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ESSAYS ON VALUE DISTRIBUTIONS IN ALL-PAY AUCTIONS

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Abstract

This dissertation consists of three chapters. The first chapter studies the value distribution adoption choice of a player when she competes against an incumbent in an all-pay auction setting. The second chapter analyzes how much would a player like to learn about her own valuation in a similar setting. Lastly, the third chapter analyzes the best information disclosure policy that an auctioneer can adopt according to different performance measures in a two-player two-stage all-pay auction setting, where the players choose their value distributions in the first stage.

The first chapter considers a two-player all-pay auction setting and modifies it by adding a technology-adoption stage at the beginning of the game. In a discrete valuations environment, assuming one player’s valuation is common knowledge, we allow the other player (informed) to pick a distribution over the valuation space. Her opponent (uninformed) observes her choice of distribution. However, her valuation is privately drawn according to this distribution. The two players then play an asymmetric all-pay auction. We show that in such a setting, the informed player adopts a distribution that assigns positive probabilities to at most two elements; that will always contain the supremum, and sometimes, the infimum of the set of available values. She pools the extreme values in order to create an information asymmetry, which then would make the uninformed player bid less aggressively. We later impose a mean condition on the distribution that the informed player could pick and observe that she still prefers to split the probability mass on in-between values to the extreme ones. As a result, she picks the same support but arranges the probability mass on these values to meet the mean condition. In other words, the informed player is first interested in including only the extreme values in the support of her value distribution, and then the probabilities assigned to those values.

The second chapter assumes that the informed player’s value distribution is common knowledge and that she cannot observe her realized value. However, she can acquire additional information about her realized value by adopting a learning experiment. She picks such an experiment in the first stage. Even though her choice of experiment is observed by the uninformed player, she privately learns the realization of the experiment. Then, they play an all-pay auction in the second stage of the game. Every learning experiment induces a posterior probability distribution over the convex hull of the set of available values. The informed player bids as if her value is drawn from this posterior distribution, where she privately observes her value. Therefore, her problem boils down to choosing a posterior distribution that stochastically dominates the prior in the second-order sense.
We show that the informed player’s motivation to split the probability mass on in-between types to the extreme types is still present. However, due to the distributional constraints, she will pick a fully informative experiment to learn her value as long as it does not result in her two lowest values bidding zero with a positive probability in the equilibrium of the all-pay auction stage. If that is the case, she would try to mimic the prior distribution for the high types, who will never bid zero, and allocate the remaining probability to only one type to meet the constraint.

One natural extension of our analysis is studying the equilibrium value distribution profiles when both players are choosing their own value distribution. When the possible values are only high and low, we show that the profile in which one player picks the high value with probability one while the other player assigns probability half to each values is the unique (up to symmetry) value distribution profile. Moreover, when we consider any set of values, we show that the profile in which one player picks the highest value with probability one, while the other player assigns probability half to the highest and the lowest values each is an equilibrium value distribution profile. Due to the lack of an analytical approach to the equilibrium bidding distributions of the all-pay auctions in an asymmetric information environment, checking whether this equilibrium is unique is left as future work.

The last chapter analyzes the best information disclosure policy that an auctioneer can adopt according to different performance measures, namely players’ payoff, prize allocation efficiency, and aggregate effort. The significant contribution of the analysis is that players have the ability to choose the distribution from which their own types are drawn. Using a two-player all-pay auction with the two-type setting, we show that the optimal disclosure policy depends on the ratio of the value of winning for a low type to the value of winning for a high type.
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Chapter 1
Endogenizing Value Distribution in All-pay Auctions

1.1 Introduction

Rent-seeking and lobbying activities, research and development races, competitions for promotions, sports competitions, and university entrance exams are examples of competitions in which the participants exert costly and irrecoverable efforts. In such economic settings, the participants compete to win a prize by expending resources regardless of whether they win or lose.

Their abundance has piqued many scholars’ interest. Economists have mostly studied such competitions to understand players’ behaviors in these environments. How the players value the prize in these competitions has been assumed to be exogenous and common knowledge. Hillman & Riley (1989) assumes asymmetric valuations but a complete information environment. In a similar setting, Baye et al. (1996) fully characterize the equilibrium. Nti (1999) considers a Buchanan et al. (1980) model and establishes the necessary and sufficient conditions for a unique pure strategy Nash equilibrium (for a broader treatment, see, e.g., Cornes & Hartley (2005) and Malheg & Yates (2006)). Che & Gale (1998) consider an exogenous cap on bids in a two-lobbyist setting, where the lobbyists’ valuations are known. Barut & Kovenock (1998) extend the analysis of the n-player all-pay auction with complete information to cover the case of \( m \leq n \) prizes. Siegel (2009) allows the contestants be asymmetrically positioned in multiple dimensions and make irreversible investments, provides a closed-form formula for players’ equilibrium payoffs, and analyzes player participation. Siegel (2014b) studies multi-prize contests that allow for a wide range of asymmetries among players, including head starts and
provides an algorithm that constructs the unique equilibrium.

Consider a research and development race in which the firm that develops the higher-quality product enjoys the dominant market position. However, the value of a dominant market position may depend on firm-specific characteristics, such as production cost and marketing expertise. The studies we mentioned above assume such firm-specific characteristics as known not only by the firm itself but also by its competitors. However, in reality, the firms may not be fully informed about their opponents’ firm-specific characteristics.

The next set of attempts in the literature is directed at the asymmetry of information in competitive settings. Krishna et al. (1997) study the war of attrition and the all-pay auction games with $n$ players and find the sufficient conditions for the existence of symmetric monotonic equilibrium when the players’ signals are affiliated and symmetrically distributed. Moldovanu & Sela (2001) study effort maximizing contests with multiple prizes, where the contestants are privately informed about their costs of effort. Amann & Leininger (1996) prove existence and uniqueness of equilibrium for a class of two bidders all-pay auctions with independent private values, while Lizzeri & Persico (2000) prove the same with a reserve price and affiliated signals.

Due to its difficulties, the analysis of asymmetric contests with incomplete information is generally limited to the contests with two players. Even though it seems limited, many real-world applications involve only two competitors. Some of the biggest business rivalries where two firms dominate the market are Coca-Cola and Pepsi in the soft drink market, NVIDIA and AMD in the graphics processing unit (GPU) market, and Apple’s iOS and Google’s Android in the mobile operating system market.

Unfortunately, there is no closed-form analytical formulation describing the players’ bidding behaviors in a general form of all-pay auction games. Siegel (2014a) presents an algorithm that constructs the unique equilibrium in an asymmetric two-player all-pay auction by constraining the signal structure to be discrete, however, allowing the valuations to be interdependent. Rentschler & Turocy (2016) consider a symmetric, two-bidder all-pay auction with interdependent values and general distribution of types and present

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1This example is adopted from Siegel (2014b).

2See https://en.wikipedia.org/wiki/Duopoly for more examples of such duopolistic markets.
a construction that characterizes all symmetric equilibria, including non-monotonic ones. Zheng (2019) develops a distributional method to solve the all-pay auction game with arbitrary type distributions in his conflict investigation. ³

One could argue that firm-specific characteristics are the results of the firms’ previous choices. For instance, a firm’s production cost is determined by her production technology choice or its workers. Workers differ in their abilities, experiences, and skills. After carefully interviewing the applicants, firms could decide not to hire the best-skilled candidate. Another example in which the firm-specific characteristics are endogenous is technology adoption decisions. When there are two types of technologies available to a firm, one develops a high-quality product with certainty while the other one with some probability, the firm could prefer the latter. Even though the reason for not adopting the technology that deterministically delivers the best quality product could be its cost, firms could prefer the presence of asymmetric information even in its absence, mainly to affect the level of the competition’s aggression because a contestant’s effort depends on her knowledge of her rival’s type. ⁴

When agents are given the option to create information asymmetry by picking a distribution that will determine their valuations against an uninformed agent, they may create asymmetry to increase their expected utility. Condorelli & Szentes (2016) analyzes a bilateral trade model where the buyer can choose a cumulative distribution function that determines her valuation and shows that the buyer can generate a higher expected utility by creating uncertainty against an uninformed monopoly seller compared to the case in which she would have the highest possible valuation with probability one. Roesler & Szentes (2017) consider an information design problem where the buyer’s valuation for the object is uncertain and she can design her own information about her value before facing a monopolist seller. Similar to Condorelli & Szentes (2016), the information structure that she adopts, in return, affects the price-setting behavior of the seller. They show that buyer-optimal signal distribution involves minimal learning.

To incorporate such endogeneity, we consider a two-player all-pay auction setting and modify it by adding a technology-adoption stage at the beginning of the game. In a discrete valuations environment, assuming one player’s valuation is common knowledge,

³See also Milgrom & Weber (1982) and Szech (2011)
⁴Serena (2017) has a detailed discussion about how the effort exertion is affected by the heterogeneity of contestants’ types.
we allow the other player (informed) to pick a distribution over some set of valuations. Her opponent observes her choice of distribution. Her valuation is privately drawn according to this distribution, and then the two players play an asymmetric all-pay auction.

We show that in such setting, the informed player adopts a distribution that assigns positive probabilities to at most two elements; that will always contain the supremum, and sometimes, the infimum of the set of available values. This is because when she has the opportunity to dominate her rival by always having the highest possible valuation, she does so. However, when she has no means to do that she prefers sometimes having the highest valuation, and she augments the economic power of the highest valuation with the information rent she creates by having the lowest valuation in the support her distribution choice.

To include the possible cost structures attached to each distribution choice, we consider a specific one; that depends only on the average quality of the technology. Constraining the informed player’s distribution choice to satisfy a given mean condition, we show that she still prefers the extreme values. This is because her motivation is first in the values, only then in the probabilities over those values.

In the subsequent section, we vary the model. The informed player’s valuation is drawn from a publicly known prior distribution. However, she cannot observe it. We allow her to acquire additional information about her own valuation by picking a learning experiment at no cost. Her problem boils down to choosing a posterior that stochastically dominates the prior in second order sense. We show that her motivation to create information asymmetry is still present. However, due the distributional assumptions she may not fully learn her prior, but she adopts a fully informative experiment as long as she bids above zero with positive probability when she draws the lowest value. Otherwise, she tries to approach the fully informative experiment as much as she could.

The rest of the paper is organized as follows. Section 1.2 studies the informed player’s distribution choice when she has no constraints. Section 1.3 introduces the mean condition. Sections 2.1 - 2.5 analyze the best learning experiment. Lastly, Section 2.6

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Section 1.2 also discusses what would happen if the distribution choice was not observable by the other player.
discusses what would happen if both players were choosing their own value distributions.

1.2 Unconstrained Value Distribution

This section analyzes the technology adoption choice of a potential entrant who scrutinizes entering a market, and facing the incumbent firm in an all-pay auction competition setting. Moreover, it assumes that everything about the incumbent, more specifically his valuation, is known. The entrant chooses a stochastic technology in the first stage, which can be interpreted as picking a CEO who may or may not be a good fit for the job. The valuation of the entrant depends on the quality of her CEO, and a CEO’s quality is defined by the probability distribution over the set of values that he could generate for the entrant. The entrant’s CEO choice, hence his quality, is observed by the incumbent. However, the entrant privately observes her realized value at the beginning of the second stage. Then the two players play an incomplete information all-pay auction where the incumbent’s valuation for winning the auction is common knowledge, while the entrant’s value is drawn according to her choice of CEO.

To illustrate, suppose that the valuations can be high or low, and the incumbent’s valuation is high. Furthermore, assume that the quality of the entrant’s CEO is $p = (p, 1 - p)$, i.e., he yields a high valuation with probability $p$ and a low valuation with the remaining probability $(1 - p)$. Then the two players play an incomplete information all-pay auction in the second stage of the game, where the incumbent’s value is high, and the entrant’s value is high with probability $p$ and low with the remaining probability $1 - p$. When $p \in \{0, 1\}$, the subsequent stage is nothing but a complete information all-pay auction game. From Hillman & Riley (1989) we know that the entrant’s payoff is zero. However, when $p \in (0, 1)$, using Siegel (2014a)’s equilibrium construction algorithm, we find that she obtains a positive payoff.

In this section we study the entrant’s optimal CEO choice when the set of values is finite, and the incumbent’s value can be anything, not necessarily an element in the set of values. The model is introduced in Section 1.2.1. Then we analyze the entrant’s optimal distribution choice when there are only two values available to her. We, then, consider finitely many values that the entrant could include in the support of her distribution choice. Later in Chapter 2, we discuss what would happen if the incumbent was allowed

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6The reason why we need $V$ being a finite set is due to the limitations of Siegel (2014a)’s algorithm.
to choose his value distribution, too.

1.2.1 Model

Consider a two-player two-stage game. In the first stage of the game, player 1 (the entrant) picks an element from the set of distributions defined over a set of values, $V = \{v_1, v_2, \ldots, v_N\}$ with $v_1 > v_2 > \ldots > v_N > 0$. Let $\mathbf{p} = (p_1, p_2, \ldots, p_N)$ be her distribution choice, where $p_j$ denotes the probability that $\mathbf{p}$ assigns on the value $v_j$ for $j = 1, 2, \ldots, N$. Players play an independent private value all-pay auction game in the second stage. Denote by $\nu_i$ player $i$’s value of winning the all-pay auction. While both players observe $\mathbf{p}$ at the end of the first stage, Player 1 privately observes her realized value $\nu_1$ at the beginning of the second stage, which is drawn from $V$ according to $\mathbf{p}$. Therefore,

$$\Pr(\nu_1 = v_j) = p_j \quad \text{for} \quad j = 1, 2, \ldots, N.$$  

Player 2’s (the incumbent) valuation, $\nu_2$, is common knowledge. For now, $\nu_2$ is not necessarily in $V$. The terms value and type are used interchangeably throughout the paper. Players simultaneously choose non-negative effort levels based on the information they have. The player who exerts the higher effort wins the price. In case of a tie, any tie-breaking procedure may be used to allocate the prize.

We are interested in the entrant’s distribution choice. For this reason, define the following reduced game.

**Definition 1** (Reduced Game). Player 1 and player 2 play a game. Player 1 picks a distribution $\mathbf{p}$ over the type space, $V$. Their payoffs are calculated as the equilibrium payoff of the independent asymmetric all-pay auction in which player 1’s type is drawn from $\mathbf{p}$, and player 2’s type is known to be $\nu_2$.

It may seem like a decision problem rather than a game. However, later in section 2.6, we will discuss what would have happened if player 2, simultaneously with player 1, chose a distribution as well. Moreover, this is only a reduced form that will allow us to study the entrants’ distribution choice behavior in the sub-game perfect equilibrium of the original game.
There is no closed-form analytical formulation describing the players’ bidding behaviors in the all-pay auction stage once we fix a distribution profile in the first stage. Fortunately, Siegel (2014a) proves the uniqueness of the equilibrium and presents an algorithm to construct it. To calculate the equilibrium payoff and check if a profitable deviation is feasible, we heavily use his algorithm. Thus, the reader is strongly encouraged to read Appendix A to understand how the algorithm works.

1.2.2 An Example with Two Types

This section analyzes the equilibrium of the reduced game in its simplest form, that is, when we have only two values available to the entrant. We explicitly depict the equilibrium construction in the all-pay auction stage, taking the entrant’s distribution choice as given. It will not only exemplify how the Siegel (2014a) works, but also grasp the main intuition behind the Propositions 1 - 4.

Let us consider a simple example where the set of values consists of only high and low ones, i.e., \( V = \{v_h, v_l\} \) with \( v_h > v_l > 0 \). Moreover, assume that the incumbent’s value is high, \( \nu_2 = v_h \). Later in this section we will consider different values for the incumbent as well. Denote by \( p \) the probability that the entrant’s distribution choice assigns on \( v_h \). To calculate the payoffs in the reduced game, consider the incomplete information all-pay auction game that the two players play in the second stage. Figure 1.1 depicts the equilibrium bidding distribution functions of the players in the all-pay auction stage, which is created using Siegel (2014a) algorithm.

Now let us try to understand how the algorithm constructs Figure 1.1. Consider the joint partition of the interval in which players mix their bids, i.e., \([0, T]\). The high types of the two players randomize their bids in the top interval. The algorithm starts constructing the equilibrium of the all-pay auction from this top interval. It first finds the bidding distributions of the players’ high types over this interval and then calculates the length of the interval. The densities of their bidding distributions are as follows:

\[
g_1(v_h, v_h) = \frac{1}{pv_h} \quad \text{and} \quad g_2(v_h, v_h) = \frac{1}{v_h},
\]

where \( g_i(\nu_1, \nu_2) \) represents the density function of player \( i \)’s bidding distribution over the interval in which player 1’s \( \nu_1 \) type bids against her opponent’s \( \nu_2 \) type. \(^7\) The length of

\(^7\)One could use \( g_i(\nu_1) \) here, however, even though we have assumed that player 2 has only one type, we will relax it later in Section 2.6 when both players choose distributions.
Figure 1.1. The equilibrium bidding distributions in the all-pay auction stage when player 2’s value is high and player 1’s value is high with probability $p$ and low with the remaining probability of $1 - p$. The two sides of the dashed line positioned below the horizontal bid axes represent the types of the players bidding against each other in the corresponding interval.

the top interval in which $v_h$ type of player 1 and the same type of player 2 mix their bids is

$$L_1 = \min \{pv_h, v_h\} = pv_h,$$

where $pv_h$ and $v_h$ are the necessary lengths of the top interval for player 1 and player 2 to exhaust their bidding probabilities, respectively. Since player 1 is the one who exhausts her bidding probability first, in the penultimate interval, her $v_\ell$ type will be bidding against $v_h$ type of player 2. Over this interval, the constant densities of their bidding distributions are as follows:

$$g_1(v_\ell, v_h) = \frac{1}{(1 - p)v_h} \quad \text{and} \quad g_2(v_\ell, v_h) = \frac{1}{v_\ell}.$$

The required length of this interval for player 1 to exhaust her bidding probability is $(1 - p)v_h$. For player 2 to exhaust his bidding probability in this interval, we must take into account that he has already exhausted some of his bidding probability in the top interval. His remaining bidding probability power could be calculated as $1 - L_1g_2(v_h, v_h) = 1 - p$. Therefore, he needs an interval with a length of $(1 - p)v_\ell$ to exhaust his remaining bidding probability. As a result, the length of the penultimate interval in which $v_\ell$ type of player
1 and \( v_h \) type of player 2 bid is

\[
L_2 = \min \left\{ (1 - p)v_h, (1 - p)v_\ell \right\} = (1 - p)v_\ell.
\]

Therefore, player 2 exhausts all his bidding probabilities, whereas player 1 is left with some to bid. So, she should put her remaining bidding probability as a mass of size \( a \) at zero, which can be calculated as:

\[
a = 1 - L_2 g_1(v_\ell, v_h) = 1 - (1 - p)v_\ell \frac{1}{(1 - p)v_h} = 1 - \frac{v_\ell}{v_h}.
\]

Adding \( L_1 \) and \( L_2 \), one could obtain the total length of the interval in which the players mix their bids.

\[
T = L_2 + L_1 = v_\ell + p(v_h - v_\ell).
\]

In equilibrium, player 1’s low type expects zero payoff while her high type obtains \((1 - p)(v_h - v_\ell)\). Hence, her expected payoff in the reduced game is \( p(1 - p)(v_h - v_\ell) \), while Player 2 expects a payoff of \((1 - p)(v_h - v_\ell)\). Therefore, the entrant will choose the payoff maximizing \( p \) in the reduced game, which is \( \frac{1}{2} \).

Even though the entrant could have chosen any distribution, including the one that makes her have the high value with certainty, she picks the one that assigns probability half to both high and low values. She does so to create information asymmetry. However, one might wonder why this probability does not depend on the values. Later when we change the incumbent’s value, we will see that it does, indeed, depend on the ratio of the high value to the incumbent’s value. Since the two happened to be the same here, we do not explicitly see the values in the optimal probability.

Now, let us consider the same problem when the incumbent’s value is \( v_\ell \). Figure 1.2 depicts the players’ equilibrium bidding distributions. This time in the top interval, the entrant’s high type mix her bids against the incumbent’s only type, \( v_\ell \). It is the entrant’s high type who first exhausts her bidding probabilities since she requires the top interval’s length to be \( pv_\ell \), while the incumbent needs it to be \( v_h \). In the next interval, the entrant’s low type bid against the incumbent. Following Siegel (2014a), one could easily find that it is the entrant, again, who exhausts her bidding probabilities. As a result, the incumbent places his remaining bidding power as a mass at zero. In equilibrium, the
incumbent obtains zero payoffs, while the entrant’s low type gets zero payoffs and her high type gets $v_h - v_l$. Therefore, the entrant expects a payoff of $p(v_h - v_l)$, and it is maximized when $p = 1$.

![Equilibrium bidding distributions](image)

**Figure 1.2.** The equilibrium bidding distributions in the all-pay auction stage when player 2’s value is low and player 1’s value is high with probability $p$ and low with the remaining probability of $1 - p$. The two sides of the dashed line positioned below the horizontal bid axes represent the types of the players bidding against each other.

In this case, the entrant does not try to create information asymmetry, and instead, she chooses to have the high value with certainty. This is because, she is facing a “weak” opponent compared to the previous case. And there is no reason to make a weak opponent bid less aggressively at the cost of losing the high economic power. There is actually one to one relationship between seeking to create information asymmetry and having some type bidding zero with positive probability in the equilibrium of the all-pay auction stage. We establish this relation considering the third and the most comprehensive scenario, that is when the incumbent’s value is not restricted to be in $V$.

To complete the analysis, assume that the incumbent’s value is some $v$, that is not necessarily in $V$. This time, in the top interval the densities of the players’ bidding distributions are as follows:

$$g_1(v_h, v) = \frac{1}{pv} \quad \text{and} \quad g_2(v_h, v) = \frac{1}{v_h},$$
The length of the top interval is, then,

\[ L_1 = \min \{ pv, v_h \} . \]

If \( pv \geq v_h \), then \( L_1 = v_h \). In other words, the incumbent exhausts his bidding probabilities first, hence, the entrant assigns all his remaining bidding power as a mass at zero. In this equilibrium, the entrant gets zero payoffs, and the incumbent’s expected payoff is \( v - v_h \). In order to avoid a zero payoffs situation, the entrant should, hence, choose a lower \( p \). The intuition of Proposition 1 relies on this. If the entire bidding took place in the top interval, and the entrant ended up placing a mass at zero bid, then she could have zero payoffs. Instead, by lowering the probability of having a high valuation, she makes the incumbent bid less aggressively. As a result, her low type could bid positive bid as well, creating the opportunity for her high type to obtain positive payoff. If \( pv < v_h \), then \( L_1 = pv \). Therefore, the entrant’s high type exhaust her bidding probabilities first. In the penultimate interval, the constant densities of the players’ bidding distributions are as follows:

\[ g_1(v_t, v) = \frac{1}{(1-p)v} \quad \text{and} \quad g_2(v_t, v) = \frac{1}{v_t} . \]

Remember that the incumbent has lost some of his bidding probabilities by randomizing his bids in the top interval. So, the length of the penultimate interval is

\[ L_2 = \min \left\{ (1-p)v, \left( 1 - \frac{pv}{v_h} \right) v_t \right\} . \]

If \( L_2 = (1-p)v \), then the incumbent gets zero payoffs in the equilibrium, while the entrant’s low type obtains \( (1 - \frac{pv}{v_h}) v_t - (1-p)v \), and her high type gets \( v_h - v \). As a result, she expects to get a payoff of

\[ U_1(p; v) = p(v_h - v) + (1 - p) \left[ \left( 1 - \frac{pv}{v_h} \right) v_t - (1-p)v \right] \]

Observe that the payoff above is maximized when \( p = 1 \) because when \( pv < v_h \), we have

\[ v_h - v > \left( 1 - \frac{pv}{v_h} \right) v_t - (1-p)v . \]

If \( L_2 = \left( 1 - \frac{pv}{v_h} \right) v_t \), then it is the incumbent who first exhausts his bidding probabilities. Entrant’s low type places all his remaining probabilities as a mass at zero. In
equilibrium, the entrant’s low type obtains zero payoffs, while her high type gets a payoff of $v_h - pv - \left(1 - \frac{pv}{v_h}\right)v$. Therefore, the entrant’s ex-ante expected payoff is

$$U_1(p; v) = p \left( v_h - pv - \left(1 - \frac{pv}{v_h}\right)v \right).$$

Taking the first order condition we obtain that the optimal probability that the entrant must assign to the high value is $p^* = \frac{v_h}{2v}$. Observe that when $p = \frac{v_h}{2v}$, if $L_2 = \left(1 - \frac{pv}{v_h}\right)v$, then we must have

$$\left(1 - \frac{v_h}{2v}\right)v < \left(1 - \frac{v_h}{2v}\right)\frac{v}{v_h}.$$

Therefore,

$$\frac{v_h + v}{2} < v.$$

Proposition 1 summarizes the analysis.

**Proposition 1.** Suppose there are two players, informed (the entrant) and uninformed (the incumbent). This is, the entrant knows her type and the type of the other player, but incumbent just knows his type. Then, fixing incumbent’s type to $v$, suppose that the entrant (informed) could pick a distribution on a set $V = \{v_\ell, v_h\}$, with $0 < v_\ell < v_h$, from where her type is drawn. She will pick a distribution that assigns probability $p^*$ on $v_h$, where

$$p^* = \begin{cases} \frac{v_h}{2v}, & \text{if } \frac{v_h + v}{2} < v \\ 1, & \text{otherwise} \end{cases}.$$

The optimal probability does not directly depend on the low value in the set of valuations. In other words, the low value could be as small as possible. Let us say that a higher probability assigned on the high value makes a player economically stronger. Furthermore call a player relatively-weak if she bids zero with positive probability in the equilibrium of the all-pay auction stage, and relatively-strong if her opponent bids zero with positive probability. Then, the intuition behind the Proposition 1 can be described as the following. Although being economically the strongest sounds appealing to a player, it motivates her opponent to fight more aggressively if he is also economically strong. Thus, an entrant chooses to have some economic power and augment it with the information rent that she creates by having the low type in the support of her
distribution choice if she is facing a relatively strong opponent. However, she prefers to be the economically strongest if she is facing a relatively weak opponent.

In fact, this intuition is valid even when \( V \) is any finite set. In the next section, we allow \( V \) to include many valuations and find the entrant’s optimal distribution choice for the incumbent’s different valuations.

### 1.2.3 Generalizing to Many Types

In this section, we generalize the example with two types to finitely many types and verify that the intuition behind Proposition 1 is still valid. There are two driving forces that the entrant takes into account when choosing her distribution: economic power and information rent. Higher types yield higher economic power, while lower types yield higher information rent, which augments the economic power. Since these forces are maximized at the extreme types, the entrant’s optimal distribution choice assigns positive probabilities to at most two elements; that will always contain the supremum of \( V \), and sometimes, the infimum of \( V \).

Remember that the set of values is \( V = \{v_1, v_2, \ldots, v_N\} \) with \( v_1 > v_2 > \ldots > v_N > 0 \), incumbent’s valuation is \( v \). Once again, \( v \) is not necessarily an element in \( V \). Consider the Reduced Game in which the entrant chooses a distribution \( p = (p_1, p_2, \ldots, p_N) \) over \( V \), where \( p_j \) is the probability that \( p \) assigns to value \( v_j \) for \( j = 1, 2, \ldots, N \). Denote by \( u_1(v_i|p) \) the conditional payoff of the entrant’s \( v_i \) type when she employs \( p \) as her distribution choice. Her ex-ante expected payoff is the average of her types’ conditional payoffs, i.e.,

\[
U_1(p) = \sum_{i=1}^{N} p_i u_1(v_i|p).
\]

Lemma 1 - 4 establish the necessary conditions for a distribution \( p \) to be an optimal one.

It is clear that if there is only one element in the support of \( p \), say \( \nu \), then either the entrant is getting zero payoffs, hence, \( \nu \) can be anything in \( V \), or, she is obtaining positive payoff, so, \( \nu \) must be \( v_1 \) to maximize that payoff.
Now, suppose there are more than one element in the support of $p$. Observe that a higher type of the entrant expects a higher payoff as long as that payoff is positive. Following lemma formalizes this observation.

**Lemma 1.** Let $p$ be a distribution over $V$, with support $\{b_1, b_2, \ldots, b_t\}$ with $t > 1$. If the entrant employs $p$, then

$$u_1(b_i|p) \geq u_1(b_{i+1}|p).$$

The inequality holds with equality when $u_1(b_i|p) = 0$.

**Proof.** The entrant’s $b_i$ and $b_{i+1}$ types have a common bid in their best response sets. Call that bid $T_{i+1}$. By bidding $T_{i+1}$, the two win with the same probability and pay the same payment. Therefore,

$$u_1(b_i|p) - u_1(b_{i+1}|p) = \Pr(\text{win} | \text{bid } T_{i+1}) (b_i - b_{i+1}),$$

which is positive as long as the winning probability is non-zero. ■

Remember that a player is called *relatively-weak* if she bids zero with positive probability in the equilibrium of the all-pay auction stage, and *relatively-strong* if her opponent bids zero with positive probability. Below, we analyze the distributions in two different classes based on whether they make the entrant relatively-weak. First, let us show that the entrant chooses the degenerate distribution that assigns probability one to $v_1$, if she is not relatively-weak. That is, a distribution that assigns probability one to the highest value in $V$ yields higher expected payoff to the entrant than any other distribution that assigns positive probabilities to at least two values in $V$, and does not make the entrant relatively-weak player. Following lemma formalizes this.

**Lemma 2.** Let $p$ be a distribution over $V$, with support $\{b_1, b_2, \ldots, b_t\}$ with $t > 1$. When the entrant employs $p$ as her distribution choice, if the incumbent bids zero with positive probability, or no player places an atom at zero in the equilibrium of the all-pay auction stage, then $p$ cannot be optimal.

**Proof.** One could consider Figure 1.3 to visualize the supports of the players’ equilibrium bidding distributions. The top two horizontal lines show the players’ types who mix their bids over the corresponding interval in the joint partition. Moreover, $d_i$ denotes the length of the interval in which the entrant’s $b_i$ type bid. For instance, in the interval $[T_3, T_2]$, the entrant’s $b_2$ type and the incumbent (his $v$ type) mix their bids, and the
length of this interval is $d_2$.

![Diagram](image)

**Figure 1.3.** A depiction of the equilibrium bidding distributions’ supports, in which the incumbent places an atom zero, and the entrant has more than one active type.

For simplicity, abusing the notation, assume that $p_i$ is the probability that $p$ assigns to value $b_i$. If the incumbent (or no players) is placing an atom at zero, then the entrant must be exhausting his bidding probabilities first (or at the same time with the incumbent). Because the constant density of her $b_i$ type’s bidding distribution is $g_1(b_i, v) = \frac{1}{p_i v}$, the length of the interval over which she mixes her bids must be $d_i = p_i v$. Therefore, the maximum bid that they bid is $T_1 = \sum_{i=1}^{t} d_i = \sum_{i=1}^{t} p_i v = v$. As a result,

$$U_1(p) = \sum_{i=1}^{t} p_i u_1(b_i|p) < u_1(b_1|p) = b_1 - v \leq v_1 - v,$$

where the strict inequality follows from Lemma 1. The degenerate distribution that assigns probability one to the highest type in $V$ yields an expected payoff of $v_1 - v$ to the entrant.

Lemma 2 shows that if the entrant is not relatively-weak, then she wants to be the relatively strongest, i.e to have the maximum economic power. Therefore, if an optimal distribution does not make the entrant relatively-weak, then it must assign probability one to the highest value in $V$.

Now, we may focus our attention to the distributions that make the entrant relatively weak, and establish the necessary conditions that they must satisfy.

Call a type an active one if that type bids above zero with positive probability, and an inactive type if she bids zero with probability one. When we have a relatively weak player with multiple types, it is not easy to calculate the payoffs of the players because not only the joint partition that we must consider has many parts, but also depending
on the distribution choice of the entrant, she might have one or more inactive types. The latter brings even more complexities when we want to check if there is a profitable deviation since we must know which types are active. However, thanks to Lemma 3, we could eliminate those further complexities, and consider the distributions that never result in the entrant having inactive types.

**Lemma 3.** Suppose the entrant employs \( p \) as her distribution choice. Let \( b_1 > b_2 > \ldots > b_t \) be the types that are in the support of \( p \). Moreover, let \( b_{i^*} \) be the entrant’s lowest type that bids above zero with positive probability in the equilibrium of the all-pay auction stage. In this environment, a game with \( i^* < t \) is payoff equivalent to one in which the entrant employs \( \tilde{p} \), where \( \tilde{p}_i = 0 \) if \( i > i^* \), \( \tilde{p}_i = p_i \) if \( i < i^* \) and \( \tilde{p}_{i^*} = \sum_{i \geq i^*} p_i \).

**Proof.** This is intuitive. If the entrant has some types who are not bidding above zero in the equilibrium, then the incumbent must be exhausting his bidding probabilities before (or at the same time as) the entrant’s \( b_{i^*} \) type does. Therefore, the change does not affect the bidding intervals, nor does it have impact on the bidding distributions of the active types. Thus, the players obtain the same expected payoffs. \( \blacksquare \)

Due to Lemma 3, we limit our attention to the distribution that assigns positive probabilities only to active types.

Remember that the entrant pools the high and the low values in her distribution choice to increase the uncertainty that her opponent is facing when there are only two values available to her and she is relatively-weak. Given a distribution, one way to create a distribution with higher uncertainty is to consider one that second order stochastically dominates it. Since it is a mean preserving spread of the original distribution and does not affect the identity of the relatively-weak player in the all-pay auction stage, it is useful in identifying our intuition that is the entrant prefers the extreme types to an in-between type. Define the following deviation.

**Definition 2** (\( \varepsilon \)-SSD deviation). Let \( p \) be a distribution over \( V \) such that \( v_k \) is in its support and \( 1 < k < N \). For sufficiently small \( \varepsilon > 0 \), \( p_k^\varepsilon \) is an \( \varepsilon \)-SSD deviation to \( p \) for type \( v_k \) if

\[
p_k^\varepsilon(v_i) = \begin{cases} 
  p(v_i) + \varepsilon(v_k - v_N) & \text{if } i = 1 \\
  p(v_i) - \varepsilon(v_1 - v_N) & \text{if } i = k \\
  p(v_i) + \varepsilon(v_1 - v_k) & \text{if } i = N \\
  p(v_i) & \text{otherwise}
\end{cases}
\]
For some $p$ and sufficiently small $\varepsilon > 0$, consider $p_k^\varepsilon$ with $1 < k < N$. Observe that $E[p] = E[p_k^\varepsilon]$ and that for all $x \in \mathbb{R}$, we have

$$\int_{-\infty}^{x} p_k^\varepsilon(s)ds \leq \int_{-\infty}^{x} p_k^\varepsilon(s)ds.$$ 

Therefore, $p_k^\varepsilon$ is a mean preserving spread of $p$.

