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EQUILIBRIUM MODELS OF THE LIMIT ORDER BOOK

A Dissertation in
Mathematics
by
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Abstract

The thesis mainly studied different equilibrium models of the Limit Order Book (LOB). And it contains two main parts. The First part starts with a "one-shot" game, and it shows that the two sides of the LOB are determined by the distribution of the random size of the incoming order, and by the maximum price accepted by external buyers (or the minimum price accepted by external sellers). Then the model is extended to an iterated game, where more agents come to the market, posting both market orders and limit orders. Equilibrium strategies are found by backward induction, in terms of a value function which depends on the current sizes of the two portions of the LOB. The existence of a unique Nash equilibrium is proved under a natural assumption, namely: the probability that the external order is so large that it wipes out the entire LOB should be sufficiently small. The second part introduces models in the infinite time horizon, allowing for random fluctuations in the fundamental value. Firstly, we consider when all agents have the same information, but the external players may anticipate random jump in the value of the stock, and what is the difference the model with constant fundamental value and with random jump in the fundamental value. Another part is considering two different cases: if the agents posting limit orders are risk-averse; and if some of the external agents have the information that there will be a jump in the fundamental value. We are curious about how the shape will be changed under these two cases.

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Acknowledgments

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Chapter 1 Introduction

1 Limit Order Book

An order is the basic trading unit of the market, and made up by a set of instructions to buy or sell stocks, currencies, options bonds and other assets. One fundamental type of orders is "market order", which is an order to trade an asset at the best available price. The market order will be filled immediately if there are willing sellers or buyers, however, the price at which will be executed is not guaranteed. The other typical type of order is called "limit order", which will control the price at which the trade is executed. This type of order will sacrifice the certainty of execution, but can help the investors avoid paying extra money than the pre-determined price.

The Limit Order Book (LOB) is a collection of all outstanding limit orders, and it is used to match buyers and sellers in the market, which is also widely used by many of electronic exchanges in the financial markets nowadays. As an active research topic both in academia and industry, more and more modern techniques has been used to study the LOB to realize different goals, [3] and [12] introduced the risk-neutral liquidity of a limit order market, [13, 14, 18] gave several stochastic models of LOB, [1, 15, 16, 19] analyzed the effect of the trading frequency on LOB, [21], [22, 23] introduced statistic properties of LOB dynamics. In particular, the spreading of information and the price impact and of a large external order have been studied in [2, 4, 5]. For a survey, we refer to [17] or [20].

A bidding game related to a continuum model of a one-sided Limit Order Book (LOB) was recently considered in [7, 8, 10, 11], proving the existence and uniqueness of a Nash equilibrium and determining the optimal strategies for the various agents, which is also the basic model of this thesis. However, this models all have the form of a "one shot game", i.e., all players' payoffs are completely determined as soon as one single incoming order is received. And this thesis will be based on these basic models, but will introduce

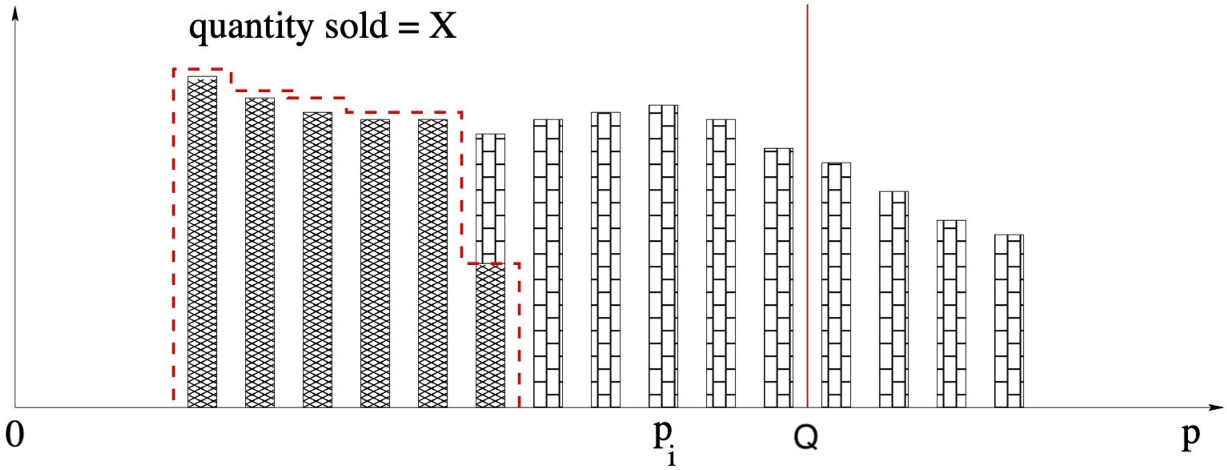

Figure 1.1. The height of the various columns indicates the amount of asset offered for sale at the various prices. The buyer will buy a random quantity X, at the lowest possible price, as long as this price does not exceed the (random) upper bound Q.

more realistic settings as well as describing the dynamics of the LOB. Besides, it can help describe the market spread in a systematically way to help quantify how much spread should be charged by the market maker.

2 The Nash equilibrium for a one-shot game

In the basic model, it is assumed that an external buyer asks for a random amount $X > 0$ of a given asset. This amount will be bought at the lowest available price, as long as this price does not exceed some (random) upper bound Q. Several sellers offer various quantities of this same asset at different prices, competing to fulfill the incoming order, whose size is not known a priori, as shown in Fig 1.1.

Having observed the prices asked by his competitors, each agent must determine an optimal strategy, maximizing his expected payoff. Because of the presence of the other sellers and of the upper bound Q on the acceptable price, asking a higher price for an asset reduces the probability of selling it.

And we define a shape of the LOB in a Nash equilibrium between the various agents as follows.

Consider n sellers competing against each other. We assume that i-th agent has an amount κ_i of assets to offer for sale. His pricing strategy will be described by the function

$\phi_i : [0, \kappa_i] \mapsto I\!R$. For every $i \in \{1, \ldots, n\}$, let

$$\Phi_i(p) \doteq \sum_{j \neq i} \sup \Big\{ \xi \in [0, \kappa_j] \,; \quad \phi_j(\xi) < p \Big\} \tag{1.1}$$

be the total amount of assets offered by all other agents $j \neq i$ at price $< p$. And the price p will not lower than the fundamental value of the asset, denoted as β. Then the expected payoff for agent i is

$$J_i(\phi_i) = \int_0^{\kappa_i} [\phi_i(\xi) - \beta] \cdot \Big[\psi\Big(\xi + \Phi_i(\phi_i(\xi))\Big) \cdot h(\phi_i(\xi)) \Big] \, d\xi \,. \tag{1.2}$$

Definition. *An n-tuple of pricing strategies $(\phi_1, \ldots, \phi_n)$ is a* **Nash equilibrium** *if each ϕ_i yields the maximum expected payoff (1.2) to the i-th player, given the function Φ_i determined by the strategies of all other players.*

To determine these equilibrium strategies, it is convenient to introduce the functions

$$U_i(p) = [\text{amount put on sale by } i\text{-th agent at price} < p] \,, \tag{1.3}$$

$$u_i(p) \doteq U_i'(p), \qquad\qquad U(p) \doteq \sum_{i=1}^{n} U_i(p).$$

Notice that the U_i provides a generalized inverse to the function $\phi_i : [0, \kappa_i] \mapsto I\!R$ describing the strategy of the i-th player. Indeed, up to sets of measure zero, one has

$$U(p) = \sup \{\xi \,; \quad \phi_i(\xi) < p\} \,, \qquad\qquad \phi_i(\xi) = \sup \{p \,; \quad U_i(p) < \xi\} \,,$$

i.e., the functions $p \mapsto U_i(p)$ and $\xi \mapsto \phi_i(\xi)$ are generalized inverses of each other.

And from [11], we guarantee a unique shape of the LOB exist under appropriate conditions.

Further, we can consider the limiting case where the number of sellers approaches infinity while the total amount of asset on sale remains bounded. More precisely, for each $n \geq 1$, consider amounts

$$0 < \kappa_1^{(n)} \leq \kappa_2^{(n)} \leq \cdots \leq \kappa_n^{(n)} \,,$$

and assume that

$$\lim_{n\to\infty}\sum_{j=1}^{n}\kappa_j^{(n)} \;=\; \kappa\,, \qquad\qquad \lim_{n\to\infty}\sup_{1\le j\le n}\kappa_j^{(n)} \;=\; 0\,. \tag{1.4}$$

Let $U^{(n)}(p)$ be the total amount of asset put on sale at price $\le p$ (by all players combined) in a Nash equilibrium. And let the function $U^\sharp$ be characterized as the unique Lipschitz continuous function such that

$$\begin{cases} U(p) \;=\; 0 \qquad \text{if} \quad p \le p_A\,, \\ U(p) \;=\; \kappa \qquad \text{if} \quad p \ge p^*\,, \end{cases} \tag{1.5}$$

$$U'(p) \;=\; -\frac{\psi\Big(U(p)\Big)}{\psi'\Big(U(p)\Big)}\cdot\left(\frac{1}{p-\beta}+\frac{h'(p)}{h(p)}\right) \qquad \text{for a.e. } p\in[p_A,p^*], \tag{1.6}$$

for a suitable value $p_A \in [\beta, p^*]$. Notice that the above equations imply that the map

$$\begin{aligned} p \;\mapsto\; & [\text{profit from the sale of an asset at price } p] \times [\text{probability of selling the asset}] \\ =\; & [p-\beta]\cdot\Big[\psi(U^\sharp(p))\cdot h(p)\Big] \end{aligned} \tag{1.7}$$

is constant over the interval $p \in [p_A, p^*]$.

We shown in [11], if the limits (1.4) hold, then one has the uniform convergence

$$U^{(n)}(p) \;\to\; U^\sharp(p)\,, \tag{1.8}$$

as $n \to \infty$

We can thus regard $U^\sharp(\cdot)$ as describing the price distribution in a Nash equilibrium with infinitely many small players. That is, the model with infinitely many small player can be viewed as a mean-filed limit, and the pricing strategy ϕ_i for each player can replaced by the LOB density function $\phi(p) = U'(p)$, as shown in Fig 1.2.

Besides, as shown in [11], under appropriate condition, we have the uniqueness and existence of such $U^\sharp$ in Nash equilibrium for the one-sided one-shot game.

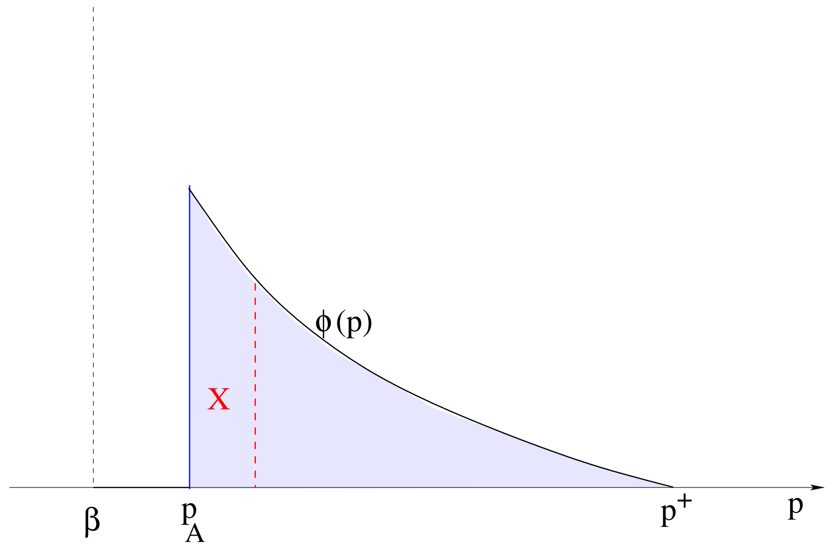

Figure 1.2. The height of the various columns indicates the amount of asset offered for sale at the various prices. The buyer will buy a random quantity X, at the lowest possible price, as long as this price does not exceed the (random) upper bound Q.

3 Outline of the dissertation

The dissertation will mainly discuss different LOB models in the mean-field limit, with the size of the LOB fixed, and it is organized as follows:

In Chapter 2, we consider a one-shot game of two-sided Limit Order Book, firstly with maximum price accepted by the external buyer and the minimum price accepted by the external seller as deterministic functions of the mean bid-ask price, with the maximum/minimum acceptable prices are random variables. For both of these two cases, we prove the existence of the unique shape, given the total number of agents, the distribution of the random size of the incoming order.

In Chapter 3, we introduce an iterated game, where more agents come to the market, posting both buy orders and sell orders. We begin by introducing the notation and the value functions, which depends on the current size of the two portions of the LOB to describe the evolution model. Then we study the equilibrium strategies in terms of the value function, and derive the conditions for the existence of a unique shape for the evolution of the two-sided LOB, together with a priori bounds on the value functions.

Starting in Chapter 4, we consider the infinite horizon with exponential discount. Consider an infinite sequence of incoming orders, each can be either a buy or sell order. We determine the shape of the LOB, and how the discounting factor of the expected profit will affect the result. Meanwhile, we also consider a case where some well informed external buyers can anticipate random jumps in the value of the stock. We study how is the strategy impacted by the volatility of the fundamental value.

Finally, in Chapter 5 we study the relation between the size of the LOB and the expected profit of the agents that post limit orders. We also consider two additional models. First. we assume that there is a probability that an external agent might receive the information of the change in the fundamental value before any of the traders have time to react and change their posted prices. In a second model, we assume that agents are risk-averse, and will give up the opportunity for a large gain in favor of safety. And we will compare the shape of the LOB and the bid-ask spread under these different models.

Chapter 2
Two-sided Limit Order Book

1 Introduction

In Chapter 2, we study a two-sided LOB, assuming that external agents will agree to the transaction only if the price is sufficiently close to the mean bid-ask price.

To simplify the analysis, we consider the limiting case of a large number of agents, each holding a small amount of cash and stock. In this setting, we prove two results showing that the shape of the two-sided LOB can be uniquely determined, depending on (i) the total amount of stocks that agents put on sale or bid to buy on the LOB, and (ii) the distribution of the random variables X, Y describing the sizes of the incoming buy or sell orders.

More precisely, in Section 2 we study the case where the maximum price accepted by an external buyer, and the minimum price accepted by an external seller, are deterministic functions of the mean bid-ask price. In Theorem 1 the existence of a unique shape for the LOB is proved under one main assumption. Namely, the probability that the incoming order is very large, wiping out the entire "buy" or "sell" portion of the order book, should be sufficiently small.

In Section 3 we study the more general case where the maximum or minimum prices acceptable to external agents are random as well. Under suitable assumptions on the distributions of these random variables, Theorem 2 provides the existence of a unique shape for the two-sided LOB. Remarkably, no assumption on the size "buy" or "sell" portion of the order book is here needed.

2 The two-sided LOB with deterministic acceptable prices

We consider a continuum model of the Limit Order Book, described by a density function $\phi = \phi(s)$, as in Fig. 2.1, right. Calling β_0 the "fundamental value" of the stock, known to all agents posting bids on the LOB, the function ϕ will describe sell orders posted on the LOB for prices $p > \beta_0$, and buy orders for $p < \beta_0$. In other words, for $\beta_0 < p_1 < p_2$, the integral

$$\int_{p_1}^{p_2} \phi(s)\, ds \tag{2.1}$$

gives the total amount of stock that the agents offer for sale at price $p \in [p_1, p_2]$. On the other hand, for $p_1 < p_2 < \beta_0$, the integral (2.1) gives the total amount of stock that the agents are willing to buy at price $p \in [p_1, p_2]$. The minimum ask price (i.e., the lowest price at which some agent offers to sell stock) is denoted by

$$p_A \doteq \inf \left\{ p > \beta_0 \,; \quad \int_{\beta_0}^{p} \phi(s)\, ds > 0 \right\}, \tag{2.2}$$

while the maximum bid price (i.e., the highest price at which some agent offers to buy stock) is denoted by

$$p_B \doteq \sup \left\{ p < \beta_0 \,; \quad \int_{p}^{\beta_0} \phi(s)\, ds > 0 \right\}. \tag{2.3}$$

Throughout the following, we denote the **mean bid-ask price** as

$$\overline{p} \doteq \frac{p_A + p_B}{2}. \tag{2.4}$$

The mean bid-ask price $\overline{p}$ can be interpreted as a reference market value of the asset. It is reasonable to assume that an external buyer will not accept to pay a price which is much higher than this reference value. We assume in the article that the maximum acceptable price is a multiple of $\overline{p}$.

In a basic model, one can assume that the maximum price that an external buyer is willing to pay (or the minimum price that an external seller is willing to accept) is a given multiple of the mean price $\overline{p}$. In this case, external agents will buy stock only at a price $p \leq (1 + \delta)\overline{p}$. Similarly, an external agent will agree to sell his stock only at a price $p \geq (1 - \delta)\overline{p}$. Here $\delta > 0$ is a small constant, given a priori.

A more general assumption, considered in [11] for a one-side LOB, is that the maximum price acceptable to external agents is random. We assume here that an external buyer

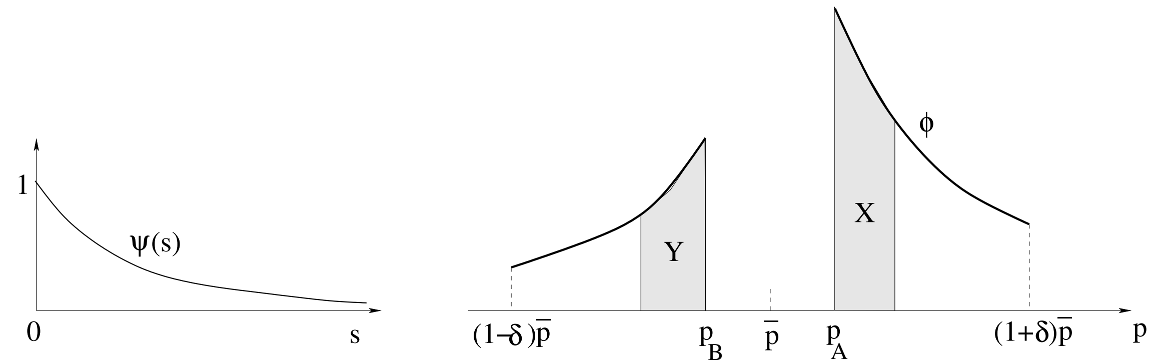

Figure 2.1. Left: a function representing the tail probability of the random variable X, describing the size of the external order. Right: a possible shape of the limit order book. If the external order is a buy order with size $X > 0$, all the stocks in the shaded region on the right (with area $= X$), will be sold. If the external order is a sell order for an amount $Y > 0$ of stocks, all the buy orders in the shaded region on the left (with area $= Y$), will be executed.

will agree to the transaction only at a price $p \leq Q^b \cdot \overline{p}$, where $Q^b \geq 1$ is a random variable. Similarly, an external seller will agree to the transaction only at a price $p \geq Q^s \cdot \overline{p}$, where Q^s is another random variable, ranging in $[0, 1]$.

In general, an external order is thus executed as follows (Fig. 2.1, right).

CASE 1: a buy order of size X. In this case the external buyer will take all stocks whose price ranges in the interval $[\beta_0,\, p(X)]$, where

$$p(X) \;=\; \sup\left\{ p > \beta_0\,;\;\; p \leq Q^b \cdot \overline{p},\;\;\; \int_{\beta_0}^{p} \phi(s)\, ds \;\leq\; X \right\}.$$

CASE 2: a sell order of size Y. In this case the external seller will fulfill all the bids whose price ranges in the interval $[p(Y),\, \beta_0]$, where

$$p(Y) \;=\; \inf\left\{ p < \beta_0\,;\;\; p \geq Q^s \cdot \overline{p},\;\;\; \int_{p}^{\beta_0} \phi(s)\, ds \;\leq\; Y \right\}.$$

Assume that, **after** the external order has been executed, the payoff for any player holding an amount c in cash and s in stock is given by

$$J \;=\; c + s\,\beta_0\,. \tag{2.5}$$

The following analysis will show that, given the distribution of the random variables X,

Y, the shape of the limit order book is entirely determined by β_0 and the quantities

$$\bar{x} \doteq \int_{\beta_0}^{+\infty} \phi(p)\,dp\,, \qquad \bar{y} \doteq \int_0^{\beta_0} \phi(p)\,dp, \tag{2.6}$$

respectively the total amount of stock in the "sell" and in the "buy" portion of the LOB.

The heart of the argument goes as follows. First, for a given mean price $\overline{p}$, we show that the both the "sell" and the "buy" portions of the LOB are uniquely determined. In particular, the minimum ask price p_A and the maximum bid price p_B are uniquely determined as functions of $\overline{p}$. In the case where

$$\frac{1}{2}\left(\frac{d}{d\overline{p}}\,p_A + \frac{d}{d\overline{p}}\,p_B\right) < 1\,, \tag{2.7}$$

the map $\overline{p} \mapsto \frac{p_A+p_B}{2}$ is a strict contraction, hence it has a unique fixed point. This will provide the unique shape of the LOB.

For sake of clarity, we first study the case where the maximum price accepted by external buyers and the minimum price accepted by external sellers are

$$p_{max} = (1+\delta_1)\overline{p}, \qquad p_{min} = (1-\delta_2)\overline{p}, \tag{2.8}$$

respectively, for some $\delta_1, \delta_2 > 0$. In the next section, we shall consider the general case where these prices are random.

(I) - Computing the "sell" portion of the LOB.

In the case of a buy order with random size X, let

$$\text{Prob.}\Big\{X > s\Big\} = \Psi(s), \qquad s \geq 0, \tag{2.9}$$

be the tail probability of this random variable, and assume

(A1) *The map $s \mapsto \Psi(s)$ is continuously differentiable and satisfies*

$$\Psi(0) = 1, \qquad \Psi(+\infty) = 0, \qquad \Psi'(s) < 0 \quad \textit{for all} \;\; s \in [0, +\infty[\,. \tag{2.10}$$

For $p > \beta_0$ we call

$$U(p) \doteq \int_{\beta_0}^{p} \phi(s)\,ds = [\text{amount of stock offered for sale at price} \leq p]. \tag{2.11}$$

The assumption that the LOB represent an equilibrium implies that the expected profit from a unit amount of stock put on sale is a constant on the support of U', i.e. on the set of prices at which some stock is offered for sale. If c and s are the amounts in cash and stock held by a player before the execution of the transaction, by (2.5) his payoff after selling a unit amount of stock at price p is

$$c + p + (s-1)\beta_0,$$

yielding a profit $p - \beta_0$. Since by (2.9), the probability that a stock offered at price $p \leq (1+\delta_1)\overline{p}$ will be sold is

$$Prob.\Big\{X > U(p)\Big\} \;=\; \Psi(U(p)), \tag{2.12}$$

the equilibrium condition implies

$$\Psi(U(p)) \cdot (p - \beta_0) \;=\; C \qquad \text{for all } p \in \text{supp}(U') \tag{2.13}$$

for some constant C independent of p. Differentiating (2.13) w.r.t. p we obtain an ODE for U, namely

$$U'(p) \;=\; -\frac{\Psi(U(p))}{\Psi'(U(p))} \cdot \frac{1}{p - \beta_0}\,. \tag{2.14}$$

Observe that by (A1) we have $\Psi' < 0$. hence the right hand side of (2.14) is non-negative for $p > \beta_0$.

As in (2.6), let $\bar{x}$ be the total amount of stock offered for sale in the LOB. Then the ODE (2.14) should be solved with the terminal condition

$$U\Big((1+\delta_1)\overline{p})\Big) \;=\; \bar{x}\,. \tag{2.15}$$

Since

$$p_A \;=\; \inf\Big\{p > \beta_0\;;\;\; U(p) > 0\Big\} \tag{2.16}$$

and, according to (2.11), the function U is absolutely continuous, we have $U(p_A) = 0$ and $\Psi(U(p_A)) = 1$. Hence, the constant C in (2.13) can be computed equivalently as

$$C \;=\; (p_A - \beta_0) \;=\; \Psi(\bar{x}) \cdot \Big((1+\delta_1)\overline{p} - \beta_0\Big). \tag{2.17}$$

This yields

$$p_A \;=\; (1+\delta_1)\Psi(\bar{x})\overline{p} + \Big(1 - \Psi(\bar{x})\Big)\beta_0, \tag{2.18}$$

$$\frac{d}{d\overline{p}}\, p_A \;=\; (1+\delta_1)\Psi(\bar{x}). \tag{2.19}$$

(II) - Computing the "buy" portion of the LOB.

Next, consider the "buy" portion of the LOB. In the case of an external sell order of random size Y, let the tail probability of this random variable be

$$\text{Prob.}\Big\{Y > s\Big\} \;=\; \Phi(s) \qquad\qquad s \geq 0\,. \tag{2.20}$$

We assume that the map $s \mapsto \Phi(s)$ satisfies the same conditions as in **(A1)**. Given a mean bid-ask price $\overline{p}$, the external agent will agree to the transaction only as long as the price ranges within an interval $[(1-\delta_2)\overline{p},\, \overline{p}]$.

In analogy with (2.11), for $p < \beta_0$ we call

$$U(p) \;\doteq\; \int_p^{\beta_0} \phi(s)\, ds \;=\; [\text{amount of stock that agents bid to buy at price} \geq p]. \tag{2.21}$$

If c and s are the amounts in cash and stock held by a player before the execution of the transaction, by (2.5) his payoff after buying at price p a quantity of stock corresponding to a unit amount of cash is

$$c - 1 + \left(s + \frac{1}{p}\right)\beta_0,$$

yielding a profit $\dfrac{\beta_0}{p} - 1$. Therefore, the expected profit from a unit amount of cash, bidding at price p, is

$$Prob.\ \{Y > U(p)\} \cdot \left(\frac{\beta_0}{p} - 1\right). \tag{2.22}$$

Since the expected profit in (2.22) is constant over the support of U', we have

$$\Phi(U(p)) \cdot \left(\frac{\beta_0}{p} - 1\right) \;=\; C \qquad \text{for all } p \in \text{supp}(U'), \tag{2.23}$$

for some constant C independent of p. Differentiating (2.23) w.r.t. p we obtain an ODE for U, namely

$$U'(p) \;=\; \frac{\Phi(U(p))}{\Phi'(U(p))} \cdot \frac{\beta_0}{p(\beta_0 - p)}\,. \tag{2.24}$$

Notice that here the right hand side is negative, because $p < \beta_0$ while $\Phi' < 0 < \Phi$. This is consistent with the definition (2.21).

Calling $\bar{y}$ the total amount of stock for which agents post buying bids, the above

ODE must be solved with the boundary condition

$$U\Big((1-\delta_2)\overline{p}\Big) \;=\; \bar{y}. \tag{2.25}$$

We have

$$p_B \;=\; \sup\Big\{p < \beta_0\;;\; U(p) > 0\Big\}. \tag{2.26}$$

Then the constant C in (2.23) can be equivalently computed as

$$C \;=\; \frac{\beta_0}{p_B} - 1 \;=\; \Phi(\bar{y})\cdot\left(\frac{\beta_0}{(1-\delta_2)\overline{p}} - 1\right). \tag{2.27}$$

This yields

$$p_B \;=\; \left(\frac{\Phi(\bar{y})}{(1-\delta_2)\overline{p}} + \frac{1-\Phi(\bar{y})}{\beta_0}\right)^{-1}. \tag{2.28}$$

Hence

$$\frac{d}{d\overline{p}}\,p_B \;=\; \frac{p_B^2}{(1-\delta_2)\overline{p}^2}\Phi(\bar{y})\,. \tag{2.29}$$

The previous analysis leads to

Theorem 1. *Assume that the random sizes X, Y of an external "buy" and a "sell" order have distributions given by (2.9), (2.20), respectively, and satisfy the assumptions* **(A1)**. *Moreover, assume that the external agent will agree to the transaction if the price is $\leq (1+\delta_1)\overline{p}$ in case of a buyer, and $\geq (1-\delta_2)\overline{p}$ in case of a seller, where $\overline{p}$ is the mean bid-ask price.*

Let $\bar{x}$, $\bar{y}$ be the total amount of stock for which selling bids and buying bids are posted on the LOB, respectively. Assume that

$$(1+\delta_1)\Psi(\bar{x}) + \frac{(1+\delta_2)^2}{1-\delta_2}\Phi(\bar{y}) \;<\; 2\,. \tag{2.30}$$

Then there exists a unique two-sided LOB satisfying (2.14) on the interval $[p_A, (1+\delta_1)\overline{p}]$ and (2.24) on $[(1-\delta_2)\overline{p},\, p_B]$.

Remark 1. In the above theorem, $\Psi(\bar{x})$ is the probability that an external buy order is so large that the entire "sell" portion of the LOB is wiped out, while $\Phi(\bar{y})$ is the probability that the external sell order is so large that the entire "buy" portion of the LOB is wiped out. In essence, the assumption (2.30) requires that the sizes $\bar{x}, \bar{y}$ of the LOB are large enough, compared with the random sizes of external orders. This is a natural assumption in our model. Indeed, in the case of a high probability of receiving large incoming orders, wiping out the entire “buy" or “sell" portion of the order book, the agents would be led to concentrate their pricing strategies “near" the limit acceptable value $(1-\delta_2)\bar{p}$ or $(1+\delta_1)\bar{p}$. This would cause an instability for the mean bid-ask price $\bar{p}$.

