ is carried by W_{p_2,p_1} at w and θ' is carried by W'_{p_3,p_2} at w', then $\theta' \circ \theta$ is carried by $(W' \circ W)_{p_3,p_1}$ at (w', w). Thus,

$$\lambda_{W'}(w', p_3, p_2)\lambda_W(w, p_2, p_1) = \lambda_{W'\circ W}((w', w), p_3, p_1)$$

so that, by Lemma 5.6.1 (3), the map $G_{\mathrm{full}}(\mathcal{F})_L \to \mathrm{Gauge}(P^k(\mathcal{F}))$ preserves the groupoid multiplication.

It remains to check the groupoid morphism $G_{\mathrm{full}}(\mathcal{F})_L \to \mathrm{Gauge}(P^k(\mathcal{F}))$ is a diffeomorphism. Because of the local nature of the problem, it suffices (by Proposition 5.3.4) to consider the case $(M, \mathcal{F}) = (\mathbb{R}^n, \mathcal{F}^k_{\mathbb{R}^n})$, $L = \mathbb{R}^\ell \times \{0\}$ which was studied in detail in Section 5.4. We do so for the remainder of the proof. In this case, the principal $J^k_{\mathbb{R}}$-bundle $P^k(\mathcal{F}) \to L$ is isomorphic to $L \times J^k_{\mathbb{R}}$, courtesy of the global section induced by the global $\mathcal{F}$-$\mathcal{F}^k_{\mathbb{R}}$-submersion $\mathrm{pr}_2 : \mathbb{R}^n \to \mathbb{R}$. Correspondingly (Example 2.1.8), there is an isomorphism of Lie groupoids:

$$L^2 \times J^k_{\mathbb{R}} \ni (x_2, x_1, \alpha) \mapsto [j^k_{x_2}(\mathrm{pr}_2), \alpha \cdot j^k_{x_2}(\mathrm{pr}_2)] \in \mathrm{Gauge}(P^k(\mathcal{F})).$$

We also have already a Lie groupoid isomorphism

$$G_{\mathrm{full}}(\mathcal{F}) \to (\mathbb{R}^\ell)^2 \times J^k_{\mathbb{R}}$$

by Theorem 5.4.15. It remains, therefore, to check that the diagram

$$\begin{array}{ccc} G_{\mathrm{full}}(\mathcal{F}) & \longrightarrow & \mathrm{Gauge}(P^k(\mathcal{F})) \\ & \searrow \quad \nearrow & \\ & L^2 \times J^k_{\mathbb{R}} & \end{array}$$

is commutative. For this purpose, recall the full holonomy atlas $\{\widetilde{\Omega}_\theta : \theta \in \mathbb{R}\}$ (Definition 5.4.9). Fix $\theta \in \mathrm{Diff}_0(\mathbb{R})$; $x_1, x_2 \in \mathbb{R}^\ell$; $t \in \mathbb{R}$ and put $w := (x_2, x_1, t, 0) \in \widetilde{\Omega}_\theta$. It is straightforward to chase $Q_{\widetilde{\Omega}_\theta}(w) \in G_{\mathrm{full}}(\mathcal{F})$ both ways around the above diagram and confirm that result is the same. □

5.7 Extracting the holonomy groupoid from the full holonomy groupoid

The holonomy groupoid of a singular foliation is contained in the full holonomy groupoid as an open subgroupoid. To be precise, $G(\mathcal{F})$ is the s-connected component $G_{\mathrm{full}}(\mathcal{F})$ (c.f. Proposition 2.3.5 and [4], Theorem 0.1).

Definition 5.7.1. The **s-connected component** of a Lie groupoid G is the subgroupoid $G_0 \subseteq G$ consisting of all $g \in G$ that can be connected to $s(g)$ by a path in $G_{s(g)}$.

In the preceding section, we obtained a description

$$G_{\mathrm{full}}(\mathcal{F}) = (M \setminus L)^2 \cup \mathrm{Gauge}(P^k(\mathcal{F}))$$

for the full holonomy groupoid (Theorem 5.6.6). As a corollary, we have:

Theorem 5.7.2. *Let $(M, \mathcal{F})$ be a transverse order $k \geq 2$ foliation with singular leaf L. If $M \setminus L$ is connected, then*

$$G(\mathcal{F}) \cong (M \setminus L)^2 \cup \mathrm{Gauge}(P^k(\mathcal{F}))_0.$$

If $M \setminus L$ has two connected components U and V, then

$$G(\mathcal{F}) \cong U^2 \cup V^2 \cup \mathrm{Gauge}(P^k(\mathcal{F}))_0.$$

Here, $\mathrm{Gauge}(P^k(\mathcal{F}))_0$ *denotes the s-connected component of* $\mathrm{Gauge}(P^k(\mathcal{F}))$*. The smooth structure of $G(\mathcal{F})$ is the one inherited from $G_{\mathrm{full}}(\mathcal{F})$.*

Although the bundle $P^k(\mathcal{F})$ is in fact a manifold in our sense (i.e. is metrizable), it is still very large; both it and its structure group have continuum-many components. One may therefore desire a more concrete description of $G(\mathcal{F})_L$ which avoids this bundle. The following proposition, whose proof we omit, shows that we in fact only need to deal with one connected component of $P_0 \subseteq P^k(\mathcal{F})$, which is automatically a second-countable manifold.

