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NONPARAMETRIC IDENTIFICATION AND SEMI-NONPARAMETRIC ESTIMATION OF FIRST-PRICE AUCTIONS

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by

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Abstract

In this thesis, we study nonparametric identification of first-price auction models and propose a semi-nonparametric simulated integrated moment estimation method to recover the underlying value distribution.[1]

In the first essay, we investigate the nonparametric identification of the first-price auction model. In most cases in the nonparametric auction literature, the support of the bidders' values is assumed to be bounded. We show via an alternative nonparametric identification proof that the boundedness assumption can be relaxed to the condition that the value distribution has a finite expectation. In the first instance, we show this for the case of independent and identical first-price auctions, and then we extend the proof to the case of first-price auctions with observed auction-specific heterogeneity. Also, we consider the case where the log of the values is modeled as a median regression model, and the case where the bidders know ex-ante the actual number of bidders rather than the number of potential bidders.

In the second essay, we propose a semi-nonparametric simulated integrated moment (SNP-SIM) to estimate the value distribution of independently repeated identical first-price auctions. First, we construct an increasing sequence of compact metric spaces of distribution functions (the sieve), based on the approach in Bierens (2007). Given a candidate value distribution function in the sieve, we simulate bids according to the equilibrium bid function involved. We take the difference of the empirical characteristic

[1]The three essays in this thesis are co-authored with Herman Bierens.

functions of the actual and simulated bids as the moment function. The objective function is then the integral of the squared moment function over an interval. Minimizing this integral to the distribution functions in the sieve then yields a uniformly consistent semi-nonparametric estimator of the actual value distribution. Also, we propose an integrated moment test for the validity of the first-price auction model, and a data-driven method for the choice of the sieve order. Finally, we conduct a few numerical experiments to check the performance of our approach.

In the third essay, we propose to estimate first-price auction models with observed auction-specific heterogeneity via a semi-nonparametric simulated integrated conditional moment (SNP-SICM) method. The auction-specific heterogeneity will be incorporated via a median regression model for the log values with unknown error distribution. The latter distribution will be modeled semi-nonparametrically using orthonormal Legendre polynomials, similar to the approach in Bierens (2007). Given a parametric specification of the median function, the semi-nonparametric conditional value distribution involved can be estimated consistently by minimizing the integrated square distance between the empirical characteristic functions of the actual bids and the simulated bids, together with the covariates, via an integrated conditional moment criterion. This approach yields as a by-product an integrated conditional moment test for the validity of the model. Moreover, we apply the SNP-SICM estimation method to the US timber auction data and test the validity of the first-price auction model for this data.

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Chapter 1

Identification of First-Price Auction Models with Unbounded Values and Observed Auction-Specific Heterogeneity

1.1 Introduction

In this paper we show the nonparametric identification of first-price sealed bid auction models under mild conditions, where the values of the potential bidders are independent and private and bidders are ex-ante identical, possibly conditional on observed auction specific covariates. This is known as the Independent Private Values (IPV) paradigm. Moreover, we assume risk neutrality. Furthermore, we assume that the bids are unsealed after the auction, and are therefore ex-post observable. In the sequel we call this type of auctions shortly "first-price auctions". Three situations will be investigated in the paper. The first one is the (not very realistic) case in which there are independent and identical auctions with the same known number of potential bidders. The second is the more realistic case where auction-specific characteristics are observed and the number of potential bidders and the reservation price are allowed to change. The third is the case where the reservation price is binding and the bidders know ex-ante the actual number of bidders rather than the number of potential bidders.

There are two seminal papers on the identification of first-price auction models, namely Donald and Paarsch (1996) and Guerre, Perrigne and Vuong (2000). Of course, parametric identification had been developed earlier. In particular, Laffont, Ossard and Vuong (1995) specify the conditional distribution of the log of the private values as normal with conditional mean a linear function of covariates.

Donald and Paarsch (1996) show the nonparametric identification of first-price auctions under the assumption that the support of the distribution $F(v)$ of the values is a known bounded interval $[\underline{v}, \overline{v}]$, i.e., $F(v)$ is absolutely continuous with density f such that $f(v) > 0$ on $(\underline{v}, \overline{v})$, and $F(\underline{v}) = 0$, $F(\overline{v}) = 1$. They identify the value distribution and the risk aversion parameter using the family of hyperbolic absolute risk aversion (HARA) utility functions. Particularly they identify the distribution $F(v)$ using a fixed upper bound.

As is well-known[1], the symmetric Nash equilibrium bid function of first-price auctions without binding reservation price takes the form

$$b = \beta(v) = v - \frac{1}{F(v)^{I-1}} \int_{\underline{v}}^{v} F(x)^{I-1} dx, \tag{1.1}$$

where I is the number of potential bidders, $F(v)$ is the value distribution and $\underline{v}$ is the lower bound of its support. In first-price auctions under the symmetric IPV paradigm, the existence and uniqueness of this symmetric Nash equilibrium is guaranteed by the finite expectation condition $\int_{\underline{v}}^{\infty} v dF(v) < \infty$, which in its turn is guaranteed by the bounded support assumption. The latter implies of course that the bids are also bounded, with lower bound $\underline{b} = \beta(\underline{v}) = \underline{v}$ and upper bound $\overline{b} = \beta(\overline{v})$.

[1] See for example Krishna (2002).

However, unlike Donald and Paarsch (1996), Guerre, Perrigne and Vuong (2000) use the inverse bid function

$$v = \beta^{-1}(b) = b + \frac{1}{I-1}\frac{\Lambda(b)}{\lambda(b)}$$

where $\Lambda(b)$ is the distribution function of the bids and $\lambda(b)$ is the corresponding density. Since the bids are observable, the bids distribution $\Lambda(b)$ and its density $\lambda(b)$ may be considered given, because they can be estimated nonparametrically. Therefore, the private values can be recovered from the bids and their distribution. Note that they do not use the boundedness of values in their identification proof even though they denote the support of the values by $[\underline{v}, \overline{v}]$. Thus, their identification proof is still valid if $\overline{v} = \infty$.

The nonparametric approach of Guerre, Perrigne and Vuong (2000) has been extended by Athey and Haile (2002, 2006a-b) to more general auction models. See Milgrom and Weber (1982) for the latter. Li, Perrigne and Vuong (2000) have extended the nonparametric approach to the conditionally independent private value (CIPV) model, under the assumption that each private value is the product of an idiosyncratic component and a common component. Li and Perrigne (2003) study first-price auctions with random reservation price and show the nonparametric identification of this model. Campo et al. (2002) consider the case of risk averse bidders.

It is frequently assumed in the nonparametric auction literature that the value distribution has a known bounded support. In this paper, we show via an alternative nonparametric identification proof that this assumption is superfluous at least for first-price auctions with independent private value, provided that the value distribution has a finite expectation. In the first instance we show this, in Section 2, for the case of independent and identical first-price auctions, and then we extend the proof in Section 3 to the case of first price auctions with observed auction-specific heterogeneity. Also, we consider the case where the log of the values is modeled as a median regression model

The standard assumption of first-price auction models is that the number of potential bidders is ex-ante known to the bidders and ex-post to the econometrician as well. The latter is often not the case in practice if the reservation price is binding. Therefore, in Section 4, we consider the case where the bidders know ex-ante the actual number of bidders, i.e., the number of bidders with a value larger than the reservation price, rather than the number of potential bidders. Finally, in Section 5, we will sketch how we plan to use these results in our research on semi-nonparametric estimation of the (conditional) value distribution of first-price auctions.

1.2 Independent identical first-price auctions

The case where identical first-price auctions are repeated independently is of limited practical interest, but we will consider this case here to illustrate the main ideas behind our alternative identification proof. The more realistic case of first-price auctions with observed auction-specific heterogeneity will be considered in the next section.

1.2.1 The bid function

Suppose there are I ex-ante identical bidders and there is an indivisible object to sell. Assume that bidders are risk-neutral. Bidders' values are assumed to be independent and private. Moreover, the bidders' values V follow a distribution $F(v)$ which is absolutely continuous. Then, given the seller's reservation price p_0 which is announced in advance, the equilibrium bid of a bidder with value v is[2]

$$\beta(v) = v - \frac{1}{F(v)^{I-1}} \int_{\max(p_0, \underline{v})}^{v} F(x)^{I-1} dx, \;\; v > \max(p_0, \underline{v}), \tag{1.2}$$

where I is the number of potential bidders, which is assumed to be known, and

$$\underline{v} = \inf_{F(v)>0} v \tag{1.3}$$

is the lower bound of the support of the private values distribution F. See Riley and Samuelson (1981) or Krishna (2002) for the derivation of (1.2).

We do not restrict $\underline{v}$ to be positive valued, nor do we assume that $\underline{v}$ is known. If $\underline{v} > 0$ and the seller sets the reserve price p_0 below $\underline{v}$, so that p_0 is non-binding, or if there is no reservation price ($p_0 = 0$), every potential bidder will enter the auction. This case is observable because then the number of bids equals the number I of potential bidders.

On the other hand, if the reservation price p_0 is binding, $p_0 > \underline{v}$, only potential bidders with value $V > p_0$ will enter the auction and make their bids. This case is observable because then the actual number of bids, I^*, is less than the number I of potential bidders.

[2]See the Appendix for the derivation of the similar expression (1.16).

Note that in the case of a binding reservation price p_0, (1.2) can be written as

$$\begin{aligned}
\beta(v) &= v - \frac{v - p_0}{F(v)^{I-1}} + \frac{1}{F(v)^{I-1}} \int_{p_0}^{v} \left(1 - F(x)^{I-1}\right) dx \\
&= \frac{vF(v)^{I-1} - v + p_0}{F(v)^{I-1}} + \frac{1}{F(v)^{I-1}} \int_{p_0}^{v} \left(1 - F(x)^{I-1}\right) dx \\
&= \frac{p_0}{F(v)^{I-1}} + \frac{\int_{p_0}^{v} \left(1 - F(x)^{I-1}\right) dx - v\left(1 - F(v)^{I-1}\right)}{F(v)^{I-1}} \\
&= \frac{\int_{p_0}^{v} \left(1 - F(x)^{I-1} - \frac{d}{dx}\left(x\left(1 - F(x)^{I-1}\right)\right)\right) dx}{F(v)^{I-1}} \\
&\qquad + \frac{p_0 - p_0(1 - F(p_0)^{I-1})}{F(v)^{I-1}} \\
&= (I-1)\frac{\int_{p_0}^{v} xF(x)^{I-2}F'(x)dx}{F(v)^{I-1}} + p_0\frac{F(p_0)^{I-1}}{F(v)^{I-1}}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\lim_{v\to\infty} \beta(v) &= (I-1)\int_{p_0}^{\infty} xF(x)^{I-2}F'(x)dx + p_0 F(p_0)^{I-1} \\
&\leq (I-1)\int_{0}^{\infty} xF'(x)dx + p_0 = (I-1)E[V] + p_0, \qquad (1.4) \\
\lim_{v\to\infty} \beta(v) &\geq (I-1)\left(E[V] - \int_{0}^{p_0} xF'(x)dx\right).F(M)^{I-2} + p_0 F(p_0)^{I-1} \text{ for } M > p_0.
\end{aligned}$$

This proves that:

Lemma 1. *If the value distribution $F(v)$ is absolutely continuous, with density $f(v)$, then $\lim_{v\to\infty} \beta(v) < \infty$ if and only if $\int_0^{\infty} vf(v)dv < \infty$.* [3]

[3] Note that Lemma 1 also follows from equation (6) in Li and Vuong (1997).

Moreover, under the conditions of Lemma 1 the expected revenue of the seller, $I\int_{p_0}^{\infty}\beta(v)$ $f(v)dv$, is finite too. Therefore, our identification analysis will be conducted under the conditions of Lemma 1:

Assumption 1. *The value distribution is absolutely continuous, with finite expected value.*

1.2.2 Non-binding reservation price

In an auction with a non-binding reservation price, we may without loss of generality assume that the seller sets $p_0 = 0$ so that the bid function (1.2) becomes (1.1). The problem is that this bid function depends on $\underline{v}$, which is unknown. However, if we replace the nonrandom argument v in (1.1) with a random drawing V from $F(v)$ we do not need to bother about $\underline{v}$, because then

$$\beta(V) = V - \frac{1}{F(V)^{I-1}}\int_{\underline{v}}^{V} F(x)^{I-1}dx = V - \frac{1}{F(V)^{I-1}}\int_{0}^{V} F(x)^{I-1}dx$$

a.s.[4], due to the fact $P[V > \underline{v}] = 1$ and thus $P[F(V) > 0] = 1$

Now suppose that there exist two distinct value distribution $F_*(v)$ different from $F(v)$ such that, with V a random drawing from $F(v)$ and V_* a random drawing from

[4]a.s. stands for almost surely, or equivalently, with probability 1.

$F_*(v)$, such that

$$\beta_*(V_*) = V_* - \frac{1}{F_*(V_*)^{I-1}} \int_0^{V_*} F_*(x)^{I-1} dx$$

has the same distribution as $\beta(V)$. In other words, $F(v)$ and $F_*(v)$ are **observationally equivalent** (see Roehrig 1988). We will show that if $F(v)$ and $F_*(v)$ are observationally equivalent then they are identical: $F(v) = F_*(v)$ on $(0, \infty)$, provided that both distributions are absolutely continuous with connected support:

Assumption 2. *In first-price sealed bid auctions, the value distributions are confined to the class $\mathcal{F}_{accs}$ of absolutely continuous distributions with connected supports.*

Connectedness of the support of $F(v)$ means that the support $\{v \in (0, \infty) : F'(v) > 0\}$ takes the form of an interval.

Note that we do not assume that the supports of $F(v)$ and $F_*(v)$ are equal, but only that they are connected.

The main reason for this assumption is the following well-known result, which follows trivially from the fact that each F is strictly monotonic and therefore invertible on its support.

Lemma 2. *Let V be a random drawing from an absolutely continuous distribution F with connected support. Then $U = F(V)$ has a uniform $[0, 1]$ distribution, and there exists an inverse function F^{-1} on $[0, 1]$ such that $V = F^{-1}(U)$ a.s.*

Under Assumption 2 it follows from Lemma 2 that $U = F(V)$ and $U_* = F_*(V_*)$ are uniformly $[0,1]$ distributed, so that

$$B = \varphi(U) = F^{-1}(U) - \frac{1}{U^{I-1}} \int_0^{F^{-1}(U_1)} F(x)^{I-1} dx$$

and

$$B_* = \varphi_*(U_*) = F_*^{-1}(U_*) - \frac{1}{U_*^{I-1}} \int_0^{F_*^{-1}(U_*)} F_*(x)^{I-1} dx$$

have the same distribution:

$$P[B \leq b] = P[B_* \leq b] = \Lambda(b), \tag{1.5}$$

say.

Since $\varphi(u)$ is monotonic increasing and therefore invertible on $(0,1)$, it follows from (1.5) that for all b in the support of $\Lambda(b)$,

$$\begin{aligned}
\varphi^{-1}(b) &= P[U \leq \varphi^{-1}(b)] = P[\varphi(U) \leq b] \\
&= P[B \leq b] = P[B_* \leq b] \\
&= P[\varphi_*(U_*) \leq b] = P[U_* \leq \varphi_*^{-1}(b)] = \varphi_*^{-1}(b).
\end{aligned}$$

Hence, $\varphi(u) = \varphi_*(u)$ a.e.[5] on $(0,1)$ and thus by continuity,

[5] a.e. stands for almost everywhere, which means that the property holds except perhaps on a set with zero Lebesgue measure.

$$F^{-1}(u) - \frac{1}{u^{I-1}} \int_0^{F^{-1}(u)} F(x)^{I-1} dx = \varphi(u)$$
$$= \varphi_*(u) = F_*^{-1}(u) - \frac{1}{u^{I-1}} \int_0^{F_*^{-1}(u)} F_*(x)^{I-1} dx$$

exactly on $(0,1)$. Multiplying both sides of this equation by u^{I-1} yields

$$u^{I-1} F^{-1}(u) - \int_0^{F^{-1}(u)} F(x)^{I-1} dx = u^{I-1} F_*^{-1}(u) - \int_0^{F_*^{-1}(u)} F_*(x)^{I-1} dx$$

and then taking the derivative to $u \in (0,1)$ yields

$$(I-1)u^{I-2} F^{-1}(u) + u^{I-1} \frac{dF^{-1}(u)}{du} - (F(F^{-1}(u)))^{I-1} \frac{dF^{-1}(u)}{du}$$
$$= (I-1)u^{I-2} F_*^{-1}(u) + u^{I-1} \frac{dF_*^{-1}(u)}{du} - (F_*(F_*^{-1}(u)))^{I-1} \frac{dF_*^{-1}(u)}{du},$$

so that $F^{-1}(u) = F_*^{-1}(u)$ for all $u \in (0,1)$. Consequently, $F(v)$ and $F_*(v)$ are equal on a common support and therefore $F(v) = F_*(v)$ on $[0, \infty)$.

1.2.3 Binding reservation price

In the binding reservation price case some bidders' values are above p_0 while some bidders' values are below p_0. The former bidders submit their bids according to the equilibrium bid function

$$\beta(v) = v - \frac{1}{F(v)^{I-1}} \int_{p_0}^{v} F(x)^{I-1} dx, \ v > p_0, \tag{1.6}$$

whereas the latter bidders do not submit any bid. In the latter case we may assume without loss of generality that these potential bidders submit zero bids. After the auction, the econometrician can observe the number of actual bids, I^*, and the number $I - I^*$ of zero bids. The number $I - I^*$ has a $\text{Bin}(I, F(p_0))$ distribution, hence $E\left[\left(I - I^*\right)/I\right] = F(p_0)$. In L repeated identical auctions, where for each auction ℓ the number of actual bidders is I_ℓ^*, $F(p_0)$ can be estimated consistently by $(1/L)\sum_{j=1}^{L}\left(I - I_\ell^*\right)/I$. Therefore,

$$\alpha = F(p_0)$$

is nonparametrically identified and may be taken as given.[6]

Now consider the conditional distribution

$$\underline{F}(v) = P\left[V \leq v | V > p_0\right] = \frac{P\left[p_0 < V \leq v\right]}{P\left[V > p_0\right]} \tag{1.7}$$

$$= \frac{F(v) - F(p_0)}{1 - F(p_0)} = \frac{F(v) - \alpha}{1 - \alpha} \text{ if } v \geq p_0,$$

$$\underline{F}(v) = 0 \text{ if } v < p_0.$$

Then

$$F(v) = \alpha + (1 - \alpha)\,\underline{F}(v) \text{ for } v \geq p_0 \tag{1.8}$$

[6] Instead of assuming that the number of potential bidders I is known, Guerre, Perrigne and Vuong (2000) identify the binomial distribution parameters I and $1 - F(p_0)$ from the distribution of the actual bids. Given these parameters, $F(v)$ is then identified on $[p_0, \overline{v}]$.

Substituting (1.8) in (1.6) yields

$$\beta(v) = F^{-1}\left(\alpha + (1-\alpha)\underline{F}(v)\right) - \frac{1}{\left(\alpha + (1-\alpha)\underline{F}(v)\right)^{I-1}}$$
$$\times \int_{p_0}^{F^{-1}(\alpha+(1-\alpha)\underline{F}(v))} \left(\alpha + (1-\alpha)\underline{F}(x)\right)^{I-1} dx, \ v > p_0.$$

Given that F satisfies Assumption 2, it follows that $\underline{F}$ also satisfies the conditions in Assumption 2, hence $\underline{F}$ is invertible on its support, with inverse denoted by $\underline{F}^{-1}(.)$. It follows therefore from Lemma 2 that for a random drawing $\underline{V}$ from $\underline{F}$, $U = \underline{F}(\underline{V})$ has a uniform $[0,1]$ distribution, and hence the bids B, including the zero bids, are distributed according to

$$B \sim \Bigg(F^{-1}\left(\alpha + (1-\alpha)U\right) - \frac{1}{\left(\alpha + (1-\alpha)U\right)^{I-1}} \times \int_{p_0}^{F^{-1}(\alpha+(1-\alpha)U)} \left(\alpha + (1-\alpha)\underline{F}(x)\right)^{I-1} dx \Bigg).D \tag{1.9}$$

where U is distributed uniform $[0,1]$, and

$$D = \mathbf{1}\left(V > p_0\right), V \sim F(v) \tag{1.10}$$

where $\mathbf{1}(.)$ is the indicator function[7], with distribution $P[D=0] = \alpha$, $P[D=1] = 1-\alpha$. Since U was actually drawn conditionally on the event $V > p_0$, it follows that U and D are independent.

[7] $\mathbf{1}(true) = 1$, $\mathbf{1}(false) = 0$.

Suppose there exists a distribution $F_*(v)$ with $F_*(p_0) = \alpha$ and corresponding conditional distribution function

$$\underline{F}_*(v) = \frac{F_*(v) - \alpha}{1 - \alpha} \text{ if } v \geq p_0, \ \underline{F}_*(v) = 0 \text{ if } v < p_0$$

such that

$$B \sim \left(F_*^{-1}(\alpha + (1-\alpha) U_*) - \frac{1}{(\alpha + (1-\alpha) U_*)^{I-1}} \right. \\ \left. \times \int_{p_0}^{F_*^{-1}(\alpha + (1-\alpha) U_*)} (\alpha + (1-\alpha) \underline{F}_*(x))^{I-1} \, dx \right) .D_*, \tag{1.11}$$

where U_* is uniformly $[0,1]$ distributed, and $D_* = \mathbf{1}(V_* > p_0)$, $V_* \sim F_*(v)$, with the same distribution as (1.10).

Again, U_* and D_* are independent. Since D and D_* have the same distribution, it suffices to compare the right-hand sides of (1.9) and (1.11) conditional on $D = 1$ and $D_* = 1$, respectively. Then similar to the non-binding reservation price case we must have that for all $u \in (0,1)$,

$$\begin{aligned} & F^{-1}(\alpha + (1-\alpha) u) \\ & - \frac{1}{(\alpha + (1-\alpha) u)^{I-1}} \int_{p_0}^{F^{-1}(\alpha + (1-\alpha) u)} (\alpha + (1-\alpha) \underline{F}(x))^{I-1} \, dx \\ & = F_*^{-1}(\alpha + (1-\alpha) u) \\ & \quad - \frac{1}{(\alpha + (1-\alpha) u)^{I-1}} \int_{p_0}^{F_*^{-1}(\alpha + (1-\alpha) u)} (\alpha + (1-\alpha) \underline{F}_*(x))^{I-1} \, dx, \end{aligned}$$

hence, by change of variables, for all $u \in (\alpha, 1)$,

$$u^{I-1} F^{-1}(u) - \int_{p_0}^{F^{-1}(u)} \left(\alpha + (1-\alpha) \underline{F}(x)\right)^{I-1} dx$$
$$= u^{I-1} F_*^{-1}(u) - \int_{p_0}^{F_*^{-1}(u)} \left(\alpha + (1-\alpha) \underline{F}_*(x)\right)^{I-1} dx\ .$$

Taking the derivative to $u \in (\alpha, 1)$ it follows that

$$(I-1)\, u^{I-2} F^{-1}(u) + u^{I-1} \frac{dF^{-1}(u)}{du}$$
$$- \left(\alpha + (1-\alpha) \underline{F}\left(F^{-1}(u)\right)\right)^{I-1} \frac{dF^{-1}(u)}{du}$$
$$= (I-1)\, u^{I-2} F_*^{-1}(u) + u^{I-1} \frac{dF_*^{-1}(u)}{du}$$
$$- \left(\alpha + (1-\alpha) \underline{F}_*\left(F_*^{-1}(u)\right)\right)^{I-1} \frac{dF_*^{-1}(u)}{du},$$

hence $F^{-1}(u) = F_*^{-1}(u)$ on $(\alpha, 1)$ and thus $F(v) = F_*(v)$ on $[p_0, \infty)$.

1.3 First-price auctions with observed auction-specific heterogeneity

Let X be the vector of auction-specific characteristics for an auctioned item, with support S_X. The number of potential bidders of an auction with characteristics $X = x \in S_X$ is a known function $I(x)$ of x, but we maintain the assumption that ex-ante $I(x)$ is known to the potential bidders and ex-post to the econometrician. The same applies to the reservation price $p_0(x)$. The conditional value distribution in each auction with characteristics $X = x \in S_X$ is denoted by

$$F(v|x) = P\left[V \le v|X = x\right],$$

which is known to each potential bidder. The values themselves are independent within and across auctions, conditional on X.

Since the non-binding reservation price case follows directly from the binding case by setting $p_0 = 0$, we will focus only on the binding reservation price case. In that case the conditional equilibrium bid function for the actual bids is

$$\beta\left(v|X\right) = v - \frac{1}{F(v|X)^{I(X)-1}} \int_{p_0(X)}^{v} F(y|X)^{I(X)-1} dy,\ \ v > p_0\left(X\right).$$

Note that Assumption 1 implies that $E[V|X] < \infty$ a.s., so that under Assumption 1, $\lim_{v\to\infty} \beta\left(v|X\right) < \infty$ a.s.

1.3.1 Nonparametric identification

In each auction with characteristics X and reservation price $p_0(X)$ the number of potential bidders $I(X)$ minus the number of actual bidders $I_*(X)$ has a conditional $\text{Bin}(I(X), F(p_0(X)|X))$ distribution, hence

$$E\left[\left.\frac{I(X) - I_*(X)}{I(X)}\right| X\right] = F(p_0(X)|X)$$

which can be consistently estimated by nonparametric kernel regression, given a random sample of auctions.[8] Therefore,

$$\alpha\left(X\right)=F(p_0(X)|X)$$

is nonparametrically identified and may be taken as given. Interpreting the non-bids as zero bids, the bids in this auction are distributed as

$$B\sim\left(V-\frac{1}{F(V|X)^{I(X)-1}}\int_{p_0(X)}^{V}F(y|X)^{I(X)-1}dy\right)\mathbf{1}\left(V>p_0\left(X\right)\right),$$

where $\mathbf{1}\left(.\right)$ is the indicator function.

