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Abstract

This thesis consists of three chapters. In the first chapter, we consider coalition formation games with externalities. A natural extension of superadditivity is not sufficient to imply that the grand coalition is efficient when externalities are present. We provide a condition, analogous to convexity, that is sufficient for the grand coalition to be efficient and show that this also implies that the (appropriately defined) core is nonempty. Moreover, we propose a mechanism which implements the most efficient partition for all coalition formation games and characterize the resulting payoff division.

The second chapter, written jointly with Vijay Krishna, studies equilibria of first- and second-price auctions with resale in a model with independent private values. With asymmetric bidders, the resulting inefficiencies create a motive for post-auction trade. In our basic model, resale takes place via monopoly pricing—the winner of the auction makes a take-it-or-leave-it offer to the loser after updating his prior beliefs based on his winning. We show that a first-price auction with resale has a unique monotonic equilibrium. Our main result is that with resale, the expected revenue from a first-price auction exceeds that from a second-price auction. The results extend to other resale mechanisms: monopsony and, more generally, probabilistic k-double auctions. The inclusion of resale possibilities thus permits a general revenue ranking of the two auctions that is not available when these are excluded.

The third chapter, written jointly with Vijay Krishna, studies first-price auctions in a model with asymmetric, independent private values. Our goal is to compare equilibria of the first-price auction without resale (FPA) with those of the first-price auction with resale (FPAR). For the two major families of distributions for which equilibria of the FPA are available in closed form, we show that resale possibilities increase the revenue of the original seller. We also show by example that, somewhat paradoxically, resale may actually decrease efficiency.

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Acknowledgments

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Dedication

This thesis is dedicated to my beloved parents and my dearest wife.

Chapter 1

Efficiency in Coalition Games with Externalities

1.1 Introduction

Much of the work on cooperative game theory tries to understand how coalitions behave in environments in which players can cooperate with each other. The central questions this body of literature asks are: first, which coalitions should form and second, how the gains of cooperation should be shared. Economic environments with no externalities (in which what a group of players can achieve by cooperating is independent of what other players do) are best modeled as *Characteristic Function Games* (*CFGs*), introduced by Von Neumann and Morgenstern (1947). Coalition formation games in economic environments with externalities (in which what a group of players can achieve by cooperating depends on what other coalitions form) were first modeled by Lucas and Thrall (1963) as *Partition Function Games* (*PFGs*).

By and large, the literature on both CFGs and PFGs has proceeded under the assumption that in cooperative settings, the *grand coalition*—the coalition of all players—will form. In this paper, we argue that in many economic environments, such an assumption is unnatural. The reason is that when there are externalities, the payoffs resulting from the formation of the grand coalition may be inefficient—total surplus may not be maximized. Maskin (2003) has also argued against the assumption that the grand coalition will form. His criticism, however, is not based on efficiency considerations but rather on the idea that when there are positive externalities, some players may be better off by remaining separate when others join a coalition. In other words, positive externalities may create an incentive for some players to free ride.

Bloch (1996), Ray and Vohra (1999), and Yi (1997) model economic environments with externalities and show that certain bargaining procedures might result in finer partitions than the grand coalition. However, they do not take the efficiency of the resulting partition into account; nor do they provide a characterization of the payoff division of the resulting games. Bloch (1996) assumes that the division of coalitional surplus is exogenously fixed: the game only determines the coalitional structures. He shows that any core stable allocation can be attained as a stationary perfect equilibrium of the game. Ray and Vohra (1999) consider a game in which the proposers offer a coalition and a contingent payoff division. They prove that there exists a stationary equilibrium of their game and provide an algorithm to determine an equilibrium partition. Yi (1997) characterizes and compares stable coalition structures under some different rules of coalition formation.

There is another strand of work which also models economic environments with externalities. Scarf (1971), Ichiishi (1981) and Zhao (1992) offer a synthesis of Nash equilibrium and the core. They model situations in which players behave cooperatively within each coalition and competitively across coalitions. Their goal is to provide sufficient conditions for the existence of an equilibrium in such settings.

Myerson (1977), Bolger (1989), Pham Do and Norde (2002), Macho-Stadler, Perez-Castrillo and Wettstein (2004) and Albizuri, Arin and Rubio (2005) give axiomatic extensions of the Shapley value for CFGs to PFGs. They all assume that the grand coalition will form, even if it is not efficient. In the first part of their paper, de Clippel and Serrano (2005) require the efficiency of the grand coalition and provide upper and lower bounds for the players' payoffs. They also characterize a value by strengthening their marginality assumption.[1] In the second part of their paper, de Clippel and Serrano (2005) consider the case in which the grand coalition does not form and characterize a payoff configuration on the basis

[1]The value they characterize coincides with the value proposed in this paper when the game is fully cohesive (see Section 4).

of Myerson's (1980) principle of balanced contributions. Their result is similar to Maskin's (2003): they argue that considerations of coalition formation may induce formation of finer partitions than the grand coalition, even if the grand coalition is efficient.

Maskin (2003) provides an axiomatic characterization of a generalized Shapley value and exhibits a mechanism that implements it. His mechanism has the interesting property that the grand coalition may not necessarily form. He axiomatizes the solution to the following noncooperative game: players enter a room sequentially and at stage k, player k enters the room and all players with the lowest index in coalitions in the room simultaneously bid for k. Player k either accepts one of the bids or makes his own (singleton) coalition and the game moves to the $k+1st$ stage. At the end of the game, the lowest index players in each coalition distribute the promised bids and keep the rest for themselves. If there are negative externalities, the grand coalition always forms, even though it may be inefficient. If there are positive externalities, this game might result in a partition finer than the grand coalition, but again this may be inefficient.

For CFGs, the assumption of superadditivity is commonly used. It says that what two coalitions can get by merging should not be less than the sum of what they get separately. This assumption implies the efficiency of the grand coalition in environments with no externalities. However, as we will show in Section 2, a straightforward extension of superadditivity does not imply efficiency of the grand coalition when externalities are present. Therefore, from an efficiency point of view, superadditive PFGs do not necessarily result in the grand coalition.

As an example, consider a symmetric Cournot oligopoly game with three firms. Assume that when two firms merge, because of cost reduction, they do better in the market. But since negative externalities are present, the other firm does worse. When the third firm also joins the coalition, superadditivity implies that their total payoff is no less than what they get separately. However, the members of the three-firm coalition do not necessarily gain over what they get when they were all separate, because of negative externalities.

In Section 2, we show that while superadditivity is not sufficient, a straightforward extension of the convexity assumption in CFGs to PFGs implies that the grand coalition is efficient. In Section 3, we then show that convex PFGs have a nonempty core (for a specific definition of the core). Note that for economic environments with externalities, there can be many definitions of the core. This is because after a deviation, the payoff of the deviating group depends on what the complementary coalition does.

In Section 4, we turn to the question of a noncooperative implementation of the efficient partition in PFGs. We propose a mechanism which gives an efficient predicted partition as well as a payoff division among the players. We also provide a characterization of this value (the resulting payoff division).

1.2 Efficiency, Superadditivity and Convexity in PFGs

The set of players is given by $N = \{1, 2, ..., n\}$. In Characteristic Function Games (CFGs), any coalition $S \subseteq N$ generates a value $v(S)$ and this value is independent of what other agents (not in S) do. In contrast, Partition Function Games (PFGs) allow for externalities, and these are captured by writing v as a function of a coalition and a partition (which has that coalition as a member). That is, in PFGs, any coalition $S \subseteq N$ generates a value $v(S; \rho)$ where ρ is a partition of N with $S \in \rho$.

Formally, given a partition ρ of N and a coalition $S \in \rho$, the pair $(S; \rho)$ is called an *embedded coalition* of N. The set of all embedded coalitions is denoted by $EC(N)$. A PFG is a function v that assigns to every embedded coalition $(S; \rho) \in EC(N)$, a real number $v(S; \rho)$. By convention, $\varnothing \in \rho$ and $v(\varnothing; \rho) = 0$ for all partitions ρ of N.

A PFG is said to have *positive externalities* if for any mutually disjoint $C, S, T \subseteq N$, and for any partition ρ of $N - (S \cup T \cup C)$, we have

$$v(C; \{S \cup T, C\} \cup \rho) > v(C; \{S, T, C\} \cup \rho).$$

Similarly a PFG is said to have *negative externalities* if

$$v(C; \{S \cup T, C\} \cup \rho) < v(C; \{S, T, C\} \cup \rho).$$

In words, a game has positive (negative) externalities if a merger between two coalitions makes other coalitions better (worse) off.

1.2.1 Superadditivity

It is well known that if a CFG is superadditive, then the grand coalition is efficient. That is, if for all $S, T \subseteq N$ with $S \cap T = \varnothing$,

$$v(S \cup T) \geq v(S) + v(T),$$

then $v(N) \geq \sum_{S \in \rho} v(S)$ for all partitions ρ of N.

A natural extension of superadditivity to PFGs used in Maskin (2003) and several others is as follows: A PFG is *superadditive* if for any $S, T \subseteq N$ with $S \cap T = \varnothing$, and any partition ρ of $N - (S \cup T)$,

$$v(S \cup T; \{S \cup T\} \cup \rho) \geq v(S; \{S, T\} \cup \rho) + v(T; \{S, T\} \cup \rho).$$

For notational convenience, let us denote $v(S \cup T; \{S \cup T\} \cup \rho)$ by $v_\rho(S \cup T; \{S \cup T\})$ and so on. With this notation, superadditivity can be written as follows: For any $S, T \subseteq N$ with $S \cap T = \varnothing$, and any partition ρ of $N - (S \cup T)$,

$$v_\rho(S \cup T; \{S \cup T\}) \geq v_\rho(S; \{S, T\}) + v_\rho(T; \{S, T\}).$$

In PFGs, superadditivity is not enough for the efficiency of the grand coalition, as the following example shows.

Example 1. *Consider the following symmetric 3-player PFG:* $N = \{1, 2, 3\}$

$$\begin{aligned} v(\{i\}; \{\{1\}, \{2\}, \{3\}\}) &= 4 \text{ for } i = 1, 2, 3; \\ v(\{j, k\}; \{\{i\}, \{j, k\}\}) &= 9 \text{ and } v(\{i\}; \{\{i\}, \{j, k\}\}) = 1 \text{ for } \{i, j, k\} = N; \\ v(N; \{N\}) &= 11. \end{aligned}$$

This game is superadditive, but the grand coalition is not efficient since $v(N, \{N\}) = 11 < \sum_{i=1}^{i=3} v(\{i\}, \{\{1\}, \{2\}, \{3\}\}) = 12.$

In this game, the grand coalition is not efficient because there are negative externalities. Although the merging coalitions benefit from merging, others are worse off and the total payoff in the grand coalition is less then the total payoff in some other partition. It can be easily shown that if the externalities are positive and the game is superadditive, then the grand coalition is always efficient.

1.2.2 Convexity

A stronger assumption on value functions in CFGs is convexity, or supermodularity. Convexity assumption in CFGs implies not only that the merging of two coalitions is beneficial for them, but also that merging with bigger coalitions is more beneficial. That is, the game is convex if there are increasing returns to cooperation (see Moulin (1988) for a detailed discussion). A natural extension of convexity to PFGs can be given as follows: A PFG is *convex* if for any $S, T \subseteq N$ and any partition ρ of $N - (S \cup T)$,

$$\begin{aligned} & v_\rho (S \cup T; \{S \cup T\}) + v_\rho (S \cap T; \{S \cap T, S - T, T - S\}) \\ \geq \; & v_\rho (S; \{S, T - S\}) + v_\rho (T; \{T, S - T\}) . \end{aligned}$$

As shown by Example 1, superadditivity by itself does not imply the efficiency of the grand coalition. We will argue below that convexity implies that any coalition can achieve at least as much as the sum of what its parts can achieve (independent of whether game has positive, negative or mixed externalities) and in particular, it implies the efficiency of the grand coalition.

Proposition 1. *If a PFG is convex, then for any coalition C, any partition ρ of $N - C$ and ρ' of C,*

$$v_\rho (C) \geq \sum_{S \in \rho'} v_\rho (S; \rho') .$$

Proof. Fix a coalition $C \subseteq N$ and a partition ρ of $N - C$. The proof is by induction on the cardinality of the partition ρ' of C. Let us denote ρ' by $\{C_1, C_2, ..., C_k\}$ with $k \leq |C|$ (suppose $C_i \neq \varnothing$).

For notational simplicity denote $C_i \cup C_{i+1} \cup ... \cup C_j$ by $\bar{S}_{ij}$ and $\{C_i, C_{i+1}, ..., C_j\}$ by $S_{i,j}$.

Induction hypothesis: For any $3 \leq l \leq k$ and any partition ρ'' of $C_{l+1} \cup C_{l+2} \cup$

$... \cup C_k$,

$$v_{\rho\cup\rho''}\left(\bar{S}_{1,l};\{\bar{S}_{1,l}\}\right) \geq v_{\rho\cup\rho''}\left(\bar{S}_{1,l-2};\{\bar{S}_{1,l-2},C_{l-1},C_l\}\right) + v_{\rho\cup\rho''}\left(C_{l-1};S_{1,l}\right) + v_{\rho\cup\rho''}\left(C_l;S_{1,l}\right)$$

Induction base: For $l = 3$, in the definition of convexity take $S = \bar{S}_{1,2}$ and $T = \bar{S}_{2,3}$ (so $S \cap T = \{C_2\}$), and get

$$\begin{aligned} & v_{\rho\cup\rho''}\left(\bar{S}_{1,3};\{\bar{S}_{1,3}\}\right) + v_{\rho\cup\rho''}\left(\{C_2\};S_{1,3}\right) \\ \geq\ & v_{\rho\cup\rho''}\left(\bar{S}_{1,2};\{\bar{S}_{1,2},C_3\}\right) + v_{\rho\cup\rho''}\left(\bar{S}_{2,3};\{\bar{S}_{2,3},C_1\}\right), \end{aligned}$$

and by superadditivity applied to the right hand side of the above inequality, obtain:

$$v_{\rho\cup\rho''}\left(\bar{S}_{1,3};\{\bar{S}_{1,3}\}\right) \geq v_{\rho\cup\rho''}\left(C_1;S_{1,3}\right) + v_{\rho\cup\rho''}\left(C_2;S_{1,3}\right) + v_{\rho\cup\rho''}\left(C_3;S_{1,3}\right).$$

Induction proof: Assume that the induction hypothesis is true for $l = t - 1$. We need to show that it is true for $l = t$ as well.

Fix a partition ρ'' of $C_{t+1} \cup C_{t+2} \cup ... \cup C_k$. For $S = \bar{S}_{1,t-1}$ and $T = \bar{S}_{2,t}$ (so $S \cap T = \bar{S}_{2,t-1}$), from convexity, obtain:

$$\begin{aligned} & v_{\rho\cup\rho''}\left(\bar{S}_{1,t};\{\bar{S}_{1,t}\}\right) + v_{\rho\cup\rho''}\left(\bar{S}_{2,t-1};\{\bar{S}_{2,t-1},C_1,C_t\}\right) \\ \geq\ & v_{\rho\cup\rho''}\left(\bar{S}_{1,t-1};\{\bar{S}_{1,t-1},C_t\}\right) + v_{\rho\cup\rho''}\left(\bar{S}_{2,t};\{\bar{S}_{2,t},C_1\}\right). \end{aligned} \tag{1.1}$$

Again from convexity, obtain (by $S = \bar{S}_{1,t-2}$ and $T = \bar{S}_{2,t-1}$, so $S \cap T = \bar{S}_{2,t-2}$):

$$\begin{aligned} & v_{\rho\cup\rho''}\left(\bar{S}_{1,t-1};\{\bar{S}_{1,t-1},C_t\}\right) + v_{\rho\cup\rho''}\left(\bar{S}_{2,t-2};\{\bar{S}_{2,t-2},C_1,C_{t-1},C_t\}\right) \\ \geq\ & v_{\rho\cup\rho''}\left(\bar{S}_{1,t-2};\{\bar{S}_{1,t-2},C_{t-1},C_t\}\right) + v_{\rho\cup\rho''}\left(\bar{S}_{2,t-1};\{\bar{S}_{2,t-1},C_1,C_t\}\right). \end{aligned} \tag{1.2}$$

Add up (1.1) and (1.2) to obtain:

$$\begin{aligned} & v_{\rho\cup\rho''}\left(\bar{S}_{1,t};\{\bar{S}_{1,t}\}\right)+v_{\rho\cup\rho''}\left(\bar{S}_{2,t-2};\{\bar{S}_{2,t-2},C_1,C_{t-1},C_t\}\right) \\ \geq\ & v_{\rho\cup\rho''}\left(\bar{S}_{1,t-2};\{\bar{S}_{1,t-2},C_{t-1},C_t\}\right)+v_{\rho\cup\rho''}\left(\bar{S}_{2,t};\{\bar{S}_{2,t},C_1\}\right). \end{aligned} \tag{1.3}$$

Use the induction hypothesis for at $l=t-1$ to obtain:

$$\begin{aligned} & v_{\rho\cup\rho''\cup\{C_1\}}\left(\bar{S}_{2,t};\{\bar{S}_{2,t}\}\right)\geq \\ & v_{\rho\cup\rho''\cup\{C_1\}}\left(\bar{S}_{2,t-2};\{\bar{S}_{2,t},C_{t-1},C_t\}\right)+v_{\rho\cup\rho''\cup\{C_1\}}\left(C_{t-1};S_{2,t}\right)+v_{\rho\cup\rho''\cup\{C_1\}}\left(C_t;S_{2,t}\right). \end{aligned} \tag{1.4}$$

Use (1.3), (1.4), and the induction hypothesis to obtain:

$$\begin{aligned} v_{\rho\cup\rho''}\left(\bar{S}_{1,t};\{\bar{S}_{1,t}\}\right) \ \geq\ & v_{\rho\cup\rho''}\left(\bar{S}_{1,t-2};\{\bar{S}_{1,t-2},C_{t-1},C_t\}\right) \\ & +v_{\rho\cup\rho''}\left(C_{t-1};S_{1,t}\right)+v_{\rho\cup\rho''}\left(C_t;S_{1,t}\right), \end{aligned}$$

which completes the induction proof.

Thus, conclude that for any partition ρ' of C, we have

$$v_\rho\left(C\right)\geq\sum_{S\in\rho'}v_{\rho\cup\rho'}\left(S\right).$$

∎

Now, we can state an immediate corollary of the above proposition, which states that convexity implies the efficiency of the grand coalition.

Corollary 1. *If a PFG is convex, then for any partition ρ of N,*

$$v\left(N;\{N\}\right)\geq\sum_{S\in\rho}v_\rho\left(S\right).$$

It should be noted that convex PFGs do not necessarily have positive externalities. Consider Example 1, with the difference that $v\left(N;\{N\}\right)=15$ instead of 11. This game is convex, yet has negative externalities.

Let us define the PFGs with the property that any coalition can achieve at least as much as the sum of what its parts can achieve by fully cohesive[2] PFGs. Formally, a PFG is *fully cohesive* if for any coalition C, any partition ρ of $N - C$ and ρ' of C,

$$v_{\rho}(C) \geq \sum_{S \in \rho'} v_{\rho}(S; \rho').$$

In other words, a fully cohesive PFG assigns more to the subset $C \subseteq N$, than to any of its partitions, for any partition of the set $N - C$. Proposition 1 shows that convexity implies full cohesiveness.

1.3 The Core

In this section, we focus on convex PFGs. Hence, in the games we consider here, the grand coalition is the most efficient partition. Therefore, any other partition can be Pareto improved by making appropriate side-transfers.

For CFGs, a vector of payoffs $x = (x_1, x_2, ..., x_n)$ is in the core if for all $S \subset N$, we have

$$\sum_{i \in S} x_i \geq v(S).$$

A nice feature of convex CFGs is that they have a nonempty core. In CFGs, when group of agents is deciding whether or not to deviate, they do not consider what other agents would do (a coalition's value is independent of what other coalitions form). However, this is not the case in PFGs. In PFGs, one has to make assumptions about what a deviating coalition conjectures about the reaction of the others while defining the core. Hence, there can be many definitions of the core. One simple definition of the core can be given by supposing that the agents in the deviating coalition S presume that agents in $N - S$ will form singletons after the deviation.

Definition 1. *A vector of payoffs $x = (x_1, x_2, ..., x_n)$ is in the* core with singleton expectations*, named* s-core*, if for all $S \subset N$, we have*

$$\sum_{i \in S} x_i \geq v(S; \{S\} \cup [N - S]).$$

[2]This term was first defined by Currarini (2003).

where $[N-S]$ *denotes the partition of* $N-S$ *to singletons.*

The next proposition shows that any convex game has a nonempty s-core.

Proposition 2. *If a PFG is convex, then the s-core is nonempty*

Proof. Define the following CFG with $\widehat{v}(S)=v(S;\{S\}\cup[N-S])$. We claim that the CFG $\widehat{v}$ is convex.

Take any $S,T\subset N$, with $|T-S|=|S-T|=1$, then from definition of convexity in the PFG (with $\rho=[N-(S\cup T)]$)

$$\begin{aligned} & v_{\rho}(S\cup T;\{S\cup T\})+v_{\rho}(S\cap T;\{S\cap T,S-T,T-S\}) \\ \geq\ & v_{\rho}(S;\{S,T-S\})+v_{\rho}(T;\{T,S-T\}), \end{aligned}$$

or in characteristic function notation,

$$\widehat{v}(S\cup T)+\widehat{v}(S\cap T)\geq\widehat{v}(S)+\widehat{v}(T). \tag{1.5}$$

Now, by using (1.5) we need to show that for all S' and T' (without the restriction $|T'-S'|=|S'-T'|=1$), (1.5) is true. This follows from the fact that weak convexity is equivalent to convexity for CFGs (see Moulin, 1988)

A very well known result of convex CFG tells us that $x_i=\widehat{v}(\{1,..,i\})-\widehat{v}(\{1,..,i-1\})$ is in the core of the game (Shapley, 1971). Hence we obtain that the s-core is nonempty for convex PFGs, since

$$x_i=v(\{1,..,i\};\{1,..,i\},\{i+1\},...,\{n\})-v(\{1,..,i-1\};\{1,..,i-1\},\{i\},...,\{n\})$$

is in the core of the PFG. ∎

Note that the s-core definition assumes that the deviating coalitions are very pessimistic in PFGs with positive externalities but are very most optimistic in PFGs with negative externalities (which makes it very easy to deviate). Therefore, we immediately have the following remark.

Remark 1. *A convex PFG with negative externalities has a nonempty core (independent of agents' conjectures about what will happen after the deviation.)*

Another natural core specification is given by the following definition.

Definition 2. *A vector of payoffs* $x = (x_1, x_2, ..., x_n)$ *is in the* core with cautious expectations*, named* c-core*, if for all* $S \subset N$*, we have*

$$\begin{aligned} \sum_{i \in S} x_i &\geq v_{\rho^*}(S, \{S\}), \\ \text{where } \rho^* &= \arg\min_{\rho} v_\rho(S, \{S\}). \end{aligned}$$

This definition of the core is analogous to the definition of α*-core* in the literature on CFGs. It is easy to see that if the game has positive externalities, then $\rho^* = [N - S]$ and if the game has negative externalities, then $\rho^* = \{N - S\}$. The following corollary is an implication of Proposition 2.

Corollary 2. *If a PFG game is convex, then the c-core is nonempty.*

There can be other definitions of the core. Maskin (2003) makes the assumption that any deviating coalition S presumes the complementary coalition $N - S$.

Definition 3. *A vector of payoffs* $x = (x_1, x_2, ..., x_n)$ *is in the* core with merging expectations*, named* m-core*, if for all* $S \subset N$*, we have*

$$\sum_{i \in S} x_i \geq v(S; \{S, N - S\}).$$

The following example shows that convexity does not imply a nonempty m-core.