Proposition 5.7.3. *Let $P \to B$ be a (smooth, left) principal H-bundle, where H is a (possibly disconnected) Lie group.*

1. *If $p \in P$, then the set of $h \in H$ for which hp belongs to the same component of P as p is a closed and open subgroup $H_p \subseteq H$.*

2. *If p and q belong to the same component of P, then $H_p = H_q$.*

3. *Let P_0 be a component of P. Then P_0 is a principal H_0-bundle, where H_0 is the stabilizer of P_0 or, equivalently, any particular point of P_0.*

4. *The inclusion $P_0 \times P_0 \to P \times P$ descends to a Lie groupoid isomorphism from* $\mathrm{Gauge}(P_0)$ *onto the s-connected component* $\mathrm{Gauge}(P)_0 \subseteq \mathrm{Gauge}(P)$. □

Corollary 5.7.4. *Let $(M, \mathcal{F})$ be a transverse order $k \geq 2$ foliation with singular leaf L. Then $G(\mathcal{F}^K)_L$ is isomorphic to* $\mathrm{Gauge}(P_0)$*, where P_0 is any connected component of the principal J^k-bundle $P^k(\mathcal{F})$.*

The following result gives additional insight into the holonomy *groups* of a transverse order k foliation at points of its singular leaf.

Theorem 5.7.5. *Let $(M, \mathcal{F})$ be a transverse order k foliation with singular leaf L. Fix $x_0 \in L$ and $q_0 \in P^{k-1}(\mathcal{F})_{x_0}$. Let $\gamma : \pi_1(L, x_0) \to J^{k-1}$ be the monodromy homomorphism determined by q_0 and let $\Gamma \subseteq J^{k-1}$ be the range of γ. Finally, let $\Gamma_{\mathbb{R}} \subseteq J^k$ be the preimage of Γ under the projection $P^k(\mathcal{F}) \to P^{k-1}(\mathcal{F})$. Then, the isotropy group of $G(\mathcal{F})$ at any point of L is isomorphic, as a Lie group, to the one-dimensional group $\Gamma_{\mathbb{R}}$. For convenience, the following diagram summarizes the relationship between the various groups at hand:*

$$
\begin{array}{ccccc}
\mathbb{R} & \longrightarrow & J^k_{\mathbb{R}} & \longrightarrow & J^{k-1}_d \\
\| & & \uparrow & & \uparrow \\
\mathbb{R} & \longrightarrow & \Gamma_{\mathbb{R}} & \longrightarrow & \Gamma \\
 & & & & {\scriptstyle\gamma}\uparrow \\
 & & & & \pi_1(L, x_0)\,.
\end{array}
$$

Proof. Let Q_0 denote the connected component of $P^{k-1}(\mathcal{F})$ containing q_0. By standard covering space theory, the subgroup Γ equals the stabilizer of Q_0 so that, by Proposition 5.7.3, Q_0 becomes a principal Γ-bundle. Let P_0 be the preimage of Q_0 under the projection $P^k(\mathcal{F}) \to P^{k-1}(\mathcal{F})$. Since the fibers of this projection are copies of $\mathbb{R}$, it follows that P_0 is a connected component of $P^k(\mathcal{F})$. Thus, by Proposition 5.7.3 (4),

$$\mathrm{Gauge}(P_0) \cong \mathrm{Gauge}(P^k(\mathcal{F}))_0 \cong G(\mathcal{F})_L.$$

The subgroup of $J^k_{\mathbb{R}}$ stabilizing P_0 is $\Gamma_{\mathbb{R}}$. By Proposition 5.7.3 (3), P_0 is a principal $\Gamma_{\mathbb{R}}$-bundle, whence every isotropy group of $\mathrm{Gauge}(P_0)$ is isomorphic to $\Gamma_{\mathbb{R}}$. □

Corollary 5.7.6. *With notation as in Theorem 5.7.5, we have*

$$C^*(G(\mathcal{F})_L) \cong C^*(\Gamma_{\mathbb{R}}) \otimes \mathbb{K},$$

where $\mathbb{K}$ *denotes the compact operators on the canonical* L^2 *space of* L.