Proof of Theorem 1. 1. By the previous analysis, both sides of the LOB are uniquely determined as soon as the mean bid-ask price $\overline{p}$ is given, specifying that no sell order (resp. buy order) is posted in the LOB when $(1+\delta_1)\bar{p} < \beta_0$ (resp. $\beta_0 < (1-\delta_2)\bar{p}$). Recalling (2.18), (2.28), we set

$$p_A = \begin{cases} \beta_0 & \text{if } \bar{p} \le \frac{\beta_0}{1+\delta_1} \\ (1+\delta_1)\Psi(\bar{x})\overline{p} + \Big(1-\Psi(\bar{x})\Big)\beta_0 & \text{if } \bar{p} > \frac{\beta_0}{1+\delta_1} \end{cases}$$

and

$$p_B = \begin{cases} \Big(\frac{\Phi(\bar{y})}{(1-\delta_2)\overline{p}} + \frac{1-\Phi(\bar{y})}{\beta_0}\Big)^{-1} & \text{if } \bar{p} < \frac{\beta_0}{1-\delta_2} \\ \beta_0 & \text{if } \bar{p} \ge \frac{\beta_0}{1-\delta_2} . \end{cases}$$

The theorem can thus be proved by showing that the continuous map

$$\overline{p} \mapsto \frac{1}{2}p_A + \frac{1}{2}p_B \doteq F(\overline{p}) \tag{2.31}$$

has a unique fixed point.

2. We claim that the function F in (2.31) maps the interval

$$I \doteq \left[\frac{\beta_0}{1+\delta_2}, \frac{\beta_0}{1-\delta_1}\right]$$

into itself. Indeed, we have

$$(1+\delta_1)\Psi(\bar{x})\overline{p} + \Big(1-\Psi(\bar{x})\Big)\beta_0 = \beta_0 + \Big[(1+\delta_1)\bar{p} - \beta_0\Big]\psi(\bar{x}) \in \Big[\beta_0, (1+\delta_1)\overline{p}\Big] \tag{2.32}$$

if $(1+\delta_1)\overline{p} \ge \beta_0$ and

$$\left(\frac{\Phi(\bar{y})}{(1-\delta_2)\overline{p}} + \frac{1-\Phi(\bar{y})}{\beta_0}\right)^{-1} = \frac{(1-\delta_2)\bar{p}\beta_0}{(1-\delta_2)\bar{p} + \Big[\beta_0 - (1-\delta_2)\bar{p}\Big]\Phi(\bar{y})} \in \Big[(1-\delta_2)\bar{p}, \beta_0\Big] \tag{2.33}$$

if $\beta_0 \geq (1-\delta_2)\bar{p}$. Hence, combining (2.31), (2.32) and (2.33), we obtain for $\bar{p} \in I$

$$\frac{\beta_0}{1+\delta_2} \;\leq\; \frac{1}{2}\Big[\beta_0 + (1-\delta_2)\overline{p}\Big] \;\leq\; F(\overline{p}) \;\leq\; \beta_0 \;\leq\; \frac{\beta_0}{1-\delta_1} \qquad \text{if } \bar{p} \leq \frac{\beta_0}{1+\delta_1}\,,$$

$$\frac{\beta_0}{1+\delta_2} \;\leq\; \frac{1}{2}\Big[\beta_0+(1-\delta_2)\overline{p}\Big] \;\leq\; F(\overline{p}) \;\leq\; \frac{1}{2}\Big[(1+\delta_1)\overline{p}+\beta_0\Big] \;\leq\; \frac{\beta_0}{1-\delta_1} \qquad \text{if } \frac{\beta_0}{1+\delta_1} < \bar{p} < \frac{\beta_0}{1-\delta_2}\,,$$

$$\frac{\beta_0}{1+\delta_2} \;\leq\; \beta_0 \;\leq\; F(\overline{p}) \;\leq\; \frac{1}{2}\Big[(1+\delta_1)\overline{p}+\beta_0\Big] \;\leq\; \frac{\beta_0}{1-\delta_1} \qquad \text{if } \frac{\beta_0}{1-\delta_2} \leq \bar{p}\,.$$

By continuity, F has a fixed point.

3. Differentiating (2.31) w.r.t. $\overline{p}$ and using (2.19), (2.29), and (2.30), one obtains

$$\frac{d}{d\overline{p}}F(\overline{p}) \;\leq\; \frac{1}{2}\left((1+\delta_1)\Psi(\bar{x}) + \frac{p_B^2}{(1-\delta_2)\overline{p}^2}\Phi(\bar{y})\right) \;\leq\; \frac{1}{2}\left((1+\delta_1)\Psi(\bar{x}) + \frac{(1+\delta_2)^2}{1-\delta_2}\Phi(\bar{y})\right) \;<\; 1\,.$$

This proves that F is a strict contraction, having a unique fixed point $\overline{p} = \frac{p_A+p_B}{2}$. □

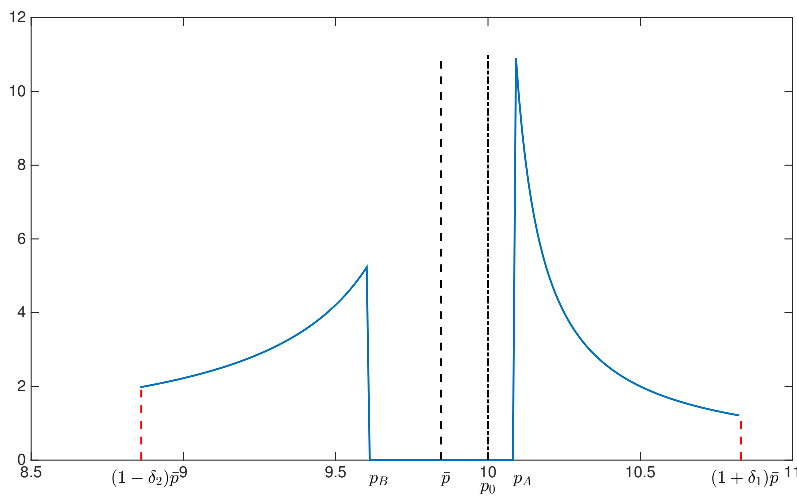

Figure 2.2. A plot of the density function ϕ, with data as in (2.39). In this case, solving (2.36)–(2.38) we find $p_A = 10.0831$, $p_B = 9.6097$, $\bar{p} = 9.8464$.

Example 1. In the case where the random incoming orders X, Y have exponential distribution, say $\Psi(s) = e^{-\mu s}$, $\Phi(s) = e^{-\eta s}$, the equations determining the shape of the LOB take a particularly simple form. Indeed, the ODE (2.14) determining the "sell" part of the LOB becomes

$$U'(p) \;=\; \frac{1}{\mu}\cdot\frac{1}{p-\beta_0}\,. \tag{2.34}$$

On the other hand, the ODE (2.24) determining the "buy" part of the LOB becomes

$$U'(p) \;=\; -\frac{1}{\eta}\cdot\frac{\beta_0}{p(\beta_0-p)}\,. \tag{2.35}$$

Let $\bar{x}$, $\bar{y}$ be the total amounts of stock on the "sell" and "buy" portions of the LOB, and let δ_1, δ_2 be given, as in (2.8).

The density function ϕ in (2.1), describing the two sides of the LOB, is here determined by

$$\phi(s) \;=\; \begin{cases} \dfrac{1}{\mu(s-\beta_0)} & \text{if} \quad s\in\Big[p_A\,,\ (1+\delta_1)\bar{p}\Big],\\ \dfrac{\beta_0}{\eta s(\beta_0-s)} & \text{if} \quad s\in\Big[(1-\delta_2)\bar{p}\,,\ p_B\Big],\\ 0 & \text{otherwise.}\end{cases}$$

The constants $\bar{p}, p_A, p_B$ are implicitly determined by the three equations

$$\bar{x} \;=\; \int_{p_A}^{(1+\delta_1)\bar{p}}\phi(s)\,ds \;=\; \frac{1}{\mu}\ln\frac{(1+\delta_1)\bar{p}-\beta_0}{p_A-\beta_0}\,, \tag{2.36}$$

$$\bar{y} \;=\; \int_{(1-\delta_2)\bar{p}}^{p_B}\phi(s)\,ds \;=\; \frac{1}{\eta}\ln\frac{\Big(\beta_0-(1-\delta_2)\bar{p}\Big)p_B}{(1-\delta_2)\bar{p}(\beta_0-p_B)}\,, \tag{2.37}$$

$$\bar{p} \;=\; \frac{p_A+p_B}{2}\,. \tag{2.38}$$

Figures 2.2 and 2.3 show the density ϕ and the integral functions U in (2.11) and (2.21), in the case where

$$\delta_1=\delta_2=\frac{1}{10}, \qquad \beta_0=10, \quad \eta=\frac{1}{2}, \quad \mu=1, \quad \bar{x}=\bar{y}=\ln 10. \tag{2.39}$$

We now consider the more general case where the maximum price $Q^b\cdot\bar{p}$ acceptable to a buyer and the minimum price $Q^s\cdot\bar{p}$ acceptable to a seller are random variables.

For example, one could let Q^b be a random variable such that

$$\text{Prob.}\Big\{Q^b>s\Big\} \;=\; h(s) \qquad\qquad s\geq 0. \tag{2.1}$$

Here $h(\cdot)$ is a continuous map, twice continuously differentiable on the open interval

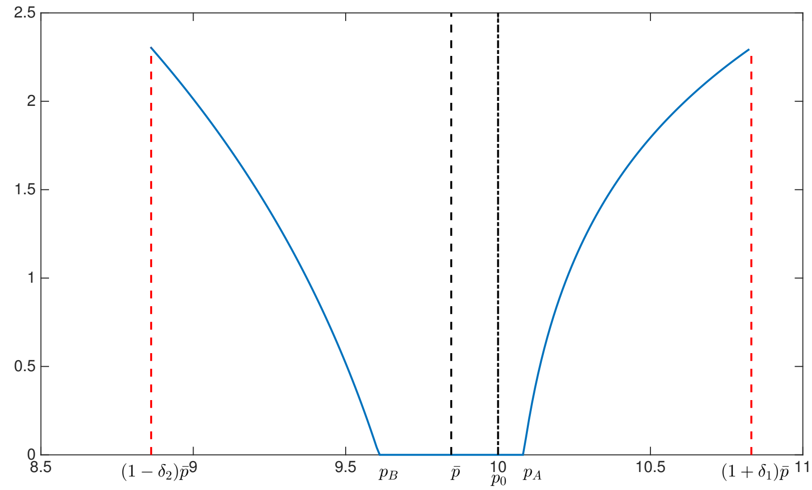

Figure 2.3. A plot of the functions $U(p)$ in (2.11) and (2.21), with data as in (2.39).

$s \in]1,\, 1+\delta_1[$ for some $\delta_1 \in]0,1[$, which satisfies

$$h(s) \;=\; 1 \quad s \in [0,1], \qquad h(s) \;=\; 0 \quad s \geq 1+\delta_1, \qquad h'(s) \;<\; 0 \quad s \in]1,\, 1+\delta_1[, \tag{2.2}$$

$$(\ln h(s))'' \leq 0 \qquad \text{for all } s \in]1,\, 1+\delta_1[\,. \tag{2.3}$$

A natural choice in (2.1) is

$$h(s) \;=\; \begin{cases} 1 & \text{if} \quad s \in [0,1], \\ 1 - \dfrac{s-1}{\delta_1} & \text{if} \quad s \in [1,\, 1+\delta_1], \\ 0 & \text{if} \quad s > 1+\delta_1\,. \end{cases} \tag{2.4}$$

In the following, we always assume that, after the external order has been executed, the payoff of any agent holding an amount c in cash and s in stock is given by (2.5).

(I) The "sell" portion of the LOB, with random acceptable prices.

As in (2.11), let $U(p)$ be the total amount of stock offered for sale at price $\leq p$. Assume that the maximum price accepted by an external buyer is $Q^b\,\overline{p}$, where Q^b is the random variable in (2.1). Moreover, assume that

$$\overline{p} \;\leq\; \left(1 + \frac{1}{\gamma - 1}\right)\beta_0\,, \tag{2.5}$$

where $\gamma > 1$ is defined by

$$\gamma \doteq \max\left\{\frac{1}{\delta_1}\,,\quad -h'(1+)\right\}. \tag{2.6}$$

The expected payoff for a seller asking a price p is

$$\Psi(U(p)) \cdot h\left(\frac{p}{\overline{p}}\right) \cdot (p - \beta_0) \;=\; C. \tag{2.7}$$

The assumption that the LOB represents an equilibrium implies that C is a constant independent of p. Differentiating (2.7) we thus obtain

$$U'(p) \;=\; -\frac{\Psi(U(p))}{\Psi'(U(p))} \cdot \left(\frac{1}{p-\beta_0} + \frac{1}{\overline{p}} \cdot \frac{h'(p/\overline{p})}{h(p/\overline{p})}\right). \tag{2.8}$$

Throughout the following we use the notation

$$a \vee b \;\doteq\; \max\{a, b\}, \qquad\qquad a \wedge b \;\doteq\; \min\{a, b\}.$$

Let $\overline{p}$ be given. For any $p \in \left]\beta_0 \vee \overline{p}\,,\ (1+\delta_1)\overline{p}\right[$, we define

$$\Lambda(p) \;\doteq\; \frac{1}{p-\beta_0} + \frac{1}{\overline{p}} \cdot \frac{h'(p/\overline{p})}{h(p/\overline{p})}\,. \tag{2.9}$$

Observe that $\Lambda(p) = \frac{1}{p-\beta_0}$ when $p < \overline{p}$. By (2.3), the map $p \mapsto \Lambda(p)$ is strictly decreasing. If $\overline{p} > \beta_0$, by (2.5) and (2.6) we have

$$\Lambda(\overline{p}+) \;=\; \frac{1}{\overline{p}-\beta_0} + \frac{1}{\overline{p}} \cdot h'(1+) \;\geq\; \frac{1}{\overline{p}-\beta_0} - \frac{1}{\overline{p}} \cdot \gamma \;=\; \frac{\beta_0\gamma - (\gamma-1)\overline{p}}{\overline{p}(\overline{p}-\beta_0)} \;\geq\; 0\,.$$

If $\overline{p} \leq \beta_0$, then $\Lambda(\beta_0+) \;=\; +\infty$. Moreover, by (2.2)-(2.3) and Gronwall's inequality it follows

$$\lim_{s \to (1+\delta_1)-} \frac{h'(s)}{h(s)} \;=\; -\infty\,.$$

Hence $\Lambda(p) \to -\infty$ as $p \to (1+\delta_1)\overline{p}-$.

By continuity and monotonicity, there exists a unique $p^\sharp \in \left[\beta_0 \vee \overline{p}\,,\ (1+\delta_1)\overline{p}\right[$ such that $\Lambda(p^\sharp) = 0$. It satisfies

$$\Lambda(p) \;>\; 0 \quad\Longleftrightarrow\quad p \in \left]\beta_0, p^\sharp\right[. \tag{2.10}$$

By the definition (2.11), the derivative of U must be positive. By (2.10), (2.8) and (2.10), no sell order can be posted at a price $p > p^\sharp$.

The ODE (2.8) should be solved with terminal condition

$$U(p^\sharp) \;=\; \bar{x}\,, \tag{2.11}$$

where $\bar{x}$ is the total amount of stocks offered for sale on the LOB. Call

$$p_A \;=\; \inf\Big\{p \in \,\Big]\beta_0\,,\, p^\sharp\Big[\,;\;\; U(p) > 0\Big\} \tag{2.12}$$

the minimum ask price. This implies $U(p_A) = 0$ and hence $\Psi(U(p_A)) = 1$. The constant C in (2.7) can be computed by taking $p = p_A$, so that

$$C \;=\; h\Big(\frac{p_A}{\overline{p}}\Big)(p_A - \beta_0). \tag{2.13}$$

Lemma 1. *Assume that, in addition to (2.2)-(2.3), the function h satisfies*

$$[(\ln h)']^2(s) + (\ln h)'(s) + (s-1)\cdot(\ln h)''(s) \;\geq\; 0\,, \tag{2.14}$$

for all $s \in]1,\; 1+\delta_1[$. Then $0 < \frac{d}{d\overline{p}}\,p_A < 1$.

Proof. For any fixed $\overline{p}$, we have

$$-\ln\Psi(\bar{x}) \;=\; \int_{p_A}^{p^\sharp} -\frac{\Psi'}{\Psi}(U(p))\cdot U'(p)\,dp \;=\; \int_{p_A}^{p^\sharp} \Lambda(p)\,dp\,. \tag{2.15}$$

Differentiating (2.15) w.r.t. $\overline{p}$ and recalling that $\Lambda(p^\sharp) = 0$, we obtain

$$0 \;=\; \int_{p_A}^{p^\sharp} \frac{\partial}{\partial\overline{p}}\Lambda(p)\,dp - \frac{d}{d\overline{p}}\,p_A\cdot\Lambda(p_A)\,. \tag{2.16}$$

The assumptions (2.2), (2.3) and the identity (2.16) imply that $0 < \frac{d}{d\overline{p}}\,p_A$. Moreover

$$\frac{d}{d\overline{p}}\,p_A \;=\; \Big[\int_{p_A}^{p^\sharp}\frac{\partial}{\partial\overline{p}}\Lambda(p)\,dp\Big]\Big/\Lambda(p_A) \;\leq\; \Big[\int_{p_A}^{p^\sharp}\frac{\partial}{\partial\overline{p}}\Lambda(p)\,dp\Big]\Big/\Lambda(p_A\vee\overline{p}) \tag{2.17}$$

$$\;=\; \Big[\int_{p_A\vee\overline{p}}^{p^\sharp}\frac{\partial}{\partial\overline{p}}\Lambda(p)\,dp\Big]\Big/\Big[\int_{p_A\vee\overline{p}}^{p^\sharp}-\frac{\partial}{\partial p}\Lambda(p)\,dp\Big]\,. \tag{2.18}$$

Fix $p \in \,]p_A \vee \overline{p}, p^\sharp[$. Since

$$0 \;<\; \Lambda(p) \;=\; \frac{1}{p-\beta_0} + \frac{1}{\overline{p}} \cdot \frac{h'}{h}\Big(\frac{p}{\overline{p}}\Big)\,,$$

one has

$$\frac{\partial}{\partial p}\Lambda(p) \;=\; -\frac{1}{(p-\beta_0)^2} + \frac{1}{\overline{p}^2}\cdot\Big(\frac{h'}{h}\Big)'\Big(\frac{p}{\overline{p}}\Big) \;<\; \frac{1}{\overline{p}^2}\cdot\Big[-\Big(\frac{h'}{h}\Big)^2\Big(\frac{p}{\overline{p}}\Big) + \Big(\frac{h'}{h}\Big)'\Big(\frac{p}{\overline{p}}\Big)\Big]. \quad (2.19)$$

By (2.19) and the assumption (2.14) we obtain

$$\frac{\partial}{\partial p}\Lambda(p) \;<\; \frac{1}{\overline{p}^2}\cdot\Big[\frac{h'}{h}\Big(\frac{p}{\overline{p}}\Big) + \frac{p}{\overline{p}}\cdot\Big(\frac{h'}{h}\Big)'\Big(\frac{p}{\overline{p}}\Big)\Big] \;=\; -\frac{\partial}{\partial \overline{p}}\Lambda(p) \qquad (2.20)$$

for every $p \in \,]p_A, p^\sharp[$. Therefore, by (2.17) and (2.20) we have $\frac{d}{d\overline{p}}\,p_A < 1$. □

Example 2. In the case where the random variable Q^b is uniformly distributed over the interval $[1, 1+\delta_1]$, i.e. the map h is given by (2.4), the condition (2.14) is satisfied whenever $\delta_1 \leq 1$. Indeed, one can compute

$$p^\sharp \;=\; \frac{1}{2}\Big(\beta_0 + (1+\delta_1)\overline{p}\Big)\,.$$

However, notice that we cannot have $\frac{d}{d\overline{p}}\,p_A < 1$ if $\delta_1 > 1$ and the total amount $\bar{x}$ of stock put on sale on the LOB is very small.

On the other hand, for any $0 < \delta_1 \leq \mu$, all the assumptions in Lemma 1 are satisfied by taking

$$h(s) \;=\; \begin{cases} 1 & \text{if} \quad s \in [0,1], \\ \Big(\dfrac{1+\delta_1 - s}{\delta_1}\Big)^{\mu} & \text{if} \quad s \in [1, 1+\delta_1], \\ 0 & \text{if} \quad s > 1+\delta_1\,. \end{cases} \qquad (2.21)$$

(II) The “buy" portion of the LOB, with random acceptable prices.

Given a mean bid-ask price $\overline{p}$, we assume that the external agent will agree to the transaction only as long as the price ranges within an interval $[Q^s\,\overline{p},\, \overline{p}]$, where $0 < Q^s < 1$ is a random variable independent of Q^b. Let

$$\text{Prob.}\Big\{Q^s < s\Big\} \;=\; g(s), \qquad\qquad s \geq 0, \qquad (2.22)$$

and assume that the map $g(\cdot)$ is continuous, $\mathcal{C}^2$ on some interval $]1-\delta_2, 1[$, with $0<\delta_2<1/3$, and satisfies

$$g(s) \;=\; 0 \quad s\in[0,1-\delta_2], \qquad g(s) \;=\; 1 \quad s\geq 1, \qquad g'(s) \;>\; 0 \quad s\in]1-\delta_2,\, 1[\,, \tag{2.23}$$

$$(\ln g(s))'' \leq 0 \qquad \text{for all } s\in]1-\delta_2, 1[\,. \tag{2.24}$$

Furthermore, we assume that $\overline{p}$ is such that

$$\overline{p} \;\geq\; \beta_0\left(1-\frac{1}{\sigma}\right), \tag{2.25}$$

where $\sigma>2$ is defined by

$$\sigma \;\doteq\; \max\left\{\frac{2(1-\delta_2)}{1-2\delta_2}\,,\; g'(1-)\right\}. \tag{2.26}$$

In particular, we have

$$(1-\delta_2)\,\overline{p} \;\geq\; \frac{\beta_0}{2}\,. \tag{2.27}$$

As in (2.21), we denote by $U(p)$ the total amount of stock that agents are offering to buy at price $>p$.

The expected profit from a unit amount of cash bidding at a price p is

$$Prob.\ \{X>U(p)\}\cdot Prob.\left\{p>Q^s\,\overline{p}\right\}\cdot\left(\frac{\beta_0}{p}-1\right). \tag{2.28}$$

Since the expected profit in (2.28) is constant over the support of U', we have

$$\Phi(U(p))\cdot g\left(\frac{p}{\overline{p}}\right)\cdot\left(\frac{\beta_0}{p}-1\right) \;=\; C, \tag{2.29}$$

for some constant C. Differentiating (2.29) w.r.t. p we obtain an ODE for U, namely

$$U'(p) \;=\; \frac{\Phi(U(p))}{\Phi'(U(p))}\cdot\left(\frac{\beta_0}{p(\beta_0-p)}-\frac{1}{\overline{p}}\cdot\frac{g'(p/\overline{p})}{g(p/\overline{p})}\right). \tag{2.30}$$

For every $p\in\Big](1-\delta_2)\overline{p}\,,\ \beta_0\wedge\overline{p}\Big[\setminus\{\overline{p}\}$, define

$$\Lambda(p) \;=\; \frac{\beta_0}{p(\beta_0-p)}-\frac{1}{\overline{p}}\cdot\frac{g'(p/\overline{p})}{g(p/\overline{p})}\,. \tag{2.31}$$

Observe that, under the assumptions (2.24) and (2.27), the map $p \mapsto \Lambda(p)$ is strictly increasing. If $\overline{p} < \beta_0$, by (2.25) and (2.26) we have

$$\Lambda(\overline{p}-) \;=\; \frac{\beta_0}{\overline{p}(\beta_0-\overline{p})} - \frac{1}{\overline{p}}\cdot g'(1-) \;\geq\; \frac{\beta_0}{\overline{p}(\beta_0-\overline{p})} - \frac{1}{\overline{p}}\cdot \sigma \;=\; \frac{\beta_0(1-\sigma)+\sigma\overline{p}}{\overline{p}(\beta_0-\overline{p})} \;\geq\; 0\,.$$

If $\overline{p} \geq \beta_0$, then $\Lambda(\beta_0-) = +\infty$. Moreover, observe that (2.23)-(2.24) and Gronwall's lemma imply that $\lim_{s\to(1-\delta_2)+} \frac{g'(s)}{g(s)} = +\infty$. Hence $\Lambda(p) \to -\infty$ as $p \to (1-\delta_2)\,\overline{p}+$.

By continuity and monotonicity, there exists a unique $p^\flat \in \Big](1-\delta_2)\overline{p},\ \beta_0\wedge\overline{p}\Big]$ such that $\Lambda(p^\flat) = 0$. One has

$$\Lambda(p) \;>\; 0 \quad\Longleftrightarrow\quad p \in \Big]p^\flat, \beta_0\Big[\,. \tag{2.32}$$

By the definition (2.21), the derivative of U must be negative. By **(A1)**, (2.30) and (2.32), no buy order can be posted at a price $p < p^\flat$.

The ODE (2.30) must be solved with terminal condition

$$U(p^\flat) \;=\; \bar{y}\,, \tag{2.33}$$

where $\bar{y}$ is the total amount of stocks for which bids are posted in the LOB. Call

$$p_B \;=\; \sup\Big\{p \in \Big]p^\flat\,,\,\beta_0\Big[\,;\;\; U(p) > 0\Big\} \tag{2.34}$$

the maximum bid price. By **(A1)**, we have that $U(p_B) = 0$. In this setting, the expected profit in (2.29) from a unit amount of cash can be computed by taking $p = p_B$, namely

$$C \;=\; g\Big(\frac{p_B}{\overline{p}}\Big)\cdot\Big(\frac{\beta_0}{p_B} - 1\Big)\,. \tag{2.35}$$

Lemma 2. *Assume that the function g in (2.22) satisfies*

$$\frac{1}{s^2} + (\ln g)''(s) - \frac{1}{4}[(\ln g)']^2(s) \;\leq\; (\ln g)'(s) + s\,(\ln g)''(s) \;<\; 0 \qquad \textit{for all } s \in]1-\delta_2, 1[\,. \tag{2.36}$$

Then $0 < \frac{d}{d\overline{p}}\, p_B < 1$.

Proof. For any $\overline{p}$, we have

$$-\ln\Phi(\bar{y}) \;=\; \int_{p^\flat}^{p_B} \frac{\Phi'}{\Phi}(U(p))\cdot U'(p)\,dp \;=\; \int_{p^\flat}^{p_B} \Lambda(p)\,dp\,. \tag{2.37}$$

Differentiating (2.37) w.r.t. $\overline{p}$ and recalling that $\Lambda(p^\flat) = 0$, we obtain

$$0 \;=\; \int_{p^\flat}^{p_B} \frac{\partial}{\partial \overline{p}} \Lambda(p)\, dp + \frac{d}{d\overline{p}}\, p_B \cdot \Lambda(p_B)\,. \tag{2.38}$$

The second inequality in (2.36) and (2.38) imply $0 < \frac{d}{d\overline{p}}\, p_B$. Moreover,

$$\frac{d}{d\overline{p}}\, p_B \;=\; \left[\int_{p^\flat}^{p_B} -\frac{\partial}{\partial \overline{p}} \Lambda(p)\, dp\right] \Big/ \Lambda(p_B) \;\leq\; \left[\int_{p^\flat}^{p_B} -\frac{\partial}{\partial \overline{p}} \Lambda(p)\, dp\right] \Big/ \Lambda(\overline{p} \wedge p_B) \tag{2.39}$$

$$\;=\; \left[\int_{p^\flat}^{\overline{p}\wedge p_B} -\frac{\partial}{\partial \overline{p}} \Lambda(p)\, dp\right] \Big/ \left[\int_{p^\flat}^{\overline{p}\wedge p_B} \frac{\partial}{\partial p} \Lambda(p)\, dp\right]. \tag{2.40}$$

Fix $p \in \left]p^\flat\,,\, \overline{p} \wedge p_B\right[$. Since

$$0 \;<\; \Lambda(p) \;=\; \frac{\beta_0}{p(\beta_0 - p)} - \frac{1}{\overline{p}} \cdot \frac{g'}{g}\left(\frac{p}{\overline{p}}\right),$$

and by (2.27), one has

$$\begin{aligned}
\frac{\partial}{\partial p} \Lambda(p) \;&=\; -\frac{1}{p^2} + \frac{1}{(\beta_0 - p)^2} - \frac{1}{\overline{p}^2} \cdot \left(\frac{g'}{g}\right)'\left(\frac{p}{\overline{p}}\right) \\
&>\; -\frac{1}{p^2} + \frac{1}{\beta_0^2} \cdot \frac{p^2}{\overline{p}^2} \left(\frac{g'}{g}\right)^2\left(\frac{p}{\overline{p}}\right) - \frac{1}{\overline{p}^2} \cdot \left(\frac{g'}{g}\right)'\left(\frac{p}{\overline{p}}\right) \\
&>\; -\frac{1}{p^2} + \frac{1}{4\overline{p}^2} \left(\frac{g'}{g}\right)^2\left(\frac{p}{\overline{p}}\right) - \frac{1}{\overline{p}^2} \cdot \left(\frac{g'}{g}\right)'\left(\frac{p}{\overline{p}}\right).
\end{aligned} \tag{2.41}$$

The inequality (2.41) and the assumption (2.36) yield

$$\frac{\partial}{\partial p} \Lambda(p) \;>\; -\frac{1}{\overline{p}^2} \cdot \left[\frac{g'(p/\overline{p})}{g(p/\overline{p})} + \frac{p}{\overline{p}} \cdot \left(\frac{g'}{g}\right)'\left(\frac{p}{\overline{p}}\right)\right] \;=\; -\frac{\partial}{\partial \overline{p}} \Lambda(p) \tag{2.42}$$

for every $p \in]p^\flat, p_B[$. Therefore, by (2.39) and (2.42) we have $\frac{d}{d\overline{p}}\, p_B < 1$. □

Example 3. Consider a random variable Q^s which is uniformly distributed over the interval $[1 - \delta_2, 1]$, so that g is given by

$$g(s) \;=\; \begin{cases} 0 & \text{if} \quad s \in [0, 1 - \delta_2], \\ \delta_2^{-1}(s - 1 + \delta_2) & \text{if} \quad s \in [1 - \delta_2, 1], \\ 1 & \text{if} \quad s > 1\,. \end{cases} \tag{2.43}$$

Then the condition (2.36) is satisfied.