Example 2. *Consider the following symmetric 3-player PFG:* $N = \{1, 2, 3\}$

$$\begin{aligned} v(\{i\}; \{\{1\}, \{2\}, \{3\}\}) &= 4 \text{ for } i = 1, 2, 3; \\ v(\{j, k\}; \{\{i\}, \{j, k\}\}) &= 9 \text{ and } v(\{i\}; \{\{i\}, \{j, k\}\}) = 6 \text{ for } \{i, j, k\} = N; \\ v(N; \{N\}) &= 16. \end{aligned}$$

One can easily verify that above game is convex: Take $S = \{1, 2\}$ and $T = \{2, 3\}$, we have:

$$\begin{aligned} 20 &= v(N; \{N\}) + v(\{2\}; \{\{1\}, \{2\}, \{3\}\}) \\ &> v(\{1, 2\}; \{\{1, 2\}, \{3\}\}) + v(\{2, 3\}; \{\{2, 3\}, \{1\}\}) = 18. \end{aligned}$$

However, this game has an empty m-core. When a singleton deviates, he presumes that the other two will make a coalition, so he can get a payoff of 6 by deviating. This implies the grand coalition should allocate at least 18, hence the m-core is empty.

Note that the above example has positive externalities. Moreover, Maskin's (2003) definition of the core relies on a very optimistic conjecture about the reactions of other agents when the externalities are positive.

One natural expectation of the deviating agents is that others will take this deviation as given and try to maximize their own payoff. We call this *rational expectations.*

Definition 4. *A vector of payoffs* $x = (x_1, x_2, ..., x_n)$ *is in the* core with rational expectations*, named* r-core*, if for all* $S \subset N$*, we have*

$$\begin{aligned} \sum_{i \in S} x_i &\geq v_{\rho^*}(S; \{S\}), \\ \textit{where } \rho^* &= \arg\max_{\rho} \sum_{C \in \rho} v_\rho(C; \{S\}). \end{aligned}$$

Although natural, convex PFGs might have an empty core with rational expectations. The game specified in Example 2 has an empty r-core. When a singleton i deviates, the best j and k can achieve is obtained by forming a coalition since they can get 9 rather than a total of 8. This then implies any core allocation should allocate at least 18, which is not possible.[3]

1.4 A noncooperative implementation

In this section, we do not impose any restrictions on PFGs. Specifically, we do not require v to have superadditivity, convexity, or positive or negative externalities.

For PFGs, Myerson (1977), Bolger (1989), Pham Do and Norde (2002), Maskin (2003), Macho-Stadler, Perez-Castrillo and Wettstein (2004), Albizuri, Arin and

[3] One could also look for a *consistent core* notion. Specifically, assume that a deviating coalition S expects that the players in $N - S$ would form a partition with an associated allocation which is in the core of the reduced game among $N - S$. For the game in Example 2, however, the consistent core is also empty. This is because when a singleton deviates, the core of the two person game contains only the grand coalition of two players.

Rubio (2005) and de Clippel and Serrano (2005) give extensions of Shapley value of CFGs to PFGs. Except for Maskin (2003) and de Clippel and Serrano (2005), other papers propose that the grand coalition will form. Most of these models consider environments in which the grand coalition is the most efficient partition. Their values and implementations are either not applicable to environments in which the grand coalition is not the most efficient or they result in inefficient partitions. Maskin (2003) proposes that for superadditive games, the grand coalition will form with negative externalities (where the grand coalition is not necessarily efficient), but might not form with positive externalities (where the grand coalition is efficient.)

Both Maskin (2003) and de Clippel and Serrano (2005) argue that inefficient outcomes may emerge in superadditive games if one introduces considerations of coalition formation, while we argue in this paper that the "nonformation" of the grand coalition might emerge from efficiency considerations. In an environment in which side payments are allowed, formation of inefficient partitions is implausible since a Pareto superior allocation would be available to the agents.

Below, we propose a game form with the property that all subgame perfect equilibria result in an efficient partition. Moreover, all subgame perfect equilibria result in the same payoff division. Consider the following simple game.

Take any ordering of the players (we consider the natural ordering, since the game will be the same for any other permutation σ of the players): players enter a room sequentially and at stage k, agent k enters the room. Suppose that before he enters, agents $1, ..., k-1$ have already formed a partition of $K-1 = \{1, ..., k-1\}$ and there are three kinds of people in the room: a boss, dependents and independents. Let $<< b, D, I >>$ denote the *state* at the end of period $k-1$.

At stage k, agent k proposes a partition of $K = \{1, .., k\}$ and a payoff division for all agents in $K-1$. After the proposal, there can be two outcomes:

- If everybody accepts the proposal, then the proposed partition forms, k becomes the new boss and all others become dependents. In this case, the state at the end of period k is given by $<< k, D \cup I \cup \{b\}, \varnothing >>$.

- If anybody rejects, the old partition and a singleton coalition of k forms and k becomes an independent. In this case, the state at the end of period k is given by $<< b, D, I \cup \{k\} >>$.

At the end of the game (stage n) independents receive their payoff from the resulting partition[4]. The boss gets the rest of the total payoffs, distributes the promised payoffs to the dependents and gets what is left. This game is played for all possible orderings of players. The *Efficient Generalized Shapley Value* (*EGSV*) of players is the average of their payoffs obtained for different orderings.

For the above game, it is not difficult to see that at every stage, the newcomer will propose an acceptable offer and become the new boss. Before showing why this is the case, let us introduce some notation.

Let ρ^k denote a partition of K. Define $V\left(\rho^k\right)$ as follows:

$$V\left(\rho^k\right) = \sum_{C \in \rho^k} v\left(C; \rho^k \cup [N-K]\right),$$

where $[N-K]$ denotes the partition of $N-K$ to singletons.

Let $\overline{\rho}^k$ be defined as follows:

$$\overline{\rho}^k = \arg\max_{\rho^k} V\left(\rho^k\right).$$

That is, $\overline{\rho}^k$ is the most efficient partition given that agents $k+1, ..., n$ remain singletons[5]. Let $\widetilde{\rho}^k$ be the proposed partition at stage k of the game. Finally, let p_i^k be the promised payoff to i at stage k.

Consider the last stage. If any of the agents reject agent n's offer, then the final partition $\rho^{n-1} \cup \{n\}$ forms (where ρ^{n-1} is equal to $\widetilde{\rho}^{n-1}$ if $n-1$'s offer was accepted at stage $n-1$), independents receive their resulting payoffs, dependents receive what has been promised, and the boss gets what is left. Then, for an independent agent i in $N-1$ to accept the offer, he needs to be promised at least $v\left(\{i\}; \rho^{n-1} \cup \{n\}\right)$. For a dependent agent i to accept the offer, he needed to be promised at least p_i^{n-1} and the boss needs to be promised at least $V\left(\rho^{n-1}\right)$ minus what independents and dependents get after a rejection. Hence, agent n can propose an acceptable offer for a sum of $V\left(\rho^{n-1}\right)$. If his offer is rejected, n

[4]Note that independents form singleton coalitions.

[5]Note that $\overline{\rho}^k$ is not necessarily unique for general class of PFGs, however $V\left(\overline{\rho}^k\right)$ is unique. If there is more than one efficient partition, let $\overline{\rho}^k$ denote a selection.

receives $v(\{n\};\rho^{n-1}\cup\{n\})$. Whereas, by proposing the partition $\overline{\rho}^n$, he can get the payoff $V(\overline{\rho}^n)-V(\rho^{n-1})$ which is never less than $v(\{n\};\rho^{n-1}\cup\{n\})$ (note that $\overline{\rho}^n$ is the most efficient partition of N). Therefore, n's (weakly) best strategy is to offer $\widetilde{\rho}^n=\overline{\rho}^n$ and promise to each agent what he gets if he rejects the offer. This actually proves that this game always results in the most efficient partition.

At stage $n-1$, if the proposal is rejected then partition $\rho^{n-2}\cup\{n-1\}$ forms, then (from backward induction) an independent agent i receives $v(\{i\};\rho^{n-2}\cup\{n-1\}\cup\{n\})$, dependents receive what has been promised, and the boss receives what is left from $V(\rho^{n-2}\cup\{n-1\})$. Agent $n-1$ can then propose an acceptable offer at a sum of $V(\rho^{n-2})$. By proposing the partition $\overline{\rho}^{n-1}$, he can get the payoff $V(\overline{\rho}^{n-1})-V(\rho^{n-2})$ which is never less than $v(\{n-1\};\rho^{n-2}\cup\{n-1\}\cup\{n\})$ (note that $\overline{\rho}^{n-1}$ is the most efficient partition of $N-1$ when n remains singleton). Therefore, $n-1$'s best strategy is to offer $\widetilde{\rho}^{n-1}=\overline{\rho}^{n-1}$ and promise each agent what he gets (at the end of the game) if he rejects.

Continuing in this fashion, and using backward induction, we can conclude that agent k at stage k proposes the partition $\overline{\rho}^k$ and promises payoffs which add up to $V\left(\overline{\rho}^{k-1}\right)$. Hence, at the end of the game agent k gets a payoff of $m^k=V\left(\overline{\rho}^k\right)-V\left(\overline{\rho}^{k-1}\right)$ (for the natural ordering) and the most efficient partition is the result of the game. Then EGSV of player i, which is denoted by $\psi_i^{Sh}(v)$ is the average of these marginal contributions over all possible permutations σ of the players. Formally, for any permutation σ of the players, let ρ_σ^k denote a partition of the set formed by the predecessors in σ, $P_\sigma^k=\{\sigma(j):j\in N \text{ and } j\leq\sigma^{-1}(k)\}$. Then,

$$V\left(\rho_\sigma^k\right)=\sum_{C\in\rho_\sigma^k}v\left(C;\rho_\sigma^k\cup\left[N-P_\sigma^k\right]\right),$$

and let $\overline{\rho}_\sigma^k$ be defined by

$$\overline{\rho}_\sigma^k=\arg\max_{\rho_\sigma^k}V\left(\rho_\sigma^k\right)$$

Then, we obtain

$$m_\sigma^k=V\left(\overline{\rho}_\sigma^k\right)-V\left(\overline{\rho}_\sigma^{\sigma\left(\sigma^{-1}(k)-1\right)}\right)$$

and EGSV is given by

$$\psi_i^{Sh}(v) = \frac{1}{n!}\sum_{i\in N} m_\sigma^i,$$

It is easy to see that this value reduces to Shapley value for superadditive CFGs.

1.4.1 Characterization of Payoffs

Before we state our axiomatic results, let us define what a *value* means.

Definition 5. *A* value *is a function ψ that assigns to every PFG v, a unique utility vector $\psi(v) \in \mathbb{R}^n$.*

When we consider a fully cohesive PFG, then $\widetilde{\rho}^k = \overline{\rho}^k = K$ and the payoff division of above game coincides with the value given by Pham Do and Norde (2002) and de Clippel and Serrano (2005). It is not difficult to see that for arbitrary games the value ψ^{Sh} is not additive. This is true even for (nonsuperadditive) CFGs. Consider the following example.

Example 3. *Consider the following two symmetric 3-player CFGs: $v(\{i\}) = 2$, $v(\{i,j\}) = 1$, $v(\{1,2,3\}) = 4$ and $w(\{i\}) = 1$, $w(\{i,j\}) = 5$, $w(\{1,2,3\}) = 4$. Any value which is efficient and symmetric should give $(2,2,2)$ to the players in both of the games. However, in the game $v+w$ this value should give $(3,3,3)$. Hence, the value is not additive.*

On the other hand, ψ^{Sh} is efficient-cover additive. That is, when we consider the efficient cover (or fully-cohesive cover) of two games, then EGSV is additive. More formally, let the *efficient cover* of the game v be defined as follows: for any $S \subset N$ and any partition ρ of $N-S$,

$$\bar{v}_\rho(S;\{S\}) = \max_{\rho'\text{: partition of } S} \sum_{C\in\rho'} v_\rho(C;\rho').$$

Definition 6. *A value ψ is* efficient-cover additive *if $\psi_i(\bar{v}) + \psi_i(\bar{w}) = \psi_i(\bar{v}+\bar{w})$ and* fully efficient *if*

$$\sum_{i\in N}\psi_i(v) = V(\overline{\rho}^n).$$

The other two axioms that will characterize ψ^{Sh} are null player property and efficient-cover anonymity.

Definition 7. *A value ψ satisfies the* null-player property *if the following holds: if $V\left(\overline{\rho}_{\sigma}^{k}\right) - V\left(\overline{\rho}_{\sigma}^{\sigma\left(\sigma^{-1}(k)-1\right)}\right)$ for all permutations, then $\psi_k(v) = 0$.*

Definition 8. *A value is* efficient-cover anonymous *if the following holds: For any permutation σ, $\psi(\sigma(\bar{v})) = \sigma(\psi(\bar{v}))$, where $\sigma(\bar{v})(S,\rho) = \bar{v}(\sigma(S);\{\sigma(T) : T \in \rho\})$ for each embedded coalition (S,ρ) and $\sigma(x)_i = x_{\sigma(i)}$ for each $x \in \mathbb{R}^n$ and each $i \in N$.*

Now, we can introduce the characterization for EGSV.

Proposition 3. *A value is efficient-cover additive, fully efficient, efficient-cover anonymous and satisfies null-player property if and only if it is Efficient Generalized Shapley Value, ψ^{Sh}.*

The proof of this proposition follows from the above observations and Proposition 3 in de Clippel and Serrano (2005).

1.4.2 An Example

In this section, we consider an example to illustrate the difference between our results and results of de Clippel and Serrano's (2005) and Maskin's (2003) in terms of the resulting partition and payoff division. Consider the following PFG, which was considered in both de Clippel and Serrano (2005) and Maskin (2003).

Example 4. *Consider the following 3-player PFG: $N = \{1,2,3\}$*

$$\begin{aligned}
v(\{i\};\{\{1\},\{2\},\{3\}\}) &= 0 \text{ for } i = 1,2,3;\\
v(\{i\};\{\{i\},\{j,k\}\}) &= 9 \text{ for } \{i,j,k\} = N\\
v(\{1,2\};\{\{1,2\},\{3\}\}) &= 12;\\
v(\{1,3\};\{\{1,3\},\{2\}\}) &= 13;\\
v(\{2,3\};\{\{2,3\},\{1\}\}) &= 14;\\
v(N;\{N\}) &= 24.
\end{aligned}$$

Note that in this game the most efficient partition is the grand coalition. However, both the balanced contributions approach of de Clippel and Serrano (2005) and the coalition formation game of Maskin (2003) result in a partition of a singleton and a coalition of two. For this game, the resulting value (the payoff division) is given by $\left(\frac{43}{6}, \frac{44}{6}, \frac{45}{6}\right)$ in de Clippel and Serrano (2005) and $\left(7, \frac{22}{3}, \frac{25}{3}\right)$ in Maskin (2003). Note that both these payoff vectors add up to less than 24, the total attainable by the grand coalition.

Let us now apply our implementation. Consider the ordering $1, 2, 3$. In the case that agent 2 is independent when agent 3 enters the room, if either agent 1 or agent 2 rejects 3's offer, they will get 0 payoff. Therefore, agent 3 will offer the partition of $\{N\}$ and will offer a payoff of 0 to both agent 1 and 2. Given this, agent 2 will offer agent 1 a payoff of 0 in the second stage (because if agent 1 rejects 2's offer, agent 2 will be an independent and agent 1 will have 0 payoff at the end of the game) and the partition of $\{1, 2\}$. Therefore, 3 will face the partition $\{1, 2\}$ in the third stage and offer payoff of 0 payoff to agent 1, payoff of 12 to agent 2 and the partition of $\{N\}$. We then can conclude that for the ordering $1, 2, 3$ the grand coalition is the resulting partition and payoff divisions are $(0, 12, 12)$. For the other orderings we can confirm that the grand coalition will form and payoff divisions are given by: $(0, 11, 13)$ if the ordering is $1, 3, 2$; $(12, 0, 12)$ if the ordering is $2, 1, 3$; $(10, 0, 14)$ if the ordering is $2, 3, 1$; $(13, 11, 0)$ if the ordering is $3, 1, 2$; and $(10, 14, 0)$ if the ordering is $3, 2, 1$. We therefore conclude that EGSV for above PFG is given by $\left(\frac{15}{2}, 8, \frac{17}{2}\right)$.

1.5 Conclusion

When externalities are present, the assumption that "two coalitions together can do better than what they can do separately" (superadditivity) is not enough to conclude that the grand coalition is the most efficient partition. We have identified a natural extension of convexity (supermodularity) to be a sufficient condition implying that "any number of coalitions together can do better than what they can do separately." We have also shown that convexity implies that a particular definition of the core is nonempty. As a remark, we noted that convex PFGs with negative externalities always have a nonempty core and the core with cautious expectations is also nonempty.

There have been different extensions of the Shapley value to PFGs that have been proposed, but except for Maskin (2003) and de Clippel and Serrano (2005), all implicitly or explicitly assume that the grand coalition will form. We have proposed a mechanism which always results in an *efficient* partition and provided a characterization of the resulting payoff division.

Applications of our game to noncooperative setups, such as bidding rings and distribution of payoffs after the bidding in the auction theory setup, are left for future work.

1.6 References

1. Albizuri M., Arin J., Rubio J., 2005. An Axiom System for a Value For Games In Partition Function Form. International Game Theory Review, 7, 63-72.

2. Bloch, F., 1996. Sequential Formation of Coalitions in Games with Externalities and Fixed Payoff Division. Games and Economic Behavior, 14, 90-123.

3. Bolger, E. M., 1989. A set of axioms for a value for partition function games. International Journal of Game Theory, 18, 37-44.

4. Currarini, S., 2003. On the Stability of Hierarchies in Games with Externalities. Working Paper, Coalition Theory Network.

5. de Clippel, G., Serrano, R., 2006. Marginal Contributions and Externalities in the Value. Working Paper, Brown University.

6. Ichiishi, T., 1981. A Social Coalitional Equilibrium Existence Lemma. Econometrica, 49, 369-377.

7. Lucas, W., Thrall, R., 1963. n-Person Games in Partition Function Form. Naval Research Logistics Quarterly X: 281-98.

8. Maskin, E., 2003. Bargaining, Coalitions and Externalities. Working Paper, Institute for Advanced Study.

9. Macho-Stadler, I., Perez-Castrillo, D., Wettstein, D., 2004. Sharing the surplus: An extension of the Shapley value for environments with externalities. Working Paper, Universitat Autònoma de Barcelona.

10. Moulin, H., 1988. Axioms of Cooperative Decision Making. Monograph of the Econometric Society, Cambridge University Press, Cambridge.

11. Myerson, R., 1977. Values for games in partition function form. International Journal of Game Theory, 6, 23-31.

12. Myerson, R., 1980. Conference Structures and Fair Allocation Rules. International Journal of Game Theory, 9, 169-182.

13. Pham Do, K., Norde, H., 2002. The Shapley value for partition function form games. Working Paper, Tilburg University, Center for Economic Research.

14. Ray, D., Vohra R., 1999. A theory of endogenous coalition structures. Games and Economic Behavior, 26, 286-336.

15. Scarf, H., 1971. On the existence of a cooperative solution for a general class of n-person games. Journal of Economic Theory, 3, 169-181.

16. Shapley, L., 1953. A Value for n-Person Games. In H. Kuhn and A. Tucker (eds.), Contributions to the Theory of Games, II, Princeton: Princeton University Press, 307-317.

17. Shapley, L., 1971. Cores of convex games. International Journal of Game Theory, 1: 11-26.

18. Von Neumann, J., Morgenstern, O., 1947. Theory of Games and Economic Behavior. Wiley, New York.

19. Yi, S., 1997. Stable Coalition Structures with Externalities. Games and Economic Behavior, 20, 201-237.

20. Zhao, J., 1992. The hybrid solutions of an N-person game. Games and Economic Behavior, 4, 145-160.

Chapter 2

Asymmetric Auctions with Resale

2.1 Introduction

In a first-price auction, asymmetries among bidders typically result in inefficient allocations—that is, the winner of the auction may not be the person who values the object the most. This inefficiency creates a motive for post-auction resale and when bidders take resale possibilities into account, their bidding behavior is affected. Standard models of such auctions, by and large, implicitly assume either that resale possibilities do not exist or that bidders do not take these into account when formulating bids.

There are at least two reasons why resale possibilities should be considered explicitly. The first one is positive. If, after the auction is over, bidders see that there are potential gains from trade, then they will naturally engage in such trade. And it seems unlikely that the seller can prevent bidders from engaging in post-auction trade, even if, for some reason, resale was deemed disadvantageous. In the auction of 3G spectrum licenses in the UK, post-auction trade was restricted by the government. The bidders, however, were easily able to circumvent these restrictions. TIW, a Canadian firm, bid successfully for the most valuable license "A" with a winning bid in excess of £4 bn. Another telecommunications company, Hutchison, then acquired the license by buying TIW itself. British Telecom created a wholly-owned subsidiary that bid in the auction and also purchased a license. After the auction, this subsidiary was floated on the stock market and sold. Thus, restrictions on the buying and selling of licenses were circumvented by the buying and selling of companies that owned the licenses. The actions of British Telecom—it created the subsidiary before the auction—suggest that bidders fully anticipated post-auction resale possibilities.

The second reason to consider auctions with resale is normative. There has been recent interest in the design of efficient auctions, especially in the context of privatization. If post-auction resale results in efficiency, however, then from the planner's perspective, an inefficient auction is just as good. Are the allocations from an inefficient auction followed by post-auction resale indeed efficient?

In this paper, we study the effects of post-auction resale in a simple model with two bidders whose private values are independently, but perhaps asymmetrically, distributed. Equilibrium allocations in first-price auctions are then inefficient and bidders have the incentive to engage in post-auction trade. In our basic model, resale takes place via monopoly pricing—the winner of the auction makes a take-it-or-leave-it offer to the loser.

We show that a first-price auction with resale has a unique monotonic equilibrium (Theorems 1 and 2). We do this by first showing that every equilibrium has the feature that, despite the asymmetries, the distributions of bids of the two bidders are identical. Symmetry would not be surprising if resale took place under complete information and so was always efficient. In that case, each bidder would bid as if his value were the maximum of the two values. In our model, the symmetry of bid distributions is striking because post-auction resale also takes place under incomplete information and so is not always efficient. Here it occurs as a result of some cost-benefit calculations at the margin. The symmetry of the bid distributions is key—it is used both to construct an equilibrium and to show that it is unique.

The possibility of resale also affects incentives in second-price auctions. It is no longer a dominant strategy to bid one's value. It is, nevertheless, a robust equilibrium—the strategies do not depend on the value distributions—and uniquely so. In this equilibrium, of course, the auction allocates efficiently and so there is no resale.

Our main result (Theorem 3) is that once resale possibilities are admitted, the expected revenue from a first-price auction exceeds that from a second-price auction. We thus obtain a *general* revenue ranking of the two auction formats. We require only that the value distributions be regular in the sense of Myerson (1981), ensuring that the monopoly pricing problem at the resale stage is well behaved. In particular, we do not assume that one of the bidders is "stronger" than the other in the sense of stochastic dominance. We remind the reader that in the absence of resale, the two auctions cannot be unambiguously ranked, even if the bidders can be classified as being "weak" and "strong" (Maskin and Riley (2000)).

The proof of Theorem 3 uses a variational technique borrowed from the calculus of variations. To the best of our knowledge, the use of this technique is new to auction theory and it will, perhaps, prove useful in other applications as well.