Proof. From [38], Theorem 3.1 it follows that, if $G \rightrightarrows M$ is any transitive Lie groupoid (hence a gauge groupoid), then $C^*(G) \cong C^*(H) \otimes \mathbb{K}(L^2(M))$, where H denotes any isotropy group of G. □

Remark 5.7.7. The homomorphisms γ in Theorem 5.7.5 are exactly the same ones used to define the holonomy invariant $h(\mathcal{F})$ (Definition 5.5.6), so Corollary 5.7.6 shows that knowledge of the invariant $h(\mathcal{F})$ is enough to compute $C^*(G(\mathcal{F})_L)$.

5.8 Transverse order k foliations on line bundles

Throughout this section, L is a connected, smooth manifold. If $E \to L$ is a smooth line bundle, we always identity L with the zero section of E, so that $L \subseteq E$.

Definition 5.8.1. Fix a positive integer r.

- If $E \to L$ is a line bundle, we write $J^r(E, \mathbb{R}) \to L$ for the smooth, left principal J^r-bundle whose fiber over $x \in L$ consists of all r-jets at x of local diffeomorphisms $E_x \to \mathbb{R}$ sending $x \mapsto 0$.

- If $U \subseteq E$ is open and $p : U \to \mathbb{R}$ is a submersion with $p^{-1}(0) = U \cap L$, we write $j^r(p) : U \cap L \to J^r(E, \mathbb{R})$ for the smooth section defined by $j^r(p)(x) := j^r_x(p|_{E_x \cap U})$.

- If $\pi_i : E_i \to L$ is a line bundle for $i = 1, 2$, we write $J^r(E_1, E_2) \to L$ for the smooth fiber bundle whose fiber over $x \in L$ consists of r-jets at x of (local) diffeomorphisms $(E_1)_x \to (E_2)_x$ sending $x \mapsto x$.

- Let $\theta : U_1 \to U_2$ be a diffeomorphism, where $U_i \subseteq E_i$ is open for $i = 1, 2$. We say that θ is **fiberwise** if $\pi_2(\theta(e)) = \pi_1(e)$ for all $e \in U_1$ and $\theta(L \cap U_1) = L \cap U_2$. If θ is fiberwise, we write $j^r(\theta) : U \cap L \to J^r(E_1, E_2)$ for the smooth section defined by $j^r(\theta)(x) = j^r_x(\theta|_{E_x \cap U})$.

The following proposition shows that all sections of these jet bundles can be lifted to honest maps.

Proposition 5.8.2. *Let* r *be a positive integer.*

1. *Let E be a line bundle over L. Then, every local section of $J^r(E,\mathbb{R})$ is equal to $j^r(p)$ for some local submersion $p : U \to L$ with $p^{-1}(0) = L \cap U$.*

2. *Let E_1 and E_2 be line bundles over L. Then, every local section of $J^r(E_1, E_2)$ is equal to $j^r(\theta)$ for some local fiberwise diffeomorphism θ.*

Proof. Both of these follow from the fact that each r-jet between a pair of one-dimensional vector spaces sending zero to zero is represented by a unique polynomial mapping of degree $\leq r$. $\square$

If $P \to M$ and $Q \to M$ are smooth principal H-bundles, we write $\mathrm{Hom}_H(P, Q)$ for the usual smooth fiber bundle over M whose fiber over $x \in M$ is the the set of H-equivariant maps $P_x \to Q_x$. We note the following without proof.

Lemma 5.8.3. *If E_1 and E_2 are line bundles over L, then there is a fiber bundle isomorphism $\alpha \mapsto \alpha_* : J^r(E_1, E_2) \to \mathrm{Hom}_{J^r}(J^r(E_1,\mathbb{R}), J^r(E_2,\mathbb{R}))$ defined by $\alpha_*(\beta) = \beta \circ \alpha^{-1}$.* $\square$

In particular, if θ is a fiberwise diffeomorphism from E_1 to E_2 defined on a neighbourhood of L, the corresponding global section $j^r(\theta) : L \to J^r(E_1, E_2)$ determines a principal bundle isomorphism $j^r(\theta)_* : J^r(E_1,\mathbb{R}) \to J^r(E_2,\mathbb{R})$. Thus, $E \mapsto J^r(E,\mathbb{R})$ is functorial for fiberwise diffeomorphisms. In fact, by Proposition 5.8.2, *every* isomorphism $J^r(E_1,\mathbb{R}) \to J^r(E_2,\mathbb{R})$ is induced by a fiberwise diffeomorphism.

The following two results are the main findings of this section.

