AN EXAMPLE OF NON-COMMUTATIVE POINCARÉ
DUALITY
ARISING FROM HYPERBOLIC DYNAMICS

A Thesis in
Mathematics
by
Heath Emerson

© 2001 Heath Emerson

Submitted in Partial Fulfillment
of the Requirements
for the Degree of

Doctor of Philosophy

August 2001
We approve the thesis of Heath Emerson.
The notion of $K$-theoretic Poincaré duality is a familiar one from the works of Connes and Kasparov, and often arises in the ‘Dirac-Dual-Dirac’ method of proving the Novikov Conjecture by non-commutative techniques. Connes has formulated what it means for a possibly non-abelian $C^*$-algebra to have Poincaré duality, and more generally for two algebras to be dual to one another; the example of the irrational rotation algebra was worked out by him. Work of Kaminker and Putnam suggested a much wider class of examples; they showed that Poincaré duality tends to arise from the $C^*$-algebras constructed out of hyperbolic dynamical systems. Motivated by this, we have constructed a large class of algebras having Poincaré Duality in $K$-theory, namely the groupoid $C^*$-algebras associated to the groupoids $\partial \Gamma \rtimes \Gamma$ where $\Gamma$ is a hyperbolic group satisfying certain hypotheses, and $\partial \Gamma$ is its Gromov boundary.
Table of Contents

Acknowledgments .................................................................................. v

Chapter 1. Introduction ................................................................. 1
  1.1 Poincaré Duality .................................................................. 1
  1.2 K-theoretic Poincaré Duality ............................................... 9
  1.3 Examples of Dual Algebras .................................................. 16
  1.4 Hyperbolic Groups and Boundaries ....................................... 23
  1.5 Open Questions .................................................................. 27

Chapter 2. Poincaré Duality and Non-commutative Manifolds ....... 29
  2.1 KK-theoretic Preliminaries .................................................. 29
  2.2 Poincaré Duality .................................................................. 34

Chapter 3. Construction of the Fundamental Class ....................... 37
  3.1 Hyperbolic groups ................................................................ 37
  3.2 The Fundamental Classes ................................................... 39

Chapter 4. Proof of the Main Theorem ........................................... 47

Chapter 5. A Remark on Poincaré Duality and Assembly .......... 68

References ......................................................................................... 73
Acknowledgments

I am indebted to my thesis advisor Nigel Higson for many useful discussions on both the subject discussed here and many others both near and far. I am also indebted to Ian Putnam and Jerry Kaminker, who suggested the line of investigation pursued here, and for subsequent useful discussions. And finally, I would like to thank all the members of my committee for their comments and suggestions.
Chapter 1

Introduction

1.1 Poincaré Duality

Poincaré Duality occupies a central place in the topology of manifolds, in the interplay between analysis and geometry expressed in the Atiyah-Singer Index theorem and its many generalizations, in the ideas circling around the Baum-Connes conjecture, and more recently, in the Non-Commutative Geometry of Alain Connes.

In perhaps its most simple and most classical expression, this duality involves a certain symmetry within the cohomology of a compact, oriented manifold.

To each such topological space \( M \) we can associate the complex of differential forms on \( M, \Omega^*(M) \) (see [5] for a discussion of this construction and most of the other material discussed in this section) and then consider the sequence of vector spaces

\[
H^k(M) = \ker(d : \Omega^k(M) \to \Omega^{k+1}(M)) / \text{ran}(d : \Omega^{k-1}(M) \to \Omega^k(M))
\]

These vector spaces are called the de Rham cohomology groups of \( M \). Each, in the case of a compact manifold, can be seen to be finite dimensional, and thus has associated to it a number, namely its rank, denoted \( \beta_k \) and called the \( k \)th Betti number of \( M \). The numbers \( \beta_k \) are homotopy invariants of the manifold. Indeed, the cohomology groups \( H^k(M) \) are themselves invariant in a suitable sense under homotopy, whence part of their importance. (The latter fact was proved by de Rham.)
It can be calculated for instance for the 2-sphere, that $\beta_0 = 1$, $\beta_1 = 0$, and $\beta_2 = 1$. For the Riemann surface of genus $g$, we have $\beta_0 = 1$, $\beta_1 = 2g$, and $\beta_2 = 1$. Already a certain symmetry is evident: namely $\beta_0 = \beta_2$. Pursuing this, consider the $n$ torus, $T^n$. By the Künneth Theorem (see [5, page 47])

$$H^k(T^n) = \oplus_{j_1+\cdots+j_n=k} H^{j_1}(S^1) \otimes \cdots \otimes H^{j_n}(S^1).$$

Since the cohomology of the circle $S^1$ is $\mathbb{R}$ in dimensions 0 and 1, and in all other dimensions is 0, the rank of the above vector space is the number of sequences of of 0's and 1's of length $n$ whose entries add up to $k$. If we have such a sequence, we can replace each 0 by a 1 and each 1 by a 0, and the result is a sequence whose entries add up to $n-k$. It follows that $\beta_k = \beta_{n-k}$. This is our first and simplest formulation of Poincaré Duality: if $M$ is a compact oriented manifold

$$\beta_k = \beta_{n-k}.$$

Another slightly more sophisticated way of formulating this duality is at the level of the complexes from which the cohomology groups (vector spaces) are formed, rather than merely at the level of their isomorphism classes as vector spaces.

Let $M$ be an oriented manifold, compact or not. Then as well as the cohomology groups $H^*(M)$ constructed from the complex $\Omega^*(M)$ of smooth differential forms on $M$, we also have compactly supported cohomology groups $H^*_c(M)$ defined in the same way except with compactly supported forms. Recall (see [5]) that any compactly supported $n$-form on $M$ can be integrated over $M$, and that by Stokes’ theorem this integration induces a well-defined map $H^n_c(M) \to \mathbb{R}$. A more precise and useful formulation of Poincaré Duality is then that the bilinear map

$$H^k(M) \times H^{n-k}_c(M) \to \mathbb{R}$$
given by $(\omega, \eta) \mapsto \int_M \omega \wedge \eta$, is nondegenerate. Note that this nondegeneracy in turn implies we have a canonical isomorphism

$$H^k(M) \to (H_c^{n-k}(M))^*$$

of vector spaces.

Of course if $M$ is compact, $H_c^{n-k}(M) \cong H^{n-k}(M)$.

Probably the simplest way of seeing that for any oriented manifold $M$ the map given above is an isomorphism, is by means of what is known as the Mayer-Vietoris Argument (again, see [5]). This proceeds as follows. Recall that if $U$ and $V$ are open sets, there is an exact sequence

$$\cdots \to H^q(U \cup V) \to H^q(U) \oplus H^q(V) \to H^q(U \cap V) \to H^{q+1}(U \cup V) \to \cdots.$$ 

On the other hand, there is also a long exact sequence

$$\cdots \leftarrow H^q_c(U \cup V) \leftarrow H^q_c(U) \oplus H^q_c(V) \leftarrow H^q_c(U \cap V) \leftarrow H^{q+1}_c(U \cup V) \leftarrow \cdots.$$ 

From the latter, by dualizing, we obtain a long exact sequence

$$\cdots \to H^q(U \cup V)^* \to H^q(U)^* \oplus H^q(V)^* \to H^q(U \cap V)^* \to H^{q+1}(U \cup V)^* \to \cdots.$$ 

We can now compare the two sequences, and, setting up an obvious commutative diagram whose vertical maps are given by the Poincaré Duality maps for $U$, $V$, $U \cup V$ and $U \cap V$, verify that the map for $U \cup V$ will be an isomorphism if the maps for $U$, $V$ and $U \cap V$ are. We are now in a position to make an induction argument. Give $M$ a Riemannian metric, cover it by a finite number of geodesically convex balls, and proceed by induction, noting the result holds for $M = \mathbb{R}^n$ by the Poincaré Lemma: $H^k(\mathbb{R}^n) = \mathbb{R}$ if $k = 0$, $H^k(\mathbb{R}^n) = 0$ else, and $H^k_c(\mathbb{R}^n) = \mathbb{R}$ if $k = n$ and $H^k_c(\mathbb{R}^n) = 0$ else.
Now the proof by the Mayer-Vietoris Argument has limited usefulness in that it has no analog in the non-commutative examples we will be looking at. However, one can take a different approach, and one more amenable to generalization. To this end, recall that to a space we can also associate homology groups $H_k(M)$, in such a way that $M \to H_k(M)$ has the opposite functoriality properties from $M \to H^k(M)$; the former represents a covariant functor, the latter contravariant. One way of describing these homology groups is by forming the dual complex $\Omega^\ast(M)^\ast$ with the dual of the de Rham differential $d$, denoted $\delta$, and then by defining $H_k(M)$ by $H_k(M) = \ker(\delta)/\text{ran}(\delta)$. We can then formulate Poincaré Duality from this point of view as an isomorphism $H^k(M) \cong H_{n-k}(M)$. The map, which we denote by $D$, from $H^k(M)$ to $H_{n-k}(M)$ is defined as follows: if $\tau$ is a $k$ form representing a class in $H^k(M)$, let $D(\tau)$ be the functional on $n-k$ forms

$$\omega \mapsto \int_M \omega \wedge \tau.$$  

In this setting, we can also write down an inverse map $\hat{D}$ to $D$. We need to produce from a functional $\phi$ on $k$ forms, an $n-k$ form $\hat{D}(\phi)$. Let $U$ be a tubular neighbourhood of the diagonally embedded copy of $M$ in $M \times M$. Then $U$ can be identified with the total space of a vector bundle $E \xrightarrow{\pi} M$. Now, to such a vector bundle $E$ there is a Thom class $\beta$, (see [5, page 64]), which in local coordinates $(x, y)$ (with $x$ the coordinate on $M$, $y$ the coordinate on the fibre) has the form $\phi(x, y)dy$, and whose integral $\int_{\pi^{-1}(x)} \phi(x, y)dy$ is equal to 1 for all $x$. This construction makes sense as long as the bundle $E$ is orientable, as is the case with $E \cong U$ above, given that $M$ itself is orientable. The form $\beta$ then defines and element of the compactly supported cohomology of $U$, which can then be pushed forward to a class in the cohomology of $M \times M$.

Now by the Eilenberg-Zilber theorem (see for example [26, page 264]) there is a natural chain equivalence $\Omega^\ast(M \times M) \cong \Omega^\ast(M) \otimes \Omega^\ast(M)$. It follows that we can represent $\beta$ in the form

$$\beta = \sum_{i+j=n} \beta_i \otimes \beta_j' \in \Omega^\ast(M) \otimes \Omega^\ast(M).$$
This decomposition enables us to evaluate the functional $\phi$ on ‘part’ of $\beta$ in such a way as to leave behind an $n - k$ form. More precisely, define

$$\hat{D}(\phi) = \sum_{i+j=n} \beta_i \phi(\beta'_j) = \beta_{n-k} \phi(\beta_k).$$

Having defined our two maps, we now are reduced to proving that the composition $\hat{D}D$ and the composition $D\hat{D}$ are the identities on respectively cohomology and homology. For this we need several observations. Firstly, note that for any vector bundle $E$ as above, we have an ‘integration over the fibre’ map (see [5, page 61])

$$\pi_v : H^k_c(E) \to H^{k-n}(M),$$

given in local coordinates by

$$\pi^* (\omega) f dy \mapsto \pi^* (\omega) \int f(\cdot, y) dy,$$

where $\omega$ is a form on $M$ and $f$ a compactly supported function on $E$. The map $\pi_v$ satisfies the following two properties:

1. $\pi_v$ commutes with the exterior derivative $d$;
2. $\pi_v(\pi^* (\omega) \wedge \tau) = \tau \wedge \phi_v(\omega)$ for every form $\omega$ on $M$, and form $\tau$ on $E$.

Let $\pi_1$ and $\pi_2$ denote the two projections $M \times M \to M$, and note that when we think of of $U$ as the total space of a bundle $E \xrightarrow{\pi} M$, the restriction of $\pi_1$ to $U$ identifies with the bundle projection $\pi$.

Consider now the composition $\hat{D}D$. If $\omega$ is a $k$ form, $\hat{D}(D(\omega))$ is the $k$ form

$$\sum_{i+j=n} \beta_i \int_M \beta'_j \wedge \omega,$$
in the above notation. This can be written more efficiently as
\[
\pi_v(\beta \wedge \pi_2^*(\omega)) = (-1)^{nk} \pi_v(\pi_2^*(\omega) \wedge \beta).
\]

Now \( U \) can be chosen in such a way that the two projections \( \pi_1 \) and \( \pi_2 \) when restricted to \( U \) are homotopic. It follows from homotopy invariance of de Rham cohomology that for some form \( \tau \) on \( U \) we have \( \pi_2^*(\omega) = \pi_1^*(\omega) + d\tau \). Thus
\[
\hat{D}D(\omega) = (-1)^{nk} \pi_v((\pi_1^*(\omega) + d\tau) \wedge \beta)
\]
\[
= (-1)^{nk}(\pi_v(\pi_1^*(\omega) \wedge \beta) + \pi_v(\beta \wedge d\tau) = (-1)^{nk}(\omega \wedge \pi_v(\beta) + d(\pi_v(\beta \wedge \tau))
\]
by properties (1) and (2) of \( \pi_v \) and the fact that \( \beta \) is a closed form, and this in turn equals
\[
(-1)^{nk}\omega + d(\pi_v\beta \wedge \tau)
\]
which is cohomologous to \( (-1)^{nk}\omega \). We note the persistant signs, which are in fact unavoidable.
Computing \( D\hat{D} \) one sees similarly that \( D\hat{D} \) is the identity map, again up to a sign of \( (-1)^{nk} \).

It seems at this point worth introducing a new idea, in order to abstract the above argument a little bit. This is the notion of a \textit{bivariant} theory. Let \( M_1 \) and \( M_2 \) be manifolds and let \( \Omega^*(M_1) \) and \( \Omega^*(M_2) \) their complexes of differential forms. A morphism between them of degree \( k \) is a map
\[
\alpha : \Omega^*(M_1) \to \Omega^*(M_2)
\]
satisfying the two conditions:

(1) \( \alpha(d(\omega)) = (-1)^k d(\alpha(\omega)) \); and

(2) \( \partial(\alpha(\omega)) = k + \partial(\omega) \).
Note that a morphism of complexes can have negative degree. Mod out by the collection of $\alpha$ of the form $\omega \mapsto d(\beta(\omega)) \pm \beta(d\omega)$, where $\beta$ is a morphism of complexes. Let the resulting group be denoted $HH^k(M_1, M_2)$. Note we have a 'product,' or composition operation

$$HH^k(M_1, M_2) \otimes HH^l(M_2, M_3) \to HH^{k+l}(M_1, M_3)$$

denoted $\alpha \otimes \beta \mapsto \alpha \otimes_M \beta$. We also have an external product

$$\sigma_N : HH(M_1, M_2) \to HH(M_1 \times N, M_2 \times N)$$

given by $\alpha \mapsto \alpha \hat{\otimes} id_N$, and similarly a map $\sigma^N : HH^k(M_1, M_2) \to HH^k(N \times M_1, N \times M_2)$.

Let $pt$ denote a point. The complex $\Omega^*(pt)$ is simply $\mathbb{R}$ with the zero differential. Any map between this complex and the complex for $M$ is simply choice of a differential form on $M$.

Since the differential for a point is 0, the conditions (1) and (2) imply this differential form is closed, and the equivalence relation ensures that $HH^k(pt, M)$ is exactly the cohomology of $M$. Remark that for $k$ negative $HH^k(pt, M)$ is 0. On the other hand, for $k \geq 0$, by arguments dual to those above, $HH^{-k}(M, pt)$ is the homology of $M$ in degree $k$.

Poincaré duality can now be formulated in terms of morphisms. Define an element $\Delta$ of $HH^{-n}(M \times M, pt)$ by first identifying $\Omega^*(M \times M)$ with $\Omega^*(M) \hat{\otimes} \Omega^*(M)$ and then mapping the latter to $\Omega^*(pt)$ by the formula

$$\omega \hat{\otimes} \tau \mapsto \int_M \omega \wedge \tau.$$

This satisfies conditions (1) and (2) by Stokes theorem, and the fact that the differential on $\Omega^*(M_1 \times M_2) \cong \Omega^*(M_1) \hat{\otimes} \Omega^*(M_2)$ is

$$\alpha \hat{\otimes} \tau \mapsto d\alpha \hat{\otimes} \tau \pm \alpha \hat{\otimes} d\tau.$$
Define an element $\beta$ of $HH^n(pt, M \times M)$ by the $n$-form $\beta$ described previously.

We can then define a map

$$\Delta_k : H^k(M) = HH^k(pt, M) \to HH^{k-n}(M, pt) = H_{n-k}(M)$$

by the composition

$$HH^k(pt, M) \xrightarrow{\sigma_M} HH^k(M, M \times M) \xrightarrow{\otimes M \times M \Delta} HH^{k-n}(M, pt).$$

Similarly we can define a map

$$\hat{\Delta}_k : H_k(M) = HH^{-k}(M, pt) \xrightarrow{\sigma_M} HH^{-k}(M \times M, M) \xrightarrow{\hat{\Delta} \otimes M \times M \Delta} HH^{-k+n}(pt, M) = H^{n-k}(M).$$

Now, in addition to these maps, we can form the morphism, denoted somewhat informally by $\hat{\Delta} \otimes_M \Delta$, which is defined by $\sigma_M(\hat{\Delta}) \otimes_M \Delta \in HH^0(M, M)$. It is not hard to compute that for $[\omega] \in HH^k(pt, M)$

$$\hat{\Delta}(\Delta([\omega])) = (-1)^{kn} [\omega] \otimes_M (\hat{\Delta} \otimes M \Delta)$$

and that for $[\tau] \in HH^{-k}(M, pt)$ we have

$$\Delta(\hat{\Delta}([\tau])) = (-1)^{nk+n} (\hat{\Delta} \otimes_M \Delta) \otimes_M [\tau].$$

Our problem is then reduced to showing that the element $\hat{\Delta} \otimes_M \Delta = 1_M$, where $1_M$ denotes the identity morphism $\Omega^*(M) \to \Omega^*(M)$. 
Although this method has little advantage over the more usual one, in other settings, the reduction of the problem to proving that a single ‘morphism’ \( M \to M \) is equivalent in the appropriate category to the identity morphism, represents a significant step, as we shall see later.

1.2 K-theoretic Poincaré Duality

As K-theory and K-homology are by their nature slightly more complicated than ordinary homology/cohomology, we will be slightly more sketchy here. For an exposition to which we will refer frequently in this dissertation, see \([20]\); alternatively, see the book by Bruce Blackadar \([3]\), or, for a more contemporary treatment of the subject, \([15]\). Let us begin by recalling that, roughly speaking, the K-theory of \( M \) is given by vector bundles over \( M \), and the K-homology by elliptic operators on \( M \).

What we need to know is firstly that there is a certain operation known as twisting, (see \([15, \text{page} 309]\)) which is defined on pairs \((D, E)\) where \( D \) is an elliptic operator on \( M \) and \( E \) is a vector bundle on \( M \), and which produces from each such pair \((D, E)\) another elliptic operator \( D_E \) on \( M \); and, secondly, that any elliptic operator on a compact manifold is Fredholm.

With these two facts in mind, suppose that \( D \) is a fixed elliptic operator on \( M \). We can then define a map from the K-theory of \( M \) to the K-homology by the assignment \( E \mapsto D_E \), which is a Poincaré Duality map of the sort we have been considering.

To describe a map backwards, from the K-homology of \( M \) to the K-theory of \( M \), is easy at the conceptual level, although to be precise requires some substantial work (see for example \([1]\)); we proceed by first noting that if \( B \) is a vector bundle over \( M \times M \), then \( B \) can be viewed as a family \( \{B_x\}_{x \in M} \) of vector bundles over \( M \) by setting \( B_x = e_x^*(B) \), where \( e_x : M \to M \times M \) is the embedding \( y \mapsto (x, y) \). Now, fix such \( B \), and define, for an elliptic operator \( F \) on \( M \), a family of elliptic operators on \( M \) by the formula \( F_x = F_{B_x} \). From this family, we can then form two families of finite-dimensional spaces parameterized by the points of \( M \); namely \( \{ \ker(F_x) \} \) and \( \{ \coker(F_x) \} \). Now a vector bundle on \( M \) is exactly a family of finite dimensional vector spaces
parameterized by the points of \( M \) and satisfying certain local triviality conditions; if we ignore these latter conditions we can consider the families \( \{ \ker(F_x) \} \) and \( \{ \coker(F_x) \} \) as bundles over \( M \), take their formal difference, and view the result as an element of the \( K \)-theory of \( M \).

