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# THE INVISCID LIMIT OF THE NAVIER STOKES EQUATIONS

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by

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# Abstract

The inviscid limit of the Navier-Stokes equations is one of the most fundamental and challenging problems in fluid dynamics. For domains with boundaries under no-slip boundary conditions, the problem is largely open due to large convection terms in the inviscid limit. On the whole space $\mathbb{R}^2$, the problem is open for irregular initial data, except for vortex patches and point vortices.

This dissertation discusses my results on the inviscid limit of the Navier-Stokes equations. My first results are to justify the inviscid limit on half-space for via a new analytic framework. The analysis is carried out for the classical no-slip boundary conditions as well as the critical boundary conditions. Finally, the thesis justifies the inviscid limit for vortex-wave data, which rigorously obtains the vortex-wave system derived in the early 90s by Marchioro-Pulvirenti as a vanishing viscosity limit of the Navier-Stokes equations.

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# Chapter 1

# Introduction

## 1.1 The model and open questions

In this thesis, we are interested in the inviscid limit of the Navier-Stokes equations for incompressible fluids

$$\begin{aligned} \partial_t u^\nu + u^\nu \cdot \nabla u^\nu + \nabla p^\nu &= \nu \Delta u^\nu, \qquad t > 0, \quad x \in \Omega, \\ \nabla \cdot u^\nu &= 0 \end{aligned} \tag{1.1}$$

posed on a smooth domain $\Omega \subset \mathbb{R}^d$ with boundary $\partial\Omega$, $d \geq 2$, with the classical no-slip boundary condition

$$u^\nu(t,x) = 0 \qquad \text{when} \quad x \in \partial\Omega. \tag{1.2}$$

Here $u^\nu = u^\nu(t,x) \in \mathbb{R}^d$ denotes the velocity vector field and $\nu > 0$ denotes the viscosity of the fluid. In physical literature, $\nu$ is the the inverse of the Reynolds number. Understanding fluids with small viscosity (or large Reynolds number) is one of the most fundamental and challenging problems in fluid dynamics. The problem is also important in understanding different phenomena in engineering and aerodynamics. Some typical Reynolds numbers are: $Re \sim 10^2$ for blood flow in the aorta, $Re \sim 10^{12}$ for the Boeing 747 wake.

In the inviscid limit: $\nu \to 0$, one would expect that solutions $u^\nu$ converge to solutions of the corresponding Euler equations for incompressible fluids.

$$\begin{aligned} \partial_t u^E + u^E \cdot \nabla u^E + \nabla p^E &= 0 \\ \nabla \cdot u^E &= 0 \\ u^E \cdot n|_{\partial\Omega} &= 0 \end{aligned}$$

where $n$ is the unit normal vector of the boundary $\partial\Omega$ of the domain $\Omega$. When $\Omega = \mathbb{R}^d$, the boundary conditions are usually replaced by suitable decay assumptions of the velocity vector field at infinity.

## 1.2 Previous results

When the fluid domain $\Omega$ has boundaries, the problem is very delicate, due to the mismatch between Euler and Navier-Stokes boundary conditions, leading to boundary layers formation near the the boundary. Justifying these phenomena mathematically remains a challenging problem. In 1904, Prandtl [56] postulated a boundary layer ansatz for the Navier-Stokes solutions, leading to the well-known Prandtl's boundary layer equation. Since then, there have been numerous works on the Prandtl equations; see [53, 54, 1, 48] regarding the local well-posedness of Prandtl equations for monotonic data, [58, 17, 30, 11] for analytic and Gevrey data, and [15, 18, 28] regarding the ill-posedness of the Prandtl equations for non-monotonic data in Sobolev spaces. Prandtl postulates that on the half-space, one could write the solution of the Navier-Stokes equations $u^\nu$ as

$$u^\nu(t,x,y) = u^E(t,x,y) + u^P\left(t, x, \frac{y}{\sqrt{\nu}}\right) + o(1)_{L^\infty} \tag{1.3}$$

in the inviscid limit. Here $o(1)_{L^\infty}$ is a term that tends to zero in $L^\infty$ norm as $\nu \to 0$, $u^E, u^P$ are solutions of the Euler and Prandtl equations respectively. The above expansion gives a very precise description of solutions to the Navier-Stokes as viscosity goes to zero. The validity of the Prandtl expansion is indeed confirmed for well-prepared analytic data [58, 59], Gevrey perturbation of some shear flows [16], domains with symmetry [19, 49, 20], for initial vorticity that is compactly away from the boundary [39]. Formally speaking, analyticity or Gevrey perturbation kills instability of the Prandtl asymptotic expansion, and if the initial vorticity is compactly supported away from the boundary, the interaction between Prandtl layer near the boundary and Euler could be control, using a careful desired norm that captures the analyticity of solutions near the boundary. For symmetric flows, the expansion (1.3) is justified in [21] for parallel, infinite straight pipe and plane-parallel channel domain. Very recently, Prandtl boundary layers expansion is proven to be unstable in $L^\infty$ for data that are Sobolev perturbation of some specific shear flows [24, 27], which is one of the key steps in understanding turbulence.

Back to the problem of inviscid limit in the $L^2$ norm, in 1984, Kato [33] proves that the Navier-Stokes solutions converge to Euler in $L^\infty(0,T,L^2(\Omega))$ if and only if the energy dissipation near the boundary of size $\nu$ vanishes in the inviscid limit. This layer of size $\nu$ is much thinner than the size of Prandtl ansatz approximation, which is of size $\sqrt{\nu}$. Up to now, the problem is still open in general. On the half-space, the inviscid limit holds for well-prepared analytic data [59], Gevrey perturbation of a class of monotone shear flows [16], for initial vorticity that is compactly supported away from the boundary [39]. In particular, Sammartino-Caflisch [59] prove that one could write the solution of the Navier-Stokes equations as (1.3), when the initial velocity is of the form

$$u^\nu_0(x,y) = u^E_0(x,y) + u^P_0\left(x, \frac{y}{\sqrt{\nu}}\right) + O(\sqrt{\nu})$$

Here $O(\sqrt{\nu})$ denotes a term of order $\sqrt{\nu}$ in a suitable analytic norm. The proof makes heavy use of the construction of Euler and Prandtl solutions in analytic spaces [58]. The remainder $u^\nu - u^E - u^P$ is then controlled, by using the analyticity of the solutions. In 2014, Maekawa [39] proves that the inviscid limit and Prandtl boundary layers expansions hold, when the initial vorticity of the Euler equations $\omega^E_0 = \nabla \times u^E_0$ is compactly supported away from the boundary for the half-space $\mathbb{R}^2_+$. The proof makes use of the Prandtl boundary expansion, and a careful chosen norm that capture the analyticity of the solutions near the boundary and Sobolev inside the interior of the half-space. The remainder is shown to be very localized near the boundary, thanks to the vanishing of the Euler vorticity near the boundary and the weak interaction between Euler and Prandtl solutions. We remark that all of the previous works rely on the Prandtl boundary expansions, which involves constructing Euler solutions, and Prandtl solutions, then controlling the remainder. It is, however, worth noting that the question of inviscid limit, could be stated without referring to the Prandtl solutions near the boundary. Very recently, in the joint work with Toan Nguyen [51], I successfully prove that the solutions of Navier-Stokes $u^\nu$ converge to Euler $u^E$ in $L^\infty(0,T,L^2(\mathbb{T}\times\mathbb{R}_+))$, when the initial data is analytic, without detailing Prandtl expansions. Using the same ideas in [51], I justify the inviscid limit for analytic data, under the critical slip boundary conditions [50].

Lastly, we refer to the readers to [29, 47] for the Navier-slip boundary conditions with constant slip length, or slip length depending on $\nu$ [55].

When the fluid domain has no boundaries, it is known that the solution of the Navier-Stokes equations tends to the Euler solution in the inviscid limit for regular initial data and data that include vortex patches [63, 46, 5, 7, 8, 62, 6, 60]. For initial data whose vorticity consists of a finite sum of point vortices (Dirac masses), Gallay [13] proved that the corresponding Navier-Stokes vorticity indeed converges weakly in the inviscid limit to the sum of point vortices whose centers evolve according to the Helmholtz-Kirchhoff point-vortex system. For vortex-sheet initial data, the problem is completely open.

## 1.3 My contribution

### 1.3.1 The inviscid limit under no-slip conditions

In [51], we prove that the inviscid limit holds for analytic data on half-space under no-slip boundary condition. The novelty in this paper, compared to the classical works [59], is that we are able to prove that the Navier-Stokes converges to Euler in a constant time interval in $L^2$ norm, and give a pointwise bound on vorticity, without detailing Prandtl boundary layers expansions. Our proof greatly simplifies [59], and justifies the inviscid limit for analytic data that are not well-prepared. In other to achieve the above goal, we make use of the boundary vorticity formulation, the pointwise bounds on the Green function, and the abstract Cauchy-Kovalevskaya theorem on boundary layer function spaces. The main theorem in our work [51] can be stated formally as follows:

**Theorem 1.3.1.** *Let $u_0^\nu$ be the initial data for the Navier-Stokes systems* (1.1) *on $\mathbb{T}\times\mathbb{R}_+$ under no-slip boundary condition $u^\nu|_{z=0} = 0$. Assume that $u_0^\nu$ is analytic, there exists a unique local solution $u^\nu(t)$ solving the Navier-Stokes equations with initial data $u_0^\nu$ and*

$$\lim_{\nu\to 0} \|u^\nu(t) - u^E(t)\|_{L^p(\mathbb{T}\times\mathbb{R}_+)} = 0 \qquad \textit{for any} \quad 1 \le p < \infty$$

*uniformly in $t \in [0,T]$.*
*Moreover, there holds the localized bound on the vorticity $\omega^\nu(t) = \nabla\times u^\nu(t)$ as follows:*

$$|\omega^\nu(t,x,z)| \le M_0 e^{-\beta z}\left(1 + \frac{1}{\sqrt{\nu}}e^{-\frac{z}{\sqrt{\nu}}} + \frac{1}{\sqrt{\nu t}}e^{-\frac{z}{\sqrt{\nu t}}}\right) \tag{1.4}$$

*for* $(t,x,z) \in (0,T] \times \mathbb{T} \times \mathbb{R}_+$, *where* $M_0 > 0$ *is independent of* $\nu$.

Ideas of the proof: In [51], we work with the vorticity formulation of the Navier-Stokes equations

$$\partial_t \omega^\nu - \nu \Delta \omega^\nu = -u^\nu \cdot \nabla \omega^\nu \tag{1.5}$$

and our goal was to derive a uniform bound on the vorticity in suitable analytic function space uniformly as $\nu \to 0$. To this end, a boundary condition is needed. To ensure the no-slip boundary condition, we impose $\partial_t u_1^\nu = 0$ on the boundary. This leads to

$$0 = \partial_t u_1^\nu = \partial_z \Delta^{-1} \partial_t \omega^\nu = \partial_z \Delta^{-1}(\nu \Delta \omega^\nu - u^\nu \cdot \nabla \omega^\nu)$$

on the boundary. Introduce $\omega_*$ so that $\Delta \omega_* = 0$ with $\omega_* = \omega^\nu$ on the boundary. This yields $\partial_z \Delta^{-1} \Delta \omega^\nu = \partial_z(\omega^\nu - \omega_*) = (\partial_z + |\partial_x|)\omega^\nu$, in which $|\partial_x|$ denotes the Dirichlet-to-Neumann operator on the half space. Thus, the boundary condition on vorticity reads

$$\nu(\partial_z + |\partial_x|)\omega^\nu_{|z=0} = [\partial_z \Delta^{-1}(u^\nu \cdot \nabla \omega^\nu)]_{|z=0}. \tag{1.6}$$

We then write the vorticity $\omega^\nu(t)$ by Duhamel principle, which involves the Green functions for the linear Stokes problem with forcing; in other words, the equation (1.5) and the boundary condition (1.6) when the right hand sides are treated as forcing terms. The difficulty we had overcome is the precise pointwise bound for the Green functions for the above problem. Then we are able to propagate boundary layers norms in analytic function spaces, by convolution estimates, elliptic estimates and estimates for the nonlinear term $u^\nu \cdot \nabla \omega^\nu$.

### 1.3.2 The inviscid limit under critical slip boundary conditions

Using the new analytic framework in [51], I prove that the solutions of the Navier-Stokes equations converge to Euler, under the critical slip boundary condition

$$u_2^\nu = 0, \qquad \partial_y u_1^\nu = \nu^{-1} u_1^\nu \qquad \text{when} \quad y = 0. \tag{1.7}$$

This boundary condition is physically derived from the hydrodynamic limit of the Boltzmann equations with the Maxwell boundary conditions by Jiang-Masmoudi in [32]. Paddick [55] proves that the solution of the Navier-Stokes equations converges to the Euler solution in $L^\infty(0,T,L^2)$ for Sobolev data under the slip condition $\partial_y u^\nu|_{y=0} = \nu^{-\beta_0} u_1^\nu$ for any positive constant $\beta_0 < 1$, with a rate of convergence $O(\nu^{1-\beta_0})$, which is not sufficient in the critical case $\beta_0 = 1$. In [50], we confirm that the inviscid limit holds the critical case $\beta_0 = 1$ for analytic data. Formally, the main theorems can be stated as follows:

**Theorem 1.3.2.** *Let* $u_0^\nu$ *be the initial velocity that is analytic on* $\mathbb{T} \times \mathbb{R}_+$. *Then there exists a positive time* $T > 0$, *independent of the viscosity* $\nu$, *so that the solution* $u^\nu(t)$ *to* *the Navier-Stokes equations under critical slip boundary condition*

$$u_2^\nu = 0, \qquad \partial_z u_1^\nu = \nu^{-1} u_1^\nu \qquad on \qquad z = 0$$

*is analytic and the inviscid limit holds, in particular:*

$$\|u^\nu(t) - u^E(t)\|_{L^2} \le \|u_0^\nu - u_0^E\|_{L^2} + C_T\sqrt{\nu} + C_T(\nu t)^{\frac14} \qquad for \quad t \in [0,T].$$

*Moreover, there holds the pointwise bound for the vorticity* $\omega^\nu = \nabla \times u^\nu$*:*

$$|\omega^\nu(t,x,z)| \le C_0 e^{-\beta z}\Big(1 + \frac{1}{\sqrt{\nu}} e^{-\frac{z}{\sqrt{\nu}}} + \frac{1}{\sqrt{\nu t}} e^{-\frac{z}{\sqrt{\nu t}}}\Big) \tag{1.8}$$

*for* $(t,x,z) \in [0,T] \times \mathbb{T} \times \mathbb{R}_+$.

While the proof is built upon the paper [51], the novelty in [50] is the precise bound on the Green function for the Stokes problem with critical-slip boundary condition (1.7).

### 1.3.3 The inviscid limit for vortex-wave data

When the initial data is a sum of a smooth compactly support function $\omega_0^E$ and point vortices that are away from the support of $\omega_0$, we recently prove that the vorticity for Navier-Stokes stays concentrated around the vortex-wave system, which was introduced by Marchiorio-Pulvirenti in 1994. More precisely, we consider the initial vorticity to be

$$\omega_0^\nu(x) = \delta_{z_0}(x) + \omega_0^E(x). \tag{1.9}$$

Here $\delta_{z_0}(x)$ denotes the Dirac delta function sitting at $z_0$ and $\omega_0^E(x)$ is smooth and compactly supported, so that $z_0 \notin \mathrm{supp}(\omega_0^E)$. In the vanishing viscousity limit, we obtained the vortex-wave system, a model that was introduced by Marchioro-Pulvirenti [43] in the early 90s. Our work [52] rigorously and physically justified the vortex-wave system as a limiting system of Navier-Stokes with the same initial data. To be precise, the vortex-wave solution $(z(t), \omega^E(t))$ is said to solve the vortex-wave system for the initial data $(z_0, \omega_0^E(x))$ if

$$\begin{aligned} &\partial_t \omega^E + (v^E + K(\cdot - z(t))) \cdot \nabla \omega^E = 0 \\ &\dot z(t) = v^E(t, z(t)), \quad v^E = K \star \omega^E, \quad K(x) = \frac{1}{2\pi}\frac{x^\perp}{|x|^2}, \\ &\omega^E_{|t=0} = \omega_0^E, \qquad z(0) = z_0. \end{aligned} \tag{1.10}$$

The first equation means that the regular component $\omega^E(t)$ of vorticity is transported by the full velocity, which is generated by the velocity of the point vortex $z(t)$ and the regular velocity $v^E = K \star \omega^E$ by Biot-Savart law. The second equation says that the location $z(t) \in \mathbb{R}^2$ of the point vortex is propagated by the velocity $v^E$ generated by the regular vorticity $\omega^E$. The theorem is as follows:

**Theorem 1.3.3.** *Let $\omega^\nu(t)$ be the unique solution to the Navier-Stokes equations on the whole space $\Omega = \mathbb{R}^2$ with initial data (1.9). Then on a time interval $[0,T]$ independent of $\nu > 0$, one can decompose the vorticity for Navier-Stokes as follows:*

$$\omega^\nu(t) = \omega^{B,\nu}(t) + \omega^{E,\nu}(t), \qquad \textit{where}$$

$$\|\omega^{E,\nu}(t) - \omega^E(t)\|_{L^4 \cap L^{4/3}(\mathbb{R}^2)} \le C_T \nu, \quad \textit{and}$$

$$\left\| \omega^{B,\nu}(t) - \frac{1}{4\pi\nu t} e^{-\frac{|x-z(t)|^2}{4\nu t}} \right\|_{L^1(\mathbb{R}^2)} \le C_T(\nu t).$$

*Here, $(\omega^E(t), z(t))$ solves the vortex-wave system (1.10). In particular, there holds*

$$\omega^\nu(t) \longrightarrow \delta_{z(t)} + \omega^E(t) \quad \textit{weakly as} \quad \nu \to 0.$$

Lastly, we remark that there were numerous works that obtain the vortex-wave system from the Euler or the Navier-Stokes equations with smooth or localized initial data ([44, 3, 40, 41, 42]). Our work [52] is the first to physically derive the vortex-wave system, directly from the vanishing viscosity limit of Navier-Stokes system with singular data (1.9).

## 1.4 Thesis outline

The thesis is organized as follows

- In Chapter 2, we introduce analytic function spaces that capture the pointwise behavior of the vorticity and the velocity. We also prove several elliptic and bilinear estimates that are important in treating the nonlinear term $-u \cdot \nabla \omega$ appearing the vorticity equation.

- In Chapter 3, we give the proof of our main theorem in the work [51]. The chapter begins with the pointwise bound on the Green function of the Stokes problem associated with the boundary conditions derived from Chapter 3.2. Then we proceed with the convolution estimates in order to propagate boundary layers norm for vorticity. Lastly, we end the chapter with the proof of inviscid limit, by using Kato criterion.

- In Chapter 4, we give the proof of our main theorem on the inviscid limit for critical-slip conditions in the work [50].

- In Chapter 5, we give the proof of our main theorem in [52]. The chapter begins with a brief introduction and outlines, then followed by a technical introduction on the approximate vortex-wave system. The chapter proceeds with Sections 5.3 and 5.4 treating the irregular and regular part respectively. We end the chapter with Section 5.5, where we give the proof of our main results.

## 1.5 Notations

We will denote $A \lesssim B$ to mean that $|A| \leq C_0|B|$ for some universal constant $C_0 > 0$ independent of the viscosity $\nu$. We write $f = O(g)$ to mean that $f \lesssim g$, or simply $O(g)$ to mean that the term can be bounded by $C_0|g|$ for some constant $C_0 > 0$ independent of $\nu$. We also write $f = O(g)_X$ for $f, g$ in a Banach space $(X, \|\cdot\|_X)$, to mean that $\|f\|_X \lesssim \|g\|_X$.

We define the norm $\|\cdot\|_{L^4 \cap L^{4/3}}$ and $\|\cdot\|_{L^1 \cap L^\infty}$ of a function $\omega(x)$ in $\mathbb{R}^2$ to be

$$\|\omega\|_{L^4 \cap L^{4/3}} = \|\omega\|_{L^4} + \|\omega\|_{L^{4/3}}, \qquad \|\omega\|_{L^1 \cap L^\infty} = \|\omega\|_{L^1} + \|\omega\|_{L^\infty}$$

We also denote by $\mathfrak{m}(\cdot)$ the Lebesgue measure on $\mathbb{R}^2$.

For a complex number $z$, we denote $\Re z$ to be the real part, and $\Im z$ to be the imaginary part of $z$ respectively. For a function depending the variable $x \in \mathbb{T}$, we denote $|\partial_x|$ to be the Fourier multiplier $|\alpha|$ in the frequency space.

# Chapter 2

## Analytic boundary layer function spaces

In this chapter, we shall deal with analytic boundary layer spaces introduced in [51, 25, 26]. These analytic function spaces are used to capture the behavior of the vorticity near the boundary. Precisely, we consider holomorphic functions on the pencil-like complex domain:

$$\Omega_\sigma = \Big\{ z \in \mathbb{C} : \quad |\Im z| < \min\{\sigma \Re z, \sigma\} \Big\}, \tag{2.1}$$

for $\sigma > 0$. Let $\delta = \sqrt{\nu}$ be the classical boundary layer thickness. We introduce the analytic boundary layer function spaces $\mathcal{B}^{\sigma,\delta}$ that consists of holomorphic functions on $\Omega_\sigma$ with a finite norm

$$\|f\|_{\sigma,\delta} = \sup_{z\in\Omega_\sigma} |f(z)| e^{\beta \Re z} \Big( 1 + \delta^{-1} \phi_P(\delta^{-1} z) \Big)^{-1} \tag{2.2}$$

for some small $\beta > 0$, and for boundary layer weight function

$$\phi_P(z) = \frac{1}{1 + |\Re z|^P}$$

for some fixed constant $P > 1$. Here, we suppress the dependence on $\beta, P$ as they are fixed throughout the paper. We expect that the vorticity function $\omega(t, x, z)$, for each fixed $t, x$, will be in $\mathcal{B}^{\sigma,\delta}$, precisely describing the behavior near the boundary and near infinity. In fact, there is an additional initial layer of thickness $\delta_t = \sqrt{\nu t}$ that appears near the boundary. To capture this, we introduce the time-dependent boundary layer norm:

$$\|f\|_{\sigma,\delta(t)} = \sup_{z\in\Omega_\sigma} |\omega(z)| e^{\beta \Re z} \Big( 1 + \delta_t^{-1} \phi_P(\delta_t^{-1} z) + \delta^{-1} \phi_P(\delta^{-1} z) \Big)^{-1}, \tag{2.3}$$

with $\delta_t = \sqrt{\nu t}$, $\delta = \sqrt{\nu}$, and with the same boundary layer weight function $\phi_P(\cdot)$. By convention, the norm $\|\cdot\|_{\sigma,\delta(0)}$ at time $t = 0$ is replaced by $\|\cdot\|_{\sigma,\delta}$, the boundary layer norm with precisely one boundary layer behavior of thickness $\delta$, and $\|\cdot\|_{\sigma,0}$ denotes the norm without the boundary layer behavior.

For functions depending on two variables $f(x, z)$, we introduce the partial Fourier transform in variable $x$

$$f(x, z) = \sum_{\alpha\in\mathbb{Z}} f_\alpha(z) e^{i\alpha x}$$

and introduce the following analytic norm

$$||f||_{\rho,\sigma,\delta(t)} = \sum_{\alpha\in\mathbb{Z}} e^{\rho|\alpha|}||f_\alpha||_{\sigma,\delta(t)}$$

for $\rho, \sigma > 0$. We denote by $B^{\rho,\sigma,\delta(t)}$ the corresponding spaces. In Section 2.1, we shall recall some basic properties of such analytic function spaces.

## 2.1 Analytic function spaces

In this section, we shall prove some basic properties of the analytic norms as well as the elliptic estimates that yield bounds on velocity in term of vorticity. These norms and estimates can be found in [51, 26]. See also [58, 59].

Let $f(x,z)$ be holomorphic functions on $\mathbb{T}\times\Omega_\sigma$, with $\Omega_\sigma$ being the pencil-like complex domain defined as in (2.1). For $\rho, \sigma > 0$ and $p \ge 1$, we introduce the analytic function spaces denoted by $\mathcal{L}^p_{\rho,\sigma}$ with the finite norm

$$\|f\|_{\mathcal{L}^p_{\rho,\sigma}} := \sum_{\alpha\in\mathbb{Z}} e^{\rho|\alpha|}\|f_\alpha\|_{L^p_\sigma}, \qquad \|f_\alpha\|_{L^p_\sigma} := \sup_{0\le\theta<\sigma}\Big(\int_{\partial\Omega_\theta}|f_\alpha(z)|^p\,|dz|\Big)^{1/p}, \tag{2.4}$$

in which $f_\alpha = f_\alpha(z)$ denotes the Fourier transform of $f(x,z)$. In the case when $p=\infty$, we replace the $L^p$ norm by the sup norm over $\Omega_\sigma$. Recalling the analytic boundary layer space $B^{\rho,\sigma,\delta(t)}$ introduced in Section 2, we have

**Lemma 2.1.** *There holds the embedding $\mathcal{B}^{\rho,\sigma,\delta(t)} \subset \mathcal{L}^1_{\rho,\sigma}$.*

*Proof.* For the holomorphic functions $f_\alpha(z)$ satisfying

$$|f_\alpha(z)| \le \|f_\alpha\|_{\sigma,\delta(t)} e^{-\beta\Re z}\Big(1 + \delta_t^{-1}\phi_P(\delta_t^{-1}z) + \delta^{-1}\phi_P(\delta^{-1}z)\Big),$$

it is clear that $\|f_\alpha\|_{L^1_\sigma} \le \|f_\alpha\|_{\sigma,\delta(t)}$. By taking the summation over $\alpha\in\mathbb{Z}$, the lemma follows. $\square$

**Lemma 2.2.** *For any $0<\sigma'<\sigma$, $0<\rho'<\rho$, and $\psi(z) = \frac{z}{1+z}$, there hold*

$$\|fg\|_{\mathcal{L}^1_{\rho,\sigma}} \le \|f\|_{\mathcal{L}^\infty_{\rho,\sigma}}\|g\|_{\mathcal{L}^1_{\rho,\sigma}}, \tag{2.5}$$

$$\|\partial_x f\|_{\mathcal{L}^1_{\rho',\sigma}} \le \frac{C}{\rho-\rho'}\|f\|_{\mathcal{L}^1_{\rho,\sigma}}, \qquad \|\psi(z)\partial_z f\|_{\mathcal{L}^1_{\rho,\sigma'}} \le \frac{C}{\sigma-\sigma'}\|f\|_{\mathcal{L}^1_{\rho,\sigma}}. \tag{2.6}$$

*The same estimates hold for boundary layer norms $\|\cdot\|_{\rho,\sigma,\delta}$ replacing $\|\cdot\|_{\mathcal{L}^1_{\rho,\sigma}}$ in the above three inequalities.*

*Proof.* By definition, we write

$$fg(x,z) = \sum_{\alpha\in\mathbb{Z}} e^{i\alpha x}\sum_{\beta\in\mathbb{Z}} f_{\alpha-\beta}(z)g_\beta(z)$$

and hence, we estimate

$$\|fg\|_{\mathcal{L}^1_{\rho,\sigma}} = \sum_{\alpha\in\mathbb{Z}} e^{\rho|\alpha|}\|\sum_{\beta\in\mathbb{Z}} f_{\alpha-\beta}(\cdot)g_\beta(\cdot)\|_{L^1_\sigma} \le \sum_{\alpha\in\mathbb{Z}}\sum_{\beta\in\mathbb{Z}} e^{\rho|\alpha-\beta|}e^{\rho|\beta|}\|f_{\alpha-\beta}\|_{L^\infty_\sigma}\|g_\beta\|_{L^1_\sigma}$$

which proves the first inequality. As for the second, we compute

$$\|\partial_x f\|_{\mathcal{L}^1_{\rho',\sigma}} \le \sum_\alpha \|f_\alpha\|_{L^1_\sigma}|\alpha|e^{\rho'|\alpha|}.$$

Using the fact that $(\rho-\rho')|\alpha|e^{(\rho'-\rho)|\alpha|}$ is bounded, the second inequality follows. Finally, we check the third inequality. By the Cauchy integral formula, we have

$$\partial_z f_\alpha(z) = \frac{1}{2\pi i}\int_{C(z,R_z)} \frac{f_\alpha(y)}{(y-z)^2}dy$$

where $C(z,R_z)$ is the circle, centered at $z$ and of radius $R_z$ so that $C(z,R_z)\in\Omega_\sigma$. Let us take

$$R_z = c_0(\sigma-\sigma')\begin{cases}\Re(z) & \text{if } \Re(z)<1\\ 1 & \text{if } \Re(z)\ge 1\end{cases}$$

for some small and positive $c_0$. Thus, using the parametrization $y = z + e^{iw}R_z$ with $0\le w\le 2\pi$, we get

$$\begin{aligned}\partial_z f_\alpha(z) &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f_\alpha(z+R_ze^{iw})}{R_z^2e^{2iw}}(R_zie^{iw})dw\\ &= \frac{1}{2\pi R_z}\int_0^{2\pi} f_\alpha(z+R_ze^{iw})e^{-iw}dw.\end{aligned}$$

Now for any $0\le\theta'<\sigma'$, we compute

$$\begin{aligned}\int_{\partial\Omega_{\theta'}}|\psi(\Re z)\partial_z f_\alpha(z)||dz| &\le \int_{\partial\Omega_{\theta'}}\int_0^{2\pi}\frac{\psi(\Re z)}{2\pi R_z}|f_\alpha(z+R_ze^{iw})||dw||dz|\\ &\le \frac{C_0}{\sigma-\sigma'}\int_{\partial\Omega_{\theta'}}\int_0^{2\pi}|f_\alpha(z+R_ze^{iw})||dw||dz|\\ &\le \frac{2\pi C_0}{\sigma-\sigma'}\sup_{0\le w\le 2\pi}\int_{\partial\Omega_{\theta'}}|f_\alpha(z+R_ze^{iw})||dz|.\end{aligned}$$

It remains to show that the above integral is bounded by $2\|f\|_{\mathcal{L}^1_{\rho,\sigma}}$. To this end, it suffices to show that for each fixed $w\in[0,2\pi]$, there is a positive constant $\theta<\sigma$ so that

$$z + R_ze^{iw}\in\partial\Omega_\theta, \qquad \forall\ z\in\partial\Omega_{\theta'}. \tag{2.7}$$

*Case 1:* $\Re(z) \le 1$. Recalling $R_z = c_0(\sigma - \sigma')\Re(z)$ and $\Im(z) = \theta'\Re(z)$ on $\partial\Omega_{\theta'}$, we compute

$$\Re(z + R_z e^{iw}) = \Re(z) + R_z \cos(w) = \Re(z)(1 + c_0(\sigma - \sigma')\cos w)$$
$$\Im(z + R_z e^{iw}) = \Im(z) + R_z \sin(w) = \Re(z)(\theta' + c_0(\sigma - \sigma')\sin w).$$

Hence, $z + R_z e^{iw} \in \partial\Omega_\theta$ for $\theta = \frac{\theta' + c_0(\sigma-\sigma')\sin w}{1 + c_0(\sigma-\sigma')\cos w}$. We now check $\theta < \sigma$. Indeed, we need

$$\theta' + c_0(\sigma - \sigma')\sin w < \sigma(1 + c_0(\sigma - \sigma')\cos w)$$

which is equivalent to

$$\sigma - \theta' > c_0(\sigma - \sigma')(\sin w - \sigma\cos w).$$

Since $\theta' < \sigma'$ and $c_0$ can be taken arbitrarily small (independent of $\sigma, \sigma'$), the above inequality follows.

*Case 2:* $\Re(z) \ge 1$. Similarly, using $R_z = c_0(\sigma - \sigma')$ and $\Im(z) = \theta'$ on $\partial\Omega_{\theta'}$, we compute

$$\Im(z + R_z e^{iw}) = \Im(z) + R_z \sin w = \theta' + c_0(\sigma - \sigma')\sin w = \widetilde{\theta},$$

where $\widetilde{\theta} < \sigma$ for sufficiently small $c_0$ and for $\theta' < \sigma' < \sigma$. This proves (2.7). □

## 2.2 Elliptic estimates

Next, we recall the elliptic estimates, which are adapted from [26, 25].

