# THE PICARD GROUP OF THE MODULI SPACE OF SHEAVES ON A QUADRIC SURFACE 

A Dissertation in<br>Mathematics<br>by<br>Dmitrii Pedchenko

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## Abstract

In this dissertation, we study the Picard group of the moduli space of semistable sheaves on a smooth quadric surface. We polarize the surface by an ample divisor close to the anticanonical class. We focus especially on moduli spaces of sheaves of small discriminant, where we observe new and interesting behavior. Our method relies on constructing certain resolutions for semistable sheaves and applying techniques of geometric invariant theory to the resulting families of sheaves.

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## Chapter 1

## Introduction

## Moduli spaces of vector bundles and sheaves

The topic of this dissertation is centered around moduli theory of vector bundles in algebraic geometry. The notion of a vector bundle is fundamental to modern geometry. Among the many applications of vector bundles, let us mention that they are useful for studying the geometry of submanifolds of a given manifold $X$, for defining maps from $X$ to other manifolds and as a tool for encoding additional geometric data on $X$ (e.g. a metric, a symplectic form, a spinor field, etc.). Interesting examples of vector bundles are the tangent bundle $T_{X}$ and various tensor bundles derived from $T_{X}$. But there are many other interesting vector bundles on $X$.

Topological vector bundles are classified by discrete invariants such as characteristic classes or homotopy classes of maps to the classifying space. In algebraic geometry, algebraic vector bundles with fixed discrete invariants further admit continuous deformations. In order to get a good (e.g. finite type or projective) deformation space one needs to work with the so-called semistable bundles which are building blocks for arbitrary vector bundles on $X$. The resulting moduli spaces are of great interest in algebraic geometry, as they provide examples of higher dimensional varieties with a rich and interesting geometry. Furthermore, they play an important role in gauge theory and topology of four-manifolds [Don90], and in mathematical physics [Wit95].

These moduli spaces have been intensively studied over the years, but many basic questions about their geometry remain open. In particular, the basic birational properties of moduli spaces of sheaves are only known for a small set of surfaces: $\mathbb{P}^{2}$, abelian surfaces, $K 3$ surfaces, and Enriques surfaces. This dissertation, the corresponding paper [Ped20], and Ryan's work [Rya18] are the first steps towards understanding the birational geometry of moduli spaces of sheaves on $X=\mathbb{P}^{1} \times \mathbb{P}^{1}$.

Before we move on, let us first recall some basic definitions and notation concerning semistable sheaves.

## Semistable sheaves

Let $X$ be a smooth projective variety and let $H$ be an ample divisor on it. For a torsion-free sheaf $\mathcal{F}$ or rank $r(\mathcal{F})$, introduce the reduced $H$-Hilbert polynomial

$$
p_{\mathcal{F}}(m)=\frac{\chi(\mathcal{F}(m H))}{r(\mathcal{F})}
$$

A torsion-free sheaf $\mathcal{V}$ on $X$ is $H$-(semi)stable if for any subsheaf $\mathcal{W} \subsetneq \mathcal{V}$ one has

$$
p_{\mathcal{W}}(m) \leq_{,} p_{\mathcal{V}}(m), \quad m \gg 0 .
$$

We will denote by $M_{X, H}(\mathbf{v})$ the moduli space of $H$-semistable sheaves on $X$ with discrete invariants $\mathbf{v} \in K(X)$. Points of $M_{X, H}(\mathbf{v})$ parameterize certain equivalence classes [ $\mathcal{V}$ ] of $H$-semistable sheaves $\mathcal{V}$ with $\mathbf{v}(\mathcal{V})=\mathbf{v}$.

For a smooth surface $X$ the discrete invariants $\mathbf{v} \in K(X)$ are usually encoded by either the rank and the Chern classes $\mathbf{v}=\left(r, c_{1}, c_{2}\right)$ or the Chern character $\mathbf{v}=\left(c h_{0}, c h_{1}, c h_{2}\right)$. However, for us it will be convenient to instead package $\mathbf{v}$ into the so called logarithmic invariants $\mathbf{v}=(r, \nu, \Delta)$ : recall that the total slope and discriminant of a Chern character $\mathbf{v} \in K(X)$ are defined by

$$
\nu=\frac{c_{1}}{r}, \quad \Delta=\frac{1}{2} \nu^{2}-\frac{c h_{2}}{r} .
$$

The convenience of the logarithmic invariants stems from the fact that they are additive on tensor products of vector bundles.

## The goal of this work

In this dissertation, we set out to compute the Picard group of the moduli space of semistable sheaves $M_{\mathbb{P}^{1} \times \mathbb{P}^{1}, H}(\mathbf{v})$ on $X=\mathbb{P}^{1} \times \mathbb{P}^{1}$ for as many Chern characters $\mathbf{v}$ as possible. Previous work of various authors already establishes basic geometric properties of $M_{\mathbb{P}^{1} \times \mathbb{P}^{1}, H}(\mathbf{v})$ such as when it is nonempty, smooth, what is its dimension, etc. Studying the Picard group was the next natural step in investigating the deeper geometry of the moduli space.

Before we explain the main results of this dissertation concerning $\operatorname{Pic}\left(M_{\mathbb{P}^{1} \times \mathbb{P}^{1}, H}(\mathbf{v})\right)$ it would be instructive to recall the historically first case of calculating the Picard group of the moduli space of sheaves on $X=\mathbb{P}^{2}$.

## The $\mathbb{P}^{2}$ case

By [DLP85], there is a fractal-like curve $\operatorname{DLP}(\nu)$ in the $(\nu, \Delta)$-plane, which we call the Drézet-Le Potier curve (DLP-curve for short), such that the moduli space $M_{\mathbb{P}^{2}}(\mathbf{v}):=M_{\mathbb{P}^{2}, \mathcal{O}(1)}(\mathbf{v})$ is positive dimensional if and only if $\Delta \geq \operatorname{DLP}(\nu)$, where $\mathbf{v}=(r, \nu, \Delta)$. The Drézet-Le Potier curve is defined piecewise, and the branches are constructed using the numerical invariants of exceptional bundles (see Figure 1.1 (a)).

Drézet [Dre88] shows that the Picard number $\rho$ of $M_{\mathbb{P}^{2}}(\mathbf{v})$ is determined by the position of $\mathbf{v}=$ $(r, \nu, \Delta)$ relative to the DLP-curve.

Theorem 1.1 ([Dre88, Theorem B]). Let $\mathbf{v}=(r, \nu, \Delta) \in K\left(\mathbb{P}^{2}\right)$ be a character with $r \geq 2$.

1. If $\mathbf{v}$ lies above the DLP-curve, then $\rho\left(M_{\mathbb{P}^{2}}(\mathbf{v})\right)=2$,
2. If $\mathbf{v}$ lies on the DLP-curve, then $\rho\left(M_{\mathbb{P}^{2}}(\mathbf{v})\right)=1$.


Figure 1.1: The graph of $\Delta=\operatorname{DLP}(\nu)$.

## The $\mathbb{P}^{1} \times \mathbb{P}^{1}$ case

Now, consider $\mathbb{P}^{1} \times \mathbb{P}^{1}$ with a generic polarization $H$ close to the anticanonical one. The choice of the polarization is motivated by the fact that for a Del Pezzo surface we are usually interested in working with the anticanonical polarization, but for $\mathbb{P}^{1} \times \mathbb{P}^{1}$ with the anticanonical polarization the moduli space of sheaves is often a highly singular variety (see $\S 2.3$ for more details). By perturbing the polarization we restore the nice local geometry of the moduli space. Concretely, since

$$
\operatorname{Pic}\left(\mathbb{P}^{1} \times \mathbb{P}^{1}\right) \cong \mathbb{Z} E \oplus \mathbb{Z} F,
$$

$E$ and $F$ being the two divisors $\mathbb{P}^{1} \times\{p t\}$ and $\{p t\} \times \mathbb{P}^{1}$, the anticanonical polarization is

$$
-\frac{1}{2} K_{X}=E+F \text {. }
$$

So we instead choose

$$
H=E+(1+\epsilon) F, \quad \epsilon \in \mathbb{Q}, \quad 0<|\epsilon| \ll 1
$$

Additionally, only for such a polarization there is a complete classification of Chern characters $\mathbf{v}$ for which the moduli space is nonempty or positive-dimensional due to Rudakov. First note that now the total slope $\nu$ is represented by a point on the plane of slopes $\mathbb{Q} E \oplus \mathbb{Q} F$. Rudakov [Rud94] constructed a fractal-like surface $\operatorname{DLP}(\nu)$ in the $(\nu, \Delta)$-space, which we continue to call the Drézet-Le Potier surface, such that again the moduli space $M(\mathbf{v}):=M_{\mathbb{P}^{1} \times \mathbb{P}^{1}, H}(\mathbf{v})$ is positive dimensional if and only if $\Delta \geq \operatorname{DLP}(\nu)$, where $\mathbf{v}=(r, \nu, \Delta)$. This surface is also comprised of branches given by the exceptional bundles on $\mathbb{P}^{1} \times \mathbb{P}^{1}$ (see Figure 1.1 (b); the details of this construction are presented in $\S 2.7$ ).

The main new feature compared with the $\mathbb{P}^{2}$ case is that now there exist characters $\mathbf{v}$ lying on the intersection of two branches of the DLP-surface. This way, a character $\mathbf{v}=(r, \nu, \Delta)$ with positive dimensional moduli space $M(\mathbf{v})$ can be positioned in three different ways with respect to the DLP-surface:

1. $\mathbf{v}$ lies above the surface,
2. $\mathbf{v}$ lies on a single branch of the surface,
3. $\mathbf{v}$ lies on the intersection of two branches of the surface.

Let us remark that characters lying on a triple intersection of branches of the DLP-surface are the characters of exceptional bundles and therefore have zero dimensional moduli spaces that are not interesting from the viewpoint of calculating the Picard group.

## Main results of the dissertation

We compute the Picard group of $M(\mathbf{v})$ for the majority of characters $\mathbf{v}$, and we encounter a new interesting behavior which initially was not expected by the analogy with the $\mathbb{P}^{2}$ case. Our key finding is that contrary to the $\mathbb{P}^{2}$ case the Picard number $\rho$ of $M(\mathbf{v})$ is not determined only by the position of $\mathbf{v}$ relative to the DLP-surface. The main results of this dissertation are summarized in the following theorem that is a streamlined but weaker version of Theorems 5.1 and 6.3.

Theorem 1.2. Let $\mathbf{v}=(r, \nu, \Delta) \in K\left(\mathbb{P}^{1} \times \mathbb{P}^{1}\right)$ be a character with $r \geq 2$ and $\Delta \geq \frac{1}{2}$.

1. If $\mathbf{v}^{\prime}=\left(r, \nu, \Delta-\frac{1}{r}\right)$ lies above the DLP-surface, then $\rho(M(\mathbf{v}))=3$.
2. If $\mathbf{v}$ lies on a single branch of the DLP-surface, then

$$
\rho(M(\mathbf{v}))=2 \text { or } \rho(M(\mathbf{v}))=1 .
$$

3. If $\mathbf{v}$ lies on the intersection of two branches of the DLP-surface, then $\rho(M(\mathbf{v}))=1$.

Furthermore, if $\mathbf{v}$ is a primitive character, then $\operatorname{Pic}(M(\mathbf{v}))$ is a free abelian group of rank $\rho$.
By analogy with the $\mathbb{P}^{2}$ case, we were expecting to have $\rho=2$ for any character $\mathbf{v}$ lying on a single branch of the DLP-surface. Case (2) of the above theorem shows that this is not the case. We split Chern characters into two groups based on their numerical invariants, calling the characters in the first group good characters and the characters in the second group bad characters, see Definition 3.8. We show that a typical character $\mathbf{v}$ lying on a single branch of the DLP-surface is a good character, and for good characters the Picard number of $M(\mathbf{v})$ is equal to 2 , see Theorem 5.1. On the other hand, we construct sporadic infinite sequences of bad characters $\mathbf{v}$ lying on a single branch of the DLPsurface whose moduli spaces $M(\mathbf{v})$ have $\rho=1$, see Examples 6.1, 6.2. In fact, for one such character $\mathbf{v}=(4,-1 / 4 E-1 / 4 F, 9 / 16)$ of the smallest rank we somewhat surprisingly show

$$
M(\mathbf{v}) \cong \mathbb{P}^{3}
$$

see Example 6.4 and Question 6.5. In general, we show that $\rho(M(\mathbf{v}))=1$ for any bad character lying on a single branch of the DLP-surface given by a line bundle, see Theorem 6.3.

We emphasize that determining which statement of the above theorem applies to a given character $\mathbf{v}=(r, \nu, \Delta)$ is a finite computational procedure and therefore can be implemented on a computer: both the computation of $\operatorname{DLP}(\nu)$ and determining whether $\mathbf{v}$ is a good or a bad character are finite computational procedures.

The proofs of the above results rely on constructing resolutions for semistable sheaves of a given character $\mathbf{v}$ and applying techniques of Mumford's geometric invariant theory to the resulting families of sheaves described by such resolutions. In the case of $\mathbb{P}^{2}$ the most powerful tool for constructing resolutions of semistable sheaves is the Beilinson-type spectral sequence coming from a choice of a full exceptional collection (see [CHW17, §5] for a detailed analysis). The main difficulty is that full exceptional collections on $\mathbb{P}^{1} \times \mathbb{P}^{1}$ require four exceptional bundles instead of three in the case of $\mathbb{P}^{2}$. As a result, for most characters $\mathbf{v}$ writing the associated Beilinson-type spectral sequence no longer gives a resolution of a semistable sheaf $\mathcal{V}$ of character $\mathbf{v}$ on $\mathbb{P}^{1} \times \mathbb{P}^{1}$ as a (co)kernel in a short exact sequence. To circumvent this difficulty we instead use the so-called Gaeta-type resolutions of Coskun and Huizenga constructed recently in [CH18].

## Full conjectural answer

Note that the above theorem does not cover all numerical invariants $\mathbf{v}$ with positive-dimensional moduli spaces: case (1) of the above theorem uses a modified character $\mathbf{v}^{\prime}$ and case (2) is formulated somewhat informally. We show in Theorem 5.1 that for good characters we have the full answer. However, with the techniques used in this dissertation we could not figure out the value of $\rho(M(\mathbf{v}))$ for bad characters $\mathbf{v}$ in certain small regions in the $(\nu, \Delta)$-space. We conjecture that for all characters $\mathbf{v}$ with positivedimensional moduli space the Picard number of $M(\mathbf{v})$ is fully determined by the relative position of $\mathbf{v}$ with respect to the DLP-surface and by whether the character is good or bad, see Question 3.11 and Conjecture 6.6.

## Previous work

Finally, let us survey the previous results on the Picard group of the moduli space of semistable sheaves on $\mathbb{P}^{1} \times \mathbb{P}^{1}$. The Picard group of $M(\mathbf{v})$ was studied by Nakashima [Nak93] and Qin [Qin92] for characters $\mathbf{v}=(r, \nu, \Delta)=\left(r, c_{1}, \chi\right) \in K\left(\mathbb{P}^{1} \times \mathbb{P}^{1}\right)$ satisfying $r=2$ and $c_{1} \cdot F=1$, by Yoshioka [Yos95] for characters $\mathbf{v}$ satisfying $r=2$, and by Yoshioka [Yos96b] for characters $\mathbf{v}$ satisfying $c_{1} \cdot F=0$ with the asymptotic polarization $H_{m}=E+m F, m \gg 0$. Yoshioka [Yos96a] also computed equivariant Picard groups of certain spaces closely related to the moduli space $M(\mathbf{v})$. This dissertation is a natural continuation of this line of work.

## Organization of the dissertation

In §2, we recall the preliminary facts and survey the known results concerning vector bundles and moduli spaces of sheaves on $\mathbb{P}^{1} \times \mathbb{P}^{1}$ needed in the rest of the dissertation.

Chapters 3-5 form the technical core of this manuscript. In §3, we study the Shatz stratification of complete families of sheaves on $\mathbb{P}^{1} \times \mathbb{P}^{1}$. We also prove the irreducibility of families parameterizing sheaves with a fixed Harder-Narasimhan filtration, which we later use to prove the irreducibility of Shatz strata in complete families of vector bundles admitting a Gaeta-type resolution. In §4, we establish basic facts about group actions in the context of Gaeta-type resolutions. In $\S 5$, we calculate the Picard group of $M(\mathbf{v})$ under the assumption that $\mathbf{v}$ is a good character.

Finally in $\S 6$, we study $\operatorname{Pic}(M(\mathbf{v}))$ for bad characters $\mathbf{v}$ that lie on a single branch of the DLP-surface given by a line bundle.

## Chapter 2

## Preliminaries

In this chapter we recall basic facts, previous results and constructions concerning moduli spaces of sheaves that will be used in the rest of the manuscript. We will denote an arbitrary variety or an arbitrary projective surface by $Y$, while $X$ will be always reserved for the quadric surface $X=\mathbb{P}^{1} \times \mathbb{P}^{1}$.

### 2.1 Chern characters

Given a torsion-free sheaf $\mathcal{V}$ on a surface $Y$ and an ample divisor $H$, the total slope $\nu$, the $H$-slope $\mu_{H}$ and the discriminant $\Delta$ are defined by

$$
\nu(\mathcal{V})=\frac{c h_{1}(\mathcal{V})}{c h_{0}(\mathcal{V})}, \quad \mu_{H}(\mathcal{V})=\frac{c h_{1}(\mathcal{V}) \cdot H}{c h_{0}(\mathcal{V})}, \quad \Delta(\mathcal{V})=\frac{1}{2} \nu^{2}-\frac{c h_{2}(\mathcal{V})}{c h_{0}} .
$$

These quantities depend only on the Chern character of $\mathcal{V}$ and not on the particular sheaf. Given a Chern character $\mathbf{v} \in K(Y)$, we define its total slope, $H$-slope and discriminant by the same formulae. We will often record Chern characters by the rank, total slope and discriminant. Note that one can recover the Chern classes from this data.

### 2.2 Stability

We refer the reader to [Hui17], [HL10] and [LP97] for more detailed discussions. Let $Y$ be a surface and $H$ be an ample divisor on it. A torsion-free coherent sheaf $\mathcal{V}$ is called $\mu_{H^{-}}$(semi)stable (or slope (semi)stable) if every proper subsheaf $0 \neq \mathcal{W} \subsetneq \mathcal{V}$ of smaller rank satisfies

$$
\mu_{H}(\mathcal{W}) \leq_{,} \mu_{H}(\mathcal{V})
$$

Define the $H$-Hilbert polynomial $P_{H, \mathcal{V}}(m)$ and the reduced $H$-Hilbert polynomial $p_{H, \mathcal{V}}(m)$ of a torsionfree sheaf $\mathcal{V}$ by

$$
P_{H, \mathcal{V}}(m)=\chi(\mathcal{V}(m H)), \quad p_{H, \mathcal{V}}(m)=\frac{P_{\mathcal{V}}(m)}{r(\mathcal{V})} .
$$

A torsion-free sheaf $\mathcal{V}$ is $H$-(semi)stable (or Gieseker (semi)stable) if for every proper subsheaf $0 \neq \mathcal{W} \subsetneq$ $\mathcal{V}$ we have

$$
p_{H, \mathcal{W}}(m)\left(\leq_{H, \mathcal{V}}(m) \text { for } m \gg 0\right.
$$

Slope stability implies Gieseker stability and Gieseker semistability implies slope semistability.
Every torsion-free sheaf $\mathcal{V}$ admits a Harder-Narasimhan filtration with respect to both $\mu_{H^{-}}$and $H$-semistability, that is there is a finite filtration

$$
0=\mathcal{V}_{0} \subset \mathcal{V}_{1} \subset \mathcal{V}_{2} \subset \ldots \subset \mathcal{V}_{n}=\mathcal{V}
$$

such that the quotients $\mathcal{W}_{i}=\mathcal{V}_{i} / \mathcal{V}_{i-1}$ are $\mu_{H}$ (respectively, $H$-Gieseker) semistable and

$$
\mu_{H}\left(\mathcal{W}_{i-1}\right)>\mu_{H}\left(\mathcal{W}_{i}\right) \quad\left(\text { respectively, } p_{H, \mathcal{W}_{i-1}}(m)>p_{H, \mathcal{W}_{i}}(m) \text { for } m \gg 0\right)
$$

for $1 \leq i \leq n$. The Harder-Narasimhan filtration is unique. A semistable sheaf further admits a JordanHölder filtration into stable sheaves. Two semistable sheaves are called $S$-equivalent if they have the same associated graded objects with respect to a Jordan-Hölder filtration.

Our main object of study will be the moduli space $M_{H}(\mathbf{v})$ parameterizing $S$-equivalence classes of $H$-Gieseker semistable sheaves of character $\mathbf{v}$ on $Y$. We refer the reader to [HL10, §4.3] for the details about the construction of $M_{H}(\mathbf{v})$ and its basic properties.

### 2.3 Choosing the polarization

For our purposes, we would like to work with a locally factorial moduli space. After recalling some definitions and results from [HL10, $\S 4 . \mathrm{C}$ ], we show that if $Y$ is rational surface other than $\mathbb{P}^{2}$, then it is always possible to vary the polarization $H$ slightly so that $M_{H}(\mathbf{v})$ becomes locally factorial.

Let $Y$ be a smooth projective surface. The intersection pairing defines a bilinear form on $\operatorname{Num}(Y)$ and the Hodge Index Theorem implies that the extension of this bilinear form to $N u m_{\mathbb{R}}(Y)$ defines the Minkowski metric on $N u m_{\mathbb{R}}$ (that is the signature of the form is $(1, N)$ ). Define the positive cone as

$$
K_{+}:=\left\{y \in N u m_{\mathbb{R}}(Y) \mid y \cdot y>0 \text { and } y \cdot H>0 \text { for some ample divisor } H\right\},
$$

and note that it contains the positive span of ample divisors as an open subcone $\operatorname{Amp}(Y)$. Since we can think of a polarization given by an ample divisor as a ray $\mathbb{R}_{>0} H \subset K_{+}$, it is convenient to introduce $\mathcal{H}$ as the set of rays in $K_{+}$. This set becomes a hyperbolic manifold if we make the identification $\mathcal{H} \cong\left\{H \in K_{+} \mid H \cdot H=1\right\}$.
Definition 2.1 ([HL10, Definition 4.C.1]). Let $r \geq 2$ be an integer and $\Delta>0$ a real number ${ }^{1}$. A class $\xi \in \operatorname{Num}(X)$ is of type $(r, \Delta)$ if $-\frac{r^{4}}{2} \Delta \leq \xi \cdot \xi<0$. The wall defined by $\xi$ is the real hypersuface

$$
W_{\xi}:=\left\{\mathbb{R}_{>0} H \in \mathcal{H} \mid \xi \cdot H=0\right\} \subset \mathcal{H}
$$

When $r \geq 2$ and $\Delta>0$, Lemma 4.C. 2 in [HL10] asserts that the set of walls of type $(r, \Delta)$ is locally finite in $\mathcal{H}$. It is therefore always possible to choose $H$ to not lie on any wall of type $(r, \Delta)$ by a small perturbation.
Lemma 2.2 ([HL10, Lemma 4.C.3]). Let $H$ be an ample divisor, let $\mathcal{V}$ be a $\mu_{H}$-semistable sheaf of rank $r$ and discriminant $\Delta$ on $Y$ and let $\mathcal{V}^{\prime} \subset \mathcal{V}$ be a subsheaf of rank $r^{\prime}, 0<r^{\prime}<r$, with $\mu_{H}\left(\mathcal{V}^{\prime}\right)=\mu_{H}(\mathcal{V})$. Then $\xi:=r c_{1}\left(\mathcal{V}^{\prime}\right)-r^{\prime} c_{1}(\mathcal{V})$ satisfies

$$
\xi \cdot H=0 \text { and }-\frac{r^{4}}{2} \Delta \leq \xi \cdot \xi \leq 0
$$

[^0]and
$$
\xi \cdot \xi=0 \Longleftrightarrow \xi=0
$$

From this we can prove that if $H$ does not lie on a wall, then the quotients in a Jordan-Hölder filtration all have the same numerical invariants.

Lemma 2.3. Given a Chern character $\mathbf{v}=(r, \nu, \Delta) \in K(Y)$ with $r \geq 2$, choose an ample divisor $H$ not on a wall of type $(r, \Delta)$. Then for any $\mu_{H}$-semistable sheaf $\mathcal{V}$ of Chern character $\mathbf{v}$ and a subsheaf $\mathcal{V}^{\prime} \subset \mathcal{V}$ of rank $r^{\prime}, 0<r^{\prime}<r$, we have

$$
\mu_{H}\left(\mathcal{V}^{\prime}\right)=\mu_{H}(\mathcal{V}) \Longleftrightarrow \nu\left(\mathcal{V}^{\prime}\right)=\nu(\mathcal{V}) .
$$

Proof. Suppose $\mathcal{V}^{\prime} \subset \mathcal{V}$ with $\mu_{H}\left(\mathcal{V}^{\prime}\right)=\mu_{H}(\mathcal{V})$, but $\nu\left(\mathcal{V}^{\prime}\right) \neq \nu(\mathcal{V})$, or equivalently $\xi:=r c_{1}\left(\mathcal{V}^{\prime}\right)-r^{\prime} c_{1}(\mathcal{V}) \neq 0$. By Lemma 2.2, we get that $\xi$ satisfies $-\frac{r^{4}}{2} \Delta \leq \xi \cdot \xi<0$. Since $\xi \cdot H=0$, we obtain that $H$ lies on a wall $W_{\xi}$ of type $(r, \Delta)$, contradicting our choice of $H$.

Corollary 2.4. Given a Chern character $\mathbf{v}=(r, \nu, \Delta) \in K(X)$ with $r \geq 2$, choose an ample divisor $H$ not on a wall of type $(r, \Delta)$. Then for any $H$-semistable sheaf $\mathcal{V}$ of Chern character $\mathbf{v}$ its Jordan-Hölder factors $\operatorname{gr}_{i}(\mathcal{V})$ satisfy

$$
\nu\left(\operatorname{gr}_{i}(\mathcal{V})\right)=\nu \quad \text { and } \quad \Delta\left(\operatorname{gr}_{i}(\mathcal{V})\right)=\Delta .
$$

Proof. By the definition of a Jordan-Hölder filtration $0 \subset F_{1} \subset F_{2} \subset \ldots \subset F_{l}=\mathcal{V}$, we have $\mu_{H}\left(F_{i}\right)=\mu_{H}(\mathcal{V})$. Then by Lemma 2.3 we get $\nu\left(F_{i}\right)=\nu$. We apply the "seesaw" property of the total slope to the short exact sequence

$$
0 \rightarrow F_{i-1} \rightarrow F_{i} \rightarrow \operatorname{gr}_{i}(\mathcal{V}) \rightarrow 0
$$

to get $\nu\left(\operatorname{gr}_{i}(\mathcal{V})\right)=\nu$. The statement about the discriminants then follows from the equality of reduced $H$-Hilbert polynomials $p_{H, \mathrm{gr}_{i}(\mathcal{V})}=p_{H, \mathcal{V}}$ and Riemann-Roch.

Now, let $(Y, H)$ be a polarized rational surface with $K_{Y} \cdot H<0$. Drézet [Dré91] calls a point in $M_{H}(\mathbf{v})$ a type 2 point if the corresponding $S$-equivalence class

$$
\left[\mathcal{V}_{1} \oplus \ldots \oplus \mathcal{V}_{k}\right]
$$

satisfies

$$
\nu\left(\mathcal{V}_{i}\right) \neq \nu\left(\mathcal{V}_{j}\right) \text { for some } 1 \leq i, j \leq k
$$

The other points are called type 1 points. Drézet shows in [Dré91, Theorem C] that the moduli space $M_{H}(\mathbf{v})$ is not locally factorial at type 2 points.

Suppose further that $Y$ is a rational surface other than $\mathbb{P}^{2}$. Then there is a morphism $Y \rightarrow \mathbb{P}^{1}$ such that the generic fiber is $\mathbb{P}^{1}$. Let $F$ be the class of a fiber. Yoshioka [Yos96a] shows that if $\left(K_{Y}+F\right) \cdot H<0$ and $\Delta(\mathbf{v})>\frac{1}{2}$, then $M_{H}(\mathbf{v})$ is locally factorial at points of type 1 . In light of Corollary 2.4, we conclude that under these assumptions $M_{H}(\mathbf{v})$ is locally factorial whenever $H$ is not on a wall of type $(r(\mathbf{v}), \Delta(\mathbf{v}))$.

### 2.4 The Donaldson homomorphism.

The Donaldson homomorphism will be our main tool for constructing line bundles on the moduli space. We briefly recall the construction while referring the reader to [HL10, §8.1] and [LP97, §18.2] for the full details.

Let $\mathcal{U} / S$ be a flat family of semistable sheaves of Chern character $\mathbf{v}$ on a smooth variety $Y$ parameterized by a variety $S$, and let $p: S \times Y \rightarrow S$ and $q: S \times Y \rightarrow Y$ be the two projections. The Donaldson homomorphism $\lambda_{\mathcal{U}}: K(Y) \rightarrow \operatorname{Pic}(S)$ is described as the composition

$$
K(Y) \xrightarrow{q^{*}} K^{0}(S \times Y) \xrightarrow{[\{\mathcal{Y}]} K^{0}(S \times Y) \xrightarrow{p_{1}} K^{0}(S) \xrightarrow{\text { det }} \operatorname{Pic}(S) .
$$

Functorial properties of $\lambda_{\mathcal{U}}$ are summarized in the following lemma.
Lemma 2.5 ([HL10, Lemma 8.1.2.] and [LP97, Lemma 18.2.1]). Let $\lambda_{\mathcal{U}}: K(Y) \rightarrow \operatorname{Pic}(S)$ be the Donaldson homomorphism constructed above.

1. If $\mathcal{U}$ is an $S$-flat family and $f: S^{\prime} \rightarrow S$ a morphism, then for any $\mathbf{u} \in K(Y)$ one has $\lambda_{f_{Y}^{*}} \mathcal{U}(\mathbf{u})=f^{*} \lambda_{\mathcal{U}}(\mathbf{u})$.
2. If $S$ is equipped with an action of an algebraic group $G$ and $\mathcal{U}$ is a $G$-linearized family over $S$, then $\lambda_{\mathcal{U}}$ factors through the group $\operatorname{Pic}^{G}(S)$ of isomorphism classes of $G$-linearized line bundles on $S$.
3. If $0 \rightarrow \mathcal{U}^{\prime} \rightarrow \mathcal{U} \rightarrow \mathcal{U}^{\prime \prime} \rightarrow 0$ is a short exact sequence of $S$-flat families of $G$-linearized coherent sheaves then $\lambda_{\mathcal{U}}(\mathbf{u})=\lambda_{\mathcal{U}^{\prime}}(\mathbf{u}) \otimes \lambda_{\mathcal{U}^{\prime \prime}}(\mathbf{u})$ in $\operatorname{Pic}^{G}(S)$.

Using the last property we can construct line bundles on the moduli space of (semi)stable sheaves $M_{H}(\mathbf{v})$. Informally, realize $M_{H}(\mathbf{v})$ as a (good) quotient $R / / G$ of a subvariety $R$ of a Quot scheme. The $G$-linearized universal family of quotient sheaves $\mathcal{U} / R$ gives a map $\lambda_{\mathcal{U}}: K(X) \rightarrow \operatorname{Pic}^{G}(R)$ and we want to descend the $G$-linearized line bundles in the image along the quotient map $R \rightarrow R / / G=M_{H}(\mathbf{v})$.

For this construction to work we, however, need to restrict the domain of $\lambda_{\mathcal{U}}$. Denote by $\mathbf{v}^{\perp} \subset K(Y)$ the complement of $\mathbf{v}$ with respect to the Euler pairing $\chi\left(\__{-}\right)$. We then get the following theorem, which shows that the above construction always produces line bundles on the stable locus $M_{H}^{s}(\mathbf{v})$ and is compatible with the universal property of the moduli space $M_{H}^{s}(\mathbf{v})$.

Theorem 2.6 ([HL10, Theorem 8.1.5]). Let $\mathbf{v}$ be a class in $K(Y)$. Then there exists a group homomorphism $\lambda^{s}: \mathbf{v}^{\perp} \rightarrow \operatorname{Pic}\left(M_{H}^{s}(\mathbf{v})\right)$ with the following property:

If $\mathcal{U}$ is a flat $G$-linearized family of stable sheaves of class $\mathbf{v}$ parameterized by a $G$-scheme $S$, and if the classifying morphism $\phi_{\mathcal{U}}: S \rightarrow M_{H}^{s}(\mathbf{v})$ is $G$-equivariant, then the following diagram commutes :


In general, for a polarized variety $(Y, H)$ one needs to further restrict the domain of the Donaldson homomorphism in order to obtain line bundles on the full semistable locus $M_{H}(\mathbf{v})$ (see the rest of [HL10, Theorem 8.1.5]). However, when $Y$ is a surface the analysis of the proof of [HL10, Theorem 8.1.5] shows that for a polarization which does not lie on a wall of type $(r(\mathbf{v}), \Delta(\mathbf{v}))$ we do not need to further shrink the domain of the Donaldson map.