An $\varepsilon$-SSD deviation will be useful in the following sections as well since it is a mean preserving spread. What is special about $\varepsilon$-SSD deviation? Remember the intuition behind the Proposition 1. In the language of stochastic dominance, we could translate it into the following. A relatively-strong player prefers a first order stochastically dominating distribution, hence, chooses the one that assigns probability one to the high type, whereas, a relatively-weak player wants to have both types in the support of her distribution choice. Having the high type gives her economic power, while the low type augments this power by creating an information rent. The higher the high type, the higher the economic power. The lower the low type, the higher the information rent. A relatively-weak player prefers a mean preserving spread. An $\varepsilon$-SSD deviation is a mean preserving spread of the original distribution. Moreover, a distribution that has only the highest and/or the lowest elements in its support cannot have a mean preserving spread. So, even if there were more types available to the entrant, she would prefer to have only the extreme types in the support of her distribution choice. Following lemma formalizes this.

**Lemma 4.** If $p$ is an optimal distribution choice for the entrant and it makes the entrant relatively-weak, then there is at most two elements in its support.

The proof is Appendix B.1. The idea behind the proof is the following. Whenever a distribution has a type that is not an extreme one, we could allocate some of the probability on it to the extreme types to increase the entrant’s payoff. Therefore, unless there are only the extreme types in its support, there is always a profitable deviation from a distribution. As a result, there can only be $v_1$ and $v_N$ in the support of an optimal distribution. Since there are only two fixed types available in the equilibrium, by Proposition 1, we have the following result.

**Proposition 2.** Suppose there are two players, informed (the entrant) and uninformed (the incumbent). This is, the entrant knows her type and the type of the other player, but incumbent just knows his type. Assuming that the incumbent’s value is $v$, suppose that the entrant (informed) could pick a distribution on a set $V = \{v_1, v_2, \ldots, v_N\}$,
$v_1 > v_2 > \ldots > v_N > 0$, from where her type is drawn. The optimal distribution $p^*$ that she picks is characterized as follows:

$$p_1^* = \begin{cases} \frac{v_1}{2v} & , \text{if } \frac{v_1 + v_N}{2} < v \\ 1 & , \text{otherwise} \end{cases}$$

where $p_N^* = 1 - p_1^*$, and $p_i^* = 0$ for all $i = 2, 3, \ldots, N - 1$.

Proposition 2 tells us that there are two driving forces that a player takes into account when choosing her distribution: economic power and information rent. Higher types yield higher economic power, while lower types yield higher information rent which augments the economic power. These forces are maximized at the extreme types. Hence, the entrant’s optimal distribution choice assigns positive probabilities to at most two elements; that will always contain the supremum of $V$, and sometimes, the infimum of $V$. The entrant tries to become economically stronger by choosing the supremum of $V$ and she augments her economic power by creating information rent on it. The best way to create the information rent is by choosing the infimum of $V$ because it makes the incumbent fight with the less aggression.

It may seem strange that reducing the informed player’s private information increases her payoff. This is because it increases the asymmetry that her opponent faces in the all-pay auction stage. If the uninformed player were not to observe the informed player’s distribution choice, he would bid as if the informed player’s value is constant in the equilibrium of the all-pay auction stage; that is the expected value of the averages of all possible distributions over $V$. Having the uninformed player observe her distribution, the informed player is able to commit to choosing low values with some probability, which then generates information rents for her.

The observability of the informed player’s distribution choice is crucial in creating the asymmetry that the uniformed player is facing. Observability increases the payoff of the informed player. If he was not to observe the informed player’s distribution choice, then the informed player would deviate to the distribution that assigns probability one to the supremum of $V$. Hence, in the unique equilibrium, they would play a complete information all-pay auction.

We have established that regardless of the incumbent’s value, the entrant will pick a distribution that has at most two elements in its support; that will always contain
the supremum of $V$, and sometimes, the infimum of $V$. For the rest of the analysis we will assume that the incumbent’s value is the supremum of $V$, and analyze the entrant’s distribution choice behavior under some constraints, more specifically one with a given mean and one that is second order stochastically dominated by a given distribution.

### 1.3 Value Distribution With Given Mean

This section assumes that the incumbent’s value is the supremum of $V$ and that the entrant can pick a distribution only if its mean is some constant $\mu$. The former assumption simplifies the math to better demonstrate our results and can easily be relaxed. The latter might eventually open the doors to include the possible cost structures attached to each distribution choice by considering a specific one; that depends only on the average quality of the technology. Moreover, it helps us to illustrate that the entrant’s interest is first in the values in her distribution choice, only then in the probabilities distributed over these values. Observe that when this constant mean is one of the extrema of $V$, the all-pay auction part of the game is still a Hillman & Riley (1989) one. So, assume that it is in the interior of $V$.

There are only two modifications to the model of the previous section. The first one is that the distribution $p$ must satisfy a mean condition, that is, $E[p] = \mu$ for some $\mu \in (v_N, v_1)$. And the second one is that the incumbent’s value is the supremum of $V$, i.e., $\nu_2 = v_1$.

Observe that the entrant is always a relatively-weak player in the all-pay auction stage. Now consider a distribution $p$ over $V$ with $E[p] = \mu$. If employing $p$ makes the entrant have more than two active types in the equilibrium of the all-pay auction stage, then by Lemma 4, it cannot be optimal because there is a profitable $\varepsilon$-SSD deviation and the deviation satisfies the mean condition. For the same reason, if the entrant has only two active types and the higher active type is not $v_1$, the same deviation is a profitable one. As a result, there can be at most two active types in the support of an optimal distribution, one of which should be supremum of $V$, i.e., $v_1$.

Let $p$ be an optimal distribution, and $v_1, b_2, \ldots, b_t$ be the types in its support. Hence,

$$p(v_1)v_1 + p(b_2)b_2 + \ldots + p(b_t)b_t = \mu$$
By the discussion above, \( b_2 \) must be the lowest active type in the equilibrium of the all-pay auction stage. Therefore, the entrant’s expected payoff could be calculated as

\[
U_1(p) = p(v_1) \left( 1 - p(v_1) \right) (v_1 - b_2)
\]

Consider another distribution \( p' \) that assigns the same probability to \( v_1 \) as \( p \), however distributes all the remaining probability to the lowest possible type, meeting the first moment constraint. Let \( \bar{b} := \frac{\mu - p(v_1) v_1}{1 - p(v_1)} \) be this type. \( \bar{b} \) may not be an element in \( V \). If it is, then \( p' \) is a better distribution than \( p \). If it is not in \( V \), then one could choose the lowest type in \( V \) that is greater than \( \bar{b} \), and distribute the probability mass of \( 1 - p(v_1) \) to this type and \( v_N \) type such that the entrant still has only two active types in the equilibrium of the all-pay auction stage. However, to better depict the idea, let us assume that \( \bar{b} \) lies in \( V \).

Employing \( p' \) will yield a payoff of \( p(v_1) \left( 1 - p(v_1) \right) (v_1 - \bar{b}) \), which is profitable than choosing \( p \) as her distribution. Hence, we could conclude that for a distribution \( p \) to be optimal for the entrant, (i) \( v_1 \) must be in its support, (ii) it must have at most two active types in the all-pay auction stage, and (iii) the lowest active type, say \( \bar{b} \), must satisfy

\[
\bar{b} \geq \frac{\mu - p(v_1)}{1 - p(v_1)v_1}.
\]

Choosing such a distribution yields the entrant an expected payoff of

\[
U_1(p) = p(v_1)(1 - p(v_1))(v_1 - \bar{b})
\]

On the other hand, consider the distribution that has only \( v_1 \) and \( v_N \) in its support. There is a unique \( p^* \) such that

\[
p^* v_1 + (1 - p^*) v_N = \mu
\]

Abusing the notation, this distribution could be represented by \( p^* \). Moreover, when the entrant employs \( p^* \) her expected payoff is

\[
U_1(p^*) = p^*(1 - p^*)(v_1 - v_N).
\]

However, any other distribution that has only two active types in its support yields a
lower payoff than $p^*$ because

$$U_1(p) = p(v_1)(1 - p(v_1))(v_1 - \bar{b})$$

$$\leq p(v_1)(1 - p(v_1)) \left( v_1 - \frac{\mu - p(v_1)v_1}{1 - p(v_1)} \right)$$

$$= p(v_1)(1 - p(v_1)) \left( v_1 - \frac{p^*v_1 + (1 - p^*)v_N - p(v_1)v_1}{1 - p(v_1)} \right)$$

$$= p(v_1)(1 - p^*)(v_1 - v_N)$$

$$< p^*(1 - p^*)(v_1 - v_N)$$

$$= U_1(p^*),$$

where the last inequality is due to fact that while the distribution $p^*$ has only $v_1$ and sometimes $v_N$ in its support, $p$ has $v_1$, $\bar{b}$, and $v_N$. Therefore, if the two has the same mean, then the weight that $p^*$ assigns on $v_1$ must be higher that that $p$ does.

$$p^*v_1 + (1 - p^*)v_N = \mu = p(v_1)v_1 + p(b_2)b_2 + p(v_N)v_N > p(v_1)v_1 + (1 - p(v_1))v_1.$$

We can now state our next proposition.

**Proposition 3.** Suppose there are two players, informed (the entrant) and uninformed (the incumbent). This is, the entrant knows her type and the type of the other player, but incumbent just knows his type. Suppose that the entrant (informed) could pick a distribution on a set $V = \{v_1, v_2, \ldots, v_N\}$, with $v_1 > \ldots > v_N > 0$, from where her type is drawn with the condition that its mean must be $\mu$ for some $\mu \in (v_N, v_1)$. Moreover, suppose that the incumbent’s type is set to be $v_1$. Then, the entrant’s optimal distribution choice $p^*$ assigns positive probabilities only to the extreme types in $V$, $v_1$ and $v_N$, by properly adjusting the probabilities on them to satisfy the mean condition.

Depending on the ratio of $\mu$ to $\nu_1$, the informed player will either choose the same value distribution or she will pick the distribution that assigns probability one to $\mu$, assuming that it is in $V$. Proposition 3 tells us that the entrant’s motivation is first in the values, then in the probabilities over those values. One could follow the same procedure to show that the result does not rely on uninformed player having the supremum value. Depending on the ratio of $\mu$ to $\nu_1$, the informed player will either choose the same value distribution or she will pick the distribution that assigns probability one to $\mu$, assuming that it is in $V$. If a distribution has a mean preserving spread that is still feasible as a choice to the entrant, then it cannot be an optimal one. Next section build on this idea,
and places a stronger condition on the distribution choice of the entrant.

Proposition 3 accommodates the cost function that depends only on the average quality of the technology, as well. Since the cost of the distribution does not affect how the players bid in the all-pay auction stage, we can simply treat it as a fixed cost. Once we fixed the cost, Proposition 3 tells us that the optimal technology will only have the extreme types in its support. Assuming that the cost of choosing a technology with an average quality level of $\mu$ is $c(\mu)$, the optimal $\mu$ would simply be the one that maximizes the entrant’s following payoff:

$$W_1(\mu) = U_1(\mu) - c(\mu) = \frac{(v_1 - \mu)(\mu - v_N)}{v_1 - v_N} - c(\mu)$$

Observe that when there is no cost, the optimal $\mu$ is obtained by the distribution that is described in Proposition 2, i.e., $\mu = \frac{1}{2}v_1 + \frac{1}{2}v_N$. However, when there is a convex cost function $c(\cdot)$, the optimal $\mu$ will be lower.

1.4 Conclusion

In a research and development race, the firm that develops the higher-quality product enjoys the dominant market position. When studying such competitions, scholars have assumed that the value of a dominant market position is exogenously given. However, it may depend on firm-specific characteristics, such as production cost and marketing expertise, which could be considered as the firm’s choice. To incorporate such choices, we consider a two-player all-pay auction game in this study. In a discrete valuations environment, assuming one player’s valuation is common knowledge, we allow the other player to pick a distribution over a finite set of values $V$. Her opponent observes her choice of distribution. However, her valuation is privately drawn according to this distribution. Then the two players play an asymmetric all-pay auction.

When there is no constraint on the informed player’s distribution choice, she picks one that has at most two elements in its support; that will always contain the supremum of $V$, and sometimes, the infimum of $V$. Even though having the highest value gives her an economic power, she prefers to augment it by creating an information rent, which she does so by sometimes drawing the lowest value if she is facing a strong opponent.

We then assume that the uninformed player’s value is fixed at the supremum of $V,$
and allow the informed player pick a distribution only if its mean equals to some $\mu$. The same intuition goes through, and she picks the distribution that assigns positive probabilities only to the extreme values in $V$. In other words, the informed player is first interested in the values, and then the probabilities assigned to those values. We will exploit this observation even further in the next chapter.
Chapter 2
Learning Your Value & Strategic Interaction

The first part of this chapter exploits the observation that the informed player is first interested in the support of her distribution choice, and then the probabilities assigned to the values in that support to answer a completely different question: Would the informed player choose to learn her own value distribution if she was able observe the realization of her value. The second part discusses what would happen if the two players were simultaneously choosing their value distributions.

2.1 Introduction

Let us revisit the hiring a CEO example. However, this time assume that the entrant does not pick her CEO; she already has one. Even though she knows her CEO’s quality level, she cannot observe the value that her CEO creates. She can invest in learning this value by picking an experiment, however, the experiment that she carries out (but not its outcome) is observed by the incumbent.

This chapter considers all such experiments and finds how much will the entrant want to learn about her own value distribution. Consider the following example as an illustration. Suppose that the possible valuations of winning the all-pay auction are low (L), medium (M), and high (H), and that the incumbent firm’s value is high (H) and known by the entrant. Suppose that prior distribution of the CEO’s quality is $p$ with the medium value being its average. Hence, if the entrant chooses to learn nothing about her CEO’s quality, she will bid as if her valuation was medium. Therefore, the incumbent and entrant will end up playing a complete information all-pay auction where their valuations
are high and medium, respectively. On the other hand, if the entrant chooses to learn perfectly, she will know her valuation for sure. The incumbent will observe how much of her valuation the entrant wants to learn, and believes that the entrant’s valuation will be drawn according to \( p \). As a result, they will play an incomplete information all-pay auction where the incumbent’s value is high and he believes that entrant’s value is drawn according to \( p \), while the entrant not only knows her own value, but also her opponent’s value, (H), and belief, \( p \). In case of a partial learning, the two will end up playing a standard asymmetric all-pay auction, where the entrant’s value is drawn according to some posterior distribution that is Bayes-plausible with his observation.

### 2.2 Model

There are two risk-neutral players, indexed by \( i \in \{1, 2\} \), who compete in an all-pay auction setting for an indivisible prize. Player 1 (entrant or informed player) values the prize at \( \nu_1 \), which is drawn from the set of values \( V = \{v_1, v_2, \ldots, v_N\} \) with \( v_1 > v_2 > \ldots > v_N > 0 \) according to the probability distribution \( p = (p_1, p_2, \ldots, p_N) \), where \( p_i = \Pr(\nu_1 = v_i) \). Player 2 (incumbent or uninformed player) values the prize at \( v_1 \). It is natural to assume that the incumbent has already fine-tuned his production technology and marketing expertise. However, this assumption is only to simplify the calculations and better illustrate our results, and could easily be relaxed. Both players only know the incumbent’s valuation and the entrant’s value distribution. More specifically, the entrant does not know her own valuation. However, she can acquire additional information about her own valuation by picking an experiment from a family of distributions \( \{F(s, v_i)\} \) over \( S \times V \), where \( S = \{s_1, s_2, \ldots, s_M\} \) is a finite set of signals. In the absence of such an information acquisition option, the entrant would bid in the all-pay auction as if her valuation was \( \bar{v} = \mathbb{E}[v_i] = \sum_i p_i v_i \). As a result, she would randomize her bids uniformly over \( [0, \bar{v}] \) with probability \( \frac{\bar{v}}{v} \) and bid zero with the remaining probability \( 1 - \frac{\bar{v}}{v} \), resulting in her obtaining a zero equilibrium payoff.

Any experiment that the entrant can adopt induces a posterior probability distribution \( \hat{p} \) over \( [v_1, v_N] \) with at most \( M \) values in its support. Because \( s \) is an unbiased signal for valuation in the sub-game, we consider the signals as valuations. Thus, the sub-game that starts upon the entrant’s experiment choice of \( F \) is identical to an asymmetric private value all-pay auction with two players where the incumbent values the object at \( v_1 \), while the entrant’s valuation is drawn from \( [v_1, v_N] \) according to \( \hat{p} \). In order for
a distribution \( \hat{p} \) to be a signal distribution it must stochastically dominate the prior distribution \( p = (p_1, p_2, \ldots, p_N) \) in the second order sense, or equivalently, \( p \) should be a mean preserving spread of \( \hat{p} \).

Following are some examples of the possible experiment choices. To simplify, we assume that \( V = \{v_h, v_\ell\} \), where \( v_h > v_\ell \), and denote by \( p = (p_h, p_\ell) \) the entrant’s prior value distribution.

**Example 1** (Uninformative Experiment). Suppose the entrant picks a posterior \( \hat{p} \) that assigns probability one to \( \mathbb{E}[p] \). Therefore, the second stage is identical to a complete information all-pay auction where the entrant values the object at \( \mathbb{E}[p] \) while the incumbent’s value is \( v_h \). In the unique equilibrium, the entrant obtains a zero payoff, while the incumbent expects a payoff of \( v_h - \mathbb{E}[p] \).

**Example 2** (Perfectly Informative Experiment). Suppose the entrant picks a posterior \( \hat{p} \) that is the same as the prior, i.e., \( \hat{p} = p \). Choosing such an experiment will let her perfectly learn her value. So, with probability \( p_h \), her value is \( v_h \), while with the remaining probability \( p_\ell = (1 - p_h) \) she draws the value \( v_\ell \). Since the incumbent can observe his opponent’s experiment choice, he will believe that the entrant will know her valuation perfectly and that with probability \( p_h \) she will have the high value and with the remaining probability \( (1 - p_h) \) she will have the low value. As a result, once the entrant picks the perfectly informative experiment, they play a standard private value asymmetric all-pay auction in the second stage. In the unique equilibrium of the all-pay auction, the entrant obtains a payoff of \( U_1 = p_h(1 - p_h)(v_h - v_\ell) \), whereas the incumbent’s payoff will be \( U_2 = (1 - p)(v_h - v_\ell) \).

A perfectly informative experiment creates the highest information asymmetry, while an uninformative one leaves the entrant with the same amount of information as the incumbent. As seen above, the entrant could generate a positive payoff by adopting the perfectly informative experiment. To see if the perfectly informative experiment, hence the maximum level of information asymmetry, is what the entrant prefers, we will first assume that there are only two values in the support of the prior and that only two signals are available to the entrant. However, since it fails to capture all the details of the distributional restrictions on the outcome of the experiments, we will generalize it to many-values and many-signals setting.
2.3 A Simple Case: Two-Values and Two-Signals Setting

This section assumes the simplest possible environment, that is when there are two values in $V$ and two signals available to the entrant. The incumbent’s value is $v_h$, and the entrant’s value is $v_h$ with probability $p$ and $v_\ell$ with the remaining probability. The entrant can pick a posterior distribution over $[v_\ell, v_h]$ that has at most two elements in its support because she only has two signals. An uninformative experiment corresponds to a posterior that assigns probability one to $\mu = pv_h + (1-p)v_\ell$, and results in player 1 obtaining zero payoff. On the other hand, a perfectly informative one is equivalent to the prior distribution and it yields $p(1-p)(v_h - v_\ell) > 0$ payoff to the entrant.

Assume that the entrant picks a partially informative experiments. Her choice of experiment results in a posterior that assigns probability $q$ to some valuation $\bar{v}$ and $(1-q)$ to some $\underline{v}$, where $v_h \geq \bar{v} \geq \underline{v} \geq v_\ell$. Bayes’ plausibility requires that the prior should be a mean preserving spread of the posterior. When we have only two elements in the support of the prior, it is equivalent to having the same mean, hence, we have the following.

$$q\bar{v} + (1-q)\underline{v} = \mu$$

Observe that even if we had more than two signals, the only requirement on the posterior would be that it should have the same mean as the prior. By Proposition 3, we have already established that the entrant should prefer a distribution that has only the extreme values in the support of her posterior distribution. Therefore, the entrant must choose the perfectly informative experiment. However, when there are two values in $V$, Bayes’ plausibility is nothing but a mean constraint on the posterior. To better understand its effects, we must have more than two values in $V$. Following section generalizes the set of values to finite ones.

2.4 Generalizing to Many-Values and Many-Signals Setting

We have seen that when there are only two possible values in her prior, the entrant prefers a perfectly informative experiment. This is mainly because a mean preserving spread constraint does not add anything to the mean constraint when there are only two
values in $V$. To better analyze its effects and see if the intuition behind Proposition 3 is still valid, we need to assume that there are more than two values in $V$. This section analyzes the entrant’s experiment choice when her prior has more than two values in its support.

Assume that $|V| > 2$ and $S$ is a finite large set. We have already established that, a weak player prefers to have a distribution that splits the probability mass on the interior types to the extreme types. Moreover, the entrant can choose any posterior distribution as long as it is a mean preserving spread of the prior. Based on our above intuition, one would expect the entrant to choose the prior as her posterior; that is she would choose the fully informative experiment. This would be true if the entrant had no inactive types when her posterior is her prior. However, when she ends up having inactive types, we will be no longer able to revoke the payoff-equivalence that Lemma 3 establishes. The $\varepsilon$-SSD deviation is a useful one only when the entrant is able to split the mass on an active interior type to the extreme active types.

The value of the entrant’s lowest active type is of critical importance. The following lemma tells us that if a relatively-weak entrant can lower the value of her lowest active type, she would be better off.

**Lemma 5.** Suppose the entrant’s posterior is $p$. Let $b_1 > b_2 > \ldots > b_t$ be the types that are in the support of $p$. Moreover, let $b_i^\ast$ with $i^\ast > t$ be the entrant’s lowest type that bids above zero with positive probability in the equilibrium of the all-pay auction stage. The entrant would be better off if she could decrease $b_i^\ast$ without affecting the higher types and the probability mass assigned to them.

The proof is intuitive. The lowest active type does not bring any direct payoff to the entrant. However, the lower it is the shorter the last interval, in which the incumbent mixes her remaining bidding probabilities. Therefore, it lowers the payment that the entrant’s other active types will make when they bid their highest bids in their best response sets. Moreover, this will not affect their winning probabilities since.

Suppose there is an inactive type under some optimal posterior $\hat{p}$. Let $v_{\hat{p}}$ be the lowest active type and $v_0$ be the highest type in $\hat{p}$’s support that is smaller than $v_{\hat{p}}$.  

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Define $v^E$ as follows:

$$
v^E = \frac{\hat{p}(v_0)v_0 + \hat{p}(v_{\hat{p}})v_{\hat{p}}}{\hat{p}(v_0) + \hat{p}(v_{\hat{p}})}
$$

Consider the distribution $\hat{p}'$ that assigns $\hat{p}(v_0) + \hat{p}(v_{G})$ probability to type $v^E$, zero probabilities to the types $v_0$ and $v_{G}$, and the same probability to the other types in the support of $\hat{p}$ as $\hat{p}$. Observe that $\hat{p}$ is a mean preserving spread of $\hat{p}'$. By Bayes’ plausibility, we know that $\hat{p}$ is a mean preserving spread of $\hat{p}'$. Therefore, $\hat{p}$ is a mean preserving spread of $\hat{p}'$, so it is also a feasible posterior distribution. By Lemma 5, we know that without changing anything above the lowest active type, if we can lower the value of the lowest active type, we could make the entrant better off. So, $\hat{p}'$ yields higher payoff than than $\hat{p}$. Therefore, if the entrant has some inactive types under some posterior distribution, then it cannot be an optimal posterior distribution. A perfectly informative experiment, then, cannot be optimal if the entrant has some inactive types in the all-pay auction stage. Following proposition builds on this idea and characterizes the optimal posterior distribution.

**Proposition 4.** Suppose when the entrant’s posterior is her prior, $p$, her lowest active type in the all-pay auction stage is $v_i^*$. Then the entrant will pick an experiment that results in a posterior which mimics the prior above $v_i^*$, while below $v_i^*$, it collects all the probability mass that the prior assigns on $v_i^*, v_i^*+1, \ldots, v_N$ to only one type, $v_{\hat{p}}$, which is defined as

$$
v_{\hat{p}} = \frac{p(v_{i^*})v_{i^*} + p(v_{i^*+1})v_{i^*+1} + \ldots + p(v_N)v_N}{p(v_{i^*}) + p(v_{i^*+1})v_{i^*+1} + \ldots + p(v_N)}
$$

Proposition 4 tells us that the fully informative experiment is what the entrant prefers as long as it does not make her have inactive types in the all-pay auction stage. Moreover, when it does, she picks a posterior distribution that mostly mimics it. The only uncertainty that the incumbent actually cares about is that which is among the active types. If the incumbent never faces some types, she cares about neither the probability masses on them nor their existence. As a result, the entrant tries to create information asymmetry only for the active types. That is why she tries to decrease her lowest active type’s value. Observe that when we only consider the posterior distributions over the active types in their supports, the distribution that is defined in Proposition 4 has the highest variance.

The entrant seeks to create information rent when she is facing a relatively-strong
opponent. One could show that the above result does not rely on uninformed player having the highest possible value in $V$. It does mainly rely on the fact that the entrant is always a relatively-weak player against the incumbent. Considering the valuations for the incumbent that will make him a relatively-weak player is not interesting because it will motivate the entrant to become a relatively-strong player, which she does so by adopting a distribution that stochastically dominates all the feasible distributions in the first order sense.

### 2.5 Conclusion

In this chapter, we assumed that the informed player’s value is drawn from a given prior distribution. However, she cannot observe her realized value. Yet, she could acquire additional information about her own valuation by picking a learning experiment. Any experiment that she picks results in a posterior. In such environment, choosing a learning experiment and a posterior is equivalent. Due to Bayes’ plausibility condition, the informed player could pick a posterior distribution only if it stochastically dominates the prior in second order sense. Moreover, we show that she bids as if her private value is drawn from $V$ according to this posterior in the equilibrium of the all-pay auction game.

We show that her motivation to maximize the uncertainty that the uninformed player faces in the all-pay auction stage is still valid. Because the uninformed player is only interested in the types against whom he is bidding in the equilibrium of the all-pay auction, the informed player maximizes the uncertainty among those types. As a result, she picks the perfectly informative experiment only if it does not make her have any inactive types. Otherwise, she picks an experiment which results in a posterior that mostly mimics the prior; it assigns the same probabilities to the types that never bid zero, and all the remaining probability to the lowest possible type to meet the distributional constraint.

In the remainder of this chapter, we will consider the reduced game an allow the incumbent adopt his value distribution as well, and discuss some of the necessary conditions that the equilibrium distribution profile must meet.
2.6 Discussion: Strategic Interaction

We have so far assumed that the incumbent’s value is known, whether it is in $V$ or not, and analyzed the entrant’s response under different setups. The natural extension to consider is allowing the incumbent to endogenize his own value distribution, simultaneously with the entrant, and finding the equilibrium distribution profiles of this version of the reduced game. The following section discusses what would happen if the two firms were choosing distributions over the same set $V$. Let us first assume that there are only high and low values in $V$, and find the reduced game’s equilibrium in pure strategies.

2.6.1 Strategic Interaction With Two Values

When there are only two values in $V$, i.e., $V = \{v_h, v_\ell\}$, constructing the equilibrium bidding distributions in the all-pay auction stage is not complicated. Let $\mathbf{p} = (p_h, p_\ell)$ be player 1’s distribution choice, where $p_h$ is the probability mass that $\mathbf{p}$ assigns to high value, $v_h$. Similarly, let $\mathbf{q} = (q_h, q_\ell)$ be player 2’s distribution choice, where $q_h$ is the probability mass that $\mathbf{q}$ assigns to high value, $v_h$. Figure 2.1 depicts the equilibrium bidding distributions in the all-pay auction stage when we employ $\mathbf{p}$’s algorithm with the distribution choice profile of $(\mathbf{p}, \mathbf{q})$, assuming that $p_h \leq q_h$.

Now let’s try to understand how the algorithm constructs Figure 2.1. Consider the joint partition of the interval in which players mix their bids. The high types of the two players randomize their bids in the top interval. The algorithm constructs the equilibrium of this stage starting from the bidding distributions of the high types of players in and the length of the top interval. The densities of their bidding distributions are as follows:

$$g_1(v_h, v_h) = \frac{1}{p_h v_h} \quad \text{and} \quad g_2(v_h, v_h) = \frac{1}{q_h v_h}.$$

Without loss of generality, suppose $p_h \leq q_h$. Therefore, the length of the top interval in which $v_h$ type of player 1 and the same type of player 2 mix their bids, $L_1$, is

$$L_1 = \min \{p_h v_h, q_h v_h\} = p_h v_h,$$

where $p_h v_h$ and $q_h v_h$ are the necessary lengths of the top interval for player 1 and player 2 to exhaust their bidding probabilities, respectively. Since player 1 is the one who exhausts his bidding probability first, in the penultimate interval, his $v_\ell$ type will be bidding against $v_h$ type of player 2. And, the constant densities of their bidding
The equilibrium bidding distributions in the all-pay auction stage of the game when \((p, q)\) with \(p_h \leq q_h\) is employed in the first stage. Again, the two sides of the dashed line positioned below the horizontal bid axes represent the types of the players bidding against each other.

The required length of this interval for player 1 to exhaust his bidding probability is simply \((1 - p_h)v_h\) since \(\frac{1}{g_1(v_L, v_H)} = (1 - p_h)v_h\). For player 2 to exhaust his bidding probability in this interval, we must take into account the fact that she has already exhausted some of her bidding probability in the top interval. Therefore, she needs an interval which has a length of \((q_h - p_h)v_L\) to exhaust her remaining bidding probability of \(1 - L_1g_2(v_H, v_h) = 1 - \frac{p_h}{q_h}\). As a result, the length of the penultimate interval in which \(v_L\) type of player 1 and \(v_H\) type of player 2 bid, \(L_2\), is

\[
L_2 = \min \{ (1 - p_h)v_h, (q_h - p_h)v_L \} = (q_h - p_h)v_L.
\]

Therefore, it is player 2 who is now switching to her low type. The last stage of the algorithm finds the interval in which their low types randomize their bids against each
other. Their bidding distributions have the following constant densities.

\[
g_1(v, v) = \frac{1}{(1 - p_h)v} \quad \text{and} \quad g_2(v, v) = \frac{1}{(1 - q_h)v}
\]

Similarly, if \( v \) type of player 2 is the one who exhausts her bidding probability in the last interval, its length should be \( \frac{1 - L_2 g_1(v, v) v}{g_1(v, v)} = (1 - p) v - (q - p) \frac{v^2}{v} \). As a result, the length of the last interval could be calculated as follows:

\[
L_3 = \min \left\{ (1 - p) v - (q - p) \frac{v^2}{v}, (1 - p) v \right\} = (1 - p) v
\]

Lastly, since player 2 exhausted her bidding probabilities of all her types, \( v \) type of player 1 should put his remaining bidding probability as a mass at zero. Let \( a \) be the size of this probability mass. It can be calculated as follows:

\[
a = 1 - L_3 g_1(v, v) - L_2 g_1(v, v) = \frac{(q - p)}{(1 - p)} \left(1 - \frac{v}{v} \right)
\]

Therefore, when \((p, q)\) is played in the first stage, player 1’s expected payoff will be the following piece-wise function.

\[
U_1(p, q) = \begin{cases} 
  p(1 - p)(v - v) & \text{if } p \leq q \\
  [p(1 - q) + (1 - p)(p - q) \frac{v^2}{v}] (v - v) & \text{if } p > q 
\end{cases} 
\]

Given (2.1), the optimal probability that player 1 should assign to the high type as a function of the probability that her opponent assigns to the high type is given by the following equation.

\[
p^*_h(q) = \begin{cases} 
  1 & \text{if } q < \frac{3}{4} \\
  \frac{1}{2} & \text{if } q = \frac{3}{4} \\
  \frac{1}{2} & \text{if } q > \frac{3}{4} 
\end{cases}
\]

Hence, the profile in which one player picks \( v \) type with probability one while the other player assigns probability \( \frac{1}{2} \) to \( v \) and \( v \) each is the unique (up to symmetry) Nash equilibrium of reduced game. We had already seen that one player was best responding to the other one in the profile described above. Now, we show that it is indeed the unique equilibrium of the reduced game. So, while one player chooses to have full economic
power whereas the other player chooses to have some economic power but augments it by creating an information rent on it.

It is not easy to conduct a similar equilibrium analysis when $V$ can be any finite set because equilibrium construction of the all-pay auction is algorithmic. Following section discusses some of the necessary conditions that a profile must have for it to be an equilibrium of the reduced game.

### 2.6.2 Strategic Interaction With Many Values

This section aims to generalize the findings in two-type setting to many-type setting. Let the set of values be $V = \{v_1, v_2, \ldots, v_N\}$ with $v_1 > v_2 > \ldots > v_N > 0$, and $\mathbf{p} = (p_1, p_2, \ldots, p_N)$ be player 1’s distribution choice, where $p_j$ is the probability mass $\mathbf{p}$ assigns to the value $v_j$ for $j = 1, 2, \ldots, N$, and player 2’s distribution choice $\mathbf{q} = (q_1, q_2, \ldots, q_N)$ is defined in a similar fashion. Fixing an equilibrium distribution profile $(\mathbf{p}, \mathbf{q})$, we will establish some of the necessary conditions that they need to meet. Then we will show that a distribution profile in which one player assigns probability one to the $v_1$ type while the other player assigns probability half to $v_1$ and $v_N$ types each constitutes an equilibrium of the reduced game.