Similar to (1.7), let

$$\underline{F}(v|X)=\frac{F\left(v|X\right)-\alpha\left(x\right)}{1-\alpha\left(x\right)}\text{ if }v\geq p_0\left(X\right),\ \underline{F}(v|X)=0\text{ if }v<p_0\left(X\right)$$

so that

$$F(v|X)=\alpha\left(X\right)+\left(1-\alpha\left(X\right)\right)\underline{F}(v|X). \tag{1.12}$$

Moreover, let $\underline{V}$ be a random drawing from $\underline{F}(v|X)$, conditional on X, and let $U=\underline{F}(\underline{V}|X)$. In order to conclude that U has a uniform $[0,1]$ distribution we need to generalize Assumption 2 to:

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[8] Guerre, Perrigne and Vuong (2000) assume that the number of potential bidders is an unknown constant I when there are heterogenous auctioned objects if the reservation price is binding. They identify the pair of binomial distribution parameters $(I,1-F(p_0|X))$ using a distribution of actual bids given X. The uniqueness of the pair $(I,1-F(p_0|X))$ gives the identification of $F(v|X)$ on $[p_0(X),\overline{v}(X)]$.

Assumption 3. *In a first-price sealed bid auction with auction-specific covariates X, the conditional value distribution given X is confined to the class $\mathcal{F}_{accs}(X)$ of absolutely continuous conditional distributions with connected supports.*

Note that in this case the endpoints of the support may be (Borel measurable) functions of X.

Now Lemma 2 can be generalized to:

Lemma 3. *Conditional on X, let V be a random drawing from a conditional distribution $F(.|X) \in \mathcal{F}_{accs}(X)$. Then $U = F(V|X)$ has a uniform $[0,1]$ distribution, and U and X are independent. Moreover, for each point x in the support of X there exists an inverse function $F^{-1}(.|x)$ on $[0,1]$ such that $V = F^{-1}(U|X)$ a.s.*

Proof: Appendix.

Similar to (1.9) we now have that the conditional distribution of the bids (including the zero bids) is

$$B|X \sim \left(F^{-1}(\alpha(X) + (1-\alpha(X))U|X) - \frac{1}{U^{I(X)-1}} \right.$$
$$\left. \times \int_{p_0(X)}^{F^{-1}(\alpha(X)+(1-\alpha(X))U|X)} F(y|X)^{I(X)-1} dy \right) .D,$$

where U is uniformly $[0,1]$ distributed, independently of X, and $D = \mathbf{1}(V > p_0(X))$. Note that U is independent of D as well, because U was actually drawn conditionally on X **and** the event $V > p_0(X)$. Now by the same argument as in the case without covariates it follows straightforwardly that conditional on X, $F(v|X)$ is nonparametrically identified on $[p_0(X), \infty)$.

1.3.2 Semi-nonparametric identification

In order to put some structure on $F(v|X)$, we will now assume that

$$\ln V = \gamma(X) + \varepsilon, \tag{1.13}$$

where

Assumption 4. *The random variable ε in* (1.13) *is independent of X, and its distribution is absolutely continuous with connected support.*

The reason for considering this case and Assumption 4 will be given at the end of this subsection.

To pin down the location of $\gamma(X)$ we will impose a quantile restriction on the distribution of ε, for example that the median of ε is zero. Moreover, to ensure that $E[V|X] = \exp(\gamma(X)) E[\exp(\varepsilon)] < \infty$ we need to require that $E[\exp(\varepsilon)] < \infty$:

Assumption 5. *The median of ε in* (1.13) *is zero: $P(\varepsilon \leq 0) = 1/2$, and $E[\exp(\varepsilon)] < \infty$.*

Thus $\gamma(X)$ is now the conditional median of $\ln V$.

It follows from (1.13) that

$$F(v|X) = P[V \leq v|X] = P[\exp(\varepsilon) \leq v\exp(-\gamma(X))\,|X] \tag{1.14}$$

$$= P[\exp(-\exp(\varepsilon)) \geq \exp(-v\exp(-\gamma(X)))\,|X]$$

$$= P[1-\exp(-\exp(\varepsilon)) \leq 1-\exp(-v\exp(-\gamma(X)))\,|X]$$

$$= H(1-\exp(-v\exp(-\gamma(X)))),$$

where $H(\cdot)$ is a distribution of $1-\exp(-v\exp(-\gamma(X)))$, i.e.,

$$H(u) = P[1-\exp(-\exp(\varepsilon)) \leq u] = P[\exp(-\exp(\varepsilon)) \geq 1-u]$$

$$= P[\varepsilon \leq \ln(\ln(1/(1-u)))]$$

which is a distribution function on $(0,1)$. Note that $H(u)$ satisfies the quantile restriction

$$H\left(1-e^{-1}\right) = 1/2. \tag{1.15}$$

The question now arises whether $\gamma(X)$ and $H(u)$ are nonparametrically identified. It suffices to establish the uniqueness of $\gamma(X)$ only, because $F(v|X)$ is nonparametrically identified on $[p_0(X),\infty)$, so that given $\gamma(X)$, $H(u)$ is identified on $[1-\exp(-p_0(X)\exp(-\gamma(X))),1]$.

The immediate proof for the uniqueness of $\gamma(X)$ is to note that $\gamma(X)$ is the conditional median of $\log V$ given X as $P[\log v \leq \gamma(X)|X] = P[\varepsilon \leq 0] = 1/2$.[9] Thus, $\gamma(X)$ is identified from the distribution of $\log V$ given X. The identification of $\gamma(X)$ implies that

$$H_*(u) = H(u) \text{ for } u \in [1 - \exp(-p_0 \exp(-\gamma(x)), 1].$$

The reason for considering the case (1.13) is that the distribution function $H(u)$ can be easily estimated semi-nonparametrically using orthonormal Legendre polynomials on the unit interval. See Bierens (2007). Given H and a parametric specification of $\gamma(X)$, for example let $\gamma(X)$ be a linear function of X, $F(v|X)$ can be determined via (1.14). Moreover, the conditional median of the computed function $F(v|X)$ can then be compared with the parametric specification, on the basis of which a test can be developed for the validity of the parametric specification of the median function. This is left for future research.

1.4 The case where the actual number of bidders is known to the bidders

The nonparametric identification of the first-price auction model with binding reservation price p_0 depends crucially on the assumption that the number of potential bidders I is known to the bidders as well as to the econometrician. But usually the econometrician only observes the actual number of bids I_*.

[9] A formal proof of the nonparametric identification of $\gamma(X)$ will be given in section 1.6.2.

To get around that problem, assume that, instead of the number of potential bidders I, the actual number of bidders $I_* \geq 2$ is ex-ante known to all the bidders. Moreover, assume that a binding reservation price p_0 is set in advance by the seller. Then it can be shown[10] that the equilibrium bid function in this case becomes

$$\beta\left(v\right) = v - \frac{1}{\underline{F}(v)^{I_*-1}} \int_{p_0}^{v} \underline{F}(x)^{I_*-1} dx, \tag{1.16}$$

where $\underline{F}(v)$ is defined in (1.7).

Similarly, in the presence of auction-specific covariates X the conditional equilibrium bid function becomes

$$\beta\left(v|X\right) = v - \frac{1}{\underline{F}(v|X)^{I_*(X)-1}} \int_{p_0(X)}^{v} \underline{F}(y|X)^{I_*(X)-1} dy$$

where

$$\underline{F}(v|X) = \frac{F\left(v|X\right) - \alpha(X)}{1 - \alpha(X)}$$

with

$$\alpha(X) = F\left(p_0(X)|X\right).$$

We will now set forth conditions under which $F\left(v|X\right)$ and $\alpha(X)$ are identified.

If we use for $F\left(v|X\right)$ the semi-nonparametric specification (1.14) with parametrized median function $\exp(\gamma\left(X, \theta_0\right))$, then

$$F\left(v|X\right) = H\left(1 - \exp\left(-v \exp\left(-\gamma\left(X, \theta_0\right)\right)\right)\right)$$

[10] See section 1.6.3.

$$= 1 - H_0\left(\exp\left(-v\exp\left(-\gamma\left(X,\theta_0\right)\right)\right)\right)$$

where

$$H_0\left(u\right) = 1 - H\left(1-u\right),$$

so that

$$\alpha(X) = 1 - H_0\left(\exp\left(-p_0(X)\exp\left(-\gamma\left(X,\theta_0\right)\right)\right)\right).$$

Note that $H_0\left(u\right)$ satisfies the quantile restriction

$$H_0\left(e^{-1}\right) = 1/2. \tag{1.17}$$

Hence

$$\underline{F}(v|X) = 1 - \frac{H_0\left(\exp\left(-v\exp\left(-\gamma\left(X,\theta_0\right)\right)\right)\right)}{H_0\left(\exp\left(-p_0(X)\exp\left(-\gamma\left(X,\theta_0\right)\right)\right)\right)},\ v \geq p_0(X)$$

Since $\underline{F}(v|X)$ is nonparametrically identified on $[p_0(X),\infty)$, it follows that for given θ_0, $H_0(u)$ is identified on $[0,\ \exp\left(-p_0(X)\exp\left(-\gamma\left(X,\theta_0\right)\right)\right)]$.

Now suppose that there exists a parameter vector $\theta_1 \neq \theta_0$ and a distribution function H_1 on [0,1] such that

$$\underline{F}(v|X) = 1 - \frac{H_1\left(\exp\left(-v\exp\left(-\gamma\left(X,\theta_1\right)\right)\right)\right)}{H_1\left(\exp\left(-p_0(X)\exp\left(-\gamma\left(X,\theta_1\right)\right)\right)\right)},\ v \geq p_0(X)$$

Then for all $v \geq p_0(X)$,

$$\frac{H_1\left(\exp\left(-v\exp\left(-\gamma\left(X,\theta_1\right)\right)\right)\right)}{H_0\left(\exp\left(-v\exp\left(-\gamma\left(X,\theta_0\right)\right)\right)\right)} = \frac{H_1\left(\exp\left(-p_0(X)\exp\left(-\gamma\left(X,\theta_1\right)\right)\right)\right)}{H_0\left(\exp\left(-p_0(X)\exp\left(-\gamma\left(X,\theta_0\right)\right)\right)\right)}$$

Since the right-hand side does not depend on v, it follows that the derivative of the left hand side to $v > p_0(X)$ is zero, which implies (after some rearrangements) that

$$\frac{h_1\left(\exp\left(-v\exp\left(-\gamma\left(X,\theta_1\right)\right)\right)\right)}{h_0\left(\exp\left(-v\exp\left(-\gamma\left(X,\theta_0\right)\right)\right)\right)} \tag{1.18}$$

$$\times\exp\left(-v\left(\exp\left(-\gamma\left(X,\theta_1\right)\right)-\exp\left(-\gamma\left(X,\theta_0\right)\right)\right)\right)$$

$$=\frac{H_1\left(\exp\left(-v\exp\left(-\gamma\left(X,\theta_1\right)\right)\right)\right)}{H_0\left(\exp\left(-v\exp\left(-\gamma\left(X,\theta_0\right)\right)\right)\right)}\times\frac{\exp\left(-\gamma\left(X,\theta_0\right)\right)}{\exp\left(-\gamma\left(X,\theta_1\right)\right)}$$

$$=\frac{H_1\left(\exp\left(-p_0(X)\exp\left(-\gamma\left(X,\theta_1\right)\right)\right)\right)}{H_0\left(\exp\left(-p_0(X)\exp\left(-\gamma\left(X,\theta_0\right)\right)\right)\right)}\times\frac{\exp\left(-\gamma\left(X,\theta_0\right)\right)}{\exp\left(-\gamma\left(X,\theta_1\right)\right)},$$

where h_1 and h_0 are the densities of H_1 and H_0, respectively.

Next, impose the condition

$$h_1(0)=h_0\left(0\right)=1 \tag{1.19}$$

and take the limit of (1.18) for $v\to\infty$. Then

$$\lim_{v\to\infty}\exp\left(-v\left(\exp\left(-\gamma\left(X,\theta_1\right)\right)-\exp\left(-\gamma\left(X,\theta_0\right)\right)\right)\right)$$

$$=\begin{cases}\infty & \text{if}\quad \gamma\left(X,\theta_1\right)>\gamma\left(X,\theta_0\right)\\ 0 & \text{if}\quad \gamma\left(X,\theta_1\right)<\gamma\left(X,\theta_0\right)\\ 1 & \text{if}\quad \gamma\left(X,\theta_1\right)=\gamma\left(X,\theta_0\right)\end{cases}$$

$$=\frac{H_1\left(\exp\left(-v\exp\left(-\gamma\left(X,\theta_1\right)\right)\right)\right)}{H_0\left(\exp\left(-v\exp\left(-\gamma\left(X,\theta_0\right)\right)\right)\right)}\times\frac{\exp\left(-\gamma\left(X,\theta_0\right)\right)}{\exp\left(-\gamma\left(X,\theta_1\right)\right)}$$

$$= \frac{H_1\left(\exp\left(-p_0(X)\exp\left(-\gamma\left(X,\theta_1\right)\right)\right)\right)}{H_0\left(\exp\left(-p_0(X)\exp\left(-\gamma\left(X,\theta_0\right)\right)\right)\right)} \times \frac{\exp\left(-\gamma\left(X,\theta_0\right)\right)}{\exp\left(-\gamma\left(X,\theta_1\right)\right)}.$$

Clearly, only the option

$$\gamma\left(X,\theta_1\right) = \gamma\left(X,\theta_0\right) \tag{1.20}$$

is possible, which implies that

$$H_1(u) = H_0\left(u\right) \text{ on } \left[0,\ \exp\left(-p_0(X)\exp\left(-\gamma\left(X,\theta_0\right)\right)\right)\right]. \tag{1.21}$$

The condition (1.19) can be implemented similar to the condition $h_0(1) = 1$ in Bierens (2007) and Bierens and Carvalho (2006). Of course, we should also impose the quantile restriction (1.17). Moreover, under Assumption 1 in Bierens and Carvalho (2006), (1.21) implies that $H_1(u) = H_0\left(u\right)$ on $\left[0,\ 1\right]$, so that $H_0\left(u\right)$ is identified on $[0,\ 1]$. Finally, we need some obvious regularity conditions on the distribution of X and the functional form of $\gamma\left(x,\theta\right)$ such that (1.20) implies $\theta_1 = \theta_0$.

1.5 Concluding remarks

In this chapter we have proved, without using the usual condition that the value distribution has known bounded support, the non-parametric and semi-nonparametric identification of various first-price auction models with and without binding a reservation price. These results, in particular the results in Sections 1.3.2 and 1.4, are the basis for our continuing research on the semi-nonparametric estimation of these models via semi-nonparametric modeling of density and distribution functions on the unit interval, along the lines in Bierens (2007) and Bierens and Carvalho (2007). See also Chen (2006) for a review of semi-nonparametric modeling and estimation. In particular, we will propose to estimate these models semi-nonparametrically via a simulated integrated conditional moment criterion, similar to the integrated conditional moment test statistic proposed by Bierens (1982) and Bierens and Ploberger (1997). In our case, the moment function is the distance between the empirical characteristic function of the observed bids and the empirical characteristic function of the corresponding simulated bids generated by the equilibrium bid function for a semi-nonparametric specification of the value distribution.

1.6 Proofs

1.6.1 Proof of Lemma 2

Let $(\underline{v}(X), \overline{v}(X))$ (or its closure) be the support of $F(.|X)$. Since $F(v|X)$ is strictly monotonic increasing on $(\underline{v}(X), \overline{v}(X))$, it is invertible: For each x in the support of X and each $u \in (0,1)$ there exists a unique $v \in (\underline{v}(x), \overline{v}(x))$ such that $F(v|x) = u$, hence there exists a conditional distribution function $F^{-1}(u|x)$ on $[0,1]$ such that $F(v|x) = u \in (0,1)$ implies $v = F^{-1}(u|x) \in (\underline{v}(x), \overline{v}(x))$. Then

$$P[U \leq u|X] = P[F(V|X) \leq u|X] = P\left[V \leq F^{-1}(u|X)|X\right]$$
$$= F(F^{-1}(u|X)|X) = u.$$

Since the right-hand side does not depend on X, U and X are independent, and therefore $P[U \leq u] = u$. Q.E.D.

1.6.2 Identification of $\gamma(X)$

To identify $\gamma(X)$, note that Assumption 5 implies that $H(u)$ is absolutely continuous with connected support, say $(\underline{u}, \overline{u}) \subset [0,1]$. Then it follows from Lemma 1 that H is invertible on $(\underline{u}, \overline{u})$, with inverse H^{-1}. Consequently, it follows from (1.14) that $1 - \exp(-v\exp(-\gamma(X))) = H^{-1}(F(v|X))$, hence

$$v = \exp(\gamma(X)) \ln\left(1/\left(1 - H^{-1}(F(v|X))\right)\right)$$

$$= \exp\left(\gamma\left(X\right)\right) \ln\left(1/\left(1 - H^{-1}\left(\alpha\left(X\right) + \left(1 - \alpha\left(X\right)\right) \underline{F}(v|X)\right)\right)\right),$$

where the latter follows from (1.12). Next, let $\underline{V}$ be a random drawing from $\underline{F}\left(v|X\right)$. Then it follows from Lemma 2 that $U = \underline{F}\left(\underline{V}|X\right)$ is uniformly $[0,1]$ distributed, and is independent of X, hence

$$\underline{V} = \exp\left(\gamma\left(X\right)\right) \ln\left(1/\left(1 - H^{-1}\left(\alpha\left(X\right) + \left(1 - \alpha\left(X\right)\right) U\right)\right)\right)$$

Suppose there exists an alternative median function $\gamma_*\left(X\right)$ and an alternative distribution function H_* with inverse H_*^{-1} for which

$$\underline{V} = \exp\left(\gamma_*\left(X\right)\right) \ln\left(1/\left(1 - H_*^{-1}\left(\alpha\left(X\right) + \left(1 - \alpha\left(X\right)\right) U\right)\right)\right)$$

Then for arbitrary $u \in (0,1)$.

$$\exp\left(\gamma_*\left(X\right) - \gamma\left(X\right)\right) = \frac{\ln\left(1/\left(1 - H^{-1}\left(\alpha\left(X\right) + \left(1 - \alpha\left(X\right)\right) u\right)\right)\right)}{\ln\left(1/\left(1 - H_*^{-1}\left(\alpha\left(X\right) + \left(1 - \alpha\left(X\right)\right) u\right)\right)\right)}$$

Since the left-hand side of this equation does not depend on u, the derivative of the right-hand side to u is zero, hence

$$\frac{\ln\left(1 - H^{-1}[\alpha(X) + (1 - \alpha(X))u]\right)}{\ln\left(1 - H_*^{-1}[\alpha(X) + (1 - \alpha(X))u]\right)} = C(X),$$

for example, and thus $\gamma_*(X) = \gamma(X) + \ln(C(X))$. But $\gamma_*(X)$ and $\gamma(X)$ are both conditional medians of $\ln V$, which is only possible if $\ln(C(X)) = 0$ a.s.:

$$\gamma_*(X) = \gamma(X).$$

This implies that

$$H_*(u) = H(u) \text{ for } u \in [1 - \exp(-p_0 \exp(-\gamma(x)), 1].$$

Q.E.D.

1.6.3 Proof of (1.16)

Let $\beta(v)$ be the strictly monotonic increasing equilibrium bid function involved, and let b be the bid of bidder 1, which corresponds to an x such that $b = \beta(x)$. Given the value $V_1 = v$ of bidder 1, the expected value for bidder 1 of the object to be auctioned off is v times the probability that he wins the object. The latter is the case if his bid $\beta(x)$ is the highest bid, which by the monotonicity of $\beta(v)$ is the case if $x > \overline{V}_2 = \max\left\{V_2, ..., V_{I_*}\right\}$. The probability of this event, conditional on $\underline{V}_2 = \min\left\{V_2, ..., V_{I_*}\right\} > p_0$, is $\underline{G}(x) = \underline{F}(x)^{I_*-1}$, where $\underline{F}(x)$ is defined by (1.7), hence the expected value is $v\underline{G}(x)$.

Given the bid $b = \beta(x)$, let $p(x)$ be the expected price to pay to the seller. At this point we do not assume yet that $p(x)$ equals $\beta(x)$ times the probability of winning the auction. Then the expected net gain for bidder 1 is $\pi(v, x) = v\underline{G}(x) - p(x)$, which is maximal if x is chosen such that

$$0 = \partial\pi\left(v, x\right)/\partial x = v\underline{G}'\left(x\right) - p'\left(x\right) \tag{1.22}$$

In order that the bid $b = \beta(x)$ is an equilibrium bid, the solution of (1.22) must be $x = v$, hence

$$p'\left(v\right) = v\underline{G}'\left(v\right). \tag{1.23}$$

Using the conditions $\underline{G}'\left(v\right) = 0$, $\pi\left(v, v\right) = 0$ for $v < p_0$, and $p(v) = 0$ at $v = p_0$, the solution of the differential equation (1.23) is $p(v) = \int_{p_0}^{v} x\underline{G}'\left(x\right)dx = v\underline{G}(v) - \int_{p_0}^{v} \underline{G}(x)dx$. For the equilibrium bid function $\beta\left(v\right)$, $p(v)$ is equal to $\beta\left(v\right)$ times the probability $\underline{G}(v)$ of winning the auction: $p(v) = \beta\left(v\right)\underline{G}(v) = v\underline{G}(v) - \int_{p_0}^{v} \underline{G}(x)dx$, hence

$$\beta\left(v\right) = v - \frac{1}{\underline{G}(v)}\int_{p_0}^{v} \underline{G}(x)dx = v - \frac{1}{\underline{F}(v)^{I_*-1}}\int_{p_0}^{v} \underline{F}(x)^{I_*-1}dx.$$

Q.E.D.

Chapter 2

Semi-Nonparametric Simulated Integrated Moments Estimation of Independently and Identically Repeated First-Price Auctions

2.1 Introduction

As Laffont and Vuong (1993) point out, the distribution of bids determines the structural elements of auction models, provided identification is achieved. In the first-price auction model with symmetric independent private values, the structural element of interest is the value distribution. Much research has been done on the identification and the estimation of the value distribution. Donald and Paarsch (1996) apply ML estimation to first-price auctions and Dutch auctions. They use a parametric specification for the value distribution to implement ML estimation. In particular, they assume in a numerical example that the value distribution is a uniform distribution on the interval $[0, \overline{v}]$ where $\overline{v} = \exp(\theta_0 + \theta_1 Z)$, with Z being an auction-specific covariate vector. Since, in this case, the support of the bid distribution involved depends on parameters, the standard consistency proof of ML estimators no longer applies. Another difficulty with ML estimation is that the equilibrium bid function is highly non-linear in the value and its distribution, which makes the implementation computationally challenging. The same applies to descending price (Dutch) auctions, because they are strategically equivalent to first-price auctions. Because of the difficulty of ML estimation of first-price auction models, Laffont and Vuong (1993) suggest a Simulated Non-Linear Least Squares (SNLLS) estimation and a Simulated Method of Moment (SMM) estimation for a descending price auction model with symmetric independent private values. Their SNLLS approach requires replacing the expectation of the winning bid with a simulated one. They also suggest that the expectation of higher moments of the winning bid can be used for SMM estimation if the expectation of the winning bids itself is not sufficient to identify all parameters. Both SNLLS and SMM approaches require a parametric specification for the value distribution. Laffont, Ossard and Vuong (1995) apply the SNLLS approach suggested by Laffont and Vuong (1993) to the eggplant auction, which is a descending price auction. They specify a log-normal value distribution conditional on covariates. Li (2005) considers first-price auctions with entry and binding reserve price. This auction consists of two stages. In the first stage, the potential bidder decides whether he or she enters the auction, with payment of entry cost. In the second stage, the bidder learns his or her value and then decides to bid according to the equilibrium bid function, which is the same function as in the first-price auction model. Li (2005) proposes a SMM approach to estimate the entry probability and the parameters of the value distribution. One of the conditional moments is a function of the upper bound of the bid support, which can be computed via the simulation approach in Laffont, Ossard and Vuong (1995). The other moment conditions are related to the number of active bidders, i.e., potential bidders who decides to participate in the auction.

Guerre, Perrigne and Vuong (2000) show the nonparametric identification[1] of value distributions with bounded support $[\underline{v}, \overline{v}]$, $\overline{v} < \infty$, and propose an indirect nonparametric kernel estimation approach. Their approach is based on the inverse bid function $v = b + (I-1)^{-1}\Lambda(b)/\lambda(b)$, where I is the number of potential bidders, v is a private value, b is a corresponding bid, Λ is the distribution function of bids and λ is the associated density function. The latter two functions are estimated via nonparametric kernel methods, as $\hat{\Lambda}(b)$ and $\hat{\lambda}(b)$, respectively. Using the pseudo-private values $\tilde{v} = b + (I-1)^{-1}\hat{\Lambda}(b)/\hat{\lambda}(b)$, the density of the private value distribution can now be estimated by kernel density estimation. However, the ratio $\hat{\Lambda}(b)/\hat{\lambda}(b)$ may be an unreliable estimate of $\Lambda(b)/\lambda(b)$ near the boundary of the support of $\lambda(b)$. To solve this problem, Guerre, Perrigne and Vuong use a trimming procedure which amounts to discarding pseudo-private values $\tilde{v}$ corresponding to bids b that are too close to the boundary of the (known) support of the bid distribution.

In Chapter 1, we have shown that the nonparametric identification of first-price auction models carries over to value distributions with support $(0, \infty)$ or smaller, as long as the values have a finite expectation and their distribution is absolutely continuous. To estimate these more general value distributions semi-nonparametrically, we propose in this paper a direct Semi-Nonparametric Simulated Integrated Moment (SNP-SIM) estimation approach as an alternative to the two-step nonparametric kernel estimation approach of Guerre, Perrigne and Vuong (2000) for independently repeated identical first-price auctions. Admittedly, this type of repeated auction is rare in practice. The

[1] For general nonparametric identification results of first-price auctions models with symmetric independent private values, see Guerre, Perrigne and Vuong (1995) or Athey and Haile (2005).

reason for considering this case is to lay the groundwork for the more realistic case of first-price auctions with auction-specific covariates, number of potential bidders and reservation prices, like the well-known US Forest Service timber auctions.

