CONCEPTUAL DEVELOPMENT OF PROSPECTIVE ELEMENTARY
TEACHERS: THE CASE OF DIVISION OF FRACTIONS

A Thesis in
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by
Ismail Ozgur Zembat

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The thesis of Ismail Ozgur Zembat was reviewed and approved* by the following:

- Martin A. Simon
  Professor of Education
  Thesis Advisor
  Chair of Committee

- Glendon W. Blume
  Professor of Education

- Thomas M. Dana
  Professor of Education

- James E. Martin
  Associate Professor of Psychology

- Patrick W. Shannon
  Professor of Education
  Coordinator for Graduate Programs in Curriculum and Instruction

*Signatures are on file in the Graduate School.
ABSTRACT

The purpose of this study is to investigate the understandings and the conceptual development of prospective elementary teachers in the area of division of fractions. The participants were two prospective elementary teachers enrolled in a teacher certification program at a northeastern U.S. university. The current study is based on teaching experiment as a vehicle to investigate the conceptual development of the participants.

The investigation conducted on the two prospective elementary teachers revealed that they had a compartmentalized understanding of division of fractions. The results of the study suggest that the participants viewed division of fractions as a sequence of arithmetic relationships and they did not have an abstraction of quotitive situations. This study also shows that the participants made an abstraction of quotitive situations by reflecting on their mental activities in a diagrammatic world.

In addition, this study investigates the mathematical structure of division of fractions as seen through these two participants’ work. As a result, division of fractions concept is based on two main operations: partitioning and quantification. In this regard, the overall goal for the division of fractions is to determine number of divisor groups through partitioning a given dividend quantity. Therefore, one needs to understand that in a given division of fractions problem, a given quantity needs to be partitioned based on a second quantity and then this partitioning needs to be quantified.

This study also shows that the participants did not have an abstraction of divisor as an intensive quantity that connects the two extensive quantities, the dividend and the quotient. Lacking such an abstraction caused the participants not be able to coordinate the quantitative relationship between the divisor, remainder and fractional part of the quotient.
under the guidance of the overall goal they set for the given division of fractions problems.

What also arises from this study is a developmental trajectory for creating an algorithm that is based on the participants’ spontaneous activity. This algorithm is common denominator algorithm as opposed to the commonly known invert-and-multiply algorithm. Using diagrams for thinking about the common denominator algorithm helped the participants make conceptual advances in their thinking.
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Chapter 1

Statement of the Problem

Current research efforts direct the focus of attention in mathematics education toward understanding mathematics and encouraging the idea of teaching meaningful mathematics. Such a direction demands knowledgeable teachers who need deep conceptual understanding about the ideas that are central to mathematics (National Council of Teachers of Mathematics, 2000). One of the areas in which teachers need to have such deep understanding is number operations. As identified by the National Council of Teachers of Mathematics (National Council of Teachers of Mathematics, 2000), students should be familiar with the meaning of number operations with whole numbers in Grades 3-5 and be able to extend that to fractions and decimals throughout the middle grades. To give students such meaningful knowledge about the number operations, teachers should have a very strong and “profound understanding” (Ma, 1999) of those operations.

One of the number operations, that is very complex in nature, is division. Here, complexity is considered as in Thompson’s (1993) description. Thompson describes complex situations as the ones that require keeping multiple relationships in mind in order to constitute them (p.165). Division of fractions is one such complex concept that in nature consists of a network of quantitative relationships and requires one to handle those multiple relationships all at once.
One classic approach to teaching the division of fractions concept in traditional school curricula is to treat it as manifestation of a meaningless rule, mostly known as the invert-and-multiply rule\(^1\). Such an approach ignores the nature of the aforesaid network of relationships and reduces the division of fractions concept to a simple memorized rule. Many students and teachers have knowledge of division (of fractions) that does not require any inquiry into this network of relationships and that is disconnected from any other mathematical idea (Armstrong & Bezuk, 1995; Simon, 1993). On the other hand, the mathematics education research area does not have a well-articulated description of the division concept within the context of fractions. Articulation is not meant to involve a basic mathematical description of the concept. On the contrary, it refers to a comprehensive treatment of the division of fractions concept that sheds light on the quantitative reasoning needed to handle such a network of quantitative relationships. Here, quantitative reasoning is “the analysis of a situation into a quantitative structure- a network of quantities and quantitative relationships” as described by Thompson (1989, March, cited in Thompson, 1993, p.165). One way to do such a detailed analysis, as intended in this study, is to inquire into learners’ understanding and development of the division of fractions concept.

\(^1\) Invert-and-multiply rule stands for conversion of a division of fractions problem into a multiplication of fractions problem by inverting the second factor in the division problem and multiplying it by the first factor. Even though this algorithm does not have a special name I use the commonly used phrase “invert-and-multiply” when referring to it.
The purpose of this study was to investigate understandings and the conceptual development of prospective elementary teachers in the area of division of fractions. To conduct this investigation, the teaching experiment methodology\(^2\) was used as a vehicle.

From their analysis of the literature Armstrong and Bezuk (1995) pointed to existence of several obstacles in doing research in the area of division and multiplication of fractions. They identified the first obstacle as the lack of a clear conceptual framework for teaching these topics. And, the second obstacle was the lack of knowledge about how students think about division and multiplication of fractions. This study contributes to these two concerns using a conceptual framework in investigating and characterizing teachers’ understanding of division within the context of fractions.

The Reflection on Activity Effect Relationship framework (here abbreviated as RAER framework) that uses Piagetian constructs of assimilation and reflective abstraction is the basis for the current study. This framework (Simon, Tzur, Heinz, & Kinzel, in press) was employed as an orienting framework during the study in examining prospective elementary teachers’ evolving understanding of division of fractions\(^3\). The current study was different from some other mathematics education research studies pertaining to division (e.g., Tirosh and Graeber, 1990) that focused in some way on use of cognitive conflict as a source to foster development.

\(^2\) A detailed description of teaching experiment methodology will be provided in Chapter 3 and Chapter 4.

\(^3\) A detailed discussion of the particulars of RAER framework will be given in Chapter 2.
1.1. Overview of the Study

The overall goal in the study was to investigate the prospective elementary teachers’ ways of reasoning about, understanding of and conceptual development concerning division of fractions. To foster the conceptual development of participants in the area of division of fractions, the participants were limited to use of diagrams, and they were not allowed to use any computation in dealing with given tasks\(^4\). Using such a basic approach, the participants were oriented toward developing a common-denominator algorithm based on their work on diagrams. This algorithm was different from the traditional invert-and-multiply algorithm.

The current study consisted of a teaching experiment spanning approximately two months of inquiry into two prospective elementary teachers’ understandings and their development of key ideas for division of fractions.

1.2. Rationale for the Study of the Development of Division of Fractions

Even though students’ and teachers’ understandings in general (Ball, 1990, 1993; Simon, 1993), the conceptual difficulties they have (Armstrong & Bezuk, 1995; Sowder, 1995; Sowder, Armstrong, Lamon, Simon, Sowder, & Thompson, 1998b; Sowder, Philipp, Armstrong, & Schappelle, 1998a), and their misconceptions (Bell, Swan, & Taylor, 1981; Fishbein, Deri, Nello, & Marino, 1985; Greer, 1987, 1992; Mack, 1993)

\(^4\) Here, use of counting and addition to find number of divisor groups within a dividend is not considered to be as use of computation.
concerning division (of fractions) have been documented to some extent, the nature of their understanding of division (of fractions) and how that understanding develops are not fully articulated. The current study tried to illuminate this area of research.

A study of the conceptual development of prospective teachers is necessary and important due to the following issues. The rationale for each issue will be provided in subsequent sections. The concept of division of fractions is a key concept and very complex in nature. A need exist for studies:

1. That investigate this complexity and identify a “knowledge package” (Ma, 1999) consisting of developmental understandings concerning division of fractions.
2. That focuses on the development of understanding of division of fractions for teachers.
3. That centers on the process by which students and teachers develop necessary understandings regarding the concept of division of fractions.

A brief elaboration of each of the above issues follows.

1.2.1. Division of Fractions Is a Key Concept

Division of fractions is a multidimensional complex mathematical structure. It lies within the intersection of two mathematical areas, division and fractions, (Armstrong & Bezuk, 1995) that are problematic both for teachers and students in elementary and middle grade levels. Therefore, it requires conceptual proficiency in both domains. Fractions belongs to a set of numbers (rational numbers) that has several different
interpretations (Kieren, 1992, p.333). On the other hand, division is an operation that acts on that set. This operation is different from others (e.g., addition, subtraction) in that different properties are true for it than are true for some other operations. This dual focus for division of fractions suggests a particular attention in school mathematics.

Division with whole numbers presents considerable difficulty for students and teachers and the use of fractions makes this concept even more complicated for the learners (Armstrong & Bezuk, 1995; Greer, 1987; Ma, 1999; Sowder, 1995; Sowder, Philipp, Armstrong, & Schappelle, 1998a). In her cross-cultural study, Ma (1999) investigated the knowledge of several U.S. and Chinese teachers to characterize their understanding of some mathematical ideas. Division of fractions was one of them. According to that study, to a problem like $\frac{3}{4} \div \frac{1}{2}$, only 43% of the U.S. teachers were able to provide a procedural answer using a correct algorithm and only 4% produced a conceptually correct explanation or story. For the Chinese teachers this result was 100% with respect to procedural proficiency and about 90% with respect to conceptual competence. Such a difference shows that there is a difficulty in this area for U.S. teachers.

What makes division of fractions a key concept is that it involves several different but interconnected ideas such as dividend, divisor, remainder, referent units, the goal of locating divisor within the dividend and coordination of all these key components. A learner needs to produce an analysis of the quantities that are involved in the division, of the relationships among those quantities, of the referent units and of the goal of the division in order to manage the level of complexity in division of fractions problems.
These issues will be explained further in the literature review (Chapter 3) and the methodology review (Chapter 4).

As illustrated in research studies, in modeling division situations students and teachers used some division models when those models were available to them. Fischbein and his colleagues (1985, p.7) described these models as partitive and quotitive (also known as measurement) models. The partitive division model refers to the missing size of the groups given the number of groups and total number of items in a division situation. The quotitive division model, on the other hand, refers to the missing number of groups that have certain size within total number of items (dividend). As indicated by many research studies, students/teachers have a strong tendency to use partitive primitive models when they encounter a division problem (Armstrong & Bezuk, 1995; Ball, 1990; Behr, Harel, Post, & Lesh, 1992; Bell, Swan, & Taylor, 1981; Fishbein, Deri, Nello, & Marino, 1985; Graeber, Tirosh, & Glover, 1986; Greer, 1992; Hiebert & Behr, 1988; Ma, 1999; Mack, 1995; Nesher, 1988; Simon, 1993; Sowder, 1995; Sowder, Armstrong, Lamon, Simon, Sowder, & Thompson, 1998b; Sowder, Philipp, Armstrong, & Schappelle, 1998a; Tirosh & Graeber, 1989, 1990). Teachers’ use of primitive models has been well documented in the research (Ball, 1990; Graeber, Tirosh, & Glover, 1986; Simon, 1993; Tirosh & Graeber, 1989, 1990). It is also known from the literature that many teachers’ are not able to reason with quotitive division situations effectively. This study promoted the quotitive understanding of division which refers to the understanding that \( \frac{a}{b} \div \frac{c}{d} \) means “How many quantities \( \frac{c}{d} \) are contained in the quantity \( \frac{a}{b} \)?” It may also be interpreted in the case where the dividend is less than the divisor as, “How much
of \( \frac{c}{d} \) can fit in \( \frac{a}{b} \)?” In either case, both fractional quantities refer to the same referent unit. “By developing rational number versions of the two division models, one’s original understanding of the two whole number models will become more comprehensive…” (Ma, 1999, p.77). Therefore, a study of the division concept in the fractional setting is an important one.

The target in this study was the quotitive interpretation of division that leads one more naturally to the common-denominator algorithm [finding the least common denominators of the dividend and divisor \( \frac{a}{b} \div \frac{c}{d} = \frac{ad}{lcm(b,d)} \div \frac{cb}{lcm(b,d)} \), and dividing the numerators, \( ad/cb \)] rather than to use the invert-and-multiply algorithm

\[
\left[ \frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc} \right]
\]
as investigated in other studies (e.g., Tzur & Timmerman, 1997).

The aforementioned reasons create a need for research studies that make a detailed articulation of the division concept as a whole by focusing on its constituent components and the connections among those components. In her analysis of characteristics of deep understanding, Ma (1999) made an attempt to address the division of fractions concept as seen through her work on in-service teachers in the U.S. and China. However, this attempt was limited to the opinions of teachers participating in the study. Even though one of the products of Ma’s study was creation of a “knowledge package” for division of fractions, this knowledge package consisted of other number operations that were linked to division in some way. What was missing from such a knowledge package was the very close investigation of crucial ideas embedded in the division of fractions concept that play an important role in helping learners make
conceptual advances. The current study attempted to articulate a “knowledge package” for division of fractions by focusing on the crucial mathematical ideas embedded in it and focusing on the development of those ideas.

1.2.2. The Conceptual Development of Learners Needs to be Studied

Student strategies, use of different models to solve division problems, and the impact of those models on student and teacher learning have been documented in the research (Kouba, 1989; Lamon, 1994, 1996; Simon, 1993; Steffe, 2002; Thompson, 1994; Tirosh & Graeber, 1989). There is also a body of research on the effects of problem types and the effects of use of different quantities on student learning (Greer, 1987; Schwartz, 1988). Another major part of research in this area articulates the operations being used (partitioning, unitizing, etc.) in solving multiplicative problems and students’ developmental levels for developing such operations (Lamon, 1994, 1996; Mack, 1990; Pothier & Sawada, 1983). In addition, research studies have documented the informal knowledge and preconceptions students bring into the classroom and the impact of those on student performance and understanding in the area of multiplicative concepts (e.g., Mack, 1990, 1995).

Although research focuses on students’ (informal or formal) conceptions of division and operations involved in it (e.g., Ball, 1990; Kouba, 1989; Pothier & Swada, 1983; Simon, 1993), the process by which students or teachers develop these conceptions is not well understood and requires further study. Therefore, there is a need for studies that investigate how the development of students’ and teachers’ conceptions of
multiplicative concepts such as division of fractions takes place. The current study pursues such an enterprise.

1.2.3. Studies of Teachers’ Development of Understandings of Division Are Needed

One body of research focuses on student strategies, student misconceptions, and understanding of mathematical ideas. Division is one such mathematical idea. Although a number of studies target division [of fractions] understanding among teachers, (e.g., Ball, 1990; Simon, 1993), new studies need to investigate teachers’ understanding of division of fractions and the development of that understanding. Further investigations of teachers’ understanding aid analyses which are helpful in investigating students’ knowledge (Tirosh & Graeber, 1989). Another imperative is the need to increase teachers’ level of understandings (Sowder, 1995; Sowder, Armstrong, Lamon, Simon, Sowder, & Thompson, 1998b; Sowder, Philipp, Armstrong, & Schappelle, 1998a) in order for them to influence students’ conceptual development in positive and meaningful ways. Results arising from such investigations which target teachers’ and students’ understandings of the mathematical concepts give teachers opportunities to identify key understandings for themselves and create a challenge to design appropriate lessons that aim at those understandings.

Since multiplicative concepts such as division (of fractions) are fundamental to students’ mathematical growth at the upper elementary and middle school levels, teachers need to have an in-depth, profound understanding (Ma, 1999; Sowder, Armstrong, Lamon, Simon, Sowder, & Thompson, 1998b; Sowder, Philipp, Armstrong, &
Schappelle, 1998a) of these concepts. However, research in this area indicates that teachers do not have adequate ways to reason with multiplicative structures such as division of fractions (Graeber, Tirosh, & Glover, 1986; Sowder, Armstrong, Lamon, Simon, Sowder, & Thompson, 1998b). This indicates that an investigation of profound understanding of such concepts at a level that is appropriate for effective instruction (Ma, 1999) is necessary. Therefore, studies like the current one provide useful ways to think about effective teacher education programs that help teachers develop key understandings in the area of multiplicative structures.

In the above argument, there is an assumption that if teachers understand something profoundly, they will be able to engage in better instruction with their students (choosing activities insightfully, responding to students’ questions appropriately, etc.). A number of studies investigated the effect of teacher knowledge on student learning. When the focus is on the number of university level courses teachers have in their repertoire, the studies (National Longitudinal Study of Mathematical abilities, 1972; Eisenberg, 1977, cited in Fennema and Franke, 1992, p.148) suggested no significant correlation between the teacher knowledge and student learning. However, these studies did not attempt to investigate the complexity of teachers’ knowledge but rather rested superficially on testing of the knowledge or the number of courses taken by the teachers (Fennema & Franke, 1992). Studies such as the Cognitively Guided Instruction (CGI) project (Fennema, Carpenter, & Peterson, 1989) suggested a direct and signification relation between teacher knowledge and student performance. Leinhardt and her colleagues, as a result of their study, identified content knowledge as one of the fundamental sources of teaching (Leinhardt, G., Putnam, R. T., Stein, M. K., & Baxter, J., 1991, cited in
As a result of her investigation of Chinese and U.S. teachers, Ma (1999) also confirmed this result and suggested that the teacher knowledge is a principal component of good teaching. In light of this literature the importance of increasing teacher knowledge and investigating the nature of that development becomes clear. The current study pursues such an agenda.

1.3. Research Questions

Regarding the aforementioned rationale, the current study aims to answer the following overarching research questions. More targeted research questions regarding conceptual development of key ideas will be given in Chapter 3. Throughout, this study investigates three main questions:

1. What does it mean to understand division of fractions? What are the key developmental understandings for division of fractions?

2. What is the nature of prospective elementary teachers’ understanding of division of fractions?
   a. What are the conceptual difficulties prospective elementary teachers face when learning division of fractions?

3. How do prospective elementary teachers develop key developmental understandings that are required for learning division of fractions?
Chapter 2

Conceptual Framework

This chapter serves the purpose of explicating the theoretical basis on which the current study relies. It also includes an articulation of the conceptual framework, Reflection on Activity-Effect Relationship (here abbreviated as “RAER framework”) which forms the basis for the current study.

2.1. Overview

In the context of a teaching experiment, to design the instructional sequence and to explain teachers’ development of division of fractions, I used the RAER framework. In designing the current study, this framework helped me in several ways: (1) to frame the overall understanding of learning, (2) to guide the instructional design, (3) to look for activities on which participants could use to base their understanding, and (4) to think about the importance of reflection on their own activities. In this sense, the RAER framework served the study as an orienting framework as opposed to an analytical framework.

In articulating the understandings central to division of fractions, I benefited from the framework, called “key developmental understandings”, formulated by Simon (2002, p.991). A description of this framework and how it served the study will also be given in this chapter. What is also included in the current chapter is a description of the reform teaching (National Council of Teachers of Mathematics, 1991, 2000) adopted in my instruction throughout this research study. This view of teaching was consistent with the
framework I used in this study as explained in the subsequent sections. What follows is a detailed description of all these different but connected aspects.

2.2. What Does a Constructivist Framework Look Like?

The RAER framework is a learning theory based on radical constructivism. Constructivism is built on two main principles that have consequences for cognitive development and learning (von Glasersfeld, 1995):

1. Knowledge is actively built up by the cognizing subject rather than passively transmitted by way of communication or through senses,

2. The function of cognition is adaptive, tending towards fit or viability, and cognition serves the subject’s organization of the experiential world, not the discovery of an objective ontological reality. (p.50)

Accepting the first principle is called trivial constructivism, whereas consideration of both principles at the same time makes one a radical constructivist (von Glasersfeld, 1995). The first principle suggests that one constructs his/her own knowledge (Jaworski, 1994), and this new knowledge is dependent on his/her prior experiences and prior knowledge. The second principle refers to one’s organization of personal experiences and closes the doors to some type of discovery of an external reality. Hence, in this sense, no reality is independent of one’s ways of knowing (Simon, 1995). However, this is not to say that there is no objective real world. On the contrary, one can never know what that reality is (Jaworski, 1994) since one only knows what one individually constructs.
The aforesaid constructivist principles guide the framework used for this study. Knowledge construction and organization of experiences are what was considered to be crucial. In the current study, the participants were provided with opportunities to make conceptual advances to higher-level understandings concerning division of fractions, based on reflection on their own mental activities.

Given these principles, we cannot say that knowledge is something that the teachers can “hand over” (Jaworski, 1994) to the students and students directly receive and absorb, which is what makes radical constructivism powerful. In this sense and based on these principles, it is not reasonable for a teacher to try to convey what is in his/her mind directly to students through language. However, students may learn some mathematical ideas in a class that adopts a “show-and-tell” method, but the depth of their understanding of these ideas is questionable. Therefore, in support of this view, the teaching sessions constituting this study involved and encouraged active engagement of the participants and their construction of mathematical ideas as opposed to the attempt to convey knowledge to the participants verbally.

von Glasersfeld (1987) identified some consequences of the adaptation of the radical constructivist perspective in education and educational research:

1. A radical separation between teaching for understanding and teaching that aims at repetition of behaviors;
2. Focus in education and research not on “overt responses” of students but on what goes on in students’ heads;
3. Linguistic communication not as a tool to transmit any knowledge to students;
4. Teacher’s interest on the errors students make; and
5. Teaching experiments focus both on students’ conceptual structures/operations and finding ways to modify them.

The teaching experiment in the current study seriously considered the issues of teaching-for-understanding, justification of ideas, students’ thinking, and conceptual structures, and modification of those structures.

Given this general background about constructivist philosophy, which was adopted as the backbone of this research study, I now elaborate on the conceptual framework used in designing tasks and analyzing the conceptual development of teachers on division of fractions. This framework was developed by Simon and his colleagues (Simon, Tzur, Heinz, & Kinzel, 2000, in press) who specified a mechanism for students’ conceptual learning.

The RAER framework borrows several constructs from the Piagetian framework. What follows is a brief summary of the Piagetian framework to compare to the one used in the current study and then an elaboration of the components of the RAER framework.

2.3. Components of the Piagetian Framework for Cognition

Piaget’s framework includes two key elements: cognitive adaptation and abstraction. In this view, the first element, adaptation, takes place through the processes of assimilation and accommodation (Gallagher & Reid, 1981, p.48). When humans perceive a situation that involves an activity (e.g., counting, grouping), they tend to incorporate elements of it into their current structures. In other words, they manage the given situation through assimilation. Meanwhile, they modify (accommodate) their
understanding of the situation to adapt to meet the demands of the environment (Gallagher & Reid, 1981, p.48). These two mechanisms together help people self-regulate their knowledge system. During this process one may encounter negative results of those activities that may cause perturbations for that person. Elimination of those perturbations results in equilibration (Gallagher & Reid, 1981; von Glasersfeld, 1995). Piaget considers equilibration as an active process of self-regulation, which is the fundamental factor in development (Gallagher & Reid, 1981). The process of this elimination, which is made possible by accommodation, results in an introduction of a new concept incompatible with earlier established concepts (von Glasersfeld, 1995).

As mentioned earlier, the second key Piagetian construct is abstraction. Piaget considers knowledge as threefold: physical (empirical) knowledge (knowledge of physical world), logicomathematical knowledge (knowledge of the relationships created among actions), and social knowledge (conventions of the social environment in which the people are). Piaget (1971) defined the mechanism of constructing relationships as abstraction. There are two types of abstraction: empirical abstraction (ranging “over physical objects or material aspects of one’s own actions” (Piaget, 2001, p.30)), reflecting abstraction [abstraction of the effects of actions (Piaget, 1983), abstracting the relationships between actions (Piaget, 1964), or abstracting the properties of action coordination (Piaget, 2001, p. 30)]. He also describes another version of reflecting abstraction as pseudoempirical abstraction, which refers to enrichment of objects by properties drawn from the coordination of one’s actions (Gallagher & Reid, 1981). The essential characteristic of empirical abstraction is that it is based on observables in the physical world (physical characteristics of the object that the child is capable of noticing
such as lightness, smoothness, color) and based on actions on those observables (Gallagher & Reid, 1981, p. 27; Piaget, 2001). On the other hand, reflecting abstraction enables humans to go beyond observables. Reflecting abstraction is based on internal structuring and always involves coordination⁵. To make the distinction between empirical abstraction and reflecting abstraction, Piaget (1983) gives his famous example as follows:

If a child, when he is counting pebbles, happens to put them in a row and … make the astonishing discovery that when he counts them from right to the left he finds the same number as when he counts from the left to the right, and again the same when he puts them in a circle, etc., he has discovered experimentally that the sum is independent of order … : What he has discovered is a relation, new to him, between the action of putting in order and the action of joining together . . . and not . . . a property belonging to pebbles (p. 119).

Piaget (2001) considers reflecting abstraction as a combination of two phases. Projection is the first phase that “projects a structure at a lower level onto a higher level” and reflection is the follow-up phase which “reorganizes the structure at the higher level” (p. 30). In addition to reflecting abstraction, Piaget (2001) called a conscious reflection on reflection itself as reflected abstraction. It is a second-order higher process than reflecting abstraction, and it is followed by a third-order process, metareflection or reflective thinking. Reflective thinking in other words takes the products of reflected abstraction as the main source and acts upon them. In identifying a basic mechanism for conceptual development, Simon and his colleagues used the word reflective abstraction to refer to both conscious and unconscious reflection as opposed to separating those two

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⁵ Gallagher and Reid (1981) explain coordination as the unifying form of the elements of an action.
like Piaget did (Simon, Tzur, Heinz, & Kinzel, in press). Hereafter, I will use the same
terminology since it covers all types of reflection Piaget discusses.

According to Piaget, reflecting abstraction is the process by which new, more
advanced conceptions develop out of existing conceptions, which, with the
disequilibrium construct, provides a solution to the “learning paradox” which will be
explained in Section 2.1.4. Even though Piaget considers reflecting abstraction as the
basis for learning new, more advanced conceptions, he did not articulate how reflecting
abstraction gives rise to conceptual advances. Hence, there is a need for a solution to the
learning paradox and cognitive conflict should not be the ultimate basis for this solution
as explained in the following section.

The subsequent section provides rationale as to why this study was not based on
cognitive conflict and articulates the advantages and disadvantages of using cognitive
conflict as a way to foster development.

2.4. Why Not Cognitive Conflict?

One body of research uses “cognitive conflict” to facilitate learning of division of
Invoking cognitive conflict to foster development is useful when teachers are able to
invoke the intended conflict on students. When trying to explain inconsistencies from a
historical and mathematical point of view, Vinner (1990) argued that:

. . . in order to convince the student who holds the potentially conflicting
ideas that something is amiss, the student must acquire the same, or nearly
the same, mathematical structure and logical principles that the teacher
holds. This might be quite difficult. If so, there is little hope that the
inconsistent ideas will contribute substantially to the mathematical development of the student. (p.89)

In addition, Wilson (1989, p.91, cited in Vinner, 1990) suggested that “contradiction is only in the teacher’s mind and not in the student’s mind.” Hence, as Vinner (1990, p.92) and others (e.g., Simon et al., in press) point out, even if the teacher is able to help the student recognize the conflict, this recognition on the student’s part does not guarantee a learning in the desired direction. Erlwanger’s (1973) study subject, Benny, is an appropriate example for such a claim. Even though the given instructional materials were intended to move Benny, a student participating in the study, in a certain developmental trajectory (based on some conflicts) he created his own world of rules and mathematical definitions and did not encounter the conflicts or dilemmas that his teachers planned.

In addition, Steffe (1990) pointed out that it is not reasonable for teachers to preplan conflicts for students but “spontaneous occurrence [of conflicts] should be expected.” In Steffe’s (1990) view, a teacher can attempt to create “doubts” for students, but she or he needs to keep in mind the consequences of doing so. Hence, creating cognitive conflict should not be the primary focus, but mathematics itself should be the central focus of teaching. On the other hand, teachers should be able to take advantage of cognitive conflict that the students generate for themselves as Steffe (1990) argued.

Because of the above reasons, the current study adopted an instructional design based on the RAER framework that does not rely on cognitive conflict. The crux of the study was about using learners’ own activities [mental or physical] to develop concepts that were within their reach. In this way participants had the opportunity to reflect on their own actions independent of an external conflict facilitator.
However, this is not to say that teachers in schools do not design activities within students’ reach. This study used a theory-based lesson design that only took into consideration what participants already had available as knowledge and helped them learn conceptions that were more complex than the ones they already had.

In this sense Piaget’s disequilibrium theory as an explanation for learning does not guarantee a sufficient basis for elaborating a theory of teaching since it may not facilitate learning in the desired direction, and since it does not articulate enough how conceptual change occurs (Simon et al., in press). Regarding these concerns, this research study used the RAER framework as a main source of facilitating conceptual development as opposed to using cognitive conflict. What follows is an articulation of the constructs of the RAER framework.

2.5. Components of RAER Framework

The basis of the RAER framework is engagement of individuals in an activity (mental) and reflecting on the result of that activity which is also the basis for Piagetian framework. However, what makes RAER framework different is that it is not based on cognitive conflict, and it makes an in-depth articulation of reflective abstraction.

The main mechanism of the RAER framework consists of assimilation and (reflecting) abstraction. Assimilation, defined by Piaget, suggests that one needs to have a concept in order to make sense of one’s experience in terms of that concept. As Simon and his colleagues mentioned, what one knows determines what his/her mental system can recognize. (Simon, Tzur, Heinz, & Kinzel, in press).” In this regard, the assimilation
construct results in the idea that one cannot take in a conceptually richer system than one already has. In spite of that, people learn new and more complex ideas everyday. For instance, a student who does not know the division concept learns the division concept through some type of instruction. This paradox is known as the learning paradox (Bereiter, 1985). The current study took the learning paradox into consideration seriously and used the RAER framework as a vehicle to overcome it when working with prospective elementary teachers.

In their theory that explains conceptual advancements, Simon and his colleagues proposed a model based on individuals’ own activities and their reflections on those activities (Simon, Tzur, Heinz, & Kinzel, in press). In this theory the learner has the central role in the learning environment in the following sense. In a given (problem) situation, the learner is the one who sets the goal and pursues it based on his or her spontaneous activity. The goal, in this framework, is the main element during the development of a new conception. However, this goal setting is dependent on the learner’s available schemes or conceptions. To reach this set goal, the learner calls on an activity (sequence) that is already a part of his or her current conceptions. As the learner engages in the activity (sequence), s/he attends to the results of it. Since the learner is the one who sets the goal, the assumption in this theory is that s/he can judge what results get closest to the goal and what results cause deviation from the goal. In going through the activity (sequence), the learner makes “conception-based” and “goal-directed”

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6 I primarily used Simon, Tzur, Heinz, Kinzel (in press) when explicating the RAER framework. The phrases stated within quotation marks are directly taken from this article unless cited otherwise.
adjustments with respect to the results to which s/he attends. These adjustments are called “effects” of the activity (sequence). Each attempt is recorded mentally as an experience (von Glasersfeld, 1995, cited in Simon et al., in press). The learner mentally compares these records of experience (or “records of activity associated with the effects of it”) which results in his or her recognition of pattern(s) or regularities. This is called “abstraction of the relationship between activity and effect.” This activity-effect relationship involves a coordination of conceptions (Piaget, 2001, cited in Simon et al., in press). The learner at this point is able to anticipate the effect of his or her activity. However, this does not guarantee that the learner can call on the activity in situations in which the activity-effect relationship would be useful. Through reflection on the activity-effect relationship, the learner can think about the relationship between the situation and the activity-effect relationship.

The above description requires definition of several terms. What is meant by activity is the mental action engaging the learner in service of reaching the goal. “An activity both generates and is a constituent of a conception” (Simon, Tzur, Heinz, & Kinzel, in press). Here, the goal set by the learner refers to the desired outcome toward which an activity is carried, (von Glasersfeld, 1995, cited in Simon et al., in press) and which is not necessarily conscious. The goal guides the focus of attention and helps the learner evaluate the degree of success for the actions being taken. To reach the goal, the learner mentally engages in an activity sequence which is considered to be the basis for building a new conception. At times the learner can employ a sequence of actions to reach the goal. This sequence of actions is called an activity sequence. Effects, on the other hand, are “structured by assimilatory conceptions that the learner
brings to the situation” (Simon, Tzur, Heinz, & Kinzel, in press). The point of interest here is the effects that give rise to the conceptual learning.

As mentioned in earlier sections, the RAER framework functions as an orienting framework as opposed to an analytic framework. I considered this framework to give me a general picture of how learning occurs. I mainly used it as a guide in instructional design to generate activities on which the participants could base their learning and to think about the importance of reflection on their spontaneous activities.

The following section describes the teaching that took place in the study and how the RAER framework operated to guide this teaching.

2.6. How Did the RAER Guide Teaching in This Study?

As emphasized by Simon and his colleagues, the hypothetical learning process consists of setting a goal, initiating an activity sequence to reach the goal, attending to the effects of this sequence, creating records of experience, and reflecting on them, which all together results in “identification of invariant aspects of the activity-effect relationship” (Simon, Tzur, Heinz, & Kinzel, in press). In this view, the learners’ conception(s) is the determinant of that to which they can attend. Hence, the teacher’s main role is to anticipate a developmental process for the students depending on the understanding of their current conceptions and activities. This process includes four main actions to be considered by teachers prior to teaching. Teachers first need to specify the learners’ current conceptions and then specify the conceptual advance that is being targeted for them. What follows in this sequence is an articulation of an activity sequence that is
already available to the learners, which has the potential to give rise to the conceptual advance being targeted. And, an appropriate task, that engages learners in the intended activity sequence to accomplish the goal set by the learners, completes the picture. Here, the activity sequence is the tool that the teacher anticipates ahead of time, and it is used by the learner in service of reaching a particular goal. Hence, the goal is not to help the learner successfully complete an activity sequence, but it is to help the learner use it to make an abstraction.

In this research study I adopted the aforesaid view about learning and used the RAER framework as a starting point to think about how people develop more advanced ideas than what they already have. Since assimilation suggests that one cannot take in what one does not already have, this notion shaped the way I teach during the teaching sessions. In designing these teaching sessions that constituted the current study, the above construction sequence was the main guide. Prior to these sessions, I was considering the fact that I needed to design them in such a way that they were dependent on an activity (sequence) that was already available to the participants and that they were giving me as the researcher enough information about the participants’ understanding and conceptual development. In this regard, I was not constantly creating cognitive conflicts for participants. Therefore, this model, as a whole, became a framework to think about learning and teaching.
Chapter 3

Review of the Literature

The purpose of this study was to examine the nature of prospective elementary teachers’ understanding of division of fractions and their conceptual development in this area. This study is situated within the research that illuminates the mathematics education field in the area of mathematical “understanding” and cognitive “development” concerning division of fractions. What is included in this chapter is a synthesis of the research literature pertinent to the mathematical area of division of fractions.

The research literature concerning division (of fractions) is quite compartmentalized and, therefore, it does not constitute a unified whole. Because of such compartmentalization in this research area, I create a synthesis that is useful for addressing the research questions. In this regard, I focus on three main components related to my research questions: literature regarding division in general and division of fractions in particular; literature pertinent to remainder concept; and literature concerning unit understanding. Research literature in these areas consists of attempts in understanding students’ or teachers’ difficulties, misconceptions, and understandings. What follows is a synthesis of this literature.

3.1. Different Ways to Think about the Division Concept

Two division models are commonly accepted by mathematics education researchers. These are the partitive division model and the quotitive division model.
a) **Partitive division model:** In this model an object or collection of objects (dividend) is divided into a number of equal subcollections (divisor). The purpose in such division type is to discover the size of a divisor group.

b) **Quotitive division model:** This model requires the determination of how many of a given quantity (divisor) is contained in another quantity (dividend). The result of this investigation is the number of divisor groups as opposed to the size of a divisor group.

Fischbein and his colleagues observed that students operated with some behavioral models (primitive models) that were intuitively linked to arithmetic operations (Fishbein, Deri, Nello, & Marino, 1985). One such primitive model was that division problems involve bigger dividends than divisors. Fishbein and his colleagues described the possible sources of such models as: (a) the negative effect of school curriculum on students; (b) resistance of the models to change; and (c) correspondence between them and “the features of human mental behavior that are primary, natural, and basic” (Fishbein, Deri, Nello, & Marino, 1985, p.14-15). In this sense, a close relationship exists between how students are taught in school and how students function as a result of that teaching. A student who always works on partitive division problems may develop a sense for division as a sharing activity which may be a limitation to the student.

One of the misconceptions among students is that the result of a division problem is always less than the dividend (Bell, Swan, & Taylor, 1981; Greer, 1987). This misunderstanding results from the use of quotitive division problems consisting of bigger dividends because students are not able to think about division problems as how much of a divisor could make up a dividend when the divisor is bigger in quantity than the
dividend. Other researchers (Graeber, Tirosh, & Glover, 1986; Ma, 1999; Sowder, Armstrong, Lamon, Simon, Sowder, & Thompson, 1998b; Sowder, Philipp, Armstrong, & Schappelle, 1998a) observed the same misconception among adult learners and teachers.

The consensus in mathematics education research pertinent to division is that the first model to which students and teachers refer when they encounter a division problem is the partitive model (Armstrong & Bezuk, 1995; Ball, 1990; Fishbein, Deri, Nello, & Marino, 1985; Graeber, Tirosh, & Glover, 1986; Greer, 1992; Hiebert & Behr, 1988; Ma, 1999; Simon, 1993; Sowder, 1995; Sowder, Armstrong, Lamon, Simon, Sowder, & Thompson, 1998b; Sowder, Philipp, Armstrong, & Schappelle, 1998a; Tirosh & Graeber, 1989, 1990). In addition, the Graeber, Tirosh, Glover (1986) study suggests that preservice teachers achieve high scores on problems that are based on the partitive model of division. Moreover, teachers and students are not familiar with the quotitive model of division and, in general, they consider division as sharing (Tirosh & Graeber, 1990).

The current study worked on a quotitive division model as a way to help participants develop a conceptual base for a division of fractions concept. One of the reasons for such focus was that it was very hard to think about contextual situations based on the partitive model in fractional settings. Another reason was that one other intent of this study was to help participants develop a new algorithm (common denominator algorithm) as opposed to the invert-and-multiply algorithm, and this kind of development was conceptually more attainable using quotitive division as opposed to partitive division. In this way, this study not only investigated the nature of teachers’
understanding of quotitive division but also it focused on the development of that understanding.

3.2. Misconceptions about Division (of Fractions)

When trying to generate profiles of mathematical understandings for teachers, Post and his colleagues (1991) found that teachers had significant difficulties with the division of fractions concept. Part of this complication and difficulty came from the fact that “division of fractions lies in the intersection of two mathematical concepts that many teachers do not have the opportunity to learn conceptually - division and fractions” (Sowder, Philipp, Armstrong, & Schappelle, 1998a, p.51).

Such complications and difficulties result in some misconceptions. The research, investigating students’ or teachers’ conceptions of division (Anghileri & Johnson, 1988; Armstrong & Bezuk, 1995; Ball, 1990; Graeber, Tirosh, & Glover, 1986; Hunting & Sharpley, 1988; Kieren, 1988; Kouba, 1989), revealed that students and teachers had many misconceptions in interpreting and comparing the quantities, dividend and divisor. For instance, the Graeber, Tirosh, Glover (1986) study indicated that preservice teachers reverse the role of the divisor and dividend when the given divisor is greater than the dividend. Additionally, as Tirosh and Graeber (1989) indicated there was an attempt among the majority of teachers to consider divisor as an whole number in given division word problems, which was also consistent with Greer’s (1987) findings based on his study on upper elementary school pupils. Even though in some division problems the teachers were given smaller dividends, they converted the units of dividends to other units that would make the dividend bigger (i.e., changing pounds to ounces). These
results were also confirmed by some other studies (Bell, Swan, & Taylor, 1981; Fishbein, Deri, Nello, & Marino, 1985; Graeber, Tirosh, & Glover, 1986).

Another difficulty concerning division (of fractions) is the notion of invariance. Harel, Post, and Lesh (1991) identified multiplicative invariance as one of the core and fundamental understandings for division and multiplication. Considering division as a relationship between three quantities (dividend, divisor, and quotient), an invariant relationship exists between these three quantities. Here, invariant relationship is meant to describe the multiplicative relationship between divisor and dividend, divisor and quotient, and dividend and quotient. These unique, fixed, multiplicative relationships depend on the size of those quantities. Abstractly thinking about this relationship among the quantities in a division situation is a difficult one for most teachers as Simon (1993) indicated.

Even though the research literature in this area articulates students’ and teachers’ difficulties and misconceptions, no well-articulated framework to help the research community explains how these difficulties and misconceptions can be overcome. The current study investigates the ways to overcome these difficulties and misconceptions, and the related developmental process.

3.3. Different Operations Used in Service of Understanding Division Units

As Hiebert and Behr (1988) indicated, a change in the nature of “unit” is what distinguishes primary and middle grade mathematics with respect to subject matter. Students need to have a solid understanding of “unit” (Carpenter, Fennema, & Romberg, 1993). However, even most teachers lack the ability to think about unit changes
embedded in the problems (Sowder, Armstrong, Lamon, Simon, Sowder, & Thompson, 1998b, p.148). In analyzing multiplication of fractions which could also be generalized to the division of fractions, Armstrong and Bezuk (1995) argued that learners needed to think about the relationships between the partition units and the whole unit, and between the sizes of the partition units and how they are created. In this regard, partitioning is another unifying factor (Carpenter, Fennema, & Romberg, 1993) that needs to be seriously considered in solving division problems.

Identifying the quantities (divisor and dividend) involved in a division problem, as Kieren (1992) pointed out, requires a process of partitioning. According to Lamon (1996, p.171), partitioning is an “experienced-based, intuitive activity” that determines “equal shares.” It is also a “multistage operation: marking objects, cutting them, and indicating one person’s share” (Lamon, 1996, p.171).

The process of partitioning yields to unitizing when solving division-of-fractions problems. Unitizing is the ability to construct a reference unit or a unit whole and then to reinterpret a situation in terms of that unit (Lamon, 1996). In other words, it is a “cognitive assignment of a unit of measurement to a given quantity” (Lamon, 1996, p.170). Also, one needs to reconfigure the unit wholes from partitions (Kieren, 1992).

These partitioning and unitizing operations relate to identification of correct referent units. The literature points out that one common difficulty among most teachers and students is the identification of referents for each fractional component of division (dividend, divisor, quotient) (Armstrong & Bezuk, 1995; Simon, 1993; Sowder, Armstrong, Lamon, Simon, Sowder, & Thompson, 1998b). In this regard, unit understanding requires particular attention in understanding the division concept.
However, how learners come to understand this key piece and how they use it to proceed in reaching their overall goal in division problems are yet to be articulated by research. The current study not only makes a detailed articulation of this key piece (unit understanding), within the context of division of fractions, but also it investigates how teachers use their unit understanding to think about division of fractions.

3.4. Difficulties Pertinent to Remainder and Understanding Remainder

From the literature, one of the difficulties prospective elementary teachers have is understanding the fractional (or decimal) part of a quotient in a division problem, and interpreting the remainder (Simon, 1993). According to Simon’s study (1993), most teachers were unable to identify the idea that the fractional (decimal) part of the quotient and the remainder gave different information.

The comparison of the fractional part of a quotient and the remainder is a major confusion for most learners regardless of their profession as illustrated by research (Armstrong & Bezuk, 1995; Sowder, Philipp, Armstrong, & Schappelle, 1998a). Computation of remainder does not create as much confusion for most learners in the whole number setting (Silver, Shapiro, & Deutsch, 1993); however, it becomes problematic within the context of fractions (Simon, 1993; Sowder, Philipp, Armstrong, & Schappelle, 1998a).

Silver (1992), as a result of his study, identified the source of this difficulty as a result of embedding division problems in context. When the given division problems were context-dependent problems, students were able to find the result procedurally, but they were not referring back to the problem context to make sense of that result and the
components of it (Silver, Shapiro, & Deutsch, 1993). Hence, division problems with
remainders require a final interpretation of the mathematical solution (Li & Silver, 2000).

Silver and his colleagues (1992) analyzed student strategies in the context-
dependent division problems with remainders. In their analysis, they concluded that a
successful solution to a division-with-remainder problem depended upon three referential
systems. These were the story text (natural language of the problem), the story situation
(context of the problem), and the mathematical model (e.g., division). According to
Silver and his colleagues:

. . . students would map successfully from the problem text to a
mathematical model, in the form of a division problem, then compute the
answer and report it as the final solution without mapping back to the
problem context or the implied story situation in order to engage in further
semantic processing to interpret the best answer. (1992, p.37)

Combination of these three referential systems is important in both constructing and
interpreting the mathematical solutions (Silver, 1992). The solution Silver (1992, 1993)
suggested was the use of “quotient-only” and “remainder-only” problems to help students
move toward more sophisticated division problems.

Research studies (Li & Silver, 2000; Silver, 1992; Silver, Shapiro, & Deutsch,
1993) that specifically focused on division-with-remainder problems are very few and
limited to whole number division. In addition, these studies investigated student
difficulties and the possible sources for such difficulties rather than how learners
developed the remainder idea and related understandings in more complex settings such
as those involving fractions. Therefore, studies that pay particular attention to the nature
of learners’ understanding and development of those understandings are important to get
a more detailed and accurate description of this mathematical area. In this sense, the
current study focused heavily on understanding and development of the remainder
concept within the context of fractions to get a clearer picture of key mathematical understandings.