#### 2.6.2.1 Some Necessary Conditions for Equilibrium

One complexity that we face when we check if a deviation is profitable is due to the very nature of Siegel (2014a)’s algorithm. It starts the equilibrium construction from the highest bid without even knowing the highest bid. Only when it stops, it learns the highest bid as well as the end points of the intervals in the joint partition. Therefore, for a pair of distributions employed by the players, we will not able to know if a player has inactive types, when the players switch their types, or which types bid against which types unless we numerically feed the distributions into the algorithm and construct the equilibrium. Thankfully, Lemma 3 is still useful to consider only the distribution profiles in which no player has an inactive type in the equilibrium of the all-pay auction stage. However, even then, a small deviation from a distribution profile might result in huge changes in the equilibrium bidding distributions. As a result, it will be almost impossible to compare the payoffs before and after a local deviation. To partially overcome this problem, we introduce an epsilon-delta deviation because it results in changes that we could track more easily.
Definition 3 (Epsilon-delta deviation). Let \( p \) be a distribution over \( V \), with support \( \{b_1, b_2, \ldots, b_t\} \), where \( b_1 > b_2 > \ldots > b_t > 0 \). For sufficiently small \( \epsilon > 0 \) and \( m < t - 1 \), the distribution \( p^{\epsilon,\delta}_m \) is called an epsilon-delta deviation to \( p \) for type \( b_m \) if

\[
p^{\epsilon,\delta}_m(v_i) = \begin{cases} 
  p(v_i) + \epsilon, & \text{if } i = m \\
  p(v_i) - \delta, & \text{if } i = m + 1 \\
  p(v_i) - \gamma, & \text{if } i = t \\
  p(v_i), & \text{otherwise}
\end{cases}
\]

where \( \delta = \frac{\epsilon}{b_m + 1} \) and \( \gamma = \epsilon - \delta \).

What makes an epsilon-delta deviation special is the following. When we want to consider a deviation from a distribution choice of a player, we will not be able to follow the corresponding changes in the equilibrium bidding functions of the players unless the deviation only affects the lowest types in its support. However, such deviations are the first order stochastically dominated ones. To follow the intuition behind Proposition 1, we consider the distributions that are either first order stochastically dominating deviations, which make the player economically stronger, or second order stochastically dominated ones, which increase the uncertainty that her opponent faces. Observe that an epsilon-delta deviation is a first order stochastically dominating one. Moreover, by applying an epsilon-delta deviation to \( p \) for some type in its support, say \( b_m \), we can keep track of the changes in the conditional payoffs of not only the types that are weakly lower than \( b_m \) but also those that are greater than \( b_m \). This is mainly because both players exhaust the same amount of probability over the two intervals that shares the lowest bid that \( b_m \) type has in her best response set before and after the deviation. Following lemma formalizes this result.

Lemma 6. Let \( p \) and \( q \) be distributions over \( V \), with supports \( \{b_1, b_2, \ldots, b_t\} \), where \( b_1 > b_2 > \ldots > b_t > 0 \), and \( \{a_1, a_2, \ldots, a_z\} \), where \( a_1 > a_2 > \ldots > a_z > 0 \), respectively. Suppose player 1 employs \( p \) in the first stage of the game while player 2 picks \( q \) as his distribution choice. Moreover, suppose that \( b_m \) type exhausts her bidding probability when she bids against \( a_k \) type. Let \( p^{\epsilon,\delta}_m \) be an epsilon-delta deviation to \( p \) for \( b_m \) type. Instead of \( p \), when we feed in Siegel (2014a)'s algorithm with \( p^{\epsilon,\delta}_m \) to construct the equilibrium bidding distributions of the all-pay auction stage, until we reach the construction of the last interval, the only change in the lengths of the intervals over which the players’ various types bid will be in the two, which share the bid in \( b_m \) and \( b_{m+1} \) types’ best response sets. Moreover, the total length of these two intervals will increase by \( \gamma a_k \), and the last interval
will shrink by $\gamma a_z$ if player 1 is not relatively-strong player, otherwise, it will not change.

The proof is given in Appendix C.1. While the identity of the relatively-weak player matters to determine whether the length of the last interval changes, the backbone of the lemma is the ratio of $\varepsilon$ to $\delta$, which is $-\frac{b_m}{b_{m+1}}$. Going over the proof, one would see that even if the signs of the $\varepsilon$ and $\delta$ were to be replaced, we would have a similar result. The only difference would be that the total length of those two intervals in the middle would decrease by $\gamma a_k$. Moreover, if player 1 is the relatively weak player, then the last interval would expand by $\gamma a_z$ instead of shrinking. Hence, to obtain some local necessary conditions, we will use the following deviation together with an epsilon-delta deviation.

**Definition 4 (Reverse-epsilon-delta deviation).** Let $\mathbf{p}$ be a distribution over $V$, with support $\{b_1, b_2, \ldots, b_t\}$, where $b_1 > b_2 > \ldots > b_t > 0$. For sufficiently small $\varepsilon > 0$ and $m < t - 1$, the distribution $\mathbf{p}^{-\varepsilon, \delta}_m$ is called a reverse-epsilon-delta deviation to $\mathbf{p}$ for type $b_m$ if

$$p^{-\varepsilon, \delta}_m(v_i) = \begin{cases} p(v_i) - \varepsilon, & \text{if } i = m \\ p(v_i) + \delta, & \text{if } i = m + 1 \\ p(v_i) + \gamma, & \text{if } i = t \\ p(v_i), & \text{otherwise} \end{cases}$$

where $\delta = \varepsilon \frac{b_{m+1}}{b_m}$ and $\gamma = \varepsilon - \delta$.

Considering epsilon-delta and reverse-epsilon-delta deviations to a distribution choice of a player, we obtain some of the necessary conditions that a best response distribution must have. The first such necessary condition puts a cap on the number of different types against which a type will bid in the equilibrium of the all-pay auction.

**Lemma 7.** No players’ three types bid against the other player’s one type in any equilibrium.

See Appendix C.2 for the proof of Lemma 7. The idea is showing that if an epsilon-delta deviation and a reverse epsilon delta deviation for the highest such type are not profitable, then there is a profitable reverse-epsilon-delta deviation for the second such highest type.

Lemmas 8 and 9 show that both players will have the supremum value in the support of their distribution choice, while a relatively-weak player’s lowest active type will be the infimum value.
Lemma 8. If a player is not relatively-strong and she has more than one active type, then, her lowest active type must be the infimum of $V$.

Proof. The proof is pretty obvious. If a player is not relatively-strong, then her opponent must be exhausting his bidding probabilities first (or at the same time in the case there is no relatively-strong player). Moreover, the density of his bidding distribution is inversely related to the magnitude of her lowest type, and her lowest type expects a zero conditional payoff. Therefore, lowering her lowest type, she would decrease the length of the last interval, lowering the payments that her higher types make when they bid the highest bid in their best response set without changing the winning probability. This change, will not affect her lowest type’s conditional payoff since she was already getting zero. As a result, her overall expected payoff will increase. ■

Lemma 9. The supremum of $V$ must be in the support of a player’s distribution choice.

Proof. We will sketch the dynamics of the proof below. For a thorough proof, one could consider the Equation C.4, which is used while proving Lemma 7. Consider an epsilon-delta deviation to the $v_1$ type. Observe that we defined epsilon-delta deviation only to the type in the support of a distribution. We could do this by carrying the $\varepsilon$ probability weight from $b_1$ to $v_1$. So, here we have $\delta = \frac{\varepsilon b_1}{v_1}$. Due to Lemma 6, there will be no change after the top interval. Consider the highest bid in $b_i$’s best response sets for $i > 1$. The deviation does not change the winning probability, while it either decreases or does not affect the level of it. Therefore, no $b_i$ with $i > 1$ is worse off after the deviation. To calculate the change in the conditional payoff of $b_1$, consider the lowest bid in her best response set. The argument above still goes through for this bid. Hence, the highest type is not worse off either. Since higher types get higher payoff and a total of $\varepsilon$ probability mass was carried from the lowest and the highest type to $v_1$, which is higher than both, the deviation is a profitable one for sufficiently small $\varepsilon > 0$. ■

\[ \Delta U_1 = Ia_z - (p_1 + \ldots + p_{m-1})a_k - u_1(b_t|p) + u_1(b_m|p) + b_{m-1}Pr(win|T_m) - \varepsilon a_k. \]

Modifying the variables for this statement, we have

\[ \Delta U_1 = Ia_z - a_1 - u_1(b_1|p) + u_1(b_1|p) + v_1Pr(win|T) - \varepsilon a_k, \]

which is positive for sufficiently small $\varepsilon > 0$ since $v_1 \geq a_1$ and $u_1(b_1|p) > u_1(b_t|p)$.\[ \frac{\Delta U_1}{\gamma} = \frac{a_z - (p_1 + \ldots + p_{m-1})a_k - u_1(b_t|p) + u_1(b_m|p) + b_{m-1}Pr(win|T_m) - \varepsilon a_k}{\gamma}. \]
Lemma 10. If a player is not relatively-weak, then her lowest type should bid against her opponent’s two lowest types.

The proof is given in Appendix C.3

Corollary 1. If no players bid zero with positive probability in the equilibrium of the all-pay auction, then the distribution profile cannot be an equilibrium of the reduced game.

Proof. Due to Lemma 10, both players cannot be not-relatively-weak at the same time. ■

Lemma 7 was putting a cap on the number of different types that a type will face in the equilibrium. Following lemma will find the minimum such number for the highest type of a relatively-strong player. Lemma 11 is extendable to the other types as well.

Lemma 11. If a player is relatively-strong, then her highest type should bid against her opponent’s two highest types.

The proof is given in Appendix C.4. To summarize our findings, for a distribution profile \((p,q)\) to constitute an equilibrium of the reduced game (i) they must include the supremum of \(V\) in their supports, while at least one of them must also have the infimum in its support. (ii) One of the players must be relatively-weak, while the other one is relatively-strong in the following all-pay auction stage. Moreover, (iii) in the equilibrium of the all-pay auction game, a player’s types won’t bid against her opponent’s three different types. Lastly, (iv) the highest type of a relatively-strong player will bid against the highest two types of her opponent. One could show that this result could easily be extended to all the types of both players except the extreme two types of the relatively-weak one; that is other than the \(v_1\) and \(v_N\) type of the relatively-weak player, all the types will bid against exactly two types of their opponent’s. Even though (i-iv) is not enough to find all the equilibria of the reduced game, they could be used to find an equilibrium.

2.6.3 An Equilibrium

This section will show that the distribution profile we studied above constitutes an equilibrium of the reduced game even when both players are allowed to pick a value distribution. For this reason, consider the distribution profile \((\bar{p}, \bar{q})\), where \(\bar{p}\) assigns probability half to \(v_1\) and \(v_N\) types each, while \(\bar{q}\) assigns probability one to the \(v_1\) type. Proposition 1 proves that \(\bar{p}\) is a best response for player 1 when player two employs \(\bar{q}\) in
the reduced game. To show that \((\bar{p}, \bar{q})\) is an equilibrium of the reduced game, we are left to show that \(\bar{q}\) is a best response to \(\bar{p}\), too.

Let \(q \neq \bar{q}\) be a best response to \(\bar{p}\) for player 2. Consider the equilibrium of the all-pay auction stage when \((\bar{p}, q)\) is the distribution profile. If player 2 is relatively-strong, then by Lemmas 11 and 7, there cannot be more than two types in the support of \(q\). Moreover, by Lemma 9, the high type has to be \(v_1\). Among such distributions, \(\bar{q}\) yields the highest payoff to player 2. ² So, for a contradiction, suppose player 2 is not relatively-strong. For this to be the case we must have \(q_1 \geq \frac{1}{2}\), otherwise player 2 will be a relatively-strong player. By Lemmas 8 - 10, \(v_1\) and \(v_N\) will be in the support of \(q\). If there is no other types in its support, by Equation 2.1, her payoff will be

\[
U_2(\bar{p}, q) = q_1(1 - q_1)(v_1 - v_N) < \frac{1}{2}(v_1 - v_N) = U_2((\bar{p}, \bar{q})),
\]

(2.2)

where \(\frac{1}{2}(v_1 - v_N)\) is the payoff that player 2 expects when he employs \(\bar{q}\) as his distribution choice, that is \(U_2(\bar{p}, \bar{q})\). If she has more than two types in her distribution’s support, consider an epsilon-delta deviation to \(v_1\) type. Similar to Equation appendix equation, the payoff from this deviation satisfies the following equality.

\[
\frac{1}{\gamma} \Delta U_2 = \frac{1}{\gamma} \left( U_2((\bar{p}, q_{i}^{-\epsilon, \delta}) - U_2((\bar{p}, q))
\right.
\]

\[=
-(q_1 + \ldots + q_1) v_1 - u_2(b_t|q) + u_2(b_m|q) + v_1 Pr(win|T_1) - \varepsilon v_1
\]

\[=-v_1 - u_2(b_t|q) + u_2(b_m|q) + v_1(1 - q_1) - \varepsilon v_1
\]

\[\overset{>}=(1 - 2q_1)v_1 - \varepsilon v_1,
\]

which is positive for sufficiently small \(\varepsilon > 0\) since \(q_1 \leq \frac{1}{2}\). So, the deviation is a profitable one.

²When player 2 is relatively-strong, if \(|\text{Supp}\{q\}| = 2\), then by Siegel (2014a) algorithm,

\[
U_2(\bar{p}, q) < u_2(v_1) = \frac{1}{2}(v_1 - v_N) = U_2((\bar{p}, \bar{q}))
\]

If \(|\text{Supp}\{q\}| = 1\), then again, \(\bar{q}\) is the best response for player 2.
2.6.4 Concluding Remarks

The natural extension of our analysis in Chapter 1 is allowing both players to pick their value distributions in the reduced game and analyzing their equilibrium value distribution choices. This section partly sheds light on this question.

We show that both players include the highest valuation in the support of their distribution choices to maximize their economic powers, while a relatively weak one also includes the lowest valuation to create information rent on her economic power.

Allowing both players to pick their value distributions in the reduced game, we find that the distribution profile we studied above is indeed an equilibrium. A further analysis might be conducted to check whether it is the unique equilibrium. Even though we do not analytically prove it, we were not able to find any other equilibrium through an extensive algorithmic search. However, this study omits such work.
Chapter 3  Disclosure Policies Under Endogeneous Value Distributions

3.1 Introduction

All-pay auctions have been commonly used in various economic problems where the players exert costly and irrecoverable efforts to win a prize such as rent-seeking, lobbying, R & D, and patent races as well as sports competitions. In such competitions, we could call players weak or strong based on their marginal cost of exerting the same amount of effort or valuations for the prize. For instance, considering a basketball game, if one team has an injured player, it is more costly to exert the same level of effort as when there are no injured players in the team. As a result, we could call a team with an injured player “weak”, whereas one without an injured player “strong” since it will be more challenging to win a game with an injured player in the squad. Technically speaking, the marginal cost of exerting effort is higher for a team when there is an injured player in the squad. In an environment where whether the teams are strong or weak is known by both teams, Hillman & Riley (1989) shows that when a basketball team is facing a strong competitor in a game, she expects a zero payoff irrespective of her strength.

The assumption that the players know whether their opponents are weak or strong is crucial for Hillman & Riley’s result. However, most of the time, the players do not know if they are facing a strong opponent. For instance, a team is not required to reveal whether they have an injured player in their squad before an NBA game. Taking a step away from the complete information environment and adding asymmetric information complicates the analysis significantly. Even though there is no analytical solution to the all-pay auctions where the players do not know their opponents’ types, yet, Siegel (2014a)
constructs an algorithm that finds the unique solution to asymmetric all-pay auctions where the signal structure is constrained to be discrete, but valuations are allowed to be interdependent.

To better understand the effect of the presence of asymmetric information, consider a research and development race between two firms where the output can be of an either high or low-quality product. A high-quality product yields more payoff than a low-quality one, thus, a higher valuation of winning the R&D competition. However, to have a high-quality product at the end of the R & D process, a firm is required to hire an expert. For the sake of our analysis, we ignore the cost of hiring an expert. In order to analyze expert hiring behavior of a firm, say firm 1, let us assume that her opponent, say firm 2, is known to be already employing an expert in his research lab, i.e., it is common knowledge that this firm has a high valuation of winning the R&D competition. If firm 1 does not hire an expert, she would not be able to produce a high-quality product; hence, she expects a zero payoff against her “strong” opponent, firm 1. If she hires an expert, although she becomes stronger in the sense that she has a high valuation of winning the R & D race, it also results in her opponent competing more aggressively, thus, yielding zero expected payoffs to her. We owe this result to the information environment of the competition. \(^1\) Now, if we allow firm 1 to hire a worker who produces a high-quality product only half of the time, she would get a positive expected payoff because not being sure about his opponent’s valuation of winning, firm 2 would play more aggressively than he would against a weak opponent, but less aggressively than he would against a strong firm. Therefore, when her “half” expert produces a low-quality product, firm 1 would obtain zero payoffs, however, when the “half” expert produces a high-quality product, she could generate a positive payoff. Some players in such competitions could actually prefer the presence of asymmetric information.

When players do not know their opponents’ types, one way to acquire information about them is through investing in information acquisition. Chen (2019) studies a two-player private value auction where each player acquires a costly, noisy, and private signal regarding the opponent’s valuation. Another way for the players to learn about their opponents’ types is when the auctioneer adopts some information disclosure policy and implements it. We consider the latter and analyze the best information disclosure

\(^1\)The first of the three building blocks that deliver Serena (2017)’s result is that effort exertion is crucially affected by the heterogeneity of contestants’ types, and in particular, the effort of a contestant - regardless of her own type - is maximum when she is competing against another equally skilled contestant.
policy that an auctioneer can adopt according to different performance measures, namely players’ payoff, prize allocation efficiency, and aggregate effort, where the players could pick the distribution that is observed publicly and later determines their types.

Going back to the basketball game example, the probability of a given player being injured is common knowledge, whereas whether that player is actually injured is known only by his team. Therefore, when a team drafts a player to its squad, its opponents know that this player will be injured on a game day with a known probability. On the other hand, according to NBA Injury Report, NBA teams must report information concerning player injuries, illnesses, and rest for all NBA games before the game starts. Even though whether having an injured player in the team is private information about that team, the NBA may decide to disclose this information to the public. However, according to its current policy, the NBA does not disclose this information. At the end of the chapter, one will be able to answer whether the NBA should keep concealing the injury information or change its policy if it must be changed.

One could consider the R & D race example, too, to clarify the objective here. Assuming that his accuracy of expertise defines the type of an expert, i.e., the probability of generating a high-quality product, the natural question that comes to one’s mind would be the type of expert decision of the two firms when they are to compete in a research and development race. One can think of this choice as an observable stochastic technology investment. Although being economically the strongest, i.e., picking an expert who always produces a high-quality product, sounds appealing to a firm, it motivates her opponent to fight more aggressively in case he is also strong. Choosing an expert who sometimes produces a low-quality product generates an information rent for the possible high-quality products that the same expert could produce. Taking firms’ technology choices into account, the question, therefore, that this study is after is the optimal information-disclosure-policy-adoptions of an auctioneer according to the performance measures, namely players’ payoffs, prize allocation efficiency, and aggregate effort, know-

\footnote{Deitch et al. (2006) uses basketball player profiles detailing each player’s position, height, weight, age, number of years of playing experience, player exposures (appearances) for games, and playing minutes to calculate the probability of a player getting injured.}

\footnote{Another interpretation of being of a high type could be having a lower bidding cost. When there are only two types in the type space, there is a strategic equivalence between a high winning value and a low bidding cost even though we are allowing players to have only one type in their type space. Observe that when we allow people to choose a distribution over the type space, this linkage would be broken if we had more than two types in the type space.}
Observe that this paper involves both asymmetric information creation and information disclosure. There is a vast literature on the asymmetry of information in competitive settings. There are usually two types of information asymmetry. One is about the physical state of the world, which determines the value of the product that the players are trying to acquire, whereas the other type of information asymmetry is about one’s potential competitors. Milgrom & Weber (1982) considers a sealed-bid auction and shows that the expected profit of the informed bidder is usually positive while his uninformed opponents have zero expected profit. When agents are given the option to create information asymmetry by picking a distribution that will determine their valuations against an uninformed agent, they create asymmetry to increase their expected utility. Condorelli & Szentes (2016) analyzes a bilateral trade model where the buyer can choose a cumulative distribution function that determines her valuation and shows that the buyer can generate a higher expected utility by creating uncertainty against an uninformed monopoly seller compared to the case in which she would have the highest possible valuation with probability one.

Most of the studies on information disclosure in contests focus on comparing no disclosure with full disclosure. Lim & Matros (2009) and Fu et al. (2011, 2014) consider how to reveal the information about the entry result when entries are stochastic. Moreover, they intensively study the information disclosing behavior of the players to their rivals. Attempts to rank disclosure policies based on the number of players has been well studied in auctions and contests with stochastic entry, including McAfee & McMillan (1987), Myerson & Wärneryd (2006), Münster (2006), Morath & Münster (2013), and Kovenock et al. (2015). Szech (2011) consider an asymmetric all-pay auction with incomplete information and shows that bidders choose to disclose information if disclosure can be partial. Denter et al. (2014) analyze the incentive for a privately informed contestant to disclose his information to his rival, the incentive for the uninformed contestant to acquire information and the incentive for the designer to mandate transparency.

This study is closely related to Lu et al. (2018) who accommodate different information disclosure policies that the auction organizer can commit to, and provide a complete ranking of all those policies according to different performance measures. However, in their model, players’ types are drawn from the same prior distribution. Following Lu et
al. (2018), this study focuses on the four exhaustive anonymous information disclosure policies that the auctioneer can commit to, and incorporates the possibility of adopting different technologies by the firms in order to describe the best way of disclosing information that will maximize players’ payoff, allocation efficiency, and total effort.

This study is also related to the literature on players’ incentives to acquire and share information in different auction formats. Persico (2000) studies the incentives to acquire information in decision problems and apply their result to the first price auction setting. Yildirim (2005) analyzes contests where players have the flexibility to add to their previous efforts after observing their rivals’ most recent effort in an intermediate stage. Compte & Jehiel (2007) investigates the procedure that generates the most revenue in the presence of endogenous information acquisition.

The disclosure policies that the auctioneer could adopt could be boiled down to the following four exhaustive anonymous information disclosure policies.

- **Policy FD** - disclose no matter what (full disclosure). Under this policy, the type profile is always disclosed; hence, the players play a complete information all-pay auction.

- **Policy FC** - conceal no matter what (full concealment). When the auctioneer adopts this policy, the firms play a standard incomplete information game.

- **Policy DL** - disclose that there is at least one low type player. Under this policy, when the auctioneer plays C, i.e., discloses nothing, players know that they are both of high type; hence, they play a complete information game. However, when the auctioneer plays D, i.e., discloses that there is at least one low type player, a low type player believes that she is facing a high type opponent with her prior belief, whereas a high type player deduces that his opponent is definitely a low type.

- **Policy DH** - disclose that there is at least one high type player. Similar to the previous policy, when the auctioneer plays C under this policy, players play a complete information game, whereas when he plays D, a high type player believes that her opponent is of high type with her prior belief; however, a low type player knows that he is facing a high type opponent.

The timeline of the events is as follows:
Auctioneer adopts an information disclosure policy. Players choose distributions (technology). Players’ types are realized according to \((p_1, p_2)\). Auctioneer implements the policy by playing either \(D\) (disclose) or \(C\) (conceal). In the last stage, players exert irrevocable effort \((b_1, b_2)\) to win the object.

**Figure 3.1.** The timing of the events: First, the auctioneer announces a disclosure policy. Then, players simultaneously pick distributions, \((p_1, p_2)\), from which their types, \((\theta_1, \theta_2)\), will be privately drawn in the next stage. Based on the realized type profile, the auctioneer implements his policy by playing either \(D\) (disclose) or \(C\) (conceal). In the last stage, players exert irrevocable effort \((b_1, b_2)\) to win the object.

To conduct performance comparison across different policies, we need to understand the players’ equilibrium bidding and technology-adoption behaviors under each policy. Fixing the players’ technology choices, we first characterize the equilibrium in the all-pay auction stage of the game. Under the full disclosure policy (FD), it is clear that for each realization of the type profile, the unique equilibrium of the auction stage is characterized by Hillman & Riley (1989). When the announced policy is the full concealment one (FC), we can use Siegel (2014a) in order to construct the unique equilibrium of the all-pay auction. However, under the partial disclosure policies DL and DH, when the auctioneer plays \(D\) (disclose), there is no existing result that we can rely on to characterize the players’ equilibrium bidding behavior in the all-pay auction stage. Therefore, we conduct our own equilibrium analysis under these policies. We find that the equilibrium is unique under policy DL and that there may be many equilibria under policy DH. However, whenever there is a multiplicity of equilibria, they all yield the same expected payoff to the players.

Once we know what will happen in the equilibrium of the following all-pay auction stage, we find the players’ optimal technology-adoption behavior. Even though we have a unique (up to symmetry) equilibrium technology profile under the policy FC, we run into a multiplicity of equilibrium technology choice profile under the remaining policies, FD, DL, and DH. To address this issue, we assume that the auctioneer is either optimist, believing that the best possible equilibrium outcome prevails, or pessimist, always considering the worst possible equilibrium scenario. Along with the equilibrium
analysis we conduct, this assumption enables us to find the best of the four disclosure policies according to performance measures of players’ payoffs, prize allocation efficiency, and total expected effort, respectively.

- **Players’ Payoff:** An optimist auctioneer would be indifferent between the full concealment policy and the partial disclosure policy DL when the low value of winning is low enough compared to the high value of winning. When it is high enough, he would announce policy DH. For the intermediate values, he is indifferent between full disclosure policy and policy partial disclosure DL. When he is a pessimist type, he is indifferent between the full concealment and policy DH or the low levels of low winning value. He will also adopt partial disclosure policy DH when the low winning value high enough. Lastly, for its intermediate levels, he will announce the full concealment policy.

- **Prize Allocation Efficiency:** An optimist auctioneer will always announce the full disclosure policy. In contrast, a pessimist one will announce partial disclosure policy DH for high levels of the ratio of low winning value to high winning value, be indifferent between partial disclosure policy DH and full concealment when for the levels of this ratio, and choose the full concealment policy for the intermediate levels.

- **Total Effort:** An optimist auctioneer will pick the full disclosure policy when he aims to maximize the aggregate effort. A pessimist one, however, will be indifferent between partial disclosure policy DH and full concealment policy. FD, for the levels of low winning value and adopt the full concealment policy, FC, otherwise.

The rest of the chapter is organized as follows. Section 3.2 introduces the model. Section 3.3 characterizes the equilibrium behavior in the all-pay auction stage while keeping the technology choice profile fixed. Section 3.4 finds the equilibrium technology-adoption behavior. Section 3.5 finds the best policy of the four across different levels of the ratio of the low winning value to the high winning value. Section 3.6 concludes with a brief comparison of the results to Lu et al. (2018).

### 3.2 Model

Consider a single prize all-pay auction of two players. The type of player $i$, $i = 1, 2$, is $\theta_i \in \{h, \ell\}$, where $h$ denotes high type while $\ell$ means low type. A high type has a high
winning value, \( v_h \), whereas a low type has a low winning value, \( v_\ell \), where \( v_h > v_\ell > 0 \). Denoting by \( c \) the ratio of the low value to the high value, we then have:

\[
0 < c := \frac{v_\ell}{v_h} < 1.
\]

The probability distribution from which \( \theta_i \) is drawn is a choice variable for player \( i \), and observed by his opponent as well. Since the valuations are independent, throughout the study, we use \( v_i \) not only as the valuation of winning but also to represent the type \( \theta_i \) for \( i = h, \ell \). Denoting by \( p_i \) the probability assigned to being a high type by the distribution choice of player \( i \), we have:

\[
p_i := \Pr (v_i = v_h)
\]

where \( v_i \) is the valuation of winning for player \( i \) for \( i = 1, 2 \).

Denote the effort exerted by player \( i \) in the all-pay auction game by \( x_i \), the winning probability of player \( i \) is, then, as follows:

\[
P_i(x_i, x_{-i}) = \begin{cases} 
1 & \text{if } x_i > x_{-i} \\
0 & \text{if } x_i < x_{-i} 
\end{cases}
\]

In the case where \( x_i = x_{-i} \), any exogenously determined tie braking rule is applied.

We assume that the auction organizer can observe the realized type profile and ex-ante commit to a disclosure policy to influence not only the bidding behavior but also the technology choices of the players, i.e., \( p_i \)'s. Focusing our attention to anonymous disclosure policies, due to Serena’s (2017) classification, it is enough to consider only the following four policies: (i) FD, disclose no matter what (full disclosure), (ii) FC, conceal no matter what (full concealment), (iii) DL, disclose that there is at least one low type player, and (iv) DH, disclose that there is at least one high type player.

In order to clarify the disclosure process, suppose the announced policy is DL. Therefore, if the prevailed type profile is \((v_h, v_h)\), then the auctioneer will play \( C \) (conceal), whereas, for all other possible type profile realizations, he will play \( D \) (disclose). Therefore, under this regime, when the auctioneer discloses that there is at least one low type player, a high type player will deduce that she is facing a low type opponent, for sure.
The timing of the events is as follows:

Time 1: Auctioneer announces a disclosure policy and commits to it while both players observe the announced policy.

Time 2: Each player chooses \( p_i \in [0, 1] \) according to which their types will be drawn in the next stage. Moreover, each player \( i \) observes her opponent’s technology choice of \( p_{-i} \).

Time 3: Each player privately learns her own type \( v_i \) without observing her opponent’s type \( v_{-i} \). Furthermore, the auctioneer observes the type profile \((v_i, v_{-i})\).

Time 4: Information disclosure policy is implemented based on the realized type profile. In other words, auctioneer either discloses (by playing D) or conceals (by playing C) in accordance with the disclosure policy he announced at the beginning of the game. Upon auctioneer’s move, players simultaneously choose non-negative effort levels based on the information they have. The winner gets the prize.

As a result, when the effort profile is \((x_1, x_2)\), a high type player 1 obtains a payoff of \( P_1(x_1, x_2)v_h - x_1 \), whereas if he were to be a low type he would expect a payoff of \( P_1(x_1, x_2)v_\ell - x_1 \).

### 3.3 Equilibrium Under Different Policies

To compare the disclosure policies according to different performance measures, we first need to characterize players’ bidding behaviors in the all-pay auction stage of the game under different disclosure policies for all possible technology choice profiles, \((p_1, p_2)\). The equilibrium analysis under the full disclosure (FD) and the full concealment (FC) policies can be carried out based on the existing results. Under the FD regime, the players play a complete information all-pay auction for every possible type profile realization. Therefore, the unique equilibrium strategies can be derived according to Hillman & Riley (1989). When the auctioneer adopts the full concealment policy (FC), they play a standard incomplete all-pay auction. The unique equilibrium strategies can be constructed by the method developed by Siegel (2014a), who studies two-player all-pay auctions with discrete, possibly correlated signals and interdependent values.
Under the partial disclosure policies, DL and DH, when the auctioneer does not disclose anything, players perfectly learn the type profile and play a complete information all-pay auction. To see this, suppose the policy that the auctioneer announces is DL, that is to say, that he discloses if there exists a low type player. Hence, when he does not disclose anything, the type profile must be \((v_h, v_h)\). However, when he discloses by playing D, a high type player knows that she is facing a low type opponent for sure, whereas a low type player, say player \(i\), believes that he is facing a high type opponent with probability \(p_i\) and a low type one with the remaining probability \(1 - p_i\). Similarly, when the DH regime is announced as the disclosure policy, if the auctioneer plays Conceal (C), players perfectly learn that the type profile is \((v_l, v_l)\). However, when he plays D, hence, discloses that there exists at least one high type player, a low type player knows that he is definitely facing a high type opponent, whereas, a high type player believes that she is facing a high type opponent according to his prior belief.

Unfortunately, there are no existing results that we can use to derive the equilibrium bidding strategies in such all-pay auctions other than Lu et al. (2018). However, in their environment, the probability of being a high type is the same for both players. Therefore, we need to conduct our own equilibrium analysis to derive the players’ equilibrium bidding behaviors under the two different partial disclosure policies, DL and DH. For this reason, we focus my attention to the sub-games that start after the players observe the distribution choice profile, \((p_1, p_2)\). More formally, define \(G_P(s_A | p_1, p_2)\) as the sub-game that starts after players observe the distribution choice profile \((p_1, p_2)\) and auctioneer’s policy implementation action \(s_A \in \{D, C\}\) under policy \(P \in \{FD, FC, DL, DH\}\).

In the remaining part of this section, we characterize players’ bidding behaviors in \(G_P(s_A | p_1, p_2)\) under all policies \(P \in \{FD, FC, DL, DH\}\).

### 3.3.1 Bidding Under Policy FD

When the auctioneer adopts the full disclosure policy, since he always discloses the types, players play a complete information all-pay auction. The solution to such auctions are presented in Hillman & Riley (1989). The following proposition summarizes the unique equilibrium under the FD regime.

**Proposition 5.** Equilibrium of \(G_{FD}(D | p_1, p_2)\) is unique and summarized as follows:
• When the players are of the same type, i.e., $v_1 = v_2$, both players bid uniformly on $[0, v_i]$, expecting a zero payoff in the equilibrium.

• When they are different, letting $v_i = h$ and $v_{-i} = \ell$, player $i$ bids uniformly on $[0, v_\ell]$, expecting a payoff of $v_h - v_\ell$, whereas player $-i$ bids uniformly on the same interval with probability $\frac{p_i}{v_h}$ and bids 0 with the remaining probability $1 - \frac{p_i}{v_h}$, expecting a zero payoff in the equilibrium.

3.3.2 Bidding Under Policy FC

Under the full concealment policy, the auctioneer always plays conceal (C). Therefore, players learn nothing new from the auctioneer. As a result, they play a standard incomplete information all-pay auction, where player $i$ believes that she is facing a high type opponent with probability $p_i - p_{-i}$ for $i = 1, 2$. Following the algorithmic approach presented in Siegel (2014a), we can derive the unique equilibrium under the FC regime as summarized in the following proposition.

**Proposition 6.** Equilibrium of $G_{FC}(C | p_1, p_2)$ is unique and, without loss of generality, letting $p_1 \leq p_2$, it is characterized as follows:

- Player 1’s high type bids uniformly on $[(1 - p_1)v_\ell, (1 - p_1)v_\ell + p_1v_h]$ and on $[(1 - p_2)v_\ell, (1 - p_1)v_\ell]$ with probability $\frac{1 - p_2}{1 - p_1}$, on $[(1 - p_2)v_\ell, (1 - p_1)v_\ell]$ with probability $\frac{p_2 - p_1}{(1 - p_1)v_h}$, and bids 0 with the remaining probability $1 - \frac{1 - p_2}{1 - p_1} - \frac{p_2 - p_1}{(1 - p_1)v_h}$. Equilibrium expected payoff for her high type is $v_h - v_\ell$, whereas her low type expects a zero payoff in the equilibrium.

- Player 2’s high type bids uniformly on $[(1 - p_1)v_\ell, (1 - p_1)v_\ell + p_1v_h]$ with probability $\frac{p_2}{p_2}$ and on $[(1 - p_2)v_\ell, (1 - p_1)v_\ell]$ with probability $1 - \frac{p_2}{p_2}$, whereas his low type mixes his bid uniformly on $[0, (1 - p_2)v_\ell]$. Equilibrium expected payoff for his high type is $v_h - v_\ell$, whereas his low type expects a payoff of $\frac{(p_2 - p_1)(v_h - v_\ell)v_\ell}{v_h}$ in the equilibrium.