Theorem 5.8.4. *Let $\pi : E \to L$ be a smooth line bundle with connected base manifold L. There is a unique bijection $\mathcal{F} \mapsto \nabla_{\mathcal{F}}$ from*

1. *the set of transverse order k foliations of E with singular leaf L to*

2. *the set of (smooth) flat connections on $J^{k-1}(E,\mathbb{R})$*

such that a submersion $p : U \to \mathbb{R}$, where $U \subseteq E$ is open, satisfying $p^{-1}(0) = U \cap L$ is a local $\mathcal{F}$-$\mathcal{F}^k_{\mathbb{R}}$-submersion if and only if the associated section $j^{k-1}(p) : U \cap L \to J^{k-1}(E,\mathbb{R})$ is parallel for $\nabla_{\mathcal{F}}$.

Proof. Suppose $U \subseteq E$ is open and $p, q : U \to \mathbb{R}$ are submersions with $p^{-1}(0) = q^{-1}(0) = U \cap L$. Up to shrinking U around $U \cap L$, we may assume that $p_x := p|_{U \cap E_x}$ and $q_x := q|_{U \cap E_x}$ are diffeomorphisms onto their images for all $x \in U \cap L$. Define $\theta_x := q_x \circ (p_x)^{-1}$ for all $x \in U \cap L$ so that, by construction, θ_x is a smooth family of local diffeomorphisms of $\mathbb{R}$ satisfying $\theta_{\pi(e)}(p(e)) = q(e)$ on a neighbourhood of

$U \cap L$. Define $h_{qp} : U \cap L \to J^{k-1}$ by $h_{qp}(x) = j_0^{k-1}(\theta_x) \in J^{k-1}$. By construction, h_{qp} is the transition function for the sections $j^{k-1}(p)$ and $j^{k-1}(q)$ (Definition 5.8.1).

Now suppose $\mathcal{F}$ is a transverse order k foliation of E with singular leaf L and $p, q : U \to \mathbb{R}$ are local $\mathcal{F}$-$\mathcal{F}^k_{\mathbb{R}}$-submersions. We claim that the transition map $h_{qp} : U \cap L \to J^{k-1}$ is locally constant. By Proposition 5.3.4, given any point $x_0 \in U \cap L$, we may choose coordinates so that x_0 is the point $(0, 0)$ in $\mathbb{R}^n = \mathbb{R}^\ell \times \mathbb{R}$ and the foliation is $\mathcal{F}^k_{\mathbb{R}^n}$. Moreover, we may perform this coordinate change in such a way that $p : E \to \mathbb{R}$ and $\pi : E \to L$ are given by $p(x, y) = y$ and $\pi(x, y) = x$ (the linear structure of the fibers is not likely to be preserved under this coordinate change, but that is irrelevant). By Theorem 5.4.6, there are constants $a_1, \ldots, a_{k-1}$ and a smooth function f such that

$$\theta_x(y) = q(x, y) = a_1 y + \ldots + a_{k-1} y^{k-1} + f(x, y) y^k$$

holds on a neighbourhood of $(0, 0)$, whence $h_{qp}(x) = j_0^{k-1}(\theta_x)$ is constant on a neighbourhood of any $x_0 \in U \cap L$. Therefore, applying Lemma 5.5.2, there is a unique J_d^{k-1}-bundle structure on $J^{k-1}(E, \mathbb{R})$ for which $j^{k-1}(p)$ is a smooth section for every local $\mathcal{F}$-$\mathcal{F}^k_{\mathbb{R}}$ submersion p. This shows the map $\mathcal{F} \mapsto \nabla_{\mathcal{F}}$ is well-defined.

Conversely, suppose that ∇ is a flat connection on $J^{k-1}(E, \mathbb{R})$. For each local submersion $p : U_p \to \mathbb{R}$ with $p^{-1}(0) = L \cap U_p$ satisfying $j^{k-1}(p) = U_p \cap L$, define $\mathcal{F}_p = p^{-1}(\mathcal{F}^k_{\mathbb{R}})$. Thus, $\mathcal{F}_p$ is a foliation of U_p. We use Proposition 2.2.14 to glue these foliations together into a foliation $\mathcal{F}$ of M. To this end, it suffices to consider p and q with $U_p = U_q = U$. It is easy to see that $\mathcal{F}_p$ and $\mathcal{F}_q$ both restrict to the trivial one leaf foliation on $U \setminus L$. It remains only to check the foliations agree on a neighbourhood of each point $x_0 \in L$. Using that $j^{k-1}(p)$ and $j^{k-1}(q)$ are ∇-parallel and shrinking U about x_0, there exists $h \in J^{k-1}$ with

$$j^{k-1}(q) = h \cdot j^{k-1}(p)$$

We may pass again to coordinates such that $x_0 = (0, 0) \in \mathbb{R}^\ell \times \mathbb{R}$ and $\pi(x, y) = x$, $p(x, y) = y$ near $(0, 0)$. Writing $h = a_1 y + \ldots + a_{k-1} y^{k-1}$, the equation above then says that $\frac{\partial^r q}{\partial y^r}(x, 0) = a_r$ holds near $(0, 0)$ for $r = 1, \ldots, k-1$. Thus, applying Theorem 5.4.6 again, we have that, in these coordinates, $p^{-1}(\mathcal{F}^k_{\mathbb{R}})$ and $q^{-1}(\mathcal{F}^k_{\mathbb{R}})$ agree with $\mathcal{F}^k_{\mathbb{R}^n}$ in a neighbourhood of $(0, 0)$.