3.5. The Importance of Algorithms from the Learners’ Point of View

One commonly used algorithm for division of fractions is the invert-and-multiply algorithm. Most traditional mathematics textbooks make their introduction to division of fractions based on this algorithm. A majority of teachers use this algorithm as a way to teach their students division of fractions.

When explaining her previous experiences on teaching division of fractions concept with the invert-and-multiply rule, a participant teacher from Sowder and her colleagues’ (1998) study commented that

… one of my students said, “why do you flip it [referring to the invert-multiply rule] and why are we multiplying? This is division.” And she [referring to the student teacher] says “Because I just told you to do it.” And I sat there and thought, “Boy that was a wonderful question, and that was a very common answer.” And I don’t know how I would … have to … think about it to give more concrete examples. (p. 46)

This teacher’s confession can be generalized to most traditional classroom teachers since they are afraid to analyze concepts such as division of fractions because of deficiency of their knowledge in this area.

Ball (1990) found that prospective elementary and secondary teachers had procedural understanding of division by fractions. In addition, Simon (1993) and Ma (1999) observed that even the procedural understanding of division that the teachers had was quite weak. In his study, Simon (1993) found that, although about 90% of the participants were able to calculate division problems correctly, only 25% of the teachers (all of whom were mathematics majors) were able to provide appropriate representations
for the given division-of-fractions problems such as $\frac{3}{4} \div \frac{1}{2}$. Furthermore, teachers are not able to connect the meaning of division with its symbolic meaning or with any underlying concept of division (Ball, 1990). For most of them, understanding division of fractions is closely associated with remembering a particular rule (invert-and-multiply) (Ball, 1990), which was very poorly understood (Borko, Eisenhart, Brown, Underhill, Jones, & Agard, 1992). Their understanding of division is disconnected and mainly dependent on rote memorization rather than any conceptual basis (Simon, 1993).

Another viewpoint for division is treating it as the inverse operation of multiplication. In most cases, teachers’ understanding of division is disconnected from other ideas about division and consists of remembering a particular rule (Ball, 1990). Their understanding depends more on memorization than conceptual understanding, and they are not able to provide concrete examples and justifications as to what makes a particular algorithm such as invert-and-multiply hold true (Ma, 1999). In fact, making sense of such a rule and conceptualizing it using the inverse relationship between multiplication and division (Contreras, 1997; Tzur & Timmerman, 1997) is very difficult even for teachers.

Given the literature in the area of division with regard to the invert-and-multiply algorithm, both students and teachers have tremendous difficulty conceptualizing this algorithm. In addition; an alternative algorithm called the “common denominator” algorithm is, in general, articulated mathematically in the literature (Flores, 2002; Siebert, 2002; Sinicrope, Mick, & Kolb, 2002) as opposed to a research-based articulation. The common denominator algorithm (hereafter labeled as CDA) has a conceptual basis depending on basic operations of partitioning, unitizing and counting, which makes it
more inventive by learners since these operations are already available to most of the learners. Also, when students are encouraged to construct this algorithm based on their work with diagrams, CDA formalizes the spontaneous work that students do with diagrams.

Even though the current literature points to the difficulties students and teachers have in understanding division of fractions and related algorithms, no research attempts to understand the nature of these difficulties and how to overcome them. The current study uses an approach to help teachers develop an understanding of CDA by referring to some of the activities (partitioning, unitizing, and counting) that are already available to them. It also investigates the ensuing developmental process in learning CDA.

3.6. Comparison of Multiplication and Division

One of the common learner difficulties existing research identifies is the confusion about division and multiplication within the context of division of fractions. Armstrong and Bezuk (1995) noted that in dealing with the division of fractions concept, teachers have the difficulty of identifying whether something is a division or multiplication problem. In addition, the magnitude of the given quantities, dividend and divisor, have a negative impact on the choice of operations to solve problems of the sort, division or multiplication (Bell, Swan, & Taylor, 1981; Greer, 1987). Hence, the difficulty in this mathematical area is multidimensional.

In his study, Simon (1993) found that about one-third of the prospective teachers confused division and multiplication operations when solving division-of-fractions problems (Simon, 1993). This result was also consistent with Ma’s (1999) and Ball’s
(1990) findings about teachers’ confusing dividing by one-half with dividing in half (i.e., dividing by two), or giving inappropriate or pedagogically wrong interpretations. All of these investigations arose from teachers’ spontaneous work in clinical interviews. Therefore, emphasis was weak regarding the reasons for how teachers think about division of fractions. Even though these studies investigated the nature of teachers’ understandings, such investigations were not based on a long term inquiry into teachers’ understanding of division (of fractions) and how that understanding could be developed. In this regard, the current study investigated the teachers’ development of mathematical ideas pertinent to division of fractions and how they overcame the conceptual difficulties they encountered in this process.

3.7. Key Aspects of the Literature

Literature in the area of division provides different mathematical analyses and ways to think about understandings of this concept as well as descriptions of students’ understandings, difficulties and misconceptions. These products, extracted from mathematical analyses of the multiplicative structures such as division, and from the analyses of student/teacher understandings, guide the mathematics education field in thinking about and conducting research on the development of multiplicative concepts. What follows is a synthesis of what currently exists in the field that can be used to conduct this study on prospective elementary teachers’ conceptual development of division of fractions.
• Different division models, (partitive, quotitive models) offered by research, are useful in articulating the different types of division structures and corresponding learner understandings.

• Learners need to grasp the mathematical structures (quantity, invariance, referents, remainder) embedded in division concept to be able to understand the division concept. However, most students and teachers lack the essence of such structures.

• Among teachers and students procedural understanding of division (of fractions), which results in learners’ use of meaningless rules, is more dominant than conceptual understanding.

• Understandings of the strategies students use to grapple with division situations and misconceptions students/teachers develop within the context of division (of fractions) are articulated to some extent.

As explained above, existing research literature illuminated the mathematics education field regarding the concept of division in a variety of ways. However, several deficiencies exist that need to be addressed with respect to division. These deficiencies require further investigations of:

1. The mathematical structure of the division concept in a fractional context and the sources of difficulties students/teachers have in learning this concept;

2. Prospective teachers’ (as teachers of future) understandings and the process by which they develop those understandings in the area of division of fractions;

3. Ways to facilitate enhanced understandings of mathematics in the area of division of fractions;
4. Efficient approaches in teacher education programs to help teachers overcome difficulties and misconceptions within the area of division of fractions; and

5. Development in learning division of fractions.

The following questions regarding the preceding aspects are yet to be considered:

1. What is the nature of prospective elementary teachers’ understanding of division?
   a. How do prospective elementary teachers develop ways to think about division of fractions as a single entity rather than as a process that consists of several procedural steps?

2. What are the difficulties prospective elementary teachers have in conceptualizing the remainder concept? How do they overcome these difficulties and learn to deal with “referent units” within the context of division of fractions?

3. How do prospective elementary teachers make sense of the interrelationship between the fractional part of the quotient, the divisor, and the dividend?

4. What are the steps that enable the participants to develop a conceptual algorithm, CDA, for division of fractions?

The overall goal for this study was to provide insight into these issues. The main focus in the study was on developing useful ways to think about these issues by articulating and increasing the understandings that are currently available in the field of mathematics education research. In this regard the main purpose was to investigate the nature of prospective elementary teachers’ understanding of division of fractions and their conceptual development in this area using a theory-based instructional sequence.
Chapter 4
Methodology for the Study

4.1. Overview

This study was based on a teaching experiment consisting of teaching sessions and pre- and post-clinical interviews to examine prospective elementary teachers’ conceptual development in the area of division of fractions. The overall goal of this study was to promote and study prospective elementary teachers’ conceptual development.

The subsequent sections consist of four main components that elaborate on the methodology used in this study: the participants, the data gathering procedure, analyses (ongoing and retrospective), and a mathematical analysis of the understandings pertinent to division of fractions.

4.2. Participants

The participants were three prospective elementary teachers from a northeastern U.S. university, who were in the fourth year of their elementary teacher certification program. As part of the program requirements, they were in a field experience during which they taught in schools of a local school district, two days (Tuesdays and Thursdays) per week throughout the Spring 2003 semester.

The participants attended different sections of a mathematics methods course (here called the pseudonym MATHED 1). Although the study used volunteers, the choice of participants was not completely random. When announcing this study in different
sections of MATHED 1 (each had about 30 students), I asked for volunteers who would commit to an up-to-two-month period. The participants would have a chance to engage in some mathematics conceptually and in-depth. They were also informed that they would be paid $5/hour if they completed the entire study. Initially, about 17 prospective elementary teachers applied for the study. However, because of the time constraints this number fell to six in about one month.

Other factors for participant choice were the extent to which they could commit to the study, and flexibility of their schedules. One other factor that affected the selection of participants was the volunteers’ knowledge of mathematics. I tried to target prospective elementary teachers who had a range of understandings as opposed to choosing the most mathematically capable ones. I looked for volunteers who had a very basic understanding\(^7\) of:

1. Fractions, including what a fraction was, knowing how to name, show and represent them, and knowing what numerator and denominator meant;
2. Carrying out basic arithmetic operations on whole numbers and knowing what they meant;
3. Equivalent fractions.

In addition, they were not to have the goal understandings\(^8\) regarding division of fractions that were targeted in the teaching sessions.

\(^7\) See Appendix A for a detailed description of the pre-interview schedule and the targeted mathematical understandings for the participants.

\(^8\) The goal mathematical understandings are explained later in this chapter.
One of the six volunteers who initially agreed to participate in the study was not chosen since this student had most of the knowledge that was planned for presentation in the study. One other volunteer, intended for the study, had to drop out before the study began because of difficulty in keeping up with course work. The third volunteer had only the minimal knowledge and promised to attend the sessions, but she dropped out during the first week of the study for reasons similar to those of the second volunteer. The other three volunteers were chosen because they had the aforesaid minimal knowledge and they did not have the targeted understandings. These three participants were average students with regard to their understanding of elementary level mathematical ideas according to information from their instructors and according to the results of the pre-interview I conducted. Since their knowledge was the base for determining whether they were appropriate subjects, I did not obtain their grades in courses or grade point average to make any assumptions about their understanding.

By the time I conducted the teaching experiment, the participants were being oriented toward reform ideas such as teaching for understanding, identification of conceptual ideas and having instances of those ideas in their current MATHED 1 course. Because of this I did not receive much reaction from the two chosen participants as to why I followed such a reformist teaching style rather than other types they experienced in the schools to which they were assigned for their field experiences. However, one of the participants (with the pseudonym Mindy) continually struggled with the notion of whether it was worth spending so much time on a conceptual idea as opposed to

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9 The results of the interviews are presented in Chapter 5.
memorizing a quick algorithm that would solve the problem much more easily. In this sense, she reacted to the sessions by wondering why one would need to devote that much time to learning a single mathematical concept.

4.3. Data Sources and Data Collection

The data consisted of videotapes and audiotapes of the teaching episodes and one-on-one interviews, the subjects’ written work produced during the teaching sessions and interviews, and the field notes taken during and after the sessions. Two interviews, the pre-interview and post-interview, were conducted. Six volunteers participated in the pre-interview each of which lasted about 70 to 90 minutes. The purpose for the pre-interview was to determine the subjects’ current understandings of fractions, the role of numerator and denominator, mixed numbers, equivalent fractions, distinction between division and multiplication, remainders, referent units, and division of fractions (see Appendix 1 for details). As mentioned earlier in Section 4.1, depending on the results of the pre-interview, subjects who had the aforementioned minimal knowledge and not the targeted understandings were chosen as participants.

Once the participants were chosen, depending on their availability, I negotiated the schedule of the teaching sessions with them. Two of the three participants (with the pseudonyms, Nancy and Wanda) agreed to meet Monday and Wednesday evenings, each for two hours, during the months of April and May of 2003. Only the second teaching session was conducted separately for Nancy and Wanda because of a time conflict. Since Mindy’s schedule did not match that of the other two, I had to teach her alone during
Monday and Thursday afternoons for two hours each from April through the beginning of June, 2003. Mindy was the one who had the most difficulty in scheduling sessions throughout the study. Therefore, at times I met with her in longer intervals than with Nancy and Wanda. Nine of the sessions with Nancy and Wanda took place during April and one session took place during May. On the other hand, I met with Mindy four times during April and three times during each of May and June.

I acted as the teacher-researcher during these teaching sessions\textsuperscript{10}. There were three other Mathematics Education graduate students (co-researchers) who helped gather the data by taping the sessions, observing the episodes, and usually making notes of participants’ evolving understandings. The observers regularly discussed their observations with me in between the teaching sessions. During the teaching episodes, all the participants had the provided worksheets. The observers were also provided with the handouts in order to see the types of questions that were being asked. The participants’ written work was collected immediately after the sessions, and it was also captured by a video camera using zooming techniques. During this whole process, the participants were not given any out-of-session assignments as part of the study. Their regular course work was separate from the work they engaged in during this experiment.

Throughout the teaching sessions, one of the three co-researchers operated a digital camcorder and an audio recorder, while at least one of the other co-researchers observed the sessions from a secluded corner of the room where s/he did not interfere

\textsuperscript{10} Discussion of the nature of the teaching experiment appears in Section 4.2.2.1.
with the recording or the implementation of the sessions. Figure 4-1 diagrams the physical configuration of the teaching sessions. The main focus for the observers was to capture students’ work as much as possible for analysis and to keep field notes pertinent to the important moments that transpired in the sessions. The observers also recorded the ideas that participants formulated and the observers’ inferences about participants’ evolving understandings. After each teaching session, there usually was a brief (up to one hour) discussion about the session and the participants’ evolving understandings. At least one of the co-researchers, depending on schedules, participated. These brief and short on-spot analyses became part of the data used to conjecture about the participants’ conceptual development and to design subsequent teaching sessions. In between each teaching session, I did ongoing analysis of the participants’ understandings.

Figure 4-1: A sketch for physical configuration of the teaching sessions.
Upon the completion of the teaching sessions, I conducted a one-on-one, 90-120 minute post-interview with each participant to identify her current understanding of division of fractions to determine the impact of the teaching that occurred in the sessions. Analysis of the post-interviews, ongoing and retrospective analyses of the teaching sessions and the analysis of the pre-interview helped me characterize participants’ understanding and conceptual development in the area of division of fractions.

In the following section, I briefly explain the nature of the teaching experiment I conducted for this research study and its main characteristics. For this study, Steffe and Thompson’s (2000) teaching experiment methodology was a great benefit.

4.4. Methodology

4.4.1. Teaching Experiment Methodology

Steffe and Thompson (2000, p.269) identified two types of mathematics pertinent to students. The first one, students’ mathematics, referred to students’ mathematical realities (separate from ours). What students say and do\textsuperscript{11} indicates their mathematics during their engagement in mathematical activities. The second type is called mathematics of students, which refers to the models of students’ mathematics that researchers generate as a result of teaching experiments and the modifications students make in their ways of operating. In this regard, the main purpose of teaching experiments is to understand students’ mathematical realities and to create models that explain these

\textsuperscript{11} What students say and do is the data on which conceptual analysis is based (von Glasersfeld, 1995).
realities. This can be accomplished by focusing on students’ mathematics and doing a conceptual analysis of that, which was my aim in this study.

Steffe and Thompson (2000) described a teaching experiment as consisting of a series of teaching episodes which include a teaching agent, one or more students, and a method of recording what goes on in the episodes. In the current study, I acted as the teacher-researcher instructing three prospective elementary teachers and benefited from three other doctoral students, observers for data gathering and analyses. To be able to make an unbiased interpretation of what occurred during the teaching episodes, it was helpful for me to include observers in the sessions. The outside observers who participated in the experiment witnessed the occurrences that took place in teaching sessions, and in this way I had another person focusing on the same issues of interest for the study.

Steffe and Thompson (2000) also emphasized the importance of hypothesis testing and hypothesis generating in teaching experiments. I, as the teacher-researcher, generated some hypotheses as to how participants developed some of the targeted mathematical ideas. Then I continuously modified them between teaching sessions through reflection on the previous episodes with the help of the observers. The next step was to make a retrospective analysis of video recordings of the sessions, and as a result, I formulated the models that explained students’ emerging mathematical realities. The purpose was to characterize how students operated and the meaning of that behavior. In this study, the main focus was on the learners’ assimilatory structures and the changes in those structures.
4.4.2. A Particular Approach to Teaching

As Simon and his colleagues indicated, learning theories do not necessarily prescribe approaches for teaching (Simon, Tzur, Heinz, & Kinzel, 2000, in press). Therefore, the need arose to adopt a teaching approach that would guide the researchers through the design and implementation of the teaching experiments. In the subsequent paragraphs, I briefly explain the view of teaching and the roles I assumed for this study.

The teaching approach I adopted for this study was consistent with the reform movement (National Council of Teachers of Mathematics, 1991) and current research on teachers. According to the National Council of Teachers of Mathematics (1991, p.18), teaching has many facets such as creating valuable mathematical tasks, classroom discourse, establishing a productive learning environment and analysis of that environment. These facets impose particular roles on teachers. In the current study, I created tasks by paying close attention to students’ prior knowledge and experiences. These tasks basically restricted the participants to using diagrams and prohibited them from using algorithms. Once the participants completed the designed tasks, the next step for them was to convince each other (as with Nancy and Wanda) and the teacher-researcher of the validity of their solutions. If there were missing parts, they made attempts to complete those missing parts without being led by the teacher. Among the participants’ roles were listening and responding to teacher, making connections, conjectures and presenting solutions to problems, providing convincing arguments of the validity of those conjectures and solutions, and relying on mathematical evidence as also

In his analysis of teaching, Simon (1995) described the roles of teachers as planning instruction, proposing learning situations within which students “seek responses to the milieu rather than responses to please the teacher.” This purpose was made clear and was negotiated with the participants during the entire course of the sessions.

The overall guiding principle during the sessions was to have a “bifocal perspective – perceiving the mathematics through the mind of the learner while perceiving the mind of the learner through mathematics” (Ball, 1993, cited in Simon, 1995). This guiding principle helped me to make conjectures about the participants’ evolving understandings and cognitive progress.

Throughout the teaching experiment, “the development of individuals’ reasoning and sense-making processes cannot be separated from their participation in the interactive constitution of taken-as-shared mathematical meanings” (Yackel & Cobb, 1996, p.460). In this regard, since the current study investigated teachers’ conceptual development in a particular mathematical area, during the teaching sessions, I took into consideration this group of teachers’ functioning together and its impact on their conceptual development. Therefore, I considered both the participants’ mathematical activities and the group’s collective activities during the teaching of Nancy and Wanda.

I designed the teaching sessions to be conducted in a particular format. Specifically, I was the teacher researcher and the participants (both Nancy and Wanda together or Mindy alone) were constantly encouraged to share their ideas, make conjectures and justify those conjectures. They were not to use any algorithm unless they
were told to do so. In all the teaching sessions they were limited to diagrams and the available materials as primary sources for reference.

At the beginning of the first teaching session\textsuperscript{12}, I tried to lay the groundwork for this experiment. The main issues were:

1. This was not to be a sequence of lecture-type sessions, and I would not directly tell or show the participants what the answers were or how to obtain them, but I would engage them in some activities that would foster and further their thinking.

2. We, as the whole group, were to focus on the mathematical understandings concerning operations on fractions as well as the underlying ideas behind some procedures regarding those operations.

3. The participants were the ones who would generate solutions, and my job was to foster the learning process. At times, the participants’ thinking might be pushed a little harder for the sake of learning conceptually.

4. Justification was to be an inevitable part of this experience.

5. It was acceptable to talk about confusions, misunderstandings or difficulties. It was okay to say, “I don’t know.”

The above points were constantly being negotiated with the participants during each teaching session. This negotiation involved not just telling them that these points were important, but it was about a process in which both parties (myself and the\textsuperscript{12} Throughout the remaining chapters, each teaching session was coded as “T(N).” For example, teaching session 1 would be referred to as T1.}
participants) came to an agreement on some of the mathematical norms such as justification, taking risks in articulating ideas, looking into the rationale underlying an algorithm or the meaning of a concept, and so on. As mentioned in the previous chapter, the participants were already enrolled in a mathematics methods course, MATHED 1, and they were experiencing a similar negotiation process in those class sessions. In this regard, few debates arose between the participants and myself with respect to the structure of the sessions.

Some of the initial ideas for the teaching sequence came from a set of problems designed by Dr. Martin A. Simon for the mathematics methods course that we were teaching. I modified this sequence, as needed, in response to my analyses of the students’ mathematical activity. I began by investigating how the participants developed the understanding that “division a÷b means figuring out the number of quantity b within quantity a.” The goal was not only to help participants develop this understanding but also to help them conceptualize it in such a way that they would be able to use it effectively when necessary.

4.5. Data Analysis

There were three types of data analysis pursued in this study: interview analysis, ongoing analysis and retrospective analysis. I explain each type in the following sub-sections.
4.5.1. Analysis of Pre-Interviews

Upon the completion of the pre-interviews, I transcribed and analyzed them to determine participants’ initial understandings of the concepts such as, fractions, whole number operations (division in particular), and some related ones as will be discussed in Chapter 5. I watched all the video recordings of the interviews. First, I marked important places in the interviews where I saw some indication of a mathematical understanding as the participants engaged in tasks or gave responses to certain questions. When investigating these responses, my foci points were: (1) the activity sequences they used to manage given tasks, (2) the way they thought about their solutions [e.g., using an algorithm, thinking about the mathematical ideas, focusing on procedure], (3) how they made sense of the provided conflicts, and (4) what resources [e.g., representations, formulas] they relied on when answering questions. Once all such incidents were marked, I watched those video segments again to identify patterns for certain ways of thinking or counter examples to such patterns. The purpose was to formulate models of participants’ mathematical knowledge. These models consisted of their understanding of some mathematical ideas, their struggles and weaknesses in understanding those ideas, and the way they handled the given tasks.

4.5.2. Ongoing Data Analysis

After analyzing the pre-interviews, I conducted the teaching sessions with the help of three co-researchers as articulated earlier in this chapter. At the conclusion of each teaching session, at least one of the co-researchers and I reflected on the session and
interpreted students’ evolving understandings of the targeted concepts. At times these discussions shifted to our own understanding of the division-of-fractions concept. This enabled us to further investigate the mathematical components of the division concept within the context of fractions. In relation to this, the investigators also focused on potential understandings to be developed by the participants, and how these understandings might be enhanced and promoted. During these meetings we continuously asked ourselves about:

1. The crucial mathematical issues that were raised during the teaching sessions and the way that students think about and handle those issues;
2. Whether there was a change in students’ thinking about those issues at the end of the sessions, and the nature of that change;
3. The weaknesses and possible explanations for such weaknesses;
4. The factors that afforded or limited the change in their conceptual understanding; and
5. The possible mathematical understandings that could be further investigated in the subsequent sessions.

In these “after teaching session meetings,” I refined these goals and subsequent teaching-session plans accordingly.

4.5.3. Retrospective Conceptual Analysis

In analyzing the data, the RAER framework and key developmental understandings were used as an interpretive framework. Simon (2002, p.963) described
“key developmental understandings” as “conceptual advances.” A conceptual advance is “a change in the learner’s ability to think about and/or perceive particular mathematical relationships” (Simon, 2002), which can be thought of as a change in the assimilatory structures of the learner. Simon (2002) also identified the notion of key developmental understandings with respect to two main characteristics: (1) they include “a conceptual advance on the part of the learner,” and (2) the learners “do not tend to acquire it as the result of an explanation and/or demonstration” (p. 963).

Before the analysis I transcribed the interviews and important pieces of the teaching episodes to be used in the analysis process. For instance, at times, the participants seemed to have a very strong understanding of a mathematical idea. However, when they were working in different mathematical domains (whole number versus fractional) their understanding seemed to be weak. In identifying portions of the data, I was looking for contradictory instances in which the participants showed different levels of understandings. For example, the participants did not have any difficulty in solving division of fractions problems numerically but they had tremendous difficulty in creating their own division of fractions word problems. Once I identified such a dilemma, then by going through the whole data, I was looking for supporting and disconfirming evidence. In this way, I investigated nature of their understanding and difficulties they had.

During the analysis, to be able to identify key developmental understandings I first observed videotapes of the sessions of the participants as they engaged in the given mathematical problems. In doing so, the main focus was on the differences in their actions and approaches they used for the same tasks or task sequences. Then I tried to
account for those differences by attempting to specify key mathematical understandings. These key understandings were the basis for identifying the developmental steps through which the participants progressed. Simon (2002) explained this as:

Understanding mathematical development involves having a sense of developmental steps and understanding how learners could progress from one step to another. Identifying key mathematical understandings was a way of specifying developmental steps. As key understandings were specified, inquiry into how these understandings develop and how they could be fostered becomes possible. (p.965)

In identifying these developmental steps, I first tried to see whether the participants were dependent only on an activity sequence to think about the provided tasks. If so, I tried to explain the reasons for that, based on their actions. In the case where they were not dependent on a certain activity sequence and thinking about the given problems without actually running the activity sequence, I investigated whether they could associate their understandings with appropriate situations.

To sum up, I first identified key developmental understandings and then made an attempt to explain the conceptual advances participants made (or could not make) with regard to those understandings in this process. RAER theory focused the attention of the investigation on the participants’ activities and how they reflected on those activities.

When going through the data, I constantly modified the research questions and made them more targeted based on the data. Meanwhile, I also tried to keep an eye on the available literature and in what ways I could contribute to that body of knowledge regarding division of fractions.

During the retrospective analysis, I did a line-by-line analysis of significant data segments. Data was considered as insignificant if there was not enough support for the
claims I made from it. The main purpose in the analysis was to formulate hypotheses about:

1. Participants’ initial understandings of division in general and of division of fractions in particular;
2. Their evolving understanding of division, unit, referents, remainders, and algorithm, and the factors [e.g., use of different types of quantities] that affect their conceptual development;
3. Critical understandings concerning division of fractions;
4. The developmental process that ensued; and
5. The understandings pertinent to division of fractions that participants had by the end of the experiment.

During the retrospective analysis, I constantly tried to formulate hypotheses about the participants’ evolving understandings and made claims and tried to support those with the data at hand. The ones for which I was able to provide considerable support were then stated as claims. Once the claims were made I also looked for counter evidence for such claims. When a hypothesis was generated or a claim made, I searched throughout all the data to check to see whether there was contradictory evidence. At times I asked other colleagues, such as doctoral students in and out of the mathematics education department, to check some parts of data and interpret the participants’ understandings. I also engaged in discussions with my advisor about the validity of the claims and hypotheses. This type of outside checking also enabled me to confirm the validity of the evidence which makes the analysis process more accurate. Finally, using the collection of claims I had, I
organized them to help model the participants’ evolving understandings pertinent to division of fractions.

As a result, I tried to formulate models for participants’ evolving knowledge that were, as Cobb and Steffe (1983, p.91) mentioned, general as well as specific. In this sense, I focused on several prospective elementary teachers with a particular instruction and built models of their knowledge that accounted for their progress. On the other hand, I tried to generalize these models in order to explain some issues related to the development of the division of fractions concept. However, as Cobb and Steffe (1983, p.92) emphasized, this modeling process was “no more than a plausible explanation of” participants’ activities.

Even though I did a detailed analysis of all three participants, at then end of this process I decided to exclude Mindy’s case from the overall analysis. The reasons for that decision were that:

1. Mindy was constantly being negative about the style of the sessions which was affecting the extent to which she participated;

2. The analysis of Mindy’s data revealed that it was not making a different contribution to the results than that which I resulted from analysis of the other two participants;

3. Since I taught Mindy alone, even though I tried to follow a similar task sequence with her, at times the nature of the instruction for Mindy’s sessions was fundamentally different from the others’ sessions, which was occasionally causing incompatibility with the data I got from the others; and
4. Since the main focus of the study was about identifying key understandings and development of those understandings concerning division, the study was not heavily dependent on the number of participants.

4.6. Mathematical Analysis of Division of Fractions and Related Components

4.6.1. Understanding Fractions

This research study centered on conceptual understanding that falls in the intersection of two mathematical domains: fractions and division. The main focus was on the concept of division of fractions. However, in order for the participants to grapple with the ideas regarding division within the context of fractions, they had to have a minimum knowledge of fractions prior to entering the study. This knowledge is articulated in the following paragraphs.

Fractions have several different but connected interpretations as investigated in the mathematics education field. One of the common interpretations for the fraction \( \frac{a}{b} \) is \( a \) partitions out of \( b \) equal partitions, which basically includes “physical or mental act of partitioning some whole into parts” (Sowder, 1995). This interpretation of fractions garners attention in current textbooks (Hiebert & Behr, 1988; Sowder, 1995) and it requires knowledge prior to the teaching sessions since use of this understanding is essential in the design of the sessions.

A fraction can also be interpreted as a number that represents a quantity, “How much” of something (Hiebert & Behr, 1988; Kieren, 1988, 1993; Sowder, 1995). In this
sense the fraction $\frac{a}{b}$ can be considered as a quantity relative to a whole. Thompson (1994) defined quantity as a conceptual entity composed of an object, a quality of the object, and an appropriate unit or dimension. The key idea here is that partitioning results in a quantity that is represented by a new type of number (Carpenter, Fennema, & Romberg, 1993; Kieren, 1988, 1993; Simon, 2002). During the teaching sessions the participants also referred to this understanding to account for the involved relationships among the quantities.

One other crucial mathematical idea that the participants used in the teaching sessions was mixed numbers and their comparison to improper fractions. Since the division problems in the teaching sessions included mixed numbers and their diagrammatic representations, the participants needed to know how to represent them and represent them as improper fractions. A mixed number, of the form $\frac{b}{c}$, consists of a whole number part and a fractional part. One way to represent a mixed number as a fractional number is to determine the number of fractional parts ($\frac{b}{c}$) within the mixed number ($\frac{a}{c}$). A shortcut for such a method is to multiply $a$ and $c$ and add $b$ to the product; hence, the equivalent improper fraction is $\frac{ac + b}{c}$.  

$^{13}$ Conceptually, one should know that this conversion formula always works because each time one makes partitions

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$^{13}$ During this chapter, the algorithm used to transform mixed numbers into improper fractions is referred to as “conversion formula.”
of a given quantity and totals those partitions. The participants in this study did not need
to have this conceptual understanding of conversion, but they needed to have some way
to convert the given mixed number into an improper fraction.

One other important and required mathematical idea the participants needed to
prior to the teaching sessions was that of equivalent fractions. Conceptually, equivalent
fractions refer to the idea that a given fractional quantity, with the help of repartitioning,
can be represented as a combination of different size units. For instance, one can easily
generate the fractions equivalent to $\frac{3}{4}$ as follows. In order to represent a fractional
quantity like $\frac{3}{4}$, one partitions a given (or arbitrary) whole into fourths and focuses on
the combination of three such partitions. It was possible to represent this new unit $\frac{3}{4}$ by
repartitioning the given whole into eighths. However, as a result of this repartitioning, six
of the new partitions ($\frac{1}{8}$) correspond to $\frac{3}{4}$.

To generate fractions equivalent to a given fraction, the commonly known
algorithm is the multiplication of numerator and denominator by the same number such
as $\frac{a}{b} = \frac{xa}{xb}$ (x is constant). The key idea here is to keep the multiplicative relationship
defined by $\frac{a}{b}$ the same when extending or shrinking the given fraction ($\frac{a}{b}$) using its
components, the numerator and the denominator. Therefore, multiplying the same
number such as 2, by both the numerator (in this case, 3) and the denominator (in this
case, 4) will keep the multiplicative relationship (that is defined by the $\frac{3}{4}$) same even though the name of the newly established fraction is $\frac{6}{8}$. In other words, the ratio of the number of total partitions (denominator) to the number of partitions at hand (numerator) should not change in order to find equivalent forms of a given fraction. The only changing quantity is the number of partitions within a quantity but the size of the quantity stays the same.

4.6.2. Understanding Division

Generally speaking, there are three quantities involved in division: dividend, divisor, and quotient. Based on these three quantities, division could be regarded as a collection of quantitative relationships among these three quantities. As Squire and Bryant (2002, p.454) suggested, to be able to understand division one needs to distinguish these three quantities and know what roles each of them plays.

Depending on what quantity is missing, a division problem can be interpreted in two different but connected ways. One way to interpret a division problem such as $a ÷ b$ is to think about it as the result of the investigation for the total number of quantity $a$ within quantity $b$ [quotitive division]. Here, $a$ or $b$ can be fractional or whole number quantities. Another commonly accepted way to think about division is to consider it as the result of a sharing activity. The same division problem of $a ÷ b$ may mean a party’s share if the

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14 Here, quantity is meant to be a conceptual entity as Thompson (1993) described.
quantity $a$ is shared among $b$ parties [partitive division]. However, the latter view is more difficult to consider within a fractional context because of the lack of contextual situations that can model it.

These different division interpretations can apply to a fractional context.

Considering a fraction division $\left[ \frac{a}{b} \div \frac{c}{d} \right]$ from a partitive-division point of view means the investigation of group size, given that there is the quantity $\frac{a}{b}$ and there are $\frac{c}{d}$ number of groups among which to share this quantity. On the other hand, from a quotitive-division point of view, the above division means the investigation of the number of the quantity $\frac{c}{d}$ within the quantity $\frac{a}{b}$. If the quantity $\frac{c}{d}$ is larger than the quantity $\frac{a}{b}$, quotitive division refers to the question, “How much of the quantity $\frac{c}{d}$ could be embedded in the quantity $\frac{a}{b}$?” Both questions of interest refer to the same conceptual idea that division is basically a multiplicative comparison of two quantities.

4.6.3. Understanding Quantities Involved in Division

The notion of quantity is an underlying factor for division. A division-of-fractions problem involves dividend and divisor that are two fractions. A fraction is a number that represents a quantity, “How much” of something (Hiebert & Behr, 1988; Kieren, 1988, 1993; Sowder, 1995). In this sense, learners should consider dividend and divisor as quantities, in other words, as totalities that are connected to each other with a certain
multiplicative relationship. This relationship determines how many of (or how much of) the quantity divisor can be located in the quantity dividend.

The ability to move beyond the question of “How many of quantity $\frac{c}{d}$ is in the quantity $\frac{a}{b}$” in quotitive division problems requires thinking about the referent units to which both the dividend and divisor refer. For example, if the given division is $1\frac{3}{4} \div \frac{1}{3}$, the question for the problem is, How many $\frac{1}{3}$ can be situated within $1\frac{3}{4}$. To attack this problem conceptually without using any algorithm, what one needs to do is to identify the dividend, in this case $1\frac{3}{4}$, and then the divisor, which is $\frac{1}{3}$, to locate within the dividend a number of times. $1\frac{3}{4}$ can be represented as a combination of a unit whole and $\frac{3}{4}$ of another same size unit whole. To determine the divisor using the units of dividend, one needs to pay very careful attention to the fact that $\frac{1}{3}$ refers to the unit whole to which $1\frac{3}{4}$ refers.

In the previous example, the stated purpose is to attack the problem in a conceptual way. Such a conceptual way is possible with the use of diagrams, which were a vehicle to help participants develop key understandings for the division of fractions in this study. For example, to represent $1\frac{3}{4}$ from the previous example, one can draw two same-size rectangular regions (two unit wholes), partition them into fourths and leave out the last fourth piece of the second rectangular region as in Figure 4.2.
Figure 4-2

After determining the dividend, as in Figure 4-2, the next step requires one to determine the divisor, $\frac{1}{3}$. This $\frac{1}{3}$ is one-third of the unit whole [a whole rectangle]. One can determine this quantity by partitioning the first and the second quantities into thirds horizontally as in Figure 4-3. The crucial point here is that both the divisor and the dividend take the same unit whole as the referent.

Figure 4-3

Figure 4-3: Division of unit wholes into thirds.

Having determined the dividend and the divisor, the next step is to determine the number of $\frac{1}{3}$-unit whole sections [first row that consists of 4 pieces of size $\frac{1}{12}$ in the first rectangle] within the dividend $1\frac{3}{4}$. A person who did not have the aforementioned unit
understanding may struggle between choosing the unit whole (one rectangle) and the given dividend (unshaded area) as a referent for the divisor.

This process includes a sub-process of partitioning, as Kieren (1992) pointed out, during the identification of the quantities of dividend and divisor. According to Lamon (1996, p.171), partitioning is an operation that generates quantity. It is an “experienced-based, intuitive activity” that determines “equal shares.” It is also a “multistage operation: marking objects, cutting them, and indicating one person’s share” (Lamon, 1996, p.170) that can be considered as different stages. In analyzing multiplication of fractions, which can also be generalized to the division of fractions, Armstrong and Bezuk (1995) argued that learners need to think about the relationships between the partition units and the whole unit, and between the sizes of the partition units and how they are created. In this regard, partitioning is another unifying factor (Carpenter, Fennema, & Romberg, 1993) that needs to be seriously taken into account in solving division of fractions problems.

This process of partitioning yields to unitizing, as described by Lamon (1996). Unitizing is the ability to construct a reference unit or a unit whole and then to reinterpret a situation in terms of that unit (Lamon, 1996). In other words, it is a “cognitive assignment of a unit of measurement to a given quantity; it refers to the size chunk one constructs in terms of which to think about a given commodity” (Lamon, 1996, p.170). Also, one needs to reconfigure the unit wholes from partitions (Kieren, 1992). For example, in the above case, once one determines the dividend, $1\frac{3}{4}$, in figuring out the divisor the horizontal partitioning enables one to unitize the first unit whole $\frac{4}{4}$ as $\frac{12}{12}$. 
This process results in the unitizing of the two quantities dividend \( \frac{3}{4} \) and divisor \( \frac{1}{3} \) as \( \frac{21}{12} \) and \( \frac{4}{12} \), respectively. This process changes the overall question from, “How many \( \frac{1}{3} \) there are in \( \frac{3}{4} \)?” into “How many \( \frac{4}{12} \) there are in \( \frac{21}{12} \)?” depending on these unitized quantities. In this regard, unit understanding is crucial for understanding the division concept.

### 4.6.4. Understanding Remainder

Sowder and her colleagues (1998b, p.132) describe sense making as “…the process of coming to a deep understanding of a situation by identifying the quantities that underlie the basic ideas presented by the problem and coming to understand the relationships that exist among these quantities.” Such sense making is necessary in interpreting division problems with “remainder” (Li & Silver, 2000; Silver, Shapiro, & Deutsch, 1993). What follows is an articulation of “remainder” within the context of division of fractions.

Consider a division-of-fractions (or division with fractions) problem such as \( \frac{a}{b} \) where \( a \) and \( b \) are fractions (\( a \) can be an integer, too). In such a division problem, one needs to determine the number of divisors (in this case, \( b \)) within the dividend (in this
case $a$). If the quantity $a$ is divisible\textsuperscript{15} by quantity $b$, then there will be no remainder. This suggests that the quantity $a$ is a multiple of quantity $b$ (e.g., $a = kb$, $k$ is a whole number). However, if $a$ is not divisible by $b$, no whole number of divisors exists within the dividend. This refers to the fact that after determining a number of $b$’s within $a$, there will be a leftover piece within the dividend that is less than $b$. In other words, $a$ can be represented as $kb + l$, $l < b$. This leftover piece $l$ is the “remainder.” “Remainder” in fact refers to the result of the process of subordinating full divisor groups into the dividend. This is a crucial understanding for one to conceptualize in the “sense-making” (Sowder, Armstrong, Lamon, Simon, Sowder, & Thompson, 1998b, p.132) process of division. A hidden but required understanding for remainder is that the units of the remainder are the same as the units of the dividend. In addition, a convention for “remainder” is that it is less than the divisor.

There are two ways to express the “leftover” part – one may be preferable in a given real world problem. To interpret the leftover, a multiplicative comparison between the leftover piece and the divisor is required in order to figure out what part of divisor would correspond to the leftover piece. As a result of this comparison, one can determine the quotient, in other words, the total number of divisors (fractional and whole) within the dividend until every bit of the dividend is used up. In other words, one needs to pay attention to the ratio between those two quantities (Squire & Bryant, 2002) since ratio is a numerical expression of how much there is of one quantity in relation to another quantity.

\textsuperscript{15} Throughout this document the phrase “divisible” refers to the fact that the result of the given division problem is an integer.
which is a result of a multiplicative comparison of those two quantities (Thompson, 1994). Therefore, one needs to know that there is a quantitative relationship between the fractional leftover piece and the divisor. One also needs to know that the result of such a multiplicative comparison between those two quantities together with the whole number of divisors gives the solution to the division problem, the quotient. In a division problem with a remainder, the quotient can consist of two parts: a whole number part and a fractional part. Hence, one also needs to know that the fractional part of the quotient refers to the part of the divisor that is equivalent to the leftover piece (or remainder).

4.6.5. Understanding the Common-Denominator Algorithm

The common denominator algorithm has a conceptual basis that depends on the basic operations of partitioning, unitizing and counting, which makes it more inventible than the invert-and-multiply algorithm by learners using a diagrammatic approach. The CDA can be considered algebraically as follows:

\[ \frac{a}{b} \div \frac{c}{d} = \frac{ad}{LCM(b,d)} \div \frac{bc}{LCM(b,d)} = \frac{ad}{bc} \]  

Eq. 4.1

There are two steps embedded in the CDA: finding the least common multiple for the dividend and divisor and dividing the numerators of those as in Eq. 4.1.

For instance, given a division problem like \( \frac{a}{b} \div \frac{c}{d} \), one can solve it based on the CDA conceptually. For clarity purposes, the assumption is that the dividend is a mixed number and it is greater than the divisor. Also it is assumed that \( b \) and \( d \) are relatively
prime numbers. One first needs to identify the dividend, \( \frac{a}{b} \). To do that one could partition some number of unit wholes and mark the number of partitions needed to identify the dividend. Also, for clarity purposes the assumption is that this partitioning is a vertical partitioning. Hence, what one has at hand is \( a \)-partitions each of which has the size \( \frac{1}{b} \). Then one needs to identify divisor \( \frac{c}{d} \) by repartitioning each unit whole. This repartitioning of each unit whole can be accomplished by horizontally partitioning each unit whole into \( d \) partitions. As a result of this unitizing process, each unit whole now consists of \( a \) number of partitions that is equal to \( bd \). Since each unit whole consists of \( bd \) partitions (each has the size of \( \frac{1}{bd} \)), one could also unitize both dividend and divisor accordingly. In this sense, partitioning the unit wholes into \( d \) more times changes the total number of partitions, as \( ad \), but each new partition now would be of the size \( \frac{1}{bd} \). Hence, the unitizing of the dividend results in \( \frac{ad}{bd} \) for the dividend. The same logic can be applied to the situation for the divisor that needs to be couched within the dividend using the unit wholes. Now each unit whole has \( bd \) partitions each of which has the size \( \frac{1}{bd} \). Partitioning each unit whole horizontally into \( d \) sections suggests each small partition now has the size \( \frac{1}{bd} \). One knows from the dividend that there were \( b \) vertical partitions and also one knows from the divisor that there are \( c \) horizontal partitions. In this sense, to identify one divisor group, one needs \( cb \) partitions each of which has the size \( \frac{1}{bd} \).
Therefore, unitizing the divisor $\frac{c}{d}$ results in $\frac{cb}{bd}$. As a result of this unitizing process the

dividend is $\frac{ad}{bd}$ and the divisor is $\frac{cb}{bd}$. The overall goal can be restated as “finding

number of $\frac{cb}{bd}$ within $\frac{ad}{bd}$.” In this sense, one is actually looking into the count of $ad$

partitions within $cb$ partitions of the same size. In a way this can also be conceptually

represented as $\frac{cb}{ad}$, which covers the second part of the algorithm.

All these aforesaid mathematical understandings and their development as seen

through participants’ work is investigated in Chapter 5.
Chapter 5

Data Analysis

In this chapter, I first describe each participant’s initial mathematical understandings arising from the pre-interviews prior to the teaching sessions. Then, I give a detailed description and analysis of some of the teaching sessions to outline the participants’ evolving understandings of division of fractions, their understandings of remainder, their understandings of common-denominator algorithm and the developmental processes that took place in each of the three areas. In doing so, I describe both participants’ understandings either separately or together within the same section using the support from post-interviews. When investigating a phenomenon, if I realize that there is a commonality in the way participants approach a certain problem, I choose the most representative one and focus on it in detail.

This investigation results in two products which will be articulated in Chapter 6: a generalization for a “knowledge package” (Ma, 1999) for division concept within the fractional domain, and a developmental trajectory of the crucial understandings and building blocks in that knowledge package.

5.1. The Nature of Participants’ Initial Understanding - Pre-Interview

The main purpose of the pre-interview was to evaluate each participant’s initial mathematical understandings of fractions and operations on whole numbers and fractions, especially multiplication and division. A detailed description of the targeted understandings and questions to evaluate those understandings for the pre-interview can
be examined in Appendix A. What follows is an articulation of participants’ understandings in fractional domain.