3.3.3 Bidding Under Policy DL

When the auctioneer adopts the partial disclosure policy DL that is to disclose if there is at least one low type, if the auctioneer does not disclose anything, i.e., plays C, players perfectly know that the type profile is $(v_h, v_h)$. Thus, they play a complete information all-pay auction. We do not need to repeat the bidding behavior in this case. However,
when the auctioneer plays D, the possible realizations of the types can be \((v_h, v_l), (v_l, v_h),\) or \((v_l, v_l).\) In other words, while high type player knows that her opponent is a low type one, a low type player believes that he is facing a high type opponent with his prior belief. Observe that if \(p_i = 1\) for some \(i,\) auctioneer’s action would reveal the type profile, resulting in the players playing a complete information all-pay auction. Hence, assuming \(p_i \neq 1\) for \(i = 1, 2,\) following proposition characterizes the unique equilibrium bidding strategies under DL regime when the auctioneer plays D.

**Proposition 7.** Wlog, letting \(p_i \leq p_2 < 1,\) equilibrium of \(G_{DL}(D | p_1, p_2)\) is unique and characterized as follows:

- **Player 2’s low type** bids uniformly on \([0, x]\) and \([x, \bar{x}]\) according to the following bidding distribution function:

  \[
  F^2_\ell = \begin{cases} \frac{x}{(1-p_2)v_h}, & x \in [0, x] \\ \frac{x}{(1-p_2)v_l} + (x - \bar{x}) \frac{1}{v_h}, & x \in [x, \bar{x}] \end{cases}
  \]

  where, \(\bar{x} = \frac{(1-p_2)v_h(v_h - v_l)}{v_h - (1-p_1)v_h} \) and \(\bar{x} = v_l - \frac{(p_2-p_1)v_l(v_h - v_l)}{v_h - (1-p_1)v_h}.\)

- **Player 2’s high type** bids uniformly on \([x, \bar{x}]\) and \([\bar{x}, v_l]\) according to the following bidding distribution function:

  \[
  F^2_h = \begin{cases} (x - \bar{x}) \frac{v_h - (1-p_2)v_l}{p_2 v_l v_h}, & x \in [x, \bar{x}] \\ (\bar{x} - x) \frac{v_h - (1-p_2)v_l}{p_2 v_l v_h} + (x - \bar{x}) \frac{1}{p_2 v_h}, & x \in [\bar{x}, v_l] \end{cases}
  \]

- **Player 1’s low type** bids uniformly on the intervals \([0, x]\) and \([x, v_l]\) (with the possibility of placing an atom at zero bid) according to the following bidding distribution function:

  \[
  F^1_\ell = \begin{cases} \frac{(p_2-p_1)(v_h - v_l)}{(1-p_1)v_h} + x \frac{1}{(1-p_1)v_l}, & x \in [0, x] \\ \frac{(p_2-p_1)(v_h - v_l)}{(1-p_1)v_h} + x \frac{1}{(1-p_1)v_h} + (x - \bar{x}) \frac{1}{v_h}, & x \in [x, v_l] \end{cases}
  \]

- **Player 1’s high type** bids uniformly on \([x, \bar{x}]\) according to following bidding distribution function:

  \[
  F^1_h = (x - \bar{x}) \frac{v_h - (1-p_1)v_l}{p_1 v_l v_h} \quad \forall x \in [x, \bar{x}]
  \]
The equilibrium expected payoffs are

\[ u^1(p_1, p_2) = p_1 \left( (v_h - v_l) + \frac{(p_2 - p_1)v_l(v_h - v_l)}{v_h - (1 - p_1)v_l} \right) \]

\[ u^2(p_1, p_2) = p_2(v_h - v_l) + (1 - p_2)\frac{(p_2 - p_1)v_l(v_h - v_l)}{v_h} \]

The proof of Proposition 7 is in the Appendix D.1.

### 3.3.4 Bidding Under Policy DH

When the auctioneer adopts the partial disclosure policy DH that is to disclose if there is a high type player, similar to Section 3.3.3, if the auctioneer plays C, the type profile is perfectly revealed; hence, the two players play a complete information all-pay auction upon observing the auctioneer playing C. However, when he plays D, the possible realizations of the types could be \((v_h, v_l)\), \((v_l, v_h)\), or \((v_h, v_h)\). Therefore, in this case, while a low type player definitely knows that his opponent is a high type one for sure, whereas a high type player believes that she is facing a high type opponent with her prior belief. The following proposition characterizes the unique equilibrium bidding strategies when the auctioneer plays D under the DH regime.

**Proposition 8.** Wlog, letting \( p_1 \geq p_2 \), equilibrium of \( G_{DH}(D | p_1, p_2) \) is unique unless \( p_1 = \frac{v_l}{v_h} \). Moreover, when there is a multiplicity of equilibria, they are all payoff equivalent.

- When \( p_1 = p_2 \geq \frac{v_l}{v_h} \), in the unique equilibrium, low type players bid zero with probability one while the bidding distributions of the high type players are the same and the following:

  \[ F_h^i(x) = \frac{x}{p_iv_h}, \quad x \in [0, p_iv_h] \quad \text{for } i = 1, 2 \]

In this equilibrium, a low type player obtains zero payoff while a high type gets \((1 - p_i)v_h\).

- When \( p_1 > p_2 \) and \( p_1 > \frac{v_l}{v_h} \), in the unique equilibrium, player 1’s low type bids zero with probability one. The bidding distribution function of his high type, \( F_h^1 \), and the
bidding distributions of his opponent’s two types are as follows:

\begin{align*}
F^1_h(x) &= \begin{cases} \frac{x}{v_\ell}, & x \in [0, m] \\ m + \frac{x-m}{p_1v_h}, & x \in [m, m+p_2v_h] \end{cases} \\
F^2_h(x) &= 1 - m \frac{v_\ell}{p_1v_h} + \frac{x}{(1-p_2)v_h}, \quad \forall x \in [0, m] \\
F^2_h(x) &= \frac{x-m}{p_2v_h}, \quad \forall x \in [m, m+p_2v_h],
\end{align*}

where \( m = \frac{(p_1-p_2)}{p_1} v_\ell \). In the equilibrium, low type players obtain zero payoff while high type players get a payoff of \((v_h - v_\ell) - p_2v_h + \frac{p_2}{p_1} v_\ell\).

- When \( p_1 < \frac{v_\ell}{v_h} \), in the unique equilibrium, for \( i = 1, 2 \), the bidding distributions are characterized as follows:

\begin{align*}
F^i_h(x) &= \frac{x}{v_\ell} \\
F^i_\ell(x) &= \frac{v_h - v_\ell}{(1-p_i)v_h} + \frac{x(v_\ell - p_1v_h)}{(1-p_1)v_h v_\ell}
\end{align*}

for all \( x \in [0, v_\ell] \). Moreover, in this case, a low type player obtains zero payoffs while a high type one gets \( v_h - v_\ell \).

- When \( p_1 = \frac{v_\ell}{v_h} \) and \( p_1 > p_2 \), there are infinitely many equilibria. However, in all of them low type players get zero payoff while high types obtain a payoff of \( v_h - v_\ell \).

The proof of Proposition 8 is in the Appendix D.2. Observe that if the case that yields to the multiplicity of equilibria occurs on an equilibrium path, we would not be able to rank the disclosure policies in terms of their performance in aggregate effort without making any further assumptions.

### 3.4 Distribution Choices Under Different Policies

This section finds out the equilibrium distribution choices of the players for various disclosure regimes that the auctioneer can possibly adopt, knowing what will happen next in the equilibrium in the following sub-game \( G_P(s_A | p_1, p_2) \), where \( G_P(s_A | p_1, p_2) \) is the sub-game
that starts after players observe the distribution choice profile \((p_1, p_2)\) and auctioneers policy implementation action \(s_A \in \{D, C\}\) under policy \(P \in \{FD, FC, DL, DH\}\). One could argue that picking a distribution that assigns probability one to being a high type sounds appealing, especially when it is costless to do so. However, having a higher probability of being a high type player might trigger the opponent to bid more aggressively, resulting in even zero payoffs. Therefore, asymmetry in the information structure in the competitive setting might be beneficial to some players. Even though the announced disclosure policy might create certain types of information asymmetry, we will see if players would like to add to it in this section. In the remaining part of the section, we find the players’ distribution choices under different disclosure policies. Because we have only two elements in our type space, picking a distribution for player \(i\) over the type space is the same as choosing a number \(p_i\) in \([0, 1]\) that represents the probability assigned to the high type. Thus, the remaining probability \(1 - p_i\) is assigned to the low type.

### 3.4.1 Distribution Choices Under Policy FD

When the auctioneer adopts the full disclosure policy, that discloses the types no matter what, players have no chance to exploit the possibility of having information rent that might arise in the presence of information asymmetry. Therefore, it is reasonable to expect them to pick a distribution that assigns probability one to being a high type. The following proposition shows that this expectation is partially fulfilled.

**Proposition 9.** When the auctioneer adopts the full disclosure policy, FD, in equilibrium, one player picks \(p_i = 1\), while her opponent chooses some \(p_{-i} \in [0, 1]\).

**Proof.** The proof is obvious. Fixing player 2’s distribution choice \(\tilde{p}_2\), by picking \(p_1\), player 1 expects a payoff of \(p_1(1 - \tilde{p}_2)(v_h - v_l)\). Therefore,

\[
p_1(\tilde{p}_2) = \begin{cases} 
[0, 1] & \text{if } \tilde{p}_2 = 1 \\
1 & \text{if } \tilde{p}_2 < 1
\end{cases}
\]

Proposition 9 tells us that under the FD regime, we have infinitely many equilibrium distribution profiles. In all equilibria, one player is always of high type while the other player is indifferent between being a high or a low type. Therefore, the indifferent player obtains zero payoff no matter what while the high type player, say player \(i\), obtains a payoff of \(p_i(1 - p_{-i})(v_h - v_l)\), which is positive when \(p_{-i} < 1\).
3.4.2 Distribution Choices Under Policy FC

Under the FC regime, because the auctioneer discloses nothing, we are in Siegel (2014a) environment. For a thought experiment, suppose that player 2 is always of high type. Under policy FD, knowing that she is going to face a high type opponent, player 1 would be indifferent between being a high or a low type because they both would yield a zero payoff to her. Observe that this would happen because player 2 would perfectly know her type and adjust his bidding behavior accordingly. However, under the FC regime, player 2 would not perfectly learn her type. Thus, she could obtain positive payoff against a high type opponent because her opponent bids less aggressively then he would bid under policy FD. In other words, her low type would generate some information rent on her high type, which would make her high type obtain positive payoff in the equilibrium.

The following proposition shows that the equilibrium distribution choice profile is independent of \( c \), the ratio of the low value to the high value, and unique up to symmetry under the FC regime.

**Proposition 10.** When the auctioneer adopts the full concealment policy, FC, in the unique (up to symmetry) pure strategy equilibrium of the game, one player assigns probability one to high type while the other player assigns probability half to each type.

The proof of Proposition 10 is in the Appendix D.3. Proposition 10 tells us that no matter how small the low valuation compared to the high one is, one player assigns probability half to being a low type. Even when \( v_l \) so close to zero while \( v_h \) is so close to the infinity. The central intuition behind this is that when facing an opponent who is always a high type one, the best way to obtain a positive payoff is by creating information asymmetry. Moreover, letting player \(-i\) be the one who is always a high type, player \( i \) makes a positive payoff when she is of a high type, which happens with probability \( p_i \), and her opponent believes that he is facing a low type, which happens with probability \( 1 - p_i \). The maximizer of \( p_i(1 - p_{-i}) \) is \( p_i = 0.5 \).

3.4.3 Distribution Choices Under Policy DL

When the auctioneer adopts the information disclosure policy DL that is to disclose if there is a low type, if the auctioneer plays C, both players know that they are both high types, hence, they play a complete information all-pay auction, resulting in both players obtaining zero payoffs. Therefore, if \((p_1, p_2)\) is the distribution choice profile, with
probability $p_1p_2$ they obtain zero payoff. With the remaining probability of $(1 - p_1p_2)$, the auctioneer will play D, and we can use Proposition 7 to calculate the payoff of player 1 in this case. The expected payoff of player 1 when $(p_1, p_2)$ is the distribution choice profile is as follows:

$$EU_1(p_1, p_2) = \begin{cases} 
(1 - p_1p_2)p_1 \left( \frac{1-c+p_2c}{1-c+p_1c} \right) v_h(1-c), & p_1 \leq p_2 < 1 \\
(1 - p_1p_2) [p_1 + (1 - p_1)(p_1 - p_2)c] v_h(1-c), & p_1 > p_2 
\end{cases}$$

Moreover, when $p_2 = 1$, whether the auctioneer plays C or D, they play a complete information all-pay auction, which leaves player 1 with zero payoff. The following proposition characterizes the equilibrium pure strategy distribution profile under DL regime.

**Proposition 11.** *When the auctioneer adopts the policy DL, in equilibrium, one player assigns probability one while the other player assigns probability $p \in \left[0, \frac{\sqrt{1-c-(1-c)}}{c} \right]$ to being a high type.*

The proof of Proposition 11 is in the Appendix D.4. Again, by Proposition 11 we know that one player is a high type player no matter what, while the other player is of high type with a probability that is, this time, lower than half. Remember that under the full concealment policy, one player chooses to be a high type one with probability half. This player lowers the probability of being a high type from half because under the DL regime her opponent knows her type for sure - if the auctioneer discloses, then she is a low type; if he does not, then she is a high type. So, benefiting from information asymmetry is not possible. However, there is a threshold probability of being a high type below which makes the opponent choose to be a high type player for sure. When a player is a high-type one with a higher probability than this threshold probability, their opponent finds it more profitable to be a probabilistically low-type player to create an information rent for her high type. As the relative value of high valuation compared to the low valuation increases, this motivation becomes even stronger.

### 3.4.4 Distribution Choices Under Policy DH

When the auctioneer adopts the disclosure policy DH that is to disclose if there is a high-type player, if he later plays C (conceal), similar to Section 3.4.3, the players play a complete information all-pay auction since they both know that they are both low types, resulting in zero payoffs for each. Therefore, letting $(p_1, p_2)$ be the distribution choice
profile, with probability \((1 - p_1)(1 - p_2)\) they obtain zero payoff. With the remaining probability of \((p_1 + p_2 - p_1p_2)\), the auctioneer plays D. Therefore, by Proposition 8, player 1’s equilibrium payoff can be calculated as follows:

\[
EU_1(p_1, p_2) = \begin{cases} 
  v_h(p_1 + p_2 - p_1p_2)p_1 (1 - c - p_2 + \frac{p_2}{p_1}c) & p_1 \geq c \text{ and } p_1 \geq p_2 \\
  v_h(p_1 + p_2 - p_1p_2)p_1 (1 - c - p_1 + \frac{p_1}{p_2}c) & p_2 \geq c \text{ and } p_1 \leq p_2 \\
  v_h(p_1 + p_2 - p_1p_2)p_1 (1 - c) & p_1 \leq c \text{ and } p_2 \leq c
\end{cases}
\]

The following proposition characterizes the equilibrium distribution profile when the auctioneer announces the policy DH.

**Proposition 12.** Under the DH regime, equilibrium distribution profile is characterized as follows:

- When \(c < \frac{1}{10}(6 - \sqrt{6})\), \((1, \frac{1}{2})\) is the only equilibrium (up to symmetry).

- When \(\frac{1}{10}(6 - \sqrt{6}) \leq c \leq \frac{3}{5}\), without loss of generality, restricting our attention to the case \(p_1 \geq p_2\), there are two types of equilibria, non-symmetric and symmetric. The non-symmetric equilibrium can be characterized as \((p_1, p_2)\) where \(p_1\) and \(p_2\) are the solution to the following equation system:

\[
0 = 2p_1(1 - p_2)(1 - p_2 - c) + p_2(1 - p_2 - cp_2) \\
0 = -3(1 - p_1)(p_1 - c)p_2^2 + 2p_1(1 - 2p_1 + cp_1)p_2 + p_1^2(1 - c),
\]

whereas the set of symmetric equilibrium distribution profiles is described as follows:

\[
\left\{(p, p) : \max \left\{c, \frac{(5 - c) - \sqrt{c^2 + 6c + 1}}{4}\right\} \leq p \leq \frac{(c + 7) - \sqrt{c^2 - 10c + 13}}{6}\right\}
\]

- When \(c > \frac{3}{5}\), the set of equilibrium distribution profiles is the following:

\[
\left\{(p, p) : c \leq p \leq \frac{(c + 7) - \sqrt{c^2 - 10c + 13}}{6}\right\}
\]

The proof of Proposition 12 is in Appendix D.5. According to Proposition 12, when the ratio of low valuation to the high valuation is low enough, i.e \(\frac{v_l}{v_h} < \frac{1}{10}(6 - \sqrt{6})\), the policies DH and FC are equivalent. DH policy becomes effective only when \(\frac{v_l}{v_h}\) is above this threshold level. Moreover, when it is greater than 0.6, the equilibrium distribution
choices are symmetric, which leaves us in Lu et al. (2018) environment in the following sub-game.

### 3.5 Ranking Disclosure Policies

This study aims to completely rank the disclosure policies according to different performance measures, namely aggregate effort, players’ payoff, and allocation value. However, in doing so we run into two different problems: multiplicity of equilibria under policies FD, DL, and DH and an equilibrium that is characterized but not solved under policy DH when \( c \in \left[\frac{6-\sqrt{6}}{10}, \frac{3}{5}\right] \), the non-symmetric equilibrium. To handle the latter, when \( c \) is in that interval, we will consider only the symmetric equilibria under policy DH, whereas we address the former issue by assuming that the auctioneer has two types, optimist and pessimist. ⁴ An optimist (pessimist) auctioneer is the one who believes that the best (worst) equilibrium outcome prevails whenever there is a multiplicity of equilibria. As a result, instead of completely ranking the disclosure regimes, one could compare the best and the worst possible equilibrium outcomes to find the best disclosure policy. In the following parts of this section, we characterize the best policy choice in terms of aggregate effort, players’ payoff, and allocation value for both types of the auctioneer as the ratio of low valuation to high one changes. Moreover, we will highlight some cases where the possibility of choosing the technology before entering the competition might switch the results in Lu et al. (2018).

#### 3.5.1 Players’ Payoff

Denoting by \( W_P \) the total expected payoff of the players under the policy \( P \in \{FD, FC, DL, DH\} \), following proposition finds the upper and the lower bounds for payers’ payoff under each policy.

**Proposition 13.** Players’ payoff under the four policies are bounded as follows:

- **Under policy FD:**
  \[
  v_h(1 - c) \geq W^{FD} \geq 0.
  \]

- **Under policy FC:**
  \[
  W^{FC} = \frac{3}{4} v_h(1 - c).
  \]

⁴Using graphical analysis tools, one could see that the results do not change when the symmetric equilibrium is excluding. However, we have not proven this analytically.
• **Under policy DL:**

\[ v_h(1 - c) \geq W^{DL} \geq v_h(1 - c) \left( \frac{1 - \sqrt{1 - c}}{c} \right). \]

• **Under policy DH:**

1. When \( c < \frac{6 - \sqrt{6}}{10} \):

\[ W^{DH} = \frac{3}{4} v_h(1 - c). \]

2. When \( c \geq \frac{6 - \sqrt{6}}{10} \):

\[ 2t^2(1 - t)(2 - t)v_h \geq W^{DH} \geq 2\bar{t}^2 \left( 1 - \bar{t} \right) \left( 2 - \bar{t} \right) v_h, \]

where \( t = \max \left\{ c, \frac{(5 - c) - \sqrt{c^2 + 6c + 1}}{4} \right\} \) and \( \bar{t} = \frac{c + 7 - \sqrt{c^2 - 10c + 13}}{6} \).

Moreover, whenever there is an interval specified above, all the values in the interval are achieved as a payoff for some equilibrium.

The proof is in Appendix D.6.

**Comparing The Policies:**

According to Proposition 13 an optimist auctioneer, who believes that the most preferred equilibrium outcome prevails, thinks that the players’ payoff would be \( v_h(1 - c) \), \( \frac{3}{4} v_h(1 - c) \), and again \( v_h(1 - c) \) under the policies FD, FC, and DL, respectively. Therefore, he never chooses the full concealment policy. Moreover, when \( c < \frac{6 - \sqrt{6}}{10} \), the equilibrium outcome under policy DH is the same as the one that full concealment policy yields, thus, he would be indifferent between announcing the policies FD and DL for such low levels of \( c \).

When \( c \geq \frac{6 - \sqrt{6}}{10} \), the maximum possible players’ payoff under policy DH is

\[ 2p^2(1 - p)(2 - p)v_h, \]

where

\[
\max \left\{ c, \frac{(5 - c) - \sqrt{c^2 + 6c + 1}}{4} \right\} \leq p \leq \frac{c + 7 - \sqrt{c^2 - 10c + 13}}{6},
\]

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whereas, it is \( v_h(1-c) \) under policy FD, equivalently under policy DL. There is a threshold level \( \tilde{c} \) such that when \( c \) is smaller than \( \tilde{c} \) policy FD is preferred to the policy DH because

\[
1 - c > \max_{p \in [0,1]} 2p^2(1-p)(2-p)
\]

for all \( c < \tilde{c} \). \(^5\) Even when we ignore the constraints that \( p \) needs to satisfy other than it being a probability, the maximum players’ payoff that can be achieved under policy DH is smaller than the maximum players’ payoff that policy FD could result in. However, when \( c > \tilde{c} \) policy DH dominates FD and DL policies from the perspective of an optimist auctioneer whose objective is maximizing the players’ payoff. To sum up, an optimist auctioneer announces policy DH when \( c > \tilde{c} \), and is indifferent in adopting the policies FD and DL when \( c < \tilde{c} \).

On the other hand, a pessimist auctioneer would never choose the policy FD because he thinks that it results in zero payoffs for both players. When \( c < \frac{6-\sqrt{6}}{10} \), he is indifferent between choosing the policies FC and DH since for such low levels of \( c \), the two regimes induce the same outcome. Moreover, he prefers these policies to policy DL because under these regimes the players’ payoff could be \( \frac{3}{4}v_h(1-c) \), while, it could be at most \( v_h(1-c) \left( \frac{1-\sqrt{1-c}}{c} \right) \) under policy DL. His indifference will be broken in favor of the full concealment policy when \( \frac{6-\sqrt{6}}{10} < c \leq \bar{c} \) for some \( \bar{c} \approx 0.633 \). \(^6\) Moreover, when \( c > \bar{c} \), he will adopt the policy DH.

Lu et al. (2018) finds that players’ payoff is highest under policy DH, whereas the other disclosure policies result in the same level of total payoff. If we allow the players to pick their technologies, when the low value of winning is close enough to the high value of winning, DH is still the best option for both optimist and pessimist types of the auctioneer. However, this is not always the case, as Figure 3.2 depicts the best disclosure policy choice for both types of auctioneers as the level of \( c \). An optimist auctioneer is indifferent between adopting the policies FD and DL when \( c \) is less than \( \tilde{c} \), whereas, he chooses policy DH when \( c \) is greater than \( \tilde{c} \). On the other hand, a pessimist auctioneer is indifferent between the policies FC and DH when \( c \) is lower than \( \frac{6-\sqrt{6}}{10} \) whereas, when \( c \) is greater than this value he adopts the policy FC if \( c \) is less than \( \bar{c} \) and DH otherwise.

\(^5\) \( f(p) = 2p^2(1-p)(2-p) \) is positive in \((0,1)\) and the unique maximum in this interval is \( f(p^*) = 1 - \tilde{c} \approx 0.4033 \).

\(^6\) See Appendix D.6 for the exact level of \( \bar{c} \).
Figure 3.2. For different levels of the ratio of the low valuation to the high one, $c$, the optimal disclosure policies for each type of auctioneers, where $\tilde{c} = \frac{363 - 51\sqrt{17}}{256} \approx 0.597$ and $\bar{c} = \approx 0.633$.

3.5.2 Allocation Efficiency

To be able to compare the disclosure policies in terms of their allocation efficiency, we first calculate the allocation value of each policy. Denoting by $\pi^P$ the probability of allocating the prize to a high type player under the policy $P \in \{FD, FC, DL, DH\}$, Proposition 14 characterizes the boundaries of allocation efficiency under the four different disclosure policies.

**Proposition 14.** Allocation efficiency under each policy lies in the corresponding boundaries specified below:

- **Under policy FD:**
  \[
  1 \geq \pi^{FD} \geq 1 - \frac{c}{2}.
  \]

- **Under policy FC:**
  \[
  \pi^{FC} = 1 - \frac{c}{8}.
  \]

- **Under policy DL:**
  \[
  1 - \frac{(1 - \sqrt{1-c})}{2} \geq \pi^{DL} \geq 1 - \frac{c}{2}.
  \]

- **Under policy DH:**
  1. When $c < \frac{1}{10}(6 - \sqrt{6})$ :
     \[
     \pi^{DH} = 1 - \frac{c}{8}
     \]
  2. When $c \geq \frac{1}{10}(6 - \sqrt{6})$ :
     \[
     \bar{t}(2 - \bar{t}) \geq \pi^{DH} \geq t(2 - t)
     \]
where $\bar{t} = \max \left\{ c, \frac{(5-c)-\sqrt{c^2+6c+1}}{4} \right\}$ and $\bar{t} = \frac{c+7-\sqrt{c^2-10c+13}}{6}$.

The proof is in the Appendix D.7.

**Comparing The Policies:**

An optimist auctioneer would adopt the full disclosure policy since it could yield the highest possible allocation efficiency: 1. On the other hand, a pessimist auctioneer would consider the level of $c$ before adopting a disclosure policy. Observe that the worst possible allocation efficiency under FC is greater than that are under the policies FD and DL. Moreover, when $c < \frac{1}{16} (6 - \sqrt{6})$, the policies FC and DH yield the same allocation efficiency. For the higher levels of $c$, we must compare $\bar{t} (2 - \bar{t})$ and $1 - \frac{c}{8}$. The former is greater when $c > \frac{17-\sqrt{33}}{16}$. Therefore, a pessimist auctioneer would announce policy DH when $c > \frac{17-\sqrt{33}}{16}$, policy FC when $\frac{1}{16} (6 - \sqrt{6}) < c < \frac{17-\sqrt{33}}{16}$, and would be indifferent between policies FC or DH when $c < \frac{1}{16} (6 - \sqrt{6})$ and at the cut-offs.

Figure 3.3 best summarizes the policy choice of an auctioneer for his both types as $c$ changes.

![Optimist Auctioneer’s Disclosure Policy Choice](image)

**Figure 3.3.** For different levels of the ratio of the low valuation to the high one, $c$, the optimal disclosure policies for each type of auctioneers.

According to Lu et al. (2018), the full concealment policy results in the highest allocation value. When $c < \frac{1}{16} (6 - \sqrt{6})$, the allocation values under the policies FC and DH are the same and greater than the one that policy DL could possibly elicit the most. However, for higher values of $c$, i.e $c > \frac{17-\sqrt{33}}{16} \approx 0.7035$, policy DH prevails greater allocation value than full concealment policy. Hence, if we allow players to choose their technology, full concealment policy is not necessarily the best when considering the allocation efficiency.
3.5.3 Total Effort

Denoting by $x^P$ the expected total effort under policy $P \in \{FD, FC, DL, DH\}$, following proposition finds $x^P$ under each policy for a generic selected equilibrium distribution profile if there are multiple.

**Proposition 15.** The expected total effort is as follows:

- **Under policy FD:** Assuming $(1, p)$ is the equilibrium distribution profile,
  \[ x^{FD} = v_h \left( p + (1 - p) \frac{c(1 + c)}{2} \right). \]
  
  Remember that, from Proposition 9, $p \in [0, 1]$.

- **Under policy FC:**
  \[ x^{FC} = v_h \frac{1}{8} \left( 2 + 5c + c^2 \right). \]

- **Under policy DL:** Assuming $(1, p)$ is the equilibrium distribution profile,
  \[ x^{DL} = v_h \left( p + (1 - p) \frac{c(1 + c)}{2} \right). \]
  
  Remember that, from Proposition 11, $p \in \left[0, \frac{\sqrt{1 - c} - (1 - c)}{c}\right]$.

- **Under policy DH:**
  
  1. When $c < \frac{1}{10}(6 - \sqrt{6})$:
     \[ x^{DH} = v_h \frac{1}{8} \left( 2 + 5c + c^2 \right). \]

  2. When $c \geq \frac{1}{10}(6 - \sqrt{6})$, assuming $(p,p)$ is the equilibrium distribution profile, we have:
     \[ x^{DH} = v_h \left( p^2 + (1 - p)^2 c \right) \]

     Remember that, from Proposition 12,
     \[ \max \left\{ c, \frac{(5 - c) - \sqrt{c^2 + 6c + 1}}{4} \right\} \leq p \leq \frac{(c + 7) - \sqrt{c^2 - 10c + 13}}{6} \]
The proof is in the Appendix D.8.

Comparing The Policies:

An optimist auctioneer would choose policy FD because no other policy could result in an aggregate effort level of $v_h$. However, a pessimist auctioneer would take the level of $c$ into account when announcing the disclosure policy. It is not difficult to show that the aggregate effort under policy FC is greater than that under policies FD and DL in the eyes of a pessimist auctioneer. Since the total effort is decreasing in $p$ under the policies FD and DL, one could simply consider the equilibrium in which the distribution profile is $(1, 0)$ to find the worst possible total effort under these policies and that is $\frac{1}{2}c(1+c)v_h$, which is smaller than $\frac{1}{8}v_h(2+5c+c^2)$. Thus, we are left with comparing the worst possible total effort levels under policies FC and DH. When $c < \frac{6-\sqrt{6}}{10}$, the pessimist auctioneer is indifferent between these two policies since they result in the same outcome.

![Figure 3.4](440x390)

**Figure 3.4.** Visual illustration of the worst possible total effort levels under the policies FC and DH as a function of $c$. The color4-colored curve represents the total expected effort under DH while american rose is for FC. In the figure, $a = \frac{6-\sqrt{6}}{10} \approx 0.3551$ and $b$ is such that $b = \frac{(5-b)-\sqrt{b^2+6b+1}}{4}$, thus, $b = \frac{7-\sqrt{13}}{6} \approx 0.5657$.

When $c \geq \frac{6-\sqrt{6}}{10}$, $x^{DH}$ is increasing in $p$, hence, we must consider the lowest possible level of $p$, that is $\max \left\{ c, \frac{(5-c)-\sqrt{c^2+6c+1}}{4} \right\}$, in order to calculate the worst possible total effort under the policy DH. If $\max \left\{ c, \frac{(5-c)-\sqrt{c^2+6c+1}}{4} \right\} = c$, then FC is preferred to DH
because
\[
\frac{1}{8} \left(2 + 5c + c^2\right) > c^2 + (1 - c^2)c.
\]
The reader is encouraged to bear the Figure 3.4 in mind when comparing the policies FC and DH while \(c\) is in \([\frac{6 - \sqrt{6}}{10}, \frac{7 - \sqrt{13}}{6}]\). The color4-colored curve in the figure represents the total expected effort under DH while the American rose-colored one under the policy FC. The total expected effort under the policy FC is increasing in \(c\), whereas, under the DH regime, it is decreasing in \(c\) when \(c \in \left[\frac{6 - \sqrt{6}}{10}, \frac{7 - \sqrt{13}}{6}\right]\). Moreover, at \(c = \frac{6 - \sqrt{6}}{10}\), DH elicits higher total expected effort, whereas, at \(c = \frac{7 - \sqrt{13}}{6}\) it is the policy FC that yields higher effort. As a result, there is some \(\hat{c}\) such that
\[
\frac{1}{8} \left(2 + 5\hat{c} + \hat{c}^2\right) = \left(\frac{5 - \hat{c}}{4} - \sqrt{\hat{c}^2 + 6\hat{c} + 1}\right)^2 + \left(1 - \left(\frac{5 - \hat{c}}{4} - \sqrt{\hat{c}^2 + 6\hat{c} + 1}\right)^2\right)^2 \hat{c}
\]
and when \(c > \hat{c}\), FC elicits higher total effort, whereas, when \(c < \hat{c}\), DH results in the highest total effort.  

**Optimist Auctioneer’s Disclosure Policy Choice**

<table>
<thead>
<tr>
<th>(c)</th>
<th>FD</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(\hat{c})</td>
</tr>
<tr>
<td>(\frac{6 - \sqrt{6}}{10})</td>
<td>FC or DH</td>
</tr>
<tr>
<td>(\frac{7 - \sqrt{13}}{6})</td>
<td>DH</td>
</tr>
<tr>
<td>1</td>
<td>FC</td>
</tr>
</tbody>
</table>

**Pessimist Auctioneer’s Disclosure Policy Choice**

**Figure 3.5.** For different levels of the ratio of the low valuation to the high one, \(c\), the optimal disclosure policies for each type of auctioneers.

Figure 3.5 depicts the optimal disclosure policies as \(c\) changes for both optimist and pessimist types of the auctioneer. In contrast to Lu et al. (2018), the full concealment policy does not always extract the maximum effort. For an optimist auctioneer, policy FD always elicits the highest total expected effort. A pessimist auctioneer is indifferent between the policies FC and DH when \(c < \frac{6 - \sqrt{6}}{10}\), prefers DH when \(\frac{6 - \sqrt{6}}{10} < c < \hat{c}\), and adopts the policy FC when \(c > \hat{c}\), where \(\hat{c} \approx 0.38684\). Moreover, at the cut off levels of \(c\), he is indifferent between the policies FC and DH.

A closed form solution for \(\hat{c}\) is not analytically possible, hence, we approximated it: \(\hat{c} \approx 0.38684\).
3.6 Conclusion

In this study, we analyze the best information disclosure policy that an auctioneer could adopt according to different performance measures, namely players’ payoff, prize allocation efficiency, and aggregate effort, where the players can choose the distribution which determines their type. Using a two-player all-pay auction with two types setting, we show that the optimal disclosure policy changes as the ratio of the value of winning for the low type to the value of winning for the high type varies. The significant feature of this analysis is that players can choose the distribution from which their own types are drawn, which we call technology-adoption. To address the multiplicity of equilibria, we assume that the auctioneer is either optimist, i.e., he believes that his most preferred equilibrium outcome prevails when there is more than one equilibrium, or pessimist, i.e., he expects the worst equilibrium outcome.