Based on the previous analysis, we can now prove

Theorem 2. *Assume that the random sizes X, Y of an external "buy" and a "sell" order have distributions given by (2.9), (2.20), respectively, and satisfy the assumptions* **(A1)**. *Moreover, assume that the external agent will agree to the transaction if the price is $\leq Q^b\overline{p}$ in case of a buyer, and $\geq Q^s\overline{p}$ in case of a seller, where $\overline{p}$ is the mean bid-ask price, Q^b is a random variable in (2.1) satisfying (2.2), (2.3), (2.14), and Q^s is a random variable in (2.22) satisfying (2.23), (2.24) and (2.36).*

Then for any given sizes $\bar{x}, \bar{y} > 0$ of the "sell" and of the "buy" portions of the LOB, the mean bid-ask price $\overline{p}$ and the two-sided LOB are uniquely determined.

Proof. 1. For any choice of the mean price $\overline{p}$, the minimum ask price p_A and the maximum bid price p_B are uniquely determined by solving the Cauchy problem (2.8), (2.11), and the Cauchy problem (2.30), (2.33), respectively.

Let γ and σ as in (2.6) and (2.26). Consider the interval

$$I \doteq \left[\Big(1 - \frac{1}{\sigma}\Big)\beta_0\,,\ \Big(1 + \frac{1}{\gamma - 1}\Big)\beta_0\right] \tag{2.44}$$

and define the map

$$\bar{p} \mapsto F(\bar{p}) = \frac{p_A + p_B}{2}\,,$$

where p_A and p_B were defined at (2.12) and (2.34), respectively. The proof will be achieved by showing that F maps I into itself and has a unique fixed point.

2. If $-h'(1+) \geq \frac{1}{\delta_1}$, then $\gamma = -h'(1+)$ and

$$F(\overline{p}) = \frac{p_B + p_A}{2} \leq \frac{\beta_0 + p^\sharp}{2}\,. \tag{2.45}$$

As in (2.10), here $p^\sharp = p^\sharp(\overline{p})$ is the unique point where the map $p \mapsto \Lambda(p)$ in (2.9) vanishes. By (2.3) we have

$$0 = \Lambda(p^\sharp) = \frac{1}{p^\sharp - \beta_0} + \frac{1}{\overline{p}} \cdot \frac{h'(p^\sharp/\overline{p})}{h(p^\sharp/\overline{p})} \leq \frac{1}{p^\sharp - \beta_0} + \frac{1}{\overline{p}} \cdot h'(1+)\,.$$

This yields

$$p^\sharp \leq \beta_0 + \frac{\overline{p}}{-h'(1+)}\,. \tag{2.46}$$

Combining (2.45) and (2.46), we have

$$F(\overline{p}) \;\leq\; \beta_0 + \frac{\overline{p}}{-2h'(1+)} \;\leq\; \beta_0\Big(1 + \frac{1}{2(\gamma-1)}\Big).$$

If $-h'(1+) \leq \frac{1}{\delta_1}$, then $\gamma = \frac{1}{\delta_1}$ and

$$F(\overline{p}) \;=\; \frac{p_B + p_A}{2} \;\leq\; \frac{\beta_0 + (1+\delta_1)\overline{p}}{2} \leq \beta_0\Big(1+\frac{\delta_1}{2}+\frac{1+\delta_1}{2(\gamma-1)}\Big) \;=\; \beta_0\Big(1+\frac{1}{\gamma-1}\Big). \tag{2.47}$$

3. Next, we prove that $F(\overline{p}) \geq \beta_0\big(1 - \frac{1}{\sigma}\big)$ for all $\overline{p} \in I$. If $g'(1-) \geq \frac{2(1-\delta_2)}{1-2\delta_2}$, then $\sigma = g'(1-)$ and

$$F(\overline{p}) \;=\; \frac{p_B + p_A}{2} \;\geq\; \frac{p^\flat + \beta_0}{2}. \tag{2.48}$$

As in (2.32), let $p^\flat = p^\flat(\overline{p})$ be the point where the function Λ in (2.31) vanishes. Since $\frac{d}{d\overline{p}}p^\flat > 0$, it will be sufficient to check that

$$p^\flat \;\geq\; \beta_0\Big(1 - \frac{2}{\sigma}\Big) \tag{2.49}$$

in the case where $\overline{p} < \beta_0$.

By (2.24), we have

$$0 \;=\; \Lambda(p^\flat) \;=\; \frac{\beta_0}{p^\flat(\beta_0 - p^\flat)} - \frac{1}{\overline{p}}\cdot\frac{g'(p^\flat/\overline{p})}{g(p^\flat/\overline{p})} \;\leq\; \frac{\beta_0}{p^\flat(\beta_0 - p^\flat)} - \frac{1}{\overline{p}}\cdot g'(1-). \tag{2.50}$$

Moreover, by (2.50), (2.27) and the definition of $p^\flat$, we obtain

$$\beta_0 - p^\flat \;\leq\; \frac{\beta_0}{\sigma}\frac{\overline{p}}{p^\flat} \;<\; \frac{\beta_0}{\sigma p^\flat}\beta_0 \;\leq\; \frac{\beta_0}{\sigma p^\flat}2(1-\delta_2)\overline{p} \;<\; \frac{2\beta_0}{\sigma}. \tag{2.51}$$

Hence (2.49) holds. Combining (2.48) and (2.49) one obtains

$$F(\overline{p}) \;\geq\; \beta_0\Big(1 - \frac{1}{\sigma}\Big).$$

In the remaining case where $g'(1-) < \frac{2(1-\delta_2)}{1-2\delta_2}$, one has $\sigma = \frac{2(1-\delta_2)}{1-2\delta_2}$ and

$$p_B \;>\; (1-\delta_2)\overline{p} \;\geq\; \beta_0\Big(1 - \delta_2 - \frac{1-\delta_2}{\sigma}\Big) \;=\; \frac{\beta_0}{2}.$$

Therefore

$$F(\overline{p}) \;=\; \frac{p_B + p_A}{2} \;\geq\; \frac{3}{4}\beta_0 \;\geq\; \frac{\beta_0}{2(1-\delta_2)} \;=\; \beta_0\Big(1 - \frac{1}{\sigma}\Big),$$

since $0 < \delta_2 < 1/3$.

4. By the previous two steps, F maps the closed interval I in (2.44) into itself. Hence it has a fixed point. By Lemmas 1 and 2 we have $0 \;<\; \frac{d}{d\bar{p}} F(\bar{p}) \;<\; 1$. Hence the map F is a strict contraction, with a unique fixed point. □

Chapter 3
The Dynamic Model

1 Introduction

In Chapter 3, we consider a time-dependent problem involving sequence of N incoming orders $X_1, \ldots, X_N$. Each can be either a buy order or a sell order. The random variables X_j, $j = 1, \ldots, N$, describing the amount of stock that the external agents want to buy (or sell), are assumed to be mutually independent.

Again, we seek conditions ensuring that, at each time step $i = 1, \ldots, N$, the shape of the two-sided LOB can be uniquely determined, by backward induction. We point out a major difference between the "one-shot" game and the dynamic model involving multiple time steps. Namely, in a game involving one single external order, the payoff for a player holding an amount c of cash and an amount s of stock is

$$J \;=\; c + \beta_0 s,$$

where β_0 is an underlying fundamental value of the stock, known to all agents posting bids on the LOB. On the other hand, at an intermediate time i, this expected payoff will have the more general form

$$J_i \;=\; c \cdot V_i^C(x, y) + s \cdot V_i^S(x, y).$$

Here $V_i^C(x, y)$ and $V_i^S(x, y)$ denote the expected payoffs to an agent that holds a unit amount of cash or stock at the i-th time step, assuming that the sizes of the "sell" and "buy" portions of the LOB at that time are x, y respectively. This reflects the fact that, during the time periods $i + 1, \ldots, N$, an agent can achieve some additional profits by repeatedly buying and selling stock on the LOB at favorable prices. As already shown in [11], these expected profits strongly depend on the size of the LOB. As the total amount of bids posted on the LOB increases, there is a stronger competition among agents, and hence a smaller expected profit for each one of them.

A detailed description the evolution model for the two-sided LOB is given in Section 2. Finally, in Section 3 we derive conditions for the existence of a unique shape of the two-sided LOB, together with a priori bounds on the value functions V_i^C, V_i^S.

2 The repeated game and its value function

We now consider a repeated game, including a sequence of N random incoming orders $X_1, \ldots, X_N$. Assume that the X_i are independent, identically distributed random variables. In addition, at each time t_i, $i = 1, 2, \ldots, N-1$, agents can post on the LOB new sell or buy orders.

If $\overline{p} = \frac{p_A + p_B}{2}$ is the mean bid-ask price, we assume that external buyers and external sellers will agree to the transaction if the price is $\leq (1+\delta_1)\overline{p}$, and $\geq (1-\delta_2)\overline{p}$, respectively.

The state variable. At each time t_i, the state is described by two positive variables: (x_i, y_i), where

- x_i is the total amount of stock in the "sell" portion of the LOB, at time t_i,
- y_i is the total amount of stock in the "buy" portion of the LOB, at time t_i.

The evolution equation. At each time t_i, an external buy order of random size X_i, or a sell order of size Y_i will arrive. After this order is executed, the corresponding part of the LOB shrinks in size, while the other portion remains unchanged. More precisely, using the notation $a_+ \doteq \max\{a, 0\}$, the new sizes are

$$\begin{cases} \tilde{x}_i &= (x_i - X_i)_+ \,, \\ \tilde{y}_i &= y_i \,, \end{cases} \qquad\qquad \begin{cases} \tilde{x}_i &= x_i, \\ \tilde{y}_i &= (y_i - Y_i)_+ \,, \end{cases}$$

in case of a buy order or a sell order, respectively.

To account for the fact that agents can post new sell or buy orders on the LOB (or remove some of the old ones), we consider a transition probability density $f(x, y;\, \tilde{x}, \tilde{y})$. Here

$$\text{Prob.}\Big\{ x_{i+1} \leq \xi, , \quad y_{i+1} \leq \eta \;\Big|\; \tilde{x}_i = \tilde{x}\,, \quad \tilde{y}_i = \tilde{y} \Big\} \;=\; \int_0^\xi \int_0^\eta f(x, y;\, \tilde{x}, \tilde{y})\, dx dy\,. \qquad (3.1)$$

If one assumes that limit orders are never removed (unless they are executed), then one has the implication

$$x < \tilde{x} \quad \text{or} \quad y < \tilde{y} \qquad \Longrightarrow \qquad f(x, y; \tilde{x}, \tilde{y}) \;=\; 0.$$

Let $P^s \in [0, 1]$ be the probability that at time t_i a "sell" order arrives, and let $P^b = 1 - P^s$ be the probability that at time t_i a "buy" order arrives. Here P^s and P^b are fixed constants. Then the sizes of the "buy" and "sell" portions of the LOB are described by a Markov process.

The value functions. Consider any point (x, y) in state space. For any $i = 1, 2, \ldots, N$, we denote by

$$V_i^C(x, y), \qquad\qquad V_i^S(x, y), \tag{3.2}$$

the maximum expected payoffs that an agent can achieve at the terminal time t_N, provided that at time t_i

- the two portions of the LOB have sizes x, y, and
- the agent owns a unit of cash, or a unit of stock, respectively

We wish to describe the evolution of the LOB, in terms of the following data:

- The random variables X, Y, describing the size of the external (buy or sell) orders.
- The transition probability density $f(\cdot, \cdot; x, y)$, describing the new limit orders posted in the LOB.
- The terminal value $\bar{b}$ of a unit of stock.

This should be solved by backward induction, computing the value functions V_i^C, V_i^S for $i = N, N-1, \ldots, 2, 1$. The terminal conditions imply that at the final time $t = t_N$ one has

$$V_N^C(x, y) \;\equiv\; 1, \qquad\qquad V_N^S(x, y) \;\equiv\; \bar{b}. \tag{3.3}$$

Assume that the value functions V_{i+1}^C, V_{i+1}^S are known. At time t_i, let the "sell" and "buy" portions of the LOB have sizes (x_i, y_i). To compute $V_i^C(x_i, y_i)$ we proceed as follows. First, assume that at time t_i a buying order arrives, of random size X_i. The portion

$\widetilde{X}_i = \min\{X_i, x_i\}$ of this order will be executed. The expected payoffs, for an agent holding a unit of stock or a unit of cash at time t_{i+1}, are thus computed as

$$E^{X_i}\left[\int V_{i+1}^S(x,y)\, f\Big(x,y;\, (x_i - X_i)_+\,, y_i\Big)\, dxdy\right], \tag{3.4}$$

$$E^{X_i}\left[\int V_{i+1}^C(x,y)\, f\Big(x,y;\, (x_i - X_i)_+\,, y_i\Big)\, dxdy\right]. \tag{3.5}$$

Next, assume that at time t_i a sell order arrives, of random size Y_i. The portion $\widetilde{Y}_i = \min\{Y_i, y_i\}$ of this order will be executed. The expected payoffs, for an agent holding a unit of stock or a unit of cash at time t_{i+1}, are then computed as

$$E^{Y_i}\left[\int V_{i+1}^S(x,y)\, f\Big(x,y;\, x_i, (y_i - Y_i)_+\Big)\, dxdy\right], \tag{3.6}$$

$$E^{Y_i}\left[\int V_{i+1}^C(x,y)\, f\Big(x,y;\, x_i, (y_i - Y_i)_+\Big)\, dxdy\right]. \tag{3.7}$$

3 Dynamic evolution of the LOB

Assume that the values of a unit of cash $V^C = V_{i+1}^C(\xi,\eta)$ and the value of a unit of stock $V^S = V_{i+1}^S(\xi,\eta)$ at time $t = t_{i+1}$ are known, depending on the sizes (ξ,η) of the two parts of the LOB at time t_{i+1}. Moreover, let x, y be the sizes of the "sell" and "buy" portions of the LOB at time t_i. We wish to find the shape of the LOB at time t_i.

3.1 The "sell" portion of the LOB.

As in (2.9), let the random variable X describe the size of the incoming "buy" order. Moreover, let $U(p)$ be the amount of stock offered for sale at price $\leq p$, as in (2.11). If a player puts on sale on the LOB at time t_i a unit of stock at price p, his payoff after the execution of the external buy order X is

$$p \cdot \iint V^C(\xi,\eta) f(\xi,\eta;\, (x-X)_+, y)\, d\xi d\eta$$

if the transaction takes place, i.e. if $X > U(p)$, or

$$\iint V^S(\xi,\eta) f(\xi,\eta;\, (x-X)_+, y)\, d\xi d\eta$$

if $X < U(p)$. Then the expected payoff by putting a unit of stock on sale at price p is

$$\begin{aligned} & p\cdot \text{Prob.}\{X > U(p)\}\cdot E\Big[V^C \;\Big|\; X > U(p)\Big] + \text{Prob.}\{X < U(p)\}\cdot E\Big[V^S \;\Big|\; X < U(p)\Big] \\ & = \; p\cdot \int_{U(p)}^{\infty}\Big(\iint V^C(\xi,\eta) f(\xi,\eta;\,(x-s)_+,y)\,d\xi d\eta\Big)\cdot\Big(-\Psi'(s)\Big)ds \\ & \quad + \int_0^{U(p)}\Big(\iint V^S(\xi,\eta) f(\xi,\eta;\,(x-s)_+,y)\,d\xi d\eta\Big)\cdot\Big(-\Psi'(s)\Big)ds\,. \end{aligned} \tag{3.8}$$

Notice that in the case where $V^C \equiv a$ and $V^S \equiv b$ are constant, the quantity in (3.8) reduces to

$$p\,\Psi(U(p))\cdot a + (1-\Psi(U(p))\cdot b\,.$$

Assuming that the LOB is a Nash equilibrium, we deduce that the quantity in (3.8) is constant on the support of U' (i.e., it is constant on the set of all prices at which some stock is actually offered for sale). Differentiating the right hand side of (3.8) w.r.t. p, one obtains

$$\begin{aligned} 0 \;=\; & \int_{U(p)}^{\infty}\Big(\iint V^C(\xi,\eta) f\Big(\xi,\eta;\,(x-s)_+,y\Big)\,d\xi d\eta\Big)\cdot\Big(-\Psi'(s)\Big)ds \\ & + U'(p)\,\Psi'(U(p))\cdot\iint\Big(pV^C(\xi,\eta)-V^S(\xi,\eta)\Big) f\Big(\xi,\eta;\,(x-U(p))_+,y\Big)\,d\xi d\eta\,. \end{aligned} \tag{3.9}$$

Notice again that, in the case where $V^C \equiv a$ and $V^S \equiv b$, the above equation reduces to

$$\Psi(U(p)) + U'(p)\Psi'(U(p))\Big(p - \frac{b}{a}\Big) \;=\; 0,$$

which yields (2.14), with $\beta_0 = b/a$.

We regard (3.9) as an ODE for the function $U(p)$, where the right hand side depends on p, x, y and on the functions V^C, V^S. This must be solved with boundary condition

$$U\Big((1+\delta_1)\overline{p}\Big) \;=\; x\,. \tag{3.10}$$

3.2 The “buy" portion of the LOB.

As in (2.20), let Y be the random size of the incoming “sell"order. Moreover, let $U(p)$ be the total amount of stock that agents bid to buy at price $\geq p$, as in (2.21). If a player bids to buy a unit of stock at price p, his payoff corresponding to a unit amount of cash after the execution of the external sell order Y is

$$\frac{1}{p} \cdot \iint V^S(\xi,\eta) f(\xi,\eta;\, x, (y-Y)_+)\, d\xi d\eta$$

if the transaction takes place, i.e. if $Y > U(p)$, or

$$\iint V^C(\xi,\eta) f(\xi,\eta;\, x, (y-Y)_+)\, d\xi d\eta$$

if $Y < U(p)$. Then the expected payoff for an agent who offers to buy a unit of stock at price p is

$$\begin{aligned}
&\text{Prob.}\{Y < U(p)\} \cdot E\Big[V^C \,\Big|\, Y < U(p)\Big] + \frac{1}{p}\text{Prob.}\{Y > U(p)\} \cdot E\Big[V^S \,\Big|\, Y > U(p)\Big] \\
= &\int_0^{U(p)} \Big(\iint V^C(\xi,\eta) f(\xi,\eta;\, x, (y-s)_+)\, d\xi d\eta\Big) \cdot \Big(-\Phi'(s)\Big) ds \\
&+\frac{1}{p} \cdot \int_{U(p)}^{+\infty} \Big(\iint V^S(\xi,\eta)(\xi,\eta;\, x, (y-s)_+)\, d\xi d\eta\Big) \cdot \Big(-\Phi'(s)\Big) ds\,.
\end{aligned} \tag{3.11}$$

Assuming that the LOB is a Nash equilibrium, we deduce that the quantity in (3.11) is constant on the support of U' (i.e., it is constant on the set of all prices at which some agent is bidding to buy the stock). Differentiating the right hand side of (3.11) w.r.t. p, one obtains

$$\begin{aligned}
0 \;=\; &-\frac{1}{p^2} \int_{U(p)}^{\infty} \Big(\iint V^S(\xi,\eta) f(\xi,\eta;\, x, (y-s)_+)\, d\xi d\eta\Big) \cdot \Big(-\Phi'(s)\Big) ds \\
&-U'(p)\,\Phi'(U(p)) \cdot \iint \Big(V^C(\xi,\eta) - \frac{1}{p} V^S(\xi,\eta)\Big) f(\xi,\eta;\, x, (y-U(p))_+)\, d\xi d\eta\,.
\end{aligned} \tag{3.12}$$

In the special case where $V^C \equiv a$ and $V^S \equiv b$, the above equation reduces to

$$-\frac{b}{p^2}\Phi(U(p)) - U'(p)\Phi'(U(p))\left(a - \frac{b}{p}\right) \;=\; 0,$$

which yields (2.24), with $\beta_0 = b/a$.

We regard (3.12) as an ODE for the function $U(p)$, where the right hand side depends on p, x, y, and on the functions V^C, V^S. This must be solved with boundary condition

$$U((1-\delta_2)\overline{p}) \;=\; y\,. \tag{3.13}$$

3.3 Existence of the two-sided LOB.

Let the mean bid-ask price $\overline{p}$ be given.

- By solving the Cauchy problem (3.9)-(3.10), we obtain the function $U(p)$ = amount of stock which agents offer for sale at price $\leq p$. Given the total amount x of stock offered for sale, the minimum ask price is then determined by the implicit equation

$$U(p_A) \;=\; 0\,. \tag{3.14}$$

- By solving the Cauchy problem (3.12)-(3.13), we obtain the function $U(p)$ = amount of stock which agents offer to buy at price $\geq p$. Given the total amount y of stock which agents bid to buy, the maximum bid price is then determined by the implicit equation

$$U(p_B) \;=\; 0\,. \tag{3.15}$$

To establish the existence and uniqueness of the two-sided LOB, we need to show that, under suitable assumptions, the map

$$\overline{p} \;\mapsto\; \frac{p_A(\overline{p}) + p_B(\overline{p})}{2} \tag{3.16}$$

is a strict contraction, hence it has a unique fixed point. As in the proof of Theorem 1, the heart of the matter is to estimate the partial derivatives $\partial p_A/\partial\overline{p}$ and $\partial p_B/\partial\overline{p}$.

To fix the ideas, assume we have a priori bounds

$$V^C_{min} \;\leq\; V^C(\xi,\eta) \;\leq\; V^C_{max}\,, \qquad\qquad V^S_{min} \;\leq\; V^S(\xi,\eta) \;\leq\; V^S_{max} \tag{3.17}$$

for all $\xi \geq 0$, $\eta \geq 0$. In connection with (3.9), these imply

$$\begin{aligned} U'(p) \;&=\; \frac{1}{-\Psi'(U(p))}\cdot\frac{\displaystyle\int_{U(p)}^{+\infty}\Big(\iint V^C(\xi,\eta) f\Big(\xi,\eta;\,(x-s)_+,y\Big)\,d\xi d\eta\Big)\cdot\Big(-\Psi'(s)\Big)ds}{\displaystyle\iint\Big(pV^C(\xi,\eta)-V^S(\xi,\eta)\Big) f\Big(\xi,\eta;\,(x-U(p))_+,y\Big)\,d\xi d\eta} \\ &\geq\; \frac{V^C_{min}}{pV^C_{max}-V^S_{min}}\cdot\frac{\Psi(U(p))}{-\Psi'(U(p))}\,. \end{aligned} \tag{3.18}$$

Notice that the right hand side of (3.18) approaches $+\infty$ as p decreases to V^S_{min}/V^C_{max}.

Introduce the functions

$$F(U) \doteq \int_U^{+\infty} \left(\iint V^C(\xi,\eta) f\Big(\xi,\eta;\, (x-s)_+, y\Big)\, d\xi d\eta \right) \cdot \Big(-\Psi'(s)\Big) ds\,,$$

$$g_C(U) \doteq \iint V^C(\xi,\eta) f\Big(\xi,\eta;\, (x-U(p))_+, y\Big)\, d\xi d\eta\,,$$

$$g_S(U) \doteq \iint V^S(\xi,\eta) f\Big(\xi,\eta;\, (x-U(p))_+, y\Big)\, d\xi d\eta.$$

Then the equation in (3.18) can be written as

$$\frac{dU}{dp} = \frac{F(U)}{-\Psi'(U)} \cdot \frac{1}{p\, g_C(U) - g_S(U)}\,. \tag{3.19}$$

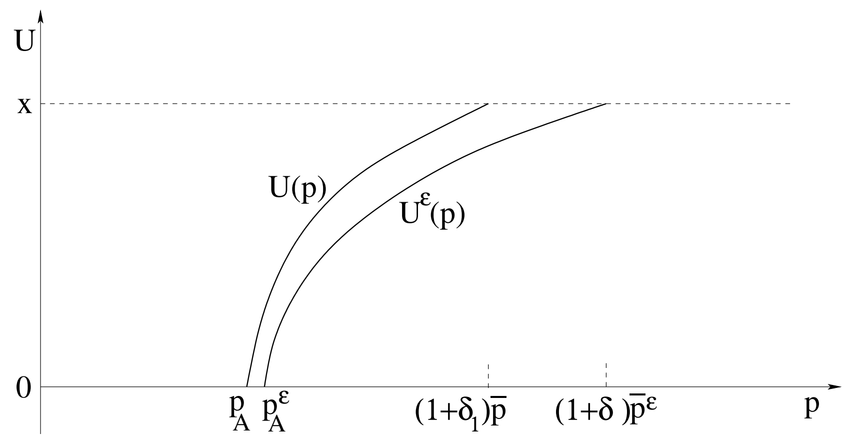

Figure 3.1. The ask price p_A is found by solving the Cauchy problem (3.18), (3.10), and finding the price at which $U = 0$. To estimate the rate at which p_A changes with the boundary data $\overline{p}$, it is convenient to invert the role of the variables U, p, thus obtaining the linear ODE (3.20) for $p = p(U)$. The figure shows how p_A changes when the value of $\overline{p}$ is increased.

Inverting the role of the two variables, we obtain the linear ODE

$$\frac{dp}{dU} = \frac{-\Psi'(U)}{F(U)} \cdot \Big[g_C(U)p - g_S(U)\Big]. \tag{3.20}$$

This should be solved with terminal data at $U = x$

$$p(x) = (1+\delta_1)\overline{p}\,. \tag{3.21}$$

The linear Cauchy problem (3.20)-(3.21) can be explicitly solved. Indeed

$$\begin{aligned} p(U) \;&=\; C_0 \exp\left\{-\int_U^x \frac{-\Psi'(w)}{F(w)} \cdot g_C(w)dw\right\} \\ &\qquad + \int_U^x \frac{-\Psi'(w)}{F(w)} \cdot g_S(w) \, \exp\left\{-\int_U^w \frac{-\Psi'(\tau)}{F(\tau)} \cdot g_C(\tau)\; d\tau\right\} dw, \end{aligned} \tag{3.22}$$

for some constant C_0. The boundary condition (3.21) yields

$$C_0 = (1+\delta_1)\overline{p}.$$

Differentiating w.r.t. $\overline{p}$ we obtain

$$\frac{\partial}{\partial \overline{p}}\, p(U) \;=\; (1+\delta_1)\exp\left\{-\int_U^x \frac{-\Psi'(w)}{F(w)}\; g_C(w)\; dw\right\}. \tag{3.23}$$

Since $p_A \;=\; p(0)$, we study the value of p at $U = 0$. Using the priori bounds (3.17) and recalling that $\ln \Psi(0) = 0$, from (3.23) we obtain

$$\begin{aligned} \frac{\partial p_A}{\partial \overline{p}} &= \; (1+\delta_1)\exp\left\{-\int_0^x \frac{-\Psi'(w)}{F(w)}\; g_C(w)\; dw\right\} \\ &\le (1+\delta_1)\exp\left\{-\int_0^x \frac{-\Psi'(w)}{\int_w^{+\infty} V^C_{max}(-\Psi'(s))ds}\; V^C_{min}\; dw\right\} \\ &= (1+\delta_1)\exp\left\{\frac{V^C_{min}}{V^C_{max}}\int_0^x \frac{\Psi'(w)}{\Psi(w)}\; dw\right\} \\ &= (1+\delta_1)\left(\Psi(x)\right)^{\lambda_C}, \end{aligned} \tag{3.24}$$

with $\lambda_C \doteq V^C_{min}/V^C_{max}$.