For appropriate choice of \( D \) and \( B \), these two maps define inverse isomorphisms. The hypothesis needed on \( M \) is that it be spin\(^c \), or, in other words, oriented for \( K \)-theory. \( D \) is then the Dirac operator associated with the spin\(^c \) structure. One can prove that the two given maps represent inverse isomorphisms via a Mayer-Vietoris argument as in the case of ordinary homology; we remark that the analog of the Poincaré Lemma in the case of ordinary homology, is Bott Periodicity.

Just as in the case of homology, these ideas can be most efficiently expressed in the language of bivariant theory, and we now pass to a discussion of this. To ease the transition to our main interest, let us work with algebras rather than spaces, which means that the manifold \( M \) (considered as a topological space) is replaced by the \( C^* \)-algebra \( C(M) \) of continuous functions on \( M \). We will sketch out the definition of a bivariant theory \( KK^i(X, Y) \), which one should think of playing the role of \( HH^i \) in the earlier discussion of classical homology. Before proceeding, for any integer \( n \) let \( C_n \) denote the complex Clifford algebra with \( n \) generators \( \epsilon_1, \ldots, \epsilon_n \) satisfying \( \epsilon_i^2 = 1 \) for all \( i \), if \( n \) is positive, and \( \epsilon_i^2 = -1 \) for all \( i \) if \( n \) is negative, and also \( [\epsilon_i, \epsilon_j] = 0 \) for all \( i, j \), where the commutators are graded. For a discussion of Clifford algebras, and of \( K \)-homology defined as below, see [15]; for a discussion of \( KK \)-theory, see [3], or [20].

Define a group \( KK^i(X, Y) \) as follows. Its cycles consist of pairs \( (E, F) \) where \( E \) is a right Hilbert \( C(Y) \otimes C_i \) module and simultaneously a left \( C(X) \) module, and \( F \) is an operator on it as a \( C(Y) \otimes C_i \)-module, satisfying the three conditions: \( f(F^2 - 1), f(F - F^*) \) and \([f, F]\) are all compact, for every \( f \in C(X) \) where again, \([\cdot, \cdot]\) denotes \textit{graded} commutator. There are several other requirements, but we prefer to list here only the most essential ones.
It is not hard to see that for a space $X$, $K^n(X) = KK^{-n}(pt, X)$. One can simply take $K_n(X) = KK^{-n}(X, pt)$ by definition; remark that, as we shall discuss below, any elliptic operator on $X$ yields in a canonical way a class in $KK$.

There is a composition law, or intersection product:

$$KK^i(X, Y) \times KK^j(Y, Z) \to KK^{i+j}(X, Z)$$

denoted $(a, b) \mapsto a \otimes_Y b$, whose description is too involved to give here (see the above references).

There are maps $\sigma_Z : KK^i(X, Y) \to KK^i(X \times Z, Y \times Z)$ and maps $\sigma^Z : KK^i(X, Y) \to KK^i(Z \times X, Z \times Y)$. Any actual map from $Y \to X$ gives an element of $KK(X, Y)$. Combining this latter fact with the existance of the intersection product, any map $f$ from $Y \to X$ gives a map $f_* : KK^i(Y, Z) \to KK^i(X, Z)$ and a map $f^* : KK^i(Z, X) \to KK^i(Z, Y)$.

Assume now $M$ is a compact manifold, and that we have an element $\Delta$ of $K_n(M) = KK^{-n}(M, pt)$ where $n = \dim(M)$. We can then define a map $\Delta_j : KK^j(M) \to K_{j+n}(M)$ by

$$a \mapsto \sigma_M(a) \otimes_{M \times M} \Delta.$$ 

Note that this formula makes sense, as $\sigma_M(a) \in KK^j(M \times M)$ and $\Delta \in KK^{-n}(M \times M, pt)$, and so $\sigma_M(a) \otimes_{M \times M} \Delta \in KK^{j-n}(M, pt)$.

Let us at least partially unravel the definitions in the case of interest (when this map actually induces Poincaré Duality), which is when the class $\Delta$ arises from the Dirac operator on $M$. Now, an elliptic operator comes equipped with a bundle, on sections of which it acts as an unbounded operator. In the case of the Dirac operator, the bundle is called the spinor bundle, and is directly related to the spin$^c$ structure on $M$. We denote it by $S$. It is well known that the Dirac operator, as an unbounded operator, is closeable, and that its closure is self-adjoint. Taking this closure and applying the function $\varphi(x) = x(1 + x^2)^{-\frac{1}{2}}$ to it in the sense of functional
calculus, one obtains a bounded operator $F$, acting on the Hilbert space $L^2(S)$. One then verifies that the pair $(L^2(S), F)$ defines a cycle in $KK^{-n}(M, pt)$. Let $[D]$ denote its class.

Let $m$ denote the diagonal embedding $M \to M \times M$ and let $\Delta = m_\ast([D]) \in KK^{-n}(M \times M, pt)$. Let $a$ denote the class of a vector bundle $E$ over $M$, so that $a$ defines a class in $KK^0(pt, M)$. Then, applying the map $\Delta_0$ to the class $a$ yields a class in $KK^{-n}(M, pt)$ which we can describe loosely by the pair $(L^2(E) \hat{\otimes} C(M) L^2(S), G)$, where $G$ is an operator which we should think of as $1 \Gamma(E) \hat{\otimes} F$, although the latter does not make sense as $F$ does not commute exactly with functions $f \in C(M)$. In order to avoid going into details as to what $G$ really is, let us assume that $E$ is the trivial bundle $\mathbb{C}^n \times M$. In this case $L^2(E) \hat{\otimes} C(M) L^2(S) \cong L^2(S) \oplus \cdots \oplus L^2(S)$ and via this interpretation $1 \hat{\otimes} F$ does make sense, as $F \oplus \cdots \oplus F$, and in fact the operator $G$ can be taken to be exactly this. The module carries an inherited right action of $C_{-n}$ as required.

To summarize, we have constructed an element of $KK^{-n}(M, pt) = K_n(M)$, from a class $a$ in $KK^0(pt, M)$.

Note this calculation gives us at least a tiny bit of information about the map: if it is to be an isomorphism, it must in particular be injective, and since the trivial bundle $[1, n] = M \times \mathbb{C}^n$ represents a non-zero element of the $K$-theory of $M$, it must be that also $[D]$ represents a non-zero element, since our map carries $[1, n]$ to $n[D]$. It is good to keep in mind for later that this ceases to be true in the non-commutative case. Indeed, for the main class of examples discussed in this dissertation, both $[1]$ (suitably interpreted) and $[D]$ (also suitably interpreted) can be 0 in their respective $K$-theory and $K$-homology groups.

We broadly sketch the definition of the map from $K$-homology to $K$-theory serving as an inverse to the above. Let $\theta_n$ denote the Bott generator of $K^{-n}(\mathbb{R}^n)$, which is a certain pair $(\mathcal{E}, \hat{D}) \in KK^n(pt, \mathbb{R}^n)$ which pairs with the Dirac operator on $\mathbb{R}^n$, an element of $KK^{-n}(\mathbb{R}^n, pt)$, to give 1 in $KK^0(pt, pt) \cong \mathbb{Z}$.

To each $x \in M$ consider a ball $B_\varepsilon(0)$ centered at 0 in $T_x(M)$ in the tangent bundle at $x$. We can view this as a ball in $\mathbb{R}^n$ and consider the Bott element $\theta_{n,x}$ centered in this ball. The spin$^c$
structure on $M$ allows us to piece together all these elements $\theta_{n,x}$ to an element $\beta \in K^{-n}(TM)$.

For small enough $\epsilon$ the support of $\beta$ can be identified with a small open neighbourhood of the
diagonal of $M$ in $M \times M$ and by a simple device known as wrong way functoriality (see [2]), we can view $\beta$ as an element of $K^{-n}(M \times M) = KK^n(pt, M \times M)$. Now one can write down a map

$$\hat{\Delta}_j : K_j(M) \hookrightarrow K^{j-n}(M)$$

by

$$b \mapsto \beta \otimes_{M \times M} \sigma^M(b)$$

and then show with a certain amount of work, that $\hat{\Delta}_* \Delta_*$ is the identity (again, up to sign), and

that similarly $\Delta_* \hat{\Delta}_*$ is the identity (again up to sign). This can be done by Mayer-Vietoris, just

as in ordinary homology, by proving the result for $M = \mathbb{R}^n$ and proceeding by induction, but a better way, and a more useful one for purposes of generalizing to the non-commutative case, where no Mayer-Vietoris argument may exist, is to show, just as in the above discussion of $HH$,

that

$$\hat{\Delta}_{j-n}(\Delta_j(a)) = (-1)^{jn} a \otimes_{M} (\hat{\Delta} \otimes_{M} \Delta),$$

where again we denote by $\hat{\Delta} \otimes_{M} \Delta$ the element $\sigma_M(\hat{\Delta}) \otimes_{M \times M} \sigma^M(\Delta) \in KK(M, M)$;

and that a similar formula exists for $\Delta_{n+j}(\hat{\Delta}_j(b))$, and then show that $\hat{\Delta} \otimes_{M} \Delta = 1_M$. Here

$1_M = (C(M), 0)$, which acts as a multiplicative unit in $KK$-theory. This latter calculation exists implicitely in the work of Kasparov (see [20]) in his paper on the Novikov conjecture, in which Poincaré Duality plays a central role.

Again, though it may seem like merely a restatement, the reduction of the problem of

showing the two maps $\Delta$ and $\hat{\Delta}$ compose to the identity to showing that a single morphism

$M \to M$ is equivalent to the identity morphism, is in fact a very important step, as we shall see

later. Indeed, without it, it would be near impossible to prove the main theorem of this paper.
But there is another idea we may draw from this discussion, which is that the analog in
$K$-theory of integration of forms

$$\Omega^n(M) \to \mathbb{R}$$
is in the commutative case the existence of an element of $KK^{-n}(M, \text{pt})$, or, in other words, an
elliptic operator on $M$.

Let us move on to the more general notion of $K$-theoretic Poincaré duality for $C^*$-algebras,
of which the above is a special case. It is possible to define groups $KK^i(A, B)$ in essentially the
same way as we constructed groups $KK^i(X, Y)$ above (for $A = C(X)$, $B = C(Y)$), for arbitrary
separable, nuclear $C^*$-algebras $A$ and $B$. See Section 2.1 for details. For the moment let us think
of cycles for $KK^j(A, B)$ as morphisms $A \to B$. There is a composition of such `morphisms,’ or
in other words a product operation

$$KK^i(A, D) \times KK^j(D, B) \to KK^{i+j}(A, B).$$

There is a map

$$\sigma_D : KK^i(A, B) \to KK^i(A \otimes D, B \otimes D)$$

which one should think of taking a morphism $x : A \to B$ and replacing it by the morphism
$x \otimes 1_D : A \otimes D \to A \otimes D$, and similarly a map $\sigma^D$, taking a morphism $x$ and replacing it by the
morphsim $1_D \otimes x$.

Specializing the first variable $A$ to the complex numbers $\mathbb{C}$, we get the $K$-theory in the
sense of Banach algebras of $B$. Specializing the second variable to $\mathbb{C}$ and setting the index $i = 1$
we get the group $\text{Ext}(A)$ of Brown, Douglas and Filmore (see [4]). In other words, the latter
group consists of homomorphisms from $A$ in to the Calkin algebra of some separable Hilbert
space.
Given the existence of this theory and our previous work, it is not a great leap to figure out what it should involve for a C*-algebra to have Poincaré Duality, although what it means — in the sense that in the classical case it meant more or less that we were dealing with a compact oriented manifold — is a little less clear. If the C*-algebra is denoted \( A \), there should exist, for some \( n \), a canonical isomorphism of groups \( K_j(A) \cong K^{j+n}(A) \), where canonical means implemented by an element of \( KK^n(\mathbb{C}, A \otimes A) \) in the same way as in the classical, commutative case; in other words, via the map 
\[
K_j(\mathbb{C}, A) \rightarrow KK^{j+n}(A, \mathbb{C})
\]

\[
a \mapsto \sigma_A(a) \otimes A \otimes A \Delta,
\]

for a fixed element \( \Delta \in KK^n(\mathbb{C}, A \otimes A) \). And we should also be given a map in the reverse direction, implemented by an element \( \hat{\Delta} \in KK^{-n}(\mathbb{C}, A \otimes A) \), inverting the previous map.

In a spirit of abstraction, one could go even further and define what it means for two C*-algebras to be dual to one another, so that the K-theory of the one is isomorphic (canonically) to the K-homology of the other. We are led to the definition of Connes (see Definition 2.6), that \( A \) and \( B \) are dual (with a dimension shift of \( n \)) if there exist \( \Delta \in KK^n(A \otimes B, \mathbb{C}) \) and \( \hat{\Delta} \in KK^{-n}(\mathbb{C}, A \otimes B) \) satisfying respectively \( \hat{\Delta} \otimes_A \Delta = 1_B \) and \( \hat{\Delta} \otimes_B \Delta = 1_A \).

For various reasons, this notion of duality between two algebras is an important one. It turns out that even when we wish to study the non-commutative analog of \( C(M) \) for \( M \) a compact, spin\(^c\) manifold, it will be useful to formulate it as a duality not quite between the algebra and itself, but its opposite algebra. See [6].

We will discuss this in greater detail in Chapter 2.

However, this notion of non-commutative duality requires some examples. We are interested in finding geometrically-arising C*-algebras \( A \) and \( B \) which are dual in the above sense; and, especially, when \( A \) is equal to \( B \) (or to \( A^{\text{op}} \)).
1.3 Examples of Dual Algebras

Connes was the first to exhibit a non-commutative $C^*$-algebra having Poincaré duality in $K$-theory ([7], pg. 590). His example was the ‘irrational rotation algebra,’ denoted $A_\theta$, where $\theta$ is an irrational number between 0 and 1. The duality is even, which is to say the $n$ appearing in our discussion at the end of the previous section, is congruent to 0 mod 2. The duality is between $A_\theta$ and its opposite algebra $A_\theta^{op}$.

To construct the algebra $A_\theta$, let us first recall that if $\Gamma$ is a group acting by homeomorphisms on a space $X$, we can form a $C^*$-algebra which plays the role of the set of continuous functions on $X/\Gamma$ but, unlike the latter, makes sense even if the action is not proper. This $C^*$-algebra is denoted $C(X) \rtimes \Gamma$, and is a certain completion of the group algebra, $C(X)\Gamma$, i.e. the set of symbols $\sum_{\gamma \in \Gamma} f_\gamma \gamma$ where the sum is finite, and where we have the relation $\gamma f = \gamma(f)\gamma$. Now let $\Gamma = \mathbb{Z}$ be the infinite cyclic group generated by a rotation $R_\theta$ of the circle by an irrational angle $\theta$. $A_\theta$ is then the algebra $C(S^1) \rtimes \mathbb{Z}$. Remark that $A_\theta$ is generated by two unitaries $U$ and $V$ satisfying the relation

$$UV = e^{2\pi i \theta} VU$$

and so in a sense is not wildly different from the commutative algebra $C(T^2)$, which is of course the case of $\theta = 0$.

Now, $A_\theta$ has two unbounded derivations $\delta_1$ and $\delta_2$ given by $\delta_1(U^nV^m) = 2\pi i n U^nV^m$ and $\delta_2(U^nV^m) = 2\pi i m U^nV^m$. Let $D_\theta = \delta_1 + i\delta_2$.

When $\theta = 0$, $A_\theta$ is, as we have said, the $C^*$-algebra of continuous functions on $T^2$, and if we identify $T^2$ with a quotient of the complex plane, then $\delta_1 + i\delta_2$ is simply $\frac{\partial}{\partial z}$, which is an elliptic operator on $T^2$, in fact the Dirac Operator for $T^2$, and so in particular gives a $K$-homology element, i.e. an element of $KK^2(A_\theta, \mathbb{C})$, c.f. the discussion of the classical case in the previous section. And in fact it is not difficult to show that even when $\theta$ is not zero, $D_\theta$ still defines, in the same way as above, an element of the $K$-homology of $A_\theta$. For the Hilbert space
on which $D_\theta$ is to act, we set $H_\theta$ equal to $A_\theta$ itself, with the Hilbert space structure obtained by applying the GNS construction to the trace $\tau : A_\theta \to \mathbb{C}$, which in group algebra notation is given by $\sum f_\gamma \gamma \mapsto \int f_\gamma$, the integral being with respect to Lebesgue measure on the circle.

This construction produces a cycle for $KK^{-2}(A_\theta, \mathbb{C})$, but for purposes of constructing a Poincaré duality, we need a cycle for $KK^2(A_\theta \otimes A_\theta^{\text{op}}, \mathbb{C})$. In the commutative case, i.e. where $\theta = 0$, we achieved this by pushing forward via the diagonal map $T^2 \to T^2 \times T^2$, or, in the language of $C^*$-algebras, pulled back via the multiplication map $C(T^2) \otimes C(T^2) \to C(T^2)$, noting that of course $C(T^2)^{\text{op}} = C(T^2)$, but in the non-commutative case this no longer works, as this multiplication map is no longer a homomorphism of $C^*$-algebras.

Nevertheless, the fact that $\tau$ is not merely a positive linear functional, but a trace, implies that $H_\theta$ has not merely a left action of $A_\theta$, but also a right action of $A_\theta$, which is compatible with $D_\theta$ and with the Hilbert space structure in an appropriate sense. This produces an element of $KK^2(A_\theta \otimes A_\theta^{\text{op}}, \mathbb{C})$.

Thus, our first example of duality arises from a simple kind of non-trivial dynamics on the circle; namely a minimal translation on a compact abelian group. The second class of known examples also arises from a type of dynamical system, which are called hyperbolic. Kaminker and Putnam (see [18]) have discovered that the $C^*$-algebras associated to the simplest and — and at the same time most prototypical — of this class, have an odd Poincaré duality.

Let $M$ be an $N \times N$ matrix. A subshift of finite type is a dynamical system of the form $(\Sigma_M, \sigma_M)$, where $\Sigma_M$ is the set of all sequences of integers between 1 and $N$ for which an $i$ is followed by a $j$ only if $M_{ij} = 1$, and where $\sigma_M$ acts on such sequences by shifting to the left. $\Sigma_M$, equipped with the product topology, is a Cantor set, and $\sigma_M$ is a homeomorphism of it. The system is hyperbolic in a certain topological sense; to see this intuitively (in fact there are axioms for such a system, see [22]) recall that a, say, $2 \times 2$, hyperbolic matrix $B$, is an invertible matrix with determinant one whose eigenvalues have the form $\lambda, \lambda^{-1}$ where $\lambda$ is real and not equal to $\pm 1$. View $B$ as a homeomorphism of the plane. Then $B$ has two linearly
independant eigenvectors, denote $v^-$ and $v^+$, these span one dimensional eigenspaces along which $B$ is respectively contracting and expanding. This is the sort of picture one should have in mind. A famous example is of course

$$B = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$ 

Now $\Sigma_M$ is a combinatorial analog of this. To see this, observe that we can put a metric on $\Sigma_M$ by setting $d(x, y) = \sum_n 2^{-|n|}|x_n - y_n|$. Suppose $x$ and $y$ are two sequences which agree for all $i \geq k$ for some $k > 0$. Then after shifting by $\sigma_M$ they agree for all $i \geq k - 1$, whence $d(\sigma_M(x), \sigma_M(y)) = \frac{1}{2}d(x, y)$. From this one sees that $d(\sigma^n_M(x), \sigma^n_M(y)) \to 0$ as $n \to \infty$. Let $x$ be any point. Then we then have two subspaces $W^s(x)$ and $W^u(x)$ of sequences agreeing with $x$ for all $i$ larger than some $k$, respectively agreeing with $x$ for all $i \leq k$, for some $k$, on which $\sigma_M$ is respectively `contracting' and `expanding', and these two subspaces should be thought of as subspaces in the `tangent bundle at $x$', corresponding to the contracting, respectively expanding, eigenspaces of the differential of a hyperbolic diffeomorphism. It is this sense in which the system is hyperbolic.

Now it is known (see [19]) how to associate to such a system a $C^*$-algebra. The procedure this time does not involve taking the cross product, but rather involves considering the equivalence relation $x \sim y$ if $\lim_{n \to \infty} d(\sigma^n_M(x), \sigma^n_M(y)) = 0$ and showing that this equivalence relation represents a topological groupoid $G^s$ with Haar system in the sense of Renault (see [23]). From such a groupoid one can form a groupoid $C^*$-algebra $C^*_r(G^s)$ in a standard way (again, see [23]); which in this case turns out to carry an action of the integers by algebra automorphisms (from the original shift homeomorphism). Taking the cross product $C^*_r(G^s) \rtimes \mathbb{Z}$, we have what is called the Ruelle algebra $R^s$. Similarly we have an unstable version, $R^u$.

Kaminker and Putnam have shown that $R^s$ and $R^u$ are dual to one another. We now discuss this duality, which motivated the work in this dissertation.
First of all, it is convenient to work in the language of Cuntz-Krieger algebras (see [9]).