![Optimist Auctioneer’s Disclosure Policy Choice](image)

**Optimist Auctioneer’s Disclosure Policy Choice**

<table>
<thead>
<tr>
<th>Performance Measure</th>
<th>Disclosure Policy</th>
</tr>
</thead>
<tbody>
<tr>
<td>PP</td>
<td>FD or DL, DH</td>
</tr>
<tr>
<td>PAE</td>
<td>FD</td>
</tr>
<tr>
<td>TE</td>
<td>FD</td>
</tr>
</tbody>
</table>

![Pessimist Auctioneer’s Disclosure Policy Choice](image)

**Pessimist Auctioneer’s Disclosure Policy Choice**

<table>
<thead>
<tr>
<th>Performance Measure</th>
<th>Disclosure Policy</th>
</tr>
</thead>
<tbody>
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<td>PP</td>
<td>FC or DH, FC, DH</td>
</tr>
<tr>
<td>PAE</td>
<td>FC or DH, FC, DH</td>
</tr>
<tr>
<td>TE</td>
<td>FC or DH, FC, DH</td>
</tr>
</tbody>
</table>

**Figure 3.6.** The winning policies for optimist (top) and pessimist (bottom) types of the auctioneer as the ratio of the value of winning for the low type to that of the high type, $c$, changes for different performance measures, namely players’ payoff (PP), prize allocation efficiency (PAE), and total effort (TE). The cutoff levels for $c$ are approximated for illustrative purposes. Actual cutoffs are presented in Section 3.5.

Figure 3.6 depicts the best disclosure policies for different values of the ratio of the value of winning for the low type to the value of winning for the high type for both
optimist and pessimist auctioneer.

Denoting by $c$ the ratio of the value of winning for the low-type to the value of winning for the high-type, we show that an optimist auctioneer prefers the policy DH for high levels of $c$, whereas he is indifferent between FD and DL for low levels of $c$. Moreover, the full disclosure policy elicits the highest aggregate effort and delivers the perfect prize allocation efficiency from his perspective.

A pessimist auctioneer mostly disagrees with the optimist one because he prefers the full disclosure policy the least. For all three performance measures, he prefers the policies FC and DH for low levels of $c$, i.e. when $c < \frac{6 - \sqrt{6}}{10}$. He believes that the players enjoy the highest payoff, and price allocation efficiency is maximized under policy DH for high levels of $c$, while, he adopts policy FC for lower levels of $c$. Lastly, in terms of total expected effort, he mostly prefers the policy FC as long as $c$ is not approximately in $(0.35, 0.38)$. 
Appendix A
Siegel (2014a) Algorithm

Let $V_i = \{v_{i1}, v_{i2}, \ldots, v_{ik_i}\}$ such that $v_{i1} > v_{i2} > \ldots > v_{ik_i}$ be player $i$’s possible valuations and $F_i$, with full support, be the probability distribution over $V_i$, from which player $i$’s valuation is drawn. Lemma 2 of the Siegel (2014a) shows that in any monotonic equilibrium, player $i$’s best response set is either a point or an interval. For any two consecutive valuations $v_i' > v_i$, the upper bound of $BR_i(v_i)$ is equal to the lower bound of $BR_i(v_i')$. Moreover,

$$\sup BR_1(v_1^1) = \sup BR_2(v_2^1) > 0 \quad \text{and} \quad \inf BR_1(v_1^{k_1}) = \inf BR_2(v_2^{k_2}) = 0$$

Defining the joint partition set as the set of intervals $[x, y]$, where $x$ and $y$ are consecutive elements in $\cup_{i=1,2,v_i \in V_i} \{\inf BR_i(v_i), \sup BR_i(v_i)\}$ and starting from the top interval of the partition, player $i$’s bidding distribution will have constant density $g_i(v_i^1, \cdot) = \frac{1}{f(v_i^1)v_i^1}$ over the top interval. The player with the greater density will exhaust her bidding probability first. Therefore, the length of the top interval, $L_1$, is going to be $L_1 = \min\{f(v_1^1)v_1^1, f(v_2^1)v_2^1\}$. In other words, $L_1$ must satisfy $L_1g_i(v_i^1, \cdot) = 1$ for the player who precisely exhaust her highest type’s bidding probability. If it is player 1 who exhausts his bidding probability in the top interval, then for player 2 we have $L_1g_2(v_2^1, \cdot) \leq 1$. That is to say that the highest type of player 2 may still have some bidding probability to spend in the penultimate interval.

The next step of the algorithm tries to find the length of the second interval from the top and the bidding densities of the players over this interval. For illustrative purpose, suppose that we have $L_1g_2(v_2^1, \cdot) < 1$. That is to say that player 2 has not exhausted her highest type’s bidding probability in the top interval. Hence, in the second top interval, the second highest type of Player 1 bids against the highest type of Player 2.
Similar to the first step, we have \( g_1(v_1^2, \cdot) = \frac{1}{f(v_1^2)v_2^1} \) and \( g_2(v_2^1, \cdot) = \frac{1}{f(v_2^1)v_1^2} \) over this interval. The necessary length of the interval for player 1 to exhaust his bidding probability is \( f(v_1^2)v_2^1 \) while it is \((1 - L_1 f(v_1^2)v_1^2) f(v_2^1)v_2^2 \) for player 2 since her highest type has already exhausted some of her winning probabilities over the top interval. So, the length of the second top interval is \( L_2 = \min\{f(v_1^2)v_2^1, (1 - L_1 f(v_1^2)v_1^2) f(v_2^1)v_2^2\} \).

Proceeding with this iterative construction, suppose that for the top \( t \) intervals of the joint partition the bidding densities of the players and the lengths of the first \( t \) top intervals have been computed. Moreover, suppose player \( i \) has exhausted the bidding probability of her \( 0 \leq t_i < k_i \) highest types. Then in the \((t + 1)\)st interval player 1’s \( v_{1i}^{t_1+1} \) type and type \( v_{2i}^{t_2+1} \) of his opponent are mixing their bids according to constant bidding density functions

\[
\begin{align*}
g_1(v_{1i}^{t_1+1}, \cdot) &= \frac{1}{f_1(v_{1i}^{t_1+1})v_{2i}^{t_2+1}} \quad \text{and} \quad g_2(v_{2i}^{t_2+1}, \cdot) &= \frac{1}{f_2(v_{2i}^{t_2+1})v_{1i}^{t_1+1}}
\end{align*}
\]

Therefore the length of the \((t + 1)\)st interval is the required length for one of the players, say \( i \), to exhaust the bidding probability of his type \( v_{1i}^{t_1+1} \). The algorithm stops when one of the players exhausts the bidding probability of her lowest type. This happens at 0, and the other player expends her remaining bidding probabilities as an atom at 0.

**Figure A.1.** An example of players’ best response structures where \( T \) denotes the common upper bound for the best response set when player 1 has two types, player 2 has four types, and player 2 has an atom at 0.
Appendix B
Proofs of Lemmas in Chapter 1

B.1 Proof of Lemma 4

Lemma 4: If \( p \) is an optimal distribution choice for the entrant and it makes the entrant relatively-weak, then there is at most two elements in its support.

Proof. Let \( p \) be an optimal distribution choice for the entrant, and \( \{b_1, b_2, \ldots, b_t\} \) with \( b_1 > b_2 > \ldots > b_t \) be its support. To the contrary, suppose that \( t > 2 \). For sufficiently small \( \varepsilon > 0 \), consider \( p_k^\varepsilon \) that is an \( \varepsilon \)-SSD deviation to \( p \) for type \( b_k \) for some \( 1 > k > t \).

Moreover, denote by \( T_i \) the highest bid the entrant’s \( b_i \) type bids in equilibrium. \([T_{i-1}, T_i]\) refers to the interval in which the entrant’s \( b_i \) type randomizes her bids. Figure B.1 could be used as a visual aid to follow the calculations. The numbers under the partition’s intervals represent the differences in weights that the distributions \( p_k^\varepsilon \) and \( p \) assign to the entrant’s types. For instance, \( p_k^\varepsilon \) assigns \( \varepsilon(b_1 - b_t) \) less mass to drawing \( d_k \) type than \( p \).

\[
\begin{array}{cccccccc}
\text{Incumbent} & & & & & & & v \\
\text{Entrant} & b_t & \ldots & b_k & \ldots & b_1 \\
\text{Partition} & d_t & \ldots & d_k & \ldots & d_1 \\
0 & T_t & \ldots & T_{k+1} & T_k & \ldots & T_2 & T_1 \\
& \varepsilon(b_1 - b_k) & \ldots & -\varepsilon(b_1 - b_t) & \ldots & \varepsilon(b_k - b_t) & \\
\end{array}
\]

\textbf{Figure B.1.} A depiction of the equilibrium bidding distributions’ supports, in which the entrant is the relatively-weak player and has more than two active types.

When the informed player employs \( p_k^\varepsilon \) instead of \( p \), the algorithm produces only three different intervals. In these intervals, the entrant’s \( b_1, b_k, \) and \( b_t \) types randomize their bids. The changes in the lengths of the first two of these intervals are simply,
\[ \Delta d_1 = \varepsilon v(b_k - b_t) \] and \[ \Delta d_k = \varepsilon v(b_1 - b_t). \] This is because the entrant switches her types in these intervals. Since it is the incumbent who exhausts his bidding probability first, we take into account the fact that the incumbent has exhausted different bidding probabilities in the intervals \([T_{k+1}, T_k]\) and \([T_2, T_1]\) when calculating the change in the last interval’s length. Since his total bidding probability does not change, we could calculate the change in the last interval’s length as follows:

\[
0 = g_1(b_t, v)\Delta d_t + g_1(b_k, v)\Delta d_k + g_1(b_1, v)\Delta d_1 = \frac{\Delta d_t}{b_t} - \frac{\varepsilon v(b_1 - b_t)}{b_k} + \frac{\varepsilon v(b_k - b_t)}{b_1} = \Delta d_t b_t - \varepsilon v(b_1 - b_t) b_k + \varepsilon v(b_k - b_t) b_1
\]  
(B.1)

Observe that \(b_1 - b_t > b_k - b_t\) and \(b_k < b_t\). Therefore, \(\Delta d_t > 0\). Because the only change is in these three intervals’ lengths and how the players bid in these intervals, we could calculate the changes in the conditional expected payoffs of the entrant’s types.

Consider a type \(b_i\) that is lower than \(b_k\), i.e., \(i > k\). Now, considering the highest bid in her best response set before and after the change, we could calculate the change in her conditional payoff as:

\[
\Delta u_i := u_1(b_i|p^\varepsilon_k) - u_1(b_i|p) = \Delta Pr \{\text{winning with bid } T_i\} b_i - \Delta T_i = \frac{\Delta d_t}{b_t} b_i - \Delta d_t = \Delta d_t \frac{b_i - b_t}{b_t} \geq 0
\]

The last inequality holds with equality only when \(i = t\). Notice that the same inequality is valid if we considered the lowest bid for the type \(b_k\). This time, however, the inequality is strict.

\[
\Delta u_k := u_1(b_k|p^\varepsilon_k) - u_1(b_k|p) = \Delta Pr \{\text{winning with bid } T_{k-1}\} b_k - \Delta T_{k-1} = \frac{\Delta d_t}{b_t} b_k - \Delta d_t = \Delta d_t \frac{b_k - b_t}{b_t} > 0
\]

We apply the same approach to calculate the change in the conditional payoffs for all the types that are greater than \(b_k\).
\[ \Delta u_i = \Delta Pr \{ \text{win with bid } T_i \} b_i - \Delta T_i \]
\[ = \left( \frac{\Delta d_t}{b_t} - \frac{\varepsilon v(b_1 - b_t)}{b_k} \right) b_i - \left( \frac{\varepsilon v(b_1 - b_t)}{b_k} \right) \]
\[ = -\frac{\varepsilon v(b_k - b_t)}{b_1} b_i + b_t \left( -\frac{\varepsilon v(b_1 - b_t)}{b_k} + \frac{\varepsilon v(b_k - b_t)}{b_1} \right) + \varepsilon v(b_t - b_t) \]
\[ = -\frac{\varepsilon v(b_k - b_t)}{b_1} (b_i - b_t) + \varepsilon v(b_1 - b_t) \frac{(b_k - b_t)}{b_k} \]
\[ = \varepsilon v(b_k - b_t) \left( \frac{b_1 - b_t}{b_k} - \frac{b_i - b_t}{b_1} \right) \]
\[ > 0, \]

where the third equality is obtained by simply plugging \( \Delta d_t \) from the Equation B.1, and the inequality is due to \( b_1 - b_t > b_i - b_t \) and \( b_k < b_1 \).

Following lemma will be useful to show that the entrant’s highest type also benefits from an \( \varepsilon \)-SSD deviation.

**Lemma 12.** Let \( x > y > z \) be three positive real numbers. Then,
\[
\frac{x - y}{z} + \frac{y - z}{x} + \frac{z - x}{y} > 0
\]

**Proof.** There exist a unique \( \alpha \in (0, 1) \) such that \( y = \alpha x + (1 - \alpha)z \). So, we have
\[
\frac{x - y}{z} + \frac{y - z}{x} + \frac{z - x}{y} = \frac{x - (\alpha x + (1 - \alpha)z)}{z} + \frac{(\alpha x + (1 - \alpha)z) - z}{x} + \frac{z - x}{\alpha x + (1 - \alpha)z}
\]
\[
= (x - z) \left( (1 - \alpha) \frac{1}{z} + \alpha \frac{1}{x} - \frac{1}{\alpha x + (1 - \alpha)z} \right)
\]
\[
> 0,
\]

where the inequality follows from the fact that the function \( f(w) = \frac{1}{w} \) is convex. ■

Observe that the \( b_1 \) type wins with probability one by bidding \( T_1 \). Therefore, when we consider her highest best response, the change in her payoff will simply be the change in her payment, i.e., her bid.

\[ \Delta u_1 = -\Delta T_1 \]
\[ = -(\Delta d_t - \varepsilon v(b_1 - b_t) + \varepsilon v(b_k - b_t)) \\
= -(\Delta d_t - \varepsilon v(b_1 - b_k)) \\
= -\varepsilon v b_t \left( \frac{b_1 - b_t}{b_k} - \frac{b_k - b_t}{b_1} - \frac{b_1 - b_k}{b_t} \right) \\
= \varepsilon v b_t \left( \frac{b_1 - b_k}{b_t} + \frac{b_k - b_t}{b_1} + \frac{b_t - b_k}{b_t} \right) \\
> 0, \]

where the inequality follows from Lemma 12. Lastly, being a relatively-weak player results in the entrant’s \( b_t \) type to obtain zero expected payoffs. Moreover, we could calculate the payoffs of her \( b_k \) and \( b_1 \) types as follows:

\[ u_k = \left( \sum_{t \geq i \geq k} \frac{d_i}{b_i} \right) b_k - \left( \sum_{t \geq i \geq k} d_i \right) = \frac{d_t}{b_t} (b_k - b_t) + \sum_{t > i \geq k} \frac{p_i v}{b_i} (b_k - b_i), \]

and, similarly,

\[ u_1 = \frac{d_t}{b_k} (b_1 - b_t) + \sum_{t > i \geq k} \frac{p_i v}{b_i} (b_1 - b_i) + \sum_{k > i \geq 1} \frac{p_i v}{b_i} (b_1 - b_i) \]

Adding \( u_1(b_t | \mathbf{p}^*_k) \), \( u_1(b_k | \mathbf{p}^*_k) \), and \( u_1(b_1 | \mathbf{p}^*_k) \) after multiplying them with \( \varepsilon(b_1 - b_k) \), \(-\varepsilon(b_1 - b_t)\), and \( \varepsilon(b_k - b_t) \), respectively, we obtain the following:

\[ S := +\varepsilon(b_1 - b_k) u_1(b_t | \mathbf{p}^*_k) - \varepsilon(b_1 - b_t) u_1(b_k | \mathbf{p}^*_k) + \varepsilon(b_k - b_t) u_1(b_1 | \mathbf{p}^*_k) \]

\[ = -\varepsilon(b_1 - b_t) \left( \frac{d_t}{b_t} (b_k - b_t) + \sum_{t > i \geq k} \frac{p_i v}{b_i} (b_k - b_i) \right) \]

\[ + \varepsilon(b_k - b_t) \left( \frac{d_t}{b_k} (b_1 - b_t) + \sum_{t > i \geq k} \frac{p_i v}{b_i} (b_1 - b_i) + \sum_{k > i \geq 1} \frac{p_i v}{b_i} (b_1 - b_i) \right) \]

\[ = \sum_{t > i \geq k} \frac{p_i v}{b_i} \varepsilon \left( - (b_1 - b_t)(b_k - b_t) + (b_k - b_t)(b_1 - b_i) \right) + \varepsilon(b_k - b_t) \sum_{k > i \geq 1} \frac{p_i v}{b_i} (b_1 - b_i) \]

\[ = \varepsilon (b_1 - b_k) \sum_{t > i \geq k} \frac{p_i v}{b_i} (b_i + b_t) + \varepsilon (b_k - b_t) \sum_{k > i \geq 1} \frac{p_i v}{b_i} (b_1 - b_i) \]

\[ > 0 \]
We, now, have all the pieces we need to check if $\varepsilon$-SSD deviation is a profitable one.

\[
U_1(p_k^\varepsilon) - U_1(p) = \sum_{t \geq i \geq 1} p_i^\varepsilon(b_i)u_1(b_i|p_k^\varepsilon) - \sum_{t \geq i \geq 1} p_iu_1(b_i|p) \\
= \sum_{t \geq i \geq 1} p_i\left(u_1(b_i|p_k^\varepsilon) - u_1(b_i|p)\right) + \varepsilon(b_1 - b_k)u_1(b_1|p_k^\varepsilon) \\
- \varepsilon(b_1 - b_t)u_1(b_t|p_k^\varepsilon) + \varepsilon(b_k - b_t)u_1(b_1|p_k^\varepsilon) \\
= \sum_{t \geq i \geq 1} p_i\Delta u_i + S \\
> 0.
\]

Therefore, $p_k^\varepsilon$ is a profitable deviation from $p$. \[\blacksquare\]
Appendix C
Proofs of Lemmas in Chapter 2

C.1 Proof of Lemma 6

Lemma 6: Let $p$ and $q$ be distributions over $V$, with supports $\{b_1, b_2, \ldots, b_t\}$, where $b_1 > b_2 > \ldots > b_t > 0$, and $\{a_1, a_2, \ldots, a_z\}$, where $a_1 > a_2 > \ldots > a_z > 0$, respectively. Suppose player 1 employs $p$ in the first stage of the game while player 2 picks $q$ as his distribution choice. Moreover, suppose that $b_m$ type exhausts her bidding probability when she bids against $a_k$ type. Let $\hat{p}_{m}^{\varepsilon, \delta}$ be an epsilon-delta deviation to $p$ for $b_m$ type. Instead of $p$, when we feed in Siegel (2014a)'s algorithm with $\hat{p}_{m}^{\varepsilon, \delta}$ to construct the equilibrium bidding distributions of the all-pay auction stage, until we reach the construction of the last interval, the only change in the lengths of the intervals over which the players' various types bid will be in the two, which share the bid in $b_m$ and $b_{m+1}$ types’ best response sets. Moreover, the total length of these two intervals will increase by $\gamma a_k$, and the last interval will shrink by $\gamma a_z$ if player 1 is not relatively-strong player, otherwise, it will not change.

Proof. We use Figure C.4 as a visual aid in proving the lemma. It depicts the supports of the equilibrium distributions in the all-pay auction stage when the player 1 employs $p$ and player 2 picks $q$ as their distribution choices. It assumes that player 1 is relatively-weak player in the all-pay auction. We will later discuss what would differ if she was not relatively-weak. For a more comprehensive case, suppose that $b_m$ and $b_{m+1}$ types bid against $a_k$ type. \footnote{One could do the same analysis when player 2 switches from his $a_k$ type to his $a_{k+1}$ type at the same time as player 1 switches from her $b_m$ type to her $b_{m+1}$ type.} Denote these two intervals by $[T_s, T_{s-1}]$ and $[T_{s+1}, T_s]$, respectively. Let $d_{i,j}$ be the length of the interval in which $b_i$ and $a_j$ types randomize their bids against
each other. We sometimes refer to those intervals as $d_{i,j}$. Therefore, Lemma 6 claims that an epsilon delta deviation will not affect $d_{i,j}$ and how players bid in these intervals unless $(i, j) \in \{(m, k), (m + 1, k), (t, z)\}$.

![Figure C.1](image)

Figure C.1. A depiction of the equilibrium bidding distributions’ supports. $d_{i,j}$ represents the length of the sub-interval in the partition line, in which player 1’s $b_t$ type and player 2’s $a_j$ type randomize their bids.

Observe that when we feed the algorithm with $(p_k^{\varepsilon,\delta}, q)$, it will produce the same outcome until it reaches at the interval in which $a_k$ and $b_m$ types bid against each other as it would when fed with $(p, q)$. We next show that when the algorithm reaches at the point where either $b_{m+1}$ or $a_k$ type exhaust their bidding probabilities first, that is the point $T_{s+1}$ in the figure above, the remaining bidding probability of the other type is the same as when we fed the algorithm with $(p, q)$. So, the algorithm will have produced the same output from this point until it reaches at the last interval in the partition line. In other words, the only difference will be in the intervals $d_{m,k}$, $d_{m+1,k}$, and, possibly, $d_{t,z}$.

At the point $T_s$, $b_m$ type was exhausting her bidding probability before the deviation. After the deviation, she has $\varepsilon$ more probability to exhaust, therefore, she needs a longer interval to exhaust her bidding probability. So, the interval $[T_s, T_{s-1}]$ will expand by $\varepsilon a_k$, i.e., $\Delta d_{m,k} = \varepsilon a_k$. Suppose that it was player 2 who was switching his type at $T_{s+1}$. For small enough $\varepsilon > 0$, it is still him who switches type. Therefore, the probability that $a_k$ type exhausts in the intervals $d_{m,k}$ and $d_{m+1,k}$ must be the same after the epsilon-delta deviation. So, we have the following equality.

\[
\frac{d_{m,k}}{q_k b_m} + \frac{d_{m+1,k}}{q_k b_{m+1}} = \frac{d'_{m,k}}{q_k b_m} + \frac{d'_{m+1,k}}{q_k b_{m+1}}.
\]

Solving for the change in $d_{m+1,k}$, we obtain the following.

\[
\Delta d_{m+1,k} = -b_{m+1} \frac{\Delta d_{m,k}}{b_m} = -\varepsilon \frac{b_{m+1}}{b_m} a_k = -\delta a_k
\]
The change in the probability that \( b_{m+1} \) type exhausts in the interval \( d_{m+1,k} \), then, is \( \frac{-\delta a_k}{a_k} = -\delta \). Since the epsilon-delta deviation assign \( \delta \) less probability to \( b_{m+1} \) type, her remaining probability power to exhaust will be the same as before after the point \( T_{s+1} \). Therefore, the change in the total length of the two intervals is as follows:

\[
\Delta d_{m,k} + \Delta d_{m+1,k} = \varepsilon a_k - \delta a_k = \gamma a_k
\]

Similarly, if it was \( b_{m+1} \) type who was exhausting her bidding probability at the point \( T_{s+1} \), then we would have

\[
\Delta d_{m+1,k} = p_k^{\varepsilon,\delta} (b_{m+1}) a_k - p (b_{m+1}) a_k = -\delta a_k.
\]

Therefore, the change in the probability that \( a_k \) type exhausts in the intervals \( d_{m,k} \) and \( d_{m+1,k} \) could be calculated as follows:

\[
\left( \frac{d'_{m,k}}{q_k b_m} + \frac{d'_{m+1,k}}{q_k b_{m+1}} \right) - \left( \frac{d_{m,k}}{q_k b_m} + \frac{d_{m+1,k}}{q_k b_{m+1}} \right) = \varepsilon a_k + \frac{-\delta a_k}{q_k b_m} + \frac{-\delta a_k}{q_k b_{m+1}} = 0.
\]

So, again, both players have the same amount of bidding power after \( T_{s+1} \) until the algorithm reaches at the construction of the last interval. Furthermore, the change in the total length of the two intervals is

\[
\Delta d_{m,k} + \Delta d_{m+1,k} = \varepsilon a_k - \delta a_k = \gamma a_k.
\]

In the last interval, if player 1 is not a relatively-strong one, then there will be no change because this length was required for \( a_z \) type to exhaust his bidding probabilities, and he still has the same bidding probability to consume. Otherwise, it will shrink by \( \gamma a_z \) because \( b_t \) type will need an interval of length \( (p(b_t) - \gamma) a_z \) instead of \( p(b_t) a_z \).

\[\blacksquare\]

C.2 Proof of Lemma 7

**Lemma 7:** No players’ three types bid against the other player’s one type in any equilibrium.

**Proof.** Let \( p \) and \( q \) be distributions over \( V \), with supports \( \{b_1, b_2, \ldots, b_t\} \), where \( b_1 > b_2 > \ldots > b_t > 0 \), and \( \{a_1, a_2, \ldots, a_z\} \), where \( a_1 > a_2 > \ldots > a_z > 0 \), respectively. Suppose that the strategy profile \((p, q)\) constitutes an equilibrium of the reduced game. To the contrary, suppose that in the equilibrium of the corresponding all-pay auction
player 1’s \( b_{m+1} \), \( b_m \), and \( b_{m-1} \) types bid against player 2’s \( a_k \) type. Figure C.4 depicts the supports of the equilibrium bidding distributions.

![Diagram of bidding distributions](image)

**Figure C.2.** A depiction of the equilibrium bidding distributions’ supports. \( d_{i,j} \) represents the length of the sub-interval in the partition line, in which player 1’s \( b_i \) type and player 2’s \( a_j \) type randomize their bids.

Consider an epsilon-delta deviation to \( p \) for type \( b_{m-1} \). This deviation cannot be profitable because \( p \) is a best response to \( q \) for player 1. Denote by \( T_i \) the highest bid in \( b_i \) type’s best response set. As the best response set changes, \( T_i \) will change as well. To calculate the change in the conditional payoff of \( b_i \) type, we will calculate the change in \( T_i \) and the winning probability by bidding \( T_i \), and Lemma 6 will be useful for that. We know that before \( T_{m-1} \) and after \( T_{m+q} \), the players have the same bidding probability power after an epsilon-delta deviation. Therefore, the winning probability will change only for \( T_m \). Let 1 take value zero if the player 2 is the relatively-weak player and one otherwise. Therefore, we have the following two equations.

\[
\Delta T_i = \begin{cases} 
-1 \gamma a_z + \gamma a_k , & i < m \\
-1 \gamma a_z - \delta a_k , & i = m \\
-1 \gamma a_z , & i > m 
\end{cases} \quad \text{and} \quad \Delta Pr (\text{win}|T_i) = \begin{cases} 
0 , & i < m \\
-\frac{\delta a_k}{b_{m-1}} , & i = m \\
0 , & i > m 
\end{cases}
\]

where \( \Delta T_i \) and \( \Delta Pr (\text{win}|T_i) \) are the change in the highest bid in \( b_i \)’s best response set and the winning probability of that bid, respectively. Using these two, we can calculate the change in \( b_i \)’s conditional payoffs as follows:

\[
\Delta u_1(b_i) = \Delta Pr (\text{win}|T_i) b_i - \Delta T_i
\]

where, \( \Delta u_1(b_i) \) is the change in \( b_i \)’s conditional payoff after the deviation. Plugging (C.1)
into (C.2), we obtain the following.

\[
\Delta u_1(b_i) = \begin{cases} 
+1 \gamma a - a_k & , i < m \\
-\frac{a_{m-1}}{b_{m-1}} b_m + 1 \gamma a + \delta a_k & , i = m \\
+1 \gamma a & , i > m
\end{cases}
\]

As a result, the payoff from the deviation is

\[
\Delta U_1 = U_1(p_{\varepsilon,\delta}^{e,-1}) - U_1(p) \\
= \sum_{t \geq i \geq 1} p_{m-1}^{e,\delta}(b_i) u_1(b_i|p_{m-1}^{e,\delta}) - \sum_{t \geq i \geq 1} p(b_i) u_1(b_i|p) \\
= \sum_{t \geq i \geq 1} p(b_i) \left( u_1(b_i|p_{m-1}^{e,\delta}) - u_1(b_i|p) \right) - \gamma u_1(b_i|p_{m-1}^{e,\delta}) - \delta u_1(b_{m}|p_{m-1}^{e,\delta}) + \varepsilon u_1(b_{m-1}|p_{m-1}^{e,\delta}) \\
= 1 \gamma a - (p_1 + \ldots + p_{m-1})a_k - \gamma u_1(b_i|p_{m-1}^{e,\delta}) - \delta u_1(b_{m}|p_{m-1}^{e,\delta}) + \varepsilon u_1(b_{m-1}|p_{m-1}^{e,\delta}) \\
= 1 \gamma a - (p_1 + \ldots + p_{m-1})a_k - \gamma u_1(b_i|p) - \delta u_1(b_{m}|p) + \varepsilon u_1(b_{m-1}|p) - \varepsilon a_k
\]

Since \(b_{m-1}\) and \(b_m\) types share the bid \(T_m\) in their best response sets, the difference in their payoff could be calculated by considering this bid.

\[
u_1(b_{m-1}|p) - u_1(b_m|p) = (b_{m-1} - b_m) \Pr(\text{win}|T_m)
\]

(C.3)

Remember that \(\gamma = \varepsilon - \delta = \varepsilon \frac{b_{m-1} - b_m}{b_{m-1}}\). Plugging (C.3) into above equation and dividing by \(\gamma\), we get

\[
\frac{\Delta U_1}{\gamma} = 1 \gamma a - (p_1 + \ldots + p_{m-1})a_k - u_1(b_i|p) + u_1(b_{m}|p) + b_{m-1} \Pr(\text{win}|T_m) - \varepsilon a_k
\]

(C.4)

For the epsilon-delta deviation not to be a profitable one, \(\Delta U_1\) must be non-positive. That is

\[
1 \gamma a - (p_1 + \ldots + p_{m-1})a_k - u_1(b_i|p) + u_1(b_{m}|p) + b_{m-1} \Pr(\text{win}|T_m) - \varepsilon a_k \leq 0
\]

(C.5)

Similar to an epsilon-delta deviation, the payoff from a reverse-epsilon-delta deviation,
hence, satisfies the following inequality.

$$\frac{\Delta U_1}{\gamma} = -1a_z + (p_1 + \ldots + p_{m-1})a_k + u_1(b_1|p) - u_1(b_m|p) - b_{m-1}Pr(win|T_m) - \varepsilon a_k$$  \hspace{1cm} (C.6)

For the reverse-epsilon-delta deviation not to be a profitable one, we must have

$$1a_z - (p_1 + \ldots + p_{m-1})a_k - u_1(b_t|p) + u_1(b_m|p) + b_{m-1}Pr(win|T_m) + \varepsilon a_k \geq 0 \quad (C.7)$$

Combining (C.5) and (C.7), we obtain the following inequality system for all sufficiently small $\varepsilon > 0$.

$$-\varepsilon a_k \leq 1a_z - (p_1 + \ldots + p_{m-1})a_k - u_1(b_t|p) + u_1(b_m|p) + b_{m-1}Pr(win|T_m) \leq \varepsilon a_k$$

Since the above inequalities hold for all sufficiently small $\varepsilon > 0$, we must have the following equality.

$$1a_z - (p_1 + \ldots + p_{m-1})a_k - u_1(b_t|p) + u_1(b_m|p) + b_{m-1}Pr(win|T_m) = 0. \quad (C.8)$$

Now, consider a reverse-epsilon-delta deviation to $p$ for type $b_m$. Similar to (C.6), the payoff from this deviation satisfies the following.

$$\frac{\Delta U_1}{\gamma} = \frac{1}{\gamma} \left( U_1(p_{\varepsilon,\delta}^m) - U_1(p) \right)$$

$$= -1a_z + (p_1 + \ldots + p_m)a_k + u_1(b_1|p) - u_1(b_{m+1}|p) - b_{m+1}Pr(win|T_{m+1}) - \varepsilon a_k$$

Plugging (C.8) into above equation we obtain

$$\frac{\Delta U_1}{\gamma} = p_m a_k + u_1(b_m|p) + b_{m-1}Pr(win|T_m) - u_1(b_{m+1}|p) - b_{m}Pr(win|T_{m+1}) - \varepsilon a_k.$$  

Because $u_1(b_m|p) > u_1(b_{m+1}|p)$, $b_{m-1} > b_m$, and $Pr(win|T_m) > Pr(win|T_{m+1})$, the reverse-epsilon-delta deviation to $p$ for type $b_m$ is profitable for sufficiently small $\varepsilon > 0$. Contradiction. Thus, in equilibrium, three types of a player won’t bid against the same type of her opponent.  \hspace{1cm} \blacksquare


C.3 Proof of Lemma 10

Lemma 10: If a player is not relatively-weak, then her lowest type should bid against her opponent’s two lowest types.

Proof. Let \( p \) and \( q \) be distributions over \( V \), with supports \( \{b_1, b_2, \ldots, b_t\} \), where \( b_1 > b_2 > \ldots > b_t > 0 \), and \( \{a_1, a_2, \ldots, a_z\} \), where \( a_1 > a_2 > \ldots > a_z > 0 \), respectively. Suppose player 1 employs \( p \) in the reduced game while player 2 picks \( q \) as his distribution choice. Moreover, assume that \((p, q)\) constitutes an equilibrium.

Suppose that player 1 is not a weak player, and in the last interval in the equilibrium construction, her \( b_t \) type bids only against her opponent’s \( a_z \) type.

Let us analyze it under two cases.

Case 1. Suppose \( b_t \) and \( a_z \) types only bid against each other.

![Figure C.3](image)

**Figure C.3.** A depiction of the equilibrium bidding distributions’ supports. \( d_{i,j} \) represents the length of the sub-interval in the partition line, in which player 1’s \( a_i \) type and player 2’s \( b_j \) type randomize their bids.

Case 2. Suppose \( b_t \) and \( b_{t-1} \) types bid against \( a_z \) type.

![Figure C.4](image)

**Figure C.4.** A depiction of the equilibrium bidding distributions’ supports. \( d_{i,j} \) represents the length of the sub-interval in the partition line, in which player 1’s \( a_i \) type and player 2’s \( b_j \) type randomize their bids.

\[\blacksquare\]
C.4 Proof of Lemma 11

Lemma 11: If a player is relatively-strong, then her highest type should bid against her opponent’s two highest types.

Proof. Let player 1 be relatively-strong player. Suppose, to the contrary, $b_1$ bids against only $a_1$.

Consider an epsilon-delta deviation to $b_1$ type. Similar to the proof of Lemma 7, the payoff from the deviation for $b_i$ is the following.

$$
\Delta u_1(b_i) = \begin{cases} 
\gamma a_z - \gamma a_k, & i = 1 \\
-\frac{\varepsilon a_k}{b_{m-1}} b_m + \gamma a_z + \delta a_k, & i = 2 \\
\gamma a_z, & i > 2
\end{cases}
$$
Appendix D
Proofs of Propositions in Chapter 3

D.1 Proof of Proposition 7

In this section, for any given distribution choice profile \((p_1, p_2)\) with \(p_i \in (0, 1)\) for \(i = 1, 2\), I characterize the equilibrium bidding behaviors of the players in the all-pay auction stage when the auctioneer plays D (disclose) under the partial disclosure policy that is to disclose if there is a low type player, i.e. \(G_{DL}(D | p_1, p_2)\).