A similar analysis applies to the "buy" portion of the LOB. Indeed, (3.12) implies

$$\begin{aligned} U'(p) \;&=\; \frac{1}{-p^2\Phi'(U(p))} \cdot \frac{\displaystyle\int_{U(p)}^{+\infty} \left(\iint V^S(\xi,\eta) f\Big(\xi,\eta;\, x,(y-s)_+\Big)\, d\xi d\eta\right) \cdot \Big(-\Phi'(s)\Big)ds}{\displaystyle\iint \left(V^C(\xi,\eta) - \frac{1}{p}V^S(\xi,\eta)\right) f\Big(\xi,\eta;\, x,(y-U(p))_+\Big)\, d\xi d\eta} \\ &\ge\; \frac{V^S_{min}}{p^2 V^C_{max} - p V^S_{min}} \cdot \frac{\Phi(U(p))}{-\Phi'(U(p))}\,. \end{aligned} \tag{3.25}$$

Notice that the right hand side of (3.19) approaches $+\infty$ as p decreases to V^S_{min}/V^C_{max}.

Introducing the functions

$$\begin{aligned} G(U) \; &\doteq \; \int_U^{+\infty} \left(\iint V^S(\xi,\eta) f\Big(\xi,\eta;\, x, (y-s)_+\Big)\, d\xi d\eta \right) \cdot \Big(-\Phi'(s)\Big) ds\,, \\ \tilde g_C(U) \; &\doteq \; \iint V^C(\xi,\eta) f\Big(\xi,\eta;\, x, (y-U(p))_+\Big)\, d\xi d\eta\,, \\ \tilde g_S(U) \; &\doteq \; \iint V^S(\xi,\eta) f\Big(\xi,\eta;\, x, (y-U(p))_+\Big)\, d\xi d\eta\,, \end{aligned}$$

the ODE in (3.25) can be written as

$$\frac{dU}{dp} \;=\; \frac{G(U)}{-\Phi'(U)} \cdot \frac{1}{p^2\, \tilde g_C(U) - p\, \tilde g_S(U)}\,. \tag{3.26}$$

Inverting the role of the two variables p and U, we obtain a Bernoulli differential equation

$$\frac{dp}{dU} \;=\; \frac{-\Phi'(U)}{G(U)} \cdot \Big[p^2\, \tilde g_C(U) - p\, \tilde g_S(U)\Big] \tag{3.27}$$

with terminal data at $U = y$ given by

$$p(y) \;=\; (1-\delta_2)\overline{p}\,. \tag{3.28}$$

Introducing the new variable $q = 1/p$, we obtain the linear Cauchy problem

$$\frac{dq}{dU} \;=\; \frac{-\Phi'(U)}{G(U)} \cdot \Big[q\, \tilde g_S(U) - \tilde g_C(U)\Big], \qquad\qquad q(y) \;=\; \frac{1}{(1-\delta_2)\overline{p}}\,. \tag{3.29}$$

An explicit computation yields

$$\begin{aligned} q(U) \;=\; & \frac{1}{(1-\delta_2)\overline{p}} \exp\left\{ -\int_U^y \frac{-\Phi'(w)}{G(w)} \cdot \tilde g_S(w) dw \right\} \\ & + \int_U^y \frac{-\Phi'(w)}{G(w)} \cdot \tilde g_C(w) \cdot \exp\left\{ -\int_U^w \frac{-\Phi'(\tau)}{G(\tau)} \cdot \tilde g_S(\tau)\; d\tau \right\} dw\,, \end{aligned} \tag{3.30}$$

Differentiating the solution $p(U)$ of (3.27)-(3.28) w.r.t. $\overline{p}$, by (3.30) we now obtain

$$\frac{\partial}{\partial \overline{p}} p(U) \;=\; \frac{\partial p}{\partial q} \cdot \frac{\partial q}{\partial \overline{p}} \;=\; \frac{p^2}{(1-\delta_2)\overline{p}^2} \exp\left\{ -\int_U^y \frac{-\Phi'(w)}{G(w)}\; \tilde g_S(w)\, dw \right\}\,,$$

Since $p_B = p(0) \leq \overline{p}$, using the a priori bounds (3.17) one obtains

$$\begin{aligned}
\frac{\partial p_B}{\partial \overline{p}} &= \frac{p_B^2}{(1-\delta_2)\,\overline{p}^2} \exp\left\{-\int_U^0 \frac{-\Phi'(w)}{G(w)}\;\tilde{g}_S(w)\;dw\right\} \\
&\leq \frac{(1+\delta_2)^2}{1-\delta_2} \exp\left\{-\int_U^0 \frac{-\Phi'(w)}{\int_w^{+\infty} V^S_{max}(-\Phi'(s))ds}\;V^S_{min}\;dw\right\} \\
&= \frac{(1+\delta_2)^2}{1-\delta_2}\Big(\Phi(y)\Big)^{\lambda_S},
\end{aligned} \tag{3.31}$$

with $\lambda_S \doteq V^S_{min}/V^S_{max}$. Combining the two inequalities (3.24) and (3.31), we obtain a sufficient condition for the existence of a unique mean bid-ask price $\overline{p}$.

Theorem 3. *Assume that the value functions V^C, V^S satisfy the a priori bounds (3.17). Moreover, assume that the total amount x of stock offered for sale and the total amount y that agents bid to buy are both large enough, so that*

$$(1+\delta_1)\Big(\Psi(x)\Big)^{\lambda_C} + \frac{(1+\delta_2)^2}{1-\delta_2}\Big(\Phi(y)\Big)^{\lambda_S} \;<\; 2\,, \tag{3.32}$$

with $\lambda_C \doteq V^C_{min}/V^C_{max}$, $\lambda_S \doteq V^S_{min}/V^S_{max}$.

Then the two-sided LOB has a unique equilibrium configuration.

Proof. Combining (3.24) and (3.31) with the assumption (3.32) one obtains

$$\frac{d}{d\overline{p}}\left(\frac{p_A+p_B}{2}\right) \leq \frac{1}{2}\;\left[(1+\delta_1)\Big(\Psi(x)\Big)^{\lambda_C} + \frac{(1+\delta_2)^2}{1-\delta_2}\Big(\Phi(y)\Big)^{\lambda_S}\right] \;<\; 1\,. \tag{3.33}$$

showing that the map $\overline{p} \mapsto \frac{1}{2}(p_A+p_B)$ is a strict contraction. Hence, a unique fixed point exists.

As soon as this unique mean bid-ask price $\overline{p}$ has been determined, the “sell" and the “buy" portions of the LOB are obtained by solving the Cauchy problems (3.9)-(3.10) and (3.12)-(3.13), respectively. □

Remark 3. In the above setting, $\Psi(x)$ is the probability that the external buy order is so large that it wipes out the entire “sell" portion of the LOB. Similarly, $\Phi(y)$ is the probability that the external sell order is so large that it wipes out the entire “buy" portion of the LOB. The key assumption of the theorem requires that these probabilities are sufficiently small. Notice that, if $V^C(\xi,\eta)$ and $V^S(\xi,\eta)$ are constants, then $\lambda_C = \lambda_S = 1$ and the assumption (3.32) is exactly the same as (2.30) in Theorem 1.

3.4 The inductive computation of the value functions.

If the existence of a unique fixed point $\overline{p}$ is known, the value functions V^C, V^S can then be inductively computed as follows. Let $P_{buy} = P$ be the probability that the external agent is a buyer, so that $P_{sell} = (1 - P)$ is the probability that the external agent is a seller.

The assumption that the LOB represents an equilibrium implies that the expected payoff for an agent holding a unit amount of stock (or a unit amount of cash) is independent of the price p he asks (or the price he bids). In particular, we can compute this payoff in the case $p = p_A$ (or $p = p_B$, respectively), where the transaction occurs with probability one.

We thus obtain the inductive relations

$$\begin{aligned} V_i^S(x,y) \;&=\; P_{buy}\cdot p_A \cdot E^{X_i}\left[\int V_{i+1}^C(\xi,\eta)\, f\Big(\xi,\eta;\, (x-X_i)_+\,,y\Big)\, d\xi d\eta\right] \\ &\qquad + P_{sell}\cdot E^{Y_i}\left[\int V_{i+1}^S(\xi,\eta)\, f\Big(\xi,\eta;\, x, (y-Y_i)_+\Big)\, d\xi d\eta\right]. \end{aligned} \tag{3.34}$$

$$\begin{aligned} V_i^C(x,y) \;&=\; P_{buy}\cdot E^{X_i}\left[\int V_{i+1}^C(\xi,\eta)\, f\Big(\xi,\eta;\, (x-X_i)_+\,,y\Big)\, d\xi d\eta\right] \\ &\qquad + P_{sell}\cdot \frac{1}{p_B}\cdot E^{Y_i}\left[\int V_{i+1}^S(\xi,\eta)\, f\Big(\xi,\eta;\, x, (y-Y_i)_+\Big)\, d\xi d\eta\right]. \end{aligned} \tag{3.35}$$

Notice that here p_A, p_B depend on x, y, and also on all values of the functions V^S, V^C.

In order to apply Theorem 3, and construct the value functions V_i^C, V_i^S for all $i = 1, \ldots, N$ by backward induction, we need to provide suitable upper and lower bounds.

Lemma 3. *Let V_i^S, V_i^C, $i = 1, 2, \ldots, N$ be a sequence of value functions satisfying the inductive relations (3.34)-(3.35), with*

$$V_N^C(\xi,\eta) \equiv 1\,, \qquad\qquad V_N^S(\xi,\eta) \equiv \bar{b}\,. \tag{3.36}$$

Then for all $i = 1, \ldots, N$

$$1 \;\le\; V_i^C(\xi,\eta) \;\le\; \overline{V}_i^C\,, \qquad\qquad \bar{b} \;\le\; V_i^S(\xi,\eta) \;\le\; \overline{V}_i^S\,, \tag{3.37}$$

where the upper bounds $\overline{V}_i^C$, $\overline{V}_i^S$ *are defined by the following inductive relations:*

$$\begin{aligned}
\overline{V}_N^C &= 1\,, \qquad \overline{V}_i^C = \left[P_{buy} + P_{sell}\cdot\frac{1+\delta_2}{1-\delta_2}\cdot\frac{\overline{V}_{i+1}^S}{\bar b}\right]\cdot\overline{V}_{i+1}^C\,, \quad i=1,\ldots,N-1\,,\\
\overline{V}_N^S &= \bar b\,, \qquad \overline{V}_i^S = \left[P_{buy}\cdot\frac{1+\delta_1}{1-\delta_1}\cdot\overline{V}_{i+1}^C + P_{sell}\right]\cdot\overline{V}_{i+1}^S\,, \quad i=1,\ldots,N-1\,.
\end{aligned} \tag{3.38}$$

Proof. By assumption, at the terminal time $i = N$ the value functions are constant and satisfy (3.36).

The proof will be achieved by backward induction. Assuming that V_{i+1}^C, V_{i+1}^S satisfy the bounds

$$1 \;\le\; V_{i+1}^C(\xi,\eta) \;\le\; \overline{V}_{i+1}^C\,, \qquad\qquad \bar b \;\le\; V_{i+1}^S(\xi,\eta) \;\le\; \overline{V}_{i+1}^S\,, \tag{3.39}$$

we will show that V_i^C, V_i^S satisfy the inequalities (3.37)-(3.38).

1. Using the functions F, g_C, g_S and G, $\tilde g_C$, $\tilde g_S$, by (3.9) and (3.12) we have

$$0 \;=\; F(U(p)) + U'(p)\Psi'(U(p))\cdot\Big(p\cdot g_C(U(p)) - g_S(U(p))\Big)$$

and

$$0 \;=\; -\frac{1}{p^2}G(U(p)) - U'(p)\Phi'(U(p))\cdot\Big(\tilde g_C(U(p)) - \frac{1}{p}\tilde g_S(U(p))\Big)\,.$$

It follows that

$$p\cdot g_C(U(p)) - g_S(U(p)) \;\ge\; 0\,, \quad \text{for all } p\in[p_A, (1+\delta_1)\overline{p}]$$

and

$$\tilde g_C(U(p)) - \frac{1}{p}\tilde g_S(U(p)) \;\le\; 0\,, \quad \text{for all } p\in[(1-\delta_2)\overline{p}, p_B]\,.$$

In particular,

$$(1+\delta_1)\overline{p}\cdot g_C(x) \;\ge\; g_S(x) \qquad \text{and} \qquad \tilde g_C(y) \;\le\; \frac{1}{(1-\delta_2)\overline{p}}\cdot\tilde g_S(y)\,. \tag{3.40}$$

Observe that the values in (3.34) and (3.35) can be expressed as

$$V_i^S(x,y) \;=\; P_{buy}\cdot\Big[(1+\delta_1)\overline{p}\int_x^{\infty} g_C(s)\Big(-\Psi'(s)\Big)ds+\int_0^x g_S(s)\Big(-\Psi'(s)\Big)ds\Big]+P_{sell}\cdot G(0). \tag{3.41}$$

$$V_i^C(x,y) \;=\; P_{buy}\cdot F(0)+P_{sell}\cdot\Big[\int_0^y \tilde{g}_C(s)\Big(-\Phi'(s)\Big)ds+\frac{1}{(1-\delta_2)\overline{p}}\int_y^{\infty}\tilde{g}_S(s)\Big(-\Phi'(s)\Big)ds\Big]. \tag{3.42}$$

Remarking that $g_{C,S}(s)=g_{C,S}(x)$ for every $s>x$ and $\tilde{g}_{C,S}(s)=\tilde{g}_{C,S}(y)$ for every $s>y$, and combining (3.40), (3.41) and (3.42), we obtain

$$V_i^S(x,y)\;\geq\;P_{buy}\cdot\int_0^{\infty} g_S(s)\Big(-\Psi'(s)\Big)ds+P_{sell}\cdot G(0)\;\geq\;\bar{b}\,,$$

$$V_i^C(x,y)\;\geq\;P_{buy}\cdot F(0)+P_{sell}\cdot\int_0^{\infty}\tilde{g}_C(s)\Big(-\Phi'(s)\Big)ds\;\geq\;1\,.$$

2. Our modeling assumptions on the maximum and minimum acceptable prices yield

$$p_A\;\leq\;(1+\delta_1)\overline{p},\qquad (1-\delta_2)\overline{p}\;\leq\;p_B\,. \tag{3.43}$$

From (3.22), (3.39) and (3.43), it follows

$$\begin{aligned}
p_A \;=\; p(0) =\; & (1+\delta_1)\overline{p}\cdot\exp\left\{-\int_0^{\bar{x}}\frac{-\Psi'(w)}{F(w)}\cdot g_C(w)dw\right\}\\
&+\int_0^{\bar{x}}\frac{-\Psi'(w)}{F(w)}\cdot g_S(w)\,\exp\left\{-\int_0^{w}\frac{-\Psi'(\tau)}{F(\tau)}\cdot g_C(\tau)\,d\tau\right\}dw\\
\geq\; & p_A\cdot\exp\left\{-\int_0^{\bar{x}}\frac{-\Psi'(w)}{F(w)}\cdot g_C(w)dw\right\}\\
&+\frac{\bar{b}}{\bar{V}_{i+1}^C}\left(1-\exp\left\{-\int_0^{\bar{x}}\frac{-\Psi'(w)}{F(w)}\cdot g_C(w)dw\right\}\right).
\end{aligned} \tag{3.44}$$

Therefore, we obtain $p_A\geq\dfrac{\bar{b}}{\bar{V}_{i+1}^C}$.

Concerning the maximum bid price p_B, from (3.30), (3.39) and (3.43), it follows

$$\begin{aligned} \frac{1}{p_B} \;=\; q(0) \;&=\; \frac{1}{(1-\delta_2)\overline{p}} \exp\left\{-\int_0^{\bar{y}} \frac{-\Phi'(w)}{G(w)} \cdot \tilde{g}_S(w)dw\right\} \\ &\quad + \int_0^{\bar{y}} \frac{-\Phi'(w)}{G(w)} \cdot \tilde{g}_C(w) \cdot \exp\left\{-\int_0^{w} \frac{-\Phi'(\tau)}{G(\tau)} \cdot \tilde{g}_S(\tau)\, d\tau\right\} dw \\ &\geq\; \frac{1}{p_B} \exp\left\{-\int_0^{\bar{y}} \frac{-\Phi'(w)}{G(w)} \cdot \tilde{g}_S(w)dw\right\} \\ &\quad + \frac{1}{\overline{V}^S_{i+1}} \left(1 - \exp\left\{-\int_0^{\bar{y}} \frac{-\Phi'(w)}{G(w)} \cdot \tilde{g}_S(w)dw\right\}\right). \end{aligned} \tag{3.45}$$

Therefore, we have $p_B \leq \overline{V}^S_{i+1}$. It follows

$$\overline{p} \;=\; \frac{p_A + p_B}{2} \;\leq\; \frac{1}{2}(1+\delta_1)\overline{p} + \frac{1}{2}\overline{V}^S_{i+1}\,,$$

so that

$$(1+\delta_1)\overline{p} \;\leq\; \frac{1+\delta_1}{1-\delta_1}\overline{V}^S_{i+1}\,.$$

Analogously, we obtain

$$(1-\delta_2)\overline{p} \;\leq\; \frac{1-\delta_2}{1+\delta_2}\frac{\overline{b}}{\overline{V}^C_{i+1}}\,.$$

By (3.41)-(3.42), for any x, y it follows

$$\begin{aligned} V^S_i(x,y) \;&\leq\; P_{buy} \cdot \left[\frac{1+\delta_1}{1-\delta_1}\overline{V}^S_{i+1}\overline{V}^C_{i+1}\Psi(x) + \overline{V}^S_{i+1}(1-\Psi(x))\right] + P_{sell} \cdot \overline{V}^S_{i+1} \\ &\leq\; \left[P_{buy} \cdot \tfrac{1+\delta_1}{1-\delta_1} \cdot \overline{V}^C_{i+1} + P_{sell}\right] \cdot \overline{V}^S_{i+1}\,, \end{aligned} \tag{3.46}$$

$$\begin{aligned} V^C_i(x,y) \;&\leq\; P_{buy} \cdot \overline{V}^C_{i+1} + P_{sell} \cdot \left[\overline{V}^C_{i+1}(1-\Phi(y)) + \frac{1+\delta_2}{1-\delta_2}\frac{\overline{V}^C_{i+1}}{\overline{b}} \cdot \overline{V}^S_{i+1}\Phi(y)\right] \\ &\leq\; \left[P_{buy} + P_{sell} \cdot \tfrac{1+\delta_2}{1-\delta_2} \cdot \tfrac{\overline{V}^S_{i+1}}{\overline{b}}\right] \cdot \overline{V}^C_{i+1}\,. \end{aligned} \tag{3.47}$$

□

Chapter 4
Infinite Horizon Models

1 Introduction

The present chapter is concerned with problems in infinite time horizon, also allowing for random fluctuations in the fundamental value $\beta = \beta(t)$ of the traded asset. We assume that all agents posting limit orders have the same information, but external buyers or sellers may occasionally anticipate random upward or downward jumps in the value of the stock. As the volatility increases, we seek to understand whether the strategy of the agents should be modified, and by how much their expected payoff will be impacted.

We consider here a situation involving an infinite sequence of incoming orders $X_1, X_2, \ldots$ Each can be either a buy order or a sell order. We assume that the random variables X_j, $j \geq 1$, describing the amount of stock that the external agents want to buy (or sell), are mutually independent and have absolutely continuous density w.r.t. Lebesgue measure.

At any given time, we describe the limit order book in terms of an absolutely continuous function $U(\cdot)$, as in Fig. 4.1.

Definition 1.1 *For $p < \beta$, we denote by $U = U_b(p)$ the total amount of stock that agents bid to buy, at price $\geq p$. For $p > \beta$, we denote $U = U_s(p)$ the total amount of stock that agents offer to sell at price $\leq p$. The maximum bid price p_B and the minimum ask price p_A are defined as*

$$p_B = \sup\{p < \beta\,;\ U_b(p) > 0\}, \qquad p_A = \inf\{p > \beta\,;\ U_s(p) > 0\}. \tag{4.1}$$

Calling K_b, K_s, the total amount of stocks that agents posting orders on th LOB affer to buy or sell, respectively, we have

$$U(s) \;=\; \begin{cases} 0 & \text{if } s \in [p_B, p_A], \\ K_b & \text{if } s \leq p^-, \\ K_s & \text{if } s \geq p^+. \end{cases} \tag{4.2}$$

Here p^-, p^+ are the lowest and highest prices at which some bid is made.

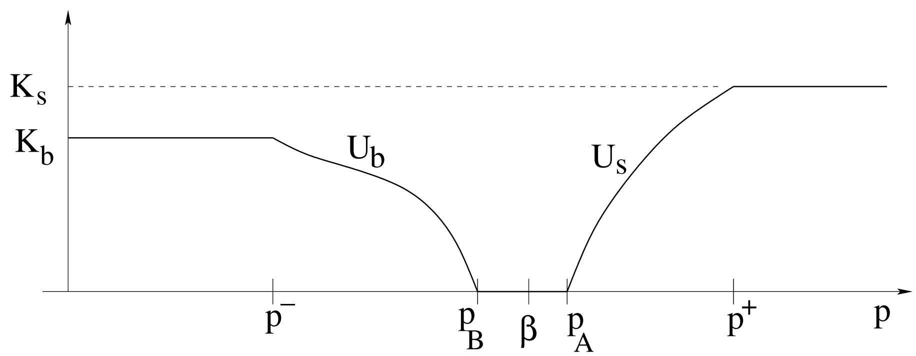

Figure 4.1. Sketch of the functions U_s, U_b introduced at (4.1)-(4.2).

As in [11], we assume that the sizes of the incoming buy or sell orders X_{buy}, X_{sell} are i.i.d. random variables, satisfying

$$\text{Prob.}\Big\{|X_{buy}| > \xi\Big\} \;=\; \Psi_b(\xi), \qquad\qquad \text{Prob.}\Big\{|X_{sell}| > \xi\Big\} \;=\; \Psi_s(\xi). \tag{4.3}$$

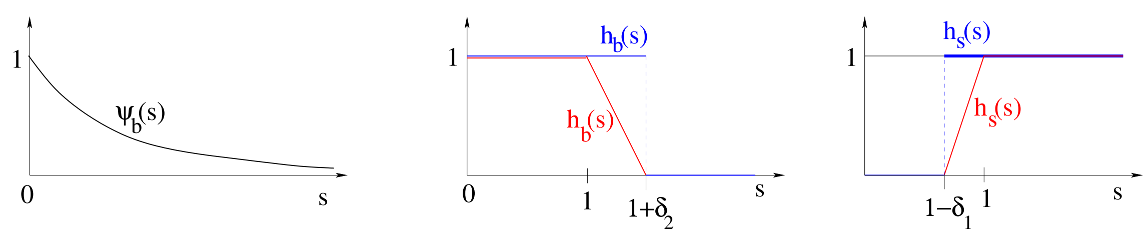

Figure 4.2. Left: an example of distribution Ψ_b of the size of external buy orders in (4.3). Center: two possible probability functions for the maximum acceptable price h_b in (4.4). Right: two possible probability functions for the minimum acceptable price h_s in (4.4).

Moreover, the maximum price that a buyer is willing to pay (or the minimum price that an external seller is willing to accept) is not known a priori. More precisely, we assume that the external agent will agree to the transaction only as long as the price ranges within an interval $\Big[Q_s\beta,\; Q_b\beta\Big]$, where also $Q_s, Q_b > 0$ are independent random variables (Fig. 4.2), say

$$\text{Prob.}\Big\{Q_b \geq s\Big\} \;=\; h_b(s)\,, \qquad\qquad \text{Prob.}\Big\{Q_s \leq s\Big\} \;=\; h_s(s)\,. \tag{4.4}$$

To fix ideas, in the following we assume that, for some $\delta_1, \delta_2 > 0$,

$$h_s(1-\delta_1) \;=\; 0 \;=\; h_b(1+\delta_2). \tag{4.5}$$

In other words, no external buyer will accept a price $p > (1+\delta_2)\beta$, and no external seller will accept a price $p < (1-\delta_1)\beta$.

An external order of size X is thus executed as follows (Fig. 4.3).

CASE 1: a buy order in the amount X_b. In this case the external buyer will take all stocks whose price ranges in the interval $[\beta,\, p(X,Q)]$, where

$$p(X,Q) \;=\; \max\Big\{p \in [\beta, Q_b\beta],\quad U_s(p) \leq X_b\Big\}. \tag{4.6}$$

CASE 2: a sell order in the amount X_s. In this case the external seller will fulfill all the bids whose price ranges in the interval $[p(X,Q),\, \beta]$, where

$$p(X,Q) \;=\; \min\Big\{p \in [Q_s\beta,\, \beta],\quad U_b(p) \leq X_s\Big\}. \tag{4.7}$$

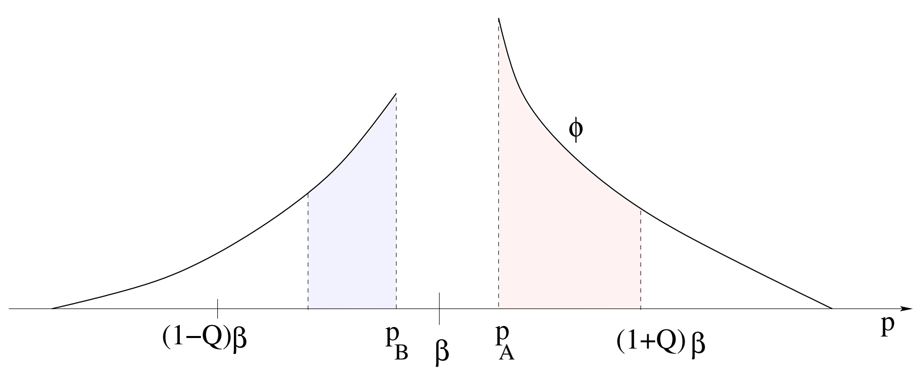

Figure 4.3. A possible shape of the limit order book. If the external order has size $X > 0$ and is a buy order, all the stocks in the red region, with price $p \in [p_A, p(X,Q)]$ as in (4.6), will be sold. If the external order is a sell order, all the buy orders in the blue region, with price $p \in [p(X,Q), p_B]$ as in (4.7), will be executed.

It is natural to assume that each agent will try to maximize the combined value of cash and stock held over time. However, since this amount may grow without bounds as $t \to +\infty$, to achieve a well defined measure of performance one needs to insert an exponential discount. Namely, denoting by $C(t)$ and $S(t)$ the amount of cash and stock held by an agent at time t, we shall seek to maximize the expected discounted payoff

$$J \doteq E\left[\int_0^{+\infty} e^{-\gamma t}\Big(C(t) + \beta(t)\, S(t)\Big)\, dt\right]. \tag{4.8}$$

2 A stock with constant fundamental value

We now consider a sequence of incoming orders $X_1, X_2 \ldots$, and price limitations $Q_1, Q_2 \ldots$. Assume that the X_j as well as the Q_j are independent, identically distributed random variables.

The arrival times $0 = t_0 < t_1 < t_2 < \cdots$ of these incoming orders are assumed to be a Poisson process. More precisely, the differences $t_j - t_{j-1}$ are independent random variables, with probability distribution

$$\text{Prob.}\{t_j - t_{j-1} > s\} = e^{-\mu s}. \tag{4.1}$$

Moreover, each order will be

- a buy order with probability $\theta \in]0,1[$,

- a sell order with probability $1-\theta$.

As a first step, in this section we consider the case where the fundamental value $\beta(t) = \beta$ of the stock remains constant in time.

To obtain a mathematically tractable problem, we make some simplifying assumptions.

(A1) The size of the LOB remains constant in time. In other words, after an external order has been executed, new agents post additional limit orders, so that the total amount K_b, K_s of buy and sell orders does not change.

(A2) The amount of buy or sell orders posted by each individual agent is small, compared with the total amounts of orders K_b, K_s on the entire LOB.

(A3) At any time $t \geq 0$ the LOB is in equilibrium. In other words, any ask price $p \in [p_A, p^+]$ or any bid price $p \in [p^-, p_B]$ yields the same expected payoff.

Thanks to **(A2)** we can make a linear approximation, and compute the expected payoff (4.8) as a linear function of the initial amount C_0 of cash and the initial amount S_0 of stock held by an individual agent:

$$J \;=\; aC_0 + bS_0\,. \tag{4.2}$$

Throughout the following, our main concern will be to determine the values of a, b, and understand how they depend on the various constants $K_b, K_s, \gamma, \ldots$ that define the model.

To obtain a system of two equations for a and b, we observe that, since the LOB is in equilibrium, the expected payoff to the various agents does not depend on the price at which they post their limit orders. We can thus compute a, b by looking at the expected payoff of the agent that posts limit orders at the prices p_A and p_B.