It is easy to see that these mappings between foliations and flat connections are inverses of each other. □

Next we show that the correspondence of the above theorem is natural with respect to fiberwise diffeomorphisms.

Theorem 5.8.5. *Let L be a connected, smooth manifold. For $i = 1, 2$, let E_i be a smooth line bundle over L equipped with a transverse order k foliation $\mathcal{F}_i$ with* *singular leaf L. Suppose that $\theta : U_1 \to U_2$ is a fiberwise diffeomorphism, where U_i is an open subset of E_i containing L_i. Then, the following are equivalent:*

1. *$\theta_*((\mathcal{F}_1)_{U_1}) = (\mathcal{F}_2)_{U_2}$.*
2. *The induced J^{k-1}-bundle isomorphism $j^{k-1}(\theta)_* : J^{k-1}(E_1, \mathbb{R}) \to J^{k-1}(E_2, \mathbb{R})$ pushes forward $\nabla_{\mathcal{F}_1}$ to $\nabla_{\mathcal{F}_2}$.*

Proof. Assume for convenience that $U_1 = E_1$, $U_2 = E_2$. Let $U \subseteq L$ be open and let $p : U \to L$ be a submersion satisfying $p^{-1}(0) = U \cap L$. Clearly (1) is equivalent to the assertion: p is a local $\mathcal{F}_1$-$\mathcal{F}^k_{\mathbb{R}}$-submersion if and only if $p \circ \theta^{-1}$ is a local $\mathcal{F}_2$-$\mathcal{F}^k_{\mathbb{R}}$-submersion. Meanwhile (applying Proposition 5.8.2 (2)), (2) is equivalent to the assertion: $j^{k-1}(p)$ is $\nabla_{\mathcal{F}_1}$ parallel if and only if $j^{k-1}(\theta)_*(j^{k-1}(p))$ is $\nabla_{\mathcal{F}_2}$ parallel. Since, $j^{k-1}(\theta)_*(j^{k-1})(p) = j^{k-1}(p) \circ j^{k-1}(\theta)^{-1} = j^{k-1}(p \circ \theta^{-1})$, the desired conclusion is an immediate consequence of Theorem 5.8.4. □

5.9 Completeness of the holonomy invariant

Let $(M, \mathcal{F})$ be a transverse order k foliation with singular leaf L. In this section, we show that the holonomy invariant (Definition 5.5.6)

$$h(\mathcal{F}) \in [\pi_1(L), J^{k-1}]$$

together with the diffeomorphism type of L is a complete invariant for the structure of $\mathcal{F}$ nearby to L.

Given a line bundle equipped with a transverse order k foliation whose singular leaf is the zero section, there are ostensibly two principal J^{k-1}_d-bundles at play: the one from Section 5.5 and the one from Section 5.8. As one would probably suspect, these two bundles are in fact the same.

Proposition 5.9.1. *Let $E \to L$ be a smooth line bundle and let $\mathcal{F}$ be transverse order k foliation of E with singular leaf L. There is an isomorphism of principal J^{k-1}_d-bundles $P^{k-1}(\mathcal{F}) \to J^{k-1}(E, \mathbb{R})$ such that, if $p : U \to \mathbb{R}$ is a local $\mathcal{F}$-$\mathcal{F}^k_{\mathbb{R}}$-submersion and $x \in U \cap L$, then $j^k_x(p) \mapsto j^k_x(p|_{E_x \cap U})$. Here, $J^k(E, \mathbb{R})$ is given the structure of a J^{k-1}_d-bundle using the flat connection $\nabla_{\mathcal{F}}$ of Theorem 5.5.*

Proof. The map is clearly well-defined and J^{k-1}-equivariant. Given a local $\mathcal{F}$-$\mathcal{F}^k_{\mathbb{R}}$-submersion $p : U \to \mathbb{R}$, the sections $x \mapsto j^{k-1}_x(p) : U \cap L \to P^{k-1}(\mathcal{F})$ and $x \mapsto j^{k-1}_x(p|_{E_x \cap U}) : U \cap L \to J^{k-1}(E, \mathbb{R})$ are smooth as part of the definition of those bundles. The first of these smooth local sections is clearly mapped to the second, and from this it follows that the map $P^{k-1}(\mathcal{F}) \to J^{k-1}(E, \mathbb{R})$ is smooth. □

We now state and prove the main result of the section. One may refer to Subsection 2.1.4 and Definition 5.5.6 for context.