With $M$ as above, the Cuntz-Krieger algebra $O_M$ is the universal $C^*$-algebra generated by $N$ isometries $S_1, \ldots, S_N$ with the relations

$$S_i^*S_i = \sum_j M_{ij}S_jS_j^*.$$ 

With some work it is possible to show that the Ruelle algebra $R^s$ associated to $(\Sigma_M, \sigma_M)$, is stably isomorphic to $O_M$. On the other hand $R^u$ is stably isomorphic to $O_{Mt}$. Since $K$-theory and $K$-homology are preserved under stable isomorphism, to prove that $R^s$ and $R^u$ are dual to one another it suffices prove the corresponding result for $O_M$ and $O_{Mt}$.

The result of Kaminker and Putnam is thus that $O_M$ and $O_{Mt}$ are dual to one another in the sense of Connes’ definition. Note that the duality is odd this time.

In what follows, as elsewhere in this dissertation, $K$ denotes the compact operators, $B$ denotes bounded operators, and $Q$ denotes the quotient $B/K$.

Let us sketch the construction of $\Delta \in KK^1(O_M \otimes O_{Mt}, \mathbb{C})$. As mentioned above, this group is isomorphic to the group $\text{Ext}(O_M \otimes O_{Mt})$ of classes of extensions of $O_M \otimes O_{Mt}$ by the compact operators; i.e. of short exact sequences of the form

$$0 \to K(H) \to E \to O_M \otimes O_{Mt} \to 0,$$

where $H$ is a Hilbert space. Such extensions (see [3]) are in one-to-one correspondence with homomorphism $O_M \otimes O_{Mt} \to Q(H)$, and it will be from the latter point of view that we will construct $\Delta$.

Let $H$ denote a fixed $N$-dimensional Hilbert space with orthonormal basis $\xi_1, \ldots, \xi_N$. For $n > 0$, let $H^\otimes n = H \otimes \cdots \otimes H$ ($n$ times), and $H^\otimes 0$ denote a fixed one dimensional space spanned by a ‘vacuum vector,’ $\Omega$. Form the now infinite dimensional Hilbert space $\mathcal{H} = \oplus_{n \geq 0} H^\otimes n$. 
There are operators $L_1, \ldots, L_N$ from $\mathcal{H}$ to itself, given by

$$L_j(\xi_{k_1} \otimes \cdots \xi_{k_n}) = \xi_j \otimes \xi_{k_1} \otimes \cdots \xi_{k_n}$$

and there are operators $R_j$ acting in the same way on the right. Note that the $L_i$ commute modulo finite rank operators with the $R_j$. Now let $P_M$ denote the projection on the subspace of $\mathcal{H}$ given by the span of all $\xi_{k_1} \otimes \cdots \xi_{k_n}$ for which $M_{k_i, k_{i+1}} = 1$ for all $i$. Let $\mathcal{H}_M$ denote the range of $P_M$. Let $L_i^M$ denote $P_M L_i$ and $R_i^M$ denote $P_M R_i$, viewed as operators on $\mathcal{H}_M$. Then it is easy to see that modulo finite rank operators, $L_i^M$ and $R_i^M$ satisfy respectively the relations for $O_M$ and $O_{Mt}$, and still commute with each other modulo finite rank operators. Consequently, we obtain two maps $O_M \to \mathcal{Q}(\mathcal{H})$ and $O_{Mt} \to \mathcal{Q}(\mathcal{H})$ given by the formulae $S_i \mapsto L_i^M$ and $S_i \mapsto R_i^M$ respectively. Remark that here as elsewhere in this dissertation, we will write an element such as $L_i^M$ in such a way as to blur the distinction between operators and their images in the Calkin algebra; this should cause no confusion, and simplifies the notation. Now, as we have observed, the $L_i^M$ and the $R_i^M$ commute as elements of $\mathcal{Q}(\mathcal{H})$, and hence the two corresponding homomorphisms into the Calkin algebra piece together to give a homomorphism

$$O_M \otimes O_{Mt} \to \mathcal{Q}(\mathcal{H}_M).$$

This is the element $\Delta$.

To prove that $\Delta$ induces a Poincaré duality between $O_M$ and $O_{Mt}$, Kaminker and Putnam write down an element $\hat{\Delta}$ of $K_1(O_M \otimes O_{Mt})$ and show that Connes’ condition is satisfied, at least at the level of maps $K_*(O_M) \to K_*(O_M)$ (and similarly $K^*(O_M) \to K^*(O_{Mt})$). In other words, the element $\hat{\Delta} \otimes O_{Mt} \Delta \in KK(O_M, O_M)$ is not itself shown to be $1_{O_M}$, but its image under the canonical map $KK(O_M, O_M) \to Hom_{\mathbb{Z}}(K_*(O_M), K_*(O_M))$ is shown to be the identity.
Part of what is required is the calculation of the $K$-theory of $O_M$, which was done previously by Cuntz (see [9]).

Now let $M$ be the matrix

$$M = \begin{pmatrix}
1 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 \\
1 & 1 & 0 & 1 \\
\end{pmatrix}.$$  

Remark $M$ is symmetric. The corresponding shift space $\Sigma_M$ is the set of all bi-infinite sequences of four symbols, which for reasons that will become apparent in a moment, we denote by $a, a^{-1}, b$ and $b^{-1}$, in which an $a$ is never followed by an $a^{-1}$ and a $b$ is never followed by a $b^{-1}$. Take such a sequence. Consider the free group $\mathbb{F}_2$ generated by $a$ and $b$ and their inverses. $\mathbb{F}_2$ has a Cayley graph $T$ which is a tree. The tree has a natural metric, in which it is geodesic in the sense that for any two points (vertices), $p$ and $q$, there is an isometry $r$ from the subinterval $[0,d(p,q)]$ of $\mathbb{Z}$ with its metric as a subset of $\mathbb{R}$, to $T$, satisfying $r(0) = p$ and $r(d(p,q)) = q$. Now any sequence in $\Sigma_M$, viewed as a sequence of generators, $(\ldots,x_{-2},x_{-1},x_0,x_1,x_2,\ldots)$ defines such an isometry $\mathbb{Z} \rightarrow T$ by viewing it as a specified sequence of edges. Thus $r(0) = e$ (where $e$ is the identity in the group), $r(1) = x_0$, $r(2) = x_0x_1$, $r(-1) = x_{-1}$, $r(-2) = x_{-2}x_{-1}$ and so on. The conditions (expressed by the matrix $M$) that $a$ never be followed by $a^{-1}$ and similarly for $b$, is exactly the condition that the path be an isometry. It is obvious that any bi-infinite ‘geodesic’ in $T$ passing through the identity $e$ arises from such a sequence. Finally, on the space of ‘geodesics’ we have an obvious ‘geodesic flow’, and this corresponds to the shift $\sigma_M$.

Consequently we see there is a connection between the geodesic flow on the tree and the system $(\Sigma_M, \sigma_M)$; indeed, the subshift should be thought of as coding the geodesic flow.

Now associated to the tree, there is a natural compact metric space $\partial\mathbb{F}_2$ which one should think of as the boundary of the tree; this is the Cantor set which can be described as the set
of infinite reduced words in the letters \(a, b\) and their inverses. \(\mathbb{F}_2\) acts on \(\partial \mathbb{F}_2\) by left multiplication. We can then form the cross product \(C^*\)-algebra \(C(\partial \mathbb{F}_2) \rtimes \Gamma\). We claim it is isomorphic to \(O_M\). For if for a generator or its inverse \(c\), we denote by \(\chi_c\) the set of infinite words beginning in the letter \(c\), then we see that the four partial isometries (in group-algebra notation) \(\chi_a a, \chi_{a^{-1}} a^{-1}, \chi_b b, \chi_{b^{-1}} b^{-1}\) satisfy exactly the relations required by \(O_M\).

The existence of this isomorphism was first noticed in the context of Fuchsian groups (of which the free group on two generators is an example) by J. Spielberg. See [28].

To summarize, the Ruelle algebra \(R^s\) associated to geodesic flow on the tree is strongly Morita equivalent (i.e. stably isomorphic to) the groupoid \(C^*\)-algebra \(C(\partial \mathbb{F}_2) \rtimes \mathbb{F}_2\) where \(\partial \mathbb{F}_2\) is a ‘boundary’ of the tree. And as the matrix \(M\) is symmetric, the Ruelle algebra \(R^w\) is also stably isomorphic to \(C(\partial \mathbb{F}_2) \rtimes \mathbb{F}_2\). Hence \(C(\partial \mathbb{F}_2) \rtimes \mathbb{F}_2\) has an odd Poincaré duality with itself.

Subshifts, trees and the like might appear to be slightly artificial objects, but the above phenomena also occur in the most classical of all hyperbolic dynamical systems: genuine geodesic flow on a negatively curved surface. In this context one can also form a Ruelle algebra, but we prefer to take Connes’ point of view, via the holonomy groupoid of a foliation. So let \(M\) be a compact Riemann surface of genus \(g \geq 2\), and \(SM\) denote its unit sphere bundle. Let \(\Gamma\) denote the fundamental group of \(M\), realized as a discrete, cocompact group of isometries of the hyperbolic plane \(\mathbb{H}\). There is a geodesic flow \(g_t\) on \(SM\), and an equivalence relation this time called weak stable equivalence, which is defined by \(v \sim w\) if \(d(g_t(v), g_t(w))\) remains bounded as \(t \to \infty\). The equivalence classes are easily seen to be 2-dimensional, and to make up a codimension 1 foliation of \(SM\). To this foliation \(\mathcal{F}\) Connes has shown how to assign a \(C^*\)-algebra; one first forms the holonomy groupoid, \(\mathcal{G}\), and then its groupoid \(C^*\)-algebra; the result is what is generally called the \(C^*\)-algebra of the foliation, and denoted \(C^*_r(\mathcal{F})\).

Now \(C^*_r(\mathcal{F})\) can be seen to be strong Morita equivalent to \(C(S^1) \rtimes \Gamma\). Remark that \(S^1\) plays the same role here as did the boundary of the tree in the context of subshifts. And again one can show that the algebra \(C^*(\mathcal{F})\), and hence also \(C(S^1) \rtimes \Gamma\), has an odd self-duality.
Hence, we have increased our list of algebras having some sort of Poincaré duality to three, and all of these arise from interesting dynamical systems, the first being a minimal rotation on the circle, and the others being hyperbolic homeomorphisms, or flows. The latter two examples can be gathered together, as we shall see in the next section, into a much more general class, which will unify some of these ideas, and which will have the property that all of its members will have Poincaré duality with themselves (or, as one might prefer, with their opposite algebras.)

1.4 Hyperbolic Groups and Boundaries

For hyperbolic groups, we invite the reader to consult [12], or [11]. We give a brief account of them here. Let $\Gamma$ be a discrete, finitely generated group, with generating set $S$, and assume $S = S^{-1}$. Let $e$ denote the identity element of the group. Give $\Gamma$ the structure of a metric space via defining $d(e, g)$ to be the minimal number of generators required to write $g$ as a word in the generators, and define $d(g, h) = d(e, g^{-1}h)$.

**Definition 1.1.** The Gromov product $(g \mid h)$ of two elements $g, h \in \Gamma$, is defined by $(g \mid h) = \frac{1}{2}(d(e, g) + d(e, h) - d(g, h))$.

**Definition 1.2.** $\Gamma$ as above is $\delta$-hyperbolic in the sense of Gromov, if there exists $\delta > 0$ for which the following inequality holds for all $g, h, k \in \Gamma$:

$$(g \mid k) \geq \min\{(g \mid h), (k \mid h)\} - \delta.$$ 

$\Gamma$ is said to be hyperbolic if there exists some $\delta \geq 0$ for which it is $\delta$-hyperbolic.

The idea is that the large scale geometry of the group is similar to that of a classical hyperbolic space $\mathbb{H}^n$ (and discrete cocompact groups of isometries of $\mathbb{H}^n$ are all in fact hyperbolic in the sense of Gromov, although this takes some work to prove). But the definition admits also
groups whose ‘curvature’ should be thought of as $-\infty$, such as the free group, a group with infinitely negative curvature in the sense that geodesic triangles are all degenerate.

Hyperbolic groups are very uniform in a large-scale sense. Part of this uniformity is reflected in their structure at infinity. To be precise, to each hyperbolic hyperbolic group $\Gamma$ we can associate a boundary $\partial \Gamma$, which is a compact metric space on which $\Gamma$ acts by homeomorphisms. This can be thought of as a generalization of the free group example above.

**Definition 1.3.** The boundary of a hyperbolic group $\Gamma$ is given by the set of all sequences $\gamma_n$ which satisfy the condition $\lim_{i,j \to \infty} (\gamma_i \mid \gamma_j) \to \infty$, modulo the equivalence relation $(\gamma_i) \sim (\gamma'_i)$ if $\lim \inf_{i,j} (\gamma_i \mid \gamma'_j) = \infty$.

The boundary is topologized using the quotient topology for the above equivalence relation on $\Pi_N \Gamma$, and can easily seen to be compact and metrizable (see [12]). One can further give $\bar{\Gamma} = \Gamma \cup \partial \Gamma$ a topology for which it is compact and metrizable, and for which $\Gamma$ is dense, whence the term ‘boundary.’ For an account of the latter facts see also [8]. Group elements $\gamma$ act as isometries on $\Gamma$ by translation, and isometries pass to homeomorphisms of $\partial \Gamma$, as is easy to see.

The boundary can be equally thought of as the collection of all geodesic rays $\mathbb{N} \to \Gamma$, where by ‘geodesic ray’ we mean simply ‘isometric map,’ modulo the relation that two rays are equivalent if they remain a bounded Hausdorff distance apart. The topology is then the quotient of the topology of uniform convergence on compact subsets of $\mathbb{N}$. By the Ascoli-Arzela theorem, the resulting space is compact. Computations in, e.g. [12], show the two descriptions are equivalent.

Remark that as we have said above, a cocompact discrete group of isometries of the hyperbolic plane (realized as a discrete subgroup of $\text{SL}_2(\mathbb{R})$) is hyperbolic in the sense of Gromov, and so acts on its boundary $\partial \Gamma$. It is not difficult to verify that this boundary is in fact $S^1$ and that the action described above is the same as the usual action by Mobius transformations.
In the case of the free group, which is also hyperbolic (for $\delta = 0$), the boundary is the set $\partial \mathbb{F}_2$ described above, and the action of $\mathbb{F}_2$ on it is the same as the left multiplication we have previously described.

To summarize, to each hyperbolic group we can canonically associate a groupoid $\partial \Gamma \rtimes \Gamma$. It is worth remarking that this groupoid is very ‘nice’ from several points of view. Though hyperbolic groups are almost never amenable, the groupoids $\partial \Gamma \rtimes \Gamma$ always are, in the sense of [10]. From the point of view of an operator algebraist, perhaps the most important consequence is that the Baum-Connes Conjecture in its full strength holds for these groupoids, because of the work of Jean-Louis Tu ([29]). As another consequence, the reduced $C^*$-algebra of the groupoid is the same as the max $C^*$-algebra, and both are nuclear.

It is a good point to remark that the definition of hyperbolic group is intended to abstract the notion of classical hyperbolic group; that is, a discrete, cocompact group of isometries of hyperbolic $n$-space; and in general, the definition is successful in that many things true of the classical groups are also true in the more general setting. However, this process of generalization is not always easy or straightforward. Our approach in this dissertation is to make certain assumptions concerning our groups, which are obviously met in the classical context, and also in the case of the free group, and to set aside for the moment the question as to in what generality our main theorem holds. We will make certain assumptions as to the existence of a space with a geodesic flow on it, and another assumption concerning existence of a fixed-point-free map from the boundary of the group to itself. We will state these as axioms (see Chapter 3).

With this proviso, our main result is that the $C^*$-algebra cross products corresponding to the groupoids $\partial \Gamma \rtimes \Gamma$, exhibit Poincaré Duality in $K$-theory. More precisely:

**Theorem 1.4.** Let $\Gamma$ be a torsion-free hyperbolic group, and $A$ the cross product $C^*$-algebra $C(\partial \Gamma) \rtimes \Gamma$. Assume that $\Gamma$ satisfies the axioms listed in Section 3.1. Then $A$ and $A^{\text{op}}$ are Poincaré dual in the sense of Definition 2.6.
We say a few words about the proof, which is somewhat long. According to Connes’
definition and the discussion above, we need find, for every hyperbolic group satisfying our
axioms, elements $\Delta$ and $\hat{\Delta}$, in respectively the $K$-homology and $K$-theory of $A = C(\partial \Gamma) \rtimes \Gamma$,
whose Kasparov product is $1_A$. In Chapter 2 we exhibit these elements, which generalize those
produced by Kaminker and Putnam in the case of $\Gamma = \mathbb{F}_2$, c.f. the above discussion.

In Chapter 3 we proceed to showing their product is 1. The arguments are motivated by
a calculation which we made in the free group case, which showed that up to unitary equivalence
and compact perturbation, $\hat{\Delta} \otimes_A \Delta$ is the image under the descent map $\lambda$ of a familiar object
lying in the equivariant group $RKK_{\mathbb{F}_2}(\partial \mathbb{F}_2; \mathbb{C}, \mathbb{C})$, namely $p^*_{\partial \mathbb{F}_2}(\gamma)$ where $\gamma \in KK_{\mathbb{F}_2}(\mathbb{C}, \mathbb{C})$ is
the notorious element constructed in this case by Julg and Valette ([17]), and where $p^*_{\partial \mathbb{F}_2}$ is a
certain natural pullback map $KK_{\mathbb{F}_2}(\mathbb{C}, \mathbb{C}) \to RKK_{\mathbb{F}_2}(\partial \mathbb{F}_2; \mathbb{C}, \mathbb{C})$. By the work of these authors,
$\gamma = 1_{\mathbb{C}}$ and it follows that $\hat{\Delta} \otimes_{A_{\text{op}}} \Delta = 1_A$ by the general mechanics of equivariant $KK$-theory.

What we show in general (for $\Gamma$ torsion-free, hyperbolic, and satisfying our axioms) is
that the product $\hat{\Delta} \otimes_{A_{\text{op}}} \Delta$ is the image under the descent map of a class $c$ in the group
$RKK_{\Gamma}(\partial \Gamma; \mathbb{C}, \mathbb{C})$, and that, under the natural map $RKK_{\Gamma}(\partial \Gamma; \mathbb{C}, \mathbb{C}) \to RKK_{\Gamma}(\partial \Gamma \times ET \mathbb{C}, \mathbb{C})$,
where $ET$ is the classifying space for proper actions of $\Gamma$, $c$ is mapped to $1_{\partial \Gamma \times ET \mathbb{C}}$. We then use
the work of Tu ([29]) to show that this implies $c$ itself is $1_{\partial \Gamma}$, and hence that $\hat{\Delta} \otimes_{A_{\text{op}}} \Delta = 1_A$
also.

In the last Chapter we present another connection with the Baum-Connes Conjecture,
generalizing an observation of Kaminker and Putnam in the case of $\Gamma$ a Fuchsian group, exhibiting
the Poincaré Duality map $\hat{\Delta}_*$ as a kind of assembly map ‘at infinity.’ A result along these lines
was noticed independently by N. Higson ([14]), who showed how to use it to give yet another
proof of the Novikov Conjecture for hyperbolic groups.
1.5 Open Questions

There are a number of ways of thinking about the above result. One, is a statement about foliations. Indeed, as we have mentioned previously, in the case of $\Gamma$ the fundamental group of a compact, negatively curved Riemannian manifold $M$, the groupoids $\partial \Gamma \rtimes \Gamma$ are equivalent in the category of groupoids to the holonomy groupoids associated with the foliation of $SM$ by weak horospheres (in other words, by the equivalence relation of ‘weak, stable equivalence’ of the geodesic flow.) Notably, even when $M$ is itself a smooth manifold it is not necessarily true that the holonomy groupoid of $F$ is smooth; indeed, $F$ itself is not necessarily smooth — and yet $C^*_r(F)$ still has Poincaré Duality, which one normally associates with smooth objects. One can ask the question: for which foliations $F$ does $C^*_r(F)$ have Poincaré Duality?

Another way of thinking about the result is from the standpoint of discrete subgroups of lie groups. Indeed, our result in the classical case (of isometries of $\mathbb{H}^n$) can be viewed as the rank-one version of a general phenomenum: let $B$ denote the Furstenburg boundary of a semi-simple lie group $G$, and let $\Gamma$ be a uniform lattice in $G$. Then $C(B) \rtimes \Gamma$ has Poincaré Duality in $K$-theory, with a dimension shift of the real rank of $G$ (the dimension of a maximal flat.) This follows from repeated application of the Thom Isomorphism to a maximal parabolic subgroup, and classical Poincaré Duality for spin$^c$ manifolds. But what is desirable in this context is not merely the fact that Poincaré Duality exists, but an explicit construction of the elements implementing it. And this latter has so far eluded me.