5.1.1. Participants’ Understandings of Fractions and Comparison of Fractions

a. The Case of Nancy

The very first question the participants were to think about was in relation to a unit fraction, $\frac{1}{4}$. Nancy’s reaction to this question was:

R: … What do you understand from one-fourth? What does that mean to you?
N: It means that there is a whole that it’s divided into four parts, and you have one of those four parts.
R: Okay, can, can you draw or show me what you mean by that?
N: Okay, if I had like pizza. The pizza would be the whole [drawing a circle] and then you’ll have it divided into four parts [partitioning the circle into four similar-size pieces], four equal parts. And then you’ll have one of those [pointing to the pieces cut-up from a given circular region] so you will have one of the four pieces.
R: So each part has to be equal?
N: Yes.
R: So can I choose another whole and uh divide it into four parts and take one and say that is one-fourth?
… [Researcher draws an arbitrary closed curve on Nancy’s worksheet]…
N: If you divide it equally, yes. But it wouldn’t be the same as this [pointing to the shaded partition in front of her] unless the wholes were of the equivalent size.
…
R: … When you said it wouldn’t be same as this, you mean the shape or?
N: Um, the actual size. The, this one-fourth and if you divided this [pointing to the closed curve the researcher drew] up into four equal parts, they wouldn’t necessarily be the same.

As seen in the above episode, Nancy was able to represent the fraction $\frac{1}{4}$ based on a diagram without any difficulty. Her very first reaction to the question suggests a
part-whole interpretation of fractions. She seemed to know that $\frac{1}{4}$ referred to one partition out of four equal partitions. When the question was changed to whether this type of identification of $\frac{1}{4}$ was generalizable to other figures, she shifted her attention to the size of the whole fraction $\frac{1}{4}$ referred to. The fact that she was able to pay attention to the referent whole and the size of that referent whole when thinking about a fractional quantity suggests awareness, on her part, that the size of a fractional quantity is determinant of how big the fractional quantity is. It also suggests that a fractional name (such as $\frac{1}{4}$) can be attributed to different referent units but it may not refer to the same size units. She seemed to know that in comparing given fractional quantities that had the same label, such as $\frac{1}{4}$, size of the quantities was the determinant factor as opposed to the appearance of the quantities. In this sense, to Nancy, regardless of the appearance of the quantities, two figures might refer to the same fraction. In addition, in deciding on whether the given shaded area in problem 1.1 would represent one-fourth, she noted that “even though these aren’t the same shape as long as they are the same like area, they’d be the same.”

Nancy was not only able to think about fractions with their part-whole and quantity interpretations when needed, but also she was able to interpret the components, numerator and denominator, and use those interpretations when comparing fractions. The
following episode illustrates her understanding of numerator and denominator and how she used them in service of comparing two fractional quantities such as $\frac{5}{17}$ and $\frac{7}{15}$.

N: The denominator tells how many pieces it is divided into. So this is divided up into seventeen pieces and this is divided up into fifteen pieces.

R: When you say this is divided up into, this refers to what?

N: The whole.

... 

R: So you are, are you using the same whole for two fractions or the different wholes for the two fractions?

N: If you are, if I want to compare them, they should be the same whole.

R: Because, why?

... 

N: Okay. As long as you have the same size whole for each fraction, then you can compare them because you can’t compare something that this big and something that this big. Because then the pieces you just can’t compare.

R: Okay, okay.

N: Um so, if you have whole and it’s divided up into seventeen pieces, there are going to be more pieces than if you divide it up into fifteen pieces. So the seventeen pieces are going to be smaller and the fifteen pieces are going to be bigger [R: Okay]. So if you have five of the smaller pieces or seven of the larger pieces, seven-fifteenths is the larger fraction because you have more pieces that are bigger in the whole.

Several points became clear from the above episode. Nancy initially noted that the denominator determines how many pieces a whole is divided into and “the numerator is the number on the top which counts how many you have…” In comparing the two quantities, she seemed to be aware that both quantities should refer to the same referent whole. However, this awareness did not seem to be at a level that would help her to generate a rationale for why the same referent wholes were being used. With the assumption that both quantities refer to the same whole, she was able to coordinate the
role of numerator and denominator to make a comparison between the two given quantities.

For a similar comparison question for the fractions $\frac{9}{22}$ and $\frac{7}{24}$, she mistakenly tried to compare $\frac{9}{22}$ and $\frac{7}{20}$, which confused her. She knew that $\frac{1}{20}$-partitions were bigger in size than $\left(\frac{1}{22}\right)$-partitions; however, she did not know how to compare nine of smaller partitions, $\left(\frac{1}{22}\right)$, to the seven of the bigger partitions, $\left(\frac{1}{20}\right)$. This ambiguity led her to use diagrams to make the comparison. Since there was a very little difference between the two fractions in size, the diagram she drew did not give her helpful information either. She made a guess that $\frac{7}{20}$ would be bigger because “It [pause] well [pause] it just seemed like more of it was shaded, more of the whole but, I can’t really rely on perfect drawings or equivalent wholes.” Even though she was making a visual judgment about the size of the given fractions, she was aware that this visual comparison would not be taken as a basis for the final answer.

Nancy was able to compare the two fractions, $\left(\frac{5}{17} \text{ vs. } \frac{7}{15}\right)$, easily but she got stuck on the comparison between $\frac{9}{22}$ and $\frac{7}{20}$. Her difficulty suggests that, in comparing fractions, her focus was on coordination of the partitioning and the meaning of numerator and denominator as opposed to the multiplicative comparison between the numerator and denominator. She seemed to be not paying attention to the key idea that the fractions
define a multiplicative relationship relative to their referent unit. On the contrary, she was using the part-whole understanding of fractions. By using the numerator and denominator as quantifiers of partitions, she was deciding on the size of the fractions at hand. This way of reasoning was working nicely for Nancy until this quantification did not clearly reveal the size difference between the two fractions.

To sum up, Nancy could think about fractions using both part-whole and quantity-understandings when needed. She also knew the functioning of numerator and denominator for a given fraction. There was also awareness of the fact that denominator is the determinant of total number of pieces a whole is partitioned into and numerator determines the number of partitions at hand. When comparing fractions, Nancy was able to coordinate the functioning of both the numerator and denominator to decide on the result of the comparison. However, she was not paying enough attention to the multiplicative comparison between the numerator and denominator, which made the comparison doubtful for her. In this sense, she was limited to the use of part-whole understanding of fractions.

b. The Case of Wanda

Wanda’s understanding of fractions was similar to Nancy’s. Wanda was aware that in order to label a piece of a whole as one-fourth, one needs to partition the whole into four equal size pieces: “I don’t mean equal as in shape necessarily but equal in amount.” Additionally, when comparing one-fourth of different size wholes, although the label for those quantities was one-fourth, the “actual amount” the partitions referred to
would be different according to Wanda. By “actual amount,” she was referring to the area the partition covered. In this sense, Wanda was focusing on the whole to which a given fraction referred and the size of it. This close attention to the size of fractions helped her get through comparison-of-fractions problems. When the problem was about comparing \( \frac{5}{17} \) to \( \frac{7}{17} \), Wanda said:

W: The seven over seventeen is larger, is a larger fraction. Because our whole is for example this is our whole, and if we divided it into seventeen equal pieces, um seven of those pieces since they are equal would be larger than only five of those pieces.

In this sense, she was focusing on the role of the numerator and the denominator. Using the denominator of 17, she seemed to know that a whole was needed to be divided into 17 equal pieces. In other words, she seemed to know that denominator determined how many equal-size sections a whole needed to be divided into. And numerator meant to Wanda how many of those sections one needed to take into consideration.

In addition, in her later responses, she emphasized the importance of having both fractions referring to the same whole in comparison problems. It appears that she knew that the size of the whole was one of the determinants of the size of the fractional pieces.

W: When you compare two fractions, you have to use the same whole, because, say, I used this whole compared to this whole [pointing to different size figures] [R: Uh huh] dividing this [pointing to the smaller figure] into seventeen pieces and dividing this [pointing to the bigger figure] into seventeen pieces um I’d say I colored seven here [pointing to the smaller figure] and five here [pointing to the bigger figure], this five in actual amount will be bigger than this seven even though seven out of seventeen usually, when you compare them in- Okay usually when you compare them in the whole, so you use the same whole, so this [pointing to 7/17] would be bigger. But like in actual amount if I used five, say this is- Represents five [R: Uh huh, okay], and seven was here, this little amount is still smaller than this. So that’s why you still need to use the same whole to compare them.
As we see in her above explanation, Wanda was aware that having two different size wholes might make the fraction \( \frac{5}{17} \) bigger than \( \frac{7}{17} \) whereas having the same referent whole for both fractions gave an opposite result. In this way, she seemed to pay close attention to the referent unit to which the given quantities referred.

When the problem was about comparing \( \frac{5}{17} \) and \( \frac{7}{15} \), Wanda first attempted to interpret the size of the fractional pieces that the denominators determined for these fractions: “we are still using the same whole to compare them. Um, there is going to be more pieces that this whole is divided into if the number seventeen- So each individual piece is going to be smaller compared to dividing these into fifteen pieces.” However, this type of determination did not help her to figure out which fraction was bigger. Hence, she wanted to draw both fractions but decided not to after some thinking. Then she realized that she already compared \( \frac{5}{17} \) and \( \frac{7}{17} \) as illustrated in the following episode:

\[
\begin{align*}
W: & \quad \text{There is bigger pieces, yeah, this one is bigger. Duh! [talking to self], okay, um. These } [\text{pointing to fifteenths in 7/15}] \text{ are bigger divisions of pieces, okay. Because it’s divided into lesser pieces, and um okay hold on. If you compared five-seventeenths to the regular seven seventeenths these pieces } [\text{pointing to the diagram for shaded 7/17}] \text{ would be bigger anyway. Okay, there is more pieces here } [\text{pointing to the diagram for 7/17}] \text{ than here } [\text{pointing to the diagram for 5/17}]. \text{ But here it is even like even bigger than that because it is divided into larger pieces than the seventeenth. I don’t know if that made any sense but it made sense to me, and this is bigger.}
\end{align*}
\]

As seen through her wording, Wanda based her judgment on a previously made comparison of \( \frac{5}{17} \) to \( \frac{7}{17} \). To her, \( \frac{7}{17} \) was already proved to be bigger than \( \frac{5}{17} \). It seemed
Wanda was thinking that since the denominator determined size of fractional pieces that made up a fractional quantity, changing the denominator to a lesser value such as 15 was going to make the fractional pieces even bigger than the seventeenths. Therefore, she announced $\frac{5}{17}$ as the bigger fraction than $\frac{7}{15}$. In this regard, Wanda showed the same two understandings concerning numerator and denominator as Nancy.

This way of reasoning was also apparent in Wanda’s solution to a similar type of problem such as comparing $\frac{9}{22}$ to $\frac{7}{24}$. She announced $\frac{9}{22}$ as the bigger fraction and the rationale for that choice was as follows:

W: Because I knew that eleven over twenty is half for this $\frac{9/22}{22}$, I knew that twelve over twenty four is half of this, for this $\frac{7/24}{24}$. And seven is further away from twelve here than nine is from eleven [laughing]
R: Okay, so you are comparing them to-
W: To their like leftovers pretty much. And now the left over, uh no I am sorry this isn’t left over. Um, in comparing them, using like different fraction of the same whole-
R: Like what kind of-
W: Like eleven, eleven of twenty two is actually one half of the whole. And then twelve of twenty fourth is one half of the whole.
R: Okay, uh huh.
W: There. And um since 11 is half, 9 pieces is only two away [looking at the numerator difference for 11/22 and 9/22]. And then 7 from twelve is five [looking at the numerator difference for 12/24 and 7/24], so-
R: Uh huh.
W: This $\frac{9/22}{22}$ is closer to the half mark than this $\frac{7/24}{24}$. So I know that this $\frac{9/22}{22}$ is bigger.

... W: Yeah [solidly]. Again this $\frac{9/22}{22}$ is more pieces of a smaller whole, less pieces of a larger whole $\frac{7/24}{24}$. So this $\frac{9/22}{22}$ is definitely bigger.

Even though what Wanda suggested was an effective way of comparing two fractions, her judgment was based on the additive comparison of the given fractions to an
5.1.2. Participants’ Understanding of Mixed Numbers

The participants had a limited understanding of mixed number conversion, which was about some formula that did not seem to have a rationale for them. For instance, when asked to change the mixed number of \( 4 \frac{2}{3} \) into an equivalent improper fraction, Nancy was calling on a prior experience that was conventional to her as follows:

\[ N: \text{… the way I grew up doing it, um, like I would multiply the denominator by the whole numbers. There are three times four which would be twelve and then add two, which is the numerator. And that would have fourteen thirds.} \]

Nancy’s response suggests that Nancy did not seem to have any difficulty in implementing the “conversion method”\(^{16}\) to the given mixed number \( 4 \frac{2}{3} \). Even though she was able to explain the process of turning mixed numbers into improper fractions, as she stated, this explanation was coming from her recollection of a recent class experience in MATHED 1. When I asked her, “according to that method, what makes four and two thirds equal to fourteen-thirds?” she provided the following response.

\[ N: \text{… Like because we are learning about it in class, like it makes sense that I just don’t think about it that way that each whole like there is four} \]

---

\(^{16}\) Throughout this chapter, the process of converting a mixed number into an improper fraction using the commonly known method, \( \frac{b}{c} \rightarrow \frac{a \cdot c + b}{c} \), is called “conversion method.”
wholes right here. So each of them is divided up into three parts, since three is the denominator and it says how many parts the whole is divided up into. So if you have four wholes and they are each divided by three, or divided into three equal parts, then you can multiply four by three and so you know you have twelve equal thirds and since you have two thirds along with the four wholes, then you’ll have twelve thirds plus two thirds, which is fourteen thirds.

From the above episode, she seemed to know that the whole number part (in this case 4) could be represented in terms of the unit fractional part (in this case, $\frac{1}{3}$). She also knew that she needed to total the number of parts to find the equivalence of a mixed number.

As a result she stated the goal of this conversion as to “find out how many … fractional parts you have [hesitantly] maybe.” These observations about Nancy imply that she knew how to implement the conversion formula and had a limited understanding of why that formula worked.

Wanda on the other hand also had a handle on the conversion formula, and without any difficulty she was able to convert the given mixed number, $4\frac{2}{3}$, into the improper fraction, $\frac{14}{3}$. When asked about the rationale for such an algorithm, Wanda created a diagram for $4\frac{2}{3}$ (four rectangles adjoined to $\frac{2}{3}$ of another rectangle) and then partitioned each unit (rectangle) into thirds. Then she counted all the thirds and found 14.

In this way, she found another way of representing $4\frac{2}{3}$ in terms of improper fractions. She was not sure why both ways were resulting in the same quantity but as she stated she knew that both ways of transforming mixed numbers into improper fractions somehow corresponded with each other.
5.1.3. Participants’ Understanding of Equivalent Fractions

The participants had varying understandings of equivalent fractions. Nancy did not have any difficulty when trying to represent $\frac{3}{4} = \frac{6}{8}$ using diagrams. She first drew $\frac{3}{4}$ and then spoke of dividing each one-fourth unit in half to re-present $\frac{3}{4}$ in terms of eighths. When asked to find an equivalent form of the same fraction without any diagram use, she brought up the algorithm of multiplying both the numerator and the denominator by the same number. However, it appears that she was aware of how the algorithm worked.

N: Because [pause] if you still have, you’ll still have the same size whole but you’ll be increasing the number of pieces that the whole is divided up into and the number of pieces that you have by the same number. The relationship between how many pieces you have and how many pieces the whole is divided up into does not change.

It is unclear whether Nancy understood why the relationship between the numerator and denominator still stayed the same or just that they did.

Wanda seemed to know about the algorithm to produce equivalent fractions. Not only she could represent the given fractional quantity $\frac{3}{4}$ diagrammatically but she was also able to produce equivalent fractions to it using the general algorithm of extending or shrinking both the numerator and the denominator accordingly. Wanda’s reasoning for how the formula worked was:

W: Well, because they are not like- Even though like the whole is getting bigger so is the number of pieces out of that same thing. So it ‘s like, it’s still, like for this reason I kept dividing you know this [referring to the shaded are in a circle equivalent to the 3/4 of the circle], I can keep
As seen in the above episode, based on the drawn diagram of $\frac{3}{4}$, she seemed to know that each equivalent form of $\frac{3}{4}$ referred to the same quantity (“amount”) and changing the number of partitions within that quantity would not affect the size. Implicit in her reasoning was that the repartitioning of a given fractional quantity (like $\frac{3}{4}$) did not affect the relationship of numerator to the denominator. She did not explicitly refer to this relationship, nor did I probe her about that relationship.

5.1.4. Participants’ Understanding of Division (of Fractions)

a. The Case of Nancy

I-Nancy’s Understanding of Division of Fractions

Observation #1. The pre-interview with Nancy revealed that she could recognize division of fractions problems.

When given a contextual problem such as “Say you have $7/2$ liters of ice cream and each cone takes up $2/3$ liter of ice cream. How many cones can I make out of the ice cream I have?” Nancy was easily able to recognize that the given problem was a division of fractions problem. This is illustrated in the following episode.

N: [reading the problem] Okay seven halves liters of ice cream. You wanna know how many times two thirds will go into seven halves.
R: Hmm.
N: So I think it would be seven halves divided by two thirds.
R: Because?
N: Because the division, you divide one set into the whole. Like one, like or two thirds into the whole which is seven halves and then you’d find out how many times two thirds can go into seven halves, which would equal how many cones you can make.

After reading the problem, she was able to name the given problem as a division problem since it asked, “How many times two thirds can go into seven halves.”

**Observation #2.** The pre-interview also shows that Nancy interprets division of fractions problems as, “How many of the second quantity are in the first quantity?” However, it seems that she could not generate division of fractions problems.

Nancy had very difficult time generating a division-by-fraction type word problem for the mathematical expression $\frac{2}{\frac{3}{4}}$. She was interpreting the problem as “How many times three-fourths can go into two?” However, she was not able to create a word problem that reflected the same structure. In other words, in her word problem, she was not setting an initial quantity and turning it into a number of groups of certain size.

Her initial reaction and the word problem she created for the given mathematical expression was:

N: Well I was thinking, two divided by three fourths means that you seem how many times three fourths can go into two. And I was thinking like if I had two, two wholes of something, three-fourths, three-fourths, I was gonna see how many times three-fourths would go into each whole [*referring to 1*]. But I guess it’s because I knew that it would only go in once. But this-
R: What, what will go in once?
N: Three-fourths.
R: Uh huh.
N: Because I guess in my problem I kind of divided like the two wholes like into two different things. And I knew three-fourths would go into each whole once. But then there would be something left over.
R: Hm hmm.
N: But I think this question might turn into a subtraction problem I’d say.
R: Would you read your question?
N: Yeah. If Jen had two candy bars and ate three-fourths of one. Then gave the other candy bar to her friend to eat three-fourths of it as well. I was gonna say how much was left over. I guess I could say how much of the two wholes would they have left over. But I don’t think that would answer two divided by three-fourths.

In her above attempt to generate a word problem, Nancy clearly stated her goal as to find the number of $\frac{3}{4}$ within 2. However, the problem she created did not mathematically correspond to the mathematical expression she was given. For this case, there seemed to be a lack of fit between the word problem that she generated and the mathematical meaning of the given expression.

She already knew that each constituent unit of 2 includes only one $\frac{3}{4}$-piece and therefore she was eating $\frac{3}{4}$ of one apple whereas another person was eating $\frac{3}{4}$ of the other apple. So far she situated two $\frac{3}{4}$-sections into the two wholes considering that each whole represented a person in her context. At this point she got lost because it seemed that she did not know what to do with the leftovers.

When we carefully look into her way of reasoning about the word problem she created, she was basically implementing each part of the word problem as in the following sense. She first translated the phrase, “Jen had two candy bars” into “if I had two wholes of something.” She then translated the second sentence, “Then gave the other candy bar to her friend to eat three-fourths of it as well” into the form “I kind of divided like the two wholes like into two different things. And I knew three-fourths would go into
each whole once.” Next, she said she was looking for leftover, perhaps an attempt to ask an appropriate question based on her word problem to that point. She was using “How many three fourths can go into 2?” However, this did not suggest a category of word problems to her. She did a rough translation of the mathematical expression into a word problem. Her question in this problem was “How much [apple] was left over?” She was aware that this problem was a subtraction problem as opposed to a problem that corresponded to the mathematical expression $2 \div \frac{3}{4}$. That is, once she had created the problem, she could think about the operations needed to solve it. Her follow-up attempt to create another word problem that corresponds to the same mathematical expression was not successful either as shown in the following episode.

R: … So what makes it the same as the given question two divided by three-fourths?
N: I don’t know that it is. I might have made a mistake.
R: I am not saying it is wrong.
N: But I don’t know if it is, if it’s right. I just thought of something else that I could maybe use. Like if said I had [pause] okay if I was going to run, should I write it down?
R: Sure. Or you can just say it. I mean if, if it is wordy. [N: Okay.] If I was going [pause] my goal for today was to run for two miles. But three fourths of that distance, after three fourths of that distance I got too tired and I stopped. How far did I run?
R: So you, you wanna run two miles.
N: Yeah, that was my goal, but-
R: Okay. After three-fourths of that distance, you got tired.
N: I got too tired and I stopped [in a confirming tone]
R: So, what was the question, the follow-up?
N: How, how far did I run?
R: [Then the researcher asked her if this problem could correspond to 2÷3/4]…
N: Well, I’d think it might be a multiplication problem.
R: Which, which might be the multiplication problem?
N: Well, normally if you have like [pause]-
R: The one you suggested or?
N: The [pause] if, the distance one.
R: Okay.
N: If I ran two miles, well if my goal is to run two miles. And I completed three-fourths of it. How far did I run? Because normally when you say “of something” you multiply.

As seen in the above dialog, one more time Nancy created a problem that did not match the structure of the division $2 \div \frac{3}{4}$.

When examined thoroughly, in both word problems Nancy was trying to directly transfer the fractional quantities from the given mathematical expression to some context. Her overall goal of finding the number of $\frac{3}{4}$ within 2 was not guiding her creation of the word problem and the use of appropriate context.

The way she chose to reach the overall goal was the use of the invert-multiply algorithm as illustrated in what follows.

R: So, if you were to solve this problem [N: Yes], the two divided by three-fourths, just solve it not a word problem. How would you do that?

N: Well, the way that I learned it would be to multiply if I do reciprocal, which would be three-fourths. And we’d have eight thirds. And if you wanted to put it [pause] like make it so it is not an improper fraction, if you want to make it a mixed number, you’d see how many times three goes into eight, which would be two. And then you’d have two thirds left over so.

... Do you know why you invert and multiply? You- Is it a rule you just remembered or?

N: It was and today in class we were working on it. But I don’t, if I remember. It is pretty much just a rule [laughing].

As a result she had this invert-and-multiply algorithm as well as the idea, “How many $a$ can fit in $b$?” However, these two sources did not seem to inform each other for Nancy. The goal, as she conceptualized it, did not seem to be abstracted from a set of word problems.
Observation #3. Nancy resorts to invert-and-multiply algorithm in order to solve a division of fractions problem.

While trying to find out the number of cones of the size $\frac{2}{3}$-liter that could be made out of $\frac{7}{2}$ liters of ice cream in problem 17, she was following the same train of thought. She first verbalized what the question was asking, “How many times two thirds will go into seven halves?” Then she labeled the problem as a division problem and applied the invert-and-multiply algorithm to figure out the answer because, as she later stated, she did not know how to reason the problem in any other way than the algorithm.

II- Nancy’ Notion of Whole Number Division

Observation #1. Nancy is quite competent with whole number division problems. She can easily identify the structure of the given whole number division problems.

Nancy showed quite a competence in whole number settings. Problem 14 (whole number multiplication), problem 15 (whole number division with missing the size of group) and problem 16 (whole number division with the missing number of the groups) were about whole number division and multiplication problems. For these problems, without doing any paper-pencil calculation, she was able, to identify the structure of the problems and to mentally formulate solutions to the problems quite easily. For instance, in thinking about problem 16 [Ali has 35 oranges and he distributed them among some friends. If each person gets seven oranges how many friends does Ali have?], she said,

N: [reading the problem distantly for 5 seconds] Well this would be a division problem as well because he have [sic] the whole and he distributed the whole among people. And you know he gave seven to
each people. So if you divide thirty five by seven, then you’d find out how many people you have.

…

N: You wanna know how many sets of seven you have so you can divide thirty five by seven.
R: Okay, okay.

Her description suggests that she was doing an analysis of the given situation in terms of the quantities on which it was based. Additionally, she was able to identify the goal for the problem as distribution of the given amount. Nancy was not only able to identify the overall goal for the given problem, but also she could think about a numerical way to reach that overall goal. She knew that she needed to find the number of sets of seven in thirty five, and she also knew, based on her whole number knowledge, that there were five sets. In other words, she was identifying the arithmetic relationship involved in the given division problem. Here, the arithmetic relationship between 35 and 7 was that there were five sets of 7 in 35. Given two whole numbers such as 35 and 7 as in the above episode, she could think about the number when multiplied by 7 gives 35. In this sense, she was benefiting from her understanding of multiplication to think about the involved numerical relationship that was required to find the result of the given division problem. And then she was relating it back to her overall goal of finding number of 7 within 35.

This kind of thinking about arithmetic relationships was also apparent in Nancy’s work when she was trying to make up problem for the answer 7R3.

N: [reading the problem for 10 seconds] Okay, um [pause] well I know that [pause] I know that seven times two is fourteen. So I know that seven can go into fourteen. Wait, wait! I know two can go into fourteen seven times. So two and the fourteen would just be seven but if I need the remainder of three I can add three more to fourteen to have seventeen. So two and the seventeen be seven. Seven times two is fourteen. Three, so it would be seven remainder of three …
R: Uhh what made you choose two? How did you come up with two?
N: I was just trying to think of multiples of seven and figure out what could go into seven, or [pause] what [pause] I was trying to think of two numbers that I can divide to get the answer of seven and then just add three to the number that was being divided into so that I would have three leftover

... N: Because I thought of two times seven is fourteen. So the two numbers I would use would be two and fourteen. But then I just added three to seven, or three to fourteen to get seventeen so that I would have three leftover when I divide seventeen by two. But then eight can go into, so it doesn’t really work.

The above episode illustrates that Nancy, when deciding on creating a division with whole number problem, was looking into the numeric relationship that would define a division expression that has the answer of 7R3. Even though her divisor (less than remainder) was not an appropriate one, she later on realized this flow and corrected herself. However, the issue here is that she could easily think about and build a numeric relationship between a given quotient and hypothetical divisor and dividend, which suggests that she had an abstraction of division of whole numbers as numeric relationships.

III-What Is the Difference Between Both Cases for Nancy?

Puzzling to me was that Nancy was able to think about division in whole number setting efficiently, but she was not able to transfer it to the fractional setting. In the fractional setting she was able to recognize division problems. In addition, when asked to solve division of fractions problems, she was able to identify the overall goal for both the given problem and the mathematical expression \((a\div b, a \text{ and } b \text{ are fractions})\) that represented that word problem as, “How many \(a\) can go into \(b\)?” However, she was not able to generate division of fractions problems.
On the other hand, in the whole number setting she could also identify the structure of the given problems as division. When asked to solve partitive (or quotitive) whole number division problems, she was able to identify the overall goal for both the given problem and the mathematical expression \(a ÷ b\) that represented that word problem as “How many \(a\) can go into \(b\)?”

At some point during the interview, both participants were asked to decide whether division problems would result in a number that was less than the dividend (Problem 8). Nancy’s first reaction was to “work out” some examples and decide whether the claim would hold true or not.

N: Can I try to work out some problems?
R: Sure, sure.
N: [using invert-and-multiply algorithm to solve 2÷1/2] Um- I think it is wrong. In this example, two divided by one-half, the answer is four. So four is larger than the dividend. Two, two is the dividend. Because when you divide you divide two by one-half you are seeing how many times one-half goes into the two wholes. And seeing one half is less than a whole, you had more of the halves going into the two. Like, you’d have more halves going into the two, because it’s less than a whole.

Here, she was using the idea of how many \(\frac{1}{2}\) goes into 2 as a way to explain why the quotient would be bigger than the dividend. She was using the example of \(2 ÷ \frac{1}{2}\) and reasoning that since \(\frac{1}{2}\) was less than 1, it would go into 2 more than twice. Her focus was on the magnitude of divisor, and since the divisor was already less than one unit, it would go into 2 more than twice according to Nancy.
In a different example, $\frac{1}{2} \div \frac{2}{3}$, Nancy was again using the same idea (how many $\frac{2}{3}$ are in $\frac{1}{2}$) to figure out whether the divisor would go into dividend more than once or not. In this case, she first made a magnitude comparison between the two fractions and since $\frac{2}{3}$, the divisor, was bigger than $\frac{1}{2}$, the dividend, she realized that the divisor would go into $\frac{1}{2}$ less than one time. Nancy’s work for such problems during the interview was consistent in the sense that she always checked for some examples or counterexamples and based her decisions on those.

In another problem, $\frac{10}{3} \div \frac{2}{5}$, when I asked about what would be a reasonable word problem for the given division expression, she responded:

N: So I need to find a problem that wants to take ten thirds and split it up into two fifths. Or find out, hmm [pause]. You have ten thirds and you wanna find how many times two fifths goes into it. But I don’t know. Um [pause], I don’t know.

She seemed to not know how to construct a word problem from a given division expression even though she was able to formulate it as, “How many 2/5 are in 10/3?” This also shows that she did not have an abstraction of quotitive situations that would help her build up a division of fractions structure for the given mathematical expression.

As a result, Nancy seemed to have the idea of, “How many $a$ can go into $b$?” for quotitive division regardless of the types of the quantities $a$ and $b$. All these evidence suggests a conclusion that Nancy did not seem to have an abstraction of quotitive
situations but just an abstraction of arithmetic situations in both fractional and whole number settings.

b. The Case of Wanda

I-Wanda’s Understanding of Division of Fractions

Observation #1. The pre-interview with Wanda shows that she could recognize division of fractions and consider the solutions to them as finding the number of one quantity that makes up another quantity.

When asked to think about the problem, “Say you have \( \frac{7}{2} \) liters of ice cream and each cone takes up \( \frac{2}{3} \) liter of ice cream. How many cones can I make out of the ice cream I have?” she reasoned as follows:

W: Um, okay we have seven [misspeaking] liters of ice cream and that turns into three and a half for this, um and then a cone takes up one-fourths liter [R: Uh huh]. And um basically I knew 1/4 went into one half twice. And then I just did that the same of a … [inaudible]… It went to one four times, so I did that three times [referring to 3 in \( \frac{3}{2} \)] like two here [pointing to 1/2 in \( \frac{3}{2} \)] because it went into half twice and then four [referring to the number of times 1/4 went into 1] plus four [referring to the number of times 1/4 went into 1] plus four [referring to the number of times 1/4 went into 1] plus two [referring to the number of times 1/4 went into 1/2] is fourteen.

She was in search of a number of \( \frac{1}{4} \) within \( \frac{3}{2} \) based on the operation of repeated combining to generate \( \frac{3}{2} \).
Observation #2. The pre-interview also reveals that Wanda could not generate division of fractions problems.

Wanda’s conceptual understanding of division of fractions (or division with a fraction) was not strong either. When asked to generate a word problem that could be represented by $2 \div \frac{3}{4}$, her first inclination was to solve the given mathematical expression using invert-and-multiply algorithm and then to think about a context as follows:

W: I have to think of \[R: \text{Uh huh}\] it’s like a, like a real life or hypothetical situation where I need to take like divide two, three-fourths into two.
R: Okay.
W: Two by three-fourths, umm … [pause about 7 seconds] … I don’t know. I can’t even think of.
R: Okay [waits for about 5 seconds].
W: This is I don’t know if this is what I really want but-

Even though Wanda was in search of a context to transfer the quantities 2 and $\frac{3}{4}$, the word problem she generated was a multiplication problem. She formulated the following problem: “I have two cookies … I wanna share with my friend but I wanna end up with more of my cookies than my friend. I wanna end up with $\frac{3}{4}$ of my two cookies. What is that?” Her question was about quantification of a part of a quantity, which was multiplication-with-fraction type of problem. Once she offered this word problem, she then tried to reason a possible solution method as follows:

R: … Uh, if you were to draw a diagram for your question…
W: Three-fourths?
R: The, two divided by three-fourths.
W: Okay. Did you mean to find the answer, like?
R: Yes.
W: Well I have two \[R: \text{Uh huh}\], two of equal \[\text{drawing two circles}\] wholes \[R: \text{Uh huh, wholes}\]. And um fraction, we are gonna have
fourths here [pointing to circles], we divide this [pointing to the first circle] into fourths [partitioning the first circle into four pieces]. So I want three-fourths of that [partitioning the second circle into four pieces]- Um, and one-fourth would be this [pointing to the last piece in the first circle], so three-fourths would be six out of eight pieces.

R: So, it, it--
W: Two, three, yeah, that’s right [talking to herself].
R: So, is it the, is it the piece you are gonna get?
W: I am getting this much [pointing to shaded pieces in both circles].
R: This much, okay.
W: Yeah [speaking simultaneously] … and my friend is getting this [pointing to unshaded pieces].
R: So the result of this uh problem is what? This much?
W: Six, it’ll be I am getting twoooo … I am getting six of the eight pieces total and so that would be the three fourths, yeah [writing 6/8 and then 3/4 next to it].
R: Okay. So-
W: Well, that doesn’t make sense, does it? Plus it is not even the same as what I came up with here [pointing to her result, 8/3, driven by using invert-and-multiply rule]. Now I am really confused. I don’t know something …[inaudible]… apparently.
R: That’s okay. So-
W: I know these diagrams are right though.
R: So, okay. So in a problem like this, what you’re saying is your goal is to figure out three-fourths of each piece?
W: Yeah, that was my goal [R: Okay]. And then add it to mean the whole two cookie.

At this point she was thinking that the word problem she created and her solution method did “not make any sense.” Neither did she state nor did she think about the overall goal (how many \( \frac{3}{4} \) are in 2) for the given division of fractions expression even when creating the diagram. It also seemed that her difficulty in creating a word problem was that she was trying to directly transfer the given quantities into a context without referring to the overall goal. This is not to say that she had to refer to the overall goal to be able to proceed because for many operations no explicit articulation of the goal is necessary to
know how to proceed. However, in division of fractions case, lack of such an articulation seemed to be an obstacle that gets in the way of creating a word problem.

**Observation #3.** Wanda has a hard time in interpreting context-free division of fractions problems, whereas this is the opposite case for contextual problems.

It seemed from her responses to different questions that she was having trouble for the given context-free division of fractions problems, whereas it was the opposite case for context-dependent problems. When asked to identify the role of quotient for a specific problem like \( \frac{5}{2} \div \frac{3}{4} = 7.3 \), she got confused as follows:

W: [using the calculator] Seven point three three three three three [7.33].
R: Seven, let’s say seven point three. Uhh, so what does this seven point three tells us?
W: The answer [laughing].
R: Okay the answer but uhh for example what does seven tell you regarding this problem? What kind of information is it giving?
W: Um…
R: You didn’t understand my question?
W: I understand but I don’t know.
…[after some pause]…
W: Like I have to [pause] I think- Say it [referring to the problem] another way maybe I [pause]
R: Uhhh…okay.
W: Maybe you can’t say it another way.

However, when there was a context in which these quantities were being embedded, she was able to then go back and write the corresponding mathematical expression. And then she stated the overall goal for the problem as “How many times \( \frac{3}{4} \) is divided into \( \frac{5}{2} \)” as follows.

W: Like I have to- I think- Say it another way maybe I-
R: Uh, okay.
W: Maybe you can’t say it another way.
R: Okay. Uh, I can. Uh, let’s say I have this many flour [pointing to 5/2], okay? [W: Okay]. This many cups of flour [W: Yep]. And each recipe calls for this many, this much [pointing to 3/4] I am sorry. And at the end how many recipes can I make?
W: Okay, and the answer is that [pointing to 7.33]?
R: Uh, the answer is this [pointing to 7.33]. Now, what does this answer tell you?
W: It tells me how many times three-fourths is divided into this [pointing to 5/2] [R: Okay]. Or how many three-fourths this-- [interrupted]
R: Okay, thinking about the recipe example like seven tells you what and point 3 [referring to 0.3] tells you what? …
W: You said how many loaves, this is the answer to how many loaves this can make, right? [R: Hm hmm.] So it tells me we can make seven point three [7.3] loaves. So not quite eight loaves but definitely seven.

As seen in the above episode, Wanda was more comfortable in thinking about the overall goal for the division of fractions problems based on context. However, when these problems were context-free mathematical expressions, she was not able to articulate the mathematical structure of the problem other than its name as division. This seems to be a result of the fact that contextual problems make explicit the referents to be focused on and the overall goal based on a real life story. In other words, as seen in the above example, when the issue was to determine number of recipes given the amount of flour at hand and the amount of flour per recipe, Wanda could easily think about the involved numerical relationship between the quantities in the problem. In this sense, Wanda was focusing on recipe-making context based on the amount of flour per recipe, and she did not have a need for identifying the mathematical structure of the problem. However, context-free cases require an abstraction of the relationship between the given quantities. This relationship helps one coordinate the two extensive units. Such coordination is done through an abstraction of divisor as an intensive quantity. In other words, this is an abstraction of divisor as an intensive quantity that relates a number of dividend units to
one quotient unit. Wanda did not seem to have such an abstraction of the relationship or divisor to work in the context-free situations. However, in contextual settings, since the coordination between the two extensive units is carried by the story in the contextual referents, she had no problem seeing $\frac{5}{2}$ as $\frac{5}{2}$ cups of flour and one bread. In contrast, when the problem is to determine the number of three-fourths in five-halves, one needs to have the abstraction that $\frac{3}{4}$ represents a group. Wanda did not seem to have such an abstraction.

II-Wanda’s Notion of Whole Number Division

Observation #1. Wanda can easily recognize the division structure in a given whole number division problem regardless of whether it is partitive or quotitive. Later in the interview, when encountering division and multiplication of whole number problems, Wanda was able to identify the operation that could be used to solve the problems and mentally go through them and solve them. She could easily identify the division operation and what made the problem a division problem as follows:

R: … Okay, now Chris has twenty five apples and she has five friends. If she wants to share those among those friends how many apples does each friend get?
W: Five [instant response].
R: And what operation did you use?
W: You said apples?
R: Yes.
W: Division.
R: Division. What made you think that it is a division problem?
W: Twenty five and she has to distribute it to five of each friends [R: Okay]. So she- For equal divisions but-
As we see in the above episode, her focus seemed to be on distribution of equal partitions to each party, which made it a division problem to her.

She pursued the follow-up question as illustrated in the next episode.

R: Okay, now. One person has thirty five oranges and he distributed them among some friends [W: Okay]. If each person [sic], if each person gets seven, how many people are there in that room?
W: Five [instant response].
R: What operation did you use?
W: What did I use? Um thirty five times some- Uh you said thirty five oranges and some friends.
R: Thirty five oranges and some friends and each friend gets seven.
W: Gets five, uh seven [R: Seven oranges.]. Yeah, I use division.
R: What made you think that it is a division problem?
W: Maybe I, hold on! Well it could also be multiplication though, I mean you could say what number [times] seven is thirty five. I forget exactly what I did initially but I mean it doesn’t have to be one or the other.

As seen through the above episodes, Wanda did not have any difficulty in identifying the structure of the given problems as division problems in whole number settings. She also did not have any difficulty in solving such whole number division problems mentally.

**Observation #2.** Wanda can generate whole number division word problems for the given mathematical expressions.

When asked to create a word problem for 38÷5, she generated the following problem:

W: Okay. Sally had 38 cookies. Um she uhh has five friends to share them with [R: Uh huh]. How many cookies...how many cookies would each of her friends get?

Without any difficulty, Wanda was able to generate a word problem for the given division expression. It was interesting that she resorted to partitive division as opposed to quotitive division.
III-What Is the Difference Between Both Cases for Wanda?

Regarding division, Wanda did not have the same comfort level with fractional settings she did with whole number settings. Her notion of whole number division was pretty strong. Similar to Nancy, Wanda was also able to transfer the given division of fractions word problems representing $\frac{a}{b}$ into a statement such as, “How many $a$ are in $b$?” and then move to the invert-and-multiply algorithm she had in her repertoire.

When asked to create a word problem for the given expressions such as $2 \div \frac{3}{4}$, Wanda’s attempts resulted in a multiplication word problem. And when she tried to solve the problem using diagrams, she created a representation of that multiplication problem. Additionally, in the case in which the problem was about interpreting a given equation such as $\frac{5}{2} \div \frac{3}{4} = 7.33$ she had great difficulty in deciding on what referents to focus on and how to use those referents for creating a word problem.

Prior to the above questions, when asked to create her own whole number division word problem, without any difficulty she was able to generate the word problem: “Sally had 38 cookies. Um she, uh, has five friends to share them with. How many cookies, how many cookies would each of her friends get?” In whole number setting, Wanda was able to set up a word problem based on the involved numeric quantities. She knew that the goal that needed to be set was about sharing a given quantity (in this case 38) among a number of parties (in this case 5). This suggests that her notion of division was not only about numeric relationships, but also it was an abstraction of partitioning. In addition, she was able to recognize quotitive division problems and was able to solve them using
invert-and-multiply algorithm. It seemed that Wanda’s reasoning was oriented by her abstraction of arithmetic relationships as opposed to an abstraction of quotitive situations. Since she had enough schooling experience with whole numbers, she could generate or solve them based on her abstraction of arithmetic situations without any difficulty.

Like Nancy, Wanda based her conclusions on specific examples when thinking about division problems too. When talking about a given statement such as, “Division always gives a result that’s less than the dividend,” Wanda indicated that if one of the factors was a proper fraction, then the given statement was not true. When making this claim, she resorted to a previously solved problem, \(2 \div \frac{3}{4}\). She first calculated the result of this problem using calculator and concluded that the result was larger than 2. Then, when asked about the underlying rationale for such a conclusion, she applied the invert-and-multiply algorithm to the division, \(2 \div \frac{3}{4}\), and realized that inverting the second factor \(\frac{3}{4}\) made this factor bigger than one (in this case \(\frac{4}{3}>1\)). To Wanda, this inversion affected the result of the problem since multiplying 2 by a number that was bigger than one would cause a bigger result than 2. This counter example led her to believe that the given statement did not hold true.

Wanda made such conclusions based on her work with examples and counter examples. Her explanation of the counter example was based on her thinking about the multiplication that resulted from the inversion of the divisor in applying invert-and-multiply algorithm. On the other hand as we see in Nancy’s case, Nancy thought about applying her overall goal in the fractional setting. She was benefiting from the example,
2÷\(\frac{1}{2}\), but her reasoning was based on application of the overall goal of putting halves into 2. Since half was a simple fraction, and since it was less than one, based on the numeric relationship between \(\frac{1}{2}\) and 2, she knew that number of halves in two could be more than two. In this, she was not focusing on the invert-and-multiply algorithm to identify the numeric relationship involved in the division problem as Wanda did.

Wanda was easily able to think about the validity of the statement “Division makes smaller” in whole number settings as seen in the following episode.

W: Um, I am just counting three different situations like nine divided by three. It is the- Okay, smaller, if you divide, two divided by … see when you are dividing something, your point is to divide it into like a bunch of pieces [R: Uh huh]. Two divided by seven, I guess what if you did like two divided by one [pause] then [pause] two divided by-

Always, I don’t like that word “always” [both laughing], um. Well, with whole numbers I guess it would be not [be] “always” [true] because two divided by one would be two so it could be equal too. But [pause] I don’t know if that makes sense to do, to say bigger because how can you get something bigger than what you have if you are dividing that. I am gonna go with: this is wrong because it could be equal two but not that’s gonna be more than.

As seen in the above episode, based on the examples she generated, Wanda realized that in whole number settings the condition of being less than divisor should have been modified as being smaller than and equal to. She did not need any algorithm to make such inference since her use of numeric values and how they were compared to each other numerically was solid.

To sum up, both participants seemed to have this idea of “How many \(a\) can go into \(b\)?” for the division problems \(a÷b\). However, their abstraction of this idea was dependent on arithmetic situations as opposed to quotitive situations. Therefore, it
seemed that Wanda’s and Nancy’s work was dependent on some numeric examples (or counter-examples). In fractional setting, Wanda’s work with numeric examples was different from Nancy’s. She benefited from the invert-and-multiply algorithm to make sense of the involved numeric relationship among the quantities, whereas Nancy used her numeric abstraction of the overall goal, “How many ___ are in ___?” to think numerically about how one quantity compares to the other.

5.1.5. Participants’ Understanding of Remainder

a. The Case of Nancy

Nancy was asked to set up a division problem for the result 7R3. This was the whole number setting in which Nancy was evaluated. She pursued the following method.

N: [reading the problem for 10 seconds] Okay, um [pause] well I know that [pause] I know that seven times two is fourteen. So I know that seven can go into fourteen wait, wait, I know two can go into fourteen seven times. So two and the fourteen would just be seven but if I need the remainder of three I can add three more to fourteen to have seventeen. So two and the seventeen give seven. Seven times two is fourteen. Three, so it would be seven remainder of three [writing each step in the long division algorithm form as in Figure 5-1].

![Figure 5.1. Nancy’s initial way of setting up a division problem for the result 7R3.](image)

It seemed that Nancy was trying to go through a certain process for finding dividend. This process was about generating a multiple of the divisor (in this case, 7) and adding...
the remainder (in this case, 3) to that multiple. She was not careful enough to choose a divisor that was more than the given remainder, 3.