Theorem 5.9.2. *For $i = 1, 2$, let $(M_i, \mathcal{F}_i)$ be transverse order k foliations with singular leaf L_i . Suppose there is a diffeomorphism $\theta_0 : L_1 \to L_2$ whose induced map $[\pi_1(L_1), J^{k-1}] \to [\pi_1(L_2), J^{k-1}]$, given by pushing forward loops, sends $h(\mathcal{F}_1)$ to $h(\mathcal{F}_2)$. Then, we can extend θ_0 to a diffeomorphism $\theta : U_1 \to U_2$, where $U_i \subseteq M_i$ are open neighbourhoods of L_i, so as to have $\theta_*(\mathcal{F}_1|_{U_1}) = \mathcal{F}_2|_{U_2}$.*

Proof. Without loss of generality, $L = L_1 = L_2$ and $\theta_0 = \mathrm{id}_L$. Thus $h(\mathcal{F}_1) = h(\mathcal{F}_2)$. By Definition 5.5.6, this means exactly that the principal J^{k-1}_d-bundles $P^{k-1}(\mathcal{F}_1)$ and $P^{k-1}(\mathcal{F}_2)$ have the same monodromy invariant so, by Proposition 2.1.11, there is an isomorphism of principal J^{k-1}_d-bundles $P^{k-1}(\mathcal{F}_1) \to P^{k-1}(\mathcal{F}_2)$ covering the identity map on L.

By passing to tubular neighbourhoods, we may assume that M_i is the total space of a line bundle $\pi_i : E_i \to L$, with L contained in E_i as the zero section. Then, applying Proposition 5.9.1, there is a J^{k-1}-bundle isomorphism $J^{k-1}(E_1, \mathbb{R}) \to J^{k-1}(E_2, \mathbb{R})$, equivalently a section of $J^{k-1}(E_1, E_2)$, which moreover maps $\nabla_{\mathcal{F}_1}$ to $\nabla_{\mathcal{F}_2}$. By Proposition 5.8.2, we can take this section of the form $j^{k-1}(\theta)$ for θ some fiberwise diffeomorphism $E_1 \to E_2$ preserving L. Then, by Theorem 5.8.5, $\theta_*(\mathcal{F}_1|_{U_1}) = \mathcal{F}_2|_{U_2}$, as desired. □

As a corollary, when L is simply connected, there is only one "L-local" isomorphism class of transverse order k foliation with singular leaf L.

Corollary 5.9.3. *Let $(M, \mathcal{F})$ be a transverse order k foliation with singular leaf L. If L is simply connected, then there exists a diffeomorphism $\theta : U \to L \times \mathbb{R}$, where $U \subseteq M$ is an open neighbourhood of L, such that $\theta|_L = \mathrm{id}_L$ and $\theta_*(\mathcal{F}_U)$ is the product the trivial one-leaf foliation of L with $\mathcal{F}^k_{\mathbb{R}}$.*

More generally, the same conclusion holds when there is no nontrivial homomorphism $\pi_1(L) \to J^{k-1}$. For example, this is the case when $\pi_1(L)$ is a simple, nonsolvable group (see Proposition 5.1.6).

5.10 Range of the holonomy invariant

In this section we show that the range of the holonomy invariant (Definition 5.5.6) is as large as one could hope it to be. Specifically, we prove the following:

Theorem 5.10.1. *Let L be a connected, smooth manifold, let $x_0 \in L$ and let $\gamma : \pi_1(L, x_0) \to J^{k-1}$ be a group homomorphism. There exists a line bundle*

$E \to L$ and a transverse order k foliation $\mathcal{F}$ on E with singular leaf L (identified with the zero section of E) whose holonomy invariant $h(\mathcal{F})$ is equal to the class of γ in $[\pi_1(L), J^{k-1}]$.

Together, Theorems 5.9.2 and 5.10.1 imply that the problem of enumerating the "L-local" isomorphism classes of transverse order k-foliations with a fixed singular leaf L is equivalent to the problem of enumerating homomorphisms $\gamma : \pi_1(L) \to J^{k-1}$ modulo a certain equivalence relation. Specifically:

$$\gamma \sim \alpha \circ \gamma \circ \beta$$

as α ranges over inner automorphisms of J^{k-1} and β ranges over automorphisms of $\pi_1(L)$ that can be induced by a diffeomorphism of L. Under a variety of assumptions on L, one is assured of being able to implement any automorphism of $\pi_1(L)$ with a diffeomorphism of L. For example, if L is a surface, then the homomorphism from the mapping class group to the group of outer automorphisms of $\pi_1(L)$ is an isomorphism (the Dehn-Nielsen-Baer theorem). Another example: for L a hyperbolic 3-manifold of dimension ≥ 3, one has that $\mathrm{Isom}(L) \to \mathrm{Out}(\pi_1(L))$ is an isomorphism by the Mostow rigidity theorem. In cases such as these, the L-local isomorphism classes of transverse order k foliations with singular leaf L can be put into correspondence with the quotient set

$$\mathrm{Inn}(J^{k-1}) \setminus \mathrm{Hom}(\pi_1(L), J^{k-1}) / \mathrm{Aut}(\pi_1(L)).$$