Proposition 2.7. Let $Y$ be a smooth projective surface. Let $\mathbf{v}=(r, \nu, \Delta)$ be a class in $K(Y)$ and let $H$ be an ample divisor not lying on a wall of type $(r, \Delta)$. Then there exists a group homomorphism $\lambda: \mathbf{v}^{\perp} \rightarrow \operatorname{Pic}\left(M_{H}(\mathbf{v})\right)$ with the following property:

If $\mathcal{U}$ is a flat $G$-linearized family of $H$-semistable sheaves of class $\mathbf{v}$ parameterized by a $G$-scheme $S$, and if the classifying morphism $\phi_{\mathcal{U}}: S \rightarrow M_{H}(\mathbf{v})$ is $G$-equivariant, then the following diagram commutes:


Proof. We follow the notation used in the proof of [HL10, Theorem 8.1.5]. Let $R \xrightarrow[\rightarrow]{\pi} M_{H}(\mathbf{v})$ be the quotient morphism in the GIT construction of the moduli space, where $R$ is a subvariety of the Quot scheme with a universal family of quotients $\mathcal{U}$. For $\mathbf{u} \in \mathbf{v}^{\perp}$, we would like to descend a $G L(V)$-linearized line bundle $L=\lambda_{\mathcal{U}}(\mathbf{u})$ to $M(\mathbf{v})$ along the quotient map $\pi$. According to the Descent Lemma [HL10, Theorem 4.2.15], we need to make sure that for any point $[q: \mathcal{H} \rightarrow F] \in R$ in a closed $G L(V)$-orbit the stabilizer $G L(V)_{[q]}$ acts trivially on the fiber $\left.L\right|_{[q]}$ of $L$ over the point $[q]$.

The orbit of $[q: \mathcal{H} \rightarrow F] \in R$ is closed if and only if $F$ is a polystable sheaf. Thus

$$
F \cong \oplus_{i}\left(F_{i} \otimes W_{i}\right)
$$

with distinct stable Jordan-Hölder factors $F_{i}$ and vector spaces $W_{i}$. The stabilizer of $[q]$ is then isomorphic to $\operatorname{Aut}(F) \cong \prod_{i} G L\left(W_{i}\right)$, and an element $\left(A_{1}, \ldots, A_{l}\right) \in \prod_{i} G L\left(W_{i}\right)$ acts on the fiber $\left.L\right|_{[q]}$ via multiplication by

$$
\begin{equation*}
\prod_{i} \operatorname{det}\left(A_{i}\right)^{\chi\left(\mathbf{u} \cdot\left[F_{i}\right]\right)} \tag{2.7.1}
\end{equation*}
$$

Let $\mathbf{v}_{i}=\left[F_{i}\right]$ and $r_{i}=r\left(F_{i}\right)$. According to Corollary 2.4, for $H$ not on a wall of type $(r, \Delta)$ we get that $\mathbf{v}_{i}=\frac{r_{i}}{r} \mathbf{v}$ for all $i$, and therefore, the exponents in (2.7.1) all vanish:

$$
\chi\left(\mathbf{u} \cdot \mathbf{v}_{i}\right)=\chi\left(\mathbf{u} \cdot \frac{r_{i}}{r} \mathbf{v}\right)=0 \text { for } \mathbf{u} \in \mathbf{v}^{\perp} .
$$

It follows that $G L(V)_{[q]}$ acts trivially on $\left.L\right|_{[q]}$.
When $Y$ is a rational surface other than $\mathbb{P}^{2}$, Yoshioka [Yos96a] analyzes the equivariant Picard group of a subvariety of the Quot scheme parameterizing a certain family of $\mathcal{O}(0,1)$-prioritary sheaves (see the next section for a review of prioritary sheaves) and proves the following result as a consequence of this analysis.
Theorem 2.8 ([Yos96a, Corollary 3.4]). Let $Y$ be a rational surface other than $\mathbb{P}^{2}$ and let $\mathbf{v}=(r, \nu, \Delta) \in K(Y)$ be a Chern character with $\Delta>1 / 2$. If $H$ is a generic polarization with $\left(K_{Y}+2 F\right) \cdot H<0$ and if $M_{H}^{s}(\mathbf{v})$ is not empty, then the Donaldson homomorphism gives a surjection

$$
\lambda: \mathbf{v}^{\perp} \rightarrow \operatorname{Pic}\left(M_{H}(\mathbf{v})\right) .
$$

Note that Proposition 2.7 ensures that for a generic polarization $H$ the Donaldson homomorphism is defined as a map $\lambda: \mathbf{v}^{\perp} \rightarrow \operatorname{Pic}\left(M_{H}(\mathbf{v})\right)$. Therefore, under these assumptions we have a bound on the Picard number of $M_{H}(\mathbf{v})$ and the computation of $\operatorname{Pic}\left(M_{H}(\mathbf{v})\right)$ boils down to the computation of the kernel of the Donaldson homomorphism.

### 2.5 Prioritary sheaves

It is often the case that the sheaves in a complete family obtained by considering various resolutions and extensions enjoy an extra cohomological property which, in particular, makes the analysis of the locus of semistable sheaves in the family much more tractable.

Definition 2.9. Let $\mathcal{L}$ be a line bundle on a projective surface $Y$. A torsion-free sheaf $\mathcal{V}$ on $Y$ is called $\mathcal{L}$-prioritary if

$$
\operatorname{Ext}^{2}\left(\mathcal{V}, \mathcal{V} \otimes \mathcal{L}^{\vee}\right)=0
$$

Let $D$ be an effective Cartier divisor on a projective surface $Y$. Denote the stack of torsion-free sheaves on $Y$ and the stack of coherent sheaves on $D$ with fixed numerical invariants by $T F_{Y}\left(r, c_{1}, c_{2}\right)$ and $\operatorname{Coh}_{D}\left(r, c_{1} \cdot D\right)$ respectively. The next result shows that the restriction of $\mathcal{O}(D)$-prioritary sheaves from $Y$ to $D$ behaves nicely in families.

Lemma 2.10. [Wal93, Lemma 4.] If $\mathcal{V}$ is an $\mathcal{O}(D)$-prioritary sheaf, then the restriction map

$$
\begin{aligned}
T F_{Y}\left(r, c_{1}, c_{2}\right) & \rightarrow C o h_{D}\left(r, c_{1} \cdot D\right) \\
\mathcal{V} & \left.\mapsto \mathcal{V}\right|_{D}
\end{aligned}
$$

is smooth (and therefore open) in a neighborhood of [V].

### 2.6 The quadric surface

Now we specialize some of the above discussion from a general variety $Y$ to the case $X=\mathbb{P}^{1} \times \mathbb{P}^{1}$.
The surface $X$ comes with two natural projections to the $\mathbb{P}^{1}$ factors. Let $F$ denote the class $\left[p r_{1}^{*}(p t)\right]$ and $E$ denote the class $\left[p r_{2}^{*}(p t)\right]$. The Picard group of $X$ and the intersection pairing is then given by

$$
\operatorname{Pic}(X)=\mathbb{Z} E \oplus \mathbb{Z} F, \quad E^{2}=F^{2}=0, \quad E \cdot F=1
$$

The canonical class of $X$ is

$$
K_{X}=-2 E-2 F .
$$

A divisor class $H=a E+b F$ is ample if and only if $a, b>0$. For $m \in \mathbb{Q}$, we consider the $\mathbb{Q}$-divisor class

$$
H_{m}=E+m F \text {. }
$$

Note that every ample divisor on $X$ is an integer multiple of some $H_{m}$ with $m>0$.
For character $\mathbf{v}=(r, \nu, \Delta)=(r, \varepsilon E+\varphi F, \Delta)$ on $X=\mathbb{P}^{1} \times \mathbb{P}^{1}$, the Riemann-Roch Theorem gives

$$
\chi(\mathbf{v})=r(P(\nu)-\Delta),
$$

where

$$
P(\nu)=(\varepsilon+1)(\varphi+1) .
$$

Given two sheaves $\mathcal{V}, \mathcal{W}$, let $\operatorname{ext}^{i}(\mathcal{V}, \mathcal{W})$ denote $\operatorname{dim} \operatorname{Ext}^{i}(\mathcal{V}, \mathcal{W})$. The Riemann-Roch Theorem says that

$$
\chi(\mathcal{V}, \mathcal{W})=\sum_{i=0}^{2}(-1)^{i} \operatorname{ext}^{i}(\mathcal{V}, \mathcal{W})=r(\mathcal{V}) r(\mathcal{W})(P(\nu(\mathcal{W})-\nu(\mathcal{V}))-\Delta(\mathcal{V})-\Delta(\mathcal{W}))
$$

Note that on $X$ with an ample divisor $H$ every $H$-semistable sheaf $\mathcal{V}$ of character $\mathbf{v}$ is both $\mathcal{O}(1,1)$ and $\mathcal{O}(0,1)$-prioritary:

$$
\begin{aligned}
\operatorname{Ext}^{2}(\mathcal{V}, \mathcal{V}(-1,-1)) & =\operatorname{Hom}(\mathcal{V}, \mathcal{V}(-1,-1))^{\vee}=0 \\
\operatorname{Ext}^{2}(\mathcal{V}, \mathcal{V}(0,-1)) & =\operatorname{Hom}(\mathcal{V}, \mathcal{V}(-2,-1))^{\vee}=0
\end{aligned}
$$

by Serre duality and semistability. Thus, if we denote the stack of $\mathcal{L}$-prioritary sheaves by $\mathcal{P}_{\mathcal{L}}(\mathbf{v})$, then we have a chain of open substacks

$$
\mathcal{M}_{H}(\mathbf{v}) \subset \mathcal{P}_{\mathcal{O}(1,1)}(\mathbf{v}) \subset \mathcal{P}_{\mathcal{O}(0,1)}(\mathbf{v})
$$

Walter's Theorem [Wal93] asserts that the stack $\mathcal{P}_{\mathcal{O}(0,1)}(\mathbf{v})$ is irreducible and smooth (if nonempty). This implies that the moduli space $M_{H}(\mathbf{v})$ is irreducible as well. Furthermore, if $r(\mathbf{v}) \geq 2$, then the general member of $\mathcal{P}_{\mathcal{O}(0,1)}(\mathbf{v})$ is locally free. Additionally, Walter shows that $M_{H}(\mathbf{v})$ is unirational.

We also have the following useful result of Yoshioka.
Theorem 2.11 ([Yos95, Proposition 5.1]). Let $\mathbf{v}=(r, \nu, \Delta)=\left(r, c_{1}, \chi\right) \in K(X)$ be a primitive $H$-semistable Chern character.

If the polarization $H$ satisfies

$$
\operatorname{gcd}\left(r, c_{1} \cdot H, \chi\right)=1
$$

then $\operatorname{Pic}\left(M_{H}(\mathbf{v})\right)$ is torsion-free.
In this paper, we will be concerned with the calculation of the Picard group of the moduli space $M_{H_{m}}(\mathbf{v})$ of $H_{m}$-semistable sheaves on $X$ when $m \in \mathbb{Q}$ is sufficiently close to 1 :

$$
H_{m}=E+m F, m=1+\epsilon, 0<|\epsilon| \ll 1 .
$$

The reason for doing so is twofold. On the one hand, as we explained above the genericity assumption makes $M_{H_{m}}(\mathbf{v})$ into a locally factorial variety with a known bound on the Picard number. On the other hand, in the next section we recall that when $H_{m}$ is close to the anticanonical class, there is a complete classification of Chern characters $\mathbf{v}$ for which the moduli space $M_{H_{m}}(\mathbf{v})$ is nonempty or positive-dimensional.

### 2.7 Exceptional bundles and existence of semistable sheaves

Let $X=\mathbb{P}^{1} \times \mathbb{P}^{1}$ polarized by an ample divisor $H$. The question of when $M_{H}(\mathbf{v})$ is nonempty was studied by Rudakov in [Rud94] and Coskun and Huizenga in [CH19] (where they studied the existence question for all Hirzebruch surfaces). We follow [CH19] in this section.

Definition 2.12. A sheaf $\mathcal{V}$ on $X$ is

1. Simple, if $\operatorname{Hom}(\mathcal{V}, \mathcal{V})=\mathbb{C}$;
2. Rigid, if $\operatorname{Ext}^{1}(\mathcal{V}, \mathcal{V})=0$;
3. Exceptional, if it is simple, rigid, and $\operatorname{Ext}^{2}(\mathcal{V}, \mathcal{V})=0$;
4. Semiexceptional, if it is a direct sum of copies of an exceptional sheaf.

We call a character $\mathbf{v} \in K(X)$ of positive rank potentially exceptional if $\chi(\mathbf{v}, \mathbf{v})=1$, and (semi)exceptional if there is a (semi)exceptional torsion-free sheaf of character $\mathbf{v}$. We also say that character $\mathbf{v}$ is $H$-(semi)stable (resp. $\mu_{H^{-}}$(semi)stable) if there is an $H$-(semi)stable (resp., $\mu_{H^{-}}$-(semi)stable) sheaf of character $\mathbf{v}$.

Any exceptional torsion-free sheaf is locally free and $\mu_{-K_{X}}$-stable by [Gor89] and therefore, remains $\mu_{H_{m}}$-stable for $m \in \mathbb{Q}$ sufficiently close to 1 by the openness of slope stability in the polarization. We reproduce a part of [CH19, Lemma 6.7] that further characterizes (potentially) exceptional bundles and characters.

Lemma 2.13 ([CH19, Lemma 6.7]). Let $\mathbf{v} \in K(X)$ be a potentially exceptional character of rank $r$.

1. The rank of $\mathbf{v}$ is odd and the discriminant of $\mathbf{v}$ is

$$
\Delta=\frac{1}{2}-\frac{1}{2 r^{2}} .
$$

2. The character $\mathbf{v}$ is primitive.
3. If $m$ is generic and $\mathcal{V}$ is an $H_{m}$-semistable sheaf of discriminant $\Delta(\mathcal{V})<\frac{1}{2}$, then $\mathcal{V}$ is semiexceptional.

Heuristically, $\mu_{H}$-stable exceptional bundles give strong bounds on the possible numerical invariants of $\mu_{H}$-semistable sheaves. In particular, if $E$ is a $\mu_{H}$-stable exceptional bundle and $\mathcal{V}$ is a $\mu_{H}$-semistable sheaf with

$$
\frac{1}{2} K_{X} \cdot H \leq \mu_{H}(\mathcal{V})-\mu_{H}(E)<0
$$

then

$$
\operatorname{Hom}(E, \mathcal{V})=0 \quad \text { and } \quad \operatorname{Ext}^{2}(E, \mathcal{V})=\operatorname{Hom}\left(\mathcal{V}, E\left(K_{X}\right)\right)^{\vee}=0
$$

by semistability and Serre duality. Therefore, $\chi(E, \mathcal{V}) \leq 0$. By the Riemann-Roch Formula, this inequality can be viewed as a lower bound on $\Delta(\mathcal{V})$ :

$$
\Delta(\mathcal{V}) \geq P(\nu(\mathcal{V})-\nu(E))-\Delta(E)
$$

Likewise, if instead

$$
0<\mu_{H}(\mathcal{V})-\mu_{H}(E) \leq-\frac{1}{2} K_{X} \cdot H
$$

then the inequality $\chi(\mathcal{V}, E) \leq 0$ provides a lower bound

$$
\Delta(\mathcal{V}) \geq P(\nu(E)-\nu(\mathcal{V}))-\Delta(E)
$$

on $\Delta(\mathcal{V})$.
This motivates the following definition.
Definition 2.14 ([CH19, Definition 6.13]). For a $\mu_{H}$-stable exceptional bundle $E$, define a function

$$
\operatorname{DLP}_{H, E}(\nu)= \begin{cases}P(\nu-\nu(E))-\Delta(E) & \text { if } \frac{1}{2} K_{X} \cdot H \leq(\nu-\nu(E)) \cdot H<0 \\ P(\nu(E)-\nu)-\Delta(E) & \text { if } 0<(\nu-\nu(E)) \cdot H \leq-\frac{1}{2} K_{X} \cdot H \\ \max \{P( \pm(\nu-\nu(E)))-\Delta(E)\} & \text { if }(\nu-\nu(E)) \cdot H=0\end{cases}
$$

on the strip of slopes $\nu=\varepsilon E+\varphi F=(\varepsilon, \varphi) \in \mathbb{Q}^{2}$ satisfying

$$
|(\nu-\nu(E)) \cdot H| \leq-\frac{1}{2} K_{X} \cdot H
$$

Let $\mathbb{E}_{H}$ be the set of $\mu_{H}$-stable exceptional bundles on $X$. Further define a function

$$
\operatorname{DLP}_{H}^{<r}(\nu)=\sup _{\substack{E \in \mathbb{E}_{H} \\|(\nu-\nu(E)) \cdot H| \leq-\frac{1}{2} K_{X} \cdot H \\ r(E)<r}} \operatorname{DLP}_{H, E}(\nu),
$$

where this time $\nu=(\varepsilon, \varphi)$ could be any point in $\mathbb{Q}^{2}$. We refer to the above functions as the Drézet-LePotier functions, or DLP-functions, for short.

One can see the graph of $\Delta=\operatorname{DLP}_{H_{m}}^{<r}(\varepsilon E+\varphi F)$ in the $(\varepsilon, \varphi, \Delta)$-space in Figure 2.1 below (for $m=1)$. In the rest of the paper we will call the graph of $\Delta=\operatorname{DLP}_{H_{m}}^{<r}(\varepsilon E+\varphi F)$ the Drézet-Le-Potier surface, the $\mathrm{DLP}_{H_{m}}^{<r}$-surface, or the $\mathrm{DLP}^{<r}$-surface for short.


Figure 2.1: The $\mathrm{DLP}^{<r}$-surface: the graph of $\Delta=\operatorname{DLP}_{H_{1}}^{<8}(\varepsilon E+\varphi F)$.
The discussion before Definition 2.14 shows that if there is a $\mu_{H}$-semistable sheaf of total slope $\nu$ and discriminant $\Delta$, then $\Delta \geq \operatorname{DLP}_{H}^{<r}(\nu)$. The next theorem shows that for a generic polarization close to the anticanonical class such inequalities also provide sufficient conditions for the existence of Gieseker semistable sheaves of total slope $\nu$ and discriminant $\Delta$.

Theorem 2.15 ([CH19, Corollary 7.6]). Let $\mathbf{v}=(r, \nu, \Delta) \in K(X)$ be a character with positive rank. Let $\epsilon \in \mathbb{Q}$ be sufficiently small (depending on $r$ ), $0<|\epsilon| \ll 1$, and set $m=1+\epsilon$.

1. If $\mathbf{v}$ is potentially exceptional, then it is exceptional if and only if

$$
\Delta \geq \operatorname{DLP}_{H_{m}}^{<r}(\nu) .
$$

2. If $\mathbf{v}$ is not semiexceptional, there is an $H_{m}$-semistable sheaf of character $\mathbf{v}$ if and only if

$$
\Delta \geq \operatorname{DLP}_{H_{m}}^{<r}(\nu) .
$$

One can also easily tell when $\mathbf{v}$ is semiexceptional. Write $\mathbf{v}=N \mathbf{v}^{\prime}$ with $N \in \mathbb{N}$ and a primitive character $\mathbf{v}^{\prime}$ of rank $r^{\prime}=r / N$. Then using (1) from the above theorem, we see that $\mathbf{v}$ is semiexceptional if and only if $\mathbf{v}^{\prime}$ is potentially exceptional and $\Delta \geq \operatorname{DLP}_{H_{m}}^{<r^{\prime}}(\nu)$.

Taken together, these statements provide a finite inductive computational procedure for determining whether the moduli space $M_{H_{m}}(\mathbf{v})$ is nonempty for a given character $\mathbf{v}$. Let us remark that for a sufficiently small $\epsilon$ we actually have $\operatorname{DLP}_{H_{m}}^{<r}(\nu)=\operatorname{DLP}_{H_{1}}^{<r}(\nu)$ by [CH19, Lemma 7.8], so one can keep using Figure 2.1 to gain insight into $\operatorname{DLP}_{H_{m}}^{<r}(\nu)$ for $m$ close to 1 .

Also note that since we are concerned with calculating the Picard group of the moduli space, we will only be interested in those characters $\mathbf{v}$ for which the moduli space $M_{H_{m}}(\mathbf{v})$ is nonempty and positive dimensional. Recall ([HL10, §4.5]) that the expected dimension of the moduli space is given by

$$
\exp \operatorname{dim} M_{H_{m}}(\mathbf{v})=r^{2}(2 \Delta-1)+1
$$

This shows that the expected dimension is positive if and only if $\Delta \geq 1 / 2$. Lemma 2.13 (1) implies that such characters are not semiexceptional.

Next, we introduce useful terminology describing the position of character $\mathbf{v}$ relative to the $\mathrm{DLP}^{<r_{-}}$ surface.

Definition 2.16. Let $\mathbf{v}=(r, \nu, \Delta) \in K(X)$ be an $H_{m}$-semistable character with $\Delta \geq \frac{1}{2}$ and $\Delta=\operatorname{DLP}_{H_{m}}^{<r}(\nu)$, where $m=1+\epsilon$ and $\epsilon \in \mathbb{Q}$ is a sufficiently small number depending on $r, 0<|\epsilon| \ll 1$.

We say that an exceptional bundle $E$ is associated to $\mathbf{v}$ if

$$
\begin{align*}
r(E) & <r, \\
\left|(\nu-\nu(E)) \cdot H_{m}\right| & \leq-\frac{1}{2} K_{X} \cdot H_{m}, \quad \text { and }  \tag{2.16.1}\\
\Delta=\operatorname{DLP}^{<r}(\nu) & =\operatorname{DLP}_{H_{m}, E}(\nu) .
\end{align*}
$$

Character $\mathbf{v}=(r, \nu, \Delta)$ can be positioned in three different ways relative to the DLP $^{<r}$-surface (see Figure 2.2):


Figure 2.2: Position of $\mathbf{v}$ relative to the $\mathrm{DLP}^{<r}$-surface.

1. If $\Delta>\operatorname{DLP}_{H_{m}}^{<r}(\nu)$, then we will say that character $\mathbf{v}$ lies above the $\mathrm{DLP}^{<r}$-surface,
2. If $\Delta=\operatorname{DLP}_{H_{m}}^{<r}(\nu)$ and there is a single exceptional bundle $E$ associated to $\mathbf{v}$, then we will say that character $\mathbf{v}$ lies on a single branch of the $\mathrm{DLP}^{<r}$-surface given by the exceptional bundle $E$,
3. If $\Delta=\operatorname{DLP}_{H_{m}}^{<r}(\nu)$ and there are at least two different exceptional bundles $E_{1}, E_{2}$ associated to $\mathbf{v}$, then we will say that character $\mathbf{v}$ lies on the intersection of branches of the DLP ${ }^{<r}$-surface given by exceptional bundles $E_{1}$ and $E_{2}$.

We finish this section with a useful result about the existence of stable bundles.
Proposition 2.17 ([CH19, Propositions 9.5 \& 9.6]). Suppose $m \in \mathbb{Q}$ is generic and $\mathbf{v}=(r, \nu, \Delta)$ is an integral Chern character.

1. If $\Delta>\frac{1}{2}$ and there are $H_{m}$-semistable sheaves of character $\mathbf{v}$, then there are $H_{m}$-stable sheaves of character $\mathbf{v}$.
2. If there are $H_{m}$-stable sheaves of of character $\mathbf{v}$, then there are $\mu_{H_{m}}$-stable sheaves of character $\mathbf{v}$.

### 2.8 Gaeta-type resolutions

These resolutions are special resolutions of sheaves on $X$ by direct sums of line bundles. Their advantage is that they are simple enough to work with and provide unirational parameterizations of moduli spaces of sheaves. Gaeta-type resolutions were studied in [CH18] for all Hirzebruch surfaces $\mathbb{F}_{e}$, but we will only need the case $X=\mathbb{F}_{0}=\mathbb{P}^{1} \times \mathbb{P}^{1}$.

Definition 2.18. Let $L$ be a line bundle on $X$. An L-Gaeta-type resolution of a sheaf $\mathcal{V}$ on $X$ is a resolution of $\mathcal{V}$ of the form

$$
\begin{equation*}
0 \rightarrow L(-1,-1)^{\alpha} \rightarrow L(-1,0)^{\beta} \oplus L(0,-1)^{\gamma} \oplus L^{\delta} \rightarrow \mathcal{V} \rightarrow 0 \tag{2.18.1}
\end{equation*}
$$

where $\alpha, \beta, \gamma, \delta$ are nonnegative integers. We say a sheaf $\mathcal{V}$ has a Gaeta-type resolution if it admits an $L$-Gaeta-type resolution for some line bundle $L$.

The results of $[\mathrm{CH} 18, \S 4]$ we will need are summarized in the following statement.
Theorem 2.19. If $\mathbf{v}$ is a $\mu_{H}$-semistable Chern character with $\Delta(\mathbf{v}) \geq \frac{1}{4}$, then a general $\mu_{H}$-semistable sheaf $\mathcal{V}$ admits a Gaeta-type resolution. More specifically:

1. Suppose $L$ is a line bundle such that the inequalities

$$
\begin{array}{r}
\alpha=-\chi\left(\mathbf{v} \otimes L^{\vee}(-1,-1)\right) \geq 0 \\
\beta=-\chi\left(\mathbf{v} \otimes L^{\vee}(-1,0)\right) \geq 0 \\
\gamma=-\chi\left(\mathbf{v} \otimes L^{\vee}(0,-1)\right) \geq 0  \tag{2.19.1}\\
\delta=\chi\left(\mathbf{v} \otimes L^{\vee}\right) \geq 0
\end{array}
$$

are satisfied. Then not all of the integers in (2.19.1) are zero and a general $\mu_{H}$-semistable sheaf $\mathcal{V}$ admits an L-Gaeta-type resolution with integers in (2.19.1) giving the exponents in (2.18.1).
2. A line bundle satisfying inequalities (2.19.1) always exists.

Gaeta-type resolutions allow us to build complete families of $\mathcal{O}(1,1)$-prioritary sheaves.
Proposition 2.20. Let $\alpha, \beta, \gamma, \delta$ be nonnegative integers satisfying

$$
r:=\beta+\gamma+\delta-\alpha>0 .
$$

For a line bundle $L$ consider the open subset

$$
U \subset \mathbb{H}:=\operatorname{Hom}\left(L(-1,-1)^{\alpha}, L(-1,0)^{\beta} \oplus L(0,-1)^{\gamma} \oplus L^{\delta}\right)
$$

parameterizing injective sheaf maps with torsion-free cokernel. For $\psi_{u} \in U$, let $\mathcal{V}_{u}$ be the cokernel:

$$
0 \rightarrow L(-1,-1)^{\alpha} \xrightarrow{\psi_{u}} L(-1,0)^{\beta} \oplus L(0,-1)^{\gamma} \oplus L^{\delta} \rightarrow \mathcal{V}_{u} \rightarrow 0
$$

If $r \geq 2$, then $U$ is nonempty, $\operatorname{codim}_{\mathbb{H}}(\mathbb{H} \backslash U) \geq 2$, and the family $\mathcal{V}_{u} / U$ is a complete family of $\mathcal{O}(1,1)$-prioritary sheaves.

Proof. Only the statement about the codimension requires a proof as the other statements are proved in [CH18, Theorem 2.10].

The statement about the codimension follows from the standard analysis of the incidence correspondence

$$
\Sigma:=\left\{(p, \psi)|\psi|_{p} \text { is not injective }\right\} \subset X \times \operatorname{Hom}\left(L(-1,-1)^{\alpha}, L(-1,0)^{\beta} \oplus L(0,-1)^{\gamma} \oplus L^{\delta}\right)
$$

using the fact that

$$
\mathscr{H} \text { om }\left(L(-1,-1)^{\alpha}, L(-1,0)^{\beta} \oplus L(0,-1)^{\gamma} \oplus L^{\delta}\right)
$$

is a globally generated vector bundle. See [LP97, pp. 238-239] and the proof of [DLP85, Theorem 4.7] for details.

We finish this section by introducing the "dual version" of a Gaeta-type resolution. Specifically, this is a resolution of the form

$$
\begin{equation*}
0 \rightarrow \mathcal{V} \rightarrow L(1,0)^{\alpha} \oplus L(0,1)^{\beta} \oplus L^{\gamma} \rightarrow L(1,1)^{\delta} \rightarrow 0 \tag{2.20.1}
\end{equation*}
$$

We have the following analogue of Proposition 2.20.
Proposition 2.21. Let $\alpha, \beta, \gamma, \delta$ be nonnegative integers satisfying

$$
r:=\alpha+\beta+\gamma-\delta>0
$$

For a line bundle $L$ consider the open subset

$$
U \subset \mathbb{H}:=\operatorname{Hom}\left(L(1,0)^{\alpha} \oplus L(0,1)^{\beta} \oplus L^{\gamma}, L(1,1)^{\delta}\right)
$$

parameterizing surjective sheaf maps. For $\psi_{u} \in U$, let $\mathcal{V}_{u}$ be the kernel:

$$
0 \rightarrow \mathcal{V}_{u} \rightarrow L(1,0)^{\alpha} \oplus L(0,1)^{\beta} \oplus L^{\gamma} \xrightarrow{\psi_{u}} L(1,1)^{\delta} \rightarrow 0
$$

If $r \geq 2$, then $U$ is nonempty, $\operatorname{codim}_{\mathbb{H}}(\mathbb{H} \backslash U) \geq 2$, and the family $\mathcal{V}_{u} / U$ is a complete family of $\mathcal{O}(1,1)$-prioritary vector bundles.

## Chapter 3

## Study of the Shatz stratification

In this chapter we use the techniques from [DLP85, §1 \& 3], [LP97, Chapter 15] and [HL10, §2.A] to detect strata of unstable sheaves of codimension one in complete families.

### 3.1 Generalities on the Shatz stratification

Given a complete family $\mathcal{V}_{t} / T$ of torsion-free sheaves of character $\mathbf{v}$, we denote by $S_{H}\left(P_{1}, P_{2}, \ldots, P_{l}\right) \subset T$ the Shatz stratum parameterizing sheaves $\mathcal{V}_{t}$ with $H$-Harder-Narasimhan filtration having quotients with $H$-Hilbert polynomials $P_{1}, P_{2}, \ldots, P_{l}$. If one further assumes that $T$ is smooth and for each $t \in T$ we have $\mathrm{Ext}^{2}\left(\mathcal{V}_{t}, \mathcal{V}_{t}\right)=0$, then the Shatz stratum $S_{H}\left(P_{1}, P_{2}, \ldots, P_{l}\right)$ is a smooth locally closed subvariety of $T$ with the normal space at point $t \in S_{H}\left(P_{1}, P_{2}, . ., P_{l}\right)$ given by $\mathrm{Ext}_{+}^{1}\left(\mathcal{V}_{t}, \mathcal{V}_{t}\right)$. We refer the reader to [DLP85, §1] for the definition of Ext ${ }_{+}^{1}$, Ext ${ }_{-}^{1}$ and the general deformation theory of filtered sheaves. We instead review here the computational aspects.