Let $\tau > 0$ be the first time when an external order arrives. We consider two cases.

For an agent who initially holds a unit amount of cash, and posted a bid to buy stocks at price p_B, the expected payoff will be

$$J_C \;=\; \int_0^\tau e^{-\gamma s}\,ds + ae^{-\gamma\tau}\Big[\theta + (1-\theta)(1-h_s(p_B/\beta))\Big] + be^{-\gamma\tau}(1-\theta)h_s(p_B/\beta)\,\frac{1}{p_B}\,. \tag{4.3}$$

Indeed, with probability θ the external order will be a buy order. Hence after time τ the agent will still hold the same amount of cash, and no stock. On the other hand, with probability $(1-\theta)$ the external order will be a sell order. When this happens, our agent will have probability $h_s(p_B/\beta)$ of actually buying stock at price p_B, while with probability $1 - h_s(p_B/\beta)$ he will remain with the original cash. This accounts for the last two terms on the right hand side of (4.3).

Next for an agent who initially holds a unit amount of stock, and posted a bid to sell it at price p_A, the expected payoff will be

$$J_S \;=\; \int_0^\tau e^{-\gamma s}\beta\,ds + ae^{-\gamma\tau}\theta h_b(p_A/\beta)\,p_A + be^{-\gamma\tau}\Big[\theta(1-h_b(p_A/\beta)) + (1-\theta)\Big]. \tag{4.4}$$

Indeed, with probability $1-\theta$ the external order will be a sell order. Hence after time τ the agent will still hold the same amount of stock, and no cash. On the other hand, with probability θ the external order will be a buy order. When this happens, our agent will have probability $h_b(p_A/\beta)$ of actually selling his stock at price p_A, while with probability $1-h_b(p_A/\beta)$ he will remain with the original amount of stock. This accounts for the last two terms on the right hand side of (4.4).

Combining the two cases (4.3)-(4.4), and using the assumption (4.1) on the distribution of the first arrival time τ, we obtain

$$\begin{aligned} aC_0 + bS_0 = & \int_0^{+\infty} \mu e^{-\mu\tau} \Bigg\{ \int_0^{\tau} e^{-\gamma s}(C_0 + \beta S_0)\, ds \\ & + e^{-\gamma\tau} \left[\theta a + (1-\theta)\big(1 - h_s(p_B/\beta)\big) a + (1-\theta) h_s(p_B/\beta)\, \frac{b}{p_B} \right] C_0 \\ & + e^{-\gamma\tau} \Big[\theta h_b(p_A/\beta)\, p_A\, a + \theta(1 - h_b(p_A/\beta)) b + (1-\theta) b \Big] S_0 \Bigg\}\, d\tau. \end{aligned} \tag{4.5}$$

Since (4.5) holds for every C_0, S_0 this yields a system of two equations for a, b, depending on β, p_A, p_B.

Using vector notation, this can be written as

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{\gamma+\mu} \begin{bmatrix} 1 \\ \beta \end{bmatrix} + \frac{\mu}{\gamma+\mu} \begin{bmatrix} \theta + (1-\theta)\big(1 - h_s(p_B/\beta)\big) & \dfrac{1-\theta}{p_B} \cdot h_s(p_B/\beta) \\ \theta p_A \cdot h_b(p_A/\beta) & (1-\theta) + \theta(1 - h_b(p_A/\beta)) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}. \tag{4.6}$$

The limiting prices p_A and p_B can be determined by finding the function U describing the LOB, in Definition 1.1. This depends on the distribution functions Ψ, h, in (4.3)-(4.4), and on the sizes of K_s, K_b of the “sell" and “buy" portions of the LOB. To determine U, we observe that, for $p \in [p_A, p^+]$ we should have

$$a\, p \cdot [\text{probability of selling at price } p] + b \cdot [\text{probability of not selling at price } p] = \text{constant}. \tag{4.7}$$

Similarly, for $p \in [p^-, p_B]$, we have

$$\frac{b}{p} \cdot [\text{probability of buying at price } p] + a \cdot [\text{probability of not buying at price } p] = \text{constant}. \tag{4.8}$$

With the above notations, we have

$$\begin{aligned} [\text{probability of selling at price } p] &= \text{Prob.}\Big\{ X_b \geq U_s(p) \text{ and } Q_b \beta \geq p \Big\} \\ &= \Psi_b(U_s(p)) \cdot h_b(p/\beta). \end{aligned}$$

$$
\begin{aligned}
\text{[probability of buying at price } p] &= \text{Prob.}\Big\{X_s \geq U_b(p) \text{ and } Q_s\beta \leq p\Big\} \\
&= \Psi_s(U_b(p)) \cdot h_s(p/\beta).
\end{aligned}
$$

Inserting these expressions in (4.7)-(4.8), we obtain

$$
\begin{cases}
a\,p\,\Psi_b(U_s(p)) \cdot h_b(p/\beta) + b\Big[1 - \Psi_b(U_s(p)) \cdot h_b(p/\beta)\Big] = a\,p_A\,h_b(p_A/\beta) + b\Big[1 - h_b(p_A/\beta)\Big], \\
\dfrac{b}{p}\,\Psi_s(U_b(p)) \cdot h_s(p/\beta) + a \cdot \Big[1 - \Psi_s(U_b(p)) \cdot h_s(p/\beta)\Big] = \dfrac{b}{p_B}\,h_s(p_B/\beta) + a \cdot \Big[1 - h_s(p_B/\beta)\Big].
\end{cases}
\tag{4.9}
$$

$$
\Psi_b(U(p)) = \frac{(ap_A - b)\,h_b(p_A/\beta)}{(ap - b)\,h_b(p/\beta)} \qquad \text{for} \quad p > \beta, \tag{4.10}
$$

$$
\Psi_s(U(p)) = \frac{\Big[(b/p_B) - a\Big]\,h_s(p_B/\beta)}{\Big[(b/p) - a\Big]\,h_s(p/\beta)} \qquad \text{for} \quad p < \beta. \tag{4.11}
$$

Remark 2.1 In general, the intervals $[p^-, p_B]$ and $[p_A, p^+]$ are not known a priori, and must be determined as part of the solution. In view of the assumption (4.5), we have the inclusions

$$
[p^-, p_B] \subseteq [(1-\delta_1)\beta,\ \beta], \qquad [p_A, p^+] \subseteq [\beta, (1+\delta_2)\beta]. \tag{4.12}
$$

2.1 Deterministic acceptable prices

The above formulas (4.9)–(4.11) can be greatly simplified if we assume that the upper and lower bounds for the prices accepted by external agents are deterministic. More precisely, for some $\delta_1, \delta_2 > 0$, assume that

$$
h_b(p) = \begin{cases} 1 & \text{if } p \leq (1+\delta_2)\beta, \\ 0 & \text{if } p > (1+\delta_2)\beta, \end{cases} \qquad h_s(p) = \begin{cases} 1 & \text{if } p \geq (1-\delta_1)\beta, \\ 0 & \text{if } p < (1-\delta_1)\beta. \end{cases} \tag{4.13}
$$

In this case, we immediately know that the maximum and minimum prices in (4.2) are

$$
p^+ = (1+\delta_2)\beta, \qquad p^- = (1-\delta_1)\beta. \tag{4.14}
$$

Using the identities

$$
U_b(p^-) = K_b\,, \qquad U_s(p^+) = K_s\,,
$$

from (4.9) we obtain

$$\begin{cases} a\,p\,\Psi_b(U_s(p)) + b\Big[1 - \Psi_b(U_s(p))\Big] & = \; a\,p^+\,\Psi_b(K_s) + b\Big[1 - \Psi_b(K_s)\Big] \;\; = \; a\,p_A\,, \\ \dfrac{b}{p}\,\Psi_s(U_b(p)) + a\,\Big[1 - \Psi_s(U_b(p))\Big] & = \; \dfrac{b}{p^-}\,\Psi_s(K_b) + a\,\Big[1 - \Psi_s(K_b)\Big] \;\; = \; \dfrac{b}{p_B}\,. \end{cases} \tag{4.15}$$

$$\begin{cases} \Psi_b(U_s(p)) \;=\; \dfrac{ap_A - b}{ap - b} & \text{for} \;\; p > \beta, \\ \Psi_s(U_b(p)) \;=\; \dfrac{(b/p_B) - a}{(b/p) - a} & \text{for} \;\; p < \beta. \end{cases} \tag{4.16}$$

Throughout the following, since these quantities will repeatedly appear in our computations, we adopt the notation

- $\Psi^\sharp \doteq \Psi_b(K_s)$ = probability that the external "buy" order is larger than the entire "sell" portion of the LOB,
- $\Psi^\flat \doteq \Psi_s(K_b)$ = probability that the external "sell" order is larger that the entire "buy" portion of the LOB.

It will be convenient to compute the left hand sides of (4.9) at $p = p^+$ and $p = p^-$, respectively. The identity (4.5) can be replaced by

$$\begin{aligned} aC_0 + bS_0 = \;& \int_0^{+\infty} \mu e^{-\mu\tau} \bigg\{ \int_0^\tau e^{-\gamma s}(C_0 + \beta S_0)\, ds \\ & + e^{-\gamma\tau} \left[\theta a + (1-\theta)(1-\Psi^\flat)a + (1-\theta)\Psi^\flat \, \frac{b}{p^-} \right] C_0 \\ & + e^{-\gamma\tau} \Big[\theta \Psi^\sharp p^+ a + \theta(1-\Psi^\sharp) b + (1-\theta) b \Big] S_0 \bigg\}\, d\tau. \end{aligned} \tag{4.17}$$

Since (4.17) holds for every C_0, S_0 this yields a system of two equations for a, b, depending on β, p^-, p^+.

Using vector notation, this can be written as

$$\begin{bmatrix} a \\ b \end{bmatrix} \;=\; \frac{1}{\gamma+\mu} \begin{bmatrix} 1 \\ \beta \end{bmatrix} + \frac{\mu}{\gamma+\mu} \begin{bmatrix} \theta + (1-\theta)(1-\Psi^\flat\big) & (1-\theta)\Psi^\flat / p^- \\ \theta\,\Psi^\sharp\, p^+ & \theta(1-\Psi^\sharp) + (1-\theta) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}. \tag{4.18}$$

Equivalently,

$$A \begin{bmatrix} a \\ b \end{bmatrix} \;=\; \begin{bmatrix} 1 \\ \beta \end{bmatrix}, \tag{4.19}$$

where

$$A \;\doteq\; \begin{bmatrix} \gamma + \mu(1-\theta)\Psi^\flat & -\mu(1-\theta)\Psi^\flat/p^- \\ -\mu\theta\,\Psi^\sharp\,p^+ & \gamma + \mu\theta\Psi^\sharp \end{bmatrix}. \tag{4.20}$$

Proposition 2.1 *The infinite horizon game described in Section 2 with exponential discount γ and deterministic acceptable prices $p^- = (1-\delta_1)\beta$, $p^+ = (1+\delta_2)\beta$ will admit a unique shape in Nash equilibrium, given the condition*

$$\gamma \;>\; \frac{\mu}{2}\left(\sqrt{\Big((1-\theta)\Psi^\flat - \theta\Psi^\sharp\Big)^2 + 4\theta(1-\theta)\Psi^\flat\Psi^\sharp\left(\frac{1+\delta_2}{1-\delta_1}\right)} - \Big((1-\theta)\Psi^\flat + \theta\Psi^\sharp\Big)\right). \tag{4.21}$$

Proof. Firstly, to solve (4.19), we notice

$$\det A \;=\; \gamma^2 + \gamma\mu\Big((1-\theta)\Psi^\flat + \theta\Psi^\sharp\Big) - \mu^2(1-\theta)\theta\Psi^\flat\Psi^\sharp\left(\frac{p^+}{p^-} - 1\right). \tag{4.22}$$

Since we are now assuming $p^+ = (1+\delta_2)\beta$, $p^- = (1-\delta_1)\beta$, when (4.21) holds, we have $\det A > 0$. Then we can invert the matrix A in (4.19) and obtain the explicit solution

$$\begin{bmatrix} a \\ b \end{bmatrix} \;=\; A^{-1}\begin{bmatrix} 1 \\ \beta \end{bmatrix} \;=\; \frac{1}{\det A}\begin{bmatrix} \gamma + \mu\theta\Psi^\sharp + \mu(1-\theta)\Psi^\flat/(1-\delta_1) \\ \Big(\gamma + \mu\theta\Psi^\sharp(1+\delta_2) + \mu(1-\theta)\Psi^\flat\Big)\beta \end{bmatrix}. \tag{4.23}$$

Next, by (4.15) we can determine the minimum ask price p_A and the maximum bid price p_B as

$$p_A \;=\; (1+\delta_2)\beta\Psi^\sharp + \frac{b}{a}(1-\Psi^\sharp), \qquad p_B \;=\; \left(\frac{1}{(1-\delta_1)\beta}\Psi^\flat + \frac{a}{b}(1-\Psi^\flat)\right)^{-1},$$

while the shape of the sell portion $U_s(p)$ and the shape of the buy portion $U_b(p)$ will be determined by (4.16) correspondingly. □

Remark 2.2 Consider the special case where

$$\Psi^\sharp \;=\; \Psi^\flat \;\doteq\; \overline{\Psi}, \qquad \theta \;=\; \frac{1}{2}, \qquad 1+\delta_2 \;=\; \frac{1}{1-\delta_1} \;\doteq\; 1+\delta\,. \tag{4.24}$$

When the above holds, the inequality (4.21) reduces to

$$\gamma \ > \ \frac{\delta\,\overline{\Psi}}{2}\cdot\mu\,. \tag{4.25}$$

Moreover, from (4.23) and (4.15) it follows

$$\frac{b}{a} = \beta \qquad\qquad \frac{p_A}{\beta} = \frac{\beta}{p_B}\,.$$

Indeed,

$$a \ = \ \frac{1}{\gamma - \mu\delta\overline{\Psi}/2}\,, \qquad b \ = \ \frac{\beta}{\gamma - \mu\delta\overline{\Psi}/2}\,.$$

Remark 2.3 When μ is small, very few external orders arrive. Hence the amount of cash or stock held by an agent will undergo little change over time. In particular, as $\mu \to 0$, the expected value of the payoff (4.8) will satisfy $a \to 1/\gamma$, $b \to \beta/\gamma$.

On the other hand, when μ is large, the incoming orders will arrive very frequently. In this case, the expected amount of cash and stock held by an agent will increase at an exponential rate. The discounting factor γ has to be large enough in order that the inequality (4.22) be satisfied. Otherwise the payoff in (4.8) will be unbounded.

2.2 Random acceptable prices

In the case where the maximum and minimum prices acceptable by external agents are random, the values p^-, p^+ are not known a priori, and must be determined as part of the solution.

Consider an agent posting a sell order at the maximum price p^+. When an external buy order arrives, there is:

- Probability $\Psi_b(K_s)\cdot h_b(p^+/\beta)$ of selling the asset. In this case, the expected payoff will be ap^+.
- Probability $1-\Psi_b(K_s)\cdot h_b(p^+/\beta)$ of not selling the asset. In this case, the expected payoff will be simply b.

Given a, b, the optimal choice of p^+ is the one that maximizes the quantity

$$\Psi_b(K_s)\cdot h_b\left(\frac{p^+}{\beta}\right)\cdot ap^+ + \left[1-\Psi_b(K_s)\cdot h_b\left(\frac{p^+}{\beta}\right)\right] b. \tag{4.26}$$

Setting $s = p^+/\beta$, we observe that the map

$$s \;\mapsto\; h_b(s) \cdot (a\beta\, s - b)$$

has a unique maximum provided that h_b is continuous and

$$h_b''(s) \cdot (a\beta s - b) + 2h_b'(s) \cdot a\beta \;<\; 0, \qquad \text{for all } s \in]1,\, 1+\delta_2[\,. \tag{4.27}$$

Assuming that (4.27) holds, the unique value of p^+ that maximizes (4.26) is uniquely determined:

$$p^+ \;=\; p^+(a,b,\beta) \;=\; \operatorname*{argmax}_{p>\beta} \left\{ h_b\left(\frac{p}{\beta}\right) \cdot (pa - b) \right\}, \tag{4.28}$$

The analysis of the lowest bid price p^- is entirely similar. The agent that posts a bid at this price seeks to maximize the expected profit

$$\frac{b}{p^-}\, \Psi_s(K_b) \cdot h_s\left(\frac{p^-}{\beta}\right) + a \cdot \left[1 - \Psi_s(K_b) \cdot h_s\left(\frac{p^-}{\beta}\right)\right]. \tag{4.29}$$

Setting $s = p^-/\beta$, we observe that the map

$$s \;\mapsto\; h_s(s) \cdot \left(\frac{b}{\beta s} - a\right)$$

has a unique maximum provided that h_s is continuous and

$$h_s''(s) \cdot \left(\frac{1}{s} - \frac{a\beta}{b}\right) - 2h_s'(s) \cdot \frac{1}{s^2} + 2h_s(s) \cdot \frac{1}{s^3} \;<\; 0 \qquad \text{for all } s \in]1-\delta_1,\, 1[\,. \tag{4.30}$$

Assuming (4.30), the unique value of p^- that maximizes (4.29) is uniquely determined:

$$p^- \;=\; p^-(a,b,\beta) \;=\; \operatorname*{argmax}_{p<\beta} \left\{ h_s\left(\frac{p}{\beta}\right) \cdot \left(\frac{b}{p} - a\right) \right\}. \tag{4.31}$$

The functions p^+, p^- in (4.28), (4.31) can be written as

$$p^+ = \beta\sigma^+\left(\frac{b}{a\beta}\right), \qquad p^- = \beta\sigma^-\left(\frac{b}{a\beta}\right),$$

where σ^+, σ^- are defined as following minimizers:

$$\sigma^+(s) \;=\; \max_{\sigma\geq 1}\left\{h_b(\sigma)\cdot\left(\sigma-\frac{b}{a\beta}\right)\right\}, \qquad \sigma^-(s) \;=\; \max_{\sigma\leq 1}\left\{h_s(\sigma)\cdot\left(\frac{1}{\sigma}-\frac{a\beta}{b}\right)\right\}. \quad (4.32)$$

We now introduce two sets of assumptions that guarantee the uniqueness of the functions σ^+, σ^- in (4.32).

(A4) *the map* $s \mapsto h_b(s)$ *is continuous, twice differentiable restricted to* $]1, 1+\delta_2[$, *and satisfies*

$$\begin{cases} h_b(s) = 1 & \text{for } s \in [0,1], \\ h_b(s) = 0 & \text{for } s \geq 1+\delta_2, \\ h_b'(s) < 0, \quad (\delta_1+\delta_2)\cdot h_b''(s) < -h_b'(s) & \text{for } \; s \in]1, 1+\delta_2[\,. \end{cases} \quad (4.33)$$

(A5) *the map* $s \mapsto h_s(s)$ *is continuous, twice differentiable restricted to* $]1-\delta_1, 1[$, *and satisfies*

$$\begin{cases} h_s(s) = 0 & \text{for } x \in [0, 1-\delta_1[\,, \\ h_s(s) = 1 & \text{for } s \geq 1, \\ h_s'(s) > 0, \quad |h_s''(s)|\cdot\left(\dfrac{1}{1-\delta_1}-\dfrac{1}{1+\delta_2}\right) < \dfrac{1}{s^3}|h_s'(s)s - 2h_s(s)| & \text{for } \; s \in]1-\delta_1, 1[\,. \end{cases} \quad (4.34)$$

We check that the above assumptions (4.33)-(4.34) imply (4.27)-(4.30). Indeed, setting $z \doteq b/a\beta$, by (4.5) we know that

$$1-\delta_1 \leq z \leq 1+\delta_2\,.$$

Therefore

$$-\delta_2 < s - z < \delta_1 + \delta_2\,.$$

Combining with **(A4)**, the above inequalities implies

$$h_b''(s)\cdot(s-z) + 2h_b'(s) < h_b''(s)\cdot(s-z) + h_b'(s) < 0\,,$$

then the condition (4.27) will be satisfied.

Similarly we can have

$$1-\frac{1}{1-\delta_1}<\frac{1}{s}-\frac{1}{z}<\frac{1}{1-\delta_1}-\frac{1}{1+\delta_2}\,,$$

and combine the above inequality with **(A5)**, we obtain

$$h_s''(s)\cdot\left(\frac{1}{s}-\frac{1}{z}\right)<\frac{1}{s^3}(h_s'(s)s-2h_s(s))<\frac{2}{s^3}(h_s'(s)s-h_s(s))\,,$$

which also means the condition (4.30) will be satisfied. Correspondingly, p^+,p^- are uniquely determined.

As soon as the prices $p^+>\beta>p^-$ have been determined, the identity (4.5) can be replaced by

$$\begin{aligned} aC_0+bS_0= &\int_0^{+\infty}\mu e^{-\mu\tau}\Bigg\{\int_0^{\tau}e^{-\gamma s}(C_0+\beta S_0)\,ds \\ &+e^{-\gamma\tau}\left[\theta a+(1-\theta)\left[1-\Psi_s(K_b)\cdot h_s\left(\frac{p^-}{\beta}\right)\right]a+(1-\theta)\Psi_s(K_b)\cdot h_s\left(\frac{p^-}{\beta}\right)\frac{b}{p^-}\right]C_0 \\ &+e^{-\gamma\tau}\left[\theta\Psi_b(K_s)\cdot h_b\left(\frac{p^+}{\beta}\right)p^+a+\theta\left[1-\Psi_b(K_s)\cdot h_s\left(\frac{p^+}{\beta}\right)\right]b+(1-\theta)b\right]S_0\Bigg\}\,d\tau. \end{aligned} \tag{4.35}$$

Adopting vector notation, (4.35) can be rewritten as

$$\begin{bmatrix}a\\b\end{bmatrix}=\frac{1}{\gamma+\mu}\begin{bmatrix}1\\\beta\end{bmatrix}+\frac{\mu}{\gamma+\mu}\begin{bmatrix}\theta+(1-\theta)(1-\Psi^\flat h_s(p^-/\beta)) & (1-\theta)\cdot\Psi^\flat\cdot h_s(p^-/\beta)/p^- \\ \theta\cdot\Psi^\sharp\cdot h_b(p^+/\beta)p^+ & (1-\theta)+\theta(1-\Psi^\sharp h_b(p^+/\beta))\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}, \tag{4.36}$$

where

$$\Psi^\sharp\doteq\Psi_b(K_s),\qquad\Psi^\flat\doteq\Psi_s(K_b)\,.$$

Again, this yields a system of the form (4.19), where now

$$A\doteq\begin{bmatrix}\gamma+\mu(1-\theta)\Psi^\flat h_s(p^-/\beta) & -\mu(1-\theta)\Psi^\flat h_s(p^-/\beta)/p^- \\ -\mu\theta\,\Psi^\sharp\,h_b(p^+/\beta)\,p^+ & \gamma+\mu\theta\Psi^\sharp h_b(p^+/\beta)\end{bmatrix}. \tag{4.37}$$

For convenience, we introduce the notation

$$\Theta_1\doteq\mu\theta\Psi^\sharp\,,\qquad\Theta_2\doteq\mu(1-\theta)\Psi^\flat\,,$$

observing that Θ_1, Θ_2 are independent of a, b. Then (4.37) can be written as

$$A = \begin{bmatrix} \gamma + \Theta_2 h_s(p^-/\beta) & -\Theta_2 h_s(p^-/\beta)/p^- \\ -\Theta_1 \, h_b(p^+/\beta) \, p^+ & \gamma + \Theta_1 h_b(p^+/\beta) \end{bmatrix} . \tag{4.38}$$

By solving the corresponding system (4.19), we can respresent $Z(p^-, p^+) \doteq b/a\beta$ as

$$Z(p^-, p^+) = \frac{\gamma + \Theta_1 h_b(p^+/\beta)p^+/\beta + \Theta_2 h_s(p^-/\beta)}{\gamma + \Theta_1 h_b(p^+/\beta) + \Theta_2 h_s(p^-/\beta)\beta/p^-} . \tag{4.39}$$

If we take appropriate distributions $h_b(s), h_s(s)$ such that assumptions **(A4)** and **(A5)** hold, for any

$$z = \frac{b}{a\beta} \in [1-\delta_1,\ 1+\delta_2]$$

we can uniquely determine the couple of prices (p^-, p^+). In turn, given (p^-, p^+), the formula (4.39) uniquely determines a value of $z = Z(p^-, p^+)$.

Lemma 4.1 *Assume the exponential discount γ is large enough such that*

$$\gamma > (1-\theta)\Psi^\flat \left(\frac{1+\delta_2}{(1-\delta_1)^2} - 1 \right) , \tag{4.40}$$

then the composite map

$$z \mapsto (p^-, p^+) \mapsto Z(p^-, p^+) \tag{4.41}$$

will admit a unique fixed point as z^*

Proof. Firstly, from (4.39) we have

$$Z(p^-, p^+) < \frac{(\gamma + \Theta_1 h_b(p^+/\beta) + \Theta_2 h_s(p^-/\beta))p^+/\beta}{\gamma + \Theta_1 h_b(p^+/\beta) + \Theta_2 h_s(p^-/\beta)} = \frac{p^+}{\beta} \leq 1 + \delta_2 ,$$

$$Z(p^-, p^+) > \frac{\gamma + \Theta_1 h_b(p^+/\beta) + \Theta_2 h_s(p^-/\beta)}{(\gamma + \Theta_1 h_b(p^+/\beta) + \Theta_2 h_s(p^-/\beta))\beta/p^-} = \frac{p^-}{\beta} \geq 1 - \delta_1 .$$

This shows that the composed map is a continuous map of the interval $[1-\delta_1, 1+\delta_2]$ into itself. Hence it has a fixed point.

Let us consider the derivative dZ/dz. By the composition in (4.41), we can use

$$\frac{dZ}{dz} = \frac{\partial Z}{\partial p^+} \cdot \frac{\partial p^+}{\partial z} + \frac{\partial Z}{\partial p^-} \cdot \frac{\partial p^-}{\partial z} .$$

Since we denote $p^+ = \beta\sigma^+(z), p^- = \beta\sigma^-(z)$ with (4.32), the above derivative can also be expanded as

$$\frac{dZ}{dz} = \frac{\partial Z}{\partial \sigma^+} \cdot \frac{\partial \sigma^+}{\partial z} + \frac{\partial Z}{\partial \sigma^-} \cdot \frac{\partial \sigma^-}{\partial z} \,.$$

Then we have

$$\begin{aligned}
\frac{\partial Z}{\partial \sigma^+} &= \frac{\Theta_1 \left(h_b'(\sigma^+)\left(\sigma^+ - Z\right) + h_b(\sigma^+)\right)}{\gamma + \Theta_1 h_b(\sigma^+) + \Theta_2 h_s(\sigma^-)/\sigma^-} \,, \\
\frac{\partial Z}{\partial \sigma^-} &= \frac{\Theta_2 \left(h_s'(\sigma^-)\left(1 - \frac{Z}{\sigma^-}\right) + Z h_s(\sigma^-)\frac{1}{(\sigma^-)^2}\right)}{\gamma + \Theta_1 h_b(\sigma^+) + \Theta_2 h_s(\sigma^-)/\sigma^-} \,,
\end{aligned}$$

directly. Further, from (4.28) we notice σ^+ should satisfy

$$h_b'(\sigma^+)(\sigma^+ - z) + h_b(\sigma^+) \;=\; 0 \,.$$

Differentiate the above formula w.r.t. z, we obtain

$$h_b''(\sigma^+)(\sigma^+ - z)\frac{\partial \sigma^+}{\partial z} + h_b'(\sigma^+)\frac{\partial \sigma^+}{\partial z} - h_b'(\sigma^+) + h_b'(\sigma^+)\frac{\partial \sigma^+}{\partial z} = 0$$

Based on (**A4**), we know

$$h_b''(\sigma^+)(\sigma^+ - z) + 2h_b'(\sigma^+) < h_b'(\sigma^+) < 0 \,,$$

which implies

$$0 < \frac{\partial \sigma^+}{\partial z} < 1 \,.$$

From (4.31), we notice σ^- should satisfy

$$h_2'(\sigma^-)\left(\frac{1}{\sigma^-} - \frac{1}{z}\right) - h_s(\sigma^-)\frac{1}{(\sigma^-)^2} \;=\; 0$$

Take derivative w.r.t. z,

$$h_s''(\sigma^-)\left(\frac{1}{\sigma^-} - \frac{1}{z}\right)\frac{\partial \sigma^-}{\partial z} - h_s'(\sigma^-)\frac{2}{(\sigma^-)^2}\frac{\partial \sigma^-}{\partial z} + h_s'(\sigma^-)\frac{1}{z^2} + h_s'(\sigma^-)\frac{2}{(\sigma^-)^2}\frac{\partial \sigma^-}{\partial z} \;=\; 0$$

Since from (**A5**) we know

$$h_s''(\sigma^-)\left(\frac{1}{\sigma^-} - \frac{1}{z}\right) - 2h_s'(\sigma^-)\frac{1}{(\sigma^-)^2} + h_s(\sigma^-)\frac{2}{(\sigma^-)^3} < h_s'(\sigma^-)\frac{1}{(\sigma^-)^2} < 0$$

which implies

$$0 < \frac{\partial \sigma^-}{\partial z} \;=\; \frac{-h_s'(\sigma^-)\dfrac{1}{z^2}}{h_s''(\sigma^-)\left(\dfrac{1}{\sigma^-}-\dfrac{1}{z}\right) - 2h_s'(\sigma^-)\dfrac{1}{(\sigma^-)^2} + h_s(\sigma^-)\dfrac{2}{(\sigma^-)^3}} < 1\,.$$

Then

$$\begin{aligned}\frac{dZ(p^+,p^-)}{dz} &< \frac{\partial Z}{\partial \sigma^+} + \frac{\partial Z}{\partial \sigma^-} \\ &= \frac{\Theta_1\left(h_b'(\sigma^+)\left(\sigma^+ - Z\right) + h_b(\sigma^+)\right) + \Theta_2\left(h_s'(\sigma^-)\left(1-\frac{Z}{\sigma^-}\right) + Zh_s(\sigma^-)\dfrac{1}{(\sigma^-)^2}\right)}{\gamma + \Theta_1 h_b(\sigma^+) + \Theta_2 h_s(\sigma^-)/\sigma^-}\,.\end{aligned} \tag{4.42}$$

Since $Z \in [\sigma^-, \sigma^+]$, and $h_b(\sigma^+) < 0, h_s(\sigma^-) > 0$, combining with (4.40), (4.42) implies

$$\frac{dZ(p^+,p^-)}{dz} < \frac{\Theta_1 h_b(\sigma^+) + \Theta_2 \cdot Zh_s(\sigma^-)\dfrac{1}{(\sigma^-)^2}}{\gamma + \Theta_1 h_b(\sigma^+) + \Theta_2 h_s(\sigma^-)/\sigma^-} < \frac{\Theta_1 + \Theta_2\dfrac{1+\delta_2}{(1-\delta_1)^2}}{\gamma + \Theta_1 + \Theta_2} < 1\,,$$

i.e., $z \mapsto Z$ is a strict contraction, and will admit a unique fixed point.