The proof of Theorem 5.10.1 will be an exercise in manipulating principal bundles and will have little to do with foliations per se. Let us first recall the associated principal bundle construction:

Definition 5.10.2. If $\varphi : G \to H$ is a homomorphism of Lie groups and P is a principal G-bundle, then $H \times_\varphi P$ denotes the principal H-bundle whose total space is the quotient of $H \times P$ by the equivalence relation defined by $(h, p) \sim (hg, g^{-1}p)$ for all $g \in G$ and equipped with the H-action defined by $h[h', p] = [hh', p]$. Here we denote the class of (h, p) by $[h, p]$.

See for example [31], Chapter 5. The following instances of this construction will be relevant:

Example 5.10.3. If $1 \to N \to G \xrightarrow{\pi} H \to 1$ is an exact sequence of Lie groups and if P is a principal G-bundle, then there is a principal H-bundle isomorphism $P/N \to H \times_\pi P$ sending $[p] \mapsto [1, p]$.

Definition 5.10.4. Given a line bundle E, the **coframe bundle** $F^*(E)$ of E is the principal $\mathrm{GL}(1) := \mathrm{GL}(1, \mathbb{R})$-bundle whose fiber at x is the collection of nonzero linear functionals $E_x \to \mathbb{R}$.

Example 5.10.5. Let E be a smooth line bundle and r a positive integer. Denote by ι the natural inclusion $a \mapsto ay : \mathrm{GL}(1) \to J^r$. Then there is a principal J^r-bundle isomorphism $J^r \times_\iota F^*(E) \to J^r(E, \mathbb{R})$ sending $[j_0^r(\theta), \varphi] \mapsto j_x^r(\theta \circ \varphi)$ for all $j_0^r(\theta) \in J^r$ and all $\varphi : E_x \to \mathbb{R}$.

Example 5.10.6. Let M be a connected smooth manifold with basepoint x_0. Let $\widetilde{M}$ be the universal cover of M with corresponding basepoint $\widetilde{x}_0$. Let Γ be a discrete group and $\varphi : \pi_1(M, x_0) \to \Gamma$ a homomorphism. View $\widetilde{M}$ as a principal $\pi_1(M, x_0)$-bundle in the usual way using deck transformations. Put $P = \Gamma \times_\varphi \widetilde{M}$ and $p_0 = [1, \widetilde{x}_0]$. Then, P is a principal Γ-bundle over M for which the monodromy homomorphism $\pi_1(M, x_0) \to \Gamma$ determined by the point p_0 is exactly φ.

Suppose we have an exact sequence of Lie groups $1 \to N \to G \xrightarrow{\pi} H \to 1$ that is split by a homomorphism $\iota : H \to G$. In other words, suppose G is presented as a semidirect product $N \rtimes H$. Then, using the fact that $\pi \circ \iota = \mathrm{id}_H$, there is a natural isomorphism

$$P \cong H \times_\pi (G \times_\iota P) \tag{5.2}$$

for every principal H-bundle P. On the other hand, $\iota \circ \pi \neq \mathrm{id}_G$ (unless π is an isomorphism), so there is no reason to expect the opposite relation

$$Q \cong G \times_\iota (H \times_\pi Q) \tag{5.3}$$

to hold for every principal G-bundle Q. With some knowledge of classifying space theory , however, one may prove the following.

Proposition 5.10.7. *Let $1 \to N \to G \xrightarrow{\pi} H \to 1$ be an exact sequence of Lie groups and let $\iota : H \to G$ satisfy $\pi \circ \iota = \mathrm{id}_H$. If either ι or π is a homotopy equivalence, then $Q \cong G \times_\iota (H \times_\pi Q)$ for every principal G-bundle Q.*

Note that, unlike the isomorphism (5.2), the isomorphism given by the above proposition is not natural.

Proof sketch. A homomorphism between G and H which is also a homotopy equivalence induces a homotopy equivalence at the level of classifying spaces. It follows that the associated bundle construction for such a homomorphism induces a bijection between isomorphism classes of G-bundles and isomorphism classes of H-bundles. The desired claim then follows from the principle that a one-sided inverse for an invertible map is the two-sided inverse. We refer the reader to [33], Chapter 4 or [47], Part II for relevant aspects of classifying space theory. □

In the following two lemmas, we give an alternative proof of a special case of the above proposition which avoids the technology of classifying space theory.