For $t \in S_{H}\left(P_{1}, P_{2}, \ldots, P_{l}\right)$ equip $\mathcal{V}_{t}$ with its $H$-Harder-Narasimhan filtration with quotients $\mathrm{gr}_{1, t}, \ldots, \mathrm{gr}_{l, t}$. Then there is a spectral sequence with $E_{1}$-term given by

$$
E_{1}^{p, q}= \begin{cases}\bigoplus_{i} \mathrm{Ext}^{p+q}\left(\mathrm{gr}_{i, t}, \mathrm{gr}_{i-p, t}\right) & \text { if } p<0 \\ 0 & \text { if } p \geq 0\end{cases}
$$

which abuts on $\operatorname{Ext}_{+}^{p+q}\left(\mathcal{V}_{t}, \mathcal{V}_{t}\right)$ in degree $p+q$. Similarly, there is a spectral sequence with $E_{1}$-term given by

$$
E_{1}^{p, q}= \begin{cases}0 & \text { if } p<0 \\ \bigoplus_{i} \operatorname{Ext}^{p+q}\left(\mathrm{gr}_{i, t}, \mathrm{gr}_{i-p, t}\right) & \text { if } p \geq 0\end{cases}
$$

which abuts on $\operatorname{Ext}_{-}^{p+q}\left(\mathcal{V}_{t}, \mathcal{V}_{t}\right)$ in degree $p+q$.
For our purposes, it would be convenient to work with a slightly refined notion of a Shatz stratum. Note that since $\operatorname{Pic}(X)$ is a discrete algebraic group scheme, for points $t$ within a connected component of a Shatz stratum $S_{H}\left(P_{1}, P_{2}, \ldots, P_{l}\right)$ the $H$-Harder-Narasimhan quotients $\mathrm{gr}_{1, t}, \mathrm{gr}_{2, t}, \ldots, \mathrm{gr}_{l, t}$ of $\mathcal{V}_{t}$ not only have the same $H$-Hilbert polynomials $P_{1}, P_{2}, \ldots, P_{l}$, but also the same numerical invariants $\mathbf{v}_{1}, \mathbf{v}_{2}, \ldots, \mathbf{v}_{l}$. Thus, $S_{H}\left(P_{1}, P_{2}, \ldots, P_{l}\right)$ breaks up into a disjoint union of strata $S_{H}\left(\mathbf{v}_{1}, \mathbf{v}_{2}, \ldots, \mathbf{v}_{l}\right)$, where each $S_{H}\left(\mathbf{v}_{1}, \mathbf{v}_{2}, \ldots, \mathbf{v}_{l}\right)$ parameterizes sheaves $\mathcal{V}_{t}$ with $H$-Harder-Narasimhan filtration having quotients with numerical invariants $\mathbf{v}_{1}, \mathbf{v}_{2}, \ldots, \mathbf{v}_{l}$. Later, when we use the notion of Shatz stratum
we will have $S_{H}\left(\mathbf{v}_{1}, \mathbf{v}_{2}, \ldots, \mathbf{v}_{l}\right)$ in mind instead of $S_{H}\left(P_{1}, P_{2}, \ldots, P_{l}\right)$. The discussion above applies to $S_{H}\left(\mathbf{v}_{1}, \mathbf{v}_{2}, \ldots, \mathbf{v}_{l}\right)$ equally well, and we conclude that when $\mathcal{V}_{t} / T$ is a smooth complete family of torsionfree sheaves satisfying $\operatorname{Ext}^{2}\left(\mathcal{V}_{t}, \mathcal{V}_{t}\right)=0$ for each $t \in T$ the stratum $S=S_{H}\left(\mathbf{v}_{1}, \mathbf{v}_{2}, \ldots, \mathbf{v}_{l}\right)$ is a smooth locally closed subvariety of $T$ with the normal space at point $t$ described as

$$
\left.N_{S / T}\right|_{t} \cong \operatorname{Ext}_{+}^{1}\left(\mathcal{V}_{t}, \mathcal{V}_{t}\right)
$$

## $3.2 \quad \Delta_{\mathrm{i}}=\frac{1}{2}$-strata

Before we proceed with the estimates, let us introduce one more definition. Careful reading of [LP97, Lemma 18.3.1], [Dre88, Proposition 2.4] and [DLP85, Lemma 4.8] suggests that in the $\mathbb{P}^{2}$ case the codimension one Shatz strata occur in complete families of $\mathcal{O}_{\mathbb{P}^{2}}(1)$-prioritary sheaves only for characters $\mathbf{v}$ on the $\mathrm{DLP}_{\mathbb{P}^{2}}$-curve and correspond to sheaves whose first or last Harder-Narasimhan quotient is semiexceptional. The next definition is created ad hoc to capture new codimension one Shatz strata which did not exist in the $\mathbb{P}^{2}$ case, but which appear in the $\mathbb{P}^{1} \times \mathbb{P}^{1}$ case due to the presence of semistable Chern characters of discriminant $\frac{1}{2}$.

Definition 3.1. Let $\mathbf{v}=(r, \nu, \Delta)$ be an integral Chern character on $X=\mathbb{P}^{1} \times \mathbb{P}^{1}$. Let $\mathcal{V}_{t} / T$ be a complete family of torsion-free sheaves parameterized by a smooth variety $T$ with $\mathbf{v}\left(\mathcal{V}_{t}\right)=\mathbf{v}$. We call Shatz stratum $S \subset T$ a $\Delta_{i}=\frac{1}{2}$-stratum if $S$ parameterizes sheaves $\mathcal{V}_{t}$ with the $H$-Harder-Narasimhan filtration of length $l=2$,

$$
S=S_{H}\left(\mathbf{v}_{1}, \mathbf{v}_{2}\right),
$$

such that the numerical invariants

$$
\mathbf{v}_{1}=\left(r_{1}, \nu_{1}, \Delta_{1}\right), \mathbf{v}_{2}=\left(r_{2}, \nu_{2}, \Delta_{2}\right)
$$

of the $H$-Harder-Narasimhan quotients of $\mathcal{V}_{t}$ satisfy the following properties:

1. $\Delta_{1}, \Delta_{2} \geq \frac{1}{2}$ with at least one $\Delta_{i}=\frac{1}{2}, i=1,2$,
2. $\nu_{2}-\nu_{1}=\frac{k}{r_{1} r_{2}} E-\frac{k}{r_{1} r_{2}} F$ for some integer $k$ with $0<|k| \leq r_{1} r_{2}$,
3. $\chi\left(\mathbf{v}_{1}, \mathbf{v}_{2}\right)=-1$.

### 3.3 Codimension of Shatz strata

In this section, we present a study of Shatz stratification through a numerical analysis involving RiemannRoch computations.

For the rest of this chapter we adopt the following convention. Consider a family $\mathcal{V}_{t} / T$ of sheaves parameterized by a variety $T$. Suppose $\mathcal{V}_{t}$ belongs to a Shatz stratum $S_{H_{m}}\left(\mathbf{v}_{1}, \mathbf{v}_{2}, \ldots, \mathbf{v}_{l}\right) \subset T$ with $H_{m^{-}}$ Harder-Narasimhan quotients $\mathrm{gr}_{1, t}, \mathrm{gr}_{2, t}, \ldots, \mathrm{gr}_{l, t}$ having numerical invariants $\mathbf{v}_{1}, \mathbf{v}_{2}, \ldots, \mathbf{v}_{l}$. To improve readability we drop the subscript $t$ in $\mathrm{gr}_{i, t}$ if any confusion is unlikely. We further write

$$
\mathbf{v}_{i}=\mathbf{v}\left(\mathrm{gr}_{i}\right)=\left(r_{i}, \nu_{i}, \Delta_{i}\right) .
$$

We start with a couple of preparatory lemmas.

Lemma 3.2. Let $\mathbf{v}=(r, \nu, \Delta) \in K(X)$ be a character with $r \geq 2$. Let $\epsilon \in \mathbb{Q}$ be a sufficiently small number (depending on $r$ ), $0<|\epsilon| \ll 1$, and set $m=1+\epsilon$.

Consider a complete family $\mathcal{V}_{t} / T$ of $\mathcal{O}(1,1)$-prioritary sheaves with $\mathbf{c h}\left(\mathcal{V}_{t}\right)=\mathbf{v}$ parameterized by a smooth variety $T$. If $\mathcal{V}_{t}$ belongs to the Shatz stratum $S=S_{H_{m}}\left(\mathbf{v}_{1}, \mathbf{v}_{2}, \ldots, \mathbf{v}_{l}\right)$ and the inequalities

$$
\begin{equation*}
\mu_{\max , H}\left(\mathcal{V}_{t}\right)-\mu_{\min , H}\left(\mathcal{V}_{t}\right) \leq 2 \tag{3.2.1}
\end{equation*}
$$

are satisfied for $H=\mathcal{O}(1,1), \mathcal{O}(1,0)$ and $\mathcal{O}(0,1)$, then

$$
\left|\left(\nu-\nu_{i}\right) \cdot H_{m}\right| \leq-\frac{1}{2} K_{X} \cdot H_{m}
$$

and

$$
\operatorname{codim}_{T, t}(S)=-\sum_{i<j} \chi\left(\mathbf{v}_{i}, \mathbf{v}_{j}\right)
$$

Proof. By the above inequalities (3.2.1), for any subsheaf $\mathcal{W} \subset \mathcal{V}_{t}$ and any quotient $\mathcal{V}_{t} \rightarrow \mathcal{E}$ of $\mathcal{V}_{t}$ the difference of the total slopes

$$
\nu(\mathcal{W})-\nu(\mathcal{E})
$$

lies in a bounded region of the $(\varepsilon, \varphi)$-plane of total slopes. Furthermore, inequality (3.2.1) for $H=\mathcal{O}(1,1)$ implies

$$
\left|\left(\nu-\nu_{i}\right) \cdot H_{1}\right|<2=-\frac{1}{2} K_{X} \cdot H_{1} .
$$

It follows that since $m$ is close enough to 1 , we have

$$
\begin{equation*}
\left|\left(\nu-\nu_{i}\right) \cdot H_{m}\right| \leq-\frac{1}{2} K_{X} \cdot H_{m} \tag{3.2.2}
\end{equation*}
$$

Let $\mathrm{gr}_{1}, \mathrm{gr}_{2}, \ldots, \mathrm{gr}_{l}$ be the quotients in the $H_{m}$-Harder-Narasimhan filtration of $\mathcal{V}_{t}$. Since

$$
\operatorname{codim}_{T, t}(S)=\left.\operatorname{dim} N_{S / T}\right|_{t}=\operatorname{ext}_{+}^{1}\left(\mathcal{V}_{t}, \mathcal{V}_{t}\right)
$$

we will use the spectral sequences for $\operatorname{Ext}_{+}^{\bullet}\left(\mathcal{V}_{t}, \mathcal{V}_{t}\right)$ and $\operatorname{Ext}-\left(\mathcal{V}_{t}, \mathcal{V}_{t}\right)$ from $\S 3.1$ to compute $\operatorname{ext}_{+}^{1}\left(\mathcal{V}_{t}, \mathcal{V}_{t}\right)$.
Since $\operatorname{Hom}\left(\mathrm{gr}_{i}, \mathrm{gr}_{j}\right)=0$ for $i<j$ by semistability, we see that $\operatorname{Ext}_{+}^{0}\left(\mathcal{V}_{t}, \mathcal{V}_{t}\right)=0$. Likewise, by our bound (3.2.2) and semistability we have

$$
\operatorname{Ext}^{2}\left(\mathrm{gr}_{i}, \mathrm{gr}_{j}\right) \cong \operatorname{Hom}\left(\mathrm{gr}_{j}, \mathrm{gr}_{i} \otimes K_{X}\right)^{\vee}=0 \quad \text { for any } i, j,
$$

so both $\operatorname{Ext}_{+}^{2}\left(\mathcal{V}_{t}, \mathcal{V}_{t}\right)=0$ and $\operatorname{Ext}_{-}^{2}\left(\mathcal{V}_{t}, \mathcal{V}_{t}\right)=0$. Therefore, the only nonzero terms in the spectral sequence for $\mathrm{Ext}_{+}^{p+q}\left(\mathcal{V}_{t}, \mathcal{V}_{t}\right)$ have $p+q=1$. We conclude

$$
\operatorname{ext}_{+}^{1}\left(\mathcal{V}_{t}, \mathcal{V}_{t}\right)=\sum_{i<j} \operatorname{ext}^{1}\left(\mathrm{gr}_{i}, \mathrm{gr}_{j}\right)=-\sum_{i<j} \chi\left(\mathrm{gr}_{i}, \mathrm{gr}_{j}\right)
$$

with $\chi\left(\operatorname{gr}_{i}, \mathrm{gr}_{j}\right) \leq 0$ for $i<j$.
Lemma 3.3. Let $\mathcal{V}_{t} / T$ be a complete family of $\mathcal{O}(1,1)$-prioritary sheaves parameterized by a smooth variety $T$. Let $H$ be the one of line bundles $\mathcal{O}(1,1), \mathcal{O}(1,0)$ or $\mathcal{O}(0,1)$. Then the set of points $t \in T$ such that

$$
\mu_{\max , H}\left(\mathcal{V}_{t}\right)-\mu_{\min , H}\left(\mathcal{V}_{t}\right)>2
$$

is a closed subset of codimension at least 2 in $T$.

Proof. To show the result for $H=\mathcal{O}(1,1)$, one follows the proof of [LP97, Corollary 15.4.4.], replacing a line $d$ on $\mathbb{P}^{2}$ with a rational curve from the complete linear series $|\mathcal{O}(1,1)|$ on $X=\mathbb{P}^{1} \times \mathbb{P}^{1}$ and using Lemma 2.10 together with the $\mathcal{O}(1,1)$-prioritariness of sheaves in the family.

For $H=\mathcal{O}(1,0)$ or $\mathcal{O}(0,1)$, we recall that $\mathcal{O}(1,1)$-prioritariness implies both $\mathcal{O}(1,0)$ - and $\mathcal{O}(0,1)$ prioritariness (see [CH19, Lemma 3.1]). The same argument as above applies in this case too.

The next two propositions describe codimension one Shatz strata in complete families of $\mathcal{O}(1,1)$ prioritary sheaves.

Proposition 3.4. Let $\mathbf{v}=(r, \nu, \Delta)$ be a Chern character satisfying

$$
\Delta>\operatorname{DLP}_{H_{m}}^{<r}(\nu),
$$

where $m=1+\epsilon$ and $\epsilon \in \mathbb{Q}$ is a sufficiently small number (depending on $r$ ), $0<|\epsilon| \ll 1$.
Consider a complete family $\mathcal{V}_{t} / T$ of $\mathcal{O}(1,1)$-prioritary sheaves with $\mathbf{v}\left(\mathcal{V}_{t}\right)=\mathbf{v}$ parameterized by a smooth variety $T$. Then the $H_{m}$-Shatz strata of codimension 1 in this family are given by (nonempty) $\Delta_{i}=\frac{1}{2}$-strata.

Proof. Since the proof is rather long, we split it into several steps for the reader's convenience.
Step 1. We start by making some preliminary reductions. By Lemma 3.3 we can pass to an open subset of points $t \in T$ where

$$
\begin{equation*}
\mu_{\max , H}\left(\mathcal{V}_{t}\right)-\mu_{\min , H}\left(\mathcal{V}_{t}\right) \leq 2, \tag{3.4.1}
\end{equation*}
$$

for $H=\mathcal{O}(1,1), \mathcal{O}(1,0)$ and $\mathcal{O}(0,1)$.
Suppose $S:=S_{H_{m}}\left(\mathbf{v}_{1}, \mathbf{v}_{2}, \ldots, \mathbf{v}_{l}\right) \subset T$ is a nonempty Shatz stratum of codimension 1 in $T$. By Lemma 3.2 for $t \in S$ we have

$$
\begin{align*}
& \left|\left(\nu-\nu_{i}\right) \cdot H_{m}\right| \leq-\frac{1}{2} K_{X} \cdot H_{m} \text { and } \\
& \operatorname{codim}_{T, t}(S)=-\sum_{i<j} \chi\left(\mathbf{v}_{i}, \mathbf{v}_{j}\right)=1 \tag{3.4.2}
\end{align*}
$$

This implies that if $\mathrm{gr}_{1}, \mathrm{gr}_{2}, \ldots, \mathrm{gr}_{l}$ are the $H_{m}$-Harder-Narasimhan quotients of $\mathcal{V}_{t}$, then we have

$$
\chi\left(\mathrm{gr}_{1}, \mathrm{gr}_{l}\right)=0 \text { or } \chi\left(\mathrm{gr}_{1}, \mathrm{gr}_{l}\right)=-1 .
$$

Below, we analyze various possibilities for what the numerical invariants $\mathbf{v}_{1}, \mathbf{v}_{2}, \ldots, \mathbf{v}_{l}$ of the $H_{m^{-}}$ Harder-Narasimhan quotients $\mathrm{gr}_{1}, \mathrm{gr}_{2}, \ldots, \mathrm{gr}_{l}$ of $\mathcal{V}_{t}$ could be and show that they must necessarily satisfy conditions (1)-(3) of Definition 3.1, i.e. $S$ must be a $\Delta_{i}=\frac{1}{2}$-stratum.

Step 2. Suppose that $\chi\left(\mathrm{gr}_{1}, \mathrm{gr}_{l}\right)=0$ holds. By the Riemann-Roch Theorem, this gives

$$
\Delta_{1}+\Delta_{l}=P\left(\nu_{l}-\nu_{1}\right) .
$$

Since $m$ is sufficiently close to 1 , we have that $P\left(\nu_{l}-\nu_{1}\right) \leq 1$ with equality holding only when $\nu_{1}=\nu_{l}$.
If $\nu_{1} \neq \nu_{l}$, we get that

$$
\Delta_{1}+\Delta_{l}=P\left(\nu_{l}-\nu_{1}\right)<1,
$$

and therefore $\Delta_{1}<\frac{1}{2}$ or $\Delta_{l}<\frac{1}{2}$.

If $\nu_{1}=\nu_{l}$, then since $\mathrm{gr}_{i}$ are the quotients in the Harder-Narasimhan filtration, we must have that $\Delta_{l}>\Delta_{1}$. Since $P\left(\nu_{l}-\nu_{1}\right)=1$ in the case, we get $\Delta_{1}<\frac{1}{2}$.

In both of these cases, Lemma 2.13 (3) implies that $\mathrm{gr}_{1}$ or $\mathrm{gr}_{l}$ is semiexceptional and we can follow the argument of [DLP85, Lemma 4.8]. Here we deal with the case where $\mathrm{gr}_{1}$ is semiexceptional. The argument for when $\mathrm{gr}_{l}$ is semiexceptional is similar. Write $\mathrm{gr}_{1} \cong E^{k}$ with exceptional bundle $E$. Then we get

$$
\chi\left(\mathbf{v}_{1}, \mathbf{v}\right)=\chi\left(E^{k}, \mathcal{V}_{t}\right)=\chi\left(\mathrm{gr}_{1}, \mathcal{V}_{t}\right)=\chi\left(\mathrm{gr}_{1}, \mathrm{gr}_{1}\right)+\sum_{2<j} \chi\left(\mathrm{gr}_{1}, \mathrm{gr}_{j}\right) \geq \chi\left(\mathrm{gr}_{1}, \mathrm{gr}_{1}\right)-1 \geq 0
$$

On the other hand, for a semistable $\mathcal{V}$ of character $\mathbf{v}$ we have $\operatorname{hom}(E, \mathcal{V})=\operatorname{ext}^{2}(E, \mathcal{V})=0$, showing that $\chi\left(\mathbf{v}_{1}, \mathbf{v}\right)=k \cdot \chi(E, \mathcal{V}) \leq 0$. Thus we have $\chi(E, \mathcal{V})=0$. Inequality (3.4.2) gives

$$
(\nu(E)-\nu) \cdot H_{m} \leq-\frac{1}{2} K_{X} \cdot H_{m},
$$

and we get $\Delta=\operatorname{DLP}_{H_{m}, E}(\nu)$. This contradicts our assumption that character $\mathbf{v}$ lies above the $\mathrm{DLP}^{<r}{ }_{-}$ surface.

Step 3. Now we know $\chi\left(\mathrm{gr}_{1}, \mathrm{gr}_{l}\right)=-1$. If one of $\mathrm{gr}_{1}, \mathrm{gr}_{l}$ is semiexceptional, we arrive to a contradiction in the same way as above. At this point, we have shown that for points $t \in S$ the $H_{m}$-Harder-Narasimhan quotients of $\mathcal{V}_{t}$ satisfy $\chi\left(\mathrm{gr}_{1}, \mathrm{gr}_{l}\right)=-1$ and $\Delta_{1}, \Delta_{l} \geq \frac{1}{2}$.

Next, we show that $l=2$. Assume on the contrary that we have $l \geq 3$. Since $\chi\left(\operatorname{gr}_{i}, \mathrm{gr}_{j}\right)=0$ for $i<j,(i, j) \neq(1, l)$, we in particular have

$$
\chi\left(\mathrm{gr}_{1}, \mathrm{gr}_{i}\right)=0 \Longrightarrow P\left(\nu_{i}-\nu_{1}\right)-\Delta_{1}-\Delta_{i}=0
$$

for $1<i<l$. Since $P\left(\nu_{i}-\nu_{1}\right) \leq 1$ and we cannot have $\nu_{1}=\nu_{i}, \Delta_{1}=\Delta_{i}$, we get that $\Delta_{i}<\frac{1}{2}$ and $\mathrm{gr}_{i}$ is semiexceptional, $\mathrm{gr}_{i} \cong E^{k}$. If $\mu_{H_{m}}\left(\mathrm{gr}_{i}\right) \geq \mu_{H_{m}}\left(\mathcal{V}_{t}\right)$, then for a semistable $\mathcal{V}$ of character $\mathbf{v}$ we have

$$
\chi\left(\mathbf{v}_{i}, \mathbf{v}\right)=\chi\left(\operatorname{gr}_{i}, \mathcal{V}\right)=k \cdot \chi(E, \mathcal{V}) \leq 0,
$$

because $\Delta \geq \operatorname{DLP}_{H_{m}}^{<r}(\nu)$. On the other hand,

$$
\begin{gathered}
\chi\left(\mathbf{v}_{i}, \mathbf{v}\right)=\chi\left(\mathrm{gr}_{i}, \mathcal{V}_{t}\right)=\sum_{j=1}^{i-1} \chi\left(\mathrm{gr}_{i}, \mathrm{gr}_{j}\right)+\chi\left(\mathrm{gr}_{i}, \mathrm{gr}_{i}\right)+\sum_{j=i+1}^{l} \chi\left(\mathrm{gr}_{i}, \mathrm{gr}_{j}\right) \\
\quad \geq \sum_{j=1}^{i-1} \chi\left(\mathrm{gr}_{j}, \mathrm{gr}_{i}\right)+\chi\left(\mathrm{gr}_{i}, \mathrm{gr}_{i}\right)+\sum_{j=i+1}^{l} \chi\left(\mathrm{gr}_{i}, \mathrm{gr}_{j}\right)=\chi\left(\mathrm{gr}_{i}, \mathrm{gr}_{i}\right)>0,
\end{gathered}
$$

which is a contradiction. Here we used that for $j<i$ we have $\mu_{H_{m}}\left(\nu_{j}\right) \geq \mu_{H_{m}}\left(\nu_{i}\right)$ and since $m$ was chosen to be close enough to 1 , this implies $\mu_{H_{1}}\left(\nu_{j}\right) \geq \mu_{H_{1}}\left(\nu_{i}\right)$. Therefore

$$
P\left(\nu_{j}-\nu_{i}\right) \geq P\left(\nu_{i}-\nu_{j}\right) \Longrightarrow \chi\left(\mathrm{gr}_{i}, \mathrm{gr}_{j}\right) \geq \chi\left(\mathrm{gr}_{j}, \mathrm{gr}_{i}\right)
$$

If $\mu_{H_{m}}\left(\mathrm{gr}_{i}\right) \leq \mu_{H_{m}}\left(\mathcal{V}_{t}\right)$, we get a contradiction by instead comparing $\chi\left(\mathcal{V}, \mathrm{gr}_{i}\right)$ for a semistable $\mathcal{V}$ to $\chi\left(\mathcal{V}_{t}, \mathrm{gr}_{i}\right)$.

Step 4. So from now on we use that $l=2$ and $\Delta_{1}, \Delta_{2} \geq \frac{1}{2}$. Expanding $\chi\left(\mathrm{gr}_{1}, \mathrm{gr}_{2}\right)=-1$ by the Riemann-Roch Theorem we get

$$
\begin{equation*}
\Delta_{1}+\Delta_{2}=P\left(\nu_{2}-\nu_{1}\right)+\frac{1}{r_{1} r_{2}} \tag{3.4.3}
\end{equation*}
$$

Below we eliminate various cases for what the values of $\Delta_{i}$ and $\nu_{2}-\nu_{1}$ could be.
Case 1. Suppose that $\Delta_{1}, \Delta_{2}>\frac{1}{2}$. Here we follow the method in [Dre88, Proposition 2.4]. In this case, the expected dimension of the moduli spaces $M\left(\mathbf{v}_{i}\right)$ for $i=1,2$ is

$$
\exp \operatorname{dim} M\left(\mathbf{v}_{i}\right)=r_{i}^{2}\left(2 \Delta_{i}-1\right)+1 \geq 2,
$$

which allows us to write

$$
\begin{equation*}
\Delta_{i} \geq \frac{1}{2}+\frac{1}{2 r_{i}^{2}} \quad \text { for } \quad i=1,2 \tag{3.4.4}
\end{equation*}
$$

Using this estimate in equation (3.4.3), we get

$$
P\left(\nu_{2}-\nu_{1}\right)+\frac{1}{r_{1} r_{2}} \geq 1+\frac{1}{2}\left(\frac{1}{r_{1}^{2}}+\frac{1}{r_{2}^{2}}\right),
$$

which simplifies to

$$
1-P\left(\nu_{2}-\nu_{1}\right)+\frac{1}{2}\left(\frac{1}{r_{1}}-\frac{1}{r_{2}}\right)^{2} \leq 0 .
$$

Since $P\left(\nu_{2}-\nu_{1}\right) \leq 1$, we in fact have

$$
1-P\left(\nu_{2}-\nu_{1}\right)=\frac{1}{r_{1}}-\frac{1}{r_{2}}=0
$$

and

$$
r_{1}=r_{2}, \nu_{1}=\nu_{2} .
$$

Comparing equations (3.4.3) and (3.4.4), we get $\Delta_{1}=\Delta_{2}$. This is a contradiction because $\mathrm{gr}_{1}$ and $\mathrm{gr}_{2}$ are quotients in the Harder-Narasimhan filtration.

Next we rule out the cases where one of $\Delta_{1}, \Delta_{2}$ is equal to $\frac{1}{2}$, but condition (2) of Definition 3.1 does not hold. We consider the case when $\Delta_{1}=\frac{1}{2}, \Delta_{2} \geq \frac{1}{2}$; the case $\Delta_{1} \geq \frac{1}{2}, \Delta_{2}=\frac{1}{2}$ is dealt with similarly.

Case 2. Assume $\Delta_{1}=\frac{1}{2}, \Delta_{2} \geq \frac{1}{2}, \nu_{1}=\nu_{2}$. Since $\Delta_{1}=\frac{1}{2}$, the rank $r_{1}$ must be even: $r_{1}=2 \bar{r}_{1}$. Equation (3.4.3) gives

$$
\begin{equation*}
\Delta_{2}=\frac{1}{2}+\frac{1}{2 \bar{r}_{1} r_{2}}=\frac{\bar{r}_{1} r_{2}+1}{2 \bar{r}_{1} r_{2}} . \tag{3.4.5}
\end{equation*}
$$

If one of $\bar{r}_{1}$ or $r_{2}$ is even, then the right hand side of equation (3.4.5) is an irreducible fraction. This means that after cancelling all the common factors in the numerator and the denominator of

$$
\begin{equation*}
\Delta_{2}=\frac{c_{2}\left(\mathrm{gr}_{2}\right)}{r_{2}}-\frac{r_{2}-1}{r_{2}^{2}} \cdot \frac{c_{1}\left(\mathrm{gr}_{2}\right)^{2}}{2}=\frac{c_{2}\left(\mathrm{gr}_{2}\right) r_{2}-\left(r_{2}-1\right) \cdot\left(c_{1}\left(\mathrm{gr}_{2}\right)^{2} / 2\right)}{r_{2}^{2}} \tag{3.4.6}
\end{equation*}
$$

the resulting denominator should be equal to $2 \overline{r_{1}} r_{2}=r_{1} r_{2}$. This implies that $r_{2}=k r_{1}$ and character $\left(r_{2}, \nu_{1}=\nu_{2}, \frac{1}{2}\right)$ is equal to $k \cdot \mathbf{v}_{1}$, so it is an integral Chern character. We can, therefore, write

$$
\Delta_{2}=\frac{\bar{r}_{1} r_{2}+1}{2 \bar{r}_{1} r_{2}}=\frac{1}{2}+\frac{N}{r_{2}}
$$

for some integer $N$. This gives equalities

$$
\bar{r}_{1} r_{2}+1=\bar{r}_{1} r_{2}+2 N \bar{r}_{1} \Longleftrightarrow 2 N \bar{r}_{1}=1,
$$

which is impossible.
Now assume both $\bar{r}_{1}$ and $r_{2}$ are odd. Then we can cancel 2 in the numerator and the denominator of (3.4.5) and the result will be an irreducible fraction with denominator $\bar{r}_{1} r_{2}$. This time, we see from equation (3.4.6) that $\bar{r}_{1}$ divides $r_{2}$, so we can write $r_{2}=\bar{r}_{1} d$ for an odd integer $d$. Since both $\chi\left(\mathrm{gr}_{1}\right)$ and $\chi\left(\mathrm{gr}_{2}\right)$ are integers, we have

$$
\begin{gathered}
\chi\left(\mathrm{gr}_{1}\right)=r_{1} \cdot\left(P\left(\nu_{1}\right)-\frac{1}{2}\right)=2 \bar{r}_{1} P\left(\nu_{1}\right)-\bar{r}_{1}=2 \bar{r}_{1} P\left(\nu_{2}\right)-\bar{r}_{1} \in \mathbb{Z} \quad\left(\text { recall } \nu_{1}=\nu_{2}\right), \\
\chi\left(\mathrm{gr}_{2}\right)=r_{2}\left(P\left(\nu_{2}\right)-\frac{1}{2}-\frac{1}{r_{1} r_{2}}\right)= \\
=\bar{r}_{1} d P\left(\nu_{2}\right)-\frac{\bar{r}_{1} d}{2}-\frac{1}{2 \bar{r}_{1}}=\frac{\bar{r}_{1} d\left(2 \bar{r}_{1} P\left(\nu_{2}\right)-\bar{r}_{1}\right)-1}{2 \bar{r}_{1}}= \\
= \\
=\frac{\bar{r}_{1} d \cdot \chi\left(\mathrm{gr}_{1}\right)-1}{2 \bar{r}_{1}} \in \mathbb{Z} .
\end{gathered}
$$

The last expression implies $\bar{r}_{1}=1$ and we get that $\mathbf{v}_{1}=\left(2, \nu_{1}, \frac{1}{2}\right)$. Now, since $r_{2}$ is odd, we can write $\nu_{1}=\nu_{2}=\frac{*}{\text { odd }} E+\frac{*}{\text { odd }} F$ with both coefficients being irreducible fractions. But explicit analysis of the $\mathrm{DLP}_{H_{m}}^{<r}$-surface shows that $H_{m}$-semistable Chern characters with $(r, \nu, \Delta)=\left(2, \nu, \frac{1}{2}\right)$ can only have $\nu=\frac{2 k+1}{2} E+l F$ or $\nu=k E+\frac{2 l+1}{2} F$ with $k, l \in \mathbb{Z}$. Thus we cannot have $\nu_{1}=\nu_{2}$ under these assumptions.