**Proposition 2.3.** *Let $\phi$ be the solution of $-\Delta\phi = \omega$ with the zero Dirichlet boundary condition, and set $u = \nabla^\perp\phi$. Then, there hold*

$$\|u_1\|_{\mathcal{L}^\infty_{\rho,\sigma}} + \|u_2\|_{\mathcal{L}^\infty_{\rho,\sigma}} \le C\|\omega\|_{\mathcal{L}^1_{\rho,\sigma}}, \tag{2.8}$$

$$\|\partial_x u_1\|_{\mathcal{L}^\infty_{\rho,\sigma}} + \|\nabla u_2\|_{\mathcal{L}^\infty_{\rho,\sigma}} + \|\psi^{-1} u_2\|_{L^\infty_{\rho,\sigma}} \le C\|\omega\|_{\mathcal{L}^1_{\rho,\sigma}} + C\|\partial_x\omega\|_{\mathcal{L}^1_{\rho,\sigma}}, \tag{2.9}$$

$$\|\nabla u_1\|_{\mathcal{L}^1_{\rho,\sigma}} + \|\nabla u_2\|_{\mathcal{L}^1_{\rho,\sigma}} \le C\|\omega\|_{\mathcal{L}^1_{\rho,\sigma}}, \tag{2.10}$$

*with $\psi(z) = z/(1+z)$, for some constant $C$.*

*Proof.* Taking the Fourier transform, it suffices to study the classical one-dimensional Laplace equation

$$\partial_z^2\phi_\alpha - \alpha^2\phi_\alpha = \omega_\alpha \tag{2.11}$$

on $\Omega_\sigma$, with the Dirichlet boundary condition $\phi_\alpha(0) = 0$, and $\alpha > 0$. For real values $z$, the solution $\phi_\alpha$ of (2.11) is explicitly given by

$$\phi_\alpha(z) = \int_0^z G_-(y,z)\omega_\alpha(y)dy + \int_z^\infty G_+(y,z)\omega_\alpha(y)dy$$

with

$$G_\pm(y,z) = -\frac{1}{2\alpha}\Big(e^{\pm\alpha(z-y)} - e^{-\alpha(y+z)}\Big).$$

This expression may be extended to complex values of $z$. Indeed, for $z \in \Omega_\sigma$, there is a positive $\theta$ so that $z \in \partial\Omega_\theta$. We then write $\partial\Omega_\theta = \gamma_-(z) \cup \gamma_+(z)$, consisting of complex numbers $y \in \partial\Omega_\theta$ so that $\Re y < \Re z$ and $\Re y > \Re z$, respectively. Then, we write

$$\phi_\alpha(z) = \int_{\gamma_-(z)} G_-(y,z)\omega_\alpha(y)dy + \int_{\gamma_+(z)} G_+(y,z)\omega_\alpha(y)dy. \tag{2.12}$$

We note in particular that for $y \in \gamma_\pm(z)$, there holds

$$|G_\pm(y,z)| \le \alpha^{-1} e^{-\alpha|y-z|}.$$

This proves that

$$|\phi_\alpha(z)| \le \int_{\partial\Omega_\theta} \alpha^{-1} e^{-\alpha|y-z|} |\omega_\alpha(y)|\,|dy| \le \alpha^{-1} \int_{\partial\Omega_\theta} |\omega_\alpha(y)|\,|dy|, \tag{2.13}$$

which by definition yields $\sup_{\Omega_\sigma} |\alpha\phi_\alpha(z)| \le \|\omega_\alpha\|_{L^1_\sigma}$. The same proof holds for $\partial_z\phi_\alpha(z)$. This completes the proof of (2.8).

Now we prove the inequality (2.9). For $\partial_x u_1, \partial_x u_2$ and $\partial_z u_2 = -\partial_x u_1$, we note that $\partial_x$ can be viewed as multiplication by $\alpha$ in the Fourier space. Hence we get

$$\|\partial_x u_1\|_{L^\infty_{\rho,\sigma}} + \|\nabla u_2\|_{L^\infty_{\rho,\sigma}} \le C\|\omega\|_{L^1_{\sigma,\rho}} + C\|\partial_x\omega\|_{L^1_{\rho,\sigma}}$$

Now for $\psi^{-1}u_2$, we have

$$\left|\psi^{-1}u_{2,\alpha}(z)\right| \le |u_{2,\alpha}(z)| + |z^{-1}u_{2,\alpha}(z)| \le C\|\omega\|_{L^1_{\rho,\sigma}} + |z^{-1}u_{2,\alpha}(z)|$$

Since $u_{2,\alpha}(0) = 0$, we have

$$u_{2,\alpha}(z) = \int_0^z \partial_y u_{2,\alpha}(y)dy$$

and thus

$$|u_{2,\alpha}(z)| \le |z| \sup_{y\in[0,z]} |\partial_y u_{2,\alpha}|_{L^\infty} \le C|z| \left(\|\omega\|_{\mathcal{L}^1_{\rho,\sigma}} + \|\partial_x\omega\|_{\mathcal{L}^1_{\rho,\sigma}}\right)$$

The proof for (2.9) is complete.

Finally, taking $L^1$ norm of the estimate (2.13) and upon noting that the kernel $\alpha e^{-\alpha|y-z|}$ is bounded in $L^1$ norm, we obtain the estimate (2.10) for $\alpha^2\phi_\alpha$. The second derivative in $z$, we use $\partial_z^2\phi_\alpha = \alpha^2\phi_\alpha + \omega_\alpha$. This completes the proof of the lemma. $\square$

## 2.3 Bilinear estimates

**Lemma 2.4.** *For any $\omega$ and $\tilde{\omega}$, denoting by $v$ the velocity related to $\omega$, we have*

$$\|v \cdot \nabla\tilde{\omega}\|_{\mathcal{L}^1_{\rho,\sigma}} \le C\|\omega\|_{\mathcal{L}^1_{\rho,\sigma}}\|\tilde{\omega}_x\|_{\mathcal{L}^1_{\rho,\sigma}} + C(\|\omega\|_{\mathcal{L}^1_{\rho,\sigma}} + \|\omega_x\|_{\mathcal{L}^1_{\rho,\sigma}})\|\psi(z)\partial_z\tilde{\omega}\|_{\mathcal{L}^1_{\rho,\sigma}}$$
$$\|v \cdot \nabla\tilde{\omega}\|_{\rho,\sigma,\delta} \le C\|\omega\|_{\rho,\sigma,\delta}\|\tilde{\omega}_x\|_{\rho,\sigma,\delta} + C(\|\omega\|_{\rho,\sigma,\delta} + \|\omega_x\|_{\rho,\sigma,\delta})\|\psi(z)\partial_z\tilde{\omega}\|_{\rho,\sigma,\delta}$$

*Proof.* We write
$$(v \cdot \nabla)\tilde{\omega} = v_1\partial_x\tilde{\omega} + v_2\partial_z\tilde{\omega}.$$
For the first term, using Lemma 2.2 and then (2.8), we have

$$\|v_1\partial_x\widetilde{\omega}\|_{L^1_{\rho,\sigma}} \lesssim \|v_1\|_{L^\infty_{\rho,\sigma}}\|\partial_x\widetilde{\omega}\|_{L^1_{\rho,\sigma}} \lesssim \|\omega\|_{L^1_{\rho,\sigma}}\|\partial_x\widetilde{\omega}\|_{L^1_{\rho,\sigma}},$$
$$\|v_1\partial_x\widetilde{\omega}\|_{\rho,\sigma,\delta} \lesssim \|v_1\|_{L^\infty_{\rho,\sigma}}\|\partial_x\widetilde{\omega}\|_{\rho,\sigma,\delta} \lesssim \|\omega\|_{\rho,\sigma,\delta}\|\partial_x\widetilde{\omega}\|_{\rho,\sigma,\delta}.$$

For the second term, using Lemma 2.2 and then (2.9), we have

$$\|v_2\partial_z\widetilde{\omega}\|_{L^1_{\rho,\sigma}} \lesssim \|\psi^{-1}v_2\|_{L^\infty_{\rho,\sigma}}\|\psi(z)\partial_z\widetilde{\omega}\|_{L^1_{\rho,\sigma}} \lesssim \left(\|\omega\|_{L^1_{\rho,\sigma}} + \|\partial_x\omega\|_{L^1_{\rho,\sigma}}\right)\|\psi(z)\partial_z\widetilde{\omega}\|_{L^1_{\rho,\sigma}},$$
$$\|v_2\partial_z\widetilde{\omega}\|_{\rho,\sigma,\delta} \lesssim \|\psi^{-1}v_2\|_{L^\infty_{\rho,\sigma}}\|\psi(z)\partial_z\widetilde{\omega}\|_{\rho,\sigma,\delta} \lesssim \left(\|\omega\|_{L^1_{\rho,\sigma}} + \|\partial_x\omega\|_{L^1_{\rho,\sigma}}\right)\|\psi(z)\partial_z\widetilde{\omega}\|_{\rho,\sigma,\delta},$$

Here, for the boundary layer norms, we note that

$$\|fg\|_{\rho,\sigma,\delta} \le \|f\|_{\mathcal{L}^\infty_{\rho,\sigma}}\|g\|_{\rho,\sigma,\delta}.$$

and $\|f\|_{\mathcal{L}^1_{\rho,\sigma}} \le \|f\|_{\rho,\sigma,\delta}$. $\square$

# Chapter 3

# The inviscid limit for no-slip conditions

## 3.1 Introduction

In this chapter, we give the proof for the inviscid limit of the Navier-Stokes equations on $\Omega = \mathbb{T}\times\mathbb{R}_+$ for analytic data, under no-slip boundary condition. The equations for velocity field is written as

$$\begin{aligned} \partial_t u^\nu + u^\nu\cdot\nabla u^\nu + \nabla p^\nu &= \nu\Delta u^\nu, \qquad (x,z)\in\mathbb{T}\times\mathbb{R}_+ \\ \nabla\cdot u^\nu &= 0, \\ u^\nu|_{z=0} &= 0. \end{aligned} \tag{3.1}$$

As mentioned before, our proof completely avoids boundary layer expansion, and we obtain a precise pointwise bound for the vorticity. Consequently, we show that the inviscid limit holds in $L^p$, where $2\le p<\infty$:

$$\sup_{0\le t\le T}\|u^\nu(t) - u^E(t)\|_{L^p}\to 0 \qquad \text{as} \quad \nu\to 0$$

for a time $T>0$ independent of the viscosity $\nu>0$.

In Section 3.2, we derive the vorticity equations with suitable boundary condition placed on vorticity. In Section 3.3, we state our main theorems on the bound for the vorticity and the inviscid limit. In Section 3.4, we study the Stokes problem with the vorticity boundary condition. In Section 3.5, we give the proof for the main theorems stated in Section 3.3.

## 3.2 Vorticity boundary formulation

Let $\omega = \partial_z u_1 - \partial_x u_2$ be the corresponding vorticity of the Navier-Stokes equations (3.1). Then, the vorticity equation reads

$$\partial_t\omega - \nu\Delta\omega = -u\cdot\nabla\omega \tag{3.2}$$

with

$$u = \nabla^\perp\Delta^{-1}\omega.$$

Here and throughout this chapter, $\Delta^{-1}$ denotes the inverse of the Laplacian operator with the Dirichlet boundary condition: precisely, $\phi=\Delta^{-1}\omega$ solves $\Delta\phi=\omega$ on the half-space $\mathbb{T}\times\mathbb{R}_+$, with $\phi_{|_{z=0}}=0$.

We shall work with the boundary vorticity formulation and the solution representation as in the recent work by Maekawa [38, 39]; see also [2].

To ensure the no-slip boundary condition, we impose $\partial_t u_1 = 0$ on the boundary. This leads to

$$0 = \partial_t u_1 = \partial_z \Delta^{-1} \partial_t \omega = \partial_z \Delta^{-1}(\nu \Delta \omega - u \cdot \nabla \omega)$$

on the boundary. Introduce $\omega_*$ so that $\Delta \omega_* = 0$ with $\omega_* = \omega$ on the boundary. This yields $\partial_z \Delta^{-1} \Delta \omega = \partial_z(\omega - \omega_*) = (\partial_z + |\partial_x|)\omega$, in which $|\partial_x|$ denotes the Dirichlet-to-Neumann operator on the half space. Thus, the boundary condition on vorticity reads

$$\nu(\partial_z + |\partial_x|)\omega_{|_{z=0}} = [\partial_z \Delta^{-1}(u \cdot \nabla \omega)]_{|_{z=0}}. \tag{3.3}$$

In the next chapters, we shall deal with the Navier-Stokes solutions that solve (3.2)-(3.3), together with the Biot-Savart law $u = \nabla^{\perp} \Delta^{-1} \omega$. Such a solution will be constructed via the Duhamel's integral representation, treating the nonlinearity as a source term.

## 3.3 Main theorems

We recall the definition of boundary layers norm in Chapter 2. Our main results are as follows.

**Theorem 3.3.1.** *Let $M_0 > 0$ and let $\omega_0$ be in $\mathcal{B}^{\rho_0,\sigma_0,\delta}$ for $\rho, \sigma > 0$ and for $\delta = \sqrt{\nu}$, with $\|\omega_0\|_{\rho_0,\sigma_0,\delta} \le M_0$. Then, there is a positive time $T$ so that the solution $\omega(t)$ to the Navier-Stokes equations* (3.2)-(3.3)*, with the initial data $\omega(0) = \omega_0$, exists in $C^1([0,T]; \mathcal{B}^{\rho,\sigma,\delta(t)})$ for $0 < \rho < \rho_0$ and $0 < \sigma < \sigma_0$. In particular, there is a $C_0$ so that the vorticity $\omega(t)$ satisfies*

$$|\omega(t,x,z)| \le C_0 e^{-\beta z}\Big(1 + \delta_t^{-1} \phi_P(\delta_t^{-1} z) + \delta^{-1} \phi_P(\delta^{-1} z)\Big) \tag{3.4}$$

*for $(t,x,z) \in [0,T] \times \mathbb{T} \times \mathbb{R}_+$, with $\delta_t = \sqrt{\nu t}$ and $\delta = \sqrt{\nu}$.*

**Theorem 3.3.2.** *Let $M_0 > 0$ and let $u_0^\nu$ be divergence-free analytic initial data so that $u_0^\nu = 0$ on the boundary and $\omega_0^\nu = \nabla \times u_0^\nu$ is in $\mathcal{B}^{\rho_0,\sigma_0,\delta}$ for $\rho, \sigma > 0$ and for $\delta = \sqrt{\nu}$, with $\|\omega_0^\nu\|_{\rho_0,\sigma_0,\delta} \le M_0$. Then, the inviscid limit holds for Navier-Stokes solutions with the initial data $u_0$. Precisely, there are unique local solutions $u^\nu(t)$ to the Navier-Stokes equations* (1.1)-(1.2)*, for small $\nu > 0$, and a unique solution $u^0(t)$ to the corresponding Euler equations, with initial data $u^0(0) = \lim_{\nu \to 0} u_0^\nu$, so that*

$$\sup_{t \in [0,T]} \|u^\nu(t) - u^0(t)\|_{L^p} \to 0$$

*for $2 \le p < \infty$, as $\nu \to 0$.*

### 3.4 The Stokes problem

In this section, we study the inhomogenous Stokes problem

$$\begin{aligned} \omega_t - \nu\Delta\omega &= f(t,x,z), \qquad &\text{in} \quad \mathbb{T}\times\Omega_\sigma, \\ \nu(\partial_z + |\partial_x|)\omega &= g(t,x), \qquad &\text{on} \quad z=0, \end{aligned} \tag{3.5}$$

together with the initial data $\omega_{|_{t=0}} = \omega_0$. Let $e^{\nu tB}$ denote the semigroup of the corresponding Stokes problem: namely, the heat equation $\partial_t\omega - \nu\Delta\omega = 0$ on $\mathbb{T}\times\Omega_\sigma$ with the homogenous boundary condition $\nu(\partial_z + |\partial_x|)\omega_{|_{z=0}} = 0$. Solutions to the linear Stokes problem is then constructed via the following Duhamel's integral representation, which will be proved in the next subsection,

$$\omega(t) = e^{\nu tB}\omega_0 + \int_0^t e^{\nu(t-s)B} f(s)\, ds + \int_0^t \Gamma(\nu(t-s))g(s)\, ds \tag{3.6}$$

in which $\Gamma(\nu t) = e^{\nu tB}(g\mathcal{H}^1_{\mathbb{T}\times\{y=0\}})$, where $\mathcal{H}^1_{\mathbb{T}\times\{y=0\}}$ is the one-dimensional Hausdorff measure restricted on the boundary; precisely, see (3.27)-(3.28) for the explicit construction of $e^{\nu tB}$ and $\Gamma(\nu t)$ in term of the Green function for the Stokes problem.

In this section, we shall derive uniform bounds for the Stokes semigroup in analytic spaces, with the analytic norm

$$\|\omega\|_{\rho,\sigma,\delta(t)} = \sum_{\alpha\in\mathbb{Z}} e^{\rho|\alpha|}\|\omega_\alpha\|_{\sigma,\delta(t)}$$

with the boundary layer norm defined by

$$\|\omega_\alpha\|_{\sigma,\delta(t)} = \sup_{z\in\Omega_\sigma} |\omega_\alpha(z)| e^{\beta\Re z}\Big(1 + \delta_t^{-1}\phi_P(\delta_t^{-1}z) + \delta^{-1}\phi_P(\delta^{-1}z)\Big)^{-1}, \tag{3.7}$$

in which the boundary thicknesses are $\delta_t = \sqrt{\nu t}$ and $\delta = \sqrt{\nu}$. As for the initial data, the norm is measured by $\|\omega_\alpha\|_{\sigma,\delta(0)}$, which consists of precisely one boundary layer behavior whose thickness is $\delta = \sqrt{\nu}$. We introduce

$$|||\omega(t)|||_{\rho,\sigma,\delta(t),k} = \sum_{j+\ell\le k} \|\partial_x^j(\psi(z)\partial_z)^\ell\omega(t)\|_{\rho,\sigma,\delta(t)}$$

and

$$|||\omega|||_{\mathcal{W}^{k,1}_{\rho,\sigma}} = \sum_{j+\ell\le k} \|\partial_x^j(\psi(z)\partial_z)^\ell\omega(t)\|_{\mathcal{L}^1_{\rho,\sigma}}.$$

We also denote $|||g|||_{\rho,k}$ the corresponding analytic norm for $g = g(x)$. We obtain the following key proposition.

**Proposition 3.1.** *Let $e^{\nu tB}$ be the semigroup for the linear Stokes problem, and $\Gamma(\nu t)$ be the operator $e^{\nu tB}(g\mathcal{H}^1_{\mathbb{T}\times\{y=0\}})$, where $\mathcal{H}^1_{\mathbb{T}\times\{y=0\}}$ is the one-dimensional Hausdorff measure restricted on the boundary. Then, $\partial_x$ commutes with both $e^{\nu tB}$ and $\Gamma(\nu t)$. In* *addition, for any $k \ge 0$, and for any $0 \le s < t \le T$, there hold*

$$|||e^{\nu tB} f|||_{\rho,\sigma,\delta(t),k} \lesssim |||f|||_{\rho,\sigma,\delta(0),k}, \qquad |||e^{\nu(t-s)B} f|||_{\rho,\sigma,\delta(t),k} \lesssim \sqrt{\frac{t}{s}}|||f|||_{\rho,\sigma,\delta(s),k},$$

$$|||\Gamma(\nu(t-s))g|||_{\rho,\sigma,\delta(t),k} \lesssim \sqrt{\frac{t}{t-s}}|||g|||_{\rho,k} + \sqrt{\nu}|||g|||_{\rho,k+1},$$

*uniformly in the inviscid limit. Similarly, we also obtain*

$$|||e^{\nu tB} f|||_{\mathcal{W}^{k,1}_{\rho,\sigma}} \lesssim |||f|||_{\mathcal{W}^{k,1}_{\rho,\sigma}}, \qquad |||\Gamma(\nu t)g|||_{\mathcal{W}^{k,1}_{\rho,\sigma}} \lesssim |||g|||_{\rho,k},$$

*uniformly in the inviscid limit.*

### 3.4.1 Duhamel principle

We first treat the Stokes problem on $\mathbb{T} \times \mathbb{R}_+$. By taking the Fourier transform in $x$, the problem is reduced to

$$\begin{aligned} \partial_t \omega_\alpha - \nu \Delta_\alpha \omega_\alpha &= f_\alpha(t,x,z), && \text{in} \quad \mathbb{R}_+ \\ \nu(\partial_z + |\alpha|)\omega_\alpha &= g_\alpha(t), && \text{on} \quad z = 0, \end{aligned} \tag{3.8}$$

in which $\omega_\alpha$ denotes the Fourier transform of $\omega$ with respect to $x$, and $\Delta_\alpha = \partial_z^2 - \alpha^2$. Let $G_\alpha(t,z;y)$ be the corresponding Green function of the linear Stokes problem (3.8). That is, for each fixed $y \ge 0$, the function $G_\alpha(t,z;y)$ solves

$$\begin{aligned} (\partial_t - \nu\Delta_\alpha)G_\alpha(t,z;y) &= 0, && \text{in} \quad \mathbb{R}_+ \\ \nu(\partial_z + |\alpha|)G_\alpha(t,z;y) &= 0, && \text{on} \quad z = 0, \end{aligned} \tag{3.9}$$

in which $\omega_\alpha$ denotes the Fourier transform of $\omega$ with respect to $x$, and $\Delta_\alpha = \partial_z^2 - \alpha^2$. Let $G_\alpha(t,z;y)$ be the corresponding Green function of the linear Stokes problem (3.8). That is, for each fixed $y \ge 0$, the function $G_\alpha(t,z;y)$ solves

$$\begin{aligned} (\partial_t - \nu\Delta_\alpha)G_\alpha(t,z;y) &= 0, && \text{in} \quad \mathbb{R}_+ \\ \nu(\partial_z + |\alpha|)G_\alpha(t,z;y) &= 0, && \text{on} \quad z = 0, \end{aligned} \tag{3.10}$$

together with the initial data $G_\alpha(0,z;y) = \delta_y(z)$. The Green function will be constructed so that $G_\alpha(t,\cdot;y) \in L^1$ for each $t, y$. It follows that

**Lemma 3.2** (Duhamel's principle)**.** *For any $T > 0$, and for any $f_\alpha \in L^\infty(0,T;L^1(\mathbb{R}_+))$ and $g_\alpha \in L^\infty(0,T)$, the unique solution to the linear Stokes problem (3.8), with the initial* *data* $\omega_\alpha(0,z) = \omega_{0,\alpha}(z)$ *in* $L^1(\mathbb{R}_+)$, *satisfies*

$$\begin{aligned}\omega_\alpha(t,z) = &\int_0^\infty G_\alpha(t,y;z)\omega_{0,\alpha}(y)\,dy + \int_0^t G_\alpha(t-s,0;z)g_\alpha(s)\,ds \\ &+ \int_0^t\int_0^\infty G_\alpha(t-s,y;z)f_\alpha(s,y)\,dyds.\end{aligned} \tag{3.11}$$

*Proof.* Using (3.8), we compute

$$\begin{aligned}&\int_0^t\int_0^\infty G_\alpha(t-s,y;z)f_\alpha(s,y)\,dyds \\ &= \int_0^t\int_0^\infty G_\alpha(t-s,y;z)(\partial_s + \nu\alpha^2 - \nu\partial_y^2)\omega_\alpha(s,y)\,dyds \\ &= \int_0^t\int_0^\infty (\partial_s + \nu\alpha^2 - \nu\partial_y^2)G_\alpha(t-s,y;z)\omega_\alpha(s,y)\,dyds + \int_0^\infty G_\alpha(0,y;z)\omega_\alpha(t,y)\,dy \\ &\quad - \int_0^\infty G_\alpha(t,y;z)\omega_{0,\alpha}(y)\,dy + \nu\int_0^t \Big(G_\alpha(t-s,y;z)\partial_y\omega_\alpha - \partial_y G_\alpha(t-s,y;z)\omega_\alpha\Big)_{|y=0}\,ds.\end{aligned}$$

The lemma follows, upon using the initial data and boundary conditions on $G_\alpha(t,y;z)$. □

### 3.4.2 The Green function for the Stokes problem

In this section, we derive sufficient pointwise bounds on the temporal Green function for the linear Stokes problem (3.8). Precisely, we prove the following.

**Proposition 3.3.** *Let* $G_\alpha(t,y;z)$ *be the Green function of the Stokes problem* (3.8). *There holds*

$$G_\alpha(t,y;z) = H_\alpha(t,y;z) + R_\alpha(t,y;z), \tag{3.12}$$

*in which* $H_\alpha(t,y;z)$ *is exactly the one-dimensional heat kernel with the homogenous Neumann boundary condition and* $R_\alpha(t,y;z)$ *is the residual kernel due to the boundary condition. Precisely, There hold*

$$\begin{aligned}H_\alpha(t,y;z) &= \frac{1}{\sqrt{\nu t}}\Big(e^{-\frac{|y-z|^2}{4\nu t}} + e^{-\frac{|y+z|^2}{4\nu t}}\Big)e^{-\alpha^2\nu t}, \\ |\partial_z^k R_\alpha(t,y;z)| &\lesssim \mu_f^{k+1}e^{-\theta_0\mu_f|y+z|} + (\nu t)^{-\frac{k+1}{2}}e^{-\theta_0\frac{|y+z|^2}{\nu t}}e^{-\frac{1}{8}\alpha^2\nu t},\end{aligned}$$

*for* $y,z\geq 0$, $k\geq 0$, *and for some* $\theta_0>0$ *and for* $\mu_f = |\alpha| + \frac{1}{\sqrt{\nu}}$.

**Remark 3.4.** We note that the residual term $R_\alpha(t,y;z)$ contains a term without viscous dissipation $e^{-\alpha^2\nu t}$, and this is precisely due to the $|\alpha|$ term in the boundary condition in the linear Stokes problem (3.10). Observe that $\omega_\alpha = \alpha e^{-\alpha z}$ is an exact stationary solution to the linear homogenous Stokes problem (3.8).

**Remark 3.5.** By the reflection method (e.g., [39]), the residual Green kernel can be explicitly defined by

$$R_\alpha(t,y;z) = 2e^{-\alpha^2\nu t}(\alpha^2 + \alpha\partial_z)(-\Delta_\alpha^{-1})G(\nu t, y+z), \tag{3.13}$$

with $G(t,z) = \frac{1}{\sqrt{4\pi t}}e^{-z^2/4t}$. The pointwise bounds as derived in Proposition 3.3 are in particular useful in propagating unbounded vorticity with boundary layer behaviors.

We proceed the construction of the Green function via the resolvent equation. Namely, for each fixed $y \ge 0$, let $G_{\lambda,\alpha}(y,z)$ be the $L^1$ solution to the resolvent problem

$$\begin{aligned} (\lambda - \nu\Delta_\alpha)G_{\lambda,\alpha}(y,z) &= \delta_y(z) \\ \nu(\partial_z + |\alpha|)G_{\lambda,\alpha}(y,0) &= 0. \end{aligned} \tag{3.14}$$

We then obtain the following.

**Lemma 3.6.** *Let $\mu = \nu^{-1/2}\sqrt{\lambda + \alpha^2\nu}$, having positive real part. There holds*

$$G_{\lambda,\alpha}(y,z) = H_{\lambda,\alpha}(y,z) + R_{\lambda,\alpha}(y,z) \tag{3.15}$$

*in which $H_{\lambda,\alpha}(y,z)$ denotes the resolvent kernel of the heat problem with homogenous Neumann boundary condition; namely,*

$$H_{\lambda,\alpha}(y,z) = \frac{1}{2\nu\mu}(e^{-\mu|y-z|} + e^{-\mu|y+z|}), \qquad R_{\lambda,\alpha}(y,z) = \frac{\alpha(\alpha+\mu)}{\lambda\mu}e^{-\mu|y+z|}.$$

*In particular, $G_{\lambda,\alpha}(y,z)$ is meromorphic with respect to $\lambda$ in $\mathbb{C} \setminus \{-\alpha^2\nu - \mathbb{R}_+\}$ with a pole at $\lambda = 0$.*

*Proof.* The construction is standard, upon noting that $G_{\lambda,\alpha}(y,z)$ is a linear combination of $e^{\pm\mu z}$ and satisfies the following jump conditions across $z = y$:

$$[G_{\lambda,\alpha}(y,z)]_{|z=y} = 0, \qquad [\nu\partial_z G_{\lambda,\alpha}(y,z)]_{|z=y} = 1.$$

The lemma follows. □

*Proof of Proposition 3.3.* The temporal Green function $G_\alpha(t,z;y)$ can then be constructed via the inverse Laplace transform:

$$G_\alpha(t,y;z) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}G_{\lambda,\alpha}(y,z)d\lambda \tag{3.16}$$

in which the contour of integration $\Gamma$ is taken such that it remains on the right of the (say, $L^2$) spectrum of the linear operator $\lambda - \nu\Delta_\alpha$, which is $-\alpha^2\nu - \mathbb{R}_+$.

In view of (3.15), we set $H_\alpha(t,y;z)$ and $R_\alpha(t,y;z)$ to be the corresponding temporal Green function of $H_{\lambda,\alpha}(y,z)$ and $R_{\lambda,\alpha}(y,z)$, respectively. It follows that $H_\alpha(t,y;z)$ is the temporal Green function of the one-dimensional heat problem with the homogenous

Neumann boundary condition, yielding

$$H_\alpha(t,y;z) = \frac{1}{\sqrt{4\pi\nu t}}\Big(e^{-\frac{|y-z|^2}{4\nu t}} + e^{-\frac{|y+z|^2}{4\nu t}}\Big)e^{-\nu\alpha^2 t}.$$

It remains to compute the residual Green function $R_\alpha(t,y;z)$:

$$R_\alpha(t,y;z) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}e^{-\mu|y+z|}\frac{\alpha(\alpha+\mu)}{\lambda\mu}\,d\lambda. \tag{3.17}$$

Note that the $z$-derivative of $R_\alpha(t,y;z)$ gains an extra $\mu$ in the above integral.

**Case 1: $\alpha^2\nu \le 1$.** By the Cauchy's theory, we may decompose the contour of integration as $\Gamma = \Gamma_\pm \cup \Gamma_c$, having

$$\begin{aligned}\Gamma_\pm &:= \Big\{\lambda = -\frac{1}{2}\nu\alpha^2 + \nu(a^2-b^2) + 2\nu iab \pm iM, \quad \pm b \in \mathbb{R}_+\Big\},\\ \Gamma_c &:= \Big\{\lambda = -\frac{1}{2}\nu\alpha^2 + \nu a^2 + Me^{i\theta}, \quad \theta\in[-\pi/2,\pi/2]\Big\},\end{aligned} \tag{3.18}$$

for some positive number $M$ and for $a = \frac{|y+z|}{2\nu t}$. Since $\alpha^2\nu\le 1$, we take $M$ large enough so that the pole $\lambda = 0$ remains on the left of the contour $\Gamma$. It is clear that $|\lambda|\gtrsim 1$ on $\Gamma$.