The results reported above concern a particular resale institution—monopoly pricing—in which the winner of the auction has all the bargaining power. In Section 2.6, we show, however, that this is inessential by first considering the monopsony mechanism in which the loser has all the bargaining power and then more generally, mechanisms in which bargaining power is shared, perhaps unequally. All of the results reported above extend to these alternative, and more general, resale institutions.

2.1.1 Related Literature

Equilibrium analysis of asymmetric first-price auctions has posed many challenges. Some of the difficulties were already pointed out by Vickrey (1961) in his pioneering paper. He constructed an example in which bidder 1's value, say a, was commonly known while the other's was uniformly distributed. In that case, there is an equilibrium of the first-price auction in which bidder 1 randomizes. Vickrey (1961) showed that for some values of a, the revenue from a first-price auction exceeded that from a second-price auction; for other values of a, the revenue ranking was reversed.

Since then, progress in the area has been sporadic at best. In asymmetric first-price auctions without resale, pure strategy equilibria exist under quite permissive conditions, as a consequence of general existence results (see, for instance, Reny (1999), Athey (2001) or Jackson and Swinkels (2005)). There is, again under weak conditions, a unique equilibrium (see, for instance, Maskin and Riley (2003) or Lebrun (2006)). But characterization results and revenue comparisons are few and far between. Griesmer, et al (1967) derive closed-form equilibrium bidding strategies in a first-price auction in which bidders draw values from uniform distributions, but over different supports. Plum (1992) extends this to situations in which the two value distributions are of the form x^n, again over different supports. Cheng (2005) identifies a class of distribution pairs which lead to linear bidding strategies. For this class, he shows that the first-price auction is revenue superior to the second-price auction. For specific examples of distribution pairs, Cantillon (2005) shows how asymmetry affects revenue in first-price auctions. In the absence of general analytic results, some researchers have resorted to numerical methods (Marshall, et al (1994)).

Maskin and Riley (2000) derive the most comprehensive characterization and revenue ranking results concerning first- and second-price auctions in the presence of asymmetries. They consider problems in which one of the bidders is unambiguously stronger than the other. Specifically, the distribution of one bidder (conditionally) stochastically dominates that of the other. Maskin and Riley (2000) are able to identify circumstances in which one or the other auction is revenue superior. For instance, the second-price auction is revenue superior if the distribution of the weak bidder is obtained from that of the strong bidder by reassigning probability mass toward lower values. Fibich, Gavious and Sela (2004) have shown that when the bidders are "nearly symmetric," the difference in revenues is of a smaller magnitude than the difference in the underlying distributions. Thus, for small asymmetries, the auctions are nearly revenue equivalent.

Gupta and Lebrun (1999) consider resale possibilities in a manner not unlike this paper. They assume, however, that at the end of the auction both *values* are announced. This means, of course, that resale is always efficient. But it is not clear how the auctioneer would come to know the values themselves. In contrast, in our model, the auctioneer knows only the bids and not the values. Haile (2003) considers resale possibilities in a *symmetric* model. At the time of bidding, however, buyers have only noisy information regarding their true values, which are revealed to them only after the auction. There is a motive for resale because although the winner of the auction may have received the highest signal, he may not have the highest true value. No general revenue ranking obtains. Zheng (2002) identifies conditions under which the outcomes of Myerson's (1981) optimal auction can be achieved when resale is permitted.

The model of Garratt and Tröger (2006) is closest to ours in spirit. The crucial difference is that they assume, as in Vickrey's (1961) example, that the value of one of the bidders is commonly known, and moreover, is equal to zero. This bidder, thus, participates in the auction for purely "speculative" reasons—he has no use value for the object. He only benefits if he can resell the object to the other bidder. In the efficient equilibrium of the second-price auction, the revenue is obviously zero. Garratt and Tröger (2006) show that there is a unique mixed strategy equilibrium in the first-price auction in which the revenue is positive. We allow for general continuous distributions and so their model may be viewed as a limiting case of ours.

2.2 Preliminaries

A single indivisible object is for sale. There are two risk-neutral buyers, labelled 1 and 2, with independently distributed private values, X_1 and X_2. Buyer i's value for the object, X_i, is distributed according to the cumulative distribution function F_i with support $[0, \omega_i]$. It is assumed that each F_i admits a continuous density $f_i \equiv F_i'$ and that this density is positive on $(0, \omega_i)$. We suppose, without loss of generality, that $\omega_1 \geq \omega_2$.

We assume that both F_i are *regular* in the sense of Myerson (1981) so that for $i = 1, 2$, the *virtual value*, defined as

$$x - \frac{1 - F_i(x)}{f_i(x)}$$

is a strictly increasing function of the actual value x. This ensures that the price at the resale stage is uniquely determined and is characterized by the first-order conditions for a maximum.[1]

In later sections we will need to consider conditional distributions of the form $F_i(x \mid X_i \leq a) = F_i(x) / F_i(a)$ with support $[0, a]$. The associated *conditional virtual value* is

[1] As shown by Bulow and Roberts (1989), the virtual value can be interpreted as the "marginal revenue" of a monopolist who faces a demand curve $1 - F_i(p)$.

$$x - \frac{1 - F_i(x \mid X_i \le a)}{f_i(x \mid X_i \le a)} = x - \frac{F_i(a) - F_i(x)}{f_i(x)}$$

The monotonicity of the virtual values implies the monotonicity of the conditional virtual values. To see this, note that if we write $q = F_i(a) \le 1$, then the latter is equivalent to the statement: if $x < x' \le a$ then

$$f_i(x') f_i(x) (x' - x) > f_i(x) [q - F_i(x')] - f_i(x') [q - F_i(x)]$$

If $f_i(x) < f_i(x')$, then the right-hand side is nonpositive for all q while the left-hand side is positive. If $f_i(x) \ge f_i(x')$, then the right-hand side is a nondecreasing function of q and since regularity implies that the inequality holds for $q = 1$, it holds for all $q < 1$. Thus, we have that for all $a < \omega_i$, the conditional virtual values are strictly increasing. In other words, if F_i is regular then the conditional distribution $F_i(\cdot \mid X_i \le a)$ is also regular.

2.3 First-Price Auction with Resale

Our model of the first-price auction with resale (FPAR) is the following. The buyers first participate in a standard sealed-bid first-price auction. The winning bid is publicly announced. We assume—as is common in real-world auctions—that the losing bid is *not* announced.[2]

In the second stage, the winner of the auction—say j—may, if he wishes to, offer to sell the object to the other bidder $i \neq j$ at some price p. If the offer is accepted by i, a sale ensues. If the offer is rejected, the original owner j retains the object. Thus resale takes place via a take-it-or-leave-it offer by the winner of the auction.[3]

[2] This assumption is discussed in more detail below in Remark 2.

[3] All bargaining power thus lies with the seller and from his perspective, this is, of course, the optimal resale mechanism. In Section 2.6 below, we show that our analysis extends to resale mechanisms in which all bargaining power lies with the buyer and then, more generally, to mechanisms in which it is shared.

2.3.1 Strategies and Beliefs

A *strategy* for bidder i in a first-price auction with resale has two components: (i) a *bidding strategy* $\beta_i : [0, \omega_i] \to \mathbb{R}$ that specifies the bid $\beta_i(x_i)$ that i will submit in the auction if his value is x_i; (ii) a *pricing strategy* $p_i(b_i, x_i)$ which specifies the asking price i will set if he wins the auction with a bid of b_i and his value is x_i. We allow i to set $p_i = \infty$, which is interpreted to be the same as a decision not to offer the object for sale.

In addition, i must specify the set of price offers $A_i(b_j, x_i)$ he will accept if j wins with a bid of b_j and his own value is x_i.

A *belief function* μ_i specifies a probability distribution $\mu_i(b_i, \cdot)$ over $[0, \omega_j]$ that represents the beliefs that i holds regarding j's values if he wins the auction with a bid of b_i.

2.3.2 Equilibrium

A *perfect Bayesian equilibrium* (henceforth, *equilibrium*) consists of a pair of bidding pricing strategies (β_i, p_i) and belief functions μ_i for $i = 1, 2$ with the property that: (i) if i loses the auction, then the set of offers he will accept is $A_i(b_j, x_i) = \{p_j : p_j \leq x_i\}$; (ii) if i wins the auction with a bid of b_i when his value is x_i, then $p_i(b_i, x_i)$ is optimal given $\mu_i(b_i, \cdot)$ and the acceptance strategy above; (iii) for each x_i, $\beta_i(x_i)$ is optimal given β_j, p_i and p_j; and (iv) the beliefs μ_i are generated from F_i and β_i using Bayes rule whenever possible.

Note that if i loses the auction, then the announcement of the winning bid b_j carries no useful information—that is, the set of price offers i will accept is independent of b_j. Thus the equilibrium would be unaffected if neither bid were announced.

As usual, we work backwards and first outline behavior in the resale stage.

2.3.3 Resale Stage

Suppose that the two bidders follow *continuous* and *strictly increasing* bidding strategies β_1 and β_2 with inverses ϕ_1 and ϕ_2, respectively.[4]

[4]We will show later that all equilibria must have these properties.

Suppose that bidder j with value x_j won the auction with a bid of b. As a result, he would infer that bidder i's value $X_i \leq \phi_i(b)$. His beliefs, therefore, are $\mu_j(b, x_i) = F_i(x_i \mid X_i \leq \phi_i(b)) = F_i(x_i)/F_i(\phi_i(b))$. If $x_j < \phi_i(b)$, then there are potential gains from trade and so bidder j will set a ("monopoly") price p that maximizes

$$[F_i(\phi_i(b)) - F_i(p)]p + F_i(p)x_j$$

The first term in the maximand is j's expected payoff from the event $X_i \geq p$ in which bidder i accepts his offer. The second term is his payoff from the event $X_i < p$, in which case bidder i rejects it.

The first-order condition for j's maximization problem can be rewritten as

$$p - \frac{F_i(\phi_i(b)) - F_i(p)}{f_i(p)} = x_j \tag{2.1}$$

Since F_i is regular, the left-hand side is increasing and so (2.1) has a unique solution. Moreover, (2.1) is also sufficient for j's maximization problem. Thus there is a unique price

$$p_j(b, x_j) \equiv \arg\max_p [F_i(\phi_i(b)) - F_i(p)]p + F_i(p)x_j \tag{2.2}$$

that maximizes j's payoff from resale and clearly, $x_j < p_j(b, x_j) < \phi_i(b)$. It follows immediately from (2.1) that the optimal price $p_j(b, x_j)$ is an increasing function of both b and x_j.

Let

$$R_j(b, x_j) = \max_p [F_i(\phi_i(b)) - F_i(p)]p + F_i(p)x_j \tag{2.3}$$

denote bidder j's optimal expected revenue from resale. For future reference, note that as a result of the envelope theorem,

$$\frac{\partial}{\partial b} R_j(b, x_j) = f_i(\phi_i(b))\, \phi_i'(b)\, p_j(b, x_j) \tag{2.4}$$

If bidder j won the auction with a bid of b and $x_j \geq \phi_i(b)$, then there are no potential gains from trade and so bidder j does not offer the object for sale.

2.3.4 Bidding Stage

We begin by deriving some necessary conditions that equilibrium bidding strategies must satisfy. At the time of the auction, both bidders anticipate that behavior in the resale stage will be as specified above.

2.3.4.1 Necessary Conditions

Suppose that, in equilibrium, each bidder i follows a *continuous* and *strictly increasing* bidding strategy $\beta_i : [0, \omega_i] \to \mathbb{R}$, so that $\beta_i (x_i)$ is the bid submitted by i when his value is x_i.

First, notice that we must have $\beta_1(0) = \beta_2(0) = 0$. Suppose that $\beta_i (0) > 0$ and without loss of generality, suppose that there is a sequence $x^n \downarrow 0$ such that $\beta_i (x^n) \geq \beta_j (x^n)$. For n large enough, x^n is less than $\beta_i (0)$ and if bidder i with value x^n wins and does not resell then his net gain is less than zero. If bidder i with value x^n wins and resells to bidder j, the price he will receive is also less than $\beta_i (0)$. This is because if he wins with bid of $\beta_i (x^n)$ then j's value $X_j \leq \beta_j^{-1} \beta_i (x^n)$ and by continuity, this is close to zero, when x^n is close to zero. Thus his gain from winning is less than $\beta_i (0)$ whether or not the object is resold. Moreover, if bidder i with value x^n loses, then $x^n \leq X_j$ and there will be no resale. Thus his payoff from losing is 0. As a result, his overall payoff is negative and it is better for bidder i with value x^n to bid 0. Hence, $\beta_i (0) > 0$ is not possible.

It is also easy to verify that $\beta_1 (\omega_1) = \beta_2 (\omega_2) \equiv \overline{b}$.

As above, let $\phi_i : \left[0, \overline{b}\right] \to [0, \omega_i]$ denote i's inverse bidding strategy in equilibrium, that is, $\phi_i = \beta_i^{-1}$. Fix a bid b and suppose that $\phi_j (b) < \phi_i (b)$. This means that if j wins with a bid of b, then there are potential gains from trade and so j will make an offer to i. If, on the other hand, i wins with bid of b, then there are no potential gains from trade and so i will not make an offer to j. Thus the bid b itself determines the direction the resale transaction, that is, the identities of the seller and the buyer.

Suppose bidder i follows ϕ_i. Bidder j's expected payoff when his value is $x_j \equiv \phi_j (b)$ and he deviates by bidding a c close to b is

$$\Pi_j (c, x_j) = R_j (c, x_j) - F_i (\phi_i (c)) c$$

where $R_j(c, x_j)$, defined in (2.3), is his expected payoff from resale if he wins the auction. If j loses the auction, then $\phi_j(c) < \phi_i(c)$ implies that bidder i will not offer to resell to him and so his payoff is 0. Since it is optimal for j to bid b, the first-order condition for maximizing Π_j, together with (2.4), results in

$$f_i(\phi_i(b))\,\phi_i'(b)\,p_j(b, x_j) - f_i(\phi_i(b))\,\phi_i'(b)\,b - F_i(\phi_i(b)) = 0$$

where $p_j(b, x_j)$ is defined in (2.2). Since $x_j = \phi_j(b)$, writing $p(b) \equiv p_j(b, \phi_j(b))$, the first-order condition results in the differential equation

$$\frac{d}{db}\ln F_i(\phi_i(b)) = \frac{1}{p(b) - b} \tag{2.5}$$

Note that p depends on both ϕ_1 and ϕ_2.

Now suppose bidder j follows an equilibrium strategy ϕ_j. Bidder i's expected payoff when his value is $x_i \equiv \phi_i(b)$ and he deviates by bidding a c close to b is

$$\Pi_i(c, x_i) = (x_i - c)F_j(\phi_j(c)) + \int_{\phi_j(c)}^{\omega_j} \max\{[x_i - p_j(\beta_j(x_j), x_j)], 0\}\, f_j(x_j)\, dx_j$$

This is because if i wins the auction, he never resells to j and so his profit is simply $x_i - c$. The second term is i's expected payoff from buying the object from j. Since it is optimal for i to bid b, the first-order condition for maximizing Π_i is

$$[x_i - b]\, f_j(\phi_j(b))\phi_j'(b) - F_j(\phi_j(b)) - [x_i - p_j(b, \phi_j(b))]\, f_j(\phi_j(b))\phi_j'(b) = 0$$

Again writing $p_j(b, \phi_j(b)) = p(b)$, the first-order condition becomes

$$\frac{d}{db}\ln F_j(\phi_j(b)) = \frac{1}{p(b) - b}$$

which is the *same* as (2.5).

We have argued that if ϕ_1, ϕ_2 are the equilibrium inverse bid functions in a first-price auction with resale, then they must both satisfy (2.5). This was derived using the first-order necessary conditions for local deviations to be unprofitable.[5]

[5] We have argued that the differential equations hold at any b such that $\phi_j(b) < \phi_i(b)$. If b is such that $\phi_j(b) = \phi_i(b)$, then whoever wins at that bid will set a price $p(b) = \phi_i(b) = \phi_j(b)$

2.3.4.2 Sufficiency

We now show that solutions to the differential equations (2.5) for $j = 1, 2$ are indeed equilibrium bidding strategies—that is, no deviations are profitable.

Proposition 4. *The strictly increasing and onto functions* $\phi_1 : \left[0, \overline{b}\right] \to [0, \omega_1]$ *and* $\phi_2 : \left[0, \overline{b}\right] \to [0, \omega_2]$ *are equilibrium inverse bidding strategies for the first-price auction with resale if and only if for all* $b \in \left[0, \overline{b}\right]$,

$$\frac{d}{db} \ln F_1 \left(\phi_1 \left(b\right)\right) = \frac{1}{p\left(b\right) - b} \tag{2.6}$$

$$\frac{d}{db} \ln F_2 \left(\phi_2 \left(b\right)\right) = \frac{1}{p\left(b\right) - b} \tag{2.7}$$

where, if $\phi_j \left(b\right) \leq \phi_i \left(b\right)$,

$$p\left(b\right) = \arg\max_p [F_i(\phi_i(b)) - F_i(p)]p + F_i(p)\phi_j\left(b\right) \tag{2.8}$$

Proof. See Appendix 2.8. ∎

Note that the boundary conditions are determined by the condition that the ϕ_i be strictly increasing and onto.

A word regarding the *equilibrium pricing function* $p\left(\cdot\right)$ and its relationship to the equilibrium inverse bidding strategies, ϕ_1 and ϕ_2, is in order. Suppose, as depicted in Figure 2.1, that $\phi_1 \left(b\right) < \phi_2 \left(b\right)$. Then if bidder 1 were to win with a bid of b, he would infer that $X_2 \leq \phi_2 \left(b\right)$ and since there are potential gains from trade, bidder 1 would make an offer $p\left(b\right)$ to bidder 2 ($j = 1$ and $i = 2$ in formula (2.8)). On the other hand, if bidder 2 were to win with a bid of b, he would infer that 1's value $X_1 \leq \phi_1 \left(b\right) < \phi_2 \left(b\right) = x_2$ and so would not make 1 an offer. But for bid b', $\phi_2 \left(b'\right) < \phi_1 \left(b'\right)$, again as in the figure, the opposite holds. Upon winning with a bid of b', bidder 2 would make an offer $p\left(b'\right)$ to bidder 1 ($j = 2$ and $i = 1$ in formula (2.8)) but not the other way around. The value of b thus uniquely determines the identity of the seller j who sets the price $p\left(b\right)$.

Recall that in any increasing equilibrium, the highest bids must be the same, say $\overline{b}$. Thus $F_1 \left(\phi_1 \left(\overline{b}\right)\right) = 1 = F_2 \left(\phi_2 \left(\overline{b}\right)\right)$. Since the boundary conditions for the two

and the same arguments as given above show that the differential equations still hold.

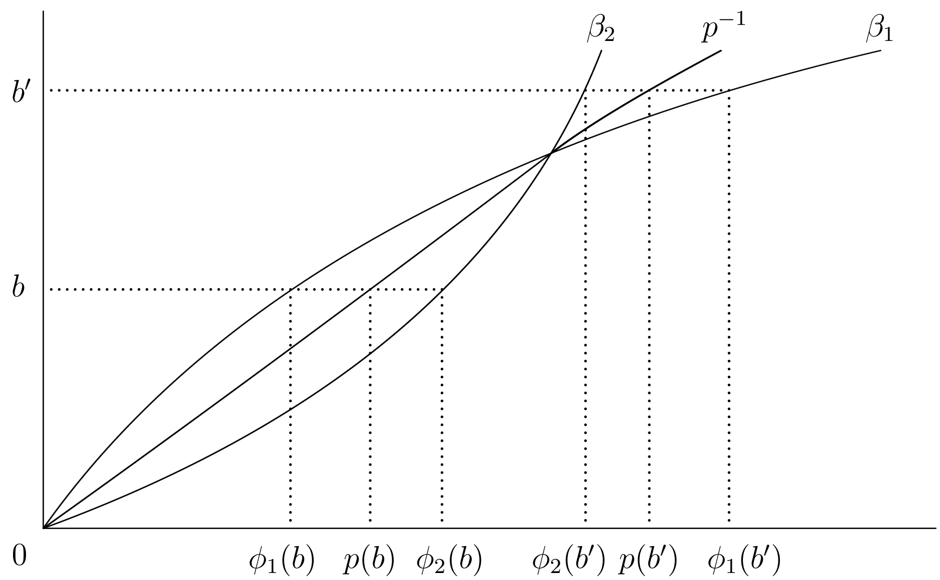

Figure 2.1. Bidding and Pricing Functions

differential equations are the same, it now follows immediately from Proposition 4 that

Corollary 3. *If* $\phi_1 : [0, \overline{b}] \to [0, \omega_1]$ *and* $\phi_2 : [0, \overline{b}] \to [0, \omega_2]$ *are strictly increasing equilibrium inverse bidding strategies, then for all* $b \in [0, \overline{b}]$,

$$F_1(\phi_1(b)) = F_2(\phi_2(b))$$

that is, the bid distributions of the two bidders are identical.

The equality of the bid distributions implies that for all $x \in [0, \omega_2]$, $F_i(x) \leq F_j(x)$ if and only if $\beta_i(x) \leq \beta_j(x)$. In particular, the bidding functions intersect at x if and only if the distributions intersect at x.

2.3.4.3 Symmetrization

Corollary 3 identifies a remarkable property of first-price auctions with resale—even though the bidders are asymmetric, in equilibrium they bid in a way that the resulting bid distributions $F_i(\phi_i(\cdot))$ are the same. In this sense, resale *symmetrizes* the auction. Since this property plays an important role in what follows, it is worth exploring the underlying reasons.[6]

As a first step, consider a standard first-price auction *without resale* (FPA) and let φ_1 and φ_2 be the equilibrium inverse bidding strategies. Suppose bidder i with value $x_i = \varphi_i(b)$ raises his bid slightly from b to $b+\varepsilon$. This make a difference only against the types of bidder j to whom bidder i loses the auction by bidding b but wins by bidding $b+\varepsilon$. By doing this bidder i gains approximately $x_i - b = \varphi_i(b) - b$ whenever $\varphi_j(b) < x_j < \varphi_j(b+\varepsilon)$. Writing the first-order condition for optimality yields the pair of differential equations: for $j = 1, 2$ and $i \neq j$,

$$\frac{d}{db} \ln F_j(\varphi_j(b)) = \frac{1}{\varphi_i(b) - b} \tag{2.9}$$

Notice that the right-hand side is the inverse of the marginal gain accruing to i from increasing his bid.

Now consider a first-price auction *with resale* (FPAR) with equilibrium inverse bidding strategies ϕ_1 and ϕ_2. Let b be such that $\phi_j(b) < \phi_i(b)$. This means that in equilibrium if j wins with a bid of b, so that his value $x_j = \phi_j(b)$, then he will try to resell the object to bidder i since there are potential gains from trade. On the other hand, if i wins with a bid of b, he will not resell the object to bidder j since there are no gains from trade.

Suppose bidder j with value $x_j = \phi_j(b)$ raises his bid slightly from b to $b+\varepsilon$. As before, we look at how much j gains against bidder i types such that $\phi_i(b) < x_i < \phi_i(b+\varepsilon)$. When he bids b, bidder j loses against these types of bidder i and since there is no resale, bidder j's payoff is 0. When he bids $b+\varepsilon$, however, he wins against these types of bidder i and is able to resell to them at a price of $p(b)$ for a gain of $p(b) - b$.