Later on, using the same method, Nancy was able to generate another division problem \((31 ÷ 4 = 7R3)\). Also she realized that her mistake in her previous formulation was the use of larger remainder than the divisor. The following episode illustrates this.

R: Actually what kind of a divisor and dividend you need to choose? Dividend is the one that’s being divided.
N: Yeah. Um, well seven needs to go into them but there has to be three leftover, but eight can’t go into them. Um [pause] seven times four is twenty eight [drawing another long division diagram and putting 4 for divisor, 31 for the dividend, and 7 for the quotient as in Figure 5-2], four goes into thirty one seven times. Seven times four is twenty eight. Thirty one less twenty eight is three. I think that works.

R: What makes it work?
N: Because four can go into thirty one seven times but yet there is still some leftover. But another whole like if I couldn’t- Oh! Here [pointing to her previous long division diagram as in Figure 5-1] my problem was that I had a remainder that was larger than my dividend.

R: Hmm hmm.
N: Or my divisor.
R: Divisor [repeating after Nancy].
N: So, since the remainder is three [pointing to her new long division diagram as in Figure 5-2] the divisor has to be larger than three.
R: Because?
N: Because [pause], the, if the remainder is larger than the divisor, then more wholes can go into the remainder. Like here [pointing to her previous long division diagram as in Figure 5-2] two, another two could go into three. But another four cannot go into three so I can’t keep computing [pointing to her new long division diagram for \(31 ÷ 4\)].
As seen in her above description, this realization of using larger divisor than remainder seemed to come from her attention to the comparison between the divisor and the remainder. In addition, this comparison was based on the overall goal for the division problem. As a result, in the whole number setting Nancy seemed to think about remainder as the leftover from the process of extracting a number of divisors (as a single unit) from dividend. She also understood the convention that divisors needed to be larger than remainders.

In addition to the whole number setting, Nancy’s understanding was also tested in a fractional setting based on a division of fractions problem. In problem 12, she was asked to find the amount of leftover flour out of \( \frac{7}{2} \) -cups of flour when a single loaf of bread required \( \frac{1}{3} \)-cup of flour. She followed the regular invert-and-multiply algorithm and found \( 10 \frac{1}{2} \) as the result. From this result, she concluded that one could only make ten full loaves of bread and the rest, \( \frac{1}{2} \), would determine the amount of leftover flour. In figuring out how much flour corresponds to one-half loaf of bread, she said:

\[ \text{N: Okay. If you have half a loaf of bread, that means that half- Okay one thirds cup of flour is used to make one loaf of bread so half of one third would make I guess half a loaf of bread. So if you find out what one half of one third is, you'll know how much flour was needed to make that one half loaf of bread.} \]

She seemed to know that the remainder meant what was leftover after identifying a number of divisor groups within the dividend in the contextual situations.
b. The Case of Wanda

To set up a problem that had the answer 7R3, Wanda was careful to choose an arbitrary divisor and followed the method below:

W: So, this is gonna be my basis for my problem I guess. Um seven remainder three, I have to think of something that would come up with the same ...[inaudible]... and I did I decided um five times seven which is thirty five. Um, to do that to come up with a remainder of three I need to do thirty eight so that when I got to the thirty five couldn’t go into three anymore, and I have the remainder of three.

R: So you first think about five times seven?
W: Yeah.
R: And you chose five arbitrarily.
W: I chose, yeah just so then I would make this ...[inaudible]... go into this. Because if I chose like seven I got forty nine, so I choose like fifty two or something [R: Okay] unless ...[inaudible]... to do, yeah. Hm hmm.

As Wanda stated in the above dialog, her focus was on finding a multiple of the whole number part of the quotient and adding the remainder to that. She seemed to know that the whole number part (in this case 7) referred to the number of divisors. Relying on this kind of logic she constructed a division problem inversely.

Context was the rescue for Wanda in fractional settings for a given problem like \( \frac{5}{2} \div \frac{3}{4} = 7.3^{17} \) for which she was asked to think about the remainder and quotient. For this problem, first she was asked to talk about what “7” means and what “.3” means to her.

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¹⁷ Note that the result of the calculation should be 3.3 as opposed to 7.3. However, I gave her the wrong result. Since the problem was about interpreting the quantities and leftover, she was not asked to make any calculation. Rather, she was to talk about the remainder and quotient in general. Therefore, my giving her the wrong answer did not affect her responses.
She did not know how to interpret the whole number and fractional part of the quotient as seen in the following episode:

R: Uhh, let’s say I … I have a question like this. Five halves divided by uhhh- Three-fourths, okay? And the result is- Can you find twenty two divided by three?

W: [using the calculator] Seven point three three three three three [7.33].

R: Seven, let’s say seven point three. Uhh, so what does this seven point three tells us?

W: The answer [laughing]

R: Okay the answer but uhh for example what does seven tell you regarding this problem? What kind of information is it giving?

W: Um-

R: You didn’t understand my question?

W: I understand but I don’t know. Like I have to- I think- Say it another way maybe I can-

R: Uhhh, okay.

W: Maybe you can’t say it another way.

When a contextual setting was added to the problem, her reaction was as follows:

R: Uhhh, let’s say I have this many [pointing to 5/2] … cups of flour [W: Yep]. And each recipe calls for … this much [pointing to 3/4] … And at the end how many recipes can I make?

W: Okay, and the answer is that [pointing to 7.3]?

R: Uhh, the answer is this [pointing to 7.3]. Now, what does this answer tell you?

W: … this [pointing to 7.3] is the answer to how many loaves this can make, right? [R: Hmm hmm.] So it tells me we can make seven point three [7.3] loaves. So not quite eight loaves but definitely seven.

W: … this [pointing to 7.3] is the answer to how many loaves this can make, right? [R: Hmm hmm.] So it tells me we can make seven point three [7.3] loaves. So not quite eight loaves but definitely seven.

R: How much extra stuff do you think is left in terms of cups of flour?

W: Well this is left, the point three [0.3] so- Oh you’re-

R: This [pointing to 5/2] was how much [sic] cups of flour we have

W: I know that.

R: And each recipe calls for this much [W: Okay]. And at the end we said uh seven point three [7.3].

W: So the point three was the leftover amount then.

R: Point three of what? Like--

W: Of the four, like thirty percent.

R: So thirty percent of bread or of recipe?

W: Of well- Of flour I guess. Because this was out of the cups of flour, and this [pointing to 7.3] is how many of them we used so [pause] I guess it
refers to flour. …[inaudible]… This [pointing to 7.3] tells you how many loaves you can make. Wait, I don’t have cups …[inaudible]… three-fourths but-

R: Per bread.
W: Okay. I guess this [pointing to .3] would have to be the whole recipe then. Or of the whole mix of including the flour …

Initially Wanda was confused when the issue was to interpret the fractional and whole number parts of the quotient in a context-free problem, $\frac{5}{2} \div \frac{3}{4} = 7.3$. In the contextual setting on the other hand, she identified the 7.3 as the number of loaves that could be made. In addition, when the issue was to think about remainder, she first said, “0.3,” but then, referring to the context, she changed her answer to the amount of flour corresponding to 30 percent of another recipe (or loaf).

She seemed to be working appropriately in a context-dependent environment when she had a dilemma regarding the remainder or fractional part of the divisor. With context, she could think about the nature of the quantities involved in the division problem and the remainder.

This same behavior was also apparent in Nancy’s thinking. Strikingly they could think about remainder in fractional settings that were situated within context. There seemed to be something about the context that made them see the structure of the problems and helped them analyze the given division of fractions problems. When thinking about the bread-making problem, the problem actually asked about the number of loaves one could make using a certain amount of flour per loaf. For example, for

**problem 12** [Johnny has $\frac{7}{2}$ cups of flour and she needs $\frac{1}{3}$ of a cup of flour to make a
loaf of bread. (a) How many whole loaves of breads can he make out of \( \frac{7}{2} \) cups of flour?

(b) How much flour is leftover?). In this sense, the overall goal they identified was, “How many ___ cups of flour (per recipe) are in ___ flour?” This means that, with context, they had the opportunity to decompose the given situation as opposed to the mathematics built into the situation. Once these quantities were identified (\( \frac{7}{2} \) and \( \frac{1}{3} \)) using the contextual labels, their next step was to think about a number of cups of flour that could be made from the given total amount flour. Once they identified enough number of full cups of flour (in this case 10 full cups) and when there was some leftover flour (in this case \( \frac{1}{6} = 0.16 \)), based on the context, they knew that they needed to determine how much of a loaf of bread (in this case \( \frac{1}{6} \div \frac{1}{3} \)) the leftover flour would correspond to.

The participants had the context to extract the quantities and referents of those quantities. The context was helping them to think about what quantities and referents they needed to consider in the solution process because their judgment was based on the story of the context. They already had the idea that \( \frac{a}{b} \) means, “How many \( a \) are in \( b \)?” Their difficulty in the solution process in context-free problems was with choosing the right referents and proceeding accordingly. Therefore, in contextual situations, they had the option to refer to the context and use given contextual labels with the quantities. Because the quantities were embedded in a real-life story, they were also able to make decisions about the right referent units when grappling with the leftover or involved multiplicative relationships. On the contrary, in context-free settings once they found leftover, they
were not able to figure out what to pay attention to since there was not any context that
would help them identify the referents. They were also having difficulty in identifying the
relationship between the divisor and remainder that helps to determine the fractional part
of the quotient. Such identification requires coordination of quantities with respect to the
referents they had. In any division problem, a divisor is an intensive quantity, whereas a
quotient and a dividend are extensive quantities. In any setting (fractional or whole
number) one needs to know that the divisor is the intensive quantity that relates the two
extensive quantities, dividend and quotient. When one is engaged in a contextual
problem, one can think about the numeric relationship between the divisor, dividend and
quotient. However, one is not supposed to think about the referents and the type of
quantities involved in the given division problem because in contextual setting, the
quantities at hand are, for example, $\frac{1}{3}$-cup of flour and $\frac{7}{2}$-cup of flour. In this sense, one
can just think about the context and think about number of $\frac{1}{3}$-cups that can be made out
of $\frac{7}{2}$-cups. In contrast, in context-free settings, since there is no story to refer to, one
needs to identify the nature of the quantities and what units they refer to. In this regard,
Nancy and Wanda had difficulty in identifying the right referents for the right quantities
in order to consider the relationship between the divisor, quotient, and dividend.
5.1.6. Participants’ Understanding of Referents

a. The Case of Nancy

Nancy’s reaction to **Problem 10** was important to highlight how she thought about referents.

N: Yeah. You have three fourths [writing \(\frac{3}{4}\) and then writing \(\frac{2}{3}\) next to it without putting any sign in between] I think it was subtracting [writing “-” sign in between the two fractions] [pause for 5 seconds]-

R: What is the problem asking and why did you choose subtraction?

N: It asks or it says that Jane has three fourths of a gallon of ice cream.

R: Hmm hmm.

N: So they have three fourths of a whole.

R: Hmm hmm.

N: And she took two thirds of the three fourths away. So what part of a gallon of ice cream does she have left.

R: Hmm hmm.

N: So it’s three fourths of a gallon minus, hmm [pause for 5 seconds]. Okay I need to find out what two thirds of three fourths is [pause]. And then take that answer and subtract it [pause], I don’t know [from] three fourths? [pause for 5 seconds] Okay. Should I try this all of it?

As she talked through the problem, she realized that \(\frac{2}{3}\) referred to \(\frac{3}{4}\) and therefore \(\frac{2}{3}\) of \(\frac{3}{4}\) should be subtracted from the \(\frac{3}{4}\). It is a subtraction problem. When she did a more careful analysis, it revealed that there was also a multiplication step.

Once she realized that she was using the wrong referents she changed her way of thinking and she multiplied \(\frac{3}{4}\) by \(\frac{2}{3}\).

N: Okay. If she has three fourths out of ice cream, and she gave Mike two thirds of the three fourths, I think I would multiply it and then I’d get six twelfths which would equal half [changing the “-“ in between 3/4 and 2/3 to “*” and writes “=6/12=1/2”]-
N: Hmm. Um so then I would think I would take away half from the three
fourths of the gallon [writing 3/4-1/2] which would be one fourth.
Because I know that one half is equal to um two fourths [writing 2/4
right below 1/2 to find common denominator of 3/4 and 1/2].
R: Hmm hmm.
N: So she would have one fourths of a gallon of ice cream left.

After multiplying \( \frac{3}{4} \) by \( \frac{2}{3} \), she found \( \frac{1}{2} \) but she did not label the one half with a
referent unit. When asked about what the \( \frac{1}{2} \) referred to, she oscillated between \( \frac{3}{4} \), and a
whole gallon. She then overcame this dilemma using a representation as follows:

N: … Okay the one half, okay they had, she had three fourths and she gave
two thirds of the three fourths to Mike, which was one half [pause and
draw a circle and partitioned it into four pieces and then draw another
circle next to the first one and partition it into three parts]. If she had
three fourths [shading in the three pieces in the first circle] and she
gave two [pause]. Yes, okay [crossing out the second drawn circle] So
she has three fourths and what she has is three equal parts [pointing to
the shaded parts in the first circle that was partitioned into four pieces
and three of those shaded] she gave two of those parts, it’s half of the
whole. So she gave half of the whole gallon to Mike.
R: So that circle, does it represent the whole gallon?
N: Yes [with a certain tone].
R: And what does the shaded area represent?
N: The, well this shaded, the shaded area represents what she has of the
whole gallon, which is three fourths.

Up until I asked her about what different fractions (e.g., \( \frac{1}{2} \), the result of

multiplying \( \frac{3}{4} \) by \( \frac{2}{3} \)) referred to in her algebraic solution, Nancy was strictly reasoning
from the algebraic symbols and manipulations she had in front of her. However, with the
question of, “\( \frac{1}{2} \) refers to what?” she felt the need to figure them out by a conceptual
representation of the given problem. Using a diagram she did not have any confusion in identifying the referent units for the quantities in the given problem.

In the diagram approach, she had the opportunity to question the, “What refers to what?” idea pictorially as in the following sense. The given fractional quantity was $\frac{3}{4}$ and she was to identify $\frac{2}{3}$ of it. Using her part-whole understanding of fractions, she knew that $\frac{3}{4}$ was three pieces out of four that constituted the whole. Therefore, she drew a whole, partitioned it into four equal parts, and shaded in three of those partitions. As a result, she already had three partitions shaded in and her new referent whole was the combination of these three shaded partitions. Determining $\frac{2}{3}$ of this new referent whole was same as identifying 2 partitions out of 3. This was easy since the new referent whole was already partitioned into three pieces, and she was to mark only two of those. In this sense, there were two levels of mathematical analysis she needed to make: (1) identification of $\frac{3}{4}$ of an arbitrary whole; (2) identification of $\frac{2}{3}$ of $\frac{3}{4}$. Diagram drawing made the transition from (1) to (2) easier since the diagram was making it easy for her to see the new referent whole consisting of three partitions and $\frac{2}{3}$ of that whole. In this case, the name of the fraction ($\frac{2}{3}$) corresponded to the size it determined ($\frac{2}{3}$) because of the new referent whole, which made it easy for her to move from first level to the second.
On the other hand, in the absence of diagrams, the context (story) of the given problem helped her to choose the right operation to apply in her solution process. The wording of the problem asked her about \( \frac{2}{3} \) “of” \( \frac{3}{4} \) and the given-away amount. The wording, “give away,” was about subtraction and the wording, “of,” reminded her of the multiplication operation. Based on the context, she sequentially used these operations and she did not have to analyze the referents since once the operation was identified as multiplication, then she referred to a rule, \( \frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d} \), for fraction multiplication.

As a result, both context and diagrams made it explicit to Nancy what referent unit to focus on, which guided her solution process.

**b. The Case of Wanda**

In dealing with **problem 9** [A plant grew \( \frac{12}{4} \) inches in a month. If \( \frac{1}{3} \) of the growth took place during daylight hours, how many inches did the plant grow during daylight hours? Does this word problem correspond to the expression \( \frac{12}{4} \div \frac{1}{3} \)], Wanda made a comparison between the results of the word problem and the given division expression rather than thinking about the mathematical structures on which both the mathematical expression and the word problem were based. She did not pay attention to the overall goals both the statement and word problem referred to, nor the referent units in both cases. Her comparison was basically based on the results she got from the use of algorithms for multiplication and division as in the following way.
W: Okay, the plant grew 12 divided by four, three inches in the year. If one-third of the growth, so I took one-third times three [R: uh huh], okay, which is one, duh, okay. And um that took place during daylight hours so right there that is wrong because you wanna take it times, times, three times the one third and not which would be the reciprocal of that three because you get nine. So I, I got one and then that was during the daylight hours. So two-thirds took place during the night hours, which would be two-- [interrupted]

As seen in her explanation, Wanda basically solved the given word problem and then found the result of the given mathematical expression, and finally she was compared the results of both.

For problem 10, Wanda became confused with what a quantity refers to and the size of a quantity.

W: Okay, Jane had three quarters of a gallon of ice-cream. …[inaudible]… she gave like two thirds of what she had, so she had this [drawing a circle, partitioning it into four pieces and shading in the three of them], and she wants to give her [looking at the diagram she drew]- Oh she has one third left. Okay, she- This is really like, this [pointing to the unshaded piece] seems like irrelevant right now because she only has this much [pointing to all three shaded pieces]. And assuming this [pointing to all three shaded pieces] is already divided into three equal pieces this much, she is giving him two thirds of it, she got one-third left.

R: One-third of what?
W: What she had. Um-

R: So she had-
W: Um. She had three here [pointing to the shaded area], she had this much [pointing to the shaded area], so, does that really make sense [talking to herself]? Yeah.

R: Okay, so she- So you said she had this much and she gave away two-thirds of it [speaking simultaneously with Wanda]
W: She never, she never had the whole gallon apparently [R: Okay]. She only had like three-fourths, so if you divide what she had into three equal pieces and she gave two-thirds, she had obviously have one-third.
Wanda was able to figure out how much of Jane’s ice-cream was to be given away. She mentioned that it should be \( \frac{1}{3} \) of what Jane had. I questioned her as to how big the given-away piece was. Looking at the diagram she said, “It looks like a quarter of a gallon.”

When I asked her whether she was sure about 1/4-gallon, her inclination was to check it with the calculator. She entered 0.660x75 into the calculator and then she said:

\[ \text{R: Why did you use multiplication?} \]
\[ \text{W: It doesn’t say one-fourth. I used because I am taking like two-thirds of something. So usually when you say “of”, you wanna multiply. But that only gives me so this, that doesn’t tell me how much she had left so. This give me like 50 [referring to the calculator result, 0.5], 50 percent, half, so…} \]

\[ \text{R: Half of what?} \]
\[ \text{W: She would have half of whatever left. That doesn’t- I don’t know this isn’t making sense to me. I can’t even do a simple problem [laughing]} \]

When checking the calculator result against the result she found earlier (\( \frac{1}{4} \)), she got confused. In the diagram, she had the opportunity to see what \( \frac{1}{4} \) of the whole figure referred to since \( \frac{1}{4} \) of the whole rectangle was already marked. She started drawing a whole rectangle and then identified \( \frac{3}{4} \) of it. Diagramming helped her to consider the \( \frac{3}{4} \) of a rectangle she drew as the new referent whole. Based on the shaded three partitions in the diagram, Wanda only focused on what she needed to focus on as a referent whole.

And then, \( \frac{2}{3} \) of the shaded new referent whole was easy since it was already partitioned into thirds and she needed to identify the 2 partitions out of 3. As was the case with
Nancy, she used the advantage of the coincidence between the fractional name $\frac{2}{3}$ and the fractional quantity $\frac{2}{3}$ to determine two-thirds of the referent whole (shaded $\frac{3}{4}$). In this way she determined one partition as the piece that was left. Since that one partition was part of the whole rectangle, referring to the diagram, she was able to determine the size of it as one-fourth. In this way, diagram was helping her to choose the right referent wholes and the focus of attention.

On the other hand, when she calculated $\frac{2}{3}$ of $\frac{3}{4}$ using calculator, she was finding the amount that was given away, not the leftover amount. Therefore, the calculator gave her 0.5 (or $\frac{1}{2}$). The problem she was dealing with had two steps into it. One of the steps was to find the fractional part of a given quantity, and the other step was to find the leftover part using subtraction. The calculator work she did only reflected the first step but not the second step. Once the first phase was taken care of, she needed to think about the referent wholes, and she, herself, had to identify the right referent wholes based on the calculator result. However, she did not move on to the subtraction phase because the calculator was not giving her a visual clue to determine the right referent units. In this case, since she could not see what 0.5 referred to on calculator screen, she got confused. In her diagram work, on the other hand, her focus was more on the referents units than the operations (multiplication and subtraction). Therefore, she was easily able to identify the right referents and base her discussion on them. In this sense she lacked the
coordination between the referents and the operations to be used. Such a lack of connection did not help her to carry over her way of thinking from diagrams to calculator.

5.2. Prospective Elementary Teachers’ Evolving Understandings of Division and Division of Fractions

Pre-interviews revealed that all the participants had at least part-whole understanding of fractions. They knew how to represent fractions with some type of drawing. In addition, the participants had the knowledge of how to partition the given quantities and determine fractions by using partitioning operation. They also had the idea, “How many $a$ could go into $b$?” for the division problems $a ÷ b$, but they did not have an abstraction of quotitive situations.

Given participants’ knowledge, in T1, they were provided with a task sequence that required them to use an activity sequence for identifying fractional quantities via partitioning, and finding how many of one quantity (divisor) could be embedded in another (dividend) through counting and comparison. They already had access to all the actions in their repertoire that made up this activity sequence. The task sequence in T1 consisted of three sections. Section I included four problems about division with or of fractions. Section II consisted of a summary question asking what was invariant among all the problems of Section I. And the first part of Section III asked the participants to judge whether the given two problems in this section fitted the general form they formulated in Section II. The second part of this section asked them to construct their own problems that fitted the general form they generated in Section II. For a detailed description see Appendix B.
5.2.1. What Happened When Solving Problems of Section I in T1

Before solving the problems in Section I, the participants were asked to use rectangular diagrams to make the communication about their drawings easier during the sessions. They were only allowed to use diagrams, nothing else (e.g., computation, algorithm). If at some point they thought that they did not have any approach but an algorithm to take care of some portions of the solutions, they were allowed to use that algorithm only if they could provide the rationale behind it. In this sense, they were limited to the diagram use. When solving the problems in Section I, they were asked to solve each problem alone, and then we did a follow-up discussion about it.

To solve the first problem of Section I [Bob has $\frac{1}{2}$ of a cup of sugar. Each recipe of cookies calls for $\frac{1}{8}$ of a cup of sugar. How many recipes could he make if he uses up every bit of sugar?], Nancy and Wanda both drew one rectangle as a whole and then partitioned it into halves. Next, they partitioned one of the halves into four pieces. Then they counted the number of those little pieces (of the size eights) in half and announced the result as 4. Wanda’s way of counting was, “Two-quarters is a half, and two-eights give a quarter, so there are four-eighths,” whereas Nancy basically counted the number of small pieces of the size $\frac{1}{8}$ within the first half of the drawn rectangle.\(^{18}\)

\(^{18}\) They were still working individually at this point.
During their explanations, they used a language that was consistent with the context. The whole drawn rectangle was considered to be a cup (rather than a whole) and $\frac{1}{2}$ and $\frac{1}{8}$ were considered as $\frac{1}{2}$-cup and $\frac{1}{8}$-cup. In this sense, they were consciously or subconsciously having both the $\frac{1}{2}$ and $\frac{1}{8}$ refer to the same referent unit, “cup.” They benefited from the context to identify the referent units. In later sessions, determination of referent units became a very confusing process for the participants when they were working in context-free settings.

Their solution to the second problem of Section I [Jan was distributing ballots for the class election. If each ballot was to be $\frac{3}{4}$ of a sheet of paper, how many ballots could she make from 3 sheets of paper (Using every piece of the papers)?] was very similar to the first one with regard to the actions that were taken in the solution process. Wanda first drew four rectangles separately whereas Nancy drew them next to each other. Then, Nancy partitioned each rectangle into four pieces and marked off groups of 3 pieces subsequently and then counted the number of three-piece groups. On the other hand, Wanda shaded in three pieces in every rectangular whole (that were already partitioned into fourths) and then counted those three-piece sections. Next, she combined the leftover pieces from each rectangle, which made up another three-piece group. Both participants announced the result as 4.

In the discussion about Problem 1 and Problem 2, Nancy and Wanda basically explained how they got their answers. At times I asked them questions such as “4 what?” and “What do you mean by this?” However, I did not ask them every little detail they
went through since the overall goal was to help them develop the aforesaid overall understanding about division.

**Problem 3** [Louisa is building a cinderblock wall; the blocks are \( \frac{2}{3} \) of a meter high. How many rows of blocks will she need for a wall 4 meters high? Assume that each block sits directly on top of each other.] was troublesome for both Nancy and Wanda. After drawing four rectangles, Nancy said that she did not know how to deal with the problem without computing it. The follow-up discussion was:

R: So, what’s the problem asking you to figure out?
N: I will have four meters and I need to figure out ohh! How many times two-thirds can fit into four meters, right?
R: Okay, so what do you need to do?
N: [drawing four wholes next to each other] I have that four meters, so now I have to figure out, I guess I have to divide each block [referring to each meter] into thirds. And then each block [referring to the whole rectangles] is two-thirds so …[inaudible]… out two of those [marking each two-thirds section and then counting the numbers of markers]. Six.
R: Six blocks?
N: Six blocks will fit into, six blocks that we need to make a wall of four meters.

She was having trouble as to whether she should divide each rectangle (referring to one meter) into thirds or not. However, she figured it out by focusing on the length of each block, which is \( \frac{2}{3} \) of a meter. My question about the overall goal for the problem oriented her to focus on the referent units. Context in combination with the overall goal helped her to solve the dilemma of what to focus on as a referent.

On the other hand, Wanda represented the four-meter long wall by drawing four rectangles sitting on top of each other assuming that they were each one-meter high. She
then partitioned each whole rectangular unit into thirds and shaded in the $\frac{2}{3}$ of the whole four-rectangle unit (which was combination of 8 pieces of the size $\frac{1}{3}$-unit each). Then she counted the number of two-piece (of the size $\frac{2}{3}$ of a meter) sections, which gave her four. Next, she said that she was just counting the number of two-thirds in $\frac{2}{3}$ of the whole four-meters, rather than number of two-thirds within the whole four-meter since she read the problem incorrectly. She was using $\frac{2}{3}$ to identify measurement of each block, the divisor, and also to identify the measurement of wall, the dividend, which confused her.

Once I realized that there was confusion in terms of the referents among the participants, I wrote on the board the following responses to question their dilemma about it: (1) “$\frac{2}{3}$ of 4m;” (2) “two-thirds in 4m.” Then I asked them about the difference between these two responses. Nancy noted that this was the source of confusion for her in the first problem when identifying $\frac{1}{2}$ of the whole and $\frac{1}{8}$ of the same whole. During the solution of the first problem she took into account the context. Even though she referred to the context to solve the confusion in the second problem, she was not able to see what was wrong until my prompt of, “What is the problem asking you to figure out?” This was because she did not coordinate the context with the overall goal of “How many two-thirds are in 4?” Once Nancy realized that she did not take the overall goal into consideration,
she was able to reflect on it and solve the dilemma, which also seemed to be the case for Wanda. On the other hand, when they were not thinking about the overall goal with respect to the context at hand, they seemed to be thinking about two things simultaneously: the divisor refers to the referent unit for the given division problem, and the divisor refers to the dividend itself. For example, for the problem \(4 \div \frac{2}{3}\), the referent unit is one. They thought that they should find \(\frac{2}{3}\) of 1 as well as \(\frac{2}{3}\) of four. This kind of dilemma confused them.

The last problem of Section I [Nowadays, the road workers have been asphalting part of W. Clinton Avenue, which is \(\frac{5}{6}\) miles-long. Each day they are able to do \(\frac{1}{3}\) of a mile of road work. With this pace, how many days would it take for them to asphalt the whole road?] was not hard for Nancy and Wanda at all. They both drew a whole rectangle (to represent one-mile), and then partitioned it into six pieces, and marked off the leftmost five of those pieces. In the problem, since each day the workers were asphalting one third of the “whole” mile, they figured out that one third corresponds to two partitions. Then they counted number of two-partition sections only within the total of five partitions, which was 2. Next, they interpreted the last one partition as one-half of the two-partition section and considered the result as \(2\frac{1}{2}\) days. Since that last partition was \(\frac{1}{6}\) of one whole, I asked them whether it was possible to consider the result as \(2\frac{1}{6}\) since that last partition was worth one-sixth. They did not have any problem identifying
the referents for 2 and $\frac{1}{6}$, and they said that it could not be the result because the problem was asking about the number of days. In this sense, context gave away what to pay attention to when interpreting the quotient. In this latter problem, the quotient referred to the number of days and since the overall goal was to find the number of days under certain conditions, they were able to think about the quotient efficiently.

5.2.2. Activity Sequence of Participants and Their First Abstraction of Division

It seemed from their actions and explanations that both Nancy and Wanda used a similar activity sequence for all the four problems. The first thing they did was to identify the overall goal. Based on the context they identified the overall goal as, “How many ___ are in/can go into ___?” for each problem. Then they more or less pursued the following activity sequence for a problem like “dividend $\div$ divisor = ?”

Table 5-1. Activity sequence followed by the participants for division of fractions

<table>
<thead>
<tr>
<th>Activity</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) Draw the given fractional (or whole number, or mixed number) quantity.</td>
<td>Dividend</td>
</tr>
<tr>
<td>(2) Partition dividend in order to identify second given quantity.</td>
<td>Divisor embedded in repartitioned dividend</td>
</tr>
<tr>
<td>(3) Mark off complete divisor sections within the dividend as much as possible.</td>
<td>Dividend consisting of shaded-in full divisor sections with some unshaded leftover pieces</td>
</tr>
<tr>
<td>(4) Compare the leftover to one divisor group to figure out how much of a divisor group it corresponds to.</td>
<td>Fractional correspondence of divisor</td>
</tr>
<tr>
<td>(5) Consider the whole number of divisor groups and fractional number of divisor groups together to announce total number of divisors within the dividend.</td>
<td>Total number of divisors</td>
</tr>
</tbody>
</table>
In all four problems they went through a very similar activity sequence. When they got stuck at some point during the implementation of the activity sequence, my probing of, “What was the question asking you to figure out?” and “What do you need to do?” helped them pay attention to the overall goal. Also such questioning helped them make conceptual modifications to their activity sequence such as redefining the referent for dividend and divisor as explained above in their solutions to problem 2 and 3 of Section I. The source of the problems at this point was that, at times, they forgot to use the goal as a guide when deciding on referents or other necessary pieces.

Even though they both used the same activity sequence, it seemed from the following episodes that they did not reflect on that sequence during their solution process. This experience was their first attempt to think about division problems within representational (diagram) and non-computational or non-algorithmic world. Also, it seemed from the pre-interviews and from some informal conversations with the participants that they had never been exposed to such a method in their previous school life. Hence, it was reasonable for them to just focus on how to solve each individual problem rather than consider the commonalities among them when solving them. Figuring out the commonality among the four problems was the most troublesome issue for them as illustrated in the following episodes:

R: What is the common thing among all those problems? And can you write a statement that tells us what that common thing is about?
W: I know what it is but I can’t put it in words [pause]. Can we talk about this?
N: You have to take part of a whole or a whole and divide it up. I don’t know.
W: Yeah part of a whole to find something else though. To find like- You need to divide something up to find something else.
R: Okay. So what operation can you use if you were to in these four problems? ... What operation are they all referring to?

W: Addition \textit{[with hesitation]}, or...is that what you mean?

R: Yes, like addition, subtraction, multiplication, which one?

W: Well, I would say. Well depending on how you looked at it but I think I used mostly addition. I could have used multiplication somewhere in some of these ...\textit{[inaudible]}... but you said it is not true or-

R: How did you use addition?

W: \textit{[referring to the “making 3/4 paper-ballots out of 3 papers” problem and mentioning that when combining 3/4 packages, she was using addition]}

Up until my question about commonality, it seemed that they did not pay attention to the structural commonality among the given four problems. Nancy’s first attempt to answer this question was to describe the first activity in the activity sequence she went through. Both participants were not sure what was important to generalize.

When I asked them about the operation that the given four questions referred to, Wanda attributed an operation to each activity or a sub-collection of activities in the activity sequence she used. According to Wanda, there was an addition operation and there was possible space for multiplication in the given problems. Nancy added to that the division operation, since they were partitioning given quantities in the way she stated.

There were two possibilities for why the participants compartmentalized the activity sequence and attributed an operation to each part of the activity sequence. Either the participants misunderstood my question about what operation to attribute to each problem as a whole or they reflected on pieces of the activity sequence and identified an appropriate operation for single activities or a subset of activities in that sequence. I was more inclined to the second interpretation than the first one.

Because they did not treat the whole activity sequence as a single entity and not did determine an appropriate operation for that whole entity, they focused on the
compartmentalization of the whole sequence. I pushed them to think about commonality in the goal and in their overall solution strategies. After a long negotiation process, they responded as follows:

W: It’s asking with this certain amount that you have, how many can you make out of- It’s like how many can you make with, and it’s a larger amount than what you have,

\ldots

N: With this certain amount that you have, how many certain amount can you make off this larger amount. How many of these [referring to divisor in general] can you use to make this larger amount [referring to dividend in general]?”

At this point, Wanda focused on the activities she went through. And by basing her discussion on the activity sequence, she made a generalization of the process she went through as a process that yields to an answer to the question of how many of something could be derived from another larger size thing. This was the first time both participants tried to give a meaning to the overall goal they set for each problem in terms of the activity sequence they went through. This reflection initially started with a search for a commonality among the goals of the problems, and then it evolved into a form by which they started to think about what made their spontaneous activities meaningful in terms of the overall goal. Their responses represented an abstraction from their work with the context problems. This was different from their earlier abstraction of arithmetic relationships as, “How many _____ are in _____?”

At this point I asked them to make the same argument for each problem and they applied this general form to each problem [e.g., how many $\frac{2}{3}$ meters are in 4 meter?].

Then I asked them “So what is the commonality?” and the response I got was:
N: How many times something can go into something else, which is division \(\text{[in a questioning manner]}\).

W: Yeah, that’s, that’s, yeah.

R: Oh, oh, okay.

... R: What makes it a division problem? ...

N: You are taking this thing \(\text{[referring to the second quantity, divisor]}\) and like you are dividing it or separating it into this thing \(\text{[referring to the first quantity, dividend]}\)

... W: There is something within something and that would be part of it so you have to be talking about yeah something is in something. The first quantity was what is in this and saying how many too. I don’t know that’s kind of …

N: But I know what you mean. If something is in something else, it’s like could I use…I mean could I use the word dividing or do I have to try to explain it?

R: Go ahead, you can use everything. I can put the constraints later so just try to explain it. So what makes it division?

N: Because you are putting…parts into a whole, so the whole is kind of being divided up, or split up.

R: Okay, so you are putting part into a whole \(\text{[drawing a whole and partitioning it into halves]}\). Okay, like one-half, one-eighth. What am I doing here? How am I putting parts into a whole? My whole is this \(\text{[marking off the drawn whole and partitioning it into eighths]}\).

N: Then you have 8 parts in the whole.

R: Okay, eight parts in the whole. Each of these is \(\text{[one]}\) eighth.

... R: So I begin with my whole, Now you are saying --

N: Then you split it into half. And then I have to count, how many parts or how many eighths are in the half.

R: So how many parts are in the part?

N: Hmm hmm [approving].

From Nancy’s wording I realized that the abstraction she made of division was the idea that, “You are taking this thing \(\text{[referring to the first quantity, dividend]}\) and like you are dividing it or separating it into this thing \(\text{[referring to the second quantity, divisor]}\).” In other words, Nancy considered division as a partitioning of a given quantity. However, this partitioning was not an arbitrary one. There was an overall goal that guided such partitioning, which was to figure out number of partitions that represented a quantity.
within another quantity. In other words, the reflection they did on the activity sequence made Nancy become aware of the quotitive structure and she associated it with the division operation.

Hidden in this argument was the assumption that the first quantity (dividend) was to be larger than the second quantity (divisor), which was also shared by Wanda. They made the abstraction that the process they went through was about finding “How many of one quantity were embedded in another bigger quantity?” This abstraction made them reconfigure their current understanding of division as an operation meaning a continuous separation of a certain size quantity from another quantity (for Nancy), or as meaning covering up a given quantity with another one (for Wanda).

As known from the pre-interviews and from the initial teaching sessions, their initial model for division was based on numerical reasoning within missing factor division problems. However, this initial model was now redefined and reconfigured as chunking a given quantity into certain size partitions and finding the number of partitions. Hence, this process had two main operations in it: partitioning a given quantity with regard to another quantity, and quantification of that partitioning. Regardless of the view, chunking or covering, the aforesaid two operations seemed to be the basis for their actions. They paid attention to the process of partitioning as seen through their wording. However, it was not clear to what extent they focused on the quantification operation.

As a consequence, they modified their understanding of division of fractions as explained in the above discussion. They were even able to distinguish between given two additional problems [1. Lindsey plants 6 acres of corn. \( \frac{2}{3} \) of the corn is sweet corn. How
much land is planted in sweet corn? 2. The temperature increased in the swimming pool at a rate of \( \frac{2}{3} \) of a degree per hour. How long did it take for it to increase 6 degrees?] as to whether their structure was the same as the previous ones. When reasoning about these two problems Nancy first identified the quantities involved in the problems, then focused on the overall goal of the problems, and then checked it against the division structure. On the other hand, Wanda went through the activity sequence and checked it against the one she abstracted to judge the type of the problems. Both of them, in a way, went through the activity sequence they already had to check whether the given problems fit the general form they created for the previous division of fractions problems.

So far, they had an abstraction of the division of fractions, and they were able to recognize that abstraction within other problem situations by comparing the activity sequence they already had with the one embedded in those new situations. Surprisingly, when asked to create their own word problems involving fraction division, both Nancy and Wanda had a very difficult time. Nancy actually created a multiplication word problem; whereas Wanda created a partitive division word problem using whole numbers. Nancy’s word problem at this point was, “If I ran \( \frac{3}{4} \) of a mile each day for seven days, how many miles did I run altogether?” whereas, Wanda created the problem, “Jess is having 8 guests over for tea. If she wants to serve her guests equal amounts, how much of one pot of tea would each guest get?” Once Wanda formulated this problem, she then said, “I think this is way too easy, I think that’s division.” Even though the
participants had an abstraction in their repertoire, they were not able to use it in service of creating their own problems.

By going through a certain activity sequence and reflecting on it, they earlier made an abstraction of the idea that \( a \div b \) means, “How many \( a \) are in \( b \)?” given that \( a \) and/or \( b \) are fractional quantities. However, this abstraction was not enough to help them generate their own word problems. As explained earlier, when asked to create word problems that were parallel to the abstraction they made, Nancy created a multiplication word problem, whereas, Wanda created a partitive-division word problem (based on whole numbers) both of which were not based at all on the abstraction they already made. On the other hand, they were able to distinguish between the problems that included division-of-fractions structure from the ones that did not (e.g., multiplication) by thinking about or actually going through the activity sequence they already used in the other division-of-fractions problems. Nancy and Wanda had not yet learned to call on their abstraction for the purpose of generating word problems.

5.2.3. Importance of comparison of activity sequences: Comparing division of fractions to multiplication with fractions

Multiplication was one topic the participants regularly confused with division in fractional number setting. In the pre-interviews and in T1, as the participants worked through some of the problems, they oscillated between the multiplication and division. These problems were the type of word problems for which the wording was not giving away the appropriate operation directly unless a detailed analysis of the overall goal and the involved quantities were produced. Therefore, T2 was intended to increase
participants’ understanding of division of fractions by having them conceptually compare the division and multiplication in a fractional number setting. In doing so, they were to compare the activity sequences that they were to go through and make abstractions about the nature of both operations within the fractional setting.

Because of the time conflicts between the participants’ schedules, I taught Nancy and Wanda separately during the second teaching session. The rules for T2 were also the same as the previous session: (1) computation was not allowed; (2) algorithm was not allowed; and (3) problems were to be pursued with the use of diagrams only. T2 consisted of three sections.

The approach I followed in designing multiplication with fractions tasks for this session was very similar to that of division of fractions. The participants were to go through a number of multiplication-with-fractions type contextual problems (of the type “(whole number) x (proper fraction)” and then look for the commonality among those. A detailed description of the task sequence can be examined in Appendix C.

The task sequence consisted of three sections. The first section was designed to facilitate the participants’ understanding of multiplication by proper fractions. It consisted of four parts each of which targeted a different level of understanding for multiplication with fractions. Section II, on the other hand, consisted of three sections and targeted understanding division of fractions at a higher level than in T1. Section III included division of fractions problems that had larger divisors than dividends, which will not be the focus of investigation in this section.

As mentioned earlier, the purpose of this session was to help participants strengthen their understanding of division of fractions. Therefore, in the following
sections, their work on multiplication with fractions will be briefly summarized. The main focus in the following sections will be on the comparison the participants made between division of fractions and multiplication with fractions, and the abstraction they derived out of that comparison.

### 5.2.4. Summary of Participants’ Work on Multiplication with Fractions

In going through the designed activity sequence for multiplication with fractions [hereafter, labeled as MwF], both participants used a particular activity sequence. The activity sequence they went through for each given problem consisted of the actions described in Table 5-2:

<table>
<thead>
<tr>
<th>Activity</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) Identifying the initial given quantity drawing repeated same-size-rectangular regions [e.g., drawing 4 rectangles to represent 4].</td>
<td>Multiplicand</td>
</tr>
<tr>
<td>(2) Partitioning the units that makes up the initial quantity according to the given fractional multiplier [e.g., partitioning each rectangle unit into thirds].</td>
<td>Multiplicand with partitions</td>
</tr>
<tr>
<td>(3) Identifying the parts of constituent multiplier pieces [e.g., shading in 2/3 of each rectangle].</td>
<td>Shaded in multiplier section in each constituent unit</td>
</tr>
<tr>
<td>(4) Combining the shaded sections from the constituent units (mentally or physically) and counting the number of identified constituent multiplier pieces [e.g., counting number of thirds].</td>
<td>Combination of colored multiplier sections</td>
</tr>
<tr>
<td>(5) Interpreting that combination of multiplier sections with respect to their size.</td>
<td>Value of the part of the multiplicand the problem was asking for</td>
</tr>
</tbody>
</table>
For each problem in Part I of Section I, the participants went through the above listed activities. After going through the above activity sequence, the problem they had to deal with was to formulate a generalization as to how the problems they went through were similar. This question pushed them to think about the actions they took and reflecting on those actions. Out of this reflection, they both abstracted the idea that the process they went through was about identification of a part of a given quantity. For instance, $\frac{2}{5} \cdot 3$ meant to them identifying part of the given quantity $3$.

5.2.5. Participants’ Work with Division of Fractions

Once the participants developed a sense for multiplication with fractions, they moved on to Section II. This section involved three parts. Part I consisted of three questions about division of/with fractions. The result of each problem in this part was either a whole number result or a mixed number result consisting of a whole number part and $\frac{1}{2}$ as the fractional part. In this sense, the participants were not to deal with the multiplicative comparison between leftover and divisor to determine the total number of divisors within the dividend. Since they already had a handle on division of fractions [hereafter labeled as DoF] problems involving $\frac{1}{2}$, they were not to be distracted or confused in determining aforesaid multiplicative relationship. This way of organizing the questions was important because their only focus was on the overall goal and they were not to be sidetracked by some intermediate reflections to be made.
a. The Case of Nancy

Nancy followed the same activity sequence as in Table 5-1 when going through the DoF problems. For instance, in solving the second problem, she drew three subsequent rectangles. She then partitioned each rectangle into fifths and added another fifth of a rectangle to her diagram. Next, Nancy marked of each \( \frac{2}{5} \)-unit section and counted the total number of \( \frac{2}{5} \) sections, which was 8. Finally, she wrote “8 items” as an answer. Her solution to other problems was very similar based on the same activity sequence as shown in Figure 5-3.

Figure 5.3. All three solutions to the problems in Part II of Section II in Nancy’s worksheet.

This time I did not interrupt her solution process until she completed all the problems. Once she did all the problems and explained them in detail I asked her about the similarity among the problems. She explained it as follows:

N: I divided the whole but then combined the parts to find wholes within like it is hard to explain. Like I would divide it up but then I would group together parts to count up a different type of whole, I guess. I don’t know if that’s right.
R: Uhh, you divide up a whole, count up the parts to find different kind of whole. What are those wholes you are talking about?

N: Like, whatever the question has asked. Like I have the liters of water but I need to find how many bottles I can use up with the water, with the liters of water. So I found the number of bottles.

R: Is it similar in the other problems?

N: Hmm hmm. I think.

R: How?

N: [reading the second problem] For the second one I have ribbon. But I need to find how many items I can pack. So I divide up the ribbon and then group it together to see how many items.

R: Third one? Is it the same?

N: I have the flour. I need to know how much, how many loaves of bread. So I divide up the flour and then group it together to count up however many loaves I can make off the flour.