Case 3. We turn our attention to those cases where $\Delta_{1}=\frac{1}{2}, \Delta_{2} \geq \frac{1}{2}$ and $\nu_{1} \neq \nu_{2}$. We can explicitly write

$$
\nu_{2}-\nu_{1}=\left(\frac{a_{2}}{r_{2}}-\frac{a_{1}}{r_{1}}\right) E+\left(\frac{b_{2}}{r_{2}}-\frac{b_{1}}{r_{1}}\right) F=\frac{a}{r_{1} r_{2}} E+\frac{b}{r_{1} r_{2}} F, \quad a, a_{i}, b, b_{i} \in \mathbb{Z},
$$

so that

$$
P\left(\nu_{2}-\nu_{1}\right)=\left(1+\frac{a}{r_{1} r_{2}}\right)\left(1+\frac{b}{r_{1} r_{2}}\right)=1+\frac{a+b}{r_{1} r_{2}}+\frac{a b}{\left(r_{1} r_{2}\right)^{2}} .
$$

Further cases depend on the values of $a$ and $b$. Note that by the definition of the Harder-Narasimhan filtration, we cannot have $a>0$ and $b>0$ simultaneously. Moreover, since $\Delta_{2} \geq \frac{1}{2}$, we get an inequality

$$
\begin{equation*}
1+\frac{a+b+1}{r_{1} r_{2}}+\frac{a b}{\left(r_{1} r_{2}\right)^{2}}=P\left(\nu_{2}-\nu_{1}\right)+\frac{1}{r_{1} r_{2}}=\Delta_{1}+\Delta_{2}=\frac{1}{2}+\Delta_{2} \geq 1, \tag{3.4.7}
\end{equation*}
$$

which we use to eliminate certain potential values of $a$ and $b$.
Case 3.1. Assume $a<0, b<0$. Rewrite (3.4.7) as

$$
\begin{equation*}
\frac{a b}{\left(r_{1} r_{2}\right)^{2}} \geq \frac{-a-b-1}{r_{1} r_{2}} \Longleftrightarrow a b \geq(-a-b-1) r_{1} r_{2} \Longleftrightarrow a\left(b+r_{1} r_{2}\right) \geq(-b-1) r_{1} r_{2} \tag{3.4.8}
\end{equation*}
$$

If $b+r_{1} r_{2}<0$, we get $a \leq \frac{-b-1}{b+r_{1} r_{2}} r_{1} r_{2}=\frac{-b-1}{-b-r_{1} r_{2}}\left(-r_{1} r_{2}\right) \leq-r_{1} r_{2}$, so that $a+r_{1} r_{2} \leq 0$. But then

$$
\mu_{H_{1}}\left(\mathrm{gr}_{1}\right)-\mu_{H_{1}}\left(\mathrm{gr}_{2}\right)=\frac{-a-b}{r_{1} r_{2}}>2,
$$

contradicting (3.4.1).
If $b+r_{1} r_{2}=0$, then, since $r_{1} r_{2} \geq 2$, the last inequality in (3.4.8) reads as

$$
0=a\left(l+r_{1} r_{2}\right) \geq(-b-1) r_{1} r_{l}>0
$$

which is a contradiction.
If $1 \leq b+r_{1} r_{2} \leq r_{1} r_{2}-1$, the last inequality in (3.4.8) reads as

$$
a \geq \frac{-b-1}{b+r_{1} r_{2}} r_{1} r_{2} \geq 0
$$

now contradicting the assumption that $a<0$.
Case 3.2. Assume $a<0, b>0,|a|>b$ or $a>0, b<0,|b|>a$. We estimate the left hand side in (3.4.7):

$$
1+\frac{a+b+1}{r_{1} r_{2}}+\frac{a b}{\left(r_{1} r_{2}\right)^{2}} \leq 1+\frac{a b}{\left(r_{1} r_{2}\right)^{2}}<1
$$

and the necessary condition (3.4.7) does not hold.
Case 3.3. Assume $a=0, b<0$ or $a<0, b=0$. Since the calculations are symmetric in $a$ and $b$, we only treat $a=0, b<0$. The left hand side of (3.4.7) now reads as

$$
1+\frac{b+1}{r_{1} r_{2}}
$$

If $b \leq-2$, then again condition (3.4.7) does not hold.
If $b=-1$, we get from (3.4.7)

$$
\Delta_{1}+\Delta_{2}=P\left(\nu_{2}-\nu_{1}\right)+\frac{1}{r_{1} r_{2}}=1+\frac{b+1}{r_{1} r_{2}}=1
$$

which gives $\Delta_{1}=\Delta_{2}=\frac{1}{2}$. This implies that now both $r_{1}$ and $r_{2}$ are even, $r_{1}=2 \bar{r}_{1}, r_{2}=2 \bar{r}_{2}$, and

$$
\nu_{2}-\nu_{1}=\left(\frac{a_{2}}{r_{2}}-\frac{a_{1}}{r_{1}}\right) E+\left(\frac{b_{2}}{r_{2}}-\frac{b_{1}}{r_{1}}\right) F=\left(\frac{2 \bar{r}_{1} a_{2}-2 \bar{r}_{2} a_{1}}{r_{1} r_{2}}\right) E+\left(\frac{2 \bar{r}_{1} b_{2}-2 \bar{r}_{2} b_{1}}{r_{1} r_{2}}\right) F
$$

which is never equal to

$$
\frac{a}{r_{1} r_{2}} E+\frac{b}{r_{1} r_{2}} F=\frac{-1}{r_{1} r_{2}} F
$$

Case 3.4. At this point observe that we have ruled out all cases for possible values of $a$ and $b$, except for the case $a<0, b>0,|a|=b$ or $a>0, b<0, a=|b|$.

If $|a|=|b|>r_{1} r_{2}$, then inequality (3.4.7) does not hold.
Finally, if $|a|=|b| \leq r_{1} r_{2}$, then this case corresponds precisely to conditions (1)-(3) of Definition 3.1. This shows that the nonempty Shatz stratum $S$ of codimension 1 in $T$ must be a $\Delta_{i}=\frac{1}{2}$-stratum.

The proof of Proposition 3.4 can be readily modified to give an analogous statement for characters $\mathbf{v}=(r, \nu, \Delta)$, which lie on a single branch of the Drézet-Le Potier surface.

Proposition 3.5. Let $\mathbf{v}=(r, \nu, \Delta)$ be a Chern satisfying

$$
\Delta=\operatorname{DLP}_{H_{m}}^{<r}(\nu)
$$

with a single exceptional bundle $E$ associated to $\mathbf{v}$, where $m=1+\epsilon$ and $\epsilon \in \mathbb{Q}$ is a sufficiently small number (depending on $r$ ), $0<|\epsilon| \ll 1$.

Consider a complete family $\mathcal{V}_{t} / T$ of $\mathcal{O}(1,1)$-prioritary sheaves parameterized by a smooth variety $T$ with $\mathbf{v}\left(\mathcal{V}_{t}\right)=\mathbf{v}$. Then $H_{m}$-Shatz strata of codimension 1 are given by

- the stratum parameterizing sheaves $\mathcal{V}_{t}$ with $H_{m}$-Harder-Narasimhan filtration

$$
\begin{array}{cc}
0 \subset E \subset \mathcal{V}_{t} & \left(\text { when } \mu_{H_{m}}(E) \geq \mu_{H_{m}}(\mathbf{v})\right), \text { or } \\
0 \subset \mathcal{F}_{1} \subset \mathcal{V}_{t}, \quad \mathcal{V}_{t} / \mathcal{F}_{1} \cong E & \left(\text { when } \mu_{H_{m}}(E) \leq \mu_{H_{m}}(\mathbf{v})\right),
\end{array}
$$

- $\Delta_{i}=\frac{1}{2}$-strata,
when these strata are nonempty for the family $\mathcal{V}_{t} / T$.
Proof. Suppose $S:=S_{H_{m}}\left(\mathbf{v}_{1}, \mathbf{v}_{2}, \ldots, \mathbf{v}_{l}\right)$ is a Shatz stratum of codimension 1. Repeating step 1 of the proof of Proposition 3.4, we have for a point point $t \in S$

$$
\begin{align*}
& \left|\left(\nu-\nu_{i}\right) \cdot H_{m}\right| \leq-\frac{1}{2} K_{X} \cdot H_{m} \text { and } \\
& \operatorname{codim}_{T, t}(S)=-\sum_{i<j} \chi\left(\mathbf{v}_{i}, \mathbf{v}_{j}\right)=1 \tag{3.5.1}
\end{align*}
$$

Again, we analyze various possibilities for what the numerical invariants $\mathbf{v}_{1}, \mathbf{v}_{2}, \ldots, \mathbf{v}_{l}$ of the $H_{m}$-HarderNarasimhan quotients $\mathrm{gr}_{1}, \mathrm{gr}_{2}, \ldots, \mathrm{gr}_{l}$ of $\mathcal{V}_{t}$ could be.

First assume that both $\mathrm{gr}_{1}$ and $\mathrm{gr}_{l}$ are not semiexceptional. Inspecting step 2 of the proof of Proposition 3.4 we see that we cannot have $\chi\left(\mathrm{gr}_{1}, \mathrm{gr}_{l}\right)=0$. We conclude that $\chi\left(\mathrm{gr}_{1}, \mathrm{gr}_{l}\right)=-1$, $\Delta_{1}, \Delta_{l} \geq \frac{1}{2}$ and $\chi\left(\mathrm{gr}_{i}, \mathrm{gr}_{j}\right)=0$ for $i<j,(i, j) \neq(1, l)$ with $\Delta_{1}, \Delta_{l} \geq \frac{1}{2}$. Now note that in steps 3 and 4 of the proof of Proposition 3.4, where we dealt with the same configuration for the numerical invariants, we only used that $\Delta \geq \operatorname{DLP}_{H_{m}}^{<r}(\nu)$. We conclude that the only type of codimension 1 Shatz strata which arise in this case are the $\Delta_{i}=\frac{1}{2}$-strata.

Now, we deal with the case when one of $\mathrm{gr}_{1}$ or $\mathrm{gr}_{l}$ is semiexceptional. As before, we show the proof for $\mathrm{gr}_{1}$ being semiexceptional, with the case where $\mathrm{gr}_{l}$ is semiexceptional being similar. Assume $\mathrm{gr}_{1} \cong F^{k}$, where $F$ is an exceptional bundle. We get an inequality

$$
\begin{equation*}
k \cdot \chi(F, \mathbf{v})=\chi\left(F^{k}, \mathbf{v}\right)=\chi\left(\operatorname{gr}_{1}, \mathbf{v}\right)=\chi\left(\operatorname{gr}_{1}, \operatorname{gr}_{1}\right)+\sum_{j=2}^{l} \chi\left(\operatorname{gr}_{1}, \operatorname{gr}_{j}\right) \geq \chi\left(\operatorname{gr}_{1}, \operatorname{gr}_{1}\right)-1 \geq 0 \tag{3.5.2}
\end{equation*}
$$

On the other hand, for an $H_{m}$-semistable $\mathcal{V}$ of character $\mathbf{v}$ we have

$$
\operatorname{Hom}(F, \mathcal{V})=\operatorname{Ext}^{2}(F, \mathcal{V})=0
$$

by semistability and Serre duality. This gives $\chi(F, \mathbf{v}) \leq 0$. Thus we obtain that all the inequalities in (3.5.2) must be equalities. We get $\chi\left(\mathrm{gr}_{1}, \mathrm{gr}_{1}\right)=1$, which is only possible when $k=1$ and $\mathrm{gr}_{1} \cong F$. We also have $\chi(F, \mathbf{v})=0$, which forces $F=E$ since by (3.5.1)

$$
(\nu(F)-\nu) \cdot H_{m} \leq-\frac{1}{2} K_{X} \cdot H_{m},
$$

and we are assuming that $\mathbf{v}$ has a single associated exceptional bundle $E$. We also remark that in this case $\mathrm{gr}_{l}$ cannot be semiexceptional, because then again $\mathbf{v}$ would have a second associated exceptional bundle different from $E$. Therefore, $\Delta_{2} \geq \frac{1}{2}$.

It remains to show that $l=2$. Assume that $l>2$. If $\chi\left(\mathrm{gr}_{1}, \mathrm{gr}_{l}\right)=-1$, then we follow the argument in Step 3 of the proof of Proposition 3.4 with minor modifications. Specifically, for $(i, j) \neq(1, l), i<j$, we have

$$
\chi\left(\mathrm{gr}_{i}, \mathrm{gr}_{j}\right)=0
$$

Taking $j=l$ and using Riemann-Roch, we get

$$
\chi\left(\mathrm{gr}_{i}, \mathrm{gr}_{l}\right)=P\left(\nu_{l}-\nu_{i}\right)-\Delta_{i}-\Delta_{l}=0 .
$$

We cannot have $P\left(\nu_{l}-\nu_{i}\right)=1$ and $\Delta_{l}=\frac{1}{2}$ because then $\nu_{i}=\nu_{l}, \Delta_{i}=\Delta_{j}$ and this contradicts the fact that $\mathbf{v}_{i}, \mathbf{v}_{l}$ are quotients in the Harder-Narasimhan filtration. It follows that $P\left(\nu_{l}-\nu_{i}\right)<1$ or $\Delta_{l}>\frac{1}{2}$. In both cases we get that $\Delta_{i}<\frac{1}{2}$ and $\mathrm{gr}_{i}$ is semiexceptional, $\mathrm{gr}_{i} \cong\left(F^{\prime \prime}\right)^{k}$. If $\mu_{H_{m}}\left(\mathrm{gr}_{i}\right) \geq \mu_{H_{m}}\left(\mathcal{V}_{t}\right)$, then for a semistable $\mathcal{V}$ of character $\mathbf{v}$ we have

$$
\chi\left(\mathbf{v}_{i}, \mathbf{v}\right)=\chi\left(\mathrm{gr}_{i}, \mathcal{V}\right)=k \cdot \chi\left(F^{\prime \prime}, \mathcal{V}\right) \leq 0
$$

because $\Delta \geq \operatorname{DLP}_{H_{m}}^{<r}(\nu)$. On the other hand,

$$
\begin{aligned}
& \chi\left(\mathbf{v}_{i}, \mathbf{v}\right)=\chi\left(\mathrm{gr}_{i}, \mathcal{V}_{t}\right)=\sum_{j=1}^{i-1} \chi\left(\mathrm{gr}_{i}, \mathrm{gr}_{j}\right)+\chi\left(\mathrm{gr}_{i}, \mathrm{gr}_{i}\right)+\sum_{j=i+1}^{l} \chi\left(\mathrm{gr}_{i}, \mathrm{gr}_{j}\right) \\
& \quad \geq \sum_{j=1}^{i-1} \chi\left(\mathrm{gr}_{j}, \mathrm{gr}_{i}\right)+\chi\left(\mathrm{gr}_{i}, \mathrm{gr}_{i}\right)+\sum_{j=i+1}^{l} \chi\left(\mathrm{gr}_{i}, \mathrm{gr}_{j}\right)=\chi\left(\mathrm{gr}_{i}, \mathrm{gr}_{i}\right)>0,
\end{aligned}
$$

where the inequality holds for the same reason as in Step 3 of Proposition 3.4. This is a contradiction. If $\mu_{H_{m}}\left(\mathrm{gr}_{i}\right)<\mu_{H_{m}}\left(\mathcal{V}_{t}\right)$, we get instead a contradiction by comparing $\chi\left(\mathcal{V}, \mathrm{gr}_{i}\right)$ for a semistable $\mathcal{V}$ to $\chi\left(\mathcal{V}_{t}, \mathrm{gr}_{i}\right)$.

Finally, we eliminate $l>2$ under the condition $\chi\left(\mathrm{gr}_{1}, \mathrm{gr}_{l}\right)=0$. If we can find $i \neq 1, l$ with $\chi\left(\mathrm{gr}_{i}, \mathrm{gr}_{l}\right)=0$, we arrive to a contradiction in the same way as in the previous paragraph. If not, then $l=3$ and the only nonzero $\chi\left(\mathrm{gr}_{i}, \mathrm{gr}_{j}\right)$ with $i<j$ is $\chi\left(\mathrm{gr}_{2}, \mathrm{gr}_{3}\right)=-1$. But then

$$
\chi\left(\mathbf{v}_{1}, \mathbf{v}\right)=\chi\left(\operatorname{gr}_{1}, \mathcal{V}_{t}\right)=\chi\left(\mathbf{v}_{1}, \mathbf{v}_{1}+\mathbf{v}_{2}+\mathbf{v}_{3}\right)=\chi\left(\mathbf{v}_{1}, \mathbf{v}_{1}\right)=\chi\left(\operatorname{gr}_{1}, \operatorname{gr}_{1}\right)=1>0,
$$

contradicting the fact that $\mathbf{v}$ is on the branch of the DLP-surface given by $\mathrm{gr}_{1} \cong E$ : for a semistable $\mathcal{V}$ of character $\mathbf{v}$ we have

$$
\chi\left(\mathbf{v}_{1}, \mathbf{v}\right)=\chi(E, \mathcal{V})=0 .
$$

The last two propositions motivate the following definition.
Definition 3.6. Let $\mathbf{v}=(r, \nu, \Delta) \in K(X)$ be an $H_{m}$-semistable Chern character, where $m=1+\epsilon$ for a sufficiently small (depending on $r$ ) number $\epsilon \in \mathbb{Q}, 0<|\epsilon| \ll 1$.

We call $\mathbf{v}$ a bad character if we can find a decomposition

$$
\mathbf{v}=\mathbf{v}_{1}+\mathbf{v}_{2},
$$

where $\mathbf{v}_{1}, \mathbf{v}_{2}$ are $H_{m}$-semistable Chern characters satisfying

1. $p_{H_{m}, \mathbf{v}_{1}}>p_{H_{m}, \mathbf{v}_{2}}$, where $p_{H_{m}, \mathbf{v}_{i}}$ is the reduced $H_{m}$-Hilbert polynomial of $\mathbf{v}_{i}$,
2. $\Delta_{1}, \Delta_{2} \geq \frac{1}{2}$ with at least one $\Delta_{i}=\frac{1}{2}, i=1,2$,
3. $\nu_{2}-\nu_{1}=\frac{k}{r_{1} r_{2}} E-\frac{k}{r_{1} r_{2}} F$, for some integer $k$ with $0<|k| \leq r_{1} r_{2}$,
4. $\chi\left(\mathbf{v}_{1}, \mathbf{v}_{2}\right)=-1$.

Otherwise, we call an $H_{m}$-semistable Chern character vagood character.
Remark 3.7. Note that a bad character $\mathbf{v}$ is always primitive. Indeed, if $\Delta_{1}=\frac{1}{2}$, then

$$
\chi\left(\mathbf{v}_{1}, \mathbf{v}\right)=\chi\left(\mathbf{v}_{1}, \mathbf{v}_{1}+\mathbf{v}_{2}\right)=\chi\left(\mathbf{v}_{1}, \mathbf{v}_{2}\right)=-1 .
$$

If instead $\Delta_{2}=\frac{1}{2}$, then $\chi\left(\mathbf{v}, \mathbf{v}_{2}\right)=-1$.
The point of this notion is that by Definition 3.1 and Propositions 3.4, 3.5 for good $H_{m}$-semistable characters $\Delta_{i}=\frac{1}{2}$-strata do not appear in smooth complete families of $\mathcal{O}(1,1)$-prioritary sheaves. This way, for good characters the study of the Shatz stratification yields results that are similar to the $\mathbb{P}^{2}$ case. On the other hand, when $\mathbf{v}$ is a bad Chern character, we get a potentially nonempty divisorial $\Delta_{i}=\frac{1}{2}$ stratum $S_{H_{m}}\left(\mathbf{v}_{1}, \mathbf{v}_{2}\right)$ in smooth complete families of $\mathcal{O}(1,1)$-prioritary sheaves for every decomposition $\mathbf{v}=\mathbf{v}_{1}+\mathbf{v}_{2}$ as in Definition 3.6.

To demonstrate this phenomenon we give an example of a bad Chern character $\mathbf{v}$ and a smooth complete family of $\mathcal{O}(1,1)$-prioritary sheaves of character $\mathbf{v}$ for which a $\Delta_{i}=\frac{1}{2}$-stratum is nonempty.

Example 3.8. Consider the character $\mathbf{v}=(r, \nu, \Delta)=\left(4,-\frac{1}{4} E-\frac{1}{4} F, \frac{9}{16}\right)$. We have

$$
\Delta(\mathbf{v})=\frac{9}{16}=\operatorname{DLP}_{H_{m}}^{<4}\left(-\frac{1}{4} E-\frac{1}{4} F\right)=\operatorname{DLP}_{H_{m}, \mathcal{O}}\left(-\frac{1}{4} E-\frac{1}{4} F\right) \text { for } m=1+\varepsilon, 0<\varepsilon \ll 1,
$$

so that $\mathbf{v}$ is $H_{m}$-semistable and the line bundle $\mathcal{O}$ is associated to $\mathbf{v}$. One checks that conditions of Definition 3.6 are met for the decomposition

$$
\mathbf{v}=\mathbf{v}_{1}+\mathbf{v}_{2}, \quad \mathbf{v}_{1}=\left(2,-\frac{1}{2} F, \frac{1}{2}\right), \mathbf{v}_{2}=\left(2,-\frac{1}{2} E, \frac{1}{2}\right),
$$

where the semistability of $\mathbf{v}_{1}, \mathbf{v}_{2}$ follows from

$$
\operatorname{DLP}_{H_{m}}^{<2}\left(-\frac{1}{2} E\right)=\operatorname{DLP}_{H_{m}}^{<2}\left(-\frac{1}{2} F\right)=\frac{1}{2} .
$$

This shows that $\mathbf{v}$ is an example of a bad Chern character.
The Beilinson-type spectral sequence (see [Dré91, Proposition 5.1]) allows one to resolve any $\mu_{H_{m}}$ semistable sheaf $\mathcal{V}$ of character $\mathbf{v}$ as

$$
\begin{equation*}
0 \rightarrow \mathcal{O}(-1,-1)^{2} \rightarrow \mathcal{O}(-1,0)^{3} \oplus \mathcal{O}(0,-1)^{3} \rightarrow \mathcal{V} \rightarrow 0 \tag{3.8.1}
\end{equation*}
$$

Note that this is precisely the $L$-Gaeta type resolution (2.18.1) with $L=\mathcal{O}$. Thus we consider the family $\mathcal{V}_{t} / T$ of $\mathcal{O}(1,1)$-prioritary sheaves admitting an $\mathcal{O}$-Gaeta type resolution

$$
0 \rightarrow \mathcal{O}(-1,-1)^{2} \xrightarrow{\psi_{t}} \mathcal{O}(-1,0)^{3} \oplus \mathcal{O}(0,-1)^{3} \rightarrow \mathcal{V}_{t} \rightarrow 0,
$$

where

$$
T \subset \mathbb{H}=\operatorname{Hom}\left(\mathcal{O}(-1,-1)^{2}, \mathcal{O}(-1,0)^{3} \oplus \mathcal{O}(0,-1)^{3}\right)
$$

is the open subset parameterizing injective sheaf maps with torsion-free cokernel. By Proposition 2.20, the subset $T$ is not empty, $\operatorname{codim}_{\mathbb{H}}(\mathbb{H} \backslash T) \geq 2$ and the family $\mathcal{V}_{t} / T$ is complete. We conclude that any $H_{m}$-semistable $\mathcal{V} \in M_{H_{m}}(\mathbf{v})$ is equal to some $\mathcal{V}_{t}$ for $t \in T$.

We demonstrate that $S_{H_{m}}\left(\mathbf{v}_{1}, \mathbf{v}_{2}\right)$ is nonempty in this complete family as follows. Note that we also have

$$
\operatorname{DLP}_{H_{1}}^{<2}\left(-\frac{1}{2} E\right)=\operatorname{DLP}_{H_{1}}^{<2}\left(-\frac{1}{2} F\right)=\frac{1}{2} .
$$

We then take

$$
F_{1} \in M_{H_{1}}\left(2,-\frac{1}{2} F, \frac{1}{2}\right) \text { and } F_{2} \in M_{H_{1}}\left(2,-\frac{1}{2} E, \frac{1}{2}\right)
$$

and consider their direct sum

$$
F_{1} \oplus F_{2},
$$

which is a $\mu_{H_{1}}$-semistable sheaf. Since the Beilinson-type spectral sequence is insensitive to small variations in the polarization, this sheaf is still resolved by (3.8.1) and, therefore, appears as $\mathcal{V}_{\tau}$ for some $\tau \in T$. For the $H_{m}$-polarization it is, however, no longer semistable:

$$
\mu_{H_{m}}\left(F_{1}\right)>\mu_{H_{m}}\left(F_{1} \oplus F_{2}\right) .
$$

Note that $F_{1}$ and $F_{2}$ are in fact $\mu_{H_{1}}$-stable ( $\mathbf{v}_{i}$ is primitive), and since slope stability is open in the polarization, they remain $\mu_{H_{m}}$-stable by our choice of $H_{m}$. It follows that the $H_{m}$-Harder-Narasimhan filtration of $\mathcal{V}_{\tau}$ is

$$
0 \subset F_{1} \subset \mathcal{V}_{\tau}
$$

so that $\mathcal{V}_{\tau}$ belongs to $S_{H_{m}}\left(\mathbf{v}_{1}, \mathbf{v}_{2}\right)$ and this divisorial $\Delta_{i}=\frac{1}{2}$-stratum is nonempty in $T$.
Example 3.9. Generalizing the previous example, we can generate an infinite sequence of bad $H_{m}{ }^{-}$ semistable Chern characters $\mathbf{w}_{k}$ such that analogous complete families of $\mathcal{O}(1,1)$-prioritary sheaves of character $\mathbf{w}_{k}$ arising from $\mathcal{O}$-Gaeta type resolutions all contain a non-empty $\Delta_{i}=\frac{1}{2}$-stratum.

Set $\mathbf{w}_{1}:=\mathbf{v}=\mathbf{v}_{1}+\mathbf{v}_{2}$, where $\mathbf{v}, \mathbf{v}_{1}, \mathbf{v}_{2}$ are the characters from the previous example. Inductively define

$$
\begin{equation*}
\mathbf{w}_{k}:=\mathbf{v}_{1}+\mathbf{w}_{k-1} \text { for } k \geq 2 . \tag{3.9.1}
\end{equation*}
$$

One inductively checks that for $m=1+\varepsilon, 0<\varepsilon \ll 1$, the character $\mathbf{w}_{k}$ is $H_{m}$-semistable and all the conditions of Definition 3.6 are satisfied for the decomposition of $\mathbf{w}_{k}$ as in (3.9.1). Below we list characters $\mathbf{w}_{k}=\left(r_{k}, \nu_{k}, \Delta_{k}\right)$ for small values of $k$ and plot their total slopes in the $(\varepsilon, \varphi)$-plane along with the top-down projection of various branches of the DLP-surface (compare to Figure 2.1 (A)):

| $k$ | $\left(r_{k}, \nu_{k}, \Delta_{k}\right)$ |
| :---: | :---: |
| 1 | $(4,-1 / 4 E-1 / 4 F, 9 / 16)$ |
| 2 | $(6,-1 / 6 E-1 / 3 F, 5 / 9)$ |
| 3 | $(8,-1 / 8 E-3 / 8 F, 35 / 64)$ |
| 4 | $(10,-1 / 10 E-2 / 5 F, 27 / 50)$ |
| 5 | $(12,-1 / 12 E-5 / 12 F, 77 / 144)$ |

Table 3.1: Sequence of bad characters from Example 3.9


Figure 3.1: Sequence of bad characters from Example 3.9

As in the previous example, the same Beilinson-type spectral sequence ([Dré91, Proposition 5.1]) allows one to resolve any $\mu_{H_{m}}$-semistable sheaf $\mathcal{W}$ of character $\mathbf{w}_{k}$ as

$$
0 \rightarrow \mathcal{O}(-1,-1)^{\alpha} \rightarrow \mathcal{O}(-1,0)^{\beta} \oplus \mathcal{O}(0,-1)^{\gamma} \rightarrow \mathcal{W} \rightarrow 0
$$

for some positive integers $\alpha, \beta, \gamma$. Arguing as above, one considers the complete family $\mathcal{W}_{t} / T$ of $\mathcal{O}(1,1)$ prioritary sheaves of character $\mathbf{w}_{k}$ admitting an $\mathcal{O}$-Gaeta type resolution

$$
0 \rightarrow \mathcal{O}(-1,-1)^{\alpha} \xrightarrow{\psi_{t}} \mathcal{O}(-1,0)^{\beta} \oplus \mathcal{O}(0,-1)^{\gamma} \rightarrow \mathcal{W}_{t} \rightarrow 0
$$

where

$$
T \subset \mathbb{H}=\operatorname{Hom}\left(\mathcal{O}(-1,-1)^{\alpha}, \mathcal{O}(-1,0)^{\beta} \oplus \mathcal{O}(0,-1)^{\gamma}\right)
$$

is the open subset parameterizing injective sheaf maps, and one shows that the $\Delta_{i}=\frac{1}{2}$-stratum $S_{H_{m}}\left(\mathbf{v}_{1}, \mathbf{w}_{k-1}\right)$ is nonempty.

Finally, note that

$$
\chi\left(\mathcal{O}, \mathbf{v}_{1}\right)=\chi\left(\mathcal{O}, \mathbf{v}_{2}\right)=0 \Longrightarrow \chi\left(\mathcal{O}, \mathbf{w}_{k}\right)=0
$$

and

$$
\Delta\left(\mathbf{w}_{k}\right)=\operatorname{DLP}_{H_{m}}^{<r\left(\mathbf{w}_{k}\right)}\left(\nu\left(\mathbf{w}_{k}\right)\right)=\operatorname{DLP}_{H_{m}, \mathcal{O}}\left(\nu\left(\mathbf{w}_{k}\right)\right),
$$

so that the characters $\mathbf{w}_{k}$ all lie on the branch of the DLP-surface given by the line bundle $\mathcal{O}$ (depicted as the upper-right circular sector in the picture above).

Example 3.10. Furthermore, we can repeat the constructions of the previous two examples starting with any pair of of primitive characters of the form $\mathbf{v}_{1}=\left(r, \varphi E+\varepsilon F, \frac{1}{2}\right), \mathbf{v}_{2}=\left(r, \varepsilon E+\varphi F, \frac{1}{2}\right)$ that lie on the branch of the DLP-surface given by the line bundle $\mathcal{O}$

$$
\mu_{H_{m}}(\mathcal{O})>\mu_{H_{m}}\left(\mathbf{v}_{i}\right), \quad \operatorname{DLP}_{H_{m}}^{<r}\left(\nu\left(\mathbf{v}_{i}\right)\right)=\operatorname{DLP}_{H_{m}, \mathcal{O}}\left(\nu\left(\mathbf{v}_{i}\right)\right)
$$

and satisfy conditions (1), (3) and (4) of Definition 3.6. This way, for each choice of $\mathbf{v}_{1}, \mathbf{v}_{2}$ as above we will get new infinite sequences

$$
\mathbf{w}_{1}:=\mathbf{v}=\mathbf{v}_{1}+\mathbf{v}_{2}, \quad \mathbf{w}_{k}:=\mathbf{v}_{1}+\mathbf{w}_{k-1} \text { for } k \geq 2
$$

of bad Chern characters with a nonempty $\Delta_{i}=\frac{1}{2}$-stratum in a similarly constructed family $\mathcal{W}_{t} / T$.
Here is a computer-generated list of such characters $\mathbf{v}_{i}$ for small rank $r\left(\mathbf{v}_{i}\right)$ :

| $r\left(\mathbf{v}_{i}\right)$ | $\mathbf{v}_{1}$ | $\mathbf{v}_{2}$ | $\mathbf{w}_{1}=\mathbf{v}_{1}+\mathbf{v}_{2}$ |
| :---: | :---: | :---: | :---: |
| 2 | $(2,-1 / 2 F,-1 / 2)$ | $(2,-1 / 2 E,-1 / 2)$ | $(4,-1 / 4 E-1 / 4 F, 9 / 16)$ |
| 12 | $(12,-1 / 4 E-1 / 3 F, 1 / 2)$ | $(12,-1 / 3 E-1 / 4 F, 1 / 2)$ | $(24,-7 / 24 E-7 / 24 F, 289 / 576)$ |
| 70 | $(70,-2 / 7 E-3 / 10 F, 1 / 2)$ | $(70,-3 / 10 E-2 / 7 E, 1 / 2)$ | $(140,-41 / 140 E-41 / 140 F, 9801 / 19600)$ |
| 408 | $(408,-7 / 24 E-5 / 17 F, 1 / 2)$ | $(408,-5 / 17 E-7 / 24 F, 1 / 2)$ | $(816,-239 / 816 E-239 / 816 F, 332929 / 665856)$ |

Table 3.2: Sequence of bad characters from Example 3.10
Further still, one can easily replace the line bundle $\mathcal{O}$ by an arbitrary line bundle $L$ and generate analogous infinite sequences of bad characters lying on the branch of the DLP-surface given by $L$.

Question 3.11. Note that the characters from the previous three examples lie on a branch of the $\mathrm{DLP}_{H_{m}}^{<r}$-surface given by a line bundle. It remains an open question whether a character lying

1. above the $\operatorname{DLP}_{H_{m}}^{<r}$-surface, or
2. on the branch of the $\operatorname{DLP}_{H_{m}}^{<r}$-surface controlled by a higher-rank exceptional bundle,
can be a bad character. The evidence coming from numerical calculations on a computer points to a negative answer to (1). The answer to (2) is most likely positive, though to construct an example one should search for a character $\mathbf{v}$ of a really high rank: computer calculations show that the rank should be taken to be $r(\mathbf{v})>4000$ to find such bad characters.

### 3.4 Irreducible families

Let $\mathbf{v}_{1}, \mathbf{v}_{2}, \ldots, \mathbf{v}_{l}$ be $H$-semistable Chern characters with

$$
p_{H, \mathbf{v}_{1}}>p_{H, \mathbf{v}_{2}}>\ldots>p_{H, \mathbf{v}_{l}} .
$$

We conclude this chapter by discussing how to build an irreducible family of sheaves containing all torsionfree sheaves whose quotients in the $H$-Harder-Narasimhan filtration have invariants $\mathbf{v}_{1}, \mathbf{v}_{2}, \ldots, \mathbf{v}_{l}$. We later use these results to show the irreducibility of Shatz strata in certain complete families of $\mathcal{O}(1,1)$ prioritary sheaves. The statements of this section are briefly mentioned in [Yos96b] without proof, and the outline of the proof of Proposition 3.13 was communicated to us by Yoshioka directly (also see the Appendix to [Yos95] for some similar constructions).