What about bidder i? Suppose bidder i with value $x_i = \phi_i(b)$ raises his bid slightly from b to $b+\varepsilon$ and again consider the benefit to i against those bidder j types such that $\phi_j(b) < x_j < \phi_j(b+\varepsilon)$. When he bids b, bidder i loses against these types of bidder j but is able to buy the object from them at a price of approximately $p(b)$. His payoff thus approximately equals $x_i - p(b)$. When he

[6] Gupta and Lebrun (1998) allude to this kind of symmetry in passing although the main thrust of their analysis is in the context of a different model—one in which values are announced at the end of the auction.

bids $b+\varepsilon$, he wins against these types of bidder j and so his payoff is $x_i - b$. The gain in payoff for i from increasing his bid from b to $b+\varepsilon$ is thus approximately equal to $(x_i - b) - (x_i - p(b)) = p(b) - b$, the same as i's gain!

In contrast to (3.2), the right-hand sides of (2.6) and (2.7) are identical.

The symmetrization effects of resale come from the fact that the *marginal gain* to both bidders from a higher bid is the same: $p(b) - b$. For bidder j (the "seller"), the marginal gain is just the profit from resale, that is, $p(b) - b$. For bidder i, (the "buyer"), the marginal gain is the difference in the "retail price" $p(b)$ he pays when he loses the auction but buys from bidder j and the "wholesale price" b that he pays when he wins the auction and buys directly from the auctioneer.

The distributions of equilibrium bids in an asymmetric first-price auction with resale are thus observationally equivalent to the distribution of equilibrium bids in a symmetric first-price auction. In other words, given F_1 and F_2, there exists a distribution F such that a first-price auction (FPA) in which both bidders draw values from F is equivalent, in terms of equilibrium bid distributions, to a first-price auction with resale (FPAR) in which bidders draw values from F_1 and F_2, respectively. This also means that the two auctions are revenue equivalent.

We now show how F may be obtained from F_1 and F_2.

Lemma 1. *Given distributions F_1 and F_2, define F as follows: if $F_i(p) \leq F_j(p)$, then*

$$F(p) = F_j\left(p - \frac{F(p) - F_i(p)}{f_i(p)}\right) \tag{2.10}$$

Then F is a uniquely determined distribution function such that $F_i(p) \leq F(p) \leq F_j(p)$. Moreover, if $F_i(p) < F_j(p)$, then $F_i(p) < F(p) < F_j(p)$.

Proof. See Appendix 2.8. ∎

The construction of F has a simple geometric interpretation, as depicted in Figure 2.2. For the point p, $F_2(p) < F_1(p)$ and so $i = 2$ and $j = 1$ in the formula (3.6). The distribution F is such that it passes through the point b, which bisects the line segment ac. The length of the line segment ab is just $p - F_1^{-1}(F(p))$. And since bd/bc is $f_2(p)$, the slope of F_2 at p, the length of bc is $[F(p) - F_2(p)]/f_2(p)$. Equation (3.6) requires that these be equal.

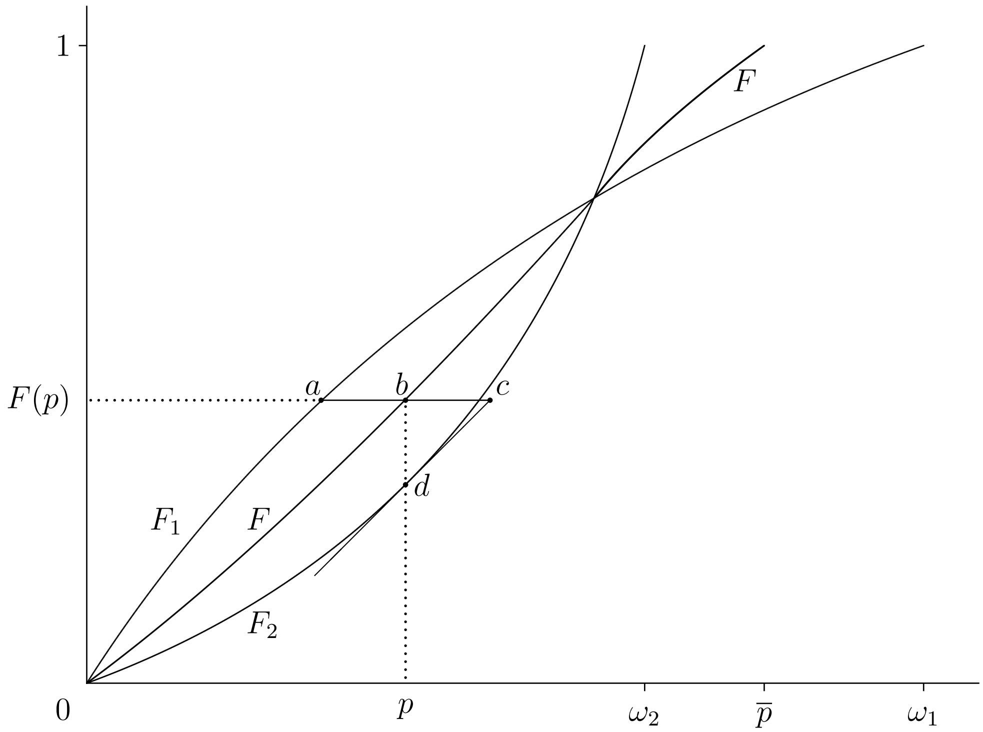

Figure 2.2. Construction of F

2.3.4.4 Equivalent Symmetric Auction

Now consider a *symmetric* first-price auction without resale in which there are two bidders and both draw values independently from the distribution function F on $[0, \overline{p}]$ as defined above in (3.6).

The equilibrium strategies in a symmetric auction can, of course, be derived explicitly and are given by

$$\beta(x) = \frac{1}{F(x)} \int_0^x y f(y)\, dy$$

Let $\overline{b} = \beta(\overline{p})$. Define the equilibrium inverse bid function for the symmetric auction as

$$\phi(b) \equiv \beta^{-1}(b) \tag{2.11}$$

so that the distribution of bids for each bidder is $F(\phi(b))$. A necessary condition for ϕ to be the equilibrium inverse bidding strategy in the symmetric auction is that

$$\frac{d}{db}\ln F\left(\phi\left(b\right)\right)=\frac{1}{\phi\left(b\right)-b} \tag{2.12}$$

2.3.5 Existence and Uniqueness of Equilibrium

In this section, we establish that the first-price auction with resale has a pure strategy equilibrium in which each bidder follows a strictly increasing bidding strategy. The equilibrium is unique in the class of pure strategy equilibria with nondecreasing bidding strategies.

The proof that there is a strictly increasing equilibrium is constructive. Given regular distribution functions F_1 and F_2, construct F as in Lemma 1. Consider a symmetric first-price auction in which each bidder draws values independently from F. In symmetric auctions, it is known that a symmetric equilibrium β exists and is strictly increasing. We will use the equilibrium β to construct equilibrium bidding strategies β_1 and β_2 for the asymmetric first-price auction with resale.

2.3.5.1 Existence of Equilibrium

Theorem 1. *Suppose F_1 and F_2 are regular. Then there exists an equilibrium in the first-price auction with resale in which the bidding strategies are strictly increasing.*

Proof. The proof is by construction.

Given F_1 and F_2, let F be determined as in the statement of Lemma 1. Let ϕ, as defined above in (2.11), be the equilibrium inverse bidding strategy in the symmetric auction in which bidders draw values from F. Let $\bar{b}$ be the maximum bid in the symmetric auction and define inverse bidding strategies $\phi_1:\left[0,\bar{b}\right]\rightarrow[0,\omega_1]$ and $\phi_2:\left[0,\bar{b}\right]\rightarrow[0,\omega_2]$ in the asymmetric first-price with resale as follows:

$$F_1\left(\phi_1\left(b\right)\right) = F(\phi\left(b\right)) \tag{2.13}$$

$$F_2\left(\phi_2\left(b\right)\right) = F(\phi\left(b\right)) \tag{2.14}$$

Then using (2.12), we have that for $i=1,2$

$$\frac{d}{db}\ln F_i\left(\phi_i\left(b\right)\right)=\frac{1}{\phi\left(b\right)-b}$$

We claim that ϕ_1 and ϕ_2 are equilibrium inverse bidding strategies in the first-price auction with resale.

The definition of F in (3.6) implies that if $F_i(\phi(b)) < F_j(\phi(b))$, then

$$F_j^{-1}(F(\phi(b))) = \phi(b) - \frac{F(\phi(b)) - F_i(\phi(b))}{f_i(\phi(b))}$$

and since $F(\phi(b)) = F_j(\phi_j(b)) = F_i(\phi_i(b))$,

$$\phi_j(b) = \phi(b) - \frac{F_i(\phi_i(b)) - F_i(\phi(b))}{f_i(\phi(b))}$$

which is precisely the first-order condition for

$$p(b) = \arg\max_p [F_i(\phi_i(b)) - F_i(p)]p + F_i(p)\phi_j(b)$$

Regularity implies that the first-order condition is both necessary and sufficient for a maximum. Thus we have that for all b

$$p(b) = \phi(b)$$

Finally, note that $F_i(p(b)) < F_j(p(b))$ is equivalent to $\phi_j(b) < \phi_i(b)$. This is because (2.13) and (2.14) imply that $F_i(p(b)) < F(p(b))$, which is equivalent to $F_i(p(b)) < F_i(\phi_i(b))$ and so also to $p(b) < \phi_i(b)$. Similarly, $F(p(b)) < F_j(p(b))$ is equivalent to $F_j(\phi_j(b)) < F_j(p(b))$ and so also to $\phi_j(b) < p(b)$. Thus, $F_i(p(b)) < F_j(p(b))$ if and only if $\phi_j(b) < \phi_i(b)$.

We have thus argued that if ϕ_1 and ϕ_2 are determined by (2.13) and (2.14), then they satisfy the differential equations (2.6) and (2.7) where $p(b)$ is determined by (2.8).

Thus as constructed, the functions ϕ_1 and ϕ_2 satisfy the conditions of Proposition 4 and so constitute equilibrium inverse bidding strategies. ∎

Remark 2. Theorem 1 relies on the assumption that at the end of the auction, the losing bid is not announced. If the losing bid is announced, the value of the losing bidder would be revealed in any strictly increasing equilibrium. This creates an incentive for a bidder to bid lower, so that if he were to lose, then the other bidder would think that his value is smaller than it actually is. This effect overwhelms the loss from not winning with a lower bid and it is known that no strictly increasing equilibrium exists (Krishna, 2002, Chapter 4). In fact, a stronger result holds: if the losing bid is announced, there is no nondecreasing equilibrium with (partial) pooling either.

2.3.5.2 Uniqueness of Equilibrium

In this section we show that the equilibrium constructed in Theorem 1 is, in fact, the only equilibrium in which bidders follow *nondecreasing* bidding strategies. We first show that it is unique in the class of equilibria with strictly increasing strategies. The proof is completed by showing that if both equilibrium bidding strategies are nondecreasing, then they must be strictly increasing and continuous.

The following proposition is a key step. It shows that the distribution of resale prices in any equilibrium is given by F. Since F is determined without reference to the equilibrium, this shows that the distribution of resale prices is uniquely determined.

Proposition 5. *Suppose ϕ_1 and ϕ_2 are strictly increasing equilibrium inverse bidding strategies. Then F, defined in* (3.6)*, is the distribution of equilibrium resale prices and for all b, $F(p(b)) = F_j(\phi_j(b))$, $j = 1, 2$.*

Proof. Let the random variable P denote the resale price resulting from ϕ_1 and ϕ_2. Fix a b such that $\phi_j(b) < \phi_i(b)$ and suppose bidder j wins the auction with a bid of b. Then we have.

$$\begin{aligned}\Pr[P \leq p(b)] &= \Pr[X_j \leq \phi_j(b)] \\ &= F_j(\phi_j(b))\end{aligned}$$

Because in equilibrium, for all b, $F_1(\phi_1(b)) = F_2(\phi_2(b))$, it is also the case that $\Pr[P \leq p(b)] = F_i(\phi_i(b))$. And since $p(b)$ is the monopoly resale price when the winning bid is b, it must satisfy the first-order condition

$$\phi_j(b) = p(b) - \frac{F_i(\phi_i(b)) - F_i(p(b))}{f_i(p(b))}$$

So

$$\begin{aligned}\Pr\left[P \leq p\left(b\right)\right] &= F_j\left(\phi_j\left(b\right)\right) \\ &= F_j\left(p\left(b\right) - \frac{\Pr\left[P \leq p\left(b\right)\right] - F_i\left(p\left(b\right)\right)}{f_i\left(p\left(b\right)\right)}\right)\end{aligned}$$

where we have used the fact that $\Pr\left[P \leq p\left(b\right)\right] = F_i\left(\phi_i\left(b\right)\right)$ also. Now (3.6) implies that

$$\Pr\left[P \leq p\left(b\right)\right] = F\left(p\left(b\right)\right)$$

and this completes the proof. ∎

Since $F\left(p\left(b\right)\right) = F_j\left(\phi_j\left(b\right)\right)$, Proposition 4 implies that

$$\frac{d}{db}\left(F\left(p\left(b\right)\right)\right) = \frac{1}{p\left(b\right) - b}$$

This means that $p\left(b\right)$ satisfies (2.12), the differential equation characterizing the equilibrium inverse bidding strategy in the symmetric first-price auction in which both bidders draw values independently from the same distribution F. But given F, the equilibrium inverse bidding strategy in a symmetric first-price auction is uniquely determined.

We have thus argued that given F_1 and F_2, the equilibrium distribution of resale prices, F, and the function $p\left(b\right)$ are uniquely determined. Now it follows from $F_i\left(\phi_i\left(b\right)\right) = F\left(p\left(b\right)\right)$ that ϕ_1 and ϕ_2 are uniquely determined. Thus there is only one equilibrium in which bidders follow *strictly increasing* bidding strategies.

In Appendix 2.8 it is shown that there are no equilibria with nondecreasing strategies that are not strictly increasing. Thus, we obtain

Theorem 2. *The first-price auction with resale has a unique equilibrium in the class of equilibria with nondecreasing bidding strategies.*

2.3.6 An Example

It is useful to consider an explicit example to illustrate the various constructs.

Example 5. *Suppose that* $F_1(x) = x/\omega_1$ *over* $[0, \omega_1]$ *and* $F_2(x) = x/\omega_2$ *over* $[0, \omega_2]$ *where* $\omega_1 \geq \omega_2$*; that is, the value distributions are both uniform, but over different supports.*

Since $F_1 \leq F_2$, it can only be that (the "weak") bidder 2 resells to (the "strong") bidder 1. It may be verified that, the distribution of resale prices, F, is also uniform. Specifically, $F(p) = 2p/(\omega_1 + \omega_2)$ over support $\left[0, \frac{1}{2}(\omega_1 + \omega_2)\right]$. The associated pricing function $p(b) = 2b$. The equilibrium inverse bidding strategies are: $\phi_1(b) = 4\omega_1 b/(\omega_1 + \omega_2)$ and $\phi_2(b) = 4\omega_2 b/(\omega_1 + \omega_2)$. The highest bid $\overline{b} = \frac{1}{4}(\omega_1 + \omega_2)$.

Notice that if $3\omega_2 < \omega_1$, then $\phi_2(b) < b$, or equivalently, $\beta_2(x) > x$; that is, bidder 2 bids more than his value in a first-price auction with resale. The reason, of course, is that he anticipates being able to resell the object to bidder 1 for a profit. Thus, bidder 2's motives have a substantial "speculative" component. The model of Garratt and Tröger (2006), in which one of the bidders is known to have a value of 0, is an extreme instance of this. There speculation is the only motive.

2.3.7 Revenue

The auxiliary symmetric first-price auction in which both bidders draw values from F has the same distribution of bids as the first-price auction with resale in which bidders draw values from F_1 and F_2, respectively. This is because $p(\cdot)$ is the equilibrium inverse bidding strategy in the auxiliary auction and from Lemma 5, for all $b \in \left[0, \overline{b}\right]$, $F(p(b)) = F_j(\phi_j(b))$.

Hence, in equilibrium, the expected revenue accruing to the auctioneer from a first-price auction with resale (FPAR) is

$$\begin{aligned} R^{FPAR}(F_1, F_2) &= R^{FPA}(F, F) \\ &= R^{SPA}(F, F) \\ &= \int_0^{\overline{p}} (1 - F(p))^2 \, dp \end{aligned} \tag{2.15}$$

where F is defined in (3.6) and $R^{SPA}(F, F)$ denotes the revenue from a symmetric second-price auction (SPA). The second equality is a consequence of the revenue equivalence principle. The third equality is a well-known formula for the expectation of the minimum of two independent random variables, both of which are distributed according to F.

Note that the expected revenue from a first-price auction with resale can thus be calculated without direct reference to the equilibrium bidding strategies.

For the asymmetric uniform distributions in Example 5, we have

$$R^{FPAR}(F_1, F_2) = \frac{1}{6}(\omega_1 + \omega_2)$$

2.4 Second-Price Auction with Resale

We now study properties of the second-price auction with resale. Our model is the same as that in previous sections except for the change in the auction format—that is, there is a second-price auction and then the winner, if he so wishes, can resell the object to the other bidder via a take-it-or-leave-it offer. There is one important difference, however. Under second-price rules, the winner of the auction inevitably knows the *losing* bid—after all this is the price he pays in the auction. Thus, unlike in a first-price auction, the winner can condition the price offered in the resale stage on the losing bid.[7] This, of course, considerably simplifies the inference problem faced by a winning bidder and puts the losing bidder in a weak position during resale.

2.4.1 Resale Stage

Suppose bidder i follows a nondecreasing bidding strategy β_i in the auction. Suppose also that bidder j wins the auction and pays a price of b_i which is in the range of β_i; that is, i's bid. He then infers that bidder i's value is in the set $\beta_i^{-1}(b_i) = \{x : \beta_i(x) = b_i\}$.

If $\beta_i^{-1}(b_i)$ is a singleton, say $\beta_i^{-1}(b_i) = \{x_i\}$, then it is optimal for j to offer the object to i only if $x_j < x_i$ and in that case, set a price $p = x_i$. If $\beta_i^{-1}(b_i)$ is an interval, then it is optimal for j to offer the object to i only if $x_j < \sup \beta_i^{-1}(b_i)$

[7] Recall that a first-price auction with resale does not have a monotonic equilibrium if the losing bid is known to the winner. See Remark 2.

and in that case, set a price $p_j(b_i, x_j)$ that maximizes

$$\left[F_i\left(\sup \beta_i^{-1}(b_i)\right) - F_i(p)\right]p + \left[F_i(p) - F_i\left(\inf \beta_i^{-1}(b_i)\right)\right]x_j$$

2.4.2 Bidding Stage

With private values, a standard second-price auction—without the possibility of resale—has some important and well-known features. First, it is a *weakly dominant* strategy for each bidder to bid his true value. Second, the resulting equilibrium is, of course, *efficient*, even in an asymmetric environment. Third, there is a continuum of other (inefficient) equilibria (see Blume and Heidhues (2004) for a complete classification).

Our first observation is that once there is the possibility of resale, it is *not* a weakly dominant strategy to bid one's value in a second-price auction. As the example below shows, if one of the bidders, say 1, bids more than his value, the other bidder may gain by bidding less than his value. This is because a lower bid in the auction may lead to a lower resale price.

Example 6. *The values* $X_1, X_2 \in [0,1]$. *Suppose that bidder* 1 *bids according to a continuous and strictly increasing strategy* β_1 *such that* $\beta_1(x) > x$, *for all* $x \in (0,1)$, *and, if he wins, has beliefs* $\mu_1(b_2, x) = 1$ *if* $x = b_2$ *and* 0 *otherwise; that is,* 1 *believes that* 2 *is following the strategy* $\beta_2(x_2) = x_2$.

Suppose that bidder 1 has value $x_1 \in (0,1)$ and bidder 2's value x_2 is such that $x_1 < x_2 < \beta_1(x_1)$. If bidder 2 bids x_2, then 1 will win the auction and will offer the object to 2 at price $p = x_2$. So bidder 2's payoff from bidding his value is 0. If bidder 2 reduces his bid to a b_2 such that $x_1 < b_2 < x_2$, then again bidder 1 will win the auction but now offer to sell the object to 2 at price $p = b_2$. By accepting this offer, bidder 2 can make a profit of $x_2 - b_2$. Thus in this situation it is strictly better for bidder 2 to bid $b_2 < x_2$ than to bid x_2.

2.4.3 Robust Equilibrium

While not weakly dominant, if both bidders bid their values and the winner prices optimally, then this nevertheless results in an equilibrium of the second-price auction with resale. In fact, it constitutes a *robust equilibrium*—that is, the proposed strategies constitute a perfect Bayesian for all distributions F_1, F_2 of values that are strictly increasing and continuous.[8]

The property that such equilibria are "distribution independent" makes them particularly attractive. Of course, every weakly dominant equilibrium is a robust equilibrium.

Proposition 6. *There is a robust equilibrium of the second-price auction with resale in which both bidders bid their values.*

Proof. Consider the following strategies. In the auction, each bidder bids his value; that is, $\beta_i(x_i) = x_i$. After the auction, the winner i believes that j's value $X_j = b_j$ and offers to sell at a price $p_i = b_j$ if and only if $b_j > x_i$; the loser responds optimally to the price offer, if any.

Suppose bidder i follows the strategy outlined above.

Suppose bidder j deviates and bids $b < x_j$. If $x_i < b < x_j$, bidder j wins for a price of x_i and so there is no resale. So his payoff is $x_j - x_i$ which is the same as if he bid x_j. If $b < x_i < x_j$, then i wins and j's payoff is zero since again there is no offer of resale. So if $b < x_i < x_j$, j's payoff is zero if he bids x and $x_j - x_i$ if he bids x_j. Finally, if $b < x_j < x_i$, bidder j loses the auction and his payoff is 0 whether he bids b or x_j. Thus underbidding is not profitable.

Now suppose bidder j bids $b > x_j$. If $x_i > b$, then j's payoff is 0 since he loses and i will not resell to him. If $b > x_i > x_j$, then again his payoff is zero, because he will pay x_i for the object and then resell to i for X_i. If $b > x_j > x_i$, then it makes no difference whether he bids b or x_j. Thus overbidding is not profitable either.

We have thus argued it is a best response for bidder j to follow the strategy $\beta_j(x) = x$, also. The optimality of the proposed strategies in the resale stage is clear.

None of the arguments above use any properties of the particular value distributions. Thus the proposed strategies constitute a robust equilibrium. ∎

[8]In normal form games, a robust equilibrium also has the ex post property—players do not suffer any regret if after the game is played, all private information is made public. Börgers and McQuade (2006) have pointed out that this equivalence does not hold in extensive form games because of the possibility of ex post regret after a deviation. They have also shown that a SPAR does not have an ex post equilibrium. Their comments helped us correct a claim we made in an earlier version of this paper.

2.4.4 Uniqueness

In a second-price auction (SPA) without resale, it is a dominant strategy for each bidder to bid his value. This, of course, is also a robust equilibrium. But in a SPA there are also *other* robust equilibria.

Example 7. *The values* $X_1, X_2 \in [0,1]$. *Bidder* 1 *bids according to the strategy* $\beta_1(x) = 1$ *and bidder* 2 *according to* $\beta_2(x) = 0$.

These strategies, while weakly dominated, nevertheless constitute a robust equilibrium. Thus in the second-price auction without resale, there is a multiplicity of robust equilibria.

When there is resale, however, there is (essentially) a unique robust equilibrium—both bidders bid their values and since the resulting allocation is efficient, there is no resale.

Proposition 7. *Suppose* β_1 *and* β_2 *are bidding strategies in a robust equilibrium of the second-price auction with resale. Then for all* $x \in [0, \omega_2]$, $\beta_1(x) = \beta_2(x) = x$.

Proof. See Appendix 2.9. ∎

If $\omega_1 > \omega_2$, then there are robust equilibria in which for $x > \omega_2$, bidder 1 bids more than x. Since bidder 1 wins for sure in these cases, all such equilibria are outcome equivalent to the one in which for all x, bidder 1 bids his value.