R: So what is the overall goal in each of these problems? What are you trying to accomplish in each of those problems?

N: To find … how much of one thing you can make from something else.

... Uh huh, what does that mean?

N: Well you have your something else, which I kind of thought of as your original whole. But you need to find groups within that whole to make something else [pause] like that that you can use for something else.

Nancy referred to three different quantities in her explanations. One is the given dividend which she called as “original whole”, and at times as “something else.” The second quantity was the divisor which she called as “groups,” and the last quantity was the result which she called, “something else.” In a sense, she was building up a meaning for division using the quantities on which her activity sequence acted. In this way she described the division with the help of its constituent quantities in light of the overall goal for it. Her current understanding was that division consisted of three quantities one of which was a result of investigating the number of one quantity within another. This was not apparent in her previous abstraction of DoF.
Once she was through with the first two sections and she formulated generalizations for both MwF and DoF, the natural extension was to ask, “How both generalizations compare to each other?” as in the following episode:

R: Okay. So now, so you described an overall goal for this section [referring to her work with MwF] and you also described an overall goal for this section [referring to her work with DoF]. In this section [referring to MwF] what was the overall goal?
N: You’re trying to find the part of a whole.
R: Part of a whole. What is the overall goal in this section [referring to DoF]?
N: You are finding --
R: How are they different or similar is what I am asking.
N: You are finding parts in each one [referring to the multiplication problems], but in these it is still the same thing. Like you are finding a part of a whole and it’s still like that whole. Like, it’s still the same thing. I don’t know how to explain it.
R: Okay [laughing].
N: Like you have six gallons of gas and you have to find 3/5 of the gas or gallons. So your answer is gonna be some number but it refers to the gallons [pause] of, of gas. But like for these your whole is the two liters of water, and you have to divide up that whole but then you are then finding, but your answer isn’t liters of water. Your answer is how many bottles the liters that you found can fill.
R: Okay, so you are trying to compare them according to the answers you are getting in each cases.
N: Hmm hmm.

From her explanations, it seemed that Nancy compared her MwF and DoF work according to the referents both resulted in. She made an observation from her activity sequence (even though this observation was not accurate) for the MwF that multiplication was a referent preserving operation whereas division was a referent transforming operation. She came to this realization by focusing and reflecting on the activity sequence she went through in each section for multiplication and division. For the MwF section, she realized that the activity sequence led her to abstract the idea of identifying part of a given quantity. Since the result was part of a given quantity, to Nancy, it referred to the
same thing as the given quantity. On the other hand, when doing the DoF problems, she realized that she was searching for a number of one type quantity within another, which had a different referent than any of the quantities involved in the problems. This kind of comparison among the activity sequences for both division and multiplication of fractions led her make a distinction between both operations with regard to their type, whether they were referent transforming or referent preserving operations. This investigation continued as follows:

R: … What makes them same or different?
N: You are finding parts of a whole. Like you have to divide up the whole, but, I don’t know how to describe, um, I don’t know. You’re, you are finding part of a whole and that’s it [referring to multiplication section]. But here [referring to division section] you are finding a part of a whole and like using it to find something else.
R: Okay, here you are finding part of a whole and you said, “that’s it.”
N: And that’s it [simultaneously speaking with R]
R: And here you are finding part of a whole, uhh --
N: And that, and that answer you are trying, to find something else.
R: … How are you coming up with that answer in that case [referring to division]?
N: By looking to see what I found [both laughing]. Um- Ah! [with a surprising tone], are these more division problems? Maybe no.
R: Okay, if you were to write expressions again for each of these problems, what would that be?
N: I think it would be two divided by two-thirds because you wanna see how many times two-thirds can fit into two wholes.
R: In the second one, how about the second one?
N: You wanna know how many two-fifths are in three-and-one-fifth.
R: Okay, is it the same thing with the third?
N: Yeah because you wanna find how many one-thirds go into three and the sixth.

Her reasoning in the above dialog suggests that she had a meaningful activity sequence attached to the idea (how many of one quantity there are in another quantity?) she had for DoF. So far, she had a certain activity sequence that supported the idea of, “How many ___ are in ___?” which altogether defined a division operation to her.
These abstractions were derived by Nancy using a constant comparison between
the activity sequences she used for MwF and DoF. This kind of comparison on the basis
of activity sequences gave her the opportunity to reflect on and abstract the crucial points
that made each operation unique. How was this abstraction developed? Her comparison
started by her focusing on the actions she took for the two types of problems. In MwF
case she knew that she was to take a quantity and identify how much some part of it was
worth. Hence, the base for the MwF was partitioning (and determining the size of it). On
the other hand in the DoF case, she realized that she not only partitioned but also she
counted the number of partitions. In this sense, the operations to be used in DoF were
partitioning of a quantity based on a different quantity and quantification of that
partitioning. By going through such a comparison on the basis of operations both cases
included, she made an abstraction of the difference between the MwF and DoF.

In addition, Nancy’s comparison of activity sequences was guided by her
understanding of the goals for both MwF and DoF. She set the goal for MwF as
identifying part of a whole which was attached to the operation of partitioning. On the
other hand, the goal for the DoF problems, she abstracted previously, was to identify
“number of parts in a whole.” However, through comparison she realized that there was
something different in the nature of the identification of number of parts in a whole. The
activities she went through included counting the number of partitions. This idea led her
to think about partitioning and counting (quantifying that partitioning) as the main
operations for DoF. Under such guidance of the goals for each case, she was able to
abstract the DoF as based on the operations of partitioning and quantifying.
b. The Case of Wanda

Wanda did not have any problem when solving the first two problems of Part I of Section II. She fluently solved the first two problems, and it took her about one-and-a-half minutes to solve them. She followed the same activity sequence she used previously for each problem. For instance, when solving Problem 1, she first drew two upright positioned rectangles to represent each liter of water. Then she partitioned each rectangle into three parts (assumed to be equal) and shaded in two of the one-third pieces from each rectangle to represent a bottle of water. Next, she marked off the leftover thirds from each rectangle to make another bottle of water. Then she identified the number of bottles as, “Here is one bottle, here is two bottles, and here is three bottles.” Finally, she wrote the result, “3 bottles” as seen in Figure 5-4. [Note that the colored numbers in Figure 5-4 represent the order in which Wanda made the drawing]

Figure 5.4. Wanda’s solution to problem 1 \( \left[ \frac{2}{3} \right] \) of part I in Section II.

After going through each problem by running the same activity sequence, Wanda moved on to Part II of Section II which asked about the commonality among all the three problems of Part I of Section I. She thought about the commonality as:

R: What is the overall goal in all of these problems?
W: Division [with an uncertain voice]. Oh wait!
R: Division is the name for that.
W: Yeah, wait. Um, how many, how many of the, how many two-fifths are in three one-fifths, three and one-fifths.

R: Is it the same in the others?

W: Yeah. How many thirds are in three and one sixths cakes, … [inaudible] … cakes [laughing]. And then how many two thirds are in two liters.

R: Okay [pause]. So how many you said this --
W: Something is, are in --
R: Are in --
W: Some other amount.
R: Some other amount, okay.

She compared the goals of each problem, and as a result, identified the similarities among those problems with regard to their goals, “How many something are in something else?”

This helped her call on the division operation since she already had this idea in her repertoire. She was also able to identify the corresponding mathematical expressions for each problem.

When asked to compare the two sets of problems (DoF and MwF) she went through, Wanda had a hard time explaining the difference.

R: What is the difference or similarity other than wordings?
W: Well, they are asking me different things.
R: How?
W: Well, they want, they are asking for like an amount of another amount in the first one [referring to the goal for multiplication with fractions]. Like in goal-two [referring to the goal for division of fractions] they want, how like how much of a certain amount… I can’t say it without saying the goals. I can’t, [murmuring], like how many times this can go into this, in number two. And then --
R: What’s that mean?
W: [pause for 17 seconds and looking at the previous multiplication problem that has the answer 5 \( \frac{1}{4} \)]. Okay, how many- See I can’t say it without that [referring to the statement for goal 1].
R: As long as you explain it you can use that. I am not saying you can’t use it [referring to the stated goals].

…
W: Three fourths of seven [pause]. Yeah! Three fourths of seven. We don’t know how much this is of this. How much this amount is of this
Wanda already made abstractions for MwF and DoF separately. However, when asked to compare one to the other she oscillated between the statement of each goal and nothing beyond those. Hence, I changed the direction of discussion to the activities she went through for both situations as follows:

R: Uhhh, how about the process you go through [long pause] what makes those things [referring to multiplication and division] different from each other?

W: Um- I am not like- I don’t know it seems like I am using all what I have in the, in the division ones, and then I am only using certain parts in the multiplication ones to get my answer.

R: Okay, so in the division one you are using all of what you have.

W: Yeah.

R: For what purpose?

W: Um, to see like [pause] how much of something is in something.

R: Okay.

W: And then like this I am not, I am not using [pause] all of what I have.

R: For the purpose of?

W: Because we just wanna see how much something is of something.

R: What is the first something, what is the second something [referring to both cases], for example?

W: Like how many thirds are in this whole amount here using with the whole, but like, here we’re only using like parts of the whole. Like a certain amount of what it is.

R: When you say whole here what do you mean by whole? What are you referring to by that whole?

W: Well, in either case like you are only using certain part of each of these wholes [referring to each rectangle used to make up multiplicand in multiplication with fraction problems] which all of represent a huge whole [referring to the combination of parts taken from each constituent unit of multiplicand] and like you’re just using like parts. You are not using all of the single wholes that make up the big whole [referring to dividend in division of fraction problems]. And here [referring to division of fractions problems] you are using all of the single wholes that make up the, the big whole [referring to dividend].

W: We are identifying like the parts of each of the whole like that’d make up, that, what we want of the big whole.
The comparison between the activity sequence of both multiplication (see Table 5-2) and division (see Table 5-1) cases in the fractional domain led Wanda toward one of the crucial distinctions between multiplication and division in the context of fractions. In MwF the given quantity (multiplicand) was not to be used up, whereas in the DoF the given quantity (dividend) needed to be fully used up, which was also what Nancy abstracted out of this session. In addition, Wanda also labeled the quantities involved in both types of problems as big whole [multiplicand or dividend] and parts [multiplier or divisor sections] by keeping the overall goals for both cases in mind.

Comparison between the activity sequences helped both participants made an abstraction because reflecting on both sequences at the same time helped them monitor what was common and uncommon between the sets. Thinking about the commonalities and differences among the activity sequences let them look into the essence of those activity sequences and what made them unique within themselves. This type of attention to the overall goals helped them to see the rationale for those goals also. They realized that they partitioned given quantities in MwF problems for the purpose of identifying part of those quantities. On the other hand, for the DoF problems, they abstracted the essence of identifying number of partitions. Since MwF did not include the operation of counting (in general quantification) this aspect of division became more apparent and dominant in the comparison they made.

Based on such comparison depending on the nature of operations involved in both cases and the nature of the goals involved in those cases (MwF vs. DoF), they made additional abstractions about properties of DoF. As mentioned above, one abstraction
they made was about the use of the given quantity (multiplicand in the multiplication problems and dividend in the division problems).

We moved on to the problems in Part I of Section III. The main problem was to mentally identify the steps that one would take if one were to use diagrams only, without actually solving the problem \( 5 \div \frac{3}{4} \). Wanda’s response was:

R: What does it ask us?
W: How many three-fourths are in five?
R: Okay.

...  
W: I think I would draw five things. Then I would divide them each into fourths.
R: Okay.
W: And then I would take three-fourths of each of those things that are divided into fourths. And then um [pause]. No, no. Um, like I would [pause] yeah, but I would, like, since each of those pieces is a fourth
R: Hm hmm.
W: I would take three fourths and then I would keep taking the three-fourths out of the five to like use all of it. ...

... 
W: Okay. And that would- And then that would and yeah I would count how many of the three-fourths are in this five and that would be my answer.

The above episode seems indicates that Wanda was able to lay out the activity sequence for solving a division with fraction problem without any difficulty. She was able to go through the activity sequence she had for the division of fractions without going through the sequence itself. At this point, Wanda was able to think about a division with fractions problem based on her focus on the partitioning and counting.
5.2.6. Meaning of Division of Fractions to the Participants as seen through their
work in Post-interviews

The first problem in the post-interview was about generating a word problem for
\[ \frac{5}{3} \div \frac{3}{4} \]
and then solving it using diagrams. Both participants were able to set the goal for
the given division of fractions problem very clearly, as finding the number of divisor
groups within the dividend. In generating the problem, they first identified the quantities
of dividend and divisor and then set up a context on those quantities by constantly
referring to the overall goal they set for the problem. In solving the problem based on
diagrams, to reach their set goal, they used their usual activity sequence: (1) identifying
the dividend with a vertical partitioning, (2) identifying the divisor with a horizontal
partitioning, (3) repartitioning the divisor and dividend to base them on the same
fractional units, (4) counting the whole number of divisor groups within the dividend, (5)
quantitatively comparing the leftover pieces and divisor group. For instance, Nancy
explained her creation of the word problem as follows:

N: All right. Well I know that my big thing is five thirds. So in my word
problem I have to have, like, something that is five thirds and then I
have to divide that into as many three fourths as I can- I think.
R: Hmm hmm.
N: So, if Jan has five thirds pounds of sugar and each batch of cookies
uses three fourths pound of sugar, how many batches of cookies can
Jan make [writing the word problem as she speaks].
R: Okay. So what make[s] it the same problem as this?
N: Because you have like your big thing, I think that’s what we called it.
Our big thing which is five thirds and like this problem is asking so
divide the five thirds into as many three fourths.
R: What’s that mean? Divide five thirds into as many three fourths?
N: You want it, you wanna group, make as many groups of three fourths
as you can within the amount of five thirds.
R: Hmm hmm. Okay.
N: So if my big thing is the amount of sugar I have and I have five thirds pounds of sugar, each batch of cookies that I am gonna make will use three fourths of a pound of sugar. So I wanna group together as many three fourths as I can within the five thirds to find out how many three fourths I have so I know how many batches of cookies I can make.

The above dialog shows that Nancy made an analysis of the given mathematical expression. She first identified the given quantities with regard to the overall goal for the expression. The overall goal was to determine number of three-fourths within five. Once the overall goal was set and the quantities to be involved were articulated, then she seemed to be in search of a context where this goal could be implemented over the quantities at hand. She was also aware that the problem was to be solved with partitioning, grouping, and counting. She chose making-up-cookie context and she identified the quantity to be partitioned as “sugar” and the quantity to be counted up as “batches of cookies” under the condition that each batch of cookie requires for some amount of sugar.

In the above thinking process, Nancy first identified the mathematical structure to be transferred to context. Then she thought about a context that would correspond to the mathematical structure she had. Previously in the earlier teaching session or even in the pre-interview, Nancy did not think about the mathematical structure first and she directly transferred the quantities involved in the given mathematical expression to some context without thinking about the operations (partitioning, grouping, counting) to which one needs to resort. However, lacking such an analysis of the given mathematics and starting with the context caused her to choose a context that was not guided by the overall goal of the problem and that did not involve any thinking as to what operations to base the
problem on. In her current state, she had the abstraction of quotitive situations and what
operations those situations were based on.

Wanda also created her word problem using a bread making context, “We have
five thirds cups of flour [writing what she said] … it takes three fourths of a cup to make
one loaf of bread [writing what she said]. How many loaves of bread can we make
[writing what she said]?” Right after she created this problem, she wanted to check it
mentally to figure out whether she built a structurally correct problem. Her reasoning at
this point was:

W: Um okay, so we have our five thirds cups of flour here.
R: Okay.
W: Can I draw it?
R: Okay, if you like.
W: Okay so we have our five thirds here [drawing two rectangles,
partitioning them into three pieces vertically and marking five of those
pieces]. And it makes three fourths loaf to make [partitioning each
rectangle into four pieces in the horizontal direction] or of one cup to
make a loaf of bread. [stop working on the diagram] Yeah, this works.
R: What is it that’s working?
W: Um [laughing], okay so, I made my- To make sure that it works I made
my five thirds and then I divided it into the same wholes here and of
fourths, and I did the rest kind of in my head. Like, I didn’t really need
to finish it out to know whether it would work or not. Basically I would
color in like the three fourths, and see how many three fourths um I can
get out of this [pointing to the drawn five-thirds] so that --

As seen in the above dialog Wanda initially made an attempt to draw a diagram.

Once she drew the dividend, and repartitioned it according to the divisor, she then
thought about the divisor and referred back to her set goal (how many three-fourths are in
five-thirds). At this point she did not need to go through the problem because she already
had the activity sequence to be followed in her mind. She did not feel a need to follow
this activity sequence physically because she was able to anticipate what would happen in that sequence.

Again, we see an analysis of the mathematical structure of DoF and the overall goal for the given mathematical expression here. Once Wanda created the word problem, the fact that she did not need to go through the process to figure out whether it would give her the right answer suggests that she could anticipate the activities to be taken and what the result of those activities would be. In her initial attempts to create such word problems (in T1 and in pre-interview), this kind of anticipation was lacking. In her current state she had the abstraction of the quotitive situations and operations to be involved to analyze those situations.

5.2.7. Why was it that their abstraction was deficient in creating word problems previously but not now?

Previously, both participants were not able to create their own contextual DoF problems; by the end of T2 they could do so without any difficulty. In the previous session, they were being oriented toward an abstraction of the overall goal for DoF. Therefore, they were engaged in some DoF problems in which they were to go through a certain activity sequence. Once they solved those problems based on this activity sequence (as in Table 5-1), when they were searching for the commonality, they actually compared the activities they relied on in each problem. Those activities were of the similar type and served a certain goal for DoF. As a result, they made a generalization for DoF that was not sufficient to create their own contextual problems. This seemed to be because they did not have an abstraction of division of fractions as coordination of two
operations (partitioning and counting) guided by the overall goal. However, during the second session, they did the same reflection in two different settings: MwF and DoF. They first made an internal comparison of the activities involved in MwF and abstracted the idea that it was about identifying part of a given quantity. Then, they reran their activity sequence for DoF and reached the abstraction they reached previously. The crucial moment in all this was when they were to compare each activity sequence against each other. Up until this point, they made an internal comparison for a certain operation (either for MwF or for DoF alone). “Internal” means the comparison of activities for different problem that lead to a unique abstraction for those problems. However, now, they were to make a cross comparison between the activity sequences of MwF and DoF. This type of cross comparison gave the participants an opportunity to check each activity sequence (and the abstractions to which they led) based on the operations and the goals. They looked for commonalities or differences among those different operations and goals. In other words, they were pushed to think about both sequences structurally regarding the mathematics they entailed. This kind of comparison helped them coordinate both operations on which the DoF was based. With such a coordination of the two operations, with regard to the overall goal, they now had an abstraction of mathematical structure of the DoF that was based on two operations and an attached goal that guided those operations.
5.3. Participants’ Development of Remainder Concept

The first two teaching sessions were heavily focused on the ideas regarding division of fractions. Some part of the third, fourth, and fifth teaching sessions were about distinguishing partitive and quotitive division problems within the context of fractions. Teaching sessions six, seven, and eight heavily focused on the remainder ideas which served as a preparation for the participants to learn a new algorithm. In this section, I articulate the developmental process that took place during the sessions concerning “remainder.” As a result of this articulation I investigated the necessary understandings to be used in developing such a concept. The purpose of this section is to investigate the nature of the barrier understandings the participants encountered for remainder concept and how those understandings were achieved and developed into a coherent notion of remainder.

What follows is the analysis of participants’ understanding of remainder concept.

5.3.1. Nancy’s and Wanda’s Evolving Understanding of Remainder in Contextual Setting

As mentioned earlier, participants’ understanding and development of remainder concept was heavily investigated in the sixth, seventh and eighth teaching sessions. Pre-interviews suggested that the participants did not have a strong conceptual understanding of remainder concept even in the whole number setting. In general, they had a view of remainder as, “Remainder means what is leftover”; however, they seemed to not have a way to reason about division problems involving remainder when they were not working with some specific numeric values or context. There were mainly two types of settings in
which the participants worked to make an abstraction of the relationship between the fractional part of the quotient and remainder, and divisor and remainder. The first setting they were engaged in was a contextual-diagram setting. It was contextual since they were going through a set of division problems that were embedded in some context. It was diagrammatic because they were to solve the given problems using diagrams. In this sense, it was contextual-diagram setting since they were working on contextual problems using diagrams as a way to reach a solution. The second setting was a context-free diagram setting. It was context-free because the set of problems were basically mathematical expressions and did not have any story behind them. This second type of problems was also diagrammatic because the participants were completely restricted to diagram use and no other source. In this sense, this second-type of problems were context-free diagram problems.

To help the participants develop an understanding of the remainder and the multiplicative relationship between the divisor, remainder, and the quotient, I engaged them in two different domains: initially whole number domain, and later on fractional domain. They worked in whole number domain based on contextual-diagram setting which was followed by their work in context-free diagram setting. They then worked in fractional domain based on contextual-diagram setting and context-free-diagram setting, respectively. The following paragraphs explain the rationale for this sequence and trace their development in this sequence.

I started T6 by focusing their attention on the whole number division problems with remainder. In doing so, the intention was to bridge the participants’ understanding of remainder in the whole number setting to that of fractional setting. However, the time
limitation of the session T6 only allowed for the treatment of remainder concept within the whole number setting.

The tasks in T6 consisted of two parts. **Part II** of T6 involved a set of whole number division problems with remainder. For this part, the participants were asked to solve the three whole number division problems (first individually) using diagrams only. All three questions in this part of the T6 asked for the possible solution options for the given division problems. One of the options was to find the quotient only (a number like 5 or 5 1/2) whereas the other option was to find a result expressed as a whole number quotient and a remainder. Hereafter, I call the first solution method as quotient-only method and the second method as quotient-with-remainder-method.

A detailed version of what solution strategy the participants pursued in Part II can be examined in **Appendix E**. Right after they solved the first problem, a discussion about the possible solutions for the problems took place, and we all agreed on the above two types of solutions: quotient-only solution and quotient-with-remainder-solution.

**Understanding #1:** There are two possible results for a given division of whole numbers problem: quotient-only result and remainder result. Both results serve the overall goal for the division problems in different ways. **Quotient-only result gives the information of total number of divisors whereas remainder result gives the full number of divisors with some remainder amount.**

Since they went through a similar activity sequence for all problems, I choose one sample problem to explain the way they approach the given division of whole numbers problems. Problem 2 was, “A 10-acre farm will be ploughed and sowed with wheat. If every day, 3 acres of farm [land] is ploughed and sown, what are the farmer’s options as
to how many days he can spend on this work? Solve the problem using diagrams.” In other words, the problem asked them to figure out $10 \div 3$ using two different but related interpretations. They both drew ten rectangles (Nancy drew one row of 10 adjacent rectangles whereas Wanda drew two rows of 5 rectangles next to each other). They then started numbering each three rectangle group as 1, 2, and 3 (hereafter this method is called numbering method), and then the last remainder part as 4 (Nancy did not label the last piece as 4). At this point Nancy wrote “3 days and 1 acre left” whereas Wanda’s worksheet showed “$3 \frac{1}{3}$ days.” Once they generated their solutions, Wanda stated the rationale for such responses as follows:

R: What did you get?
N: Three days and one acre.
...
W: And the other side of that would be three and third days.
R: Three and third days [writing both responses on the board]. Okay. How did you find three and third days?
W: Um, it said three acres can be done in one day so I marked off three acres for day 1, and then I marked off three day, three acres for day 2, and then three for day 3. I saw there was one left, one acre left out of, out of the three that would fill a whole day. So that was one out of three, so one third.

After the first problem, they used the numbering method to figure out divisor sections within the given dividend. They also followed the same method in problem 2. As seen in the above episode and as a result of their work for all three problems, it seemed that they used a similar activity sequence to solve the problems. However, when they came to a point where they had to make a decision about the final answer they considered one of the answers but not both. In the follow-up discussions after individual work,
though, they easily talked about how both options could be derived. In this sense, their attention was taken to the two possibilities one can have for a division problem.

In solving the problems, the activity sequence they used was:

(1) Identifying dividend,

(2) Identifying divisor by grouping units in dividend,

(3) Either

   (a) stating the leftover part directly with the number of full divisor sections

   or

   (b) making a multiplicative comparison between the remainder and one divisor section and stating the quotient as a whole.

Based on this activity sequence, they now knew that there were two possibilities to think about division problems in a whole number setting. One option was to give the information about the number of divisors directly whereas the other option was to give the full divisor groups with some part of the dividend leftover. In their solutions; they seemed to only focus on one of the answers and they were not paying attention to the relationship between the two answers, yet.

**Understanding #2:** *Remainder and the fractional part of the quotient refer to the same quantity expressed as a different magnitude and different units. One can be converted into the other.*

To orient them toward conceptualizing the commonality among all three problems and have them reflect on the relationship between both interpretations, I wrote their responses on the board as in Figure 5-5, and then I asked them whether both answers were same or not. They considered both options as the answers to the given problems
since the problems asked about possible options but they also considered the left column, consisting of the quotient (that is a combination of a whole number and a fractional part), as the actual answer to the given questions, since it was directly giving the number of one party within another.

Figure 5.5. The two results of each problem as written on the board. Each row refers to the results of a single problem sequentially.

Once, they expressed their opinions about the quotient for the given problems, the follow-up question was about the relation between two answers. The response at this point was:

N: Well, like for the first one, you can either fill up four and a half bags, or you fill four bags and have a kilogram of flour left over but that one-kilogram can fill up half of a bag. So you still have like the same amount on each side.

R: Do you understand what Nancy said?
W: I understand what she is saying.

R: How does it apply to the second one?
W: Because you have 3 days … that you can fill up total like fully with ploughing acres of land. But you’ll have one acre left over which you can either just like leave it or it’ll take this amount of day to do that.

R: Okay, how about the next one?
W: You have four hats that you can create like full hats of, and then you have two thirds of a ball of string left, which you can either leave like that or you can start your next hat …

R: So these are the same? [circling remainder pieces and the fractional part of the quotients in the answers]

N: You can make it so that they like equate to each other.
W: Well, point five bags is, it refers to like how much flour you can, you can fill the bag with.
N: Hmm hmm.
R: Alright.
W: But I mean, I guess the units are bags though but you can still fill that. I mean that’s how much of a bag is full with that. But it’s fill up with that extra one half kilogram.
R: Hmm hmm. So half a bag … takes up one kilogram of flour.
W-N: Yeah.

Using the context, they began to pay attention to the mathematical relationship between the remainder and the fractional part of the quotient. In addition, to Wanda and Nancy, those two pieces of information now referred to the different units but same quantity. For example, for them “0.5 of a bag refers to how much flour will fill up the bag.” By reflecting on similar problems where they analyzed one quantity (remainder) with regard to another (fractional part of the quotient), their focus was oriented toward the relationship between those two quantities. With my question about the nature of the relationship between those two quantities, they were encouraged to think about and reflect on the multiplicative relationship between the two quantities, divisor and remainder, which resulted in the fractional part of the quotient. Out of this reflection in context, they realized that the fractional part of the quotient was a result of the multiplicative comparison between the remainder and the divisor. Now, to Nancy and Wanda, fractional part of the quotient meant a partial divisor group that corresponded to the remainder. In other words, their current state of knowledge was that the fractional part of the quotient determines what part of the divisor would correspond to the remainder. Because they were working in a context, it was unclear whether they had abstracted the multiplicative relationship between the divisor and remainder.
This realization arose as a result of participants’ work through a set of whole number division problems where they had to identify two possible answers for those problems. Once they generated the two types of results (quotient only and quotient-with-remainder only), they were pushed to look for commonality among their activities in this setting. I constantly asked them to think about the relationship between those two answers. As a result of searching for commonality among those answers for different problems, they realized that both answers were mathematically related. Going through such whole number problems in context allowed them to work toward the abstraction of this relationship between the fractional part of the quotient and remainder in context. Note that the participants did not work on one single problem. They went through a sequence of same structure problems so that they could search for what was common among their activities.

After this type of reflection, they were pushed to think about this abstraction of the relationship in context-free-diagram setting as explained in the following sections.

**Understanding #3:** *Remainder refers to part of the dividend whereas fractional part of the quotient refers to the part of an additional divisor-sized group that can be made. Fractional part of the quotient is different from remainder.*

Even though both participants were becoming comfortable in thinking about the multiplicative relationship between the two types of answers, identification of the relationship between the fractional part of the quotient and the remainder was not that much clearer to them in context-free settings. In the same teaching session, as a follow-up, when they were asked to figure out the remainder for the division problem 1354÷38 using a calculator, they were stuck. However, they solved this puzzle by paying close
attention to “referents” and diagram use in service of identifying the remainder as
detailed below.

When they were asked to find the remainder using calculator, as a result of
entering 1354÷28 into the calculator, they found 35.6315. After finding this result,
Wanda’s first reaction was that the fractional part (0.6315) was the remainder. It was
interesting that previously, Wanda was able to generate two different interpretations for
the given division of whole numbers problems and she was also able to generate her own
problem of the same type [Joe has 6 and a quarter bags of chips for parties. If at each
party three quarters bags of chips are eaten, how many parties can Joe have?] without
any trouble; yet, she thought that the fractional part of the quotient was actually the
remainder. This suggests that she was only able to think about the distinction between the
fractional part of the quotient and the remainder part in the presence of an activity
sequence derived out of her work in context. It seemed that Wanda was not taking into
consideration the goal of the division problem, and thus, was unable to interpret the
fractional part of the quotient. This also suggests that she did not abstract the relationship
between the fractional part of the quotient and the remainder. On the other hand, Nancy
raised the following issue:

N: It’s point six three one five of the whole.
W: Right, of another whole that would make thirty six [sic] [referring to
dividend, 38] but it’s not thirty six [sic] [referring to dividend, 38] so.
Right? [no response or cue from researcher] Wait I am, I am confused
now.

Nancy’s idea got Wanda confused. It was not clear from Nancy’s wording whether the
“whole” she referred to was the divisor as a whole or the constituent units of dividend. It
appeared that Wanda was thinking about “whole” as the constituent units of the divisor.
Nancy cleared up this issue later on when she compared the problem at hand to the previous party problem \( \left[ 6 \frac{1}{4} \div \frac{3}{4} = \frac{25}{3} \right] \) as follows:

\[ \begin{align*}
N: \quad & \ldots \text{So we have to, we have to find the remainder. Like how, we got one fourth because oh, we have to find point six three one five of thirty eight} \ [\text{laughing}], \text{I think. Or something like that.}
\end{align*} \]

Even though Nancy was not so sure about it, she made a comparison between the current problem and the previous one (party problem) they solved together. Nancy turned the party problem \( 6 \frac{1}{4} \div \frac{3}{4} \Rightarrow 8 \frac{1}{3} \text{ parties or “8 parties with} \frac{1}{4} \text{ bags of chips leftover”} \) into the decimal form \( 6.25 \div 0.75 = 8.333 \) and made an item by item comparison between that problem and the current one \( 1354 \div 28 = 35.6315 \) as illustrated in the following episode.

\[ \begin{align*}
N: \quad & \text{Okay in your problem [referring to Wanda’s party problem], this, when we had six and a quarter, like what we actually did was six and a quarter divided by three fourths, right?}
W: \quad & \text{Yeah.}
N: \quad & \text{So I just changed that to --}
R: \quad & \text{This was the problem [writing} \ 6 \frac{1}{4} \div \frac{3}{4} \text{ on the board].}
N: \quad & \text{Yeah.}
\ldots \quad & \text{I changed it to decimals just like that.}
W: \quad & \text{Okay.}
N: \quad & \text{So you have six and a quarter divided by point seven five it’s equivalent, equals to eight point three three three. [on the worksheet Nancy has} 6.25\div0.75=8.333\text{]}
R: \quad & \text{Hmm hmm.}
N: \quad & \text{And the answers we came up with, we have eight and a third parties --}
W: \quad & \text{Which where we got the eight point three three three, right.}
N: \quad & \text{… which or it’s equal to eight remainder one fourth bags, or yeah bags.}
W: \quad & \text{Okay. Hm hmm.}
N: \quad & \text{So, you want to know what the remainder is.}
R: \quad & \text{Yeah.}
W: \quad & \text{So relate those for me.}
\end{align*} \]
N: So we have thirteen fifty four [1354] which is like the six point two five [6.25]. And the thirty eight [38] which is like the point seven five [0.75].
W: Okay.
N: And the answer is thirty five point six three one five [35.6315].
...  
R: What information do you have from thirty five point six three one five?
W: That thirty eight [38] can go into uh thirteen fifty four [1354] at least thirty five [35] times and then some other amount of times.
N: Yeah.
W: Which I thought the remainder was point six three one five. But that’s what you are trying to convince me.
N: Our remainder there is one fourth.
W: Hmm hmm.
N: So if our remainder is one fourth [referring to Wanda’s previous problem], point three three three three [fractional part of the previous quotient] should be equal to one fourth but it’s not so our remainder isn’t- I mean it’s related to point six three one five [fractional part of the current quotient] but it’s not point six three one five. Like it’s --

It seemed that Nancy tried to match the information given in the current problem with the party problem so that she could determine the meaning of the decimal part of the quotient. This comparison helped her to point out that there was a difference between the fractional part of the quotient and the remainder in the current division problem. Also, this type of comparison was a way of reflection for Nancy by which she was getting the idea, at least on the numerical level, that the fractional part of the quotient, 0.6315, helped one figure out the remainder, but it was not identical to remainder. Here, this one-to-one parallel structure aided Nancy to get through this problem, but the fact that she said, “It’s related to point six three one five but it’s not point six three one five,” suggests an awareness of some type of relationship between the fractional part of the quotient and the divisor. This was not new to Nancy since she already did it in the previous problem, too. Nancy’s explanations also convinced Wanda that the fractional part of the quotient was related but different from the remainder.
Understanding #4: Fractional part of the quotient and remainder are multiplicatively related to each other.

Now, the issue for both participants was to figure out a way to find the remainder in the previous problem and apply that to the current problem. In doing so, Wanda proposed that in the previous problem, \( \frac{1}{4} \div \frac{3}{4} = 8 \frac{1}{3} \), the remainder was found by dividing \( \frac{3}{4} \) [the divisor] by \( \frac{1}{4} \) [the remainder part from the dividend]. The rationale she had for this was that the divisor, \( \frac{3}{4} \), could go into dividend, \( \frac{1}{4} \), so many full times until it could not go anymore. Hence, in order to find what part of divisor corresponded to that remainder piece (\( \frac{1}{4} \)) Wanda made a mental comparison between the remainder (part of the dividend), and the divisor to make sense of the fractional part of the quotient as follows:

W: Like this [pointing to the drawn figure as shown in Figure 5-6] is what we are working with. We have this whole [pointing to the whole rectangle in Figure 5-6] here but of a bag. And we only have three, three fourths of this whole --

N: Wait what is this whole [pointing to the whole rectangle in Figure 5-6]?

W: This is a bag [pointing to the whole rectangle in Figure 5-6]. And these are the fourths [pointing to each fourth piece in the whole rectangle in Figure 5-6], and we are using three fourths for a party [marking the first three pieces in the whole rectangle as shown Figure 5-6]. Okay? So this one fourth here [pointing to the last one fourth piece and then marking the first one fourth piece in the rectangle] which was remainder of the bags of chips is one third of this amount here [pointing to the marked first three fourth section in the drawn rectangle], this three fourths.
Wanda was on the right track in identifying the numeric relationship between the divisor \( \frac{3}{4} \) and remainder \( \frac{1}{4} \) but the way she set up her algorithm to find the fractional part of the quotient \( \left[ \frac{3}{4} \div \frac{1}{4} \right] \) was not directly resulting in the fractional part of the quotient \( \frac{1}{3} \) in quantity. She seemed to know that there needed to be a multiplicative comparison between the divisor and remainder. She chose the division operation to make sense of that multiplicative comparison. As a result of applying this operation to remainder and divisor (as divisor \div remainder), she found that divisor was three times as large as the remainder. This is shown in what follows:

\[ \frac{3}{4} \div \frac{1}{4} = \frac{3}{1} \]

As seen through her wording, she knew that she was looking to identify the part of the divisor within the remainder. Therefore, she said, “one fourth bags can only serve one third of a party,” which was giving away the fractional part of the quotient as \( \frac{1}{3} \). In this
situation, Wanda was able to see the multiplicative relationship with these juxtaposed
numbers ($\frac{3}{4}$ and $\frac{1}{4}$), but we cannot assume that she had an abstraction of that
relationship. In fact such a comparison did not require one to call on an abstraction of
multiplicative comparison since the comparison between fractional quantities at hand was
obvious. Also, Wanda’s current way of reasoning, later on, will show us a lack of this
abstraction on Wanda’s part.

Even though at times Nancy was getting confused by Wanda’s interruptions, she
considered $\frac{1}{4} \div \frac{3}{4}$ as the way to find the remainder. In other words, she was in search of
quantifying the remainder in terms of the divisor. To Nancy, this kind of search meant
finding the number of divisors (or part of divisor) within the remainder piece because
remainder piece was part of the dividend and the overall goal was to figure out the
number of divisors within every piece of dividend including the remainder part. In doing
such reasoning, she was calling on her previous understanding, on the numerical level,
that the fractional part of the quotient helps one determine the remainder but it is not
identical to remainder.

**Understanding #5:** Understanding the relationship between the remainder and
the fractional part of the quotient seems to be dependent on coordination of the
referents for three quantities (dividend, divisor, quotient) under the guidance of
overall goal (how many divisors are in dividend).

Later on in the session, the participants began to pay more attention to the
referents as in the following episode.
N: So, these are bags \([wrote \text{ bags underneath } 6.25, \text{ the dividend}]\) and these \([pointing \text{ to } 0.75, \text{ divisor}]\) are bags per party.

…

N: This was three fourths of a bag for a party, right? \([pointing \text{ to } 0.75 \text{ and asking Wanda}]\)

W: Right, right.

N: So does that make sense \([pointing \text{ to the phrase bag/party}]\)?

W: Okay.

…

N: So this point three three three is third of a party.

R: Okay.

N: So it refers to this \([pointing \text{ to } 0.75, \text{ the divisor}]\).

R: What do you think Wanda?

W: Say that again? Wait, why does it refer to this one? Oh oh, yeah \(\text{enthusiastically}\) because we did the remainder thing with the, the quarter bags of party left is one third party, okay.

R: Wait, wait, wait, say it again. \(\text{all laughing}\)

W: No. It refers to this one \([pointing \text{ to } 0.75]\) because this one is bags per party and that’s why our answer we said that eight remainder of fourth is equal to eight-thirds party, or eight parties with remainder of one fourth bag is equal to eight and one thirds party that can be served because of one fourth serves one third of another party. So that’s why this \([pointing \text{ to the fractional part of the quotient, } 0.333]\) can go back to these like per party thing \([pointing \text{ to divisor, } 0.75]\).

N: So, now I lost where I was. So I said okay this one third refers to parties, like we have eight and a third or point three, three, three parties.

W: Okay.

…

N: And we got the third or point three, three, three by dividing one fourth by three fourths. So our remainder here is one fourth but we divided it by this \([pointing \text{ to } 0.75]\). You know how you said, like you said do we have to divide that by thirty eight.

…

N: It’s one fourth divided by three fourths. \([scratching 3/4 \div 1/4]\)

As also seen in the above discussion, they began to pay close attention to the referents involved in the division based on the context. Even though Nancy was proposing that \(\frac{1}{4} \div \frac{3}{4}\) corresponded to the fractional part of the quotient, she was also trying to make sense of the division operation. They solved this puzzle using the referents and the overall goal as follows:
N: So if you have your bag, wait. You have your bags. Okay you have point two five of a bag, but you need point seven five of a bag per party. So you wanna find out how much of a party this \[0.25\] is. So you take the quarter and divide it, divide it by how much you need for a party to find out how much you have of that. Does that make sense? You have --

W: This \[0.25/0.75\] is how many more times you …[pause]…

R: Yes, how many more times?

N: You have --

W: How many more times point seven five \[0.75\] can go into six point two five \[6.25\].

N: Because this \[0.75\] refers to how much you need for each party.

R: Okay.

N: And you wanna know how much of a party this \[0.25\] will give you. So if you divide this \[0.25\] by how much you need \[0.75\] you’ll find out how much of this \[0.25\] will --

W: Support another party.

N: Yeah. Or how much of a party you can support with this \[0.25\].

W: Yeah with that \[0.25\].

R: You were using some other thing. You were saying I am trying to figure out how many point seven fives can go into six point two five \[speaking simultaneously\]. How is it related?

W: It can go eight times with this extra here which would be the point three, three, three.

R: I can understand the eight times, why is it that extra?

W: Because it can’t go into it a full another time. But for this extra here it can go into it, it can go into it that many more, that much more.

R: Okay I understand that and where does one third come from? Why are you dividing those?

…

N: I mean this \[pointing to the remainder 1/4\] divided by this \[pointing to the divisor 3/4\] gives you this.

W: That’s what it’s [sic] asking why.

N: Yeah. But he asks where the one third comes from. And that’s because you are taking what you have and dividing it by what you need.

W: By what you really need for a whole party.

R: For what purpose?

N: To figure out --

W: How much of a party you can support how much like because you know you can’t support a full load of party, you know.

N: You have less than what you need.

W: Yeah.
As seen in the above episode and the previous ones, context was very helpful to the participants in deciding on what to pay attention to with regard to the referents and involved multiplicative relationships. For instance, within the context, the participants were able to state the overall goal clearly and identify the involved quantities. In the party problem, the overall goal was to serve a number of parties with the amount of chips at hand. The involved quantities were number of parties (to be determined), the total number of bags of chips and number of bags of chips per party. In this sense, they were able to identify the two extensive quantities (total number of bags of chips [dividend] and number of parties to be determined [quotient]) and an intensive quantity (number of bags of chips per party [divisor]). Such identification helped them think about the overall goal of partitioning the total number of bags of chips based on “bags of chips per party” and finding the number of parties. Meanwhile, they both focused on the involved relationships among the quantities. In this way, they realized that they needed to pay attention to the relationship between the remainder and divisor.

In the contextual situations, the participants were to coordinate the overall goal, the activity sequence they were going through, and the relationship between the referents all at once. The contextual referents carried a lot of the thinking about what was the relationship between the quantities. Therefore, there was less attention to the multiplicative relationship among the quantities than to the referents and the activity sequence in the contextual situations because of the contextual referents. Out of their work in context, they abstracted the activity sequence they went through and the referents; however, as we will see in the following section, their abstraction of the
5.3.2. An Attempt to Have Participants Think about Remainder and the Multiplicative Relationship in Context-Free Diagram Setting

At this point, the participants knew that there was a mathematical relationship between the fractional part of quotient and the remainder. They also realized that this relationship could be identified by dividing the remainder by the fractional part of the quotient or vice versa. They were able to think about this relationship within a context by coordinating the referents for divisor, quotient and remainder using the contextual referents and quantities. Both of them knew at this point that the fractional part of the divisor was a different quantity than the remainder even though both were related multiplicatively. However, they were not able to think about this multiplicative relationship in the absence of the context. They did not know how to coordinate the referents for the involved quantities to make sense of this multiplicative relationship in context-free diagram setting yet.

With this above knowledge they moved on to working on the given division problem 1354÷38.

N: Well, we know the remainder is something divided by thirty-eight because we have to somehow take that remainder which refers to the one thousand three hundred fifty four and get it to, make it refer to the thirty eight. So if we take the remainder divided by thirty-eight we’ll get. I don’t know --

W: Other than that I don’t know how you get it.

N: Yeah, I don’t know how else to explain it.
As seen in the above dialog, they referred to their previous abstraction that the multiplicative relationship between the divisor and remainder could be identified by division operation.

Then, I asked them to generate a word problem that would fit into the given mathematical expression, 1354÷38. Together, they formulated a word problem such as, “It’s Halloween and we are going egging. We use thirty eight eggs per house. How many houses can we egg?” (hereafter, called as Halloween problem). Their answer to the problem was what they found earlier, 35.6315 houses. They also stated it as 35 houses and “point three one five of another house.”

N: I mean if you have point six three one five of a house. So [pause] point six three one five. You wanna know point, you wanna know point six three one five of thirty eight. Because that’s part of your house and thirty-eight is number of eggs per house [turning to Wanda]. Does that make sense?

W: [murmuring] How many, how many eggs are, yeah it makes sense. How many eggs are used to egg the point six of the house.

R: Let me write that down [writing their statement on the board].

N: Uhh because I was trying to think like point, point six three one five of thirty-eight eggs per house will give you the number of eggs [talking to W]. Because I mean with the labels, per eggs [sic] will drop out and just give you, I mean house and give you eggs.

W: Yeah.

Previously in this section, it was mentioned that they became aware of the coordination between the referents for divisor, remainder, and quotient under the guidance of the overall goal. As a result of such coordination, they were interpreting the quotient with its two parts: whole number part and fractional part. This was being done with the help of context because they were considering 35.6315 houses as 35 full houses and 0.6315 of another house to be egged. Hence, both the whole number and the fractional part of the quotient referred to the houses for them. However, as we see in the above episode, now,
they considered the division operation as being applied to the referents as opposed to the quantities. In other words, Nancy’s above way of reasoning suggests that she was focusing on the unit cancellation rather than thinking about the relationship between the quantities. They continued as follows:

N: So it would be point six three one five times thirty-eight eggs per house.
R: Because?
N: Because thirty-eight eggs for a house but you wanna know point six three one five of that house.
W: What is it of for thirty-eight, point six three one five [speaking simultaneously with Nancy]?
N: Which is thirty-eight eggs. Does that make sense?
W: We have --
R: You were gonna say something [talking to Nancy].
N: Well I was just gonna say, this is your part of the house and this is your whole house. So and you wanna find out point six three one five of thirty eight, which is equivalent to the one house. So you multiply that by thirty-eight and that’ll give how much of that thirty-eight you’ll need or will be able to egg that part of the house.