Finally, there is the circle of ideas proposed by Connes concerning Poincaré Duality and the notion of non-commutative manifold. Connes has written down a number of axioms for a $C^*$-algebra (see [6]) which, if satisfied, and if the $C^*$-algebra is also commutative, imply that it is in fact the algebra of continuous functions on a spin$^c$ manifold. The most important of these axioms is the existance of a so called fundamental class: i.e. an element $\nu \in KK(A \otimes A^{op}, \mathbb{C})$, inducing Poincaré Duality in $K$-theory.
Although the example with which we are concerned in this dissertation does not satisfy all of his axioms, it does satisfy this latter — and perhaps most crucial — one. So it bears asking: in what respect (if any) is the $C^*$-algebra $A = C(\partial \Gamma) \rtimes \Gamma$ a non-commutative manifold?
Chapter 2

Poincare Duality and
Non-commutative Manifolds

2.1 KK-theoretic Preliminaries

Let $A$ and $B$ be two separable, nuclear, graded $C^*$-algebras. We will be using $KK$-theory throughout, and so let us recall some basic facts concerning this. See, for example [20] for an exposition, or the other references listed in the Introduction.

$KK(A, B)$ is given by pairs $(E, F)$ where $E$ is a Hilbert $A - B$ bimodule, and $F$ is an operator on it as a $B$-module, satisfying: $a(F^2 - 1)$, $a(F - F^*)$ and $[a, F]$ are compact, for every $a \in A$ where $[,]$ denotes graded commutator. With respect to the grading, the $A$ action is even and the operator $F$ is odd.

Elements of $KK(A, B)$ should be thought of as generalized homomorphisms from $A$ to $B$, and the basic properties of $KK$ can be summarized by the assertion that formally, they behave exactly like homomorphisms.

In fact, Kasparov’s $KK$-theory can conveniently be thought of as an additive category in which the objects are $C^*$-algebras, and the morphisms from $A \to B$ are the elements of $KK(A, B)$. There is a functor from the category of $C^*$-algebras and $*$-homomorphisms into the $KK$-category which is the identity on objects. And there is a tensor product operation on the $KK$-category which is compatible with the tensor product operation, and with $*$-homomorphisms, in the $C^*$-category. See [16] for details.

Thus, we have the following operations. If $\alpha \in KK(A, D)$ and $\beta \in KK(D, B)$ then we can ‘compose’ them, forming what is denoted $\alpha \otimes_D \beta$ and called the intersection product of $\alpha$
and $\beta$, and which we should think of as the composition

$$A \xrightarrow{\alpha} D \xrightarrow{\beta} B.$$ 

Secondly, any actual homomorphism of $C^*$-algebras $A \xrightarrow{\phi} B$ gives an element $[\phi]$ of $KK(A, B)$. Combining this with the intersection product above, we see that $\phi$ also gives maps $\phi_* : KK(D, A) \to KK(D, B)$ given by intersection product on the right, and similarly a map $\phi^* : KK(B, D) \to KK(A, D)$ by intersection product on the left.

Let $\hat{\otimes}$ denote graded tensor product.

If $\alpha \in KK(A, B)$ and $D$ is a $C^*$-algebra, we can form $\sigma^D(\alpha) \in KK(D \hat{\otimes} A, D \hat{\otimes} B)$, which we should think of as the homomorphism $D \hat{\otimes} A \to D \hat{\otimes} B$, $d \hat{\otimes} a \mapsto d \hat{\otimes} \alpha(a)$. Similarly we can form $\sigma_D(\beta) \in KK(A \hat{\otimes} D, B \hat{\otimes} D)$.

In what follows, we will let $\sigma_{ij}$ denote the flip

$$A_1 \hat{\otimes} \cdots A_i \hat{\otimes} \cdots A_j \hat{\otimes} \cdots A_n \to A_1 \hat{\otimes} \cdots A_j \hat{\otimes} \cdots A_i \hat{\otimes} \cdots A_n$$

obtained by flipping the two factors.

If $\alpha \in KK(A_1, B_1)$ and $\beta \in KK(A_2, B_2)$, we can form their external product, denoted $\alpha \otimes_C \beta$, which lies in $KK(A_1 \hat{\otimes} A_2, B_1 \hat{\otimes} B_2)$, and which is defined to be $\sigma_{A_2}(\alpha) \otimes_{B_1} \sigma_{A_2}(\beta)$. One should think of $\alpha \otimes_C \beta$ as a simple tensor product of two homomorphisms $A_1 \hat{\otimes} B_1 \to A_2 \hat{\otimes} B_2$

$$a_1 \hat{\otimes} b_1 \mapsto \alpha(a_1) \hat{\otimes} \beta(b_1).$$

Now, if as claimed, $KK$-elements really do behave like homomorphisms, it should be true that $\alpha \otimes_C \beta$ is the same as $\sigma_{12}^*(\sigma_{12})_*(\beta \otimes_C \alpha)$, for the latter corresponds to the homomorphism

$$a_1 \hat{\otimes} b_1 \to b_1 \hat{\otimes} a_1 \to \beta(b_1) \hat{\otimes} \alpha(a_1) \to \alpha(a_1) \hat{\otimes} \beta(b_1),$$
and indeed this is true (see [20], pg.159).

There are higher $KK$-groups defined in the following way. For any integer $n$ let $C_n$ denote the complex Clifford algebra with $n$ generators $\epsilon_1, \ldots, \epsilon_n$ satisfying $\epsilon_i^2 = 1$ for all $i$, if $n$ is positive, and $\epsilon_i^2 = -1$ for all $i$ if $n$ is negative, and also $[\epsilon_i, \epsilon_j] = 0$ for all $i, j$ (where, again the commutators are graded). We then set, for $C^*$-algebras $A$ and $B$,

$$KK^n(A, B) = KK(A, C_n \hat{\otimes} B) \cong KK(C_{-n} \hat{\otimes} A, B).$$

The latter isomorphism is a special case of periodicity of Clifford algebras, which we will express in the following way: in the $KK$ category, $C_i \hat{\otimes} C_j \cong C_{i+j}$.

Periodicity permits us to describe the most general form of the product as a pairing

$$KK^i(A_1, B_1 \hat{\otimes} D) \times KK^j(D \hat{\otimes} B_2, A_2) \rightarrow KK^{i+j}(A_1 \hat{\otimes} B_2, B_1 \hat{\otimes} A_2).$$

To define this pairing requires establishing certain conventions. We proceed as follows. Let $\alpha \in KK^i(A_1, B_1 \hat{\otimes} D) = KK(A_1, C_i \hat{\otimes} B_1 \hat{\otimes} D)$ and $\beta \in KK^j(D \hat{\otimes} B_2, A_2) = KK(D \hat{\otimes} B_2, C_i \hat{\otimes} A_2)$.

Reverting to the intuition of homomorphisms, we think of $\alpha$ as a homomorphism $A_1 \rightarrow C_i \hat{\otimes} B_1 \hat{\otimes} D$ and $\beta$ as a homomorphism $D \hat{\otimes} B_2 \rightarrow C_j \hat{\otimes} A_2$. To define their ‘composition’ $\alpha \otimes_D \beta$, we first replace $\alpha$ by

$$\sigma_{B_2}(\alpha) : A_1 \hat{\otimes} B_2 \xrightarrow{\alpha \otimes 1_{B_2}} C_i \hat{\otimes} B_1 \hat{\otimes} D \hat{\otimes} B_2$$

and $\beta$ by

$$\sigma_{C_i \hat{\otimes} B_1} : C_i \hat{\otimes} B_1 \hat{\otimes} D \hat{\otimes} B_2 \xrightarrow{1_{C_i \hat{\otimes} B_1} \hat{\otimes} \beta} C_i \hat{\otimes} B_1 \hat{\otimes} C_j \hat{\otimes} A_2 \cong C_{i+j} \hat{\otimes} B_1 \hat{\otimes} A_2.$$

These two $KK$-elements are now compatible, and so we can take their intersection product, which produces an element of $KK(A_1 \hat{\otimes} B_2, C_{i+j} \hat{\otimes} B_1 \hat{\otimes} A_2)$. 
Lemma 2.1. If $\alpha \in KK^i(A_1, B_1)$ and $\beta \in KK^j(A_2, B_2)$, then

$$\alpha \otimes_C \beta = (-1)^{ij} (\sigma_{12})_* \sigma_{12}^*(\beta \otimes \alpha) \in KK(A_1 \hat{\otimes} A_2, B_1 \hat{\otimes} B_2).$$

Proof. By definition $\alpha \otimes_C \beta$ is the morphism

$$A_1 \hat{\otimes} A_2 \xrightarrow{\alpha \otimes 1 A_2} C_i \hat{\otimes} B_1 \hat{\otimes} A_2,$$

while $(\sigma_{12})_* \sigma_{12}^*(\beta \hat{\otimes} \alpha)$ is the morphism

$$A_1 \hat{\otimes} A_2 \xrightarrow{\sigma_{12} A_1 \hat{\otimes} A_2} \sigma_{12} B_2 \hat{\otimes} A_1 \xrightarrow{\beta \otimes 1 A_2} C_j \hat{\otimes} B_2 \hat{\otimes} C_i \hat{\otimes} B_1 \hat{\otimes} B_2 \cong C_i \hat{\otimes} C_j \hat{\otimes} B_1 \hat{\otimes} B_2 \cong C_{i+j} \hat{\otimes} B_1 \hat{\otimes} B_2,$$

But by commutativity of tensoring over $C$, these are the same, up to the automorphism of $C_{i+j}$ which flips the first $i$ generators with the second $j$ generators. The result now follows from noting (see [3, page 179]) that this latter automorphism induces multiplication by $(-1)^{ij}$,

$$KK^{i+j}(A_1 \hat{\otimes} A_2, B_1 \hat{\otimes} B_2) \to KK^{i+j}(A_1 \hat{\otimes} A_2, B_1 \hat{\otimes} B_2).$$

From this point onwards, all our algebras will be ungraded.

Corresponding to this, we may make certain simplifications in the definitions of the $KK$ groups (see [3]). With such ungraded $A$ and $B$, elements of $KK(A, B)$ are given simply by pairs $(\mathcal{E}, F)$ as in the original definition, except where we omit all issues of grading, $a(F^* - F)$ is
no longer required to be compact, and the condition $a(F^2 - 1)$ compact is replaced by the two conditions: $a(F^*F - 1)$ is compact, and $a(FF^* - 1)$ is compact.

Elements of $KK^1(A, B)$ are given by pairs $(\mathcal{E}, P)$ for which $P$ is as before an operator on $\mathcal{E}$ as a right $B$ module, commuting essentially with the left $A$-action, and satisfying $a(P^2 - P)$ is compact. Such pairs are equivalently given by extensions, i.e. homomorphisms $A \mapsto Q(\mathcal{E})$ where recall here and throughout, $Q$ denotes the Calkin algebra, $B$ bounded operators, and $K$ denotes compact operators, all on a given Hilbert modules. Finally, we will simply define $KK^{-1}(C, A)$, for any $C^*$-algebra $A$, to be $KK(C, C^*(\mathbb{R}))$. The latter simplification is justified by the existence of an invertible element in $KK^{-1}(C, C^*(\mathbb{R}))$, namely the class of the Dirac Operator on the line.

The following lemma involves a verifications of the definitions (see [3]), and we state it without proof. Let $\psi$ be the function on $\mathbb{R}$ whose Fourier transform is the function $t \mapsto \frac{2i}{t+i}$. Remark that $\psi + 1 \in C^*(\mathbb{R})^+$ is unitary, where the $+$ denotes unitization. If $D$ is a $C^*$-algebra, denote by $M(D)$ its multiplier algebra (see [30]).

**Lemma 2.2.** Suppose an element $\alpha \in KK^{-1}(C, D)$ is given by a homomorphism $\tau : C^*(\mathbb{R}) \to D$, and another element $\beta \in KK^1(D, A)$ is given by an extension

$$0 \to A \otimes K(V) \to E \to D \to 0$$

or, equivalently, a homomorphism $\tau : D \to Q(A \otimes K(V))$. Then the class of $\alpha \otimes D \beta \in KK(C, A)$ is represented by the cycle $(A \otimes V, F)$ where $F - 1$ is any a lift of $\tau(\psi)$ to $M(A \otimes K(V)) \cong B(A \otimes V)$.

**Corollary 2.3.** Define an extension $[\tau]$, given by a homomorphism $\tau : C^*(\mathbb{R}) \to Q(L^2(\mathbb{R}))$ by the formula $f \mapsto \chi \cdot \lambda(f)$, where $\lambda$ is the left regular representation of $C^*(\mathbb{R})$ and $\chi$ is the characteristic function of the left half-line. Then under the pairing

$$KK^{-1}(C, C^*(\mathbb{R})) \times KK^1(C^*(\mathbb{R}), C) \to KK(C, C) \cong \mathbb{Z}$$
the classes represented by the identity homomorphism \( C^*(\mathbb{R}) \to C^*(\mathbb{R}) \) and the \( KK \) class given by \( \tau \) above pair to 1.

**Proof.** This follows from Lemma 2.2 and a calculation; one checks simply that \( \chi \cdot \psi \) as an operator on \( L^2(\mathbb{R}) \) has index 1. One can do this by solving a simple differential equation. (See [27]).

\[ \square \]

### 2.2 Poincaré Duality

**Theorem 2.4.** Let \( A \) and \( B \) be \( C^* \)-algebras and let \( \Delta \) and \( \hat{\Delta} \) be two elements in \( KK^i(A \otimes B, C) \) and \( KK^{-i}(A \otimes B, C) \) respectively. Define maps \( \hat{\Delta}_j : K^j(B) \mapsto K^{j-i}(A) \) by

\[ \hat{\Delta}_j(x) = \Delta \otimes_B x \]

and maps \( \Delta_j : K^j(A) \mapsto K^{j+i}(B) \) by

\[ \Delta_j(y) = y \otimes_A \Delta. \]

Then:

1. \( \Delta_{j-i}(\hat{\Delta}_j(x)) = (-1)^{ij} (\Delta \otimes_A \Delta) \otimes_B x, \ x \in K^j(B); \)
2. \( \hat{\Delta}_{j+i}(\Delta_j(y)) = (-1)^{ij} y \otimes_A (\Delta \otimes_B \Delta), \ y \in K^j(A). \)

**Remark 2.5.** By \( \hat{\Delta} \otimes_B \Delta \) we actually mean \( \hat{\Delta} \otimes_B \sigma_{12}^*(\Delta) \). We omit the flip so as to minimize notation. Similarly, by \( \hat{\Delta} \otimes_A \Delta \) we actually mean \( (\sigma_{12})_*(\hat{\Delta}) \otimes_A \Delta \). These conventions will hold throughout this dissertation.

**Proof.** Let \( x \in K^j(B) \). Then it follows from the definitions that

\[ \Delta_{j-i}(\hat{\Delta}_j(x)) = \sigma_B(\hat{\Delta}) \otimes_A \Delta \otimes_B \sigma_{AB} \sigma_B(x) \otimes_A \Delta. \]
Now, note that $\sigma^A \sigma_B(x) \otimes_{A \otimes B} \Delta = \sigma^*_{23} (x \otimes_C \Delta)$, and hence by skew-commutativity over $C$ (Lemma 2.1)

$$\sigma^A \sigma_B(x) \otimes_{A \otimes B} \Delta = (-1)^{ij} \sigma^*_{23} (\sigma_B(\Delta) \otimes_B x).$$

To verify (1) it remains to check that

$$\sigma_B(\hat{\Delta}) \otimes_{A \otimes B \otimes B} \sigma^*_{23} (\sigma_B(\Delta)) = \hat{\Delta} \otimes_A \Delta.$$

But, noting that $\sigma^B(\Delta) = \sigma^*_{12} \sigma^*_{23} (\sigma_B(\Delta))$, and that $\hat{\Delta} \otimes_A \Delta$ is by definition $\sigma_B((\sigma_1), \hat{\Delta}) \otimes_{B \otimes A \otimes B} \sigma^B(\Delta)$, we see that

$$\hat{\Delta} \otimes_A \Delta = \sigma_B(\hat{\Delta}) \otimes_{A \otimes B \otimes B} \sigma^*_{12} \sigma^*_{12} \sigma^*_{23} (\sigma_B(\Delta)) = \sigma_B(\hat{\Delta}) \otimes_{A \otimes B \otimes B} \sigma^*_{23} (\sigma_B(\Delta))$$

and we are done. The second assertion follows in exactly the same way.

\[\Box\]

In view of this theorem, we will take as the definition of duality between two $C^*$-algebras the following (compare [7, page 588]):

**Definition 2.6.** Two separable, unital, and nuclear $C^*$-algebras $A$ and $B$ are dual with a dimension shift of $i$ if there exists $\Delta \in KK^i(A \otimes B, C)$, $\hat{\Delta} \in KK^{-i}(C, A \otimes B)$ such that

$$\hat{\Delta} \otimes_B \Delta = 1_A$$

and

$$\hat{\Delta} \otimes_A \Delta = (-1)^i 1_B.$$
**Corollary 2.7.** If $A$ and $B$ are dual in the sense of Definition 2.6, then the maps $\hat{\Delta}_*$ and $\Delta_*$ defined in Theorem 2.4 induce inverse isomorphisms (up to the signs specified there) $K^j(A) \cong K^{j+1}(B)$ and $K^j(B) \cong K_{j-i}(A)$.

For the purposes of this dissertation, we will define ‘non-commutative manifold’ in the following way:

**Definition 2.8.** A separable, nuclear $C^*$-algebra $A$ is a non-commutative manifold of dimension $i$ if there exists an element $\Delta \in K_i(A \otimes A^{\text{op}})$ implementing Poincaré duality in the sense of Definition 2.6, between $A$ and $A^{\text{op}}$. The element $\Delta$ will called the fundamental class of $A$.

**Remark 2.9.** We have focused on only one of Connes’ axioms for a non-commutative geometry (see [6]). We will see later that the operator $J$ appearing in his axioms, appears also in the example on which we are focussed on in this paper.
Chapter 3

Construction of the Fundamental Class

3.1 Hyperbolic groups

We will be working with a class of hyperbolic groups. See Chapter 1, Section 1.4 for the definition of hyperbolic group, and for a discussion of the boundary. We will be using certain other slightly deeper properties, which we describe in this section. So let us fix a hyperbolic group $\Gamma$, with a fixed generating set, yielding as per the discussion in Section 1.4 a metric giving $\Gamma$ the structure of a metric space. We can compactify this metric space by adding the boundary $\partial \Gamma$ of $\Gamma$, also discussed in Section 1.4, obtaining thereby a compact metric space $\bar{\Gamma}$ in which $\Gamma$ is contained as a dense subset.

For convenience, let us also fix, once and for all, metrics $d_{\bar{\Gamma}}$ on $\bar{\Gamma}$, and $d_{\partial \Gamma}$ on $\partial \Gamma$, which generate the topology on $\bar{\Gamma}$ and $\partial \Gamma$ respectively.

A remark on notation: frequently when thinking of $\Gamma$ as a geometric space, we will use $x, y, \text{etc}$ to denote points.

The following Lemma follows from the definition of the topology on $\bar{\Gamma}$.

Lemma 3.1. (Visibility) If $\epsilon > 0$, there exists $R \geq 0$ such that if $x, y \in \bar{\Gamma}$ and $d_{\bar{\Gamma}}(x, y) \geq \epsilon$, then every geodesic from $x$ to $y$ passes through the ball of radius $R$ with respect to the usual word metric in $\Gamma$. Conversely, if $R \geq 0$, there exists $\epsilon > 0$ such that if every geodesic passes between $x$ and $y$ passes through the ball of radius $R$ in $\Gamma$ then $d_{\bar{\Gamma}}(x, y) \geq \epsilon$.

The Rips complex for $\Gamma$ of dimension $N$, $P_N(\Gamma)$, is the simplicial complex whose vertices are the points of $\Gamma$, and whose $k$-simplices are the sets of cardinality $k$ of diameter less than or equal to $N$. 
We let $X_N$ denote the realization of the Rips Complex. It can also be viewed as the collection of finitely supported probability measures on $\Gamma$ whose support has diameter $\leq N$, a point of view useful later on the proof, when some linear interpolation is needed from $\Gamma$ to $X_N$. (Note $\Gamma$ is embedded naturally in $X_N$.) Clearly $X_N$ carries a free, simplicial, isometric, proper, cocompact action of $\Gamma$.

The following is well known, and can be found in for example [12].

**Lemma 3.2.** For large enough $N$, $X_N$ is contractible.

From now on, we will fix $N$ sufficiently large as provided by the lemma, and in order to minimize subscripts, denote $X_N$ simply by $X$.