Given $H$-semistable Chern characters $\mathbf{v}_{1}, \mathbf{v}_{2}, \ldots, \mathbf{v}_{l}$ with

$$
p_{H, \mathbf{v}_{1}}>p_{H, \mathbf{v}_{2}}>\ldots>p_{H, \mathbf{v}_{l}}
$$

consider the family $\mathbb{F}\left(\mathbf{v}_{1}, \mathbf{v}_{2}, \ldots, \mathbf{v}_{l}\right)$ of isomorphism classes of torsion-free sheaves $\mathcal{V}$ whose $H$-HarderNarasimhan filtration

$$
0 \subset \mathcal{F}_{1} \subset \mathcal{F}_{2} \subset \ldots \subset \mathcal{F}_{l}=\mathcal{V}
$$

is of length $l$ and whose quotients satisfy $\mathbf{v}\left(\mathrm{gr}_{i}\right)=\mathbf{v}_{i}$. Note that when $l=1, \mathbb{F}\left(\mathbf{v}_{1}\right)$ is just the family of isomorphism classes of $H$-semistable sheaves with Chern character $\mathbf{v}_{1}$.

We first recall how to construct irreducible families for $H$-semistable sheaves of Chern character $\mathbf{v}$.
Lemma 3.12. Let $\mathbf{v}=(r, \nu, \Delta)$ be an $H$-semistable Chern character. Then there exists a family $\mathcal{V}_{s} / S$ of sheaves over an irreducible base $S$ with the following property:

$$
\begin{equation*}
\mathcal{V}_{s} \in \mathbb{F}\left(\mathbf{v}_{1}\right) \text { for every } s \in S \text {, and for any } \mathcal{V} \in \mathbb{F}\left(\mathbf{v}_{1}\right) \text { there exists } s \in S \text { with } \mathcal{V}=\mathcal{V}_{s} \tag{P}
\end{equation*}
$$

Proof. When $r=1$, the moduli space $M_{H}\left(\mathbf{v}_{1}\right)$ for $\mathbf{v}_{1}=(1, \nu, n)$ is a fine moduli space with a universal family $\mathcal{U}$. For $\mathbf{v}_{1}=(1, \nu, n)$ the moduli space $M_{H}\left(\mathbf{v}_{1}\right)$ is isomorphic to the Hilbert scheme $X^{[n]}$ of $n$ points on $X$. Therefore, it is irreducible and we take $S:=M_{H}\left(\mathbf{v}_{1}\right)$.

When $r \geq 2$, take $S$ to be an open subset of the Quot scheme parameterizing $H$-semistable quotients $V \otimes \mathcal{O}\left(-N H_{m}\right) \rightarrow \mathcal{V}$ for $N \gg 0$ as in the GIT construction of $M_{H}(\mathbf{v})$ ([HL10, paragraph 4.3]). Walter shows in the proof of [Wal93, Theorem 1] that $S$ is irreducible as a consequence of his more general result which says that the stack of $\mathcal{O}(0,1)$-prioritary sheaves is irreducible (see our discussion in $\S 2.6$ ).

Now we prove the analogous result for the family $\mathbb{F}\left(\mathbf{v}_{1}, \mathbf{v}_{2}, \ldots, \mathbf{v}_{l}\right)$.

Proposition 3.13. Let $\mathbf{v}_{1}, \mathbf{v}_{2}, \ldots, \mathbf{v}_{l}$ be $H$-semistable Chern characters with

$$
\begin{equation*}
p_{H, \mathbf{v}_{1}}>p_{H, \mathbf{v}_{2}}>\ldots>p_{H, \mathbf{v}_{l}} . \tag{3.13.1}
\end{equation*}
$$

Then there exists a family $\mathcal{V}_{s} / S$ of sheaves over an irreducible base $S$ with the following property: $\mathcal{V}_{s} \in \mathbb{F}\left(\mathbf{v}_{1}, \mathbf{v}_{2}, \ldots, \mathbf{v}_{l}\right)$ for every $s \in S$, and for any $\mathcal{V} \in \mathbb{F}\left(\mathbf{v}_{1}, \mathbf{v}_{2}, \ldots, \mathbf{v}_{l}\right)$ there exists $s \in S$ with $\mathcal{V}=\mathcal{V}_{s}$.

Proof. We use induction on $l$. Case $l=1$ is Lemma 3.12.
For $l \geq 2$, take $\mathcal{V} \in \mathbb{F}\left(\mathbf{v}_{1}, \mathbf{v}_{2}, \ldots, \mathbf{v}_{l}\right)$. It fits into a short exact sequence

$$
\begin{equation*}
0 \rightarrow \mathcal{F} \rightarrow \mathcal{V} \rightarrow \mathcal{E} \rightarrow 0 \tag{3.13.2}
\end{equation*}
$$

with $\mathcal{F} \in \mathbb{F}\left(\mathbf{v}_{1}\right)$ and $\mathcal{E} \in \mathbb{F}\left(\mathbf{v}_{2}, \ldots, \mathbf{v}_{l}\right)$. By the induction assumption we have a family $\mathcal{F}_{t} / T$ over an irreducible base $T$ satisfying Property (P) with respect to $\mathbb{F}\left(\mathbf{v}_{1}\right)$, and a family $\mathcal{E}_{r} / R$ over an irreducible base $R$ satisfying Property $(\mathrm{P})$ with respect to $\mathbb{F}\left(\mathbf{v}_{2}, \ldots, \mathbf{v}_{l}\right)$. Intuitively, we want to build $S$ by taking all possible extensions of $\mathcal{E}_{r}$ by $\mathcal{F}_{t}$ for all possible $t \in T$ and $r \in R$. However, since ext ${ }^{1}\left(\mathcal{E}_{r}, \mathcal{F}_{t}\right)$ may not be constant for different $t \in T, r \in R$, we will have to enlarge $S$ in a certain sense.

To this end, since by the induction assumption $\mathbb{F}\left(\mathbf{v}_{1}\right)$ and $\mathbb{F}\left(\mathbf{v}_{2}, \ldots, \mathbf{v}_{l}\right)$ are bounded families, we can choose $N \gg 0$ so that

$$
H^{i}\left(X, \mathcal{E}_{r}\left(N H_{m}\right)\right)=0 \text { for } i>0 \text { and all } r \in R
$$

and

$$
\begin{equation*}
H^{i}\left(X, \mathcal{F}_{t}\left(N H_{m}\right)\right)=0 \text { for } i>0 \text { and all } t \in T \text {. } \tag{3.13.3}
\end{equation*}
$$

Taking $V$ to be a vector space of dimension $h^{0}\left(X, \mathcal{E}_{r}\left(N H_{m}\right)\right)$, we have a surjection

$$
\begin{equation*}
V \otimes \mathcal{O}\left(-N H_{m}\right) \rightarrow \mathcal{E}_{r} \text { for each } r \in R \tag{3.13.4}
\end{equation*}
$$

Since

$$
\operatorname{hom}\left(V \otimes \mathcal{O}\left(-N H_{m}\right), \mathcal{E}_{r}\right)=h^{0}\left(X, V^{\vee} \otimes \mathcal{E}_{r}\left(N H_{m}\right)\right)
$$

is constant as a function of $r$, we get that $p_{*}\left(\mathscr{H o m}\left(V \otimes q^{*} \mathcal{O}\left(-N H_{m}\right), \mathcal{E}\right)\right.$ is a vector bundle on $R$.
Let

$$
\mathbb{V} \xrightarrow{\pi} R
$$

be the corresponding geometric vector bundle. Note that $\mathbb{V}$ remains irreducible. On $\mathbb{V} \times X$ we have a universal morphism

$$
V \otimes \pi_{X}^{*} q^{*} \mathcal{O}\left(-N H_{m}\right) \xrightarrow{\Phi} \pi_{X}^{*} \mathcal{E}
$$

and we let $\mathbb{U} \subset \mathbb{V}$ be the open subset parameterizing surjective morphisms. Observe that due to (3.13.4) $\mathbb{U} \xrightarrow{\pi} R$ remains surjective and that $\mathbb{U}$ is irreducible. Let $A=\operatorname{ker}\left(\left.\Phi\right|_{\mathbb{U} \times X}\right), \quad B=V \otimes \pi_{X}^{*} q^{*} \mathcal{O}\left(-N H_{m}\right)$, and consider the exact sequence of sheaves over $\mathbb{U} \times X$ :

$$
\begin{equation*}
0 \rightarrow A \xrightarrow{\Xi} B \xrightarrow{\Phi} \pi_{X}^{*} \mathcal{E} \rightarrow 0 \tag{3.13.5}
\end{equation*}
$$

By (3.13.3) we have $\operatorname{Ext}^{i}\left(B_{u}, \mathcal{F}_{t}\right)=0$ for $i>0$ and all $u \in \mathbb{U}, t \in T$. By (3.13.1) and semistability we also have that

$$
\left.\operatorname{Ext}^{2}\left(\mathcal{E}_{r}, \mathcal{F}_{t}\right)=\operatorname{Hom}\left(\mathcal{F}_{t}, \mathcal{E}_{r} \otimes K_{X}\right)\right)^{\vee}=0 \text { for all } r \in R, t \in T
$$

Applying $\operatorname{Hom}\left(\_, \mathcal{F}_{t}\right)$ to (3.13.5) at point $u \in \mathbb{U}$ we get that $\operatorname{Ext}^{i}\left(A_{u}, \mathcal{F}_{t}\right)=0$ for $i>0$ and all $u \in \mathbb{U}, t \in T$. Thus hom $\left(A_{u}, \mathcal{F}_{t}\right)$ is constant for all $u \in \mathbb{U}, t \in T$ and we have

$$
\begin{equation*}
0 \rightarrow \operatorname{Hom}\left(\mathcal{E}_{\pi(u)}, \mathcal{F}_{t}\right) \rightarrow \operatorname{Hom}\left(B_{u}, \mathcal{F}_{t}\right) \rightarrow \operatorname{Hom}\left(A_{u}, \mathcal{F}_{t}\right) \rightarrow \operatorname{Ext}^{1}\left(\mathcal{E}_{\pi(u)}, \mathcal{F}_{t}\right) \rightarrow 0 \text { for all } u \in \mathbb{U}, t \in T \tag{3.13.6}
\end{equation*}
$$

Recall that in our intuitive explanation we mentioned that parameterising extensions of $\mathcal{E}_{r}$ by $\mathcal{F}_{t}$ might be problematic due to jumping values of $\operatorname{ext}^{1}\left(\mathcal{E}_{r}, \mathcal{F}_{t}\right)$ for different $r \in R, t \in T$. Now (3.13.6) shows that $\operatorname{Hom}\left(A_{u}, \mathcal{F}_{t}\right)$ is a vector space of constant dimension for different $u \in \mathbb{U}, t \in T$, so we can build an irreducible space parameterizing all homomorphisms $A_{u} \rightarrow \mathbb{F}_{t}$ for all $u \in \mathbb{U}, t \in T$. Since $\operatorname{Hom}\left(A_{u}, \mathcal{F}_{t}\right)$ surjects onto $\operatorname{Ext}{ }^{1}\left(\mathcal{E}_{\pi(u)}, \mathcal{F}_{t}\right)$, this irreducible space will be a "cover" for the naive "space of extensions."

To this end, consider the natural projections

$$
\begin{array}{ll}
p r_{\mathbb{U} \times X}: & \mathbb{U} \times T \times X \rightarrow \mathbb{U} \times X, \\
p r_{T \times X}: & \mathbb{U} \times T \times X \rightarrow T \times X, \\
p r_{\mathbb{U} \times T}: & \mathbb{U} \times T \times X \rightarrow \mathbb{U} \times T .
\end{array}
$$

By the above discussion,

$$
p r_{\mathbb{U} \times T, *}\left(\mathscr{H} o m\left(p r_{\mathbb{U} \times X}^{*} A, p r_{T \times X}^{*} \mathcal{F}\right)\right)
$$

is a vector bundle over the irreducible base $\mathbb{U} \times T$, therefore the associated geometric vector bundle $S \xrightarrow{\rho} \mathbb{U} \times T$ is irreducible too. Consider the universal morphism over $S \times X$

$$
\rho_{X}^{*} p r_{\mathbb{U} \times X}^{*} A \xrightarrow{\Psi} \rho_{X}^{*} p r_{T \times X}^{*} \mathcal{F},
$$

as well as the induced morphism

$$
\rho_{X}^{*} p r_{U \times X}^{*} A \xrightarrow{\rho_{X}^{*} p r_{U \times X}^{*}(\Xi)} \rho_{X}^{*} p r_{U \times X}^{*} B,
$$

where $\rho_{X}:=\rho \times I d_{X}$. Taking the direct sum of these maps and calling the resulting cokernel sheaf by $\mathcal{V}$, we obtain the following short exact sequence of sheaves on $S \times X$ :

$$
\begin{equation*}
0 \rightarrow \rho_{X}^{*} p r_{U \times X}^{*} A \xrightarrow{\rho_{X}^{*} p r_{U \times X}^{*}(\Xi) \oplus \Psi} \rho_{X}^{*} p r_{U \times X}^{*} B \oplus \rho_{X}^{*} p r_{T \times X}^{*} \mathcal{F} \xrightarrow{\Omega} \mathcal{V} \rightarrow 0 \tag{3.13.7}
\end{equation*}
$$

For a point $s \in S$, this short exact sequence can be expanded into the following commutative diagram:


The row in the middle corresponds to (3.13.7), while the row at the bottom corresponds to (3.13.5). The column in the middle is the trivial extension. By construction, the fiber of $S$ over point ( $u, t) \in \mathbb{U} \times T$ is the vector space $\operatorname{Hom}\left(A_{u}, \mathcal{F}_{t}\right)$ which by (3.13.6) surjects onto $\operatorname{Ext}{ }^{1}\left(\mathcal{E}_{\pi(u)}, \mathcal{F}_{t}\right)$. For a given $s \in S$ with corresponding $\Psi_{s} \in \operatorname{Hom}\left(A_{u}, \mathcal{F}_{t}\right)$, the resulting extension in $\operatorname{Ext}^{1}\left(\mathcal{E}_{\pi(u)}, \mathcal{F}_{t}\right)$ is displayed in the right column in the above diagram. This way, as $s$ varies over $S$, we parameterize all possible extensions (3.13.2) and the Property ( P ) is satisfied.

## Chapter 4

## Group actions and Gaeta-type resolutions

In this chapter, we recall some basic facts about the Picard group of $G$-linearized line bundles on a variety $Y$, and discuss how to compute with the Donaldson homomorphism when working with the family of $\mathcal{O}(1,1)$-prioritary sheaves admitting an $L$-Gaeta type resolution constructed above in Propositions 2.20 and 2.21.

### 4.1 Characters and linearized line bundles

Let $G$ be an algebraic group acting on a variety $Y$. A crossed morphism is a morphism of varieties

$$
\theta: G \times Y \rightarrow \mathbb{C}^{*},
$$

satisfying

$$
\theta\left(g g^{\prime}, y\right)=\theta\left(g, g^{\prime} y\right) \theta\left(g^{\prime}, y\right) \text { for any } g, g^{\prime} \in G, y \in Y
$$

Crossed morphisms are in bijection with the linearizations of the trivial bundle $\mathcal{O}_{Y}$. Indeed, given a crossed morphism $\theta$ define the action of $G$ on the total space $Y \times \mathbb{C}$ of $\mathcal{O}_{Y}$ over the action of $G$ on $Y$ by

$$
g \cdot(y, a)=(g \cdot y, \theta(g, y) a) .
$$

A crossed morphism $\theta$ is said to be principal if there exists $f \in \mathcal{O}^{*}(Y)$ such that

$$
\theta(g, y)=\frac{f(g \cdot y)}{f(y)} \text { for any } g \in G, y \in Y
$$

Observe that for a principal crossed morphism $\theta$ coming from $f \in \mathcal{O}^{*}(Y)$ the trivial line bundle $\mathcal{O}_{Y}$ with a trivial linearization is isomorphic as $G$-bundles to the bundle $\left(\mathcal{O}_{Y}, \theta\right)$ via

$$
(y, a) \mapsto(y, f(y) a),
$$

which is easily seen to be a $G$-equivariant map.
In summary, we get an exact sequence

$$
\mathcal{O}^{*}(Y) \rightarrow \operatorname{CrMor}(Y, G) \rightarrow \operatorname{Pic}^{G}(Y) \rightarrow \operatorname{Pic}(Y)^{G}
$$

where the second term is the group of crossed morphisms and the last term denotes $G$-invariant line bundles. Now note that any character $\eta \in \operatorname{Char}(G)$ can be viewed as a crossed morphism via

$$
\theta_{\eta}(g, y)=\eta(g) .
$$

Drezét shows in [Dré87, Proposition 14] that for those algebraic groups $G$ for which any invertible function on $G$ can be written as a product of a constant and a character of $G$ we in fact have an isomorphism

$$
\operatorname{Char}(G) \rightarrow \operatorname{CrMor}(Y, G) .
$$

Therefore, for such groups we have the following result.
Proposition 4.1. Let $Y$ be an integral variety equipped with an action of an algebraic group $G$. Further suppose that any invertible function on $G$ can be written as a product of a constant and a character of $G$. Then we have the following exact sequence

$$
\begin{equation*}
\mathcal{O}^{*}(Y) \rightarrow \operatorname{Char}(G) \rightarrow \operatorname{Pic}^{G}(Y) \rightarrow \operatorname{Pic}(Y)^{G} . \tag{4.1.1}
\end{equation*}
$$

We remark that in the first map $\left(f \mapsto \eta_{f}\right)$ the resulting character $\eta_{f}$ is described by the equality

$$
\eta_{f}(g)=\frac{f(g \cdot y)}{f(y)} \quad \text { for any } g \in G, y \in Y
$$

### 4.2 Characters of the general linear group

In the context of the Gaeta-type resolutions we will be interested in the action of the general linear group and groups closely related to it. These groups will satisfy the assumption of Proposition 4.1. In view of exact sequence (4.1.1), we now recall how to describe characters for such groups.

For a fixed positive integer $n$, consider the homomorphism

$$
\begin{gathered}
\mathbb{Z} \rightarrow \operatorname{Char}(G L(n)) \\
a \mapsto\left[\eta_{a}: A \mapsto \operatorname{det}(A)^{a}\right] .
\end{gathered}
$$

Since the coordinate ring of $G L(\alpha)$ is the localization $\mathbb{C}\left[\left\{x_{i j}\right\}\right]_{\text {det }}$, the only invertible functions mapping $I d \in G L(n)$ to $1 \in \mathbb{C}$ are of the form $\left[A \rightarrow \operatorname{det}(A)^{a}\right]$ for some $a \in \mathbb{Z}$. It follows that the above homomorphism is in fact an isomorphism.

More generally, for $k$ positive integers $n_{1}, n_{2}, \ldots, n_{k}$ let $G:=G L\left(n_{1}\right) \times G L\left(n_{2}\right) \times \ldots \times G L\left(n_{k}\right)$. We have an isomorphism

$$
\begin{equation*}
\mathbb{Z}^{k} \rightarrow \operatorname{Char}(G) \tag{4.1.2}
\end{equation*}
$$

given by

$$
\left(a_{1}, a_{2}, \ldots, a_{k}\right) \mapsto\left[\eta_{\left(a_{1}, a_{2}, \ldots, a_{k}\right)}:\left(A_{1}, A_{2}, \ldots, A_{k}\right) \mapsto \operatorname{det}\left(A_{1}\right)^{a_{1}} \operatorname{det}\left(A_{2}\right)^{a_{2}} \ldots \operatorname{det}\left(A_{k}\right)^{a_{k}}\right] .
$$

Finally, let $\bar{G}:=\left(G L\left(n_{1}\right) \times G L\left(n_{2}\right) \times \ldots \times G L\left(n_{k}\right)\right) / \mathbb{C}^{*}(I d, I d, \ldots, I d)$. Under the above isomorphism $\operatorname{Char}(\bar{G})$ can be described as

$$
\begin{equation*}
\operatorname{Char}(\bar{G})=\left\{\left(a_{1}, a_{2}, \ldots, a_{k}\right) \in \mathbb{Z}^{k} \mid a_{1} n_{1}+a_{2} n_{2}+\ldots+a_{k} n_{k}=0\right\} \subset \mathbb{Z}^{k} \tag{4.1.3}
\end{equation*}
$$

### 4.3 Natural action of $G$ on Gaeta-type resolutions

We return back to the case $X=\mathbb{P}^{1} \times \mathbb{P}^{1}$. Consider the family $\mathcal{V}_{t} / T$ of $\mathcal{O}(1,1)$-prioritary sheaves of Chern character $\mathbf{v}$ with $r(\mathbf{v}) \geq 2$ over $T=U \subset \mathbb{H}$ admitting an $L$-Gaeta type resolution (2.18.1)

$$
0 \rightarrow L(-1,-1)^{\alpha} \xrightarrow{\psi_{t}} L(-1,0)^{\beta} \oplus L(0,-1)^{\gamma} \oplus L^{\delta} \rightarrow \mathcal{V}_{t} \rightarrow 0,
$$

as in Proposition 2.20. We first treat the case where all integers $\alpha, \beta, \gamma, \delta$ are not zero and say how to modify the argument when some of the exponents vanish later.

In this case, there is a natural group action of

$$
G=G L(\alpha) \times G L(\beta) \times G L(\gamma) \times G L(\delta)
$$

on $T$ : for point $\psi_{t} \in T$ the point

$$
\left(g_{\alpha}, g_{\beta}, g_{\gamma}, g_{\delta}\right) \cdot \psi_{t}
$$

corresponds to the morphism

$$
\left(g_{\beta} \oplus g_{\gamma} \oplus g_{\delta}\right) \circ \psi_{t} \circ\left(g_{\alpha}\right)^{-1}
$$

Note that since

$$
c(I d, I d, I d, I d) \cdot \psi_{t}=\psi_{t}, c \in \mathbb{C}
$$

there is also an induced action of

$$
\bar{G}=(G L(\alpha) \times G L(\beta) \times G L(\gamma) \times G L(\delta)) / \mathbb{C}^{*}(I d, I d, I d, I d)
$$

on $T$.
We extend both actions onto $T \times X$. On $T \times X$, there is a universal short exact sequence of sheaves

$$
\begin{equation*}
0 \rightarrow q^{*}(L(-1,-1))^{\alpha} \xrightarrow{\psi} q^{*}(L(-1,0))^{\beta} \oplus q^{*}(L(0,-1))^{\gamma} \oplus q^{*} L^{\delta} \rightarrow \mathcal{V} \rightarrow 0 . \tag{4.1.4}
\end{equation*}
$$

We endow the trivial families with a natural $G$-linearization as follows. Let $g=\left(g_{\alpha}, g_{\beta}, g_{\gamma}, g_{\delta}\right) \in G$. The action of $g$ is described as

$$
\begin{aligned}
& \left(q^{*}(L(-1,-1))^{\alpha}\right)_{t}=L(-1,-1)^{\alpha} \xrightarrow{g_{\alpha}} L(-1,-1)^{\alpha}=\left(q^{*}(L(-1,-1))^{\alpha}\right)_{g \cdot t} \\
& \left(q^{*}(L(-1,0))^{\beta}\right)_{t}=\quad L(-1,0)^{\beta} \xrightarrow{g_{\beta}} \quad L(-1,0)^{\beta}=\left(q^{*}(L(-1,0))^{\beta}\right)_{g \cdot t} \\
& \left(q^{*}(L(0,-1))^{\gamma}\right)_{t}=\quad L(0,-1)^{\gamma} \xrightarrow{g_{\gamma}} \quad L(0,-1)^{\gamma}=\left(q^{*}(L(0,-1))^{\gamma}\right)_{g \cdot t} \\
& \left(q^{*} L^{\delta}\right)_{t}=\quad L^{\delta} \xrightarrow{g_{\delta}} \quad L^{\delta}=\left(q^{*} L^{\delta}\right)_{g \cdot t}
\end{aligned}
$$

There is then a unique $G$-linearization of $\mathcal{V}$ making (4.1.4) a short exact sequence of $G$-linearized sheaves. For $g \in G, \psi_{t} \in T$ as above, it is described as the unique isomorphism $\Phi_{(g, t)}$ completing the diagram


This allows us to use Lemma 2.5 (3) and compute the Donaldson homomorphism

$$
\lambda_{\mathcal{V}_{t}}: K(X) \rightarrow \operatorname{Pic}^{G}(T)
$$

explicitly, taking into account that both $K(X)$ and $\operatorname{Pic}^{G}(T)$ are free $\mathbb{Z}$-modules. Specifically, we identify $K(X) \cong \mathbb{Z}^{4}$ by choosing the following $\mathbb{Z}$-basis

$$
\mathbf{e}_{1}:=\left[L^{\vee}(-1,-1)\right], \mathbf{e}_{2}:=\left[L^{\vee}(-1,0)\right], \mathbf{e}_{3}:=\left[L^{\vee}(0,-1)\right], \mathbf{e}_{4}:=\left[L^{\vee}\right] .
$$

On the other hand, by Proposition 2.20

$$
\operatorname{codim}_{\mathbb{H}}(\mathbb{H} \backslash T) \geq 2
$$

and, since $\mathbb{H}$ is an affine space, it follows that

$$
\mathcal{O}^{*}(T)=\mathbb{C}^{*} \text { and } \operatorname{Pic}(T)=0 .
$$

Note that $G$ satisfies the assumptions of Proposition 4.1, so we get

$$
\begin{equation*}
\operatorname{Char}(G) \xrightarrow{\sim} \operatorname{Pic}^{G}(T) \tag{4.1.5}
\end{equation*}
$$

and the former group was shown to be $\mathbb{Z}^{4}$ in (4.1.2).
Proposition 4.2. Consider the family $\mathcal{V}_{t} / T$ of $\mathcal{O}(1,1)$-prioritary sheaves of character $\mathbf{v}$ with $r(\mathbf{v}) \geq 2$ admitting an L-Gaeta type resolution

$$
0 \rightarrow L(-1,-1)^{\alpha} \xrightarrow{\psi_{t}} L(-1,0)^{\beta} \oplus L(0,-1)^{\gamma} \oplus L^{\delta} \rightarrow \mathcal{V}_{t} \rightarrow 0
$$

where

$$
T \subset \mathbb{H}=\operatorname{Hom}\left(L(-1,-1)^{\alpha}, L(-1,0)^{\beta} \oplus L(0,-1)^{\gamma} \oplus L^{\delta}\right)
$$

is the open subset parameterizing injective sheaf maps with torsion-free cokernel and all the exponents $\alpha, \beta, \gamma, \delta$ are nonzero.

Then the Donaldson homomorphism

$$
\lambda_{\mathcal{V}_{t}}: K(X) \rightarrow \operatorname{Pic}^{G}(T)
$$

is an isomorphism, and the image of $\mathbf{v}^{\perp}$ is equal to $\operatorname{Char}(\bar{G}) \subset \operatorname{Char}(G) \cong \operatorname{Pic}^{G}(T)$.
Proof. Let

$$
A=q^{*}(L(-1,1))^{\alpha}=q^{*}(L(-1,-1)) \otimes V_{\alpha}
$$

and

$$
B=q^{*}(L(-1,0))^{\beta} \oplus q^{*}(L(0,-1))^{\gamma} \oplus q^{*} L^{\delta}=\left(q^{*}(L(-1,0)) \otimes V_{\beta}\right) \oplus\left(q^{*}(L(0,-1)) \otimes V_{\gamma}\right) \oplus\left(q^{*} L \otimes V_{\delta}\right),
$$

where $V_{\alpha}, V_{\beta}, V_{\gamma}, V_{\delta}$ are vector spaces of dimension $\alpha, \beta, \gamma$ and $\delta$ respectively. Since the universal short exact sequence (4.1.4) is a sequence of $G$-linearized sheaves, we have that for $\mathbf{u} \in K(X)$,

$$
\lambda_{\mathcal{V}}(\mathbf{u})=\lambda_{B}(\mathbf{u}) \otimes \lambda_{A}(\mathbf{u})^{\vee} \text { as elements of } \operatorname{Pic}^{G}(T),
$$

or

$$
\lambda_{\mathcal{V}}(\mathbf{u})=\lambda_{B}(\mathbf{u})-\lambda_{A}(\mathbf{u}) \text { as elements of } \operatorname{Char}(G)
$$

under the isomorphism (4.1.5).
Using this, one readily checks that
$p_{!}\left(B \otimes q^{*} L^{\vee}(-1,-1)\right)=p_{!}\left(\left(q^{*} \mathcal{O}(-1,-2) \otimes V_{\beta}\right) \oplus\left(q^{*} \mathcal{O}(-2,-1) \otimes V_{\gamma}\right) \oplus\left(q^{*} \mathcal{O}(-1,-1) \otimes V_{\delta}\right)\right)=0$
and

$$
p_{!}\left(A \otimes q^{*} L^{\vee}(-1,-1)\right)=p_{!}\left(q^{*} \mathcal{O}(-2,-2) \otimes V_{\alpha}\right)=\left[\mathcal{O}_{T} \otimes V_{\alpha}\right],
$$

viewed as elements in $K^{G}(X)$. Thus

$$
\lambda_{\mathcal{V}_{t}}\left(\mathbf{e}_{1}\right)=\operatorname{det}\left(\mathcal{O}_{T} \otimes V_{\alpha}\right)^{\vee}
$$

which corresponds to character

$$
\eta_{(-1,0,0,0)} \in \operatorname{Char}(G)
$$

under the isomorphism (4.1.5). Similar calculations show that

$$
\begin{aligned}
& \lambda_{\mathcal{V}}\left(\mathbf{e}_{2}\right) \text { corresponds to } \eta_{(0,-1,0,0)}, \\
& \lambda_{\mathcal{V}}\left(\mathbf{e}_{3}\right) \text { corresponds to } \eta_{(0,0,-1,0)}, \\
& \lambda_{\mathcal{V}}\left(\mathbf{e}_{4}\right) \text { corresponds to } \eta_{(0,0,0,1)} .
\end{aligned}
$$

In summary, the Donaldson homomorphism $\lambda_{\mathcal{V}_{t}}$ viewed as a map $K(X) \rightarrow \operatorname{Char}(G)$ is given by

$$
\begin{equation*}
\mathbf{u}=a_{1} \mathbf{e}_{1}+a_{2} \mathbf{e}_{2}+a_{3} \mathbf{e}_{3}+a_{4} \mathbf{e}_{4} \mapsto \eta=\eta_{\left(-a_{1},-a_{2},-a_{3}, a_{4}\right)} \tag{4.2.1}
\end{equation*}
$$

Alternatively, it has matrix

$$
\left[\begin{array}{cccc}
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right]
$$

when viewed as a map $\mathbb{Z}^{4} \rightarrow \mathbb{Z}^{4}$.
Now we turn to the second statement in the proposition. One checks that

$$
\mathbf{v}=-\alpha[L(-1,-1)]+\beta[L(-1,0)]+\gamma[L(0,-1)]+\delta[L] \text { in } K(X)
$$

by applying $\chi\left({ }_{-} \cdot \mathbf{e}_{i}\right)$ to both sides and using (2.19.1) along with the fact that

$$
\overline{\mathbf{e}}_{1}=[L(-1,-1)], \overline{\mathbf{e}}_{2}=[L(-1,0)], \overline{\mathbf{e}}_{3}=[L(0,-1)], \overline{\mathbf{e}}_{4}=[L]
$$

is a $\chi\left({ }_{-}{ }^{\cdot}\right)$-orthogonal basis to $\mathbf{e}_{1}, \mathbf{e}_{2}, \mathbf{e}_{3}, \mathbf{e}_{4}$ :

$$
\chi\left(\overline{\mathbf{e}}_{i} \cdot \mathbf{e}_{j}\right)=0
$$

for $i \neq j$ and

$$
\chi\left(\overline{\mathbf{e}}_{1} \cdot \mathbf{e}_{1}\right)=\chi\left(\overline{\mathbf{e}}_{4} \cdot \mathbf{e}_{4}\right)=1, \chi\left(\overline{\mathbf{e}}_{2} \cdot \mathbf{e}_{2}\right)=\chi\left(\overline{\mathbf{e}}_{3} \cdot \mathbf{e}_{3}\right)=-1 .
$$

One further checks that the condition

$$
\mathbf{u} \in \mathbf{v}^{\perp} \Longleftrightarrow \chi(\mathbf{v} \cdot \mathbf{u})=0
$$

is equivalent to

$$
-a_{1} \alpha-a_{2} \beta-a_{3} \gamma+a_{4} \delta=0
$$

By (4.1.3), this last condition is precisely equivalent to $\eta=\eta_{\left(-a_{1},-a_{2},-a_{3}, a_{4}\right)} \in \operatorname{Char}(\bar{G})$.