In what follows, we restrict attention to the unique robust equilibrium outcome of the second-price auction with resale in which bidders bid their values and the outcome is efficient. Henceforth, we will refer to this as the *efficient* equilibrium.

2.4.5 Revenue

The expected revenue from the efficient equilibrium of a second-price auction with resale (SPAR) is

$$R^{SPAR}(F_1, F_2) = \int_0^{\omega_2} (1 - F_1(x))(1 - F_2(x))\, dx \tag{2.16}$$

The right-hand side of the formula above is simply $E[\min\{X_1, X_2\}]$.

It is easily verified that for the asymmetric uniform distributions in Example 5, we have

$$R^{SPAR}(F_1, F_2) = \frac{\omega_2 (3\omega_1 - \omega_2)}{6\omega_1}$$

and thus,

$$R^{FPAR}(F_1, F_2) - R^{SPAR}(F_1, F_2) = \frac{(\omega_1 - \omega_2)^2}{6\omega_1}$$

and this is positive as long as $\omega_1 > \omega_2$.

In the next section, we show that the revenue superiority of the first-price auction over its second-price counterpart is general.

2.5 Revenue Comparison

In this section we establish our main result.

Theorem 3. *The seller's revenue from a first-price auction with resale is at least as great as that from a second-price auction with resale.*

Before proving Theorem 3, some preliminaries are in order.

2.5.1 Calculus of Variations

In what follows, we will make use of a simple technique from the calculus of variations that is used to derive the *Euler equation.* (See, for instance, Section 3 in Kamien and Schwartz, 1981).

Consider the integral

$$\Delta = \int_0^{\overline{p}} \Phi(p, M(p), m(p))\, dp$$

where $M : [0, \overline{p}] \to \mathbb{R}$ and $m(p) = M'(p)$. Suppose $Z(p) : [0, \overline{p}] \to \mathbb{R}$ is a *variation* satisfying $Z(0) = Z(\overline{p}) = 0$ and let $z(p) \equiv Z'(p)$. Define

$$\Delta(\varepsilon) = \int_0^{\overline{p}} \Phi(p, M + \varepsilon Z, m + \varepsilon z)\, dp$$

to be the value of the integral when M is perturbed by εZ. Differentiating with respect to ε,

$$\Delta'(\varepsilon) = \int_0^{\overline{p}} [\Phi_M Z + \Phi_m z] \, dp$$

where $\Phi_M \equiv \partial\Phi/\partial M$ and $\Phi_m \equiv \partial\Phi/\partial m$. Integrating by parts,

$$\int_0^{\overline{p}} \Phi_m z dp = \Phi_m Z\Big|_0^{\overline{p}} - \int_0^{\overline{p}} \frac{d}{dp}(\Phi_m) Z dp = -\int_0^{\overline{p}} \frac{d}{dp}(\Phi_m) Z dp$$

since $Z(0) = Z(\overline{p}) = 0$.

Thus we obtain,

$$\Delta'(0) = \int_0^{\overline{p}} \left[\Phi_M - \frac{d}{dp}(\Phi_m)\right] Z dp \tag{2.17}$$

2.5.2 Notation

In the proof below it is convenient to reformulate the problem in terms of the *decumulative* distribution functions. Let

$$H_1(p) \equiv 1 - F_1(p), \;\; H_2(p) \equiv 1 - F_2(p) \;\text{ and }\; H(p) \equiv 1 - F(p)$$

and

$$h_1(p) = H_1'(p), \;\; h_2(p) = H_2'(p) \;\text{ and }\; h(p) = H'(p)$$

Notice that in terms of decumulative functions, (3.6) can be rewritten as: if $H_i(p) \geq H_j(p)$, then

$$H(p) = H_j\left(p - \frac{H(p) - H_i(p)}{h_i(p)}\right)$$

or equivalently,

$$H(p) = H_i(p) + \left[p - H_j^{-1} H(p)\right] h_i(p) \tag{2.18}$$

The regularity assumption; that is, the monotonicity of virtual values is equivalent to

$$2h_i(p)^2 - H_i(p) h_i'(p) > 0 \tag{2.19}$$

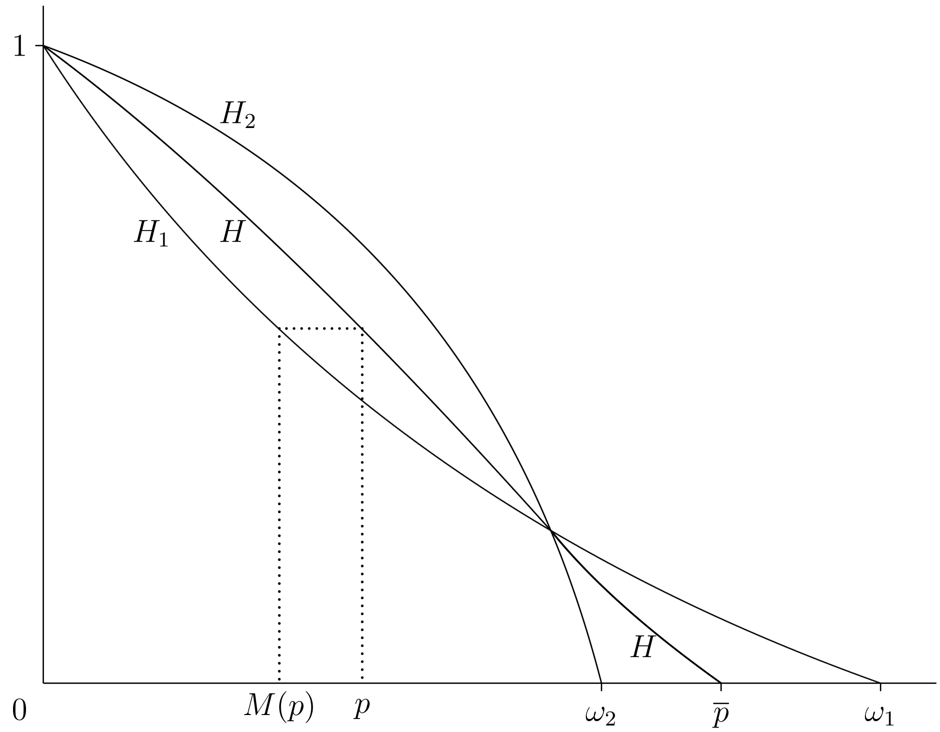

Figure 2.3. The Function M

2.5.3 Proof of Theorem 3.

In terms of the decumulative functions, the difference between the revenue from a FPAR (2.15) and the revenue from a SPAR (2.16) is

$$\Delta = \int_0^{\overline{p}} H(p)^2 \, dp - \int_0^{\omega_2} H_i(x) \, H_j(x) \, dx$$

Define the function $M : [0, \overline{p}] \to [0, \omega_2]$ as follows: if $H_i(p) \geq H_j(p)$, then

$$M(p) = H_j^{-1}(H(p))$$

M is an increasing function satisfying $M(0) = 0$, $M(\overline{p}) = \omega_2$ and since for all p, $H_i(p) \geq H(p) \geq H_j(p)$ it is the case that $M(p) \leq p$. (See Figure 2.3.) Define $m(p) = M'(p)$ wherever this is well defined. By changing the variable of integration from $x \in [0, \omega_2]$ to $p = M^{-1}(x) \in [0, \overline{p}]$ in the second integral in the expression for Δ, we obtain

$$\begin{aligned}\Delta &= \int_0^{\overline{p}} H(p)^2\,dp - \int_0^{\overline{p}} H_i(M(p))\,H_j(M(p))\,m(p)\,dp \\ &= \int_0^{\overline{p}} \left[H(p)^2 - H_i(M(p))\,H(p)\,m(p)\right] dp\end{aligned}$$

where $H(p)$ is given by

$$H(p) = H_i(p) + [p - M(p)]\,h_i(p) \tag{2.20}$$

which, in turn, is directly obtained from (2.18).

Lemma 1 implies that any H_i and H_j uniquely determine an H satisfying (2.18) and this in turn uniquely determines an M. Conversely, given an H_i and an M, (2.20) determines a unique H and this in turn determines a unique H_j since $H_j(x) = H(M^{-1}(x))$.

Note that $H_i = H_j$ if and only if $M(p) = p$ for all p, and in that case, $\Delta = 0$. We will argue that, in general; that is, for all H_i and H_j, $\Delta \geq 0$.

Consider the integrand in the expression for Δ, that is, the function[9]

$$\Phi(p, M, m) = H^2 - H_i(M)\,Hm$$

and the perturbation $Z(p) \equiv p - M(p) \geq 0$. Then define

$$\Delta(\varepsilon) = \int_0^{\overline{p}} \Phi(p, M + \varepsilon Z, m + \varepsilon z)\,dp$$

to be the revenue difference when M is perturbed slightly in the direction of p. (Figure 2.4 shows how a variation of M in the direction of p moves both H_j and H closer to H_i; that is, in the direction of increased symmetry.)

We now use (2.17) to evaluate $\Delta'(0)$. For this we need

$$\begin{aligned}\Phi_M &= 2H\frac{\partial H}{\partial M} - h_i(M)\,Hm - H_i(M)\,\frac{\partial H}{\partial M}m \\ &= -2Hh_i - h_i(M)\,Hm + H_i(M)\,h_i m \qquad (2.21)\end{aligned}$$

[9] To ease the notational burden, we suppress the argument p for the remainder of the proof.

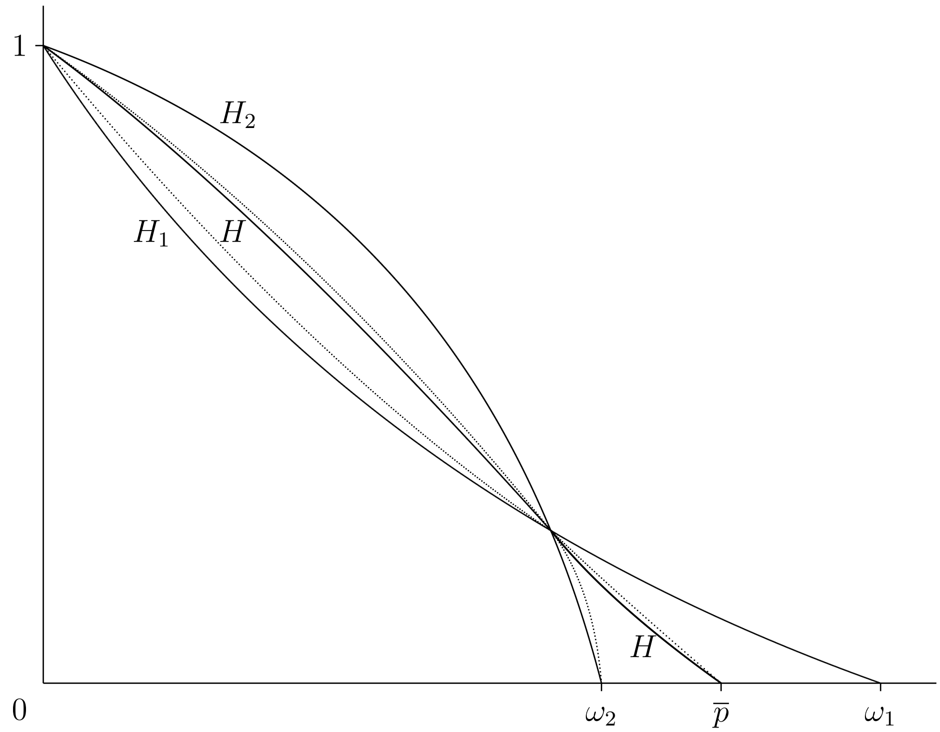

Figure 2.4. Variation of M

using $\partial H/\partial M = -h_i$, derived from (2.20).

And since,

$$\Phi_m = -H_i(M)\,H$$

we have

$$\frac{d}{dp}(\Phi_m) = -h_i(M)\,Hm - H_i(M)\,h \tag{2.22}$$

Now (2.21) and (2.22) result in

$$\begin{aligned}
\Phi_M - \frac{d}{dp}(\Phi_m) &= -2Hh_i + H_i(M)\,h_i m + H_i(M)\,h \\
&= -2Hh_i + H_i(M)\,h_i m + H_i(M)\left[2h_i + [p-M]\,h_i' - h_i m\right] \\
&= 2\left[H_i(M) - H\right]h_i + H_i(M)\,[p-M]\,h_i' \\
&= H_i(M)\left(2\left[1 - \frac{H}{H_i(M)}\right]h_i - \left[\frac{H_i - H}{h_i}\right]h_i'\right)
\end{aligned}$$

where the second equality is obtained by substituting $h = 2h_i + [p-M]\,h_i' - h_i m$, which is derived from (2.20). The fourth equality is obtained by substituting

$[p-M] = -(H_i - H)/h_i$, again from (2.20).

But since H_i is decreasing and $M(p) \leq p$, we have $H_i(M(p)) \geq H_i(p)$ and using the fact that $h_i < 0$,

$$\begin{aligned}\Phi_M - \frac{d}{dp}(\Phi_m) &\leq H_i(M)\left(2\left[1-\frac{H}{H_i}\right]h_i - \left[\frac{H-H_i}{h_i}\right]h_i'\right) \\ &= H_i(M)\left[\frac{H_i - H}{H_i h_i}\right]\left(2h_i^2 - H_i h_i'\right)\end{aligned}$$

Since F_i is regular, $2h_i^2 - H_i h_i' > 0$, as in (2.19). Together with $H_i > H$ and $h_i < 0$, this implies

$$\Phi_M - \frac{d}{dp}(\Phi_m) < 0$$

whenever $M(p) < p$.

If $M(p) < p$ for all p, then $Z(p) = p - M(p) > 0$, and so (2.17) now implies

$$\Delta'(0) = \int_0^{\overline{p}} \left[\Phi_M - \frac{d}{dp}(\Phi_m)\right][p-M]\,dp < 0$$

We have shown that given any M such that $M(p) < p$ for all p, a small perturbation $(1-\varepsilon)M(p) + \varepsilon p$ in the direction of p always decreases Δ. Thus, Δ is minimized at $M = p$. But when $M(p) = p$ for all p, $H_i = H_j$ and this means that $F_1 = F_2$, and in that case, the revenue equivalence principle implies that $\Delta = 0$. We have thus shown that for all F_1 and F_2 that are regular, $\Delta \geq 0$.

This completes the proof of Theorem 3. ■

Remark 3. The proof of Theorem 3 actually shows not only that the difference in revenues between the two auctions is nonnegative but moreover that it is, in fact, increasing in the degree of asymmetry. Suppose that $F_1 \leq F_2$ so that $i = 1$ and $j = 2$ holds everywhere. For fixed F_1, if F_2 is perturbed in the direction of F_1, thereby bringing the situation closer to symmetry, then the difference in revenues decreases.

Remark 4. Can Theorem 3 be strengthened so that its conclusion holds for all equilibria (not necessarily robust) of the second-price auction? The answer is no. With resale, there may exist inefficient equilibria of the second-price auction which are not robust, but result in a higher expected revenue than the first-price auction. An example is available from the authors.

2.6 Other Resale Mechanisms

In our analysis of asymmetric auctions with resale, we assumed that post-auction trade took place via a take-it-or-leave-it offer from the winner of the auction. Since all bargaining power resides in the hands of the seller, it is natural to refer to this as the *monopoly* mechanism.

In this section we show that our results are quite robust. We first show that they continue to hold if post-auction trade takes place via a take-it-or-leave-it offer from the loser of the auction. Now all bargaining power lies with the buyer and so by analogy, we will refer to this as the *monopsony* mechanism.

We then extend our results to a class of mechanisms in which the bargaining power is shared, perhaps unequally. Specifically, we consider a mechanism in which with probability k, the seller makes a take-it-or-leave-it offer and with probability $1-k$, the buyer makes a take-it-or-leave-it offer. We refer to this as the (probabilistic) *k-double auction*.[10]

Recall that in our basic model, at the end of the auction, only the winning bid was announced. When resale takes place via the monopoly mechanism, this is the same as if no bid were announced. This is because information about the winner's bid is irrelevant to the other bidder—he faces a take-it-or-leave-it offer from the winner. In other resale mechanisms, say when the buyer makes a take-it-or leave-it offer, information regarding the winner's value is no longer irrelevant—for instance, if the winning bid revealed the winner's value, then the loser could extract all surplus from the winner during resale. It can be shown that if the winning bid is announced when resale takes place via monopsony, then there is no monotonic equilibrium.

In the extensions of the basic model that follow, we assume that *no bids are announced* at the end of the auction. Thus only the identity of the winner is

[10] The term k-double auction usually refers to a situation in which the price is a weighted average of the price demanded by the seller and that offered by the buyer. This mechanism is known to have a continuum of equilibria and so is not useful for our analysis.

commonly known.

2.6.1 Monopsony Resale

In this case, the loser of the auction can make an offer to buy the object from the winner.

Suppose that the two bidders follow *continuous* and *strictly increasing* inverse bidding strategies ϕ_1 and ϕ_2, respectively.

Suppose bidder i with value x_i bids b and loses the auction. He infers that bidder j's value $X_j \geq \phi_j(b)$. If $x_i > \phi_j(b)$, there are potential gains from trade. Let $r_i(b, x_i)$ be the optimal (monopsony) price set by bidder i with value x_i when he bids b and loses. This results in a resale profits of

$$S_i(b, x_i) = \max_r [F_j(r) - F_j(\phi_j(b))](x_i - r) \tag{2.23}$$

and the optimal price $r_i(b, x_i)$ must satisfy the first-order condition

$$r - \frac{F_j(\phi_j(b)) - F_j(r)}{f_j(r)} = x_i \tag{2.24}$$

Regularity again implies that the left-hand side is increasing and so there is a unique solution.

Notice that the monopsony pricing formula in (2.24) is the *same* as the monopoly pricing formula in (2.1). The only difference is that now the optimal price $r_i(b, x_i) < x_i$. Notice also that the envelope theorem implies

$$\frac{\partial}{\partial b} S_i(b, x_i) = -f_j(\phi_j(b))\, \phi_j'(b)\,(x_i - r_i(b, x_i))$$

Now fix a b satisfying $\phi_j(b) < \phi_i(b)$. Suppose bidder j follows ϕ_j. Bidder i's expected payoff when his value is $x_i \equiv \phi_i(b)$ and he deviates by bidding a c close to b is

$$\Pi_i(c, x_i) = F_j(\phi_j(c))(x_i - c) + S_i(c, x_i)$$

where $S_i(c, x_i)$ as defined above, is his expected payoff from resale if he loses the auction. Maximizing Π_i, using the envelope theorem to evaluate the derivative of

$S_i(c, x_i)$ and writing $r(b) \equiv r_i(b, \phi_i(b))$ leads to

$$\frac{d}{db} \ln F_j(\phi_j(b)) = \frac{1}{r(b) - b}$$

Now consider bidder j. A bid of c satisfying $\phi_j(c) < \phi_i(c)$ when his value is $x_j = \phi_j(b)$ results in an expected payoff of

$$\Pi_j(c, x_j) = F_i(z_i)\, x_j + \int_{z_i}^{\phi_i(c)} r(\beta_i(x_i))\, f_i(x_i)\, dx_i - F_i(\phi_i(c))\, c$$

where z_i is highest type of bidder i whose offer is accepted, that is, $r(\beta_i(z_i)) = x_j$. Since it is optimal for j to bid b, the first-order condition for maximizing Π_j results in the same differential equation as above.

Thus, just as (2.6), (2.7) and (2.8) characterize the equilibrium when resale is via the monopoly mechanism, the following characterize the equilibrium bidding when resale is via the monopsony mechanism.

$$\frac{d}{db} \ln F_1(\phi_1(b)) = \frac{1}{r(b) - b} \tag{2.25}$$

$$\frac{d}{db} \ln F_2(\phi_2(b)) = \frac{1}{r(b) - b} \tag{2.26}$$

where if $\phi_j(b) \leq \phi_i(b)$,

$$r(b) = \arg\max_r \left[F_j(r) - F_j(\phi_j(b))\right](\phi_i(b) - r) \tag{2.27}$$

It should be noted that although the formulae defining the two are similar, in general, $r(b) \neq p(b)$. This implies of course that the equilibrium bidding strategies when resale is via the monopsony mechanism—that is, the solutions to (2.25) and (2.26)—are different from the equilibrium bidding strategies when resale is via the monopoly mechanism. So that there is no ambiguity, let us denote by $\widetilde{\varphi}_1$ and $\widetilde{\varphi}_2$ the equilibrium inverse bidding strategies in a first-price auction in which resale is via monopsony.

The remainder of the analysis parallels that in the case of monopoly *exactly* once the monopoly pricing function $p(b)$ is replaced by the monopsony pricing function $r(b)$. Specifically,

1. As in Theorem 4, the differential equations (2.25) and (2.26), together with (2.27), are both necessary and sufficient for $\widetilde{\varphi}_1$ and $\widetilde{\varphi}_2$ to be an equilibrium.

2. As in Lemma 1, F_1 and F_2 uniquely determine a distribution G of monopsony resale prices where if $F_i(r) \leq F_j(r)$, then

$$G(r) = F_i\left(r - \frac{G(r) - F_j(r)}{f_j(r)}\right) \tag{2.28}$$

   The distribution G has a geometric interpretation similar to that of F in Figure 2.2. In general, $G \neq F$.

3. As in Theorem 1, there exists an equilibrium with strictly increasing bidding strategies $\widetilde{\varphi}_1$ and $\widetilde{\varphi}_2$ and it is unique in the class of all nondecreasing equilibria.

4. As in Theorem 3, the revenue from a first-price auction with monopsony resale is at least as large as that from a second-price auction with resale.

2.6.2 Probabilistic k-Double Auctions

In this mechanism, resale takes place as follows. With probability k, the winner of the auction makes a take-it-or-leave-it offer to the loser and with probability $1-k$ the loser makes a take-it-or-leave-it offer to the winner. Either side may decide not to participate, in which case no transaction takes place.

When $k = 1$, this reduces to the monopoly resale mechanism considered earlier. When $k = 0$, it reduces to the monopsony mechanism of the previous subsection.

Again, fix a b satisfying $\phi_j(b) < \phi_i(b)$. Suppose bidder j follows ϕ_j. Bidder i's expected payoff when his value is $x_i \equiv \phi_i(b)$ and he deviates by bidding a c close to b is

$$\begin{aligned}\Pi_i(c, x_i) &= (x_i - c) F_j(\phi_j(c)) \\ &\quad + k \int_{\phi_j(c)}^{\omega_j} \max\{x_i - p(\beta_j(x_j)), 0\} f_j(x_j)\, dx_j + (1-k) S_i(c, x_i)\end{aligned}$$

This is because if i loses, with probability k bidder j will offer to sell him the object at price $p(\beta_j(x_j))$. With probability $1-k$, he will offer to buy from the other bidder and obtain a profit of $S_i(c, x_i)$, as defined in (2.23). Differentiating with respect to c and using the equilibrium condition $x_i = \phi_i(b)$ results in

$$(kp(b) + (1-k)r(b) - b) f_j(\phi_j(b)) \phi_j'(b) - F_j(\phi_j(b)) = 0$$

If we write $s(b) = kp(b) + (1-k)r(b)$, then

$$\frac{d}{db} \ln F_j(\phi_j(b)) = \frac{1}{s(b) - b}$$

Now consider bidder j. A bid of c satisfying $\phi_j(c) < \phi_i(c)$ when his value is $x_j = \phi_j(b)$ results in an expected payoff of

$$\Pi_j(c, x_j) = kR_j(c, x_j) + (1-k) \int_{z_i}^{\phi_i(c)} r(\beta_i(x_i)) f_i(x_i) dx_i - F_i(\phi_i(c)) c$$

where z_i is lowest type of bidder i whose offer is accepted, that is, $r(\beta_i(z_i)) = x_j$. Differentiating, and using the equilibrium condition $x_j = \phi_j(b)$, leads to

$$[kp(b) + (1-k)r(b)] f_i(\phi_i(b)) \phi_i'(b) - b f_i(\phi_i(b)) \phi_i'(b) - F_i(\phi_i(b)) = 0$$

which again results in

$$\frac{d}{db} \ln F_i(\phi_i(b)) = \frac{1}{s(b) - b}$$

Thus the equilibrium when resale is via a k-double auction is characterized by

$$\begin{aligned} \frac{d}{db} \ln F_1(\phi_1(b)) &= \frac{1}{s(b) - b} \\ \frac{d}{db} \ln F_2(\phi_2(b)) &= \frac{1}{s(b) - b} \end{aligned}$$

where $s(b) = kp(b) + (1-k)r(b)$ and $p(b)$ and $r(b)$ are given in (2.8) and (2.27), respectively.