Our SNP-SIM methodology is different from the SMM approach of Laffont and Vuong (1993) and Li (2005) in that the latter two approaches require parametric specification of the value distribution while ours does not. Moreover, our approach uses an infinite number of moment conditions while Laffont and Vuong (1993) and Li (2005) use a finite number of moment conditions.

First, based on the approach in Bierens (2007), we construct an increasing sequence $\mathcal{F}_n$ (the sieve[2]) of compact metric spaces of absolutely continuous distribution functions $F(v)$ on $(0, \infty)$, where the distribution functions in each space $\mathcal{F}_n$ can be represented by functions of Legendre polynomials of order n.

Given a distribution function $F \in \mathcal{F}_n$, we simulate bids according to the equilibrium bid function involved. Motivated by the well-known fact that distributions are equal if and only if their characteristic functions are identical, we take the difference of the empirical characteristic functions of the actual and simulated bids as the moment function. Since the actual and simulated bids are bounded random variables, and characteristic functions of bounded random variables coincide everywhere if and only if they coincide on an arbitrary interval around zero, we take the integral of the squared moment function over such an interval as our objective function, similar to the Integrated Conditional Moment (ICM) test statistic of Bierens and Ploberger (1997). Minimizing this objective function to the distribution functions in $\mathcal{F}_n$ and letting n increase with

[2]See Shen(1997) and Chen (2004).

the sample size N then yield a uniformly consistent SNP sieve estimator of the actual value distribution. This approach yields as by-products an Integrated Moment (IM) test of the validity of the first-price auction model, together with an information criterion similar to the information criteria of Hannan-Quinn (1979) and Schwarz (1978) for likelihood models, which can be used to estimate the sieve order n consistently if it is finite, and otherwise yields a data-driven sequence n_N for which the SNP sieve estimator is uniformly consistent as well. Finally, we conduct a few numerical experiments to check the performance of our SNP-SIM approach.

Throughout this paper we confine our analysis to first-price sealed bid auctions where values are independent, private and bidders are symmetric and risk-neutral.

The paper is organized as follows. In section 2 we introduce our SNP-SIM estimation methodology and show how to draw simulated values from a candidate value distribution and its associated bid distribution. In section 3 we show how to construct a metric space $\mathcal{F}$ and corresponding sieve $\mathcal{F}_n$ of absolutely continuous distribution functions on $(0, \infty)$, and in section 4, we show the uniform strong consistency of the SNP-SIM estimator. In section 5, we propose a consistent integrated moment (IM) test of the validity of the first-price auction model with symmetric independent values and risk neutrality. In section 6, we show the performance of our proposed SNP-SIM estimation and testing approach via a few numerical experiments. In the concluding remarks, we suggest directions for further research.

Throughout the paper, we denote random variables in upper-case and non-random variables in lower-case. The indicator function is denoted by $I(\cdot)$.[3] Almost sure (a.s.)

[3] $I(True) = 1$, $I(False) = 0$.

convergence is denoted by $X_n \to X$ a.s.[4] Similarly, convergence in probability will be denoted by $X_n \to_p X$ or $p\lim_{n\to\infty} X_n = X$, and $X_n \to_d X$ indicates that X_n converges in distribution to X. In the case that X_n and X are random functions we use the notation $X_n \Rightarrow X$ to indicate that $X_n(\cdot)$ converges weakly to $X(\cdot)$. See for example Billingsley (1999) for the meaning of the notion of weak convergence.

2.2 Semi-Nonparametric Simulated Integrated Moment Estimation of First-Price Auctions

2.2.1 Data-Generating Process

As is well-known, the equilibrium bid function of first-price sealed bid auctions where values are independent, private and bidders are symmetric and risk-neutral takes the form

$$\beta(v|F) = v - \frac{1}{F(v)^{I-1}} \int_{p_0}^{v} F(x)^{I-1} dx \text{ for } v > p_0, \tag{2.1}$$

if the reservation price p_0 is binding, and

$$\beta(v|F) = v - \frac{1}{F(v)^{I-1}} \int_{0}^{v} F(x)^{I-1} dx \text{ for } v > \underline{v}, \tag{2.2}$$

if the reservation price p_0 is non-binding, where $F(v)$ is the value distribution, $\underline{v} \geq 0$ is the lower bound of its support, $I \geq 2$ is the number of potential bidders, and p_0 is the reservation price. See for example Riley and Samuelson (1981) or Krishna (2002). Note that the equilibrium bid is different depending on whether the seller's reservation

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[4]This means that $P[\lim_{n\to\infty} X_n = X] = 1$.

price is binding, $p_0 > \underline{v}$, or not. If p_0 is binding, only potential bidders whose values are greater than p_0 participate in the auction, and issue a bid $B > p_0$. For notational convenience we will assume that the other potential bidders issue a zero bid: $B = 0$.

Note that (2.1) can be written as

$$\beta\left(v|F\right) = v - \frac{v - p_0}{F(v)^{I-1}} \int_0^1 F\left(p_0 + u(v - p_0)\right)^{I-1} du \text{ for } v > p_0, \tag{2.3}$$

by substituting $x = p_0 + u(v - p_0)$ in the integral in (2.1), and similarly, (2.2) can be written as

$$\beta\left(v|F\right) = v - \frac{v}{F(v)^{I-1}} \int_0^1 F\left(u.v\right)^{I-1} du \text{ for } v > \underline{v}, \tag{2.4}$$

In this paper we will consider the case where this auction is repeated independently L times, with the same value distribution $F_0(v)$, the same fixed number of potential bidders I, and the same reservation price p_0. Thus, we observe $N = I \times L$ bids B_j generated independently according to

$$B_j = \begin{cases} \beta(V_j|F_0) & \text{if} \quad V_j > p_0, \\ 0 & \text{if} \quad V_j \leq p_0, \end{cases} \quad , \ j = 1, 2, ..., N = I \times L, \tag{2.5}$$

in the case of a binding reservation price p_0, or

$$B_j = \beta(V_j|F_0), \ j = 1, 2, ..., N = I \times L, \tag{2.6}$$

in the case of a non-binding reservation price, where the values V_j are independent random drawings from the unknown true value distribution $F_0(v)$. The asymptotic results will be derived for $L \to \infty$, under the assumption that

Assumption 1. *The true value distribution $F_0(v)$ is absolutely continuous with density $f_0(v)$ and finite expectation, $\int_0^\infty v f_0(v) dv < \infty$.*

2.2.2 Identification

It has been shown in Chapter 1 that under Assumption 1 and the additional condition that the support of $f_0(v)$ is connected, the value distribution $F_0(v)$ is identified on (p_0, ∞) from the distribution of the bids B_j, and that these bids are **bounded** random variables:

$$P[B_j \le \bar{b}_0] = 1 \tag{2.7}$$

where

$$\begin{aligned}
\bar{b}_0 &= \sup_{v>0} \beta(v|F_0) = (I-1)\int_{p_0}^{\infty} x F_0(x)^{I-2} f_0(x) dx + p_0 F_0(p_0)^{I-1} \qquad (2.8)\\
&= (I-1)\int_0^{\infty} x F_0(x)^{I-2} f_0(x) dx + \int_0^{p_0} F_0(x)^{I-1} dx\\
&\le (I-1)\int_0^{\infty} x f_0(x) dx + p_0
\end{aligned}$$

The significance of (2.7) is that the bid distribution $\Lambda_0(b) = P[B_j \leq b]$ is then completely determined by the shape of its characteristic function $\varphi(t)$,

$$\varphi(t) = E\left[\exp(i.t.B_j)\right] = \int_0^\infty \exp(i.t.b)\, d\Lambda_0(b),\ i = \sqrt{-1}, \tag{2.9}$$

in an arbitrary neighborhood of $t = 0$. More formally:

Lemma 1. *Let B be a bounded random variable with distribution function $\Lambda_0(b)$ and characteristic function $\varphi(t)$. Let $\psi(t)$ be the characteristic function of a distribution function $\Lambda(b)$. Then $\Lambda(b) = \Lambda_0(b)$ for all $b \in \mathbb{R}$ if and only if for an arbitrary $\kappa > 0$, $\varphi(t) = \psi(t)$ for all $t \in (-\kappa, \kappa)$.*

This is a well-known result[5], which is based on the fact that due to the boundedness condition $\varphi(t)$ can be written as $\varphi(t) = \sum_{m=0}^{\infty} \frac{i^m}{m!} t^m E\left[B^m\right]$, hence $\varphi(t) = \psi(t)$ on $(-\kappa, \kappa)$ implies that $i^{-m}\, d^m\psi(t) / (dt)^m\big|_{t=0} = i^{-m}\, d^m\varphi(t) / (dt)^m\big|_{t=0} = E\left[B^m\right]$ for $m = 0, 1, 2, ...$, so that $\psi(t) = \varphi(t) = \sum_{m=0}^{\infty} \frac{i^m}{m!} t^m E\left[B^m\right]$ for all $t \in \mathbb{R}$. As is well-known, the latter implies that the two distributions involved are identical.

Note that we do not need to assume from the outset that $\Lambda(b)$ is a distribution function of a bounded random variable. The condition $\varphi(t) = \psi(t)$ on $(-\kappa, \kappa)$ automatically implies boundedness of this distribution.

The connected support condition in Chapter 1 guarantees that F_0 is invertible, i.e., for each $u \in (0, 1)$ there exists a unique v such that $F_0(v) = u$, so that with U

[5] Usually stated for moment generating functions rather than characteristic functions.

a random drawing from the uniform $[0,1]$ distribution, $V = F_0^{-1}(U)$ has distribution $F_0(v)$. This well-known result was used in Chapter 1 to prove the identification of $F_0(v)$. However, invertibility of $F_0(v)$ on its support is not necessary for identification, and neither is absolute continuity, due to the following lemma:

Lemma 2. *Let $F(v)$ be a continuous distribution function. If V is a random drawing from $F(v)$ then $F(V)$ is uniformly $[0,1]$ distributed. Moreover, given a random drawing U from the uniform $[0,1]$ distribution, the solution V of $F(V) = U$ is almost surely unique, and $P[V \leq v] = F(v)$.*

Proof: Section 2.9.1.

Consequently, the identification of F_0 does not hinge on the connected support condition; only continuity matters!

2.2.3 Simulated Integrated Moment using Empirical Characteristic Functions

Let F be a potential candidate (henceforth called a *candidate value distribution*) for the true value distribution F_0, and let $\left\{\widetilde{V}_j\right\}_{j=1}^{N}$ be a random sample drawn from F.[6]

[6] The method for generating these simulated values will be addressed below.

Next, generate simulated bids $\widetilde{B}_j$ similar to (2.5):

$$\widetilde{B}_j = \begin{cases} \beta(\widetilde{V}_j|F) & \text{if} \quad \widetilde{V}_j > p_0, \\ 0 & \text{if} \quad \widetilde{V}_j \leq p_0, \end{cases} \quad , \ j = 1, 2, ..., N = I \times L, \tag{2.10}$$

Let

$$\widehat{\Psi}(t|F) = \widehat{\varphi}(t) - \widehat{\psi}(t|F), \ t \in \mathbb{R}, \tag{2.11}$$

where

$$\widehat{\varphi}(t) = \frac{1}{N} \sum_{j=1}^{N} \exp(i.t.B_j) \tag{2.12}$$

is the empirical characteristic function of the actual bids and

$$\begin{aligned} \widehat{\psi}(t|F) &= \frac{1}{N} \sum_{j=1}^{N} \exp(i.t.\widetilde{B}_j) \\ &= \frac{1}{N} \sum_{j=1}^{N} \exp(i.t.\beta(\widetilde{V}_j|F)) I\left(\widetilde{V}_j > p_0\right) + \frac{1}{N} \sum_{j=1}^{N} I\left(\widetilde{V}_j \leq p_0\right) \end{aligned} \tag{2.13}$$

is the empirical characteristic function of the simulated bids.

By the strong law of large numbers for i.i.d. random variables,

$$\widehat{\Psi}(t|F) \to \Psi(t|F) \text{ a.s., pointwise in } t \text{ and } F \tag{2.14}$$

where

$$\Psi(t|F) = \varphi(t) - \psi(t|F)$$

with $\varphi(t)$ defined by (2.9) and

$$\psi(t|F) = E\left[\exp(i.t.\widetilde{B}_j)\right] = \int_{p_0}^{\infty} \exp\left(i.t.\beta(v|F)\right) dF(v) + F(p_0) \tag{2.15}$$

the characteristic function of the simulated bids distribution.

Since by Lemma 1, $F \equiv F_0$ if and only if $\Psi(t|F) = 0$ for all t in an arbitrary interval $(-\kappa, \kappa)$, $\kappa > 0$, and $\Psi(t|F)$ is continuous in t, it follows that $F \equiv F_0$ if and only if

$$\overline{Q}(F) = \frac{1}{2\kappa} \int_{-\kappa}^{\kappa} |\Psi(t|F)|^2 \, dt \tag{2.16}$$

is equal to zero. Moreover, it follows from the bounded convergence theorem and (2.14) that

$$\widehat{Q}(F) = \frac{1}{2\kappa} \int_{-\kappa}^{\kappa} \left|\widehat{\Psi}(t|F)\right|^2 \, dt \tag{2.17}$$

converges a.s. to $\overline{Q}(F)$, pointwise in F. However, we will set forth conditions such that $\overline{Q}(F)$ is continuous in F, and $\widehat{Q}(F) \to \overline{Q}(F)$ a.s. **uniformly** on a compact space of distribution functions F containing the true value distribution F_0. Therefore, F_0 can be estimated consistently by minimizing $\widehat{Q}(F)$ to F, in some way to be discussed below.

Note that the objective function $\widehat{Q}(F)$ has a closed form expression in terms of the actual bids B_j and the simulated bids $\widetilde{B}_j$:

$$\widehat{Q}(F) = \frac{2}{N^2} \sum_{j_1=1}^{N-1} \sum_{j_2=j_1+1}^{N} \frac{\sin\left(\kappa.(B_{j_1} - B_{j_2})\right)}{\kappa.(B_{j_1} - B_{j_2})}$$

$$+\frac{2}{N^2}\sum_{j_1=1}^{N-1}\sum_{j_2=j_1+1}^{N}\frac{\sin\left(\kappa.(\widetilde{B}_{j_1}-\widetilde{B}_{j_2})\right)}{\kappa.(\widetilde{B}_{j_1}-\widetilde{B}_{j_2})}$$

$$-\frac{2}{N^2}\sum_{j_1=1}^{N}\sum_{j_2=1}^{N}\frac{\sin\left(\kappa.(B_{j_1}-\widetilde{B}_{j_2})\right)}{\kappa.(B_{j_1}-\widetilde{B}_{j_2})}. \tag{2.18}$$

See section 2.9.2 for the derivation.

2.2.4 Representation of the Value Distribution by a Distribution on the Unit Interval

Any absolutely continuous distribution function $F(v)$ can be expressed as

$$F(v) = H(G(v)), \tag{2.19}$$

where $G(v)$ is a given absolutely continuous distribution function with connected support[7] containing the support of F, and H is an absolutely continuous distribution function on the unit interval, namely $H(u) = F\left(G^{-1}(u)\right)$. The density $f(v)$ of $F(v)$ then takes the form

$$f(v) = h(G(v))g(v), \tag{2.20}$$

where $g(v)$ is the density of $G(v)$ and $h(u)$ is the density of $H(u)$, i.e.,

$$H(u) = \int_0^u h(x)dx. \tag{2.21}$$

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[7]So that $G(v)$ is invertible: $v = G^{-1}(u)$, $u \in [0,1]$, with support $(\underline{v},\overline{v})$, where $\underline{v} = \lim_{u\downarrow 0} G^{-1}(u)$ and $\overline{v} = \lim_{u\uparrow 1} G^{-1}(u)$.

Therefore, we can estimate f and F by estimating h given G.

In our case, where $F(v)$ is candidate value distribution, it is advisable to choose for G a distribution function with support $(0, \infty)$, for example the exponential or log-normal distribution, because in general the support of $F(v)$ is unknown.

2.2.5 Generation of Simulated Values and Bids

For SNP-SIM estimation we need to generate simulated bids from a candidate value distribution F. This can be done in two steps. First, we draw simulated values $\widetilde{V}$ from the density f of F, using an accept-reject method, and then we obtain the corresponding simulated bids by plugging the simulated values in the bid function $\beta(v|F)$, where the integral involved is computed numerically or by Monte Carlo integration.

The following lemma states the well-known[8] accept-reject random drawing method.

Lemma 3. *Let $f(\cdot)$ be a density function from which we want to draw a random variable X, and let $g(\cdot)$ be a density function from which it is easy to draw a random variable X_0. The result of the proposed accept-reject method below (steps $1-4$) then generates X.*

Step 1: *Find a constant $\overline{c} \geq 1$ such that $f(x) \leq \overline{c}g(x)$ for all x.*

Step 2: *Draw an X_0 from $g(x)$.*

Step 3: *Draw a U from the uniform distribution on* $[0, 1]$.

Step 4: *If $U \leq \overline{c}^{-1} f(X_0)/g(X_0)$ then set $X = X_0$, else redo steps $2-4$.*[9]

[8] See, for example, Devroye (1986) or Rubinstein (1981).

[9] It is important to restart from step 2, because X_0 and U need to be mutually independent.

Proof: Section 2.9.3.

This method can be used to generate random drawings $\widetilde{V}$ from densities of the form $f(v) = h(G(v))g(v)$, where $G(v)$ is the distribution function corresponding to $g(x)$ for which the inverse G^{-1} can be computed, and $h(u)$ is a continuous density on the unit interval. Step 1 can be conducted by computing

$$\bar{c} = \sup_{0 \leq u \leq 1} h(u), \tag{2.22}$$

for example by grid search, and step 2 can be done by setting $X_0 = G^{-1}(U_0)$, where U_0 is a random drawing from the uniform $[0,1]$ distribution. The uniform random variable U in step 3 has to be drawn independently of U_0, so that X_0 and U are independent. Then step 4 yields a random drawing $\widetilde{V}$ from the distribution function $F(v)$ of $f(v)$.

In other words, draw independently a sequence U_k of uniformly $[0,1]$ distributed random variables, let

$$X_k = G^{-1}(U_{2k-1})\, I\left(U_{2k} \leq \frac{f\left(G^{-1}(U_{2k-1})\right)}{\bar{c}.g\left(G^{-1}(U_{2k-1})\right)}\right).I\left(\sum_{m=1}^{k-1} X_m = 0\right) \tag{2.23}$$

so that $X_k > 0$ for only one k, say $\widetilde{k}$, and set $\widetilde{V} = X_{\widetilde{k}}$.

Observe from the proof of Lemma 3 in section 2.9.3 that the accept-reject method does not require that the support of f is connected, nor that the expectation of f is finite.

Given the random drawings $\widetilde{V}_j$, $j = 1, ..., N$, from a candidate value distribution $F(v)$, the corresponding simulated bids are

$$\widetilde{B}_j = \left(\widetilde{V}_j - \frac{\widetilde{V}_j - p_0}{F(\widetilde{V}_j)^{I-1}} \int_0^1 F\left(p_0 + u(\widetilde{V}_j - p_0)\right)^{I-1} du \right) .I\left(\widetilde{V}_j > p_0\right) \qquad (2.24)$$

if the reservation price p_0 is binding, and

$$\widetilde{B}_j = \widetilde{V}_j - \frac{\widetilde{V}_j}{F(\widetilde{V}_j)^{I-1}} \int_0^1 F\left(u.\widetilde{V}_j\right)^{I-1} du \qquad (2.25)$$

if the reservation price p_0 is non-binding. See (2.3) and (2.4). The integral involved can be computed numerically or via Monte Carlo integration. However, the asymptotic theory in this paper will be based on the assumption that this integral is computed exactly.

To guarantee the smoothness of the empirical characteristic function $\widehat{\psi}(t|F)$ of the simulated bids $\widetilde{B}_j$ in F, for each candidate value distribution F the random drawings $\widetilde{V}_j$, $j = 1, ..., N$, from F should be based on a common sequence $\{U_k\}_{k=1}^{\infty}$ of independent and uniformly $[0, 1]$ distributed random variables in (2.23), by generating this sequence numerically from the same seed.

In view of Lemma 2, an alternative way to generate simulated values $\widetilde{V}_j$ is the following:

Assumption 2. *Given a sequence $\widetilde{U}_1, ..., \widetilde{U}_N, ...$ of independent random drawings from the uniform $[0,1]$ distribution, for each candidate value distribution F the corresponding simulated values $\widetilde{V}_1, ..., \widetilde{V}_N$ are generated by solving $\widetilde{U}_j = F(\widetilde{V}_j)$ for $j = 1, ..., N$.*

Since any distribution function $F(v)$ with support contained in the support of a distribution function G can be written as $F(v) = H(G(v))$, where H is a distribution function on $[0,1]$, the procedure in Assumption 2 can be conducted in the following way. For each $\widetilde{U}_j$ find first a U_j such that $H(U_j) = \widetilde{U}_j$, and then let $\widetilde{V}_j = G^{-1}(U_j)$. This U_j can be computed iteratively as follows. Starting from the initial interval $I_0 = (a_0, b_0] = (0,1]$, let for $n = 1, 2, ...$,

$$I_n = (a_n, b_n] = \begin{cases} (a_{n-1}, a_{n-1} + (b_{n-1} - a_{n-1})/2] & if \quad H\left((b_{n-1} - a_{n-1})/2\right) \geq \widetilde{U}_j, \\ ((b_{n-1} - a_{n-1})/2, b_{n-1}] & if \quad H\left((b_{n-1} - a_{n-1})/2\right) < \widetilde{U}_j, \end{cases} \tag{2.26}$$

until $b_n - a_n < \varepsilon$ for some a priori chosen small $\varepsilon > 0$. Then $U_j \approx a_n + (b_n - a_n)/2$.

Admittedly, this procedure is more computationally intensive than the accept-reject method, because for the latter we only need to compute the upper bound (2.22) once for each h, whereas in the former case we have to do the iteration (2.26) for each $\widetilde{U}_j$. Nevertheless, for the reason given in the next subsection our preferred method is the one in Assumption 2.

2.2.6 Continuity of the Simulated Values and Bids in the Candidate Value Distribution

The simulation procedure in Assumption 2 has the advantage that it is easier to prove that the simulated values and bids involved are continuous in F, in the following sense:

Lemma 4. *Let F_n and F be candidate value distributions such that*

$$\lim_{n\to\infty}\sup_{v>0}|F_n(v)-F(v)|=0 \tag{2.27}$$

For a given random drawing $\widetilde{U}$ from the uniform $[0,1]$ distribution, let $\widetilde{V}_n$ and $\widetilde{V}$ be the solutions of $F_n\left(\widetilde{V}_n\right)=\widetilde{U}$ and $F\left(\widetilde{V}\right)=\widetilde{U}$, respectively. Then conditional on $\widetilde{U}$,

$$\lim_{n\to\infty}\widetilde{V}_n=\widetilde{V}. \tag{2.28}$$

The simulated bids corresponding to F_n and F are, respectively,

$$\widetilde{B}_n=\left(\widetilde{V}_n-\left(\widetilde{V}_n-p_0\right)\widetilde{U}^{1-I}\int_0^1 F_n\left(p_0+u\left(\widetilde{V}_n-p_0\right)\right)^{I-1}du\right).I\left(\widetilde{U}>F_n\left(p_0\right)\right)$$
$$\widetilde{B}=\left(\widetilde{V}-\left(\widetilde{V}-p_0\right)\widetilde{U}^{1-I}\int_0^1 F\left(p_0+u\left(\widetilde{V}-p_0\right)\right)^{I-1}du\right).I\left(\widetilde{U}>F\left(p_0\right)\right)$$

if the reservation price p_0 is binding, and

$$\widetilde{B}_n=\widetilde{V}_n.\left(1-\widetilde{U}^{1-I}\int_0^1 F_n\left(u.\widetilde{V}_n\right)^{I-1}du\right),$$

$$\widetilde{B} = \widetilde{V}.\left(1 - \widetilde{U}^{1-I}\int_0^1 F\left(u.\widetilde{V}\right)^{I-1} du\right),$$

if the reservation price p_0 *is non-binding. Then conditional on* $\widetilde{U}$,

$$\lim_{n\to\infty} \widetilde{B}_n = \widetilde{B}. \tag{2.29}$$

Proof: Section 2.9.4.

Note that the bids $\widetilde{B}_n$ and $\widetilde{B}$ in Lemma 4 are functions of $\widetilde{U}$ and F_n and F, respectively:

$$\widetilde{B}_n = \eta\left(F_n, \widetilde{U}\right),\ \widetilde{B} = \eta\left(F, \widetilde{U}\right), \tag{2.30}$$

say, where $\eta(F, u)$ is continuous in F, in the sense that (2.27) implies that

$$\lim_{n\to\infty} \eta(F_n, u) = \eta(F, u) \text{ a.e. in } u \in [0, 1] \tag{2.31}$$

Consequently, under Assumption 2 the empirical characteristic function of the simulated bids can be written as

$$\widehat{\psi}(t|F) = \frac{1}{N}\sum_{j=1}^{N} \exp\left(i.t.\eta\left(F, \widetilde{U}_j\right)\right) \tag{2.32}$$

which for given t is an average of almost surely continuous functions of F. The latter plays a crucial role in the proof that pointwise in t, $\widehat{\psi}(t|F) \to \psi(t|F)$ a.s., uniformly on a metric space of distribution functions F endowed with the sup metric $\sup_{v>0} |F_1(v) - F_2(v)|$. See Theorem 1 below. Therefore, the asymptotic results in this paper will be based on the condition that the simulated values are generated according to the procedure in Assumption 2.