Let $\partial^2 \Gamma$ denote the set $\{(a, b) \in \partial \Gamma \times \partial \Gamma \mid a \neq b\}$. This space will play an important role in what will follow.

We now state the axioms we will require $\Gamma$ to satisfy. Note also that, for convenience, our groups are throughout assumed to be torsion free. For a discussion of $GX$ see [11].

**AXIOM 1.** There exists a locally compact space $GX$ satisfying:

1. $GX$ is a locally trivial principal $\mathbb{R}$-bundle over $\partial^2 \Gamma$. The projection $GX \to \partial^2 \Gamma$ is denoted $r \mapsto (r(-\infty), r(+\infty))$.

2. $\Gamma$ acts on $GX$ freely properly and cocompactly, and its action commutes with the $\mathbb{R}$ action.

3. There is a continuous involution $GX \to GX$ (reversal of geodesics) denoted $r \mapsto \hat{r}$, which commutes with the $\Gamma$ action, and is compatible with the $\mathbb{R}$ action in the sense that $g_t(\hat{r}) = g_{-t}(r)$ for all $t$.

4. There exists a proper $\Gamma$-equivariant map $GX \to X$, denoted $r \mapsto r(0)$ and satisfying $\lim_{t \to \infty} g_t(r)(0) = r(+\infty)$ and $\lim_{t \to -\infty} g_t(r)(0) = r(-\infty)$, where the limits are taken in the compact space $\bar{X}$. 
AXIOM 2. There exists a continuous, Γ-equivariant map $S : \partial \Gamma \times \Gamma \to \partial \Gamma$ satisfying: for every $z \in \Gamma$, the map $a \mapsto S(a, z)$ has no fixed points.

**Remark 3.3.** A point $r$ of $GX$ is to be thought of as a bi-infinite geodesic in $X$. From now on we will call such an $r$ a pseudo-geodesic.

**Remark 3.4.** The existence of the map $S$ is equivalent to the existence of a map $S_0 : \partial \Gamma \to \partial \Gamma$ with no fixed points, and we should think of it as the analog in the classical case of the antipodal map on spheres. For the free group, we can define $S_0$ in several ways. One of them is as follows. Define first $S_0$ on $\Gamma$ by setting $S_0(g_1 \cdot g_2 \cdots g_n) = g_n \cdot g_{n-1} \cdots g_1$ where $g_1 \cdot g_2 \cdots g_n$ is a reduced word in the generators. Observe that $S_0$ is isometric. Hence it extends to the boundary. It clearly has no fixed points.

We will require the following. Recall $(x \mid y)$ denotes the Gromov product, which was discussed in Chapter 1. For the proof, see for example [25].

**Lemma 3.5.** If $f$ be a bounded function on $\Gamma$, then $f$ extends to a continuous function on $\bar{\Gamma}$ if and only if for all $\epsilon > 0$ there exists $K \geq 0$ such that $(x \mid y) > K \Rightarrow |f(x) - f(y)| < \epsilon$.

### 3.2 The Fundamental Classes

Our object is to show that the $C^*$-algebra $A = C(\partial \Gamma) \rtimes \Gamma$ is a noncommutative manifold in the sense of Definition 2 of the previous chapter. To this end, we will now construct the Fundamental Class, $\Delta$, which will lie, from the real point of view, in $KK^1(A \otimes A^{\text{op}}, \mathbb{C})$.

It is worth pausing at this point to meditate on the $1$. The Dirac operator on the line $\mathbb{R}$ defines an element in the group $K_1(\mathbb{R}) = KK^{-1}(C_0(\mathbb{R}), \mathbb{C})$, the Bott element in $K^{-1}(\mathbb{R}) = KK^1(\mathbb{C}, C_0(\mathbb{R}))$. On the other hand, the Toeplitz extension, though equivalent in complex $K$-homology to the class defined by the Dirac operator, lies in $K_{-1}(\mathbb{R})$. Our fundamental class, which is given by an extension, is thus the analog not of the Dirac operator, but of the Toeplitz extension.
Remark 3.6. The action of $\Gamma$ on $\partial \Gamma$ is well-known to be amenable (see, e.g. [10]). As a consequence, the algebra $A = C(\partial \Gamma) \rtimes \Gamma$ is nuclear. Furthermore, the reduced and max cross products are the same, which is why our notation does not indicate which cross product we are using.

The class $\Delta$ itself will be given by an extension; i.e. as a map $A \otimes A^{\text{op}} \to \mathcal{Q}(H)$ for a Hilbert space $H$. Since, by Remark 3.6, $A$ and thus also $A^{\text{op}}$ (which, as we show later, is isomorphic to $A$) are nuclear, the Stinespring Theorem (see for example [3] for a discussion of this) tells us that such an extension defines a class in $KK^1(A \otimes A^{\text{op}}, \mathbb{C})$.

Let $\gamma \in \Gamma$. We will denote by $u_\gamma$ the unitary operator on $l^2(\Gamma)$ given by left translation by $\gamma$, and by $v_\gamma$ the unitary operator given by right translation by $\gamma$. Thus, if we denote by $e_x$, $x \in \Gamma$, the canonical basis of $l^2(\Gamma)$, $u_\gamma(e_x) = e_{\gamma x}$ and $v_\gamma(e_x) = e_{x\gamma}$. Let $f$ be a function on $\partial \Gamma$. Then by the Tietze extension theorem $f$ extends to a continuous function on $\bar{\Gamma}$. Let $\tilde{f}$ denote any such extension. Define then a map

$$\lambda : A \to \mathcal{Q}(l^2(\Gamma))$$

by

$$\lambda(f)e_x = \tilde{f}(x)e_x$$

and

$$\lambda(\gamma)e_x = u_\gamma(e_x) = e_{\gamma x}$$

where these expressions are to be understood modulo compact operators. Remark that the map $f \mapsto \tilde{f}$ is well-defined modulo functions vanishing on $\partial \Gamma$. Since multiplication operators corresponding to such functions are compact operators, the map $f \to \tilde{f}$ is also well-defined modulo compact operators, i.e. well-defined as a map into the Calkin algebra.
The formulae above define a covariant pair which therefore extends to a homomorphism denoted $\lambda$, from the cross product $A$ into the Calkin algebra. Remark that $\lambda$ so extends not only because it is induced by a covariant pair, but because the max and reduced cross products are the same (Remark 3.6).

Next, define a map $\lambda^\text{op}$ from $A^\text{op}$ into the Calkin algebra by the pair

$$
\lambda^\text{op}(f)e_x = \tilde{f}(x^{-1})e_x
$$

and

$$
\lambda^\text{op}(\gamma)e_x = v_\gamma(e_x) = e_{x\gamma}.
$$

It is not difficult to calculate that this is also a covariant pair, this time with respect to the opposite action of $\Gamma$ on $C(\partial\Gamma)$, and so induces a map $A^\text{op} \to Q(l^2(\Gamma))$.

**Lemma 3.7.** Let $\tilde{f}$ be a function on $\Gamma$, viewed as a multiplication operator on $l^2(\Gamma)$, and let $\gamma \in \Gamma$.

1. If $x \mapsto \tilde{f}(x)$ is continuous on $\bar{\Gamma}$, then $[v_\gamma, \tilde{f}]$ is a compact operator.

2. If $x \mapsto \tilde{f}(x^{-1})$ is continuous, then $[u_\gamma, \tilde{f}]$ is a compact operator.

**Proof.** Let $\tilde{f}$ be as in (1). Choose $\epsilon > 0$. Remark if $x, \gamma \in \Gamma$ we have $(x, x\gamma) \geq |x| - |\gamma|$. From this and Lemma 3.3 we see: for every $\epsilon > 0$ there exists $K$ such that $|x| > K \Rightarrow |f(x) - f(x\gamma)| < \epsilon$. It follows immediately that $v_\gamma f v_{\gamma^{-1}} - f$ is compact; for this operator is multiplication by the function $f(x) - f(x\gamma)$ which, from above, vanishes at infinity. It follows that

$$
(v_\gamma f v_{\gamma^{-1}} - f)v_\gamma = [v_\gamma, \tilde{f}]
$$

is also a compact operator. (2) follows in exactly the same way. 

From Lemma 3.4, the following is immediate.
Corollary 3.8. The maps $\lambda$ and $\lambda^{op}$ commute as maps into the Calkin algebra.

We may thus make the following

Definition 3.9. The Fundamental Class of the $C^*$-algebra $A = C(\partial \Gamma) \rtimes \Gamma$ is the class in $KK^1(A \otimes A^{op}, \mathbb{C})$ of the homomorphism $A \otimes A^{op} \rightarrow \mathcal{Q}(l^2(\Gamma))$ induced by the two commuting maps $\lambda$ and $\lambda^{op}$.

Remark 3.10. Define a conjugate linear operator $J : l^2(\Gamma) \hookrightarrow l^2(\Gamma)$ by $J(\sum \alpha_{\gamma} e_\gamma) = \sum \overline{a}_{\gamma} e_{\gamma}$.

Then it is easy to calculate that $J \pi(a^*) J^* = \pi^{op}(a)$ which is one of Connes’ axioms for a non-commutative geometry. See [6].

Next, we show that there exists $\hat{\Delta} \in KK^{-1}(\mathbb{C}, A \otimes A^{op})$ for which

$$\hat{\Delta} \otimes_{A^{op}} \Delta = 1_A$$

and

$$\hat{\Delta} \otimes_A \Delta = 1_{A^{op}}$$

As $\hat{\Delta}$ does not occur most naturally as an element of $A \otimes A^{op}$, but rather of $A \otimes A$, we will need the following

Lemma 3.11. For $A$ as above, $A \cong A^{op}$.

Proof. Define a map $j : A \rightarrow A^{op}$ by $j(f) = f$ and $j(\gamma) = \gamma^{-1}$. Then $j$ defines a covariant pair inducing the required isomorphism.

We are now going to use Axiom 1 of the previous section. But first we review the notion of strong Morita equivalence. See for instance [24] for a slightly more classical definition, and [7], pg.155 for the one we will be using. Recall that an $M-N$ $C^*$-bimodule $E$ is given by a right Hilbert $N$-module, together with a $C^*$-algebra homomorphism $M \rightarrow \mathcal{B}(E)$. 
**Definition 3.12.** Let $M$ and $N$ be $C^*$-algebras. $M$ and $N$ are strongly Morita equivalent if there exists an $M$-$N$ $C^*$-bimodule $E$ and an $N$-$M$ $C^*$-bimodule $\bar{E}$ satisfying $E \otimes_N \bar{E} \cong 1_M$ and $\bar{E} \otimes_M E \cong 1_N$, where $1_M$ is the trivial $M$-$M$ $C^*$-bimodule given by $M$ itself, and similarly for $1_N$.

We will use the following theorem, due to Rieffel ([24]).

**Theorem 3.13.** If $P$ is a locally compact space with two free and proper commuting actions of groups $H$ and $K$, then the groupoid $C^*$-algebras $C(P/K) \rtimes H$ and $C(P/H) \rtimes K$ are strongly Morita equivalent.

The main consequence of Theorem 3.1 is that the $C^*$-algebras $M$ and $N$ have the same $K$-theory and $K$-homology. More precisely:

**Theorem 3.14.** Let $M$, $N$, $E$, etc be as in Theorem 3.1 Then, product with the class of $E \in KK(M, N)$ on the right induces a canonical isomorphism

$$KK^i(C, M) \to KK^i(C, N)$$

with inverse given by product on the left with $\bar{E}$. Similar statements hold for $K$-homology.

Consider the space $GX$. It is a free and proper $\Gamma$ space and a free and proper $\mathbb{R}$ space and the two actions commute and are proper. Therefore we may apply theorem 3.1 to construct a strong Morita equivalence bimodule between $C(GX/\Gamma) \rtimes \mathbb{R}$ and $C_0(\partial^2 \Gamma) \rtimes \Gamma$. Let us provide here an explicit construction of $E$. It consists of a certain completion of $C_c(GX)$ with the obvious left action of $C(GX/\Gamma) \rtimes \mathbb{R}$, and the right $C_0(\partial^2 \Gamma) \rtimes \Gamma$-valued inner product

$$\langle \xi, \eta \rangle_{C_0(\partial^2 \Gamma) \rtimes \Gamma}([r], \gamma) = \int_{\mathbb{R}} \bar{\xi}(g_t(r))\eta(g_t\gamma^{-1}(r))dt,$$

where $r$ is a lift of $[r] \in GX/\Gamma$ to $GX$. 
Remark for later purposes: we will sometimes denote by \( r_{a,b} \) a choice of pseudo-geodesic from \( a \) to \( b \), in other words a point \( r \) of \( GX \) for which \( r(-\infty) = a \) and \( r(+\infty) = b \). With this notation the above inner product becomes

\[
< \xi, \eta >_{C_0(\partial^2\Gamma) \rtimes \Gamma} ((a, b), \gamma) = \int_{\mathbb{R}} \xi(g_t(r_{a,b}))\eta(g_t\gamma^{-1}(r_{a,b}))dt.
\]

Now by Axiom 1(2) the action of \( \Gamma \) on \( GX \) is cocompact, and hence \( C(GX) \) is a unital algebra. Consequently there is a canonical inclusion \( u : C^*(\mathbb{R}) \to C(GX/\Gamma) \rtimes \mathbb{R} \). Pulling back by \( u \) the strong Morita equivalence bimodule \( E \) gives an element \( u^*([E]) \) of \( KK(C^*(\mathbb{R}), C_0(\partial^2\Gamma) \rtimes \Gamma) \cong KK^{-1}(\mathbb{C}, C_0(\partial^2\Gamma) \rtimes \Gamma) \), which we denote by \([D]\). Now let \( i \) be the inclusion

\[
C_0(\partial^2\Gamma) \rtimes \Gamma \to A \otimes A
\]

Finally, recall the map \( j \) defined in Lemma 3.5.

**Definition 3.15.** We define the *dual* element \( \hat{\Delta} \in KK^{-1}(\mathbb{C}, A \otimes A^{op}) \) to be

\[
\hat{\Delta} = (1_A \otimes j)_* i_* ([D]) \in KK^{-1}(\mathbb{C}, A \otimes A^{op}).
\]

The following lemma will be used to show that in our example, only one of the products need be computed (see Corollary 1).

**Lemma 3.16.** The elements \( \hat{\Delta}_1 = i_* ([D]) \in KK^{-1}(\mathbb{C}, A \otimes A) \), and \( \Delta_1 = (1_A \otimes j)^* (\Delta) \in KK^1(A \otimes A, \mathbb{C}) \), are respectively skew-invariant and invariant under the flip \( \sigma_{12} : A \otimes A \to A \otimes A \).

**Proof.** Since \( (\sigma_{12})_* (\hat{\Delta}) = \sigma_{12}^* i_* (\Delta_0) = (\sigma_{12} \circ i)_* ([D]) \) it suffices to show \( (\sigma_{12} \circ i)_* ([D]) = -i_* ([D]) \). Under the strong Morita equivalence the flip corresponds to the map

\[
s : C(GX/\Gamma) \rtimes \mathbb{R} \to C(GX/\Gamma) \rtimes \mathbb{R}
\]
given by the covariant pair \( r \mapsto \hat{r} \) and \( t \mapsto -t \), where \( \hat{r} \) denotes the reversal of \( r \), and pulling back by \( u \) gives the map \( C^*(\mathbb{R}) \to C^*(\mathbb{R}) \) induced by the group homomorphism also denoted \( s \), \( t \mapsto -t \).

In other words, we see that \( (\sigma_{12} \circ i)_*((u)^*([E])) = i_*((u \circ s)^*([E])) = -i_*((u^*([E])) \)
as required. For the second statement note the unitary \( l^2(\Gamma) \to l^2(\Gamma) \) induced from inversion in the group conjugates the homomorphism \( A \to \mathbb{Q}(l^2(\Gamma)), a \mapsto \lambda(a), \) to the homomorphism \( a \mapsto \lambda^{\text{op}}(j(a)) \).

Hence the maps \( a \otimes b \mapsto \lambda(a)\lambda^{\text{op}}(j(b)) \) and \( a \otimes b \mapsto \lambda^{\text{op}}(j(a))\lambda(b) \) are conjugate, and so give the same element of \( KK^1(C, A \otimes A) \).

\[ \square \]

**Corollary 3.17.** To show \( A \) is a non-commutative manifold, it suffices to show

\[ \hat{\Delta} A^{\text{op}} \Delta = 1_A. \]

**Proof.** For clarity of presentation we work from the point of view of morphisms. Assume that

\( \hat{\Delta} A^{\text{op}} \Delta = 1_A \). Consider first \( \hat{\Delta} A^{\text{op}} \Delta \). This is the 'composition' of the morphism

\[
A \xrightarrow{\hat{\Delta}_1 \otimes 1_A} A \otimes A \otimes A \xrightarrow{1_A \otimes j \otimes 1_A} A \otimes A^{\text{op}} \otimes A
\]

and the morphism

\[
A \otimes A^{\text{op}} \otimes A \xrightarrow{\sigma_{23}} A \otimes A \otimes A^{\text{op}} \xrightarrow{1_A \otimes 1_A \otimes j^{-1}} A \otimes A \otimes A \xrightarrow{1_A \otimes \Delta_1} A.
\]

This composition is clearly the same as the morphism

\[
A \xrightarrow{\hat{\Delta}_1 \otimes 1_A} A \otimes A \otimes A \xrightarrow{\sigma_{23}} A \otimes A \otimes A \xrightarrow{1_A \otimes \Delta_1} A.
\]

In other words \( \hat{\Delta} A^{\text{op}} \Delta = \sigma_A(\hat{\Delta}_1) \otimes A \otimes A \sigma_A(\Delta_1) \), where we have used symmetry of \( \Delta_1 \) (Lemma 3.6).
On the other hand consider $\hat{\Delta} \otimes_A \Delta$. This is the composition of the morphism

$$A^{\text{op}} \xrightarrow{1_A \otimes (\sigma_{12})_{\hat{\Delta}_1}} A \otimes A \otimes A^{\text{op}} \xrightarrow{j \otimes 1_A \otimes j^{-1}} A^{\text{op}} \otimes A \otimes A$$

and the morphism

$$A^{\text{op}} \xrightarrow{1_A \otimes \Delta_1} A^{\text{op}}.$$ 

From the latter two formulae, it follows that $j^*(j^{-1})_*(\hat{\Delta} \otimes_A \Delta)$ is the composition of the morphism

$$A \xrightarrow{1_A \otimes (\sigma_{12})_{\hat{\Delta}_1}} A \otimes A \otimes A \xrightarrow{j \otimes 1_A \otimes 1_A} A^{\text{op}} \otimes A \otimes A$$

and the morphism

$$A^{\text{op}} \otimes A \otimes A \xrightarrow{j^{-1} \otimes \Delta_1} A,$$

which of course is the same as the composition

$$A \xrightarrow{1_A \otimes (\sigma_{12})_{\hat{\Delta}_1}} A \otimes A \otimes A \xrightarrow{1_A \otimes \Delta_1} A.$$ 

In other words, we have

$$j^*(j^{-1})_*(\hat{\Delta} \otimes_A \Delta) = \sigma_A((\sigma_{12})_* (\hat{\Delta}_1)) \otimes_A \otimes_A A \sigma_A(\Delta) = (-1)^i \hat{\Delta} \otimes_{A^{\text{op}}} \Delta = (-1)^i 1_A.$$

$\square$
Chapter 4

Proof of the Main Theorem

In this chapter we proceed to the proof that the elements $\Delta$ and $\hat{\Delta}$ given above implement a Poincaré Duality between $A = C(\partial \Gamma) \rtimes \Gamma$ and its opposite algebra. Precisely, we will prove:

**Theorem 4.1.** Suppose $\Gamma$ is a torsion-free hyperbolic group satisfying Axioms 1 and 2 of Section 3.1. Let $A$ denote the cross product $C^*$-algebra $C(\partial \Gamma) \rtimes \Gamma$. Let $\Delta \in KK^1(A \otimes A^{\text{op}}, C)$ and $\hat{\Delta} \in KK^{-1}(C, A \otimes A^{\text{op}})$ the $KK$-classes specified in respectively Definition 3.9 and Definition 3.15. Then $\hat{\Delta} \otimes A^{\text{op}} \Delta = 1_A$ and $\hat{\Delta} \otimes_A \hat{\Delta} = 1_{A^{\text{op}}}$. Hence the $C^*$-algebra $A$ is a non-commutative manifold in the sense of Definition 2.8.

The strategy of the proof is to associate to the product $\hat{\Delta} \otimes A^{\text{op}} \Delta \in KK(A, A)$ an equivariant class $c$. We will then use the proof of the Baum-Connes Conjecture with coefficients to show that this equivariant class is 1. This will imply the original class is also 1.