Once again, in general, $p(b) \neq r(b)$ and so for $k \in (0,1)$, $s(b)$ is distinct from both $p(b)$ and $r(b)$. Thus the equilibrium bidding strategies are now different from both those in the case of monopoly resale and those in the case of monopsony resale. This in turn implies that the pricing functions $p(b)$ and $r(b)$ in the case of a k-double auction are also different from those resulting in the case of a pure monopoly or a pure monopsony.

Define $\widehat{\varphi}_1, \widehat{\varphi}_2$ to be the equilibrium inverse bidding strategies in a first-price auction in which resale is via the k-double auction. Similarly, define $\widehat{p}(b)$ and $\widehat{r}(b)$ to be the monopoly and monopsony pricing functions in the k-double auction. Thus $\widehat{\varphi}_1, \widehat{\varphi}_2, \widehat{p}, \widehat{r}$ are the simultaneous solutions to the two differential equations above together with (2.8) and (2.27). For notational consistency, let $\widehat{s}(b) = k\widehat{p}(b) + (1-k)\widehat{r}(b)$.

2.6.2.1 Distribution of $\widehat{s}(b)$

We wish to determine the distribution of the random variable $\widehat{s}(b)$. As before, the differential equations imply once again that the distributions of bids of the two bidders are identical; that is, $F_1(\widehat{\varphi}_1(b)) = F_2(\widehat{\varphi}_2(b))$. Let $A(b) \equiv F_j(\widehat{\varphi}_j(b))$ be the common distribution of bids.

Assume, without loss of generality, that $\widehat{\varphi}_j(b) < \widehat{\varphi}_i(b)$ and notice that $\widehat{p}(b)$, $\widehat{r}(b)$ and $\widehat{s}(b)$ satisfy:

$$\widehat{\varphi}_j(b) = \widehat{p}(b) - \frac{F_i(\widehat{\varphi}_i(b)) - F_i(\widehat{p}(b))}{f_i(\widehat{p}(b))} \tag{2.29}$$

$$\widehat{\varphi}_i(b) = \widehat{r}(b) - \frac{F_j(\widehat{\varphi}_j(b)) - F_j(\widehat{r}(b))}{f_j(\widehat{r}(b))} \tag{2.30}$$

$$\widehat{s}(b) = k\widehat{p}(b) + (1-k)\widehat{r}(b) \tag{2.31}$$

Define distributions $\widehat{F}, \widehat{G}$ and L as follows

$$\widehat{F}(\widehat{p}(b)) = \widehat{G}(\widehat{r}(b)) = L(\widehat{s}(b)) = A(b) \tag{2.32}$$

F, G and L are well defined since $\widehat{p}(\cdot)$, $\widehat{r}(\cdot)$ and $\widehat{s}(\cdot)$ are increasing functions.

Note that (2.29) and (2.30) then can be rewritten as follows:

$$\widehat{F}(p) = F_j\left(p - \frac{\widehat{F}(p) - F_i(p)}{f_i(p)}\right) \tag{2.33}$$

$$\widehat{G}(r) = F_i\left(r - \frac{\widehat{G}(r) - F_j(r)}{f_j(r)}\right) \tag{2.34}$$

But now notice that since (2.33) is the same as (3.6) and F was uniquely determined there, $\widehat{F} = F$, the distribution of prices when resale is via monopoly. Similarly, (2.34) is the same as (2.28) and so $\widehat{G} = G$, the distribution of prices when resale is via monopsony. We thus obtain the conclusion that even though, in general, $\widehat{p} \neq p$ and $\widehat{\varphi}_i \neq \phi_i$, we have

$$F(\widehat{p}(b)) = F_i(\widehat{\varphi}_i(b)) \text{ and } F(p(b)) = F_i(\phi_i(b))$$

Similarly, even though, in general, $\widehat{r} \neq r$ and $\widehat{\varphi}_i \neq \widetilde{\varphi}_i$,

$$G(\widehat{r}(b)) = F_i(\widehat{\varphi}_i(b)) \text{ and } G(r(b)) = F_i(\widetilde{\varphi}_i(b))$$

Finally, since $F = \widehat{F}$ and $G = \widehat{G}$, (2.32) implies that for all b,

$$L^{-1}(A(b)) = kF^{-1}(A(b)) + (1-k)G^{-1}(A(b))$$

Since $A(b)$ varies from 0 to 1, we obtain

$$L^{-1}(q) = kF^{-1}(q) + (1-k)G^{-1}(q) \tag{2.35}$$

for all $q \in [0,1]$, where F and G satisfy (2.33) and (2.34), respectively.

2.6.2.2 Revenue from FPA with Resale via k-Double Auction

Lemma 2.

$$\int_0^{\overline{s}} (1 - L(s))^2\, ds = k\int_0^{\overline{p}} (1 - F(p))^2\, dp + (1-k)\int_0^{\overline{r}} (1 - G(r))^2\, dr$$

Proof. Let $L(s) = q$, then from (2.35) we obtain

$$s = L^{-1}(q) = kF^{-1}(q) + (1-k)G^{-1}(q)$$

so that

$$ds = \left(\frac{k}{f\left(F^{-1}\left(q\right)\right)} + \frac{1-k}{g\left(G^{-1}\left(q\right)\right)} \right) dq$$

By changing the variable of integration from $s \in [0, \overline{s}]$ to $q = L(s) \in [0, 1]$ we obtain

$$\begin{aligned} \int_0^{\overline{s}} (1 - L(s))^2 \, ds &= \int_0^1 (1-q)^2 \left(\frac{k}{f\left(F^{-1}\left(q\right)\right)} + \frac{1-k}{g\left(G^{-1}\left(q\right)\right)} \right) dq \\ &= k \int_0^1 \frac{(1-q)^2}{f\left(F^{-1}\left(q\right)\right)} dq + (1-k) \int_0^1 \frac{(1-q)^2}{g\left(G^{-1}\left(q\right)\right)} dq \end{aligned}$$

Changing the variables again from $q \in [0, 1]$ to $p = F^{-1}(q) \in [0, \overline{p}]$ in the first integral and $r = G^{-1}(q) \in [0, \overline{r}]$ in the second, we obtain the required equality. ∎

The expected revenue of the original seller when resale is via the k-double auction is given by

$$\int_0^{\overline{s}} (1 - L(s))^2 \, ds$$

where L satisfies (2.35).

We have already shown in Theorem 3 that

$$\int_0^{\overline{p}} (1 - F(p))^2 \, dp \geq \int_0^{\omega_2} (1 - F_1(x)) (1 - F_2(x)) \, dx$$

and it is similarly the case that

$$\int_0^{\overline{r}} (1 - G(r))^2 \, dr \geq \int_0^{\omega_2} (1 - F_1(x)) (1 - F_2(x)) \, dx$$

Lemma 2 now implies that the revenue from a first-price auction with resale via the k-double auction is also greater than or equal to the revenue from a second-price auction.

The following theorem extends our main result.

Theorem 4. *The seller's revenue from a first-price auction with resale via a probabilistic k-double auction is at least as great as that from a second-price auction.*

Theorem 4 subsumes the case of monopoly (when $k = 1$) and the case of monopsony (when $k = 0$). The reader may thus wonder why we did not establish only the more general result. The proof of Theorem 4, however, makes use of both Theorem 3 and its counterpart for the monopsony mechanism.

Finally, one may well ask whether the revenue ranking is valid for arbitrary (balanced-budget) resale mechanisms. In other words, suppose that the resale mechanism is of the direct form (Q, T), where $Q(z_b, z_s)$ is the probability that the object is transferred from the seller to the buyer when (z_b, z_s) are the values reported by the two parties and $T(z_b, z_s)$ is the payment from the buyer to the seller. As usual, we ask that the mechanism be incentive compatible and individually rational. Does Theorem 4 extend to this general specification? Clearly, the answer is no. For instance, the result does not hold for the "no-trade" mechanism ($Q = 0$ and $T = 0$) that, in effect, bans resale—as Vickrey's (1961) example shows. This suggests that the revenue ranking to hold, the probability of a resale transaction should not be "too low."

2.7 Conclusion

We have shown that a consideration of resale possibilities allows for a simpler characterization of equilibrium strategies in first-price auctions than available when resale is not admitted. In our model, equilibrium strategies can be explicitly computed in a relatively simple manner as in the proof of Theorem 1. Moreover, we obtain a general revenue ranking result between first- and second-price auctions that is not available in the standard model. Thus this appears to be one of those happy circumstances where complicating the model with a real-world feature—resale—actually simplifies the analysis.

In this paper, we have restricted attention to the case of two bidders. Considering resale when there are three or more bidders poses some conceptual difficulties.[11] There are simply too many modelling choices and the challenge is to model resale in a way that is both realistic and analytically tractable. To fix ideas, consider a first-price auction with three bidders and suppose that bidder 3 wins the auction. At the resale stage, among the many available options are:

1. Bidder 3 invites both bidders 1 and 2 to participate in an auction.

[11] Zheng (2002) also finds that when there are three or more bidders, Myerson's optimal auction is robust to resale only under stringent conditions (see Mylovanov and Tröger (2005)).

2. Bidder 3 approaches one of the bidders, say bidder 1, and makes a take-it-or-leave-it offer.

The results of this paper have a direct bearing on any analysis of the first option. This is because the resale auction involves only two bidders. A model with these features could, however, be legitimately criticized on the grounds that most resale transactions are bilateral—we simply do not observe multiple rounds of auctions with the winner in each round holding an auction for the remaining bidders. But it is not clear why the bilateral transaction should consist of only one take-it-or-leave-it offer. Bidder 3 could approach bidder 1 and if his offer is refused, then approach bidder 2. And this raises the question of the order in which bidders 1 and 2 should be approached. We hope to consider some of these issues in future work.

2.8 Appendix: Proofs from Section 3

This appendix contains proofs of results on first-price auctions with resale.

2.8.1 Proof of Proposition 4

Proof. The necessity of (2.6) and (2.7) for all $b \in \left[0, \overline{b}\right]$ has already been shown. It remains to show that these are sufficient.

Suppose bidder j follows the equilibrium inverse bidding strategy ϕ_j. We will argue that when bidder i has a value of x_i, he cannot do better than to bid b such that $\phi_i(b) = x_i$. We do this by showing that neither underbidding nor overbidding can be profitable.

Notice that the differential equations can be rewritten as: for $j = 1, 2$

$$(p(b) - b) f_j(\phi_j(b)) \phi_j'(b) - F_j(\phi_j(b)) = 0 \tag{2.36}$$

Case A (Underbidding): Suppose bidder i bids b such that $\phi_i(b) < x_i$.

Case A1: $\phi_j(b) < \phi_i(b) < x_i$. If i wins the auction with a bid of b, then his payoff is simply $(x_i - b)$ since there are no benefits to reselling. If i loses, however, j will offer to sell the object to him for a price of $p(\beta_j(x_j))$ and so i's payoff is

$\max\{x_i - p(\beta_j(x_j)), 0\}$. Thus i's expected payoff is

$$\Pi_i(b, x_i) = (x_i - b) F_j(\phi_j(b)) + \int_{\phi_j(b)}^{\omega_j} \max\{x_i - p(\beta_j(x_j)), 0\} f_j(x_j)\, dx_j$$

Differentiating with respect to b and using (2.36), results in

$$\frac{\partial \Pi_i}{\partial b} = (p(b) - b) f_j(\phi_j(b)) \phi_j'(b) - F_j(\phi_j(b)) = 0$$

Case A2: $\phi_i(b) \leq \phi_j(b) < x_i$. If i wins the auction with a bid of b, then his payoff is simply $(x_i - b)$ since again there are no benefits to reselling. Similarly, if i loses, bidder j will not offer to sell the object to him since from j's perspective, there appear to be no benefits to selling to i. Thus i's expected payoff is simply

$$\Pi_i(b, x_i) = (x_i - b) F_j(\phi_j(b))$$

and so again by using (2.36),

$$\begin{aligned} \frac{\partial \Pi_i}{\partial b} &= (x_i - b) f_j(\phi_j(b)) \phi_j'(b) - F_j(\phi_j(b)) \\ &\geq (\phi_j(b) - b) f_j(\phi_j(b)) \phi_j'(b) - F_j(\phi_j(b)) \\ &\geq (p(b) - b) f_j(\phi_j(b)) \phi_j'(b) - F_j(\phi_j(b)) \\ &= 0 \end{aligned}$$

Case A3: $\phi_i(b) < x_i \leq \phi_j(b)$. If i wins the auction with a bid of b, then he may resell it to bidder j since again there are potential gains from trade. His expected payoff from winning is

$$R_i(b, x_i) = \max[F_j(\phi_j(b)) - F_j(p)]p + F_j(p) x_i$$

If i loses, bidder j will not offer to sell the object to him since from j's perspective, there appear to be no gains from trade. Thus i's expected payoff from bidding b is

$$\Pi_i(b, x_i) = R_i(b, x_i) - F_j(\phi_j(b))\, b$$

and using the envelope theorem as in (2.4) and the fact that $p_i(b, x_i) \geq p_i(b, \phi_i(b)) \equiv$

$p(b)$,

$$\begin{aligned}\frac{\partial \Pi_i}{\partial b} &= (p_i(b,x_i) - b) f_j(\phi_j(b)) \phi_j'(b) - F_j(\phi_j(b)) \\ &\geq (p(b) - b) f_j(\phi_j(b)) \phi_j'(b) - F_j(\phi_j(b)) \\ &= 0\end{aligned}$$

Case B (Overbidding): Suppose bidder i bids b such that $x_i < \phi_i(b)$.

Case B1: $\phi_j(b) < x_i < \phi_i(b)$. If i wins the auction with a bid of b, then his payoff is simply $(x_i - b)$ since there is no benefit from reselling to j. On the other hand, if i loses, j will offer to sell the object to him for a price of $p(\beta_j(x_j))$ and so i's payoff if he loses is $\max\{x_i - p(\beta_j(x_j)), 0\}$. Thus i's expected payoff from bidding b is

$$\Pi_i(b,x_i) = (x_i - b) F_j(\phi_j(b)) + \int_{\phi_j(b)}^{\omega_j} \max\{x_i - p(\beta_j(x_j)), 0\} f_j(x_j) \, dx_j$$

Differentiating with respect to b,

$$\begin{aligned}\frac{\partial \Pi_i}{\partial b} &= (x_i - b) f_j(\phi_j(b)) \phi_j'(b) - F_j(\phi_j(b)) - \max\{x_i - p(b), 0\} f_j(\phi_j(b)) \phi_j'(b) \\ &\leq (x_i - b) f_j(\phi_j(b)) \phi_j'(b) - F_j(\phi_j(b)) - (x_i - p(b)) f_j(\phi_j(b)) \phi_j'(b) \\ &= (p(b) - b) f_j(\phi_j(b)) \phi_j'(b) - F_j(\phi_j(b)) \\ &= 0\end{aligned}$$

Case B2: $x_i \leq \phi_j(b) < \phi_i(b)$. If i wins the auction with a bid of b, then he may resell it to bidder j since again there are potential gains from trade. If he loses, bidder j will offer to sell the object to him for a price of $p(\beta_j(x_j))$ but this price will always exceed x_i and so i will refuse the offer. Thus i's expected payoff from bidding b is just

$$\Pi_i(b,x_i) = R_i(b,x_i) - F_j(\phi_j(b)) b$$

and again using (2.4) and the fact that $p_i(b,x_i) \leq \phi_j(b) \leq p_j(b,\phi_j(b)) \equiv p(b)$,

$$\frac{\partial \Pi_i}{\partial b} = (p_i(b,x_i) - b) f_j(\phi_j(b)) \phi_j'(b) - F_j(\phi_j(b))$$

$$\begin{aligned} &\leq \quad (p(b) - b) f_j(\phi_j(b)) \phi_j'(b) - F_j(\phi_j(b)) \\ &= \quad 0 \end{aligned}$$

Case B3: $x_i < \phi_i(b) \leq \phi_j(b)$. If i wins the auction with a bid of b, then he may resell it to bidder j since again there are potential gains from trade. His expected payoff from winning is the monopoly profit $R_i(b, x_i)$. If he loses, bidder j will not offer to sell the object to him since from j's perspective, there appear to be no gains from trade. Thus i's expected payoff from bidding b is again

$$\Pi_i(b, x_i) = R_i(b, x_i) - F_j(\phi_j(b)) b$$

and the argument is the same as in Case B2, except that now $p_i(b, x_i) \leq p_i(b, \phi_i(b)) \equiv p(b)$.

We have thus argued that for all b such that $\phi_i(b) < x_i$, $\frac{\partial \Pi_i}{\partial b} \geq 0$ and for all b such that $\phi_i(b) > x_i$, $\frac{\partial \Pi_i}{\partial b} \leq 0$. Thus bidding a b such that $\phi_i(b) = x_i$ is a best response to ϕ_j. ∎

2.8.2 Proof of Lemma 1

Proof. Fix a p and note that

$$\Psi(p, q) \equiv F_j\left(p - \frac{q - F_i(p)}{f_i(p)}\right)$$

is a strictly decreasing function of q, where $F_i(p) \leq q \leq 1$.

If $F_i(p) = F_j(p)$, then $F(p) = F_i(p)$ also. If $F_i(p) < F_j(p)$, then $\Psi(p, F_i(p)) > F_i(p)$ and $\Psi(p, F_j(p)) < F_j(p)$. Thus for every p, there exists a unique fixed-point $q \in (F_i(p), F_j(p))$ such that $\Psi(p, q) = q$ and by (3.6) $F(p) \equiv q$. Thus, $F_i(p) \leq F(p) \leq F_j(p)$ and the inequalities are strict if $F_i(p) < F_j(p)$.

Clearly, $F(0) = 0$. We now argue that F is strictly increasing. It suffices to show that $F(p') < F(p'')$ for $p' < p''$ such that for all $p \in [p', p'']$, $F_i(p) \leq F_j(p)$. The regularity of F_i (see Section 2.2) implies that for all q,

$$p'' - \frac{q - F_i(p'')}{f_i(p'')} > p' - \frac{q - F_i(p')}{f_i(p')}$$

Thus, if $p' < p''$, then for all q, $\Psi(p'', q) > \Psi(p', q)$. This implies that if $\Psi(p', q') = q'$ and $\Psi(p'', q'') = q''$, then $q'' > q'$. So $F(p'') > F(p')$. Finally, note that since $\omega_1 \geq \omega_2$, F has support $[0, \overline{p}]$, where $\overline{p}$ satisfies

$$\omega_2 = \overline{p} - \frac{1 - F_1(\overline{p})}{f_1(\overline{p})}$$

We have thus shown that F is a well-defined distribution function over $[0, \overline{p}]$. ∎

2.8.3 Proof of Theorem 2

It has already been established that the equilibrium constructed in Theorem 1 is unique in the class of equilibria with strictly increasing strategies. Here we complete the proof Theorem 2 by showing that every equilibrium in nondecreasing strategies must, in fact, have strictly increasing bidding strategies.

Lemma 3. *If β_1 and β_2 are nondecreasing equilibrium bidding strategies in the first-price auction with resale, then β_1 and β_2 are continuous.*

Proof. Suppose that there exists an $x_i > 0$ such that $\lim_{x \uparrow x_i} \beta_i(x) = b' < b'' = \lim_{x \downarrow x_i} \beta_i(x)$. First, note that in that case bidder j also does not bid between b' and b''; that is, there does not exist an x_j such that $b' < \beta_j(x_j) < b''$. Otherwise, bidder j with value x_j could increase his payoff by decreasing his bid to $\beta_j(x_j) - \varepsilon > b'$. This change does not affect his payoff if he were to lose but increases it by ε if he were to win (which happens with positive probability since $x_i > 0$). Second, note that bidder j bids b' or lower with positive probability. Otherwise, $\lim_{x \to 0} \beta_j(x) \geq b''$ and bidder j with value x_j close to zero can improve his payoff by reducing his bid to b'. This is because bidder j would gain at least $b'' - b'$ whenever bidder i's value was between 0 and x_i and suffer only a small loss in the cases when bidder i's value is just slightly above x_i. So there does not exist an x_j such that $b' < \beta_j(x_j) < b''$.

Now consider bidder i with a value slightly greater than x_i, say $x_i + \delta$. By reducing his bid from $\beta_i(x_i + \delta) \geq b''$ to b', bidder i could increase his payoff. Once again, this change does not affect his payoff if he were to lose and increases it by at least $b'' - b'$ if he were to win (which happens with positive probability). Thus for δ small enough, bidder i with value $x_i + \delta$ has a profitable deviation.

We have argued that the bidding strategies β_i are continuous at any $x_i > 0$. It remains to argue that they are also continuous at 0.

Suppose that $\lim_{x\to 0} \beta_i(x) = b_0 > 0$ and without loss of generality, suppose that for some small δ, it is the case that for all $x \in (0,\delta)$, $\beta_i(x) \geq \beta_j(x)$. Then we must have that $\lim_{x\to 0} \beta_j(x) = b_0$ also. Otherwise, bidder i with a value close to zero could reduce his bid an improve his payoff. If β_i is increasing in $(0,\delta)$, then the same argument that shows that $\beta_i(0) = 0$ when both strategies are increasing shows that this is impossible. If both β_i and β_j are constant over $(0,\delta)$, then bidder i with value close to zero can improve his payoff by increasing his bid slightly. Thus $\lim_{x\to 0} \beta_i(x) = 0$. ∎

Lemma 4. *If β_1 and β_2 are nondecreasing equilibrium bidding strategies in the first-price auction with resale, then β_1 and β_2 are strictly increasing.*

Proof. Suppose that there is an interval $[x', x'']$ such that for all $x_i \in (x', x'')$, $\beta_i(x_i) = b > 0$; that is, bidder i's strategy is constant. Consider bidder j with value x_j such that $\lim_{x\uparrow x_j} \beta_j(x) = b$. For x close to x_j, bidder j can improve his payoff by bidding higher than b. This is because he then wins whenever $X_i \in (x', x'')$ and the loss is arbitrarily small. If $b = 0$, then bidder j with a small value x_j can improve his payoff by reducing his bid. ∎

2.9 Appendix: Proofs from Section 4

2.9.1 Proof of Proposition 7

The proof that there is a unique robust equilibrium in the second-price auction with resale follows from the lemmas below.

We suppose that equilibrium bidding strategies are right continuous. Since the expected payoff functions of the bidders are continuous in equilibrium, if there is a discontinuity in the bidding strategies, bidders must be indifferent between bidding the right and left limits. Therefore, focusing on equilibria with right-continuous bidding strategies is without loss of generality.

Consider a robust equilibrium of the SPAR: $(\beta_1(\cdot), p_1(\cdot,\cdot), \beta_2(\cdot), p_2(\cdot,\cdot))$.