On $\Gamma_c$, we note that

$$\begin{aligned}\Re\mu &= \nu^{-1/2}\Re\sqrt{\frac{1}{2}\nu\alpha^2 + \nu a^2 + Me^{i\theta}} \ge \nu^{-1/2}\sqrt{\frac{1}{2}\nu\alpha^2+\nu a^2} \ge a,\\ \Re\mu &= \nu^{-1/2}\Re\sqrt{\frac{1}{2}\nu\alpha^2 + \nu a^2 + Me^{i\theta}} \ge \nu^{-1/2}\sqrt{M}.\end{aligned}$$

This implies that $\Re\mu \ge \frac{a}{2} + \theta_0\mu_f$ for some positive constant $\theta_0$, recalling $\mu_f = \alpha + 1/\sqrt{\nu}$ and $\alpha\le\nu^{-1/2}$. In particular, $|\mu|\gtrsim\mu_f\ge\alpha$. This proves that

$$\begin{aligned}\Big|\int_{\Gamma_c} e^{\lambda t}e^{-\mu|y+z|}\frac{\alpha(\alpha+\mu)}{\lambda\mu}\,d\lambda\Big| &\lesssim \int_{-\pi/2}^{\pi/2} e^{Mt}e^{a^2\nu t}e^{-\frac{a}{2}|y+z|}e^{-\theta_0\mu_f|y+z|}\mu_f d\theta\\ &\lesssim \mu_f e^{-\theta_0\mu_f|y+z|}e^{a^2\nu t}e^{-\frac{a}{2}|y+z|}\\ &\lesssim \mu_f e^{-\theta_0\mu_f|y+z|},\end{aligned}$$

in which we used $e^{a^2\nu t}e^{-\frac{a}{2}|y+z|} = 1$ by definition of $a$. As for derivatives, we estimate

$$\begin{aligned}\Big|\int_{\Gamma_c} e^{\lambda t}e^{-\mu|y+z|}\frac{\alpha(\alpha+\mu)}{\lambda}\,d\lambda\Big| &= \Big|\int_{\Gamma_c} e^{\lambda t}e^{-\mu|y+z|}\frac{\alpha}{\nu(\mu-\alpha)}\,d\lambda\Big|\\ &\lesssim \nu^{-1}\int_{-\pi/2}^{\pi/2} e^{Mt}e^{a^2\nu t}e^{-\frac{a}{2}|y+z|}e^{-\theta_0\mu_f|y+z|}d\theta\\ &\lesssim \mu_f^2 e^{-\theta_0\mu_f|y+z|},\end{aligned}$$

upon recalling $\mu_f = \alpha + 1/\sqrt{\nu}$.

Meanwhile, on $\Gamma_\pm$, we note that

$$\Re\mu = \Re\sqrt{\frac{1}{2}\alpha^2 + (a+ib)^2 \pm i\nu^{-1}M} \ge \Re\sqrt{(a+ib)^2} = a,$$

upon noting that the sign of $b$ and $\pm M$ is the same on $\Gamma_\pm$. Similarly, we note that $\Re\mu \gtrsim M/\sqrt{\nu}$. By definition of $a$, we have

$$|e^{\lambda t}e^{-\mu|y+z|}| \le e^{-\frac{1}{2}\nu\alpha^2 t}e^{-\frac{|y+z|^2}{4\nu t}}e^{-\nu b^2 t},$$

and together with the fact that $\lambda = \nu(\mu^2 - \alpha^2)$, we compute

$$\frac{\alpha(\alpha+\mu)}{\lambda\mu}\,d\lambda = \frac{2i\alpha(a+ib)}{\mu(\mu-\alpha)}db.$$

Since $\alpha^2\nu \lesssim 1$, we have $\alpha \lesssim |\mu|$. In addition, we have $(a+ib)^2 \pm i\nu^{-1/2}M = \mu^2 - \alpha^2$ on $\Gamma_\pm$ with $b$ having the same sign as does $\pm M$. This implies that $|a+ib|^2 \le |\mu^2 - \alpha^2| \lesssim |\mu|^2$. Putting the above computations together, we obtain

$$\begin{aligned}\Big|\int_{\Gamma_\pm} e^{\lambda t}e^{-\mu|y+z|}\frac{\alpha(\alpha+\mu)}{\lambda\mu}\,d\lambda\Big| &\le C_0 e^{-\frac{1}{2}\nu\alpha^2 t}e^{-\frac{|y+z|^2}{4\nu t}}\int_{\mathbb{R}} e^{-\nu b^2 t}\frac{\alpha|a+ib|}{|\mu(\mu-\alpha)|}db\\ &\le C_0 e^{-\frac{1}{2}\nu\alpha^2 t}e^{-\frac{|y+z|^2}{4\nu t}}\int_{\mathbb{R}} e^{-\nu b^2 t}db\\ &\le C_0(\nu t)^{-1/2}e^{-\frac{1}{2}\nu\alpha^2 t}e^{-\frac{|y+z|^2}{4\nu t}}.\end{aligned}$$

As for derivatives, we estimate

$$\begin{aligned}\Big|\int_{\Gamma_\pm} e^{\lambda t}e^{-\mu|y+z|}\frac{\alpha(\alpha+\mu)}{\lambda}\,d\lambda\Big| &\le C_0 e^{-\frac{1}{2}\nu\alpha^2 t}e^{-\frac{|y+z|^2}{4\nu t}}\int_{\mathbb{R}} e^{-\nu b^2 t}(a+|b|)db\\ &\le C_0(\nu t)^{-1}\Big(1+\frac{|x+y|}{\sqrt{\nu t}}\Big)e^{-\frac{1}{2}\nu\alpha^2 t}e^{-\frac{|y+z|^2}{4\nu t}}\\ &\le C_0(\nu t)^{-1}e^{-\frac{1}{2}\nu\alpha^2 t}e^{-\frac{|y+z|^2}{8\nu t}}.\end{aligned}$$

**Case 2:** $\alpha^2\nu \ge 1$. Take $a = \frac{|y+z|}{2\nu t}$ as in the previous case. Consider first the case when $|a-\alpha| \ge \frac{1}{2}\alpha$. In this case, we move the contour of integration to

$$\Gamma_1 := \Big\{\lambda = -\nu\alpha^2 + \nu(a^2 - b^2) + 2\nu iab, \quad \pm b \in \mathbb{R}_+\Big\}$$

which may pass the pole at $\lambda = 0$ (precisely, it does when $a = \alpha$). By the Cauchy's theory, we have

$$R_\alpha(t, y; z) = \frac{1}{2\pi i} \int_{\Gamma_1} e^{\lambda t} e^{-\mu|y+z|} \frac{\alpha(\alpha+\mu)}{\lambda\mu}\, d\lambda + \mathrm{Res}_0$$

in which the residue at the pole $\lambda = 0$ is given by

$$\mathrm{Res}_0 = 2\alpha e^{-\alpha|y+z|} \tag{3.19}$$

when $a < \alpha$. We take $\mathrm{Res}_0 = 0$ when $a > \alpha$. Note that the residue does not decay in time. This accounts for the contribution of the inhomogenous Neumann boundary condition. Since $\alpha^2\nu \ge 1$, we have $\mu_f = \alpha + 1/\sqrt{\nu} \le 2\alpha$, and hence

$$\mathrm{Res}_0 \le 2\mu_f e^{-\frac{1}{2}\mu_f|y+z|}.$$

As for the integral term, we note that $\mu = a + ib$ and hence

$$\frac{\alpha(\alpha+\mu)}{\lambda\mu}\, d\lambda = \frac{2i\nu\alpha(\alpha+\mu)}{\lambda} db.$$

Note that $\Gamma_1$ cuts the real axis at $\nu(a^2 - \alpha^2)$ and the imaginary axis at $2\nu a b_0$ (when $a > \alpha$), with $b_0 = \pm\sqrt{a^2 - \alpha^2}$. This in particular yields $|\lambda| \gtrsim \nu\alpha(a+\alpha)$ and $|\mu| = \nu^{-1/2}|\sqrt{\lambda + \alpha^2\nu}| \lesssim \nu^{-1/2}|\lambda|^{1/2}$. In particular,

$$\Big|\frac{\alpha(\alpha+\mu)}{\lambda\mu}\, d\lambda\Big| \le db.$$

We thus obtain

$$\begin{aligned} \Big|\int_{\Gamma_1} e^{\lambda t} e^{-\mu|y+z|} \frac{\alpha(\alpha+\mu)}{\lambda\mu}\, d\lambda\Big| &\le C_0 e^{-\nu\alpha^2 t} e^{-\frac{|y+z|^2}{4\nu t}} \int_{\mathbb{R}} e^{-\nu b^2 t} db \\ &\le C_0(\nu t)^{-1/2} e^{-\nu\alpha^2 t} e^{-\frac{|y+z|^2}{4\nu t}}. \end{aligned} \tag{3.20}$$

It remains to consider the case when $|a - \alpha| \le \frac{1}{2}\alpha$ and $\alpha^2\nu \ge 1$. We note in particular that $\frac{1}{2}\alpha \le a \le \frac{3}{2}\alpha$. In this case, we simply modify the contour of integration as follows: we take

$$\Gamma_1 := \Big\{\lambda = -\frac{1}{8}\nu\alpha^2 + \nu(a^2 - b^2) + 2\nu i a b, \quad \pm b \in \mathbb{R}_+\Big\}.$$

Observe that the contour $\Gamma_1$ leaves the origin on the left, with $|\lambda| \gtrsim \nu\alpha^2$. The integral is thus estimated exactly as done in (3.20). The derivative estimates follow as in the previous case.

The proof of Proposition 3.3 is complete. □

### 3.4.3 The Green function on $\Omega_\sigma$

The Green function constructed in Proposition 3.3 can be easily extended to the complex domain $\Omega_\sigma$ defined by

$$\Omega_\sigma = \Big\{ z \in \mathbb{C} : \quad |\Im z| < \min\{\sigma |\Re z|, \sigma\}\Big\},$$

for some small $\sigma > 0$. Indeed, in view of (3.13), the Green function involves precisely the heat kernel $G(t,z) = \frac{1}{\sqrt{4\pi t}} e^{-z^2/4t}$, which is extended to the complex domain. In addition, we note that for $z \in \Gamma_\sigma$, there holds $\Im z \le \sigma \Re z$, which implies that

$$|e^{-z^2/4t}| \le e^{-|\Re z|^2/4t + |\Im z|^2/4t} \le e^{-(1-\sigma^2)|\Re z|^2/4t}.$$

Similar estimates hold for the other terms in the Green function $G_\alpha(t,y;z) = H_\alpha(t,y;z) + R_\alpha(t,y;z)$, yielding

$$\begin{aligned} H_\alpha(t,y;z) &\lesssim \frac{1}{\sqrt{\nu t}} \Big( e^{-(1-\sigma^2)\frac{|\Re y - \Re z|^2}{4\nu t}} + e^{-(1-\sigma^2)\frac{|\Re y + \Re z|^2}{4\nu t}} \Big) e^{-\frac18 \alpha^2 \nu t}, \\ R_\alpha(t,y;z) &\lesssim \mu_f e^{-\theta_0 (1-\sigma)\mu_f |\Re y + \Re z|}, \end{aligned} \tag{3.21}$$

for $y, z \in \Gamma_\sigma$, and for some $\theta_0 > 0$ and for $\mu_f = |\alpha| + \frac{1}{\sqrt{\nu}}$.

The solution $\omega_\alpha(t,z)$ to the Stokes problem can now be constructed on $\Omega_\sigma$ in the similar manner as done in (2.12). Precisely, for any $z \in \Omega_\sigma$, let $\theta$ be the positive constant so that $z \in \partial\Omega_\theta$. The Duhamel principle (3.11) then becomes

$$\begin{aligned} \omega_\alpha(t,z) = &\int_{\partial\Omega_\theta} G_\alpha(t,y;z)\omega_{0,\alpha}(y)\,dy + \int_0^t G_\alpha(t-s,0;z) g_\alpha(s)\,ds \\ &+ \int_0^t \int_{\partial\Omega_\theta} G_\alpha(t-s,y;z) f_\alpha(s,y)\,dyds, \end{aligned} \tag{3.22}$$

which is well-defined for $z \in \Omega_\sigma$, having the Green function $G_\alpha(t,y;z)$ satisfies the pointwise estimates (3.21), similar to those on the real line. For this reason, it suffices to derive convolution estimates for real values $y, z$.

### 3.4.4 Convolution estimates

We now derive convolution estimates. We start with the analytic $L^1$ norms. For $k \ge 0$, we introduce

$$\|\omega_\alpha\|_{\mathcal{W}^{k,1}_\sigma} = \sum_{j=0}^k \|(\psi(z)\partial_z)^j \omega_\alpha\|_{L^1_\sigma}.$$

We prove the following.

**Proposition 3.7.** *Let $T > 0$ and let $G_\alpha(t,y;z)$ be the Green function of the Stokes problem* (3.8)*, constructed in Proposition 3.3. Then, for any $0 \le s < t \le T$ and $k \ge 0$,* *there is a universal constant $C_T$ so that*

$$\Big\| \int_0^\infty G_\alpha(t,y;\cdot)\omega_\alpha(y)\, dy \Big\|_{\mathcal{W}^{k,1}_\sigma} \le C_T \|\omega_\alpha\|_{\mathcal{W}^{k,1}_\sigma},$$
$$\Big\| \int_0^\infty G_\alpha(t-s,y;\cdot)\omega_\alpha(y)\, dy \Big\|_{\mathcal{W}^{k,1}_\sigma} \le C_T \|\omega_\alpha\|_{\mathcal{W}^{k,1}_\sigma},$$

*uniformly in the inviscid limit.*

*Proof.* We shall prove the convolution for real values $y,z$. For the complex extension, see Section 3.4.3. Recall from Proposition 3.3 that $G_\alpha(t,y;z) = H_\alpha(t,y;z) + R_\alpha(t,y;z)$, with

$$H_\alpha(t,y;z) = \frac{1}{\sqrt{\nu t}}\Big(e^{-\frac{|y-z|^2}{4\nu t}} + e^{-\frac{|y+z|^2}{4\nu t}}\Big)e^{-\alpha^2\nu t},$$
$$|\partial_z^k R_\alpha(t,y;z)| \lesssim \mu_f^{k+1} e^{-\theta_0\mu_f|y+z|} + (\nu t)^{-\frac{k+1}{2}} e^{-\theta_0\frac{|y+z|^2}{\nu t}} e^{-\frac{1}{8}\alpha^2\nu t}$$

for $k \ge 0$. In particular, $\|G_\alpha(t,y;\cdot)\|_{L^1_\sigma} \lesssim 1$, for each fixed $y,t$. The $L^1$ convolution estimate is thus straightforward. We now check the estimates for derivatives. We estimate

$$\begin{aligned}\psi(z)\partial_z R_\alpha(t,y;z) &\lesssim |z|\mu_f^2 e^{-\theta_0\mu_f|y+z|} + \frac{|z|}{\nu t} e^{-\theta_0\frac{|y+z|^2}{\nu t}} e^{-\frac{1}{8}\alpha^2\nu t} \\ &\lesssim \mu_f e^{-\frac{1}{2}\theta_0\mu_f|y+z|} + \frac{1}{\sqrt{\nu t}} e^{-\theta_0\frac{|y+z|^2}{2\nu t}} e^{-\frac{1}{8}\alpha^2\nu t}.\end{aligned}$$

That is, $\psi(z)\partial_z R_\alpha(t,y;z)$ obeys essentially the same bound as does $R_\alpha(t,y;z)$. The convolution estimates for derivatives of $R_\alpha(t,y;z)$ follow.

Next, we treat the integral involving $H_\alpha(t,y;z)$. Precisely, we set $H(t,y;z) = \frac{1}{\sqrt{\nu t}} e^{-\frac{|y-z|^2}{4\nu t}}$. Note that $\partial_z H(t,y;z) = -\partial_y H(t,y;z)$. Hence, we compute

$$\begin{aligned}&\int_0^\infty \psi(z)\partial_z H(t-s,y;\cdot)\omega_\alpha(y)\, dy \\ &= \int_0^{z/2} \psi(z)\partial_z H(t-s,y;\cdot)\omega_\alpha(y)\, dy - \psi(z)H(t-s,z/2;z)\omega_\alpha(z/2) \\ &\quad + \int_{z/2}^\infty \psi(z)H(t-s,y;\cdot)\partial_y\omega_\alpha(y)\, dy.\end{aligned}$$

We now estimate each term on the right. Since $\psi(z) \le 2\psi(y)$ for $y \ge z/2$, the last integral on the right is already estimated in the previous case with $\omega_\alpha(y)$ replaced by

$\psi(y)\partial_y\omega_\alpha(y)$. As for the first integral, since $y \le z/2$, we compute

$$\begin{aligned}\psi(z)\partial_z H(t-s,y;z) &\lesssim \frac{z}{1+z}(\nu(t-s))^{-1}e^{-\frac{|y-z|^2}{8\nu(t-s)}}\\ &\lesssim |y-z|(\nu(t-s))^{-1}e^{-\frac{|y-z|^2}{8\nu(t-s)}}\\ &\lesssim (\nu(t-s))^{-1/2}e^{-\frac{|y-z|^2}{16\nu(t-s)}}.\end{aligned}$$

Thus, the integral over $[0, z/2]$ is again already estimated in the previous case. Finally, we compute

$$|\psi(z)H(t-s,z/2;z)\omega_\alpha(z/2)| \lesssim z(\nu(t-s))^{-1/2}e^{-\frac{|z|^2}{16\nu(t-s)}}|\omega_\alpha(z/2)| \lesssim |\omega_\alpha(z/2)|,$$

whose $L^1_\sigma$ norm is clearly bounded by $\|\omega_\alpha\|_{L^1_\sigma}$. □

#### 3.4.5 Convolution estimates with boundary layer behaviors

In this section, we provide the convolution estimates of the Green function against functions in the boundary layer spaces, whose norm is defined by

$$\|\omega_\alpha\|_{\sigma,\delta(t)} = \sup_{z\in\Omega_\sigma}|\omega_\alpha(z)|e^{\beta\Re z}\Big(1+\delta_t^{-1}\phi_P(\delta_t^{-1}z)+\delta^{-1}\phi_P(\delta^{-1}z)\Big)^{-1}, \tag{3.23}$$

for $t > 0$ and $\beta > 0$, in which the boundary thicknesses are $\delta_t = \sqrt{\nu t}$ and $\delta = \sqrt{\nu}$ and for boundary layer weight $\phi_P(z) = \frac{1}{1+|\Re z|^P}$, $P > 1$. We also introduce the boundary norm for derivatives:

$$\|\omega_\alpha\|_{\sigma,\delta(t),k} = \sum_{j=0}^k \|(\psi(z)\partial_z)^j\omega_\alpha\|_{\sigma,\delta(t)}$$

for $k \ge 0$. In the case $t = 0$, the norm $\|\cdot\|_{\sigma,\delta(0)}$ is defined to consist of precisely one boundary layer with thickness $\delta = \sqrt{\nu}$.

We prove the following.

**Proposition 3.8.** *Let $T > 0$ and let $G_\alpha(t,y;z)$ be the Green function of the Stokes problem* (3.8)*, constructed in Proposition 3.3. Then, for any $0 \le s < t \le T$ and $k \ge 0$, there is a universal constant $C_T$ so that*

$$\begin{aligned}\Big\|\int_0^\infty G_\alpha(t,y;\cdot)\omega_\alpha(y)\,dy\Big\|_{\sigma,\delta(t),k} &\le C_T\|\omega_\alpha\|_{\sigma,\delta(0),k},\\ \Big\|\int_0^\infty G_\alpha(t-s,y;\cdot)\omega_\alpha(y)\,dy\Big\|_{\sigma,\delta(t),k} &\le C_T\sqrt{\frac{t}{s}}\|\omega_\alpha\|_{\sigma,\delta(s),k},\end{aligned}$$

*uniformly in the inviscid limit.*

We shall prove the convolution estimates for real values $y, z$. The complex extension follows from the similar estimates on the Green function obtained in (3.21). As a consequence, Proposition 3.8 is a direct combination of the following two lemmas.

**Lemma 3.9.** *Let $R(t,y;z) := \mu_f e^{-\mu_f|y+z|}$, with $\mu_f = \alpha + \frac{1}{\sqrt{\nu}}$. Then, for any $s,t$, and $k \ge 0$, there is a universal constant $C_0$ so that*

$$\Big\| \int_0^\infty R(t-s,y;z)\omega_\alpha(y)\, dy \Big\|_{\sigma,\delta(t),k} \le C_0 \|\omega_\alpha\|_{\sigma,\delta(s)}.$$

*Proof.* The estimate for $k=0$ follows directly from the fact that $\omega_\alpha$ belongs to $L^1$: $\|\omega_\alpha\|_{L^1} \le C_0\|\omega_\alpha\|_{\sigma,\delta(s)}$. The convolution in fact belongs to the boundary layer space with finite norm $\|\cdot\|_{\sigma,\delta(0)}$. As for derivatives, we note that by induction

$$|(\psi(z)\partial_z)^k R(t,y;z)| \lesssim \sum_{i=0}^k |z|^i \mu_f^{i+1} e^{-\mu_f|y+z|} \lesssim \mu_f e^{-\frac{1}{2}\mu_f z}.$$

The proof is complete. □

**Lemma 3.10.** *Let $H(t,y;z) := (\nu t)^{-1/2} e^{-\frac{|y\pm z|^2}{M\nu t}}$, for some positive $M$, and let $T>0$. Then, for any $0 \le s < t \le T$, $\epsilon_0 > 0$, and $k \ge 0$, there is a universal constant $C_T$ so that*

$$\Big\| \int_0^\infty H(t,y;\cdot)\omega_\alpha(y)\, dy \Big\|_{\sigma,\delta(t),k} \le C_T \|\omega_\alpha\|_{\sigma,\delta(0),k},$$
$$\Big\| \int_0^\infty H(t-s,y;\cdot)\omega_\alpha(y)\, dy \Big\|_{\sigma,\delta(t),k} \le C_T \sqrt{\frac{t}{s}} \|\omega_\alpha\|_{\sigma,\delta(s),k},$$

*uniformly in the inviscid limit.*

*Proof.* It suffices to prove the convolution for $H(t,y;z) = (\nu t)^{-1/2} e^{-\frac{|y-z|^2}{M\nu t}}$. We start with the case $k=0$. Let $0 \le s < t$. For $|y-z| \ge M\beta\nu(t-s)$, it is clear that

$$e^{-\frac{|y-z|^2}{M\nu(t-s)}} e^{-\beta|y|} \le e^{-\beta|z|} e^{-|y-z|\left(\frac{|y-z|}{M\nu(t-s)} - \beta\right)} \le e^{-\beta|z|}.$$

Whereas, for $|y-z| \le M\beta\nu(t-s)$, we note that

$$e^{-M\beta^2\nu(t-s)} e^{-\beta|y|} \le e^{-\beta|y-z|} e^{-\beta|y|} \le e^{-\beta|z|}.$$

That is, the exponential decay $e^{-\beta z}$ is recovered at an expense of a slowly growing term in time: $e^{M\beta^2\nu(t-s)}$, which is bounded in finite time. Precisely, this proves

$$e^{-\frac{|y-z|^2}{M\nu(t-s)}} e^{-\beta y} \le e^{M\beta^2\nu(t-s)} e^{-\beta|z|}, \qquad \forall y,z \in \mathbb{R}. \tag{3.24}$$

It remains to study the integral

$$\int_0^\infty (\nu(t-s))^{-1/2} e^{-\frac{|y-z|^2}{M\nu(t-s)}} \Big(1 + \delta_s^{-1}\phi_P(\delta_s^{-1}y) + \delta^{-1}\phi_P(\delta^{-1}y)\Big)\, dy. \tag{3.25}$$

First, without the boundary layer behavior, the integral is clearly bounded. We now treat the boundary layer terms. Using the fact that $\phi_P(\cdot)$ is decreasing, we have

$$\begin{aligned}
&\int_{z/2}^\infty (\nu(t-s))^{-1/2} e^{-\frac{|y-z|^2}{M\nu(t-s)}} \delta^{-1}\phi_P(\delta^{-1}y)\, dy \\
&\quad \le C_0\delta^{-1}\phi_P(\delta^{-1}z) \int_{z/2}^\infty (\nu(t-s))^{-1/2} e^{-\frac{|y-z|^2}{M\nu(t-s)}}\, dy \\
&\quad \le C_0\delta^{-1}\phi_P(\delta^{-1}z)
\end{aligned}$$

and, upon noting that $y/\delta_s \ge z/2\delta_t$, we obtain

$$\begin{aligned}
&\int_{z/2}^\infty (\nu(t-s))^{-1/2} e^{-\frac{|y-z|^2}{M\nu(t-s)}} \delta_s^{-1}\phi_P(\delta_s^{-1}y)\, dy \\
&\quad \le C_0\delta_s^{-1}\phi_P(\delta_t^{-1}z) \int_{z/2}^\infty (\nu(t-s))^{-1/2} e^{-\frac{|y-z|^2}{M\nu(t-s)}}\, dy \\
&\quad \le C_0\sqrt{t/s}\,\delta_t^{-1}\phi_P(\delta_t^{-1}z).
\end{aligned}$$

Whereas on $y \in (0, \frac{z}{2})$, we have $|y-z| \ge \frac{z}{2}$ and $\phi_P \le 1$. Hence, we have

$$\begin{aligned}
&\int_0^{z/2} (\nu(t-s))^{-1/2} e^{-\frac{|y-z|^2}{M\nu(t-s)}} \delta_s^{-1}\phi_P(\delta_s^{-1}y)\, dy \\
&\quad \le e^{-\frac{|z|^2}{8M\nu(t-s)}} \int_0^{z/2} (\nu(t-s))^{-1/2} e^{-\frac{|y-z|^2}{2M\nu(t-s)}} \delta_s^{-1}\phi_P(\delta_s^{-1}y)\, dy \\
&\quad \le C_0 e^{-\frac{|z|^2}{8M\nu t}} \min\{\delta_s^{-1}, \delta_{t-s}^{-1}\}.
\end{aligned}$$

Note that $\min\{\delta_s^{-1}, \delta_{t-s}^{-1}\} \le 2\delta_t^{-1}$. Hence, the above integral is bounded in $\|\cdot\|_{\sigma,\delta(t)}$ norm. Similarly, we estimate

$$\int_0^{z/2} (\nu(t-s))^{-1/2} e^{-\frac{|y-z|^2}{M\nu(t-s)}} \delta^{-1}\phi_P(\delta^{-1}y)\, dy \le C_0 e^{-\frac{|z|^2}{8M\nu(t-s)}}\delta^{-1}.$$

To estimate this, we will prove that

$$e^{-\frac{|z|^2}{8M\nu t}} \le C_0 e^{\epsilon_0 t}\phi_P(\delta^{-1}z), \tag{3.26}$$

for arbitrarily small $\epsilon_0$ (and hence, $C_0$ depends on $\epsilon_0$). Indeed, when $|z| \ge \epsilon_0\sqrt{\nu}t$, it is clear that $e^{-\frac{|z|^2}{8M\nu t}} \lesssim \phi_P(\delta^{-1}z)$. On the other hand, when $|z| \le \epsilon_0\sqrt{\nu}t$, we note that

$e^{z/\delta} \le e^{\epsilon_0 t}$, which implies that $1 \le e^{\epsilon_0 t} e^{-z/\delta} \le C_0 e^{\epsilon_0 t}\phi_P(\delta^{-1}z)$. The estimate (3.26) follows, and hence the claimed estimate for $k = 0$.

Next, we consider the derivative estimate. Note that $\partial_z H(t, y; z) = -\partial_y H(t, y; z)$. Hence, we compute

$$\begin{aligned}
&\int_0^\infty \psi(z)\partial_z H(t-s, y; \cdot)\omega_\alpha(y)\, dy \\
&= \int_0^{z/2} \psi(z)\partial_z H(t-s, y; \cdot)\omega_\alpha(y)\, dy - \psi(z)H(t-s, z/2; z)\omega_\alpha(z/2) \\
&\quad + \int_{z/2}^\infty \psi(z)H(t-s, y; \cdot)\partial_y\omega_\alpha(y)\, dy.
\end{aligned}$$

We now estimate each term on the right. Since $\psi(z) \le 2\psi(y)$ for $y \ge z/2$, the last integral on the right is already estimated in the previous case with $\omega_\alpha(y)$ replaced by $\psi(y)\partial_y\omega_\alpha(y)$. As for the first integral, since $y \le z/2$, we compute

$$\begin{aligned}
\psi(z)\partial_z H(t-s, y; z) &\lesssim \frac{z}{1+z}(\nu(t-s))^{-1} e^{-\frac{|y-z|^2}{2M\nu(t-s)}} \\
&\lesssim |y-z|(\nu(t-s))^{-1} e^{-\frac{|y-z|^2}{2M\nu(t-s)}} \\
&\lesssim (\nu(t-s))^{-1/2} e^{-\frac{|y-z|^2}{4M\nu(t-s)}}.
\end{aligned}$$

Thus, the integral over $[0, z/2]$ is again already estimated in the previous case. Finally, we compute

$$\begin{aligned}
&|\psi(z)H(t-s, z/2; z)\omega_\alpha(z/2)| \\
&\lesssim z(\nu(t-s))^{-1/2} e^{-\frac{|z|^2}{4M\nu(t-s)}} \Big(1 + \delta_s^{-1}\phi_P(\delta_s^{-1}z) + \delta^{-1}\phi_P(\delta^{-1}z)\Big) \\
&\lesssim 1 + \delta_s^{-1}\phi_P(\delta_s^{-1}z) + \delta^{-1}\phi_P(\delta^{-1}z) \\
&\lesssim 1 + \sqrt{t/s}\,\delta_t^{-1}\phi_P(\delta_t^{-1}z) + \delta^{-1}\phi_P(\delta^{-1}z),
\end{aligned}$$

in which again the last inequality is due to the decreasing property of $\phi_P$ and the fact that $\delta_s \le \delta_t$. This completes the proof of the lemma. □

### 3.4.6 Semigroup bounds in analytic spaces

In this section, we shall prove Proposition 3.1 on deriving uniform bounds for the Stokes semigroup in analytic spaces, with the analytic norm

$$\|\omega\|_{\rho,\sigma,\delta(t)} = \sum_{\alpha\in\mathbb{Z}} e^{\rho|\alpha|}\|\omega_\alpha\|_{\sigma,\delta(t)}$$

We first write the Stokes semigroup $e^{tB}$ and the operator $\Gamma(t)$ in the Fourier series:

$$e^{\nu tB}\omega = \sum_{\alpha\in\mathbb{Z}} e^{i\alpha x}(e^{\nu tB}\omega)_\alpha, \qquad \Gamma(\nu t)g = \sum_{\alpha\in\mathbb{Z}} e^{i\alpha x}(\Gamma(\nu t)g)_\alpha \tag{3.27}$$

in which

$$(e^{\nu tB}\omega)_\alpha = \int_0^\infty G_\alpha(t,y;z)\omega_\alpha(y)\,dy, \qquad (\Gamma(\nu t)g)_\alpha = G_\alpha(t,0;z)g_\alpha, \tag{3.28}$$

with the Green kernel $G_\alpha(t,y;z)$ constructed in Proposition 3.3. The convolution estimates obtained in Proposition 3.8 yield

$$\|(e^{\nu(t-s)B}\omega)_\alpha\|_{\sigma,\delta(t),k} \le C_0\sqrt{\frac{t}{s}}\|\omega_\alpha\|_{\sigma,\delta(s),k}.$$

These prove the claimed estimates on $e^{\nu tB}$. As for the trace operator, using Proposition 3.3, we have

$$\begin{aligned} G_\alpha(t-s,0;z) &\lesssim (\nu(t-s))^{-1/2}e^{-\frac{z^2}{4\nu(t-s)}} + \mu_f e^{-\mu_f z} \\ &\lesssim (\nu(t-s))^{-1/2}e^{-\frac{z^2}{4\nu t}} + (|\alpha| + \nu^{-1/2})e^{-\frac{z}{\sqrt{\nu}}}, \end{aligned} \tag{3.29}$$

upon recalling that $\mu_f = |\alpha| + \nu^{-1/2}$. By definition, $\|G_\alpha(t-s,0;z)\|_{\sigma,\delta(t)} \lesssim \sqrt{\frac{t}{t-s}} + 1 + \alpha\sqrt{\nu}$. The proof of Proposition 3.1 is complete.