**Lemma 13.** There is no bid at which both low type players have an atom.

**Proof.** Suppose, to the contrary, there is a bid \(x \geq 0\) at which both low types place an atom. Moreover, suppose that player \(i\)'s low type does not win the tie with probability 1. Then she could be better of by slightly increasing her bid. So, \(x\) is not a best response. ■

**Lemma 14.** There is no positive bid at which some player has an atom.

**Proof.** Suppose \(v_i\) type of player \(i\) bids \(x > 0\) with positive probability. Therefore, no types of player \(-i\), who are playing against player \(i\)'s \(v_i\) type are bidding \(x\) with positive probability, neither do they have any best responses in \((x - \varepsilon, x)\) for sufficiently small \(\varepsilon > 0\). But, then player \(i\) obtains strictly higher payoff by bidding slightly less than \(x\) with the same probability. Hence, \(x\) cannot be a best response for player \(i\). ■

**Corollary 2.** In any equilibrium, each type has continuous bidding distributions above zero.

No-gap property: In any equilibrium, there is no interval \((x, y)\) with \(x < y\), in which both players bid with probability zero and some player bids above \(y\) with positive
probability. \(^1\)

**Proof.** To the contrary, suppose there exists such an interval. Consider such a maximal interval \((x, y)\). A player would only bid in interval \([y', y)\) with \(y < y'\), if her opponent bids \(y\) with positive probability. However, her opponent would be strictly better off by slightly increasing his bid if his opponent bids \(y\) and wins the tie, or by decreasing his bid if his opponent does not bid \(y\) with positive probability. ■

**Corollary 3.** If player \(i\) bids in some interval \((x, y)\), so does player \(-i\).

Now that we have these tools, we can utilize them in proving Proposition 7.

When the auctioneer adopts the disclosure policy DL that is to disclose if there is at least one low type player, upon the auctioneer’s disclosing move, a high type player knows that she is facing a low type opponent for sure, while a low type player thinks that he is facing a high type player with his prior belief. Following lemma states that since a high type player knows that she is facing a low type opponent for sure, she will not play any bid that her opponent’s low type is not bidding.

**Lemma 15 (Lu et al. (2018) - Lemma 5).** \(\operatorname{Supp} [F_h^i] \subseteq \operatorname{Supp} [F_{\ell-i}].\)

**Proof.** Suppose there exists a bid \(x > 0\) such that \(x \in \operatorname{Supp} [F_h^i]\) and \(x \not\in \operatorname{Supp} [F_{\ell-i}].\) Because the equilibrium bidding distributions are continuous, there is an open ball around \(x\), \(B(x)\), such that for all \(x^* \in B(x)\), \(F_{\ell-i}(x) = F_{\ell-i}(x^*).\) Hence, deviating to some \(x^* < x\) is a profitable deviation for player \(i\) because it does not change the her winning probability while decreasing her payment. Therefore, \(x \not\in \operatorname{Supp} [F_h^i].\) Contradiction. On the other hand, when \(x = 0\), since a high type bids only against her opponent’s low type, player \(i\)’s high type would obtain a zero payoff if \(x \not\in \operatorname{Supp} [F_{\ell-i}].\) However, player \(i\)’s high type would guarantee herself a payoff of \(v_h - v_{\ell}\) by bidding \(v_{\ell}\) because player \(-i\)’s low type would not bid any bid above \(v_{\ell}\) in equilibrium. Therefore, \(x \in \operatorname{Supp} [F_{\ell-i}].\) ■

Following lemma shows that no type has a “hole” in her equilibrium bidding distribution.

**Lemma 16.** In any equilibrium, the support of \(F_{\theta}^i\) is an interval for all \(i \in \{1, 2\}\) and \(\theta \in \{h, \ell\}.\)

\(^1\)In Siegel (2014)’s environment, we have this property for many players and many prizes.
Proof. Let us first show that the equilibrium bidding distribution of a low type has no “hole” in its support. Suppose to the contrary and consider such a maximal interval \((a, b)\) in which \(F^i_\ell\) is constant, while \(a\) and \(b\) belong to its support. Due to Lemma 15, \(F^{-i}_\ell\) is also constant over \((a, b)\). Because both \(a\) and \(b\) are best responses for player \(i\)’s low type, they must yield the same payoff to her. Therefore,

\[
0 = u^i_\ell(b) - u^i_\ell(a) = \left(p_{-i}F^{-i}_h(b)v_\ell + (1 - p_{-i})F^{-i}_\ell(b)v_\ell - b\right) - \left(p_{-i}F^{-i}_h(a)v_\ell + (1 - p_{-i})F^{-i}_\ell(a)v_\ell - a\right) = (1 - p_{-i})v_\ell \left(F^{-i}_\ell(b) - F^{-i}_\ell(a)\right) - (b - a) \tag{D.1}
\]

On the other hand, because of the no-gap property, along with the fact that the bidding distributions are continuous, both \(F^i_\ell\) and \(F^{-i}_\ell\) have full support in \([a, b]\). As a result,

\[
0 = u^i_h(b) - u^i_h(a) = \left[F^{-i}_\ell(b)v_h - b\right] - \left[F^{-i}_\ell(a)v_h - a\right] = v_h \left(F^{-i}_\ell(b) - F^{-i}_\ell(a)\right) - (b - a) \tag{D.2}
\]

Combining (D.1) and (D.2), we obtain

\[
(1 - p_{-i})v_\ell \left(F^{-i}_\ell(b) - F^{-i}_\ell(a)\right) = v_h \left(F^{-i}_\ell(b) - F^{-i}_\ell(a)\right)
\]

Since \(F^{-i}_\ell\) has full support in \([a, b]\), \(F^{-i}_\ell(b) > F^{-i}_\ell(a)\). Contradiction. Thus, no low type has a “hole” in the support of her equilibrium bidding distribution.

Now, we can show that the bidding distribution of a high type has no “hole” in its support either. Similarly, to the contrary, suppose \(F^i_h\) has a “hole” in its support. Consider such a maximal “hole” \((c, d)\) in which \(F^i_h\) is constant, while \(c\) and \(d\) belong to its support. Lemma 15 implies that \(c\) and \(d\) also belong to the support of \(F^{-i}_\ell\). Since no low type has a “hole” in its support, \(F^{-i}_\ell\) must have full support in \([c, d]\). Because player \(-i\) (his low type) bids in \([c, d]\), Corollary 3 implies that player \(i\) must be bidding in the same interval as well. By assumption, her high type does not bid in \((a, b)\), therefore, it must be her low type bidding in that interval, i.e \(F^i_\ell\) has full support in \((c, d)\). Now, consider some bid \(x \in (c, d)\). All other bids \(x' \in (c, d)\) must yield the same payoff as \(x\) to player \(-i\)’s low type. In other words, we have the following:

\[
0 = u^{-i}_\ell(x) - u^{-i}_\ell(x')
\]
\[ p_i F^i_h(x)v_\ell + (1 - p_i) F^i_\ell(x)v_\ell - x \] \[ = (1 - p_i) v_\ell \left( F^i_h(x) - F^i_\ell(x') \right) - (x - x') \]

Rearranging the terms, one could obtain:

\[ \frac{F^i_h(x) - F^i_\ell(x')}{x - x'} = \frac{1}{(1 - p_i)v_\ell} \]

Thus, as \( x' \) approaches to \( x \), we get:

\[ f^i_\ell(x) = \frac{1}{(1 - p_i)v_\ell} \quad (D.3) \]

for all \( x \in (c, d) \), where \( f^i_\ell \) is the density of \( F^i_\ell \). If there was a bid \( b \) in \( (c, d) \) that is in the support of \( F^i_h \), by slightly increasing his bid from \( b \) to some \( b' < d \), player \(-i\)'s high type could obtain a higher payoff. As a result, \( F^i_h \) is constant in \( (c, d) \). Therefore, similar to equation (D.3), considering the payoff of player \( i \)'s low type, we obtain

\[ f^{-i}_\ell(x) = \frac{1}{(1 - p^{-i})v_\ell} \]

for all \( x \in (c, d) \), where \( f^{-i}_\ell \) is the density of \( F^{-i}_\ell \). However, this time, by decreasing her bid from \( d \) to \( c \), player \( i \)'s high type could obtain a higher payoff, contradicting with \( d \) being in the support of her bidding distribution. Therefore, no high type has a “hole” in the support of her equilibrium bidding distributions.

We have established that the bidding distributions of each type of each player are interval. Denote by \([0, m]\) the union of the supports of the bidding distributions of the players. Lemma 15, along with Corollary 1, implies that a low type bids \( m \) for sure. Say it is player \( i \) whose low type bids \( m \) in equilibrium. Depending on whether the low type of player \(-i\) bids \( m \), too, we have either (i) \( \max \{ \text{Supp} \left[ F^{-i}_\ell \right] \} = \max \{ \text{Supp} \left[ F^i_\ell \right] \} \) or (ii) \( \max \{ \text{Supp} \left[ F^{-i}_\ell \right] \} < \max \{ \text{Supp} \left[ F^i_\ell \right] \} \). In the following two sections we will separately analyze these two cases.

\textbf{D.1.0.1} \( \max \{ \text{Supp} \left[ F^{-i}_\ell \right] \} = \max \{ \text{Supp} \left[ F^i_\ell \right] \} \):

Suppose that the maximum bids in the supports of the low type players’ bidding distributions are the same, that is \( \max \{ \text{Supp} \left[ F^{-i}_\ell \right] \} = \max \{ \text{Supp} \left[ F^i_\ell \right] \} \).
Claim 1. The maximum bid that each type of each player bids in equilibrium is the same, hence, it is $m$.

Proof. Let $\bar{y}$ be the maximum bid that a player’s high type, say player $i$, bids. If $\bar{y} < m$, then Corollary 3 would imply that $F_i^\ell$ had full support in $(\bar{y}, m]$. Moreover, due to Lemma 15, $\bar{y}$ would be a best reply for player $-i$’s low type, that is $\bar{y} \in \text{Supp} \left[F_i^{-\ell}\right]$. Therefore, $F_i^{-\ell}$ has full support in $[\bar{y}, m]$ since none of the bidding distributions have a “hole” in their support. Considering the payoff of player $-i$’s low type when she bids a bid $x \in (\bar{y}, m]$, similar to equation (D.3), we obtain

$$f_{i}^{\ell}(x) = \frac{1}{(1 - p_i)v_{\ell}}. \quad (D.4)$$

If there was a bid $b$ in $(\bar{y}, m]$ that is in the support of $F_h^{-i}$, by slightly increasing his bid from $b$ to some $b' < m$, player $-i$’s high type could obtain a higher payoff because

$$u_h^{-i}(b') - u_h^{-i}(b) = \left(F_i^{\ell}(b')v_h - b'\right) - \left(F_i^{\ell}(b)v_h - b\right) = (b' - b)\left(\frac{v_h}{(1 - p_i)v_{\ell}} - 1\right) > 0.$$

Hence, $F_h^{-i}$ is constant in $(\bar{y}, m]$. As a result, considering the payoff of player $i$’s low type when she bids in $(\bar{y}, m]$, one would obtain

$$f_{i}^{-\ell}(x) = \frac{1}{(1 - p_i)v_{\ell}}. \quad (D.5)$$

But, now, player $i$’s high type could obtain a higher payoff by increasing her bid from $\bar{y}$ to $m$ since

$$u_h^i(m) - u_h^i(\bar{y}) = \left(F_i^{-i}(m)v_h - m\right) - \left(F_i^{-i}(\bar{y})v_h - \bar{y}\right) = (m - \bar{y})\left(\frac{v_h}{(1 - p_i)v_{\ell}} - 1\right) > 0.$$

Thus, the maximum bid that each type of each player bids in equilibrium must be the same. □

So, we have shown the following:

$$\max \{\text{Supp} \left[F_i^\ell\right]\} = \max \{\text{Supp} \left[F_i^{-\ell}\right]\} = \max \{\text{Supp} \left[F_h^i\right]\} = \max \{\text{Supp} \left[F_h^{-i}\right]\}$$

Since bidding $m$ is a best response for both low types and no type places an atom at $m$, they could obtain the same payoff in equilibrium. If zero was a best response for a high type player, say $i$, then it would also be a best response for her opponent’s low type.
Therefore, no “hole” would imply

$$\text{Supp} \left[F^i_h\right] = \text{Supp} \left[F^{-i}_\ell\right] = [0, \ m].$$

In this case, since a high type player can guarantee a positive payoff, player $-i$’s low type must be bidding zero with positive probability. Since both types cannot place atom at zero at the same time, his equilibrium payoff must be zero. However, player $i$’s low type would obtain a positive payoff by bidding zero since player $-i$’s low type has an atom at zero. Contradicting with above observation that the two low types have the same equilibrium payoff. Hence, zero is not a best response for neither of the high type players.

If high types do not bid zero, then no-gap property, along with Lemma 16, implies that the low types bid in $[0, \ m]$. Since, by bidding zero a low type player obtains a zero payoff, we must have $m = v_\ell$. Hence,

$$\text{Supp} \left[F^i_\ell\right] = \text{Supp} \left[F^{-i}_\ell\right] = [0, \ v_\ell]$$

Denote by $\bar{x}^i$ the minimum bid that the high type of player $i$ bids in equilibrium. Without loss, suppose that $\bar{x}^i < \bar{x}^{-i}$. Considering the payoff of player $i$’s high type when she bids in $x \in (\bar{x}^i, \bar{x}^{-i})$, similar to Equation (D.3), we obtain the following

$$f^{-i}_\ell(x) = \frac{1}{v_h} , \quad \forall \ x \in (\bar{x}^i, \bar{x}^{-i}). \quad (D.6)$$

Similarly, considering the payoff of her low type when she bids in the same interval, one could obtain the following

$$f^{-i}_\ell(x) = \frac{1}{(1 - p_{-i})v_\ell} , \quad \forall \ x \in (\bar{x}^i, \bar{x}^{-i}),$$

which contradicts with (D.6). Therefore, $\bar{x}^i = \bar{x}^{-i}$. We can, hence, conclude that for some $\bar{x} \in (0, \ v_\ell)$, we have

$$\text{Supp} \left[F^i_\ell\right] = \text{Supp} \left[F^{-i}_\ell\right] = [0, \ v_\ell] \quad \text{and} \quad \text{Supp} \left[F^i_h\right] = \text{Supp} \left[F^{-i}_h\right] = [\bar{x}, \ v_\ell]$$

Now, we can find the bidding distributions starting with pinning down the low types’. Consider the payoff of a low type player, say player $-i$, when she bids $x \in [0, \bar{x}]$.

$$0 = u^{-i}_\ell(x) = p_i F^i_h(x)v_\ell + (1 - p_\ell) F^i_\ell(x)v_\ell - x = (1 - p_{-i}) F^{-i}_\ell(x)v_\ell - x$$
where the last equality follows from the fact that the lowest bid in the support of a high
type’s bidding distribution is \( \bar{x} \) and that it is continuous. As a result, for \( x \in [0, \bar{x}] \),
we have \( F_i^h(x) = \frac{x}{(1-p_i)v_e} \). Now, considering the payoff of a high type player with a bid
\( x \in [\bar{x}, v_e] \),
\[
v_h - v_e = u_h^i(x) = F_i^h(\bar{x})v_h - x
\]
As a result, for \( x \in [\bar{x}, v_e] \), we have \( F_i^h(x) = \frac{x}{v_h} + \frac{v_h - v_e}{v_h} \). Continuity of \( F_i^h \) implies that
\[
\frac{\bar{x}}{(1-p_i)v_e} = \frac{\bar{x}}{v_h} + \frac{v_h - v_e}{v_h}
\]
Thus \( \bar{x} \) is pinned down as follows:

\[
\bar{x} = \frac{(1-p_i)(v_h - v_e)}{v_h - (1-p_i)v_e} v_e
\]
Lastly, considering the payoff of a low type player with a bid \( x \) in \([\bar{x}, v_e]\),
\[
0 = u_e^i(x) = p_iF_h^i(x)v_e + (1-p_i)F_e^i(x)v_e - x = p_iF_h^i(x)v_e + (1-p_i)\left( \frac{x}{v_h} + \frac{v_h - v_e}{v_h} \right) - x
\]
Therefore,
\[
F_h^i(x) = \frac{1}{p_i} \left( \left( \frac{1}{v_e - \frac{(1-p_i)}{v_h}} \right) x - (1-p_i)\frac{v_h - v_e}{v_h} \right)
\]
for all \( x \in [\bar{x}, v_e] \). So, for \( i = 1, 2 \), we must have
\[
F_i^h(x) = \begin{cases} 
\frac{x}{(1-p_i)v_e}, & x \in [0, \bar{x}] \\
\frac{x}{v_h} + \frac{v_h - v_e}{v_h}, & x \in [\bar{x}, v_e]
\end{cases}
\]
and
\[
F_i^e(x) = \begin{cases} 
0, & x \in [0, \bar{x}] \\
\frac{1}{p_i} \left( \left( \frac{1}{v_e - \frac{(1-p_i)}{v_h}} \right) x - (1-p_i)\frac{v_h - v_e}{v_h} \right), & x \in [\bar{x}, v_e]
\end{cases}
\]
where
\[
\bar{x} = \frac{(1-p_i)(v_h - v_e)}{v_h - (1-p_i)v_e} v_e.
\]
Observe that this would require that \( p_1 = p_2 \), which is the case in Lu et al. (2018).
Moreover, when \( p_1 = p_2 \) they show that the bidding distribution profile described above
is the unique equilibrium.

D.1.0.2 $\max \{ \text{Supp} [F_{\ell}^{-i}] \} < \max \{ \text{Supp} [F_{\ell}^i] \}$:

Remember that $[0, m]$ is the union of the supports of the bidding distributions of the players. Moreover, a low type bids $m$ for sure and we assumed that it is player $i$ whose low type bids $m$ in equilibrium. In this section, we will analyze the equilibrium bidding behavior when bidding $m$ is not a best response for player $-i$’s low type, in other words $\max \{ \text{Supp} [F_{\ell}^{-i}] \} < m$. Lemma 15 and no-gap property imply that $\max \{ \text{Supp} [F_{h}^{-i}] \} = \max \{ \text{Supp} [F_{\ell}^i] \} = m$. Similar to the the previous case, one could show that $\max \{ \text{Supp} [F_{h}^i] \} = \max \{ \text{Supp} [F_{\ell}^{-i}] \}$. Observe that by bidding $m$ the low type of player $-i$ could have obtained the same payoff as his opponent’s low type, hence, his equilibrium payoff must be weakly better than that of player $i$’s low type, i.e

$$u_{\ell}^{-i} \geq u_{\ell}^i.$$  \hspace{1cm} (D.7)

On the other hand, no-gap property, along with Lemma 15, implies that zero must lie in the support of at least one low type’s bidding distribution, resulting in at least one low type having zero payoff in equilibrium. Combining with (D.7), we conclude that the low type of player $i$ expects zero payoff in equilibrium, which would require $m$ to be equal to her valuation of winning since

$$0 = u_{\ell}^i(m) = \left( p_{-i} F_{h}^{-i}(m) + (1 - p_{-i}) F_{\ell}^{-i}(m) \right) v_{h} - m = v_{\ell} - m.$$

Claim 2. Zero is a best response for both low types.

Proof. Let $x_{\ell}^{-i}$ denote the minimum bid that the low type of player $-i$ bids in equilibrium. For the sake of a contradiction, suppose $x_{\ell}^{-i} > 0$. Then, no-gap property would imply that $F_{h}^{-i}$ has full support in $[0, x_{\ell}^{-i}]$. Therefore, we would have

$$0 = u_{h}^{-i}(x_{\ell}^{-i}) - u_{h}^{-i}(0) = \left[ F_{\ell}^i(x_{\ell}^{-i}) - F_{\ell}^i(0) \right] v_{h} - x_{\ell}^{-i}. \hspace{1cm} (D.8)$$

On the other hand, however, since zero is not in the support of $F_{\ell}^{-i}$ while $x_{\ell}^{-i}$ is, we must have

$$0 \leq u_{\ell}^{-i}(x_{\ell}^{-i}) - u_{\ell}^{-i}(0) = p_{i} \left[ F_{\ell}^i(x_{\ell}^{-i}) - F_{\ell}^i(0) \right] v_{\ell} - x_{\ell}^{-i}. \hspace{1cm} (D.9)$$

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Combining (D.8) and (D.9),

\[ 0 \leq p_i v_{i\ell} \frac{\bar{x}_{i\ell}}{v_h} - \bar{x}_{i\ell} = \left( \frac{p_i v_{i\ell}}{v_h} - 1 \right) \bar{x}_{i\ell} < 0 \]

Contradiction. So it must be the case that \( \bar{x}_{i\ell} = 0 \). One could use the same reasoning to show that \( \bar{x}_{i\ell} = 0 \), too.

Therefore, we could summarize what we have established so far as the following:

\[
\text{Supp } F^i_{\ell} = [0, v_{i\ell}] \quad \text{and} \quad \text{Supp } F^{-i}_{\ell} = [0, \bar{x}]
\]

for some \( \bar{x} \in (0, v_{i\ell}) \). Similar to the same maxima case, it can easily be shown that the minimum bids in the supports of the high types’ bidding distributions are the same, i.e \( \min \{\text{Supp} [F^j_h]\} = \min \{\text{Supp} [F^{-j}_{h}]\} \). Let \( x \) denote this minimum bid. So, we have

\[
\text{Supp } F^j_h = [x, v_h] \quad \text{and} \quad \text{Supp } F^{-j}_{h} = [x, \bar{x}]
\]

Observe that equation (D.7) may require the low type of player \( i \) to bid zero with positive probability. The supports of equilibrium bidding distributions can be illustrated by the following diagram.

Figure D.1. The supports of the equilibrium bidding distributions.

Next, we pin down the bidding distributions. Observe that the low type of player \( i \) expects zero payoff while her high type obtains a payoff of \( v_h - \bar{x} \), and the low type of player \( -i \) expects a non-negative (even positive for sure when player \( i \)'s low type places an atom at zero) payoff while his high type obtains a payoff of \( v_h - v_{i\ell} \). Considering the
payoff of player $i$’s high type with a bid $x$ such that $x \in [\underline{x}, \bar{x}]$, we obtain
\[
v_h - \bar{x} = u^i_h(x) = F^{-i}_\ell(x)v_h - x
\]
Thus, for all $x \in [\underline{x}, \bar{x}]$, we have
\[
F^{-i}_\ell(x) = \frac{x}{v_h} + \frac{v_h - \bar{x}}{v_h}. \tag{D.10}
\]
Now, considering her low type’s payoff with a bid $x$ that is lower than $\underline{x}$, we get
\[
0 = u^i_\ell(x) = p_{-i}F^{-i}_h(x)v_\ell + (1 - p_i)F^{-i}_\ell(x)v_\ell - x = (1 - p_i)F^{-i}_\ell(x)v_\ell - x
\]
So, this time, for all $x \in [0, \underline{x}]$ we obtain
\[
F^{-i}_\ell(x) = \frac{x}{(1 - p_{-i})v_\ell}. \tag{D.11}
\]
Continuity of $F^{-i}_\ell$ at $\underline{x}$ implies that
\[
\frac{\underline{x}}{(1 - p_{-i})v_\ell} = \frac{\underline{x}}{v_h} + \frac{v_h - \bar{x}}{v_h}. \tag{D.12}
\]
Rearranging the terms, we obtain
\[
(\bar{x} - x) = v_h \left(1 - \frac{x}{(1 - p_{-i})v_\ell}\right)
\]
Repeating the same procedure this time by considering the payoffs of player $-i$’s high type with a bid in $[0, \underline{x}]$ and of her low type with a bid above $\underline{x}$, one could obtain the bidding distribution of player $i$’s low type as follows:
\[
F^i_\ell = \begin{cases}
\frac{(p_{-i} - p_i)(v_h - v_\ell)}{(1 - p_i)v_h} + x \frac{1}{(1 - p_i)v_\ell}, & x \in [0, \underline{x}] \\
\frac{(p_{-i} - p_i)(v_h - v_\ell)}{(1 - p_i)v_h} + x \frac{1}{(1 - p_i)v_\ell} + (x - \underline{x}) \frac{1}{v_h}, & x \in [\underline{x}, v_\ell]
\end{cases} \tag{D.13}
\]
Now we can obtain the bidding distributions of the high types. Only the low type of player $i$ and the high type of player $-i$ bid above $\bar{x}$. Considering the payoff of player $i$’s low type with a bid $x$ in $[\bar{x}, v_\ell]$, we have
\[ 0 = u_{i}^{e}(x) = \left( p_{-i}F_{h}^{-i}(x) + (1 - p_{-i})F_{l}^{-i}(x) \right) v_{l} - x = \left( p_{-i}F_{h}^{-i}(x) + (1 - p_{-i}) \right) v_{l} - x, \]

where the last inequality follows from \( \bar{x} \) being the highest bid in the support of \( F_{l}^{-i} \).

Rewriting the equation, get

\[ F_{h}^{-i}(x) = \frac{x}{p_{-i}v_{l}} - \frac{1 - p_{-i}}{p_{-i}} \]

for all \( x \in [\bar{x}, v_{l}] \). Lastly, considering the payoffs of the low types of players \( i \) and \(-i\), respectively, with a bid in \([x, \bar{x}]\), one can obtain

\[ F_{h}^{i}(x) = (x - \bar{x}) \frac{p_{i}v_{l}v_{h}}{v_{h} - (1 - p_{i})v_{l}} \quad \text{and} \quad F_{l}^{-i} = (x - \bar{x}) \frac{v_{h} - (1 - p_{i})v_{l}}{p_{-i}v_{l}v_{h}}. \]

for all \( x \in [\bar{x}, \bar{x}] \). Since \( 1 = F_{h}^{i}(\bar{x}) \), we have

\[ \bar{x} - x = \frac{p_{i}v_{l}v_{h}}{v_{h} - (1 - p_{i})v_{l}}. \quad (D.14) \]

Combining Equations (D.12) and (D.14), we get

\[ v_{h} \left( 1 - \frac{x}{(1 - p_{-i})v_{l}} \right) = \frac{p_{i}v_{l}v_{h}}{v_{h} - (1 - p_{i})v_{l}}. \]

Rearranging the terms, we get

\[ \bar{x} = \frac{(1 - p_{-i})v_{l}(v_{h} - v_{l})}{v_{h} - (1 - p_{i})v_{l}}. \quad (D.15) \]

Plugging (D.15) back to (D.14), we get

\[ \bar{x} = v_{l} - \frac{(p_{2} - p_{1})v_{l}(v_{h} - v_{l})}{v_{h} - (1 - p_{1})v_{l}}. \]

These conditions characterize the necessary conditions for the equilibrium bidding distributions. Observe that another necessary condition in order for the bidding distributions above to constitute an equilibrium is that \( p_{i} < p_{-i} < 1 \). We are left with showing that these bidding distributions actually constitute an equilibrium when they satisfy the necessary conditions, however, it is obvious to show that the players are best responding to their opponent in the profile described above. Thus, when \( p_{i} < p_{-i} < 1 \), the equilibrium
is unique and the expected payoffs in this unique equilibrium could be calculated as the following:

\[
\begin{align*}
    u^i_\ell &= 0 \\
    u^i_h &= v_h - v_\ell + \frac{(p_{-i} - p_i)(v_h - v_\ell)v_\ell}{v_h - (1 - p_i)v_\ell} \\
    u^{-i}_\ell &= \frac{(p_{-i} - p_i)(v_h - v_\ell)v_\ell}{v_h} \\
    u^{-i}_h &= v_h - v_\ell.
\end{align*}
\]

**D.2 Proof of Proposition 8**

In this section, for any given distribution choice profile \((p_1, p_2)\) with \(p_i \in (0, 1)\) for \(i = 1, 2\), I characterize the equilibrium bidding behaviors of the players in the all-pay auction stage when the auctioneer plays D (disclose) under the partial disclosure policy that is to disclose if there is a high type player, i.e \(G_{DH}(D \mid p_1, p_2)\). When we consider the game \(G_{DH}(D \mid p_1, p_2)\), a low type player knows that he is facing a high type opponent for sure, while, a high type opponent believes that she is facing a high type player with her prior beliefs. Lemma 14 still holds, however, Lemma 13 needs to be slightly modified since this time two low types could place an atom at zero because they never bid against each other. Yet, still, equilibrium bidding distributions of each type of each player are continuous. Moreover, a low type should not bid a bid that her opponent’s high type is not bidding unless the bid is zero. Following lemma is the counter-part of Lemma 15.

**Lemma 17.** For all \(x > 0\) such that \(x \in \text{Supp}[F^i_\ell]\), \(x \in \text{Supp}[F^{-i}_h]\).

**Proof.** Suppose there exists a bid \(x > 0\) such that \(x \in \text{Supp}[F^i_\ell]\) and \(x \not\in \text{Supp}[F^{-i}_h]\). Because the equilibrium bidding distributions are continuous, there is an open ball around \(x\), \(B(x)\) such that for all \(x^* \in B(x)\), \(F^{-i}_h(x) = F^{-i}_h(x^*)\). Hence, deviating to some \(x^* < x\) is a profitable deviation for player \(i\)’s low type. Therefore, \(x\) cannot be a best response for player \(i\)’s low type, i.e \(x \not\in \text{Supp}[F^i_\ell]\). Contradiction. \(\blacksquare\)

A low type player can choose not to participate at all, i.e always bids zero. However, Lemma 17 tells us whenever she bids a positive bid, in equilibrium, the high type of her opponent must be bidding that bid, too.

Following lemma tells us the condition under which the equilibrium bidding distributions cannot have 'holes' in their support.
Lemma 18. When \( p_i \neq \frac{v_i}{v_h} \), player i’s high type does not have a hole in its bidding distribution’s support for \( i = 1, 2 \). Moreover, when both \( p_1 \) and \( p_2 \) are different than \( \frac{v_i}{v_h} \), the bidding distributions of the low types have no “holes” above zero, i.e. there are no \( c \) and \( d \) with \( d > c > 0 \) such that a low type bids \( c \) and \( d \), while she does not bid any bid in \((c,d)\).

Proof. Suppose \( p_i \neq \frac{v_i}{v_h} \) and that \( F^i_h \) has a hole. In other words \( F^i_h \) is constant over some interval \((a, b)\) while \( a \) and \( b \) belong to its support. Because of Lemma 17, \( F^{-i}_\ell \) is also constant in the same interval. As a result,

\[
0 = u^i_h(b) - u^i_h(a) = \left[ p_i \left( F^{-i}_h(b) - F^{-i}_h(a) \right) + (1 - p_i) \left( F^{-i}_\ell(b) - F^{-i}_\ell(a) \right) \right] v_h - (b - a) = p_i v_h \left( F^{-i}_h(b) - F^{-i}_h(a) \right) - (b - a) \tag{D.16}
\]

On the other hand, due to the no-gap property, both \( F^i_\ell \) and \( F^{-i}_h \) have full support on \((a, b)\). Considering the payoff of the low type of player \( i \) with a bid \( x \in (a, b) \), similar to Equation (D.3), we obtain

\[
f^{-i}_h(x) = \frac{1}{v_\ell}
\]

for all \( x \in (a, b) \). Therefore,

\[
F^{-i}_h(b) - F^{-i}_h(a) = \frac{(b - a)}{v_\ell}.
\]

Combining this with Equation (D.16), we obtain \( p_i = \frac{v_i}{v_h} \). Contradiction. So, when \( p_i \neq \frac{v_i}{v_h} \), the support of bidding distribution of player i’s high type is an interval.

Now, assume that both \( p_1 \) and \( p_2 \) are different than \( \frac{v_i}{v_h} \). Moreover, to the contrary, suppose \( F^i_\ell \) is constant over some \((c, d)\) while \( c \) and \( d \) belong to its support and \( d > c > 0 \). Therefore, we have

\[
0 = u^i_\ell(d) - u^i_\ell(c) = \left( F^{-i}_h(d) - F^{-i}_h(c) \right) v_\ell - (d - c) \tag{D.17}
\]

Lemma 17 implies that both \( c \) and \( d \) belong to the support of \( F^{-i}_h \). Because of the first part of Lemma 18 we know that \( F^{-i}_h \) has no “holes”. Therefore, we must have \([c, d] \subseteq \text{Supp} \left[ F^{-i}_h \right] \). Furthermore, along with the continuity, Corollary 3 would then imply that every bid in \([c, d]\) belongs to the support of \( F^i_h \). Considering the payoff of
player $-i$’s high type with a bid $x \in [c, d]$, similar to Equation (D.3), one would obtain

$$f^i_h(x) = \frac{1}{p_iv_h}$$

for all $x \in [c, d]$. Since $p_i \neq \frac{v_h}{v_h}$, no bid in $(c, d)$ could be a best response for the low type of player $-i$ because bidding $d$ if $p_i < \frac{v_h}{v_h}$ or $c$ if $p_i > \frac{v_h}{v_h}$ would yield a strictly higher payoff to him. As a result, $F^{-i}_h$ is constant over $(c, d)$. Thus, considering the payoff of player $i$’s high type with a bid $x \in [c, d]$, we obtain

$$f^{-i}_h(x) = \frac{1}{p-i v_h}$$

for all $x \in [c, d]$. However, then we would have

$$F^{-i}_h(d) - F^{-i}_h(c) = \frac{d-c}{p-i v_h} = \frac{d-c}{v_\ell}.$$ 

where the second equality follows from (D.17). However, we assumed that $p_i \neq \frac{v_h}{v_h}$ and $d-c > 0$. Contradiction.

We can now establish the fact that a high type will never expect a non-positive payoff in equilibrium.