Lemma 5. *For all x_i and for all $b \leq \beta_i(x_i)$, the set $\{x_j : \beta_j(x_j) = b$ and $x_j > x_i\}$ is either a singleton or empty.*

Proof. Suppose there are two points, say x'_j and x''_j are in the set $\{x_j : \beta_j(x_j) = b$ and $x_j > x_i\}$. Upon winning, the optimal price $p_i(b, x_i)$ that i will set must then depend on the distribution F_j, contradicting the definition of a robust equilibrium. ∎

A simple but important consequence of Lemma 5 is that if j loses in the auction, his payoff must be zero. Either bidder i makes no offer to him or makes an offer equal to x_j.

Lemma 6. $\inf_{x_1} \beta_1(x_1) = \inf_{x_2} \beta_2(x_2)$

Proof. Suppose $\inf_{x_1} \beta_1(x_1) < \inf_{x_2} \beta_2(x_2)$. By right continuity, there is an open interval I of values X_1 such that for all $x_1 \in I$, $\beta_1(x_1) < \inf_{x_2} \beta_2(x_2)$. Consider $x'_1, x''_1 \in I$ such that $0 < x'_1 < x''_1$. For all $x_2 < x'_1$, bidder 2 with value x_2 wins against both x'_1 and x''_1 and from Lemma 5, offers to sell to both at prices equal to their values. But this means that bidder 1 with value x''_1 is better off by bidding $\beta_1(x'_1) \neq \beta_1(x''_1)$. ∎

Lemma 7. *For $i = 1, 2$ there exists an x_i^0 such that bidder i with value x_i^0 makes an overall expected payoff of* 0.

Proof. From Lemma 6, $\inf_{x_1} \beta_1(x_1) = \inf_{x_2} \beta_2(x_2) = m$, say. Let $x_i^0 = \inf\{x_i : \beta_i(x_i) = m\}$. Since $\beta_i(0) \geq m$, from Lemma 5 there is at most one x_j such that $\beta_j(x_j) = m$. This means that bidder i with value x_i^0 wins the object with probability 0. Since the payoff from losing is also 0, his overall payoff is 0. ∎

Lemma 8. $\beta_1(x) = \beta_2(x) = x$, *for all* $x \in [0, \omega_2]$.

Proof. Suppose that $\beta_j(x'_j) < x'_j$ for some x'_j. By right continuity, there exists a δ such that for all $x_j \in I = [x'_j, x'_j + \delta]$, $\beta_j(x_j) < x'_j$. Thus there exists an $\varepsilon > 0$ such that $x'_j - \beta_j(x_j) \geq \varepsilon$ for all $x_j \in I$. Suppose now that bidder i with value x_i^0 (whose overall profit is zero), bids x'_j. Whenever bidder j's value $x_j \in I$, i will win and make a profit of at least ε. Consider a distribution of values F_j such that

$\Pr[X_j \in I]$ is close enough to 1 so that the overall expected profit of x_i^0 is also positive, which is a contradiction. So $\beta_j(x_j') < x_j'$ is impossible.

Similarly, suppose $\beta_j(x_j') > x_j'$ for some x_j'. Again, by right continuity, there exists a δ such that for all $x_j \in I = [x_j', x_j' + \delta]$, $\beta_j(x_j) > x_j' + \delta$. For all $x_i \in I$, if $\beta_i(x_i) > x_j' + \delta$, then bidder i with value x_i makes a negative profit when he faces bidder j with value $X_j \in I$. Consider a distribution of values F_j such that $\Pr[X_j \in I]$ close enough to 1 so that his overall expected profit is also negative, which is a contradiction. This shows that for all $x_i \in I$, $\beta_i(x_i) \leq x_j' + \delta$. Now consider x_i' and x_i'' such that $x_j' < x_i' < x_i'' < x_j' + \delta$. Bidder j with value x_j such that $x_j' < x_j < x_i'$ wins against both x_i' and x_i'' and from Lemma 5, offers to sell to both at prices equal to their values. But this means if the distribution of values F_j is such that $\Pr[x_j' < x_j < x_i']$ is close to 1, then bidder i with value x_i'' is better off by bidding $\beta_i(x_i') \neq \beta_i(x_i'')$. ∎

2.10 References

1. Athey, S. (2001): "Single Crossing Properties and the Existence of Pure Strategy Equilibria in Games of Incomplete Information," *Econometrica*, 69, 861–889.

2. Blume, A. and P. Heidhues (2004): "All Equilibria of the Vickrey Auction," *Journal of Economic Theory*, 114, 170–177.

3. Börgers, T. and T. McQuade (2006): "Information Invariant Equilibria of Extensive Games," mimeo, Department of Economics, University of Michigan, September.

4. Bulow, J., and J. Roberts (1989): "The Simple Economics of Optimal Auctions," *Journal of Political Economy*, 97, 1060–1090.

5. Cantillon, E. (2005): "The Effects of Bidder Asymmetries on Expected Revenue in Auctions," Working Paper, Harvard Business School.

6. Cheng, H. (2005): "Ranking Sealed High-Bid and Open Asymmetric Auctions," IEPR Working Paper 05.26, University of Southern California.

7. Fibich, G., A. Gavious and A. Sela (2004): "Revenue Equivalence in Asymmetric Auctions," *Journal of Economic Theory*, 115, 309–321.

8. Garratt, R. and T. Tröger (2006): "Speculation in Standard Auctions with Resale," *Econometrica*, 74, 753–770.

9. Griesmer, J., R. Levitan and M. Shubik (1967): "Toward a Study of Bidding Processes, Part IV—Games with Unknown Costs," *Naval Research Logistics Quarterly*, 14, 415–434.

10. Gupta, M. and B. Lebrun (1999): "First Price Auctions with Resale," *Economics Letters*, 64, 181–185.

11. Haile, P. (2003): "Auctions with Private Uncertainty and Resale Opportunities," *Journal of Economic Theory*, 108, 72–110.

12. Jackson, M. and J. Swinkels (2005): "Existence of Equilibrium in Single and Double Private Value Auctions," *Econometrica*, 73, 93–139.

13. Kamien, M. and N. Schwartz (1981): *Dynamic Optimization*, New York: Elsevier.

14. Krishna, V. (2002): *Auction Theory*, San Diego: Academic Press.

15. Lebrun, B. (2006): "Uniqueness of Equilibrium in First-Price Auctions," *Games and Economic Behavior* (forthcoming).

16. Marshall, R., M. Meurer, J.-F. Richard, and W. Stromquist (1994): "Numerical Analysis of Asymmetric First Price Auctions," *Games and Economic Behavior*, 7, 193–220.

17. Maskin, E. and J. Riley (2000): "Asymmetric Auctions," *Review of Economic Studies*, 67, No. 3, 413–438.

18. Maskin, E. and J. Riley (2003): "Uniqueness of Equilibrium in Sealed High-Bid Auctions," *Games and Economic Behavior*, 45, 395–409.

19. Mylovanov, T. and T. Tröger (2006): "A Characterization of the Conditions for Optimal Auction with Resale," Working Paper, University of Bonn, June.

20. Myerson, R. (1981): "Optimal Auction Design," *Mathematics of Operations Research*, 6, No. 1, 58–73.

21. Plum, M. (1992): "Characterization and Computation of Nash Equilibria for Auctions with Incomplete Information," *International Journal of Game Theory*, 20, 393–418.

22. Reny, P. (1999): "On the Existence of Pure and Mixed Strategy Nash Equilibria in Discontinuous Games," *Econometrica*, 67, 1029–1056.

23. Vickrey, W. (1961): "Counterspeculation, Auctions and Competitive Sealed Tenders," *Journal of Finance*, 16, 8–37.

24. Zheng, C. (2002): "Optimal Auctions with Resale," *Econometrica*, 70, 2197–2224.

Chapter 3

Revenue and Efficiency Effects of Resale in First-Price Auctions

3.1 Introduction

This paper studies how resale possibilities affect the performance of first-price auctions in terms of revenue and efficiency. Resale has an important role to play only when the equilibrium allocation of the auction is inefficient so that unrealized gains from trade remain. One important source of inefficiency in auctions is the presence of ex ante asymmetries among bidders. For instance, one of the bidders may be inherently strong relative to the other in the sense that his values are stochastically higher. Such asymmetries imply that equilibria of first-price auctions are inefficient—this is because the two bidders would use different bidding strategies and it may be that the person who wins the auction is not the one with the higher realized value.

It is commonly argued that resale possibilities are detrimental to the original seller. This intuition comes from the fact that resale is thought to dilute the market power of a seller. For instance, a price discriminating monopolist would be hurt by resale.

It is also argued that resale markets are good for efficiency because they allow buyers to reallocate the object after the auction is over. While this is certainly true in an ex post sense, this argument does not take into account that if buyers anticipate that resale may take place, this will affect their bidding strategies and the resulting allocations.

The purpose of this paper is to examine the validity of these arguments in the context of first-price auctions. Here we study a simple model of resale with two buyers who bid in a first-price auction to obtain a single indivisible object. We postulate an environment with *asymmetric* independent private values. In the benchmark model of a first-price auction (FPA), there is no resale and the allocation of the auction is final. In the model of a first-price auction with resale (FPAR), the winner of the auction may, if he so wishes, sell the object to the other buyer. The resale transaction is assumed to take place via a take-it-or-leave-it offer from the winner to the loser. In effect, the winner acts as a monopolist during the resale transaction.

In an earlier paper, Hafalir and Krishna (2006), we showed that the first-price auction with resale has an equilibrium in which the bidders use monotone bidding strategies and moreover, that the equilibrium is unique in this class. The equilibrium has a key characteristic: even though the bidders are asymmetric, in equilibrium their bid distributions are identical. This symmetry property proves to be very convenient in characterizing the equilibrium and evaluating its performance.

Finding equilibria of first-price auctions without resale, however, is known to be a thorny problem. An equilibrium exists and is typically unique but is characterized by a pair of linked differential equations which can be analytically solved only rarely. To the best of our knowledge, analytic expressions for the bidding strategies in asymmetric first-price auctions are available for only two families of distributions:

1. The family $\mathcal{P}$, identified by Plum (1992), consists of distributions F_1 and F_2 which are both proportional to x^a for some fixed $a > 0$ but have differing supports.

2. The family $\mathcal{C}$, identified by Cheng (2006), consists of distribution F_1 and F_2 which are proportional to $x^{a_1}$and x^{a_2}, respectively. It is assumed that $a_1 > a_2$ and that the ratio of the supports is a particular constant depending on a_1 and a_2.

Our main finding below is that in all known cases where the bidding strategies can be calculated in the first-price auction without resale, resale *increases* revenue—the FPAR is revenue superior to the FPA. Thus contrary to the intuition derived from the price discriminating monopolist, resale actually improves the revenue of the seller. We conjecture that the FPAR is revenue superior to the FPA in general.

We also show a somewhat paradoxical result. The presence of resale may actually *decrease* social surplus. An auction with resale may lead to an allocation that is so inefficient that the gains from post-auction trade are unable to recover these losses relative to a situation without resale. Again this is contrary to the intuition that resale promotes efficiency.

3.1.1 Related Literature

Asymmetric first-price auctions were already studied by Vickrey (1961). He studied environments in which bidder 1's value, say v_1, was commonly known while the other bidder's value was uniformly distributed. In that case, there is no pure strategy equilibrium. Vickrey (1961) showed that there was a mixed strategy equilibrium in which only bidder 1 randomizes.

Asymmetric first-price auctions without resale are known to have pure strategy equilibria under quite general conditions (see, for instance, Athey, 2001). Moreover, the equilibrium is typically unique (see, for instance, Maskin and Riley, 2003). Closed-form expressions for bidding strategies and revenue comparisons are rare, however. Griesmer, et al. (1967) derive closed-form equilibrium bidding strategies in a first-price auction in which bidders draw values from uniform distributions, but over different supports. Plum (1992) extends this to the class of distribution pairs $\mathcal{P}$ in which the two value distributions are of the form x^a, again over different supports. Cheng (2005) identifies a class of distribution pairs $\mathcal{C}$ and show that in this class, the bidding strategies are linear. For these two classes—the only ones for which bidding strategies are known—it is the case that the first-price auction is revenue superior to the second-price auction. Working with distributions from $\mathcal{P}$, Cantillon (2005) shows how asymmetry affects revenue in first-price auctions. In this paper, we will also consider distributions from $\mathcal{P}$ and $\mathcal{C}$.

In the absence of general analytic results, some researchers have resorted to numerical methods (Marshall, et al. 1994).

There is a small but growing literature on auctions with resale. Gupta and Lebrun (1999) consider resale possibilities but assume that at the end of the auction both *values* are announced. In contrast, in our model, the auctioneer knows only the bids and not the values and so these cannot be made public. Haile (2003) considers resale possibilities in a *symmetric* model. At the time of bidding, however, buyers have only noisy information regarding their true values, which are revealed only after the auction. Zheng (2002) identifies conditions under which the outcomes of Myerson's (1981) optimal auction can be achieved when resale is permitted. Garratt and Tröger (2006) consider resale in Vickrey's asymmetric auction in which one of the bidder's value is commonly known to be 0. Garratt and Tröger (2006) show that there is a unique mixed strategy equilibrium in the first-price auction in which the revenue is positive.

In an earlier paper (Hafalir and Krishna, 2006), discussed below, we have compared the revenue from the first-price auction with resale to the second-price auction, showing that the former is superior for all regular distributions.

3.2 Preliminaries

There are two risk-neutral buyers, labelled 1 and 2, who seek to buy a single object. Buyers' values are X_1 and X_2 are private and independently distributed. Buyer i's value for the object, X_i, is distributed according to the cumulative distribution function F_i with support $[0, \omega_i]$. It is assumed that each F_i admits a continuous density $f_i \equiv F_i'$ and that this density is positive on $(0, \omega_i)$.

We suppose throughout that bidder 1 is "strong" (and bidder 2 is "weak") in the sense that for all x, $F_1(x) \leq F_2(x)$.

We assume that both F_i are *regular* in the sense of Myerson (1981); that is, the functions

$$x - \frac{1 - F_i(x)}{f_i(x)} \tag{3.1}$$

are increasing.

We consider two environments.

3.2.1 First-price auctions without resale (FPA)

In the first, the object is sold via a first-price auction and there is no further trade. In this case, there is typically a unique equilibrium with strictly increasing bidding strategies β_i. The inverse bid functions ϕ_i satisfy the following system of differential equations: for $j = 1, 2$ and $i \neq j$,

$$\frac{d}{db} \ln F_j\left(\phi_j\left(b\right)\right) = \frac{1}{\phi_i\left(b\right) - b} \tag{3.2}$$

See, for instance, Maskin and Riley (2000).

3.2.2 First-price auctions with resale (FPAR)

In the second, the object is again sold via a first-price auction but now the winner of auction may sell the object to the loser by making a take-it-or-leave-it offer. We suppose that at the end of the auction, the losing bid is not announced.[1] In this case, there is a unique monotonic equilibrium in which the inverse bid functions ϕ_j, $j = 1, 2$, satisfy the system:

$$\frac{d}{db} \ln F_j\left(\phi_j\left(b\right)\right) = \frac{1}{p\left(b\right) - b} \tag{3.3}$$

where

$$p\left(b\right) = \arg\max_{p} \left[F_1\left(\phi_1\left(b\right)\right) - F_1\left(p\right)\right] p + F_1\left(p\right) \phi_2\left(b\right) \tag{3.4}$$

is the *pricing function* that determines the monopoly price set in equilibrium by bidder 2 when he wins with a bid of b. An implication of this is that for all bids b that occur in equilibrium

$$F_1\left(\phi_1\left(b\right)\right) = F_2\left(\phi_2\left(b\right)\right) \tag{3.5}$$

[1] It can be shown that if the losing bid is announced, then there does not exist a monotonic equilibrium (see Krishna, 2002). For other specifications of the resale stage, see Hafalir and Krishna (2006).

that is, the bid distributions are identical. It can be shown that if we define a distribution F by

$$F(p) = F_2\left(p - \frac{F(p) - F_1(p)}{f_1(p)}\right) \tag{3.6}$$

then the revenue to the original seller is just

$$R^{FPAR} = \int_0^{\omega_1} (1 - F(p))^2 \, dp \tag{3.7}$$

See Hafalir and Krishna (2006) for details.

3.3 Asymmetric First-Price Auctions

When bidders are asymmetric, closed-form solutions for the equilibrium bidding strategies are difficult to obtain. To the best of our knowledge, there are only two families of distribution-pairs for which equilibria in the FPA are explicitly known.

1. Plum (1992) derived the bidding strategies in a first-price auction when the distributions of values belonged to the class $\mathcal{P}$ consisting of F_1 and F_2 such that:
$$F_1(x) = \left(\frac{x}{\omega_1}\right)^a \text{ and } F_2(x) = \left(\frac{x}{\omega_2}\right)^a$$
where $a > 0$ and $\omega_1 > \omega_2$.

2. Cheng (2006) derived the bidding strategies in a first-price auction when the distributions of values belonged to the class $\mathcal{C}$ consisting of F_1 and F_2 such that:
$$F_1(x) = \left(\frac{x}{\omega_1}\right)^{a_1} \text{ and } F_2(x) = \left(\frac{x}{\omega_2}\right)^{a_2}$$
where $a_1 > a_2 > 0$ and $\omega_2 = \frac{a_2}{a_2+1}\frac{a_1+1}{a_1}\omega_1$.

3.4 Distribution Class $\mathcal{P}$

3.4.1 Equilibrium without resale

For this family, Plum (1992) finds that the equilibrium strategies $\beta_i^N : [0, \omega_i] \to \mathbb{R}$ are:

$$\beta_1^N(x) = \frac{(1+kx^{a+1})^{\frac{a}{a+1}} - 1}{kx^a} \text{ and } \beta_2^N(x) = \frac{1-(1-kx^{a+1})^{\frac{a}{a+1}}}{kx^a} \tag{3.8}$$

where

$$k \equiv \frac{1}{\omega_2^{a+1}} - \frac{1}{\omega_1^{a+1}} > 0$$

The maximum amount bid by either bidder is

$$\bar{b}^N = \omega_1\omega_2 \frac{\omega_1^a - \omega_2^a}{\omega_1^{a+1} - \omega_2^{a+1}}$$

The inverse bid functions $\phi_i^N : [0, \bar{b}^N] \to [0, \omega_i]$, however, cannot be written in closed form. Nevertheless, if we define $A(x) = (1+kx^{a+1})^{\frac{1}{a+1}}$ and $B(x) = (1-kx^{a+1})^{\frac{1}{a+1}}$, then it can be easily verified that

$$\beta_1^N(x) = \beta_2^N\left(\frac{x}{A(x)}\right) \text{ and } \beta_2^N(x) = \beta_1^N\left(\frac{x}{B(x)}\right)$$

if we write $y = \phi_2^N\left(\beta_1^N(x)\right)$ as the value for 2 such that he bids the same as 1 does when the latter's value is x, then

$$b = \beta_1^N(x) = \beta_2^N\left(\frac{x}{A(x)}\right) = \beta_2^N(y)$$

Since β_1^N and β_2^N are both increasing, we obtain the identity

$$y = \frac{x}{A(x)} \tag{3.9}$$

3.4.2 Equilibrium with resale

We now turn to first-price auctions with resale. From (3.5) we have that the inverse bidding strategies with resale satisfy

$$F_1(\phi_1(b)) = \left(\frac{\phi_1(b)}{\omega_1}\right)^a = \left(\frac{\phi_2(b)}{\omega_2}\right)^a = F_2(\phi_2(b))$$

so that

$$\phi_2(b) = \frac{\omega_2}{\omega_1}\phi_1(b)$$

Next we determine the pricing function $p(b)$ making use of (3.4):

$$p\left(b\right)=\arg\max\frac{1}{\omega_1^a}\left[\phi_1(b)^a-p^a\right]\left[p-\frac{\omega_2}{\omega_1}\phi_1(b)\right]$$

If we let $\phi_1(b)=x$, then the regularity of F_1 (see (3.1)) guarantees that $p\left(b\right)$ is the unique solution to the first-order condition

$$ap^{a-1}\frac{\omega_2}{\omega_1}x+x^a-\left(a+1\right)p^a=0$$

Notice that if we substitute $p=cx$ for some c, then the first-order condition becomes

$$x^a\left[\frac{\omega_2}{\omega_1}ac^{a-1}+1-\left(a+1\right)c^a\right]=0$$

The left-hand side of the equation above is decreasing in c. When $c=1$ it is negative. When $c=\sqrt{\frac{\omega_2}{\omega_1}}$, the bracketed expression on the left-hand side is

$$a\left(\frac{\omega_2}{\omega_1}\right)^{\frac{a+1}{2}}+1-\left(a+1\right)\left(\frac{\omega_2}{\omega_1}\right)^{\frac{a}{2}}$$

and we claim that this is positive. To see this, note that the function $at^{\frac{a+1}{2}}+1-\left(a+1\right)t^{\frac{a}{2}}$ is minimized at $t=1$ and its value there is 0.

Thus we have verified that there exists a c satisfying $\sqrt{\frac{\omega_2}{\omega_1}}<c<1$ such that the monopoly pricing function is

$$p(b)=c\phi_1(b)$$

where $\frac{\omega_2}{\omega_1}ac^{a-1}+1-\left(a+1\right)c^a=0$.

To find the inverse bidding strategies in the FPAR consider the differential equation (3.3), which now becomes

$$\frac{a\phi_1'(b)}{\phi_1(b)}=\frac{1}{c\phi_1(b)-b}$$

The solution to this is linear:

$$\phi_1^R(b)=\frac{a+1}{ac}b \tag{3.10}$$

and so we have

$$\phi_2^R(b) = \frac{a+1}{ac}\frac{\omega_2}{\omega_1}b \tag{3.11}$$

and as a result, $p(b) = \frac{a+1}{a}b$.

Thus equilibrium bidding strategies in the first-price auction with resale are

$$\beta_1^R(x) = \frac{ac}{a+1}x \text{ and } \beta_2^R(x) = \frac{\omega_1}{\omega_2}\frac{ac}{a+1}x$$

The maximum amount bid is $\bar{b}^R = \frac{ac}{a+1}\omega_1$.

3.4.3 Revenue comparison

The distribution of revenues—the highest bid—in the FPA is:

$$\begin{aligned} L^N(b) &= F_1\left(\phi_1^N(b)\right)F_2\left(\phi_2^N(b)\right) \\ &= \left(\frac{\phi_1^N(b)\,\phi_2^N(b)}{\omega_1\omega_2}\right)^a \end{aligned}$$

and similarly, the distribution of revenues in the FPAR is

$$L^R(b) = \left(\frac{\phi_1^R(b)\,\phi_2^R(b)}{\omega_1\omega_2}\right)^a$$

We will argue that for all b, $L^N(b) > L^R(b)$ which will imply that the revenues in the FPAR stochastic dominate the revenues in the FPA. This is equivalent to showing that for all b,

$$\frac{\left(\phi_1^N(b)\,\phi_2^N(b)\right)^{\frac{1}{2}}}{b} > \frac{\left(\phi_1^R(b)\,\phi_2^R(b)\right)^{\frac{1}{2}}}{b}$$

Making use of (3.9), (3.10) and (3.11), we obtain

$$\begin{aligned} \frac{\left(\phi_1^N(b)\,\phi_2^N(b)\right)^{\frac{1}{2}}}{b} &= \frac{\phi_1^N(b)}{\left(A\left(\phi_1^N(b)\right)\right)^{\frac{1}{2}}}\frac{k\left(\phi_1^N(b)\right)^a}{A(\phi_1^N(b))^a - 1} \\ \frac{\left(\phi_1^R(b)\,\phi_2^R(b)\right)^{\frac{1}{2}}}{b} &= \frac{a+1}{ac}\left(\frac{\omega_2}{\omega_1}\right)^{\frac{1}{2}} \end{aligned}$$

Since $\sqrt{\frac{\omega_2}{\omega_1}} < c$,

$$\frac{a+1}{a} > \frac{a+1}{ac}\left(\frac{\omega_2}{\omega_1}\right)^{\frac{1}{2}}$$

Therefore, it suffices to show that

$$\frac{1}{\left(A\left(\phi_1^N\left(b\right)\right)\right)^{\frac{1}{2}}}\frac{k\left(\phi_1^N\left(b\right)\right)^{a+1}}{A(\phi_1^N\left(b\right))^a - 1} \geq \frac{a+1}{a}$$

for all b.