2.3 Semi-Nonparametric Density and Distribution Functions

2.3.1 Legendre Polynomials

In this subsection, we show how to approximate any density function $h(u)$ on the unit interval arbitrary close by using orthonormal Legendre polynomials.

Legendre polynomials of order $n \geq 2$ on the unit interval $[0,1]$ can be constructed recursively by

$$\rho_n(u) = \frac{\sqrt{2n-1}\sqrt{2n+1}}{n}(2u-1)\rho_{n-1}(u) - \frac{(n-1)\sqrt{2n+1}}{n\sqrt{2n-3}}\rho_{n-2}(u)$$

starting from

$$\rho_0(u) = 1,\ \rho_1(u) = \sqrt{3}(2u-1).$$

They are orthonormal, in the sense that

$$\int_0^1 \rho_m(u)\rho_k(u)du = \begin{cases} 1 & \text{for } m = k \\ 0 & \text{otherwise} \end{cases}$$

2.3.2 Compact Spaces of Density and Distribution Functions

Theorem 1 in Bierens (2007) states that the Legendre polynomials $\rho_k(u)$ form a complete orthonormal basis for the Hilbert space $L^2_B(0,1)$ of square-integrable Borel measurable real functions on $[0,1]$, endowed with the inner product $\langle f,g\rangle = \int_0^1 f(u)g(u)du$ and associated norm $||f||_2 = \sqrt{\langle f,f\rangle}$ and metric $||f-g||_2$. Hence, any square-integrable Borel measurable real function $q(u)$ on $[0,1]$ can be represented as $q(u) = \sum_{k=0}^{\infty} \gamma_k \rho_k(u)$ a.e. on $[0,1]$ where the γ_k's are the Fourier coefficients: $\gamma_k = \int_0^1 \rho_k(u)q(u)du$, satisfying $\sum_{k=0}^{\infty} \gamma_k^2 < \infty$. Therefore, every density function $h(u)$ on $[0,1]$ can be written as $h(u) = q(u)^2$ where $q(u) \in L^2_B(0,1)$, with $\int_0^1 q(u)^2 du = \sum_{k=0}^{\infty} \gamma_k^2 = 1$.

Without loss of generality we may assume that $\gamma_0 \in (0,1)$, because given $h(u)$ we may assume that $q(u) = \sqrt{h(u)}$. Therefore, the restriction $\sum_{k=0}^{\infty} \gamma_k^2 = 1$ can be imposed by reparameterizing the γ_k's as

$$\gamma_0 = \frac{1}{\sqrt{1+\sum_{k=1}^{\infty}\delta_k^2}}, \ \gamma_k = \frac{\delta_k}{\sqrt{1+\sum_{k=1}^{\infty}\delta_k^2}} \text{ for } k = 1,2,3,\ldots$$

Thus, any density function $h(u)$ on $[0,1]$ can be represented as

$$h(u) = \frac{(1+\sum_{k=1}^{\infty}\delta_k\rho_k(u))^2}{1+\sum_{k=1}^{\infty}\delta_k^2}, \text{ where } \sum_{k=1}^{\infty}\delta_k^2 < \infty, \tag{2.33}$$

and therefore any absolutely continuous distribution function $H(u)$ on $[0,1]$ takes the form (2.21), where $h(u)$ is of the form (2.33).

The standard consistency proof for parameter estimators of nonlinear parametric models requires that the parameters are confined to a compact subset of a Euclidean space. Since, indirectly, the density h in (2.20) plays the role of unknown parameter, we will first construct a compact metric space of densities on the unit interval. This can be done by imposing restrictions on the parameters δ_k in (2.33), as follows.

Lemma 5. *Let $\mathcal{D}$ be the space of density function $h(u)$ on $[0,1]$ of the form (2.33), where the parameters δ_k are restricted by the inequality*

$$|\delta_k| \leq c\left(1+\sqrt{k}\ln k\right)^{-1},\ k=1,2,3,.... \tag{2.34}$$

for an a priori chosen constant $c>0$. If we endow $\mathcal{D}$ with the L^1 metric

$$||h_1 - h_2||_1 = \int_0^1 |h_1(u) - h_2(u)|\, du, \tag{2.35}$$

then $\mathcal{D}$ is a compact metric space. Consequently, the corresponding space of absolutely continuous distribution functions on $[0,1]$,

$$\mathcal{H} = \left\{H(u) = \int_0^u h(x)dx,\ h \in \mathcal{D}\right\},$$

endowed with the "sup" metric $\sup_{0\leq u\leq 1}|H_1(u) - H_2(u)|$, is a compact metric space as well.

Proof: Bierens (2007).

Actually, the L^1 metric (2.35) is applicable to $\mathcal{H}$ as well. Endowing $\mathcal{H}$ with the L^1 metric has the advantage that then the two spaces $\mathcal{D}$ and $\mathcal{H}$ are in essence identical.

To construct compact spaces of densities and distribution functions on $(0, \infty)$,

Assumption 3. *Choose an absolutely continuous distribution function $G(v)$ with density $g(v)$, finite expectation $\int_0^\infty v g(v) dv < \infty$, and support $(0, \infty)$, as initial guess of the true value distribution.*

Then it follows straightforwardly from Lemma 5 that:

Lemma 6. *With $G(v)$ and $g(v)$ as in Assumption 3, the space*

$$\mathcal{D}(G) = \{f(v) = h(G(v))g(v),\ h \in \mathcal{D}\} \tag{2.36}$$

of densities on $(0, \infty)$, endowed with the L^1 metric

$$||f_1 - f_2|| = \int_0^\infty |f_1(v) - f_2(v)|\, dv. \tag{2.37}$$

is a compact metric space. Moreover, the corresponding space

$$\mathcal{F}(G) = \left\{F(v) = \int_0^v f(x)dx,\ f \in \mathcal{D}(G)\right\} \tag{2.38}$$

of absolutely continuous distribution functions on $(0, \infty)$, *endowed with the* sup *metric*

$$||F_1 - F_2|| = \sup_{v>0} |F(v) - F(v)|,$$

is a compact metric space as well.

Now $\mathcal{F}(G)$ is the "parameter" space of candidate value distributions $F(v)$, provided that:

Assumption 4. *The constant* $c > 0$ *in* (2.34) *is chosen so large that the density* $f_0(v)$ *of the true value distribution* $F_0(v)$ *is contained in* $\mathcal{D}(G)$.

Note that not all the densities in $\mathcal{D}(G)$ will have finite expectations. The reason is that it is always possible to select a sequence $f_n \in \mathcal{D}(G)$ with finite expectations such that for a density $f \in \mathcal{D}(G)$ with infinite expectation, $\lim_{n\to\infty} \int_0^\infty |f_n(v) - f(v)|\, dv = 0$. However, this is of no consequence, as long as the true value distribution F_0 has finite expectation, and Assumption 4 holds, because then the true bid distribution $\Lambda_0(b)$ has bounded support, so that Lemma 1 is applicable[10] for all bid distributions $\Lambda(b)$ corresponding to an $F \in \mathcal{F}(G)$.

[10]Recall that Lemma 1 does not require that the other bid distribution $\Lambda(b)$ has bounded support.

2.3.3 The Sieve Spaces

For a density function $h(u)$ in (2.33) and its associated parameter sequence $\{\delta_k\}_{k=1}^{\infty}$, let

$$h_n(u) = h(u|\delta_{\mathbf{n}}) = \frac{\left(1 + \sum_{k=1}^{n} \delta_k \rho_k(u)\right)^2}{1 + \sum_{k=1}^{n} \delta_k^2}, \text{ where } \delta_{\mathbf{n}} = (\delta_1, \dots, \delta_n)', \tag{2.39}$$

be the n-th order truncation of $h(u)$. The case $n = 0$ corresponds to the uniform density:

$$h_0(u) = 1.$$

Following Gallant and Nychka (1987) we will call this truncated density a SNP density function. It has been shown by Bierens (2007) that

$$\lim_{n \to \infty} \int_0^1 |h_n(u) - h(u)|\, du = 0. \tag{2.40}$$

Thus, defining the space of n-th order truncations of $h(u)$ by

$$\mathcal{D}_n = \left\{ h_n(u) = \frac{\left(1 + \sum_{k=1}^{n} \delta_k \rho_k(u)\right)^2}{1 + \sum_{k=1}^{n} \delta_k^2},\ |\delta_k| \leq c\left(1 + \sqrt{k} \ln k\right)^{-1} \text{ for } k \geq 1. \right\}, \tag{2.41}$$

it follows that for each $h \in \mathcal{D}$ there exists a sequence $h_n \in \mathcal{D}_n$ of SNP densities such that (2.40) holds. Consequently, defining

$$\mathcal{H}_n = \left\{ H_n(u) = \int_0^u h_n(v) du,\ h_n \in \mathcal{D}_n \right\} \tag{2.42}$$

it follows that for each distribution function $H \in \mathcal{H}$ there exists a sequence of SNP distribution functions $H_n \in \mathcal{H}_n$ such that

$$\lim_{n\to\infty} \sup_{0\leq u\leq 1} |H_n(u) - H(u)| = 0. \tag{2.43}$$

Note that the distribution functions $H_n(u)$ can easily be computed as an quadratic form in $\delta_{\mathbf{n}} = (\delta_1, \ldots, \delta_n)'$, using the approach in Bierens (2007).

The densities $h_n \in \mathcal{D}_n$ will be used to construct densities $f_n(v) = h_n(G(v))g(v)$ of candidate value distributions, where $G(v)$ and its density $g(v)$ are chosen in advance according to Assumption 3. The latter implies that

$$\int_0^\infty v f_n(v) dv = \int_0^\infty v h_n(G(v))g(v) dv \leq \sup_{0\leq u\leq 1} h_n(u) \int_0^\infty v g(v) dv < \infty$$

because each $h_n(u) \in \mathcal{D}_n$ is a squared polynomial of order n, with bounded coefficients, and therefore is bounded itself:

$$\overline{h}_n = \sup_{h_n \in \mathcal{D}_n} \sup_{0\leq u\leq 1} h_n(u) < \infty,$$

although it is possible that $\lim_{n\to\infty} \overline{h}_n = \infty$.

Similar to (2.36) and (2.38), define the increasing sets

$$\mathcal{D}_n(G) = \{f_n(v) = h_n(G(v))g(v),\ h_n \in \mathcal{D}_n\}, \tag{2.44}$$

$$\mathcal{F}_n(G) = \{F_n(v) = H_n(G(v)),\ H_n \in \mathcal{H}_n\}. \tag{2.45}$$

Then

Lemma 7. *Choose G as in Assumption 3. Then all the densities $f_n \in \mathcal{D}_n(G)$ have finite expectation. Moreover, for each density $f \in \mathcal{D}(G)$ there exists a sequence of densities $f_n \in \mathcal{D}_n(G)$ such that $\lim_{n\to\infty} \int_0^\infty |f_n(v) - f(v)|\, dv = 0$, and for each distribution function $F \in \mathcal{F}(G)$ there exists a sequence of distribution functions $F_n \in \mathcal{F}_n(G)$ such that $\lim_{n\to\infty} \sup_{v>0} |F_n(v) - F(v)| \leq \lim_{n\to\infty} \int_0^\infty |f_n(v) - f(v)|\, dv = 0$. Consequently, $\bigcup_{n=0}^\infty \mathcal{D}_n(G)$ is dense in $\mathcal{D}(G)$, and $\bigcup_{n=0}^\infty \mathcal{F}_n(G)$ is dense in $\mathcal{F}(G)$.*

The latter means that $\mathcal{D}(G)$ is the closure of $\bigcup_{n=0}^\infty \mathcal{D}_n(G)$, denoted by $\mathcal{D}(G) = \overline{\bigcup_{n=0}^\infty \mathcal{D}_n(G)}$, and similar, $\mathcal{F}(G) = \overline{\bigcup_{n=0}^\infty \mathcal{F}_n(G)}$.

The sequence of spaces $\mathcal{F}_n(G)$ now forms the sieve. Since the distribution functions in $\mathcal{F}_n(G)$ are parametric, with parameters $\delta_{\mathbf{n}} = (\delta_1, \ldots, \delta_n)'$, the computation of $\widehat{F}_n = \arg\min_{F\in\mathcal{F}_n(G)} \widehat{Q}(F)$ is feasible. In particular, $\widehat{F}_n$ can be computed via the simplex method of Nelder and Mead (1965).

2.4 Strong Consistency

2.4.1 General Almost Sure Convergence Results

To prove strong consistency of semi-nonparametric sieve estimators of non-Euclidean parameters, we need to generalize the standard consistency proof for parametric estimators to the non-Euclidean case. The uniform strong law of large numbers plays a key-role in proving consistency of parameter estimators. Therefore, we first generalize Jennrich's

(1969) uniform strong law of large numbers to random functions on compact metric spaces:

Theorem 1. *Let Θ be a compact metric space, and let $\Psi_j(\theta)$, $j = 1, 2, ..., N, ...$ be a sequence of real valued almost surely continuous i.i.d. random functions on Θ. If in addition*

$$E\left[\sup_{\theta\in\Theta} |\Psi_1(\theta)|\right] < \infty \tag{2.46}$$

then

$$\sup_{\theta\in\Theta} \left| \frac{1}{N} \sum_{j=1}^{N} \Psi_j(\theta) - \overline{\Psi}(\theta) \right| \to 0 \tag{2.47}$$

a.s., where $\overline{\Psi}(\theta) = E\left[\Psi_1(\theta)\right]$. [11] *This result carries over to complex-valued random functions $\Psi_j(\theta)$ if the conditions involved hold for* $\mathrm{Re}\left[\Psi_j(\theta)\right]$ *and* $\mathrm{Im}\left[\Psi_j(\theta)\right]$.

Proof: Section 2.9.5.

We need this result to prove that for fixed t the empirical characteristic function $\widehat{\psi}(t|F)$ of the simulated bids converges a.s. to the actual characteristic function $\psi(t|F)$ of the simulated bids, uniformly on the space $\mathcal{F}(G)$.

Note that by the standard strong law of large numbers the empirical characteristic function $\widehat{\varphi}(t)$ of the actual bids converges a.s. to the characteristic function $\varphi(t)$ of the bid distribution. Then it follows from the bounded convergence theorem

[11] Note that $\overline{\Psi}(\theta)$ is continuous.

that $\sup_{F\in\Theta}\left|\widehat{Q}(F)-\overline{Q}(F)\right| \to 0$ a.s. Using the fact that by Lemma 1, $\overline{Q}(F) = \overline{Q}(F_0)$ if and only if $F = F_0$, the strong consistency of the (infeasible) estimator $\widehat{F} = \arg\min_{F\in\Theta}\widehat{Q}(F)$ follows then from Theorem 2 below, which is a generalization of a standard consistency result for parametric estimators. See Jennrich (1969).

Theorem 2. *Let* $\widehat{Q}_N(\theta)$ *be a sequence of real valued random functions on a compact metric space* Θ *with metric* $\rho(\theta_1,\theta_2)$, *such that* $\sup_{\theta\in\Theta}\left|\widehat{Q}_N(\theta)-\overline{Q}(\theta)\right| \to 0$ *a.s., where* $\overline{Q}(\theta)$ *is a continuous real function on* Θ. *Let* $\widehat{\theta}_N = \arg\min_{\theta\in\Theta}\widehat{Q}_N(\theta)$ *and* $\theta_0 = \arg\min_{\theta\in\Theta}\overline{Q}(\theta)$. *Then for* $N\to\infty$,

$$\overline{Q}(\widehat{\theta}_N) \to \overline{Q}(\theta_0)\ a.s. \tag{2.48}$$

If θ_0 *is unique then* (2.48) *implies* $\rho(\widehat{\theta}_N,\theta_0)\to 0$ *a.s.*

Proof: Section 2.9.6.

Since in our case the parameter space Θ is a compact metric space of functions, the estimator $\widehat{\theta}_N$ cannot be computed in practice. The following sieve estimation result provides the solution to this problem:

Theorem 3. *Let the conditions of Theorem 2 be satisfied, including the uniqueness of* θ_0. *Let* $\{\Theta_n\}_{n=0}^{\infty}$ *be an increasing sequence of compact subspaces of* Θ *for which the* *computation of*

$$\widetilde{\theta}_{n,N} = \arg\min_{\theta \in \Theta_n} \widehat{Q}_N(\theta)$$

is feasible. Suppose that for each $\theta \in \Theta$ there exists a sequence $\theta_n \in \Theta_n$ such that $\lim_{n\to\infty} \rho(\theta_n, \theta) = 0$*. Let n_N be an arbitrary subsequence of n satisfying* $\lim_{N\to\infty} n_N = \infty$*, and denote the sieve estimator involved by $\widetilde{\theta}_N = \widetilde{\theta}_{n_N,N}$. Then $\rho(\widetilde{\theta}_N, \theta_0) \to 0$ a.s.*

Proof: Section 2.9.7.

2.4.2 Uniform Strong Consistency of the SNP-SIM Estimator of the Value Distribution

Given Assumption 2, the conditions of Theorem 1 are satisfied for the empirical characteristic function of the simulated bids in the form (2.32), for fixed t: The function $\Psi_j(\theta)$ in Theorem 1 takes the form

$$\Psi_j(\theta) = \exp\left(i.t.\eta\left(F, \widetilde{U}_j\right)\right),\ \theta = F,$$

which by (2.31) is a.s. continuous on $\mathcal{F}(G)$ ($= \Theta$ in Theorem 1). The limit function involved,

$$E\left[\Psi_1(\theta)\right] = \int_0^1 \exp\left(i.t.\eta\left(F, u\right)\right) du = \psi(t|F),\ \theta = F,$$

is of course continuous on $\mathcal{F}(G)$ as well. Thus, it follows from Theorem 1 that

$$\sup_{F\in\mathcal{F}(G)}\left|\widehat{\psi}(t|F)-\psi(t|F)\right|\to 0 \text{ a.s., pointwise in } t.$$

Moreover, by the standard strong law of large numbers,

$$\widehat{\varphi}(t)\to\varphi(t) \text{ a.s., pointwise in } t.$$

It follows now from the bounded convergence theorem that

$$\sup_{F\in\mathcal{F}(G)}\left|\widehat{Q}(F)-\overline{Q}(F)\right|\to 0 \text{ a.s.}$$

Furthermore, it follows from Lemma 1 that $\overline{Q}(F)=\overline{Q}(F_0)$ $(=0)$ if and only if $F=F_0$. Thus, all the conditions of Theorem 2 are satisfied: Denoting $\widehat{F}=\arg\min_{F\in\mathcal{F}(G)}\widehat{Q}(F)$ with corresponding density $\widehat{f}$, we have $\int_0^\infty\left|\widehat{f}(v)-f_0(v)\right|dv\to 0$ a.s., and therefore $\sup_{v>0}\left|\widehat{F}(v)-F_0(v)\right|\to 0$ a.s.

Of course, $\widehat{F}$ cannot be computed in practice. However, due to the results of Lemma 7, Theorem 3 is applicable:

Theorem 4. *Let n_N be an arbitrary subsequence of n such that $\lim_{N\to\infty}n_N=\infty$, and let*

$$\widetilde{F}=\arg\min_{F\in\mathcal{F}_{n_N}(G)}\widehat{Q}(F).$$

Then under Assumptions 1-4, $\sup_{v>0}\left|\widetilde{F}(v)-F_0(v)\right|\to 0$ *a.s.*

2.5 An Integrated Moment Test of the Validity of the First-Price Auction Model

2.5.1 The Test

If the IPV and/or the risk neutrality assumptions do not hold, the bid functions (2.1) and (2.2) no longer apply to the actual bids. Since the simulated bids are derived from these bid functions, we then have

$$\widehat{Q}(\widetilde{F}) \to \inf_{F \in \mathcal{F}(G)} \overline{Q}(F) > 0 \text{ a.s.}, \tag{2.49}$$

where $\widetilde{F}$ is the sieve estimator. This suggests to use $\widehat{Q}(\widetilde{F})$ as a basis for a consistent IM test of the null hypothesis that

H_0: *the actual bids come from a first-price sealed bid auction where values are independent, private and bidders are symmetric and risk-neutral,*

against the general alternative that

H_1: *the null hypothesis H_0 is false.*

The Integrated Moment (IM) test we will propose is based on the fact that similar to the results in Bierens (1990) and Bierens and Ploberger (1997) for the Integrated Conditional Moment (ICM) test,

Theorem 5. *Under* H_0,

$$\widehat{W}_N(.) = \sqrt{N}\left(\widehat{\psi}(.|F_0) - \widehat{\varphi}(.)\right) \Rightarrow W(.)$$

on $[-\kappa,\kappa]$, *where* $W(t)$ *is a complex-valued zero-mean Gaussian process on* $[-\kappa,\kappa]$ *with covariance function*

$$\Gamma(t_1,t_2) = E\left[\widehat{W}_N(t_1)\overline{\widehat{W}_N(t_2)}\right].$$

Hence[12]

$$N.\widehat{Q}(F_0) = \frac{1}{2\kappa}\int_{-\kappa}^{\kappa}\left|\widehat{W}_N(t)\right|^2 dt \to_d \frac{1}{2\kappa}\int_{-\kappa}^{\kappa}|W(t)|^2\, dt.$$

Note that this result does not imply that $N.\widehat{Q}(\widetilde{F}) \to_d \frac{1}{2\kappa}\int_{-\kappa}^{\kappa}|W(t)|^2\, dt$, because this requires that $\sqrt{N}\left(\widehat{\psi}(t|\widetilde{F}) - \widehat{\psi}(t|F_0)\right) \to_d 0$, which in its turn requires that the subsequence n_N in Theorem 4 is chosen such that $\sup_{v>0}\left|\widetilde{F}(v) - F_0(v)\right| = o_p\left(1/\sqrt{N}\right)$, together with some further conditions.[13] However, if

Assumption 5. *The true value distribution* F_0 *is of the SNP type itself*: $F_0 \in \cup_{n=1}^{\infty}\mathcal{F}_n(G)$,

[12] By the continuous mapping theorem.

[13] The latter is a conjecture based in the proof of Lemma 4.

then there exists a smallest natural number n_0 such that $F_0 \in \mathcal{F}_{n_0}(G)$, so that

$$N.\widehat{Q}(\widetilde{F}) \leq N.\widehat{Q}(F_0) \text{ for } n_N \geq n_0.$$

This suggests that upper bounds of the critical values of the test can be based on the limiting distribution of $N.\widehat{Q}(F_0)$. The consistency of this IM tests then follows from (2.49).

The proof that (2.49) holds under H_1 follows straightforwardly from the proof of Theorem 3. The result under H_0 follows from the fact that

Lemma 8. *The process* $\widehat{W}_N(.)$ *is tight*[14] *on* $[-\kappa, \kappa]$.

Proof: Section 2.9.8.

2.5.2 Bootstrap Critical Values

The function $\widehat{W}_N(t) = \sqrt{N}\left(\widehat{\psi}(t|F_0) - \widehat{\varphi}(t)\right)$ takes the form

$$\widehat{W}_N(t) = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \left(\exp\left(i.t.\widehat{B}_j\right) - \exp\left(i.t.B_j\right)\right)$$

where the $\widehat{B}_j$'s are the simulated bids corresponding to the true value distribution F_0, and the B_j's are the actual bids. Thus, the $\widehat{B}_j$'s are independent of the B_j's, but come from the same distribution. The problem in approximating the limiting process $W(t)$ is

[14]See Billingsley (1999) for the definition of tightness.

two-fold, namely that we cannot increase $N \to \infty$ because the B_j's are only observable for $j = 1, ..., N$, and F_0 is unknown. To overcome these problems, generate for large M simulated bids $\widetilde{B}_j$, $j = 1, 2, ..., 2.M$, from the bid distribution corresponding to the sieve estimator $\widetilde{F}$ of F_0, according to the approach in Assumption 2. Thus, draw $\widetilde{U}_j$, $j = 1, 2, ..., 2.M$, independently from the uniform $[0, 1]$ distribution, and generate the corresponding simulated bids $\widetilde{B}_j$ as in Lemma 4. Denote

$$\begin{aligned}\widetilde{W}_M\left(t|\widetilde{F}\right) &= \frac{1}{\sqrt{M}}\sum_{j=1}^{M}\exp\left(i.t.\widetilde{B}_j\right) - \frac{1}{\sqrt{M}}\sum_{j=M+1}^{2M}\exp\left(i.t.\widetilde{B}_j\right) \\ &= \frac{1}{\sqrt{M}}\sum_{j=1}^{M}\exp\left(i.t.\eta\left(\widetilde{F},\widetilde{U}_j\right)\right) - \frac{1}{\sqrt{M}}\sum_{j=M+1}^{2M}\exp\left(i.t.\eta\left(\widetilde{F},\widetilde{U}_j\right)\right).\end{aligned}$$

where $\eta(F, u)$ is (implicitly) defined by (2.30). Then similar to Lemma 8, $\widetilde{W}_M\left(t|\widetilde{F}\right)$ is tight on $[-\kappa, \kappa]$, conditional on $\widetilde{F}$. Hence for $M \to \infty$,

$$\widetilde{W}_M\left(.|\widetilde{F}\right) \Rightarrow W\left(.|\widetilde{F}\right) \text{ on } [-\kappa,\kappa], \text{ conditional on } \widetilde{F}$$

where $W\left(.|\widetilde{F}\right)$ is a complex-valued zero-mean Gaussian process with conditional covariance function

$$\widetilde{\Gamma}(t_1, t_2|\widetilde{F}) = E\left[\widetilde{W}_M\left(t_1|\widetilde{F}\right)\overline{\widetilde{W}_M\left(t_2|\widetilde{F}\right)}\middle|\widetilde{F}\right]$$

Lemma 9. *Under H_0 and the conditions of Theorem 4,*

$$\sup_{(t_1,t_2)\in[-\kappa,\kappa]\times[-\kappa,\kappa]}\left|\widetilde{\Gamma}(t_1,t_2|\widetilde{F}) - \Gamma\left(t_1,t_2\right)\right| \to 0 \text{ a.s.} \tag{2.50}$$

as $N \to \infty$, and consequently

$$\frac{1}{2\kappa}\int_{-\kappa}^{\kappa}\left|W\left(t|\widetilde{F}\right)\right|^{2}dt \to_{d} \frac{1}{2\kappa}\int_{-\kappa}^{\kappa}|W(t)|^{2}dt. \tag{2.51}$$

Hence, for $M \to \infty$ first, and then $N \to \infty$,

$$\frac{1}{2\kappa}\int_{-\kappa}^{\kappa}\left|\widetilde{W}_{M}\left(t|\widetilde{F}\right)\right|^{2}dt \to_{d} \frac{1}{2\kappa}\int_{-\kappa}^{\kappa}|W(t)|^{2}dt. \tag{2.52}$$

Proof: Section 2.9.9.