□

As soon as this fixed point z^* is found, we can insert the values $p^- = p^-(z^*)$ and $p^+ = p^+(z^*)$ in (4.38), and solve (4.19) for a, b. More explicitly, this yields

$$\begin{cases} a \;=\; \dfrac{1}{\det A}\left[\gamma + \Theta_1 h_b\left(\dfrac{p^+}{\beta}\right) + \Theta_2 h_s\left(\dfrac{p^-}{\beta}\right)\dfrac{\beta}{p^-}\right], \\ b \;=\; \dfrac{1}{\det A}\left[\gamma + \Theta_1 h_b\left(\dfrac{p^+}{\beta}\right)p^+ + \Theta_2 h_s\left(\dfrac{p^-}{\beta}\right)\right]. \end{cases} \tag{4.43}$$

given

$$\begin{aligned}\det A(z^*) \;&=\; \left(\gamma + \Theta_2 h_s\left(\tfrac{p^-(z^*)}{\beta}\right)\right)\cdot\left(\gamma + \Theta_1 h_b\left(\tfrac{p^+(z^*)}{\beta}\right)\right) \\ &\quad -\Theta_1\Theta_2 h_s\left(\tfrac{p^-(z^*)}{\beta}\right) h_b\left(\tfrac{p^+(z^*)}{\beta}\right)\frac{p^+(z^*)}{p^-(z^*)} \\ &> \; 0\end{aligned} \tag{4.44}$$

holds.

Proposition 2.2 *Assume the distributions h_b, h_s of random variable p^+, p^- satify the assumptions* **(A4-A5)**, *then the shape of the infinite horizon game described in Section 2 with exponential discount γ and random acceptable prices p^-, p^+ will be uniquely* *determined given the condition*

$$\gamma > \frac{1}{2}\sqrt{(\Theta_1 h_b^* - \Theta_2 h_s^*)^2 + 2\Theta_1\Theta_2 h_b^* h_s^* \frac{p^+(z^*)}{p^-(z^*)}} - \frac{1}{2}(\Theta_1 h_b^* + \Theta_2 h_s^*)\,, \tag{4.45}$$

where

$$h_b^* \doteq h_b(p^+(z^*)/\beta)\,, \qquad h_s^* \doteq h_s(p^-(z^*))\,,$$

and z^* is the fixed point of the map (4.41).

3 An asset with fundamental value modeled by a jump process

In this section we assume that the fundamental value $\beta(t)$ of the asset is piecewise constant in time, and jumps at random times $0 < \tau_1 < \tau_2 < \cdots$ These times will be modeled by a Poisson arrival process with rate λ. Setting

$$\beta(t) \;=\; \beta_j \qquad t \in [\tau_j,\, \tau_{j+1}[\,,$$

we assume that every time a jump occurs, the new value satisfies

$$\ln\left(\frac{\beta_{j+1}}{\beta_j}\right) \;\sim\; \mathcal{N}(0,\sigma^2). \tag{4.1}$$

The logarithm of the ratio between the new value and the previous one thus follows a normal distribution with mean zero and variance σ^2, for some given $\sigma > 0$.

As in the previous models, we assume that external orders arrive with rate μ. Moreover, to simplify the analysis, we assume that the maximum and minimum acceptable prices are deterministic:

$$p^- \;=\; (1-\delta_1)\beta, \qquad p^+ \;=\; (1+\delta_2)\beta. \tag{4.2}$$

Combining the two independent arrival processes, we obtain a Poisson arrival process with intensity $\lambda + \mu$. Call t_i the random time when the i-th event occurs. This event can now be of three types:

1) The fundamental value β of the stock has a jump.

2) A buy order arrives,

3) A sell order arrives,

The probabilities of these three events are, respectively,

$$\theta_1 = \frac{\lambda}{\lambda+\mu}\,, \qquad \theta_2 = \frac{\mu\theta}{\lambda+\mu}\,, \qquad \theta_3 = \frac{\mu(1-\theta)}{\lambda+\mu}\,, \tag{4.3}$$

Assuming that the two arrival processes are independent, we have

$$\text{Prob.}\{t_i - t_{i-1} \;>\; s\} \;=\; e^{-(\lambda+\mu)s}. \tag{4.4}$$

Let $V^C(\beta_0)$ denote the expected payoff of a unit amount of cash, $V^S(\beta_0)$ denote the expected payoff of one unit amount of stock, given that the initial fundamental value of the stock as β_0.

By the same arguments used in Section 2, these values satisfy

$$\begin{aligned} V^C(\beta_0)C_0 + V^S(\beta_0)S_0 \;&=\; E\left[\int_0^{+\infty} e^{-\gamma t}(C(t)+\beta(t)S(t))\;dt\right] \\ =\; E\Bigg[\int_0^{t_1} e^{-\gamma t}&(C_0+\beta_0 S_0)\;dt + e^{-\gamma t_1}[V^C(\beta(t_1))C_1 + V^S(\beta(t_1))S_1]\Bigg]\,. \end{aligned} \tag{4.5}$$

Here C_1, S_1 denote the random amount of cash or stock held at time t_1.

Since the LOB is in equilibrium, the expected payoff is the same for all agents, regardless of the price at which they post their limit orders. We can thus compute the above expectations at $p = p^-$ and $p = p^+$ with p^+ determined as $(1+\delta_2)\beta$, and p^- as $(1-\delta_1)\beta$. This yields

$$\begin{aligned} V^C(\beta_0)C_0 + V^S(\beta_0)S_0 \;&=\; \int_0^{+\infty} e^{-(\lambda+\mu)t_1}\Bigg\{\int_0^{t_1} e^{-\gamma t}(C_0+\beta_0 S_0)\;dt \\ &+\theta_1 e^{-\gamma t_1}\cdot E\Big[V^C(\beta_1)C_0 + V^S(\beta_1)S_0 \;\Big|\; \beta_0\Big] \\ &+\theta_2 e^{-\gamma t_1}\cdot\Big[(V^C(\beta_0)C_0 + V^C(\beta_0)\Psi^\sharp p^+ S_0 + (1-\Psi^\sharp)V^S(\beta_0)S_0\Big] \\ &+\theta_3 e^{-\gamma t_1}\cdot\Big[V^S(\beta_0)\Psi^\flat\frac{1}{p^-}C_0 + V^C(\beta_0)(1-\Psi^\flat)C_0 + V^S(\beta_0)S_0)\Big]\Bigg\}\,dt_1\,. \end{aligned} \tag{4.6}$$

As before, we denoted by

$$\Psi^\sharp \;\doteq\; \Psi_b(K_s), \qquad \Psi^\flat \;\doteq\; \Psi_s(K_b),$$

the probabilities that a bid at the maximum price p^+ or at the minimum price p^- will be fulfilled, respectively.

We seek a solution in the form

$$V^C(\beta) = a, \qquad V^S(\beta) = b\beta\,.$$

In view of (4.3), from (4.6) we obtain

$$\begin{aligned}\begin{bmatrix} a \\ b \end{bmatrix} &= \frac{1}{\gamma+\mu+\lambda}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{\lambda}{\gamma+\mu+\lambda}\begin{bmatrix} a \\ bE[\beta_1|\beta_0]/\beta_0 \end{bmatrix} \\ &+ \frac{\mu}{\gamma+\mu+\lambda}\begin{bmatrix} \theta+(1-\theta)(1-\Psi^\flat) & (1-\theta)\Psi^\flat\dfrac{\beta_0}{p^-} \\ \theta\Psi^\sharp\dfrac{p^+}{\beta_0} & \theta(1-\Psi^\sharp)+(1-\theta) \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}.\end{aligned} \tag{4.7}$$

Since $E[\beta_1|\beta_0] = e^{\sigma^2/2}\beta_0$, (4.7) can be simplified as

$$\begin{bmatrix} \gamma+\mu(1-\theta)\Psi^\flat & -\mu(1-\theta)\Psi^\flat\dfrac{\beta_0}{p^+} \\ -\mu\theta\Psi^\sharp\dfrac{p^+}{\beta_0} & \gamma+\lambda(1-e^{\sigma^2/2})+\mu\theta\Psi^\sharp \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Proposition 3.1 *The above biding game with random jumps on the fundamental value $\beta(t)$ will admit a unique shape in Nash equilibrium, given the condition*

Assume that the max/min acceptable prices are deterministic. When the discount factor γ is large enough:

$$\begin{aligned}\gamma \;&> \frac{\mu}{2}\sqrt{\left((1-\theta)\Psi^\flat - \theta\Psi^\sharp + (e^{\frac{\sigma 2}{2}}-1)\tfrac{\lambda}{\mu}\right)^2 + 4\theta(1-\theta)\Psi^\flat\Psi^\sharp\left(\frac{p^+}{p^-}\right)} \\ &-\frac{\mu}{2}\Big((1-\theta)\Psi^\flat+\theta\Psi^\sharp\Big) + \frac{\lambda}{2}\left(e^{\frac{\sigma^2}{2}}-1\right)\,,\end{aligned} \tag{4.8}$$

where $p^+/p^- = (1+\delta_2)/(1-\delta_1)$, one can uniquely determine the value functions in the form $V^C(\beta) = a$, $V^S(\beta) = b\beta$. The above bidding game with random jumps will admit a unique shape of the LOB, in Nash equilibrium.

Proof. Denote the coefficient matrix as

$$A \doteq \begin{bmatrix} \gamma+\mu(1-\theta)\Psi^\flat & -\mu(1-\theta)\Psi^\flat\dfrac{\beta_0}{p^+} \\ -\mu\theta\Psi^\sharp\dfrac{p^+}{\beta_0} & \gamma+\lambda(1-e^{\sigma^2/2})+\mu\theta\Psi^\sharp \end{bmatrix}$$

If (4.8) holds, then

$$\det A = \gamma^2+\gamma\mu\Big((1-\theta)\Psi^\flat+\theta\Psi^\sharp\Big)-\mu^2\theta(1-\theta)\Psi^\sharp\Psi^\flat\left(\frac{p^+}{p^-}-1\right)+\Big(\gamma+\mu(1-\theta)\Psi^\flat\Big)(1-e^{\sigma^2/2})\lambda > 0\,. \tag{4.9}$$

Further, we can obtain the explicit solution for (a, b) as

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{\det A}A^{-1}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\det A}\begin{bmatrix} \gamma+\mu\theta\Psi^\sharp+\mu(1-\theta)\Psi^\flat\frac{1}{1-\delta_1}+\lambda(1-e^{\sigma^2/2}) \\ \gamma+\mu(1-\theta)\Psi^\flat+\mu\theta\Psi^\sharp(1+\delta_2) \end{bmatrix}\,. \tag{4.10}$$

And we notice a, b are independent of β_0, then $V^C(\beta) = a, V^S(\beta) = b\beta$ is an appropriate solution for equation (4.6). Further, based on (4.16), we can determine the minimum ask price p_A and maximum bid price p_B as well as the the shape of the LOB. □

Remark 2

- Compared with the model where the value of the stock is constant, under the same values of the parameters $(\gamma, \theta, \delta_1, \delta_2, K_s, K_b)$ and β, the ratio z between stock value V^S and cash value V^C will be greater than the one in the constant model. And based on

$$p_A = p^+\Psi^\sharp + z(1-\Psi^\sharp)\,, \quad p_B = \left(\frac{1}{p^-}\Psi^\flat + \frac{1}{z}(1-\Psi^\flat)\right)^{-1}\,,$$

  p_A, p_B will be greater than the ones in section 3 as well.

- The value function of the cash V^C will be independent of the fundamental value, and the value function of the stock V^S will be a linear function of the initial fundamental value β_0.

- If $K_s = K_b, \theta = 1/2, 1+\delta_2 = 1/(1-\delta_1)$, the ratio z of the stock value and the cash value will always greater than β_0. Because the expectation of β will keep increasing.

- It is obvious that either $\sigma \to 0$ or $\lambda \to 0$, this model will coincide with the previous one in section 3.

- As $\mu \to 0$, $V^C \to 1/\gamma$ and $V^S \to \beta_0/(\gamma + \lambda(1-e^{\sigma^2/2}))$, which means when there are no incoming orders, we need the discounting factor γ satisfying

$$\gamma > \lambda(e^{\sigma^2/2} - 1)\,. \tag{4.11}$$

Indeed, we can calculate the expected value of the fundament value β_t at time t as $E[\beta_t|\beta_0] = \exp\{\lambda e^{\sigma^2 t/2} - 1\}\beta_0$, and (4.11) will make sure γ is large enough to cancel out the increment of the fundamental value, *i.e.*, let $\int_0^{+\infty} \exp\{-\gamma t\}\beta_t \; dt < \infty$.

Chapter 5
Some Additional Models

1 Size of the LOB

Based on the analysis in the previous Chapter, we observe that the rate of growth, for the wealth of the agents posting limit orders, strongly depends on the sizes K_s, K_b of the LOB. Indeed, when K_s, K_b are large, there is more competition among agents and their profit decreases.

This leads to a natural question. Assume that all agents expect a certain minimum payoff (after the exponential discount). If they cannot achieve it, they simply leave the LOB and look for other investments. For simplicity, assume that the right and left portions of the LOB have the same size: $K_s = K_b = K$. Is there a unique value K that will provide this given payoff ?

We formulate this question in a more precise way.

1.1 Agents' profit vs. size of the LOB

Based on the analysis in the previous sections, we observe that the expected profit, for the agents posting limit orders, strongly depends on the sizes K_s, K_b of the LOB. Indeed, when K_s, K_b are large, there is more competition among agents and their profit decreases.

This leads to a natural question. Assume that all agents expect a certain minimum payoff (after the exponential discount). If they cannot achieve it, they simply leave the LOB and look for other investments. For simplicity, assume that the right and left portions of the LOB have the same size: $K_s = K_b = K$. What is the largest value of K that will provide this given payoff ?

We formulate this question in a more precise way.

Given a discount factor γ and a size K for the two sides of the LOB, let $a(K)$ and

$b(K)$ be the discounted expected payoffs for an agent holding a unit amount of cash or of stock, respectively. Having fixed an expected profit J, we seek to determine the size K that will yield

$$a(K) + b(K) \;=\; J. \tag{5.1}$$

According to the analysis in Chapter 4, for a given discount factor γ and a size K of the LOB, the expected discounted profit from a unit amount of cash and a unit amount of stock are given by

$$\begin{bmatrix} a(K) \\ b(K) \end{bmatrix} \;=\; \frac{1}{\det A} \begin{bmatrix} \gamma + \mu\theta\Psi(K) + \mu(1-\theta)\Psi(K)/(1-\delta_1) \\ \Big(\gamma + \mu\theta\Psi(K)(1+\delta_2) + \mu(1-\theta)\Psi(K)\Big)\beta \end{bmatrix}. \tag{5.2}$$

We recall that the determinant of the matrix A is computed by

$$D(K) \;\doteq\; \det A \;=\; \gamma^2 + \gamma\mu\Psi(K) - \mu^2(1-\theta)\theta\Psi^2(K)\left(\frac{p^+}{p^-} - 1\right).$$

Differentiating $D(K)$ w.r.t. K one finds

$$D'(K) \;=\; \gamma\mu\Psi'(K) - 2\mu^2(1-\theta)\theta\frac{\delta_1+\delta_2}{1-\delta_1}\Psi(K)\Psi'(K).$$

In turn, this yields

$$a'(K) \;=\; \frac{\Psi'(K)}{D^2(K)} \cdot \left[\gamma^2\mu\left(\frac{(1-\theta)\delta_1}{1-\delta_1}\right) + \left(2\gamma + \mu\theta\Psi(K) + \mu\Psi(K)\frac{1-\theta}{1-\delta_1}\right)\left(\mu^2(1-\theta)\theta\left(\frac{\delta_1+\delta_2}{1-\delta_1}\right)\right)\right],$$

$$b'(K) \;=\; \frac{\Psi'(K)}{D^2(K)} \cdot \left[\gamma^2\mu\cdot\theta\delta_2 + (2\gamma + \mu\theta\Psi(K)(1+\delta_2) + \mu\Psi(K)(1-\theta))\left(\mu^2(1-\theta)\theta\left(\frac{\delta_1+\delta_2}{1-\delta_1}\right)\right)\right].$$

Since $\Psi'(K) < 0$, we clearly $a'(K) < 0$ and $b'(K) < 0$. This shows that both $a(K)$ and $b(K)$ are monotone decreasing functions of K, that is

$$\frac{d}{K}J(K) < 0,$$

which implies more competitors in the market will let each agent have a lower expected profit.

Next, we consider the limits of a, b as $K \to +\infty$ or $K \to 0$.

(i) As $K \to +\infty$, we have $\Psi(K) \to 0$. Therefore

$$a(K) \;\to\; \frac{1}{\gamma}, \qquad\qquad b(K) \;\to\; \frac{\gamma}{\beta}.$$

In this case, the expected discounted payoff for an agent holding a unit amount of cash and a unit amount of stock will approach

$$J_{min} \;=\; \frac{1+\beta}{\gamma}.$$

Hence, for any agent posting limit orders, the rate of increase of the wealth approaches 0.

We observe that, as $K \to +\infty$, both the minimum ask price p_A and the maximum bid price p_B will approach β, i.e., the spread of the limit order book will approach zero. In this case, the agents do not make any profit from the market.

(ii) As $K_s, K_b \to 0$, we have $\Psi(K) \to 1$. In this case

$$a(K) \to \frac{\gamma + \mu\theta + \mu(1-\theta)\dfrac{1}{1-\delta_1}}{\gamma^2 + \gamma\mu - \mu^2\theta(1-\theta)\dfrac{\delta_1+\delta_2}{1-\delta_1}}, \qquad b(K) \to \frac{\gamma + \mu\theta(1+\delta_2) + \mu(1-\theta)}{\gamma^2 + \gamma\mu - \mu^2\theta(1-\theta)\dfrac{\delta_1+\delta_2}{1-\delta_1}}\beta.$$

The expected profit J of the agent holding a unit amount of cash and a unit amount of stock will now approaches

$$J_{max} \;=\; \frac{\gamma(1+\beta) + \mu\theta(1+(1+\delta_1)\beta) + \mu(1-\theta)\left(\dfrac{1}{1-\delta_1}+\beta\right)}{\gamma^2 + \gamma\mu - \mu^2\theta(1-\theta)\dfrac{\delta_1+\delta_2}{1-\delta_1}}$$

Since both a, b are monotone decreasing functions of K, the same holds for the expected (exponentially discounted) profit J. We thus conclude

Proposition 1.1 *In the previous setting, for every $J \in [J_{min}, J_{max}]$ there exists a unique common size K for the two parts of the LOB such that the expected (exponentially discounted) payoff for an agent holding a unit amount of cash and a unit amount of stock is precisely J.*

Remark 1.1 *For the symmetric case, where $\theta = 1/2$, $1+\delta_2 = 1/(1-\delta_1)$, then as*

$K \to 0$, we have

$$a(K) \to \frac{1}{\gamma - \mu\delta_2/2} > \frac{1}{\gamma}, \qquad b(K) \to \frac{\beta}{\gamma - \mu\delta_2/2} > \frac{\beta}{\gamma},$$

which implies when $K \to 0$, we did achieve a better expected profit than the case where $K \to +\infty$.

1.2 LOB with size as random variable.

We also curious about what happens if the sizes of the sell and buy parts of the LOB are not constant, but at each step these sizes are i.i.d. random variables, uniformly distributed over the intervals $[K_s^-, K_s^+]$ and $[K_b^-, K_b^+]$, respectively?

Since if a pair of K_s, K_b is given, we can uniquely determine a pair of prices p_A, p_B. Then when K_s, K_b becomes random variable with p.d.f

$$\text{Prob.}\{K_s = x\} = g_s(x)\,, \qquad \text{Prob.}\{K_b = x\} = g_b(x)\,,$$

To determine a, b, we need to take the expectation of the previous expected profit w.r.t K_s and K_b, then (2.38) in Chapter 4 becomes

$$\begin{aligned} aC_0 + bS_0 &= E^{K_s,K_b}\Bigg[\int_0^{+\infty} \mu e^{-\mu\tau} \Bigg\{ \int_0^{\tau} e^{-\gamma s}(C_0 + \beta S_0)\, ds \\ &\quad + e^{-\gamma\tau}\left[\theta a + (1-\theta)(1-\Psi^\flat)a + (1-\theta)\Psi^\flat \frac{b}{p^-}\right] C_0 \\ &\quad + e^{-\gamma\tau}\left[\theta\Psi^\sharp p^+ a + \theta(1-\Psi^\sharp)b + (1-\theta)b\right] S_0 \Bigg\}\, d\tau \Bigg] \\ &= \int_0^{+\infty} \mu e^{-\mu\tau} \Bigg\{ \int_0^{\tau} e^{-\gamma s}(C_0 + \beta S_0)\, ds \\ &\quad + e^{-\gamma\tau}\left[\theta a + (1-\theta)(1 - E^{K_b}[\Psi_s(K_b)])a + (1-\theta)E^{K_b}[\Psi_s(K_b)]\, \frac{b}{p^-}\right] C_0 \\ &\quad + e^{-\gamma\tau}\left[\theta E^{K_s}[\Psi_b(K_s)]p^+ a + \theta(1 - E^{K_s}[\Psi_b(K_s)])b + (1-\theta)b\right] S_0 \Bigg\}\, d\tau. \end{aligned} \tag{5.3}$$

Then the system of equations of a, b can be written as

$$A \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ \beta \end{bmatrix},$$

where

$$A \doteq \begin{bmatrix} \gamma + \mu(1-\theta)(1 - E^{K_b}[\Psi_s(K_b)]) & -\mu(1-\theta)E^{K_b}[\Psi_s(K_b)]\dfrac{1}{p^-} \\ -\mu\theta E^{K_s}[\Psi_b(K_s)]p^+ & \gamma + \mu\theta E^{K_s}[\Psi_b(K_s)] \end{bmatrix}.$$

Assume

$$\begin{aligned} \det A \;=\;& \gamma^2 + \gamma\mu\Big((1-\theta)E^{K_b}[\Psi_s(K_b)] + \theta E^{K_s}[\Psi_b(K_s)]\Big) \\ & -\mu^2(1-\theta)\theta E^{K_b}[\Psi_s(K_b)]E^{K_s}[\Psi_b(K_s)]\left(\frac{p^+}{p^-} - 1\right) \;>\; 0\,. \end{aligned}$$

Since $p^+ = (1+\delta_2)\beta$, $p^- = (1-\delta_1)\beta$, the inequality in (4.22) will be satisfied provided that

$$\begin{aligned} \gamma \;>\; & \frac{\mu}{2}\left(\sqrt{\Big((1-\theta)E^{K_b}[\Psi_s(K_b)] - \theta E^{K_s}[\Psi_b(K_s)]\Big)^2 + 4\theta(1-\theta)E^{K_b}[\Psi_s(K_b)]E^{K_s}[\Psi_b(K_s)]\left(\frac{1+\delta_2}{1-\delta_1}\right)}\right. \\ & \left. -\Big((1-\theta)E^{K_b}[\Psi_s(K_b)] + \theta E^{K_s}[\Psi_b(K_s)]\Big)\right)\,. \end{aligned} \tag{5.4}$$

When (5.4) holds, we can invert the matrix A and obtain the explicit solution

$$\begin{bmatrix} a \\ b \end{bmatrix} \;=\; A^{-1}\begin{bmatrix} 1 \\ \beta \end{bmatrix} \;=\; \frac{1}{\det A}\begin{bmatrix} \gamma + \mu\theta E^{K_s}[\Psi_b(K_s)] + \mu(1-\theta)E^{K_b}[\Psi_s(K_b)]/(1-\delta_1) \\ \Big(\gamma + \mu\theta E^{K_s}[\Psi_b(K_s)](1+\delta_2) + \mu(1-\theta)E^{K_b}[\Psi_s(K_b)]\Big)\beta \end{bmatrix}. \tag{5.5}$$

Indeed, for the case with deterministic acceptable prices, compare with (4.17), the only difference here is that the last lines are replaced by an expected value, depending on the probability distribution of the variables K_s, K_b, which didn't change the linear relationship in (4.18).

2 Risk-averse agents

In (4.7)-(4.8), we assumed that all agents simply maximized the expected payoff. In this section, we analyze how the shape of the LOB would change, some risk-aversion were taken into account.

This issue can be best studied in the setting of a single-shot game. Consider the "sell part" of the LOB, and let K_s be the total amount of stock offered for sale. For $\xi \in [0, K_s]$, let $p(\xi)$ be the price asked by agent ξ. After a rearrangement, it is not restrictive to assume that the function $p : [0, K_s] \mapsto I\!R$ is monotone increasing. Call β the fundamental value of the stock. To simplify the analysis, we assume that the maximum acceptable price for the external buyer is deterministic: $p_{max} = (1 + \delta_2)\beta$. The expected profit for agent ξ is

$$(p(\xi) - \beta)\Psi_b(\xi).$$

If the agents seek to maximize this expected profit, this implies that at an equilibrium the quantity

$$m(\xi) \doteq (p(\xi) - \beta)\Psi_b(\xi) \tag{5.6}$$

is a constant, independent of $\xi \in [0, K_s]$. Evaluating this constant at $\xi = K_s$, we obtain

$$p(\xi)\Psi_b(\xi) - \beta\Psi_b(\xi) = \beta(1 + \delta_2)\Psi_b(K_s) - \beta\Psi_b(K_s), \tag{5.7}$$

This yields

$$p(\xi) = \frac{\delta_2 \Psi_b(K_s)}{\Psi_b(\xi)} \beta\,. \tag{5.8}$$

Next, assume instead that the agents seek to maximize the difference

[expected profit] - $\varepsilon\cdot$ [variance].

At a Nash equilibrium, this implies that the above quantity is constant for all $\xi \in [0, K_s]$. Calling $m(\xi)$ the mean value, defined at (5.6), this leads to the identity

$$m(\xi) - \varepsilon\Big[\Big(p(\xi) - m(\xi)\Big)^2 \Psi_b(\xi) + \Big(m(\xi) - \beta\Big)^2 (1 - \Psi_b(\xi))\Big] = \text{constant.} \tag{5.9}$$

Once again, the value of the constant can be computed explicitly at $\xi = K_s$.