### 3.5 Proof of the main theorem

As mentioned in the introduction, we construct the solutions to the Navier-Stokes equation via the vorticity formulation:

$$\begin{aligned} \partial_t\omega - \nu\Delta\omega &= -u\cdot\nabla\omega \\ \nu(\partial_z + |\partial_x|)\omega_{|z=0} &= [\partial_z\Delta^{-1}(u\cdot\nabla\omega)]_{|z=0}, \end{aligned} \tag{3.30}$$

in which $u = \nabla^\perp\Delta^{-1}\omega$, with $\Delta^{-1}$ being the inverse of Laplacian with the Dirichlet boundary condition. For convenience, we set $N = u\cdot\nabla\omega$. The solution to (3.30) is then constructed via the Duhamel's principle:

$$\omega(t) = e^{\nu tB}\omega_0 - \int_0^t e^{\nu(t-s)B}N(s)\,ds + \int_0^t \Gamma(\nu(t-s))(\partial_z\Delta^{-1}N(s))_{|z=0}\,ds \tag{3.31}$$

with $\omega_0 \in \mathcal{B}^{\rho_0,\sigma_0,\delta}$, for some $\rho_0, \sigma_0 > 0$.

### 3.5.1 Nonlinear iterations

Let us fix positive numbers $\gamma, \zeta$, and $\rho_0$, and introduce the following nonlinear iterative norm for vorticity:

$$A(\gamma) = \sup_{0<\gamma t<\rho_0} \sup_{\rho<\rho_0-\gamma t} \Big\{ |||\omega(t)|||_{\mathcal{W}^{1,1}_{\rho,\rho}} + |||\omega(t)|||_{\mathcal{W}^{2,1}_{\rho,\rho}} (\rho_0 - \rho - \gamma t)^{\zeta} \Big\} \tag{3.32}$$

with recalling

$$|||\omega(t)|||_{\mathcal{W}^{k,1}_{\rho,\rho}} = \sum_{j+\ell\le k} \|\partial_x^j (\psi(z)\partial_z)^\ell \omega(t)\|_{L^1_{\rho,\rho}}.$$

Here, for sake of presentation, we take the same analyticity radius in $x$ and $z$; namely, $\sigma = \rho < \rho_0$. Thanks to Lemma 2.1, $\omega_0 \in \mathcal{W}^{k,1}_{\rho,\rho}$, for any $k \ge 0$.

We shall show that the vorticity norm remains finite for sufficiently large $\gamma$. The weight $(\rho_0 - \rho - \gamma t)^\zeta$, with a small $\zeta > 0$, is standard to avoid time singularity when recovering the loss of derivatives ([4, 57]). Let $\rho < \rho_0 - \gamma t$. Thanks to Lemma 2.4, we have

$$\begin{aligned} |||N(t)|||_{\mathcal{W}^{0,1}_{\rho,\rho}} &\lesssim |||\omega(t)|||^2_{\mathcal{W}^{1,1}_{\rho,\rho}} \le A(\gamma)^2 \\ |||N(t)|||_{\mathcal{W}^{1,1}_{\rho,\rho}} &\lesssim |||\omega(t)|||_{\mathcal{W}^{1,1}_{\rho,\rho}} |||\omega(t)|||_{\mathcal{W}^{2,1}_{\rho,\rho}} \le A(\gamma)^2 (\rho_0 - \rho - \gamma t)^{-\zeta}. \end{aligned} \tag{3.33}$$

In addition, using the elliptic estimates, we have

$$|||(\partial_z \Delta^{-1} N(t))_{|z=0}|||_{\rho,k} \lesssim |||N(t)|||_{\mathcal{W}^{k,1}_{\rho,\rho}}. \tag{3.34}$$

Now, using the Duhamel integral formula (3.31), we estimate

$$\begin{aligned} |||\omega(t)|||_{\mathcal{W}^{k,1}_{\rho,\rho}} \le |||e^{\nu t B}\omega_0|||_{\mathcal{W}^{k,1}_{\rho,\rho}} &+ \int_0^t |||e^{\nu(t-s)B} N(s)|||_{\mathcal{W}^{k,1}_{\rho,\rho}}\, ds \\ &+ \int_0^t |||\Gamma(\nu(t-s))(\partial_z \Delta^{-1} N(s))_{|z=0}|||_{\mathcal{W}^{k,1}_{\rho,\rho}}\, ds. \end{aligned}$$

In view of Proposition 3.1, the term from the initial data is already estimated, giving $|||e^{\nu t B}\omega_0|||_{\mathcal{W}^{k,1}_{\rho,\rho}} \le \|\omega_0\|_{\mathcal{W}^{k,1}_{\rho,\rho}}$. As for the integral terms, we estimate

$$\begin{aligned} \int_0^t |||e^{\nu(t-s)B} N(s)|||_{\mathcal{W}^{1,1}_{\rho,\rho}}\, ds &\le C_0 \int_0^t |||N(s)|||_{\mathcal{W}^{1,1}_{\rho,\rho}}\, ds \\ &\le C_0 A(\gamma)^2 \int_0^t (\rho_0 - \rho - \gamma s)^{-\zeta}\, ds \\ &\le C_0 \gamma^{-1} A(\gamma)^2. \end{aligned}$$

Similarly,

$$\int_0^t |||\Gamma(\nu(t-s))(\partial_z \Delta^{-1} N(s))_{|z=0}|||_{\mathcal{W}^{1,1}_{\rho,\rho}}\, ds \le C_0 \int_0^t |||N(s)|||_{\mathcal{W}^{1,1}_{\rho,\rho}}\, ds,$$

which is again bounded by $C_0\gamma^{-1}A(\gamma)^2$. Next, we give estimates for $k=2$. Noting that $\rho<\rho_0-\gamma t\le\rho_0-\gamma s$, we take $\rho'=\frac{\rho+\rho_0-\gamma s}{2}$ and compute

$$\begin{aligned}\int_0^t |||e^{\nu(t-s)B}N(s)|||_{\mathcal{W}^{2,1}_{\rho,\rho}}\,ds &\le C_0\int_0^t |||N(s)|||_{\mathcal{W}^{2,1}_{\rho,\rho}}\,ds\\ &\le C_0\int_0^t \frac{1}{\rho'-\rho}|||N(s)|||_{\mathcal{W}^{1,1}_{\rho',\rho'}}\,ds\\ &\le C_0A(\gamma)^2\int_0^t(\rho_0-\rho-\gamma s)^{-1-\zeta}\,ds\\ &\le C_0\gamma^{-1}A(\gamma)^2(\rho_0-\rho-\gamma t)^{-\zeta}.\end{aligned}$$

Same computation holds for the trace operator $\Gamma(\nu t)$, yielding

$$A(\gamma)\le C_0\|\omega_0\|_{\mathcal{W}^{2,1}_{\rho,\rho}}+C_0\gamma^{-1}A(\gamma)^2.$$

By taking $\gamma$ sufficiently large, the above yields the uniform bound on the iterative norm in term of initial data. This yields the local solution in $L^1_{\rho,\rho}$ for $t\in[0,T]$, with $T=\gamma^{-1}\rho_0$.

### 3.5.2 Propagation of boundary layers

It remains to prove that the constructed solution has the boundary layer behavior as expected, having already constructed solutions in $L^1_{\rho,\rho}$ spaces. Indeed, we now introduce the following nonlinear iterative norm for vorticity:

$$B(\gamma)=\sup_{0<\gamma t<\rho_0}\sup_{\rho<\rho_0-\gamma t}\Big\{|||\omega(t)|||_{\rho,\delta(t),1}+|||\omega(t)|||_{\rho,\delta(t),2}(\rho_0-\rho-\gamma t)^{\zeta}\Big\} \tag{3.35}$$

with the boundary layer norm

$$|||\omega(t)|||_{\rho,\delta(t),k}=\sum_{j+\ell\le k}\|\partial_x^j(\psi(z)\partial_z)^\ell\omega(t)\|_{\rho,\rho,\delta(t)}.$$

Thanks to Lemma 2.4, we estimate

$$\begin{aligned}|||N(t)|||_{\rho,\delta(t),0}&\lesssim|||\omega(t)|||^2_{\rho,\delta(t),1}\le B(\gamma)^2\\ |||N(t)|||_{\rho,\delta(t),1}&\lesssim|||\omega(t)|||_{\rho,\delta(t),1}|||\omega(t)|||_{\rho,\delta(t),2}\le B(\gamma)^2(\rho_0-\rho-\gamma t)^{-\zeta}.\end{aligned} \tag{3.36}$$

Now, using the Duhamel integral formula (3.31), we estimate

$$\begin{aligned}|||\omega(t)|||_{\rho,\delta(t),k}\le|||e^{\nu tB}\omega_0|||_{\rho,\delta(t),k}&+\int_0^t|||e^{\nu(t-s)B}N(s)|||_{\rho,\delta(t),k}\,ds\\ &+\int_0^t|||\Gamma(\nu(t-s))(\partial_z\Delta^{-1}N(s))_{|z=0}|||_{\rho,\delta(t),k}\,ds.\end{aligned}$$

In view of Proposition 3.1, the term from the initial data is already estimated, giving $|||e^{\nu tB}\omega_0|||_{\rho,\delta(t),k} \le \|\omega_0\|_{\rho,\delta(0),k}$. In addition, using the estimates on $\Gamma(\nu t)$ from Proposition 3.1 and (4.43), we have

$$\begin{aligned}
&\int_0^t |||\Gamma(\nu(t-s))(\partial_z\Delta^{-1}N(s))_{|z=0}|||_{\rho,\delta(t),k}ds \\
&\lesssim \int_0^t \sqrt{t}(t-s)^{-1/2}\|(\partial_z\Delta^{-1}N(s))_{|z=0}\|_{\rho,k}\,ds + \sqrt{\nu}\int_0^t \|(\partial_z\Delta^{-1}N(s))_{|z=0}\|_{\rho,k+1}\,ds \\
&\lesssim \int_0^t \sqrt{t}(t-s)^{-1/2}|||N(s)|||_{\mathcal{W}^{k,1}_{\rho,\rho}}\,ds + \sqrt{\nu}\int_0^t |||N(s)|||_{\mathcal{W}^{k+1,1}_{\rho,\rho}}\,ds \\
&\lesssim t\sup_{0\le s\le t}|||N(s)|||_{\mathcal{W}^{k+1,1}_{\rho,\rho}}
\end{aligned}$$

in which $\|N(s)\|_{\mathcal{W}^{k+1,1}_{\rho,\rho}}$ is bounded, thanks to the (independent) iteration obtained in the previous subsection. It remains to estimate

$$\begin{aligned}
&\int_0^t |||e^{\nu(t-s)B}N(s)|||_{\rho,\delta(t),1}\,ds \\
&\le C_0\int_0^t \sqrt{\frac{t}{s}}|||N(s)|||_{\rho,\delta(s),1}\,ds \le C_0B(\gamma)^2\int_0^t \sqrt{\frac{t}{s}}(\rho_0-\rho-\gamma s)^{-\zeta}\,ds \\
&\le C_0B(\gamma)^2\Big(\int_0^{t/2}+\int_{t/2}^t\Big)\sqrt{\frac{t}{s}}(\rho_0-\rho-\gamma s)^{-\zeta}\,ds \\
&\le C_0B(\gamma)^2\Big(t(\rho_0-\rho-\frac{1}{2}\gamma t)^{-\zeta}+\frac{1}{\gamma}(\rho_0-\rho-\frac{1}{2}\gamma t)^{1-\zeta}\Big) \\
&\le C_0\gamma^{-1}B(\gamma)^2(\rho_0-\rho)^{-\zeta},
\end{aligned}$$

in which we used $\gamma t \le \rho_0$ and $\gamma t < \rho_0-\rho$. Next, noting that $\rho < \rho_0-\gamma t \le \rho_0-\gamma s$, we take $\rho' = \frac{\rho+\rho_0-\gamma s}{2}$ and compute

$$\begin{aligned}
&\int_0^t |||e^{\nu(t-s)B}N(s)|||_{\rho,\delta(t),2}\,ds \\
&\le C_0\int_0^t \sqrt{\frac{t}{s}}|||N(s)|||_{\rho,\delta(s),2}\,ds \le C_0\int_0^t \sqrt{\frac{t}{s}}\frac{1}{\rho'-\rho}|||N(s)|||_{\rho',\delta(s),1}\,ds \\
&\le C_0B(\gamma)^2\int_0^t \sqrt{\frac{t}{s}}(\rho_0-\rho-\gamma s)^{-1-\zeta}\,ds \\
&\le C_0B(\gamma)^2\Big(\int_0^{t/2}+\int_{t/2}^t\Big)\sqrt{\frac{t}{s}}(\rho_0-\rho-\gamma s)^{-1-\zeta}\,ds \\
&\le C_0B(\gamma)^2\Big(t(\rho_0-\rho-\frac{1}{2}\gamma t)^{-1-\zeta}+\frac{1}{\gamma}(\rho_0-\rho-\gamma t)^{-\zeta}\Big) \\
&\le C_0\gamma^{-1}B(\gamma)^2(\rho_0-\rho-\gamma t)^{-\zeta}.
\end{aligned}$$

This proves the boundedness of the iterative norm $B(\gamma)$, and hence the propagation of the boundary layer behaviors. Theorem 3.3.1 follows.

### 3.5.3 The inviscid limit

Let us now prove Theorem 3.3.2. Since $\omega_0^\nu \in \mathcal{B}^{\rho_0,\sigma_0,\delta}$, the velocity $u_0^\nu$ and its conormal derivatives $\partial_x^k u_0^\nu, (\psi(z)\partial_z)^j u_0^\nu$, for $k, j, \geq 0$ and $\psi = \frac{z}{1+z}$, are all uniformly bounded on $\mathbb{T} \times \mathbb{R}_+$, thanks to the elliptic estimates; see Proposition 2.3. In particular, the limit of $u_0^\nu$ exists in the classical sense: $u^0(0) = \lim_{\nu\to 0} u_0^\nu$. From Theorem 3.3.1, we check the validity of the Kato's condition:

$$\begin{aligned}
&\nu \int_0^T \iint_{\mathbb{T}\times\mathbb{R}_+} |\nabla u(t,x,z)|^2 \, dxdzdt \\
&= \nu \int_0^T \iint_{\mathbb{T}\times\mathbb{R}_+} |\omega(t,x,z)|^2 \, dxdzdt \\
&\leq C\nu \int_0^T \iint_{\mathbb{T}\times\mathbb{R}_+} e^{-2\beta z}\Big(1 + \delta_t^{-1}\phi_P(\delta_t^{-1} z) + \delta^{-1}\phi_P(\delta^{-1} z)\Big)^2 \, dxdzdt \\
&\leq C\nu \int_0^T \Big(1 + \delta_t^{-1} + \delta^{-1}\Big) dt \leq C_T \sqrt{\nu}
\end{aligned}$$

which tends to zero as $\nu \to 0$. This proves that $\sup_{t\in[0,T]} \|u^\nu(t) - u^0(t)\|_{L^2} \to 0$. The $L^p$ convergence follows from the interpolation between $L^2$ and $L^\infty$ norms and the fact that $u^\nu$ is bounded in $L^\infty$ thanks to the elliptic estimate (2.8).

# Chapter 4

# The inviscid limit for critical-slip conditions

## 4.1 Introduction

In this chapter, we give the proof of the inviscid limit of the Navier-Stokes equations on $\Omega = \mathbb{T} \times \mathbb{R}_+$ for analytic data, under critical-slip boundary condition. The equations for velocity field is written as

$$\begin{aligned} \partial_t u + u \cdot \nabla u + \nabla p = \nu \Delta u, \qquad (x,z) \in \mathbb{T} \times \mathbb{R}_+ \\ \nabla \cdot u = 0, \\ u_2|_{z=0} = 0, \qquad (\nu \partial_z u_1 - u_1)|_{z=0} = 0. \end{aligned} \tag{4.1}$$

Our proof completely avoids boundary layer expansion, and we obtain a precise pointwise bound for the vorticity. In the vanishing viscosity limit, we prove that the solution of Navier-Stokes converges to solution of the Euler equations, and obtain the rate of convergence in $L^2$ norm as follows:

$$\|u^\nu(t) - u^E(t)\|_{L^2} \lesssim (\nu t)^{1/4} + \sqrt{\nu} + \|u_0^\nu - u_0^E\|_{L^2}.$$

Moreover, we prove that the vorticity $\omega^\nu$ satisfies the pointwise bound

$$\omega^\nu(t,x,z) \lesssim e^{-\beta z} \left(1 + \delta^{-1} \phi_P(\delta^{-1} z) + \delta_t^{-1} \phi_P(\delta_t^{-1} z)\right)$$

where $\delta = \sqrt{\nu}$, $\delta_t = \sqrt{\nu t}$ and $\phi_P(z) = \frac{1}{1+z^P}$ for $z > 0$.

This chapter is organized as follows: In Section 4.2, we derive the boundary condition placed on the vorticity, using from Biot-Savart law. In Section 4.3, we state our main theorems. In Section 4.4, we study the Stokes problem under the critical slip boundary condition for the vorticity, namely, we compute the resolvent Green kernel, derive a sharp pointwise bound for the Green function, then prove several convolutions estimates in boundary layers norm. In Section 4.5, we give the proof for our main theorems, which is stated in Section 4.3.

## 4.2 The vorticity formulation

To ensure the critical slip boundary condition, we impose $\nu\omega = u_1$ on the boundary. Taking Fourier transform in $x$, namely $\omega(x,z) = \sum_{\alpha \in \mathbb{Z}} \omega_\alpha(z) e^{i\alpha x}$, we impose the following boundary condition

$$\nu\omega_\alpha(0) = -\int_0^\infty e^{-\alpha y}\omega_\alpha(y)dy \tag{4.2}$$

which follows from the following lemma:

**Lemma 4.1.** *Let $u_{1,\alpha}$ be the Fourier transform of the tangential component $u_1$ and $\omega_\alpha$ the Fourier transform of $\omega$. Then the value of $u_{1,\alpha}$ on the boundary $z = 0$ is given by:*

$$u_{1,\alpha}(0) = -\int_0^\infty e^{-\alpha y}\omega_\alpha(y)dy.$$

*Proof.* Since $\partial_x u_1 + \partial_z u_2 = 0$, one can write $u_1 = \partial_z\phi$ and $u_2 = -\partial_x\phi$ for some stream function $\phi$. Since $\Delta\phi = \omega$ on $\mathbb{T}\times\mathbb{R}_+$ and $\phi|_{z=0} = 0$, we have

$$(\partial_z^2 - \alpha^2)\phi_\alpha = \omega_\alpha, \qquad \phi_\alpha(0) = 0$$

where $\phi_\alpha$ and $\omega_\alpha$ are the Fourier transform of $\phi$ and $\omega$. The solution of the above equation is given explicitly by

$$\begin{aligned}\phi_\alpha(z) &= \frac{1}{2\alpha}\int_0^\infty \left(e^{-\alpha|y+z|} - e^{-\alpha|y-z|}\right)\omega_\alpha(y)dy \\ &= \frac{1}{2\alpha}\left(\int_0^\infty e^{-\alpha(y+z)}\omega_\alpha(y)dy - \int_0^z e^{\alpha(y-z)}\omega_\alpha(y)dy - \int_z^\infty e^{\alpha(z-y)}\omega_\alpha(y)dy\right).\end{aligned}$$

Since $u_{1,\alpha} = \partial_z\phi_\alpha$, a direct calculation yields

$$u_{1,\alpha}(z) = \frac{1}{2\alpha}\left(-\alpha\int_0^\infty e^{-\alpha(y+z)}\omega_\alpha(y)dy + \alpha\int_0^z e^{\alpha(y-z)}\omega_\alpha(y)dy - \alpha\int_z^\infty e^{\alpha(z-y)}\omega_\alpha(y)dy\right)$$

The lemma follows, after evaluating $u_{1,\alpha}$ at $z = 0$. □

## 4.3 Main results

We recall the definition of boundary layers norm in Chapter 2. Our main results are as follows.

**Theorem 4.3.1.** *Let $M_0 > 0$ and let $\omega_0$ be in $\mathcal{B}^{\rho_0,\sigma_0,\delta}$ for $\rho, \sigma > 0$ and for $\delta = \sqrt{\nu}$, with $\|\omega_0\|_{\rho_0,\sigma_0,\delta} \le M_0$. Then, there is a positive time $T$, independent of $\nu > 0$, so that the solution $\omega(t)$ to the Navier-Stokes equations under critical slip boundary condition*

$$u_2^\nu = 0, \qquad \partial_z u_1^\nu = \nu^{-1}u_1 \qquad on \qquad z = 0$$

*, with the initial data $\omega(0) = \omega_0$, exists in $C^1([0,T];\mathcal{B}^{\rho,\sigma,\delta(t)})$ for $0 < \rho < \rho_0$ and $0 < \sigma < \sigma_0$. In particular, there is a $C_0$ so that the vorticity $\omega(t)$ satisfies*

$$|\omega(t,x,z)| \le C_0 e^{-\beta z}\left(1 + \delta_t^{-1}\phi_P(\delta_t^{-1}z) + \delta^{-1}\phi_P(\delta^{-1}z)\right) \tag{4.3}$$

*for* $(t,x,z) \in [0,T] \times \mathbb{T} \times \mathbb{R}_+$, *with* $\delta_t = \sqrt{\nu t}$ *and* $\delta = \sqrt{\nu}$.

**Theorem 4.3.2.** *Let* $M_0 > 0$ *and let* $u_0^\nu$ *be divergence-free analytic initial data so that* $\omega_0^\nu = \nabla \times u_0^\nu$ *is in* $\mathcal{B}^{\rho_0,\sigma_0,\delta}$ *for* $\rho, \sigma > 0$ *and for* $\delta = \sqrt{\nu}$, *with* $\|\omega_0^\nu\|_{\rho_0,\sigma_0,\delta} \le M_0$. *Then, the inviscid limit holds for Navier-Stokes solutions with the initial data* $u_0^\nu$, *with the time scale set by Theorem 4.3.1. Precisely, there are unique local solutions* $u^\nu(t)$ *to the Navier-Stokes equations under critical slip boundary condition, for small* $\nu > 0$, *and a unique solution* $u^E(t)$ *to the corresponding Euler equations, with initial data* $u_0^E = \lim_{\nu\to 0} u_0^\nu$, *so that*

$$\|u^\nu(t) - u^E(t)\|_{L^2} \le \|u_0^\nu - u_0^E\|_{L^2} + C_T\sqrt{\nu} + C_T(\nu t)^{\frac{1}{4}} \qquad \text{for} \quad t \in [0,T],$$

*where* $C_T$ *is a constant that only depends on the solution of Euler and* $T$. *In particular, we have*

$$\sup_{0\le t\le T} \|u^\nu(t) - u^E(t)\|_{L^p} \to 0 \qquad \text{as} \quad \nu \to 0$$

*for any* $2 \le p < \infty$.

As mentioned, the proof of the main theorems is direct, using the vorticity formulation (3.2)-(4.2). For Theorem 4.3.1, we first prove the local existence of solutions in the analytic space $L^1_{\rho,\sigma}$ (see (2.4)), and then in boundary layer spaces in order to establish the precise pointwise behavior of the vorticity (see Section 4.5.2). Theorem 4.3.1 applies in particular for well-prepared analytic data that satisfy the Prandtl's ansatz of size $\sqrt{\nu}$. For general analytic data, beside the Prandtl's layers, the initial layers whose thickness is of order $\sqrt{\nu t}$ appear as captured in (4.3). After proving Theorem 4.3.1, we establish Theorem 4.3.2 by a direct energy estimate; see Section 4.5.3.

## 4.4 The Stokes problem

### 4.4.1 Main propositions

In this section, we state our Proposition 4.2 for the inhomogenous Stokes problem

$$\begin{cases} \omega_t - \nu\Delta\omega &= f(t,x,y), \qquad \text{in} \quad \mathbb{T} \times \Omega_\sigma, \\ \nu\omega &= u_1, \qquad \text{on} \quad y = 0, \\ u_2|_{y=0} &= 0, \\ \omega|_{t=0} &= \omega_0. \end{cases} \tag{4.4}$$

Let $e^{\nu t B}$ denote the semigroup of the corresponding Stokes problem: namely, the heat equation $\partial_t\omega - \nu\Delta\omega = 0$ on $\mathbb{T} \times \Omega_\sigma$ with the homogenous boundary condition

$$(\nu\omega - u_1)|_{y=0} = 0.$$

Solutions to the linear Stokes problem is then constructed via the following Duhamel's integral representation:

$$\omega(t) = e^{\nu tB}\omega_0 + \int_0^t e^{\nu(t-s)B} f(s)\, ds. \tag{4.5}$$

In this section, we shall derive uniform bounds for the Stokes semigroup in analytic spaces, with the analytic norm

$$\|\omega\|_{\rho,\sigma,\delta(t)} = \sum_{\alpha\in\mathbb{Z}} e^{\rho|\alpha|}\|\omega_\alpha\|_{\sigma,\delta(t)}$$

with the boundary layer norm defined by

$$\|\omega_\alpha\|_{\sigma,\delta(t)} = \sup_{z\in\Omega_\sigma} |\omega_\alpha(z)|e^{\beta\Re z}\Big(1 + \delta_t^{-1}\phi_P(\delta_t^{-1}z) + \delta^{-1}\phi_P(\delta^{-1}z)\Big)^{-1}, \tag{4.6}$$

in which the boundary thicknesses are $\delta_t = \sqrt{\nu t}$ and $\delta = \sqrt{\nu}$. As for the initial data, the norm is measured by $\|\omega_\alpha\|_{\sigma,\delta(0)}$, which consists of precisely one boundary layer behavior whose thickness is $\delta = \sqrt{\nu}$. We introduce

$$|||\omega(t)|||_{\rho,\sigma,\delta(t),k} = \sum_{j+\ell\le k} \|\partial_x^j(\psi(z)\partial_z)^\ell\omega(t)\|_{\rho,\sigma,\delta(t)}$$

and

$$|||\omega|||_{\mathcal{W}^{k,1}_{\rho,\sigma}} = \sum_{j+\ell\le k} \|\partial_x^j(\psi(z)\partial_z)^\ell\omega(t)\|_{\mathcal{L}^1_{\rho,\sigma}}.$$

Next, we state our main proposition, which will be proved in Section 4.4.6:

**Proposition 4.2.** *Let $e^{\nu tB}$ be the semigroup for the linear Stokes problem. Then, $\partial_x$ commutes with $e^{\nu tB}$. In addition, for any $k\ge 0$, and for any $0\le s<t\le T$, there hold*

$$|||e^{\nu tB}f|||_{\rho,\sigma,\delta(t),k} \lesssim |||f|||_{\rho,\sigma,\delta(0),k},$$
$$|||e^{\nu(t-s)B}f|||_{\rho,\sigma,\delta(t),k} \lesssim \sqrt{\frac{t}{t-s}}|||f|||_{\mathcal{W}^{k,1}_{\rho,\sigma}} + \sqrt{\frac{t}{s}}|||f|||_{\rho,\sigma,\delta(s),k},$$

*uniformly in the inviscid limit. Similarly, we also obtain*

$$|||e^{\nu tB}f|||_{\mathcal{W}^{k,1}_{\rho,\sigma}} \lesssim |||f|||_{\mathcal{W}^{k,1}_{\rho,\sigma}},$$

*uniformly in the inviscid limit.*

### 4.4.2 Duhamel principle

We first treat the Stokes problem on $\mathbb{T}\times\mathbb{R}_+$. By taking the Fourier transform in $x$, the problem is reduced to

$$\begin{cases} \partial_t\omega_\alpha - \nu\Delta_\alpha\omega_\alpha & = f_\alpha(t,z), \qquad \text{in} \quad \mathbb{R}_+ \\ \nu\omega_\alpha(0) & = -\int_0^\infty e^{-\alpha y}\omega_\alpha(y)dy. \end{cases} \tag{4.7}$$

in which $\omega_\alpha$ denotes the Fourier transform of $\omega$ with respect to $x$, and $\Delta_\alpha = \partial_z^2 - \alpha^2$. Let $G_\alpha(t,z,y)$ be the corresponding Green function of the linear Stokes problem (4.7), together with the initial data $G_\alpha(0,z,y) = \delta_y(z)$. For each fixed $y \ge 0$, the function $G_\alpha(t,z,y)$ solves

$$\begin{aligned} (\partial_t - \nu\Delta_\alpha)G_\alpha(t,z,y) = 0, \qquad \text{in} \quad \mathbb{R}_+, \\ \nu G_\alpha(t,0,y) = -\int_0^\infty e^{-\alpha z}G_\alpha(t,z,y)dz \end{aligned} \tag{4.8}$$

together with the initial data $G_\alpha(0,z,y) = \delta_y(z)$.

**Proposition 4.3.** *The solution to* (4.7) *is constructed via Duhamel's principle:*

$$\omega_\alpha(t,z) = \int_0^\infty G_\alpha(t,z,y)\omega_{0,\alpha}(y)\,dy + \int_0^t\int_0^\infty G_\alpha(t-s,z,y)f_\alpha(s,y)\,dyds. \tag{4.9}$$

*Proof.* We first show that $\partial_t\omega_\alpha - \nu\Delta_\alpha\omega_\alpha = f_\alpha$. Without loss of generality, we can assume $\omega_{0,\alpha} = 0$. We have

$$\begin{aligned} \partial_t\omega_\alpha &= \frac{d}{dt}\left(\int_0^t\int_0^\infty G_\alpha(t-s,z,y)f_\alpha(s,y)dyds\right) \\ &= \int_0^\infty G_\alpha(0,z,y)f_\alpha(t,y)dy + \int_0^t\int_0^\infty \big(\partial_t G_\alpha(t-s,z,y)\big)\,f_\alpha(y,s)dyds \\ &= f_\alpha(t,z) + \nu\int_0^t\int_0^\infty \Delta_\alpha G_\alpha(t-s,z,y)f_\alpha(y,s)dyds \\ &= f_\alpha(t,z) + \nu\Delta_\alpha\omega_\alpha. \end{aligned}$$

We now check the boundary condition in (4.7). Let $z = 0$, we have

$$\begin{aligned} \nu\omega_\alpha(t,0) &= \nu\int_0^\infty G_\alpha(t,0,y)\omega_{0,\alpha}(y)dy + \nu\int_0^t\int_0^\infty G_\alpha(t-s,0,y)f_\alpha(s,y)dyds \\ &= -\int_0^\infty\int_0^\infty \Big(e^{-\alpha z}G_\alpha(t,0,y)dz\Big)\,\omega_{0,\alpha}(y)dy \\ &\quad - \int_0^t\int_0^\infty\left(\int_0^\infty e^{-\alpha z}G_\alpha(t-s,z,y)dz\right)f_\alpha(s,y)dyds \\ &= -\int_0^\infty e^{-\alpha z}\omega_\alpha(t,z)dz. \end{aligned}$$

The proof is complete. □

### 4.4.3 The Green function for the Stokes problem

In this section, we derive sufficient pointwise bounds on the temporal Green function for the linear Stokes problem (4.7). Precisely, we prove the following. In this section, we derive sufficient pointwise bounds on the temporal Green function for the linear Stokes problem (4.7). Precisely, we prove the following.