Therefore, bootstrap critical values of $\frac{1}{2\kappa}\int_{-\kappa}^{\kappa}|W(t)|^{2}dt$ can be computed as follows. First, choose a large M, say $M = 1000$. Next, generate $\widetilde{T}_{k} = \frac{1}{2\kappa}\int_{-\kappa}^{\kappa}\left|\widetilde{W}_{M}\left(t|\widetilde{F}\right)\right|^{2}dt$ independently for $k = 1, ..., K$, say $K = 500$, and sort the statistics $\widetilde{T}_{k}$ in increasing order. The $\alpha \times 100\%$ bootstrap critical value is then $\widetilde{T}_{(1-\alpha)K}$.

2.5.3 Critical Values Based on a Further Upper Bound

It has been shown by Bierens and Ploberger (1997) that

$$\frac{\frac{1}{2\kappa}\int_{-\kappa}^{\kappa}|W(t)|^{2}dt}{\frac{1}{2\kappa}\int_{-\kappa}^{\kappa}\Gamma(t,t)dt} \leq \sup\frac{1}{m}\sum_{k=1}^{m}\varepsilon_{j}^{2} = \overline{T},$$

say, where the ε_j's are independently $N(0,1)$ distributed. Therefore, with $\widehat{\Gamma}(t,t)$ a consistent estimator of $\Gamma(t,t)$, we have

$$\widetilde{T} = \frac{N.\widehat{Q}(\widetilde{F})}{\frac{1}{2\kappa}\int_{-\kappa}^{\kappa}\widehat{\Gamma}(t,t)dt} \leq \frac{N.\widehat{Q}(F_0)}{\frac{1}{2\kappa}\int_{-\kappa}^{\kappa}\widehat{\Gamma}(t,t)dt} \to_d \frac{\frac{1}{2\kappa}\int_{-\kappa}^{\kappa}|W(t)|^2\,dt}{\frac{1}{2\kappa}\int_{-\kappa}^{\kappa}\Gamma(t,t)dt} \leq \overline{T}.$$

The 5% and 10% critical values based on $\overline{T}$ are 4.26 and 3.23, respectively.

As to the choice of $\widehat{\Gamma}(t,t)$, note that

Lemma 10. $\Gamma(t,t) = 2 - 2|\varphi(t)|^2$, *where* $\varphi(t)$ *is the characteristic function of the actual bid distribution. Let* $\widehat{\varphi}(t)$ *be the empirical characteristic function involved. Then* $\widehat{\Gamma}(t,t) = 2 - 2|\widehat{\varphi}(t)|^2 \to \Gamma(t,t)$ *a.s., pointwise in* t.

Proof: Section 2.9.10.

2.6 Determination of the Sieve Order Via an Information Criterion

Recall that under Assumption 5 there exists a smallest natural number n_0 such that $F_0 \in \mathcal{F}_{n_0}(G)$. The question now arises how to estimate n_0 consistently.

For nested likelihood models this can be done via information criteria, for example the Hannan-Quinn (1979) or Schwarz (1978) information criteria. These information criteria are of the form

$$C_N(n) = \frac{-2}{N}\ln\left(L_N(n)\right) + n.\frac{\phi(N)}{N}$$

where $L_N(n)$ is the maximum likelihood of a models with n parameters, with $\phi(N) = \ln(N)$ for the Schwarz criterion and $\phi(N) = 2.\ln(\ln(N))$ for the Hannan-Quinn criterion. Then for $2 \leq n \leq n_0$,

$$p\lim_{N\to\infty}(C_N(n) - C_N(n-1)) = p\lim_{N\to\infty}\frac{2}{N}\ln(L_N(n-1)) - p\lim_{N\to\infty}\frac{2}{N}\ln(L_N(n))$$
$$< 0$$

whereas for $n > n_0$, $-2(\ln(L_N(n_0)) - \ln(L_N(n))) \to_d \chi^2_{n-n_0}$, hence

$$p\lim_{N\to\infty}\frac{N}{\phi(N)}(C_N(n) - C_N(n_0)) = n - n_0$$

The latter result only hinges on $-2(\ln(L_N(n_0)) - \ln(L_N(n))) = O_p(1)$.

Since under Assumption 5,

$$N\left(\inf_{F\in\mathcal{F}_n(G)}\widehat{Q}(F) - \inf_{F\in\mathcal{F}_{n_0}(G)}\widehat{Q}(F)\right) = O_p(1) \text{ if } n > n_0,$$

whereas for $2 \leq n \leq n_0$,

$$p\lim_{N\to\infty}\left(\inf_{F\in\mathcal{F}_n(G)}\widehat{Q}(F) - \inf_{F\in\mathcal{F}_{n-1}(G)}\widehat{Q}(F)\right) < 0$$

it seems that in our case we may replace $\frac{-2}{N}\ln(L_N(n))$ by $\inf_{F\in\mathcal{F}_n(G)}\widehat{Q}(F)$:

$$\widehat{C}_N(n) = \inf_{F\in\mathcal{F}_n(G)}\widehat{Q}(F) + n.\frac{\phi(N)}{N},$$

and estimate n_0 by $\widehat{n}_N = \arg\min \widehat{C}_N(n)$. Asymptotically that will work: $\lim_{N\to\infty} P[\widehat{n}_N = n_0] = 1$. However, in practice it will not, due to the fact that $\widehat{Q}(F)$ is bounded: $\sup_F \widehat{Q}(F) \leq 4$, and that $\inf_{F\in\mathcal{F}_n(G)} \widehat{Q}(F)$ will be close to zero if $n < n_0$ but not too far away from n_0, so that in small samples the penalty term $n.\phi(N)/N$ may dominate $\inf_{F\in\mathcal{F}_n(G)} \widehat{Q}(F)$ too much. Therefore, we propose the following modification of $\widehat{C}_N(n)$:

$$\widetilde{C}_N(n) = \inf_{F\in\mathcal{F}_n(G)} \widehat{Q}(F) + \Phi(n).\frac{\phi(N)}{N}, \tag{2.53}$$

$$\phi(N) = o(N), \quad \lim_{N\to\infty} \phi(N) = \infty.$$

where $\Phi(n)$ is an increasing but bounded function of n. For example, let for some $\alpha \in (0,1)$,

$$\Phi(n) = 1 - (n+1)^{-\alpha}.$$

Then similar to the Hannan-Quinn and Schwarz information criteria we have:

Theorem 6. *Let* $\widetilde{n}_N = \max_{s.t.\ \widetilde{C}_N(n)\leq\widetilde{C}_N(n-1)} n$ *and* $\widetilde{F} = \arg\min_{F\in\mathcal{F}_{\widetilde{n}_N}(G)} \widehat{Q}(F)$. *Under Assumption 5,* $\lim_{N\to\infty} P[\widetilde{n}_N = n_0] = 1$, *hence*

$$\sup_{v>0} \left| \widetilde{F}(v) - F_0(v) \right| \to 0 \ a.s.$$

If Assumption 5 is not true then $p\lim_{N\to\infty} \widetilde{n}_N = \infty$, *hence*

$$p\lim_{N\to\infty} \sup_{v>0} \left| \widetilde{F}(v) - F_0(v) \right| = 0.$$

Proof: Section 2.9.11.

2.7 Some Numerical Experiments

In this section we check the performance of the IM test of the validity of the first-price auction model, and the fit of SNP-SIM density estimator with estimated the truncation order $\widetilde{n}_N$, via a few numerical experiments. In all experiments we use the exponential distribution

$$G(v) = 1 - \exp(-v/3),\ g(v) = \frac{1}{3}\exp(-v/3) \tag{2.54}$$

as the initial guess for the value distribution, and the truncation order $\widetilde{n}_N$ is determined via the approach in Theorem 6, with information criterion

$$\widetilde{C}_N(n) = \inf_{F \in \mathcal{F}_n(G)} \widehat{Q}(F) + \left(1 - (n+1)^{-1/3}\right) \frac{\log_{10}(\log_{10}(N))}{N}.$$

The 5% and 10% bootstrap critical values of the IM test will be based on $K = 500$ bootstrap samples.

2.7.1 The IM Test

In this subsection, we check the performance of the IM test by two numerical examples. The first is the case where then null hypothesis that the observed bids can be rationalized by the first-price sealed bid auction model with independent private values (IPV) is false, and the second case is where this null hypothesis is true. In both cases we have generated bids from 500 identical and independent auctions, each with two sealed bids and no reservation price. In both cases the true value distribution is exponential, although different from the initial guess (2.54), namely

$$F_0(v) = 1 - \exp(-v),\ f_0(v) = \exp(-v).$$

In the first case the observed bids come from a second price auction with IPV and two symmetric, risk-neutral bidders. As is well known, in a second price auction, it is a weakly dominant strategy to bid the true value. See Krishna (2002). Therefore, the actual bids are drawn directly from the value distribution $F_0(v)$. As to the results, the estimate of the truncation order is $\widetilde{n}_N = 4$, and the value of the corresponding IM test statistic is $\widetilde{T} = 3.0531$. The 5% and 10% bootstrap critical values are $\widetilde{T}_{0.95K} = 1.1882$ and $\widetilde{T}_{0.90K} = 0.9447$, respectively.[15] Consequently, the null hypothesis is firmly rejected at the 5% significance level, as expected.

The second example is a first-price auction with risk-averse bidders. Again, we have generated bids from 500 auctions, where each auction has two risk-averse bidders, with utility function $U(x) = x^{1/2}$. In this case the equilibrium bid function has a closed form. See example 4.1 in Krishna (2002).[16] But, as it is pointed out in Krishna (2002), a first-price auction with two risk-bidders and value distribution $F_0(v)$ is observationally

[15]Thus, under the null hypothesis, $P(\widetilde{T} \geq \widetilde{T}_{0.95}) = 0.05$ and $P(\widetilde{T} \geq \widetilde{T}_{0.90}) = 0.1$.

[16]Note that if the number of risk-averse bidders is greater than two the equilibrium bid function no longer has a closed form.

equivalent to a first-price auction with two risk-averse neutral bidders with value distribution $F_0(v)^2$.[17] Therefore, in this case we may expect that our IM test will not reject the null hypothesis. The IM test statistic now takes the value $\widetilde{T} = 0.0004$, with 5% and 10% bootstrap critical values $\widetilde{T}_{0.95K} = 0.3630$ and $\widetilde{T}_{0.90K} = 0.2661$, respectively, based on the estimated truncation order $\widetilde{n}_N = 2$. Thus, the (true) null hypothesis is not rejected, which is the anticipated result.

2.7.2 The Fit

In the previous experiments the estimated truncation orders $\widetilde{n}_N = 4$ and $\widetilde{n}_N = 2$ are small, so the question arises whether for such a small truncation order the value density can be adequately approximated. In this section we check this for three cases. In each case we generate independently 200 auctions without a reservation price, where each auction consists of 5 bids whose private values come from a chi-square distribution, so in each case we have 1000 i.i.d. sample bids. The three cases only differ with respect to the degrees of freedom r of the chi-square distribution, namely $r = 3, 4, 5$, respectively.

In these cases the true value densities $f_0(v)$ are quite different from the density $g(v)$ of the initial guess (2.54), in particular the left tails, as shown in Figure 2.1: Thus, the SNP density $h_n(u)$ needs to convert the exponential density $g(v)$ into an approximation $f_n(v) = h_n(G(v))\,g(v)$ of a χ^2_r density, so that $h_n(u)$ needs to bend down the left tail of $g(v)$ towards zero. This seems challenging. However, it appears that the SNP density $h_n(u)$ has no problem doing that, even for small values of n.

[17] This holds when the utility function is a constant relative risk aversion (CRRA) utility function, $U(x) = x^{\alpha}$ with $0 < \alpha < 1$ and $U(0) = 0$. See example 4.1 in Krishna (2002).

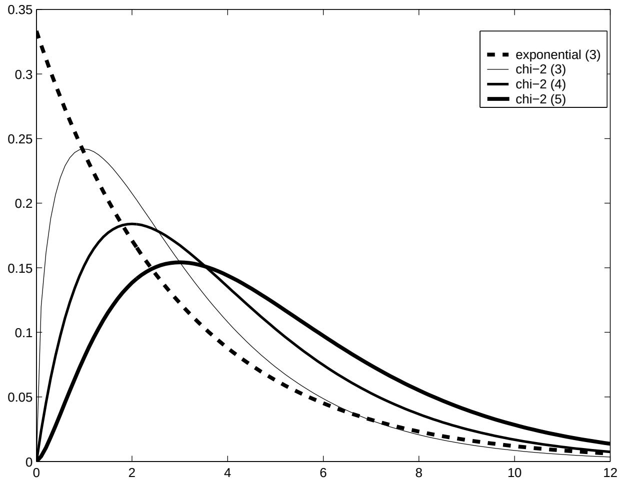

Fig. 2.1. $g(v) = \exp(-v/3)/3$ compared with the χ^2_r densities for $r = 3, 4, 5$

First, the estimated truncation orders are small: See Table 2.1 below.

To see whether these truncation orders are too small or not, we compare in Figures 2.2-2.4 the SNP sieve density estimators $f_{\widetilde{n}_N}(v) = h_{\widetilde{n}_N}(G(v))\, g(v)$ with the true χ^2_r value densities $f_0(v)$ for $r = 3, 4, 5$.

These figures show that our SNP-SIM estimation approach works remarkably well, certainly in view of the bad choice of the initial guess $g(v)$ for $f_0(v)$ (see Figure 2.1) and the small truncation orders. On the other hand, it seems from Figure 2.3 that the truncation order $\widetilde{n}_N = 2$ is somewhat too small, as the fit of $f_{\widetilde{n}_N}(v)$ for $\widetilde{n}_N = 4$ in Figures 2.2 and 2.4 looks better than in Figure 2.3.

Table 2.1.
Estimated Truncation Orders

r	3	4	5
$\widetilde{n}_N$	4	2	4

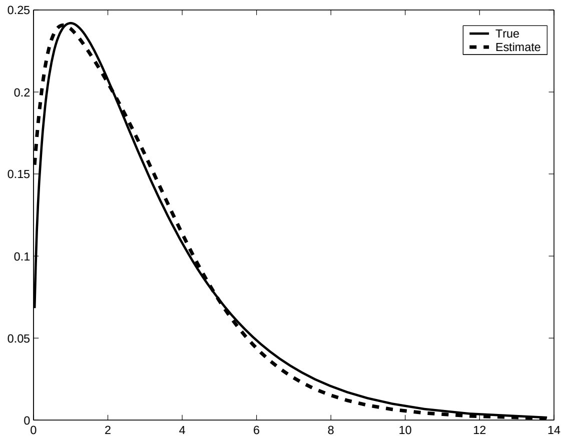

Fig. 2.2. $f_{\widetilde{n}_N}(v)$ (dashed curve) compared with the true χ_3^2 density $f_0(v)$

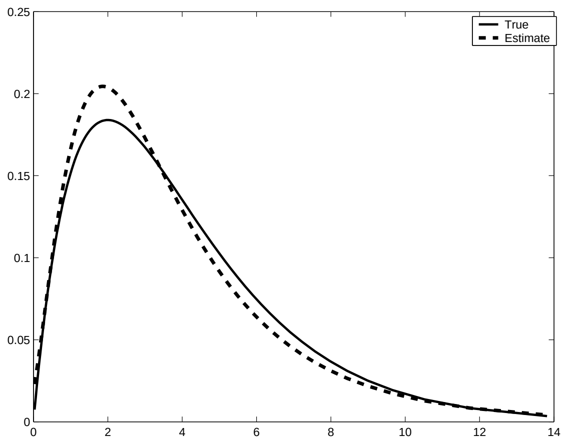

Fig. 2.3. $f_{\widetilde{n}_N}(v)$ (dashed curve) compared with the true χ_4^2 density $f_0(v)$

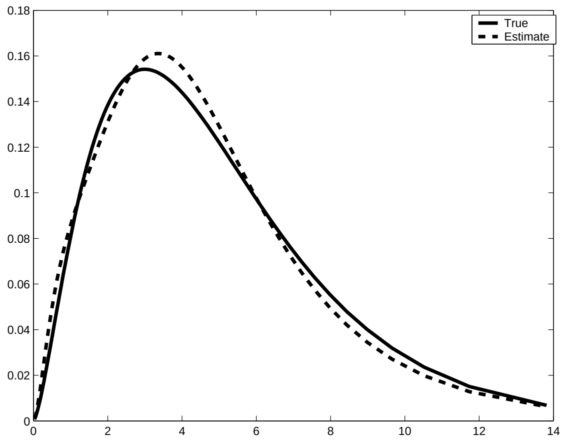

Fig. 2.4. $f_{\widetilde{n}_N}(v)$ (dashed curve) compared with the true χ_5^2 density $f_0(v)$

2.8 Concluding Remarks

In this chapter we have proposed a SNP-SIM method to estimate the value distribution of the first-price auction, based on the identification results in Chapter 1. Our SNP-SIM estimation method differs fundamentally from the nonparametric estimation approaches in the literature in that we estimate the value distribution directly, whereas in the nonparametric auction literature, the value distribution is estimated indirectly via kernel estimation of the inverse bid function. For general nonparametric approaches, see Athey and Haile (2005). Another novelty of our approach is that it yields, as a by-products an integrated moment test for the validity of the first-price auction model.

The approach in this paper can be extended to Dutch auctions and auctions with auction-specific heterogeneity. This will be left for future research.

2.9 Proofs

2.9.1 Proof of Lemma 2

Let F be continuous distribution function with support contained in $(\underline{v}, \overline{v})$, where $\underline{v} = \arg\min_{F(v)>0} v$ and $\overline{v} = \arg\max_{F(v)<1} v$. Suppose that F is invertible on $(\underline{v}, \overline{v})$, i.e., for each $u \in (0,1)$ there exists a unique $v \in (\underline{v}, \overline{v})$ such that $u = F(v)$. It is a standard textbook exercise to verify that then for a random drawing V from F,

$$F(V) \sim \text{Uniform}[0,1]. \tag{2.55}$$

If F is not invertible then there exists a $u \in (0,1)$ such that $F(v) = u$ for more than one $v \in (\underline{v}, \overline{v})$. In particular, for such a u let

$$v_1(u) = \inf_{u=F(v)} v, \;\; v_2(u) = \sup_{u=F(v)} v. \tag{2.56}$$

Note that by the continuity of $F(v)$, $F(v_1(u)) = F(v_2(u)) = u$, hence $F(v) = u$ for all $v \in [v_1(u), v_2(u)]$. Moreover, $F(v) < u$ for $v \in (\underline{v}, v_1(u))$ and $F(v) > u$ for $v \in (v_2(u), \overline{v})$. Then for such a u,

$$\begin{aligned}
P[F(V) \leq u] = E[\mathbf{I}(F(V) \leq u)] \\
&= \int_{\underline{v}}^{v_1(u)} \mathbf{I}(F(v) \leq u) f(v)\, dv + \int_{v_1(u)}^{v_2(u)} \mathbf{I}(F(v) \leq u) f(v)\, dv \\
&+ \int_{v_2(u)}^{\overline{v}} \mathbf{I}(F(v) \leq u) f(v)\, dv
\end{aligned}$$

$$= \int_{\underline{v}}^{v_1(u)} f(v)\, dv + \int_{v_1(u)}^{v_2(u)} f(v)\, dv = F(v_2(u)) = u$$

Since this result also holds if $v_1(u) = v_2(u)$, it follows that for all $u \in (0,1)$,

$$P[F(V) \leq u] = u.$$

Thus, the only requirement for (2.55) is that F is continuous.

To prove that, with U a random drawing from the uniform $[0,1]$ distribution, the solution V of $U = F(V)$ is a.s. unique, it suffices to prove that the set $S = \{u \in (0,1) : v_1(u) < v_2(u)\}$ has Lebesgue measure zero. The latter follows from the fact that for any pair $u_1, u_2 \in S$, $u_1 \neq u_2$, the intervals $(v_1(u_1), v_2(u_1))$ and $(v_1(u_2), v_2(u_2))$ are disjoint, which implies that S is countable because any collection of disjoint open intervals is countable.

Finally, $P[V \leq v] = F(v)$ follows trivially from

$$P[V \leq v] = P[F(V) \leq F(v)] = P[U \leq F(v)] = F(v).$$

Q.E.D.

2.9.2 Derivation of (2.18)

Note that $\left|\widehat{\Psi}(t|F)\right|^2 = \widehat{\Psi}(t|F)\overline{\widehat{\Psi}(t|F)}$, where $\overline{\widehat{\Psi}(t|F)} = \widehat{\Psi}(-t|F) = \widehat{\varphi}(-t) - \widehat{\psi}(-t|F)$ is the complex-conjugate of $\widehat{\Psi}(t|F) = \widehat{\varphi}(t) - \widehat{\psi}(t|F)$. Then it follows from (2.12) and (2.13) that

$$\widehat{Q}(F) = \frac{1}{2\kappa}\int_{-\kappa}^{\kappa} \widehat{\Psi}(t|F)\overline{\widehat{\Psi}(t|F)}dt$$

$$= \frac{1}{2\kappa}\int_{-\kappa}^{\kappa}\left(\frac{1}{N}\sum_{j=1}^{N}\exp(i.t.B_j) - \frac{1}{N}\sum_{j=1}^{N}\exp(i.t.\widetilde{B}_j)\right)$$

$$\times\left(\frac{1}{N}\sum_{j=1}^{N}\exp(-i.t.B_j) - \frac{1}{N}\sum_{j=1}^{N}\exp(-i.t.\widetilde{B}_j)\right)dt$$

$$= \frac{1}{N^2}\sum_{j_1=1}^{N}\sum_{j_2=1}^{N}\frac{1}{2\kappa}\int_{-\kappa}^{\kappa}\exp(i.t.(B_{j_1} - B_{j_2}))dt$$

$$- \frac{1}{N^2}\sum_{j_1=1}^{N}\sum_{j_2=1}^{N}\frac{1}{2\kappa}\int_{-\kappa}^{\kappa}\exp(i.t.(B_{j_1} - \widetilde{B}_{j_2}))dt$$

$$- \frac{1}{N^2}\sum_{j_1=1}^{N}\sum_{j_2=1}^{N}\frac{1}{2\kappa}\int_{-\kappa}^{\kappa}\exp(i.t.(\widetilde{B}_{j_1} - B_{j_2}))dt$$

$$+ \frac{1}{N^2}\sum_{j_1=1}^{N}\sum_{j_2=1}^{N}\frac{1}{2\kappa}\int_{-\kappa}^{\kappa}\exp(i.t.(\widetilde{B}_{j_1} - \widetilde{B}_{j_2}))dt$$

$$= \frac{2}{N^2}\sum_{j_1=1}^{N-1}\sum_{j_2=j_1+1}^{N}\frac{\sin\left(\kappa.(B_{j_1} - B_{j_2})\right)}{\kappa.(B_{j_1} - B_{j_2})}$$

$$+ \frac{2}{N^2}\sum_{j_1=1}^{N-1}\sum_{j_2=j_1+1}^{N}\frac{\sin\left(\kappa.(\widetilde{B}_{j_1} - \widetilde{B}_{j_2})\right)}{\kappa.(\widetilde{B}_{j_1} - \widetilde{B}_{j_2})}$$

$$- \frac{2}{N^2}\sum_{j_1=1}^{N}\sum_{j_2=1}^{N}\frac{\sin\left(\kappa.(B_{j_1} - \widetilde{B}_{j_2})\right)}{\kappa.(B_{j_1} - \widetilde{B}_{j_2})}.$$

Q.E.D.

2.9.3 Proof of Lemma 3

Since $X = X_0$ conditional on $U \le c^{-1} f(X_0)/g(X_0)$, and U and X_0 are independent, we have

$$\begin{aligned} P[X \le a] &= P\left[X_0 \le a \middle| U \le c^{-1} f(X_0)/g(X_0)\right] \\ &= \frac{P[X_0 \le a,\ U \le c^{-1} f(X_0)/g(X_0)]}{P[U \le c^{-1} f(X_0)/g(X_0)]} \\ &= \frac{E\left(E\left[\mathbf{I}(X_0 \le a) I\left(U \le c^{-1} f(X_0)/g(X_0)\right) \middle| X_0\right]\right)}{E\left(E\left[\mathbf{I}\left(U \le c^{-1} f(X_0)/g(X_0)\right) \middle| X_0\right]\right)} \\ &= \frac{E\left[\mathbf{I}(X_0 \le a)\left(f(X_0)/g(X_0)\right)\right]}{E\left[f(X_0)/g(X_0)\right]} \\ &= \frac{\int_{-\infty}^{a} f(x)dx}{\int_{-\infty}^{\infty} f(x)dx} = F(a) \end{aligned}$$

Q.E.D.

2.9.4 Proof of Lemma 4

We only consider the binding reservation price case. First, we show that

$$\lim_{n\to\infty} \widetilde{B}_n = 0 \text{ if and only if } \widetilde{B} = 0, \tag{2.57}$$

as follows. Suppose that $\widetilde{B} = 0$ and $\widetilde{B}_n > p_0$, so that $F(p_0) \leq \widetilde{U} < F_n(p_0)$. Then it follows from (2.27) that there exists an $n_0(\widetilde{U})$ depending on $\widetilde{U}$ such that for all $n \geq n_0(\widetilde{U})$,

$$I\left(F(p_0) \leq \widetilde{U} < F_n(p_0)\right) = 0$$

Thus, if $\widetilde{B} = 0$ then $\lim_{n\to\infty} \widetilde{B}_n = 0$. Similarly, $\lim_{n\to\infty} \widetilde{B}_n = 0$ only if $\widetilde{B} = 0$.