**Lemma 19.** Both high types obtain positive payoff in equilibrium.

**Proof.** Suppose to the contrary that the high type of some player, say player 1, gets a zero expected payoff in an equilibrium. If the maximum bid that the high type of her opponent bids in equilibrium were less than $v_h$, by bidding his maximum bid, the high type of player 1 could obtain positive payoff. Hence, we must have $\max \{\text{Supp}\left[F^2_h\right]\} = v_h$. Since a low type player never bids a bid that is greater than $v_\ell$, Corollary 3 implies that $\max \{\text{Supp}\left[F^1_h\right]\} = \max \{\text{Supp}\left[F^2_h\right]\} = v_h$. As a result both high types obtain zero equilibrium payoff. Now, consider the bidding distributions of the low types and denote by $\bar{m}$ be the highest bid in the union of their supports. Without loss, say $\max \{\text{Supp}\left[F^2_\ell\right]\} = \bar{m}$. Observe that $\bar{m} < v_\ell$ and that the low type of player 2 should be obtaining non-negative payoff by bidding $\bar{m}$, i.e

$$0 \leq u^2_\ell(\bar{m}) = F^1_h(\bar{m})v_\ell - \bar{m}$$

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Leaving $F^1_h(\bar{m})$ alone, we obtain

$$F^1_h(\bar{m}) \geq \frac{\bar{m}}{v_\ell} \quad (D.18)$$

On the other hand, Corollary 3 implies the following set inclusions:

$$[\bar{m}, v_h] \subseteq \text{Supp}[F_h] \quad \text{and} \quad [\bar{m}, v_h] \subseteq \text{Supp}[G_h]$$

Considering, the payoff of player 2’s high type with a bid $x \in [\bar{m}, v_h]$, we get

$$f^h_1(x) = \frac{1}{p_1v_h}$$

for all $x \in [\bar{m}, v_h]$. If $\frac{1}{p_1v_h} > \frac{1}{v_\ell}$ then bidding $v_h$ would yield player 2’s low type a payoff that is at least as good as the bid $\bar{m}$. However, her payoff when she bids $v_h$ is negative. $\bar{m}$ is a best response for him by assumption. Therefore, we must have

$$\frac{1}{p_1v_h} \leq \frac{1}{v_\ell}. \quad (D.19)$$

Consequently, combining (D.18) and (D.19), we get

$$1 = F^1_h(v_h) = F^1_h(\bar{m}) + \left( F^1_h(v_h) - F^1_h(\bar{m}) \right)$$

$$= F^1_h(\bar{m}) + \frac{v_h - \bar{m}}{p_1v_h}$$

$$\geq \frac{\bar{m}}{v_\ell} + \frac{v_h - \bar{m}}{p_1v_h}$$

$$\geq \frac{\bar{m}}{p_1v_h} + \frac{v_h - \bar{m}}{p_1v_h}$$

$$= \frac{1}{p_1}$$

$$> 1$$

Contradiction. Hence, the equilibrium payoff of a high type player must be positive. ■

Because Lemma 17 does not characterize a complete inclusion between the supports of the bidding distributions unlike Lemma 15, we will analyze the equilibrium in three different cases depending on whether if a low type player always bids zero. Following three cases are completely covers all the possible shapes that an equilibrium could take.
D.2.0.1 Suppose \( \text{Supp} \left[ F_i^\ell \right] = \text{Supp} \left[ F_i^{h-} \right] = \{0\} \)

Remember that a low type player knows that he is facing a high type opponent. In order for a low type to bid zero with probability one in equilibrium, the maximum bid that his opponent’s high type bids should be at least \( v_\ell \). Otherwise, by bidding that maximum bid the low type would obtain a positive payoff. Moreover, Corollary 3 implies that the support of the high types’ bidding distributions are the same. Denoting by \( y \) the maximum bid in the supports of the high types’ bidding distributions, we have

\[
\text{Supp} \left[ F_h^i \right] = \text{Supp} \left[ F_h^{i-} \right] = [0, y].
\]

The equilibrium bidding distributions of the high types could, then, be obtained uniquely from their equilibrium payoffs. Considering the payoff of player \( i \)’s high type, for \( i = 1, 2 \), with a bid \( x \in [0, y] \) we get

\[
f_h^{-i}(x) = \frac{1}{p^{-iv_h}}
\]

for all \( x \in [0, y] \). Since a high type does not bid zero with positive probability, we have

\[
1 = F_h^{-i}(y) = \frac{y}{p^{-iv_h}} \geq \frac{v_\ell}{p^{-iv_h}}
\]

As a result, in order for both low types to always bid zero in an equilibrium, we must have

\[
p_1 = p_2 \geq \frac{v_\ell}{v_h}
\]

and the high types must be bidding according to the same bidding distribution function, \( F \) that is defined as follows:

\[
F(x) = \frac{x}{pv_h}
\]

for all \( x \in [0, y] \), where \( p \equiv p_1 = p_2 \). Moreover, in this equilibrium, a low type player obtains zero payoff while a high type obtains \((1 - p)v_h\). It is obvious that neither player has an incentive to deviate from this profile. \(^2\) The equilibrium payoffs, then, are as follows:

\[
u_\ell^i = u_\ell^{i-} = 0, \quad u_h^i = u_h^{i-} = (1 - p)v_h, \quad \text{and} \quad u_i = u^{i-} = p(1 - p)v_h.
\]

\(^2\)This is, in fact, one of the two cases in Lu et al. (2018).
D.2.0.2 Suppose \( \text{Supp} [F^i] = \{0\} \), but \( [F^{-i}] \neq \{0\} \)

When only one player’s low type, say player \( i \), always bids zero, from the case above, the maximum bid that the high type of player \(-i\) bids in equilibrium should be at least \( v_\ell \). Denoting this maximum bid by \( \tilde{m} \), along with Lemma 17, Corollary 3 implies

\[
\text{Supp} [F^i] = [0, \tilde{m}].
\]

Since bidding zero and \( \tilde{m} \) are best replies for player \( i \)’s high type, they must yield the same payoff to her, hence, we have

\[
(1-p_i)v_h = u_h^{-i}(0) \leq u_h^{-i}(\tilde{m}) = v_h - \tilde{m} = u_h^i(\tilde{m}) = u_h^i(0) = (1-p_{-i})F^{-i}_\ell(0)v_h < (1-p_{-i})v_h,
\]

where the first inequality follows from \( m \) being a best response for the high type of player \(-i\), whereas zero is not necessarily one; and the last inequality is due to the fact that \( F^{-i}_\ell(0) < 0 \). Therefore, we must have

\[
p_i > p_{-i}.
\]

Lemma 17 says that all the positive bids that the low type of player \(-i\) bids in equilibrium are a subset of \([0, m]\). Denoting by \( m \) and \( \bar{b} \) the minimum bid in the support of \( F^{-i}_h \) and the maximum bid in the support of \( F^{-i}_\ell \), respectively, no-gap property implies \( m \leq \bar{b} \) because otherwise she would not be bidding in \((\bar{b}, m)\) interval. We can further our analysis in two different cases where (i) \( m = \bar{b} \) and (ii) \( m < \bar{b} \).

**Case i:** \( m = \bar{b} \).

Let the minimum bid in the support of \( F^{-i}_h \) and the maximum bid in the support of \( F^{-i}_\ell \) be the same, i.e \( m = \bar{b} \). Due to Corollary 3, the supports of the bidding distribution of each type of each player will be as follows:

\[
\text{Supp} [F^i] = \{0\} , \text{Supp} [F^i_h] = [0, \tilde{m}] , \text{Supp} [F^{-i}_\ell] = [0, m] , \text{and Supp} [F^{-i}_h] = [m, \tilde{m}].
\]

(D.20)

By considering the payoff of the low type of player \(-i\) with a bid in \([0, m]\) and the payoff
of his high type with a bid in \([m, m]\), one can uniquely pin down \(F^i_h\) as follows:

\[
F^i_h(x) = \begin{cases} 
\frac{x}{v_\ell} & x \in [0, m] \\
\frac{m}{v_\ell} + \frac{x-m}{p_i v_h} & x \in [m, \bar{m}]. 
\end{cases}
\]

Similarly, we can uniquely pin down \(F^{-i}_h\) and \(F^{-i}_\ell\) as follows:

\[
F^{-i}_\ell(x) = F^{-i}_\ell(0) + \frac{x}{(1-p_{-i})v_h}
\]
for all \(x \in [0, m]\) and

\[
F^{-i}_h(x) = \frac{x-m}{p_{-i}v_h}
\]
for all \(x \in [m, m]\). Since \(F^{-i}_h(\bar{m}) = 1\), we must have \(\bar{m} - m = p_{-i}v_h\). Moreover, \(F^i_h(\bar{m}) = 1\) implies that

\[
1 = \frac{m}{v_\ell} + \frac{x-m}{p_i v_h} = \frac{m}{v_\ell} + \frac{p_{-i}v_h}{p_i v_h} \Rightarrow m = \frac{(p_i - p_{-i})}{p_i} v_\ell
\]

Plugging \(m\) and \(\bar{m}\) values back into the bidding distributions, we obtain

\[
F^i_h(0) = 1
\]

\[
F^i_h(x) = \begin{cases} 
\frac{x}{v_\ell} & x \in \left[0, \frac{(p_i - p_{-i})}{p_i} v_\ell\right] \\
\frac{(p_i - p_{-i})}{p_i} v_\ell + \frac{x}{p_i (1-p_{-i}) v_h} & x \in \left[\frac{(p_i - p_{-i})}{p_i} v_\ell, \frac{(p_i - p_{-i})}{p_i} v_\ell + p_{-i}v_h\right]
\end{cases}
\]

\[
F^{-i}_\ell(x) = 1 - \frac{(p_i - p_{-i})v_\ell}{p_i (1-p_{-i}) v_h} + \frac{x}{(1-p_{-i})v_h}, \quad \forall x \in \left[0, \frac{(p_i - p_{-i})}{p_i} v_\ell\right]
\]

\[
F^{-i}_h(x) = \frac{x-v_\ell}{p_{-i}v_h} + \frac{v_\ell}{p_i v_h}, \quad \forall x \in \left[\frac{(p_i - p_{-i})}{p_i} v_\ell, \frac{(p_i - p_{-i})}{p_i} v_\ell + p_{-i}v_h\right]
\]

It is clear that this profile constitutes an equilibrium given the following two necessary conditions are satisfied. The first one is that \(p_i > p_{-i}\) and the second one is \(\bar{m} \geq v_\ell\). The second one condition can be rewritten as

\[
v_\ell \leq \bar{m} = p_{-i}v_h + m = p_{-i}v_h + \frac{(p_i - p_{-i})}{p_i} v_\ell = p_{-i}v_h + v_\ell - \frac{p_{-i}}{p_i} v_\ell
\]

Therefore, it can be replaced with

\[
p_i \geq \frac{v_\ell}{v_h}.
\]
Therefore, the unique equilibrium in the shape of Case (ii) is the bidding distribution profile in (D.21) and it constitutes an equilibrium only if $p_i > p_{-i}$ and $p_i \geq \frac{v_h}{v_h}$. In this equilibrium, a low type player obtains zero payoff whereas a high type gets a payoff of $(1 - p_{-i})v_h - \frac{(p_i - p_{-i})}{p_i}v_\ell$. As a result

$$u^i = p_i(1 - p_{-i})v_h - (p_i - p_{-i})v_\ell$$

$$u^{-i} = p_{-i}(1 - p_{-i})v_h - p_{-i}\frac{(p_i - p_{-i})}{p_i}v_\ell$$

**Case ii:** $m < \bar{b}$.

Let the minimum bid in the support of $F_{-i}^h$ be smaller than the maximum bid in the support of $F_{-i}^\ell$ be the same, i.e $m < \bar{b}$. In this case, we would replace equation (D.20) with the following:

$$\text{Supp} \left[F_{-i}^h \right] = \{0\} , \text{Supp} \left[F_{-i}^\ell \right] = [0, m] , \text{Supp} \left[F_{-i}^{\ell-i} \right] \subseteq \text{Supp} \left[F_{-i}^i \right] , \text{and } [\bar{b}, m] \subseteq \text{Supp} \left[F_{i}^{\ell-i} \right].$$

If we can pin down the equilibrium behavior of player $-i$ in $(m, \bar{b})$, we will be done with this case. Observe that continuity requires both types of player $-i$ to be indifferent between bidding $m$ and $\bar{b}$. Therefore, we must have the following

$$0 = u_{-i}^\ell(\bar{b}) - u_{-i}^\ell(m) = \left(F_{i}^\ell(\bar{b}) - F_{i}^\ell(m)\right)v_\ell - \left(\bar{b} - m\right) \Rightarrow \bar{b} - m = \left(F_{i}^\ell(\bar{b}) - F_{i}^\ell(m)\right)v_\ell$$

and

$$0 = u_{-i}^i(\bar{b}) - u_{-i}^i(m) = p_i \left(F_{i}^i(\bar{b}) - F_{i}^i(m)\right)v_h - \left(\bar{b} - m\right) \Rightarrow \bar{b} - m = \left(F_{i}^i(\bar{b}) - F_{i}^i(m)\right)p_i v_h$$

Combining the two, we obtain $p_i = \frac{v_\ell}{v_h}$. Hence, when $m < \bar{b}$, we have

$$p_i = \frac{v_\ell}{v_h} > p_{-i}.$$ 

By Lemma 18, the bidding distributions of player $-i$’s both high and low types do not necessarily have interval supports. Since $\bar{b} \geq v_\ell$, equilibrium payoff of player $-i$’s high type cannot exceed $v_h - v_\ell$. Moreover, he chooses not to bid zero. So, we must have

$$v_h - v_\ell \geq v_h - \bar{b} = u_{-i}^i(0) = (1 - p_i)v_h = v_h - v_\ell$$

Therefore, $\bar{b} = v_\ell$. Hence, when $p_i = \frac{v_\ell}{v_h} > p_{-i}$, there are infinitely many equilibria such
that the high type of player $i$ bids uniformly on $[0, v_\ell]$ while her low type bids zero with probability one. The bidding distributions of player $-i$’s types do not necessarily have convex support. Even though there are infinitely many equilibria, they are all payoff equivalent. In all those equilibria, low types expect zero payoff while high types obtain a payoff of $v_h - v_\ell$. As a result

$$u^i = p_i(v_h - v_\ell) \quad \text{and} \quad u^{-i} = p_{-i}(v_h - v_\ell).$$

**D.2.0.3 Suppose $\text{Supp}[F^i] \neq \{0\}$ and $\text{Supp}[F^{-i}] \neq \{0\}$**

The last possibility is that both low types are active bidders, that is to say they bid positive bids as well. Since any positive bid that is a best response for a low type player is also best response for her opponent’s high type (Lemma 17), zero must be in the support of one of the high types’ bidding distributions. To see this, suppose, to the contrary, no high type bids zero. Due to continuity, for some $\varepsilon > 0$, no high type bids a bid $b \in [0, \varepsilon)$. Hence, Corollary 3 implies that both low types bid in $[0, \varepsilon)$, which contradicts with Lemma 17. Say it is player $i$ who has zero in the support of her high type’s bidding distribution. Therefore, in order for her high type to obtain a positive equilibrium payoff, player $-i$’s low type must be bidding zero with positive probability. Now, similar to the previous cases, because the equilibrium payoff of player $-i$’s low type is zero, the maximum bid that the high type of player $i$ bids should be at least $v_\ell$. Let $\bar{m}$ denotes the maximum bid that one of the two high types bid in equilibrium, hence, $\bar{m} \geq v_\ell$.

**Claim 3. Zero is also in the support of $F^{-i}_h$.**

**Proof.** Let us prove this claim in multiple steps. Denote by $\omega_1$ and $\omega_2$ the lowest bid that player $-i$’s high type and the lowest positive bid that player $i$’s low type bid. Observe that we want to prove that $\omega_1 = 0$. Suppose the contrary, i.e $\omega_1 > 0$. $\omega_2$ is, by assumption, a best response for the low type of player $i$. Lemma 17 implies that $\omega_2 \in \text{Supp}[F^{-i}_h]$. Hence, $\omega_2 \geq \omega_1$. Corollary 3 requires that $[0, \omega_1] \subseteq \text{Supp}[F^{-i}_h]$ and $[0, \omega_2] \subseteq \text{Supp}[F^i]$. Since $\omega_2$ is the lowest positive bid that the low type of player $i$ bids, $F^i$ is constant in $[0, \omega_2]$ and bidding $\omega_2$ yields non-negative payoff to the low type of player $i$. Hence,

$$0 \leq u^i_\ell(\omega_2) = F^{-i}_h(\omega_2)v_\ell - \omega_2 \quad \Rightarrow \quad F^{-i}_h(\omega_2) \geq \frac{\omega_2}{v_\ell}. \quad (D.22)$$

Using the fact that player $i$’s high type is indifferent between bidding $\omega_1$ and $\omega_2$, we have

$$0 = u^i_h(\omega_2) - u^i_h(\omega_1) = p_{-i}v_h F^{-i}_h(\omega_2) + (1 - p_{-i})v_h \left( F^{-i}_h(\omega_2) - F^{-i}_h(\omega_1) \right) - (\omega_2 - \omega_1)$$

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\[ \geq p_i v_h F_{h}^{-i}(\omega_2) - (\omega_2 - \omega_1) \]
\[ \geq p_i v_h \frac{\omega_2}{v_\ell} - (\omega_2 - \omega_1) \]
\[ > p_i v_h \frac{(\omega_2 - \omega_1)}{v_\ell} - (\omega_2 - \omega_1), \]

where the first inequality follows from \( F_{i}^{-i} \) being a cdf and \( \omega_2 \geq \omega_1 \), the second one is from Equation (D.22), and the last one from the supposition that \( \omega_1 > 0 \). Therefore,

\[ p_i < \frac{v_\ell}{v_h} \quad \text{and} \quad \omega_2 > \omega_1 \quad (D.23) \]

Furthermore, while bidding \( \omega_1 \) is in the support of \( F_{h}^{-i} \), zero is not. So, we must have

\[ 0 \leq u_{h}^{-i}(\omega_1) - u_{h}^{-i}(0) = p_i v_h \left[ F_{h}^{-i}(\omega_1) - F_{h}^{-i}(0) \right] + (1 - p_i) v_h \left[ F_{\ell}^{-i}(\omega_1) - F_{\ell}^{-i}(0) \right] - \omega_1 \]
\[ = p_i v_h F_{h}^{-i}(\omega_1) - \omega_1, \quad (D.24) \]

where the first equality is due to \( F_{\ell}^{-i}(\omega_1) = F_{\ell}^{-i}(0) \) and \( F_{h}^{-i}(0) = 0 \). Furthermore, remember that player \(-i\)'s low type obtains zero payoff by bidding \( \omega_1 \), hence,

\[ 0 = u_{\ell}^{-i}(\omega_1) = F_{h}^{-i}(\omega_1) v_\ell - \omega_1 \quad (D.25) \]

Combining the equations (D.24) and (D.25), we obtain

\[ p_i \geq \frac{v_\ell}{v_h} \quad (D.26) \]

If the inequality were holding with equality, we would have \( p_i = \frac{\omega_1}{v_h} \). Denoting by \( \bar{z} \) the maximum bid that a low type bids in equilibrium, we would have (i) only high types bidding in \([\bar{z}, \bar{m}]\), (ii) for some player \( j \), \( F_{h}^{j}(\bar{z}) \geq \frac{\bar{z}}{v_\ell} \), and therefore,

\[ 1 = F_{h}^{j}(\bar{m}) = \left( F_{h}^{j}(\bar{m}) - F_{h}^{j}(\bar{z}) \right) + F_{h}^{j}(\bar{z}) \]
\[ = \frac{\bar{m} - \bar{z}}{p_j v_h} + F_{h}^{j}(\bar{z}) \]
\[ \geq \frac{\bar{m} - \bar{z}}{p_j v_h} + \frac{\bar{z}}{v_\ell} \]
\[ \geq \frac{\bar{m} - \bar{z}}{v_\ell} + \frac{\bar{z}}{v_\ell} \]
\[ = \frac{\bar{m}}{v_\ell} \geq 1 \]
In order for the equality to hold we would have \( \bar{m} = v_\ell \) and \( p_j = \frac{\omega_j}{v_h} \). Therefore, \( j = i \).

As a result, we would have

\[
\begin{align*}
v_h - v_\ell &= u_h^{-i}(\bar{m}) \\
&= u_h^{-i}(\omega_1) \\
&= p_1 v_h F_h^i(\omega_1) + (1 - p_1) v_h F_\ell^i(\omega_1) - \omega_1 \\
&= v_\ell \frac{\omega_1}{v_\ell} + (1 - p_1) v_h F_\ell^i(0) - \omega_1 \\
&= (1 - p_1) v_h F_\ell^i(0) \\
&< (1 - p_1) v_h = v_h - v_\ell
\end{align*}
\]

Contradiction As a result, Equation (D.26) must hold with strict inequality:

\[ p_i > \frac{v_\ell}{v_h}. \] (D.27)

Now, considering the inequalities (D.23) and (D.26) we can revoke Lemma 18 and conclude that the supports of the high types’ bidding distributions are intervals while positive bids in the supports of the low types’ bidding distributions are convex in \( \mathbb{R}^{++} \).

Denoting by \( \bar{b} \) the maximum bid in the support of \( F_\ell^i \), we then have \( \bar{b} \leq \bar{m} \) and

\[
[\omega_2, \bar{b}] \subseteq \text{Supp} \left[ F_\ell^i \right], \quad [0, \bar{m}] = \text{Supp} \left[ F_h^i \right], \quad \text{and} \quad [\omega_1, \bar{m}] = \text{Supp} \left[ F_h^{-i} \right].
\] (D.28)

Considering the payoff of player \(-i\)’s high type with a bid \( x \in [\omega_1, \omega_2] \), we obtain

\[
f_h^i(x) = \frac{1}{p_i v_h}
\]

for all \( x \in (\omega_1, \omega_2) \). Thus, no bid in \((\omega_1, \omega_2)\) can be a best response for player \(-i\)’s low type because bidding \( \omega_1 \) is strictly better. Hence, by Corollary 3 and Lemma 18 we must have

\[
\text{Supp} \left[ F_\ell^{-i} \right] = [0, \omega_1].
\]

Considering the payoffs of the low and high types of player \( i \) with a bid \( x \in [\omega_2, \bar{b}] \), we obtain

\[
f_h^{-i}(x) = \frac{1}{v_\ell} \quad \text{and} \quad f_h^{-i}(x) = \frac{1}{p_{-i} v_h}
\]

for all \( x \in [\omega_2, \bar{b}] \). Contradicting with \( p_{-i} < \frac{\omega_2}{v_h} \). So, zero must be in the support of \( F_h^{-i} \).
Because the high type of player $-i$ obtains positive payoff in equilibrium, player $i$’s low type must be bidding zero with positive probability, resulting in him obtaining zero payoff in equilibrium. Moreover, the maximum bid that player $-i$’s high type bids should be at least $v_\ell$. On the other hand, a low type player will never bid any bid above $v_\ell$. Therefore, Corollary 3 implies that

$$\max \{\text{Supp} \left[ F_h^i \right] \} = \max \{\text{Supp} \left[ F_{-i}^i \right] \} = \bar{m} \geq v_\ell$$

To summarize this case, we must have (i) both high types bidding zero, (ii) both low types bidding zero with positive probability, and (iii) $\bar{m} \geq v_\ell$. We now further our analysis by separately studying cases in part (iii).

**D.2.0.3.1 :** $\bar{m} = v_\ell$.

When the maximum bid that the high types bid in equilibrium is $v_\ell$, they obtain a payoff of $v_h - v_\ell$ in equilibrium. However, zero is also a best response for them. Thus,

$$v_h - v_\ell = u_h^i(0) = (1 - p_i)v_h F^i_\ell(0) < (1 - p_i)v_h,$$

where the inequality follows from the assumption that $\text{Supp} \left[ F^i_\ell \right] \neq \{0\}$. Therefore, for $i = 1, 2$,

$$p_i < \frac{v_\ell}{v_h}. \quad (D.29)$$

When $p_i < \frac{v_\ell}{v_h}$ for $i = 1, 2$, low types obtain zero payoff while high types obtain $v_h - v_\ell$. In the remaining of this part, we will derive the equilibrium bidding distributions.

**Claim 4.** $\text{Supp} \left[ F^i_\ell \right] = \text{Supp} \left[ F^{-i}_\ell \right] = [0, v_\ell]$.

**Proof.** Suppose, to the contrary, that a low type, say player $i$’s, bids at most $\bar{x} < v_\ell$. Then the payoff of player $-i$’s high type with a bid $x \in [\bar{x}, v_\ell]$, we obtain

$$f_h^i = \frac{1}{p_i v_h} \quad (D.30)$$

for all $x \in [\bar{x}, v_\ell]$. Observe that by bidding $v_\ell$ player $-i$’s low type guarantees herself her equilibrium payoff of zero. On the other hand, when she bids a bid $x \in [\bar{x}, v_\ell]$, she
obtains a negative payoff because
\[
    u^{i-\ell}_i(v_\ell) - u^{i-\ell}_i(x) = v_\ell \left[ F^i_h(v_\ell) - F^i_h(x) \right] - (v_\ell - x) \\
    \quad = v_\ell \frac{(v_\ell - x)}{p_i v_h} - (v_\ell - x) \\
    \quad = (v_\ell - x) \left( \frac{v_\ell}{p_i v_h} - 1 \right) \\
    \quad > 0 ,
\]
where the second equality follows from Equation (D.30) and the inequality from Equation (D.29). Therefore, player \( -i \)'s low type does not bid above \( \bar{x} \) either. As a result, similar to (D.30), we must have
\[
    f^{i-\ell}_{-i} = \frac{1}{p_{-i} v_h} \tag{D.31}
\]
for all \( x \in [\bar{x}, v_\ell] \). However, when we calculate player \( i \)'s payoff with the bid \( \bar{x} \), we get
\[
    0 = u^i_\ell(v_\ell) - u^i_\ell(\bar{x}) = v_\ell \left[ F^{-i}_h(v_\ell) - F^{-i}_h(\bar{x}) \right] - (v_\ell - \bar{x}) \\
    \quad = v_\ell \frac{(v_\ell - \bar{x})}{p_{-i} v_h} - (v_\ell - \bar{x}) \\
    \quad = (v_\ell - \bar{x}) \left( \frac{v_\ell}{p_{-i} v_h} - 1 \right) \\
    \quad > 0 ,
\]
where the first equality is due to \( \bar{x} \) being a best reply (by assumption) and \( v_\ell \) yielding the equilibrium payoff of zero, the third equality follows from Equation (D.31) and the inequality from Equation (D.29). Contradiction. \( \blacksquare \)

As a result,
\[
    \text{Supp} \left[ F^i_\ell \right] = \text{Supp} \left[ F^i_h \right] = \text{Supp} \left[ F^{-i}_\ell \right] = \text{Supp} \left[ F^{-i}_h \right] = [0, v_\ell]
\]
Since any bid \( x \in [0, v_\ell] \) yields zero payoff to a low type, say player \( -i \)'s, we have
\[
    0 = u^{i-\ell}_i(x) = F^i_h(x)v_\ell - x
\]
Rearranging the terms, we obtain

$$F_h^i(x) = \frac{x}{v_\ell}$$  \hspace{1cm} (D.32)

for all $x \in [0, v_\ell]$ and for $i = 1, 2$. On the other hand, every bid $x \in [0, v_\ell]$ yields a payoff of $v_h - v_\ell$ to a high type player. Considering player $-i$’s high type, we get

$$v_h - v_\ell = u_h^i(x) = p_i v_h F_h^i(x) + (1 - p_i) v_h F_\ell^i(x) - x = p_i v_h \frac{x}{v_\ell} + (1 - p_i) v_h F_\ell^i(x) - x,$$

where the last equality follows from (D.32). Rearranging the term, we obtain

$$F_\ell^i(x) = \frac{v_h - v_\ell}{(1 - p_i)v_h} + \frac{x(v_\ell - p_i v_h)}{(1 - p_i)v_h v_\ell}$$  \hspace{1cm} (D.33)

for all $x \in [0, v_\ell]$ and for $i = 1, 2$. It is obvious that the bidding distribution characterized in (D.32) and (D.33) constitute equilibrium when (D.29) is satisfied.

**D.2.0.3.2 : $\bar{m} > v_\ell$.**

Letting $\bar{b}$ be the maximum bid that a low type player bids in equilibrium, without loss say it is player $i$’s low type who has $\bar{b}$ in his bidding distribution’s support, we must have

$$0 = u_\ell^i(\bar{b}) = F_h^{-i}(\bar{b}) v_\ell - \bar{b}$$

Because both high types will be bidding in $[\bar{b}, \bar{m}]$, using the fact that the high type of player $i$ is indifferent between bidding $\bar{b}$ and $\bar{m}$, we obtain the following:

$$0 = u_h^i(\bar{m}) - u_h^i(\bar{b}) = p_{-i} v_h \left( F_h^{-i}(\bar{m}) - F_h^{-i}(\bar{b}) \right) - (\bar{m} - \bar{b})$$

Combining the equations, we get

$$1 = F_h^{-i}(\bar{m}) = \left( F_h^{-i}(\bar{m}) - F_h^{-i}(\bar{b}) \right) + F_h^{-i}(\bar{b}) = \frac{(\bar{m} - \bar{b})}{p_{-i} v_h} + \frac{\bar{b}}{v_\ell}$$  \hspace{1cm} (D.34)

As a result,

$$p_{-i} > \frac{v_\ell}{v_h}.$$  

If the maximum bid that the low type of player $-i$ bids in equilibrium were to be less than $\bar{b}$, say it was $\bar{b} < \bar{b}$, the high type of player $i$ would be indifferent between bidding $\bar{b}$
and \( \bar{b} \). Therefore, we would have

\[
0 = u^i_h(\bar{b}) - u^i_h(b) = p_{-ih} \left( F_{-i}^h(\bar{b}) - F_{-i}^h(b) \right) - (\bar{b} - b).
\]

However, this time by bidding \( b \) the low type of player \( i \) would obtain a higher payoff than by bidding \( \bar{b} \) because

\[
u^i_{\ell}(b) - u^i_{\ell}(\bar{b}) = v_{\ell} \left( F_{-i}^h(\bar{b}) - F_{-i}^h(b) \right) - (\bar{b} - b) < 0.
\]

So, the maximum bid that the low type players bid are the same, \( \bar{b} \), and it is less than \( v_{\ell} \). But, now, similar to (D.34), we would obtain

\[
1 = \frac{(\bar{m} - \bar{b})}{p_{ih}} + \frac{\bar{b}}{v_{\ell}}.
\]

Therefore,

\[
p_1 = p_2 > \frac{v_{\ell}}{v_{h}}.
\]

This case is already solved in Lu et al. (2018). And, in the unique equilibrium, a low type must bid zero with probability one. However, we assumed that they do not. Hence, there is no equilibrium in this case.

### D.3 Proof of Proposition 10

When the Auctioneer adopts the information disclosure policy that is never disclosing, if \((p_1, p_2)\) is being employed by the players as their distribution choices, player 1’s expected payoff can be calculated using Siegel (2014) algorithm as:

\[
EU_1(p_1, p_2) = \begin{cases} 
    p_1(1 - p_1)(v_h - v_{\ell}) & \text{if } p_1 \leq p_2 \\
    \left[ p_1(1 - p_2) + (1 - p_1)(p_1 - p_2) \frac{v_{\ell}}{v_h} \right] (v_h - v_{\ell}) & \text{if } p_1 > p_2
\end{cases}
\]

Therefore, the optimal \( p_1 \) for different levels of \( p_2 \) can be obtained as following

\[
p_1^*(p_2) = \begin{cases} 
    \frac{1}{2} & \text{if } p_2 > \frac{3}{4} \\
    \left\{ \frac{1}{2}, 1 \right\} & \text{if } p_2 = \frac{3}{4} \\
    1 & \text{if } p_2 < \frac{3}{4}
\end{cases}
\]
Since the best response functions are symmetric for the two players, in the unique equilibrium (up to symmetry) of the game, one player will choose to be of high type with probability one, whereas her opponent chooses to be of high type with probability half and of low type with the remaining probability half.

### D.4 Proof of Proposition 11

Letting \( c = \frac{v_h}{v_h} \), we rewrite \( EU_1(p_1, p_2) \) as follows:

\[
EU_1(p_1, p_2) = \begin{cases} 
  v_h(1-c)(1-p_1p_2) p_1 \left( \frac{1-c+p_2c}{1-c+p_1c} \right), & p_1 \leq p_2 \\
  v_h(1-c)(1-p_1p_2) (p_1 + (1-p_1)(p_1-p_2)c), & p_1 > p_2
\end{cases}
\]

The following lemma shows the optimal distribution choice of player 1 as an iterative function of his opponent’s distribution choice.

**Lemma 20.** The optimal \( p_1 \) satisfies the following equality:

\[
p_1^*(p_2) = \begin{cases} 
  \min \left\{ p_2, \frac{-(1-c)p_2+\sqrt{(1-c)p_2(c+p_2(1-c))}}{cp_2} \right\}, & \text{if } p_1 \leq p_2 \\
  \min \left\{ 1, \max \left\{ p_2, \frac{(c^2p_2^2+c+p_2+1)-\sqrt{(c^2p_2^2+c+p_2+1)^2-3p_2(c^2p_2^2+c+p_2+1)}}{3p_2c} \right\} \right\}, & \text{if } p_1 > p_2
\end{cases}
\]

**Proof.** Taking the derivative of \( EU_1(x, y) \) with respect to \( x \), we obtain

\[
\frac{1}{v_h(1-c)} \frac{\partial EU_1(x, y)}{\partial x} = \begin{cases} 
  \frac{1-c+cy}{(1-c+cy)^2} (-cyx^2 - 2xy(1-c) + (1-c)) , & x \leq y \\
  3cyx^2 - 2(cy^2 + cy + c + y)x + (cy^2 + cy + c + 1) , & x > y
\end{cases}
\]

Define \( f(x, y) \) and \( g(x, y) \) as follows:

\[
f(x, y) = -cyx^2 - 2xy(1-c) + (1-c)
\]
\[
g(x, y) = 3cyx^2 - 2(cy^2 + cy + c + y)x + (cy^2 + cy + c + 1)
\]

The quadratic equation \( f(x, y) = 0 \) has two roots, say \( x_1(y) < x_2(y) \) are these roots. While \( x_1(y) \) is negative, we have

\[
x_2(y) = \frac{-(1-c)y + \sqrt{(1-c)y(c+y(1-c))}}{cy} > 0
\]
And, $f(x, y) > 0$ when $x \in (x_1(y), x_2(y))$. As a result, when $p_1$ is not greater than $p_2$,

$$p_1^*(p_2) = \min\{p_2, x_2(p_2)\}$$

On the other hand, denoting by $x_3(y)$ and $x_4(y)$ the solutions to $g(x, y) = 0$, we have

$$x_{3,4}(y) = \frac{(cy^2 + cy + c + y) \pm \sqrt{(cy^2 + cy + c + y)^2 - 3cy(cy^2 + cy + c + 1)}}{3cy}$$

Because $(cy^2 + cy + c + y)^2 - 3cy(cy^2 + cy + c + 1) > 0$ and $cy^2 + cy + c + y > 3cy$, we have $x_3(y) > 1$ and $x_3(y) > x_4(y)$. Moreover, $g(x, y)$ is negative when $x \in (x_4(y), x_3(y))$. Observe that $x_3(y)$ can be greater 1. Hence, when $p_1$ is greater than $p_2$, we have

$$p_1^*(p_2) = \min\{1, \max\{p_2, x_4(p_2)\}\}$$

\[\blacksquare\]

**Claim D.4.1:** $\left(1, \frac{\sqrt{1-c}-(1-c)}{c}\right)$ is an equilibrium.

**Proof.** When $p_1 = 1$ we have

$$\min \left\{ p_1, -\frac{(1-c)p_1 + \sqrt{(1-c)p_1(c + p_1(1-c))}}{cp_1} \right\} = \frac{\sqrt{1-c}-(1-c)}{c}$$

So, $p_2 = \frac{\sqrt{1-c}-(1-c)}{c}$ is the best response for player 2 when $p_1 = 1$. On the other hand, when $p_2 = \frac{\sqrt{1-c}-(1-c)}{c}$,

$$\min \left\{ 1, \max \left\{ p_2, \frac{(p_2^2c + p_2c + c + p_2) - \sqrt{(p_2^2c + p_2c + c + p_2)^2 - 3p_2c(p_2^2c + p_2c + c + 1)}}{3p_2c} \right\} \right\} = 1$$

and

$$\min \left\{ p_2, -\frac{(1-c)p_2 + \sqrt{(1-c)p_2(c + p_2(1-c))}}{cp_2} \right\} = p_2 = \frac{\sqrt{1-c}-(1-c)}{c}.$$ 

Because

$$EU_1 \left(1, \frac{\sqrt{1-c}-(1-c)}{c}\right) > EU_1 \left(\frac{\sqrt{1-c}-(1-c)}{c}, \frac{\sqrt{1-c}-(1-c)}{c}\right),$$

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Thus, the profile \( (1, \frac{\sqrt{1-c-(1-c)}}{c}) \) constitutes an equilibrium. ■

**Claim D.4.2:** For all \( p < \frac{\sqrt{1-c-(1-c)}}{c} \), the profile \( (1, p) \) constitutes an equilibrium, whereas \( (p_1, p) \) is not an equilibrium for any \( p_1 \) that is less than 1.