When examined carefully, the problem they were working on was based on messier quantities (0.6315 versus 38) than the ones they worked on before (\(\frac{1}{4}\) versus \(\frac{3}{4}\)).

Previously (for party problem) they made a numeric comparison between \(\frac{3}{4}\) and \(\frac{1}{4}\).

However, they did not select multiplicative comparison as the needed operation to do such a comparison. The fractions they had did not require them to think about the relationship between those two quantities consciously. In other words, to the participants, the numeric relationship was such that they did not have to think about the quantitative relationship between the divisor and the remainder. They were only coordinating the overall goal for the problem with the numeric relationship between the \(\frac{3}{4}\) and \(\frac{1}{4}\) but this
coordination did not require them to call on a multiplicative operation because the relationship between them was obvious. On the other hand, in the above case, they dealt with 0.6315 and 38. These numbers were not the kind that could easily be compared multiplicatively mentally. And also these numbers required calling on a multiplicative operation. Using messier numbers helped them pay their full attention to the multiplicative relationship between the divisor and remainder and reflect on it. In addition, for both problems, there was a basic way of using diagrams in the solution processes. In this regard they were using the same diagram approach for both problems, which was also the commonality for them at this point.

With the messier numbers, they were at least having the experience of reasoning about the involved quantitative relationship. With this kind of experience there was a chance that they could make an abstraction. In this way, later on, for the similar problems, they would call on such a conception so that they could begin to abstract across situations.

As a follow up, when I asked them to figure out what was 0.6315 of 38, they mentioned that they needed to multiply those two numbers. I announced the result of that multiplication as 24. They considered 24 as the remainder. Next, I asked the participants to find a way to check mentally whether 24 for the remainder was correct, using diagrams. The response I got was:

R: If you were to draw the pictures what would you do? … Just tell us.
W: Well I would draw 1354 eggs and then I would wipe off thirty-eight, thirty-eight, thirty-eight, thirty-eight, thirty-eight, thirty-eight until I got to thirty five \([times]\) and I saw that twenty four was left.
R: You would make up \([pause]\) groups of thirty-eight. Okay how do you know that how many such groups you can make? …
W: You count them.
R: Okay what would be the result of that counting?
W: Thirty-five.
R: Thirty-five. Okay so you can make thirty five groups --
W: And twenty four little eggs left over that would make part of another group.

Thinking about the given division problem in light of the context and diagrams at the same time seemed to make things clearer for the participants in the following sense. The quantity at hand was 1354 and they were thinking about it as 1354 units (each unit represents one egg). The overall goal in this case was to partition these units in such a way that each partition was worth a 38-egg group. Hence, they were to identify 38 eggs per group and then count the number of groups until all the units were fully taken into consideration. Each grouping of 38 eggs resulted in one extra group which needed to be added to the count of groups. Upon finishing up the identification of the full groups, the issue was to make sense of the extensive quantity (remainder, a number of partitions that were not enough to make another group) in terms of an intensive quantity (divisor, a number of partitions per group), which was possible by identifying the multiplicative relationship between those two. Therefore, the focus of attention was moved from counting number of groups to the multiplicative relationship between the size of one group and the remainder.

Later on, Nancy identified the issue that if the only given information was the remainder, 24, and the divisor, then she could have divided it by 38 to find the fractional part of the quotient. She was also able to connect it to the overall goal to explain why such a calculation would result in the fractional part of the quotient. In this sense, they were able to make sense of this situation in both directions [given the remainder amount and divisor or given the quotient amount and divisor]. In other words, previously their
notion of multiplicative relationship included only two quantities, remainder and fractional part of the quotient; however, now, this multiplicative comparison between the divisor and remainder results in fractional part of the quotient. These above realizations on the participants’ part resulted from their work in a contextual setting. What follows explains how these understandings carried over to the context-free setting and became more solid for the participants.

5.3.3. Thinking about the Remainder and the Multiplicative Relationship in Diagram Setting Only

Use of the diagram approach was also helpful in T7 when they were asked to think about another similar type division with remainder problem, 1379÷28. They described what the problem asked about as follows:

N: I said how many times can twenty-eight go into one thousand three hundred seventy nine.
W: And I said how many twenty-eighths are in thirteen seventy-nine.

Once they explained what 1379÷28 actually meant, they used their calculators to find the result of the division problem as 49.25. When the question was about what the remainder was, the automatic response I got was “0.25 times 28.” To explain why that calculation would give the remainder, they wanted to go back to some context and reason within the context, but this time the limitation I set for them was “no context.” In this way they were not to be distracted with contextual details and were to focus on actual quantities and the multiplicative relationship between them in the given problem. Their response was:

W: Okay, well point, twenty-eight can go into thirteen seventy-nine, forty-nine times and point two five times of another whole time.
R: Okay, so how is it helping you to figure out remainder?
W: Because um --

...  
N: Well, the point two five is like point two five of a time, right?
W: Yeah.
R: Point two five of a what?
N: Of a --
W: A whole time.
N: Which is twenty-eight.
W: Which is twenty-eight.
N: So we need to find what --
W: Point, exactly point two five times twenty-eight is [speaking simultaneously with Nancy].
N: Ohh [exhale noisily], it became clear right now.
W: There you go.
R: What became clear? I mean you just told me point two five times twenty-eight, you didn’t tell me why.
N: Because point two five. Twenty-eight I mean twenty-eight is in the one thousand three hundred seventy nine, forty-nine times and then point two five of another time. So we don’t have a whole twenty-eight, we have point two five of twenty-eight. So in order to find what point two five of twenty-eight is we can multiply them by each other.

As mentioned above, they were about to jump to some context to think about the rationale for multiplying the fractional part of the quotient and the divisor. However, they were limited with the diagram use, and they began to carry over their thinking about remainder in context to thinking about remainder in a context-free-diagram setting. They already knew that the divisor would go into the dividend so many full times (in this case 49). They needed to make sense of the fractional part of the quotient. Since they considered each action of making up a full divisor group within the dividend as “one time,” they now thought about the “0.25 times divisor” as “0.25 of another time.” In other words, by “0.25 of another time,” they were referring to the partial divisor group. Their intensive quantity here was “numeric divisor value per time” and the extensive quantity was “times.” One can consider “time” as one group. To find part of a time (or group),
they knew that they needed to multiply 0.25 and the value of divisor. In this sense, they carried over their understanding of quantities and referents into the context-free setting.

As they suggested, I calculated “28 times 0.25” and announced the result as 7. In addition to this, I asked them to think about a way to check mentally whether 7 was actually the dividend or not. The method they offered for such a request was:

W: Oh, oh. You can take twenty-eight times forty-nine plus um and see what you get and then add seven [Speaking simultaneously with Nancy]
R: So you are saying, find twenty-eight times forty-nine.
W: Can we do the --
R: Uh before that, we are gonna do, don’t worry [Nancy laughing]. Plus seven. How did you come up with this? Why would this give us 1379?
W: Because it can go into it forty-nine times at least we know but it can’t go into it a whole like twenty-eight can’t go into it a whole other time.
R: Uh huh.
W: So you just [wanna see] what the difference is between twenty-eight times forty-nine and um thirteen seventy-nine, and that would be seven.

They were now also able to think about the process by which remainder could be found using the activity sequence they went through. Their current understanding was more advanced than their previous understanding of remainder as explained in the following paragraphs.

The way they thought about quotient (with its whole number part and fractional part) in context was carried over to a context-free-diagram setting. They already knew their goal was to identify the number of 28 within 1379. Since they already had the abstraction of this idea, they knew what constituted reaching that goal. Therefore, thinking about the diagrams, they focused on the partitioning process of 1379 with regard to 28. They also knew that in their diagrammatic approach (which was mentally done), there needed to be a repetition of the quantity 28 within 1379. Each repetition was called a “time.” They also knew that this repetition of full 28 units per group was not to cover
all 1379 units since there was some leftover (because the given result was 49.25). In this sense, the activity of partitioning the given quantity, 1379 units, with respect to 28 units per group repeatedly until it could not be done anymore was being associated with the meaning of quotient. The quotient referred to some whole number of partitions (of the size 29 units/group) in combination with some fractional number of partitions. In this sense, they associated the meaning of quotient with the meaning of the partitioning activity they were going through. Such an association resulted in the realization that 1379 units could embed 49 full groups of 29 units and an additional $\frac{1}{4}$ (0.25) of another group of units. Hence, they were considered making up 29 units per groups as many times as they could and finding a number of units that was not enough to constitute another group. Since these leftover units were not enough to generate another whole group, they called on their understanding of the convention that the remainder was to be less than the divisor and considered this partial group (0.25 of a group) as the amount that referred to the remainder. The size of this group was giving them the remainder.

To sum, there were several actions and coordination of those actions the participants relied on. First thing is to set the overall goal as figuring out number of one quantity within another. And then the issue is to think about the process of partitioning the overall goal refers to. Next, the challenge is to identify and decompose the nature of the quantification of this partitioning. In doing so, there needs to be an attention to the nature of quotient. If it has a fractional part, then it means that there is a remainder. In this sense, the quotient informs the quantification process as to how to measure the number of divisors. Hence, this quantification process is enabled in two phases: thinking
about the number of full divisor groups, and thinking about excessive amount as part of another group. Once the nature of the quantification process is revealed this way, the next and final issue is to think about the fractional group itself as part of the quantification process that corresponds to the fractional part of the quotient and to think about the size of the fractional group as remainder. In other words, what is required is the identification of the multiplicative relationship between the divisor and remainder that results in fractional part of the quotient.

At this stage, since the participants could coordinate the quantification operation with the quotient and interpret that coordination, they were able to think about the remainder in context-free diagram settings. Such coordination helped them make sense of the involved referents also. However, most of this was done in whole number setting.

5.3.4. Affordances and Constraints of Contextual-diagram setting versus Context-free-diagram Setting

The participants initially worked on the contextual-diagram setting in whole number domain. Out of this work, they abstracted the notion that the fractional part of the quotient is mathematically related to the remainder but not identical to the remainder. As they worked through similar structure problems, they were also oriented toward paying attention to the multiplicative relationship between the divisor and remainder which resulted in the fractional part of the quotient. As a result of this work they abstracted an activity sequence and referents involved in whole number division problems. Once they worked on contextual problems and made the aforesaid abstractions, they moved onto
context-free problems, and they started working in context-free-diagram setting to think about the given whole number division problems.

Whether they worked in contextual-diagram setting or context-free diagram setting, they reflected on the same idea: the connection between the two types of answers to the given division problems. However, there was something different about the contextual-diagram settings than the context-free-diagram settings. How do context-free-diagram settings afford and foster development? I explore this question in the following paragraphs.

As mentioned earlier, in both cases (contextual vs. context-free) the participants reflected on the relationship between the two types of answers that could be generated for a division problem. However, the nature of the reflection they did in a contextual-diagram setting was different from a context-free-diagram setting since a context-free-diagram setting allowed them to see the commonality among their own activities. The reason for such a difference was that context-free-diagram situations were already an abstraction of contextual-diagram situations. When they started working in a context-free-diagram setting, they already had an abstraction of their “activity sequence” and “referents.” This is further explained in what follows.

In a context-free-diagram setting, they compared quantities in the given mathematical situations. They went through a certain activity sequence and one of the activities in that sequence was to compare a number of partitions they had to a single group of partitions. In this way, they worked with generic quantities, and actually every problem I gave them had involved such generic quantities. Therefore, the language they used to describe these quantities and the relationships among them was the same as the
language for the abstraction they were after. For example, they were looking for certain-
size partitions and at some point they found out that they did not have enough of the
partitions to make up another group of partitions. At this point, they questioned what they
were supposed to do with the number of partitions (remainder) at hand. This questioning
required them to refer back to their abstraction of the referents and activity sequence that
they had from the contextual-diagram setting. In this way, they had to focus on the
multiplicative relationship between the divisor and the remainder that resulted in
fractional part of the quotient. And as they looked across multiple situations with
diagrams, they made an abstraction of the quantitative relationships (which was the
commonality they were looking for) involved in division. Furthermore, the participants
used a contextual-diagram setting to come to a point where they did not need diagrams
anymore.

The multiplicative relationship toward which they were working was not obvious
in the contextual-diagram setting, because the contextual clues carry a lot of the thinking
about what was the relationship between the quantities. However, context-free-diagram
setting requires an abstraction of the relationship between the divisor, remainder and
quotient. In order to think about this relationship, they were to have an abstraction of
divisor as an intensive quantity that connects two extensive quantities (dividend and
quotient). In a context-free-diagram setting, one already needs to have such an abstraction
of the divisor as an intensive quantity in order to think about the multiplicative
relationship between the divisor and remainder. These key features were being abstracted
in a context-free-diagram setting by looking across the same structure problems.
5.3.5. Moving to the Fractional Setting and Thinking about the Remainder in Fractional Setting

Later on, in the rest of the T7 and at the beginning of the T8, they were having struggles as to what to choose as remainder and the fractional part of the quotient. However, T8 was where they straightened out their notion of remainder in fractional settings.

The tasks of T8 consisted of four parts. Part I was similar to what they did in T7 to test and enhance their understandings regarding the so called multiplicative relationship and the remainder in the fractional domain. Part II, Part III and Part IV were designed to help them move toward the common denominator algorithm. However, because of the time limitations the participants were only able to go through Part I and Part II, and in the subsequent sessions a philosophically similar but different task sequences were applied.

During the first third of T8, Wanda and Nancy worked on the problems given in Part I. There were mainly two problems with the provided diagrams to be used in the solution process. They went through the first problem, \( \frac{7}{3} \div \frac{3}{4} \), using the two-way partitioning method. When they felt that they were getting confused, they referred back

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19 Part II, Part III and Part IV were very similar to the task sequence Prof. Martin Simon designed for his students when teaching division of fractions. The similar structure was used for this part of the study with his permission.

20 This partitioning method was invented by the participants during their work in T7. It refers to the combination of vertical and horizontal partitioning of the unit wholes that makes up the dividend. Vertical partitioning is used to identify dividend and the horizontal partitioning is used to unitize unit wholes and
to their overall goal and reevaluated their referent units. They used the same partitioning style for the second given problem, $\frac{9}{5} \div \frac{2}{3}$. The vertical partitioning was already provided in a given diagram for the problem. Using the given diagram for dividend, they horizontally partitioned each unit whole and unitize each unit whole with respect to fifteenths. Each unit whole now had fifteen fifteenths. They then grouped each $\frac{10}{15}$ section, and they came up with 2 full such groups and 7 pieces (of the size $\frac{1}{15}$ each) as remainder. They knew that 7 pieces were to be compared to the 10 pieces that made up a full group so that they could complete the quantification of number of divisors within the dividend. They also knew that each of the 7 pieces had the size $\frac{1}{15}$ of a whole rectangle.

In this way, the two responses they got were $2 \frac{7}{10}$ and $2R \frac{7}{15}$. Their interpretation of these two results and remainder was as follows:

W: ... This is how much it’s really like worth [writing the word “worth” next to 2/15 in her worksheet] in terms of the whole.
R: Okay so which one is my answer?
W: Two and seven um tenths.
R: But you said this [pointing to Wanda’s paper and circling $2 \frac{7}{15}$] is also my answer.
W: But that’s different. That’s two plus seven fifteenths.
...
R: So it’s not two and seven fifteenth, it’s two plus seven fifteenth?

identify divisor. This partitioning style was used by the participants to manage with different referent units in T7. A detailed version of this is given in the subsequent section.
W: Right, it’s two, um two whole two-thirds plus, and seven fifteenths like remainder so. And then the top one [referring to $2\frac{7}{10}$] is two whole two-thirds and seven-tenths of another two-thirds.

R: So this many [pointing to $2\frac{7}{10}$] two-thirds?

W: Yes.

R: And this is seven-fifteenth of what?

N: Of a whole.

R: Your whole is?

N: [pointing to the first rectangle]

R: This whole thing [pointing to the second rectangle].

W: Fifteen fifteenths.

R: Fifteen fifteenths, okay. So how is seven-fifteenths related to seven-tenths?

N: It’s the same amount but it’s like in different context. Like seven-tenths of two-thirds is equivalent to seven-fifteenths of the whole. Yeah.

R: You agree?

W: Hmm hmm.

As we saw in the above episode, they were able to talk about the two different but related results for the given division of fractions problem. They were also able to identify the remainder, and the distinction (and relation) between the remainder and the fractional part of the quotient without any difficulty. How come they carried over what they abstracted from the whole number setting to fractional setting? The only concrete source they had in the fractional setting was diagrams (not context). They did not have any other source to work on. And, they already had the abstractions from the whole number setting that when the quotient was a mixed number, there was to be a remainder amount, and there was a multiplicative relationship between this remainder amount and the divisor which resulted in the fractional part of the quotient.

In the diagram setting, they identified the dividend and divisor based on partitioning. This was almost automatic for them. They also went through the
quantification process. In the diagram setting, this quantification process was not different from the one in the whole number setting. Once the dividend and divisor were identified, what they had in front of them were partitions. In other words, their overall goal was to look for the number of “10 pieces per group” within the drawn quantity dividend. Hence, they either labeled each ten-piece with a certain numeral or just circled each ten-piece group and labeled it as a group number until they could not do so any more. In this way, they determined the number of the intensive quantity divisor within the drawn extensive quantity dividend. The type of the quantities was not making a difference for them because diagram use was taking care of the difference. When they exhausted all the full divisor groups, they had a number of leftover partitions to be interpreted with respect to divisor. At this point they referred back to the overall goal they had, which was to find total number of divisors within dividend. This overall goal helped them to focus only on the multiplicative comparison between the divisor (10-piece-group) and the leftover (7 pieces). This multiplicative comparison was also feasible for them since they were to compare 10-piece-group to 7-piece leftover section. They knew that 7-piece section was 7/10 of the full divisor group, which together with 2 results in $2 \frac{7}{10}$. This $2 \frac{7}{10}$ was the quantity that tells the number of divisors within the dividend.

In the above explanation we see a reduction of fractional dimension to the whole number dimension in a diagram setting. Once one sets the overall goal and identifies the quantities, dividend and divisor, then one is actually dealing with some whole number of partitions and grouping them. In this way, one is in the whole number domain until
finding the result. The result refers to the number of divisors which in this case was \( \frac{7}{10} \) two-thirds (or ten-fifteenths). In this sense, they were basing their whole argument on their understanding of division of whole numbers in a diagram setting.

To see to what extent they related this to calculator calculations I asked them:

R: So what am I gonna get when I use my calculator?
W: [sigh]
N: I said two point seven \([2.7]\). I am not positive about that.
R: Two point seven. Why is that? What do you say [asking Wanda]?
W: Because that’s the tenths, and we are doing tenths so we have seven tenths. So that’s right.
R: How did you find this \([pointing to \“.7”]?)
W: Because --
N: Well, like uhh okay. To answer how many two-thirds is two-and-seven-tenths and not like from just what we’ve been discussing and that’s what we know a calculator gives you is like that answer, not a remainder.

... Why in the world would calculator give us that \([referring to the answer 2.7]?)

... Because I think that’s the actual answer. That tells us how many two-thirds are in nine-fifths.

It seemed from the above episode that, according to the participants, calculators gave one type of answer because that answer was the one informing them about the number of one quantity within another. When I reversed the question as, “Why not \( \frac{7}{15} \)’’ their response was about the referents as follows:

N: Because it’s two different things. Like it’s two of the two-thirds but yeah the seven-fifteenths is not of the two-thirds, it’s of the whole. Like it’s of a different. I don’t know if calculators can like compare those two things.
W: Yeah, I don’t know that. Well, I don’t know calculators can do a lot of things these days. It might be able to give you both answers in some programs.
...  
R: But why not this?  
W: Because that’s like, like she was saying, that’s out of the two different things and it’s not answering that question.  
...
R: So it cannot do two things at the same time. But it can do this [pointing to the answer 2 2/10] even though this has two parts [referring to the 2 and 2/10 as two parts]. Why?  
N: Because --  
W: They are the same. They refer to the same thing.  
R: They both refer to the --  
W: Two-thirds.  

They attributed the calculator calculations to the referent differences in answers. This suggests that they had an abstraction of the quotient and the referents it refers to in a context-free setting. In the follow up discussion, when I asked them about what each unit whole was a part of, they referred to the dividend. Also they knew at this point that remainder was part of the dividend. In addition, they knew that the \( \frac{7}{10} \), the fractional part of the divisor referred to the divisor and \( \frac{7}{15} \) (remainder) actually told how big that remainder was.

At this point they were able to anticipate the results of the activities they needed to go through without actually going through those activities for the given division problems. And they were also able to set up situations for division problems (in the whole number setting) where they could easily identify the remainder.

Post-interview results also show the advances in their thinking about the remainder and referents as illustrated in the following section.
5.3.6. Meaning of remainder to the participants as seen through their work in post-interview

According to the post-interview results, both participants seemed to have a handle on the understanding that the whole number part of the quotient refers to the divisor and the remainder refers to the dividend. In answering problem 4 (which was asking about the remainder for the division problem \[ \frac{86099}{13246} \div \frac{11725}{2345} = 1.3 \]), Nancy approached the problem as follows:

N: I meant three-tenths of this is wait, three tenths of this is what is, is how many, how much of this [pointing to the divisor] can go in what is left of that [pointing to the dividend].

R: How much of this can go in?

N: To what’s left of this.

R: Oh okay. So does it mean- so you are saying but they are not same?

N: The remainder and this [pointing to fractional part of the quotient, .3]?

R: The remainder and three tenths of this.

N: No, they are the same [laughing]. I mean [laughing-pause], the remain, the remainder is what’s remainder from this [pointing to dividend].

R: Uh huh.

N: Three-tenths is taking the divisor and putting it into that remainder.

R: Okay.

N: So the remainder is, okay so the remainder divided by this [pointing to divisor] gives you the three tenths.

R: Now, how did you come up with that idea?

N: I think! [pause] Because that’s how we find out.

R: Is it another rule or?

N: Well because you wanna see how much of your divisor goes into your dividend. But you only have part of your dividend left. But you still wanna know how much of the divisor can go into whatever is up to the dividend so that you know how much a whole, like it’s not going to be a whole but how much it can go in.

Her response suggests that Nancy made a crucial distinction between the fractional part of the quotient and the remainder. She was able to think about what those parts meant without actually going through and solving the problem. She was also using her goal,
which she identified for the given problem to make this distinction, and coordinating it with the referents of three quantities.

In addition to Nancy, Wanda was also aware of such a distinction between the fractional part of the quotient and remainder. Since she already referred to this distinction for different problems, I did not ask her the question 4 again. When she worked with the third problem [Problem 3a. Johnny says that the answer to a division of fractions problem is $2 \frac{1}{2}$ (assume that both factors in the problem are fractions). Jack says that the answer to the same problem is $2 \frac{2}{3}R$ (Here R means remainder). Can they both be correct?], she reasoned as follows:

W: [reading the problem]. Yeah [answering the question].
R: Yes?
W: Yeah.
... Yeah.
R: Why?
W: Because we came up with two different situations in several of the problems that we did [referring to the problems investigated in the previous class sessions]. And two and a half referred to the same thing, usually the second factor and then the two thirds was referring back to like the first thing in the word problem usually like, which is different than the actual like the answer. …
R: First you said two and a half refers to the second factor. What is that second factor you were referring to?
W: Well, like um like in the here. Eight thirds in one half or something [writing “8/3 1/2” on her worksheet], okay?
R: Okay.
W: Um two and a half would be both refer to like this half [pointing to the $1/2$ in the division problem she wrote] here but then like two with the remainder of two thirds, two would refer to this [pointing to $1/2$] but then two thirds would be how much like just extras leftover not um related to the answer.
R: So is there a division sign between eight thirds and one half?
W: Okay, yeah. Just I mean this obviously isn’t the problem for this but it’s just an example [referring to $8/3 \div 1/2$].
R: Okay, okay.
W: So yeah.

Wanda seemed to have a conceptualization of the difference between the fractional part of the quotient and the remainder. She also understood that fractional part of the quotient referred to the divisor, whereas the remainder was part of the dividend and referred to the same unit as dividend. She also seemed to have the abstraction of the multiplicative relationship between the divisor and the remainder.

5.3.7. Meaning of Referent Units to the Participants

In distinguishing referents I asked the participants a similar question I had asked in the pre-interview. It was the plant question as follows:

Problem 7: Would the following be a division problem corresponding to the expression: \( \frac{137}{135} \div \frac{428}{744} \)? “The plant grew \( \frac{137}{135} \) inches in a month. If \( \frac{428}{744} \) of the growth took place during the first two weeks of April, how many inches did the plant grow during those first two weeks?”

Wanda reacted to this problem as:

W: And the plant grew this much on a month, if two thirds of the growth took place during the first two weeks of April, two thirds, growth, we need two thirds in one and a fourth inches [talking to self]? [pause for 10 seconds] I don’t know.
R: So you changed the question into something like one and one fourth and two thirds.
W: Yeah.
R: So if the plant grew that much in a month --
W: Yeah and the plant grew this much in a month and this much of the growth took place during that first two weeks, two thirds of - what is two thirds of one and one fourth. I am gonna say no.
R: Because?
W: What is two thirds of- Because I would, if what I am saying that, I would be multiplying not dividing.
R: Because?
W: Because I wanna know what, what is two thirds of one and a fourths inch. To know how many inches during that first two weeks. And this isn’t saying multiplication.

It seemed that Wanda was initially making the given quantities simpler by using simpler fractions such as $\frac{1}{4}$ and then thinking about the goal for each problem (the contextual problem and the given mathematical expression) and making a comparison between the goals. In this way, she determined the operations necessary for those problems. Her comments were based on mathematical structure of the two problems. This suggested that she knew about what the involved quantities in a division problem should refer to.

Nancy followed a similar pattern of thought without actually using simpler fractions for the quantities in the problem.

N: If only a portion of this growth took place during the first two week of April, how many inches did the plant grow during those first two weeks.
R: Those first two weeks.
N: I think this might be multiplication too. Because if you’d have, you have how much it grew in a month but you only wanna know, you wanna know how much this is of like how many inches it’s equivalent to. I think you multiply that by that.
R: Hmm. So you are not dividing?
N: I don’t think so. [sighing]. You have how much it grew and you have a part of what it grew, grew during the first three weeks. You wanna know how much that portion of the growth is. So I would take how much it grew- because I think if you divided this $\frac{137}{135}$, or this $\frac{137}{135}$ by this $\frac{428}{744}$ that’s finding out how many groups of this $\frac{428}{744}$ are in here. And I don’t think you’d do it.

Once the quantities and the goal for the problem were identified, Nancy also thought about the structure of the both problems and then fitted the right operation into them.
It seemed that the participants used the goals and the involved referents they identified as a guide and coordinated these goals and referents to think about the appropriate operation for the problems. This was because they thought about the involved quantities qualitatively as opposed to numerically. In relation to this, they called on their abstraction of the multiplicative relationships between the quantities in the given problems as opposed to getting lost in the involved arithmetic meaning of the given situations.

5.4. Participants’ Development of Common Denominator Algorithm

T9 and T10 specifically targeted development of common denominator algorithm. The tasks that made up these sessions did not include any context. Almost all of the given problems were context-free problems. Also, I designed the problems in such a way that the denominators of the fractions were relatively prime numbers. In this way, they were not to be distracted with different partitioning strategies. The disadvantages of such a decision will be discussed later in this chapter. The main goal was to help the participants abstract a brand new algorithm with related understandings and to make an investigation of such an abstraction.

The common denominator algorithm has a conceptual basis depending on basic operations of partitioning, unitizing and counting, which makes it more inventible by learners using diagrammatic approach. The common denominator algorithm can be considered algebraically as follows given that the denominators of the dividend and
divisor are relatively prime. The participants initially worked in this restricted interpretation of common denominator algorithm.

Eq. 5.1

\[
\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bd} \div \frac{cb}{bd} = \frac{ad \div cb}{bd}
\]  

There are two steps embedded in common denominator algorithm (hereafter abbreviated as CDA) finding the common denominator for the dividend and divisor, and dividing the numerators of those as shown in Eq. 5-1.

Prior to going into the algorithm discussion in T9, the participants worked on division of fractions from different perspectives and made several abstractions throughout the first eight sessions. Upon entering the algorithm sessions, they already had an abstraction of division of fractions as a whole. They had the abstraction that division of fractions is about investigating the number of one quantity within another. In other words, it is about multiplicative comparison of two quantities. They also had the abstraction that there is a network of multiplicative relationships among the divisor, remainder, and quotient. As we know from the pre-interviews and from some of the teaching sessions, they had an abstraction of what equivalent fractions were about. Additionally, they had the abstraction of referents and coordination of referents.

With respect to all these understandings, T9 and T10 served the purpose of helping the participants coordinate these understandings in such a way that they would develop an algorithm based on their spontaneous activities. The following section illustrates how the participants develop CDA using the understandings they abstracted from their previous work.
5.4.1. How Did Nancy and Wanda Make Sense of CDA?

In T9, the participants were provided four context-free problems in **Part I of T9** to develop a sense for the creation of CDA process. In fact this task sequence was something they had gone through previously. However, this time the main focus was on the given task sequence to develop an algorithm. At the beginning of the session, they were reminded about the specific goal of this session (T9). Their solution processes were very similar, and they solved all the problems without any difficulty in about ten minutes. A detailed version of their solution methods can be seen in **Appendix G**.

Nancy and Wanda first solved the given problems alone one at a time. Then, once they announced their results, one of them explained her result on the board with a discussion. Since they all followed the same activity sequence for each problem, I will give one example to show their approaches and how they thought about the relationships in the problems.

W:  *Drawing three rectangles and partitioning each into four pieces vertically*  Okay so we have our dividend. I am not using these three because we only have nine [shading in the right most three pieces in the last drawn rectangle] fourths. And another ...(mumbling)...[partitioning each whole rectangle into five pieces horizontally]. Okay. So I divided [each whole rectangle] into fifths because we wanna know how many three fifths are in nine fourths. So um since these are fifths [pointing to horizontal sections in the first rectangle], we wanna count by three fifths so here is one thing of three fifths [circling the upper three rows of the first rectangle] --

R:  Uh huh.

W:  And here is two things [referring to the word “groups”] of three fifths [marking the bottom two row of the first rectangle together with the utmost row of the second rectangle], and here is three things of three-fifths [circling the second, third and fourth row of the second rectangle together] and we don’t have enough so there is three [referring to 3 divisor groups]. And we don’t have enough to make um another three-fifths so in three fifths, there is [pointing to the pieces of the size 1/20 in
the first marked 3/5 group] twelve of these little things. And we only have nine, so there is nine twelfths of another three-fifths left. And --

R: What’s the? Okay, and what?
W: And um why divide like something into twelfths when you can have it simpler [referring to 9/12 and its simpler form 3/4] so there is three, it could be three [and] three fourths instead [considering the answer as 3 ¾ instead of 3 9/12].

R: Oh okay. What’s the remainder here?
W: Remainder? Uh, you should have told me this early [laughing]. Hold on, how many, yeah, nine here [pointing to the last row of the third rectangle together with the unshaded pieces in the last rectangle], nine out of twenty [emphasis on the last word], nine twentieths.

R: So nine-twentieths?
W: Yeah.

R: How do you find that?
W: Um because in a whole [there] is twenty and we only have nine, so we only have --
R: There is twenty what and nine what?

W: Well, this is divided into twen..., the twentieths, these things.
R: So they are twentieths?
W: Yeah.

W: We only have nine. So nine-twentieths.

The overall goal Wanda set for the problem was finding the number of three fifths within nine fourths. The activity sequence Wanda used in this problem and in all other problems (so was Nancy) of Part I (in T9) was:

1. Using the given diagram, shade out the necessary (vertical) partitions to identify dividend;

2. Partition horizontally each unit whole (rectangle) to allow for marking divisor-size groups;

3. Unitize divisor and/or dividend according to the new partitioning;
(4) Identify full divisor groups within dividend by either numbering partitions that makes up a group with the same numeral, or by grouping the partitions first and numbering each group as a whole;

(5) When there is not enough partitions to make another divisor group, multiplicatively compare the number of remainder partitions with the number of partitions that make up a divisor group;

(6) Identify quotient using the results of activity 5 and activity 4.

It seemed from their accurateness in choosing the appropriate referents and in referring to the important multiplicative relationships, they followed this activity sequence without any trouble (Wanda was also able to identify the remainder correctly). It also seemed that they thought about this activity sequence as a single entity to reach the goal of identifying the number of one quantity within another.

Part II of T9 was about having the participants reflect on the activity sequence they went through on the basis of other similar type problems without actually solving the problems. The purpose in doing so was to help them reflect on the activity sequence they had and move them toward an algorithmic thinking about the sequence. They were to think about the steps they would go through to solve the problems as if they were using diagrams, but they were not allowed to draw any diagram or use an algorithm. This way of operating was thought to be helpful for them to develop an anticipation of the activities they would go through (see Appendix G for details).

In Part II, the problems consisted of fractional quantities for both dividend and divisor and those fractional quantities were relatively prime and getting increased from one question to the other. The participants were to write down every step they were to
take if they were to use diagrams to solve the given division problems. In this way, they were to think about what should be drawn first and then to note the corresponding result of that action. The purpose was to help them reflect on the activity sequence mentally, and as the fractional quantities got larger, they were pushed to think about how their activities affected the size of the dividend and divisor, and the overall goal. I wanted them to learn two things in this process: (1) knowing that common denominator results in same size units for divisor and dividend which give rise to the second part of the algorithm, dividing numerators; (2) knowing that when the dividend and divisor are based on same size partitions, quantifying the number of partitions (that make up the divisor) within the dividend is same as dividing the numerators. They already had an activity sequence to go through mentally. By increasing the numbers drastically, they were pushed to condense the activities they went through. For instance, their activity sequence includes several activities to generate dividend. First they needed to draw several rectangles (or one rectangle depending on the size of the dividend they were to determine) and partition them based on the denominator of the dividend and shade in the number of partitions based on the numerator of the dividend. All these activities had certain results, but all of them served the purpose of identification of dividend. Hence, they could think about all these activities as a single entity (as identifying the dividend) attached to a single result, the dividend. When the given fractional quantities (dividend and divisor) were increased drastically, they were pushed to think about ways to condense their activity sequence as explained in the previous sentences. As a result of this condensation, they were pushed toward thinking about the main steps to be taken in finding the result of a given division of fractions problem. These main steps are: (1)
identifying the dividend, (2) identifying a single divisor; (3) quantify the number of
divisors within every part of dividend. The first two condensed activities result in
dividend and divisor quantities with same denominator. The third condensed activity
results in division of numerators.

In this way, by condensing the activity sequence they had and generating as few
steps as possible, their activity sequence would match with the two aforementioned goals
and reflection on these goals resulted in the learning of the CDA. This learning occurred
as explained in the following paragraphs.

5.4.2. Steps in Developing the CDA

Once the participants solved the first problem, I asked them to tell me what they
wrote for each activity and the corresponding result, and then I was their hands drawing
what they directed me to draw. After the first problem, the discussion revolved around
the activities and the corresponding results without going into actual drawings. Their
solutions consisted of two-way partitioning (horizontal and vertical partitioning of the
dividend) and they paid considerable mental attention to the referent units and the
involved multiplicative relationships. They were able to anticipate the results of the
hypothetical activities to be taken in representational world without physically working in
that environment. For example, for problem 3 of Part II \[ \frac{14}{15} \div \frac{3}{8} \], the participants
explained what steps to take and the corresponding results as follows:

R: Okay, what did you do?
W: Draw a whole [Nancy laughing].
R: Okay Nancy?
N: And then divide the whole into fifteen equal parts vertically.
R: Uh huh.
W: And we have fifteen-fifteenths. And we shaded out one fifteenth, so we have fourteen-fifteenths.
R: Okay.
N: Then you divide the whole into eighths horizontally. ... you need to use entire whole. So then you have a hundred-twenty hundred-twentieths in the whole, in the entire whole.
R: Okay.
W: And I said to disinclude [sic] the hundred twentieths in one fifteenth, which is [eight] then you would end up with hundred and twelve one twentieths [112/120].
R: Okay.
N: And then I said count the number of one hundred twentieths in three eighths. And that would be forty five one twentieths [45/120].
R: Okay.
W: Hmm hmm, yeah. That’s what I have. And then mark off every forty-five one-twentieths [1/120] in one twelve one twentieths [112/120] and I got two whole like forty five one [hundred twentieths].
N: Hmm hmm. And then I said ... how much of a, wait, of thirty-five twentieths [25/20] is remainder. And then I said it was twenty-two one-twentieths [22/120]. And then figure out how much of three, well are we allowed to use three eighths? Or no? I guess it doesn’t matter.
R: It doesn’t matter.
N: Yeah.
R: So you have twenty-two one-twentieths remainder [22/120]?
N: Hmm hmm.
R: Once you count out those twenty-five ... one-twentieths.
W: Well if you do that, that would be the, that would be the remainder. But at the end, you should have twenty-two forty-fifths [22/45].
R: Why?
W: Because that’s how much of another three-eighths that we have like extra. So it’s two twenty-two forty-fifths [referring to $\frac{22}{45}$] is our answer but if you wanna bring in neither, it would be twenty-two one-twentieth [22/120].
R: So, it is either two twenty-two forty-fifths [referring to $\frac{22}{45}$] what?
W-N: Three eighths.
R: This many three-eighths.
W: Yeah. And then two plus your other twenty-two one-twentieths [referring to the $2R(22/120)$], is your, which is your remainder.
R: Or two remainder [referring to the format $2R(22/120)$]. Okay, now uhhh we are gonna create an algorithm out of this, okay?
W: Hmm, okay.

As seen in the above dialog, even though the numbers were increased, the participants were still able to think about the activity sequence they had and apply it efficiently to the problem $\frac{14}{15} \div \frac{3}{8}$. However, they were still working in the numerical domain and not attempting to think about ways to consider the problems with an algorithm. They were able to go through the same activity sequence even for an additional problem like $\frac{64}{21} \div \frac{4}{7}$. At times, when they thought they were having trouble, they reminded themselves about the overall goal for the problem and refocused on the solution process. In this last additional problem, they did not find the common denominator by focusing on the common factors both denominators shared. On the contrary, they used one way partitioning (vertical) to think about $\frac{1}{21}$ 'ths and another way partitioning (horizontal) to unitize the dividend. In this sense, it seemed that they were able to execute the activity sequence they already had. However, they did not reflect on the parts of the activity sequence to formulate a way to think about the division-of-fractions problems more efficiently since they did not have an abstraction of the CDA yet.

5.4.3. Taking the Risk of Going into Algebraic Formulation

To move them toward a more reflective path on the activities, I asked them to formulate a generalization to think about such division problems. They chose the hard
way to go by focusing on a general form $\frac{a}{b} \div \frac{c}{d}$. They tried to think about this form pictorially and I let them go through this process of making sense of $\frac{a}{b} \div \frac{c}{d}$. After some discussion and negotiation, they were able to identify $\frac{a}{b}$ and the nested $\frac{c}{d}$ within $\frac{a}{b}$, but they did not have a way to pursue the solution after this point.

There was an instructional flaw about the setup of this part of the session. I let the participants use algebraic approach to develop the CDA. However, such an algebraic formulation requires that they already have, the abstraction and they are just in need of representing it. They did not have the abstraction of CDA that I was leading them toward. In other words, they did not have the abstraction that the process they went through was about finding the common denominator of fractional quantities and then dividing the numerators. Without having such an understanding, when they were asked to proceed after they represented the fractional quantities (dividend and divisor), they did not have a representation for how to make sense of the rest of the process.

Also another problem with my instruction was that at different points in time I was leading them step-by-step without thinking about a model of how such a development could have taken place. Therefore, they were trying to represent every activity they would go through algebraically, as opposed to stepping back and reflecting on the goals for the activities and how those goals were impacting the results they were getting. I was leading them toward a compartmentalization of the activity sequence they had, as opposed to helping them reflect on actions they engaged in.
By the end of T9, they were able to represent both the numerator and denominator algebraically, but they were not able to proceed further. Because of the time limitation, I ended T9 at that point. The focus on CDA continued in the T10 as explained in the following section.

5.4.4. Focusing on the Numeric Aspects of the Algorithm

T10 consisted of three initial problems in Part I, followed by another three in Part II. See Appendix H for a detailed description of the tasks asked during T10. For Part I, the participants were not allowed to solve the problems. On the contrary, similar to T9, they were asked to think about the steps they would take if they were using diagrams in solving the given division of fractions problems. When they solved the problems, they were to answer several questions as illustrated in Table 5-3.

Table 5.3: A sample question from T10.

<table>
<thead>
<tr>
<th>( \frac{3}{2} \div \frac{2}{5} )</th>
<th>What is the goal of this question?</th>
<th>What needs to be done?</th>
<th>For what purpose?</th>
</tr>
</thead>
<tbody>
<tr>
<td>STEP 1</td>
<td>...</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

For each step, they were to identify the specific goal, the action to be taken and the purpose of that action. If the change in the type of the quantities affected the goal, they needed to restate the goal in the appropriate column. For example, when the common denominator for the given fractions was found, the initial overall goal (finding
number of \( \frac{2}{5} \) within \( \frac{3}{2} \) was to be changed to “finding the number of \( \frac{4}{10} \) within \( \frac{15}{10} \).”

The purpose for following such a method was to push them to think about why they were doing what they were doing rather than having them go through additional similar type problems. Previously, they went through certain activities but they did not question the rationale for those activities until I intervened. This task sequence was intended to have them reflect on those rationales for their actions. What follows is an example of how they went through one of the problems, \( \frac{8}{3} ÷ \frac{3}{4} \).

R: Okay you tell me I am gonna write it down [talking to Wanda].

W: How many three fourth in eight thirds.

R: Okay. What are your first couple steps?

W: Um draw three wholes and divide it [referring to each] into thirds.

… You shade out one third. [the result was stated as to get 8/3] … Um. Then you still have the same goal.

…

R: Okay. Next?

N: I said. Wait, okay I said, to find, my goal is to find the number of fourths in eight thirds.

…

N: So I drew horizontally, or I split them into fourths horizontally. All those.

R: … Okay why?

N: Um to have one fourths to group together to make groups of three fourths.

R: So to make groups of three fourths [writing on the third column]. But your overall goal still stays the same?

N: [approving]

R: Okay. Wanda what [is] next?

W: Um, count how many twelfths in eight thirds.

…

R: Actually you don’t have eight thirds anymore. What do you have?

W: Twelfths.

R: How many twelfths do you have?

W: Thirty-two twelfths.

R: In the whole [referring to combination of all rectangles] or just this piece [pointing to the dividend part of diagram]?
W: In just that eight thirds.

... 

R: Okay, what is next?
N: I said, find the number of nine twelfths in thirty two twelfths.
R: [writing on the board]. Okay. What?
N: So I grouped together nine twelfths as many times as possible.

As seen in the above episode, the participants basically went through certain mental activities. They initially identified the dividend [drawing enough rectangles, partitioning them, and marking enough of them to identify dividend], identifying the divisor [repartitioning the dividend and grouping enough partitions within the dividend to identify divisor], counting the number of divisors within the dividend [grouping a number of partitions that make a full divisor group, counting such full groups], and if there was a remainder, they made a multiplicative comparison between the divisor and leftover, and noted the result of that comparison as the fractional part of the quotient. Once they identified the dividend and then the divisor by unitizing the dividend, they actually found the common denominator of the divisor and the dividend. When they counted the number of divisor groups (certain number of partition groups) within the dividend (total number of partitions in dividend), they actually counted a number of partitions within total number of partitions, which was same as dividing the numerators of the dividend and divisor.

As they went through this sequence, they began to see some numeric pattern among the results.

R: So you began there [pointing to $8/3 \div 3/4$] you ended up here [pointing to $32/12 \div 9/12$]. Then you said this is three plus five fifths or thirty two nineths.
N: [laughing] Well I see some connections between some of the numbers.
R: What are they?
N: Well, I don’t know if it is just coincidence but it’s thirty two over nine and there is a thirty two, like you can cross out the twelfths and then there would be thirty two divided by nine.

R: So like this [circling 32 and 9 in the division 32/12 ÷ 9/12] so we began with 8/3 divided by 3/4 it turned out that it is thirty two twelfths divided by nine twelfths. And then we said thirty ninths or this. Okay. Are we allowed to cross out those twelfths?

N: No. Oh wait.

R: What does it mean to cross out those twelfths in this case?

N: Well, since you are both being divided by the same thing, can you just divide them by each other?

W: It works.

N: But.

W: … I mean thirty two twelfths and thirty two are definitely- but I mean it’s still- I don’t know.

This realization was based on their attention to the numeric patterns among the results of their spontaneous activities since they also agreed that they did not know why there would be such numerical pattern. Another instructional deficiency at this point was that I let them look for a numeric pattern among the results they had, which was leading them toward empirical abstraction as opposed to reflective abstraction.