The above proof easily carries over to the case when one of the exponents in an $L$-Gaeta-type resolution is zero. In particular, we will later work with the case when $\delta=0$. In this case, set

$$
\begin{align*}
& G_{\hat{\delta}}=G L(\alpha) \times G L(\beta) \times G L(\gamma) \\
& \bar{G}_{\hat{\delta}}=(G L(\alpha) \times G L(\beta) \times G L(\gamma)) / \mathbb{C}^{*}(I d, I d, I d) . \tag{4.2.2}
\end{align*}
$$

Proposition 4.3. Consider the family $\mathcal{V}_{t} / T$ of $\mathcal{O}(1,1)$-prioritary sheaves of character $\mathbf{v}$ with $r(\mathbf{v}) \geq 2$ admitting an L-Gaeta-type resolution

$$
0 \rightarrow L(-1,-1)^{\alpha} \xrightarrow{\phi_{t}} L(-1,0)^{\beta} \oplus L(0,-1)^{\gamma} \rightarrow \mathcal{V}_{t} \rightarrow 0,
$$

where

$$
T \subset \mathbb{H}=\operatorname{Hom}\left(L(-1,-1)^{\alpha}, L(-1,0)^{\beta} \oplus L(0,-1)^{\gamma}\right)
$$

is the open subset parameterizing injective sheaf maps with torsion-free cokernel and the exponents $\alpha, \beta, \gamma$ are nonzero.

Then the Donaldson homomorphism

$$
\lambda_{\mathcal{V}_{t}}: K(X) \rightarrow \operatorname{Pic}^{G_{\hat{\gamma}}}(T)
$$

is an epimorphism, and the image of $\mathbf{v}^{\perp}$ is equal to $\operatorname{Char}\left(\bar{G}_{\hat{\delta}}\right) \subset \operatorname{Char}\left(G_{\hat{\delta}}\right) \cong \operatorname{Pic}^{G_{\hat{\delta}}}(T)$.
Finally, we can repeat the discussion of this chapter for the "dual version" of a Gaeta-type resolution. Consider the family $\mathcal{V}_{t} / T$ of $\mathcal{O}(1,1)$-prioritary sheaves of Chern character $\mathbf{v}$ with $r(\mathbf{v}) \geq 2$ over $T=U \subset \mathbb{H}$ admitting an $L$-Gaeta type resolution (2.20.1)

$$
0 \rightarrow \mathcal{V}_{t} \rightarrow L(1,0)^{\alpha} \oplus L(0,1)^{\beta} \oplus L^{\gamma} \xrightarrow{\psi_{t}} L(1,1)^{\delta} \rightarrow 0
$$

as in Proposition 2.21. There is a natural action of $G=G L(\alpha) \times G L(\beta) \times G L(\gamma) \times G L(\delta)$ and $\bar{G}=(G L(\alpha) \times G L(\beta) \times G L(\gamma) \times G L(\delta)) / \mathbb{C}^{*}(I d, I d, I d, I d)$ on $T$ and $T \times X$ if all the exponents $\alpha, \beta, \gamma, \delta$ are nonzero, and of

$$
G_{\hat{\gamma}}=G L(\alpha) \times G L(\beta) \times G L(\delta) \text { and } \bar{G}_{\hat{\gamma}}=(G L(\alpha) \times G L(\beta) \times G L(\delta)) / \mathbb{C}^{*}(I d, I d, I d)
$$

if $\alpha, \beta, \delta>0$, but $\gamma=0$.
As before, the action of $G$ (resp. $G_{\hat{\gamma}}$ ) on $T \times X$ lifts to a linearization of the universal families of sheaves and we have the obvious analogues of Propositions 4.2 and 4.3.

## Chapter 5

## The Picard group of the moduli space

### 5.1 Associated exceptional bundles and the Donaldson homomorphism

Exceptional bundles associated to an $H_{m}$-semistable character $\mathbf{v}$ (see Definition 2.16) give rise to easy-to-describe classes in the kernel of the the Donaldson homomorphism $\lambda: \mathbf{v}^{\perp} \rightarrow \operatorname{Pic}\left(M_{H_{m}}(\mathbf{v})\right)$.

Specifically, suppose $E$ is associated to a nonsemiexceptional $H_{m}$-semistable character $\mathbf{v}$ and $\mu_{H_{m}}(E) \geq \mu_{H_{m}}(\mathbf{v})$. By semistability and Serre duality, for any semistable $\mathcal{V}$ of character $\mathbf{v}$ we have

$$
\operatorname{Hom}(E, \mathcal{V})=\operatorname{Ext}^{2}(E, \mathcal{V})=0,
$$

and since $\Delta(\mathcal{V})=\operatorname{DLP}_{H_{m}, E}(\nu(\mathcal{V}))$, we also have

$$
\chi(E, \mathcal{V})=0 \quad \text { and } \quad E x t^{1}(E, \mathcal{V})=0
$$

This way, we see that if $\mathcal{U}_{r} / R$ is the family of $H_{m}$-semistable sheaves parameterized by a subset $R$ of the Quot scheme used in the GIT construction of $M_{H_{m}}(\mathbf{v})$, then in the notation of $\S 2.4$ we have

$$
p_{!}\left(q^{*}\left[E^{\vee}\right] \cdot[\mathcal{U}]\right)=0 .
$$

Proposition 2.7 then shows that

$$
\lambda\left(\left[E^{\vee}\right]\right)=0
$$

Similarly, if $\mu_{H_{m}}(E)<\mu_{H_{m}}(\mathbf{v})$, then

$$
\lambda\left(\left[E^{\vee} \otimes K_{X}\right]\right)=0
$$

For that matter, we introduce the following uniform notation: for an exceptional bundle $E$ associated to character $\mathbf{v}$, we define the following class in $K(X)$

$$
[\bar{E}]= \begin{cases}{\left[E^{\vee}\right]} & \text { if } \mu_{H_{m}}(E) \geq \mu_{H_{m}}(\mathbf{v}) \\ {\left[E^{\vee} \otimes K_{X}\right]} & \text { if } \mu_{H_{m}}(E)<\mu_{H_{m}}(\mathbf{v})\end{cases}
$$

### 5.2 The main theorem

Finally, we are ready to state our first main result about the Picard group of the moduli space $M_{H_{m}}(\mathbf{v})$. We recall that $\lambda: \mathbf{v}^{\perp} \rightarrow \operatorname{Pic}\left(M_{H_{m}}(\mathbf{v})\right)$ denotes the Donaldson homomorphism constructed in Proposition 2.7.

Theorem 5.1. Let $\mathbf{v}=(r, \nu, \Delta) \in K(X)$ be a character with $r \geq 2$ and $\Delta \geq \frac{1}{2}$. Let $\epsilon \in \mathbb{Q}$ be sufficiently small (depending on $r$ ), $0<|\epsilon| \ll 1$, and set $m=1+\epsilon$.

1. (v lies above the $\mathrm{DLP}^{<r}$-surface) If $\Delta>\frac{1}{2}, \Delta>\operatorname{DLP}_{H_{m}}^{<r}(\nu)$ and either
(a) $\Delta-\frac{1}{r} \geq \operatorname{DLP}_{H_{m}}^{<r}(\nu)$ and $\Delta-\frac{1}{r}>\frac{1}{2}$, or
(b) $\Delta-\frac{1}{r} \geq \operatorname{DLP}_{H_{m}}^{<r}(\nu)$ and $\mathrm{v}^{\prime}=\left(r, \nu, \Delta-\frac{1}{r}\right)$ is primitive, or
(c) $\mathbf{v}$ is a good character,
then

$$
\operatorname{Pic}\left(M_{H_{m}}(\mathbf{v})\right) \cong \mathbb{Z}^{3}
$$

and $\lambda$ is an isomorphism.
2. ( $\mathbf{v}$ lies on a single branch of the $\mathrm{DLP}^{<r}$-surface) If $\mathbf{v}$ is a good character with $\Delta=\operatorname{DLP}_{H_{m}}^{<r}(\nu)>\frac{1}{2}$ with a single exceptional bundle $E$ associated to $\mathbf{v}$, then

$$
\operatorname{Pic}\left(M_{H_{m}}(\mathbf{v})\right) \cong \mathbb{Z}^{2}
$$

and $\lambda$ is an epimorphism with

$$
\operatorname{ker} \lambda=\mathbb{Z}[\bar{E}] .
$$

3. ( v lies on an intersection of two branches of the $\mathrm{DLP}^{<r}$-surface)
(a) If $\Delta=\operatorname{DLP}_{H_{m}}^{<r}(\nu)>\frac{1}{2}$ with two exceptional bundles $E_{1}, E_{2}$ associated to $\mathbf{v}$ and
$\mathbf{v}$ is primitive or $\left(E_{1}, E_{2}\right)$ is an exceptional pair,
then

$$
\operatorname{Pic}\left(M_{H}(\mathbf{v})\right) \cong \mathbb{Z}
$$

and $\lambda$ is an epimorphism with

$$
\operatorname{ker} \lambda=\mathbb{Z}\left[\overline{E_{1}}\right]+\mathbb{Z}\left[\overline{E_{2}}\right] .
$$

(b) If $\Delta=\frac{1}{2}$, then $M_{H_{m}}(\mathbf{v})$ is a projective space and

$$
\operatorname{Pic}\left(M_{H_{m}}(\mathbf{v})\right) \cong \mathbb{Z} .
$$

We draw the reader's attention to statements (1.a) and (2) of the above theorem which use the notion of a good Chern character from Definition 3.6. This assumption is substantial: as we show in Theorem 6.3 below for certain bad characters lying on a single branch of the DLP-surface the Picard number drops to 1 . We also emphasize that determining which statement of the theorem applies to a given character $\mathbf{v}=(r, \nu, \Delta)$ is a finite computational procedure and, therefore, can be implemented on a computer: the computation of $\operatorname{DLP}_{H_{m}}^{<r}(\nu)$ is finite, and to check whether $\mathbf{v}$ is a good character one needs to test finitely many candidate characters with discriminant $\frac{1}{2}$ for whether they give a decomposition $\mathbf{v}=\mathbf{v}_{\mathbf{1}}+\mathbf{v}_{\mathbf{2}}$ as in Definition 3.6.

Let us mention that the classification in the case $r=1$ also fits the above pattern in a certain sense. For $\mathbf{v}=(1, a E+b F, n)$ the moduli space $M_{H_{m}}(\mathbf{v})$ is isomorphic to the Hilbert scheme of $n$ points $X^{[n]}$. Therefore, when

$$
n=1=\operatorname{DLP}_{H_{m}, \mathcal{O}(a, b)}(a E+b F)
$$

we have $X^{[1]} \cong X$ and $\operatorname{Pic}\left(X^{[1]}\right) \cong \mathbb{Z}^{2}$. When

$$
n>1=\operatorname{DLP}_{H_{m}, \mathcal{O}(a, b)}(a E+b F)
$$

we have $\operatorname{Pic}\left(X^{[n]}\right) \cong \mathbb{Z}^{3}$ by the Theorem of Fogarty [Fog73].
The proof of Theorem 5.1 occupies the rest of this chapter. For the convenience of the reader, we will prove the theorem in a series of propositions according to how the theorem is stated. Cases (1.a), (1.b) and (3) of the theorem have relatively simple proofs. We then prove a part of case (2) so that at that point the theorem will be proved for characters $\mathbf{v}$ in a large region in the $(r, \nu, \Delta)$-space. The remaining characters $\mathbf{v}$ have their discriminant in a narrow range $\frac{1}{2}<\Delta<1$ and have no line bundles associated to them. These conditions allow us to deal with case (1.c) and the remainder of case (2) in a uniform fashion though the proofs become considerably more involved.

### 5.3 Proof of the main theorem

We start by proving case (3.b) of the theorem to be able to assume $\Delta>\frac{1}{2}$ in the rest of the proof and use the surjectivity of the Donaldson morphism from Theorem 2.8.

Proposition 5.2. Let $\mathbf{v}=(r, \nu, \Delta) \in K(X)$ be a character with $r \geq 2$ and $\Delta=\frac{1}{2}$. Let $\epsilon \in \mathbb{Q}$ be sufficiently small (depending on $r$ ), $0<|\epsilon| \ll 1$, and set $m=1+\epsilon$.

If $\mathbf{v}$ is $H_{m}$-semistable, then $M_{H_{m}}(\mathbf{v})$ is a projective space and

$$
\operatorname{Pic}\left(M_{H_{m}}(\mathbf{v})\right) \cong \mathbb{Z}
$$

Proof. First, if $\mathbf{v}=\left(r, \nu, \frac{1}{2}\right)=\left(r, c_{1}, \chi\right)$ is a primitive character, then for an appropriate (generic) choice of $m=\frac{p}{q}$ one checks that

$$
\operatorname{gcd}\left(r, c_{1} \cdot\left(q H_{m}\right), \chi\right)=1
$$

In this case, $M_{H_{m}}(\mathbf{v})=M_{H_{m}}^{s}(\mathbf{v})$ is a smooth projective variety of dimension

$$
\operatorname{dim} M_{H_{m}}(\mathbf{v})=\exp \operatorname{dim} M_{H_{m}}(\mathbf{v})=1
$$

Moreover, Walter shows in [Wal93] that $M_{H_{m}}^{s}(\mathbf{v})$ is irreducible and unirational (we briefly mentioned that in §2.6). It follows that in this case $M_{H_{m}}(\mathbf{v}) \cong \mathbb{P}^{1}$.

Now, assume $\mathbf{v}$ is not primitive. We can write $\mathbf{v}=N \mathbf{v}^{\prime}$ with $N \in \mathbb{N}$ and $\mathbf{v}^{\prime}$ primitive. In this case $M_{H_{m}}(\mathbf{v})$ consists of strictly semistable sheaves. But since $M_{H_{m}}\left(\mathbf{v}^{\prime}\right)=M_{H_{m}}^{s}\left(\mathbf{v}^{\prime}\right)$ carries a universal family of $H_{m}$-stable sheaves $\mathcal{U}$, we can take its $N$-fold sum to get a morphism

$$
M_{H_{m}}\left(\mathbf{v}^{\prime}\right) \times M_{H_{m}}\left(\mathbf{v}^{\prime}\right) \times \ldots \times M_{H_{m}}\left(\mathbf{v}^{\prime}\right) \rightarrow M_{H_{m}}(\mathbf{v})
$$

This morphism is surjective on closed points and invariant under permutation of factors, i.e. factors through the symmetric product

$$
\begin{equation*}
S^{N}\left(M_{H_{m}}\left(\mathbf{v}^{\prime}\right)\right) \rightarrow M_{H_{m}}(\mathbf{v}) \tag{5.2.1}
\end{equation*}
$$

which is now bijective at closed points. Note that since $M_{H_{m}}\left(\mathbf{v}^{\prime}\right) \cong \mathbb{P}^{1}$, the symmetric product is just a projective space

$$
S^{N}\left(M_{H_{m}}\left(\mathbf{v}^{\prime}\right)\right) \cong \mathbb{P}^{N} .
$$

Now recall that $M_{H_{m}}(\mathbf{v})$ is a good quotient $R / / G$ of a smooth subvariety $R$ of the Quot scheme. In particular, $R$ is normal. Since normality is preserved under taking categorical quotients (see [MFK94, Page 5]), $M_{H_{m}}(\mathbf{v})$ is normal too. It follows that (5.2.1) is an isomorphism.

From now on, we will be working with characters $\mathbf{v}$ with $\Delta(\mathbf{v})>\frac{1}{2}$. Note that by Proposition 2.17 (1) the stable locus $M_{H_{m}}^{s}(\mathbf{v})$ will be nonempty for $H_{m}$-semistable Chern characters with $\Delta(\mathbf{v})>\frac{1}{2}$. Applying Theorem 2.8, we know that the Donaldson homomorphism is surjective

$$
\lambda: \mathbf{v}^{\perp} \rightarrow \operatorname{Pic}\left(M_{H_{m}}(\mathbf{v})\right),
$$

and we need to study its kernel.
The next proposition corresponds to cases (1.a) and (1.b) of Theorem 5.1.
Proposition 5.3. Let $\mathbf{v}=(r, \nu, \Delta) \in K(X)$ be a character with $r \geq 2$ and $\Delta>\frac{1}{2}$. Let $\epsilon \in \mathbb{Q}$ be sufficiently small (depending on $r$ ), $0<|\epsilon| \ll 1$, and set $m=1+\epsilon$.

$$
\begin{aligned}
& \text { If } \Delta-\frac{1}{r} \geq \operatorname{DLP}_{H_{m}}^{<r}(\nu) \text { and either } \\
& \qquad\left(\Delta-\frac{1}{r}>\frac{1}{2}\right) \quad \text { or } \quad\left(\mathrm{v}^{\prime}=\left(r, \nu, \Delta-\frac{1}{r}\right) \text { is primitive }\right),
\end{aligned}
$$

then

$$
\operatorname{Pic}\left(M_{H_{m}}(\mathbf{v})\right) \cong \mathbb{Z}^{3}
$$

and $\lambda$ is an isomorphism.
Proof. Since $\Delta-\frac{1}{r} \geq \operatorname{DLP}_{H_{m}}^{<r}(\nu)$, Theorem 2.15 implies that for $\mathbf{v}^{\prime}=\left(r, \nu, \Delta-\frac{1}{r}\right)$ the moduli space $M_{H_{m}}\left(\mathbf{v}^{\prime}\right)$ is non-empty. If character $\mathbf{v}$ is primitive, then as in the proof of Proposition 5.2 above we get that $M_{H_{m}}\left(\mathbf{v}^{\prime}\right)=M_{H_{m}}^{s}\left(\mathbf{v}^{\prime}\right)$ and there are $H_{m}$-stable sheaves of character $\mathbf{v}^{\prime}$. If

$$
\Delta\left(\mathbf{v}^{\prime}\right)=\Delta-\frac{1}{r}>\frac{1}{2},
$$

then Proposition 2.17 (1) guarantees the existence of $H_{m}$-stable sheaves. In both cases, Proposition 2.17 (2) implies that there are $\mu_{H_{m}}$-stable sheaves of character $\mathbf{v}^{\prime}$. By the results of Walter discussed in §2.6, we can find a $\mu_{H_{m}}$-stable vector bundle $\mathcal{V}^{\prime}$ of character $\mathbf{v}^{\prime}$.

Now, taking elementary modifications of $\mathcal{V}^{\prime}$ as described in [HL10, Example 8.1.7] one can show that $\operatorname{Pic}\left(M_{H_{m}}(\mathbf{v})\right)$ contains $\mathbb{Z} \oplus \operatorname{Pic}(X) \cong \mathbb{Z}^{3}$. It follows that

$$
\mathbb{Z}^{3} \cong \mathbf{v}^{\perp} \xrightarrow{\lambda} \operatorname{Pic}\left(M_{H_{m}}(\mathbf{v})\right)
$$

is an isomorphism, for if it had a nontrivial kernel, Theorem 2.8 would imply that the Picard number $\rho\left(M_{H_{m}}(\mathbf{v})\right) \leq 2$, yielding a contradiction.

The next proposition corresponds to case (3.a) of Theorem 5.1.

Proposition 5.4. Let $\mathbf{v}=(r, \nu, \Delta) \in K(X)$ be a character with $r \geq 2$ and $\Delta>\frac{1}{2}$. Let $\epsilon \in \mathbb{Q}$ be sufficiently small (depending on $r$ ), $0<|\epsilon| \ll 1$, and set $m=1+\epsilon$.

$$
\begin{aligned}
& \text { If } \Delta=\operatorname{DLP}_{H_{m}}^{<r}(\nu)>\frac{1}{2} \text { with two exceptional bundles } E_{1}, E_{2} \text { associated to } \mathbf{v} \text { and } \\
& \qquad \mathbf{v} \text { is primitive or }\left(E_{1}, E_{2}\right) \text { is an exceptional pair, }
\end{aligned}
$$

then

$$
\operatorname{Pic}\left(M_{H}(\mathbf{v})\right) \cong \mathbb{Z}
$$

and $\lambda$ is an epimorphism with

$$
\operatorname{ker} \lambda=\mathbb{Z}\left[\overline{E_{1}}\right]+\mathbb{Z}\left[\overline{E_{2}}\right]
$$

Proof. In this case, the discussion in $\oint 5.1$ shows that the subgroup $\mathbb{Z}\left[\overline{E_{1}}\right]+\mathbb{Z}\left[\overline{E_{2}}\right]$ lies in the kernel of the Donaldson homomorphism, which now factors as

$$
\mathbf{v}^{\perp} / \mathbb{Z}\left[\overline{E_{1}}\right] \oplus \mathbb{Z}\left[\overline{E_{2}}\right] \rightarrow \operatorname{Pic}\left(M_{H_{m}}(\mathbf{v})\right) .
$$

Since the ample bundle generates a free $\mathbb{Z}$-submodule inside $\operatorname{Pic}\left(M_{H_{m}}(\mathbf{v})\right)$, it follows that the Picard number is equal to one

$$
\rho\left(M_{H_{m}}(\mathbf{v})\right)=1
$$

If $\mathbf{v}$ is primitive, then for a generic choice of $m=\frac{p}{q}$ we have

$$
\operatorname{gcd}\left(r, c_{1} \cdot\left(q H_{m}\right), \chi\right)=1
$$

Applying Theorem 2.11 we get that $\operatorname{Pic}\left(M_{H_{m}}(\mathbf{v})\right)$ is torsion-free and, therefore,

$$
\operatorname{Pic}\left(M_{H_{m}}(\mathbf{v})\right) \cong \mathbb{Z} .
$$

Now assume $\left(E_{1}, E_{2}\right)$ or $\left(E_{2}, E_{1}\right)$ forms an exceptional pair. One checks that ( $\left.\bar{E}_{1}, \bar{E}_{2}\right)$ or $\left(\overline{E_{2}}, \overline{E_{1}}\right)$ is still an exceptional pair. Zyuzina [Zyu94] shows that any exceptional pair on $\mathbb{P}^{1} \times \mathbb{P}^{1}$ can be completed to a full exceptional collection. Since a full exceptional collection forms a $\mathbb{Z}$-basis for $K(X)$, we see that $\mathbb{Z}\left[\overline{E_{1}}\right] \oplus \mathbb{Z}\left[\overline{E_{2}}\right]$ is a primitive lattice inside $\mathbf{v}^{\perp} \subset K(X)$. This way, the Donaldson homomorphism induces

$$
\mathbb{Z} \cong \mathbf{v}^{\perp} / \mathbb{Z}\left[\overline{E_{1}}\right] \oplus \mathbb{Z}\left[\overline{E_{2}}\right] \rightarrow \operatorname{Pic}\left(M_{H_{m}}(\mathbf{v})\right) .
$$

The result follows.
The arguments above worked equally well for both good and bad $H_{m}$-semistable Chern characters. However, for the remaining cases (1.c) and (2) of Theorem 5.1 the assumption that character $\mathbf{v}$ is good is essential.

It will be convenient to separate the proof of case (2) of Theorem 5.1 into the following two subcases (keeping the notation and the assumptions of the theorem):
(2.a) If $\mathbf{v}$ is a good character with $\Delta=\operatorname{DLP}_{H_{m}}^{<r}(\nu)>\frac{1}{2}$ with a single exceptional bundle $L$ associated to $\mathbf{v}$ and $r(L)=1$, then

$$
\operatorname{Pic}\left(M_{H_{m}}(\mathbf{v})\right) \cong \mathbb{Z}^{2}
$$

and $\lambda$ is an epimorphism with

$$
\operatorname{ker} \lambda=\mathbb{Z}[\bar{L}] \text {. }
$$

(2.b) If v is a good character with $\Delta=\operatorname{DLP}_{H_{m}}^{<r}(\nu)>\frac{1}{2}$ with a single exceptional bundle $E$ associated to $\mathbf{v}$ and $r(E)>1$, then

$$
\operatorname{Pic}\left(M_{H_{m}}(\mathbf{v})\right) \cong \mathbb{Z}^{2}
$$

and $\lambda$ is an epimorphism with

$$
\operatorname{ker} \lambda=\mathbb{Z}[\overline{\bar{E}}]
$$

We prove the case (2.a) first.
Proposition 5.5. Let $\mathbf{v}=(r, \nu, \Delta) \in K(X)$ be a character with $r \geq 2$ and $\Delta>\frac{1}{2}$. Let $\epsilon \in \mathbb{Q}$ be sufficiently small (depending on $r$ ), $0<|\epsilon| \ll 1$, and set $m=1+\epsilon$.

If $\mathbf{v}$ is a good character with $\Delta=\operatorname{DLP}_{H_{m}}^{<r}(\nu)$ with a single exceptional bundle $L$ associated to $\mathbf{v}$ and $r(L)=1$, then

$$
\operatorname{Pic}\left(M_{H_{m}}(\mathbf{v})\right) \cong \mathbb{Z}^{2}
$$

and $\lambda$ is an epimorphism with

$$
\operatorname{ker} \lambda=\mathbb{Z}[\bar{L}] .
$$

Proof. We treat the case

$$
\mu_{H_{m}}(\mathbf{v}) \leq \mu_{H_{m}}(L)
$$

and say how to modify the argument in the other case at the end of the proof.
In this situation, the general $H_{m}$-semistable sheaf $\mathcal{V}$ admits an $L$-Gaeta type resolution with exponents

$$
\begin{array}{r}
\alpha=-\chi\left(\mathbf{v} \otimes L^{\vee}(-1,-1)\right)>0 \\
\beta=-\chi\left(\mathbf{v} \otimes L^{\vee}(-1,0)\right)>0 \\
\gamma=-\chi\left(\mathbf{v} \otimes L^{\vee}(0,-1)\right)>0 \\
\delta=\chi\left(\mathbf{v} \otimes L^{\vee}\right)=0
\end{array}
$$

Note that none of $\alpha, \beta, \gamma$ can be equal to 0 . For otherwise, one checks using the semistability of $\mathcal{V}$ that one of the bundles $L(1,1), L(1,0), L(0,1)$ or their Serre twists would also be associated to $\mathbf{v}$, contradicting our assumption.

Consider the family $\mathcal{V}_{t} / T$ of $\mathcal{O}(1,1)$-prioritary sheaves admitting the $L$-Gaeta type resolution

$$
\begin{equation*}
0 \rightarrow L(-1,-1)^{\alpha} \xrightarrow{\psi_{t}} L(-1,0)^{\beta} \oplus L(0,-1)^{\gamma} \rightarrow \mathcal{V}_{t} \rightarrow 0 \tag{5.5.1}
\end{equation*}
$$

where

$$
T \subset \mathbb{H}=\operatorname{Hom}\left(L(-1,-1)^{\alpha}, L(-1,0)^{\beta} \oplus L(0,-1)^{\gamma}\right)
$$

is the open subset parameterizing injective sheaf maps with torsion-free cokernel from Proposition 2.20. Since we are assuming that $\mathbf{v}$ is a good character, there is no $\Delta_{i}=\frac{1}{2}$ strata in this family by Proposition 3.5. The other potential divisorial Shatz stratum should consist of sheaves $\mathcal{V}_{t}$ admitting the $H_{m}$-HarderNarasimhan filtration

$$
0 \subset L \subset \mathcal{V}_{t}
$$

Applying $\operatorname{Hom}\left(L,{ }_{-}\right)$to the short exact sequence (5.5.1), we see that

$$
\operatorname{Hom}\left(L, \mathcal{V}_{t}\right)=0
$$

for all $\psi_{t} \in T$, so this potential stratum is empty. Combined with Proposition 2.20, we get that

$$
\operatorname{codim}_{T}\left(T \backslash T^{s s}\right) \geq 2
$$

Functorial properties of the Donaldson homomorphism from Lemma 2.5 give the following commutative diagram (recall our notation from (4.2.2) and Proposition 4.3):


By Proposition 4.3 we know that the image of $\left(\right.$ res $\left.^{-1} \circ \phi_{\mathcal{V}_{t \mid T s s}}^{*}\right)$ is

$$
\operatorname{Char}\left(\bar{G}_{\hat{\delta}}\right) \subset \operatorname{Char}\left(G_{\hat{\delta}}\right) \subset \operatorname{Pic}^{G_{\hat{\delta}}}(T),
$$

which is a free $\mathbb{Z}$-module of rank 2 . On the other hand, by the discussion in $\S 5.1$ we know that $[\bar{L}]$ lies in the kernel of the Donaldson homomorphism $\lambda$. Putting these together, we get that $\phi_{\left.\mathcal{V}_{t}\right|_{T} s s}^{*} \circ \lambda$ factors as

$$
\mathbb{Z}^{2} \cong \mathbf{v}^{\perp} / \mathbb{Z}[\bar{L}] \rightarrow \operatorname{Pic}\left(M_{H_{m}}(\mathbf{v})\right) \rightarrow \mathbb{Z}^{2}
$$

so both maps are isomorphisms and $\operatorname{Pic}\left(M_{H_{m}}(\mathbf{v})\right) \cong \mathbb{Z}^{2}$.
In the other case when

$$
\mu_{H_{m}}(\mathbf{v})>\mu_{H_{m}}(L),
$$

one modifies the above proof by using the dual version of a Gaeta-type resolution

$$
0 \rightarrow \mathcal{V} \rightarrow L(1,0)^{\beta} \oplus L(0,1)^{\gamma} \rightarrow L(1,1)^{\delta} \rightarrow 0
$$

and replacing Propositions 2.20 and 4.3 by Proposition 2.21 and the dual version of Proposition 4.3.
At this point let us make a couple of useful observations. First, note that the previous cases fully establish Theorem 5.1 for characters $\mathbf{v}=(r, \nu, \Delta)$ with $r=2$. Indeed, $\left(r=2, \Delta=\frac{1}{2}\right)$ was covered by Proposition 5.2 and for ( $r=2, \Delta>\frac{1}{2}$ ) we have four different cases:

1. $\mathbf{v}=(2, \nu, \Delta)=(2, \varepsilon E+\varphi F, \Delta)$ with $\varepsilon, \varphi \in \mathbb{Z}$ and

$$
\Delta>\operatorname{DLP}_{H_{m}}^{<2}(\varepsilon E+\varphi F)=\operatorname{DLP}_{H_{m}, \mathcal{O}(\varepsilon, \varphi)}(\varepsilon E+\varphi F)=1 .
$$

The inequality implies that $\Delta \geq \frac{3}{2}$. Therefore,

$$
\Delta-\frac{1}{2} \geq \operatorname{DLP}_{H_{m}, \mathcal{O}(\varepsilon, \varphi)}(\varepsilon E+\varphi F)=1
$$

and this is covered by Proposition 5.3.
2. $\mathbf{v}=(2, \nu, \Delta)=(2, \varepsilon E+\varphi F, \Delta)$ with $\varepsilon, \varphi \in \mathbb{Z}$ and

$$
\Delta=\operatorname{DLP}_{H_{m}}^{<2}(\varepsilon E+\varphi F)=\operatorname{DLP}_{H_{m}, \mathcal{O}(\varepsilon, \varphi)}(\varepsilon E+\varphi F)=1
$$

The line bundle $\mathcal{O}(\varepsilon, \varphi)$ is the only exceptional bundle associated to $\mathbf{v}$ in this case. Since characters $\mathbf{v}$ with $r=2$ are always good characters (see Definition 3.6), this case is covered by Proposition 5.5 above.
3. $\mathbf{v}=(2, \nu, \Delta)=(2, \varepsilon E+\varphi F, \Delta)$ with $\varepsilon \in\left(\mathbb{Z}\left[\frac{1}{2}\right] \backslash \mathbb{Z}\right)$, or $\varphi \in\left(\mathbb{Z}\left[\frac{1}{2}\right] \backslash \mathbb{Z}\right)$ (or both) and

$$
\Delta>\operatorname{DLP}_{H_{m}}^{<2}(\varepsilon E+\varphi F)
$$

As before, one shows that then

$$
\Delta-\frac{1}{2} \geq \operatorname{DLP}_{H_{m}}^{<2}(\varepsilon E+\varphi F) \quad \text { and } \quad \operatorname{DLP}_{H_{m}}^{<2}(\varepsilon E+\varphi F) \geq \frac{1}{2}
$$

In case one of the inequalities is strict, we have $\Delta-\frac{1}{2}>\frac{1}{2}$. If $\Delta-\frac{1}{2}=\frac{1}{2}$, then $\mathbf{v}^{\prime}=\left(2, \nu, \frac{1}{2}\right)=$ ( $2, c_{1}, \chi$ ) is primitive, because by our assumption $c_{1}=(2 \varepsilon) E+(2 \varphi) F$ has an odd component and therefore is not divisible by 2 . We conclude that this case is covered by Proposition 5.3 above.
4. $\mathbf{v}=(2, \nu, \Delta)=(2, \varepsilon E+\varphi F, \Delta)$ with $\varepsilon \in\left(\mathbb{Z}\left[\frac{1}{2}\right] \backslash \mathbb{Z}\right)$, or $\varphi \in\left(\mathbb{Z}\left[\frac{1}{2}\right] \backslash \mathbb{Z}\right)$ (or both) and

$$
\Delta=\operatorname{DLP}_{H_{m}}^{<2}(\varepsilon E+\varphi F) .
$$

From Figure 2.1, one sees that $\mathbf{v}$ lies on two branches of the $\operatorname{DLP}_{H_{m}}^{<2}$-surface

$$
\Delta=\operatorname{DLP}_{H_{m}, L_{1}}(\varepsilon E+\varphi F)=\operatorname{DLP}_{H_{m}, L_{2}}(\varepsilon E+\varphi F)=\frac{3}{4}
$$

with $\left.L_{1}=\mathcal{O}((\lfloor\varepsilon\rfloor+1) E+\lfloor\varphi\rfloor F)\right), L_{2}=\mathcal{O}(\lfloor\varepsilon\rfloor E+(\lfloor\varphi\rfloor+1) F)$, so there are two line bundles associated to $\mathbf{v}$ and they form an exceptional pair. This is covered by Proposition 5.4 above.