**Proof.** The proof directly follows from the proof of Claim 1. ■

**Claim D.4.3:** There is no equilibrium with \( 1 > p_1 > p_2 \).

**Proof.** Suppose to the contrary, \( (p_1, p_2) \) with \( 1 > p_1 > p_2 \) is an equilibrium. In order for \( p_2 < p_1 \) to be a best response to \( p_1 \), we must have

\[
0 = f(p_2, p_1) = -cp_1 p_2^2 - 2p_1 p_2 (1 - c) + (1 - c)
\]

Leaving the term \( cp_2^2 \) alone, we obtain

\[
cp_2^2 = -2p_2 (1 - c) + \frac{(1 - c)}{p_1} \tag{D.35}
\]

Similarly, if \( p_1 > p_2 \) is a best response to \( p_2 \), then we have

\[
0 = g(p_1, p_2) = 3cp_2 p_1^2 - 2p_1 (cp_2^2 + cp_2 + c + p_2) + (cp_2^2 + c + c + 1). \tag{D.36}
\]

Plugging (D.35) into (D.36), we get

\[
0 = 3cp_2 p_1^2 + (1 - 2p_1) \left( cp_2^2 + cp_2 + c \right) - 2p_1 p_2 + 1
\]

\[
= 3cp_2 p_1^2 + (1 - 2p_1) \left( -2p_2 (1 - c) + \frac{(1 - c)}{p_1} + cp_2 + c \right) - 2p_1 p_2 + 1
\]

\[
= 3cp_2 p_1^2 + (1 - 2p_1) \left( c + 3cp_2 + \frac{(1 - c)}{p_1} - p_2 \right) + 1 - p_2
\]

\[
= 3cp_2 p_1^2 + p_2 (1 - 2p_1)(3c - 1) - p_2 + (1 - 2p_1) \left( c + \frac{(1 - c)}{p_1} \right) + 1
\]

\[
= p_2 \left( 3cp_1^2 + (1 - 2p_1)(3c - 1) - 1 \right) + (1 - 2p_1) \left( c + \frac{(1 - c)}{p_1} \right) + 1
\]

\[
= p_2 \left( 3cp_1^2 - 6cp_1 + 3c - 2 + 2p_1 \right) + (1 - 2p_1) \left( c + \frac{(1 - c)}{p_1} \right) + 1
\]

\[
= p_2 \left( 3c(1 - p_1)^2 - 2(1 - p_1) \right) + (1 - 2p_1) \left( c + \frac{(1 - c)}{p_1} \right) + 1
\]

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Because $1 > p_1 \geq 0$, dividing both sides by $(1 - p_1)$, we obtain

$$0 = p_2(3c(1 - p_1) - 2) + \left(2c + \frac{1 - c}{p_1}\right)$$  \hfill (D.37)

Due to Lemma 20, we have

$$p_2 = -\frac{(1 - c)p_1 + \sqrt{(1 - c)p_1(c + p_1(1 - c))}}{cp_1}$$  \hfill (D.38)

Plugging (D.38) into (D.37), we obtain

$$c(2cp_1 + (1 - c)) = (2 - 3c(1 - p_1)) \left(-(1 - c)p_1 + \sqrt{(1 - c)p_1(c + p_1(1 - c))}\right)$$
$$\leq (2 - 3c(1 - p_1))\frac{c}{2}$$  \hfill (D.39)

where inequality holds because $(2 - 3c(1 - p_1))$ is positive for (D.37) to hold and the following inequality $^3$

$$\left(-(1 - c)p_1 + \sqrt{(1 - c)p_1(c + p_1(1 - c))}\right) \leq \frac{c}{2}.$$

Rewriting (D.39), we get

$$\frac{1}{2}c \leq -\frac{1}{2}cp_1$$

Contradiction. Thus, there is no equilibrium in which $1 > p_1 > p_2$. $\blacksquare$

**Claim D.4.4:** There is no equilibrium with $1 > p_1 = p_2$. $^3$

---

$^3$Following is the proof of the inequality:

$$(1 - c)p_1(c + p_1(1 - c)) \leq (1 - c)p_1(c + p_1(1 - c)) + \frac{c^2}{4} = \left(\frac{c}{2} + p_1(1 - c)\right)^2$$
$$\iff \sqrt{(1 - c)p_1(c + p_1(1 - c))} \leq \frac{c}{2} + p_1(1 - c)$$
$$\iff -p_1(1 - c) + \sqrt{(1 - c)p_1(c + p_1(1 - c))} \leq \frac{c}{2}$$
Proof. Suppose to the contrary, \((p_1, p_2)\) with \(1 > p_1 = p_2 = p\) is an equilibrium for some \(p\). If no \(p_1 < p\) is a best response to \(p\), then

\[
p \leq \frac{-(1 - c)p + \sqrt{(1 - c)p(c + p_2(1 - c))}}{cp} \leq \frac{c}{2} \cdot \frac{1}{cp} = \frac{1}{2p}
\]

where the inequality follows from previous case. So, we must have

\[
p^2 \leq \frac{1}{2}
\]

However, then player 1 can be better of by choosing some \(p_1\) that is slightly above \(p\) because

\[
g(p, p) = 3cp^3 - 2p(cp^2 + c + p) + (cp^2 + c + 1)
\]

\[
= cp^3 - cp^2 - cp - 2p^2 + c + 1
\]

\[
= 1 - 2p^2 + c(1 - p)(1 - p^2)
\]

\[
> 0.
\]

Hence, there is no equilibrium in which \(1 > p_1 = p_2\). □

Combining the claims D.4.1 - D.4.4 proves the proposition.

### D.5 Proof of Proposition 12

When \((p_1, p_2)\) is the distribution profile employed by the two players, player 1 expects a payoff of

\[
EU_1(p_1, p_2) = \begin{cases} 
  v_h(p_1 + p_2 - p_1^2)p_1 \left(1 - c - p_2 + \frac{p_2}{p_1}c\right) & p_1 \geq c \text{ and } p_1 \geq p_2 \\
  v_h(p_1 + p_2 - p_1p_2)p_1 \left(1 - c - p_1 + \frac{p_1}{p_2}c\right) & p_2 \geq c \text{ and } p_1 \leq p_2 \\
  v_h(p_1 + p_2 - p_1p_2)p_1(1 - c) & p_1 \leq c \text{ and } p_2 \leq c
\end{cases}
\]

Therefore, taking its derivative with respective to \(p_1\), we obtain:

\[
\frac{1}{v_h} \frac{\partial EU_1(p_1, p_2)}{\partial p_1} = \begin{cases} 
  2p_1(1 - p_2)(1 - p_2 - c) + p_2(1 - p_2 - cp_2) & p_1 \geq c \text{ and } p_1 \geq p_2 \\
  -\frac{3(p_2 - c)}{p_2}p_2^2 + 2(1 - 2p_2 + cp_2)p_1 + p_2(1 - c) & p_2 \geq c \text{ and } p_1 \leq p_2 \\
  2p_1(1 - p_2)(1 - c) + p_2(1 - c) & p_1 \leq c \text{ and } p_2 \leq c
\end{cases}
\]

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Now, let $\tilde{c} = \frac{1}{10} \left( 6 - \sqrt{6} \right)$ and assume that $(p_1, p_2)$ with $1 \geq p_1 \geq p_2$ is an equilibrium.

**Claim D.5.1:** $p_1 \geq c$.

*Proof.* For player $i$, $p_i < c$ is dominated by $p_i = c$ when $p_{-i} \leq c$. ■

**Corollary 4.** No $(p_1, p_2) \in [0, c]^2 \setminus \{c, c\}$ is an equilibrium.

Next, define $f$ and $g$ as follows:

\[
\begin{align*}
f(x, y) &= 2x(1 - y)(1 - y - c) + y(1 - y - cy) \\
g(x, y) &= -3(1 - y)(y - c)x^2 + 2y(1 - 2y + cy)x + y^2(1 - c)
\end{align*}
\]  

**Claim D.5.2:** No $(p_1, p_2)$ with $p_1 > p_2$ is an equilibrium when $c > \frac{3}{5}$ or $\tilde{c} < c$.

*Proof.* In order for such $(p_1, p_2)$ to constitute an equilibrium, it is necessary that the following system of equations has solution:

\[
\begin{align*}
f(p_1, p_2) &= 0 \\
g(p_1, p_2) &= 0 \\
p_1 &> p_2
\end{align*}
\]

Lemma D.42 of Appendix 22 shows that when $c \not\in (\tilde{c}, \frac{3}{5})$, it has no solution. ■

**Claim D.5.3:** $(p_1, p_2)$ with $p_1 = p_2 = p$ is an equilibrium when $c > \frac{3}{5}$ and $c \leq p \leq \bar{p} \leq 1$, where $\bar{p}$ solves:

\[
0 = 3p^2 - (c + 7)p + 3 + 2c
\]

*Proof.* $p \geq c$ because of Claim D.5.1. Let $p$ and $\bar{p}$ be such that $0 = f(q, q)$ and $0 = g(\bar{q}, \bar{q})$. Observe that $f(q', q') < 0$ for all $q' > q$ while $g(q'', q'') > 0$ for all $q'' < \bar{q}$. Furthermore, because $c > \frac{3}{5}$, we have the following ordering:

\[q < c < \bar{q}\]

Hence, for all $p \in [c, \bar{q}]$, $(p, p)$ constitutes an equilibrium, where $\bar{q} > 0$ solves:

\[
0 = g(\bar{q}, \bar{q}) = -3(1 - \bar{q})(\bar{q} - c)\bar{q}^2 + 2\bar{q}(1 - 2\bar{q} + c\bar{q})\bar{q} + \bar{q}^2(1 - c)
\]

■
Claim D.5.4: No \((p_1, p_2)\) with \(p_1 = p_2\) is an equilibrium when \(c < \bar{c}\).

**Proof.** To the contrary, suppose \((p, p)\) is an equilibrium for some \(c\) that is less than \(\bar{c}\). Hence, we must have

\[
f(p, p) \leq 0 \quad \text{and} \quad g(p, p) \geq 0 \tag{D.41}
\]

First of all, observe that there are unique \(x^*_f \in (0, 1)\) and \(x^*_g \in (0, 1)\) such that \(f(x^*_f, x^*_f) = 0\) and \(g(x^*_g, x^*_g) = 0\). Moreover, \(f(x, x) < 0\) only when \(x < x^*_f\) and \(g(x, x) > 0\) only when \(x < x^*_g\). Taking the partial derivatives of \(x^*_f\) and \(x^*_g\) with respect to \(c\), we obtain

\[
\frac{\partial x^*_f}{\partial c} < 0 \quad \text{and} \quad \frac{\partial x^*_g}{\partial c} > 0.
\]

Lastly, \(x^*_f(\bar{c}) = x^*_g(\bar{c})\). Therefore, Equation (D.41) does not hold, thus, there is no equilibrium in which both players choose the same distribution over the type set when \(c < \bar{c}\). \(\blacksquare\)

Claim D.5.5: \((1, \frac{1}{2})\) is an equilibrium when \(c \leq \frac{3}{5}\).

**Proof.** When \(p_i = 1\) for some player \(i\), playing \(p_{-i} = \frac{1}{2}\) is best response for player \(-i\) because

\[
0 = g\left(\frac{1}{2}, 1\right)
\]

Furthermore, as long as \(c \geq \frac{3}{5}\), we have

\[
f\left(1, \frac{1}{2}\right) \geq 0
\]

where the inequality holds with equality when \(c = \frac{3}{5}\). And, observe that playing \(p_i = 1\) yields higher payoff than playing some \(p_i < \frac{1}{2}\). Combining the claims D.5.1 - D.5.5 completes the proof. \(\blacksquare\)

**D.6 Proof of Proposition 13**

Under policy FD, by Proposition 9 the distribution profile in an equilibrium is \((1, p)\), where \(p\) can be any real number in \([0, 1]\). Therefore, one player obtains a payoff \((1 - p)(v_h - v_l)\) while the other player gets zero payoff. Thus, the total players’ payoff is \((1 - p)v_h(1 - c)\). By changing \(p\) we obtain

\[
v_h(1 - c) \geq W^{FD} \geq 0.
\]
Under policy FC, by Proposition 10, the unique (up to symmetry) equilibrium distribution profile is \((1, 0.5)\). In this equilibrium, one player obtains \(\frac{1}{2}(v_h - v_l)\) and the other player gets \(\frac{1}{4}(v_h - v_l)\). As a result the total players’ payoff under FC regime is 

\[ W^{FC} = \frac{3}{4}v_h(1 - c). \]

Similar to FD regime, Under policy DL, by Proposition 11 the distribution profile in an equilibrium is \((1, p)\), where \(p\) can be any real number in \([0, \frac{\sqrt{1-c-(1-c)}}{c}]\). Thus, the total players’ payoff is \((1-p)v_h(1-c)\). By changing \(p\) we obtain 

\[ v_h(1 - c) \geq W^{DL} \geq v_h(1 - c) \left(\frac{1 - \sqrt{1 - c}}{c}\right). \]

Lastly, under policy DH, when \(c < \frac{6 - \sqrt{6}}{10}\), the outcome is the same as it is under FC. When \(c \geq \frac{6 - \sqrt{6}}{10}\), paying our attention to the symmetric equilibria, by the second part of Proposition 12 the distribution profile in an equilibrium is \((p, p)\), where \(p\) can be any real number in \([\max\{c, \frac{(5-c) - \sqrt{c^2 + 6c + 1}}{4}, \frac{(c+7) - \sqrt{c^2 - 10c + 13}}{6}\} \) \]. Moreover, in equilibrium both players obtain a payoff of \(p^2(1-p)(2-p)v_h\).

Now, for a given level of \(c \geq \frac{6 - \sqrt{6}}{10}\), let us find the extreme values of \(f(p) = 2p^2(1 - p)(2 - p)\) when \(p \in \left[\max\{c, \frac{(5-c) - \sqrt{c^2 + 6c + 1}}{4}, \frac{(c+7) - \sqrt{c^2 - 10c + 13}}{6}\}\right)\) so that we can compare the policy DH with the other policies. \(f(0) = 0, f(1) = 1\), and \(f\) is positive over \([0, 1]\) with only one maximizer in it. Figure D.2 illustrates the possible levels of players’ payoff under the policy DH for some level of \(c\) in \([\frac{6 - \sqrt{6}}{10}, 1]\), i.e \(f(p)\). 

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Figure D.2. The shape of \( f(p) = 2p^2(1 - p)(2 - p) \) when \( p \in [0, 1] \). \( p^* \) is the maximizer of \( f(p) \). For a given level of \( c \) \((p, p)\) is an equilibrium distribution profile under policy DH for all \( p \in [\underline{t}, \bar{t}] \), where \( \underline{t} = \max \left\{ c, \frac{(5-c)-\sqrt{c^2+6c+1}}{4} \right\} \) and \( \bar{t} = \frac{c+7-\sqrt{c^2-10c+13}}{6} \). Therefore, all the red points on \( f(p) \) correspond to a possible equilibrium players’ payoff.

For any \( c \) such that \( 1 - c \) is greater than the maximum of \( f(p) \) for \( p \in [0, 1] \), an optimist auctioneer would adopt policy FD. Let us first find \( \max_{p \in [0, 1]} f(p) \). Taking the first order condition we have

\[
0 = \frac{\partial f(p)}{\partial p} = 2p \left( 4p^2 - 9p + 4 \right)
\]

hence, \( p^* = \frac{9-\sqrt{17}}{8} \) is the maximizer. Plugging this into \( f(p) \) we obtain \( f(p^*) = \frac{51\sqrt{17} - 107}{256} \approx 0.403 \). Therefore, for \( c < \tilde{c} = \frac{363-51\sqrt{17}}{256} \approx 0.597 \), policy FD dominates policy DH from the perspective of an optimist auctioneer whose objective is maximizing the players’ payoff.

I will next show that for any \( c > \tilde{c} \), an optimist auctioneer’s desired policy switches. When \( \tilde{c} \leq c \leq p^* \), observe that \( p^* \in [\underline{t}, \bar{t}] \). Therefore the maximum players’ payoff could be \( f(p^*)v_h \) under policy DH while it is \( (1-c)v_h < f(p^*)v_h \) under policy FD. Moreover, when \( c > p^* \), \((c, c)\) is an equilibrium distribution profile under policy DH, hence, \( f(c)v_h \) is a possible players’ payoff and \( f(c)v_h < (1-c)v_h \).

When we consider a pessimist auctioneer’s disclosure policy adoption problem, we must compare the worst possible players’ payoffs under each policy for a given \( c \). When \( c < \frac{6-\sqrt{5}}{10} \), the policies FC and DH are preferred since

\[
\frac{3}{4}(1-c)v_h > (1-c) \left( \frac{1-\sqrt{1-\tilde{c}}}{\tilde{c}} \right) v_h > 0,
\]

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where each term above is the worst players' payoff under policies FC and DH, DL, and FD, respectively. When \( c \geq \frac{6 - \sqrt{6}}{10} \), the worst players' payoff under policy DH is \( f(\bar{t})v_h \).

Now, I will show that there is \( \bar{c} \in \left( \frac{6 - \sqrt{6}}{10}, 1 \right) \) such that the worst players' payoff under policy FC is greater than the others, whereas, it policy DH that dominates the others when \( c > \bar{c} \). Moreover, at \( \bar{c} \), FC and DH yield the same players' payoff. In other words, \( \bar{c} \) is the solution to \( f(\bar{t}(\bar{c})) = \frac{3}{4}(1 - \bar{c}) \).

### D.7 Proof of Proposition 14

When there is no high type player, the prize will definitely be allocated to a low type player; whereas when both players are of high type, it will for sure be allocated to a high type. Therefore, we need to consider the probability of the prize being allocated to a high type when the type profile \((h, \ell)\) prevails. Denoting by \( \Pr_P(h, \ell) \) this probability under policy \( P \in \{FD, FC, DL, DH\} \), we can calculate allocation efficiency for different disclosure policies as follows:

Under the full disclosure policy, FD, due to Proposition 5, letting \((1, p)\) be the equilibrium distribution choice profile, we have

\[
\Pr^{FD}(h, \ell) = \Pr(x_\ell < x_h) = \mathbb{E}_{x_h}[F_\ell(x_h)] = \frac{1}{2} \frac{v_h - v_\ell}{v_h} = \frac{2 - c}{2}
\]

where \( x_\theta \) is the bid that a type \( \theta \) player bids for \( \theta \in \{h, \ell\} \). Therefore, the allocation efficiency under the full disclosure policy could be calculated as

\[
\pi^{FD} = p \cdot 1 + (1 - p) \cdot \Pr^{FD}(h, \ell) = p + (1 - p) \frac{2 - c}{2} = 1 - \frac{(1 - p)c}{2}
\]

Since \( p \) can be anything in \([0, 1]\), we have

\[
1 \geq \pi^{FD} \geq 1 - \frac{1}{2}c.
\]

Under the full concealment policy, FC, we have one player being definitely of high type, while the other player is a high type player with probability half. Therefore, with probability half, the prize is allocated to a high type for sure, while with the remaining

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probability of one half, it is being allocated to a high type with probability $\Pr^{FC}(h, \ell)$, which can be calculated as follows:

$$\Pr^{FC}(h, \ell) = \Pr(x_\ell < x_h)$$

$$= \mathbb{E}_{x_h}[F_\ell(x_h)]$$

$$= \int_0^{\frac{1}{2}v_\ell} \left( \frac{v_h - v_\ell}{v_h} + \frac{x_h}{v_h} \right) \cdot \frac{1}{v_\ell} dx_h + \int_{\frac{1}{2}v_\ell}^{\frac{1}{2}v_h + \frac{1}{2}v_h} \frac{1}{v_h} dx_h$$

$$= 1 - \frac{c}{4}.$$  

Therefore, the allocation efficiency under the full concealment policy is

$$\pi^{FC} = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \Pr^{FC}(h, \ell) = 1 - \frac{c}{8}.$$  

When the auctioneer adopts the partial disclosure policy that is to disclose if there is a low type, DL, Proposition 11 tells us that the equilibrium distribution profile is $(1, p)$, where $0 \leq p \leq \frac{\sqrt{1 - c} - (1 - c)}{c}$. Using Proposition 7, we can calculate the allocation efficiency under policy DL. With probability $p$ both players are high types, hence, the good is allocated to a high type for sure. However, with probability $(1 - p)$, the type profile is $(h, \ell)$ and the players play a complete information all-pay auction. Thus, we have

$$\Pr^{DL}(h, \ell) = \Pr^{FD}(h, \ell) = \frac{2 - c}{2}.$$  

As a result, the allocation efficiency under the policy DL could be calculated as

$$\pi^{DL} = p \cdot 1 + (1 - p) \Pr(h, \ell) = p + (1 - p) \frac{2 - c}{2} = 1 - \frac{1}{2}(1 - p)c$$

Because $0 \leq p \leq \frac{\sqrt{1 - c} - (1 - c)}{c}$,

$$1 - \frac{1}{2} \left( 1 - \sqrt{1 - c} \right) \geq \pi^{DL} \geq 1 - \frac{c}{2}$$

Lastly, under the the partial disclosure policy that is to disclose if there is a high type, DH, when $c < \frac{6 - \sqrt{6}}{10}$, the outcome is the same as the one under the policy FC, whereas, when $c \geq \frac{6 - \sqrt{6}}{10}$, because the equilibrium distribution profile is symmetric, we could utilize Lu, Ma, Wang (2016) to conclude that $\pi^{DH} = p(2 - p)$ if $(p, p)$ is the distribution profile. By Proposition 8, we have $p \in [\ell, \bar{\ell}]$, therefore,
• When $c < \frac{1}{10}(6 - \sqrt{6})$:

$$\pi^{DH} = 1 - \frac{c}{8}$$

• When $c \geq \frac{1}{10}(6 - \sqrt{6})$:

$$\bar{t}(2 - \bar{t}) \geq \pi^{DH} \geq t(2 - t),$$

where $\bar{t} = \max \left\{ c, \frac{5 - c - \sqrt{c^2 + 6c + 1}}{4} \right\}$ and $t = \frac{c + 7 - \sqrt{c^2 - 10c + 13}}{6}$.

### D.8 Proof of Proposition 15

Let $x^P_i$ and $x^P$ denote the player i’s equilibrium effort (bid) and the total expected effort under policy $P \in \{FD, FC, DL, DH\}$ for an equilibrium. Under the full disclosure policy, when the equilibrium distribution profile is $(1, p)$, using the bidding distributions described in Proposition 5, we obtain

$$\mathbb{E}[x^{FD}_1] = p \frac{v_h}{2} + (1 - p) \frac{v_\ell}{2}$$

$$\mathbb{E}[x^{FD}_2] = p \frac{v_h}{2} + (1 - p) \frac{v_\ell}{2}$$

and that $p \in [0, 1]$. Therefore,

$$x^{FD} = \mathbb{E}[x^{FD}_1] + \mathbb{E}[x^{FD}_2] = pv_h + (1 - p) \frac{v_\ell(v_h + v_\ell)}{2v_h} = v_h \left( p + (1 - p) \frac{c(1 + c)}{2} \right).$$

When the auctioneer adopts the full concealment policy, the unique equilibrium distribution profile is $(1, \frac{1}{2})$. By using the bidding behavior of the players in this equilibrium, which is characterized in Proposition 6, we obtain the following:

$$\mathbb{E}[x^{FC}_1] = \frac{1}{2} \frac{v_\ell}{4} + \frac{1}{2} \left( \frac{v_\ell}{2} + \frac{v_h}{4} \right) = \frac{3v_\ell}{8} + \frac{v_h}{8}$$

$$\mathbb{E}[x^{FC}_2] = \frac{1}{2} \left( \frac{v_\ell}{2} + \frac{v_h}{4} \right) + \frac{1}{2} \frac{v_\ell v_\ell}{4} = \frac{2v_h}{8} + \frac{5v_\ell}{8} + \frac{v_\ell^2}{8v_h}$$

Thus,

$$x^{FC} = \mathbb{E}[x^{FC}_1] + \mathbb{E}[x^{FC}_2] = v_h \frac{1}{8} \left( c^2 + 5c + 2 \right).$$

When the announced policy is DL, in equilibrium, the bidding behavior is identical to that under policy FD. Fixing an equilibrium distribution profile, say $(1, p)$, by Proposition
7, we could calculate the players’ expected effort levels as

\[
\mathbb{E}[x_{1}^{DL}] = \frac{1}{2} v_\ell + \frac{1}{2} \left( \frac{v_\ell}{4} + \frac{v_h}{4} \right) = \frac{3v_\ell}{8} + \frac{v_h}{8}
\]
\[
\mathbb{E}[x_{2}^{DL}] = \frac{1}{2} \left( \frac{v_\ell}{2} + \frac{v_h}{4} \right) + \frac{1}{2} \frac{v_\ell v_h}{4} = \frac{2v_h}{8} + \frac{5v_\ell}{8} + \frac{v_\ell^2}{8v_h}
\]

Summing up the two, we get

\[
x^{DL} = \mathbb{E}[x_{1}^{DL}] + \mathbb{E}[x_{2}^{DL}] = v_h \left( p + (1 - p) \frac{c(1 + c)}{2} \right),
\]

where \(0 \leq p \leq \frac{\sqrt{1-c}-(1-c)}{c}\).

Lastly considering the policy DH, when \(c < \frac{6-\sqrt{6}}{10}\), the equilibrium outcome is identical to that under FC. When \(c \geq \frac{6-\sqrt{6}}{10}\), because the equilibrium distribution profile is \((p, p)\), we could utilize Lu, Ma, Wang (2016)’s findings to conclude that

\[
x^{DH} = v_h \left( p^2 + (1 - p)^2 c \right),
\]

where \(\max \left\{ c, \frac{(5-c)-\sqrt{c^2+6c+1}}{4} \right\} \leq p \leq \frac{(c+7)-\sqrt{c^2-10c+13}}{6}\) by Proposition 12.

### D.9 Supplementary Lemma

**Lemma 21.** When \(1 > x > y\), the following system of equations has no solution when \(c \not\in \left[ \frac{1}{10}(6 - \sqrt{6}), \frac{3}{5} \right]\).

\[
0 = h_1(x, y) \equiv 2x(1 - y)(1 - y - c) + y(1 - y - cy)
\]
\[
0 = h_2(x, y) \equiv -3(1 - x)(x - c)y^2 + 2x(1 - 2x + cx)y + x^2(1 - c)
\]

**Proof.** Solve the first equation for \(x\) by assuming that \(y\) is different than \((1 - c)\) and 1.

\[
x = -\frac{y(1 - y - cy)}{2(1 - y)(1 - y - c)}
\]

Equation (D.43) has two critical points, which are: \(y_{c1} = (1 - c)\) and \(y_{c2} = 1\). Because
we are given that $x > y$, we must have the following:

$$-\frac{y(1 - y - cy)}{2(1 - y)(1 - y - c)} - y > 0.$$  

Observe that the zeros of above equation are the zeros of

$$y(1 - y(1 - c)) + 2y(1 - y)(1 - y - c) = 0$$

So, $y = 0$ is one root while the other roots are the zeros of the following quadratic function

$$(1 - y(1 - c)) + 2(1 - y)(1 - y - c) = 0$$

Hence, the remaining two roots are as follows:

$$y_1 = \frac{1}{4} \left( 5 - c - \sqrt{c^2 + 6c + 1} \right)$$
$$y_2 = \frac{1}{4} \left( 5 - c + \sqrt{c^2 + 6c + 1} \right).$$

Note that $y_2 > 1$ when $c \in [0, 1]$, hence, we can disregard it. Furthermore, we have the following ranking among these critical points:

$$0 < y_{c1} < y_1 < y_{c2} = 1.$$  

Therefore, in order to determine when $x > y$ we need to determine the sign of

$$z(y) = -\frac{y(1 - y - cy)}{2(1 - y)(1 - y - c)} - y$$

in the regions $(0, y_{c1})$, $(y_{c1}, y_1)$, and $(y_1, 1)$. By checking each region, we find that $x > y$ only when

$$y \in (y_{c1}, y_1) = \left( 1 - c, \frac{1}{4} \left( 5 - c - \sqrt{c^2 + 6c + 1} \right) \right).$$

We also need $x$ to be less than 1. By looking at the fractional form of $x$, that is (D.43), one can show that $x$ is monotone decreasing in $y$ with $x \left( \frac{1}{1 + c} \right) = 0$. Therefore,

$$-\frac{y(1 - y - cy)}{2(1 - y)(1 - y - c)} - 1$$

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must have unique real root in the interval \((1 - c, \frac{1}{1+c})\), and it is:

\[
P = \frac{\sqrt{-4c^2 + 4c + 1 + 2c - 3}}{2(c - 1)}.
\]

Note that

\[
1 - c = y_{c1} < P < y_1 < \frac{1}{1+c}.
\]

Consequently, \(1 > x > y\) when \(y \in (P, y_1)\).

Next, assume \(x \neq 0\) and multiply \(h_2\) by \(x\). Then Plug (D.43) into it and set equal to zero to find that:

\[
\begin{align*}
-\frac{1}{4(-1+y)^2(-1+c+y)^2} & y^2(3(-1+y)^4 - c(-1+y)^2(15 - 20y + 6y^2) \\
+ & c^3(-12 + 24y - 17y^2 + 4y^3) + c^2(24 - 64y + 61y^2 - 24y^3 + 3y^4) \) = 0
\end{align*}
\]

Because \(y\) is different than \(1 - c\) and 1, we can multiply this by \(-4(-1 + y)^2(-1 + c + y)^2\) to obtain that

\[
(3c^2 - 6c + 3)y^6 + (4c^3 - 24c^2 + 32c - 12)y^5 + (-17c^3 + 61c^2 - 61c + 18)y^4
+ (24c^3 - 64c^2 + 50c - 12)y^3 + (-12c^3 + 24c^2 - 15c + 3)y^2 = 0
\]

If \(y \neq 0\) then the above is a fourth order polynomial:

\[
t(y) := (3c^2 - 6c + 3)y^4 + (4c^3 - 24c^2 + 32c - 12)y^3 + (-17c^3 + 61c^2 - 61c + 18)y^2
+ (24c^3 - 64c^2 + 50c - 12)y + (-12c^3 + 24c^2 - 15c + 3) = 0
\]

We are seeking the conditions on \(c\) that ensures this equation has a solution in the interval \((P, y_1)\). When \(c \in (0, 1)\), \(t(y)\) has two real roots. The smallest root is:

\[
t_{r1} = \frac{\sqrt{-4c^2 + 4c + 1 + 2c - 3}}{2(c - 1)}
\]

which can be disregarded since it is smaller than \(y_{c1}\). Because \(t(y)\) is monotone decreasing in the interval \((P, y_1)\), in order to conclude that \(t(y)\) has a root in the interval \((P, y_1)\),
we need only to check the end points and to see if they are of opposite signs.

\[ t(P) = \frac{1}{2(c - 1)^2} \left( 8c^5 + (17 - 3\sqrt{-4c^2 + 4c + 1}) c^2 + (19 - 13\sqrt{-4c^2 + 4c + 1}) c \right. \\
-3\sqrt{-4c^2 + 4c + 1} - 2 \left( \sqrt{-4c^2 + 4c + 1} + 2 \right) c^4 + \left( 13\sqrt{-4c^2 + 4c + 1} - 35 \right) c^3 + 3 \]

and

\[ t(y_1) = \frac{1}{32} \left( -5c^6 + (14\sqrt{c^2 + 6c + 1} + 1) c^2 - 9 \left( \sqrt{c^2 + 6c + 1} - 2 \right) c - 3\sqrt{c^2 + 6c + 1} \right. \\
- \left( 5\sqrt{c^2 + 6c + 1} + 14 \right) c^5 \left( \sqrt{c^2 + 6c + 1} + 25 \right) c^4 + 2 \left( \sqrt{c^2 + 6c + 1} - 22 \right) c^3 + 3 \]

Now, \( t(P) = 0 \) when \( c = 0 \) or \( c = \frac{3}{5} \) and \( t(y_1) = 0 \) when \( c = 0 \) or \( c = \frac{1}{10}(6 - \sqrt{6}) \). Furthermore, \( t(P) > 0 \) when \( c \in \left( 0, \frac{3}{5} \right) \) and \( t(P) < 0 \) when \( c \in \left( \frac{3}{5}, 1 \right) \), whereas, \( t(y_1) > 0 \) when \( c \in \left( 0, \frac{1}{10}(6 - \sqrt{6}) \right) \) and \( t(y_1) < 0 \) when \( c \in \left( \frac{1}{10}(6 - \sqrt{6}), 1 \right) \). Therefore, the signs of \( t(P) \) and \( t(y_1) \) are opposite only when \( c \in \left( \frac{1}{10}(6 - \sqrt{6}), \frac{3}{5} \right) \). Consequently, if \( x > y \), the equation system (D.42) has no solution when \( c \not\in \left( \frac{1}{10}(6 - \sqrt{6}), \frac{3}{5} \right) \).

\[ \blacksquare \]
Bibliography


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