Next we show that (2.28) is true, by contradiction. Suppose that $\limsup_{n\to\infty} \widetilde{V}_n > \widetilde{V}$. Then there exists a subsequence n_m and an $\varepsilon > 0$ such that for all m, $\widetilde{V}_{n_m} > \widetilde{V} + \varepsilon$. But then by (2.27)

$$\widetilde{U} = F_{n_m}\left(\widetilde{V}_{n_m}\right) \geq F_{n_m}\left(\widetilde{V} + \varepsilon\right) > F_{n_m}\left(\widetilde{V}\right) \to F\left(\widetilde{V}\right) = \widetilde{U}$$

which is impossible. Thus, $\limsup_{n\to\infty} \widetilde{V}_n \leq \widetilde{V}$. Similarly, it follows that $\liminf_{n\to\infty} \widetilde{V}_n \geq \widetilde{V}$. Thus (2.28) is true.

Finally, it follows straightforwardly from (2.27) and (2.28) that

$$\lim_{n\to\infty} \int_0^1 F_n\left(p_0 + u(\widetilde{V}_n - p_0)\right)^{I-1} du = \int_0^1 F\left(p_0 + u(\widetilde{V} - p_0)\right)^{I-1} du. \qquad (2.58)$$

The result (2.29) now follows from (2.57), (2.28) and (2.58). Q.E.D.

2.9.5 Proof of Theorem 1

Originally this uniform strong law was derived by Jennrich (1969, Theorem 2) for the case that Θ is a compact subset of a Euclidean space and $\Psi_j(\theta) = \Psi(X_j, \theta)$, where X_j is an i.i.d. sequence of random vectors in a Euclidean space with support $\mathcal{X}$, $\Psi(x,\theta)$ is Borel measurable in x for each $\theta \in \Theta$, and $\Psi(x,\theta)$ is continuous in θ for each $x \in \mathcal{X}$. However, it is not hard to verify from the more detailed proof in Bierens (2004, Appendix to Chapter 6) of Jennrich's result that this law carries over to a.s. continuous random functions on a compact metric space Θ with metric $\rho(\theta_1,\theta_2)$, provided that for each $\theta_0 \in \Theta$ and arbitrary $\delta > 0$, $\sup_{\theta\in\Theta,\rho(\theta,\theta_0)\leq\delta} \Psi_j(\theta)$ and $\inf_{\theta\in\Theta,\rho(\theta,\theta_0)\leq\delta} \Psi_j(\theta)$ are measurable, because then by the a.s. continuity condition,

$$\lim_{\delta\downarrow 0}\left(E\left[\sup_{\theta\in\Theta,\rho(\theta,\theta_0)\leq\delta} \Psi_j(\theta)\right] - E\left[\inf_{\theta\in\Theta,\rho(\theta,\theta_0)\leq\delta} \Psi_j(\theta)\right]\right) = 0,$$

$$\lim_{\delta\downarrow 0}\left(\sup_{\theta\in\Theta,\rho(\theta,\theta_0)\leq\delta} E\left[\Psi_j(\theta)\right] - \inf_{\theta\in\Theta,\rho(\theta,\theta_0)\leq\delta} E\left[\Psi_j(\theta)\right]\right) = 0,$$

where the expectations are well-defined. These results play a key-role in the proof.

To prove the measurability of $\sup_{\theta\in\Theta,\rho(\theta,\theta_0)\leq\delta} \Psi_j(\theta)$ and $\inf_{\theta\in\Theta,\rho(\theta,\theta_0)\leq\delta} \Psi_j(\theta)$ along the lines of the proof of Lemma 2 of Jennrich(1969), we first establish the existence of an increasing sequence of **finite** subsets Θ_n of Θ which is dense in Θ, i.e., Θ is the closure of $\bigcup_{n=1}^{\infty}\Theta_n$. These sets Θ_n can be constructed as follows. For each $\theta \in \Theta$ and n, let $U_n(\theta) = \{\theta_* \in \Theta : \rho(\theta,\theta_*) < 1/n\}$. Then $\bigcup_{\theta\in\Theta} U_n(\theta)$ is a open covering of Θ hence by the definition of compactness there exists a finite set $\Theta_n = \left\{\theta_{1,n}, ..., \theta_{M_n,n}\right\}$ such that $\Theta \subset \bigcup_{\theta\in\Theta_n} U_n(\theta)$. To show that $\bigcup_{n=1}^{\infty}\Theta_n$ is dense in Θ, pick an arbitrary $\theta \in \Theta$, and observe that for each n there exists an $\theta_n \in \Theta_n$ such that $\rho(\theta,\theta_n) < 1/n$. Therefore, for each $\theta \in \Theta$ there exists a sequence $\{\theta_n\}$ in $\bigcup_{n=1}^{\infty}\Theta_n$ such that $\lim_{n\to\infty} \rho(\theta,\theta_n)$, hence $\bigcup_{n=1}^{\infty}\Theta_n$ is dense in Θ. Consequently

$$\sup_{\theta\in\Theta}\Psi_1(\theta)=\lim_{n\to\infty}\sup_{\theta\in\Theta_n}\Psi_1(\theta)=\lim_{n\to\infty}\sup\max_{\theta\in\Theta_n}\Psi_1(\theta).$$

Since Θ_n is finite, $\max_{\theta\in\Theta_n}\Psi_1(\theta)$ is measurable, hence $\sup_{\theta\in\Theta}\Psi_1(\theta)$ is measurable. The same holds for the "inf" case, and for $\sup_{\theta\in\Theta,\rho(\theta,\theta_0)\leq\delta}\Psi_j(\theta)$ and $\inf_{\theta\in\Theta,\rho(\theta,\theta_0)\leq\delta}\Psi_j(\theta)$, because the sets $\{\theta\in\Theta,\rho(\theta,\theta_0)\leq\delta\}$ are compact. Q.E.D.

2.9.6 Proof of Theorem 2

The key of the proof of Theorem 2 is the easy inequality

$$\begin{aligned}0&\leq\overline{Q}\left(\widehat{\theta}_N\right)-\overline{Q}\left(\theta_0\right)\leq\overline{Q}\left(\widehat{\theta}_N\right)-\widehat{Q}_N\left(\widehat{\theta}_N\right)+\widehat{Q}_N\left(\theta_0\right)-\overline{Q}\left(\theta_0\right)\\&\leq 2\sup_{\theta\in\Theta}\left|\widehat{Q}_N(\theta)-\overline{Q}(\theta)\right|,\end{aligned}$$

so that

$$\overline{Q}\left(\widehat{\theta}_N\right)\to\overline{Q}\left(\theta_0\right)\ \text{a.s.}\tag{2.59}$$

The rest of the proof is now similar to the case where Θ is a compact subset of a Euclidean space:[18] Since $\left\{\widehat{\theta}_N\right\}$ is an infinite sequence in a compact space Θ, it has at least one limit point θ_*, say, and all the limit points are contained in Θ. But (2.59) implies that all the limit points θ_* of $\left\{\widehat{\theta}_N\right\}$ are equal to θ_0, a.s., which proves the theorem. Q.E.D.

[18] See Jennrich (1969) or the proof of Theorem 6.14 in Bierens (2004, p. 174).

2.9.7 Proof of Theorem 3

It suffices to show that $\overline{Q}(\widetilde{\theta}_N) \to \overline{Q}(\theta_0)$ a.s., because then the rest of the proof is the same as for Theorem 2.

We can choose a sequence $\theta_n \in \Theta_n$ such that

$$\lim_{n\to\infty} \rho(\theta_n, \theta_0). \tag{2.60}$$

Then

$$0 \le \overline{Q}(\widetilde{\theta}_N) - \overline{Q}(\theta_0) = \overline{Q}(\widetilde{\theta}_N) - \widehat{Q}_N(\widetilde{\theta}_N) + \widehat{Q}_N(\widetilde{\theta}_N) - \overline{Q}(\theta_0)$$

$$\le \sup_{\theta\in\Theta_{n_N}} \left|\widehat{Q}_N(\theta) - \overline{Q}(\theta)\right| + \widehat{Q}_N(\theta_{n_N}) - \overline{Q}(\theta_{n_N}) + \overline{Q}(\theta_{n_N}) - \overline{Q}(\theta_0)$$

$$\le 2 \sup_{\theta\in\Theta_{n_N}} \left|\widehat{Q}_N(\theta) - \overline{Q}(\theta)\right| + \overline{Q}(\theta_{n_N}) - \overline{Q}(\theta_0)$$

$$\le 2 \sup_{\theta\in\Theta} \left|\widehat{Q}_N(\theta) - \overline{Q}(\theta)\right| + \overline{Q}(\theta_{n_N}) - \overline{Q}(\theta_0) \to 0 \text{ a.s.}$$

because $\sup_{\theta\in\Theta} \left|\widehat{Q}_N(\theta) - \overline{Q}(\theta)\right| \to 0$ a.s. by the conditions of Theorem 2, and $\lim_{N\to\infty} \overline{Q}(\theta_{n_N}) = \overline{Q}(\theta_0)$ by (2.60) and the continuity of $\overline{Q}(\theta)$ on Θ. Q.E.D.

2.9.8 Proof of Lemma 8

Similar to the proof of Lemma 4 in Bierens (1990), we need to show that the following two conditions hold:

(i) For each $\delta > 0$ and an arbitrary $t_0 \in [-\kappa, \kappa]$, there exists an ε such that

$$\sup_N P\left(\widehat{W}_N(t_0) > \varepsilon\right) \leq \delta$$

(ii) For each $\delta > 0$ and $\varepsilon > 0$, there exists an $\xi > 0$ such that for $t_1, t_2 \in [-\kappa, \kappa]$,

$$\sup_N P\left(\sup_{|t_1 - t_2| < \xi} |\widehat{W}_N(t_1) - \widehat{W}_N(t_2)| \geq \varepsilon\right) \leq \delta.$$

Condition (i) follows from the fact that for arbitrary $t \in [-\kappa, \kappa]$,

$$\left(\mathrm{Re}\left[\widehat{W}_N(t)\right], \mathrm{Im}\left[\widehat{W}_N(t)\right]\right)'$$

converges in distribution to a bivariate normal distribution. Condition (ii) follows from Chebishev's inequality:

$$P\left(\sup_{|t_1 - t_2| < \xi} |\widehat{W}_N(t_1) - \widehat{W}_N(t_2)| \geq \varepsilon\right) \leq \frac{E\left[\sup_{|t_1 - t_2| < \xi} |\widehat{W}_N(t_1) - \widehat{W}_N(t_2)|\right]}{|\varepsilon|}$$

$$\leq 2.\xi E\left[|B_1|\right]/|\varepsilon| = \delta$$

and the fact that $E\left[|B_1|\right] < \infty$, where the last inequality follows from

$$E\left[\sup_{|t_1-t_2|<\xi} |\hat{W}_N(t_1) - \hat{W}_N(t_2)|\right]$$

$$\leq \frac{1}{N}\sum_{j=1}^{N} E\left[\sup_{|t_1-t_2|<\xi} \left|\left(\exp\left(i.t_1 B_j\right) - \exp\left(i.t_1 \widehat{B}_j\right)\right) - \left(\exp\left(i.t_2 B_j\right) - \exp\left(i.t_2 \widehat{B}_j\right)\right)\right|\right]$$

$$\leq E\left[\sup_{|t_1-t_2|<\xi} |\exp\left(i.t_1 B_1\right) - \exp\left(i.t_2 B_1\right)|\right] + E\left[\sup_{|t_1-t_2|<\xi} \left|\exp\left(i.t_1 \widehat{B}_1\right) - \exp\left(i.t_2 \widehat{B}_1\right)\right|\right]$$

$$= 2.E\left[\sup_{|t_1-t_2|<\xi} |\exp\left(i.t_1 B_1\right) - \exp\left(i.t_2 B_1\right)|\right] \leq 2.\xi E\left[|B_1|\right]$$

Q.E.D.

2.9.9 Proof of Lemma 9

Part (2.50) of Lemma 9 follows from

$$\widetilde{\Gamma}(t_1, t_2|\widetilde{F}) = \tag{2.61}$$

$$E\left[\left(\frac{1}{\sqrt{M}}\sum_{j=1}^{M}\left(\exp\left(i.t_1.\eta\left(\widetilde{F}, \widetilde{U}_j\right)\right) - \exp\left(i.t_1.\eta\left(\widetilde{F}, \widetilde{U}_{j+M}\right)\right)\right)\right)\right.$$

$$\left.\times\left(\frac{1}{\sqrt{M}}\sum_{j=1}^{M}\left(\exp\left(-i.t_2.\eta\left(\widetilde{F}, \widetilde{U}_j\right)\right) - \exp\left(-i.t_2.\eta\left(\widetilde{F}, \widetilde{U}_{j+M}\right)\right)\right)\right)\middle|\, \widetilde{F}\right]$$

$$= \frac{1}{M}\sum_{j=1}^{M} E\left[\left(\exp\left(i.t_1.\eta\left(\widetilde{F}, \widetilde{U}_j\right)\right) - \exp\left(i.t_1.\eta\left(\widetilde{F}, \widetilde{U}_{j+M}\right)\right)\right)\right.$$

$$\left.\times\left(\exp\left(-i.t_2.\eta\left(\widetilde{F}, \widetilde{U}_j\right)\right) - \exp\left(-i.t_2.\eta\left(\widetilde{F}, \widetilde{U}_{j+M}\right)\right)\right)\middle|\, \widetilde{F}\right]$$

$$= 2\int_0^1 \exp\left(i.\left(t_1 - t_2\right).\eta\left(\widetilde{F}, u\right)\right) du$$

$$-2\int_0^1 \exp\left(i.t_1.\eta\left(\widetilde{F},u\right)\right)du\int_0^1 \exp\left(-i.t_2.\eta\left(\widetilde{F},u\right)\right)du$$
$$= 2.\int_0^1 \cos\left(\left(t_1-t_2\right).\eta\left(\widetilde{F},u\right)\right)du + 2.i.\int_0^1 \sin\left(\left(t_1-t_2\right).\eta\left(\widetilde{F},u\right)\right)du$$
$$-2\left(\int_0^1 \cos\left(t_1.\eta\left(\widetilde{F},u\right)\right)du + i.\int_0^1 \sin\left(t_1.\eta\left(\widetilde{F},u\right)\right)du\right)$$
$$\times\left(\int_0^1 \cos\left(t_2.\eta\left(\widetilde{F},u\right)\right)du - i.\int_0^1 \sin\left(t_2.\eta\left(\widetilde{F},u\right)\right)du\right),$$

the fact that similar to (2.31), $\eta\left(\widetilde{F},u\right) \to \eta\left(F_0,u\right)$ a.s., (a.e.) pointwise in $u \in [0,1]$, and the bounded convergence theorem. The results (2.51) and (2.52) follow now from the continuous mapping theorem and the fact that zero-mean Gaussian processes are completely determined by their covariance functions. Q.E.D.

2.9.10 Proof of Lemma 10

Similar to (2.61) it follows that

$$\begin{aligned}\Gamma(t,t) &= 2-2\left(E\left[\cos\left(t.B_1\right)\right]+i.E\left[\sin\left(t.B_1\right)\right]\right)\\ &\quad\times\left(E\left[\cos\left(t.B_1\right)\right]-i.E\left[\sin\left(t.B_1\right)\right]\right)\\ &= 2-2\left(E\left[\cos\left(t.B_1\right)\right]\right)^2-2\left(E\left[\sin\left(t.B_1\right)\right]\right)^2\\ &= 2-2\left|\varphi\left(t\right)\right|^2\end{aligned}$$

where $\varphi\left(t\right)$ is the characteristic function of B_1. Therefore, $\widehat{\Gamma}(t,t) = 2-2\left|\widehat{\varphi}\left(t\right)\right|^2$ is a consistent estimator of $\Gamma(t,t)$. Q.E.D.

2.9.11 Proof of Theorem 6

The event $\widetilde{n}_N = n_0$ is equivalent to

$$\max_{1\leq n\leq n_0}\left(\widetilde{C}_N(n)-\widetilde{C}_N(n-1)\right)\leq 0 \text{ and } \widetilde{C}_N(n_0+1)-\widetilde{C}_N(n_0)>0.$$

so that

$$\begin{aligned}P\left[\widetilde{n}_N\neq n_0\right]\leq P&\left[\max_{1\leq n\leq n_0}\left(\widetilde{C}_N(n)-\widetilde{C}_N(n-1)\right)>0\right] \\ &+P\left[\widetilde{C}_N(n_0+1)-\widetilde{C}_N(n_0)\leq 0\right]\end{aligned} \tag{2.62}$$

For fixed $n\leq n_0$,

$$\widetilde{C}_N(n)-\widetilde{C}_N(n-1)\rightarrow \inf_{F\in\mathcal{F}_n(G)}\overline{Q}(F)-\inf_{F\in\mathcal{F}_{n-1}(G)}\overline{Q}(F)\leq 0 \text{ a.s.}$$

hence

$$\lim_{N\to\infty}P\left[\max_{1\leq n\leq n_0}\left(\widetilde{C}_N(n)-\widetilde{C}_N(n-1)\right)>0\right]=0. \tag{2.63}$$

For $n=n_0+1$,

$$\begin{aligned}&\left|N\left(\widetilde{C}_N(n_0+1)-\widetilde{C}_N(n_0)\right)-\phi(N)\left(\Phi(n_0+1)-\Phi(n_0)\right)\right|\\ &=N\left(\inf_{F\in\mathcal{F}_{n_0}(G)}\widehat{Q}(F)-\inf_{F\in\mathcal{F}_{n_0+1}(G)}\widehat{Q}(F)\right)\\ &\leq N\left(\inf_{F\in\mathcal{F}_{n_0}(G)}\widehat{Q}(F)+\inf_{F\in\mathcal{F}_{n_0+1}(G)}\widehat{Q}(F)\right)\leq 2.N.\widehat{Q}(F_0)\end{aligned}$$

so that with probability 1,

$$\frac{N}{\phi(N)}\left(\widetilde{C}_N(n_0+1)-\widetilde{C}_N(n_0)\right)\geq \Phi(n_0+1)-\Phi(n_0)-2.\frac{N.\widehat{Q}(F_0)}{\phi(N)}$$

Therefore,

$$\lim_{N\to\infty} P\left[\widetilde{C}_N(n_0+1)-\widetilde{C}_N(n_0)\leq 0\right]$$
$$\leq \lim_{N\to\infty} P\left[\frac{N.\widehat{Q}(F_0)}{\phi(N)}\geq \frac{1}{2}\left(\Phi(n_0+1)-\Phi(n_0)\right)\right]=0 \qquad (2.64)$$

because $N.\widehat{Q}(F_0)/\phi(N) = O_p\left(1/\phi(N)\right) = o_p(1)$. It follows now from (2.62), (2.63) and (2.64) that $\lim_{N\to\infty} P\left[\widetilde{n}_N = n_0\right] = 1$.

In the case $n_0 = \infty$ it follows from (2.63) that for any $\overline{n}\geq 1$, $\lim_{N\to\infty} P\left[\widetilde{n}_N \geq \overline{n}\right]$ $= 1$, which implies that $p\lim_{N\to\infty}\widetilde{n}_N = \infty$. Since for each n we can choose an $F_n \in \mathcal{F}_n(G)$ such that $\lim_{n\to\infty}\sup_{v>0}\left|F_n(v)-F_0(v)\right| = 0$, it follows that for this sequence F_n, $p\lim_{N\to\infty}\sup_{v>0}\left|F_{\widetilde{n}_N}(v)-F_0(v)\right| = 0$. Hence

$$p\lim_{N\to\infty}\overline{Q}\left(F_{\widetilde{n}_N}\right)=\overline{Q}\left(F_0\right)$$

The result in Theorem 6 for the case $n_0 = \infty$ now follows from the proof of Theorem 3, adapted to the “plim” case. Q.E.D.

Chapter 3

Semi-Nonparametric Simulated Integrated Conditional Moments Estimation of First-Price Auctions with Auction-Specific Heterogeneity

3.1 Introduction

In many repeated auctions, the objects to be auctioned off are different across auctions. Consequently, the value distributions are different across auctions. However, if we observe the auction-specific characteristics in the form of covariates, and the value distributions conditional on these covariates have the same functional form, the conditional bid distribution given the auction-specific covariates will be the same for all auctions. Then the question of how to incorporate the observable characteristics into the auction model arises. Laffont, Ossard and Vuong (1995) incorporate covariates in the value distribution by specifying a linear regression model for the log of values with zero-mean normal errors. Donald and Paarsch (1996) parameterize the upper bound of the values as a function of covariates. Li (2005) specifies the value distribution as the exponential distribution with a mean of a linear function of covariates. Guerre, Perrigne and Vuong (2000) propose a two-stage nonparametric kernel density estimation approach. In the first stage the bid distribution and density conditional on the covariates are estimated nonparametrically, which are then used in inverse form to generate values given the actual bids and the covariates. The generated values are then used to estimate the conditional value distribution nonparametrically.

In this chapter, we propose an alternative semi-nonparametric approach to estimating first-price auction models with observed auction-specific heterogeneity and private, symmetric and independent values conditional on the auction specific covariates.[1] This approach extends the semi-nonparametric simulated integrated moments estimation (SNP-SIM) method of Chapter 2 to the heterogenous first-price auction model with observable auction-specific covariates. We consider a first-price auction model where the log value takes the form of a median regression model conditional on covariates, with unknown error distribution. The latter distribution is modeled semi-nonparametrically using orthonormal Legendre polynomials, similar to the approach in Bierens (2007). Given a parametric specification of the median function, we generate for each auction artificial bids conditional on the auction-specific covariates. Next, we take the difference of the empirical characteristic functions of the actual bids and the simulated bids, both jointly with the covariates, as the moment conditions. Integrating the squared difference of these empirical characteristic functions yields an integrated conditional moment (ICM) objective function, similar to the ICM test statistic proposed by Bierens (1982) and Bierens and Ploberger (1997). Minimizing this ICM objective function to the median regression parameters and the corresponding semi-nonparametric error distribution via a sieve method then yields a consistent estimator of the conditional value distribution. Similar to Chapter 2 we propose a data-driven sieve order selection procedure based on an information criterion. Moreover, the minimum value of the ICM objective function

[1] Thus, asymmetry and risk-aversion is beyond the scope of this paper.

can be used as a test statistic of a consistent ICM test for the validity of the model, similar to Bierens (1990) and Bierens and Ploberger (1997).

The parametric specification of the median regression function of the log values is a matter of convenience rather than a necessity. In Chapter 1, it has been have shown that this median regression function is nonparametrically identified, provided that the errors of the model are independent of the covariates. Therefore, it is possible in principle to estimate the median regression function semi-nonparametrically as well.

Throughout the paper, we denote random variables in upper-case and non-random variables in lower-case. The indicator function is denoted by $\mathbf{I}(\cdot)$.[2] Almost sure (a.s.) convergence is denoted by $X_n \to X$ a.s.[3] Similarly, convergence in probability will be denoted by $X_n \to_p X$ or $p\lim_{n\to\infty} X_n = X$, and $X_n \to_d X$ indicates that X_n converges in distribution to X. In the case that X_n and X are random functions we use the notation $X_n \Rightarrow X$ to indicate that $X_n(\cdot)$ converges weakly to $X(\cdot)$. See for example Billingsley (1999) for the meaning of the notion of weak convergence.

3.2 Model and Data-Generating Process

3.2.1 The Equilibrium Bid Function

Given a vector X of auction-specific characteristics, let $F(v|X)$ be the conditional distribution of the private value V that each potential bidder has for the object to be auctioned off, and let

$$\underline{v}(X) = \inf_{F(v|X)>0} v$$

[2] $\mathbf{I}(True) = 1$, $\mathbf{I}(False) = 0$.

[3] This means that $P\left[\lim_{n\to\infty} X_n = X\right] = 1$.

be the lower bound of the support of $F(v|X)$. We do not restrict $\underline{v}(X)$ to be positive.[4] As is well-known, the equilibrium bid function of first-price sealed bid auctions where values are independent and private, and bidders are symmetric and risk-neutral, takes the form

$$\beta\left(v|X\right) = v - \frac{1}{F(v|X)^{I(X)-1}} \int_{\max(p_0(X),\underline{v}(X))}^{v} F(y|X)^{I(X)-1} dy \qquad (3.1)$$

$$\text{for } v > \max\left(p_0\left(X\right), \underline{v}(X)\right),$$

where $I(X) \geq 2$ is the number of potential bidder[5] and $p_0(X)$ is the seller's reservation price. This is a unique symmetric Nash equilibrium for an actual bidder, i.e., a potential bidder whose private value V is greater or equal to the reservation price $p_0(X)$. See for example Riley and Samuelson (1981) or Krishna (2002). Note that we allow the number of potential bidders and the reservation price to depend on the auction-specific characteristics. We will assume that $p_0(X)$ and $I(X)$ are observed. Since we condition on X, we therefore do not need to bother about the functional form of $p_0(X)$ and $I(X)$.

Note that in the binding reservation price case, $p_0\left(X\right) > \underline{v}(X)$, the equilibrium bid function (3.1) can also be written as

$$\beta\left(v|X\right) = v - \frac{v - p_0\left(X\right)}{F(v|X)^{I(X)-1}} \int_0^1 F\left(p_0\left(X\right) + u(v - p_0\left(X\right))|X\right)^{I(X)-1} du, \qquad (3.2)$$

$$v > p_0\left(X\right).$$

---

[4] In the empirical auction literature it is usually assumed that the value distribution $F(v)$ has bounded support $\left[\underline{v}, \overline{v}\right]$, with $\underline{v} > 0$, $\overline{v} < \infty$. However, for our approach we only need the condition that the expectation of the values is finite. See Chapter 1 and Chapter 2.

[5] Which is assumed to be known to each potential bidder as well to the econometrician.