It is convenient to write $z = A\left(\phi_1^N\left(b\right)\right)$ in above inequality. We claim that the function,

$$H\left(z\right) = \frac{z^{a+1}-1}{z^{a+\frac{1}{2}} - z^{\frac{1}{2}}}$$

which is defined for $z \geq 1$ is bounded below from $\frac{a+1}{a}$. We show this by establishing that $H\left(1\right) = \frac{a+1}{a}$ and that H is increasing.

Lemma 9. $H\left(1\right) = \frac{a+1}{a}$

Proof. At $z = 1$, we have a $\frac{0}{0}$ indeterminacy, but we can use L'Hopital's rule to conclude

$$\lim_{z\to 1}\frac{z^{a+1}-1}{z^{a+\frac{1}{2}} - z^{\frac{1}{2}}} = \lim_{z\to 1}\frac{\left(a+1\right)z^a}{\left(a+\frac{1}{2}\right)z^{a-\frac{1}{2}} - \frac{1}{2}z^{-\frac{1}{2}}} = \frac{a+1}{a}$$

∎

Lemma 10. $H'\left(z\right) > 0$ *for* $z > 1$.

Proof. Note that

$$H'\left(z\right) = \frac{z^{2a+1} - 1 - \left(z^{a+1} - z^a\right)\left(2a+1\right)}{2z^{\frac{3}{2}}\left(z^a - 1\right)^2}$$

We want to show that for all $z > 1$,

$$\gamma\left(z\right) \equiv \left(z^{2a+1} - 1\right) - \left(z^{a+1} - z^a\right)\left(2a+1\right) > 0$$

Note that $\gamma\left(1\right) = 0$ and

$$\gamma'\left(z\right) = \left(2a+1\right)\left(z^{2a} - \left(a+1\right)z^a - az^{a-1}\right)$$

Again $\gamma'(1) = 0$ and

$$\begin{aligned}\gamma''(z) &= (2a+1)\left(2az^{2a-1} - a(a+1)z^{a-1} - a(a-1)z^{a-2}\right) \\ &= (2a+1)az^{a-2}\left(2z^{a+1} - (a+1)z - (a-1)\right)\end{aligned}$$

Now note that the function $\psi(z) = 2z^{a+1} - (a+1)z - (a-1) > 0$ for all $z > 1$ since $\psi(1) = 0$ and $\psi'(z) = 2(a+1)z^a - (a+1) > 0$ for all $z > 1$. Thus we have argued that $\gamma''(z) > 0$. Now the fact that $\gamma'(1) = 0$ implies that for all $z > 1$, $\gamma'(z) > 0$. Finally, the fact that $\gamma(1) = 0$ now implies that for all $z > 1$, $\gamma(z) > 0$. ∎

Thus we have shown that when F_1 and F_2 belong to the family studied by Plum (1992), then the FPAR results in a higher revenue than the FPA.[2]

Proposition 8. *When the value distributions belong to the class $\mathcal{P}$, the revenue from a first-price auction with resale is greater than that from a first-price auction without resale.*

3.5 Distribution Class $\mathcal{C}$

3.5.1 Equilibrium without resale

For this family Cheng (2006) finds that the equilibrium strategies in the FPA are in fact linear:

$$\beta_1^N(x) = \frac{a_2}{a_2+1}x \text{ and } \beta_2^N(x) = \frac{a_1}{a_1+1}x$$

which gives the maximum bid of

$$\bar{b}^N = \frac{a_2}{a_2+1}\omega_1 = \frac{a_1}{a_1+1}\omega_2$$

[2]This proof shows that $\phi_1^R(b)\,\phi_2^R(b) < \phi_1^N(b)\,\phi_1^N(b)$ for all $b \in \left[0, \min\{\bar{b}^R, \bar{b}^N\}\right]$ which in turn implies that $\bar{b}^R > \bar{b}^N$ (as otherwise we should have $\phi_1^N(\bar{b}^R)\phi_1^N(\bar{b}^R) > \omega_1\omega_2$, which is not possible). This fact can be also shown by using direct arguments.

The resulting inverse bid functions are

$$\phi_1^N(b) = \frac{a_2+1}{a_2} b \text{ and } \phi_2^N(b) = \frac{a_1+1}{a_1} b$$

3.5.2 Equilibrium with resale

For Cheng's family, equilibrium strategies in the FPAR can only be determined implicitly.

Nevertheless, because of (3.7), it is sufficient to determine the distribution F and because of (3.6) this can be done without an explicit expression for the equilibrium strategies. For this family, condition (3.6) that determines the distribution F becomes

$$T(F(p)) \equiv \frac{a_2(a_1+1)}{(a_2+1)} F(p)^{\frac{1}{a_2}} \left(\frac{p}{\omega_1}\right)^{a_1-1} + F(p) - (a_1+1)\left(\frac{p}{\omega_1}\right)^{a_1} = 0 \quad (3.12)$$

3.5.3 Revenue comparison

The distribution of revenues in the FPA is:

$$L^N(b) = \left(\frac{a_2+1}{a_2}\frac{b}{\omega_1}\right)^{a_1}\left(\frac{a_1+1}{a_1}\frac{b}{\omega_2}\right)^{a_2} = \left(\frac{b}{\bar{b}^N}\right)^{a_1+a_2}$$

The revenue from the FPA is then given by

$$R^{FPA} = \bar{b}^N - \int_0^{\bar{b}^N} \left(\frac{b}{\bar{b}^N}\right)^{a_1+a_2} db = \frac{a_1+a_2}{a_1+a_2+1}\bar{b}^N = \frac{a_1+a_2}{a_1+a_2+1}\frac{a_2}{a_2+1}\omega_1$$

Consider the distribution function

$$G(p) = \left(\frac{p}{\omega_1}\frac{(a_1+a_2)(a_2+1)}{a_2(2+a_2+a_1)}\right)^{\frac{a_1+a_2}{2}}$$

It is routine to confirm that

$$\int_0^{\omega_1 \frac{a_2(2+a_2+a_1)}{(a_1+a_2)(a_2+1)}} (1-G(p))^2 \, dp = \frac{a_1+a_2}{a_1+a_2+1}\frac{a_2}{a_2+1}\omega_1 = R^{FPA}$$

In other words, G is a distribution such that a symmetric first-price auction in which both bidders draw values from G is *revenue equivalent* to an asymmetric first-price auction in which the bidders draw values from F_1 and F_2, respectively.

We will show that F determined in (3.12) *stochastically dominates* G; that is, for all $p > 0$, $F(p) < G(p)$. First, note that T, defined in (3.12), is an increasing function. Therefore it suffices to show that $T(G(p)) > 0$.

Lemma 11. $T(G(p)) > 0$.

Proof.

$$\begin{aligned} T(G(p)) &= \frac{a_2(a_1+1)}{(a_2+1)}\left(\frac{(a_2+1)(a_1+a_2)}{a_2(2+a_2+a_1)}\frac{p}{\omega_1}\right)^{\frac{a_1+a_2}{2a_2}}\left(\frac{p}{\omega_1}\right)^{a_1-1} \\ &\quad + \left(\frac{(a_2+1)(a_1+a_2)}{a_2(2+a_2+a_1)}\frac{p}{\omega_1}\right)^{\frac{a_1+a_2}{2}} - (a_1+1)\left(\frac{p}{\omega_1}\right)^{a_1} \\ &= \left(\frac{p}{\omega_1}\right)^{\frac{a_1+a_2}{2}}\left(\frac{(a_1+1)a_2}{a_2+1}\left(\frac{(a_2+1)(a_1+a_2)}{a_2(2+a_2+a_1)}\right)^{\frac{a_1+a_2}{2a_2}}\left(\frac{p}{\omega_1}\right)^{\frac{a_1-a_2}{2a_2}+\frac{a_1-a_2}{2}}\right. \\ &\quad \left. + \left(\frac{(a_2+1)(a_1+a_2)}{a_2(2+a_2+a_1)}\right)^{\frac{a_1+a_2}{2}} - (a_1+1)\left(\frac{p}{\omega_1}\right)^{\frac{a_1-a_2}{2}}\right) \end{aligned}$$

Let $\frac{p}{\omega_1} = r$, $\frac{(a_2+1)(a_1+a_2)}{a_2(2+a_2+a_1)} = m$, since $\left(\frac{p}{\omega_1}\right)^{\frac{a_1+a_2}{2}} > 0$, it suffices to show that

$$D(r) = m^{\frac{a_1+a_2}{2}} - (a_1+1)r^{\frac{a_1-a_2}{2}}\left(1 - \frac{a_2}{a_2+1}m^{\frac{a_1+a_2}{2a_2}}r^{\frac{a_1-a_2}{2a_2}}\right) > 0$$

Note that the function $r^{\frac{a_1-a_2}{2}}\left(1 - \frac{a_2}{a_2+1}m^{\frac{a_1+a_2}{2a_2}}r^{\frac{a_1-a_2}{2a_2}}\right)$ is maximized at

$$r = m^{-\frac{2a_2}{a_1-a_2}\frac{a_1+a_2}{2a_2}}$$

Therefore the minimum value of D is given by

$$\begin{aligned} \underline{D} &= m^{\frac{a_1+a_2}{2}} - (a_1+1)m^{-\frac{a_1+a_2}{2}}\left(1 - \frac{a_2}{a_2+1}\right) \\ &= m^{-\frac{a_1+a_2}{2}}\left(m^{a_1+a_2} - \frac{a_1+1}{a_2+1}\right) \end{aligned}$$

Therefore, it suffices to show that

$$\left(\frac{(a_2+1)(a_1+a_2)}{a_2(2+a_2+a_1)}\right)^{a_1+a_2} - \frac{a_1+1}{a_2+1} > 0$$

or equivalently that

$$\left(1+\frac{a_1-a_2}{a_2(2+a_2+a_1)}\right)^{a_1+a_2} > 1+\frac{a_1-a_2}{a_2+1}$$

and this follows from the fact that[3]

$$\left(1+\frac{a_1-a_2}{a_2(2+a_2+a_1)}\right)^{a_1+a_2} > 1+(a_1+a_2)\frac{a_1-a_2}{a_2(2+a_2+a_1)} > 1+\frac{a_1-a_2}{a_2+1}$$

∎

Thus we have shown that FPAR gives more revenue than FPA for Cheng's family.

Proposition 9. *When the value distributions belong to the class* $\mathcal{C}$, *the revenue from a first-price auction with resale is greater than that from a first-price auction without resale.*

3.6 Resale and efficiency

In this section, we examine the effects of resale as they pertain to efficiency. One line of thought, loosely associated with the "Chicago School," suggests that if the allocation from an auction is inefficient, then resale markets will reallocate in a way so as to ensure full efficiency. For resale markets to be fully efficient requires at the very least, as is now well understood, the absence of market power and the absence of incomplete information.

In the model we have formulated resale does not result in fully efficient outcomes. This is because it also takes place under incomplete information—the seller is unsure of the precise value of the buyer—and exercises his monopoly power.

[3] Since $(1+x)^k > 1+kx$ for $x>0$. This can be seen by noting the derivative of the former function is always greater than the latter.

With positive probability, an inefficient allocation remains inefficient even after the resale stage is over.

But does the presence of resale markets *enhance* efficiency? In other words, is the total social surplus higher with resale than without, even if neither reaches full efficiency levels? Ex post, of course, resale can only help to increase social surplus—with positive probability the object is transferred to the buyer with the higher value. Its ex ante effects, on the other hand, are not so clear since the possibility of resale affects bidding behavior and hence also how the object is allocated by the auction.

3.6.1 Surplus without resale

For a particular realization of values (x_1, x_2), bidder 2 wins if and only if (neglecting ties since they occur with zero probability):

$$\beta_2^N(x_2) > \beta_1^N(x_1)$$

or equivalently, when

$$x_2 > \phi_2^N(\beta_1^N(x_1)) \equiv Q^N(x_1) \tag{3.13}$$

Bidder 2 with value $Q^N(x_1)$ would bid the same amount as bidder 1 with value x_1.

The ex ante expected *social surplus* in the first-price auction without resale is

$$S^{FPA} = \int_0^{\omega_1} \left(\int_0^{Q^N(x_1)} x_1 dF_2(x_2) + \int_{Q^N(x_1)}^{\omega_2} x_2 dF_2(x_2) \right) dF_1(x_1) \tag{3.14}$$

For fixed x_1, the first integral in the parentheses is the surplus when bidder 1 wins and the second is the surplus when bidder 2 wins.

3.6.2 Surplus with resale

In the model with resale, the results of the auction do not, of course, represent the final allocation. Again, for a particular realization of the values (x_1, x_2), bidder 2 wins if and only if:

$$\beta_2^R(x_2) > \beta_1^R(x_1)$$

or equivalently, when

$$x_2 > \phi_2^R(\beta_1^R(x_1)) \equiv Q^R(x_1) \tag{3.15}$$

Suppose that 2 wins the auction when his value is x_2. In equilibrium, he will set a monopoly price of $p\left(\beta_2^R(x_2)\right)$. Bidder 1 will accept this offer if $p\left(\beta_2^R(x_2)\right) < x_1$, or equivalently if

$$x_2 < \min \phi_2^R\left(p^{-1}(x_1)\right) \equiv P(x_1) \tag{3.16}$$

Otherwise, the offer will be refused and the object will remain with bidder 2. Bidder 2 with value $P(x_1)$ would offer to sell the object for a price equal to x_1.

The ex ante expected *social surplus* in the first-price auction with resale isThe ex ante expected *social surplus* in the first-price auction with resale is

$$S^{FPAR} = \int_0^{\omega_1} \left(\begin{array}{c} \int_0^{Q^R(x_1)} x_1 dF_2(x_2) + \int_{Q^R(x_1)}^{P(x_1)} x_1 dF_2(x_2) \\ + \int_{P(x_1)}^{\omega_2} x_2 dF_2(x_2) \end{array} \right) dF_1(x_1)$$

For fixed x_1, the first integral in the parentheses is the surplus when bidder 1 wins and so there is no resale. The second integral is the surplus when 2 wins and sets a price low enough so that 1 accepts the offer and there is resale. The third integral is the surplus when 2 wins but sets a price so high that 1 rejects the offer.

The expression above can be simplified to

$$S^{FPAR} = \int_0^{\omega_1} \left(\int_0^{P(x_1)} x_1 dF_2(x_2) + \int_{P(x_1)}^{\omega_2} x_2 dF_2(x_2) \right) dF_1(x_1) \tag{3.17}$$

3.6.3 An example

Consider the following example in which the distributions belong to the class $\mathcal{P}$. Specifically, suppose that the two distributions are uniform over different supports, that is,

$$F_1(x) = \frac{x}{\omega_1} \text{ and } F_2(x) = x$$

for some $\omega_1 > 1$.

3.6.4 Equilibrium without resale

The equilibrium strategies in the FPA are determined as in (3.8):

$$\beta_1^N(x) = \frac{\sqrt{1+kx^2}-1}{kx} \text{ and } \beta_2^N(x) = \frac{1-\sqrt{1-kx^2}}{kx}$$

where

$$k \equiv 1 - \frac{1}{\omega_1^2} > 0$$

For the case when $\omega_1 = \frac{3}{2}$, these are depicted in the figure below as the thin curved lines. As always, the weak bidder 2 bids higher than bidder 1. The inverse bidding strategies are

$$\phi_1^N(b) = \frac{2b}{1-kb^2} \text{ and } \phi_2^N(b) = \frac{2b}{1+kb^2}$$

Thus the function Q^N defined in (3.13) is

$$\begin{aligned} Q^N(x_1) &= \phi_2^N(\beta_1^N(x_1)) \\ &= \frac{x_1}{\sqrt{1+kx_1^2}} \end{aligned}$$

3.6.5 Equilibrium with resale

Using (3.10) and (3.11) the inverse bidding strategies in the FPAR are

$$\phi_1^R(b) = \frac{4\omega_1}{\omega_1+1}b \text{ and } \phi_2^R(b) = \frac{4}{\omega_1+1}b$$

The pricing function is

$$p(b) = 2b$$

and the bidding strategies are

$$\beta_1^R(x) = \frac{\omega_1+1}{4\omega_1}x \text{ and } \beta_2^R(x) = \frac{\omega_1+1}{4}x$$

Once again, for the case of $\omega_1 = \frac{3}{2}$, these are depicted in the figure below as the thick straight lines (the thin lines are the bidding strategies without resale). Once again the weak bidder 2 bids higher than the strong bidder 1. Notice also that resale causes the weak bidder to bid higher, that is, for all $x > 0$,

$$\beta_2^R(x) > \beta_2^N(x)$$

This, of course, occurs because resale possibilities increase the value of winning for the weak bidder—he has the option of reselling it to the the strong bidder. Once the asymmetries are significant, say if $\omega_1 > 3$, then $\beta_2^R(x) > x$. That is, the weak bidder submits bids *higher* than his own value. In this case, the "speculative" motive of resale dominates any value he derives from direct consumption.

The functions Q^R and P, defined in (3.15) and (3.16), respectively, are

$$Q^R(x_1) = \frac{x_1}{\omega_1}$$

and

$$P(x_1) = \min\left\{\frac{2x_1}{\omega_1 + 1}, 1\right\}$$

since $P(x_1)$ can never exceed $\omega_2 = 1$.

3.6.6 Surplus comparison

Using (3.14), the expected surplus in the first-price auction without resale is

$$\begin{aligned} S^{FPA} &= \frac{1}{\omega_1}\int_0^{\omega_1}\left(\int_0^{Q^N(x_1)} x_1 dx_2 + \int_{Q^N(x_1)}^1 x_2 dx_2\right) dx_1 \\ &= \frac{1}{\omega_1}\int_0^{\omega_1}\left(\frac{x_1^2\omega_1}{\sqrt{\omega_1^2 - x_1^2 + x_1^2\omega_1^2}} - \frac{1}{2}\frac{x_1^2\omega_1^2}{(\omega_1^2 - x_1^2 + x_1^2\omega_1^2)} + \frac{1}{2}\right) dx_1 \end{aligned}$$

since $Q^N(x_1) = \frac{x_1}{\sqrt{1+kx_1^2}} = \frac{\omega_1 x_1}{\sqrt{\omega_1^2 - x_1^2 + x_1^2\omega_1^2}}$.

Using (3.17), the expected surplus in the first-price auction with resale is

$$\begin{aligned} S^{FPAR} &= \frac{1}{\omega_1}\int_0^{\frac{\omega_1+1}{2}}\left(\int_0^{\frac{2}{\omega_1+1}x_1} x_1 dx_2 + \int_{\frac{2}{\omega_1+1}x_1}^1 x_2 dx_2\right) dx_1 + \frac{1}{\omega_1}\int_{\frac{\omega_1+1}{2}}^{\omega_1} x_1 dx_1 \\ &= \frac{11}{24}\omega_1 + \frac{1}{8\omega_1} + \frac{1}{12} \end{aligned}$$

While a comparison of S^{FPA} and S^{FPAR} cannot be made analytically, these can be evaluated numerically.

For this example, the degree of asymmetry can be measured by one parameter alone, ω_1. When the degree of asymmetry is small, that is, $\omega_1 - \omega_2 = \omega_1 - 1$ is small, it is indeed the case that resale improves efficiency—the expected social surplus under resale is higher than the expected social surplus without resale.

When the degree of asymmetry is somewhat large, that is, $\omega_1 - \omega_2 = \omega_1 - 1$ is somewhat larger, a somewhat surprising phenomenon occurs—the social surplus under resale is *smaller* than the social surplus without resale. Resale may decrease ex ante efficiency!

The reason for this is that with large asymmetries, the weak bidder bids more aggressively since the expected profits from resale are also large. This causes the allocation at the end of auction to be so inefficient that even post-auction resale is unable to compensate enough.

Specifically, numerical analysis reveals that for all $\omega_1 \in (1, \omega_1^*)$ where $\omega_1^* \simeq 1.95$

$$S^{FPA} < S^{FPAR}$$

and for all $\omega_1 \in (\omega_1^*, 10)$

$$S^{FPA} > S^{FPAR}$$

3.7 Conclusion

We have compared the performance of first-price auctions with and without resale. This comparison has been carried out for the two main families of distributions $\mathcal{P}$ and $\mathcal{C}$ for which the equilibrium of the first-price auction can be explicitly characterized. We have shown that in both cases, resale improves the revenue of the original seller. In an earlier paper, Hafalir and Krishna (2006), we have shown that the first-price auction with resale (FPAR) is revenue superior to the second-price auction (SPA) whenever the distributions are regular. Thus it is also true that for the family of distributions for which the SPA is revenue superior to the FPA (see Maskin and Riley, 2000 for examples), the FPAR is revenue superior to the FPA. Thus we are led to conjecture:

Conjecture 1. *For all regular distributions, the revenue from the first-price auction with resale is higher than that from the first-price auction, that is,*

$$R^{FPAR} > R^{FPA}$$

We also examined the efficiency properties of resale. While resale does not restore efficiency to a first-price auction, we showed that, in fact, it may decrease efficiency.

3.8 References

1. Athey, S. (2001): "Single Crossing Properties and the Existence of Pure Strategy Equilibria in Games of Incomplete Information," *Econometrica*, 69, 861–889.

2. Cantillon, E. (2005): "The Effects of Bidder Asymmetries on Expected Revenue in Auctions," *Games and Economic Behavior*, forthcoming.

3. Cheng, H. (2006): "Ranking Sealed High-Bid and Open Asymmetric Auctions," *Journal of Mathematical Economics*, 42, 471–498.

4. Garratt, R. and T. Tröger (2006): "Speculation in Standard Auctions with Resale," *Econometrica*, 74, 753–770.

5. Griesmer, J., R. Levitan and M. Shubik (1967): "Toward a Study of Bidding Processes, Part IV—Games with Unknown Costs," *Naval Research Logistics Quarterly*, 14, 415–434.

6. Hafalir, I. and V. Krishna (2006): "Asymmetric Auctions with Resale," mimeo, Penn State University, October.

7. Krishna, V. (2002): *Auction Theory*, San Diego: Academic Press.

8. Marshall, R., M. Meurer, J.-F. Richard, and W. Stromquist (1994): "Numerical Analysis of Asymmetric First Price Auctions," *Games and Economic Behavior*, 7, 193–220.

9. Maskin, E. and J. Riley (2000): "Asymmetric Auctions," *Review of Economic Studies*, 67, No. 3, 413–438.

10. Maskin, E. and J. Riley (2003): "Uniqueness of Equilibrium in Sealed High-Bid Auctions," *Games and Economic Behavior*, 45, 395–409.

11. Myerson, R. (1981): "Optimal Auction Design," *Mathematics of Operations Research*, 6, No. 1, 58–73.

12. Plum, M. (1992): "Characterization and Computation of Nash Equilibria for Auctions with Incomplete Information," *International Journal of Game Theory*, 20, 393–418.

13. Vickrey, W. (1961): "Counterspeculation, Auctions and Competitive Sealed Tenders," *Journal of Finance*, 16, 8–37.

Vita

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