**Proposition 4.4.** *Let $G_\alpha(t,z,y)$ be the Green function of the Stokes problem* (4.7)*. There holds*

$$G_\alpha(t,z,y) = H_\alpha(t,z,y) + R_\alpha(t,z,y), \tag{4.10}$$

*in which $H_\alpha(t,z,y)$ is exactly the one-dimensional heat kernel with the homogenous Neumann boundary condition and $R_\alpha(t,z,y)$ is the residual kernel due to the boundary condition. Precisely, There hold*

$$H_\alpha(t,z,y) = \frac{1}{\sqrt{4\pi\nu t}}\Big(e^{-\frac{|y-z|^2}{4\nu t}} + e^{-\frac{|y+z|^2}{4\nu t}}\Big)e^{-\alpha^2\nu t},$$

$$|\partial_z^k R_\alpha(t,z,y)| \lesssim (\nu t)^{-k/2} e^{-\theta_0\alpha^2\nu t}\cdot(\nu t)^{-1/2}e^{-\theta_0\frac{z^2}{4\nu t}}$$

*for $y,z\ge 0$, $k\ge 0$, and for some $\theta_0>0$.*

We proceed the construction of the Green function via the resolvent equation. Namely, for each fixed $y\ge 0$, let $G_{\lambda,\alpha}(y,z)$ be the $L^1$ solution to the resolvent problem

$$\begin{aligned}(\lambda-\nu\Delta_\alpha)G_{\lambda,\alpha}(y,z) &= \delta_y(z)\\ \nu G_{\lambda,\alpha}(0,y) &= -\int_0^\infty e^{-\alpha z}G_{\lambda,\alpha}(z,y)dz.\end{aligned} \tag{4.11}$$

Here, the second non-local boundary condition is derived as follows: Given a forcing term $f(y)$, we look for solution of the form:

$$\omega_{\lambda,\alpha}(z) = \int_0^\infty G_{\lambda,\alpha}(z,y)f(y)dy$$

We define the operator $L$ to be $L=-\nu(\partial_z^2-\mu^2)$, where $\mu=\sqrt{\frac{\lambda}{\nu}+\alpha^2}$ with positive real part. Then we get

$$L\omega_{\lambda,\alpha}(z) = \int_0^\infty LG_{\lambda,\alpha}(y,z)f(y)dy = \int_0^\infty \delta(y-z)f(y)dy = f(z).$$

Putting this in the boundary condition (4.7) we get

$$\nu\int_0^\infty G_{\lambda,\alpha}(0,y)f(y)dy = -\int_0^\infty e^{-\alpha z}\left(\int_0^\infty G_{\lambda,\alpha}(z,y)f(y)dy\right)dz.$$

Hence we have

$$\int_0^\infty (\nu G_{\lambda,\alpha}(0,y))f(y)dy = -\int_0^\infty \left(\int_0^\infty e^{-\alpha z} G_{\lambda,\alpha}(z,y)dz\right) f(y)dy.$$

Thus we take the following condition on the Green function

$$\nu G_{\lambda,\alpha}(0,y) = -\int_0^\infty e^{-\alpha z} G_{\lambda,\alpha}(z,y)dz \qquad \text{for any} \qquad y \geq 0 \tag{4.12}$$

We then obtain the following:

**Lemma 4.5.** *Let $\mu = \nu^{-1/2}\sqrt{\lambda + \alpha^2\nu}$, having positive real part. There holds*

$$G_{\lambda,\alpha}(z,y) = H_{\lambda,\alpha}(z,y) + R_{\lambda,\alpha}(z,y) \tag{4.13}$$

*in which $H_{\lambda,\alpha}(y,z)$ denotes the resolvent kernel of the heat problem with homogenous Neumann boundary condition and $R_{\alpha,\lambda}(z,y)$ denotes the residue resolvent kernel; namely,*

$$\begin{cases} H_{\lambda,\alpha}(z,y) &= \frac{1}{2\mu\nu}\left(e^{-\mu|y-z|} + e^{-\mu(y+z)}\right), \\ R_{\lambda,\alpha}(z,y) &= \frac{1}{\mu\nu}\frac{\alpha-\lambda}{\lambda+\mu-\alpha}e^{-\mu(y+z)} - \frac{1}{\nu(\lambda+\mu-\alpha)}e^{-\alpha y-\mu z}. \end{cases}$$

*In particular, $G_{\lambda,\alpha}(y,z)$ is meromorphic with respect to $\lambda$ in $\mathbb{C}\setminus\{-\alpha^2\nu - \mathbb{R}_+\}$ with a pole at $\lambda = 0$.*

*Proof.* We have

$$G_{\lambda,\alpha}(z,y) = \begin{cases} c_1(y)e^{\mu z} + c_2(y)e^{-\mu z} & , \qquad z < y \\ c_3(y)e^{-\mu z} & , \qquad z > y \end{cases} \tag{4.14}$$

The continuity of $G_{\lambda,\alpha}$ at $z = y$ gives

$$c_1(y)e^{2\mu y} + c_2(y) = c_3(y). \tag{4.15}$$

Now, the jump condition of $-\nu\partial_z G_{\lambda,\alpha}$ at $z = y$ gives

$$c_3(y) = -c_1(y)e^{2\mu y} + c_2(y) + \frac{1}{\mu\nu}e^{\mu y}. \tag{4.16}$$

Combining (4.15) and (4.16), we get

$$c_1(y) = \frac{1}{2\mu\nu}e^{-\mu y}. \tag{4.17}$$

and hence (4.16) becomes

$$c_3(y) = c_2(y) + \frac{1}{2\mu\nu}e^{\mu y}. \tag{4.18}$$

Now we find $c_2$. Using the boundary condition (4.12) and the form of $G_{\lambda,\alpha}$ in (4.14), we have

$$-\nu(c_1(y)+c_2(y)) = \int_0^y e^{-\alpha z}\left(c_1(y)e^{\mu z}+c_2(y)e^{-\mu z}\right)dz + \int_y^\infty e^{-\alpha z}\left(c_3(y)e^{-\mu z}\right)dz.$$

By a direct calculation, we get

$$c_2(y) = \frac{1}{2\mu\nu}\frac{\mu+\alpha-\lambda}{\lambda+\mu-\alpha}e^{-\mu y} - \frac{1}{\nu(\lambda+\mu-\alpha)}e^{-\alpha y}. \tag{4.19}$$

Combining the above equation with (4.18), we have

$$c_3(y) = \frac{1}{2\mu\nu}e^{\mu y} + \frac{1}{2\mu\nu}\frac{\mu+\alpha-\lambda}{\lambda+\mu-\alpha}e^{-\mu y} - \frac{1}{\nu(\lambda+\mu-\alpha)}e^{-\alpha y}. \tag{4.20}$$

Hence, putting $c_1, c_2, c_3$, computed in (4.17),(4.19),(4.20), in the formula of $G_{\lambda,\alpha}(z,y)$ in (4.14), we get

$$G_\lambda(z,y) = H_{\lambda,\alpha}(z,y) + R_{\lambda,\alpha}(z,y),$$

where

$$\begin{cases} H_{\lambda,\alpha}(z,y) &= \frac{1}{2\mu\nu}\left(e^{-\mu|y-z|}+e^{-\mu(y+z)}\right) \\ R_{\lambda,\alpha}(z,y) &= \frac{1}{\mu\nu}\frac{\alpha-\lambda}{\lambda+\mu-\alpha}e^{-\mu(y+z)} - \frac{1}{\nu(\lambda+\mu-\alpha)}e^{-\alpha y-\mu z}. \end{cases}$$

This completes the proof. □

*Proof of Proposition 4.4.* The temporal Green function $G_\alpha(t,z,y)$ can then be constructed via the inverse Laplace transform:

$$G_\alpha(t,z,y) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}G_{\lambda,\alpha}(z,y)d\lambda \tag{4.21}$$

in which the contour of integration $\Gamma$ is taken such that it remains on the right of the (say, $L^2$) spectrum of the linear operator $\lambda-\nu\Delta_\alpha$, which is $-\alpha^2\nu-\mathbb{R}_+$. In view of (4.13), we set $H_\alpha(t,z,y)$ and $R_\alpha(t,z,y)$ to be the corresponding temporal Green function of $H_{\lambda,\alpha}(z,y)$ and $R_{\lambda,\alpha}(z,y)$, respectively. It follows that $H_\alpha(t,z,y)$ is the temporal Green function of the one-dimensional heat problem with the homogenous Neumann boundary condition, yielding

$$H_\alpha(t,z,y) = \frac{1}{\sqrt{4\pi\nu t}}\left(e^{-\frac{|y-z|^2}{4\nu t}}+e^{-\frac{|y+z|^2}{4\nu t}}\right)e^{-\nu\alpha^2 t}.$$

It remains to compute the residual Green function

$$\begin{aligned} R_\alpha(t,z,y) &= \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}R_{\lambda,\alpha}(z,y)d\lambda, \\ R_{\lambda,\alpha}(z,y) &= \frac{1}{\mu\nu}\frac{\alpha-\lambda}{\lambda+\mu-\alpha}e^{-\mu(y+z)} - \frac{1}{\nu(\lambda+\mu-\alpha)}e^{-\alpha y-\mu z}. \end{aligned} \tag{4.22}$$

We note that $R_{\lambda,\alpha}$ has a pole when $\lambda + \mu - \alpha = 0$, which happens only when $\lambda = 0$. We consider two cases: when $\alpha^2\nu \leq 1$ and when $\alpha^2\nu \geq 1$.

**Case 1:** $\alpha^2\nu \leq 1$.
Let us give a bound on the first part of the kernel $R_{\lambda,\alpha}$ in (4.22):

$$R^1_{\lambda,\alpha}(z,y) = \frac{1}{\mu\nu} \cdot \frac{\alpha - \lambda}{\lambda + \mu - \alpha} e^{-\mu(y+z)}.$$

By Cauchy's theory, we may decompose the contour of integration as $\Gamma = \Gamma_\pm \cup \Gamma_c$, having

$$\Gamma_\pm = \left\{ \lambda = -\frac{1}{2}\alpha^2\nu + \nu(a^2 - b^2) + 2ab\nu i \pm iM, \qquad \pm b \in \mathbb{R}_+ \right\},$$
$$\Gamma_c = \left\{ \lambda = -\frac{1}{2}\alpha^2\nu + \nu a^2 + Me^{i\theta}, \quad \theta \in [-\pi/2, \pi/2] \right\}.$$

for some positive number $M$ and $a = \frac{|y+z|}{2\nu t}$. Since $\alpha^2\nu \leq 1$, we can take $M$ large so that the pole $\lambda = 0$ remains on the left of the contour $\Gamma$. It is clear that $|\lambda| \gtrsim 1$ on $\Gamma$.
On $\Gamma_c$, we note that

$$\Re\mu = \nu^{-1/2}\Re\sqrt{\frac{1}{2}\nu\alpha^2 + \nu a^2 + Me^{i\theta}} \geq \nu^{-1/2}\sqrt{\frac{1}{2}\nu\alpha^2 + \nu a^2} \geq a,$$
$$\Re\mu = \nu^{-1/2}\Re\sqrt{\frac{1}{2}\nu\alpha^2 + \nu a^2 + Me^{i\theta}} \geq c_0\nu^{-1/2}\sqrt{M}$$

for some $c_0 > 0$.
This implies that $\Re\mu \geq \frac{a}{2} + \frac{a}{2}$ and $|\mu|\nu \geq c_0\nu^{1/2}$. This proves that

$$\begin{aligned}
&\Big|\int_{\Gamma_c} e^{\lambda t} e^{-\mu(y+z)} \left(\frac{1}{\mu\nu}\frac{\alpha - \lambda}{\lambda + \mu - \alpha}\right) d\lambda\Big| \\
&\lesssim \int_{-\pi/2}^{\pi/2} e^{Mt - \frac{1}{2}\alpha^2\nu t} e^{a^2\nu t} e^{-\frac{a}{2}|y+z|} e^{-\frac{a}{2}|y+z|} \nu^{-1/2} d\theta \cdot \sup_{\lambda\in\Gamma_c}\left|\frac{\alpha - \lambda}{\lambda + \mu - \alpha}\right| \\
&\lesssim \nu^{-1/2} e^{-\frac{a}{2}|y+z|} e^{a^2\nu t} e^{-\frac{a}{2}|y+z|} e^{-\frac{1}{2}\alpha^2\nu t} \\
&\lesssim \nu^{-1/2} e^{-\frac{a}{2}|y+z|} e^{-\frac{1}{2}\alpha^2\nu t} \\
&\lesssim (\nu t)^{-1/2} e^{-\frac{|y+z|^2}{4\nu t}} e^{-\frac{1}{2}\alpha^2\nu t}
\end{aligned}$$

in which we used $e^{a^2\nu t} e^{-\frac{a}{2}|y+z|} = 1$ by definition of $a$, and the fact that $\left|\frac{\alpha-\lambda}{\lambda+\mu-\alpha}\right|$ is bounded on $\Gamma_c$. Indeed, we write

$$\left|\frac{\alpha - \lambda}{\lambda + \mu - \alpha}\right| = \left|-1 + \frac{\mu}{\lambda + \mu - \alpha}\right| \leq 1 + \left|\frac{\mu}{\lambda + \mu - \alpha}\right|.$$

It suffices to estimate $\left|\frac{\mu}{\lambda+\mu-\alpha}\right|$ when $\lambda \in \Gamma_c$. Using the fact that $\lambda = \nu(\mu^2 - \alpha^2)$, we can rewrite this term as follows:

$$\left(1 + \frac{\alpha}{\mu - \alpha}\right) \frac{1}{\nu(\mu + \alpha) + 1}. \tag{4.23}$$

First we see that $\frac{\alpha}{\mu-\alpha}$ is bounded, since

$$|\mu - \alpha| \geq \Re\mu - \alpha \geq c_0\sqrt{M}\nu^{-1/2} - \alpha \geq c_0\sqrt{M}\alpha - \alpha = (c_0\sqrt{M} - 1)\alpha \qquad (\text{since} \quad \alpha^2\nu \leq 1). \tag{4.24}$$

Moreover, we get

$$|\nu(\mu + \alpha) + 1| \geq 1 + \alpha\nu + \nu\Re\mu \geq 1. \tag{4.25}$$

Hence the quantity (4.23) is uniformly bounded when $\lambda \in \Gamma_c$. This implies that

$$\sup_{\lambda \in \Gamma_c} \left|\frac{\alpha - \lambda}{\lambda + \mu - \alpha}\right| \lesssim 1$$

as claimed.

Now we estimate the term

$$\int_{\Gamma_\pm} e^{\lambda t} \frac{1}{\mu\nu} \cdot \frac{\alpha - \lambda}{\lambda + \mu - \alpha} e^{-\mu(y+z)} d\lambda. \tag{4.26}$$

On $\Gamma_\pm$, we note that

$$\Re\mu = \Re\sqrt{\frac{1}{2}\alpha^2 + (a + ib)^2 \pm i\nu^{-1}M} \geq \Re\sqrt{(a + ib)^2} = a,$$

upon noting that the sign of $b$ and $\pm M$ is the same on $\Gamma_\pm$. Similarly, we note that $\Re\mu \gtrsim \sqrt{M}/\sqrt{\nu}$. By definition of $a$, we have

$$|e^{\lambda t} e^{-\mu|y+z|}| \leq e^{-\frac{1}{2}\nu\alpha^2 t} e^{-\frac{|y+z|^2}{4\nu t}} e^{-\nu b^2 t},$$

Moreover, by a similar argument as in (4.23),(4.24) and (4.25), we get

$$\sup_{\lambda \in \Gamma_\pm} \left|\frac{\alpha - \lambda}{\lambda + \mu - \alpha}\right| \lesssim 1.$$

Thus we get the following bound for the term in (4.26) as follows:

$$\left|\int_{\Gamma_\pm} e^{\lambda t} \frac{1}{\mu\nu} \frac{\alpha - \lambda}{\lambda + \mu - \alpha} e^{-\mu(y+z)} d\lambda\right| \lesssim (\nu t)^{-1/2} e^{-\frac{1}{2}\alpha^2\nu t} e^{-\frac{|y+z|^2}{4\nu t}}.$$

The proof of the bound for $\int_\Gamma e^{\lambda t} R^1_{\lambda,\alpha}(y,z)d\lambda$ is complete. Similarly, we get the following bound for the second term in the kernel (4.22)

$$\Big|\frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\frac{1}{\nu(\lambda+\alpha-\mu)}e^{-\alpha y-\mu z}d\lambda\Big| \lesssim e^{-\alpha y}(\nu t)^{-1/2}e^{-\frac12\alpha^2\nu t}e^{-\frac{z^2}{4\nu t}},$$

which we skip the details. This completes the proof the case $\alpha^2\nu\le 1$.

**Case 2:** $\alpha^2\nu\ge 1$.
Take $a=\frac{z}{2\nu t}$. Consider first the case when $|a-\alpha|\ge\frac12\alpha$. In this case, we move the contour of integration to

$$\Gamma_1 := \Big\{\lambda = -\nu\alpha^2+\nu(a^2-b^2)+2\nu iab, \quad \pm b\in\mathbb{R}_+\Big\}$$

which may pass the pole at $\lambda=0$ (precisely, it does when $a=\alpha$). By the Cauchy's theory, we have

$$R_\alpha(t,z,y)=\frac{1}{2\pi i}\int_{\Gamma_1} e^{\lambda t}R_{\lambda,\alpha}(z,y)\,d\lambda + \mathrm{Res}_0$$

in which the residue at the pole $\lambda=0$ is computed explicitly by

$$\mathrm{Res}_0 = 0. \tag{4.27}$$

Indeed, at the pole $\lambda=0$, we have $\mu=\alpha$. Hence

$$(\lambda+\mu-\alpha)R_{\lambda,\alpha}=\frac{\alpha}{\mu\nu}e^{-\mu(y+z)}-\frac{1}{\nu}e^{-\alpha y-\mu z}=0, \qquad \text{since} \qquad \mu=\alpha.$$

Hence, we have

$$R_\alpha(t,z,y)=\frac{1}{2\pi i}\int_{\Gamma_1} e^{\lambda t}R_{\lambda,\alpha}(y,z)d\lambda$$

where

$$R^1_{\lambda,\alpha}(z,y)=\frac{1}{\mu\nu}\cdot\frac{\alpha-\lambda}{\lambda+\mu-\alpha}e^{-\mu(y+z)}, \qquad R^2_{\lambda,\alpha}(z,y)=-\frac{1}{\nu(\lambda+\mu-\alpha)}e^{-\alpha y-\mu z}.$$

Now we estimate

$$\begin{aligned}\Big|\frac{1}{2\pi i}\int_{\Gamma_1} e^{\lambda t}R^1_{\lambda,\alpha}(z,y)d\lambda\Big| &\lesssim \int_{\Gamma_1} e^{\Re\lambda t}\frac{1}{\nu|\mu|}\Big|\frac{\alpha-\lambda}{\lambda+\mu-\alpha}\Big|e^{-\Re\mu(y+z)}|d\lambda| \\ &\lesssim \sup_{\lambda\in\Gamma_1}\Big|\frac{\alpha-\lambda}{\lambda+\mu-\alpha}\Big|\int_{\mathbb{R}} e^{-\alpha^2\nu t+\nu a^2 t-\nu b^2 t}\Big(e^{-\frac{a}{2}z}e^{-\frac{a}{2}z}\Big)\,db \\ &\lesssim (\nu t)^{-1/2}e^{-\frac{z^2}{4\nu t}}e^{-\alpha^2\nu t}.\end{aligned} \tag{4.28}$$

Here, we used the fact that $e^{\nu a^2 t} e^{-\frac{a}{2}|y+z|} = 1, |d\lambda| = \nu |d\mu|$ and

$$\sup_{\lambda\in\Gamma_1} \left|\frac{\alpha-\lambda}{\lambda+\mu-\alpha}\right| \lesssim 1. \tag{4.29}$$

Indeed, we have

$$\begin{aligned}\left|\frac{\alpha-\lambda}{\lambda+\mu-\alpha}\right| &= \left|-1+\frac{\mu}{\lambda+\mu-\alpha}\right| \le 1+\left|\frac{\mu}{\lambda+\mu-\alpha}\right| = 1+\left|\frac{\mu}{(\mu-\alpha)(\nu(\mu+\alpha)+1)}\right| \\ &\le 1+\left|\left(1+\frac{\alpha}{\mu-\alpha}\right)\frac{1}{\nu(\mu+\alpha)+1}\right| \le 1+\left(1+\frac{\alpha}{|\mu-\alpha|}\right)\frac{1}{|\nu\mu+\alpha\nu+1|} \\ &\le 1+\left(1+\frac{\alpha}{|\mu-\alpha|}\right) \lesssim 1,\end{aligned} \tag{4.30}$$

since $|\mu-\alpha| \ge |\Re\mu - \alpha| = |a-\alpha| \ge \frac{1}{2}\alpha$. The bound for $\left|\int_{\Gamma_1} e^{\lambda t} R^1_{\lambda,\alpha}(z,y)d\lambda\right|$ is complete. Similarly, one can obtain the following bound

$$\left|\frac{1}{2\pi i}\int_{\Gamma_1} e^{\lambda t} R^2_{\lambda,\alpha}(z,y)d\lambda\right| \lesssim (\nu t)^{-1/2} e^{-\frac{z^2}{4\nu t}} e^{-\alpha^2\nu t} e^{-\alpha y},$$

which we skip the details. Combining the above bounds for $R^1_{\lambda,\alpha}$ and $R^2_{\lambda,\alpha}$, we have

$$R_\alpha(t,z,y) \lesssim (\nu t)^{-1/2} e^{-\alpha^2 \nu t} e^{-\frac{z^2}{4\nu t}}.$$

It remains to consider the case when $|a-\alpha| \le \frac{1}{2}\alpha$ and $\alpha^2\nu \ge 1$. We note in particular that $\frac{1}{2}\alpha \le a \le \frac{3}{2}\alpha$. In this case, we take the contour of integration as follows

$$\Gamma_2 := \left\{\lambda = -\frac{1}{8}\nu\alpha^2 + \nu(a^2-b^2) + 2\nu iab, \quad \pm b \in \mathbb{R}_+\right\}.$$

Observe that the contour $\Gamma_1$ always leaves the origin on the left, hence the pole at the origin does not appear. Proceeding as in the estimate (4.36) and (4.30), it suffices to check that

$$\sup_{\lambda\in\Gamma_2}\left|\frac{\alpha}{\mu-\alpha}\right| \lesssim 1 \tag{4.31}$$

in order to conclude

$$\left|\frac{1}{2\pi i}\int_{\Gamma_2} e^{\lambda t} R^1_{\lambda,\alpha}(z,y)d\lambda\right| \lesssim (\nu t)^{-1/2} e^{-\frac{z^2}{4\nu t}} e^{-\frac{1}{8}\alpha^2\nu}. \tag{4.32}$$

To check (4.31), we first see that the contour $\Gamma_2$ cuts the real axis at $\nu\left(a^2 - \frac{1}{8}\alpha^2\right)$ and cuts the imaginary axis at $\pm 2a\nu\sqrt{a^2 - \frac{1}{8}\alpha^2}$. In particular this implies

$$|\lambda| \ge \nu\left(a^2 - \frac{1}{8}\alpha^2\right) \ge \nu\left(\frac{1}{4}\alpha^2 - \frac{1}{8}\alpha^2\right) \ge \frac{1}{8}\alpha^2\nu, \qquad \text{since} \quad a \ge \frac{1}{2}\alpha.$$

Hence we have

$$|\lambda| \ge \frac{1}{8}\alpha^2\nu. \tag{4.33}$$

Now using the fact $\lambda = \nu(\mu^2 - \alpha^2)$ and (4.33), we see that

$$\left|\frac{\alpha}{\mu - \alpha}\right| = \left|\frac{\alpha\nu(\mu + \alpha)}{\nu(\mu^2 - \alpha^2)}\right| = \frac{|\alpha^2\nu + \alpha\nu\mu|}{|\lambda|} \le \frac{\alpha^2\nu}{|\lambda|} + \frac{\alpha\nu|\mu|}{|\lambda|} \le 8 + \frac{\alpha\nu|\mu|}{|\lambda|}.$$

Now to bound $\frac{\alpha\nu|\mu|}{|\lambda|}$, we note that $\lambda = \nu(\mu^2 - \alpha^2)$ and (4.33), and hence

$$\nu|\mu|^2 \le |\lambda| + \alpha^2\nu \le 9|\lambda|.$$

Thus

$$\frac{\alpha\nu|\mu|}{|\lambda|} \lesssim \frac{\alpha\nu|\mu|}{\nu|\mu|^2} = \frac{\alpha}{|\mu|} \lesssim \frac{\alpha}{\Re\mu} \lesssim \frac{\alpha}{a} \lesssim 1.$$

This completes the proof of the bound stated in (4.32). As for the derivatives bound, it is straight forward that

$$|\partial_z^k H_\alpha(t,z,y)| \lesssim (\nu t)^{-\frac{k}{2}} \frac{1}{\sqrt{\nu t}}\left(e^{-\theta_0\frac{|y-z|^2}{4\nu t}} + e^{-\theta_0\frac{|y+z|^2}{4\nu t}}\right), \qquad k \ge 1$$

for some $\theta_0 > 0$. For the residue kernel $R_\alpha(t,z,y) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t} R_{\lambda,\alpha}(z,y)d\lambda$, we note that

$$\partial_z\left(\frac{1}{2\pi i}\int_\Gamma e^{\lambda t} R_{\lambda,\alpha}(z,y)d\lambda\right) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\mu R_\alpha(z,y)d\lambda$$

Hence, we get

$$|\partial_z R_\alpha(t,z,y)| \lesssim (\nu t)^{-1/2} \cdot \frac{1}{\sqrt{\nu t}} e^{-\theta_0\frac{z^2}{4\nu t}} e^{-\theta_0\alpha^2\nu t}$$

by the exact same argument represented for the bound $R_\alpha(t,z,y)$, and the fact that $\int_{\mathbb{R}} be^{-\nu t b^2}db \lesssim (\nu t)^{-1/2}$ and $\frac{z}{\nu t}e^{-\frac{z^2}{4\nu t}} \lesssim (\nu t)^{-1/2}e^{-\theta_0\frac{z^2}{4\nu t}}$, which we skip the details (see also [51]). □

### 4.4.4 The Green function on $\Omega_\sigma$

The Green function constructed in Proposition 4.4 can be directly extended to the complex domain $\Omega_\sigma$ defined by

$$\Omega_\sigma = \Big\{ z \in \mathbb{C} : \quad |\Im z| < \min\{\sigma|\Re z|, \sigma\}\Big\},$$

for some small $\sigma > 0$. Indeed, the Green function involves precisely the heat kernel $G(t,z) = \frac{1}{\sqrt{4\pi t}} e^{-z^2/4t}$, which is extended to the complex domain. In addition, we note that for $z \in \Gamma_\sigma$, there holds $\Im z \le \sigma \Re z$, which implies that

$$|e^{-z^2/4t}| \le e^{-|\Re z|^2/4t + |\Im z|^2/4t} \le e^{-(1-\sigma^2)|\Re z|^2/4t}.$$

Similar estimates hold for the other terms in the Green function $G_\alpha(t,z,y) = H_\alpha(t,z,y) + R_\alpha(t,z,y)$, yielding

$$\begin{aligned} H_\alpha(t,z,y) &\lesssim \frac{1}{\sqrt{\nu t}} \Big( e^{-(1-\sigma^2)\frac{|\Re y - \Re z|^2}{4\nu t}} + e^{-(1-\sigma^2)\frac{|\Re y + \Re z|^2}{4\nu t}} \Big) e^{-\frac{1}{8}\alpha^2 \nu t}, \\ R_\alpha(t,z,y) &\lesssim e^{-\theta_0 \alpha^2 \nu t} (\nu t)^{-1/2} e^{-\theta_0 (1-\sigma^2)\frac{(\Re z)^2}{4\nu t}}, \end{aligned} \tag{4.34}$$

for $y, z \in \Gamma_\sigma$, and for some $\theta_0 > 0$. Precisely, for any $z \in \Omega_\sigma$, let $\theta$ be the positive constant so that $z \in \partial\Omega_\theta$. The Duhamel principle (4.9) then becomes

$$\omega_\alpha(t,z) = \int_{\partial\Omega_\theta} G_\alpha(t,z,y)\omega_{0,\alpha}(y)\, dy + \int_0^t \int_{\partial\Omega_\theta} G_\alpha(t-s,z,y) f_\alpha(s,y)\, dyds, \tag{4.35}$$

which is well-defined for $z \in \Omega_\sigma$, having the Green function $G_\alpha(t,z,y)$ satisfies the pointwise estimates (4.34), similar to those on the real line. For this reason, it suffices to derive convolution estimates for real values $y, z$.

### 4.4.5 Convolution estimates

We now derive convolution estimates. We start with the analytic $L^1$ norms. For $k \ge 0$, we introduce

$$\|\omega_\alpha\|_{\mathcal{W}^{k,1}_\sigma} = \sum_{j=0}^{k} \|(\psi(z)\partial_z)^j \omega_\alpha\|_{L^1_\sigma}.$$

We prove the following.

**Proposition 4.6.** *Let $T > 0$ and let $G_\alpha(t,z,y)$ be the Green function of the Stokes problem (3.8), constructed in Proposition 4.4. Then, for any $0 \le s < t \le T$ and $k \ge 0$,* *there is a universal constant $C_T$ so that*

$$\Big\| \int_0^\infty G_\alpha(t,\cdot,y)\omega_\alpha(y)\,dy \Big\|_{\mathcal{W}^{k,1}_\sigma} \le C_T \|\omega_\alpha\|_{\mathcal{W}^{k,1}_\sigma},$$
$$\Big\| \int_0^\infty G_\alpha(t-s,\cdot,y)\omega_\alpha(y,s)\,dy \Big\|_{\mathcal{W}^{k,1}_\sigma} \le C_T \|\omega_\alpha(s)\|_{\mathcal{W}^{k,1}_\sigma},$$

*uniformly in the inviscid limit.*

*Proof.* We shall prove the convolution for real values $y, z$. For the complex extension, see Section 4.4.4. Recall from Proposition 4.4 that $G_\alpha(t,z,y) = H_\alpha(t,z,y) + R_\alpha(t,z,y)$, with

$$\begin{cases} H_\alpha(t,z,y) & = \frac{1}{\sqrt{4\pi\nu t}}\Big(e^{-\frac{|y-z|^2}{4\nu t}} + e^{-\frac{|y+z|^2}{4\nu t}}\Big)e^{-\alpha^2\nu t}, \\ R_\alpha(t,z,y) & \lesssim e^{-\theta_0\alpha^2\nu t}(\nu t)^{-\frac{1}{2}} e^{-\theta_0\frac{z^2}{\nu t}}. \end{cases}$$

For $H_\alpha$, we apply ([51], Proposition 3.7) to get:

$$\Big\| \int_0^\infty H_\alpha(t-s,\cdot,y)\omega_\alpha(y,s)\,dy \Big\|_{\mathcal{W}^{k,1}_\sigma} \le C_T \|\omega_\alpha(s)\|_{\mathcal{W}^{k,1}_\sigma}.$$

Now we will prove that

$$\Big\| \int_0^\infty R_\alpha(t-s,\cdot,y)\omega_\alpha(y,s)\,dy \Big\|_{\mathcal{W}^{k,1}_\sigma} \le C_T \|\omega_\alpha(s)\|_{\mathcal{W}^{k,1}_\sigma}.$$

Using the pointwise bound of $R_\alpha(t-s,z,y)$ in Proposition 4.4, we have

$$\left|\int_0^\infty R_\alpha(t-s,z,y)\omega_\alpha(s,y)dy\right| \lesssim e^{-\theta_0\alpha^2\nu(t-s)} e^{-\theta_0\frac{z^2}{4\nu(t-s)}} \big(\nu(t-s)\big)^{-1/2} \int_0^\infty |\omega_\alpha(s,y)|dy.$$

Integrating in $z$, we have

$$\left\|\int_0^\infty R_\alpha(t-s,z,y)\omega_\alpha(s,y)dy\right\|_{L^1_z} \lesssim \|\omega_\alpha(s)\|_{L^1_y}.$$

As for derivatives, we have

$$\begin{aligned} &\left|(\psi(z)\partial_z)^k \left(\int_0^\infty R_\alpha(t-s,y,z)\omega_\alpha(s,y)dy\right)\right| \\ &\lesssim \left(\frac{z^2}{\nu(t-s)}\right)^k \big(\nu(t-s)\big)^{-1/2} e^{-\theta_0\frac{z^2}{\nu(t-s)}} \int_0^\infty |\omega_\alpha(s,y)|dy \qquad (4.36) \\ &\lesssim \big(\nu(t-s)\big)^{-1/2} e^{-\theta_0\frac{z^2}{\nu(t-s)}} \|\omega_\alpha(s)\|_{L^1_y}. \end{aligned}$$

From here, we get

$$\left\|(\psi(z)\partial_z)^k \int_0^\infty R_\alpha(t-s,y,z)\omega_\alpha(s,y)dy\right\|_{L^1_z} \lesssim \|\omega_\alpha(s)\|_{L^1_y}.$$

The proof is complete. $\square$

### 4.4.6 Convolution estimates with boundary layer behaviors

In this section, we provide the convolution estimates of the Green function against functions in the boundary layer spaces, whose norm is defined by

$$\|\omega_\alpha\|_{\sigma,\delta(t)} = \sup_{z\in\Omega_\sigma} |\omega_\alpha(z)|e^{\beta\Re z}\Big(1+\delta_t^{-1}\phi_P(\delta_t^{-1}z)+\delta^{-1}\phi_P(\delta^{-1}z)\Big)^{-1}, \tag{4.37}$$

for $t>0$ and $\beta>0$, in which the boundary thicknesses are $\delta_t=\sqrt{\nu t}$ and $\delta=\sqrt{\nu}$ and for boundary layer weight $\phi_P(z)=\frac{1}{1+|\Re z|^P}$, $P>1$. We also introduce the boundary norm for derivatives:

$$\|\omega_\alpha\|_{\sigma,\delta(t),k} = \sum_{j=0}^k \|(\psi(z)\partial_z)^j\omega_\alpha\|_{\sigma,\delta(t)}$$

for $k\geq 0$. In the case $t=0$, the norm $\|\cdot\|_{\sigma,\delta(0)}$ is defined to consist of precisely one boundary layer with thickness $\delta=\sqrt{\nu}$.