Only those potential bidders whose private values V are greater or equal to the reservation price $p_0(X)$ will issue a bid $B = \beta(V|X)$. However, since the number of potential bidders $I(X)$ in each auction is considered to be known, we may without loss of generality assume that the non-bidders issue a zero bid:

$$B = \begin{cases} \beta(V|X) & \text{if} \quad V \geq p_0(X), \\ 0 & \text{if} \quad V < p_0(X). \end{cases}$$

Note that the number of zero bids has a Binomial $(I(X), F(p_0(X)|X))$ distribution, conditional on X.

If the reservation price is not binding, that is, $p_0(X) \leq \underline{v}(X)$ or equivalently, $F(p_0(X)|X) = 0$, the equilibrium bid function simplifies to

$$\begin{aligned} \beta(v|X) &= v - \frac{1}{F(v|X)^{I(X)-1}} \int_0^v F(y|X)^{I(X)-1} dy \qquad (3.3) \\ &= v - \frac{v}{F(v|X)^{I(X)-1}} \int_0^1 F(u.v|X)^{I(X)-1} du, \; v > \underline{v}(X) \end{aligned}$$

In this case each potential bidder with private value V will issue a bid $B = \beta(V|X)$.

3.2.2 Data-Generating Process

As argued before, we may without loss of generality assume that the potential bidders with a lower value than the reservation price issue a zero bid. Thus, for each auction $\ell = 1, ..., L$ which auction-specific covariates X_ℓ we observe $I_\ell = I(X_\ell)$ bids

(including zero bids)

$$B_{\ell,j} = \begin{cases} \beta_0(V_{\ell,j}|X_\ell) & \text{if} \quad V_{\ell,j} \geq p_0\left(X_\ell\right) \\ \\ 0 & \text{if} \quad V_{\ell,j} < p_0\left(X_\ell\right) \end{cases} , \; j = 1, ..., I_\ell, \tag{3.4}$$

where the values $V_{\ell,j}$, $j = 1, ..., I_\ell$ are independent random drawings from the true value distribution $F_0(v|X_\ell)$, and $\beta_0(v|X_\ell)$ is the corresponding true bid function. Conditional on X_ℓ the bids $B_{\ell,j}$, $j = 1, ..., I_\ell$ are independent.

We will also assume that the auctions themselves are independent. In particular,

Assumption 1. *The covariate vectors X_ℓ are independently and identically distributed as $X \in \mathbb{X} \subset \mathbb{R}^d$, where $\mathbb{X}$ is the support of X,*

so that all the bids $B_{\ell,j}$ are independently distributed.

3.2.3 Conditional Boundedness of the Bids

It has been shown in Lemma 1 of Chapter 1, that if the value distribution is absolutely continuous then the support of the bid distribution is bounded if and only if the value distribution has a finite expectation. This result carries over to our case, conditional on X:

Lemma 1. *If conditional on the vector X of the auction-specific covariates the value distribution $F_0(v|X)$ is absolutely continuous then* $\sup_{v>0} \beta_0 (v|X) < \infty$ *if and only if* $\int_0^\infty v dF_0(v|X) < \infty$.

Proof: Section 3.10.1.

The conditional boundedness of the actual bids is crucial for our approach, because the conditional bid distribution $\Lambda_0(b|X)$ has then finite conditional moments $\int_0^\infty b^n d\Lambda_0(b|X)$ of any order n. As is well-known, this implies that $\Lambda_0(b|X)$ is completely identified by the shape of its conditional characteristic function

$$\varphi_0(t|X) = \int_0^\infty \exp(i.t.b)\, d\Lambda_0(b|X),\ i = \sqrt{-1}, \tag{3.5}$$

in an arbitrary neighborhood of $t = 0$.

3.2.4 The conditional value distribution

To incorporate auction-specific heterogeneity in the conditional value distribution we need to put some structure on $F(v|X)$. We will do that by assuming that

Assumption 2. *There exists a function $\gamma(X)$ such that*

$$\ln V = \gamma(X) + \varepsilon, \tag{3.6}$$

where V is a random drawing from the true conditional value distribution $F_0(v|X)$. The random variable ε in (3.6) *is independent of X, and its distribution is absolutely continuous.*

Then

$$\begin{aligned} F_0(v|X) &= P\left[V \leq v|X\right] \\ &= P\left[\exp(\gamma(X)+\varepsilon) \leq v|X\right] \\ &= P\left[\exp(\varepsilon) \leq v\exp(-\gamma(X))|X\right] \\ &= \Gamma\left(v\exp(-\gamma(X))\right) \end{aligned} \tag{3.7}$$

for example, where Γ is the distribution of $\exp(\varepsilon)$: $\Gamma(x) = P[\exp(\varepsilon) \leq x]$.

Since without loss of generality we may add a constant to $\gamma(X)$ and subtract this constant from ε, we need to pin down the location of ε. For example, assume that $E[\varepsilon] = 0$, or that the median of ε is zero, $P[\varepsilon \leq 0] = 0.5$. Moreover it follows from Chapter 1 that a necessary condition for the nonparametric identification of the first-price auction model is that $E[V|X] = \exp(\gamma(X))\,E[\exp(\varepsilon)] < \infty$, so that we need to require that $E[\exp(\varepsilon)] < \infty$. However, the latter condition does not guarantee that $E[\varepsilon]$ is finite; it is possible that $E[\varepsilon] = -\infty$ whereas $E[\exp(\varepsilon)] < \infty$.[6] Therefore we assume that

[6] For example, let $\varepsilon = \min(Z_1, Z_2/|Z_3|)$, where Z_1, Z_2 and Z_3 are independent $N(0,1)$ distributed. Note that $Z_2/|Z_3|$ is standard Cauchy distributed.

Assumption 3. $E\left[\exp\left(\varepsilon\right)\right] < \infty$, *and the median of* ε *in* (3.6) *is zero.*

Thus $\exp\left(\gamma\left(X\right)\right)$ is now the conditional median of V given X.

It follows trivially that

Lemma 2. *Under Assumptions 2 and 3 the true conditional value distribution* $F_0(v|X)$ *is absolutely continuous with density* $f_0(v|X)$ *and finite expectation* $\int v f_0(v|X)dv < \infty$.

Given the median function $\gamma\left(X\right)$, $F_0(v|X)$ is now determined by the distribution function $\Gamma(x) = P[\exp(\varepsilon) \leq x]$:

$$F_0(v|X) = \Gamma\left(v \exp\left(-\gamma\left(X\right)\right)\right)$$

Since by Assumption 2, $\Gamma(x)$ is absolutely continuous, it follows that, given an a priori chosen absolutely continuous distribution function $G(x)$ with support $(0, \infty)$, there exists an absolutely continuous distribution function $H_0(u)$ on $[0, 1]$ such that $\Gamma(x) = H_0\left(G(x)\right)$, namely $H_0\left(u\right) = \Gamma(G^{-1}(u))$. Then

$$F_0(v|X) = H_0\left(G\left(v. \exp\left(-\gamma\left(X\right)\right)\right)\right)$$

For example, if we choose for $G(x)$ the standard exponential distribution

$$G(x) = 1 - \exp(-x), \;\; x \geq 0, \tag{3.8}$$

then

$$F_0(v|X) = H_0\left(1 - \exp(-v\exp(-\gamma(X)))\right) \tag{3.9}$$

$$= H_0(\overline{G}(v|X)) \tag{3.10}$$

where

$$\overline{G}(v|X) = 1 - \exp(-v\exp(-\gamma(X))). \tag{3.11}$$

We may consider $\overline{G}(v|X)$ as an initial guess for $F(v|X)$, which is right if $H_0(u)$ is the uniform $[0,1]$ distribution: $H_0(u) = u$, and if wrong we can correct that by estimating $H_0(u)$ semi-nonparametrically, similar to the approach in Bierens (2007).

The median restriction $F_0\left(\exp\left(\gamma(X)\right)|X\right) = 1/2$ can be implemented by imposing the quantile restriction

$$H_0\left(G\left(1\right)\right) = 1/2. \tag{3.12}$$

If the reservation price is non-binding, this quantile restriction identifies $\gamma(X)$ nonparametrically because then $F_0(v|X)$ is completely identified by the conditional distribution of the bids. See Chapter 1. In the binding reservation price case $F_0(v|X)$ is only identified on $(p_0(X), \infty)$. However, if $P\left[p_0(X) < \exp\left(\gamma(X)\right)\right] = 1$ then $\gamma(X)$ is also nonparametrically identified. Thus, in principle, it is possible to estimate $\gamma(X)$ nonparametrically. Nevertheless, we will use a parametric specification for $\gamma(X)$, for example the linear specification $\gamma(X) = \left(1, X'\right)\theta = \gamma_0\left(X, \theta\right)$, say. Thus,

Assumption 4. *Given an a priori chosen absolutely continuous distribution function G with density g and support $(0,\infty)$, the true conditional value distribution $F_0(v|X)$ and its density $f_0(v|X)$ take the forms*

$$F_0(v|X) = H_0\left(G\left(v\exp\left(-\gamma_0\left(X,\theta_0\right)\right)\right)\right), \tag{3.13}$$

$$\begin{aligned} f_0(v|X) = {} & h_0\left(G\left(v\exp\left(-\gamma_0\left(X,\theta_0\right)\right)\right)\right) g\left(v\exp\left(-\gamma_0\left(X,\theta_0\right)\right)\right) \\ & \times \exp\left(-\gamma_0\left(X,\theta_0\right)\right), \ \ \theta_0 \in \Theta, \end{aligned} \tag{3.14}$$

respectively, where

- $\gamma_0\left(x,\theta\right)$ *with* $(x,\theta) \in \mathbb{X}\times\Theta$ *is a parametric specification of the conditional median* $\gamma(X)$ *of* $\ln(V)$, *satisfying* $P\left[\gamma(X) = \gamma_0\left(X,\theta_0\right)\right] = 1$.
- $\Theta \subset \mathbb{R}^p$ *is a given compact parameter space for* θ *containing* θ_0.
- *For each* $x \in \mathbb{X}$, $\gamma_0\left(x,\theta\right)$ *is continuous on* Θ, *and for each* $\theta \in \Theta$, $\gamma_0\left(x,\theta\right)$ *is Borel measurable on* $\mathbb{X}$,
- θ_0 *is unique*: $P\left[\gamma(X) = \gamma_0\left(X,\theta\right)\right] < 1$ *for all* $\theta = \Theta\backslash\left\{\theta_0\right\}$.
- H_0 *is an absolutely continuous distribution function on* $[0,1]$ *with density* h_0, *satisfying the quantile restriction* (3.12).

3.3 Simulated Integrated Conditional Moments Sieve Estimation

3.3.1 Characteristic Functions as Moment Conditions

Recall that, since by Lemmas 1-2 the bids $B_{\ell,j}$ are bounded random variables, the actual conditional bid distribution

$$\Lambda_0\left(b|X_\ell\right) = P\left[B_{\ell,j} \leq b|X_\ell\right]$$

is completely identified by the shape if its conditional characteristic function

$$\begin{aligned}\varphi_0\left(t|X_\ell\right) &= E\left[\frac{1}{I_\ell}\sum_{j=1}^{I_\ell}\exp\left(i.t.B_{\ell,j}\right)\middle| X_\ell\right] \\ &= \int_{p_0(X_\ell)}^{\infty}\exp\left(i.t.\beta_0(v|X_\ell)\right)dF_0\left(v|X_\ell\right) + F_0\left(p_0\left(X_\ell\right)|X_\ell\right)\end{aligned}$$

in an arbitrary neighborhood of $t = 0$.

Let $F(v|X)$ be a potential candidate (henceforth called a *candidate conditional value distribution*) for the true conditional value distribution $F_0(v|X)$. Similar to Assumption 4 we assume that

Assumption 5. *The candidate conditional value distributions take the form*

$$F(v|X,H,\theta) = H\left(G\left(v\exp(-\gamma_0\left(X,\theta\right))\right)\right), \quad (H,\theta) \in \mathcal{H}\times\Theta, \tag{3.15}$$

where

- *G, $\gamma_0(x,\theta)$, Θ are the same as in Assumption 4.*

- *$\mathcal{H}$ is a compact metric space of absolutely continuous distribution functions H on $[0,1]$ endowed with the* sup *metric*

$$\|H_1 - H_2\| = \sup_{0\leq u\leq 1} |H_1(u) - H_2(u)|,$$

  containing H_0.

- *Each $H \in \mathcal{H}$ satisfies the quantile restriction*

$$H(G(1)) = 1/2. \tag{3.16}$$

For each auction ℓ, draw a random sample $\widetilde{V}_{\ell,1}, ..., \widetilde{V}_{\ell,I_\ell}$ from $F(v|X_\ell, H, \theta)$,[7] and generate simulated bids $\widetilde{B}_{\ell,j}$ similar to (3.4) by:

$$\widetilde{B}_{\ell,j} = \begin{cases} \beta(\widetilde{V}_{\ell,j}|X_\ell) & \text{if} \quad \widetilde{V}_{\ell,j} \geq p_0(X_\ell) \\ 0 & \text{if} \quad \widetilde{V}_{\ell,j} < p_0(X_\ell) \end{cases}, \ j = 1, ..., I_\ell. \tag{3.17}$$

Then

$$\varphi(t|X_\ell, H, \theta) = E\left[\left.\frac{1}{I_\ell}\sum_{j=1}^{I_\ell} \exp\left(i.t.\widetilde{B}_{\ell,j}\right)\right| X_\ell\right]$$

[7]The method for generating these simulated values will be discussed in the next section.

$$= \int_{p_0(X_\ell)}^{\infty} \exp\left(i.t.\beta(v|X_\ell)\right) dF\left(v|X_\ell, H, \theta\right) + F\left(p_0\left(X_\ell\right)|X_\ell, H, \theta\right)$$

is the conditional characteristic function of $\widetilde{B}_{\ell,j}$, which depend on $(H, \theta) \in \mathcal{H} \times \Theta$ via (3.15).

Lemma 3. *Let Assumptions 1-5 hold. Then* $\|H - H_0\| = 0$ *and* $\theta = \theta_0$ *if and only if* $\varphi\left(t|X, H, \theta\right) = \varphi_0\left(t|X\right)$ *a.s. for all* t *in an arbitrary open interval.*

Proof: Section 3.10.2.

The next result is a straightforward corollary of Theorem 1 in Bierens and Ploberger (1997):

Lemma 4. *Let* $\Phi : \mathbb{R}^d \rightarrow \mathbb{R}^d$ *be a bounded one-to-one mapping. Suppose that for some* (*possibly random*) $t \in \mathbb{R}$, $P\left[\varphi\left(t|X, H, \theta\right) = \varphi_0\left(t|X\right)\right] < 1$. *Then, the set*

$$\left\{\varsigma \in \mathbb{R}^d : E\left[\left(\varphi\left(t|X, H, \theta\right) - \varphi_0\left(t|X\right)\right)\exp\left(i.\varsigma'\Phi\left(X\right)\right)\right] = 0\right\}$$

has Lebesgue measure zero and is nowhere dense in $\mathbb{R}^d$.

Of course, if X is already bounded we may choose for Φ the identity matrix I_d.

3.3.2 Simulated Integrated Conditional Moments Estimation

Denote

$$\overline{Q}\left(H, \theta\right) = \frac{1}{\mu\left(\Xi\right)} \int_{\Xi} \left|\psi\left(\xi|H, \theta\right)\right|^2 d\xi, \tag{3.18}$$

where

$$\psi(\xi|H,\theta) = E\left[(\varphi(t|X,H,\theta) - \varphi_0(t|X))\exp\left(i.\varsigma'\Phi(X)\right)\right],$$

$$\xi = \binom{t}{\varsigma} \in \Xi \subset \mathbb{R}^{d+1},\ \mu(\Xi) = \int_\Xi 1.d\xi > 0.$$

with $\Xi \subset \mathbb{R}^{d+1}$ a set with positive Lebesgue measure $\mu(\Xi)$. Then Lemmas 3 and 4 imply that

$$(H_0,\theta_0) = \arg\min_{(H,\theta)\in\mathcal{H}\times\Theta} \overline{Q}(H,\theta) \text{ is unique.}$$

This suggests to estimate (H_0,θ_0) by the simulated integrated conditional moments method:

$$\left(\widehat{H},\widehat{\theta}\right) = \arg\min_{(H,\theta)\in\mathcal{H}\times\Theta} \widehat{Q}(H,\theta). \tag{3.19}$$

where

$$\widehat{Q}(H,\theta) = \frac{1}{\mu(\Xi)}\int_\Xi \left|\widehat{\psi}(\xi|H,\theta)\right|^2 d\xi \tag{3.20}$$

with

$$\widehat{\psi}(\xi|H,\theta) = \frac{1}{L}\sum_{\ell=1}^{L}\left(\widetilde{\varphi}_\ell(t|H,\theta) - \widehat{\varphi}_\ell(t)\right)\exp\left(i.\varsigma'\Phi(X_\ell)\right),$$

$$\widehat{\varphi}_\ell(t) = \frac{1}{I_\ell}\sum_{j=1}^{I_\ell}\exp\left(i.t.B_{\ell,j}\right),$$

$$\widetilde{\varphi}_\ell(t|H,\theta) = \frac{1}{I_\ell}\sum_{j=1}^{I_\ell}\exp\left(i.t.\widetilde{B}_{\ell,j}\right).$$

Note that if we choose

$$\Xi = [-\kappa, \kappa]^{d+1} \text{ for a } \kappa > 0, \tag{3.21}$$

the function $\widehat{Q}(H, \theta)$ has a closed form. See Chapter 2.

Of course, the estimator (3.19) is not feasible because the metric space $\mathcal{H}$ is infinite-dimensional. Therefore, the actual estimation will be done by sieve estimation, discussed in the next subsection.

3.3.3 Sieve Estimation

The idea of sieve estimation[8] is to construct an increasing sequence of subspaces $\mathcal{H}_n$ of $\mathcal{H}$ such that the computation of the sieve estimator

$$\left(\widetilde{H}_n, \widetilde{\theta}_n\right) = \arg \min_{(H,\theta)\in\mathcal{H}_n\times\Theta} \widehat{Q}(h, \theta). \tag{3.22}$$

is feasible, and with $n = n_L \to \infty$ as $L \to \infty$, $\left(\widetilde{H}_{n_L}, \widetilde{\theta}_{n_L}\right)$ is strongly consistent.

The latter can be shown by proving that the conditions (3.23), (3.24) and (3.25) in the following theorem hold:

Theorem 1. *Let n_L be an arbitrary subsequence of L such that $\lim_{L\to\infty} n_L = \infty$. Under Assumptions 1-5 and the conditions*

$$\overline{Q}(H, \theta) \text{ is continuous on } \mathcal{H} \times \Theta, \tag{3.23}$$

[8]See for example Chen (2004) and the references therein.

$$P\left[\lim_{L\to\infty}\sup_{(H,\theta)\in\mathcal{H}\times\Theta}\left|\widehat{Q}\left(H,\theta\right)-\overline{Q}\left(H,\theta\right)\right|=0\right]=1, \tag{3.24}$$

$$\{\mathcal{H}_n\}_{n=1}^{\infty} \textit{ is dense in } \mathcal{H}, \tag{3.25}$$

the sieve estimator $\left(\widetilde{H}_{n_L},\widetilde{\theta}_{n_L}\right)$ *is strongly consistent*:

$$\sup_{0\le u\le 1}\left|\widetilde{H}_{n_L}(u)-H_0(u)\right|\to 0 \;\; a.s., \; \widetilde{\theta}_{n_L}\to\theta_0 \;\; a.s.$$

Proof: Theorem 4 in Chapter 2.

Note that condition (3.25) means that $\mathcal{H}=\overline{\cup_{n=1}^{\infty}\mathcal{H}_n}$, where the bar indicates the closure. This condition holds if for each $H\in\mathcal{H}$ there exists a sequence of distribution functions $H_n\in\mathcal{H}_n$ such that

$$\lim_{n\to\infty}\sup_{0\le u\le 1}\left|H_n\left(u\right)-H\left(u\right)\right|=0. \tag{3.26}$$

3.4 Continuity of $\overline{Q}\left(H,\theta\right)$

3.4.1 Generation of Simulated Values and Bids

There are various ways to generate random drawing $\widetilde{V}_{\ell,j}$ from a candidate value distribution $F(v|X_\ell,H,\theta)$. A convenient way is the well-known accept-reject method. See, for example, Devroye (1986) and Rubinstein (1981). However, it is difficult to prove that then condition (3.23) holds. Following the approach in Chapter 2, we will therefore assume that the random drawing $\widetilde{V}_{\ell,j}$ are generated as follows.

Assumption 6. *For $\ell = 1, ..., L$, draw a random sample $\widetilde{U}_{\ell,1}, ..., \widetilde{U}_{\ell,I_\ell}$ from the uniform $[0,1]$ distribution, and compute for each candidate value distribution $F(v|X_\ell, H, \theta)$ defined in Assumption 5,*

$$\widetilde{V}_{\ell,j} = \exp\left(\gamma_0\left(X,\theta\right)\right).G^{-1}\left(H^{-1}\left(\widetilde{U}_{\ell,j}\right)\right). \tag{3.27}$$

Then it follows from (3.15) that $\widetilde{U}_{\ell,j} = F\left(\widetilde{V}_{\ell,j}|X_\ell, H, \theta\right)$, which implies that $\widetilde{V}_{\ell,j}$ is a random drawing from $F(v|X_\ell, H, \theta)$. The computation of $H^{-1}\left(\widetilde{U}_{\ell,j}\right)$ can be done numerically, and G can be chosen such that G^{-1} has a closed form.

3.4.2 Continuity of the Simulated Values and Bids

The simulation procedure in Assumption 6 has the advantage that it is easier to prove that the simulated values and bids involved are continuous in H and θ, in the following sense:

Lemma 5. *Let $F\left(v|X, H_n, \theta_n\right)$ and $F\left(v|X, H, \theta\right)$ be candidate value distributions conditional on X such that*

$$\lim_{n\to\infty}\sup_{0\leq u\leq 1}\left|H_n\left(u\right) - H\left(u\right)\right| = 0, \quad \lim_{n\to\infty}\theta_n = \theta. \tag{3.28}$$

For a given random drawing $\widetilde{U}$ *from the uniform* $[0,1]$ *distribution, let* $\widetilde{V}_n$ *and* $\widetilde{V}$ *be the solutions of* $F\left(\widetilde{V}_n|X,H_n,\theta_n\right)=\widetilde{U}$ *and* $F\left(\widetilde{V}|X,H,\theta\right)=\widetilde{U}$*, respectively. Then under Assumptions 1-5,*

$$P\left[\lim_{n\to\infty}\widetilde{V}_n=\widetilde{V}\right]=1 \tag{3.29}$$

Consequently, the corresponding simulated bids $\widetilde{B}_n$ *and* $\widetilde{B}$ *satisfy*

$$P\left[\lim_{n\to\infty}\widetilde{B}_n=\widetilde{B}\right]=1 \tag{3.30}$$

as well.

Proof: Section 3.10.3.

The results of Lemma 5 now imply that:

Theorem 2. *Under Assumptions 1-6 the conditions* (3.23) *and* (3.24) *in Theorem 1 hold.*

Proof: The continuity condition (3.23) follows straightforwardly from Lemma 5, and condition (3.24) is not too hard to verify from Lemma 5 and Theorem 1 in Chapter 2. Q.E.D.

3.5 The Compact Metric Space $\mathcal{H}$ and its Sieve Spaces

3.5.1 The Space $\mathcal{H}$

It has been shown by Bierens (2007) that any density function $h(u)$ on $[0,1]$ can be written as

$$h(u) = \frac{(1 + \sum_{k=1}^{\infty} \delta_k \rho_k(u))^2}{1 + \sum_{k=1}^{\infty} \delta_k^2} \text{ a.e. on } [0,1] \tag{3.31}$$

where $\sum_{k=1}^{\infty} \delta_k^2 < \infty$ and the $\rho_k(u)$'s are orthonormal Legendre polynomials of order k. These polynomials can be constructed recursively by the three-term recursive relation

$$\rho_k(u) = \frac{\sqrt{2k-1}\sqrt{2k+1}}{n}(2u-1)\rho_{k-1}(u) - \frac{(k-1)\sqrt{2k+1}}{k\sqrt{2k-3}}\rho_{k-2}(u)$$

for $k \geq 2$, starting from $\rho_0(u) = 1$, $\rho_1(u) = \sqrt{3}(2u-1)$.

The standard consistency proof for parameter estimators of nonlinear parametric models requires that the parameters are confined to a compact subset of a Euclidean space. Since the distribution H in (3.18) plays the role of unknown parameter, we need to construct a compact metric space of distributions on the unit interval. This can be done by imposing restrictions on the parameters δ_k in (3.31), as follows.

Theorem 3. *Let $\mathcal{D}$ be the space of density function $h(u)$ on $[0,1]$ of the form* (3.31), *where the parameters δ_k are restricted by the inequality*

$$|\delta_k| \leq c\left(1 + \sqrt{k}\ln k\right)^{-1}, \ k = 1, 2, 3, ... \tag{3.32}$$

for an a priori chosen constant $c > 0$*. If we endow* $\mathcal{D}$ *with the* L^1 *metric*

$$||h_1 - h_2||_1 = \int_0^1 |h_1(u) - h_2(u)|\, du, \tag{3.33}$$

then $\mathcal{D}$ *is a compact metric space. Consequently, the corresponding space of absolutely continuous distribution functions on* $[0, 1]$,

$$\mathcal{H} = \left\{ H(u) = \int_0^u h(x)dx,\ h \in \mathcal{D} \right\},$$

endowed with the "sup" metric $\sup_{0 \leq u \leq 1} |H_1(u) - H_2(u)|$, *is a compact metric space as well.*

Proof: Bierens (2007).

Of course, we need to assume that

Assumption 7. *The constant* c *in* (3.32) *is chosen so large that* $h_0 \in \mathcal{D}$, *so that* $H_0 \in \mathcal{H}$.