It remains to prove statements (1.c) and (2.b). By the above discussion we can assume that $r(\mathbf{v}) \geq 3$. We show next that for the remaining characters $\mathbf{v}=(r, \nu, \Delta)$ there is a strong upper bound on the discriminant $\Delta$.

Lemma 5.6. Let $\mathbf{v}=(r, \nu, \Delta) \in K(X)$ be a character with $r \geq 3$ and $\Delta>\frac{1}{2}$. Let $\epsilon \in \mathbb{Q}$ be sufficiently small (depending on $r$ ), $0<|\epsilon| \ll 1$, and set $m=1+\epsilon$.

If $\mathbf{v}$ is a character satisfying either

- $\Delta>\operatorname{DLP}_{H_{m}}^{<r}(\nu)$, and

$$
\left(\Delta-\frac{1}{r}<\operatorname{DLP}_{H_{m}}^{<r}(\nu)\right) \text { or }\left(\Delta-\frac{1}{r} \leq \frac{1}{2}\right),
$$

or

- $\Delta=\operatorname{DLP}_{H_{m}}^{<r}(\nu)$ with a single exceptional bundle $E$ associated to $\mathbf{v}$ with $r(E)>1$,
then $\Delta<1$.

Proof. Suppose first

$$
\Delta-\frac{1}{r} \leq \frac{1}{2}
$$

We immediately get

$$
\Delta \leq \frac{1}{2}+\frac{1}{r} \leq \frac{1}{2}+\frac{1}{3}=\frac{5}{6}<1 \quad(\text { recall } r \geq 3)
$$

Now suppose that

$$
\Delta>\operatorname{DLP}_{H_{m}}^{<r}(\nu) \quad \text { and } \quad\left(\Delta-\frac{1}{r}<\operatorname{DLP}_{H_{m}}^{<r}(\nu)\right)
$$

Note that then $\operatorname{DLP}_{H_{m}}^{<r}(\nu)$ cannot be equal to $\operatorname{DLP}_{H_{m}, L}(\nu)$ for a line bundle $L$. Indeed,

$$
\Delta>\operatorname{DLP}_{H_{m}, L}(\nu)
$$

is equivalent to

$$
r(P(\nu-\nu(L))-\Delta)<0 \text { or } r(P(\nu(L)-\nu)-\Delta)<0 .
$$

This implies

$$
r\left(P(\nu-\nu(L))-\Delta+\frac{1}{r}\right) \leq 0 \text { or } r\left(P(\nu(L)-\nu)-\Delta+\frac{1}{r}\right) \leq 0,
$$

which is equivalent to

$$
\Delta-\frac{1}{r} \geq \operatorname{DLP}_{H_{m}, L}^{<r}(\nu)
$$

contradicting our assumption. Thus $\operatorname{DLP}_{H_{m}}^{<r}(\nu)=\operatorname{DLP}_{H_{m}, E}(\nu)$ with $r(E)>1$. By Lemma 2.13 (1)

$$
\begin{equation*}
r(E) \geq 3 \quad \text { and } \quad \operatorname{DLP}_{H_{m}, E}(\nu) \leq \operatorname{DLP}_{H_{m}, E}(\nu(E))=1-\Delta(E)=\frac{1}{2}+\frac{1}{2 r(E)^{2}} \leq \frac{5}{9} \tag{5.6.1}
\end{equation*}
$$

which implies

$$
\Delta<\operatorname{DLP}_{H_{m}}^{<r}(\nu)+\frac{1}{r}=\operatorname{DLP}_{H_{m}, E}(\nu)+\frac{1}{r} \leq \frac{5}{9}+\frac{1}{3}=\frac{8}{9}<1
$$

Finally, suppose character $\mathbf{v}=(r, \nu, \Delta)$ satisfies

$$
\Delta=\operatorname{DLP}_{H_{m}}^{<r}(\nu)=\operatorname{DLP}_{H_{m}, E}(\nu),
$$

where $r(E) \geq 3$. From (5.6.1) we get

$$
\Delta=\operatorname{DLP}_{H_{m}, E}(\nu) \leq \frac{5}{9}<1
$$

Let us make one last observation: for the remaining cases we can assume that in $\nu(\mathbf{v})=\varepsilon E+\varphi F$ we have

$$
\varepsilon+\varphi \neq\lfloor\varepsilon\rfloor+\lfloor\varphi\rfloor+1
$$

Indeed, otherwise the total slope $\nu$ would lie on the "antidiagona" in the $(\varepsilon, \varphi)$-plane of total slopes along which the DLP-surface is given by Line bundles (see Figure 2.1). This is covered by Propositions 5.3, 5.4 and 5.5.

So in the proof of the remaining cases (1.c) and (2.b) of the theorem we can assume that $\mathbf{v}=(r, \nu, \Delta)$ satisfies

$$
\begin{align*}
& r \geq 3,  \tag{1}\\
& \frac{1}{2}<\Delta<1,  \tag{2}\\
& \text { (3) }(\lfloor\varepsilon\rfloor+\lfloor\varphi\rfloor \leq \varepsilon+\varphi<\lfloor\varepsilon\rfloor+\lfloor\varphi\rfloor+1) \text { or }(\lfloor\varepsilon\rfloor+\lfloor\varphi\rfloor+1<\varepsilon+\varphi<\lfloor\varepsilon\rfloor+\lfloor\varphi\rfloor+2) \text {, }  \tag{5.6.2}\\
& \text { There is no line bundle associated to } \mathbf{v} \text {. } \tag{4}
\end{align*}
$$

These assumptions make it possible to resolve a general $H_{m}$-semistable sheaf of character $\mathbf{v}$ via a Gaeta-type resolution with all exponents nonzero.

Lemma 5.7. Let $\mathbf{v}=(r, \nu, \Delta)=(r, \varepsilon E+\varphi F, \Delta) \in K(X)$ be a character satisfying conditions (5.6.2). Let $\epsilon \in \mathbb{Q}$ be sufficiently small (depending on $r$ ), $0<|\epsilon| \ll 1$, and set $m=1+\epsilon$.

If

$$
\lfloor\varepsilon\rfloor+\lfloor\varphi\rfloor \leq \varepsilon+\varphi<\lfloor\varepsilon\rfloor+\lfloor\varphi\rfloor+1
$$

then a general $H_{m}$-semistable sheaf $\mathcal{V}$ of character $\mathbf{v}$ admits an L-Gaeta-type resolution

$$
0 \rightarrow L(-1,-1)^{\alpha} \rightarrow L(-1,0)^{\beta} \oplus L(0,-1)^{\gamma} \oplus L^{\delta} \rightarrow \mathcal{V} \rightarrow 0
$$

where $L:=L_{\lfloor\epsilon\rfloor,\lfloor\phi\rfloor}$ and all the exponents are nonzero.
If

$$
\lfloor\phi\rfloor+1<\epsilon+\phi<\lfloor\epsilon\rfloor+\lfloor\phi\rfloor+2
$$

then a general $H_{m}$-semistable sheaf $\mathcal{V}$ of character $\mathbf{v}$ admits a dual L-Gaeta-type resolution

$$
0 \rightarrow \mathcal{V} \rightarrow L(1,0)^{\alpha} \oplus L(0,1)^{\beta} \oplus L^{\gamma} \rightarrow L(1,1)^{\delta} \rightarrow 0
$$

where $L:=L_{\lfloor\epsilon\rfloor,\lfloor\phi\rfloor}$ and all the exponents are nonzero.
Proof. We prove the case

$$
\begin{equation*}
\lfloor\varepsilon\rfloor+\lfloor\varphi\rfloor \leq \varepsilon+\varphi<\lfloor\varepsilon\rfloor+\lfloor\varphi\rfloor+1 \tag{5.7.1}
\end{equation*}
$$

and say how to modify the argument for the other case at the end of the proof.
With the notation of $\S 2.8$, we use the bound (5.6.2) (2) and formally compute using Riemann-Roch:

$$
\chi\left(\mathbf{v}\left(-L_{\varepsilon, \varphi}\right)\right)=r(1-\Delta)>0
$$

Thus, in the $(a, b)$-plane $\mathbb{R}^{2}$ the point $(\varepsilon, \varphi)$ lies below the lower-left branch $Q$ of the hyperbola $\chi\left(\mathbf{v}\left(-L_{a, b}\right)\right)=0$. Therefore, the integral point $(\lfloor\varepsilon\rfloor,\lfloor\varphi\rfloor)$ also lies below $Q$ and we have that

$$
\chi\left(\mathbf{v}\left(-L_{\lfloor\varepsilon\rfloor,\lfloor\varphi\rfloor}\right)\right)>0
$$

For a sufficiently small $\epsilon$ condition (5.7.1) translates into a condition on $\mu_{H_{m}}$-slopes:

$$
\mu_{H_{m}}\left(L_{\lfloor\varepsilon\rfloor,\lfloor\varphi\rfloor}\right) \leq \mu_{H_{m}}(\mathbf{v})<\mu_{H_{m}}\left(L_{\lfloor\varepsilon\rfloor+k,\lfloor\varphi\rfloor+l}\right)+1 \text { for }(k, l) \in\{(1,0),(0,1),(1,1)\}
$$

Therefore, for $\mathcal{V} \in M_{H_{m}}(\mathbf{v})$ we have

$$
\operatorname{Hom}\left(L_{\lfloor\varepsilon\rfloor+k,\lfloor\varphi\rfloor+l}, \mathcal{V}\right)=\operatorname{Ext}^{2}\left(L_{\lfloor\varepsilon\rfloor+k,\lfloor\varphi\rfloor+l}, \mathcal{V}\right)=0
$$

resulting in

$$
-\chi\left(\mathbf{v}\left(-L_{\lfloor\varepsilon\rfloor+k,\lfloor\varphi\rfloor+l}\right)\right)=-\chi\left(L_{\lfloor\varepsilon\rfloor+k,\lfloor\varphi\rfloor+l}, \mathbf{v}\right)=\operatorname{ext}^{1}\left(L_{\lfloor\varepsilon\rfloor+k,\lfloor\varphi\rfloor+l}, \mathcal{V}\right) \geq 0
$$

In fact, the inequalities are strict, for otherwise $L_{\lfloor\varepsilon\rfloor+k,\lfloor\varphi\rfloor+l}$ (or their Serre twists) would be associated to $\mathbf{v}$ contradicting assumption (5.6.2) (4). This shows that the line bundle $L:=L_{\lfloor\varepsilon\rfloor,\lfloor\varphi\rfloor}$ satisfies (2.19.1) of Theorem 2.19 with all integers $\alpha, \beta, \gamma, \delta$ being nonzero.

In the case

$$
\lfloor\varepsilon\rfloor+\lfloor\varphi\rfloor+1<\varepsilon+\varphi<\lfloor\epsilon\rfloor+\lfloor\phi\rfloor+2,
$$

one can first pass to the dual character $\mathbf{v}^{\prime}$, resolve a generic $\mu_{H_{m}}$-stable vector bundle by a Gaeta-type resolution with all exponents nonzero as above, and then take the dual of the whole resolution. Here we use Proposition 2.17 to guarantee the existence $\mu_{H_{m}}$-stable bundles.

Thus, to study $M_{H_{m}}(\mathbf{v})$ for $\mathbf{v}$ satisfying conditions (5.6.2) with

$$
\lfloor\varepsilon\rfloor+\lfloor\varphi\rfloor \leq \varepsilon+\varphi<\lfloor\varepsilon\rfloor+\lfloor\varphi\rfloor+1
$$

we consider the complete family $\mathcal{V}_{t} / T$ of $\mathcal{O}(1,1)$-prioritary sheaves admitting an $L$-Gaeta type resolution

$$
\begin{equation*}
0 \rightarrow L(-1,-1)^{\alpha} \xrightarrow{\psi_{t}} L(-1,0)^{\beta} \oplus L(0,-1)^{\gamma} \oplus L^{\delta} \rightarrow \mathcal{V}_{t} \rightarrow 0, \tag{5.7.2}
\end{equation*}
$$

where $L=L_{\lfloor\varepsilon\rfloor,\lfloor\varphi\rfloor}$, all the exponents are nonzero, and

$$
\begin{equation*}
T \subset \mathbb{H}=\operatorname{Hom}\left(L(-1,-1)^{\alpha}, L(-1,0)^{\beta} \oplus L(0,-1)^{\gamma} \oplus L^{\delta}\right) \tag{5.7.3}
\end{equation*}
$$

is the open subset parameterizing injective sheaf maps with torsion-free cokernel. By Proposition 2.20

$$
\begin{equation*}
\operatorname{codim}_{\mathbb{H}}(\mathbb{H} \backslash T) \geq 2 \tag{5.7.4}
\end{equation*}
$$

Likewise, to study $M_{H_{m}}(\mathbf{v})$ for $\mathbf{v}$ satisfying conditions (5.6.2) with

$$
\lfloor\varphi\rfloor+1<\varepsilon+\phi<\lfloor\varepsilon\rfloor+\lfloor\varphi\rfloor+2
$$

we consider the complete family $\mathcal{V}_{t} / T$ of $\mathcal{O}(1,1)$-prioritary vector bundles admitting the dual version of an $L$-Gaeta type resolution

$$
0 \rightarrow \mathcal{V}_{t} \rightarrow L(1,0)^{\alpha} \oplus L(0,1)^{\beta} \oplus L^{\gamma} \xrightarrow{\psi_{t}} L(1,1)^{\delta} \rightarrow 0
$$

where $L=L_{\lfloor\varepsilon\rfloor,\lfloor\varphi\rfloor}$, all the exponents are nonzero, and

$$
T \subset \mathbb{H}=\operatorname{Hom}\left(L(1,0)^{\alpha} \oplus L(0,1)^{\beta} \oplus L^{\gamma}, L(1,1)^{\delta}\right)
$$

is the open subset parameterizing surjective sheaf maps. By the dual version of Proposition 2.21

$$
\operatorname{codim}_{\mathbb{H}}(\mathbb{H} \backslash T) \geq 2
$$

Next, we analyze the Shatz stratification of $T$ and apply the irreducibility results of $\S 3.4$ to compute the kernel of the Donaldson homomorphism in the remaining cases. In what follows, we treat the case

$$
\lfloor\varepsilon\rfloor+\lfloor\varphi\rfloor \leq \varepsilon+\varphi<\lfloor\varepsilon\rfloor+\lfloor\varphi\rfloor+1,
$$

leaving the necessary modifications of the proof in the other case to the reader.
The next proposition proves case (1.c) of Theorem 5.1.

Proposition 5.8. Let $\mathbf{v}=(r, \nu, \Delta) \in K(X)$ be a character with $r \geq 3$ and $\Delta>\frac{1}{2}$. Let $\epsilon \in \mathbb{Q}$ be sufficiently small (depending on $r$ ), $0<|\epsilon| \ll 1$, and set $m=1+\epsilon$.

If $\mathbf{v}$ is a good character with

$$
\Delta>\operatorname{DLP}_{H_{m}}^{<r}(\nu)
$$

and

$$
\left(\Delta-\frac{1}{r}<\operatorname{DLP}_{H_{m}}^{<r}(\nu)\right) \text { or }\left(\Delta-\frac{1}{r} \leq \frac{1}{2}\right),
$$

then

$$
\operatorname{Pic}\left(M_{H}(\mathbf{v})\right) \cong \mathbb{Z}^{3}
$$

and $\lambda$ is an isomorphism.
Proof. By the discussion above, we can assume that character $\mathbf{v}$ satisfies (5.6.2) with

$$
\lfloor\varepsilon\rfloor+\lfloor\varphi\rfloor \leq \varepsilon+\varphi<\lfloor\varepsilon\rfloor+\lfloor\varphi\rfloor+1 \text {. }
$$

Consider the family $T$ from (5.7.3). Proposition 3.4 says that potential divisorial Shatz strata of $T$ are given by the $\Delta_{i}=\frac{1}{2}$-strata. Since we are assuming that $\mathbf{v}$ is a good Chern character, it follows that there is no Shatz strata of codimension 1 in $T$. Therefore, by (5.7.4) the semistable locus $T^{s s} \subset T \subset \mathbb{H}$ satisfies

$$
\operatorname{codim}_{\mathbb{H}}\left(\mathbb{H} \backslash T^{s s}\right) \geq 2
$$

The Donaldson morphism fits into the following commutative diagram


By Proposition 4.2, the bottom map is an isomorphism and it follows that $\lambda$ is injective.
Finally, we finish proving Theorem 5.1 by considering the last remaining case (2.b).
Proposition 5.9. Let $\mathbf{v}=(r, \nu, \Delta) \in K(X)$ be a character with $r \geq 3$ and $\Delta>\frac{1}{2}$. Let $\epsilon \in \mathbb{Q}$ be sufficiently small (depending on $r$ ), $0<|\epsilon| \ll 1$, and set $m=1+\epsilon$.

If $\mathbf{v}$ is a good character with

$$
\Delta=\operatorname{DLP}_{H_{m}}^{<r}(\nu)
$$

with a single exceptional bundle $E$ associated to $\mathbf{v}$ and $r(E)>1$, then

$$
\operatorname{Pic}\left(M_{H_{m}}(\mathbf{v})\right) \cong \mathbb{Z}^{2}
$$

and $\lambda$ is an epimorphism with

$$
\operatorname{ker} \lambda=\mathbb{Z}[\bar{E}] .
$$

Proof. By the discussion above, we can again assume that character $\mathbf{v}$ satisfies (5.6.2) with

$$
\lfloor\varepsilon\rfloor+\lfloor\varphi\rfloor \leq \varepsilon+\varphi<\lfloor\varepsilon\rfloor+\lfloor\varphi\rfloor+1
$$

We further assume

$$
\mu_{H_{m}}(\mathbf{v}) \leq \mu_{H_{m}}(E)
$$

leaving the necessary modifications of the proof in the other case to the reader. By the discussion in $\S 5.1$, we know that $[\bar{E}]$ lies in the kernel of the Donaldson homomorphism. We show that in fact

$$
\operatorname{ker}(\lambda)=\mathbb{Z}[E]
$$

Step 1. Once again we analyze the Shatz stratification of the family $\mathcal{V}_{t} / T$ from (5.7.3) which parameterizes (1,1)-prioritary sheaves admitting an $L$-Gaeta type resolution (5.7.2):

$$
\begin{equation*}
0 \rightarrow L(-1,-1)^{\alpha} \xrightarrow{\psi_{t}} L(-1,0)^{\beta} \oplus L(0,-1)^{\gamma} \oplus L^{\delta} \rightarrow \mathcal{V}_{t} \rightarrow 0 . \tag{5.7.2}
\end{equation*}
$$

This time, Proposition 3.5 says that there is at most one possible divisorial Shatz stratum $S_{T}$ since $\Delta_{i}=\frac{1}{2}$-strata are excluded by the assumption that $\mathbf{v}$ is good. In fact, it must be nonempty for our family $\mathcal{V}_{t} / T$. For otherwise arguing as in the Proposition 5.8 , we would show that $\lambda$ is injective, in contradiction to $\mathbb{Z}[\bar{E}] \subset \operatorname{ker}(\lambda)$.

According to Proposition 3.5, this stratum $S_{T}=S_{T, H_{m}}\left(\mathbf{v}_{1}, \mathbf{v}_{2}\right)$ consists of points $\psi_{t} \in T$ such that the corresponding sheaf $\mathcal{V}_{t}$ admits the $H_{m}$-Harder-Narasimhan filtration of length $l=2$

$$
\begin{equation*}
0 \subset E \subset \mathcal{V}_{t} \tag{5.9.1}
\end{equation*}
$$

where $E$ is the exceptional bundle associated to $\mathbf{v}$ and $\mathbf{v}_{1}:=\mathbf{v}(E), \mathbf{v}_{2}:=\mathbf{v}\left(\mathcal{V}_{t} / E\right)$.
Step 2. We claim that this Shatz stratum is irreducible. Set $B=L(-1,0)^{\beta} \oplus L(0,-1)^{\gamma} \oplus L^{\delta}$. Consider the Quot scheme $\operatorname{Quot}(B, \mathbf{v})$ parameterizing quotients

$$
q=\left[B \rightarrow \mathcal{E}_{q}\right], \quad q \in \operatorname{Quot}(B, \mathbf{v})
$$

where $\mathbf{v}\left(\mathcal{E}_{q}\right)=\mathbf{v}$. First, restrict the family $\left\{\mathcal{E}_{q}\right\}$ to the open subset of $\operatorname{Quot}(B, \mathbf{v})$ parameterizing torsion-free $\mathcal{E}_{q}$. Over this subset we have a universal family of sheaves

$$
0 \rightarrow K \rightarrow q^{*} B \rightarrow \mathcal{E} \rightarrow 0
$$

Note that by (5.7.2)

$$
\mathbf{v}\left(K_{q}\right)=\mathbf{v}\left(L(-1,-1)^{\alpha}\right)
$$

We further restrict to the open subset $Q \subset Q u o t(B, \mathbf{v})$ parameterizing those quotients $q$ for which $K_{q}$ is a semistable vector bundle. Since $\mathbf{v}\left(L(-1,-1)^{\alpha}\right)$ is a semiexceptional Chern character,

$$
K_{q} \cong L(-1,-1)^{\alpha} \text { for each } q \in Q
$$

By the universal property of Quot schemes the family of quotients $\left\{B \rightarrow \mathcal{V}_{t}\right\}_{\psi_{t} \in T}$ of (5.7.2) gives a surjective morphism

$$
T \stackrel{\Omega}{\rightarrow} Q \subset Q u o t(B, \mathbf{v}),
$$

whose fibers are isomorphic to $G L(\alpha)$. Denote the Shatz stratum of points $q \in Q$ such that the corresponding $\mathcal{E}_{q}$ has the $H_{m}$-Harder-Narasimhan filtration

$$
0 \subset E \subset \mathcal{E}_{q}
$$

by $S_{Q}=S_{Q, H_{m}}\left(\mathbf{v}_{1}, \mathbf{v}_{2}\right)$. From the Cartesian diagram

it follows that the irreducibility of $S_{T}$ is equivalent to the irreducibility of $S_{Q}$. Indeed, $S_{T}$ is equidimensional and the fibers of $S_{T} \rightarrow S_{Q}$ are all irreducible and isomorphic to $G L(\alpha)$. We can apply the following version of the irreducibility criterion: if $Y \rightarrow X$ is a finite type surjective morphism from an equidimensional Noetherian $\mathbb{C}$-scheme to an irreducible Noetherian $\mathbb{C}$-scheme, and all fibers over the closed points are irreducible of the same dimension, then $Y$ is irreducible.

To show the irreducibility of $S_{Q}$, consider the family $\mathcal{W}_{s} / S$ over irreducible $S$ having Property (P) with respect to $\mathbb{F}\left(\mathbf{v}_{1}, \mathbf{v}_{2}\right)$ that was constructed in Proposition 3.13 (we denote the sheaves in this family by $\mathcal{W}_{s}$ instead of $\mathcal{V}_{s}$ to avoid confusion with the sheaves $\mathcal{V}_{t}$ from (5.7.2)). Recall that heuristically $\mathcal{W}_{s} / S$ parameterizes all torsion-free sheaves whose $H_{m}$-Harder Narasimhan filtration is of lenght 2 and has quotients of characters $\mathbf{v}_{1}, \mathbf{v}_{2}$, possibly with repetition. Intuitively, we are going to build a family of quotients over an irreducible base out of $\mathcal{W}_{s} / S$ that will surject onto $S_{Q}$ under the universal morphism to the Quot scheme $Q$.

Note that for $\psi_{t} \in T$ the Gaeta-type resolution (5.7.2) implies

$$
\operatorname{Ext}^{i}\left(B, \mathcal{V}_{t}\right)=0 \text { for } i>0 \Longrightarrow \operatorname{hom}\left(B, \mathcal{V}_{t}\right)=\chi(B, \mathbf{v})
$$

Thus, consider the open subset $U \subset S$ parameterizing those $\mathcal{W}_{s}$ for which

$$
\operatorname{Ext}^{i}\left(B, \mathcal{W}_{s}\right)=0 \text { for } i>0
$$

It is non-empty because we concluded above that $S_{T}$ is non-empty, and irreducible. It follows that

$$
\left(p_{U}\right)_{*} \mathscr{H} \operatorname{Com}\left(q^{*} B, \mathcal{W}\right)
$$

is a vector bundle on $U$. Denote the corresponding geometric vector bundle by

$$
\mathbb{V} \xrightarrow{\pi} U,
$$

so that over $\mathbb{V} \times X$ we have a universal morphism

$$
\pi^{*} q^{*} B \xrightarrow{\Psi} \pi^{*} \mathcal{W} .
$$

We further restrict to an open subset $\mathbb{U} \subset \mathbb{V}$ parameterizing surjective maps with an $H_{m}$-semistable kernel, so that for $u \in \mathbb{U}$ we have an exact sequence

$$
0 \rightarrow L(-1,-1)^{\alpha} \rightarrow B \xrightarrow{\Psi_{u}} \mathcal{W}_{\pi(u)} \rightarrow 0
$$

By the universal property of Quot schemes, we obtain a map

$$
\mathbb{U} \rightarrow Q,
$$

whose image is equal to $S_{Q}$ because of the Property (P). Since $\mathbb{U}$ is irreducible, it follows that $S_{Q}$ is irreducible too. We summarize the discussion in the following diagram


Step 3. We return to the problem of describing the kernel of the Donaldson homomorphism. Let $T^{\prime}=T \backslash \overline{S_{T}}$ and note that this is a $G$-invariant open subset of $T$. Because of the irreducibility proved at the previous step, $\overline{S_{T}}=V(f)$ for some irreducible polynomial

$$
\begin{equation*}
f \in \mathbb{C}\left[\left\{x_{i j}\right\}\right], \tag{5.9.2}
\end{equation*}
$$

where $\mathbb{C}\left[\left\{x_{i j}\right\}\right]$ is the coordinate algebra of $\mathbb{H}$. The sequence (4.1.1) for $Y=T^{\prime}$

$$
\begin{equation*}
\left\{a f^{k} \mid a \in \mathbb{C}^{*}, k \in \mathbb{Z}\right\}=\mathcal{O}^{*}\left(T^{\prime}\right) \xrightarrow{a f^{k} \mapsto k \eta_{f}} \operatorname{Char}(G) \rightarrow \operatorname{Pic}^{G}\left(T^{\prime}\right) \rightarrow 0 \tag{5.9.3}
\end{equation*}
$$

now implies that

$$
\operatorname{Pic}^{G}\left(T^{\prime}\right) \cong \operatorname{Char}(G) / \mathbb{Z} \cdot \eta_{f} .
$$

Since $S_{T}$ was the only divisorial Shatz stratum, we have that

$$
\operatorname{codim}_{T^{\prime}}\left(T^{\prime} \backslash T^{s s}\right) \geq 2
$$

and we obtain the following commutative diagram


Chasing this diagram shows that integer multiples of $[\bar{E}] \in \mathbf{v}^{\perp}$ are the only elements in the kernel of $\lambda$. For if there was $\mathbf{u} \in\left(\mathbf{v}^{\perp} \backslash \mathbb{Z}[\bar{E}]\right)$ with

$$
\lambda(\mathbf{u})=0,
$$

then going around the outer lower part of the diagram would imply that two $\mathbb{Z}$-linearly independent elements, $\lambda_{\mathcal{\nu}_{t}}(\mathbf{u})$ and $\lambda_{\mathcal{\nu}_{t}}([\bar{E}])$, lie in the kernel of the restriction

$$
\operatorname{Pic}^{G}(T) \xrightarrow{\text { res }} \operatorname{Pic}^{G}\left(T^{\prime}\right) .
$$

But this then contradicts the fact that the kernel of this restriction is a cyclic subgroup, that could be seen from looking at sequence (4.1.1) for the inclusion $T^{\prime} \subset T$ :


This finishes the proof of statement (3.b) of the theorem.
The main Theorem 5.1 is now fully proved.
Remark 5.10. Note that we can describe the polynomial $f$ appearing in (5.9.2) in such a way so that one can explicitly compute the character $\eta_{f}$ appearing in (5.9.3). Recall that for $\psi_{t} \in S_{T}$ the corresponding sheaf $\mathcal{V}_{t}$ comes equipped with a filtration

$$
0 \subset E \subset \mathcal{V}_{t}
$$

while for an $H_{m}$-semistable $\mathcal{V}_{\tau}$ we have

$$
\operatorname{Hom}\left(E, \mathcal{V}_{\tau}\right)=\operatorname{Ext}^{1}\left(E, \mathcal{V}_{\tau}\right)=0
$$

Therefore, $S_{T} \subset\left\{\psi_{t} \in T \mid \operatorname{Hom}\left(E, \mathcal{V}_{t}\right) \neq 0\right\}$. The long exact sequence in cohomology coming from (5.7.2)

$$
0 \rightarrow \operatorname{Hom}\left(E, \mathcal{V}_{t}\right) \rightarrow \operatorname{Ext}^{1}\left(E, L(-1,-1)^{\alpha}\right) \xrightarrow{\left(\psi_{t}\right)_{*}} \operatorname{Ext}^{1}(E, B) \rightarrow \operatorname{Ext}^{1}\left(E, \mathcal{V}_{t}\right) \rightarrow 0
$$

shows that $\left\{\psi_{t} \in T \mid \operatorname{Hom}\left(E, \mathcal{V}_{t}\right) \neq 0\right\}$ is a determinantal divisor given as the vanishing locus of

$$
\psi_{t} \mapsto \operatorname{det}\left(\left(\psi_{t}\right)_{*}\right)
$$

As $\overline{S_{T}}=\left\{\psi_{t} \in T \mid \operatorname{Hom}\left(E, \mathcal{V}_{t}\right) \neq 0\right\}$, this describes $f$ as

$$
f\left(\psi_{t}\right)=\operatorname{det}\left(\left(\psi_{t}\right)_{*}\right) .
$$

Since $\eta_{f}$ is defined by the equation

$$
f\left(g \cdot \psi_{t}\right)=\eta_{f}(g) f\left(\psi_{t}\right),
$$

we can explicitly recover $\eta_{f}$ from the following computation

$$
\begin{aligned}
f\left(g \cdot \psi_{t}\right) & =\operatorname{det}\left(\left(g \cdot \psi_{t}\right)_{*}\right) \\
& =\operatorname{det}\left[\left(\left(g_{\beta} \oplus g_{\gamma} \oplus g_{\delta}\right) \circ \psi_{t} \circ\left(g_{\alpha}\right)^{-1}\right)_{*}\right] \\
& =\left[\operatorname{det}\left(\left(g_{\beta}\right)_{*}\right) \operatorname{det}\left(\left(g_{\gamma}\right)_{*}\right) \operatorname{det}\left(\left(g_{\gamma}\right)_{*}\right) \operatorname{det}\left(\left(g_{\alpha}\right)_{*}\right)^{-1}\right] \operatorname{det}\left(\left(\psi_{t}\right)_{*}\right) \\
& =\left[\operatorname{det}\left(g_{\beta}\right)^{-\chi\left(E, L(-1,0)^{\beta}\right)} \operatorname{det}\left(g_{\gamma}\right)^{-\chi\left(E, L(0,-1)^{\gamma}\right)} \operatorname{det}\left(g_{\delta}\right)^{-\chi\left(E, L^{\delta}\right)} \operatorname{det}\left(g_{\alpha}\right)^{\chi\left(E, L(-1,1)^{\alpha}\right)}\right] \operatorname{det}\left(\left(\psi_{t}\right)_{*}\right) .
\end{aligned}
$$

Thus

$$
\eta_{f}=\eta_{a, b, c, d}
$$

with

$$
\begin{aligned}
a & =\chi\left(E, L(-1,1)^{\alpha}\right), \\
b & =-\chi\left(E, L(-1,0)^{\beta}\right), \\
c & =-\chi\left(E, L(0,-1)^{\gamma}\right), \\
d & =-\chi\left(E, L^{\delta}\right) .
\end{aligned}
$$

### 5.4 Corollaries of Theorem 5.1

We conclude this chapter by exploring some immediate corollaries of Theorem 5.1.
First, we can get rid of some of the assumptions in Proposition 5.4 at the expense of loosing the information about torsion in $\operatorname{Pic}\left(M_{H_{m}}(\mathbf{v})\right)$.