We denote the variance of the profit for the agent ξ as

$$\begin{aligned} Var(\xi) &= \Big(p(\xi) - m(\xi)\Big)^2 \Psi_b(\xi) + \Big(m(\xi) - \beta\Big)^2 (1 - \Psi_b(\xi)) \\ &= (p(\xi) - \beta)^2 \Psi_b(\xi)(1 - \Psi_b(\xi))\,. \end{aligned}$$

At $p = p^+ = (1 + \delta_2)\beta$, we have $\xi = K_s$ and

$$m(K_s) = (1+\delta_2)\beta\Psi_b(K_s) + \beta(1-\Psi_b(K_s)), \qquad Var(K_s) = ((1+\delta_2)\beta-\beta)^2\Psi_b(K_s)(1-\Psi_b(K_s))$$

And from the equilibrium (5.9), we require

$$m(\xi) - \varepsilon Var(\xi) \;=\; m(K_s) - \varepsilon Var(K_s) \qquad \text{for every } \xi \in [0, K_s].$$

This implies

$$\Psi_b(\xi) \cdot \Big(1 - \varepsilon(p(\xi) - \beta)(1 - \Psi_b(\xi)\Big) \cdot (p(\xi) - \beta) \;=\; \delta_2 \beta \Psi_b(K_s)\Big(1 - \varepsilon\delta_2\beta(1 - \Psi_b(K_s))\Big), \tag{5.10}$$

for any $\xi \in [0, K_s]$, since

$$p(K_s) \;=\; (1 + \delta_2)\beta\,.$$

Differentiating (5.10) w.r.t. ξ, we obtain

$$p'(\xi) \;=\; -\frac{\Psi_b'(\xi)}{\Psi_b(\xi)} \cdot \frac{1 - \varepsilon\Big(p(\xi) - \beta\Big)\Big(1 - 2\Psi_b(\xi)\Big)}{1 - 2\varepsilon\Big(p(\xi) - \beta\Big)\Big(1 - \Psi_b(\xi)\Big)} \cdot (p(\xi) - \beta)\,, \qquad \xi \in [0, K_s], \tag{5.11}$$

We notice that to guarantee the price p will be increasing w.r.t. ξ, we require ε to be small enough. That is, ε should satisfy

$$\varepsilon < \frac{1}{2\delta_2\beta(1 - \Psi_b(K_s)} \tag{5.12}$$

to make sure $p'(\xi) > 0$ for all $\xi \in [0, K_s]$. Then the shape of the sell portion and the minimum ask price p_A will be determined by (5.10) immediately as

$$p_A \;=\; \beta + \delta_2\beta\Psi^\sharp(1 - \varepsilon\delta_2\beta(1 - \Psi^\sharp))\,.$$

Further, differentiating (5.10) w.r.t. ε, we obatin

$$\frac{\partial p}{\partial \varepsilon} \;=\; \frac{\Psi_b(\xi)(1 - \Psi_b(\xi))(p(\xi) - \beta)^2 - \Psi_b(K_s)(1 - \Psi_b(K_s)\delta_2^2\beta^2}{\Psi_b(\xi)\Big(1 - 2\varepsilon(1 - \Psi_b(\xi))(p(\xi) - \beta)\Big)}\,, \text{ for any } \xi \in [0,\ K_s]\,. \tag{5.13}$$

From (5.12), we notice the denominator in (5.13) is > 0, while the numerator is equivalent to

$$\frac{\delta_2\beta\Psi_b(K_s)}{\varepsilon}\left(\frac{1 - \varepsilon\delta_2\beta(1 - \Psi_b(K_s)}{1 - \varepsilon(p - \beta)(1 - \Psi_b(\xi))} - 1\right) \;<\; 0\,,$$

then we proved that $\partial p/\partial\varepsilon < 0$. It implies that as ε increases the price charged by each agent decreases. Especially, the minimum ask price p_A for the agents posting selling orders will be closer to β, as shown in Fig 5.1.

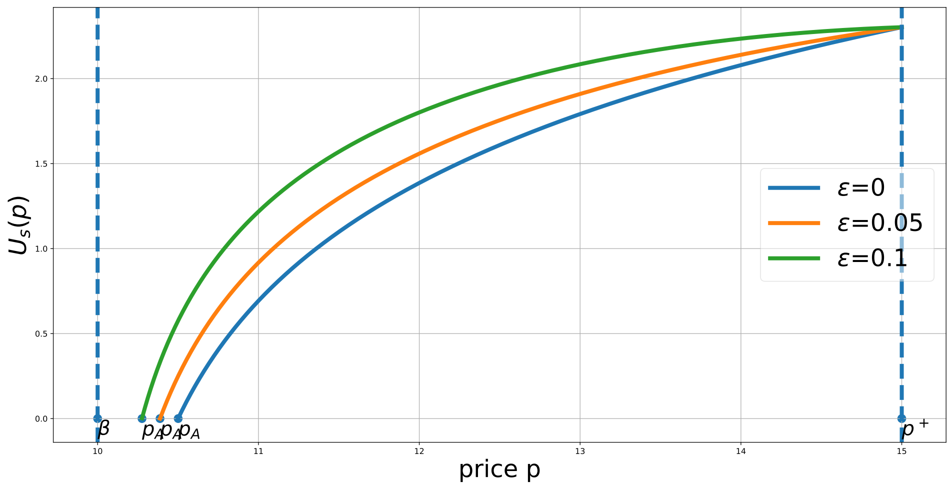

Figure 5.1. The shape of the "sell portion" of the limit order book with risk-averse agents, with 3 different risk-averse factors $\varepsilon = 0,\ 0.05,\ 0.1$. More parameters chosen as $a = 1$, $b = \beta = 10$, $p^+ = 1.5\beta$, $K_s = \ln 10$. The distribution $\Psi_b(s)$ is taken as e^{-s}.

2.1 "Buy" portion

Similarly, we can consider the buy portion of the LOB.

Denote the expected profit of the agent posting a bid to buy stock as $M_b(\xi)$, and the variance as $Var_b(\xi)$. Then

$$M_b(\xi) = \frac{\beta}{p(\xi)}\Psi_s(\xi) + (1 - \Psi_s(\xi))\,,$$

$$Var_b(\xi) = \left(M_b(\xi) - \frac{\beta}{p}\right)^2 \Psi_s(\xi) + (M_b(\xi) - 1)^2(1 - \Psi_s(\xi)\,.$$

Further, $Var_b(p)$ can be simplified as

$$Var_b(\xi) \;=\; \left(\frac{\beta}{p(\xi)} - 1\right)^2 \Psi_s(\xi)(1 - \Psi_s(\xi))\,.$$

At $\xi = K_b$, we have

$$M_b(K_b) = \frac{\beta}{p^-}\,\Psi_s(K_b) + (1 - \Psi_s(K_b)), \qquad Var_b(K_b) = \left(\frac{\beta}{p^-} - 1\right)^2 \Psi_s(K_b)(1 - \Psi_s(K_b))\,.$$

And based on the equilibrium

$$M_b(\xi) - \varepsilon Var_b(\xi) = \text{ constant}\,,$$

we require

$$M_b(\xi) - \varepsilon Var_b(\xi) = M_b(K_b) - \varepsilon Var_b(K_b) \qquad \text{for any } \xi \in [0, K_b]\,,$$

which implies

$$\begin{aligned} &\Psi_s(\xi)\left(1 - \varepsilon\left(\frac{\beta}{p(\xi)} - 1\right)\Big(1 - \Psi_s(\xi)\Big)\right)\left(\frac{\beta}{p(\xi)} - 1\right) \\ &\quad = \Psi_s(K_b)\left(1 - \varepsilon\left(\frac{\delta_1}{1-\delta_1}\right)(1 - \Psi_s(K_b))\right)\left(\frac{\delta_1}{1-\delta_1}\right) \end{aligned} \tag{5.14}$$

since

$$p(K_b) = (1 - \delta_2)\beta\,.$$

Differentiating (5.14) w.r.t. ξ, we can obtain

$$p'(\xi) \;=\; \frac{\Psi_s'(\xi)}{\Psi_s(\xi)} \cdot \frac{1 - \varepsilon(b/p - a)(1 - 2\Psi_s(\xi))}{1 - 2\varepsilon(b/p - a)(1 - \Psi_s(\xi))} \cdot \frac{p(\xi)(\beta - p(\xi))}{\beta}\,, \quad \xi \in [0, K_b]\,. \tag{5.15}$$

To guarantee $p'(\xi) < 0$, we require

$$\varepsilon < \frac{1}{2\Big(\beta/p(\xi) - 1\Big)\Big(1 - \Psi_s(\xi)\Big)}\,, \tag{5.16}$$

for any $\xi \in [0, K)s]$.

Then the shape of the buy portion and the maximum bid price p_B will be determined by (5.14) as

$$p_B \;=\; \left[1 - \Psi_s(K_b)\left(1 - \varepsilon\left(\frac{\delta_1}{1-\delta_1}\right)(1 - \Psi_s(K_b))\right)\left(\frac{\delta_1}{1-\delta_1}\right)\right]^{-1}\beta\,.$$

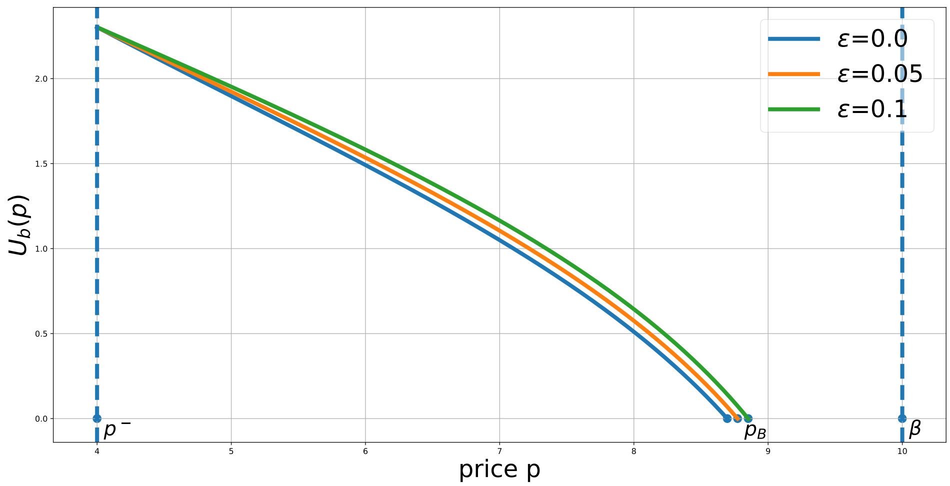

Figure 5.2. The shape of the "buy portion" of the limit order book with risk-averse agents, with 3 different risk-averse factors $\varepsilon = 0,\ 0.05,\ 0.1$. More parameters chosen as $a = 1$, $b = \beta = 10$, $p^- = 0.4\beta$, $K_b = \ln 10$. The distribution $\Psi_s(s)$ is taken as e^{-s}.

Differentiating (5.10) w.r.t. ε, we obatin

$$\frac{\partial p}{\partial \varepsilon} = -\frac{p^2}{\beta} \cdot \frac{\Psi_s(\xi)\left(\dfrac{\beta}{p(\xi)} - 1\right) - \Psi_s(K_b)\dfrac{\delta_1}{1-\delta_1}}{\varepsilon \Psi_s(\xi)\left(1 - 2\varepsilon(1-\Psi_s(\xi))\left(\dfrac{\beta}{p(\xi)} - 1\right)\right)}\,, \text{ for any } \xi \in [0,\, K_s]\,.$$

Combining with the condition (5.16), it implies $\dfrac{\partial p}{\partial \varepsilon} > 0$, i.e., as ε increases the price each agent paid also increases. Especially, the maximum bid price p_B for the agents posting buy orders will be closer to β, as shown in Fig 5.2.

Proposition 2.1 *Given the condition that the risk averse factor ε is small enough, i.e., satisfies (5.12) and (5.16), the limit order book with above setting will admit a unique shape in Nash equilibrium. Further, both the minimum ask price p_A and the maximum bid price p_B will be closer to β as ε increases, which also implies the bid-ask spread will decrease.*

3 Informed external agents

Consider the following model. The fundamental value of the asset is piecewise constant in time, and jumps at random times $0 < \tau_1 < \tau_2 < \cdots$ (modeled by a Poisson arrival process with parameter λ). Every time a jump occurs, the new value satisfies

$$\ln\left(\frac{\beta_{j+1}}{\beta_j}\right) \sim \mathcal{N}(0, \sigma^2). \tag{5.17}$$

Whenever this happens, there is a positive probability $\varepsilon > 0$ that an external agent gets hold of the information *before* any of the traders have time to react and change their priced posted on the LOB. As a consequence:

- If the fundamental value of the stock increases, i.e. $\beta_{j+1} > \beta_j$, then all assets put on sale at a price $< \beta_{j+1}$ are immediately bought.
- If the value decreases, i.e. $\beta_{j+1} < \beta_j$, then all of the assets that the traders offered to buy at a price $> \beta_{j+1}$ are immediately sold.

In this new setting, the agents posting limit orders on the LOB must take this additional possible scenario in consideration. We thus seek to understand how their strategy should change, and what will be the new shape of the LOB.

3.1 The expected payoff

To study this model, we consider the random times $0 < t_1 < t_2 < \cdots$, where at each t_i

- either an incoming buy or sell order arrives,
- or else the fundamental value of the stock has a jump.

Assuming that the external orders come with intensity μ, while jumps in the stock value occur with intensity λ, the time increments $Y_i = t_i - t_{i-i}$ between one event and the next one are i.i.d. Poisson random variables, with the probability

$$\text{Prob.}\{Y_i > s\} = e^{-(\lambda+\mu)s},$$

i.e., the random arrival time t_i is a mix of two Poisson process.

At each of these random times, four possibilities can now occur, with probabilities θ_1, θ_2, θ_3,θ_4, respectively.

1) β has a jump, and nothing else happens,

2) β has a jump, and an informed external agent wipes out one side of the LOB,

3) a buy order arrives,

4) a sell order arrives.

The corresponding probabilities are

$$\theta_1 = \frac{(1-\varepsilon)\lambda}{\lambda+\mu}\,, \qquad \theta_2 = \frac{\varepsilon\lambda}{\lambda+\mu}\,, \qquad \theta_3 = \frac{\mu\theta}{\lambda+\mu}\,, \qquad \theta_4 = \frac{\mu(1-\theta)}{\lambda+\mu}\,. \tag{5.18}$$

As before, we call

$$aC_0 + b\beta_0 S_0 \;=\; V^C(\beta_0)\,C_0 + V^S(\beta_0)\,S_0$$

the expected (exponentially discounted) payoff to an agent that initially has an amount C_0 of cash ans S_0 of stock, assuming that the fundamental value of the stock is initially $\beta(0) = \beta_0$.

To determine a and b, we proceed in the same way as in (4.5)-(4.6), but now taking into account the possible presence of an external informed agent. As before, since the LOB is supposed to be in equilibrium, we can compute V^C, V^S in connection with the bid prices p^-, p^+. In case 2), where the fundamental value of the stock has a jump and an external informed trader arrives, then the expected payoff of an agent initially with C_0 in cash and S_0 in stock can be computed as

$$\begin{aligned}
&E\Big[V^C(\beta_1)C_1 + V^S(\beta_1)S_1 \;\Big|\; \text{an informed agent arrives}\Big] \\
&\quad= \Big(\text{Prob.}\{\beta_1 < p^-\}V^S(\beta_1)\frac{1}{p^-} + \text{Prob.}\{\beta_1 \geq p^-\}V^C(\beta_1)\Big)C_0 \\
&\qquad+\Big(\text{Prob.}\{\beta_1 > p^+\}p^+V^C(\beta_1) + \text{Prob.}\{\beta_1 \leq p^+\}V^S(\beta_1)\Big)S_0 \\
&\quad= \Big(\text{Prob.}\{\beta_1 \leq p^-\}\cdot\frac{b\cdot E[\beta_1|\beta_1 \leq p^-]}{p^-} + \text{Prob.}\{\beta_1 > p^-\}\cdot a\Big)C_0 \\
&\qquad+\Big(\text{Prob.}\{\beta_1 > p^+\}p^+a + \text{Prob.}\{\beta_1 \leq p^+\}\cdot b\cdot E[\beta_1|\beta_1 \leq p^+]\Big)S_0\,.
\end{aligned} \tag{5.19}$$

Denote the cumulative distribution function of the standard normal distribution as

$$\Phi(x) \doteq \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \exp\left\{-\frac{s^2}{2}\right\}\,ds.$$

Consider the distribution function $F_{\beta_1}(p) = \text{Prob.}\{\beta_1 < p\}$. Since the assumption (5.17) implies $\ln\left(\frac{\beta_1}{\beta_0}\right) \sim \mathcal{N}(0, \sigma^2)$, we have

$$F_{\beta_1}(p) \;=\; \int_{-\infty}^{\ln(p/\beta_0)} \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2\sigma^2}\right\} \, dx \;=\; \Phi\left(\frac{\ln p - \ln\beta_0}{\sigma}\right).$$

In particular, at $p^+ = (1+\delta_2)\beta_0$ and at $p^- = (1-\delta_1)\beta_0$ we have

$$F_{\beta_1}(p^+) \;=\; \Phi\left(\frac{\ln(1+\delta_2)}{\sigma}\right), \qquad F_{\beta_1}(p^-) \;=\; \Phi\left(\frac{\ln(1-\delta_1)}{\sigma}\right),$$

respectively. Notice that these values are independent of β_0.

And we notice that the conditional expectation

$$E[\beta_1|\beta_1 \le p] = \frac{1}{\text{Prob.}\{\beta_1 \le p\}} \int_0^p x F'_{\beta_1}(x) \, dx \;=\; \frac{M_{\beta_1}(p)}{F_{\beta_1}(p)},$$

where

$$M_{\beta_1}(p) \;\doteq\; \int_0^p x \frac{1}{x\sigma\sqrt{2\pi}} \exp\left\{-\frac{(\ln x - \ln\beta_0)^2}{2\sigma^2}\right\} \, dx \;=\; e^{\frac{\sigma^2}{2}} \beta_0 \cdot \Phi\left(\frac{\ln p - \ln\beta_0 - \sigma^2}{\sigma}\right).$$

The identity (4.6) is now replaced by

$$\begin{aligned}
V^C(\beta_0)C_0 + V^S(\beta_0)S_0 \;=\; & \int_0^{+\infty} (\lambda+\mu)e^{-(\lambda+\mu)t_1} \Bigg\{ \int_0^{t_1} e^{-\gamma t}(C_0 + \beta_0 S_0) \, dt \\
& + \theta_1 e^{-\gamma t_1} \cdot \Big[V^C(\beta_1)C_0 + V^S(\beta_1)\, S_0\Big] \\
& + \theta_2 e^{-\gamma t_1} \cdot E\Big[V^C(\beta_1)C_1 + V^S(\beta_1)S_1 \;\Big|\; \text{an informed agent arrives}\Big] \\
& + \theta_3 e^{-\gamma t_1} \cdot \Big[(V^C(\beta_0)C_0 + V^C(\beta_0)\Psi^\sharp p^+ S_0 + (1-\Psi^\sharp)V^S(\beta_0)S_0\Big] \\
& + \theta_4 e^{-\gamma t_1} \cdot \Big[V^S(\beta_0)\Psi^\flat \frac{1}{p^-} C_0 + V^C(\beta_0)(1-\Psi^\flat)C_0 + V^S(\beta_0)S_0)\Big]\Bigg\} \, dt_1 .
\end{aligned} \tag{5.20}$$

We recall that the constants $\Psi^\sharp, \Psi^\flat$, introduced in Chapter 4, denote the probability that, when an external agent comes, a limit order at price p^+, or p^-, respectively, will be executed.

Adopting matrix notation, (5.20) leads to

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{\gamma+\mu+\lambda}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{\mu}{\gamma+\mu+\lambda}\begin{bmatrix} \theta+(1-\theta)(1-\Psi^\flat) & (1-\theta)\Psi^\flat\dfrac{\beta_0}{p^-} \\ \theta\Psi^\sharp\dfrac{p^+}{\beta_0} & \theta(1-\Psi^\sharp)+(1-\theta) \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$$
$$+\frac{\lambda}{\gamma+\lambda+\mu}\begin{bmatrix} 1-\varepsilon F_{\beta_1}(p^-) & \varepsilon F_{\beta_1}(p^-)\dfrac{E[\beta_1|\beta_1\le p^-]}{p^-} \\ \varepsilon(1-F_{\beta_1}(p^+)\dfrac{p^+}{\beta_0} & (1-\varepsilon)\dfrac{E[\beta_1]}{\beta_0}+\varepsilon F_{\beta_1}(p^+))\dfrac{E[\beta_1|\beta_1\le p^+]}{\beta_0} \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}. \tag{5.21}$$

Equivalently,

$$A\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \tag{5.22}$$

where

$$A \doteq A_0 + \varepsilon\lambda\, B\,, \tag{5.23}$$

with

$$A_0 \doteq \begin{bmatrix} \gamma+\mu(1-\theta)\Psi^\flat & -\Big(\mu(1-\theta)\Psi^\flat\Big)\dfrac{\beta_0}{p^-} \\ -\mu\theta\Psi^\sharp\dfrac{p^+}{\beta_0} & \gamma+\mu\theta\Psi^\sharp+\lambda\left(1-e^{\sigma^2/2}\right), \end{bmatrix}$$

$$B \doteq \begin{bmatrix} F_{\beta_1}(p^-) & -M_{\beta_1}(p^-)\dfrac{1}{p^-} \\ -\Big(1-F_{\beta_1}(p^+)\Big)\dfrac{p^+}{\beta_0} & e^{\sigma^2/2}-\dfrac{M_{\beta_1}(p^+)}{\beta_0} \end{bmatrix}.$$

Since $p^+ = (1+\delta_2)\beta_0$, $p^- = (1-\delta_1)\beta_0$, we have

$$A_0 = \begin{bmatrix} \gamma+\mu(1-\theta)\Psi^\flat & -\dfrac{\mu(1-\theta)\Psi^\flat}{1-\delta_1} \\ -\mu\theta\Psi^\sharp(1+\delta_2) & \gamma+\mu\theta\Psi^\sharp+\lambda\left(1-e^{\sigma^2/2}\right), \end{bmatrix}$$

$$B = \begin{bmatrix} \Phi_1 & -\dfrac{e^{\sigma^2/2}}{1-\delta_1}\Phi_2 \\ -(1+\delta_2)\Phi_3 & e^{\sigma^2/2}\Phi_4 \end{bmatrix},$$

where

$$\Phi_1 \doteq \Phi\left(\frac{\ln(1-\delta_1)}{\sigma}\right), \qquad \Phi_2 \doteq \Phi\left(\frac{\ln(1-\delta_1)-\sigma^2}{\sigma}\right),$$
$$\Phi_3 \doteq \Phi\left(\frac{-\ln(1+\delta_2)}{\sigma}\right), \qquad \Phi_4 \doteq \Phi\left(\frac{-\ln(1+\delta_2)+\sigma^2}{\sigma}\right)$$

And we notice $\Phi_2 < \Phi_1 < 1/2$, $\Phi_3 < \Phi_4$.

We seek conditions on the above parameters defining the LOB, so that the equation (5.22) uniquely determines the values of $a = V^C(\beta_0)$ and $b = V^S(\beta_0)/\beta_0$. Toward this goal, we denote

$$\begin{aligned}
C_1 &\doteq \mu\Big((1-\theta)\Psi^\flat + \theta\Psi^\sharp\Big) + \lambda\left(1 - e^{\sigma^2/2}\right) + \varepsilon\lambda\left(\Phi_1 + e^{\sigma^2/2}\Phi_4\right) \\
C_2 &\doteq \mu^2\theta(1-\theta)\Psi^\sharp\Psi^\flat\left(\frac{\delta_1+\delta_2}{1-\delta_1}\right) + \lambda\left(e^{\sigma^2/2}-1\right)\mu(1-\theta)\Psi^\flat \\
&\quad +\varepsilon\lambda\left[\mu\theta\Psi^\sharp\left(e^{\sigma^2/2}\frac{1+\delta_2}{1-\delta_1}\Phi_2 - \Phi_1\right) + \mu(1-\theta)\Psi^\flat\left(\frac{1+\delta_2}{1-\delta_1}\Phi_3 - e^{\sigma^2/2}\Phi_4\right) + \lambda\left(e^{\sigma^2/2}-1\right)\Phi_1\right] \\
&\quad +\varepsilon^2\lambda^2 e^{\sigma^2/2}\left(-\Phi_1\Phi_4 + \frac{1+\delta_2}{1-\delta_1}\Phi_2\Phi_3\right),
\end{aligned}$$

so that

$$\det A \;=\; \gamma^2 + C_1\gamma - C_2\,. \tag{5.24}$$

Proposition 3.1 *Assume we take appropriate volatility σ such that*

$$\Phi_1 \;<\; \frac{1}{1-\delta_1}\Phi_2\,, \qquad \Phi_4 \;<\; (1+\delta_2)\Phi_3\,, \tag{5.25}$$

and let the exponential discount factor γ satisfy

$$\gamma > -\frac{C_1}{2} + \sqrt{C_2 + C_1^2/4}\,, \tag{5.26}$$

then the equations (5.21) in the above settings determine unique values of a, b.

Proof. If (5.25) holds, then we have

$$\det B \;=\; e^{\sigma^2/2}\left(-\Phi_1\Phi_4 + \frac{1+\delta_2}{1-\delta_1}\Phi_2\Phi_3\right) > 0\,,$$

and

$$e^{\sigma^2/2}\frac{1+\delta_2}{1-\delta_1}\Phi_2 - \Phi_1 \;>\; 0\,, \qquad \frac{1+\delta_2}{1-\delta_1}\Phi_3 - e^{\sigma^2/2}\Phi_4 \;>\; 0\,,$$

which implies $C_2 > 0$. Combining it with the assumption (5.26), we obtain $\det A > 0$.

We thus have

$$\begin{bmatrix} a \\ \\ b \end{bmatrix} = \frac{1}{\det A} \begin{bmatrix} \gamma + \mu\Big(\theta\Psi^\sharp + (1-\theta)\Psi^\flat \dfrac{1}{1-\delta_1}\Big) + \lambda\left(1 - e^{\sigma^2/2}\right) + \varepsilon\lambda e^{\sigma^2/2}\Big(\dfrac{\Phi_2}{1-\delta_1} + \Phi_4\Big) \\ \gamma + \mu\Big((1-\theta)\Psi^\flat + \theta\Psi^\sharp(1+\delta_2)\Big) + \varepsilon\lambda\Big(\Phi_1 + (1+\delta_2)\Phi_3\Big) \end{bmatrix}. \tag{5.27}$$

□

Notice that both a and b are independent of β_0.

Remark 3.1 *Compare the ratio z between stock value b and cash value a with the one in section 8 we notice that if*

$$\begin{aligned} &\Big(\gamma + \mu\theta\Psi^\sharp + \mu(1-\theta)\Psi^\flat \frac{1}{1-\delta_1} + \lambda\left(1 - e^{\sigma^2/2}\right)\Big)\Big(\Phi_1 + (1+\delta_2)\Phi_3\Big) \\ &\quad > \Big(\gamma + \mu(1-\theta)\Psi^\flat + \mu\theta\Psi^\sharp(1+\delta_2)\Big)\Big(\frac{e^{\sigma^2/2}}{1-\delta_1}\Phi_2 + e^{\sigma^2/2}\Phi_4\Big), \end{aligned} \tag{5.28}$$

then $z = b/a$ is greater the one without informed external agents, otherwise, z will be less than the one without informed external agents. If change the inequality in (5.28) into less than, then we have z is less than the one without informed external agents. Besides,

- *as $\gamma \to +\infty$, we have $a \to 1/\gamma$, $b \to 1/\gamma$, which also implies when γ is large, z will be quite close to 1.*
- *as $\sigma \to 0$, the result will coincide the model in section 3*
- *σ cannot be too large, which is implied by condition (5.25).*

Next, we analyze how the shape of the LOB is affected by the possible arrival of a better informed external agent. In particular, we study how the bid-ask spread $p_A - p_B$ (see Fig. 4.1) can increase.