We prove the following.

**Proposition 4.7.** *Let $T>0$ and let $G_\alpha(t,z,y)$ be the Green function of the Stokes problem* (4.7)*, constructed in Proposition 4.4. Then, for any $0\leq s<t\leq T$ and $k\geq 0$, there is a universal constant $C_T$ so that*

$$\Big\|\int_0^\infty G_\alpha(t,\cdot,y)\omega_\alpha(y)\,dy\Big\|_{\sigma,\delta(t),k} \leq C_T\|\omega_\alpha\|_{\sigma,\delta(0),k},$$

$$\Big\|\int_0^\infty G_\alpha(t-s,\cdot,y)\omega_\alpha(s,y)\,dy\Big\|_{\sigma,\delta(t),k} \leq C_T\sqrt{\frac{t}{s}}\|\omega_\alpha(s)\|_{\sigma,\delta(s),k} + C_T\sqrt{\frac{t}{t-s}}\|\omega_\alpha(s)\|_{\mathcal{W}^{k,1}_\sigma}$$

*uniformly in the inviscid limit.*

*Proof.* Since $G_\alpha(t-s,z,y)=H_\alpha(t-s,z,y)+R_\alpha(t-s,z,y)$, the convolution estimates are needed for the heat kernel $H_\alpha$ and $R_\alpha$. For $H_\alpha$, we apply ([51], Lemma 3.10) to get

$$\Big\|\int_0^\infty H_\alpha(t-s,\cdot,y)\omega_\alpha(s,y)\,dy\Big\|_{\sigma,\delta(t),k} \leq C_T\sqrt{\frac{t}{s}}\|\omega_\alpha(s)\|_{\sigma,\delta(s),k}.$$

Now we will prove that

$$\Big\|\int_0^\infty R_\alpha(t-s,\cdot,y)\omega_\alpha(s,y)\,dy\Big\|_{\sigma,\delta(t),k} \leq C_T\sqrt{\frac{t}{t-s}}\|\omega_\alpha(s)\|_{\sigma,\delta(s),k}.$$

By the estimate (4.36), it suffices to check that

$$(\nu(t-s))^{-1/2} e^{-\theta_0 \alpha^2 \nu(t-s)} e^{-\theta_0 \frac{z^2}{4\nu(t-s)}} \lesssim \sqrt{\frac{t}{t-s}} e^{-\beta_0 z} \left( \delta_t^{-1} \phi_P(\delta_t^{-1} z) \right).$$

To this end, we have

$$\begin{aligned}
&(\nu(t-s))^{-1/2} e^{-\theta_0 \alpha^2 \nu(t-s)} e^{-\theta_0 \frac{z^2}{4\nu(t-s)}} \\
&= \sqrt{\frac{t}{t-s}} \delta_t^{-1} e^{-\theta_0 \frac{z^2}{8\nu(t-s)}} e^{-\theta_0 \frac{z^2}{8\nu(t-s)}} e^{-\theta_0 \alpha^2 \nu(t-s)} \\
&\lesssim \sqrt{\frac{t}{t-s}} \left( \delta_t^{-1} e^{-\theta_0 \frac{z^2}{8\nu t}} \right) e^{-\theta_0 \frac{z^2}{8\nu(t-s)}} e^{-32\cdot\theta_0 \nu(t-s)} e^{32\cdot\theta_0 \nu(t-s)} \\
&\lesssim \sqrt{\frac{t}{t-s}} \left( \delta_t^{-1} \phi_P(\delta_t^{-1} z) \right) e^{-\beta_0 z}
\end{aligned}$$

as long as $\beta_0 \le 2\theta_0$, by a simple Cauchy inequality $\frac{z^2}{8\nu(t-s)} + 32\nu(t-s) \ge 2z$. The proof is complete. □

## 4.5 Proof of the main theorem

In this section, we give a proof for Theorem 4.3.1, stated in the introduction. We construct the solutions to the Navier-Stokes equation via the vorticity formulation:

$$\begin{aligned}
\partial_t \omega - \nu \Delta \omega &= -u \cdot \nabla \omega \\
(\nu \omega - u_1)|_{z=0} &= 0,
\end{aligned} \tag{4.38}$$

in which $u = \nabla^\perp \Delta^{-1} \omega$, with $\Delta^{-1}$ being the inverse of Laplacian with the Dirichlet boundary condition. For convenience, we set $N = u \cdot \nabla \omega$. The solution to the Navier-Stokes is then constructed via the Duhamel's principle:

$$\omega(t) = e^{\nu t B} \omega_0 - \int_0^t e^{\nu(t-s)B} N(s)\, ds \tag{4.39}$$

with $\omega_0 \in \mathcal{B}^{\rho_0, \sigma_0, \delta}$, for some $\rho_0, \sigma_0 > 0$.

### 4.5.1 Nonlinear iteration

Let us fix positive numbers $\gamma, \zeta$, and $\rho_0$, and introduce the following nonlinear iterative norm for vorticity:

$$A(\gamma) = \sup_{0<\gamma t<\rho_0} \sup_{\rho<\rho_0-\gamma t} \left\{ |||\omega(t)|||_{\mathcal{W}^{1,1}_{\rho,\rho}} + |||\omega(t)|||_{\mathcal{W}^{2,1}_{\rho,\rho}} (\rho_0 - \rho - \gamma t)^\zeta \right\} \tag{4.40}$$

with recalling

$$|||\omega(t)|||_{\mathcal{W}^{k,1}_{\rho,\rho}} = \sum_{j+\ell \le k} \|\partial_x^j (\psi(z)\partial_z)^\ell \omega(t)\|_{L^1_{\rho,\rho}}.$$

Here, for sake of presentation, we take the same analyticity radius in $x$ and $z$; namely, $\sigma = \rho < \rho_0$. Thanks to Lemma 2.1, $\omega_0 \in \mathcal{W}^{k,1}_{\rho,\rho}$, for any $k \geq 0$.

We shall show that the vorticity norm remains finite for sufficiently large $\gamma$. The weight $(\rho_0 - \rho - \gamma t)^{\zeta}$, with a small $\zeta > 0$, is standard to avoid time singularity when recovering the loss of derivatives ([4, 57]). Let $\rho < \rho_0 - \gamma t$. Thanks to Lemma 2.4, we have

$$\begin{aligned} &|||N(t)|||_{\mathcal{W}^{0,1}_{\rho,\rho}} \lesssim |||\omega(t)|||^2_{\mathcal{W}^{1,1}_{\rho,\rho}} \leq A(\gamma)^2, \\ &|||N(t)|||_{\mathcal{W}^{1,1}_{\rho,\rho}} \lesssim |||\omega(t)|||_{\mathcal{W}^{1,1}_{\rho,\rho}} |||\omega(t)|||_{\mathcal{W}^{2,1}_{\rho,\rho}} \leq A(\gamma)^2 (\rho_0 - \rho - \gamma t)^{-\zeta}. \end{aligned} \tag{4.41}$$

Now, using the Duhamel integral formula (4.39), we estimate

$$|||\omega(t)|||_{\mathcal{W}^{k,1}_{\rho,\rho}} \leq |||e^{\nu t B}\omega_0|||_{\mathcal{W}^{k,1}_{\rho,\rho}} + \int_0^t |||e^{\nu(t-s)B} N(s)|||_{\mathcal{W}^{k,1}_{\rho,\rho}}\, ds.$$

In view of Proposition 4.2, the term from the initial data is already estimated, giving $|||e^{\nu t B}\omega_0|||_{\mathcal{W}^{k,1}_{\rho,\rho}} \leq \|\omega_0\|_{\mathcal{W}^{k,1}_{\rho,\rho}}$. As for the integral terms, we estimate

$$\begin{aligned} \int_0^t |||e^{\nu(t-s)B} N(s)|||_{\mathcal{W}^{1,1}_{\rho,\rho}}\, ds &\leq C_0 \int_0^t |||N(s)|||_{\mathcal{W}^{1,1}_{\rho,\rho}}\, ds \\ &\leq C_0 A(\gamma)^2 \int_0^t (\rho_0 - \rho - \gamma s)^{-\zeta}\, ds \\ &\leq C_0 \gamma^{-1} A(\gamma)^2. \end{aligned}$$

Next, we give estimates for $k = 2$. Noting that $\rho < \rho_0 - \gamma t \leq \rho_0 - \gamma s$, we take $\rho' = \frac{\rho + \rho_0 - \gamma s}{2}$ and compute

$$\begin{aligned} \int_0^t |||e^{\nu(t-s)B} N(s)|||_{\mathcal{W}^{2,1}_{\rho,\rho}}\, ds &\leq C_0 \int_0^t |||N(s)|||_{\mathcal{W}^{2,1}_{\rho,\rho}}\, ds \\ &\leq C_0 \int_0^t \frac{1}{\rho' - \rho} |||N(s)|||_{\mathcal{W}^{1,1}_{\rho',\rho'}}\, ds \\ &\leq C_0 A(\gamma)^2 \int_0^t (\rho_0 - \rho - \gamma s)^{-1-\zeta}\, ds \\ &\leq C_0 \gamma^{-1} A(\gamma)^2 (\rho_0 - \rho - \gamma t)^{-\zeta}. \end{aligned}$$

Same computation holds for the trace operator $\Gamma(\nu t)$, yielding

$$A(\gamma) \leq C_0 \|\omega_0\|_{\mathcal{W}^{2,1}_{\rho,\rho}} + C_0 \gamma^{-1} A(\gamma)^2.$$

By taking $\gamma$ sufficiently large, the above yields the uniform bound on the iterative norm in term of initial data. This yields the local solution in $L^1_{\rho,\rho}$ for $t \in [0, T]$, with $T = \gamma^{-1}\rho_0$.

### 4.5.2 Propagation of boundary layers

It remains to prove that the constructed solution has the boundary layer behavior as expected, having already constructed solutions in $L^1_{\rho,\rho}$ spaces. Indeed, we now introduce the following nonlinear iterative norm for vorticity:

$$B(\gamma) = \sup_{0<\gamma t<\rho_0} \sup_{\rho<\rho_0-\gamma t} \Big\{ |||\omega(t)|||_{\rho,\delta(t),1} + |||\omega(t)|||_{\rho,\delta(t),2}(\rho_0 - \rho - \gamma t)^{\zeta} \Big\} \tag{4.42}$$

with the boundary layer norm

$$|||\omega(t)|||_{\rho,\delta(t),k} = \sum_{j+\ell\le k} \|\partial_x^j (\psi(z)\partial_z)^\ell \omega(t)\|_{\rho,\rho,\delta(t)}.$$

Thanks to Lemma 2.4, we estimate

$$\begin{aligned} |||N(t)|||_{\rho,\delta(t),0} &\lesssim |||\omega(t)|||^2_{\rho,\delta(t),1} \le B(\gamma)^2 \\ |||N(t)|||_{\rho,\delta(t),1} &\lesssim |||\omega(t)|||_{\rho,\delta(t),1}|||\omega(t)|||_{\rho,\delta(t),2} \le B(\gamma)^2(\rho_0 - \rho - \gamma t)^{-\zeta}. \end{aligned} \tag{4.43}$$

Now, using the Duhamel integral formula (4.39), we estimate

$$|||\omega(t)|||_{\rho,\delta(t),k} \le |||e^{\nu t B}\omega_0|||_{\rho,\delta(t),k} + \int_0^t |||e^{\nu(t-s)B}N(s)|||_{\rho,\delta(t),k}\, ds$$

In view of Proposition 4.2, the term from the initial data is already estimated, giving $|||e^{\nu t B}\omega_0|||_{\rho,\delta(t),k} \le \|\omega_0\|_{\rho,\delta(0),k}$. We estimate

$$\begin{aligned} &\int_0^t |||e^{\nu(t-s)B}N(s)|||_{\rho,\delta(t),1}\, ds \\ &\lesssim \int_0^t \left( \sqrt{\frac{t}{s}}|||N(s)|||_{\rho,\delta(s),1} + \sqrt{\frac{t}{t-s}}|||N(s)|||_{\mathcal{W}^{1,1}_{\rho,\rho}} \right) ds \\ &\lesssim B(\gamma)^2 \int_0^t \sqrt{\frac{t}{s}}(\rho_0 - \rho - \gamma s)^{-\zeta} ds + \sup_{0\le s\le T} |||N(s)|||_{\mathcal{W}^{1,1}_{\rho,\rho}} \int_0^t \sqrt{\frac{t}{t-s}}\, ds \\ &\lesssim B(\gamma)^2 \Big( \int_0^{t/2} + \int_{t/2}^t \Big) \sqrt{\frac{t}{s}} (\rho_0 - \rho - \gamma s)^{-\zeta}\, ds + t \cdot \sup_{0\le s\le T} |||N(s)|||_{\mathcal{W}^{1,1}_{\rho,\rho}} \\ &\le C_0 B(\gamma)^2 \Big( t(\rho_0 - \rho - \frac{1}{2}\gamma t)^{-\zeta} + \frac{1}{\gamma}(\rho_0 - \rho - \frac{1}{2}\gamma t)^{1-\zeta} \Big) + t \cdot \sup_{0\le s\le T} |||N(s)|||_{\mathcal{W}^{1,1}_{\rho,\rho}} \\ &\le C_0 \gamma^{-1} B(\gamma)^2 (\rho_0 - \rho)^{-\zeta} + t \cdot \sup_{0\le s\le T} |||N(s)|||_{\mathcal{W}^{1,1}_{\rho,\rho}}, \end{aligned}$$

in which we used $\gamma t \le \rho_0$ and $\gamma t < \rho_0 - \rho$. Next, noting that $\rho < \rho_0 - \gamma t \le \rho_0 - \gamma s$, we take $\rho' = \frac{\rho+\rho_0-\gamma s}{2}$ and compute

$$\begin{aligned}
&\int_0^t |||e^{\nu(t-s)B} N(s)|||_{\rho,\delta(t),2}\, ds \\
&\lesssim \int_0^t \left( \sqrt{\frac{t}{s}} |||N(s)|||_{\rho,\delta(s),2} + \sqrt{\frac{t}{t-s}} |||N(s)|||_{\mathcal{W}^{2,1}_{\rho,\rho}} \right) ds \\
&\lesssim \int_0^t \sqrt{\frac{t}{s}} \frac{1}{\rho'-\rho} |||N(s)|||_{\rho',\delta(s),1}\, ds + t \cdot \sup_{0\le s\le T} |||N(s)|||_{\mathcal{W}^{2,1}_{\rho,\rho}} \\
&\lesssim B(\gamma)^2 \int_0^t \sqrt{\frac{t}{s}} (\rho_0 - \rho - \gamma s)^{-1-\zeta}\, ds + t \cdot \sup_{0\le s\le T} |||N(s)|||_{\mathcal{W}^{2,1}_{\rho,\rho}} \\
&\le C_0 B(\gamma)^2 \Big( \int_0^{t/2} + \int_{t/2}^t \Big) \sqrt{\frac{t}{s}} (\rho_0 - \rho - \gamma s)^{-1-\zeta}\, ds + t \cdot \sup_{0\le s\le T} |||N(s)|||_{\mathcal{W}^{2,1}_{\rho,\rho}} \\
&\le C_0 B(\gamma)^2 \Big( t(\rho_0 - \rho - \frac{1}{2}\gamma t)^{-1-\zeta} + \frac{1}{\gamma}(\rho_0 - \rho - \gamma t)^{-\zeta} \Big) + t \cdot \sup_{0\le s\le T} |||N(s)|||_{\mathcal{W}^{2,1}_{\rho,\rho}} \\
&\le C_0 \gamma^{-1} B(\gamma)^2 (\rho_0 - \rho - \gamma t)^{-\zeta} + t \cdot \sup_{0\le s\le T} |||N(s)|||_{\mathcal{W}^{2,1}_{\rho,\rho}}.
\end{aligned}$$

This proves the boundedness of the iterative norm $B(\gamma)$, and hence the propagation of the boundary layer behaviors. Theorem 4.3.1 follows.

### 4.5.3 The inviscid limit

*Proof of Theorem 4.3.2.* Let $u^E \in W^{2,\infty}(\Omega) \cap W^{2,2}(\Omega)$ be the solution to Euler (in our case, $u^E$ is even analytic). As in (**??**), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|v\|^2_{L^2} + \int_\Omega (v\cdot\nabla u^E)\cdot v + \nu\int_\Omega \nabla u^E \cdot \nabla v + \nu \int_\Omega |\nabla v|^2 \\
&+ \int_{\mathbb{T}} |u^\nu_1(t,x,0)|^2 dx - \nu \int_{\mathbb{T}} \omega^\nu(t,x,0) u^E_1(t,x,0) = 0.
\end{aligned} \tag{4.44}$$

By Cauchy inequality, we have

$$\frac{d}{dt}\|v\|^2_{L^2} \lesssim C_E \left( \|v\|^2_{L^2} + \nu + \nu \int_{\mathbb{T}} |\omega^\nu(t,x,0)| dx \right),$$

where $C_E$ is a constant only depending on $u^E$. Now, since $\|\omega^\nu(t)\|_{\sigma,\rho,\delta(t)}$ is uniformly bounded in $\nu$, there exists $C_0 > 0$ such that

$$|\omega^\nu(t,x,y)| \le C_0 e^{-\beta_0 y} \left( 1 + \delta^{-1}\phi_P(\delta^{-1}y) + \delta_t^{-1}\phi_P(\delta_t^{-1}y) \right).$$

Putting $y = 0$, we get

$$|\omega^\nu(t,x,0)| \lesssim \delta_t^{-1}. \tag{4.45}$$

Combining (4.44) and (4.45), we get

$$\frac{d}{dt}\|v(t)\|_{L^2}^2 \lesssim \|v(t)\|_{L^2}^2 + \frac{\sqrt{\nu}}{\sqrt{t}} + \nu.$$

Hence, by Gronwall inequality, we get

$$\|v(t)\|_{L^2} \lesssim (\nu t)^{1/4} + \sqrt{\nu} + \|v(0)\|_{L^2}.$$

The proof is complete.

# Chapter 5

# The inviscid limit for vortex wave data

## 5.1 Introduction and main theorem

In this chapter, we give the proof for the inviscid limit of the Navier-Stokes equations

$$\begin{aligned} &\partial_t\omega^\nu + u^\nu \cdot \nabla\omega^\nu = \nu\Delta\omega^\nu \\ &u^\nu = K \star \omega^\nu, \quad K(x) = \frac{1}{2\pi} \cdot \frac{x^\perp}{|x|^2} \end{aligned} \tag{5.1}$$

with initial vorticity

$$\omega^\nu|_{t=0} = \delta_{z_0} + \omega_0^E(x)$$

Here $\delta_{z_0}$ denotes the Dirac-delta function at $z_0 \in \mathbb{R}^2$ and $\omega_0^E \in W^{4,4}(\mathbb{R}^2)$ is the compactly support function so that $z_0$ is not in the support of $\omega_0^E$. In the inviscid limit $\nu \to 0$, we prove that

$$\omega^\nu(t) \rightharpoonup \delta_{z(t)} + \omega^E(t)$$

for $t \in [0, T]$ independent of $\nu$, and $(z(t), \omega^E(t))$ is the solution to the vortex-wave system:

$$\begin{cases} \partial_t\omega^E + (v^E + H) \cdot \nabla\omega^E = 0 \\ \dot{z}(t) = v^E(t, z(t)), \quad v^E = K \star \omega^E, \quad H(\cdot) = K(\cdot - z(t)) \\ \omega^E_{|t=0} = \omega_0^E, \qquad z(0) = z_0. \end{cases} \tag{5.2}$$

We refer the readers to the works [43, 45, 37, 10, 61, 36] about the existence and uniqueness of the vortex wave system, and [9, 22, 34, 12] for existence and uniqueness of 2D Navier-Stokes for singular initial data. Now, let us recall the following theorem.

**Theorem 5.1.1** ([36]). *Consider initial data $z_0 \in \mathbb{R}$ and $\omega_0^E \in L^1 \cap L^\infty(\mathbb{R}^2)$. Assume that $\omega_0^E$ has compact support and is constant in a neighborhood of $z_0$. Then, there are a unique global solution $(z(t), \omega^E(t))$ to (5.2) and a positive function $R(t)$ so that $\omega^E(t)$ remains constant in the ball centered at the point vortex $z(t)$ with radius $R(t)$ for all times $t \geq 0$. If we assume in addition that $\omega_0^E \in W^{k,p}$ for $kp > 2$ and $p > 1$, then for any $T \geq 0$, there holds*

$$\sup_{0\le t\le T} \|\omega^E(t)\|_{W^{k,p}} \le C_T \tag{5.3}$$

*for some constant $C_T$.*

Our main theorem is as follows:

**Theorem 5.1.2.** *Let $z_0 \in \mathbb{R}$ and $\omega_0^E \in W^{4,4}(\mathbb{R}^2)$ that has compact support and vanishes in a neighborhood of $z_0$, and let $(z(t), \omega^E(t))$ and $\omega^\nu(t)$ be the unique solution to the vortex-wave system* (5.2) *and to the Navier-Stokes equation* (5.1)*, respectively, with initial data $\omega_0 = \omega_0^E + \delta_{z_0}$. Then, there exists a time $T > 0$, independent of $\nu$, such that the vorticity $\omega^\nu(t)$ can be written as*

$$\omega^\nu(x,t) = \omega^{E,\nu}(x,t) + \omega^{B,\nu}(x,t),$$

*where $\omega^{E,\nu}(t), \omega^{B,\nu}(t)$ satisfy*

$$\sup_{0\le t\le T} \|\omega^{E,\nu}(t) - \omega^E(t)\|_{L^4\cap L^{4/3}(\mathbb{R}^2)} \le C_T \nu,$$
$$\sup_{0\le t\le T} t^{-1} \left\| \omega^{B,\nu}(t,x) - \frac{1}{4\pi\nu t} e^{-\frac{|x-z(t)|^2}{4\nu t}} \right\|_{L^1(\mathbb{R}^2)} \le C_T \nu,$$

*for some constant $C_T$ independent of $\nu$. In particular, $\omega^{E,\nu}(t) \to \omega^E(t)$ strongly in $L^4 \cap L^{4/3}$ and $\omega^{B,\nu}(t,\cdot) \rightharpoonup \delta_{z(t)}(\cdot)$ weakly in the sense of finite measures in the inviscid limit.*

**Remark 5.1.** *Theorem 5.1.2 derives the vortex-wave system* (5.2) *as the inviscid limit of Navier-Stokes flows on the whole plane, complementing the earlier derivation [44, 3, 23] from Euler equations. In addition, we obtain:*

$$T \ge \min\left\{T_*^-, \frac{1}{5\|\nabla v^E\|_{L^\infty}}\right\}$$

*for $T_*$ being the smallest time when the point vortex $z(t)$ meets the support of $\omega^E(s)$ for some $s \in [0,t]$, recalling from Theorem 5.1.1 that $z(t)$ never meets the support of $\omega^E(t)$ for all times. See Proposition 5.2 and Remark 5.18.*

Let us now discuss some difficulties in proving the theorem. First of all, the initial data containing a Dirac mass are too singular to perform a direct proof from the standard $L^2$ energy estimates. One then needs to construct a good approximation of solutions to treat the singular part, and control the remainder. The difficulty arises due to the presence of an vortex-wave interaction term of the form

$$v^{E,\nu}(t,x) \cdot \nabla_x \left( \frac{1}{4\pi\nu t} e^{-\frac{|x-z(t)|^2}{4\nu t}} \right). \tag{5.4}$$

Formally, this term blows up when $x$ is near the point vortex $z(t)$ and $\nu t \to 0$. To treat this singularity, we follow [13] to work in the vortex scaling variable, construct approximate solutions, and perform weighted energy estimates to control the remainder.

Lastly, we remark that we assume the initial vorticity to be $\delta_{z_0} + \omega_0^E$, where $\omega_0^E$ is smooth and compactly supported away from the point vortex $z_0$. The regularity is needed in the construction of the high order approximation of solutions. It would be interesting to further combine our analysis with the viscous approximation near vortex-patch solutions constructed in [29] to treat the case when $\omega_0^E \in L^1 \cap L^\infty$. However, the weighted energy estimates with the scaling variable $\xi = \frac{x-z(t)}{\sqrt{\nu t}}$ used in [13] are not enough to treat the interaction term (5.4), as it leaves a remainder of order one, but not smaller. To overcome this difficulty, we introduce an *approximate viscous* vortex wave system (Section 5.2), along with the new point vortex $\widetilde{z}(t) = z(t) + O(\nu t)$ and the scaled variable $\xi = \frac{x-\widetilde{z}(t)}{\sqrt{\nu t}}$ in order to close the estimate.

Following [12, 13], we first decompose the vorticity into the so-called regular part $\omega^{E,\nu}$ and irregular part $\omega^{B,\nu}$, both of which are advected by the full velocity vector field $u^\nu = K \star \omega^\nu$. Precisely, we write

$$\omega^\nu = \omega^{E,\nu} + \omega^{B,\nu}, \tag{5.5}$$

where $\omega^{E,\nu}$ and $\omega^{B,\nu}$ solve

$$\begin{aligned} \partial_t \omega^{E,\nu} + u^\nu \cdot \nabla \omega^{E,\nu} &= \nu \Delta \omega^{E,\nu}, \\ \omega^{E,\nu}|_{t=0} &= \omega_0^E, \end{aligned} \tag{5.6}$$

and

$$\begin{aligned} \partial_t \omega^{B,\nu} + u^\nu \cdot \nabla \omega^{B,\nu} &= \nu \Delta \omega^{B,\nu}, \\ \omega^{B,\nu}(t) \quad &\rightharpoonup \quad \delta_{z_0} \quad \text{as} \quad t \to 0^+. \end{aligned} \tag{5.7}$$

Here and in what follows, the weak convergence for finite measures is understood in the following sense: $\mu_n \rightharpoonup \mu$ if and only if

$$\int_{\mathbb{R}^2} \phi d\mu_n \to \int_{\mathbb{R}^2} \phi d\mu,$$

for all the continuous functions $\phi$ that vanish at infinity. A direct computation shows that the decomposition preserves the mass:

$$\int_{\mathbb{R}^2} \omega^{E,\nu}(x,t)dx = \int_{\mathbb{R}^2} \omega_0^E(x)dx, \qquad \int_{\mathbb{R}^2} \omega^{B,\nu}(x,t)dx = 1, \tag{5.8}$$

for all positive times. We shall prove that in the inviscid limit $\omega^{E,\nu} \to \omega^E$ and $\omega^{B,\nu}$ is concentrated near the point vortex $z(t)$, transported by $v^E$, yielding weak solutions to the vortex wave system with the same initial data $(\omega_0^E, z_0)$.

In Section 5.2, we construct the solution $(\widetilde{z}(t), \widetilde{\omega}^E(t))$ to the approximate vortex wave system, which is introduced to justify the asymptotic expansion for the regular part $\omega^{E,\nu}$:

$$\begin{cases} \omega^{E,\nu}(t,x) &= \widetilde{\omega}^E(t,x) + O(\nu^{3/2})_{L^4 \cap L^{4/3}}, \\ v^{E,\nu}(t,x) &= \widetilde{v}^E(t,x) + O(\nu^{3/2})_{L^\infty}. \end{cases} \tag{5.9}$$

Moreover, the introduced point vortex $\widetilde{z}(t)$ in the approximate vortex wave system is justified to stay within a radius of order $(\nu t)$ the original point vortex $z(t)$ of the vortex wave system.

In Section 5.3, we study the irregular part $\omega^{B,\nu}$, solving the advection heat equation driven by $u^\nu$ with initial data $\delta_{z_0}$. Since we expect $\omega^{B,\nu}$ to behave like the Lamb-Oseen vortex sitting at $\widetilde{z}(t)$, we perform the change of variables (see also [13]) $\xi = \frac{x-\widetilde{z}(t)}{\sqrt{\nu t}}$, and rewrite the equation of $\omega^{B,\nu}$ in the new rescaled variable $\xi$. We then construct approximate solution by using many established properties of the linearized operator around the point vortex and the error coming from the interaction between the point vortex and the vortex-wave system. Next, we control the remainder in the weighted $L^2$ norm in $\xi$, with weight $p(\xi) = e^{|\xi|^2/4}$, which has the crucial cancellation properties for the bad interaction term [1] between the remainder and the Oseen vortex (see Lemma 5.8). We conclude Section 5.3 by an apriori estimate (see Proposition 5.10) for the remainder of term, which is the difference between the approximate solution and the irregular part.

In Section 5.4, we derive apriori estimate for the remainder of the regular part $\omega^{E,\nu}$. We conclude the chapter with Section 5.5, we rigorously justify the asymptotic expansion (5.9) for the regular part and the expansion

$$\omega^{B,\nu}(t,x) = \frac{1}{4\pi\nu t} e^{-\frac{|x-z(t)|^2}{4\nu t}} + O(\nu t)_{L^1(\mathbb{R}^2)}$$

for the irregular part.

## 5.2 Approximate vortex wave system

### 5.2.1 Motivation

In this section, we introduce the approximate vortex wave system, which can be thought of as a small perturbation of the vortex wave system (5.2). This is one of the technical point in proving the inviscid limit in our work [52]. Let us explain briefly this point before proceeding to the detailed introduction of the approximate vortex wave system. As mentioned before, we construct higher order approximation to both the equation for $\omega^{E,\nu}$ and $\omega^{B,\nu}$, and then control the remainder in suitable norm, which is $L^4 \cap L^{4/3}$ for $\omega^{E,\nu}$ and $L^2$ weighted space for $\omega^{B,\nu}$. It is very natural to use the vortex wave system (5.2) as a first approximation for $\omega^{E,\nu}$, without introducing the approximate vortex-wave system, and expect that

$$\omega^{E,\nu}(t,x) = \omega^E(t,x) + O(\nu)_{L^4 \cap L^{4/3}}.$$

However, this approximation is not good enough when we try to close the estimate for the irregular part $\omega^{B,\nu}$ in $L^2$ weighted space. In order to circumvent this difficulty, we introduce the approximate system to kill the error term $\nu\Delta\omega^E$, coming from the first

[1] this term has a pre-factor $\nu^{-1}$ and is linear.

order approximation, leaving a finer expansion of the regular part as follows:

$$\omega^{E,\nu}(t,x) = \omega^E(t,x) + O(\nu)_{L^4\cap L^{4/3}} + O(\nu^{3/2})_{L^4\cap L^{4/3}}.$$

This turns out to be good enough in our analysis when controlling the remainder. Next, we apply our entire analysis, but now on the approximate vortex-wave system $(\widetilde{\omega}^E(t), \widetilde{z}(t))$. In order to prove that the inviscid limit holds for vortex-wave system, we also need to check, in this section, that the introduced system should stay near the vortex-wave system, at least for a time interval independent of the viscosity. Namely, we will prove, later in this section, that

$$\widetilde{z}(t) = z(t) + O(\nu t), \qquad \text{and} \quad \widetilde{\omega}^E(t) = \omega^E(t) + O(\nu)_{L^4\cap L^{4/3}}.$$

Lastly, let us mention that the norm $L^4 \cap L^{4/3}$ for the vorticity control is natural, as it allows us to control the corresponding velocity in $L^\infty$ (see Lemma 5.26 for details).