More precisely, recall (see [20]) that there is a group $RKK_\Gamma(\partial \Gamma; C, C)$ and a descent map $\lambda : RKK_\Gamma(\partial \Gamma; C, C) \to KK(A, A)$. The descent map has the property that $\lambda(1_{\partial \Gamma}) = 1_A$. We shall show that $\hat{\Delta} \otimes A^{\text{op}} \Delta = 1_A$ by constructing a class $c$ in $RKK_\Gamma(\partial \Gamma; C, C)$ such that $\hat{\Delta} \otimes A^{\text{op}} \Delta = \lambda(c)$, and then show that $c = 1_{\partial \Gamma}$. It will follow that $\hat{\Delta} \otimes A^{\text{op}} \Delta = \lambda(c) = \lambda(1_{\partial \Gamma}) = 1_A$.

We will prove that the class $c$ is $1_{\partial \Gamma}$ by using the work of Tu ([29]). Namely, we will prove (Lemma 4.13) that his proof of the Baum-Connes Conjecture for the groupoid $\partial \Gamma \rtimes \Gamma$ implies that a naturally defined map

$$p^*_X : RKK_\Gamma(\partial \Gamma; C, C) \to RKK_\Gamma(\partial \Gamma \times X; C, C)$$
is a ring isomorphism, and secondly, that $p_X^*(c) = 1_{\partial \Gamma \times X}$. This will yield the result, as it must then be that $c = 1_{\partial \Gamma}$ also.

Throughout this chapter, as in the last, we will let $A$ denote $C(\partial \Gamma) \rtimes \Gamma$ and let $B$ denote the $C^*$-algebra $C_0(\partial^2 \Gamma) \rtimes \Gamma$, which will play an important auxiliary role. Our first goal, as per the above discussion, is to show that $\hat{\Delta} \otimes A^{\text{op}} \Delta$ is in the range of the descent map $\lambda$.

Let us first expand $\hat{\Delta} \otimes A^{\text{op}} \Delta$. Recall that $\hat{\Delta}$ was constructed from an element $[D] \in KK^{-1}(\mathbb{C}, B) = KK(C^*(\mathbb{R}), B)$ by means of pushing forward via

$$(1_A \otimes j) \circ i : B \to A \otimes A \to A \otimes A^{\text{op}}$$

(see Definition 3.15 of the previous chapter.) In other words $\hat{\Delta} = (1_A \otimes j)_* i_*(D)$. We have

$$\hat{\Delta} \otimes A^{\text{op}} \Delta = \sigma_A(\hat{\Delta}) \otimes_{A \otimes A^{\text{op}} \otimes A} \sigma^A(\sigma_{12}^*(\Delta))$$

and expanding the term involving $\hat{\Delta}$ we can write this

$$(1_A \otimes j \otimes 1_A)_* (i \otimes 1_A)_*(D) \otimes_{A \otimes A^{\text{op}} \otimes A} \sigma^A(\sigma_{12}^*(\Delta))$$

$$= \sigma_A(D) \otimes_{B \otimes A} (1_A \otimes j \otimes 1_A)^* (\sigma^A(\sigma_{12}^*(\Delta)))$$

To simplify our formulas, let $[\tilde{\tau}]$ denote the class $((1_A \otimes j \otimes 1_A))_* (\sigma^A(\sigma_{12}^*(\Delta)))$, so that $\hat{\Delta} \otimes A^{\text{op}} \Delta$ becomes $\sigma_A(D) \otimes_{B \otimes A} [\tilde{\tau}]$. A straightforward computation verifies that $\tilde{\tau}$ is the map $B \otimes A \to Q(l^2(\Gamma))$ given by

$$a_1 \otimes a_2 \otimes a_3 \mapsto a_1 \otimes \lambda^{\text{op}}(j(a_2)) \lambda(a_3)$$

where we understand $a_1 \otimes a_2$ as in $B \subset A \otimes A$, suppressing the inclusion $i$. 

It will be convenient to work with a unitarily equivalent extension, which we denote $\tau$, defined as follows.

Denoting by $e_x$ the canonical basis of $l^2(\Gamma)$, for $x \in \Gamma$, define a unitary map of Hilbert $A$-modules $U : A \otimes l^2(\Gamma) \to A \otimes l^2(\Gamma)$ by $a \otimes e_x \mapsto x \cdot a \otimes e_x$. We then define $\tau$ by the map $B \otimes A \to \mathbb{Q}(l^2(\Gamma))$

$$b \otimes a \mapsto U\hat{\tau}(b \otimes a)U^*.$$  

By definition of the equivalence relation(s) in $KK^1(B \otimes A, A)$, $[\hat{\tau}] = [\tau]$.

Next, it will be convenient to note that $B \otimes A \cong C(\partial^2 \Gamma \times \Gamma) \rtimes \Gamma \times \Gamma$. With this identification, we can view $\tau$ as given by a covariant pair. A calculation shows this covariant pair is given as follows. If $F$ is a function on $\partial^2 \Gamma \times \Gamma \subset \partial \Gamma \times \partial \Gamma \times \partial \Gamma$, let $\hat{F}$ denote an extension of $F$ to a continuous function on $\partial \Gamma \times \bar{\Gamma} \times \bar{\Gamma}$. Remark that a function on $\partial \Gamma \times \Gamma$ continuous in the first variable gives an element of $A \otimes \mathbf{B}(l^2(\Gamma)) \subset \mathbf{B}(A \otimes l^2(\Gamma))$. With this in mind, $\tau(F)$ is the operator corresponding to the function on $\partial \Gamma \times \Gamma$

$$\tau(F)(a, x) = \hat{F}(x^{-1}(a), x^{-1}, x).$$

As for group elements $(\gamma_1, \gamma_2) \in \Gamma \times \Gamma$, we have

$$\tau(\gamma_1, \gamma_2) = \gamma_1 \otimes \lambda(\gamma_2)\lambda^{\text{op}}(\gamma_1^{-1}) \in A \otimes \mathbf{B}(l^2(\Gamma)) \subset \mathbf{B}(A \otimes l^2(\Gamma)).$$

As usual, these expressions are to be understood modulo $A \otimes K(l^2(\Gamma))$.

The following lemma is the crucial step in showing that $\hat{\Delta} \otimes A^{\text{op}} \Delta$ is in the range of the descent map.

**Lemma 4.2.** Let $F \in C_\infty(\partial^2 \Gamma \times \Gamma)$, and $\hat{F}$ an extension of $F$ to a continuous function on $\partial \Gamma \times \bar{\Gamma} \times \bar{\Gamma}$. Then modulo $A \otimes K(l^2(\Gamma))$, the operators corresponding to the two functions on
\( \partial \Gamma \times \Gamma \), respectively

\[(a, x) \mapsto \tilde{F}(x^{-1}(a), x^{-1}, x) \]

and

\[(a, x) \mapsto \tilde{F}(x^{-1}(a), x^{-1}, a) \]

are the same.

Proof. Let \( F \) be as in the statement of the lemma. Then for some \( \epsilon > 0 \), \( F(a, b, x) = 0 \) if \( d(a, b) \leq \epsilon \). Extend \( F \) to a function \( \tilde{F} \) on \( \partial \Gamma \times \bar{\Gamma} \). Then its image under \( \tau_1 \) is the operator corresponding to multiplication by \( (a, g) \mapsto \tilde{F}(g^{-1}(a), g^{-1}, g) \). We need only show this is a compact perturbation of the function \( (a, g) \mapsto \tilde{F}(g^{-1}(a), g^{-1}, a) \). Note that \( F \) can be lifted to a function \( \tilde{F} \) supported on those \( (a, g_1, g_2) \) satisfying \( d_{\tilde{\Gamma}}(a, g_1) \geq \epsilon \). Consider the difference

\[(a, g) \mapsto \tilde{F}(g^{-1}(a), g^{-1}, g) - \tilde{F}(g^{-1}(a), g^{-1}, a) \]

We claim this converges to 0 as \( g = g_n \to \infty \). By compactness we can assume that \( g_n \) converges to a boundary point. By the support assumption we can assume that \( d_{\tilde{\Gamma}}(g_n^{-1}(a), g_n^{-1}) \geq \epsilon \) for all \( n \) (else both terms in the difference and hence the difference itself converges to 0.) Then, for some \( R \) large enough, and for all \( n \) large, we have by Lemma 3.1. \( d(g_n, [e, a]) \leq R \), where \([e, a]\) denotes any geodesic ray from \( e \) to \( a \). Hence \( g_n \to a \), and the result follows from continuity of \( \tilde{F} \) in the third variable.

Now for what follows, it will be convenient to view \([D]\) as given by a homomorphism

\[ C^*(\mathbb{R}) \to B \otimes K(V) \]
where \( V \) is a separable Hilbert space, and \( \mathbf{K} \) denotes as always the compact operators on \( V \), as we may do via the standard arguments linking Kasparov theory with ordinary \( K \)-theory for \( C^* \)-algebras. Alternatively, we can assume there exists \( w \in B \otimes \mathbf{K}(V) \) for which \( [D] \) is the class of \( w + 1 \in \mathcal{U}(B^+) \), and for which \( \hat{\Delta} \) is the class of \( ((1_A \otimes j) \circ i)(w) + 1 \).

We will use this presentation of \( \hat{\Delta} \) only for purposes of Lemma 4.5; later, we will revert to the original description of \( [D] \) in terms of the strong Morita equivalence bimodule \( E \).

**Lemma 4.3.** The class \( \hat{\Delta} \otimes_{A^\text{op}} \Delta \) is given by the following cycle for the group \( KK(A, A) \) (see the discussion following Lemma 2.2 for our standard simplications of \( KK \) for ungraded \( C^* \)-algebras):

- The module is \( A \otimes V \otimes l^2(\Gamma) \). The \( A \)-action on the left is by left multiplication in the \( A \)-factor. The \( A \)-action on the right is by right multiplication on the \( A \)-factor. The operator is the essentially unitary operator given by any lift \( F \) of \( (\tau \otimes 1_A)(w \otimes 1) + 1 \otimes \text{id}_{V \otimes l^2(\Gamma)} \).

**Proof.** The product \( \hat{\Delta} \otimes_{A^\text{op}} \Delta \) has been shown to be the product of the class of the extension \( \tau : B \otimes A \hookrightarrow A \otimes Q(l^2(\Gamma)) \) described just prior to Lemma 4.2, and the class of a homomorphism \( A \otimes C^*(\mathbb{R}) \to B \otimes \mathbf{K}(V) \otimes A \). That the cycle corresponding to the product can be so written follows from Lemma 2.2.

\[ \square \]

Now for what follows we will need to recall (see [20]) the definition and properties of the group \( RKK_{\Gamma}(\partial \Gamma; \mathbb{C}, \mathbb{C}) \). Recall that its cycles are the same as the cycles for \( KK_{\Gamma}(C(\partial \Gamma), C(\partial \Gamma)) \) except that the left action of \( C(\partial \Gamma) \) on the relevant \( C(\partial \Gamma) \)-module is required to be unital (that is, the function 1 acts as the operator 1), and secondly, that this left action agrees with the right action. We will frequently understand these as continuous fields \( (H_a, F_a) \) of \( KK(\mathbb{C}, \mathbb{C}) \) cycles over \( \partial \Gamma \), with \( \gamma \in \Gamma \) acting as a map \( H_a \to H_{\gamma(a)} \), and with the equivariance condition \( \gamma(F_a) - F_{\gamma(a)} \) is compact.
If $Z$ is any space (and in particular $Z = X_N$)) there exists a natural map

$$p^*_Z : RKK_\Gamma(\partial\Gamma; \mathbb{C}, \mathbb{C}) \to RKK_\Gamma(\partial\Gamma \times Z; \mathbb{C}, \mathbb{C}).$$

Also, there is a descent map $RKK_\Gamma(\partial\Gamma; \mathbb{C}, \mathbb{C}) \mapsto KK(A, A)$ (see [20]) which is defined by the composition

$$RKK_\Gamma(\partial\Gamma; \mathbb{C}, \mathbb{C}) \to KK_\Gamma(C(\partial\Gamma), C(\partial\Gamma)) \xrightarrow{\lambda} KK(A, A),$$

where the second map is the usual descent map, which in turn is defined more generally on the group $KK_\Gamma(D, D)$ for any $C^*$-algebra $D$, as follows.

If $(\mathcal{E}, \mathcal{F})$ is a cycle in $KK_\Gamma(D, D)$, then its image under descent $\lambda$ is the cycle whose module is a completion $\mathcal{E} \rtimes \Gamma$ of $C_c(\Gamma, \mathcal{E})$ with the $D \rtimes \Gamma$-valued inner product

$$<\xi, \eta>_{D \rtimes \Gamma}(\gamma) = \sum_{\Gamma} \gamma^{-1}(<\xi(1), \eta(\gamma_1^{-1})>_D).$$

The right action of $D \rtimes \Gamma$ on $E$ is given by $\xi(\gamma) \cdot f = \sum_{\Gamma} \xi(\gamma_1) \cdot \gamma_1 (f(\gamma_1^{-1}))$ and the left action of $D \rtimes \Gamma$ is given by the covariant pair $(\gamma_1 \cdot \xi)(\gamma) = \gamma_1 (\xi(\gamma_1^{-1}))$, and $d \cdot \xi(\gamma) = d \cdot (\xi(\gamma))$. The operator $F$ is $F \cdot \xi(\gamma) = F(\xi(\gamma))$.

Finally, recall that for any $Z$, $RKK_\Gamma(Z; \mathbb{C}, \mathbb{C})$ has a unit, which we denote by $1_Z$. Under the descent map, $1_Z$ is mapped to $1_{C_0(Z) \rtimes \Gamma}$.

For what follows, we will be interested in $Z = \partial\Gamma$. Recall we are denoting $C(\partial\Gamma) \rtimes \Gamma$ by $A$.

We first define the cycle $c$, of whose class $\hat{\Delta} \otimes_{A^{op}} \Delta$ will be the image under the descent map.
**Definition 4.4.** Let $c$ be given by the pair $(C(\partial \Gamma) \otimes l^2(\Gamma) \otimes V, F)$, where $C(\partial \Gamma) \otimes l^2(\Gamma) \otimes V$ has its obvious $C(\partial \Gamma)$-bimodule structure, and the action of $\Gamma$ given by $\gamma (f \otimes e_x \otimes v) = \gamma (f) \otimes e_{\gamma x} \otimes v$. Let $F$ be an essentially unitary lift of $(\tau \otimes id_{K(V)})(w) + id_{C(\partial \Gamma) \otimes l^2(\Gamma) \otimes V}$.

**Lemma 4.5.** $c$ defines a cycle in $KK_\Gamma(\partial \Gamma; \mathbb{C}, \mathbb{C})$.

*Proof.* $F - 1$ is a limit of finite linear combinations of operators of the form

$$f \otimes e_x \otimes v \mapsto (h_1 \circ x^{-1})h_3 f \otimes \tilde{h}_2(x^{-1})e_x \otimes T(v)$$

where $T$ is compact, and $h_1 \otimes h_2 \otimes h_3 \in C_0(\partial^2 \Gamma \times \partial \Gamma)$, and where $\tilde{h}_2$ denotes a lift of $h_2$ to a continuous function on $\bar{\Gamma}$; and also of the right translation operators

$$f \otimes e_x \otimes v \mapsto f \otimes e_{x\gamma} \otimes v.$$

All of these operators commute modulo compacts with the $\Gamma$ action on $C(\partial \Gamma) \otimes l^2(\Gamma) \otimes V$, and so $F - 1$ does also, and hence so does $F$. Thus $c$ defines a cycle for $KK_\Gamma(C(\partial \Gamma), C(\partial \Gamma))$. Furthermore, since the $C(\partial \Gamma)$ action on the module is the same on the left and on the right, and the operator $F$ actually commutes with the action of these functions, $c$ actually defines a class in the group $RKK_\Gamma(\partial \Gamma; \mathbb{C}, \mathbb{C})$.

The descent map, whose definition immediately precedes Definition 4.4, when applied to $c$ given in Definition 4.3, produces the cycle presented in Lemma 4.3. To summarize, we have:

**Lemma 4.6.** $\lambda(c) = \hat{\Delta} \otimes A^\text{op} \Delta$.

From now on we will make use of Lemma 4.5 and the discussion immediately following Lemma 4.3 to view $c$ as a field of $KK(\mathbb{C}, \mathbb{C})$ cycles over $\partial \Gamma$ given by the field of Hilbert spaces $\{H_a = l^2(\Gamma) \otimes V\}_{a \in \partial \Gamma}$, and the field of operators $\{F_a\}_{a \in \partial \Gamma}$ acting on it fibrewise. We understand
the $\Gamma$ action as mapping $H_a \to H_\gamma(a)$. By Lemma 4.5 and the fact that $\lambda(1_{\partial \Gamma}) = 1_A$, to show that $\hat{\Delta} \otimes_{A^{op}} \Delta$ is $1_A$ it suffices to show $c = 1_{\partial \Gamma}$ in the sense of $RKK_\Gamma(\partial \Gamma; \mathbb{C}, \mathbb{C})$. For this we use the Baum-Connes conjecture in a crucial way. We will pull back by the map $p^*_X$ (see the discussion just following Lemma 4.3) the element $c$ to a field over $\partial \Gamma \times X$ (recall that by $X = X_N$ we mean the realization of the Rips complex $P_N(\Gamma)$ for sufficiently large $N$) and show that the pulled-back element $p^*_X(c)$ is $1_{\partial \Gamma \times X}$. We will then make use of the Baum-Connes conjecture to show the original element must be $1_{\partial \Gamma}$. We would like to thank Nigel Higson for suggesting an approach along these lines.

To clarify the steps involved in the first part of the argument (the second part is abstract and general) let us describe the basic steps involved. We will first give a description of the class $1_{\partial \Gamma \times X}$ as a cycle, denoted, say, $\gamma \in RKK_\Gamma(\partial \Gamma; \mathbb{C}, \mathbb{C})$, which will be useful for our purposes. We will then construct two elements $a$ and $b \in RKK_\Gamma^{-1}(\partial \Gamma; \mathbb{C}, B)$ and $RKK_\Gamma^1(\partial \Gamma; B, \mathbb{C})$ and verify that $a \otimes_{\partial \Gamma \times X} b = \gamma$ by checking that $\gamma$ verifies the axioms for a Kasparov product of $a$ and $b$. Finally, we will show that $p^*_X(c) = a \otimes_{\partial \Gamma \times X} b = \gamma$ and therefore that $p^*_X(c) = 1$.

The convoluted nature of this argument arises from the fact that although the product $\hat{\Delta} \otimes_{A^{op}} \Delta$ is ‘equivariant’ it is not true that it is the product of equivariant elements (the problem arises with $\Delta$, and with the general difficulty in finding, for an equivariant extension, a completely positive equivariant lifting.) However, it so happens that when we pull back the product (or, more precisely $c$) to $\partial \Gamma \times X$, we can write it equivariantly as a product of the $a$ and $b$ above. This gives us enough flexibility to show the result.

We start by defining a function $Q(a, z, x)$ on $\partial \Gamma \times \Gamma \times \bar{\Gamma}$ as follows: let first $Q = Q(a, x)$ be a function on $\partial \Gamma \times \bar{\Gamma}$ which is 1 on an $\epsilon/2$ neighbourhood of the diagonal in $\partial \Gamma \times \partial \Gamma \subset \partial \Gamma \times \bar{\Gamma}$ and 0 outside an $\epsilon$ neighbourhood of it, where $\epsilon$ is to be determined by the proof of Lemma 4.7. Then define $Q(a, z, x) = Q(z^{-1}a, z^{-1}x)$ for $z \in \Gamma$.

Recall the map $S$ provided by Axiom 2. Let $S_z : \partial \Gamma \to \partial \Gamma$ denote the map $S_z(a) = S(a, z)$. Define, for $(a, z) \in \partial \Gamma \times \Gamma$, homomorphisms $\phi_{(a, z)} : B \to B(\ell^2(\Gamma))$ by $\phi_{(a, z)}(F)(x) =$
$F(x^{-1}(a), x^{-1}(S_z(a)))$ and $\phi_{(a,z)}(\gamma) = \lambda^{op}(\gamma^{-1})$. This defines a covariant pair, and so a homomorphism as required.

**Lemma 4.7.** The function $Q$ can be chosen in such a way that the following hold: if $x_n$ is a sequence of points in $\Gamma$ and $x_n \to S_z(a)$, then for every $w \in \Gamma$ with $d(w,z) \leq N$, we have $Q(a,w,x_n) \to 0$.

**Proof.** We claim it suffices to show this for $z = x_0$. For, if the result holds for $x_0$, and $z$ is arbitrary, and $w$ such that $d(z,w) \leq N$, and if $x_n \to S_z(a) = zS(z^{-1}(a))$, then $z^{-1}x_n \to S(z^{-1}(a))$. Now $d(z^{-1}x_n, x_0) \leq N$ so $Q(z^{-1}(a), z^{-1}w, z^{-1}x_n) \to 0$. But $Q(z^{-1}(a), z^{-1}w, z^{-1}x_n) = Q(a,w,x_n)$, and this proves the claim.