3.2 The shape of the LOB.

"Sell portion"

Similar as in section 4.3, for any $p \in [p_A, p^+]$, the equilibrium requires the expected profit of the agent asking price p to be a constant. The difference here is that when the fundamental value jumps, it will also wipe out a portion of the LOB, so we need to take the effect of this part into the equilibrium,i.e.,

$$
\begin{aligned}
\text{constant} \;=\;& ap\cdot[\text{probability of selling at price } p] \\
&+b\beta_0\cdot[\text{probability of } \beta_t \text{ not changed and not selling at price p}] \\
&+bE[\beta_1|\beta_0]\cdot[\text{probability of } \beta_t \text{ changed and no informed agent arrives}] \\
&+bE[\beta_1|\beta_1\le p]\cdot[\text{probability of informed agent arrives but not buying}]\,.
\end{aligned}
\tag{5.29}
$$

And we have

$$
\begin{aligned}
[\text{probability of selling at price } p] &= \theta_3\cdot\text{Prob.}\Big\{X_b > U_s(p)\Big\} + \theta_2\cdot\text{Prob.}\Big\{\beta_1 > p\Big\} \\
&= \frac{\mu\theta}{\lambda+\mu}\Psi_b(U_s(p)) + \frac{\varepsilon\lambda}{\lambda+\mu}(1-F_{\beta_1}(p))\,,
\end{aligned}
$$

where the first term denotes the expected payoff when the external buyer arrives before the fundamental value changes, and the second term denotes the fundamental value changes first and will wipe the LOB off, and

$$
\begin{aligned}
[\text{probability of } \beta_t \text{ not changed and not selling at price p}] &= \frac{\mu}{\lambda+\mu}\Big(1-\theta\Psi_b(U_s(p))\Big)\,, \\
[\text{probability of } \beta_t \text{ changed and no informed agent arrives}] &= \frac{(1-\varepsilon)\lambda}{\lambda+\mu}\,, \\
[\text{probability of } \beta_t \text{ changed and informed agent arrives but not selling}] &= \frac{\varepsilon\lambda}{\lambda+\mu}F_{\beta_1}(p)\,.
\end{aligned}
$$

Substitute these expressions back to (5.29), we have

$$
\begin{aligned}
&\left(\frac{\mu\theta\Psi_b(U_s)}{\lambda+\mu} + \frac{\varepsilon\lambda\Big(1-F_{\beta_1}(p)\Big)}{\lambda+\mu}\right)\cdot ap + \frac{\mu(1-\theta\Psi_b(U_s))}{\lambda+\mu}\cdot b\beta_0 \\
&\quad + \frac{(1-\varepsilon)\lambda}{\lambda+\mu}\cdot b\beta_0 e^{\sigma^2/2} + \frac{\varepsilon\lambda}{\lambda+\mu}bM_{\beta_1}(p) \;=\; \text{constant}\,,
\end{aligned}
\tag{5.30}
$$

equivalently,

$$
\mu\theta\Big(\Psi_b(U_s(p))\,(ap-b\beta_0)\Big)+\varepsilon\lambda\left(\Phi\left(-\frac{\ln(p/\beta_0)}{\sigma}\right)ap + e^{\sigma^2/2}b\beta_0\Phi\left(\frac{\ln(p/\beta_0)-\sigma^2}{\sigma}\right)\right) \;=\; \text{constant.}
\tag{5.31}
$$

Proposition 3.2 *Assume that the exponential discount factor γ satisfies (5.26), and take appropriate volatility σ satisfies (5.25), then the above model admits a unique shape* *of the "sell" portion.*

Proof. From proposition 9.1, we know values of a, b are determined given condition (5.25) and (5.26). Besides, since at $p^+ = (1+\delta_2)\beta_0$, we have

$$U_s\Big(p^+\Big) = K_s\,,$$

from (5.31), we obtain the minimum ask price p_A is determined by

$$\begin{aligned}&\mu\theta\left[\Psi_b(K_s)(ap^+ - b\beta_0)\right] + \varepsilon\lambda\left[\Phi\left(\tfrac{-\ln(1+\delta_2)}{\sigma}\right)ap^+ + e^{\sigma^2/2}b\beta_0\Phi\left(\tfrac{\ln(1+\delta_2)-\sigma^2}{\sigma}\right)\right]\\ &\qquad = \ \mu\theta(ap_A - b\beta_0) + \varepsilon\lambda\left[\Phi\left(\tfrac{-\ln(p_A/\beta_0)}{\sigma}\right)ap_A + e^{\sigma^2/2}b\beta_0\Phi\left(\tfrac{\ln(p_A/\beta_0)-\sigma^2}{\sigma}\right)\right].\end{aligned} \tag{5.32}$$

For simplicity, denote

$$g(p) = \Phi\left(\frac{-\ln(p/\beta_0)}{\sigma}\right)\cdot p + \Phi\left(\frac{\ln(p/\beta_0)-\sigma^2}{\sigma}\right)\cdot\frac{e^{\sigma^2/2}b\beta_0}{a}\,.$$

Differentiating (5.31) w.r.t. p, we obtain

$$U_s'(p) = -\frac{\Psi_b(U_s(p))}{\Psi_b'(U_s(p))\cdot(p - b\beta_0/a)} - \frac{\varepsilon\lambda g'(p)}{\mu\theta\cdot\Psi'(U_s(p))\cdot(p - b\beta_0/a)}\,. \tag{5.33}$$

Since $\Psi_b'(s) < 0$, then we know when ε is small, $U_s(p)$ is strictly increasing on $[\beta_0, p^+]$. Correspondingly, p_A will be uniquely determined by (5.32), so as the shape of the sell portion in Nash equilibrium. □

"Buy portion"

Similarly, for the agents posting buy orders on the LOB, the Nash equilibrium requires that for any $p \in [p^-, p_B]$,

$$\begin{aligned}\text{constant} \ = \ &\frac{b\beta_0}{p}\cdot[\text{probability of } \beta_t \text{ changed and buying at price } p]\\ &+\frac{bE[\beta_1|\beta_1 \le p]}{p}\cdot[\text{probability of } \beta_t \text{ changed and buying at price } p]\\ &+a\cdot[\text{probability of not buying at price } p]\,,\end{aligned} \tag{5.34}$$

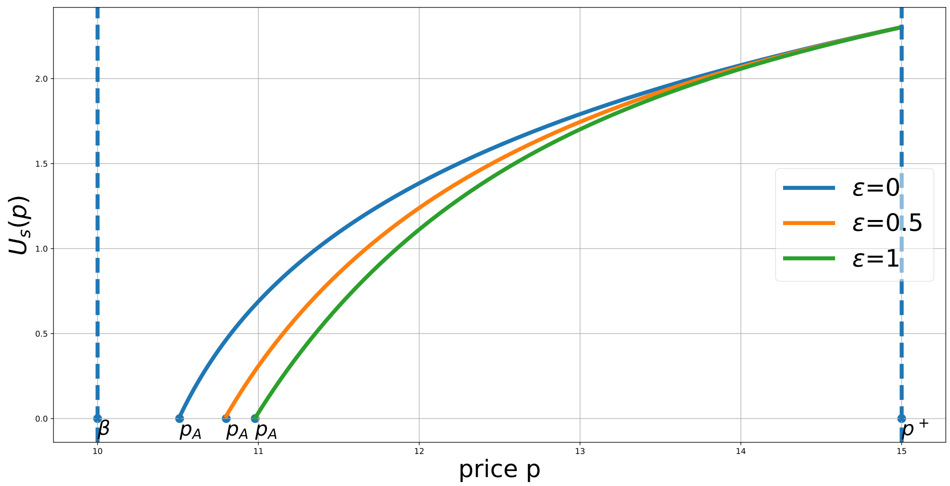

Figure 5.3. The shape of the "sell portion" of the limit order book with informed agents, with 3 different probabilities, without informed agents($\varepsilon = 0$), half of the time will have an informed agents ($\varepsilon = 0.5$), informed agents always arrive ($\varepsilon = 1$). More parameters chosen as $a = 1$, $b = \beta = 10$, $p^+ = 1.5\beta$, $K_s = \ln 10$. The distribution $\Psi_b(s)$ is taken as e^{-s}. $\lambda = 0.5$, $\sigma = 3$.

where

$$\begin{aligned}
&[\text{probability of } \beta_t \text{ changed and buying at price } p] = \theta_4 \cdot \text{Prob.}\Big\{X_s > U_b(p)\Big\} \\
&\qquad = \frac{\mu(1-\theta)}{\lambda+\mu}\Psi_s(U_b(p))\,, \\
&[\text{probability of } \beta_t \text{ changed and buying at price } p] = \theta_2 \cdot \text{Prob.}\{\beta_1 < p\} = \frac{\varepsilon\lambda}{\lambda+\mu}F_{\beta_1}(p)\,, \\
&[\text{probability of not buying at price } p] = 1 - \frac{\mu(1-\theta)}{\lambda+\mu}\Psi_s(U_b(p)) - \frac{\varepsilon\lambda}{\lambda+\mu}F_{\beta_1}(p)\,.
\end{aligned}$$

Substitute these expressions back to (5.34), we have

$$\begin{aligned}
&\frac{\mu(1-\theta)}{\lambda+\mu}\Psi_s(U_b(p)) \cdot \frac{b\beta_0}{p} + \frac{\varepsilon\lambda}{\lambda+\mu}\frac{bM_{\beta_1}(p)}{p} \\
&\quad + \left(1 - \frac{\mu(1-\theta)}{\lambda+\mu}\Psi_s(U_b(p)) - \frac{\varepsilon\lambda}{\lambda+\mu}F_{\beta_1}(p)\right) \cdot a = \text{constant}\,,
\end{aligned} \tag{5.35}$$

equivalently,

$$\mu(1-\theta)\Psi_s(U_b(p))\left(\frac{b\beta_0}{p} - a\right) + \varepsilon\lambda\left(e^{\sigma^2/2}\Phi\left(\frac{\ln(p/\beta_0)-\sigma^2}{\sigma}\right)\frac{b\beta_0}{p} - \Phi\left(\frac{\ln(p/\beta_0)}{\sigma}\right)a\right) = \text{constant}. \tag{5.36}$$

Proposition 3.3 *Assume that the exponential discount factor γ satisfies (5.26), and take appropriate volatility σ satisfies (5.25), then the above model admits a unique shape of the "buy" portion.*

Proof. From proposition 9.1, we know values of a, b are determined given condition (5.25) and (5.26). Besides, since at $p^- = (1-\delta_1)\beta_0$, we have

$$U_s\big(p^-\big) = K_b\,,$$

from (5.36), we obtain the maximum bid price p_B is determined by

$$\begin{aligned} \mu(1-\theta)\Psi^b\left(\frac{b}{1-\delta_1} - a\right) + \varepsilon\lambda\left(e^{\sigma^2/2}\Phi\left(\frac{\ln(1-\delta_1)-\sigma^2}{\sigma}\right)\frac{b}{1-\delta_1} - \Phi\left(\frac{\ln(1-\delta_1)}{\sigma}\right)a\right) \\ = \mu(1-\theta)\left(\frac{b\beta_0}{p_B} - a\right) + \varepsilon\lambda\left(e^{\sigma^2/2}\Phi\left(\frac{\ln(p_B/\beta_0)-\sigma^2}{\sigma}\right)\frac{b\beta_0}{p_B} - \Phi\left(\frac{\ln(p_B/\beta_0)}{\sigma}\right)a\right). \end{aligned} \tag{5.37}$$

For simplicity, denote

$$g(p) = e^{\sigma^2/2}\Phi\left(\frac{\ln(p/\beta_0)-\sigma^2}{\sigma}\right)\frac{1}{p} - \Phi\left(\frac{\ln(p/\beta_0)}{\sigma}\right)\frac{a}{b\beta_0}\,.$$

Differentiating (5.36) w.r.t. p,

$$U_b'(p) = \frac{\Psi_s(U_b(p))}{\Psi_s'(U_b(p))p^2\cdot\left(\dfrac{1}{p} - \dfrac{a}{b\beta_0}\right)} - \frac{\varepsilon\lambda g'(p)}{\mu(1-\theta)\cdot\Psi_s'(U_b(p))\cdot\left(\dfrac{1}{p} - \dfrac{a}{b\beta_0}\right)}\,. \tag{5.38}$$

Since $\Psi_s'(s) < 0$, when ε is small, $U_b(p)$ is strictly increasing on $[p^-,\, \beta_0]$. Correspondingly, p_B will be uniquely determined by (5.37), so as the shape of the buy portion in Nash equilibrium. □

3.3 The effects of the jump on the bid-ask spread.

From (5.32), we observe that as ε increases, we obtain a lower p_A. Indeed, denote $z = b/a$, we have

$$\frac{\partial p_A}{\partial\varepsilon} = \frac{\lambda\Big(g(p^+) - g(p_A)\Big) + \Big[\mu\theta(1-\Psi_b(K_s)) + \varepsilon\lambda e^{\frac{\sigma^2}{2}}z\beta_0 d_1\Big]\frac{\partial z}{\partial\varepsilon}}{\mu\theta + \varepsilon\lambda g'(p_A)}\,,$$

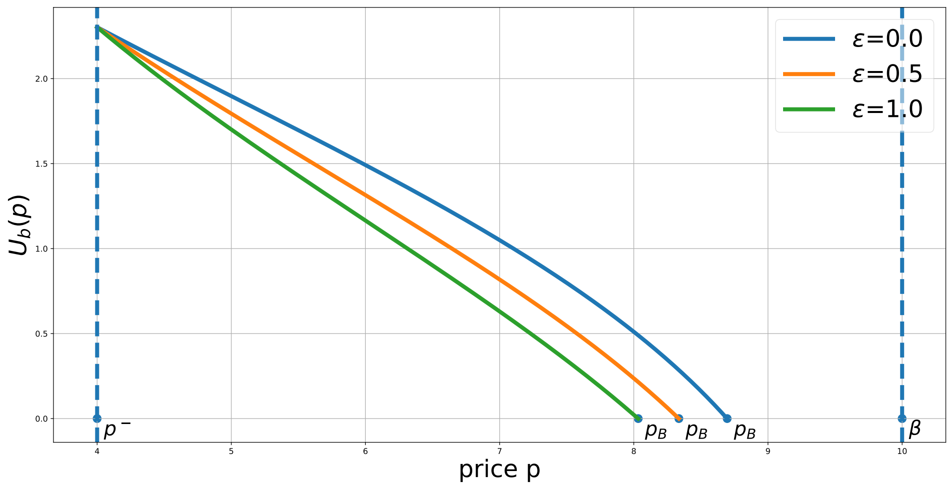

Figure 5.4. The shape of the "buy portion" of the limit order book with informed agents, with 3 different probabilities, without informed agents($\varepsilon = 0$), half of the time will have an informed agents ($\varepsilon = 0.5$), informed agents always arrive ($\varepsilon = 1$). More parameters chosen as $a = 1$, $b = \beta = 10$, $p^- = 0.4\beta$, $K_b = \ln 10$. The distribution $\Psi_s(s)$ is taken as e^{-s}. $\lambda = 0.5$, $\sigma = 3$.

where

$$d_1 \; = \; \left(\Phi\left(\frac{\ln((1+\delta_2) - \sigma^2}{\sigma}\right) - \Phi\left(\frac{\ln(p_A/\beta_0) - \sigma^2}{\sigma}\right)\right),$$

$$\frac{\partial z}{\partial \varepsilon} \; = \; \frac{\lambda}{a^2}\Bigg(\left(\gamma + \mu\theta\Psi^\sharp + \mu(1-\theta)\Psi^\flat \frac{1}{1-\delta_1} + \lambda\left(1 - e^{\sigma^2/2}\right)\right)\left(\Phi_1 + (1+\delta_2)\Phi_3\right)$$
$$- \left(\gamma + \mu(1-\theta)\Psi^\flat + \mu\theta\Psi^\sharp(1+\delta_2)\right)\left(\frac{e^{\sigma^2/2}}{1-\delta_1}\Phi_2 + e^{\sigma^2/2}\Phi_4\right)\Bigg).$$

As mentioned in remark 9.2, when γ is large enough, we have b/a close to 1, i.e.,

$$\exists\; M_1 > 0 \quad s.t. \text{ when } \gamma > M_1, \qquad \text{we have} \qquad |z - 1| < \sigma\Phi \exp\left\{\frac{(\ln p/\beta_0)^2}{2\sigma}\right\}.$$

Since

$$g'(p) \; = \; \Phi\left(\frac{\ln(p/\beta_0) - \sigma^2}{\sigma}\right) + \frac{1}{\sigma}\exp\left\{-\frac{(\ln p/\beta_0)^2}{2\sigma}\right\}(z-1),$$

then $g'(p) > 0$ for $p \in [\beta_0, p^+]$, which also implies $g(p^+) - g(p_A) > 0$. Besides, $\partial z/\partial\varepsilon$ will be close to 0 when γ is large. Correspondingly, we will obtain

$$\frac{\partial p_A}{\partial \varepsilon} \; > \; 0\,.$$

It means under such cases the existence of the informed agent will increase the minimum ask price p_A, as shown in Fig 5.3.

As for the maximum bid price p_B, we notice that

$$\frac{\partial p_B}{\partial \varepsilon} = \frac{-\lambda\Big(g(p^-) - g(p_B)\Big) + [\mu(1-\theta)(1-\Psi_s(K_b)) + \varepsilon\lambda d_2]\,\frac{1}{z^2\beta_0^2}\frac{\partial z}{\partial \varepsilon}}{\mu(1-\theta)z\beta_0/p_B^2 - \varepsilon\lambda g'(p_B)},$$

where

$$d_2 = \left(\Phi\left(\frac{\ln(p_B/\beta_0)}{\sigma}\right) - \Phi\left(\frac{\ln(1-\delta_1)}{\sigma}\right)\right).$$

We notice

$$\exists \text{ constant } M_2 \quad s.t. \quad \text{when } \gamma > M_2, \quad \left|\frac{1}{z} - \frac{1}{p}\right| < \sigma\beta_0\Phi\left(\frac{\ln(p/\beta_0) - \sigma^2}{\sigma}\right)\exp\left\{\frac{(-\ln p/\beta_0)^2 + \sigma^2}{2\sigma}\right\}.$$

Then

$$g'(p) = -\frac{1}{p^2}e^{\frac{\sigma^2}{2}}\Phi\left(\frac{\ln(p/\beta_0) - \sigma^2}{\sigma}\right) - \frac{1}{\sigma p\beta_0}\left(\frac{1}{z} - \frac{1}{p}\right)\exp\left\{-\frac{(\ln p/\beta_0)^2}{2\sigma}\right\} < 0,$$

for any $p \in [p^-, \beta_0]$, which also implies $g(p^-) - g(p_B) > 0$. Besides, $\partial z/\partial\varepsilon$ will be close to 0 when γ is large. Correspondingly, we will obtain

$$\frac{\partial p_B}{\partial \varepsilon} < 0.$$

It means under such cases the existence of the informed agent will decrease the maximum bid price p_B.

In conclusion, the model with informed external agents will have a larger bid-ask spread when ε (the probability of the appearance of informed agents) increases.

The parameter λ which represents the frequency of the jump in the fundamental value will generate same effect, since

$$\partial p_A/\partial\lambda > 0, \qquad \partial p_B/\partial\lambda < 0,$$

when γ is large enough. That is, if the fundamental value β_t jumps more frequently, i.e., if λ increases, then the minimum ask price p_A will increase and the maximum bid price p_B will decrease, then the bid-ask spreader will be wider.

As for σ, when $\varepsilon = 0$, we have

$$\frac{\partial p_A}{\partial \sigma} = (1 - \Psi_b(K_s))\beta_0 \frac{\partial z}{\partial \sigma}, \qquad \frac{\partial p_B}{\partial \sigma} = \frac{(1 - \Psi_s(K_b))p_B^2}{z^2 \beta_0} \cdot \frac{\partial z}{\partial \sigma}, \tag{5.39}$$

where $\partial z/\partial \sigma > 0$. It implies that

$$\partial p_A/\partial \sigma > 0, \quad \partial p_B/\partial \sigma > 0 .$$

However, if we compare formulas in (5.39), we notice that the increment of p_B will be less than p_A when we have same size for the sell portion and buy portion, i.e., $K_s = K_b$. That is, the bid-ask spread will increase as σ increases under such a case.

When ε is large, it is hard to predict how the volatility will effect p_A and p_B. Since, a higher volatility means that when the informed agent exists, there might be a higher probability the external informed agent will wipe out the sell portion, however, it also means the seller sells his asset at prices lower than the market more frequently.

In general, with the existence of the informed agent, the agents posting sell orders on the LOB will increase the minimum ask price, and the agents posting buy orders will decrease the maximum bid price, to reduce their potential losses.

Proposition 3.4 *If γ satisfies*

$$\gamma > \max\left\{M_1,\ M_2,\ -\frac{C_1}{2} + \sqrt{C_2 + C_1^2/4}\right\},$$

then the model with informed external agents will have a larger bid-ask spread than the one without informed agents.

3.4 Liquidity reduction.

In this section, we study that if an external better informed agent is more likely to come, by how much should the LOB size decrease, in order to maintain the same profit to all agents posting limit orders.

This would be a way to quantify the “liquidity crisis", which occurs when posting limit orders becomes too risky and many not-well-informed agents shy away from the market.

For simplicity, in the following, let us assume

$$\Psi_s(K_b) = \Psi_b(K_s) \doteq \Psi(K), \quad ,\theta = \frac{1}{2}, \quad (1+\delta_2)\cdot\frac{1}{1-\delta_1} = 1 .$$

Denote the expected profit of an agent holding a unit amount of cash and a unit amount of stock when the probability of an informed external agent appear as ε, and the total size of the sell portion as K by

$$J(\varepsilon, K) \;=\; a(\varepsilon, K) + b(\varepsilon, K)\beta_0\,,$$

where

$$\begin{bmatrix} a(\varepsilon, K) \\ b(\varepsilon, K) \end{bmatrix} = A^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

and A is defined in (5.27), but with $\Psi^\sharp = \Psi^\flat = \Psi(K)$. Denote $D(\varepsilon, K) = \det A$. When $\varepsilon = 0$, it means there are no external buyers, which coincides with the result in section 8. While if $\lambda = 0$, it becomes the case we discussed in section 5, where the fundamental value will stay as constant.

When $\varepsilon = 0$, i.e., there are no informed external agents, we notice that if γ stasifies (4.8), we have

$$A_0 \;=\; \begin{bmatrix} \gamma + \frac{\mu}{2}\Psi(K) & -\frac{\mu(1+\delta)}{2}\Psi(K) \\ -\frac{\mu(1+\delta)}{2}\Psi(K) & \gamma + \frac{\mu}{2}\Psi(K) + \lambda\left(1 - e^{\sigma^2/2}\right), \end{bmatrix}, \tag{5.40}$$

and

$$D(0, K) = \gamma^2 + \gamma\mu\Psi(K) - \frac{\mu^2}{4}\Psi^2(K)(1+\delta)\delta + (1 - e^{\sigma^2/2})\lambda(\gamma + \mu\Psi(K)/2)$$

$$\frac{\partial D(0, K)}{\partial K} \;=\; \left[\gamma - \frac{1}{2}\left(\mu\Psi(K)(\delta^2 + 2\delta) + \lambda(1 - e^{\sigma^2/2})\right)\right]\mu\Psi'(K) < 0\,.$$

Besides,

$$\frac{\partial a(0, K)}{\partial K} \;=\; \frac{\Psi'(K)}{D^2(0, K)}\left(\frac{\delta\mu}{2}\gamma^2 + k_1\gamma + k_2\right),$$

$$\frac{\partial b(0, K)}{\partial K} \;=\; \frac{\Psi'(K)}{D^2(0, K)}\left(\frac{\delta\mu}{2}\gamma^2 + k_3\gamma + k_4\right),$$

where parameters k_1, k_2, k_3, k_4 are not depend on γ. Due to the continuity w.r.t γ, there should be a constant M_0 s.t. when $\gamma > M_0$,

$$\frac{\delta\mu}{2}\gamma^2 + k_1\gamma + k_2 \;>\; 0, \text{ and } \frac{\delta\mu}{2}\gamma^2 + k_3\gamma + k_4 \;>\; 0\,.$$

Since $\Psi'(K) < 0$, we obtain

$$\frac{\partial a(0,K)}{\partial K} \;<\; 0, \text{ and } \frac{\partial b(0,K)}{\partial K} \;<\; 0\,.$$

That is, the expected profit $J(0,K)$ without informed external agents will decrease as the size of the LOB increases. It also coincides with real case, when there are more competitors in the market, each agent will have a lower expected profit.

Similarly, for the model with $\varepsilon \neq 0$, we also can find a constant N_0 which ensure γ large enough to obtain

$$\frac{\partial a(\varepsilon,K)}{\partial K} \;<\; 0, \text{ and } \frac{\partial b(\varepsilon,K)}{\partial K} \;<\; 0\,, \text{ when } \gamma > N_0\,.$$

To compare the difference with and without the informed external agents, let us differentiate $a(\varepsilon,K)$ and $b(\varepsilon,K)$ w.r.t. ε. Follow the notations in (5.23), we have $A = A_0 + \varepsilon\lambda B$. Applying Taylor expansion around A_0, we obtain

$$A^{-1} \;=\; A_0 - \varepsilon\lambda \cdot A_0^{-1} B A_0^{-1} + \mathcal{O}(\varepsilon^2)\,,$$

then

$$\begin{bmatrix} \dfrac{\partial a}{\partial \varepsilon} \\ \dfrac{\partial b}{\partial \varepsilon} \end{bmatrix} \;=\; \frac{\partial A^{-1}}{\partial \varepsilon} \cdot \mathbf{1} \Bigg|_{\varepsilon=0} = -\lambda A_0^{-1} B A_0^{-1} \cdot \mathbf{1}\,,$$

where A_0 as in (5.40), $\mathbf{1} = [1,1]^T$ and

$$B \;=\; \begin{bmatrix} \Phi_1 & -e^{\sigma^2/2}(1+\delta)\Phi_2 \\ -(1+\delta)\Phi_3 & e^{\sigma^2/2}\Phi_4 \end{bmatrix}.$$

Then

$$BA_0^{-1} \cdot \mathbf{1} \;=\; \frac{1}{\det A_0} \begin{bmatrix} \Big(\gamma + \mu\Psi(K)(1+\delta/2)\Big) d_1 - \lambda(e^{\sigma^2/2} - 1)\Phi_1 \\ \Big(\gamma + \mu\Psi(K)(1+\delta/2)\Big) d_2 + (1+\delta)\lambda(e^{\sigma^2/2} - 1)\Phi_3\,, \end{bmatrix}$$

where

$$d_1 = \Big(\Phi_1 - e^{\sigma^2/2}(1+\delta)\Phi_2\Big) < 0, \quad d_2 = \Big(e^{\sigma^2/2}\Phi_4 - (1+\delta)\Phi_3\Big) < 0\,.$$

Further,

$$A_0^{-1}BA_0^{-1}\cdot\mathbf{1} \;=\; \frac{1}{(\det A_0)^2}\begin{bmatrix} d_1\gamma^2+\Phi_1(e^{\sigma^2/2}-1)^2\lambda^2+g_1(\gamma,\lambda,\sigma,\delta,\mu,K) \\ d_2\gamma^2+(e^{\sigma^2/2}-1)(1+\delta)\Phi_1\gamma\lambda+g_2(\gamma,\sigma,\delta,\mu,K)\end{bmatrix},$$

where g_1 only contains terms with power of γ less than 2 and power of λ less than 2, and g_2 contains terms with power of γ less than 2 and without λ.

Then we can conclude

- When λ is fixed and γ is large enough, we have $A_0^{-1}BA_0^{-1}\cdot\mathbf{1}<\mathbf{0}$, which implies $\partial J(\varepsilon,K)/\partial\varepsilon|_{\varepsilon=0}>0$. Since as $\gamma\to+\infty$, the infinite horizon game tends to a one-shot game. And the existence of the informed external agents contributes a probability of additional transaction, it might not be executed in the most ideal price, but still increase the expected profit than the one without any additional transactions.

- When λ is large, the condition (5.26) requires

  $$\gamma>\varepsilon e^{\frac{\sigma^2}{2}}\Big((1+\delta^2)\Phi_2\Phi_3-\Phi_1\Phi_4\Big)\sqrt{\lambda}+o(\sqrt{\lambda})\,.$$

  If we fix γ as $2\varepsilon e^{\frac{\sigma^2}{2}}\Big((1+\delta^2)\Phi_2\Phi_3-\Phi_1\Phi_4\Big)\sqrt{\lambda}$,then

  $$A_0^{-1}BA_0^{-1}\cdot\mathbf{1}=\frac{1}{(\det A_0)^2}\begin{bmatrix}\Phi_1(e^{\sigma^2/2}-1)^2\lambda^2+o\left(\lambda^{3/2}\right)\\ 2\varepsilon e^{\frac{\sigma^2}{2}}\Big((1+\delta^2)\Phi_2\Phi_3-\Phi_1\Phi_4\Big)(e^{\sigma^2/2}-1)(1+\delta)\Phi_1\lambda^{3/2}+o(\lambda)\end{bmatrix}>\mathbf{0},$$

  which implies $\partial J(\varepsilon,K)/\partial\varepsilon|_{\varepsilon=0}<0$, as $\lambda\to\infty$. Since it indicates when the fundamental value jumps frequently, and each time the shape of the LOB will change, however, the informed external agents always try to use a price lower than the market to buy and a price higher than market to sell, which will extremely hurt the expected profits of the agents use the old LOB.

When γ is large enough, we notice that for each size $K>0$ we can fix an expected profit $J(\varepsilon,K)$ and J is monotone increasing w.r.t. K, so $K\mapsto J$ is one-to-one and we obtain

$$\frac{\partial K}{\partial\varepsilon}\bigg|_{\varepsilon=0} \;=\; \frac{1}{\frac{\partial J(\varepsilon,K)}{\partial\varepsilon}\Big|_{\varepsilon=0}} \;=\; -\frac{1}{\lambda[1,\beta_0]\cdot A_0^{-1}BA_0^{-1}\cdot\mathbf{1}}<0\,.$$

Since the players posting limit orders on the LOB lose profit because of the arrival of external informed agents, if they want to keep the same expected profit the size of the LOB must shrink. This model thus explains the "liquidity crisis", i.e. the fact that few agents are willing to trade stocks when the volatility of the market is too large.

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