Lemma 5.10.8. *Let $1 \to N \to G \to H \to 1$ be short exact exact sequence of Lie groups. Let $P \to B$ and $Q \to B$ be smooth principal G-bundles. If $P/N \cong Q/N$ as principal H-bundles and N is diffeomorphic to a finite-dimensional Euclidean space, then $P \cong Q$ as principal G-bundles.*

Proof. Recall $\mathrm{Hom}_G(P, Q)$ is defined as the fiber bundle over B whose fiber at $x \in B$ is the set of G-equivariant maps from P_x to Q_x. Reformulated slightly, the statement to be proven is that $\mathrm{Hom}_G(P, Q) \to B$ admits a (smooth) global section. We are given that the fiber bundle $\mathrm{Hom}_H(P/N, Q/N) \to B$ has a global section. We may instead view $\mathrm{Hom}_G(P, Q)$ as a fiber bundle over $\mathrm{Hom}_H(P/N, Q/N)$ with typical fiber N. Since N is locally Euclidean, it follows that the latter fiber bundle has a global section. Indeed, the Tietze extension theorem can be used to show that every fiber bundle over a manifold with Euclidean space fibers has a continuous section (see [47], Theorem 12.2) and routine smoothing arguments can then be applied. Composing the section $B \to \mathrm{Hom}_H(P/N, Q/N)$ with the section $\mathrm{Hom}_H(P/N, Q/N) \to \mathrm{Hom}_G(P, Q)$ gives the result. □

Lemma 5.10.9. *Let $1 \to N \to G \to H \to 1$ be a short exact exact sequence of Lie groups and let $\iota : H \to G$ satisfy $\pi \circ \iota = \mathrm{id}_H$. Let $P \to B$ be a smooth principal G bundle. If is N diffeomorphic to a finite-dimensional Euclidean space, then there is a smooth isomorphism of principal G-bundles from P to $G \times_\iota (H \times_\pi P)$.*

Proof. Let $Q := H \times_\pi P$. Since $\pi \circ \iota = \mathrm{id}_H$, we have $Q \cong H \times_\pi (G \times_\iota Q)$ as principal H-bundles. Noting that $Q \cong P/N$ and $H \times_\pi (G \times_\iota Q) \cong (G \times_\iota Q)/N$ (see (see Example 5.10.3), we have $P/N \cong (G \times_\iota Q)/N$. Thus, applying Lemma 5.10.8, P is isomorphic to $G \times_\iota Q$, as desired. □

In particular, the above lemma applies to the extension

$$1 \to N \to J^r \xrightarrow{\pi} \mathrm{GL}(1) \to 1,$$

where quotient map is $a_1 y + \ldots + a_r y^r \mapsto a_1$, N is its kernel and $\iota(a) = ay$ provides a splitting on the right. From this we obtain:

Proposition 5.10.10. *Let $P \to B$ be any principal J^r bundle, where r is a positive integer. Then, there exists a line bundle $E \to B$ such that $P \cong J^r(E, \mathbb{R})$ as principal J^r-bundles.*

Proof. Applying Lemma 5.10.9 to P with the extension described above, we obtain $P \cong J^r \times_\iota Q$ where $Q := \mathrm{GL}(1) \times_\pi P$. Any $\mathrm{GL}(1)$-bundle is isomorphic to the coframe bundle of its associated vector bundle, so we have $Q \cong F^*(E)$ where $E := \mathbb{R} \times_{\mathrm{GL}(1)} Q$. Thus, $P \cong J^r \times_\iota F^*(E) \cong J^r(E, \mathbb{R})$ (see Example 5.10.5). □

We conclude with the proof of the main result of this section.

Proof of Theorem 5.10.1. Let $\gamma : \pi_1(L, x_0) \to J^{k-1}$ be any homomorphism. Let $P \to L$ be any principal J_d^{k-1}-bundle such that the monodromy mapping associated to some point $p_0 \in P_{x_0}$ is γ. For instance, we may use $P = J^{k-1} \times_\gamma \widetilde{L}$, as in Example 5.10.6. We may equivalently view P as a principal J^{k-1}-bundle equipped with a flat connection. According to the preceding lemma, there exists a line bundle $E \to L$ and an isomorphism $P \to J^{k-1}(E, \mathbb{R})$ of principal J^{k-1}-bundles. Push forward the flat connection on P to $J^{k-1}(E, \mathbb{R})$ through this isomorphism, and then use Theorem 5.8.4 to produce a corresponding transverse order k foliation $\mathcal{F}$ of E with singular leaf L. By Proposition 5.9.1, $J^{k-1}(E, \mathbb{R})$ and $P^{k-1}(\mathcal{F})$ are isomorphic as J_d^{k-1}-bundles, so the holonomy invariant of $\mathcal{F}$ is as desired. □

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