Now, by compactness of $\partial \Gamma$, $\inf\{d_{\cal F}(a, S(a)) \mid a \in \partial \Gamma\} > 0$. Hence $\sup\{d(x_0, [a, S(a)]) \mid a \in \partial \Gamma\} < \infty$ by Lemma 2.1. Let $R$ be such that $R > 2N + 2\sup\{d(x_0, [a, S(a)]) \mid a \in \partial \Gamma\}$ and $Q$ such that $Q(a,x) = 0$ unless $d(x_0, [x,a]) \geq R$. (This determines $\epsilon$ as corresponding to $R$ via Lemma 2.1.) Let then $x_n \to S(a)$ and let $w \in B_N(x_0)$. Then if $Q(a,w,x_n) = Q(w^{-1}(a), w^{-1}x_n)$ does not converge to 0 it follows for $n \to \infty$ $d(w, [x_n, a]) \geq R$. Since $x_n \to S(a)$ it follows $d(w, [a, S(a)]) \geq R \frac{R}{2}$ and hence $d(x_0, [a, S(a)]) \geq \frac{R}{2} - N > \sup\{d(x_0, S(a)) \mid a \in \partial \Gamma\}$, a contradiction.

Let $Q_{(a,z)}$ denote the operator on $l^2(\Gamma)$ of pointwise multiplication by the function $Q(a,z,\cdot)$, and recall that the space $X$ can be viewed as probability measures on $\Gamma$ whose supports have diameter $\leq N$.

Define a continuous field of Hilbert spaces and operators over $\partial \Gamma \times X$ in the following way. For $\mu \in X$ let $H_{(a,\mu)} = L^2_{\mu}(\Gamma; l^2(\Gamma))$, for all $(a,\mu)$, with left $B$-action given by the homomorphism

$$(\phi_{(a,\mu)}(b)\xi)(z) = \phi_{(a,z)}(\xi(z)).$$
Note that the sections of this field are provided by the map $C_c(\Gamma; l^2(\Gamma)) \to L^2_\mu(\Gamma; l^2(\Gamma))$ taking a compactly supported function to its class in $L^2_\mu$. Next, define a field $P_{a,\mu}$ of operators on the Hilbert spaces $H_{(a,\mu)}$ by

$$(P_{(a,\mu)}\xi)(z)(x) = \int_{\Gamma} Q_{(a,z)}(\xi(z)(x))d\mu(z).$$

Finally, give the $C^*$-algebra $B$ the structure of a $\Gamma$-algebra by giving it the trivial action of $\Gamma$.

**Lemma 4.8.** The field of pairs $\{(H_{(a,\mu)}, P_{(a,\mu)})\}$ defines an element $b$ of the group $RKK^1_\Gamma(\partial\Gamma \times X; B, \mathbb{C})$.

**Remark 4.9.** It is useful to think of the case when $\mu \in X$ is a point mass $\delta_z$, where $z$ is a point of $\Gamma$ — i.e. a vertex of $P_N(X)$. In this case the Hilbert space is simply $l^2(\Gamma)$, and the operator is simply pointwise multiplication by the function $Q_{(a,z)}$, which is everywhere between 0 and 1, is 1 near $a$, and is supported near $a$. Note that $\phi_{(a,z)}(b)(Q_{(a,z)}^2 - Q_{(a,z)})$ is a compact operator, for $b \in B$, as we discuss below, although it is certainly false that $Q_{(a,z)}^2 - Q_{(a,z)}$ itself is a compact operator.

**Proof.** We need check firstly that:

(1) $\gamma \phi_{(a,\mu)}(b)\gamma^{-1} = \phi_{(\gamma(a),\gamma(\mu))}(b)$;

(2) $\gamma P_{(a,\mu)}\gamma^{-1} = P_{(\gamma(a),\gamma(\mu))}$.

For the second assertion, we have

$$[\gamma P_{(a,\mu)}\gamma^{-1}\xi](z)(x) = \int_{\Gamma} Q(a, w, \gamma^{-1}x)\xi(\gamma w)(x)d\mu(w) = \int_{\Gamma} Q(\gamma(a), \gamma w, x)\xi(\gamma w)(x)d\mu(w)$$

and by equivariance of $Q$ this equals $[P_{(\gamma(a),\gamma(\mu))}\xi](z)(x)$, as required. Similar calculations verify the first assertion.
Next, we show that $P_{(a,\mu)}$ commutes mod compacts with the action of $B$. First let $F \in C_c(\partial^2 \Gamma)$. Then

$$\left[ P_{(a,\mu)} \phi_{(a,\mu)} (F) \right] (z)(x) = \int_{\Gamma} Q(a, w, x) \phi_{(a,\mu)}(F)(x) \xi(w)(x) d\mu(w)$$

and

$$\left[ \phi_{(a,\mu)}(F) P_{(a,\mu)} \right] (z)(x) = \phi_{(a,\mu)}(F)(x) \int_{\Gamma} Q(a, w, x) \xi(w)(x) d\mu(w).$$

Now let $x \to \infty$. For any $w \in \text{supp}(\mu)$ we have $d(z, w) \leq N$ and, noting this, if the scalar $\phi_{(a,\mu)}(F)(x) - \phi_{(a,\mu)}(F)(x) = F(x^{-1}(a), x^{-1}S_z(a)) - F(x^{-1}(a), x^{-1}S_w(a))$ does not converge to 0, it follows from $F \in C_c(\partial^2 \Gamma)$ and the usual argument, that the distance from $x$ to the geodesic $[S_z(a), S_w(a)]$ remains bounded, and hence that either $x \to S_z(a)$ or $x \to S_w(a)$. But in either case it follows from Lemma 4.7 that both $Q(a, z, x_n) \to 0$ and $Q(a, w, x_n) \to 0$. Hence the difference

$$\int_{\Gamma} Q(a, w, x) \phi_{(a,\mu)}(F)(x) \xi(w)(x) d\mu(w) - \phi_{(a,\mu)}(F)(x) \int_{\Gamma} Q(a, w, x) \xi(w)(x) d\mu(w)$$

converges to 0. To show the commutator $[\phi_{(a,\mu)}(\gamma), P_{(a,\mu)}]$ is compact follows immediately from the equivariance of all terms involved. In fact the commutator is 0.

Finally, we need show that $\phi_{(a,\mu)}(b)(P_{(a,\mu)}^2 - P_{(a,\mu)})$ is compact for all $b \in B$. It is sufficient to show this for $b = F \in C_c(\partial^2 \Gamma)$). We have:

$$\phi_{(a,\mu)}(F)P_{(a,\mu)}^2 \xi(x) = \int_{\Gamma} \int_{\Gamma} F(x^{-1}(a), x^{-1}(S_z(a))Q_{(a,\mu)}(x)Q_{(a,\mu)}(x)\xi(w')(x) d\mu(w) d\mu(w').$$

Let $x \to \infty$. Then

$$\int_{\Gamma} \int_{\Gamma} F(x^{-1}(a), x^{-1}(S_z(a))Q_{(a,\mu)}(x)Q_{(a,\mu)}(x)\xi(w')(x) d\mu(w) d\mu(w').$$
converges to 0 unless \( x \to a \) or \( x \to S_w(a) \) for at least one \( w \) appearing in the integrand. By Lemma 4.7 if the latter holds — that is, if \( x \to S_w(a) \) — then for every \( w \) in the integrand, \( Q_{(a,w)}(x) \to 0 \), since all such \( w \)'s are at distance \( \le N \) apart, and so in this case the entire integral converges to 0. So we may assume \( x \to a \). But then \( Q_{(a,w)}(x) \to 1 \) for every \( w \) and so the difference between the above integral and the integral

\[
\int_{\Gamma} F(x^{-1}(a), x^{-1}(S_z(a)))Q_{(a,w')}^1(x)\xi(w')(x)d\mu(w')
\]

converges to 0. But the latter integral is just

\[
\phi_{(a,\mu)}(F)P_{(a,\mu)}\xi(z)(x)
\]

and so we are done.

**Lemma 4.10.** Define an element \( a \in RKK_{-1}(\partial \Gamma \times X; \mathbb{C}, B) \) by the constant field of \( C^*(\mathbb{R}) \)-\( B \) bimodules \( E_{(a,z)} = u^*(E) \) where \( u : C^*(\mathbb{R}) \to C(GX/\Gamma) \otimes \mathbb{R} \) is the canonical inclusion. Then

\[
a \otimes_{\partial \Gamma \times X, B} b = 1_{\partial \Gamma \times X}.
\]

Before beginning the proof, let us introduce some notation. For distinct boundary points \( a \) and \( b \), let us denote by \([a, b]\) the fibre over \((a, b)\) in the map \( GX \to \partial^2 \Gamma \) provided by Axiom 1(4). Note that \([a, b]\) has a canonical affine structure, and hence there is in particular a canonical translation invariant measure on it corresponding to Lebesgue measure on \( \mathbb{R} \).

**Proof.** First we describe a certain (geometrically defined) cycle \( \gamma \) in \( RKK_{1}(\partial \Gamma; \mathbb{C}, \mathbb{C}) \), then argue that its class is the same as the class of \( 1_{\partial \Gamma \times X} \); lastly, we will verify that it also represents the product of \( a \) and \( b \). To simplify we will use field notation. We will first define the cycle as a
field of cycles over \( \partial \Gamma \times \Gamma \subset \partial \Gamma \times X \), then interpolate this field linearly over each simplex of the Rips complex to a field of cycles over \( \partial \Gamma \times X \), just as we did with the cycle \( b \) above.

For \( (a, z) \in \partial \Gamma \times \Gamma \) let \( \mathcal{H}_{(a, z)} = L^2([a, S_z(a)]) \) where we are using the measure space structure of \([a, S_z(a)]\) mentioned above. Each \( \mathcal{H}_{(a, z)} \) has a natural \( C^* (\mathbb{R}) \) action on the left, the assignment \( (a, z) \mapsto \mathcal{H}_{(a, z)} \) is continuous, and \( \Gamma \) acts on whole field \( \{ \mathcal{H}_{(a, z)} \mid (a, z) \in \partial \Gamma \times X \} \), with \( \gamma(\mathcal{H}_{(a, z)}) = \mathcal{H}(\gamma(a), \gamma z) \), because \( \gamma(S_z(a)) = S_{\gamma z}(\gamma(a)) \). Define an operator \( \tilde{P}_{(a, z)} \) on each \( \mathcal{H}_{(a, z)} \) by multiplication by the function

\[
\tilde{Q}_{(a, z)}(r) = Q_{(a, z)}(r(0))
\]

where \( Q \) is as above, and where we recall that \( r \mapsto r(0) \) is the map \( GX \to X \) provided by Axiom 1(4). Now as \( t \to -\infty \), \( g_t(r)(0) \to a \), and hence \( \tilde{Q}_{(a, z)}(g_t(r)) \to 1 \), by construction of \( Q_{(a, z)} \). Similarly as \( t \to \infty \), \( \tilde{Q}_{(a, z)}(g_t(r)) \to 0 \). It follows that for every \( a \) and every \( z \) and every \( \varphi \in C^* (\mathbb{R}) \), \( \varphi(\tilde{Q}^2 - \tilde{Q}) \) is a compact operator on \( \mathcal{H}_{(a, z)} \). The collection \( \{ (\mathcal{H}_{(a, z)}, \tilde{Q}_{(a, z)}) \} \) defines our field over the space \( \partial \Gamma \times \Gamma \).

It will be convenient, before interpolating this field of spaces and operators to a field of spaces and operators over \( X \), to first trivialize the field of Hilbert spaces. To this end, note that there exists a global section \( s : \partial^2 \Gamma \to GX \) of the map \( GX \to \partial^2 \Gamma \), by Axiom 1(1). Indeed, the fibres of this map have a canonical affine structure, and this allows us to form partitions of unity and thus manufacture out of a collection of local sections a global section. It follows that there is a continuous choice of a point \( r_a \in [a, S_e(a)] \) for each \( a \in \partial \Gamma \), \( e \) being the identity point in the group \( \Gamma \). From the assignment \( a \mapsto r_a \) we can construct an equivariant assignment \( (a, z) \mapsto r_{(a, z)} = z(r_a) \) of a point (pseudo-geodesic) on each \([a, S_z(a)]\). Since this selection is \( \Gamma \)-equivariant we can canonically (and equivariantly) identify each \([a, S_z(a)]\) with \( \mathbb{R} \), and hence \( \mathcal{H}_{(a, z)} \) with \( L^2 (\mathbb{R}) \). Under this identification, the \( \Gamma \) action becomes trivial, and the action of \( \tilde{Q}_{(a, z)} \) becomes multiplication by a function \( \chi_{(a, z)}(t) \) which for every \( (a, z) \) is 1 near \( -\infty \) and 0
near $+\infty$. The $C^*(\mathbb{R})$ action on each $L^2(\mathbb{R})$ is via the left regular representation. The field of pairs $\{(L^2(\mathbb{R}), \tilde{Q}_{(a,z)})\}$ forms a cycle for $RKK_1^\Gamma(\partial \Gamma \times \Gamma; C^*(\mathbb{R}), \mathbb{C})$, by a $\partial \Gamma \times \Gamma$-parameterized application of Corollary 2.3 in Section 2.1.

We are now ready to interpolate this field to a field over $\partial \Gamma \times X$ and show that the resulting element of $RKK_1(\partial \Gamma \times X; C^*(\mathbb{R}), \mathbb{C}) \cong RKK(\partial \Gamma \times X; \mathbb{C}, \mathbb{C})$ is $1_{\partial \Gamma \times X}$.

As previously, view points of $X$, i.e. of the realization of the Rips complex, as probability measures on $\Gamma$ with (finite) support of a fixed, bounded diameter $\leq N$. For one of these measures $\mu$ let $\mathcal{H}_{(a,\mu)} = L^2(\mathbb{R}) \otimes L^2_\mu(\Gamma) \cong L^2(\Gamma; L^2(\mathbb{R}))$. Define, for each $(a,\mu)$, an operator $\chi_{(a,\mu)}$ by $(\chi_{(a,\mu)}\xi)(z) = \chi_{(a,z)}(\xi(z))$, $\xi \in L^2(\Gamma; L^2_\mu(\mathbb{R}))$, where $\chi_{(a,z)}$ is as above. Finally, letting $P_\mu$ be projection onto the constant functions in $L^2_\mu(\Gamma)$, define $\tilde{P}_{(a,\mu)} = \chi_{(a,\mu)} \cdot (1 \otimes P_\mu)$. The action of $C^*(\mathbb{R})$ on $\mathcal{H}_{(a,\mu)}$ is $(\varphi\xi)(z) = \varphi(\xi(z))$, where the right hand side of this formula indicates the previously defined action of $\varphi$ via the left regular representation. This whole construction clearly is $\Gamma$-equivariant with respect to the action on the realization of the Rips complex. It is clear that, if $\varphi \in C^*(\mathbb{R})$, then $\varphi \cdot (\tilde{P}_{(a,\mu)}^2 - \tilde{P}_{(a,\mu)})$ is compact. Hence our field defines an element of $RKK_1(\partial \Gamma \times X; C^*(\mathbb{R}), \mathbb{C}) \cong RKK_1(\partial \Gamma \times X; \mathbb{C}, \mathbb{C})$ as claimed. We now show it is $1_{\partial \Gamma \times X}$, using a $\partial \Gamma \times X$-parameterized version of Corollary 2.3.

We first deform the field as follows. Form a homotopy of operators $\tilde{P}_{(a,\mu)}^t$ by the formula

$$[\tilde{P}_{(a,\mu)}^t \xi](z) = [(1 - t)\chi_{(a,z)} + t\chi_{(-\infty,0)}](\xi(z)).$$

It is easy to check this formula defines an operator homotopy in $RKK_1^\Gamma(\partial \Gamma \times X; C^*(\mathbb{R}), \mathbb{C})$, deforming the cycle $\gamma$ to the cycle given by the same field of Hilbert spaces, but with the field of operators given on $\mathcal{H}_{(a,\mu)}$ by $\chi_{(-\infty,0]} \otimes P_\mu$. By abuse of notation, we denote this cycle also by $\gamma$. Now, $P_\mu$ is a rank one projection. Let $\mu \mapsto \xi_\mu$ denote a continuous selection of a unit vector
in $L^2_\mu(\Gamma)$ for which $P_\mu \xi_\mu = \xi_\mu$. We have

$$\mathcal{H}_{(a,\mu)} = L^2(\mathbb{R}) \otimes [\xi_\mu] \oplus L^2(\mathbb{R}) \otimes L^2_\mu(\Gamma)^0,$$

where $L^2_\mu(\Gamma)^0$ denotes the functions in $L^2_\mu(\Gamma)$ with $\mu$-integral 0, and $[\xi_\mu]$ denotes the one dimensional linear subspace generated by $\xi_\mu$. With respect to this decomposition, the operator corresponding to our new deformed cycle is simply $\chi_{(-\infty,0]} \otimes 1 \oplus 0$, and the $C^*(\mathbb{R})$-action is diagonal. It follows that the deformed cycle is the direct sum of a degenerate cycle and the cycle given by the constant field of Hilbert spaces $L^2(\mathbb{R})$, and operators $\chi_{(-\infty,0]}$, with the usual $C^*(\mathbb{R})$-action. The class of the latter is $1_{\partial \Gamma \times X}$, by Corollary 2.3 in Section 2.1, and the class of the former is 0 in $RKK$, and so we are done: the class of $\gamma$ is $1_{\partial \Gamma \times X}$.

Next we show, using the axioms for a Kasparov product (see [20]), that the class of $\gamma$ is in fact the product of $a$ and $b$. Taking the product of the modules involved in $a$ and in $b$, we get the module $E \otimes_B L^2_\mu(\Gamma; l^2(\Gamma))$ with the inherited left action of $C^*(\mathbb{R})$ and a certain operator constructed out of the operators for $a$ and for $b$. Now consideration of the inner product on this module shows that it is in fact isomorphic to $\mathcal{H}_{(a,\mu)}$. The isomorphism is defined on the dense subset $C_c(GX) \otimes_B L^2_\mu(\Gamma; \mathbb{C})$ of $E \otimes_B L^2_\mu(\Gamma; l^2(\Gamma))$ by the composition

$$C_c(GX) \otimes_B L^2_\mu(\Gamma; \mathbb{C}) \cong (C_c(GX) \otimes_B \mathbb{C}) \otimes L^2_\mu(\Gamma)

\rightarrow C_c(GX) \otimes_B L^2_\mu(\Gamma) \rightarrow \oplus_{z \in \text{supp} \mu} C_c([a, S_z(a)]) \rightarrow \mathcal{H}_{(a,\mu)},$$

where the second to last map is restriction of a function $\xi \in C_c(GX)$ to $\cup_{z \in \text{supp} \mu} [a, S_z(a)]$. This composition is isometric with respect to the various Hilbert module norms.

Hence, since the operator for the element $a$ is 0, it suffices to show that $\tilde{P}_{(a,\mu)}$ is a $P_{(a,\mu)}$-connection. Let $\xi \in C_c(GX) \subset E$ and $\theta_\xi$ denote the operator $l^2(\Gamma) \otimes L^2_\mu(\Gamma) \rightarrow E \otimes_B (l^2(\Gamma) \otimes$
\(L^2_\mu(\Gamma)\), \(\eta \mapsto \xi \otimes_B \eta\). By [20] we need show that

\[\eta \mapsto \tilde{P}_{(a,\mu)}(\xi \otimes_B \eta) - \xi \otimes_B P_{(a,\mu)}(\eta)\]

is a compact operator. Assume \(\eta \in L^2_\mu(\Gamma; C^c_c(\Gamma))\) for the purpose of this calculation. Thus, for every \(z \in \text{supp}(\mu)\), \(\eta(z) \in \mathbb{C}\Gamma\). Let \(\zeta = \tilde{P}_{(a,\mu)}(\xi \otimes_B \eta) - \xi \otimes_B P_{(a,\mu)}(\eta)\). We have

\[\zeta(w)(r) = \int_{\Gamma} Q_{(a,z)}(r(0))(\xi \cdot \eta(z))(r)d\mu(z) - (\xi \cdot P_{(a,\mu)}\eta)(z)(r)\]

Let us write \(\eta(z) = \sum_i \alpha_i(z)e_{x_i}\). Then \(\zeta(w)(r)\) can be written

\[\sum_i \alpha_i(w)\int_{\Gamma} (Q_{(a,z)}(r(0)) - Q_{(a,z)}(x_i)\xi(x^{-1}_i(r))d\mu(z)\]

It is now evident that it suffices to show that for \(x \to \infty\), the \(L^2\)-norm of the function \(h = h(r) = (Q_{(a,z)}(r(0)) - Q_{(a,z)}(x))\xi(x^{-1}(r))\) of \(H_{(a,z)}\) converges to 0, as \(x \to \infty\). Choose \(\epsilon > 0\). As \(\xi \in C^c_c(GX)\), there exists some \(R\) for which \(\xi(r) = 0\) unless \(r(0) \in B_R(x_0)\). On the other hand, there exists \(C > 0\) such that if \(d(e, x) \geq C\) and \(r(0) \in B_R(x)\), then \(|Q_{(a,z)}(r(0)) - Q_{(a,z)}(x)| < \epsilon\), by uniform continuity of \(Q_{(a,z)}\). It follows that for \(d(e, x) \geq C\) and \(r \in [a, S_x(a)]\), either \(h(r) = 0\) or \(|h(r)| < \epsilon\|\xi\|_E\). Consequently, for \(x\) sufficiently large, \(\|h\|_{H_{(a,z)}} < \epsilon\|\xi\|_E\), and we are done.