3.5.2 The Sieve Spaces $\mathcal{H}_n$

For a density function $h(u)$ in (3.31) and its associated parameter sequence $\{\delta_k\}_{k=1}^{\infty}$, let

$$h_n(u) = h(u|\delta_{\mathbf{n}}) = \frac{(1 + \sum_{k=1}^{n} \delta_k \rho_k(u))^2}{1 + \sum_{k=1}^{n} \delta_k^2}, \ \delta_{\mathbf{n}} = (\delta_1, \ldots, \delta_n)', \tag{3.34}$$

be the n-th order truncation of $h(u)$. The case $n = 0$ corresponds to the uniform density: $h_0(u) = 1$. Following Gallant and Nychka (1987) we will call this truncated density a SNP density function.

It has been shown by Bierens (2007) that

$$\lim_{n\to\infty} \int_0^1 |h_n(u) - h(u)|\, du = 0. \tag{3.35}$$

Therefore,

Theorem 4. *Let $\mathcal{D}_n$ be the space of all densities of the type* (3.34), *subject to the same restrictions on the δ_k's as in Theorem 3. Then $\{\mathcal{D}_n\}_{n=1}^{\infty}$ is dense in $\mathcal{D}$. Consequently, defining*

$$\mathcal{H}_n = \left\{ H_n(u) = \int_0^u h_n(v) du, \ h_n \in \mathcal{D}_n \right\} \tag{3.36}$$

it follows that $\{\mathcal{H}_n\}_{n=1}^{\infty}$ is dense in $\mathcal{H}$.

Note that the distribution functions $H_n(u)$ can be computed by the method proposed in Bierens (2007).

Since the density functions in $\mathcal{D}_n$ and distributions functions in $\mathcal{H}_n$ are parametric, with parameters $\delta_{\mathbf{n}} = (\delta_1, \ldots, \delta_n)'$, the computation of the sieve estimator (3.22) is feasible. In particular, the parameter vector $\widetilde{\delta}_{\mathbf{n}}$ for which $\widetilde{h}_n(u) = h\left(u|\widetilde{\delta}_{\mathbf{n}}\right)$, together with $\widetilde{\theta}_n$, can be computed via the simplex method of Nelder and Mead (1965), penalized for violations of the restrictions (3.32) and the quantile restriction (3.16).

3.5.3 Strong Consistency of the Sieve Estimator

Summarizing, we have shown that

Theorem 5. *Under Assumptions 1-7 all the conditions of Theorem 1 hold, so that the sieve estimator* $\left(\widetilde{H}_{n_L}, \widetilde{\theta}_{n_L}\right)$ *is strongly consistent.*

3.6 An ICM Test of the Validity of First-Price Auction Models with Heterogeneity

If the assumptions of symmetric independent private values with risk neutral bidders do not hold, the bid functions (3.1) and (3.3) no longer apply to the actual bids. The same applies if the functional form of the median function $\gamma_0(X, \theta)$ is misspecified. If so,

$$\widehat{Q}\left(\widetilde{H}_{n_L}, \widetilde{\theta}_{n_L}\right) \to \inf_{(H,\theta)\in\mathcal{H}\times\Theta} \overline{Q}(H, \theta) > 0 \text{ a.s.}, \tag{3.37}$$

This suggests to use $\widehat{Q}\left(\widetilde{H}_{n_L}, \widetilde{\theta}_{n_L}\right)$ as a basis for a consistent ICM test of the null hypothesis that

H_0: *the actual bids come from a first-price sealed bid auctions with auction-specific heterogeneity where values are symmetric, independent, private and bidders are risk-neutral, and the functional specification of the median function* $\gamma_0(X, \theta)$ *is correct,*

against the general alternative that

H_1: *the null hypothesis* H_0 *is false.*

The ICM test we will propose is based on the fact that similar to the results in Bierens (1990) and Bierens and Ploberger (1997), the following results hold:

Theorem 6. *Let* $\Xi \subset \mathbb{R}^{d+1}$ *be compact. Under* H_0*,*

$$\widehat{W}_L(.) = \frac{1}{\sqrt{L}} \sum_{\ell=1}^{L} \left(\widetilde{\varphi}(.|H_0, \theta_0) - \widehat{\varphi}(.)\right) \exp\left(i.\varsigma' \Phi\left(X_\ell\right)\right) \Rightarrow W(.)$$

on Ξ*, hence*[9]

$$L.\widehat{Q}(H_0, \theta_0) = \frac{1}{\mu(\Xi)} \int_{\Xi} \left|\widehat{W}_L(\xi)\right|^2 d\xi \to_d \frac{1}{\mu(\Xi)} \int_{\Xi} |W(\xi)|^2 d\xi,$$

[9]By the continuous mapping theorem.

where $W(\zeta)$ *is a complex-valued zero-mean Gaussian process on* Ξ *with covariance function* $\Gamma(\xi_1, \xi_2) = E\left[\widehat{W}_L(\xi_1)\overline{\widehat{W}_L(\xi_2)}\right]$. *Under* H_1, (3.37) *holds.*

Proof: Similar to Theorem 5 in Chapter 2.

Note that the result in Theorem 6 does **not** imply that

$$L.\widehat{Q}\left(\widetilde{H}_{n_L}, \widetilde{\theta}_{n_L}\right) \to_d \frac{1}{\mu(\Xi)} \int_{\Xi} |W(\xi)|^2 \, d\xi.$$

However, if

Assumption 8. *The distribution* H_0 *is of the SNP type itself*: $H_0 \in \bigcup_{n=0}^{\infty} \mathcal{H}_n$,

then for $L \to \infty$,

$$L.\widehat{Q}\left(\widetilde{H}_{n_L}, \widetilde{\theta}_{n_L}\right) \leq L.\widehat{Q}(H_0, \theta_0)$$

so that upper bounds of the critical values of the test can be based on the limiting distribution of the upper bound $L.\widehat{Q}(H_0, \theta_0)$. These critical values can be derived by a bootstrap approach, similar to Chapter 2.

3.7 Determination of the Sieve Order via an Information Criterion

Under Assumption 8 there exists a smallest natural number n_0 such that $H_0 \in \mathcal{H}_{n_0}$. Similar to Chapter 2 we propose to estimate n_0 by minimizing a criterion function of the type

$$\widetilde{C}_L(n) = \inf_{(H,\theta)\in\mathcal{H}_n\times\Theta} \widehat{Q}(H,\theta) + \Phi(n).\frac{\phi(L)}{L}, \tag{3.38}$$

$$\phi(L) = o(L), \quad \lim_{L\to\infty} \phi(L) = \infty.$$

where $\Phi(n)$ is an increasing but bounded function of n. For example, let for some $\alpha \in (0,1)$,

$$\Phi(n) = 1 - (n+1)^{-\alpha}.$$

Then similar to the Hannan-Quinn and Schwarz information criteria we have:

Theorem 7. *Let* $\widetilde{n}_L = \max_{s.t.\ \widetilde{C}_L(n)\le\widetilde{C}_L(n-1)} n$ *and*

$$\left(\widetilde{H},\widetilde{\theta}\right) = \arg \min_{(H,\theta)\in\mathcal{F}_{\widetilde{n}_L}\times\Theta} \widehat{Q}(H,\theta).$$

Under Assumption 8, $\lim_{L\to\infty} P\left[\widetilde{n}_L = n_0\right] = 1$, *hence*

$$\sup_{0\le u\le 1} \left|\widetilde{H}\left(u\right) - H_0(u)\right| \to 0 \ a.s. \ and \ \widetilde{\theta} \to \theta_0 \ a.s.$$

If Assumption 8 is not true then $p\lim_{L\to\infty} \widetilde{n}_L = \infty$, *hence*

$$p \lim_{L\to\infty} \sup_{0\le u\le 1} \left|\widetilde{H}\left(u\right) - H_0(u)\right| = 0 \ and \ p \lim_{L\to\infty} \widetilde{\theta} = \theta_0.$$

Proof: Theorem 6 in Chapter 2.

3.8 An Application of SNP-SICM Estimation to the USFS Timber Auctions

For the application, we use the USFS timber auction data for region 9. Region 9 covers Illinois, Indiana, Maine, Michigan, Minnesota, Missouri, New Hampshire, Ohio, Pennsylvania, Vermont, West Virginia and Wisconsin.[10]The time horizon of the data is 1982-1993.[11] The data set consists of 949 auctions and 3238 bids.[12] As covariates, we use the fraction of saw timbers and the log of acres. Depending on future products, the timbers are mainly classified into two types of timbers: saw timbers and pulp timbers. The latter are used to make paper while the former are made into products which are necessary for general constructions, houses and furniture. These volume estimates depending on future products are the main characteristics in timber auctions. One interesting thing in USFS timber auctions is that the reservation price is known to be non-binding. For details, see Baldwin, Marshall and Richard (1997).[13]

The objective function used in the application is

$$\widehat{Q}(H,\theta) = \frac{1}{L^2}\sum_{\ell_1=1}^{L}\sum_{\ell_2=1}^{L}\frac{1}{I_{\ell_1}I_{\ell_2}}\sum_{j_1=1}^{I_{\ell_1}}\sum_{j_2=1}^{I_{\ell_2}}\left[\left(\Pi_{m=1}^{d}\frac{\sin\left(\left(X^*_{\ell_1,m}-X^*_{\ell_2,m}\right)\right)}{\left(X^*_{\ell_1,m}-X^*_{\ell_2,m}\right)}\right)\right.$$

---

[10]The data are available at http://www.econ.yale.edu/~pah29/timber/timber.htm.

[11]There were some policy changes during 1979-1981. To take it into account, we selected the data after 1981. For more details of policy changes, see Haile (2001).

[12]Sales which are related to the salvage, SBA set aside and road construction are excluded.

[13]They raise some reasons why the reservation price in timber auctions is not binding. One reason they quoted is that the advertised rate for the timber species is low unrealistically.

Table 3.1. Summary Statistics

variable	min	max	average	standard deviation
bid (unit: million dollars)	0.13	36.40	3.14	4.09
fraction of saw timbers	0.00	1.00	0.60	0.38
log acres of a timber lot	0.69	8.42	5.19	1.17

$$\left(\frac{\sin\left(\left(B^*_{\ell_1,j_1} - B^*_{\ell_2,j_2}\right)\right)}{\left(B^*_{\ell_1,j_1} - B^*_{\ell_2,j_2}\right)} - \frac{\sin\left(\left(B^*_{\ell_1,j_1} - \widetilde{B}^*_{\ell_2,j_2}\right)\right)}{\left(B^*_{\ell_1,j_1} - \widetilde{B}^*_{\ell_2,j_2}\right)} - \frac{\sin\left(\left(\widetilde{B}^*_{\ell_1,j_1} - B^*_{\ell_2,j_2}\right)\right)}{\left(\widetilde{B}^*_{\ell_1,j_1} - B^*_{\ell_2,j_2}\right)} + \frac{\sin\left(\left(\widetilde{B}^*_{\ell_1,j_1} - \widetilde{B}^*_{\ell_2,j_2}\right)\right)}{\left(\widetilde{B}^*_{\ell_1,j_1} - \widetilde{B}^*_{\ell_2,j_2}\right)} \right) \Bigg] \tag{3.39}$$

where X_ℓ is a covariate vector in auction ℓ and $X_{\ell,m}$ is an m_{th} element in X_ℓ. Note that bids and covariates are normalized by the sample mean and sample standard deviation.

In this application, we use candidate conditional value distributions of the form

$$F(v|X) = H\left(1 - \exp\left(-v \exp\left(-(\theta_0 + \theta_1 X_1 + \theta_2 X_2)\right)\right)\right).$$

Moreover, in the information criterion we have chosen $\alpha = 1/4$ and $\phi(L) = \log_{10}(\log_{10} L)$.

3.8.1 Preliminary Estimation Results

To save computation time, we only have used data of 100 randomly chosen auctions out of 949 auctions. The penalty $L(H(1-\exp(-1)) - 0.5)^2$ has been added to the objective function (3.39) to enforce the quantile restriction (3.16). Our information criterion function selected sieve order $\tilde{n}_L = 2$. The estimated conditional median function is

$$\gamma(X, \tilde{\theta}) = -1.75 - 0.63X_1 + 0.58X_2$$

where X_1 is the fraction of saw timbers and X_2 is the log of acres of a timber lot. The estimated SNP density and the estimated conditional density function of the value at the sample mean of covariates $\overline{X}$ are plotted below. The value at the average of covariates is shown to have a right skewed density.

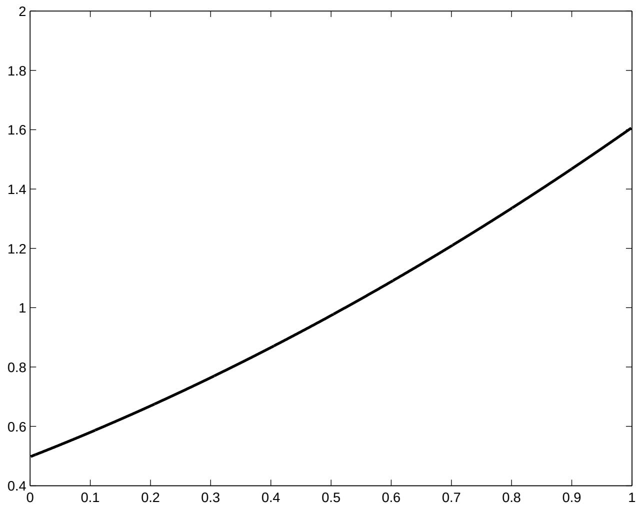

Fig. 3.1. Estimated SNP density $\hat{h}_2(u)$

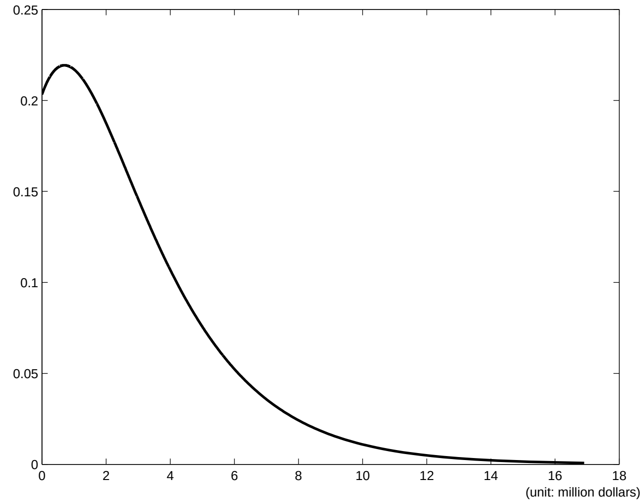

Fig. 3.2. Estimated conditional value density $\widehat{f}_2(v|\overline{X})$ for $\overline{X} =$ average of $X_\ell's$

3.8.1.1 ICM Test

Next we have conducted the ICM test. Based on estimation results $\left(\widetilde{H}_{n_L}, \widetilde{\theta}_{n_L}\right)$, we can generate L auctions $2M$ times independently via random drawings of simulated values from a distribution estimator $\widetilde{F}(v|X) = \widetilde{H}_{n_L}\left(1 - \exp\left(-v \exp\left(-\gamma\left(X, \widetilde{\theta}_{n_L}\right)\right)\right)\right)$. Two consecutive auctions constitute one simulated empirical process $\widetilde{W}_m(.)$, $m = 1, ..., M$, where

$$\widetilde{W}_m(\xi) = \frac{1}{\sqrt{L}} \sum_{\ell=1}^{L} \left(\frac{1}{I_\ell} \sum_{j=1}^{I_\ell} \exp\left(i.t.\widetilde{B}_{\ell,j,2m-1}\right) - \frac{1}{I_\ell} \sum_{j=1}^{I_\ell} \exp\left(i.t.\widetilde{B}_{\ell,j,2m}\right) \right) \exp\left(i.\varsigma'\Phi\left(X_\ell\right)\right).$$

By computing $\widetilde{T}_m = \frac{1}{\mu(\Xi)} \int_\Xi \left|\widetilde{W}_m(\xi)\right|^2 d\xi$ and sorting them increasing order, we can determine the $\alpha \times 100\%$ bootstrap value $\widetilde{T}_{(1-\alpha)M}$. By comparing our original statistic $\widehat{T} = \frac{1}{\mu(\Xi)} \int_\Xi \left|\widehat{W}_L(\xi)\right|^2 d\xi$ with $\widetilde{T}_{(1-\alpha)M}$, we can test the null hypothesis.

The ICM test statistic takes the value $\hat{T} = 0.121$. The 10% and 5% bootstrap critical values are $\tilde{T}_{0.90M} = 0.133$, $\tilde{T}_{0.95M} = 0.1822$ respectively, where $M = 100$. Therefore, the null hypothesis is not rejected at the 10% significance level.

3.8.1.2 Conditional Expectation of the Value given the Covariates

Given that the model is correct, we can compute the conditional expectation of the value on the basis of the estimated conditional value distribution $F_0(v|X)$ as follows. First, note that

$$
\begin{aligned}
& E\left[V|X\right] \\
&= \int_0^\infty v f_0(v|X) dv \\
&= \int_0^\infty v h_0 \left(1 - \exp\left(-v \exp\left(-\gamma_0(X,\theta)\right)\right)\right) \exp\left(-v \exp\left(-\gamma_0(X,\theta)\right)\right) \exp\left(-\gamma_0(X,\theta)\right) dv \\
&= \exp\left(\gamma_0(X,\theta)\right) \int_0^1 \ln\left(\frac{1}{1-u}\right) h_0\left(u\right) du
\end{aligned}
$$

The last equation can be obtained by letting $u = 1 - \exp\left(-v\exp\left(-\gamma_0(X,\theta)\right)\right)$. Plugging in the estimates of $\gamma_0(X,\theta)$ and $h_0(u)$, then yields the estimated conditional expectation:

$$\widehat{E}\left[V|X\right] = 1.286 \times \exp(-1.75 - 0.63X_1 + 0.58X_2).$$

3.9 Concluding Remarks

In this chapter we have proposed a semi-nonparametric simulated integrated conditional moments estimation method for first-price auctions with observed auction-specific heterogeneity, and applied it to USFS timber auction data for region 9. In particular, we use the variation of the auction-specific characteristics to infer the conditional value distribution. This application is preliminary. We plan to extend this analysis by including more regions and more detailed covariates. Moreover, we also need to derive the asymptotic normality of median regression coefficients.

3.10 Proofs

3.10.1 Proof of Lemma 1

It follows from (3.1) and integration by parts that for $v \to \infty$,

$$\beta_0 \left(v|X\right) \to \max\left(p_0\left(X\right), \underline{v}(X)\right) F_0(\max\left(p_0\left(X\right), \underline{v}(X)\right)|X)^{I(X)-1}$$
$$+ \left(I\left(X\right)-1\right) \int_{\max(p_0(X),\underline{v}(X))}^{\infty} vF_0(v|X)^{I(X)-2} dF(v|X)$$

The integral involved is bounded from above by $\int_0^\infty vdF_0(v|X)$ and bounded from below by $\left(\int_0^\infty vdF_0(v|X) - \int_0^M ydF_0(y|X)\right).F_0(M|X)^{I(X)-2}$, for any $M > \max\left(p_0\left(X\right), \underline{v}(X)\right)$.
Q.E.D.

3.10.2 Proof of Lemma 3

First, let the arbitrary open interval involved be $(-\kappa, \kappa)$ for some $\kappa > 0$. Since conditional on X_ℓ, $\varphi_0\left(t|X_\ell\right)$ is the characteristic function of a bounded random variable $B_{\ell,1}$ we can write

$$\varphi_0\left(t|X_\ell\right) = E\left[\exp\left(i.t.B_{\ell,1}\right)|X_\ell\right] = \sum_{m=0}^{\infty} \frac{i^m t^m}{m!} E\left[B_{\ell,1}^m|X_\ell\right].$$

Recall that $\varphi\left(t|X_\ell, H, \theta\right)$ is the conditional characteristic function of the simulated bid $\widetilde{B}_{\ell,1}$. Then the equality $\varphi\left(t|X_\ell, h, \theta\right) = \varphi_0\left(t|X_\ell\right)$ a.s. for all $t \in (-\kappa, \kappa)$ implies that

$$\partial^m \varphi\left(t|X_\ell, H, \theta\right) / \left(\partial t\right)^m\Big|_{t=0} = \partial^m \varphi_0\left(t|X_\ell\right)\left(\partial t\right)^m\Big|_{t=0}$$
$$= i^m E\left[B_{\ell,1}^m|X_\ell\right]$$

a.s. for all $m \geq 0$, which in its turn implies that $E\left[\widetilde{B}_{\ell,1}^{m}|X_{\ell}\right] = E\left[B_{\ell,1}^{m}|X_{\ell}\right]$ a.s. for all $m \geq 0$, so that

$$\varphi_0\left(t|X_\ell\right) = \varphi\left(t|X_\ell, H, \theta\right) = \sum_{m=0}^{\infty} \frac{i^m t^m}{m!} E\left[B_{\ell,1}^{m}|X_\ell\right]$$

for all $t \in \mathbb{R}$.

Next, suppose that $\varphi\left(t|X_\ell, H, \theta\right) = \varphi_0\left(t|X_\ell\right)$ a.s. for all $t \in \left(t_* - \kappa, t_* + \kappa\right)$, where $t_* \in \mathbb{R}$ and $\kappa > 0$ are arbitrary. Then by the same argument as for the case $t_* = 0$,

$$\varphi_0\left(t|X_\ell\right) = \varphi\left(t|X_\ell, H, \theta\right) = \sum_{m=0}^{\infty} \frac{i^m\left(t - t_*\right)^m}{m!} E\left[\exp\left(i.t_* B_{\ell,1}\right) B_{\ell,1}^{m}|X_\ell\right]$$

for all $t \in \mathbb{R}$. The result involved now follows straightforwardly from the well-known fact that distributions are equal if and only if their characteristic functions are equal. Q.E.D.

3.10.3 Proof of Lemma 5

It follows from (3.27) that

$$\widetilde{V}_n = -\exp\left(\gamma_0\left(X, \theta_n\right)\right).G^{-1}\left(H_n^{-1}\left(\widetilde{U}\right)\right),$$

$$\widetilde{V} = -\exp\left(\gamma_0\left(X, \theta\right)\right).G^{-1}\left(H^{-1}\left(\widetilde{U}\right)\right)$$

Let $U_n = H_n^{-1}\left(\widetilde{U}\right)$ and $U = H^{-1}\left(\widetilde{U}\right)$,so that

$$H_n\left(U_n\right) = H(U) = \widetilde{U}. \tag{3.40}$$

Since by (3.28),

$$|H_n(U_n) - H(U_n)| \to 0 \text{ a.s.}$$

it follows from (3.40) that $H(U_n) \to H(U)$ a.s., which by the continuity of $H(u)$ implies that $U_n \to U$ a.s., hence

$$G^{-1}\left(H_n^{-1}\left(\widetilde{U}\right)\right) \to G^{-1}\left(H^{-1}\left(\widetilde{U}\right)\right) \text{ a.s.} \tag{3.41}$$

Moreover, it follows trivially from (3.28) and the continuity of $\gamma_0(X,\theta)$ that

$$\exp\left(\gamma_0\left(X,\theta_n\right)\right) \to \exp\left(\gamma_0\left(X,\theta\right)\right) \text{ a.s.} \tag{3.42}$$

The result (3.29) now follows from (3.41) and (3.42).

It follows from (3.2) and (3.3) that the simulated bids $\widetilde{B}_n$ and $\widetilde{B}$ can be generated by, respectively,

$$\begin{aligned}
\widetilde{B}_n &= \mathbf{I}\left(\widetilde{U} > F\left(p_0(X)|X, H_n, \theta_n\right)\right) . \left[\widetilde{V}_n - \left(\widetilde{V}_n - p_0(X)\right)\widetilde{U}^{1-I(X)}\right. \\
&\quad \left. \times \int_0^1 F\left(p_0(X) + u\left(\widetilde{V}_n - p_0(X)\right)|X, H_n, \theta_n\right)^{I(X)-1} du\right] \\
\widetilde{B} &= \mathbf{I}\left(\widetilde{U} > F\left(p_0(X)|X, H, \theta\right)\right)\left[\widetilde{V} - \left(\widetilde{V} - p_0(X)\right)\widetilde{U}^{1-I(X)}\right. . \\
&\quad \left. \times \int_0^1 F\left(p_0(X) + u\left(\widetilde{V} - p_0(X)\right)|X, H, \theta\right)^{I(X)-1} du\right]
\end{aligned}$$

in the binding reservation price case, and by

$$\widetilde{B}_n = \widetilde{V}_n . \left(1 - \widetilde{U}^{1-I} \int_0^1 F\left(u.\widetilde{V}_n|X, H_n, \theta_n\right)^{I(X)-1} du\right),$$

$$\widetilde{B} = \widetilde{V}.\left(1 - \widetilde{U}^{1-I}\int_0^1 F\left(u.\widetilde{V}|X,H,\theta\right)^{I(X)-1} du\right),$$

in the non-binding case. Moreover, it follows straightforwardly from (3.28) and (3.29) that

$$F\left(p_0(X)|X,H_n,\theta_n\right) \to F\left(p_0(X)|X,H,\theta\right) \text{ a.s.}$$

and pointwise in $u \in [0,1]$,

$$F\left(p_0(X) + u\left(\widetilde{V}_n - p_0(X)\right)|X,H_n,\theta_n\right)$$

$$\to F\left(p_0(X) + u\left(\widetilde{V} - p_0(X)\right)|X,H,\theta\right) \text{ a.s.}$$

so that (3.30) follows from the bounded convergence theorem. Q.E.D.

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Vita

I was born in Seoul, Korea in 1972. Most of my life has been lived there until my PhD life in America. I received a BA in Economics from Seoul National University in 1996. After serving my military service, I received in 1999 an MA in Economics from Seoul National University. I have worked for the central bank of Korea as an Economist from January 2000 to August 2002, and since then I have been enrolled in the PhD program in Economics of the Pennsylvania State University. Meanwhile, I got married to Dajeong Byun and now Dajeong and I have two lovely sons, Jayhahn and Jeanhahn.