### 5.2.2 Construction of the approximate system

Let us now turn to our construction of the approximate vortex wave system. Let $(z(t), \omega^E)$ be the global solution to the vortex-wave system (5.2) with initial data $\omega_0^E \in W^{4,4}$ that has compact support and vanishes in a neighborhood of $z_0$. We introduce an *approximate viscous* vortex-wave system $(\widetilde{z}(t), \widetilde{\omega}^E)$, given by

$$\begin{aligned} \widetilde{\omega}^E(x,t) &= \omega^E(x,t) + \nu w_{1,a}(x,t) \\ \partial_t \widetilde{z} = \widetilde{v}^E(\widetilde{z}(t),t) &= K \star \widetilde{\omega}^E(\widetilde{z}(t),t), \qquad \widetilde{z}(0) = z_0, \end{aligned} \tag{5.10}$$

where the added vorticity component $w_{1,a}$ solves

$$\partial_t w_{1,a} + \left( v^E + \frac{1}{\sqrt{\nu t}} v^G \left( \frac{x - z(t)}{\sqrt{\nu t}} \right) \right) \cdot \nabla w_{1,a} + v_{1,a} \cdot \nabla \omega^E = \Delta \omega^E \tag{5.11}$$

with zero initial data. Here and in what follows, velocity and vorticity are defined through the Biot-Savart law (5.1). For instance, $v_{1,a} = K \star w_{1,a}$ and $v^G(\xi) = \frac{1}{2\pi} \frac{\xi^\perp}{|\xi|^2}(1 - e^{-|\xi|^2/4})$. We obtain the following simple proposition.

**Proposition 5.2.** *Let $T_*$ be defined by*

$$T_* = \inf_{t\geq 0} \Big\{ t \ : \ z(t) \in \cup_{0\leq s\leq t} supp(\omega^E(s)) \Big\}, \tag{5.12}$$

*with $T_* = \infty$ if $z(t)$ never meets the support of $\omega^E(s)$ for $s \in [0,t]$. Then, for any $T < T_*$, the unique smooth solution $w_{1,a}(t)$ of (5.11) exists on $[0,T]$, has compact support, vanishes in a neighborhood of $z(t)$, and satisfies*

$$\mathfrak{m}\Big( supp(w_{1,a}(t)) \Big) + \|w_{1,a}(t)\|_{W^{2,4}(\mathbb{R}^2)} + \|\partial_t w_{1,a}(t)\|_{L^\infty(\mathbb{R}^2)} + \|v_{1,a}(t)\|_{W^{2,\infty}(\mathbb{R}^2)} \leq C_T, \tag{5.13}$$

*for $t \in [0,T]$ and for some constant $C_T$ independent of $\nu$. In addition, there holds*

$$|\widetilde{z}(t) - z(t)| \le C_T \nu t \qquad \text{for any} \quad t \in [0,T]. \tag{5.14}$$

*Here, $\mathfrak{m}$ denotes the Lebesgue measure on $\mathbb{R}^2$.*

**Corollary 5.3.** *Let $T_*$ be defined as in (5.12). For any $T < T_*$, $\widetilde{\omega}^E(t)$ has compact support, vanishes in a neighborhood of $\widetilde{z}(t)$, and satisfies*

$$\mathfrak{m}\left(supp(\widetilde{\omega}^E(t))\right) + \|\widetilde{\omega}^E(t)\|_{W^{2,4}(\mathbb{R}^2)} + \|\partial_t \widetilde{\omega}^E(t)\|_{L^\infty(\mathbb{R}^2)} + \|\widetilde{v}^E(t)\|_{W^{2,\infty}(\mathbb{R}^2)} \le C_T, \tag{5.15}$$

*for $t \in [0,T]$ and for some constant $C_T$ independent of $\nu$.*

*Proof.* The corollary is a direct consequence of Proposition 5.2 and Theorem 5.1.1. □

*Proof of Proposition 5.2.* Recall from Theorem 5.1.1 that $\omega^E(t)$ has compact support and vanishes in a neighborhood of $z(t)$. This remains valid for $w_{1,a}(t)$ for small times, due to the transport structure of (5.11). Precisely, $w_{1,a}(t)$ is supported in $\cup_{0\le s\le t}\mathrm{supp}(\omega^E(s))$. Since $z(t) \notin \mathrm{supp}(\omega^E(t))$ for all positive times, we have $T_* > 0$ by continuity. Thus, for any $T < T_*$, there is a positive distance $d_T$ so that

$$|x - z(t)| \ge d_T > 0 \tag{5.16}$$

for all $x \in \mathrm{supp}\left(w_{1,a}(t)\right)$ and $0 \le t \le T$, which yields

$$\left|\frac{1}{\sqrt{\nu t}} v^G\left(\frac{x - z(t)}{\sqrt{\nu t}}\right)\right| = \frac{1}{2\pi |x - z(t)|}\left(1 - e^{-\frac{|x-z(t)|^2}{4\nu t}}\right) \le \frac{1}{2\pi|x - z(t)|} \le \frac{1}{2\pi d_T}.$$

Similar estimates hold for derivatives of $v^G(\cdot)$ for $x$ away from $z(t)$. It follows from (5.11) that

$$\begin{aligned}\|w_{1,a}(t)\|_{L^4} &\le \int_0^t \left(\|\Delta\omega^E(s)\|_{L^4} + \|v_{1,a}(s)\|_{L^\infty}\|\nabla\omega^E(s)\|_{L^4}\right) ds \\ &\lesssim \int_0^t (1 + \|v_{1,a}(s)\|_{L^\infty}) ds,\end{aligned}$$

which yields the estimate on $w_{1,a}$, upon using the elliptic estimate $\|v_{1,a}\|_{L^\infty} \lesssim \|w_{1,a}\|_{L^4 \cap L^{4/3}}$ and the fact that $w_{1,a}$ is compactly supported. The derivative estimates follow similarly. Finally, let us prove the estimate on $\widetilde{z}(t)$. By definition, we write

$$\begin{cases}\widetilde{z}(t) &= z_0 + \int_0^t \left(v^E(\widetilde{z}(s), s) + \nu v_{1,a}(\widetilde{z}(s), s)\right) ds, \\ z(t) &= z_0 + \int_0^t v^E(z(s), s) ds,\end{cases} \tag{5.17}$$

which gives

$$\begin{aligned} |\widetilde{z}(t) - z(t)| &\le \int_0^t \Big|(v^E(\widetilde{z}(s), s) - v^E(z(s), s))\Big|\, ds + \nu \int_0^t |v_{1,a}(\widetilde{z}(s), s)| ds \\ &\le \int_0^t \|\nabla v^E(s)\|_{L^\infty} |\widetilde{z}(s) - z(s)|\, ds + \nu t \sup_{0\le s\le t} \|v_{1,a}(s)\|_{L^\infty}. \end{aligned} \tag{5.18}$$

Applying the Gronwall's lemma gives (5.14). □

## 5.3 Inviscid limit for the irregular part

In this section, we give estimates on the irregular part of vorticity $\omega^{B,\nu}$, solving (5.7). Let us recall the equation:

$$\begin{aligned} \partial_t \omega^{B,\nu} + u^\nu \cdot \nabla \omega^{B,\nu} &= \nu \Delta \omega^{B,\nu}, \\ \omega^{B,\nu}|_{t=0} &= \delta_{z_0}. \end{aligned} \tag{5.19}$$

Here $u^\nu = v^{E,\nu} + v^{B,\nu}$ is the velocity field for the full Navier-Stokes equations. Following [13], we introduce the change of variables

$$\xi = \frac{x - \widetilde{z}(t)}{\sqrt{\nu t}}$$

and write

$$v^{B,\nu}(x,t) = \frac{1}{\sqrt{\nu t}} v_2(\xi, t), \qquad \omega^{B,\nu}(x,t) = \frac{1}{\nu t} w_2(\xi, t). \tag{5.20}$$

Here, we recall that $\widetilde{z}(t)$ to be the solution to the approximate vortex wave system, given in (5.10). Note that the change of variables is consistent with the Biot-Savart law: $v_2 = K \star_\xi w_2$. Putting the Ansatz into the equation (5.7) for $\omega^{B,\nu}$, we get the following equation

$$\Phi(w_2, v^{E,\nu}) := \big(t\partial_t - \mathcal{L}\big)\, w_2 + \sqrt{\frac{t}{\nu}} (v^{E,\nu}(\widetilde{z}(t) + \xi\sqrt{\nu t}, t) - \partial_t \widetilde{z}(t)) \cdot \nabla_\xi w_2 + \frac{1}{\nu} v_2 \cdot \nabla_\xi w_2 = 0, \tag{5.21}$$

where $\mathcal{L}$ is defined by

$$\mathcal{L} w_2 := \Delta_\xi w_2 + \frac{1}{2} \xi \cdot \nabla_\xi w_2 + w_2.$$

In the vanishing viscosity limit, we expect that the viscous regular velocity remains close to the inviscid one: $v^{E,\nu} \to v^E$, and hence the irregular part should tend to the so-called Lamb-Oseen vortex, which is defined by

$$G(\xi) = \frac{1}{4\pi} e^{-|\xi|^2/4}, \qquad v^G(\xi) = \frac{1}{2\pi} \frac{\xi^\perp}{|\xi|^2} \Big(1 - e^{-|\xi|^2/4}\Big).$$

It follows that $\mathcal{L}G = 0$ and $v^G \cdot \nabla_\xi G = 0$. Therefore, the pair $(G(\xi), v^{E,\nu})$ solves (5.21), up to the following error term $tR_1(\xi,t)$, with

$$R(\xi,t) := \frac{1}{\sqrt{\nu t}}\left(v^{E,\nu}(\widetilde{z}(t)+\xi\sqrt{\nu t},t) - \widetilde{v}^E(\widetilde{z}(t),t)\right)\cdot\nabla G \tag{5.22}$$

which does not vanish in the inviscid limit, upon recalling that $\partial_t\widetilde{z}(t) = \widetilde{v}^E(\widetilde{z}(t),t)$. Roughly speaking, $R = \mathcal{O}(1)$ in the small viscosity limit.

We shall construct better approximate solutions to the equation (5.21). Here we stress that the equation (5.21) involves two unknown functions $w_2, v^{E,\nu}$ which are coupled through the full velocity $u^\nu$. To leading order, let us take $v^{E,\nu}_{app} = \widetilde{v}^E$ for $\widetilde{v}^E$ solving the approximate vortex-wave system (5.10) and

$$w_{2,\text{app}}(\xi,t) = G(\xi) + (\nu t)w_{2,a}(\xi,t) \tag{5.23}$$

where $w_{2,a}$ to be defined later. The pair $(w_{2,\text{app}}, v^{E,\nu}_{app})$ thus solves (5.21), leaving an error of the form

$$\begin{aligned}\Phi(w_{2,\text{app}}, v^{E,\nu}_{app}) &= t(\Lambda+\nu(1-\mathcal{L}))w_{2,a} + \nu t^2\partial_t w_{2,a} + \nu t^2 v_{2,a}\cdot\nabla w_{2,a} \\ &\quad + \sqrt{\nu}t^{3/2}(\widetilde{v}^E(\widetilde{z}(t)+\xi\sqrt{\nu t},t) - \widetilde{v}^E(\widetilde{z}(t),t))\cdot\nabla w_{2,a} + tR_1(\xi,t),\end{aligned} \tag{5.24}$$

where $R_1(\xi,t)$ is defined as in (5.22) with $v^{E,\nu}_{app} = \widetilde{v}^E$, and

$$\Lambda w := v^G\cdot\nabla_\xi w + v\cdot\nabla_\xi G, \qquad v = K\star w.$$

To treat the order one remainder $R_1(\xi,t)$, we first solve $(\Lambda+\nu(1-\mathcal{L}))w_{2,a} = -R_1$ to leading order in $\nu$. We recall the following proposition from [13], Lemma 5 and Remark 1.

**Proposition 5.4.** *Let $z = z(\xi)$ be a function of the form*

$$z(\xi) = a_1(r)\cos(2\theta) + a_2(r)\sin(2\theta) + a_3(r)\cos(3\theta) + a_4(r)\sin(3\theta)$$

*for $\xi = re^{i\theta}$. Assume that the coefficients satisfy*

$$\sum_{i=1}^4(|a_i(r)| + |a_i'(r)|) \le C_0P(r)e^{-r^2/4} \qquad \forall r > 0.$$

*for some polynomial $P(r)$. Then for any $\nu > 0$, there exists a unique solution $w^\nu$ to the elliptic equation*

$$\Lambda w^\nu + \nu(1-\mathcal{L})w^\nu = z$$

*such that*

$$|w^\nu(\xi)| + |\nabla w^\nu(\xi)| \le C_\gamma e^{-\gamma|\xi|^2/4}$$

*for any $\gamma \in (0,1)$ and for some constant $C_\gamma$ that is independent of $\nu$.*

### 5.3.1 Vortex-wave reaction term

In this section, we show that the leading term in the reaction term in (5.22) satisfies the assumption of Proposition 5.4. Precisely, we introduce

$$R_1(\xi,t) = \frac{1}{\sqrt{\nu t}}(\widetilde{v}^E(\widetilde{z}(t)+\xi\sqrt{\nu t},t) - \widetilde{v}^E(\widetilde{z}(t),t))\cdot\nabla G. \tag{5.25}$$

We have the following lemma.

**Lemma 5.5.** *For any $T>0$, there is a constant $C_T$ so that*

$$|R_1(\xi,t) - A_0(\xi,t)| \le C_T(\nu t)|\xi|^4 e^{-|\xi|^2/4},$$

*where*

$$\begin{aligned} A_0(\xi,t) &= \frac{1}{16\pi^2}|\xi|^2 e^{-|\xi|^2/4}\int_{\mathbb{R}^2}\frac{\sin(2\psi)}{|\widetilde{z}(t)-y|^2}\widetilde{\omega}^E(y,t)dy \\ &\quad - \frac{1}{16\pi^2}\sqrt{\nu t}|\xi|^3 e^{-|\xi|^2/4}\int_{\mathbb{R}^2}\frac{\sin(3\psi)}{|\widetilde{z}(t)-y|^3}\widetilde{\omega}^E(y,t)dy. \end{aligned} \tag{5.26}$$

*Here, $\psi$ denotes the angle between $\xi$ and $\widetilde{z}(t)-y$.*

*Proof.* Recalling (5.25) and $G = \frac{1}{4\pi}e^{-|\xi|^2/4}$, and using the Biot-Savart law (**??**), we have

$$\begin{aligned} R_1(\xi,t) &= \frac{-1}{8\pi\sqrt{\nu t}}(\widetilde{v}^E(\widetilde{z}(t)+\xi\sqrt{\nu t},t) - \widetilde{v}^E(\widetilde{z}(t),t))\cdot\xi e^{-|\xi|^2/4} \\ &= \frac{-e^{-|\xi|^2/4}}{16\pi^2\sqrt{\nu t}}\int_{\mathbb{R}^2}\xi\cdot\left(\frac{(\widetilde{z}(t)+\xi\sqrt{\nu t}-y)^\perp}{|\widetilde{z}(t)+\xi\sqrt{\nu t}-y|^2} - \frac{(\widetilde{z}(t)-y)^\perp}{|\widetilde{z}(t)-y|^2}\right)\widetilde{\omega}^E(y,t)dy \\ &= \frac{-e^{-|\xi|^2/4}}{16\pi^2\sqrt{\nu t}}\int_{\mathbb{R}^2}\xi\cdot(\widetilde{z}(t)-y)^\perp\left(\frac{1}{|\widetilde{z}(t)+\xi\sqrt{\nu t}-y|^2} - \frac{1}{|\widetilde{z}(t)-y|^2}\right)\widetilde{\omega}^E(y,t)dy \\ &=: A_1(\xi,t) + A_2(\xi,t), \end{aligned}$$

where $A_1(\xi,t), A_2(\xi,t)$ denote the integral over $\{|\xi|\sqrt{\nu t} \le \frac{1}{2}|\widetilde{z}(t)-y|\}$ and $\{|\xi|\sqrt{\nu t} \ge \frac{1}{2}|\widetilde{z}(t)-y|\}$, respectively. Let us first treat $A_1(\xi,t)$. Applying Lemma 5.27 for $|\xi|\sqrt{\nu t} \le \frac{1}{2}|\widetilde{z}(t)-y|$, we have

$$\frac{1}{|\widetilde{z}(t)+\xi\sqrt{\nu t}-y|^2} - \frac{1}{|\widetilde{z}(t)-y|^2} = \frac{1}{|\widetilde{z}(t)-y|^2}\sum_{n=1}^{\infty}(-1)^n\frac{|\xi|^n\sqrt{\nu t}^n}{|\widetilde{z}(t)-y|^n}\frac{\sin((n+1)\psi)}{\sin(\psi)}.$$

Here $\psi$ is the angle between $\xi$ and $\widetilde{z}(t)-y$. Thus we get

$$\begin{aligned}
&\xi\cdot(\widetilde{z}(t)-y)^{\perp}\left(\frac{1}{|\widetilde{z}(t)+\xi\sqrt{\nu t}-y|^{2}}-\frac{1}{|\widetilde{z}(t)-y|^{2}}\right)\\
&=\sum_{n=2}^{\infty}(-1)^{n+1}(\nu t)^{\frac{n-1}{2}}\frac{|\xi|^{n}}{|\widetilde{z}(t)-y|^{n}}\sin(n\psi)\\
&=-(\nu t)^{1/2}\frac{|\xi|^{2}}{|\widetilde{z}(t)-y|^{2}}\sin(2\psi)+(\nu t)\frac{|\xi|^{3}}{|\widetilde{z}(t)-y|^{3}}\sin(3\psi)+\frac{1}{\sqrt{\nu t}}\sum_{n\geq 4}(-1)^{n+1}\frac{(|\xi|\sqrt{\nu t})^{n}}{|\widetilde{z}(t)-y|^{n}}\sin(n\psi),
\end{aligned}$$

in which we can estimate

$$\Big|\frac{1}{\sqrt{\nu t}}\sum_{n\geq 4}(-1)^{n+1}\frac{(|\xi|\sqrt{\nu t})^{n}}{|\widetilde{z}(t)-y|^{n}}\sin(n\psi)\Big|\leq 2\frac{(\nu t)^{3/2}|\xi|^{4}}{|\widetilde{z}(t)-y|^{4}},$$

since $|\xi|\sqrt{\nu t}\leq\frac{1}{2}|\widetilde{z}(t)-y|$. Hence, we have

$$\begin{aligned}
A_{1}(\xi,t)=&\frac{|\xi|^{2}e^{-|\xi|^{2}/4}}{16\pi^{2}}\int_{|\xi|\sqrt{\nu t}\leq\frac{1}{2}|\widetilde{z}(t)-y|}\frac{1}{|\widetilde{z}(t)-y|^{2}}\sin(2\psi)\widetilde{\omega}^{E}(y,t)dy\\
&-\frac{\sqrt{\nu t}|\xi|^{3}e^{-|\xi|^{2}/4}}{16\pi^{2}}\int_{|\xi|\sqrt{\nu t}\leq\frac{1}{2}|\widetilde{z}(t)-y|}\frac{1}{|\widetilde{z}(t)-y|^{3}}\sin(3\psi)\widetilde{\omega}^{E}(y,t)dy\\
&+\mathcal{O}(\nu t|\xi|^{4}e^{-|\xi|^{2}/4})\int_{|\xi|\sqrt{\nu t}\leq\frac{1}{2}|\widetilde{z}(t)-y|}\frac{1}{|\widetilde{z}(t)-y|^{4}}\sin(4\psi)\widetilde{\omega}^{E}(y,t)dy.
\end{aligned}$$

We note that all the integrals above are bounded by $\|\widetilde{\omega}^{E}(t)\|_{L^{1}}$, since $\widetilde{z}(t)$ is bounded away from the support of $\widetilde{\omega}^{E}(t)$ by Corollary 5.3. Therefore, defining $A_{0}(\xi,t)$ as in (5.26), we can write

$$\begin{aligned}
A_{1}(\xi,t)=A_{0}(\xi,t)-&\frac{|\xi|^{2}e^{-|\xi|^{2}/4}}{16\pi^{2}}\int_{|\xi|\sqrt{\nu t}\geq\frac{1}{2}|\widetilde{z}(t)-y|}\frac{1}{|\widetilde{z}(t)-y|^{2}}\sin(2\psi)\widetilde{\omega}^{E}(y,t)dy\\
&+\frac{\sqrt{\nu t}|\xi|^{3}e^{-|\xi|^{2}/4}}{16\pi^{2}}\int_{|\xi|\sqrt{\nu t}\geq\frac{1}{2}|\widetilde{z}(t)-y|}\frac{1}{|\widetilde{z}(t)-y|^{3}}\sin(3\psi)\widetilde{\omega}^{E}(y,t)dy+\mathcal{O}(\nu t|\xi|^{4}e^{-|\xi|^{2}/4}).
\end{aligned}$$

It remains to treat the integral over the domain $\{|\xi|\sqrt{\nu t}>\frac{1}{2}|\widetilde{z}(t)-y|\}$. Since $\widetilde{z}(t)$ is bounded away from the support of $\widetilde{\omega}^{E}(t)$, the above (explicitly written) integrals vanish for $|\xi|\sqrt{\nu t}\leq c_{T}$ for all $t\in[0,T]$, for some constant $c_{T}$. On the other hand, for $|\xi|\sqrt{\nu t}\geq c_{T}$, we have

$$\Big|\frac{|\xi|^{2}e^{-|\xi|^{2}/4}}{16\pi^{2}}\int_{|\xi|\sqrt{\nu t}\geq\frac{1}{2}|\widetilde{z}(t)-y|}\frac{1}{|\widetilde{z}(t)-y|^{2}}\sin(2\psi)\widetilde{\omega}^{E}(y,t)dy\Big|\leq C_{T}\nu t|\xi|^{4}e^{-|\xi|^{2}/4}\|\widetilde{\omega}^{E}(t)\|_{L^{1}},$$

for some constant $C_T$. Similarly, we also have $A_2(\xi,t) = 0$ for $|\xi|\sqrt{\nu t} \le c_T$ for all $t\in[0,T]$, for some constant $c_T$, while for $|\xi|\sqrt{\nu t}\ge c_T$, we have

$$\begin{aligned}|A_2(\xi,t)| &\le |A_1(\xi,t)| + |A(\xi,t)|\\ &\le C_T|\xi|^2(1+\nu t|\xi|^2)e^{-|\xi|^2/4}\|\widetilde{\omega}^E(t)\|_{L^1} + C_T(\nu t)^{-1/2}|\xi|e^{-|\xi|^2/4}\|\widetilde{v}^E\|_{L^\infty}\\ &\le C_T(\nu t)|\xi|^4 e^{-|\xi|^2/4},\end{aligned}$$

upon using Corollary 5.3 to bound $\widetilde{v}^E$ and $\widetilde{\omega}^E$. The lemma follows. □

### 5.3.2 Construction of an approximation solution

We now construct $w_{2,a}$ that solves the following elliptic equation

$$\Lambda w_{2,a} + \nu(1-\mathcal{L})w_{2,a} = -A_0(\xi,t) \tag{5.27}$$

with $A_0(\xi,t)$ defined as in (5.26). We have the following.

**Lemma 5.6.** *There exists a solution $w_{2,a}$ to (5.27) so that, for any $\gamma\in(0,1)$, there holds*

$$|w_{2,a}(t,\xi)| + |\nabla w_{2,a}(\xi,t)| \le C_\gamma e^{-\gamma|\xi|^2/4}$$

*uniformly in $\nu>0$. In particular, we have*

$$\|v_{2,a}(t)\|_{L^\infty} + \int_{\mathbb{R}^2}|w_{2,a}(\xi,t)|^2 e^{|\xi|^2/4}d\xi + \int_{\mathbb{R}^2}|\nabla w_{2,a}(\xi,t)|^2 e^{|\xi|^2/4}d\xi \lesssim 1. \tag{5.28}$$

*Proof.* For each $y\in\mathbb{R}^2$, we introduce

$$B_0(\xi,y,t) = \frac{-1}{16\pi^2}|\xi|^2 e^{-|\xi|^2/4}\frac{\sin(2\psi)}{|\widetilde{z}(t)-y|^2}\widetilde{\omega}^E(y,t) + \frac{1}{16\pi^2}\sqrt{\nu t}|\xi|^3 e^{-|\xi|^2/4}\frac{\sin(3\psi)}{|\widetilde{z}(t)-y|^3}\widetilde{\omega}^E(y,t), \tag{5.29}$$

recalling $\psi$ the angle between $\xi$ and $\widetilde{z}(t)-y$. If follows from (5.26) that $A_0(\xi,t) = \int_{\mathbb{R}^2} B_0(\xi,y,t)\,dy$. It is clear that for each $y$, $B_0(\xi,y,t)$ satisfies the assumption of Proposition 5.4 and hence we can define

$$W_{2,a}(\xi,y,t) := \Big(\Lambda + \nu(1-\mathcal{L})\Big)^{-1}B_0(\xi,y,t),$$

stressing that $y\in\mathbb{R}^2$ and $t\ge 0$ play a role as independent parameters. The solution $w_{2,a}$ is thus defined by the average of $W_{2,a}(\xi,y,t)$ with respect to $y$. The pointwise estimates follow directly from Proposition 5.4 and the estimates on $\widetilde{\omega}^E$. Taking $\gamma>1/2$ and using the elliptic estimate $\|v_{2,a}\|_{L^\infty}\lesssim\|w_{2,a}\|_{L^1\cap L^\infty}$, we obtain the estimates (5.28). □

### 5.3.3 Estimating the error term

Construct $w_{2,a}$ as in Lemma 5.6. Then, $w_{2,\text{app}} = G(\xi) + \nu t w_{2,a}$ and $v^{E,\nu}_{\text{app}} = \widetilde{v}^E$ approximately solves (5.21) in the following sense.

**Proposition 5.7.** *For any $\gamma \in (0,1)$, there holds*

$$\left|\Phi(w_{2,app}, v^{E,\nu}_{app})(\xi,t)\right| \le C_\gamma \nu t^{3/2} e^{-\gamma|\xi|^2/4} \tag{5.30}$$

*for some constant $C_\gamma$.*

*Proof.* Fix a $\gamma \in (0,1)$. Using (5.27) into (5.24), we write

$$\begin{aligned}\Phi(w_{2,\mathrm{app}}, v^{E,\nu}_{app})(\xi,t) &= \nu t^2 v_{2,a}\cdot\nabla w_{2,a} + \sqrt{\nu} t^{3/2}(\widetilde{v}^E(\widetilde{z}(t)+\xi\sqrt{\nu t},t) - \widetilde{v}^E(\widetilde{z}(t),t))\cdot\nabla w_{2,a} \\ &\quad + \nu t^2 \partial_t w_{2,a} + t(R_1(\xi,t) - A_0(\xi,t)) \\ &=: \sum_{i=1}^4 \Phi_i(\xi,t).\end{aligned}$$

Let us estimate each term on the right. Using Proposition 5.2 and Lemma 5.6, we get

$$|\Phi_1(\xi,t)| \le \nu t^2 \|v_{2,a}(t)\|_{L^\infty} |\nabla w_{2,a}(\xi,t)| \lesssim \nu t^2 e^{-\gamma|\xi|^2/4}.$$

Similarly, using Corollary 5.3, we bound

$$|\widetilde{v}^E(\xi\sqrt{\nu t}+\widetilde{z}(t),t) - \widetilde{v}^E(\widetilde{z}(t),t)| \lesssim |\xi|\sqrt{\nu t}\|\nabla\widetilde{v}^E\|_{L^\infty}$$

and hence

$$\begin{aligned}|\Phi_2(\xi,t)| &\le \sqrt{\nu} t^{3/2} |\widetilde{v}^E(\xi\sqrt{\nu t}+\widetilde{z}(t),t) - \widetilde{v}^E(\widetilde{z}(t),t)||\nabla w_{2,a}(\xi,t)| \\ &\lesssim \nu t^2 |\xi| e^{-\gamma'|\xi|^2/4} \\ &\lesssim \nu t^2 e^{-\gamma|\xi|^2/4},\end{aligned}$$

upon taking $\gamma'$ from Lemma 5.6 so that $\gamma' > \gamma$.
Next, we treat $\Phi_3(\xi,t) = \nu t^2 \partial_t w_{2,a}$. Since $\sqrt{t}\partial_t$ commutes with $\Lambda$ and $\mathcal{L}$, the equation (5.27) gives

$$(\nu(1-\mathcal{L}) + \Lambda)\,(\sqrt{t}\partial_t w_{2,a}) = -\sqrt{t}\partial_t A_0(\xi,t).$$

To apply Proposition 5.4, it suffices to prove that

$$\sqrt{t}|\partial_t A_0(\xi,t)| \lesssim |\xi|^2(1+|\xi|)e^{-|\xi|^2/4}. \tag{5.31}$$

Indeed, we recall from (5.29) that

$$\begin{cases} A_0(\xi,t) &= \int_{\mathbb{R}^2} B_0(\xi,y,t)dy \\ B_0(\xi,y,t) &= \frac{-1}{16\pi^2}|\xi|^2 e^{-|\xi|^2/4}\frac{\sin(2\psi)}{|\widetilde{z}(t)-y|^2}\widetilde{\omega}^E(y,t) + \frac{1}{16\pi^2}\sqrt{\nu t}|\xi|^3 e^{-|\xi|^2/4}\dfrac{\sin(3\psi)}{|\widetilde{z}(t)-y|^3}\widetilde{\omega}^E(y,t), \end{cases} \tag{5.32}$$

where $\psi$ is the angle between $\xi$ and $\widetilde{z}(t)-y$. By Corollary 5.3, $\widetilde{\omega}^E(t)$ and $\partial_t\widetilde{\omega}^E(t)$ are both bounded, compactly supported, and vanishing in a neighborhood of $\widetilde{z}(t)$. In particular,

$|\widetilde{z}(t) - y|$ is bounded below away from zero for $y$ in the support of $\widetilde{\omega}^E(t)$. The estimate (5.31) thus follows, upon recalling that $\partial_t \widetilde{z}(t) = \widetilde{v}^E(\widetilde{z}(t), t)$ and $\widetilde{v}^E$ is bounded (Corollary 5.3). Arguing similarly as in Lemma 5.6, we obtain

$$|\sqrt{t}\partial_t w_{2,a}(\xi, t)| \le C_\gamma e^{-\gamma|\xi|^2/4}.$$

Finally, the last term $\Phi_4(\xi, t) = t(R_1(\xi, t) - A_0(\xi, t))$ is already treated in Lemma 5.5. This concludes the proof. □

### 5.3.4 Equations for the remainder

Having introduced the approximate solutions $w_{2,\mathrm{app}}$ and $v^{E,\nu}_{\mathrm{app}}$, let us now study the remainder. Precisely, we search for solutions of (5.21) in the following form

$$\begin{cases} w_2 & = G(\xi) + (\nu t) w_{2,a} + (\nu t)\bar{w}_2 \\ v^{E,\nu} & = \widetilde{v}^E + \nu^{3/2}\bar{v}_1, \end{cases} \tag{5.33}$$

in which $\widetilde{v}^E$ and $w_{2,a}$ are constructed in the previous sections. Putting this Ansatz into (5.21), we have

$$\begin{aligned} &(t\partial_t - \mathcal{L} + 1)\bar{w}_2 + \frac{1}{\nu}\Lambda\bar{w}_2 + \sqrt{\frac{t}{\nu}}(\widetilde{v}^E - \dot{\widetilde{z}}) \cdot \nabla\bar{w}_2 + t(\bar{v}_2 \cdot \nabla w_{2,a} + v_{2,a} \cdot \nabla\bar{w}_2) \\ &+ \frac{1}{\sqrt{t}}(\bar{v}_1 \cdot \nabla G) + \nu\sqrt{t}(\bar{v}_1 \cdot \nabla w_{2,a}) + t(\bar{v}_2 \cdot \nabla\bar{w}_2) + \nu\sqrt{t}(\bar{v}_1 \cdot \nabla\bar{w}_2) + \frac{1}{\nu t}\Phi(w_{2,app}, v^{E,\nu}_{app}) = 0, \end{aligned} \tag{5.34}$$

in which we stress that $\widetilde{v}^E$ and $\bar{v}_1$ are functions of $(x, t)$, while $G, w_{2,a}$, and $\bar{w}_2$ are functions of $\xi, t$. Again, velocity and vorticity are defined through the Biot-Savart law in their respective variables.