The rule they used, at this point, consisted of finding the common denominator and canceling out the common denominators. They derived this rule from the numerical pattern, but they did not know the rationale for such a rule yet. For another problem, \( \frac{22}{5} \div \frac{2}{3} \), they went through a similar activity sequence and generated a result. When they were asked about the reason for changing the nature of dividend and divisor (through unitizing), they reasoned as in the following episode:

N: So you are working with the same like the wholes that are divided into same number of parts.

W: Hmm hmm.

R: Like in this case, fifteenths?

N: Yeah. Instead of working with fifths and thirds.

R: So this [pointing to 66/15] tells us what?
N: That tells us what twenty two fifths.
R: Twenty two fifths and two thirds [writing \( \frac{2}{3} \) next to \( \frac{10}{15} \)]. Why didn’t we focus on these [pointing to \( \frac{2}{3} \) and \( \frac{22}{5} \) in \( \frac{22}{5} ÷ \frac{2}{3} \)] and moved to here [pointing to \( \frac{66}{15} \) and \( \frac{10}{15} \)]?
W: What she said.
N: Because there it was just hard to figure out like equate thirds and fifths together.

As seen in the above episode, they seemed to understand the rationale for finding the common denominators as generating same size units on which the divisor and dividend were based.

5.4.5. Focusing on the Rationale for the Explored Numeric Pattern

When they went through the individual activities and the results of those, the numerical values they encountered for the divisor and dividend seemed to have the same denominators. They realized that the denominators of both quantities were being equated numerically.

As illustrated in the last episode, it was coming together for Nancy as she made some reflection on the activities and the results associated with those. She came to realize that the comparison between \( \frac{2}{3} \) and \( \frac{22}{5} \) was not as easy as the comparison between \( \frac{66}{15} \) and \( \frac{10}{15} \). In one case there was a common denominator whereas in the other case there was no common ground to compare the two fractions. In the first case it was easier for them to think about the involved multiplicative relationships whereas in the latter case identification of the multiplicative relationship was impossible in a diagrammatic
approach. When I asked them about the relation between $\frac{66}{15}$ and $\frac{22}{5}$ in a diagrammatic environment, their response was:

N: Well you divided the twenty two fifths into thirds, so there is three times as many pieces. Does that make sense?
W: Yeah. Hmm. Interesting, interesting.
R: What is interesting?
W: I don’t know I didn’t think about it that way.
R: What way?
W: What she just said.
N: Like we have here, we have five fifths but you divide each fifth into three parts.
R: Hmm hmm.
N: So, and so the fifths, you have fifteen and so that twenty two pieces we have sixty six pieces. But because both of them stay the same I mean.

They were basically evaluating their understanding of equivalent fractions concept. In this setting, Nancy realized that repartitioning an already partitioned quantity by a certain factor (e.g., 7) requires a relative proportional increase between numerator and denominator. Their call on equivalent fractions was important since it was the basis for understanding the rationale for finding a common denominator. The next step for them was to develop an understanding of the second part of the algorithm: dividing the numerators. This is investigated in the following section.

5.4.6. Making Sense of Dividing the Numerators

Nancy and Wanda observed that dividing the new equivalent forms of dividend by divisor would give the same result as dividing the numerators if the denominators were same. They initially were thinking about a canceling method that they had familiarity with from probably their early schooling. However, I encouraged them to
think back to their diagram activity so they could abstract an understanding of why this relationship existed.

R: Okay, now. Why are we dividing sixty six by ten? You are saying we are canceling these out, how does it appear in the diagram?

W: I don’t know we kind of know we are working in fifteenths so.

R: Okay you are working with fifteenths but why would you divide sixty six by ten?

N: Well, because there are sixty six total pieces that we’re working with. And we are grouping uh ten pieces together.

W: As many times as we can.

N: As many times as we can.

R: Okay. How is it related to sixty six divided by ten?  

N: Because that would be the same thing as dividing sixty six by ten.

R: What does sixty six \(\text{divided}\) by 10 tell us?

W: It says how many groups of ten are in sixty six.

R: Okay [writing on what Wanda said on the board].

N: It’s kind of exciting to me.

... 

N: Like everything is in fifteenths. Like both when we look at sixty six, it’s sixty six fifteenths in the whole thing. And we want groups of tens, ten fifteenths, so.

R: So you are trying to figure out number of ten fifteenths in sixteen \(\text{sic}\) fifteenths, which is same as--

N: How many tens are in sixty six.

Based on the diagrammatic representation of grouping \(\frac{10}{15}\) partition within \(\frac{66}{15}\), they seemed to think that the actions both divisions required \(\left(\frac{66}{15} \div \frac{10}{15}\right)\) versus \(66 \div 10\) were the same. In each case, there was a grouping action of ten pieces. And therefore they thought that both divisions resulted in the same answer. They considered the denominator as the common size of the pieces. The “How is \(\frac{66}{15} \div \frac{10}{15}\) related to 66\(\div10\)?” question pushed them to think about the relationship between those two expressions. They did the comparison based on the goals for both fractions. But then, since they were working with
same size pieces, this realization led them to think about the process for both division cases as investigating 10 objects within 66 objects since the size of those objects was already same.

The subsequent problem I gave them was \( \frac{23}{24} \div \frac{3}{7} \). Their first inclination was to identify the given dividend so they noted that they needed a whole that was partitioned into 24, and 23 partitions out of it were marked. And then they needed to repartition the same whole into 7 parts and the number of pieces in the marked area (24x7-7) would give the number of pieces in the dividend. The reason for such result was:

R: Why is that again?
N: Because if you had your whole divided in twenty fourths [making vertical partitions in the air with her hand] and you divide it by sevenths this way [making horizontal partitions in the air with her hand], you’d be like taking away one of those twenty fourths which has seven pieces in it. So--
W: So subtract seven [from] one sixty eight.
R: So one-sixty-one one-sixty-eighths. What is that?
N: The dividend [speaking simultaneously with Wanda].
W: That is twenty-three twenty fourths.
...
R: Okay. Is there any other way to figure that out other than subtraction?
N: By using those numbers?
R: By using those numbers, how would you do that?
W: Seven [distantly- pause]. Twenty three times seven, right?
N: Yeah.
R: Why?
W: Because you only are working with twenty three of the twenty fourths and each of those are in seventh so that would encompass everything except for the twenty fourth.

Once they suggested identifying the dividend, the next step they confronted was the identification of divisor.

R: Okay. So this is your dividend. What else do you need to do?
N: Find the divisor.
W: Find.
Okay. What is the divisor? Think about the diagram.

Three sevenths [thinking aloud]

It’s three of the sevenths that we make horizontally and--

It’s seven times twenty four.

It’s seven times twenty four because we have twenty fourths [making vertical movements in the air] and like if they are divided horizontally [making horizontal movements in the air], you just want three of those [seven] twenty fourths.

Yeah whatever that encompasses.

... But it’s twenty four across so there are twenty four in each line and we want three of those [referring to each row of twenty four pieces].

Yeah.

Okay, so, so.

Eighty two. Oops. Seventy two.

Seventy two what?

Seventy two one sixty-eighths.

Okay.

What is it?

That’s our divisor.

And the final step for them was to group each divisor group \(\frac{72}{168}\) within the dividend \(\frac{161}{168}\). They explained the rationale for it as follows:

Okay what do you need to do now?

Um, group them.

What is the goal here now?

How many seventy-two one-sixty-eighths \([\frac{72}{168}]\) are in one-sixty-one one-sixty-eighths \([\frac{161}{168}]\).

[writing it on the board] How would you figure that out?

We would keep adding seventy two until like go over and then like go back.

You want to see how many seventy twos are in one sixty one. So divide one-sixty-one by seventy-two. I mean it’s the same thing.

Yeah, yeah.

Why are you dividing it?

It’s the same thing as counting seventy two in like counting in the one-sixty-one.

To see how many times seventy two goes into it.
Once they explained their activities and the results they would get from those activities, they mainly pointed to two outcomes: finding the common denominators and dividing the numerators. This realization came from their treatment of the activities to generate dividend and divisor as single entities. They knew that their initial goal was to determine dividend even though it might include several steps to reach that goal. The next goal for them was to identify the divisor even though it might mean a new set of activities. Once the dividend and divisor were determined this way, their new goal was to identify the multiplicative relationship between the divisor and dividend. However, this time such identification was easier since both quantities were based on same size partitions. The partitioning they did so far to figure out divisor and dividend resulted in two quantities of the same type to be multiplicatively compared. Their abstraction at this point was that the identification process was about simplifying the multiplicative comparison between the given two quantities (divisor and dividend). They also had dividend and divisor as two single entities to be compared. And the problem at this point was to make sense of that multiplicative comparison. Conceptualizing the divisor and dividend as single entities led them to abstract the multiplicative relationship between those two quantities as manifestations of finding one object (of a certain size) within another object (of the same size), which was about division of numerators.

5.4.7. Coordinating Both Parts of CDA

Later on in the session, with my leading, we set up an algebraic representation of what they were describing as \( \frac{a}{b} \div \frac{c}{d} = \frac{ad}{bd} \div \frac{cb}{bd} = \frac{ad}{cb} \). Now the issue was to have them
reflect on “whys?” as opposed to “hows?” for such an algorithm. Therefore, I asked them about the reason for finding the common denominators and wanted them to base their argument on what they had done so far. Their response referred to the multiplicative relationship as follows:

W: Like we were working with like least common multiples pretty much. Like the twenty four and the … seven. But uh if you did like, like the next [pause] It would be the next thing both twenty four and seven go to right. It would be the next but the numbers would be not the same but they would be like same --

N: Proportionate.

W: Yeah, they would be proportionate. And you could still end up with--

R: What does that mean? You are using that word I am not familiar with?

N: What?

R: Proportionate, what does that mean?

W: Like they would end up still with [pause]-

N: Like if you have a given unit, I don’t know. [pause] If you have the same whole, if you have a whole and just divided it into twenty fourths … and then you have the same size whole and it’s divided into one-sixty-eighths. Twenty-three of the twenty-fourths is equal to one-sixty-one of one sixty-eighths. Like they would cover the same amount.

R: So you are saying that--

W: So like there are other fractions, you could still divide that further that would still cover that same amount in both kind of fractions.

…

W: Yeah like that’s the same amount ultimately.

R: So both shaded pieces have the same size?

N: Hmm hmm.

W: Yeah.

As seen through their wording in the above discussion, their new focus was on the proportion defined by the given fractions as well as the size of those fractions. They referred back to their understanding of equivalent fractions and re-articulated that understanding in this context. In this sense, to the participants, working with two fractions was the same as working with equivalent forms of those same fractions. As a result of this articulation they were able to make sense of the first part of the algorithm.
The second part of the algorithm was about dividing the numerators. They already explained that “grouping certain size subdivisions” was the basis for division. As long as one was grouping certain size pieces like \( \frac{10}{15} \) within \( \frac{66}{15} \), grouping \( \frac{10}{15} \) within \( \frac{66}{15} \) was conceptually the same as grouping 10 within 66 to Nancy and Wanda. Also as they stated earlier, their diagram gave this away, too, because they were grouping 10-piece sections within a 66-piece section where the pieces were of the same size.

To sum up, they had a well articulated activity sequence that they could refer to, whenever needed, prior to this session. The only problem seemed to be that they did not reflect on the purpose of each activity in their activity sequence. On the contrary they seemed to be reflecting on the results of the activities and the pattern among those results. They had following activity sequence:

1. Identify dividend by partitioning enough number of units → resulted dividend
2. Unitize dividend → resulted unitized dividend
3. Identify divisor according to new partitioning → resulted divisor
4. Group divisor sections within dividend → partitioned dividend with respect to divisor groups
5. Find (e.g., count) the number complete divisor groups within the dividend → number of complete divisor sections
6. Make a multiplicative comparison between divisor and remainder → fractional part of the quotient
7. Combine whole number result with the fractional result from steps (5) and (6) → quotient.
When those results were evaluated in the numeric domain, they were able to identify a pattern based on those numerical values. However, without any reflection on the relationship between the activities and the results they were just generating ways to think about the transition between different steps.

When they were directed to think about what they did and how that made sense in lieu of the purpose, they seemed to generate a rationale for each activity and for the pattern among the results. Therefore, it seemed that it was important for the participants to keep in mind the goal of each step and make their comparison based on the tri-set: goal-activity-result. Once they each had this kind of reflection, then, they were able to think about the numerically generated algorithm independently of its numeric base.

The results of post-interviews also revealed the participants’ understandings of the CDA. The following section articulates their understandings as seen through their work in post-interviews.

### 5.4.8. The Participants Understanding of the Meaning of Algorithm as Seen Though Their Work in Post-Interview

The participants were asked a question that consisted of an algorithm for a specific example as follows:

**Problem 8:** Mary claimed that to divide two fractions, you change all mixed numbers into improper fractions, find common denominators, and then divide the numerators. For example: \( \frac{4}{5} \div \frac{2}{3} = \frac{19}{5} \div \frac{2}{3} = \frac{57}{15} \div \frac{10}{15} = \frac{57}{10} = 5 \frac{7}{10} \). Will this method always work?
In order to answer such a question, they needed to know that there was a reduction of fractional division to the whole number division. And they also needed to know that this reduction was possible by making both the divisor and dividend quantities which referred to the same entities (unit types).

Wanda was aware that the first part of the algorithm was about equivalent fractions and she explained it as:

W: [pause for 5 seconds] Um, because nineteen fifths and three and four fifths. Although they are in different forms they still represent the same amount of something.
R: Okay.
W: And two thirds and two thirds obviously still represent the same amount.
R: Okay.
W: And since those represent the same amount you need to put them in like the same proportion than like so that you can like see them like side by side like as equal things. So that’s when like the fifteen, finding the common denominator would do that.
R: Would do what? Why are we finding common denominator?
W: Would make it, would make the um the parts, the pieces like equal. Like one of the fifty seven is equal to one of the ten. Here. Do you know what I mean? They are the same size. But there is just more of those here than here, so we are just dividing now and I think that way --
R: Why are we finding common denominator?
W: So that we can look at them in the same way. Compare the quantities because we know that they are the same size pieces, you know what I mean, in each like --

As seen through her wording, she referred back to her equivalent fraction understanding. In a sense, she was also referring back to the diagrammatic approach she would use for such a division problem. In this way she knew that finding equivalent form of a fractional quantity did not affect the size of the quantity at all. In this manner, to Wanda, it was possible to turn the given dividend and divisor into improper fraction mode. And since the goal for the division problem stayed the same, this change would
not affect the result. She also seemed to be aware that she needed to make a multiplicative comparison between the two quantities, and the comparison could easily be done when the involved quantities are based on the same size fractional units.

Nancy’s reaction was not much different from Wanda’s in interpreting the common denominator step:

N: Like before it was fifths and thirds. Fifths of, if you look at the same whole, fifths are smaller than thirds. So you can’t really compare fifths and thirds. But fifteenths, I mean if you find the common denominator so you do change the numerators but they remain equivalent like the new numerators here, they are equivalent to the prior fractions but now you have the same base. So the fifteenths are the same size as these fifteenths. So you don’t even really have to worry about the size of them. Just the number of things that are being divided. Like the fifty seven divided by ten.

In this above episode, she referred to the fact that changing the number of partitions proportionally for a fraction did not affect the size of the fraction.

When the issue was to explain the rationale for the second part of the algorithm, Wanda reasoned as follows:

W: So we just, up here [pointing to second equality sign in the given solution] just tells us how many of these fifteenths we have and since we are using those we don’t need the fifteenths anymore since we know that we can just do this.
R: Why do we not need the fifteenths anymore? Say it again.
W: Because we just needed that to make our, sought our pieces, our things of equal amount out [referring to groups] of the same here. To see how these two [pointing to new divisor and dividend] relate in terms of like quantity and amount, or whatever.
R: Okay you are using several different things I need to make sense of. So fifty seven fifteenths versus ten fifteenths. All of a sudden you are disregarding fifteenths. Why?
W: Because I mean since out of fifteenths, one of these ten fifteenths is equal to one of these fifty seven fifteenths.
R: Okay, but why are you disregarding those fifteenths in the problem?
W: Because we don’t need them [sarcastically].
R: Why do we not need them?
W: Because we just needed them to, to, to find out where an equal amount would be out of, or where the same size pieces would be between these so that we could compare these two numbers.

R: So if you just look at this problem fifty seven fifteenths divided by ten fifteenths what is it asking?

W: How many ten fifteenths are in fifty seven fifteenths?
R: Okay. So you are telling me how many ten fifteenths are in fifty seven fifteenths –

W: Right and since and since there are fifteenths all we are doing is counting how many tens anymore. We're not saying okay here is one ten fifteenth, I mean we could say that but we don’t need to, we can just count by tens and see how many times.

Even though it was not so strong, Wanda based her rationale for dividing the numerators on the fact that she was using the same size pieces. I think Wanda was thinking about her diagrammatic approach and thinking about the activity she would go through for such a step. Since she already identified the dividend and divisor, she seemed to know that the division $57 \div 10$ was conceptually and procedurally same as the division $\frac{57}{10} \div \frac{15}{15}$. In addition to Wanda, Nancy pursued a similar reasoning to make sense of the division of numerators.

As a result, it seemed from their behaviors, when doing the post-interview tasks, that the participants had an understanding of the common denominator algorithm and were able to provide rationale for each step of the algorithm. They seemed to know that having a common ground for both given fractional quantities was a way to reduce the complexity in the given division of fractions problems. They also knew that in this way, one could think about the quantities (divisor and dividend) as objects of a certain size. And, as long as the size of the objects matched with each other, they would think about
the investigation of number of one object within another in different ways ("number of \( \frac{2}{3} \) in \( \frac{4}{5} \)" = “number of \( \frac{10}{15} \) in \( \frac{57}{15} \)” = “number of 10 in 57”). In this sense, the appearance of the object did not affect the overall goal and functioning of the operation for the problem.

Thinking about the \( a \div b \) as “\( a \) of something within \( b \) of something of the same type” regardless of whether they are whole numbers, complex numbers or fractions seems to be a significant way of interpreting division problems. This kind of view in which one considers the quantities as entities that needs to be compared makes division conceptually richer than thinking about it as “How many \( b \) are in \( a \)” only. These “\( a \)” and “\( b \)” need to be considered in two ways: the value they hold and the entity they refer to. When one considers these two ways in advance, then, this reduction process from fractional dimension to the whole number dimension becomes easier.
Chapter 6
Discussion of Conclusions

The purpose of this study was to investigate prospective elementary teachers’ understanding and conceptual development concerning division of fractions. This investigation resulted in: (1) a characterization of prospective elementary teachers’ understanding of division of fractions; (2) a characterization of the developmental trajectory for division of fractions concept; (3) a detailed analysis of remainder concept and how it can be conceptually developed; (4) development of an algorithm using the case of division of fractions; (5) an articulation of the key understandings and operations for division of fractions; and (6) instructional design.

This chapter serves the purpose of discussing the major findings of the current study. What follows is an examination of each of the above points.

6.1. How do prospective elementary teachers conceive of division of fractions?

The current study reveals that the prospective elementary teachers have a compartmentalized understanding of division of fractions. The way they operate in whole number division situations and fraction division situations helped me characterize their understanding in general. Their conceptions in fractional settings are outlined below:

1. They have a general definition for such DoF problems as: “division of fractions means how many of one quantity can go into another quantity.” In addition to this definition, they also have an algorithm (invert-and-multiply algorithm) to which they continuously refer.
2. When they encounter mathematical expressions such as $a \div b$, they interpret them as “How many $a$ can go into $b$?” and then move on to using the invert-and-multiply algorithm (hereafter called as IMA) to generate a solution. In contextual settings, when pushed to think about the given situations without using IMA, they either follow the same method they do for mathematical expressions, or based on the contextual referents and the given two factors (dividend and divisor) they think about repeated combination of one factor to make up the second one. For instance, when the problem is about finding the number of loaves of bread that can be made from $x$ pounds of flour given that each loaf of bread requires $y$ pounds of flour ($x$ and $y$ are fractions), they can think about repeated combinations of quantity $y$ to use up the quantity $x$. In this sense, they benefit from the numeric relationship involved between the $x$ and $y$ as opposed to thinking about involved quotitive structure. At times, they use their understanding of multiplication to think about this numeric relationship between the two factors that gives the third one.

3. They can recognize division of fractions problems based on their definition. When solving such problems, they state their definition as an overall goal, and then resort to IMA as a way to reach that goal. Even though they refer to IMA to generate a solution to the given division of fractions problems, they do not know the rationale underlying it. These two sources (definition and IMA) seem to be the main sources they rely on to think about the division of fractions problems.

Although they have this certain way of operating when working on mathematical expressions that are based on DoF, they have tremendous difficulty in creating their own DoF word problems. Their attempts to generate such word problems result in a transfer of
the given mathematical expression to some context that is not consistent with the required mathematical meaning.

On the other hand, their comfort level with whole number division problems is adequate as detailed below:

1. They can easily recognize division structure in the given whole number problems. Whether the problems they encounter are partitive, quotitive, or contextual, they can identify the structure of the problems as division.

2. They can work with both the partitive and quotitive division problems in whole number settings. Creation of such problems in whole number settings does not cause any difficulty for them either.

3. In whole number settings, their focus is on the arithmetic relationships. Based on the given two of the three factors, they are able to determine the third one using the involved numeric relationships.

From all the aforementioned clues and in light of the data, it seems that prospective elementary teachers’ abstraction of the division concept is dependent on arithmetic relationships as opposed to quotitive ones. They do not seem to have an abstraction of quotitive situations. Although, at times, they can conjecture a way to find solutions to DoF word problems using the contextual quantities, the way they approach problems is based on their understanding of arithmetic relationship between the given two quantities, dividend and divisor.

This kind of lack of abstraction of the quotitive situations causes difficulty for the teachers in creating division of fractions problems. Since they do not have an abstraction
of quotitive situations, they cannot identify such situations based on the provided quantities (divisor and dividend). What they can do is to think about the relationship between those two quantities numerically and try to fit an operation (mostly multiplication) to identify it. In addition, the deficiency in their abstraction of quotitive situations results in their inability to have a conceptual representation for division of fractions problems.

### 6.2. Developmental trajectory for division of fractions

1. Initially, the participants’ abstraction of division of fractions was an abstraction of numerical relationships among the quantities as mentioned in the previous section. They did not have the abstraction of quotitive situations. They also had a general definition for division of fractions to which they initially referred when starting their solution process. To help them have an abstraction of this definition that they had, I designed a task sequence that restricted their activities to the use of diagrams and nothing else (no computation, no algorithm). This task sequence included several DoF word problems that referred to the common goal of, “How many ___ are in ___?” In going through such contextual problems, the participants were implementing a specific activity sequence as in Table 6-1.

<table>
<thead>
<tr>
<th>Activity</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) Draw the given fractional, or whole number, or mixed number quantity.</td>
<td>Dividend</td>
</tr>
<tr>
<td>(2) Partition dividend in order to identify second given quantity.</td>
<td>Divisor embedded in repartitioned dividend</td>
</tr>
<tr>
<td>(3) Mark off complete divisor sections</td>
<td>Dividend consisting of shaded-in full</td>
</tr>
</tbody>
</table>
Each time they completed the same activity sequence for the given problems, their attention was directed toward the overall goal in the problems. In this activity sequence, they were basically proceeding each contextual problem and determining the number of one quantity within another based on the diagrams. However, they did not reflect on what was common among the activities they implemented. This was due to their overall goal: solve the problems based on the context (e.g., looking for number of $\frac{1}{8}$-cup of sugar in $\frac{1}{2}$-cup). They continued to focus on the numeric relationships based on the diagram work they did while experiencing this set of contextual DoF problems.

After running through such a sequence, they were asked to look for the commonality among the given problems. When looking for the commonality, they reflected on each activity in the sequence and how those activities served them to reach their results. The mental activities they engaged in the first problem were parallel to that of the second problem and so on. This type of reflection helped them to identify the main operations of the process they pursued. In context, they abstracted the idea that each time to reach their goal for each problem; they partitioned a given quantity and counted the number of partitions in that quantity. They did not have such an abstraction before

<table>
<thead>
<tr>
<th>Within the dividend as much as possible.</th>
<th>Divisor sections with some unshaded leftover pieces</th>
</tr>
</thead>
<tbody>
<tr>
<td>(4) Compare the leftover piece to one divisor group to figure out how much of a divisor group it corresponds to.</td>
<td>Fractional correspondence of divisor</td>
</tr>
<tr>
<td>(5) Consider the whole number of divisor groups and fractional number of divisor groups together to announce total number of divisors within the dividend.</td>
<td>Total number of divisors</td>
</tr>
</tbody>
</table>
entering the sessions. They also reflected on the commonality among the goals for each problem. Through comparison of the goals, the participants abstracted the notion that each problem has the goal of identifying the number of one quantity within another (based on the real world quotitive situations). However, they did not see them as coordinated yet. It was just a process they completed to reach a certain goal.

As a result, this process, based on the activity sequence of the participants, involves two main operations: partitioning a given quantity with regard to another quantity, and quantification of that partitioning. These two operations are the basis on which to reflect in order to get an abstraction of division of fractions. The Figure 6-1 illustrates the structure of division of fractions.

![Figure 6.1: General structural scheme of division of fractions.](image)

In this structure, central to DoF, is the multiplicative relationships between quantities. A more detailed discussion about these multiplicative relationships will be given in the subsequent section. These multiplicative relationships make sense based on two operations: partitioning of one quantity with regard to the other, and quantification of
that partitioning. And these operations are guided by the set overall goal for the division of fractions problems.

2. Even though they abstracted the idea that partitioning and the quantification were the main operations that served one to reach the goal of identifying the number of one quantity in another quantity, their abstraction was based on contextual problems. They did not learn to call on their abstraction for the purpose of generating word problems yet. However, they increased their level of understanding based on the comparison between the two activity sequences they worked on. The following paragraphs articulate this conceptual advance on their part.

Previously, as a result of their work with DoF problems, they made a generalization for DoF that was not enough to create their own contextual problems. This seemed to be because they did not have an abstraction of division of fractions as coordination of two operations (partitioning and counting) guided by the overall goal. However, during the subsequent session (T2), they did the same type of reflection on two different settings: MwF and DoF. They first made an internal comparison of the activities involved in MwF and abstracted the idea that it was about identifying part of a given quantity. Then, they reran their activity sequence for DoF and reached the abstraction they obtained previously. Then, they compared each activity sequence against each other. Up until this point, they were comparing the activities in similar-structure problems for the purpose of making an abstraction (either for MwF or for DoF alone). However, to compare different abstractions and to make an abstraction of the difference between MwF
and DoF, they were to make a cross comparison between the activity sequences of MwF and of DoF.

Comparison between the different activity sequences helped both participants make an abstraction because reflecting on both sequences at the same time helped them monitor what was common and uncommon between the sets. Thinking about the commonalities and differences among the activity sequences let them look into the essence of those activity sequences and what made the sequences unique within themselves. They were also paying close attention to the overall goals for these two sequences. This type of attention to the overall goals helped them to understand the rationale for those goals. Through comparison, they realized that they were partitioning given quantities in MwF problems for the purpose of identifying part of those quantities. On the other hand for the DoF problems, they abstracted the essence of the identifying number of partitions. Since MwF did not include the operation of counting (in general quantification) this aspect of division became more apparent and dominant in the comparison they made.

This kind of comparison helped them coordinate both operations of partitioning and quantification on which the DoF is based. In this sense, to the participants, the combination of these two operations became an entity that serves the purpose of reaching the overall goal of, “How many ____ are in ____?"
6.3. Understanding and development of remainder concept in fraction division

For participants, the remainder basically is the amount that is leftover after subdividing a quantity (dividend) into divisor groups. To the participants, this process of subdividing to find remainder is easier in a whole number setting than in a fractional setting. In a fractional setting, identification of remainder becomes complex and problematic.

The concept of remainder is a conceptually rich area, especially in fractional domain. It involves understanding unit, referents, multiplicative comparison and coordination of those ideas. School mathematics does not deal with the idea of remainder other than in whole number settings since, in fractional settings, the IMA eliminates thinking about remainder as in the following sense. One using the IMA only inverts the second factor and multiplies it with the first one. In the case that the quotient is a mixed number, this does not give one an opportunity to see the involved multiplicative relationships between the divisor, remainder and quotient. Therefore, one does not have the opportunity to reflect on such relationship.

The most troublesome idea about remainder is that it involves coordination of several intertwined relationships. The main difficulty learners encounter is the difficulty of distinguishing the fractional part of the divisor from the remainder and abstracting the multiplicative relationship between the remainder and divisor. This is crucial in understanding division of fractions concept. There are some required actions, and the coordination of those help one overcome these difficulties as outlined below.
The participants relied on several actions and coordination of those in developing an abstraction of remainder and the involved multiplicative relationship in DoF problems. The first thing is to set the overall goal as: determining the number of one quantity within another. And then the issue is to think about the process of partitioning to which the overall goal refers. The next challenge is to identify and decompose the nature of the quantification of this partitioning. In doing so, there needs to be attention paid to the nature of quotient. If it has a fractional part, then it means that there is a remainder. Hence, the quotient informs the quantification process as to how to measure the number of divisors. In this sense, this quantification process consists of two phases: (1) thinking about the number of full divisor groups; and (2) thinking about the excessive amount as part of another group. Once the nature of the quantification process is completed this way, the final issue is to think about the fractional group itself as part of the quantification process that corresponds to the fractional part of the quotient and to think about the size of the fractional group as remainder. To be able to think about the remainder in a context-free setting, there needs to be a coordination of quantification operation with the quotient and this coordination needs to be interpreted. Such coordination also helps one to make sense of the involved referents.

When going through the above sequence of activities the participants developed some abstractions over time:

1. There are two possible results for a given division of whole numbers problem: quotient-only result (giving the information of total number of
divisors) and remainder result (giving the full number of divisors with
some remainder amount).

2. Remainder and the fractional part of the quotient refer to the same quantity
expressed as a different magnitude and different units.

3. Remainder or fractional part of the quotient can be converted to each other
but they are different quantities.

4. Fractional part of the quotient and remainder are multiplicatively related to
each other.

5. The relationship between the remainder and the fractional part of the
quotient is dependent on coordination of the referents for three quantities
(dividend, divisor, quotient) under the guidance of the overall goal, “How
many divisors are in dividend?”

6. The coordination expressed in (5) requires abstracting divisor as an
intensive quantity that connects two extensive quantities, the dividend and
quotient. This abstraction is the key in understanding the multiplicative
relationship between the quantities in DoF problems.

These abstractions were achieved through the participants work in two specific
settings: contextual-diagram setting and context-free diagram setting. First, they worked
in a contextual-diagram setting and then moved to the context-free-diagram setting in
whole number domain. Then, they followed the same sequence in the fractional domain.
Whether they worked in contextual-diagram setting or context-free diagram setting, they
reflected on the same idea: the connection between the remainder and the divisor and the
involved multiplicative relationship in the given DoF problems. However, the context-
free-diagram setting afforded and fostered development differently than the contextual-
diagram setting. The nature of their reflection in the contextual-diagram setting was
different from the context-free-diagram setting since context-free-diagram setting
allowed them to see the commonality in their own activities. The reason for such a
difference was that context-free-diagram situations were already an abstraction of
contextual-diagram situations. When they started working in the context-free-diagram
setting, they already had an abstraction of their activity sequence and referents.

In the context-free-diagram setting, they compared quantities in the given
mathematical situations. They went through a certain activity sequence, and one of the
activities in that sequence was to compare a number of partitions, they had, to a single
group of partitions. In this way, they worked with generic quantities. For instance, they
looked for certain-size partitions and at some point they found out that they did not have
enough partitions to make up another group of partitions. This led them to question what
to do with the number of partitions (remainder) at hand. This questioning required them
refer back to their abstraction of the referents and activity sequence that they had from
the contextual-diagram setting. In this way, they had to focus on the multiplicative
relationship between the divisor and the remainder that resulted in a fractional part of the
quotient. And as they looked across multiple situations with diagrams, they made an
abstraction of the quantitative relationships (which was the commonality they were
looking for) involved in division.
Since the contextual clues carried a lot of the thinking about what was the relationship between the quantities, the multiplicative relationship toward which they were working was not obvious in contextual-diagram setting. However, a context-free-diagram setting requires an abstraction of the relationship between the divisor, remainder and quotient. In order to think about this relationship, they were to have an abstraction of divisor as an intensive quantity that connects two extensive quantities (dividend and quotient). In a context-free-diagram setting, one already needs to have such an abstraction of the divisor as an intensive quantity in order to think about the multiplicative relationship between the divisor and the remainder.

6.4. Developing CDA

This section explains development of an algorithm using the case of division of fractions. There are studies (e.g., Tzur & Timmerman, 1997) that investigated the rationale for invert-and-multiply algorithm, but they did not investigate development of an algorithm that is based on students’ own activities. This section contributes to understanding how development of CDA occurs.

Prior to developing CDA, the participants worked on division of fractions from different perspectives and made several abstractions. They had abstracted: (1) division of fractions as a whole; (2) that division of fractions is about investigating number of one quantity within another (in other words, multiplicative comparison of two quantities); (3) that there is a network of multiplicative relationship among the divisor, remainder, and
quotient; (4) what equivalent fractions are about (known from pre-interviews); (5) referents and coordination of referents.

The overall purpose for developing the CDA was to help participants coordinate these understandings based on their spontaneous activities. In this regard, they were encouraged to think about and abstract two main ideas in this process: (1) knowing that common denominator results in same size units for divisor and dividend which give rise to the second part of the algorithm, dividing numerators; (2) knowing that when the dividend and divisor are based on same size partitions, quantifying the number of partitions (that make up the divisor) within the dividend is same as dividing the numerators.

To help participants develop the above conceptions, the instructional sequence I used consisted of tasks initially asking them to solve the given DoF problems based on diagrams. In this way, they were to pursue a certain activity sequence, and because the magnitude of the numerator and denominator was increasing from one problem to another, they were to pay attention to the size of the fractional pieces and the quantification process.

Once they went through the problems using diagrams, they were led to think about the algebraic representation of what they did, and then, they were allowed to look for numeric patterns. First, I let the participants use an algebraic approach to develop the CDA. However, such an algebraic formulation requires that they already had the abstraction, and they were just in need of representing it. They did not have the abstraction of CDA that I was leading them toward. In other words, they did not have the
abstraction that the process they went through was about finding the common
denominator of fractional quantities (dividend and divisor) and then dividing the
numerators. Also another problem during my instruction was that at different points in
time I was leading them step-by-step without thinking about a model of how such a
development could have taken place. Therefore, they were trying to represent every
activity they would go through algebraically as opposed to stepping back and reflecting
on the goals for the activities and how those goals impacted the results they got. This way
of operating led them toward a compartmentalization of the activity sequence they had, as
opposed to helping them reflect on actions they engaged in, which was affecting the
process of reflective abstraction.

After going through these two unsuccessful venues, they became aware that there
was a numeric relationship (pattern) between the results of their activities. Since they also
agreed that they did not know why there would be such numerical pattern, this realization
seemed to be based on their attention to the numeric pattern among the results of their
spontaneous activities.

The abstractions they made beyond this point are articulated in the following list:

1. The multiplicative relationship between the given two quantities (dividend and
divisor) is hard to identify without having both quantities based on same size units.

In solving the problems mentally by thinking about the activities they would go
through using diagrams, and the results of those activities, their descriptions had a pattern
in terms of the activity sequence they followed. This pattern in their activity sequence is
outlined below. The left side of the arrow refers to the activity they went through, whereas the right side is the corresponding result.

(1) Identify the dividend by partitioning a number of units and considering a combination of those partitions → Dividend,

(2) Repartition the dividend (unitizing) to identify divisor within it → Unitized dividend,

(3) Identify the divisor within the dividend based on the repartitioning → Divisor,

(4) Group full divisor sections in the dividend → Partitioned dividend with respect to divisor groups,

(5) Find (e.g., count) the number of full divisor groups within the dividend → Number of full divisor sections,

(6) Make a multiplicative comparison between the divisor and the remainder → Fractional part of the quotient,

(7) Combine whole number result with the fractional result from steps (5) and (6) → Quotient.

Each given question in the task sequence had fractional quantities that had much larger divisor and dividend than the preceding problems. The goal was to help the participants think about the essence of this process. In other words, each time they dealt with a larger dividend, they generated much smaller fractional units. However, as the number of fractional units drastically increased, the process of partitioning and quantifying became more and more exhaustive. Therefore, the use of such large numbers
for the denominators encouraged the participants to go back to the process they went through and reflect on that process. Then, they looked for ways to make that process more condensed in an algorithmic way.

In the follow-up session, I modified the instruction in such a way that they also had to think about reasons for why they went through certain activities. In this way, I encouraged them to reflect on their spontaneous activities, results of those and the reasons underlying those activities. The purpose was to have them coordinate these three factors when formalizing an abstraction of CDA. However, at some point, they saw a numerical pattern in the results they had for each problem and they were convinced, numerically, that the algorithm was about finding the common denominator (of divisor and dividend) and then dividing the numerators.

Once they figured out the numerical pattern, I questioned how the algorithm they generated made sense. Looking through their activity sequence and reflecting on the goal-activity-result tri-set, they realized that in order to multiplicatively compare two quantities (divisor and dividend), those quantities have to be compatible, meaning that they should be based on the same referent unit. This was the reason why they were finding the common denominator.

2. *Dividend and divisor are based on same size fractional pieces.*

Once they figured out this rationale, they started to think about a diagram that included a divisor and dividend that were represented by the same size units (e.g., $\frac{10}{15}$ vs. $\frac{66}{15}$). They became aware that the denominators of divisor and dividend represented the
same size pieces within the divisor and dividend. This realization came out as they worked through the given problems mentally with the help of diagrams. In those problems, they were to anticipate the results of their actions. In doing so, they were paying attention to the possible actions they could take and the results of those actions. Going through the same activity sequence several times, they began to outline the main points in their activity sequence. In other words, the actions of drawing rectangular wholes, partitioning them, and choosing the number of partitions to represent dividend, resulted in dividend. Hence, they were getting more conscientious about the activities they went through, and they were grouping their activities according to the intermediate goals they had. For example, in the above outline of actions, the initial goal was to identify the dividend. Therefore, the three actions they took on were meant to identify the dividend. This kind of thinking led them toward considering action groups as single entities serving a particular goal.

Based on this train of thought, the second action group that they outlined resulted in identification of the divisor. At this point, the diagram use helped them to think about and reflect on these two results (the dividend and divisor) and to come up with the idea that they were both dependent on same size fractional pieces.

3. Dividing two fractions is same as dividing equivalent forms of those two fractions.

After realizing that the fractional pieces on which the dividend and divisor are based, the questions such as, “How is $\frac{66}{15} \div \frac{10}{15}$ related to $66\div10$?” pushed them to think
about and reflect on the relationship between those two expressions based on their activity. They did the comparison based on the goals for both division cases. But then, based on their understanding that they were working with same size pieces, they conceptualized the process for both division cases as investigating, for example, 10 objects within 66 objects since the size of those objects were already same. In this sense, the dividend and divisor became objects for them to compare to each other multiplicatively. This type of realization let them make a transition from thinking about

\[
\frac{66}{15} \div \frac{10}{15}
\]
to thinking about 66÷10 serving the same purpose.

To sum up, this process of developing CDA consists of several developmental steps that are based on learners’ spontaneous activities. First, it requires a multiplicative comparison between the given two quantities. To do such a comparison one needs to simplify the given quantities if they are not easily comparable to each other. This simplification process is based on identifying the given quantities and unitizing them to make them refer to the same referents so that they can be easily multiplicatively comparable. This type of unitizing results in the modification of the initially set overall goal. If the initially given problem is \( \frac{3}{4} \div \frac{3}{7} \), for example, then after simplification process one gets \( \frac{21}{28} \div \frac{12}{28} \) which leads one to modify the initial goal according to these newly unitized quantities as: “How many \( \frac{12}{28} \) are in \( \frac{21}{28} \)?” which is same as, “How many \( \frac{3}{4} \) are in \( \frac{3}{7} \)?” Modification of the overall goal sheds light to the multiplicative
comparison to be done between the unitized divisor and dividend. Here, one other
developmental step is that the multiplicative comparison between the \( \frac{21}{28} \) and \( \frac{12}{28} \) is the
same as the one between the numerators 21 and 12 since both are based on the same
overall goal of “finding 12-partition groups in 21-partition (same size) groups.” The
result of such realization takes care of developing a sense for the second part of CDA,
dividing the numerators. Since the investigation of \( \frac{12}{28} \) in \( \frac{21}{28} \) is based on same size
fractional units (\( \frac{1}{28} \)), the same investigation can be considered when looking for 12 units
within 21 units of the same size. In other words, it requires one to think about both
fractional quantities as objects to be compared multiplicatively.

6.5. What are the key developmental understandings and operations for division of
fractions?

The work with two prospective elementary teachers revealed that there are several
key understandings that are important in developing an understanding for division of
fractions concept. Since the real world situations mostly fit into quotitive division
problems in a fractional setting, the investigation conducted for this study is about
quotitive division of fractional numbers.

Division of fractions is a network of multiplicative relationships that are based on
the operations of partitioning and quantification of that partitioning. This section details
the particulars of the structure of division of fractions in a more thorough way than that of Section 6.1.

Division of fractions requires an abstraction of the multiplicative relationship between the divisor and dividend. The initial understanding one needs to have is that the dividend and the divisor are objects to be compared multiplicatively. This multiplicative comparison can be done based on the operations of partitioning and quantifying that partitioning. Section 6.4 also suggests that unitizing is another fundamental operation in this process of identifying the multiplicative relationship. Once the given quantity dividend is partitioned, depending on how it compares to the divisor, there may be a need for repartitioning of the dividend (unitizing). Unitizing is like a bridge between one form of multiplicative comparison (between the initially given dividend and divisor) and its transformed form (comparison between the re-partitioned dividend and divisor). However, at times, use of unitizing is not essential since the given quantities (divisor and dividend) can share similar denominators (e.g., one is a multiple of the other). As a result, partitioning, unitizing, and quantification are the main operations one needs to conceptualize in order to conceptually make sense of the multiplicative relationship between the dividend and the divisor. Use of these three operations serves one to simplify the multiplicative comparison between the dividend and divisor.

The second main process is the quantification process. Quantification refers to the process of identification of number of the unitized (or not) divisors within the dividend. It becomes messy if there is a remainder. Having a remainder after quantifying full divisor groups requires a multiplicative comparison between the remainder and one divisor group
to quantify how much of a divisor group corresponds to the leftover. This requires an abstraction of referent units. In other words, there needs to be coordination between the quotient and remainder (which is part of the dividend) and this coordination is enabled by abstracting the divisor as an intensive quantity. Such abstraction of coordination is another crucial understanding in the quantification process of partitions.

To sum up, division of fractions is about networking multiplicative relationships between three quantities (dividend, divisor, and quotient) based on certain operations as outlined above.

6.6. Instructional Sequence to Develop an Understanding of DoF and CDA

To help participants develop an understanding of division of fractions and a new algorithm, the instructional sequence consisted of several interventions. First, the participants were oriented toward abstracting the main idea of DoF, which is, an investigation of number of one quantity within another. They already had this idea in numeric domain. Also, they did not have an abstraction of quotitive situations, and therefore, helping them have an abstraction of this idea was important.

The second step was to help them increase their understanding of this aforementioned idea for DoF to an extent which would be enough for them to create their own DoF problems. This was needed because their abstraction of this idea out of the first step was not solid enough for them to create their own DoF word problems. One other reason was that, at times, their understanding of multiplication in the fractional domain interrupted their way of operating in DoF problems. Therefore, they were engaged in
some tasks by which they would have an abstraction of the difference between MwF and DoF. And, in this way, they would have an understanding of DoF at a level for which they would generate their own DoF word problems. To foster such a development, they engaged in two different task sequences: one, for them to develop an understanding for MwF and another, for them to increase their understanding of DoF. Once they went through such two sequences, and once they developed meanings for MwF and DoF, they were encouraged to make a comparison between their experiences for each sequence. Such a comparison approach helped them to abstract the crucial difference between MwF and DoF, and also helped them to understand DoF with its two important operations: partitioning and quantification. They already abstracted partitioning from the first step earlier on. Such a comparison helped them to abstract the quantification operation and to coordinate both operations to think about DoF problems.

The third step was to help them develop an understanding of remainder and the multiplicative relationship between the remainder and the divisor. This was accomplished in four sub-steps. They first focused on developing an understanding of remainder in a contextual-diagram setting in whole number domain. They were then encouraged to think about remainder in a context-free-diagram setting in the whole number domain. Third, their attention was directed to a contextual-diagram setting in the fractional domain to deal with remainder. Finally, they worked on context-free-diagram settings in fractional domain to think about remainder. This sequence was significant in their developing an abstraction of remainder and the multiplicative relationship between remainder, divisor and quotient.
The final and fourth step in the experiment was about developing CDA. This step consisted of two sub-steps. The first one was to help the participants develop an understanding for use of same size pieces to multiplicatively compare two given fractional quantities. Then the next step was to help them develop an understanding of the idea that dividing two fractional quantities had the same structure as dividing numerators of those two quantities as long as the quantities were all based on the same size partitions. The reason for choosing this algorithm was that it represented the activity participants pursued. In proceeding the activity sequence they had, there was not too much curtailment and CDA was inventable, based on their spontaneous activity.