Corollary 5.11. Let $\mathbf{v}=(r, \nu, \Delta) \in K(X)$ be a character with $r \geq 2$ and $\Delta>\frac{1}{2}$. Let $\epsilon \in \mathbb{Q}$ be sufficiently small (depending on $r$ ), $0<|\epsilon| \ll 1$, and set $m=1+\epsilon$.

If $\Delta=\operatorname{DLP}_{H_{m}}^{<r}(\nu)$ with two exceptional bundles $E_{1}, E_{2}$ associated to $\mathbf{v}$, then

$$
\rho\left(M_{H}(\mathbf{v})\right) \cong \mathbb{Z}
$$

and $\lambda$ is an epimorphism.
The next corollary is concerned with the position of certain good $H_{m}$-semistable characters $\mathbf{v}$ relative to the branches of the DLP-surface given by exceptional bundles of rank higher than the rank of $\mathbf{v}$.

Corollary 5.12. Let $\mathbf{v}=(r, \nu, \Delta)$ be a good $H_{m}$-semistable Chern character with $\Delta>\frac{1}{2}$, where $m=1+\epsilon$ and $\epsilon \in \mathbb{Q}$ is a sufficiently small number depending on $r, 0<|\epsilon| \ll 1$.

If $\mathbf{v}$ lies above the $\mathrm{DLP}_{H_{m}}^{<r}$-surface or has a single exceptional bundle $E$ associated to $\mathbf{v}$, then for any exceptional bundle $F$ with $r(F)>r$ satisfying

$$
\left|(\nu-\nu(F)) \cdot H_{m}\right| \leq-\frac{1}{2} K_{X} \cdot H_{m}
$$

we have

$$
\Delta>\mathrm{DLP}_{H_{m}, F}(\nu) .
$$

In other words, such $H_{m}$-semistable character $\mathbf{v}$ of rank $r$ does not lie on any branch of the DLPsurface given by an exceptional bundle of rank higher than $r$.

Proof. Suppose on the contrary that

$$
\Delta=\operatorname{DLP}_{H_{m}, F}(\nu)
$$

for an exceptional bundle $F$ with $r(F)>r$ and

$$
\left|(\nu-\nu(F)) \cdot H_{m}\right| \leq-\frac{1}{2} K_{X} \cdot H_{m}
$$

By $\S 5.1$, the class $[\bar{F}]$ lies in the kernel of the Donaldson homomorphism.
Case 1: $\Delta>\operatorname{DLP}_{H_{m}}^{<r}(\nu)$. Existence of $F$ as above then contradicts the fact that by cases (1.c) Theorem 5.1 the Donaldson homomorphism $\lambda$ has a trivial kernel.

Case 2: $\Delta=\operatorname{DLP}_{H_{m}}^{<r}(\nu)$ with a single exceptional bundle $E$ associated to $\mathbf{v}$. According to case (2) of Theorem 5.1 we get that

$$
[\bar{F}]=n[\bar{E}], \quad r(\bar{E})<r
$$

Since $r(\bar{F})>r$, we conclude $n>1$. This shows that $[\bar{F}]$ is not a primitive Chern character. This is a contradiction, since $\bar{F}$ is an exceptional bundle and characters of exceptional bundles are primitive by Lemma 2.13 (2).

Finally, our last corollary states the conditions on character $\mathbf{v}$ under which the set of $H_{m}$-semistable sheaves of character $\mathbf{v}$ admitting a Gaeta-type resolution forms an open subset of the moduli space $M_{H_{m}}(\mathbf{v})$ whose complement has codimension at least 2 .

Corollary 5.13. Let $\mathbf{v}=(r, \nu, \Delta)=(r, \varepsilon E+\varphi F, \Delta) \in K(X)$ be a good $H_{m}$-semistable Chern character with $r \geq 2, \Delta>\frac{1}{2}$ satisfying either

- $\Delta=\operatorname{DLP}_{H_{m}}^{<r}(\nu)$ with a single exceptional bundle $L$ associated to $\mathbf{v}$ with $r(L)=1$ and $\mu_{H_{m}}(L) \geq \mu_{H_{m}}(\mathbf{v})$, or
- conditions (5.6.2) with

$$
\lfloor\varepsilon\rfloor+\lfloor\varphi\rfloor \leq \varepsilon+\varphi<\lfloor\varepsilon\rfloor+\lfloor\varphi\rfloor+1
$$

where $m=1+\epsilon$ and $\epsilon \in \mathbb{Q}$ is a sufficiently small number depending on $r, 0<|\epsilon| \ll 1$.
Then one can choose a line bundle $L$ such that for the complete family $\mathcal{V}_{t} / T$ of $\mathcal{O}(1,1)$-prioritary sheaves admitting an L-Gaeta type resolution (2.18.1)

$$
0 \rightarrow L(-1,-1)^{\alpha} \xrightarrow{\psi_{t}} L(-1,0)^{\beta} \oplus L(0,-1)^{\gamma} \oplus L^{\delta} \rightarrow \mathcal{V}_{t} \rightarrow 0,
$$

over the open subset

$$
T \subset \mathbb{H}=\operatorname{Hom}\left(L(-1,-1)^{\alpha}, L(-1,0)^{\beta} \oplus L(0,-1)^{\gamma} \oplus L^{\delta}\right)
$$

parameterizing injective sheaf maps with torsion-free cokernel we have $T^{s s} \neq \emptyset$ and the image of the classifying morphism

$$
T^{s s} \xrightarrow{\phi \mathcal{V}_{t} \mid T_{T} s s} M_{H_{m}}(\mathbf{v})
$$

is an open set whose complement has codimension $\geq 2$.
Similarly, if $\mathbf{v}=(r, \nu, \Delta)$ satisfies either

- $\Delta=\operatorname{DLP}_{H_{m}}^{<r}(\nu)$ with a single exceptional bundle $L$ associated to $\mathbf{v}$ with $r(L)=1$ and $\mu_{H_{m}}(L)<\mu_{H_{m}}(\mathbf{v})$, or
- conditions (5.6.2) with

$$
\lfloor\epsilon\rfloor+\lfloor\phi\rfloor+1<\epsilon+\phi<\lfloor\epsilon\rfloor+\lfloor\phi\rfloor+2,
$$

then one can choose a line bundle $L$ such that for the complete family $\mathcal{V}_{t} / T$ of $\mathcal{O}(1,1)$-prioritary vector bundles admitting the dual version of an L-Gaeta type resolution (2.20.1)

$$
0 \rightarrow \mathcal{V}_{t} \rightarrow L(1,0)^{\alpha} \oplus L(0,1)^{\beta} \oplus L^{\gamma} \xrightarrow{\psi_{t}} L(1,1)^{\delta} \rightarrow 0,
$$

over the open subset

$$
T \subset \mathbb{H}=\operatorname{Hom}\left(L(1,0)^{\alpha} \oplus L(0,1)^{\beta} \oplus L^{\gamma}, L(1,1)^{\delta}\right)
$$

parameterizing surjective sheaf maps we have $T^{s s} \neq \emptyset$ and the image of the classifying morphism

$$
T^{s s} \xrightarrow{\phi_{\left.\mathcal{V}_{t}\right|_{T} s s}^{s s}} M_{H_{m}}(\mathbf{v})
$$

is an open set whose complement has codimension $\geq 2$.

Proof. Choose $L$ and $\mathcal{V}_{t} / T$ as in the proof of Proposition 5.5 or 5.8 or 5.9 depending on which case we are considering. Suppose there is an irreducible Weil divisor $Z$ in the complement of the image. Then the corresponding line bundle $\mathcal{O}(Z)$ lies in the kernel of

$$
\operatorname{Pic}\left(M_{H_{m}}(\mathbf{v})\right) \xrightarrow{\phi_{\left.\mathcal{V}_{t}\right|_{T} ^{s s}}^{* s}} \operatorname{Pic}^{G}\left(T^{s s}\right) .
$$

But in the proofs of Propositions 5.5, 5.8, 5.9 we showed that the above map is injective, a contradiction.

Remark 5.14. The restrictions on the numerical invariants in Corollary 5.13 are substantial conditions. When these conditions are not satisfied some of the exponents in a Gaeta-type resolution may become zero. As a result, we can no longer ensure that we can find $L$ such that for the resulting complete family $\mathcal{V}_{t} / T$ of $\mathcal{O}(1,1)$-prioritary sheaves admitting an $L$-Gaeta-type resolution we have

$$
\operatorname{rk}\left(\operatorname{Pic}^{G}\left(T^{s s}\right)\right) \geq \rho\left(M_{H_{m}}(\mathbf{v})\right) .
$$

This way, the homomorphism

$$
\operatorname{Pic}\left(M_{H_{m}}(\mathbf{v})\right) \xrightarrow{\left.\phi_{v_{t}}^{*}\right|_{T^{s s}}} \operatorname{Pic}^{G}\left(T^{s s}\right)
$$

may no longer be injective.

## Chapter 6

## Bad Chern characters

In this chapter we show that when an $H_{m}$-semistable character $\mathbf{v}$ is bad, the Picard number of $M_{H_{m}}(\mathbf{v})$ is no longer controlled only by the position of $\mathbf{v}$ relative to the $\mathrm{DLP}^{<r}$-surface. One also needs to take into account the presence of the the $\Delta_{i}=\frac{1}{2}$-strata in complete families that force additional characters to be in the kernel of the Donaldson homomorphism $\lambda$.

### 6.1 Drop in the Picard number

We start with a continuation of Example 3.8.
Example 6.1. Let $m=1+\epsilon$, where $\epsilon \in \mathbb{Q}$ and $0<\epsilon \ll 1$. Consider character $\mathbf{v}=\left(4,-\frac{1}{4} E-\frac{1}{4} F, \frac{9}{16}\right)$ from Example 3.8. In that example we considered the family $\mathcal{V}_{t} / T$ of $\mathcal{O}(1,1)$-prioritary sheaves of character $\mathbf{v}$ admitting an $\mathcal{O}$-Gaeta-type resolution

$$
\begin{equation*}
0 \rightarrow \mathcal{O}(-1,-1)^{2} \xrightarrow{\psi_{t}} \mathcal{O}(-1,0)^{3} \oplus \mathcal{O}(0,-1)^{3} \rightarrow \mathcal{V}_{t} \rightarrow 0, \tag{6.1.1}
\end{equation*}
$$

where

$$
T \subset \mathbb{H}=\operatorname{Hom}\left(\mathcal{O}(-1,-1)^{2}, \mathcal{O}(-1,0)^{3} \oplus \mathcal{O}(0,-1)^{3}\right)
$$

is the open subset parameterizing injective sheaf maps with torsion-free cokernel. We showed that $T$ is not empty, $\operatorname{codim}_{\mathbb{H}}(\mathbb{H} \backslash T) \geq 2$, the family $\mathcal{V}_{t} / T$ is complete and any $H_{m}$-semistable $\mathcal{V} \in M_{H_{m}}(\mathbf{v})$ is equal to $\mathcal{V}_{t}$ for some $t \in T$. This last property implies that the classifying morphism

$$
T^{s s} \xrightarrow{\phi{\left.\mathcal{V}_{t}\right|_{T} ^{s s}}} M_{H_{m}}(\mathbf{v})
$$

realizes $M_{H_{m}}(\mathbf{v})=M_{H_{m}}^{s}(\mathbf{v})$ as a geometric quotient of $T^{s s}$ under the action of

$$
\bar{G}=(G L(2) \times G L(3) \times G L(3)) / \mathbb{C}^{*}(I d, I d, I d)=G / \mathbb{C}^{*}(I d, I d, I d),
$$

see [DLP85, Proposition 2.6]. Thus, by [MFK94, p. 32]

$$
\operatorname{Pic}\left(M_{H_{m}}(\mathbf{v})\right) \cong \operatorname{Pic}^{\bar{G}}\left(T^{s s}\right) .
$$

As before, we can compute the latter group using the exact sequence from Proposition 4.1:

$$
\begin{equation*}
\mathcal{O}^{*}\left(T^{s s}\right) \rightarrow \operatorname{Char}(\bar{G}) \rightarrow \operatorname{Pic}^{\bar{G}}\left(T^{s s}\right) \rightarrow 0 \tag{6.1.2}
\end{equation*}
$$

We claim that the first map is not zero. Take the $\Delta_{i}=\frac{1}{2}$-stratum $S_{T}=S_{H_{m}}\left(\mathbf{v}_{1}, \mathbf{v}_{2}\right)$ described in Example 3.8. Its closure $\overline{S_{T}}$ is a Weil divisor in $\mathbb{H}$, so it is given by a polynomial $f$ :

$$
\overline{S_{T}}=V(f), f \in \mathbb{C}\left[\left\{x_{i j}\right\}\right] .
$$

Since $\overline{S_{T}}$ is $\bar{G}$-invariant, the complement $\mathbb{H} \backslash \overline{S_{T}}$ is $\bar{G}$-invariant too, and the polynomial $f$ defines an invertible function on it, which by a remark after Proposition 4.1 satisfies

$$
f(\bar{g} h)=\eta_{f}(\bar{g}) f(h) \text { for some } \eta_{f} \in \operatorname{Char}(\bar{G}) \text { and any } \bar{g} \in \bar{G}, h \in \mathbb{H} \backslash \overline{S_{T}} .
$$

Note that since $f(h)=f(\bar{g} h)=0$ for $h \in \bar{S}$ the above equation in fact holds for all $h \in \mathbb{H}$.
We show that $\eta_{f}$ is a nontrivial character, which would establish our claim. Assume, on the contrary, that $\eta_{f}$ is a trivial character so that $f$ is $\bar{G}$-invariant and, consequently, $G$-invariant:

$$
f \in \mathbb{C}\left[\left\{x_{i j}\right\}\right]^{G} .
$$

As the closure of any $G$-orbit contains the zero morphism $0 \in \mathbb{H}$, all $G$-invariant functions are constant

$$
\mathbb{C}\left[\left\{x_{i j}\right\}\right]^{G}=\mathbb{C}^{*} .
$$

But $f$ defines a non-empty divisor, so this is a contradiction.
Now, sequence (6.1.2) gives

$$
\mathbb{Z}^{2} / \mathbb{Z} \eta_{f} \rightarrow \operatorname{Pic}^{\bar{G}}\left(T^{s s}\right) \cong \operatorname{Pic}\left(M_{H_{m}}(\mathbf{v})\right) .
$$

Since the ample bundle generates a free $\mathbb{Z}$-submodule inside $\operatorname{Pic}\left(M_{H_{m}}(\mathbf{v})\right)$, it follows that the Picard number is equal to one

$$
\rho\left(M_{H_{m}}(\mathbf{v})\right)=1
$$

Note, that an explicit computation of $\eta_{f}$ along the lines of remark 5.10 does not work in this case. The closure of the divisorial Shatz stratum is now described as

$$
\overline{S_{T}}=\left\{\psi_{t} \in T \mid \operatorname{Hom}\left(F_{1}, \mathcal{V}_{t}\right) \neq 0 \text { for some } F_{1} \in M_{H_{m}}\left(\mathbf{v}_{1}\right)\right\}
$$

and compared to Remark 5.10 the computation is obstructed by the fact that $F_{1}$ is not a fixed bundle, but varies along its one-dimensional moduli space.

However, note that by Remark 3.7, character $\mathbf{v}=\left(r, c_{1}, \chi\right)$ is primitive, so for a generic choice of $m=\frac{p}{q}$ we have

$$
\operatorname{gcd}\left(r, c_{1} \cdot\left(q H_{m}\right), \chi\right)=1
$$

Applying Proposition 2.11 we get that $\operatorname{Pic}\left(M_{H_{m}}(\mathbf{v})\right)$ is torsion-free and therefore

$$
\operatorname{Pic}\left(M_{H_{m}}(\mathbf{v})\right) \cong \mathbb{Z} .
$$

Let us also remark that using Proposition 3.13 we can argue as in the proof of Proposition 5.9 and show that $S_{T}$ is an irreducible subvariety of $T$.
Example 6.2. Let $m=1+\epsilon$, where $\epsilon \in \mathbb{Q}$ and $0<\epsilon \ll 1$. Let $\left\{\mathbf{w}_{k}\right\}_{k \in \mathbb{N}}$ be one of the infinite sequences of bad Chern characters constructed in Examples 3.9 and 3.10. The same argument can be applied verbatim to the complete family $\mathcal{W}_{t} / T$ from Example 3.9 to conclude that

$$
\operatorname{Pic}\left(M_{H_{m}}\left(\mathbf{w}_{k}\right)\right) \cong \mathbb{Z}
$$

for any $k \in \mathbb{N}$.

It turns out that the techniques of the previous two examples allow us to tackle bad $H_{m}$-semistable Chern characters whenever they lie on a branch of the DLP-surface given by a line bundle. Note that $H_{m}$-semistable characters $\mathbf{v}$ with $r=2$ are always good (see Definition 3.8), so we can assume $r \geq 3$.
Theorem 6.3. Let $\mathbf{v}=(r, \nu, \Delta) \in K(X)$ be a character with $r \geq 3, \Delta>\frac{1}{2}$. Let $\epsilon \in \mathbb{Q}$ be sufficiently small (depending on $r$ ), $0<|\epsilon| \ll 1$, and set $m=1+\epsilon$.

If $\mathbf{v}$ is a bad character with $\Delta=\operatorname{DLP}_{H}^{<r}(\nu)$ with a single exceptional bundle $L$ associated to $\mathbf{v}$ with $r(L)=1$, then

$$
\operatorname{Pic}\left(M_{H_{m}}(\mathbf{v})\right) \cong \mathbb{Z}
$$

and $\lambda$ is an epimorphism with

$$
\operatorname{ker} \lambda \supsetneq \mathbb{Z}[\bar{L}]
$$

Proof. Assume first that $\mu_{H_{m}}(L) \geq \mu_{H_{m}}(\mathbf{v})$. For a semistable $\mathcal{V}$ of character $\mathbf{v}$ we have

$$
\operatorname{Ext}^{i}(L, \mathcal{V})=0, i=0,1,2,
$$

by semistability and the fact that $L$ is associated to $\mathbf{v}$.
Using this, one checks that the Beilinson-type resolution from [Dré91, Proposition 5.1] coincides with the $L$-Gaeta-type resolution and every $H_{m}$-semistable sheaf $\mathcal{V}$ of character $\mathbf{v}$ is resolved as

$$
0 \rightarrow L(-1,-1)^{\alpha} \rightarrow L(-1,0)^{\beta} \oplus L(0,-1)^{\gamma} \rightarrow \mathcal{V} \rightarrow 0
$$

We can then repeat the argument of Example 6.1 to conclude

$$
\operatorname{Pic}\left(M_{H_{m}}(\mathbf{v})\right) \cong \mathbb{Z}
$$

When $\mu_{H_{m}}(L)<\mu_{H_{m}}(\mathbf{v})$, the Beilinson-type resolution coincides with the dual version of the $L$ Gaeta type resolution and every $H_{m}$-semistable sheaf $\mathcal{V}$ of character $\mathbf{v}$ is resolved as

$$
0 \rightarrow \mathcal{V} \rightarrow L(1,0)^{\alpha} \oplus L(0,1)^{\beta} \rightarrow L(1,1)^{\delta} \rightarrow 0
$$

We can also repeat the argument of Example 6.1 with straightforward modifications.
It is interesting to further explore the geometry of $M_{H_{m}}(\mathbf{v})$ for bad characters $\mathbf{v}$ as in the previous theorem, taking into account that these are unirational varieties (see §2.6) with Picard number $\rho=1$. As a step in this direction, we consider the character $\mathbf{v}$ from example 6.1.

Example 6.4. We claim that for $\mathbf{v}=\left(4,-\frac{1}{4} E-\frac{1}{4} F, \frac{9}{16}\right)$ and $m=1+\epsilon$ with $\epsilon \in \mathbb{Q}, 0<\epsilon \ll 1$, we in fact have

$$
M_{H_{m}}(\mathbf{v}) \cong \mathbb{P}^{3} .
$$

First, note that for a generic $\mathcal{V} \in M_{H_{m}}(\mathbf{v})$ with the corresponding Gaeta-type resolution

$$
0 \rightarrow \mathcal{O}(-1,-1)^{2} \xrightarrow{\psi} \mathcal{O}(-1,0)^{3} \oplus \mathcal{O}(0,-1)^{3} \rightarrow \mathcal{V} \rightarrow 0
$$

the map

$$
p r_{\mathcal{O}(0,-1)^{3}} \circ \psi: \mathcal{O}(-1,-1)^{2} \rightarrow \mathcal{O}(-1,0)^{3}
$$

is an injective map of vector bundles. Therefore, we can expand the Gaeta-type resolution of $\mathcal{V}$ into the following commutative diagram


So, next we consider extensions

$$
\xi:=\left[0 \rightarrow \mathcal{O}(0,-1)^{3} \rightarrow \mathcal{V}_{\xi} \rightarrow \mathcal{O}(-1,2) \rightarrow 0\right]
$$

with $\xi=\left(\xi_{1}, \xi_{2}, \xi_{3}\right) \in \operatorname{Ext}^{1}(\mathcal{O}(-1,2), \mathcal{O}(0,-1))^{\oplus 3}$. We assert that $\mathcal{V}_{\xi}$ is $H_{m}$-semistable if and only if the corresponding vectors $\xi_{1}, \xi_{2}, \xi_{3}$ are linearly independent.

Indeed, suppose without loss of generality that $\xi_{1}=a \xi_{2}+b \xi_{3}$. Consider the morphism

$$
\mathcal{O}(0,-1)^{2} \xrightarrow{A} \mathcal{O}(0,-1)^{3}
$$

given by the matrix

$$
A=\left[\begin{array}{lll}
a & 1 & 0 \\
b & 0 & 1
\end{array}\right]
$$

Then the induced map

$$
\operatorname{Ext}^{1}(\mathcal{O}(-1,2), \mathcal{O}(0,-1))^{\oplus 2} \xrightarrow{A_{*}} \operatorname{Ext}^{1}(\mathcal{O}(-1,2), \mathcal{O}(0,-1))^{\oplus 3}
$$

sends $\left(\xi_{2}, \xi_{3}\right)$ to $\left(\xi_{1}, \xi_{2}, \xi_{3}\right)$. This fact translates into the following commutative diagram


Since

$$
\mu_{H_{m}}\left(\mathcal{V}_{\left(\xi_{2}, \xi_{3}\right)}\right)=-\frac{1}{3}-\frac{\varepsilon}{3}>-\frac{1}{2}-\frac{\varepsilon}{4}=\mu_{H_{m}}\left(\mathcal{V}_{\left(\xi_{1}, \xi_{2}, \xi_{3}\right)}\right),
$$

we conclude that $\mathcal{V}_{\left(\xi_{1}, \xi_{2}, \xi_{3}\right)}$ is unstable.
Conversely, suppose $\mathcal{V}_{\xi}$ is unstable. A rank 4 bundle can be destabilized by subbundles or quotient bundles of rank 1 or 2 . We will only sketch the argument in the case of a destabilizing subbundle of rank 2 and leave the similar routine checks for the other cases to the reader. Suppose there is a destabilizing subbundle

$$
\mathcal{W} \subset \mathcal{V}_{\xi}
$$

with $r(\mathcal{W})=2$. Since

$$
\begin{equation*}
\mu_{H_{m}}(\mathcal{W}) \geq \mu_{H_{m}}\left(\mathcal{V}_{\xi}\right)>\mu_{H_{m}}\left(\mathcal{O}(0,-1)^{3}\right) \tag{6.4.1}
\end{equation*}
$$

there is no maps $\mathcal{W} \rightarrow \mathcal{O}(0,-1)^{3}$ and, therefore, the composition $\mathcal{W} \rightarrow \mathcal{V}_{\xi} \rightarrow \mathcal{O}(-1,2)$ is not zero. We will further assume that this composition is surjective, leaving the check in the other case to the reader. In this case we have the following commutative diagram

with $a \leq 0$ and $b \leq-1$. Furthermore, one checks that (6.4.1) is satisfied only if $(a, b)=(0,-1)$. In this case, denote the extension defining $\mathcal{W}$ by $\zeta$ and write

$$
B=\left[\begin{array}{l}
b_{1} \\
b_{2} \\
b_{3}
\end{array}\right], \quad b_{i} \in \mathbb{C}
$$

The induced map

$$
\operatorname{Ext}^{1}(\mathcal{O}(-1,2), \mathcal{O}(0,-1)) \xrightarrow{B_{*}} \operatorname{Ext}^{1}(\mathcal{O}(-1,2), \mathcal{O}(0,-1))^{\oplus 3}
$$

sends $\zeta$ to $\xi=\left(\xi_{1}, \xi_{2}, \xi_{3}\right)=\left(b_{1} \eta, b_{2} \eta, b_{3} \eta\right)$. Thus, we see that $\xi_{1}, \xi_{2}, \xi_{3}$ are linearly dependent.
Denote the locus of $\xi$ with linearly independent component vectors $\xi_{1}, \xi_{2}, \xi_{3}$ by

$$
U \subset \operatorname{Ext}^{1}(\mathcal{O}(-1,2), \mathcal{O}(0,-1))^{\oplus 3}
$$

By the above discussion, the universal extension over $U \times X$ defines a dominant morphism

$$
U \rightarrow M_{H_{m}}(\mathbf{v})
$$

Note that the isomorphism class of $\mathcal{V}_{\xi}$ only depends on the hyperplane spanned by $\xi_{1}, \xi_{2}, \xi_{3}$ in the four-dimensional space $\operatorname{Ext}^{1}(\mathcal{O}(-1,2), \mathcal{O}(0,-1))$, so the above map factors through

$$
\mathbb{P}\left(\operatorname{Ext}^{1}(\mathcal{O}(-1,2), \mathcal{O}(0,-1))^{\vee}\right) \rightarrow M_{H_{m}}(\mathbf{v})
$$

as claimed.
It is also interesting to note that the extensions

$$
0 \rightarrow F_{2} \rightarrow \mathcal{V} \rightarrow F_{1} \rightarrow 0
$$

with $F_{i} \in M_{H_{m}}\left(\mathbf{v}_{i}\right), \mathbf{v}_{1}=\left(2,-\frac{1}{2} F, \frac{1}{2}\right), \mathbf{v}_{2}=\left(2,-\frac{1}{2} E, \frac{1}{2}\right)$, give an embedding of a quadric into the moduli space $M_{H_{m}}(\mathbf{v}) \cong \mathbb{P}^{3}$.

$$
\mathbb{P}^{1} \times \mathbb{P}^{1} \hookrightarrow \mathbb{P}^{3}
$$

Indeed, $M_{H_{m}}\left(\mathbf{v}_{i}\right) \cong \mathbb{P}^{1}$ by Theorem 5.1 (3.b). The isomorphism class of $\mathcal{V}$ is uniquely determined by $F_{1}$ and $F_{2}$ because $\operatorname{ext}^{1}\left(F_{1}, F_{2}\right)=1$. Finally, the stability of $\mathcal{V}$ follows from [CH19, Lemma 10.8] and

$$
\mu_{H_{m}}\left(F_{2}\right)<\mu_{H_{m}}(\mathcal{V})<\mu_{H_{m}}\left(F_{1}\right)
$$

Question 6.5. One can also show that $M_{H_{m}}(\mathbf{v})$ is isomorphic to a projective space for the bad characters v of small rank listed in Example 3.9 using the same method as in the previous example. However, it takes more and more work to directly check semistability for characters of higher and higher rank. An interesting question is whether $M_{H_{m}}(\mathbf{v})$ is isomorphic to a projective space for all bad characters in the infinite sequence of Example 3.9. One can pose the same question for the bad characters constructed in Example 3.10 and, more generally, for all bad characters lying on a single branch of the DLP ${ }^{<r}$-surface (keeping in mind our discussion in Question 3.11).

### 6.2 Conjectural picture

We finish this dissertation by making the following conjecture about Pic $\left(M_{H_{m}}(\mathbf{v})\right)$ for all characters $\mathbf{v}$ with positive-dimensional moduli space.

Conjecture 6.6. Let $\mathbf{v}=(r, \nu, \Delta) \in K\left(\mathbb{P}^{1} \times \mathbb{P}^{1}\right)$ be a character with $r \geq 2$ and $\Delta \geq \frac{1}{2}$. Let $\epsilon \in \mathbb{Q}$ be sufficiently small (depending on $r$ ), $0<|\epsilon| \ll 1$, and set $m=1+\epsilon$.

1. ( $\mathbf{v}$ lies above the $\mathrm{DLP}^{<r}$-surface) If $\Delta>\operatorname{DLP}_{H_{m}}^{<r}(\nu)$, then

$$
\operatorname{Pic}\left(M_{H_{m}}(\mathbf{v})\right) \cong \mathbb{Z}^{3}
$$

and $\lambda$ is an isomorphism.
2. ( $\mathbf{v}$ lies on a single branch of the DLP ${ }^{<r}$-surface)
(a) If $\mathbf{v}$ is a good character with $\Delta=\operatorname{DLP}_{H_{m}}^{<r}(\nu)$ with a single exceptional bundle $E$ associated to $\mathbf{v}$, then

$$
\operatorname{Pic}\left(M_{H_{m}}(\mathbf{v})\right) \cong \mathbb{Z}^{2}
$$

and $\lambda$ is an epimorphism with

$$
\operatorname{ker} \lambda \cong \mathbb{Z}[\bar{E}]
$$

(b) If $\mathbf{v}$ is a bad character with $\Delta=\operatorname{DLP}_{H_{m}}^{<r}(\nu)$ with a single exceptional bundle $E$ associated to $\mathbf{v}$, then

$$
\operatorname{Pic}\left(M_{H_{m}}(\mathbf{v})\right) \cong \mathbb{Z}
$$

and $\lambda$ is an epimorphism with

$$
\operatorname{ker} \lambda \supsetneq \mathbb{Z}[\bar{E}]
$$

3. ( $\mathbf{v}$ lies on an intersection of two branches of the DLP ${ }^{<r}$-surface)
(a) If $\Delta=\operatorname{DLP}_{H_{m}}^{<r}(\nu)>\frac{1}{2}$ with at least two different exceptional bundles $E_{1}, E_{2}$ associated to $\mathbf{v}$, then

$$
\operatorname{Pic}\left(M_{H_{m}}(\mathbf{v})\right) \cong \mathbb{Z}
$$

and $\lambda$ is an epimorphism with

$$
\operatorname{ker} \lambda \cong \mathbb{Z}\left[\overline{E_{1}}\right]+\mathbb{Z}\left[\overline{E_{2}}\right] .
$$

(b) If $\Delta=\frac{1}{2}$, then $M_{H_{m}}(\mathbf{v})$ is a projective space and

$$
\operatorname{Pic}\left(M_{H_{m}}(\mathbf{v})\right) \cong \mathbb{Z}
$$

In other words, this conjecture states that the Picard number of $M_{H_{m}}(\mathbf{v})$ is determined by the position of $\mathbf{v}$ relative to the $\mathrm{DLP}^{<r}$-surface and by whether character $\mathbf{v}$ is good or bad. Conjecturally, $\mathbf{v}$ can be a bad character only if it lies on a single branch of the DLP ${ }^{<r}$-surface, see our discussion in Question 3.11.

It is likely that in order to verify this conjecture one needs to study some fine properties of full exceptional collections on $\mathbb{P}^{1} \times \mathbb{P}^{1}$ along the lines of [Rud89], which could allow one to build Beilinsontype resolutions better suited for studying semistable sheaves of a given Chern character v. Another interesting problem in this direction is the problem of classifying $H_{m}$-semistable Chern characters with $\Delta=\frac{1}{2}$. This could further lead to a classification of bad $H_{m}$-semistable Chern characters, thus answering Question 3.11. Finally, it remains an open question how to explicitly describe the second generator of ker $\lambda$ in Theorem 6.3 and Conjecture 6.6 (2.b).

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2. D.Pedchenko, The Picard group of moduli spaces of sheaves on a quadric surface, submitted (2020) arXiv:2007.11666
3. D. Pedchenko, The Arens-Michael envelopes of the Jordanian plane and $U_{q}(\mathfrak{s l}(2))$, submitted (2020) arXiv:2009.06477

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[^0]:    ${ }^{1}$ Note that the definition of the discriminant $\Delta$ we are using in this paper differs from the definition of discriminant $\tilde{\Delta}$ in [HL10]: $\tilde{\Delta}=2 r^{2} \Delta$. That is why some formulas in this section differ by a factor $2 r^{2}$ compared to the formulas in [HL10, §4.C].