**Lemma 4.11.** We have

\[a \otimes_{\partial \Gamma \times X, B} b = p^*_X(c) \in RKK_{\Gamma}(\partial \Gamma \times X; \mathbb{C}, \mathbb{C})\]

and hence \(p^*_X(c) = 1_{\partial \Gamma \times X}\).
Proof. The class of $c$ can be represented by an equivariant map $C^*(\mathbb{R}) \otimes C(\partial \Gamma) \to C(\partial \Gamma) \otimes Q(l^2(\Gamma) \otimes V)$, namely, $\varphi \otimes f \mapsto (f \otimes id_{l^2(\Gamma) \otimes V}) \cdot (\tau \otimes id_{K(V)})(\hat{\Delta}(\varphi))$ where $\hat{\Delta}$ is viewed as a homomorphism $C^*(\mathbb{R}) \to B \otimes K(V)$.

Hence $p_X^*(c)$ can be viewed as a map $C_0(\partial \Gamma \times X) \otimes C^*(\mathbb{R}) \to Q(C_0(\partial \Gamma \times X) \otimes l^2(\Gamma) \otimes V)$ via $f \otimes \varphi \mapsto (f \otimes id_{l^2(\Gamma) \otimes V}) \cdot (\tau_1 \otimes id_{K(V)})(\hat{\Delta}(\varphi))$, where $\tau_1 : B \to Q(C_0(\partial \Gamma \times X) \otimes l^2(\Gamma))$ is the composition

$$B \xrightarrow{\tau} C(\partial \Gamma) \otimes Q(l^2(\Gamma)) \to C_0(\partial \Gamma \times X) \otimes Q(l^2(\Gamma))$$

induced by the map $C(\partial \Gamma) \to C(\partial \Gamma \times X)$ taking a function $f$ on $\partial \Gamma$ to the function $f(a, x) = f(a)$ on $\partial \Gamma \times X$.

Now we show that this decomposes into two equivariant cycles. The first is the homomorphism $C_0(\partial \Gamma \times X) \otimes C^*(\mathbb{R}) \to C_0(\partial \Gamma \times X) \otimes B \otimes K(V)$ given by $1_{C_0(\partial \Gamma \times X)} \otimes \hat{\Delta}$. Here the $\Gamma$ action is trivial on $B \otimes K(V)$ and on $C^*(\mathbb{R})$, and the usual one on $\partial \Gamma \times X$.

As for the second, it is clear that there is a $\Gamma$ equivariant homomorphism $C_0(\partial \Gamma \times X) \otimes B \otimes K(V) \to Q(C_0(\partial \Gamma \times X) \otimes l^2(\Gamma) \otimes V)$ given essentially by the above formula: $f \otimes b \otimes T \mapsto (f \otimes id_{l^2(\Gamma) \otimes V}) \cdot (\tau_1(b) \otimes T)$. This defines an equivariant cycle in, say, $E$-theory; but to show it defines one in $KK$ (in fact in $RKK$) it is necessary to produce an equivariant completely positive lifting, or, what is equivalent, a larger Hilbert space with a homomorphic lifting, and a projection compressing it to the original map in the Calkin algebra (see, e.g. [3].) This can however be accomplished precisely by means of the cycle $b$ described earlier. We need, in otherwords, show only that $b$ defines the same homomorphism $C_0(\partial \Gamma \times X) \otimes B \otimes K(V) \to Q(C_0(\partial \Gamma \times X) \otimes l^2(\Gamma \otimes V))$ as the above formula. It is clear that both cycles consist of fields over $\partial \Gamma \times X$ (the former is the constant field) and so we work in terms of this fields. Fix a point $(a, \mu) \in \partial \Gamma \times X$. We need compare the homomorphisms $B \otimes K(V) \to Q(l^2(\Gamma) \otimes V)$, $b \otimes T \to \tau_1(b) \otimes T$, given above, and the homomorphism $B \to Q(l^2(\Gamma) \otimes l^2(\Gamma))$ given by $b \mapsto P_{(a, \mu)}^0(b)P_{(a, \mu)}$. For purposes of comparing them we can drop the $K(V)$ term in the first homomorphism, so that we are reduced
to comparing the homorphisms into the Calkin algebra of \( l^2(\Gamma) \), \( b \mapsto P_{(a,\mu)} \phi_{(a,\mu)} P_{(a,\mu)} \) and \( b \mapsto \tau_1(b) \). Define an isometry \( U_{(a,\mu)} : l^2(\Gamma) \to H_{(a,\mu)} \) by \( (U_{(a,\mu)} \xi)(w) = \xi \) for all \( w \in \text{supp} \mu \).

We want to show that the operator

\[
T_b(\xi) = U_{(a,\mu)}^* P_{(a,\mu)} \phi_{(a,\mu)}(b) P_{(a,\mu)} U_{(a,\mu)} \xi - \tilde{\tau}(b) \xi
\]

is a compact operator on \( l^2(\Gamma) \). If \( b = \gamma \), \( T_b \) is the zero operator, and so we can pass to the case \( b = F \in C_c(\partial^2 \Gamma) \). In this case, a short calculation shows that \( T_b \) is a diagonal operator, and, moreover, that to show it is compact it is enough to show that as \( x \to \infty \),

\[
\int_{\Gamma} Q(a, w, x) F(x^{-1}(a), x^{-1} S_w(a)) \, d\mu(w) - \tilde{F}(x^{-1}(a), x^{-1}) \to 0.
\]

Firstly, if \( x \to a \), then for large enough \( x \), \( Q(a, w, x) = 1 \) for all \( w \in \text{supp} \mu \), and hence the difference between the above integral and the integral

\[
\int F(x^{-1}(a), x^{-1} S_w(a)) - \tilde{F}(x^{-1}(a), x^{-1}) \, d\mu(w)
\]

converges to 0 as \( x \to a \). Considering the latter integral, for every \( w \) in the integrand we certainly have \( d_F(x^{-1} S_w(a), x^{-1}) \to 0 \) as \( x \to \infty \), else we would have by the usual argument that for some \( w \), the distance from \( x \) to the ray \([e, S_w(a)]\) remains bounded, which would imply \( x \to S_w(a) \), thus contradicting \( x \to a \) and \( a \neq S_w(a) \). Hence for every \( w \) in the integrand \( d_F(x^{-1} S_w(a), x^{-1}) \to 0 \) as claimed, and so the integral converges to 0 by continuity of \( \tilde{F} \) in the second variable. If \( x \) does not converge to \( a \), it follows that \( \tilde{F}(x^{-1}(a), x^{-1}) \to 0 \), and we need only show the integral also converges to 0. If it does not, for at least one \( w \), say \( w_1 \), \( d_F(x^{-1}(a), x^{-1} S_{w_1}(a)) \) does not converge to 0, whence \( x \to a \) or \( S_{w_1}(a) \). By assumption \( x \) does not converge to \( a \) so it converges to \( S_{w_1}(a) \). But then by Lemma 4.7, for all \( w \) in the support of \( \mu \), \( Q(a, w, x) \to 0 \), and we are done.
We now conclude the proof by some general arguments.

**Definition 4.12.** Let \( Y \) and \( Z \) be locally compact \( \Gamma \) spaces (where \( \Gamma \) is an arbitrary discrete group.) Define a map

\[
p^*_Y : RKK(Z; \mathbb{C}, \mathbb{C}) \to RKK(Y \times Z; \mathbb{C}, \mathbb{C})
\]

by replacing a cycle \((H, F)\) by the cycle \((C_0(Y) \otimes H, 1_{C_0(Y)} \otimes F)\).

**Lemma 4.13.** Let \( \Gamma \) be a torsion-free hyperbolic group and \( X \) denote its classifying space and \( \partial \Gamma \) its Gromov boundary. Then the map \( p^*_X \) defines a ring isomorphism

\[
RKK_{\Gamma}(\partial \Gamma; \mathbb{C}, \mathbb{C}) \to RKK_{\Gamma}(X \times \partial \Gamma; \mathbb{C}, \mathbb{C})
\]

**Proof.** By Tu’s Theorem (see [29]) there exists a \( \Gamma - C_0(X \times \partial \Gamma) \)-algebra \( D \) and elements \( \alpha \in RKK_{\Gamma}(\partial \Gamma; C(\partial \Gamma), D) \), \( \beta \in RKK_{\Gamma}(\partial \Gamma; D, C_0(X \times \partial \Gamma)) \), satisfying \( \alpha \otimes_{\partial \Gamma, D} \beta = 1_{\partial \Gamma} \in RKK_{\Gamma}(\partial \Gamma; C(\partial \Gamma), C(\partial \Gamma)) \cong RKK_{\Gamma}(\partial \Gamma; \mathbb{C}, \mathbb{C}) \) and \( \beta \otimes_{\partial \Gamma} \alpha = 1_{\partial \Gamma, D} \in RKK_{\Gamma}(\partial \Gamma; D, D) \). We can use \( \alpha \) and \( \beta \) to define a map backwards by the following composition. First map

\[
\sigma_{X \times \partial \Gamma, D} : RKK_{\Gamma}(X \times \partial \Gamma; \mathbb{C}, \mathbb{C}) \to RKK_{\Gamma}(\partial \Gamma, D, D)
\]

by replacing a pair \((H, F)\) by the pair \((H \otimes_{C_0(\partial \Gamma \times X)} D, F \otimes 1)\). Then define a map

\[
RKK_{\Gamma}(\partial \Gamma; D, D) \to RKK_{\Gamma}(\partial \Gamma; C(\partial \Gamma), C(\partial \Gamma)) \cong RKK_{\Gamma}(\partial \Gamma; \mathbb{C}, \mathbb{C})
\]

by product on the left by \( \alpha \), followed by product on the right by \( \beta \). We claim the composition of these two maps provides an inverse to \( p^*_X \). Because the proof is simply a \( \partial \Gamma \)-parameterized
version of the corresponding statement for $\partial \Gamma = \text{pt}$ we prove the latter for simplicity. From this point of view we have $\alpha \in KK_{\Gamma}(\mathbb{C}, D)$, $\beta \in KK_{\Gamma}(D, \mathbb{C})$, and we have mapsto $p^*_X : KK_{\Gamma}(\mathbb{C}, \mathbb{C}) \to KK_{\Gamma}(C_0(X), C_0(X))$ and backwards given by product with $\alpha$ and $\beta$ as above. The composition

$$KK_{\Gamma}(\mathbb{C}, \mathbb{C}) \to KK_{\Gamma}(C_0(X), C_0(X)) \to KK_{\Gamma}(\mathbb{C}, \mathbb{C})$$

is easily seen to be the identity; one checks if $\epsilon \in KK_{\Gamma}(\mathbb{C}, \mathbb{C})$, that $\alpha \otimes_D \sigma_{X,D}(p^*_X(\epsilon) \otimes_D \beta)$ is simply $\alpha \otimes_D \sigma_D(\epsilon) \otimes_D \beta = \alpha \otimes_D \beta \otimes_C \epsilon = \epsilon$ by commutativity of the external tensor product. The composition

$$KK_{\Gamma}(C_0(X), C_0(X)) \to KK_{\Gamma}(\mathbb{C}, \mathbb{C}) \to KK_{\Gamma}(C_0(X), C_0(X))$$

is slightly more elaborate. Let $\eta \in RKK_{\Gamma}(X; \mathbb{C}, \mathbb{C}) \cong KK_{\Gamma}(\mathbb{C}, C_0(X))$. Remark that $\alpha$ has the form (see [29]) $(D, M)$ where $M$ is a certain self-adjoint multiplier of $D$. Now it is easy to see that

$$\alpha \otimes_D \sigma_{X,D}(\eta) = \eta \otimes_{C_0(X)} \hat{\alpha}$$

where $\hat{\alpha} \in KK_{\Gamma}(C_0(X), D)$ is again represented by the pair $(D, M)$ but with the canonical left action of $C_0(X)$ given by the $C(X)$ structure of $D$. Now, consider

$$p^*_X(\alpha \otimes_D \sigma_{X,D}(\eta) \otimes_D \beta) = p^*_X(\alpha \otimes_D \sigma_{X,D}(\eta)) \otimes_D p^*_X(\beta)$$

and consider in particular the term

$$p^*_X(\alpha \otimes_D \sigma_{X,D}(\eta)) = p^*_X(\eta \otimes_{C_0(X \times X)} p^*_X(\hat{\alpha})) \in RKK_{\Gamma}(X; \mathbb{C}, D)$$

We would like to say that this is equal to $\eta \otimes_{C_0(X)} p^*_X(\alpha)$ and then would be done. However, the two modules involved are different. Namely, if the module for $\eta$ is $E$, the former has module
\[ E \otimes C_0(X) \otimes (C_0(X) \otimes D) \] and the latter has the module given by the same formula, except that the tensoring over \( C_0(X) \) involves in the former case the homomorphism

\[ C_0(X) \to \mathcal{M}(C_0(X \times X)) \]

given by multiplication in the second coordinate, and in the latter case, in the first coordinate.

However, the two projection maps \( X \times X \to X \) in the case of \( X \) the classifying space for \( \Gamma \), are \( \Gamma \)-equivariantly homotopic, and this gives a homotopy of Hilbert modules, and the result. (See [20], pg. 179 for the identical sort of argument.)

\[ \square \]

**Corollary 4.14.** The element \( c \) is \( 1_{\partial \Gamma} \in RKK_\Gamma(\mathbb{C}, \mathbb{C}) \).

*Proof.* By Lemma 4.11, \( p_X^*(c) = 1_X \) and by Lemma 4.13 \( p_X^* \) is a ring isomorphism and in particular takes units to units. Hence \( c = 1_{\partial \Gamma} \) also.

\[ \square \]

To summarize: we have proven that \( \hat{\Delta} \otimes A^{\text{op}} \Delta = 1_A \). To restate Theorem 4.1, we have:

**Theorem 4.15.** The elements \( \hat{\Delta} \) and \( \Delta \) implement a Poincaré Duality isomorphism between \( A \) and \( A^{\text{op}} \). Hence the \( C^* \)-algebra \( A \) is a non-commutative manifold in the sense of Definition 2.8.

*Proof.* This follows from our calculation that \( \hat{\Delta} \otimes A^{\text{op}} \Delta = 1_A \), together with Corollary 3.17.

\[ \square \]
Chapter 5

A Remark on Poincaré Duality and Assembly

In this chapter we discuss a relationship between the duality maps defined above, and the Baum-Connes map without coefficients. The relationship was first observed by Jerry Kaminker and Ian Putnam in the context of geodesic flow on a Riemann surface; Poincare Duality can be reformulated in these terms using Ruelle algebras (see the introduction) and the commutativity of the diagram in Theorem 5.1 was verified by them. The only contribution of the author in this context is to use a calculation of Nigel Higson [14] to verify this commutativity in the general context of hyperbolic groups. As a consequence, one sees that the Poincaré Duality maps defined in this paper represent a sort of assembly map ‘at infinity.’ This idea is further born out by Higson’s proof of the Novikov conjecture ([14]).

Observe that the exact sequence

$$0 \longrightarrow C_0(EG) \overset{i}{\longrightarrow} C(EG) \overset{r}{\longrightarrow} C(\partial G) \longrightarrow 0$$

is $\Gamma$-equivariant, and so induces a sequence of $K$-homology groups

$$\cdots \rightarrow KK^*_\Gamma(C(EG), \mathbb{C}) \rightarrow KK^*_\Gamma(C_0(EG), \mathbb{C}) \overset{\partial}{\rightarrow} KK^{*+1}_\Gamma(C(\partial G), \mathbb{C}) \rightarrow \cdots$$

which is exact by virtue, firstly, of the fact that for any separable, nuclear $C^*$-algebra $D$, $KK_\Gamma(D, \mathbb{C}) \cong KK(D \rtimes \Gamma, \mathbb{C})$ (see [21]), and secondly that $C(EG) \rtimes \Gamma$, $C_0(EG) \rtimes \Gamma$ and $C(\partial G) \rtimes \Gamma$ are all nuclear (see [3].)
**Theorem 5.1.** The following diagram commutes, where \( \Gamma \) is a torsion free hyperbolic group, \( \partial \) is the connecting map in the above long exact sequence of \( K \)-homology groups, and \( i \) is the inclusion \( C^*_r(\Gamma) \to C(\partial \Gamma) \times \Gamma \)

\[
\begin{array}{ccl}
K_*(B\Gamma) & \xrightarrow{\mu} & K_*(C_r(\Gamma)) \\
\downarrow \partial & & \downarrow i \\
K_{*+1}(C(\partial \Gamma) \rtimes \Gamma) & \xrightarrow{\hat{\Delta}^*} & K_*(C(\partial \Gamma) \rtimes \Gamma)
\end{array}
\]

**Proof.** First remark it suffices by Theorem 4.1 to show that the following diagram commutes:

\[
\begin{array}{ccl}
K_*(B\Gamma) & \xrightarrow{\mu} & K_*(C_r(\Gamma)) \\
\downarrow \partial & & \downarrow i \\
K_{*+1}(C(\partial \Gamma) \rtimes \Gamma) & \xleftarrow{\Delta^*} & K_*(C(\partial \Gamma) \rtimes \Gamma)
\end{array}
\]

By [14] the following diagram commutes:

\[
\begin{array}{ccl}
KK^*\Gamma(C_0(ET), \mathbb{C}) & \xrightarrow{\mu} & K_*(C_r(\Gamma)) \\
\downarrow = & & \downarrow \otimes K^*_{r}(\Gamma) \hat{\Delta} \\
KK^*\Gamma(C_0(ET), \mathbb{C}) & \xleftarrow{\Delta} & KK_{r+1}(C(\partial \Gamma), \mathbb{C})
\end{array}
\]

where \( \hat{\Delta} \) is the equivariant map \( C^*_r(\Gamma) \otimes C(\partial \Gamma) \to \mathbb{Q}(l^2(\Gamma), x \otimes f \mapsto \rho(x)\lambda(f), \) using the notations of the previous section. In other words we have \( \mu(x) \otimes C^*_r(\Gamma) \hat{\Delta} = \partial(x). \)

We wish to show

\[
\nu([\partial]) \otimes_{C_0(ET) \rtimes \Gamma} u_*(x) = \sigma_1(\mu(x)) \otimes C^*_r(\Gamma) \otimes A \ (i \otimes 1_A)^*(\Delta).
\]

Now clearly \( (i \otimes 1_{C(\partial \Gamma)})^*(u_*^{-1}(\Delta)) = \hat{\Delta}. \) Hence \( \mu(x) \otimes C^*_r(\Gamma) \ (i \otimes 1_{C(\partial \Gamma)})^*(u_*^{-1}(\Delta)) = \partial(x). \)

The right hand side of this equation is clearly \( u_*^{-1}(\nu([\partial])) \otimes_{C_0(ET) \rtimes \Gamma} u_*(x) \) and the left hand side is \( u_*^{-1}(\sigma_1(\partial \Gamma) \otimes C^*_r(\Gamma) \otimes C(\partial \Gamma) \otimes (i \otimes 1_A)^*(\Delta)) \), and so we are done.

\[\square\]
References


Vita

Heath Emerson was born in Edmonton, Albert, Canada in 1970. After beginning his academic career in Economics, he became interested in philosophy and mathematical logic and subsequently switched his principal field to Mathematics. He has since earned a Bachelor’s of Science degree in mathematics with a minor in Economics from the University of Victoria, Canada (1994) and a Master’s of Science degree also from the University of Victoria (1996). He has been awarded two consecutive National Science and Engineering Council of Canada Fellowships, and a Curry Fellowship from Penn State University. In the year 2000 he married Sarah Hoyon, and they are as yet still married. They plan to live in France. Heath has a great penchant for travel, and enjoys the piano. He also has an interest in literature, which he hopes to pursue following the completion of his doctoral studies. Since 1996 he has been pursuing an elusive PhD from the Pennsylvannia State University.