To help them develop these two sub-steps for an algorithm, the designed task sequence engaged them in mentally solving the given division of fractions problems as if they were using diagrams. This type of work helped them come to a point where they anticipated what to do next and focus on what to pay attention to. In this way, they were encouraged to think about their thought processes to obtain an abstraction. By going through the activity sequence they already had from the previous sessions, in light of diagram use, they were also encouraged to think about the reason as to why the given fractional quantities transformed into another form. In this way, they realized that the purpose was to have equal size partitions so that the multiplicative comparison between the divisor and the dividend was easily identifiable. Then, based on their diagram work, they realized that they were counting a certain number of partitions within some total number of partitions, which was equivalent to thinking about dividing the numerators of the fractional quantities at hand.
Bibliography


Normal, IL: Eric Clearinghouse for Science, Mathematics, and Environmental Education.


Appendix A
Pre-Interview Schedule

Problem 1. What does $\frac{1}{4}$ mean?

Problem 1.1. Would the following shape represent $\frac{1}{4}$ too?

![Diagram with shaded part]

Problem 2. Consider the fraction $\frac{5}{17}$. If I changed the numerator to 7, do I get a bigger/smaller fraction?

Problem 2.1. What if I changed the numerator to 3?

Problem 2.2. How about changing the numerator to 7 and the denominator to 15?

Problem 3. Without doing any written or calculator computation explain how you know which is the larger fraction $\frac{9}{22}$ or $\frac{7}{24}$?

Problem 4. How would you change a mixed number like $4\frac{2}{3}$ into an improper fraction?

Problem 5. Write a word problem for $2\div\frac{3}{4}$

Problem 6. Can you show $1\div\frac{3}{4}$ drawing a diagram?
Problem 7. Little Johnny came to his teacher and said multiplication with a fraction can give an answer that is smaller than one of the factors.” Would Johnny be right? If so under what conditions would he be right?

Problem 8. Little Johnny also thinks that “division always gives a result that is less than the dividend [the number being divided].” Would Johnny be right? If so under what conditions would he be right?

Problem 9. Would the following be a division problem corresponding to the expression:

$$\frac{10}{3} \div \frac{2}{5}.$$  

“The plant grew $\frac{10}{3}$ inches in a month. If $\frac{2}{5}$ of the growth took place during daylight hours, how many inches did the plant grow during daylight hours?”

Note that in the original problem sheet the given fractions were different and messy. However, the researcher changed them by hand before handing in the worksheet to the interviewee.

Problem 10. Jane had $\frac{3}{4}$ of a gallon of ice cream and she gave Mike $\frac{2}{3}$ of what she had.

What part of a gallon of ice cream does she have left?

Problem 11. If the answer to a division problem is 7R3 [Here R means remainder] can you make up a problem that gives this kind of answer?

Problem 12: Johnny has $\frac{7}{2}$ cups of flavor and she needs $\frac{1}{3}$ of a cup of flavor to make a loaf of bread.

a) How many whole loaves of breads can he make out of $\frac{7}{2}$ cups of flavor?

b) How much flavor is leftover?
Problem 13. Using a diagram draw the fraction \( \frac{3}{4} \) and find an equivalence of the fraction \( \frac{3}{4} \) only using the diagram you draw.

Problem 14. Johnny has 10 bags of pencils and each bag has 5 pencils in it. How many pencils does Johnny have in total?

Problem 15. Chris has 25 apples and she has 5 friends. If she wants to share those apples among those friends, how many apples does each person get?

Problem 16. Ali has 35 oranges and he distributed them among some friends. If each person gets seven oranges how many friends does Ali have?

Problem 17. Say you have \( \frac{7}{2} \) liters of ice cream and each cone takes up \( \frac{2}{3} \) liter of ice cream. How many cones can I make out of the ice cream I have?

Note that in the original problem it was indicated that “each cone takes up \( \frac{1}{4} \) liter” but the researcher scratched it and changed it to “\( \frac{2}{3} \)” by pen.

Targeted Mathematical Understandings with this Schedule

I. Understanding the meaning of fractions

- A fraction can be thought of as parts of wholes or as a quantity in addition to other ways of thinking about fractions. The overall goal is to see the limitations of the subject’s fraction-understanding and to see which sub-con structs (Kieran, 1990) the subject relies on.

Questions to evaluate this understanding:
Problem 1. What does $\frac{1}{4}$ mean?

*If the subject says “it is one part out of four equal parts” ask:*

☐ Can you show what you mean by that [by a diagram]?

*If the interviewee draws either of the following and shade in one piece in either case to represent $\frac{1}{4}$ ask:*

**Figure A-1**

![Diagram](image1)

Figure A-1: The diagram used to ask about meaning of 1/4.

Problem 1.1. Would the following shape represent $\frac{1}{4}$ too?

**Figure A-2**

![Diagram](image2)

Figure A-2: What does the shaded area represent?

☐ How can that be possible?

II. Understanding the role of numerator and denominator in a fraction
To be able to understand how a fraction functions, one needs to know about the roles of numerator and denominator. In a fraction, the numerator refers to the number of partitions being identified, and the denominator refers to the total number of partitions that make up the whole that the fractional quantity refers to and the size of those partitions. To make comparisons among fractions, one needs to focus on how the change in the numerator and the denominator affect the size of the given fractions.

Questions to evaluate this understanding:

Problem 2. Consider the fraction $\frac{5}{17}$. If I changed the numerator to 7, do I get a bigger/smaller fraction?

Problem 2.1. What if I changed the numerator to 3?

Problem 2.2. How about changing the numerator to 7 and the denominator to 15?

Problem 3. Without doing any written or calculator computation explain how you know which is the larger fraction $\frac{9}{22}$ or $\frac{7}{24}$?

III. Understanding what changing mixed numbers into fractions means

The goal of this question is to figure out whether the subjects know that the conversion formula works because each time one makes partitions and total them.

Questions to evaluate this understanding:

Problem 4. How would you change a mixed number like $4\frac{2}{3}$ into an improper fraction?

If the student uses the method “multiply the denominator by the whole number add the result to the numerator and put it over the denominator”, then ask

- Why do you think it works?
IV. Understanding equivalent fractions

- Knowing that a given quantity [fraction] can be represented by combination of different size units by repartitioning the whole that the given quantity refers to. In order to represent a fractional quantity like \( \frac{3}{4} \), one partitions a given [or arbitrary] one whole into fourths and focus on the combination of 3 such partitions. By repartitioning the given whole into eights, one can represent the given \( \frac{3}{4} \) using this new unit. The second goal of this section is to figure out whether the student knows as to why algorithm for equivalence \[
\frac{a}{b} = \frac{ax}{bx}, \text{ x is constant}
\] works.

Questions to evaluate this understanding:

Problem 13. Using a diagram draw the fraction \( \frac{3}{4} \) and find an equivalence of the fraction \( \frac{3}{4} \) only using the diagram you draw.

\( \frac{3}{4} \) only using the diagram you draw.

*If the student cannot suggest a way to do that ask:*

- Can you find an equivalent form of \( \frac{3}{4} \) without using a diagram [e.g., an algorithm]?

Then follow up by asking:

- Why is that a reasonable thing to do?

V. Understanding the meaning of division and multiplication

- Division means that given two quantities how many of one quantity there are in the other quantity. The question of interest with this understanding is that
“do they have an abstraction of this idea or is this something that they memorize and do not refer back to when solving division problems?”

Questions to evaluate this understanding in the fractional context:

Problem 5. Write a word problem for \(2 \div \frac{3}{4}\)

Problem 6. Can you show \(1 \div \frac{3}{4}\) drawing a diagram?

Problem 12: Johnny has \(\frac{7}{2}\) cups of flavor and she needs \(\frac{1}{3}\) of a cup of flavor to make a loaf of bread.

c) How many whole loaves of breads can he make out of \(\frac{7}{2}\) cups of flavor?

d) How much flavor is leftover?

Problem 17. Say you have \(\frac{7}{2}\) liters of ice cream and each cone takes up \(\frac{2}{3}\) liter of ice cream. How many cones can I make out of the ice cream I have?

- **Multiplication means combination of groups of certain size, and division means number of certain size groups within another group.**

Questions to evaluate this understanding:

Problem 7. Little Johnny came to his teacher and said multiplication with a fraction can give an answer that is smaller than one of the factors.” Would Johnny be right? If so, under what conditions would he be right?
Problem 8. Little Johnny also thinks that “division always gives a result that is less than the dividend [the number being divided].” Would Johnny be right? If so, under what conditions would he be right?

Questions to evaluate above understandings in the whole number context:

Problem 14. Johnny has 10 bags of pencils and each bag has 5 pencils in it. How many pencils does Johnny have in total?

Problem 15. Chris has 25 apples and she has 5 friends. If she wants to share those apples among those friends, how many apples does each person get?

Problem 16. Ali has 35 oranges and he distributed them among some friends. If each person gets seven oranges how many friends does Ali have?

VI. Understanding remainder

- In division problems, within the activity of counting the number of certain size divisor sections in the given dividend, one needs to understand that the left over piece can either be interpreted in terms of the divisor or in terms of the dividend.

Questions to evaluate this understanding:

Problem 11. If the answer to a division problem is 7R3 [Here R means remainder] can you make up a problem that gives this kind of answer?

If the student says yes, ask:

☐ How can that be?

If the student seems puzzled, ask:

☐ What does 7 mean in this context?
☐ What does remainder 3 mean in this context?
The understanding that is being tested in this question is that \( \frac{3}{4} \) and \( \frac{2}{3} \) both refer to different wholes. In solving a division of fraction problem one needs to pay close attention to what the three constructs of a division [dividend, divisor, and quotient] refer to. Identifying what construct refer to what unit is crucial in distinguishing division operation from multiplication operation.

Questions to evaluate this understanding:

Problem 10. Jane had \( \frac{3}{4} \) of a gallon of ice cream and she gave Mike \( \frac{2}{3} \) of what she had. What part of a gallon of ice cream does she have left?

VII. Understanding referents

- Dividend and divisor have different referents.
- Quotient identifies the number of divisors.

The main goals are to look for:
1. Whether the subject can distinguish between multiplication and division,
2. Whether the subject is able to make this distinction by focusing on the idea of what refers to what in the question.

Problem 9. Would the following be a division problem corresponding to the expression:

\[
\frac{10}{3} \div \frac{2}{5}.
\]

“The plant grew \( \frac{10}{3} \) inches in a month. If \( \frac{2}{5} \) of the growth took place during daylight hours, how many inches did the plant grow during daylight hours?”
Appendix B

Task Sequence Used in Teaching Session 1

Part I

**Draw diagrams** to solve each of the following word problems. **DO NOT** use any computation or algorithm while solving the problems.

1. Bob has \( \frac{1}{2} \) of a cup of sugar. Each recipe of cookies calls for \( \frac{1}{8} \) of a cup of sugar. How many recipes could he make if he uses up every bit of sugar?

2. Jan was distributing ballots for the class election. If each ballot was to be \( \frac{3}{4} \) of a sheet of paper, how many ballots could she make from 3 sheets of paper [Using every piece of the papers]?

3. Louisa is building a cinderblock wall; the blocks are \( \frac{2}{3} \) of a meter high. How many rows of blocks will she need for a wall 4 meters high? [Assume that each block sits directly on top of each other.]

4. Nowadays, the road workers have been asphalting part of W. Clinton Avenue, which is \( \frac{5}{6} \) miles-long. Each day they are able to do \( \frac{1}{3} \) of a mile of road work. With this pace, how many days would it take for them to asphalt the whole road?

Part II

a. What is the commonality among the problems in Part I?

b. Write a general form defining that commonality for the problems in Part I.

Part III

A.1. Do the following problems fit the general form that you identified for the first four problems?
A.2. Write down an explanation for each of the problems as to why they are the same or different.

1. Lindsey plants 6 acres of corn. \( \frac{2}{3} \) of the corn is sweet corn. How much land is planted in sweet corn?

2. The temperature increased in the swimming pool at a rate of \( \frac{2}{3} \) of a degree per hour. How long did it take for it to increase 6 degrees?

B. Make up a problem that fits the general form described in Part II. You should use a context that is different from those already used in this lesson.
Appendix C
Tasks Implemented in Teaching Session 2

Section I

Solve the following problems drawing diagrams. DO NOT USE any computation or algorithm when solving these problems.

<table>
<thead>
<tr>
<th>Part I</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Alyssa has 3 cookies and gave ( \frac{2}{5} ) of it to her friend Joe. How much cookie does Joe have?</td>
<td></td>
</tr>
<tr>
<td>2. Bob has 4 liters of water. His neighbors borrowed ( \frac{2}{3} ) of what Bob had. How much of water does his neighbor borrow?</td>
<td></td>
</tr>
<tr>
<td>3. John has 6 gallons of gas stored in the basement of his house. He used ( \frac{3}{5} ) of it to clean up the greasy bumper of his car. How much gas did he use?</td>
<td></td>
</tr>
<tr>
<td>4. Betul has 7 meters long wooden stick. She used ( \frac{3}{4} ) of what she has to make a table cloth. How much fabric did she use?</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Part II</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Write down the steps you would take [and the results of those steps] if you were to use diagrams to solve the following problem.</td>
<td></td>
</tr>
<tr>
<td>Problem: Gulden has a 18-feet-long string. She used ( \frac{2}{7} ) of it to tie her husband Ismail. How much string did she use to tie Ismail?</td>
<td></td>
</tr>
</tbody>
</table>
Part III

Write down the overall goal for the above problems as to how they are similar. [Do not describe the process you would go through for each of those problems]

Part IV

For each of the above problems, write down an expression that would solve the problem. [Use the second column to put expressions].

Section II

Part I

Solve the following problems drawing diagrams. DO NOT USE any computation or algorithm when solving these problems.

1. Joe has 2 liters of water. If every bottle takes up \( \frac{2}{3} \) of a liter, how many bottles can he fill up using every bit of water at hand?

2. Tracy has \( \frac{5}{13} \) meter long ribbon. If she uses \( \frac{2}{5} \) of a meter of ribbon to pack an item, how many items can she pack using every bit of the ribbon she has.

3. Molly bought \( 3 \frac{1}{6} \) kg flour. If a loaf of bread calls for \( \frac{1}{3} \) kg flour, how many loaves of breads can she make using every bit of flour?

Part II

Write down the overall goal for the above problems as to how they are similar. [Do not describe the process you would go through for each of those problems]
Part III

What is the similarity/difference between the overall goal that you identified in Part III of Section I and the overall goal that you identified in Part II of Section II?

Section III

Part I

1. Write a statement to represent the expression $5 \div \frac{3}{4}$.

2. Without actually finding a solution to the above expression, write down the steps that you would do if you were to draw diagrams to solve the problem.

Part II

1. Write a statement to represent the expression $\frac{3}{4} \div 4$

2. Solve Problem 1 using diagrams.

3. Write a statement to represent the expression $\frac{2}{5} \div 3$

4. Solve Problem 3 using diagrams.

5. Write a statement to represent the expression $\frac{2}{3} \div 5$

6. Without actually finding a solution to the above expression, write down the steps that you would do if you were to draw diagrams to solve the problem.
Appendix D Tasks for Teaching Session 6

Tasks Used in Teaching Session 6

Part I

Sort the following problems into categories without actually solving them.

1. Tim has $\frac{3}{3}$ acre of land that he inherited from his grandma. He wants to split up this land into small gardens. Each garden is $\frac{5}{6}$ of an acre. How many such gardens can he have?

2. Ken has 204 apples and he distributed it among 12 friends. How many apples does each of his friends get?

3. Jesse bought $\frac{63}{8}$ square-yard-fabric to do patchwork quilts. The materials she has are enough to cover only $3\frac{1}{2}$ blankets. How much fabric does she spend for each blanket?

4. In the UniMart where Jimmy works, they are selling candies in packs of 6. Jimmy was told that UniMart just bought 34212 total candies. How many packs of candies do you think UniMart bought?
Part II

Solve the following problems using diagrams:

1. Kelly has 9 kg of flour and she wants to put them in bags. If each bag can take 2 kg of flour, what are her options as to how many bags she can use to store the flour? Solve the problem using diagrams.

2. A 10-acre farm will be ploughed and sowed with wheat. If every day 3 acre of farm is being ploughed and sown, what are the farmer’s options as to how many days he can spend on this work? Solve the problem using diagrams.

3. Jane is knitting a hat for his baby using a special type of string. She has 6 string balls. For each hat she uses $\frac{1}{3}$ of string ball. What are the alternatives that Jane has regarding the number of hats she can knit? Solve the problem using diagrams.

4. Using the calculator find 1354 : 38.

5. What information does the result give you about?

   What does $35.6315$ mean?

   How can we find what the remainder is in this case?

   So how can we define leftover/remainder?
Tasks Used in Teaching Session 7

Part I

1379 ÷ 28.

A) Write a statement as to what the above expression asking you to figure out?

B) Write down the steps that you would take to figure out the remainder for the above problem?

Part II

Solve the following problems using diagrams only. No computation is allowed.

1. Karen has 6 1/2 pounds of brown sugar and she wants to put them in bags. If each bag can take 3/4 of a pound of brown sugar, what are her options as to how many bags she can use to store the sugar? Solve the problem using diagrams.

2. A mowing company has a land of 6 2/3-acre area covered with grass that needs to be mowed. If every day the company has the capacity to mow 1 1/6 acre of land, what are the company’s options as to how many days it can spend on this work? Solve the problem using diagrams.

3. a. Solve the following problem using the given diagram: \( \frac{5}{2} \div \frac{3}{4} \)
b. What is the leftover piece in this problem?

c. If you were to find the solution to the same problem using your calculator, what would you get as an answer? How is it related to the answer you have from your solution?

4. a. Solve the following problem using the given diagram: $\frac{7}{3} \div \frac{5}{9}$

b. What is the leftover piece in this problem?

c. If you were to find the solution to the same problem using your calculator, what would you get as an answer? How is it related to the answer you have from your solution?
Tasks Used in Teaching Session 8

Part I

1. a. Solve the following problem using the given diagram: \( \frac{7}{3} \div \frac{3}{4} \)

b. What is the leftover piece in this problem?

c. If you were to find the solution to the same problem using your calculator, what would you expect as an answer? How is it related to the answer you have from your solution?

2. a. Solve the following problem using the given diagram: \( \frac{9}{5} \div \frac{2}{3} \)

b. What is the leftover piece in this problem?

c. If you were to find the solution to the same problem using your calculator, what would you expect as an answer? How is it related to the answer you have from your solution?
Part II

Solve the following problems using diagrams.

1. Cindy is baking a cake. She has \( \frac{3}{4} \) cups of sugar. Each recipe calls for \( \frac{1}{8} \) of a cup. How many recipes can she make using every bit of the sugar she has?

2. Matt was making ribbons for celebrating his sister’s birthday. If each ribbon was to be \( \frac{2}{9} \) of a yard in length, how many ribbons can he make from \( \frac{7}{3} \) yards of ribbon?

3. \( \frac{7}{2} \div \frac{5}{6} \)

4. \( \frac{4}{3} \div \frac{7}{12} \)
5. \( \frac{1}{2} \div \frac{4}{5} \)

Part III -- Intended to be given to the participants but time did not permit!

For the following problems, do not draw diagrams. Instead write down in words each step that you would do if you were to draw diagrams.

- **Problem 1:** \( \frac{4}{3} \div \frac{2}{9} \)
- **Problem 2:** \( \frac{9}{4} \div \frac{3}{8} \)
- **Problem 3:** \( \frac{5}{6} \div \frac{2}{9} \)
- **Problem 4:** \( \frac{14}{15} \div \frac{3}{8} \)

Part IV - Intended to be given to the participants but time did not permit!

Based on your work in problems in Part II and Part III, write a general procedure for

\[
\frac{\frac{a}{b}}{\frac{c}{d}}
\]
**Appendix E**

Wanda and Nancy’s Solutions to Problems in Part I of Teaching Session 6

<table>
<thead>
<tr>
<th>Wanda’s Solution Process</th>
<th>Nancy’s Solution Process</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Problem-1: 9 ÷ 2</strong></td>
<td>Drawing 9 rectangular same size wholes. [first row has 4, second row has 5 wholes]</td>
</tr>
<tr>
<td>Drawing 9 rectangular same size wholes in two rows [each row has 3 rectangles].</td>
<td>Circling two wholes at a time</td>
</tr>
<tr>
<td>Circling two rectangles at a time to make up a bag as in the following Figure E-1.</td>
<td>Writing “4 bags will be full of 2 kg” and “1 bag will be half full with one kg”, and right under it writing “5 bags”[See Figure E-3].</td>
</tr>
<tr>
<td><img src="image" alt="Figure E-1" /></td>
<td><img src="image" alt="Figure E-5" /></td>
</tr>
<tr>
<td>Figure E-1. Wanda’s way of reasoning for problem 1.</td>
<td>Figure E-5. Nancy’s solution to problem 1.</td>
</tr>
<tr>
<td>Writing what is shown in Figure E-2.</td>
<td></td>
</tr>
<tr>
<td><img src="image" alt="Figure E-2" /></td>
<td></td>
</tr>
<tr>
<td>Figure E-2. Wanda’s solution to problem 1.</td>
<td></td>
</tr>
<tr>
<td>Nancy’s Solution Process</td>
<td>Wanda’s Solution Process</td>
</tr>
<tr>
<td>-------------------------</td>
<td>--------------------------</td>
</tr>
<tr>
<td><strong>Problem 2: 10÷3</strong></td>
<td></td>
</tr>
<tr>
<td>Drawing two rectangles adjacent to each other.</td>
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</tr>
<tr>
<td>Partitioning each of the rectangles into five equal sections.</td>
<td></td>
</tr>
<tr>
<td>Labeling each partition with a number ranging from 1 through 3 respectively except for the last piece as in the Figure E-4.</td>
<td></td>
</tr>
<tr>
<td>Figure E-3. Nancy’s numbering of partitions in problem 2.</td>
<td></td>
</tr>
<tr>
<td>Writing “3 days and 1 acre left.”</td>
<td></td>
</tr>
<tr>
<td>I asked, “what did you get”, and she responded, “Three days and one acre left.”</td>
<td></td>
</tr>
<tr>
<td>Figure E-6. Wanda’s drawing for the solution of p2.</td>
<td></td>
</tr>
<tr>
<td>Since the camera was focused only on Nancy at this point, there was not a record of the order in which Wanda did things. However, it was obvious from her drawing in her worksheet that she followed a similar way as Nancy did.</td>
<td></td>
</tr>
<tr>
<td>Nancy’s Solution Process</td>
<td>Wanda’s Solution Process</td>
</tr>
<tr>
<td>--------------------------</td>
<td>--------------------------</td>
</tr>
<tr>
<td><strong>Problem 3: 6 ÷ 1 1/3</strong></td>
<td><strong>Problem 3: 6 ÷ 1 1/3</strong></td>
</tr>
<tr>
<td>Drawing six circles and then partitioning each of them into three sections.</td>
<td>Labeling first four partitions [all in the first circle and one in the second circle] with the numeral 1. Then continuing in the same manner to label the rest of the partitions.</td>
</tr>
</tbody>
</table>

Figure E-4. Nancy’s solution to problem 3.

Writing “4 hats with 2/3 lball left OR 4.”

Figure E-7. Wanda’s solution to problem 3.
## Appendix F

### Nancy and Wanda’s Solutions for the First Two Problems in Part II of T7

<table>
<thead>
<tr>
<th>Wanda Solution Method</th>
<th>Nancy’s Solution Method</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Part II - PROBLEM 1: ( \frac{3}{4} \div \frac{1}{2} )</strong></td>
<td>Drawing 7 rectangles as Wanda did. Then partitioning them into fourths and labeling each piece as 1,1,1; 2,2,2; etc. sequentially. Then labeling the last two pieces in the seventh rectangle as 1, 1.</td>
</tr>
<tr>
<td>Drawing 6 rectangles and then one more rectangle. Then, partitioning the last rectangle into halves and shading in the last half as in Figure F-1.</td>
<td>Partitioning each rectangle into four pieces and then circling three pieces at a time until the last rectangle as in Figure F-1.</td>
</tr>
<tr>
<td>Partitioning each rectangle into four pieces and then circling three pieces at a time until the last rectangle as in Figure F-1.</td>
<td></td>
</tr>
<tr>
<td>Figure F-1. Wanda’s grouping of fourths.</td>
<td>Figure F-6. Nancy’s drawing for problem 1.</td>
</tr>
<tr>
<td>Pointing at each circled 3/4-section in a linear fashion and then writing “8”.</td>
<td>Writing “8 full bags”.</td>
</tr>
<tr>
<td>Pointing to the last half-shaded rectangle and putting a marker at the end of the third piece, and then underlining the first two pieces as in Figure F-2.</td>
<td>Then changing the arrangement of pieces in the last rectangle as follows:</td>
</tr>
<tr>
<td>Figure F-2. Wanda’s treatment of the last rectangle.</td>
<td>Figure F-7. Nancy’s work R7.</td>
</tr>
<tr>
<td>Writing “2/3 bags” next to the previously written “8”.</td>
<td></td>
</tr>
<tr>
<td>Wanda’s final answer for the</td>
<td></td>
</tr>
</tbody>
</table>
The problem is as follows:
“8 2/3 bags or 8 bags and ½ pound of sugar.”

**Part II - PROBLEM 2:** $6\frac{2}{3} \div 1\frac{1}{6}$

<table>
<thead>
<tr>
<th>Drawing 7 pieces adjacent to each other and then partitioning the last piece into thirds and shading in the most bottom one as in Figure F-3.</th>
<th>[Time: 19:18] Drawing seven rectangles and then partitioning the last rectangle into thirds. Next, marking off at the end of the second third-piece in that rectangle as follows:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Figure F-3. Wanda’s representation of 6 and 2/3.</td>
<td>Figure F-8. Nancy’s representation of $6\frac{2}{3}$ using diagrams.</td>
</tr>
</tbody>
</table>

[Time: 17:13] Note that irrelevant pieces in the picture are smudged for the sake of clarity.

Next, horizontally partitioning the whole shape into six long pieces as in Figure F-4.

Next, partitioning the first rectangle into halves and then the first half of that into thirds and doing the same thing for the second half. Following the same procedure for all the other rectangles. For the last rectangle, copying the rectangle in another place and vertically partitioning it into thirds and horizontally partitioning it into sixths.

Figure F-4. Wanda’s partitioning of 6 2/3 into six pieces. No partitioning of last piece into sixths.

Figure F-9. Nancy’s final partitioning for the problem 2.

Extending the partitioning lines as follows and also partitioning the last piece as in F-5.

Next, counting the number of pieces in the shaded area of last rectangle [this was apparent from her movement of the pen in the air].

Figure F-5. Wanda’s new drawing.

Trying to circle 7-little-piece groups with her hand without marking but since it was

Then labeling each part as ones, twos, etc., until hitting the last rectangle.
confusing, numbering those little pieces.

<table>
<thead>
<tr>
<th>Drawing the last rectangle for the third time. This time partitioning it into sixths vertically only and marking off the fourth piece.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Figure F-10. Last rectangle redrawn by Nancy.</td>
</tr>
<tr>
<td>Continuing to number the first four pieces.</td>
</tr>
<tr>
<td>Writing $\frac{5}{6}$ right underneath this last rectangle.</td>
</tr>
</tbody>
</table>
## Appendix G

### Nancy and Wanda’s Solutions to Part I of T9

<table>
<thead>
<tr>
<th>Wanda’s Solution Process</th>
<th>Nancy’s Solution Process</th>
</tr>
</thead>
<tbody>
<tr>
<td>Problem 1: (\frac{2}{3} \div \frac{1}{4})</td>
<td></td>
</tr>
<tr>
<td><img src="image" alt="Diagram" /></td>
<td><img src="image" alt="Diagram" /></td>
</tr>
<tr>
<td>Shading in the last third piece in the given diagram.</td>
<td>Shading in the last third piece in the given diagram.</td>
</tr>
<tr>
<td>Partitioning the whole rectangle into fourths sequentially [not half-half method].</td>
<td>Dividing the whole rectangle into half horizontally and then partitioning each horizontal half into halves horizontally again to get fourths.</td>
</tr>
<tr>
<td><img src="image" alt="Diagram" /></td>
<td></td>
</tr>
<tr>
<td>The above diagram is similar to what is on her worksheet at this moment.</td>
<td></td>
</tr>
<tr>
<td>Moving her hand from the upper corner of the first column toward the bottom in a counting fashion. Then moving her index finger along with the first row [as if she was counting the number of pieces in the first row that is one fourth of the whole rectangle].</td>
<td>Numbering each three-piece section [starting from the bottom right corner adjacent to the shaded pieces of the bottom row and moving left-up-right-up-left-up-right direction] until all the pieces are used up. There are only two pieces numbered as 3.</td>
</tr>
<tr>
<td><img src="image" alt="Diagram" /></td>
<td></td>
</tr>
<tr>
<td>Numbering each three-piece section [starting from the upper left corner of the first column and moving zig zag] until all the pieces are used up. There are only two pieces numbered as 3.</td>
<td></td>
</tr>
<tr>
<td>Writing “2” right above the rectangle and then adding (\frac{2}{3}) next to it.</td>
<td></td>
</tr>
<tr>
<td>Writing “2” right above the rectangle and then adding (\frac{2}{3}) next to it.</td>
<td></td>
</tr>
<tr>
<td>Finishing numbering the pieces and first writing (2 \frac{2}{3}) and then right next to it writing “2 + (\frac{2}{12})” of the whole” as an alternative answer.</td>
<td></td>
</tr>
<tr>
<td>Wanda’s Solution Process</td>
<td>Nancy’s Solution Process</td>
</tr>
<tr>
<td>--------------------------</td>
<td>--------------------------</td>
</tr>
<tr>
<td><strong>Problem 2:</strong> ( \frac{13}{7} \div \frac{2}{3} )</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>Time: 7:02</td>
<td>Horizontally partitioning the given whole rectangles into thirds.</td>
</tr>
<tr>
<td>Shading in the last piece of the second rectangle and then horizontally partitioning the given whole rectangles into thirds.</td>
<td></td>
</tr>
<tr>
<td>Then marking off the first two horizontal section of the first rectangle completely.</td>
<td></td>
</tr>
<tr>
<td>Figure G-1. Wanda’s partitioning of the rectangles.</td>
<td></td>
</tr>
<tr>
<td>Marking off the bottom row of the first rectangle, and the first row of the second rectangle up to the shaded parts, and the last piece [before the shaded part] of the second row of the second rectangle.</td>
<td></td>
</tr>
<tr>
<td>Counting the number of pieces that are not marked and not shaded in.</td>
<td></td>
</tr>
<tr>
<td>Writing ( \frac{11}{14} ) as the answer right below the second rectangle.</td>
<td>There was not a complete video record of how Nancy progressed. However, as far as it was seen from her paper, once she partitioned the rectangles horizontally into thirds, then she numbered each fourteen ( \frac{1}{21} )-section from one through three until all the pieces are numbered. In her paper, the answer she put was ( \frac{11}{14} ).</td>
</tr>
<tr>
<td>[Time: 8:45]</td>
<td></td>
</tr>
</tbody>
</table>
### Problem 3: \( \frac{11}{3} \div \frac{3}{4} \)

<table>
<thead>
<tr>
<th>Wanda’s Solution Process</th>
<th>Nancy’s Solution Process</th>
</tr>
</thead>
<tbody>
<tr>
<td>Horizontally partitioning the given rectangles into fourths.</td>
<td>Horizontally partitioning the given rectangles into fourths.</td>
</tr>
<tr>
<td>She then said, “one, two, three” by pointing at each row of the first rectangle and she circled them as one group.</td>
<td>Marking and numbering nine of ( \frac{1}{12} ) sections with the same numeral sequentially until the last rectangle.</td>
</tr>
<tr>
<td>Following the same method for the bottom row of first rectangle and the upper two rows of second rectangle.</td>
<td>Leaving all the pieces in the last rectangle unnumbered.</td>
</tr>
<tr>
<td>Continuing to group ( \frac{3}{4} ) [of a whole rectangle] sections until the last rectangle and then writing ( \frac{8}{9} ) as the answer.</td>
<td>Writing ( 4\frac{8}{9} ) as the answer.</td>
</tr>
</tbody>
</table>

![Figure G-2. Wanda’s Solution to Problem 3.](image)

[Time: 11:32]

Note that later on she realized that she miscounted the number of divisor sections even though she marked four sections.

### Problem 4: \( \frac{9}{4} \div \frac{3}{5} \)

<table>
<thead>
<tr>
<th>Horizontally partitioning the given rectangles into fourths.</th>
<th>Horizontally partitioning the given rectangles into fourths.</th>
</tr>
</thead>
<tbody>
<tr>
<td>She then said, “one, two, three” by pointing at each row of the first rectangle and she circled them as one group.</td>
<td>Marking and numbering nine of ( \frac{1}{12} ) sections with the same numeral sequentially until the last rectangle.</td>
</tr>
<tr>
<td>Following the same method for the bottom row of first rectangle and the upper two rows of second rectangle.</td>
<td>Leaving all the pieces in the last rectangle unnumbered.</td>
</tr>
<tr>
<td>Continuing to group ( \frac{3}{4} ) [of a whole rectangle] sections until the last rectangle and then writing ( \frac{8}{9} ) as the answer.</td>
<td>Writing ( 4\frac{8}{9} ) as the answer.</td>
</tr>
</tbody>
</table>

[Time: 12:50]

Note that later on she realized that she miscounted the number of divisor sections even though she marked four sections.

[Time: 13:00]
<table>
<thead>
<tr>
<th>Partitioning the given rectangles into fifths horizontally.</th>
<th>Partitioning the given rectangles into fifths horizontally.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Marking off each three-row section throughout the rectangles.</td>
<td>Numbering each 12-piece section with the same number until the pieces are exhausted sequentially.</td>
</tr>
<tr>
<td>Writing “( \frac{9}{12} \ldots \frac{3}{4} )” on her sheet.</td>
<td>Writing “( \frac{9}{12} (3 \frac{3}{4}) )” on her sheet.</td>
</tr>
</tbody>
</table>
### Nancy and Wanda’s Solutions to the Problems given in Part II

**Wanda’s Scanned Work**

**W-Problem 1:**

#### Part II

For the following problems, do not draw diagrams. Instead write down in words each step that you would do if you were to draw diagrams.

**Problem 1:** \( \frac{7}{2} ÷ \frac{2}{3} \)

<table>
<thead>
<tr>
<th>Steps You Would Take</th>
<th>Results of Those Steps</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Draw 4 wholes</td>
<td>1. 4 wholes</td>
</tr>
<tr>
<td>2. Divide wholes into halves</td>
<td>2. 5 halves</td>
</tr>
<tr>
<td>3. Shade 1 out</td>
<td>3. 7 halves</td>
</tr>
<tr>
<td>4. Divide all wholes into 3 parts</td>
<td>4. 12/3</td>
</tr>
<tr>
<td>5. Count how many ( \frac{1}{3} ) are in ( \frac{7}{2} ). Not the ( \frac{1}{2} ) shaded out</td>
<td>5. ( \frac{5}{4} )</td>
</tr>
</tbody>
</table>
W-Problem 2:

<table>
<thead>
<tr>
<th>Problem 2:</th>
<th>$\frac{5}{8} \div \frac{2}{7}$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Steps You Would Take</strong></td>
<td><strong>Results of Those Steps</strong></td>
</tr>
<tr>
<td>1. <strong>Draw 1 whole</strong></td>
<td>1. <strong>1 whole</strong></td>
</tr>
<tr>
<td>2. <strong>Divide into 8ths vertically</strong></td>
<td>2. <strong>$\frac{8}{56}$</strong></td>
</tr>
<tr>
<td>3. <strong>Shade out $\frac{3}{8}$ths</strong></td>
<td>3. <strong>$\frac{5}{56}$ths</strong></td>
</tr>
<tr>
<td>4. <strong>Divide same whole into sevenths horizontally</strong></td>
<td>4. <strong>$\frac{5}{6}$ths</strong></td>
</tr>
<tr>
<td>5. <strong>Only need $\frac{35}{56}$s so shade out 21 $\frac{5}{6}$ths</strong></td>
<td>5. <strong>$\frac{35}{56}$s</strong></td>
</tr>
<tr>
<td>6. Find how many $\frac{56}{3}$s are in $\frac{2}{7}$</td>
<td>6. <strong>$\frac{16}{56}$s in $\frac{2}{7}$</strong></td>
</tr>
</tbody>
</table>
| 7. **Mark off every 16 $\frac{56}{3}$s as possible in $\frac{35}{56}$** | 8. $\frac{3}{76}$ }
### Problem 3: \( \frac{14}{15} + \frac{3}{8} \)

<table>
<thead>
<tr>
<th>Steps You Would Take</th>
<th>Results of Those Steps</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Draw 1 whole</td>
<td>1. 1 whole</td>
</tr>
<tr>
<td>2. Divide into 15ths</td>
<td>2. ( \frac{15}{15} \times )</td>
</tr>
<tr>
<td></td>
<td>3. Divide second piece into 8ths horiz.</td>
</tr>
<tr>
<td></td>
<td>4. Find out how many 720ms are in 1</td>
</tr>
<tr>
<td></td>
<td>3. Shade out ( \frac{3}{5} )</td>
</tr>
<tr>
<td></td>
<td>4. Divide remaining piece into 8ths horiz.</td>
</tr>
<tr>
<td>5. Divide 120s into 1/15</td>
<td>5. ( \frac{120}{120} = 6 \frac{1}{15} )</td>
</tr>
<tr>
<td>6. How many 120s in 3/8ths?</td>
<td>6. ( \frac{45}{120} \times )</td>
</tr>
<tr>
<td>7. Mark off every 45/120s in 10/120s</td>
<td>7. ( 2 )</td>
</tr>
<tr>
<td>8. Count how much more ( \text{another} \frac{3}{120} \times )</td>
<td>8. ( \frac{23}{120} \text{ left} )</td>
</tr>
<tr>
<td>9. ( \frac{2}{120} = \frac{1}{60} )</td>
<td>9. ( \frac{2}{120} = \frac{1}{60} \text{ or} \frac{1}{11} )</td>
</tr>
<tr>
<td>10. Find out ( 45 \times )</td>
<td>10. ( \frac{3}{45} = \frac{1}{15} )</td>
</tr>
</tbody>
</table>
Nancy’s Scanned Work

N-Problem 1:

### Part II

For the following problems, do not draw diagrams. Instead write down in words each step that you would do if you were to draw diagrams.

<table>
<thead>
<tr>
<th>Problem 1: ( \frac{7}{2} + \frac{2}{3} )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Steps You Would Take</strong></td>
</tr>
<tr>
<td>1. Draw 4 wholes</td>
</tr>
<tr>
<td>2. Divide wholes into halves</td>
</tr>
<tr>
<td>3. Cross out last half</td>
</tr>
<tr>
<td>4. Divide each whole into thirds</td>
</tr>
<tr>
<td>5. Group together as many ( \frac{2}{3} )'s as possible</td>
</tr>
<tr>
<td>6. Figure out remaining amount</td>
</tr>
<tr>
<td>7. State answer</td>
</tr>
</tbody>
</table>
N-Problem 2:

<table>
<thead>
<tr>
<th>Steps You Would Take</th>
<th>Results of Those Steps</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Draw 1 whole</td>
<td>1. 1 whole</td>
</tr>
<tr>
<td>2. Divide whole into 8 parts vertically</td>
<td>2. 8 eighths in 1 whole</td>
</tr>
<tr>
<td>3. Cross out last 3 eighths</td>
<td>3. 5 eighths left</td>
</tr>
<tr>
<td>4. Divide whole into sevenths, horizontally</td>
<td>4. 56 sevenths in 1 whole</td>
</tr>
<tr>
<td>5. Group together 16, 56 more (16 make 3/8)</td>
<td>5. 2 groups of 16 with same leftover</td>
</tr>
<tr>
<td>6. Derive amount of 16 56ths left over</td>
<td>6. $\frac{3}{7}$ of $\frac{16}{56}$ left</td>
</tr>
<tr>
<td>7. State answer</td>
<td>7. $\frac{3}{16}$ $\frac{5}{8}$ left</td>
</tr>
</tbody>
</table>

$\frac{5}{8} \div \frac{2}{7}$
## Problem 3: \( \frac{14}{15} + \frac{3}{8} \)

<table>
<thead>
<tr>
<th>Steps You Would Take</th>
<th>Results of Those Steps</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Draw 1 whole</td>
<td>1. 1 whole</td>
</tr>
<tr>
<td>2. Divide the whole</td>
<td>2. ( \frac{15}{15} ) in whole</td>
</tr>
<tr>
<td>into 15 equal parts</td>
<td></td>
</tr>
<tr>
<td>vertically</td>
<td>3. 14 ( \frac{14}{15} )</td>
</tr>
<tr>
<td>3. Cross out 15( \times )</td>
<td>4. 120 ( \text{cm} )</td>
</tr>
<tr>
<td>4. Divide whole into</td>
<td></td>
</tr>
<tr>
<td>eighths horizontally</td>
<td>5. 112 ( \text{cm} )</td>
</tr>
<tr>
<td>5. Note what can</td>
<td>6. 45 ( \text{cm} )</td>
</tr>
<tr>
<td>be used</td>
<td></td>
</tr>
<tr>
<td>6. Figure out the</td>
<td>7. 2 full 45</td>
</tr>
<tr>
<td>number of 120( \times ) in ( \frac{1}{8} )</td>
<td>120( \text{cm} ) with</td>
</tr>
<tr>
<td>7. Group 45 ( \text{cm} ) many</td>
<td>left over</td>
</tr>
<tr>
<td>8. Determine how much</td>
<td>8. 22 ( \text{cm} )</td>
</tr>
<tr>
<td>of a ( \frac{1}{8} ) is left</td>
<td></td>
</tr>
<tr>
<td>9. State answer</td>
<td>9. ( 2 \frac{2}{8} ) ( \text{cm} )</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>
## Appendix H

### Tasks for T10

#### Problem 1

\[
\frac{3}{2} \div \frac{2}{5}
\]

<table>
<thead>
<tr>
<th>What is the goal of this question?</th>
<th>What needs to be done?</th>
<th>For what purpose?</th>
</tr>
</thead>
<tbody>
<tr>
<td>STEP 1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>…</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

#### Problem 2

\[
\frac{8}{3} \div \frac{3}{4}
\]

<table>
<thead>
<tr>
<th>What is the goal of this question?</th>
<th>What needs to be done?</th>
<th>For what purpose?</th>
</tr>
</thead>
<tbody>
<tr>
<td>STEP 1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>…</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

#### Problem 3

\[
\frac{22}{5} \div \frac{2}{3}
\]

<table>
<thead>
<tr>
<th>What is the goal of this question?</th>
<th>What needs to be done?</th>
<th>For what purpose?</th>
</tr>
</thead>
<tbody>
<tr>
<td>STEP 1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>…</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Part II

For the following problems, do not draw diagrams. Instead write down in words each step that you would do if you were to draw diagrams.

| Problem 1: $\frac{19}{4} \div \frac{3}{5}$ [This problem was intended to be asked but skipped during the instruction.]
<table>
<thead>
<tr>
<th>绺Steps You Would Take</th>
<th>Results of Those Steps</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td></td>
</tr>
</tbody>
</table>

<p>| Problem 2: $\frac{23}{24} \div \frac{3}{7}$ |</p>
<table>
<thead>
<tr>
<th>绺Steps You Would Take</th>
<th>Results of Those Steps</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td></td>
</tr>
</tbody>
</table>

| Problem 3: $\frac{19}{11} \div \frac{3}{8}$ [This question was intended to be asked but skipped during the instruction.]

Additional Problem: Solve $\frac{21}{38} \div \frac{7}{98}$ using common denominator algorithm.
VITA

Ismail Ozgur Zembat

Place of Birth: Ankara, Turkey
Date of Birth: December 7, 1975
Education: The Pennsylvania State University, University Park, Pennsylvania
Ph.D. in Mathematics Education, Curriculum and Instruction
Candidacy exam passed on:
Comprehensive exam passes on:
Ankara University, Ankara, Turkey
Master of Science (ABD), Mathematics, 1996-1998
Ankara University, Ankara, Turkey
Bachelor of Science, Mathematics, June 1996

Professional Activities/Services
Manuscript Reviewer, Pennsylvania Council of Teachers of Mathematics, 2001, 2004
Research Assistant, The Pennsylvania State University, August 2000 - July 2004
Teaching Assistant, The Pennsylvania State University, August 2002 - May 2003
Fixed-term Instructor, Gazi University, Faculty of Commercial and Tourism Training, Ankara, Turkey, September 1998 - December 1998
Fixed-term Instructor, Gazi University, Faculty of Technical Training, Ankara, Turkey, February 1997 - June 1997
Mathematics Teacher, Tevfik Ileri Anadolu High School, Ankara, Turkey, September 1996 - December 1996

Selected Conference Presentations and Publications

Membership in Professional Organizations
Member of National Council of Teachers of Mathematics, 1999-current
International Group for the Psychology of Mathematics Education, 2002-current