Our goal is to derive estimates for the remainder solution $(\bar{w}_2, \bar{v}_1)$ in suitable function spaces. Precisely, we shall work with the following weighted $L^2$ norm

$$\|\omega\|^2_{L^2_p} := \int_{\mathbb{R}^2} |\omega(\xi)|^2 p(\xi)d\xi, \qquad p(\xi) = e^{|\xi|^2/4}.$$

The weight function is natural in view of the following lemma.

**Lemma 5.8.** *The operator $\mathcal{L}$ is self-adjoint in $L^2_p$, while $\Lambda$ is skew-symmetric in $L^2_p$. In particular, we have $\mathcal{L} \le 0$ and*

$$\langle \Lambda\omega, \omega\rangle_{L^2_p} = 0$$

*for any $\omega(\xi)$ in the domain of $\Lambda$.*

*Proof.* The lemma follows from a direct calculation; see [14, Lemma 4.8]. □

**Lemma 5.9** (Elliptic estimates)**.** *Let $\bar{v}_2 = K \star_\xi \bar{w}_2$ be the velocity obtained from $\bar{w}_2$ by the Biot-Savart law. There holds*

$$\|\bar{v}_2\|_{L^\infty} \lesssim \|\bar{w}_2\|_{L^2_p} + \|\bar{w}_2\|^{1/2}_{L^2_p}\|\nabla\bar{w}_2\|^{1/2}_{L^2_p}.$$

*Proof.* By Hölder inequality and Sobolev embeddings, we have

$$\begin{aligned}\|\bar{v}_2\|_{L^\infty} &\lesssim \|\bar{w}_2\|^{1/2}_{L^{4/3}}\|\bar{w}_2\|^{1/2}_{L^4} \lesssim \|\bar{w}_2\|^{1/2}_{L^2_p}\left(\|\bar{w}_2\|_{L^2_p} + \|\nabla\bar{w}_2\|_{L^2_p}\right)^{1/2} \\ &\lesssim \|\bar{w}_2\|^{1/2}_{L^2_p}\left(\|\bar{w}_2\|^{1/2}_{L^2_p} + \|\nabla\bar{w}_2\|^{1/2}_{L^2_p}\right) \\ &= \|\bar{w}_2\|_{L^2_p} + \|\bar{w}_2\|^{1/2}_{L^2_p}\|\nabla\bar{w}_2\|^{1/2}_{L^2_p}.\end{aligned}$$

The proof is complete. □

### 5.3.5 Estimates for the remainder

This section is devoted to prove the following proposition.

**Proposition 5.10.** *There are a positive constant $\kappa$ and a positive time $T$ so that*

$$\begin{aligned}t\frac{d}{dt}\|\bar{w}_2(t)\|^2_{L^2_p} + \kappa(\|(1+|\xi|)\bar{w}_2(t)\|^2_{L^2_p} + \|\nabla\bar{w}_2(t)\|^2_{L^2_p}) \\ \lesssim t\|\bar{w}_2(t)\|^5_{L^2_p} + \nu t\|\bar{v}_1(t)\|^4_{L^\infty} + t^{-1}\|\bar{v}_1(t)\|^2_{L^\infty}\end{aligned} \tag{5.35}$$

*uniformly in $\nu$ and in $t \in [0, T]$.*

The proposition follows from weighted energy estimates. To proceed, using the equation (5.34) for $t\partial_t\bar{w}_2$, we compute

$$t\frac{d}{dt}\|\bar{w}_2(t)\|^2_{L^2_p} = \int_{\mathbb{R}^2}(t\partial_t\bar{w}_2(\xi,t))\bar{w}_2(\xi,t)p(\xi)d\xi = \sum_{i=1}^{9}\mathcal{E}_i(t), \tag{5.36}$$

where

$$\begin{cases}\mathcal{E}_1(t) &= \int_{\mathbb{R}^2} p(\xi)(\mathcal{L}\bar{w}_2 - \bar{w}_2)(\xi,t)d\xi, \\ \mathcal{E}_2(t) &= -\frac{1}{\nu}\int_{\mathbb{R}^2}\Lambda\bar{w}_2(\xi,t)\bar{w}_2(\xi,t)p(\xi)d\xi, \\ \mathcal{E}_3(t) &= -\sqrt{\frac{t}{\nu}}\int_{\mathbb{R}^2}((\tilde{v}^E - \dot{\tilde{z}})\cdot\nabla\bar{w}_2)\bar{w}_2(\xi,t)p(\xi)d\xi, \\ \mathcal{E}_4(t) &= -t\int_{\mathbb{R}^2}(\bar{v}_2\cdot\nabla w_{2,a} + v_{2,a}\cdot\nabla\bar{w}_2)\bar{w}_2(\xi,t)p(\xi)d\xi, \\ \mathcal{E}_5(t) &= -t\int_{\mathbb{R}^2}\left(\bar{v}_2\cdot\nabla\bar{w}_2\right)\bar{w}_2(\xi,t)p(\xi)d\xi, \\ \mathcal{E}_6(t) &= -\nu\sqrt{t}\int_{\mathbb{R}^2}\left(\bar{v}_1\cdot\nabla\bar{w}_2\right)\bar{w}_2(\xi,t)p(\xi)d\xi, \\ \mathcal{E}_7(t) &= -\frac{1}{\nu t}\int_{\mathbb{R}^2}\Phi_{\text{app}}(\xi,t)\bar{w}_2(\xi,t)p(\xi)d\xi, \\ \mathcal{E}_8(t) &= -\frac{1}{\sqrt{t}}\int_{\mathbb{R}^2}(\bar{v}_1\cdot\nabla G)\bar{w}_2(\xi,t)p(\xi)d\xi, \\ \mathcal{E}_9(t) &= -\nu\sqrt{t}\int_{\mathbb{R}^2}\left(\bar{v}_1\cdot\nabla w_{2,a}\right)\bar{w}_2(\xi,t)p(\xi)d\xi.\end{cases}$$

Let us estimate each term $\mathcal{E}_i$. Thanks to Lemma 5.8, we have $\mathcal{E}_2(t) = 0$, while $\mathcal{E}_1(t) \le -\|\bar{w}_2(t)\|^2_{L^2_p}$. In fact, the following lemma gives a better coercive estimate for $\mathcal{E}_1(t)$.

**Lemma 5.11** (Diffusive term)**.** *There holds*

$$\mathcal{E}_1(t) \le -\frac{1}{24}\Big(\|\nabla \bar{w}_2(t)\|^2_{L^2_p} + \|(1+|\xi|)\bar{w}_2(t)\|^2_{L^2_p}\Big).$$

*Proof.* Recalling $\mathcal{L} = 1 + \frac{1}{2}\xi \cdot \nabla + \Delta$ and integrating by parts, we compute

$$\begin{aligned}
&\int_{\mathbb{R}^2} (\mathcal{L}\bar{w}_2 - \bar{w}_2)(\xi,t)p(\xi)\bar{w}_2(\xi,t)d\xi \\
&= \int_{\mathbb{R}^2} \left(\Delta \bar{w}_2 + \frac{1}{2}\xi \cdot \nabla \bar{w}_2\right) \bar{w}_2(\xi,t)p(\xi)d\xi, \\
&= -\int_{\mathbb{R}^2} |\nabla \bar{w}_2|^2 p(\xi)d\xi - \int_{\mathbb{R}^2} \bar{w}_2(\nabla p \cdot \nabla \bar{w}_2)d\xi + \frac{1}{4}\int_{\mathbb{R}^2} \Big(\xi \cdot \nabla(|\bar{w}_2|^2)\Big)\, p(\xi,t)d\xi \\
&= -\int_{\mathbb{R}^2} |\nabla \bar{w}_2|^2 p(\xi,t)d\xi - \int_{\mathbb{R}^2} \bar{w}_2(\nabla p \cdot \nabla \bar{w}_2)d\xi - \frac{1}{2}\int_{\mathbb{R}^2} |\bar{w}_2|^2 p(\xi,t)d\xi - \frac{1}{4}\int_{\mathbb{R}^2} |\bar{w}_2|^2(\xi \cdot \nabla p)d\xi.
\end{aligned}$$

The second integral is treated by

$$-\int_{\mathbb{R}^2} \bar{w}_2(\nabla p \cdot \nabla \bar{w}_2)d\xi \le \frac{3}{4}\int_{\mathbb{R}^2} |\nabla \bar{w}_2|^2 p(\xi,t) + \frac{1}{3}\int_{\mathbb{R}^2} \frac{|\nabla p|^2}{p^2}|\bar{w}_2|^2 p(\xi)d\xi.$$

Recalling now the weight function $p(\xi) = e^{|\xi|^2/4}$, we obtain the lemma at once. □

**Lemma 5.12.** *There holds*

$$\mathcal{E}_3(t) \lesssim t\|\xi \bar{w}_2(t)\|^2_{L^2_p}.$$

*Proof.* Integrating by parts and using the fact that $\widetilde{v}^E - \dot{\widetilde{z}}$ is divergence free, we have

$$\begin{aligned}
\mathcal{E}_3(t) &= -\sqrt{\frac{t}{\nu}}\int_{\mathbb{R}^2} \Big((\widetilde{v}^E - \dot{\widetilde{z}}) \cdot \nabla \bar{w}_2\Big)\, \bar{w}_2(\xi,t)p(\xi)d\xi \\
&= \frac{1}{2}\sqrt{\frac{t}{\nu}}\int_{\mathbb{R}^2} (\widetilde{v}^E - \dot{\widetilde{z}}) \cdot \nabla p(\xi)|\bar{w}_2(\xi,t)|^2 d\xi.
\end{aligned}$$

Recalling $\dot{\widetilde{z}} = \widetilde{v}^E(\widetilde{z}(t),t)$ and using Corollary 5.3, we estimate

$$|\widetilde{v}^E(\xi\sqrt{\nu t} + \widetilde{z}(t),t) - \dot{\widetilde{z}}(t)| = |\widetilde{v}^E(\xi\sqrt{\nu t} + \widetilde{z}(t),t) - \widetilde{v}^E(\widetilde{z}(t),t)| \lesssim \sqrt{\nu t}|\xi|.$$

The lemma follows, upon using $\nabla p = \frac{1}{2}\xi p(\xi)$. □

**Lemma 5.13.** *There holds*

$$\mathcal{E}_4(t) \lesssim t\left(\|\bar{w}_2(t)\|^2_{L^2_p} + \|\nabla \bar{w}_2(t)\|^2_{L^2_p}\right).$$

*Proof.* We write $\mathcal{E}_4(t) = -t\left(\mathcal{E}_{41}(t) + \mathcal{E}_{42}(t)\right)$, where

$$\begin{cases} \mathcal{E}_{41}(t) & = \int_{\mathbb{R}^2} \left(\bar{v}_2 \cdot \nabla w_{2,a}\right) \bar{w}_2(\xi,t) p(\xi) d\xi, \\ \mathcal{E}_{42}(t) & = \int_{\mathbb{R}^2} \left(v_{2,a} \cdot \nabla \bar{w}_2\right) \bar{w}_2(\xi,t) p(\xi) d\xi. \end{cases}$$

Using Hölder's inequality, we estimate

$$|\mathcal{E}_{41}(t)| \le \|\bar{v}_2(t)\|_{L^\infty} \|\bar{w}_2(t)\|_{L^2_p} \left(\int_{\mathbb{R}^2} |\nabla w_{2,a}(\xi,t)|^2 p(\xi) d\xi\right)^{1/2},$$

in which the integral is bounded by Lemma 5.6. As for $\|\bar{v}_2(t)\|_{L^\infty}$, we use the elliptic estimate and Sobolev embedding, giving

$$\|\bar{v}_2\|_{L^\infty}^2 \lesssim \|\bar{w}_2\|_{L^{4/3}} \|\bar{w}_2\|_{L^4} \lesssim \|\bar{w}_2\|_{L^{4/3}} \|\bar{w}_2\|_{L^2}^{1/2} (\|\bar{w}_2\|_{L^2} + \|\nabla \bar{w}_2\|_{L^2})^{1/2}.$$

Recalling the weight function $p = e^{|\xi|^2/4}$, we have $\|\bar{w}_2\|_{L^{4/3}} \lesssim \|\bar{w}_2\|_{L^2_p}$. Thus, we get

$$\|\bar{v}_2\|_{L^\infty}^2 \lesssim \|\bar{w}_2\|_{L^2_p}^{3/2} (\|\bar{w}_2\|_{L^2_p} + \|\nabla \bar{w}_2\|_{L^2_p})^{1/2} \lesssim \|\bar{w}_2\|_{L^2_p}^2 + \|\nabla \bar{w}_2\|_{L^2_p}^2, \tag{5.37}$$

and so

$$|\mathcal{E}_{41}(t)| \lesssim \|\bar{w}_2(t)\|_{L^2_p} (\|\bar{w}_2(t)\|_{L^2_p} + \|\nabla \bar{w}_2(t)\|_{L^2_p}) \lesssim \|\bar{w}_2(t)\|_{L^2_p}^2 + \|\nabla \bar{w}_2(t)\|_{L^2_p}^2.$$

On the other hand, the estimate on $\mathcal{E}_{42}(t)$ is direct, since $v_{2,a}$ is bounded. The lemma follows. □

**Lemma 5.14.** *There holds*

$$\mathcal{E}_5(t) \lesssim t\left(\|\bar{w}_2(t)\|_{L^2_p}^2 + \|\bar{w}_2(t)\|_{L^2_p}^5 + \|\nabla \bar{w}_2(t)\|_{L^2_p}^2\right).$$

*Proof.* By Hölder's inequality and (5.37), we get

$$\begin{aligned} |\mathcal{E}_5(t)| &= t\left| \int_{\mathbb{R}^2} \left(\bar{v}_2 \cdot \nabla \bar{w}_2\right) \bar{w}_2(\xi,t) p(\xi) d\xi \right| \\ &\le t \|\bar{v}_2(t)\|_{L^\infty} \|\bar{w}_2(t)\|_{L^2_p} \|\nabla \bar{w}_2(t)\|_{L^2_p} \\ &\lesssim t\left(\|\bar{w}_2(t)\|_{L^2_p} + \|\nabla \bar{w}_2(t)\|_{L^2_p}\right)^{1/4} \|\bar{w}_2(t)\|_{L^2_p}^{7/4} \|\nabla \bar{w}_2(t)\|_{L^2_p}. \end{aligned}$$

The lemma follows upon using Young's inequality. □

**Lemma 5.15.** *There holds*

$$\mathcal{E}_6(t) \lesssim \nu t \|\bar{v}_1(t)\|_{L^\infty}^4 + \nu t \|\bar{w}_2(t)\|_{L^2_p}^4 + \nu \|\nabla \bar{w}_2(t)\|_{L^2_p}^2.$$

*Proof.* Again by Hölder inequality, we get

$$\begin{aligned}|\mathcal{E}_6(t)| &= \nu\sqrt{t}\Big|\int_{\mathbb{R}^2}\left(\bar{v}_1\cdot\nabla\bar{w}_2\right)\bar{w}_2(\xi,t)p(\xi)d\xi\Big| \\ &\lesssim \nu t^{1/2}\|\bar{v}_1(t)\|_{L^\infty}\|\bar{w}_2(t)\|_{L^2_p}\|\nabla\bar{w}_2(t)\|_{L^2_p},\end{aligned}$$

which yields the lemma upon using Young's inequality. □

**Lemma 5.16.** *There holds*

$$\mathcal{E}_7(t) \lesssim t^{1/2}\|\bar{w}_2(t)\|_{L^2_p}.$$

*Proof.* Using the estimates from (5.30) for a fixed $\gamma \in \left(\frac{1}{2},1\right)$ and Hölder inequality, we get

$$\begin{aligned}|\mathcal{E}_7(t)| &\le (\nu t)^{-1}\int_{\mathbb{R}^2}|\Phi_{\text{app}}(\xi,t)||\bar{w}_2(\xi,t)|p(\xi)d\xi \\ &\le (\nu t)^{-1}\int_{\mathbb{R}^2}(\nu t^{3/2})C_\gamma e^{-\gamma|\xi|^2/4}|\bar{w}_2(\xi,t)|p(\xi)d\xi \\ &\le C_\gamma t^{1/2}\left(\int_{\mathbb{R}^2}e^{-2\gamma|\xi|^2/4}p(\xi)d\xi\right)^{1/2}\left(\int_{\mathbb{R}^2}|\bar{w}_2(\xi,t)|^2p(\xi)d\xi\right)^{1/2} \\ &\lesssim t^{1/2}\|\bar{w}_2(t)\|_{L^2_p},\end{aligned}$$

where we used $\gamma > 1/2$. This concludes the proof. □

**Lemma 5.17.** *There hold*

$$\mathcal{E}_8(t) \lesssim t^{-1/2}\|\bar{v}_1(t)\|_{L^\infty}\|\bar{w}_2(t)\|_{L^2_p}, \qquad \mathcal{E}_9(t) \lesssim \nu t^{1/2}\|\bar{v}_1(t)\|_{L^\infty}\|\bar{w}_2(t)\|_{L^2_p}.$$

*Proof.* We recall that

$$\mathcal{E}_8(t) = -\frac{1}{\sqrt{t}}\int_{\mathbb{R}^2}(\bar{v}_1(\xi,t)\cdot\nabla G(\xi))\bar{w}_2(\xi,t)p(\xi)d\xi$$

where $G(\xi) = \frac{1}{4\pi}e^{-|\xi|^2/4}$ and $p(\xi) = e^{|\xi|^2/4}$. We have

$$|\mathcal{E}_8(t)| \lesssim t^{-1/2}\|\bar{v}_1(t)\|_{L^\infty}\int_{\mathbb{R}^2}|\xi||\bar{w}_2(\xi,t)|d\xi \lesssim t^{-1/2}\|\bar{v}_1(t)\|_{L^\infty}\|\bar{w}_2(t)\|_{L^2_p}.$$

The proof for $\mathcal{E}_9(t)$ is identical, upon recalling the pointwise bound on $\nabla w_{2,a}$ from Lemma 5.6. □

*Proof of Proposition 5.10.* We are now ready to prove Proposition 5.10. Collecting and combining all the estimates from the previous lemmas, we get

$$\begin{aligned} &t\frac{d}{dt}\|\bar w_2(t)\|_{L^2_p}^2+\kappa(\|(1+|\xi|)\bar w_2(t)\|_{L^2_p}^2+\|\nabla\bar w_2(t)\|_{L^2_p}^2) \\ &\lesssim t\left(\|(1+|\xi|)\bar w_2(t)\|_{L^2_p}^2+\|\bar w_2(t)\|_{L^2_p}^5+\|\nabla\bar w_2(t)\|_{L^2_p}^2\right)+t^{1/2}\|\bar w_2(t)\|_{L^2_p} \\ &\quad+\nu t\|\bar v_1(t)\|_{L^\infty}^4+\nu t\|\bar w_2(t)\|_{L^2_p}^4+\nu\|\nabla\bar w_2(t)\|_{L^2_p}^2+t^{-1/2}\|\bar v_1(t)\|_{L^\infty}\|\bar w_2(t)\|_{L^2_p}, \end{aligned} \tag{5.38}$$

for $\kappa=1/24$. Taking $t$ and $\nu$ sufficiently small and using Young's inequality, we obtain

$$\begin{aligned} t\frac{d}{dt}\|\bar w_2(t)\|_{L^2_p}^2&+\frac{\kappa}{2}(\|(1+|\xi|)\bar w_2(t)\|_{L^2_p}^2+\|\nabla\bar w_2(t)\|_{L^2_p}^2) \\ &\lesssim t\|\bar w_2(t)\|_{L^2_p}^5+\nu t\|\bar v_1(t)\|_{L^\infty}^4+t^{-1}\|\bar v_1(t)\|_{L^\infty}^2. \end{aligned} \tag{5.39}$$

This completes the proof of the proposition. □

**Remark 5.18.** *The constraint on the smallness of times $T$ is precisely due to the term $\mathcal{E}_3(t)$ treated in Lemma 5.12. The remaining terms are treated using the standard Young's inequality. Hence, we in fact obtain*

$$\begin{aligned} t\frac{d}{dt}\|\bar w_2(t)\|_{L^2_p}^2&+\kappa\Big(\|\bar w_2(t)\|_{L^2_p}^2+\|\nabla\bar w_2(t)\|_{L^2_p}^2+(1-5t\|\nabla v^E(t)\|_{L^\infty})\|\xi\bar w_2(t)\|_{L^2_p}^2\Big) \\ &\lesssim t(\|\bar w_2(t)\|_{L^2_p}^2+\|\bar w_2(t)\|_{L^2_p}^5)+\nu t\|\bar v_1(t)\|_{L^\infty}^4+t^{-1}\|\bar v_1(t)\|_{L^\infty}^2 \end{aligned} \tag{5.40}$$

*for all positive times, as long as the estimates from Proposition 5.2 and Corollary 5.3 on the approximate vortex-wave solutions are valid. This yields a lower bound on the smallness of $T$ so that $\sup_{0\le t\le T}5t\|\nabla v^E(t)\|_{L^\infty}\le 1$.*

**Remark 5.19.** *One may try to improve the time interval by introducing a new weight function, as done similarly in [13], $p_{new}(\xi)=p(\xi)(1+\nu t q(\xi,t))$, where $q(\xi,t)$ solves*

$$v^G(\xi)\cdot\nabla_\xi q=\frac{1}{\sqrt{\nu t}}\Big(v^E(z(t)+\xi\sqrt{\nu t},t)-v^E(z(t),t)\Big)\cdot\xi,$$

*whose solution is however unclear for large $\xi\sqrt{\nu t}$.*

## 5.4 Inviscid limit for the regular part

In the previous section, we have proved the apriori estimate for $\omega^{B,\nu}$ and $v^{E,\nu}$ in the weighted energy space with the re-scaled variable $\xi=\frac{x-\tilde z(t)}{\sqrt{\nu t}}$. In this section, we derive estimates on the regular vorticity component $\omega^{E,\nu}$, which solves

$$\partial_t\omega^{E,\nu}+u^\nu\cdot\nabla\omega^{E,\nu}=\nu\Delta\omega^{E,\nu} \tag{5.41}$$

with the initial data $\omega_0^E$. We write:

$$\begin{cases} \omega^{E,\nu}(t,x) & = \widetilde{\omega}^E(t,x) + \nu^{3/2}\bar{w}_1(t,x), \\ v^{E,\nu}(t,x) & = \widetilde{v}^E(t,x) + \nu^{3/2}\bar{v}_1(t,x), \\ v^{B,\nu}(t,x) & = \frac{1}{\sqrt{\nu t}} v^G\left(\frac{x-\widetilde{z}(t)}{\sqrt{\nu t}}\right) + \sqrt{\nu t}(v_{2,a}+\bar{v}_2)\left(\frac{x-\widetilde{z}(t)}{\sqrt{\nu t}}, t\right), \\ u^\nu(t,x) & = v^{E,\nu}(t,x) + v^{B,\nu}(t,x), \end{cases} \tag{5.42}$$

where $(\widetilde{z}(t), \widetilde{\omega}^E)$ is the solution to the viscous vortex-wave system introduced in Section 5.2, while $v^G$ and $v_{2,a}$ are constructed in Section 5.3. Here, we note that the form of the common velocity $u^\nu(t,x)$ is compatible with the form in (5.33) and (5.20) in the scaled variable $\xi$. The velocity $\bar{v}_2$ is kept the same as in the previous section, with $\xi$ is replaced by $\frac{x-\widetilde{z}(t)}{\sqrt{\nu t}}$ and $\bar{v}_2 = K\star_\xi \bar{w}_2$. It is natural to work in the original variables $(x,t)$ instead of $(\xi, t)$, since $\omega^{E,\nu}(t)$ solves (5.41) with regular initial data $\omega_0^E$. Hence one does not expect $\omega^{E,\nu}$ to have the localized behavior near the point vortex. Roughly speaking, we want to get an apriori bound on $\|\bar{v}_1(t)\|_{L^\infty}$ (in terms of $\bar{w}_2(t)$) on a time interval independent of $\nu$. Precisely, we shall prove the following proposition.

**Proposition 5.20.** *Let $\bar{w}_1$ solve the equations* (5.41) *and* (5.42)*. There exists a positive time $T$, independent of $\nu > 0$, such that*

$$\|\bar{w}_1(t)\|_{L^4\cap L^{4/3}} \lesssim \int_0^t s^{3/2}(\|\bar{w}_2(t)\|_{L^2_p} + \|\nabla\bar{w}_2(t)\|_{L^2_p})ds + \nu^{1/2}t$$

*for $t \in [0,T]$.*

### 5.4.1 Equations for the remainder

In this subsection, we derive the equations for the remainder $\bar{w}_1$ as well as $\bar{v}_2$ appearing in (5.41) and (5.42). Putting the Ansatz (5.42) into equation (5.41) and using equation (5.11), we obtain the following transport-diffusion equation for $\bar{w}_1$:

$$\partial_t \bar{w}_1 + u^\nu \cdot \nabla\bar{w}_1 - \nu\Delta\bar{w}_1 = f(x,t),$$

where $f(x,t)$ are given by

$$\begin{aligned} f(x,t) = &-\frac{1}{\nu\sqrt{t}}\left(v^G\left(\frac{x-\widetilde{z}(t)}{\sqrt{\nu t}}\right) - v^G\left(\frac{x-z(t)}{\sqrt{\nu t}}\right)\right)\cdot\nabla w_{1,a} - \bar{v}_1\cdot\nabla\widetilde{\omega}^E - \frac{\sqrt{t}}{\nu}\bar{v}_2\cdot\nabla\widetilde{\omega}^E \\ &- \sqrt{\nu}(v_{1,a}\cdot\nabla w_{1,a}) + \frac{1}{2\pi\nu^{3/2}}\frac{(x-z(t))^\perp}{|x-z(t)|^2}e^{-\frac{|x-z(t)|^2}{4\nu t}}\cdot\nabla\omega^E + \sqrt{\nu}\Delta w_{1,a}. \end{aligned} \tag{5.43}$$

### 5.4.2 Estimating the forcing term $f(x,t)$

In this subsection, we prove the following proposition

**Proposition 5.21.** *Let $f(x,t)$ be defined as in* (5.43)*. There holds*

$$\|f(t)\|_{L^4\cap L^{4/3}} \lesssim \|\bar{w}_1(t)\|_{L^4\cap L^{4/3}} + t^{3/2}\left(\|\bar{w}_2(t)\|_{L^2_p} + \|\nabla \bar{w}_2(t)\|_{L^2_p}\right) + \sqrt{\nu}.$$

We will give a proof at the end of this subsection, after proving some useful lemmas. First, let us write $f$ as:

$$f(x,t) = f_1(x,t) + f_2(x,t) + f_3(x,t)$$

where

$$\begin{cases} f_1(x,t) &= -\frac{1}{\nu\sqrt{t}}\left(v^G\left(\frac{x-\widetilde{z}(t)}{\sqrt{\nu t}}\right) - v^G\left(\frac{x-z(t)}{\sqrt{\nu t}}\right)\right)\cdot \nabla w_{1,a} - \sqrt{\nu}(v_{1,a}\cdot\nabla w_{1,a}) \\ &\quad + \frac{1}{2\pi\nu^{3/2}}\frac{(x-z(t))^\perp}{|x-z(t)|^2}e^{-\frac{|x-z(t)|^2}{4\nu t}}\cdot\nabla\omega^E + \sqrt{\nu}\Delta w_{1,a}, \\ f_2(x,t) &= -\bar{v}_1\cdot\nabla\widetilde{\omega}^E, \\ f_3(x,t) &= -\frac{\sqrt{t}}{\nu}\bar{v}_2\cdot\nabla\widetilde{\omega}^E, \end{cases}$$

In what follows, we bound $\|f_i(t)\|_{L^4\cap L^{4/3}}$ for each $i\in\{1,2,3\}$.

**Lemma 5.22.** *There holds*

$$\|f_1(t)\|_{L^4\cap L^{4/3}} \lesssim \sqrt{\nu}$$

*uniformly in $\nu > 0$.*

*Proof.* First we see that

$$\left\| -\sqrt{\nu}(v_{1,a}\cdot\nabla w_{1,a}) - \frac{1}{2\pi\nu^{3/2}}\frac{(x-z(t))^\perp}{|x-z(t)|^2}e^{-\frac{|x-z(t)|^2}{4\nu t}}\cdot\nabla\omega^E + \sqrt{\nu}\Delta\omega_{1,a}\right\|_{L^4\cap L^{4/3}} \lesssim \sqrt{\nu}$$

thanks to the fact that $\omega^E$ is supported away from $z(t)$ and $\widetilde{z}(t)$, and $w_{1,a}$ is bounded in $W^{2,4}$, by Proposition 5.2. Now for the first term in $f_1$, it suffices to prove that

$$\frac{1}{\sqrt{\nu t}}\left|v^G\left(\frac{x-\widetilde{z}(t)}{\sqrt{\nu t}}\right) - v^G\left(\frac{x-z(t)}{\sqrt{\nu t}}\right)\right| \lesssim \nu t \qquad \text{for all} \quad x\in\operatorname{supp}(w_{1,a}). \tag{5.44}$$

As long as the above claim is proved, we would get

$$\begin{aligned} &\left\|\frac{1}{\nu\sqrt{t}}\left(v^G\left(\frac{x-\widetilde{z}(t)}{\sqrt{\nu t}}\right) - v^G\left(\frac{x-z(t)}{\sqrt{\nu t}}\right)\right)\cdot\nabla w_{1,a}\right\|_{L^4\cap L^{4/3}} \\ &\quad\lesssim \sqrt{\nu}\|\nabla w_{1,a}(t)\|_{L^4\cap L^{4/3}(\operatorname{supp}(w_{1,a}))} \lesssim \sqrt{\nu} \end{aligned}$$

by Proposition 5.2.

Now we shall prove the inequality (5.44). To this end, let us denote

$$\eta_1 = x - \widetilde{z}(t), \qquad \text{and} \quad \eta_2 = x - z(t) \tag{5.45}$$

The left hand side of (5.44) can be re-written as:

$$\frac{1}{\sqrt{\nu t}}\left(v^G(\frac{\eta_1}{\sqrt{\nu t}}) - v^G(\frac{\eta_2}{\sqrt{\nu t}})\right) = \frac{1}{2\pi}\left(V_1(\eta_1,\eta_2) + V_2(\eta_1,\eta_2)\right) \tag{5.46}$$

where

$$\begin{cases} V_1(\eta_1,\eta_2) &= \left(\frac{\eta_1^\perp}{|\eta_1|^2} - \frac{\eta_2^\perp}{|\eta_2|^2}\right), \\ V_2(\eta_1,\eta_2) &= \left(\frac{\eta_2^\perp}{|\eta_2|^2}e^{-|\eta_2|^2/4\nu t} - \frac{\eta_1^\perp}{|\eta_1|^2}e^{-|\eta_1|^2/4\nu t}\right). \end{cases}$$

When $x \in \mathrm{supp}(\widetilde{\omega}^E(t))$, by the properties established in Section 5.2, we have a positive constant $c_T$, independent of $\nu$, such that

$$|x - z(t)| \geq c_T \qquad \text{and} \qquad |x - \widetilde{z}(t)| \geq c_T \qquad \forall t \in [0,T]. \tag{5.47}$$

This implies that $|\eta_1| \geq c_T$ and $|\eta_2| \geq c_T$, upon recalling the notations (5.45). Thus, we get

$$\begin{aligned} |V_1(\eta_1,\eta_2)| &= \left|\frac{\eta_1^\perp}{|\eta_1|^2} - \frac{\eta_2^\perp}{|\eta_2|^2}\right| \leq \left|\frac{\eta_1^\perp}{|\eta_1|^2} - \frac{\eta_2^\perp}{|\eta_1|^2}\right| + \left|\frac{\eta_2^\perp}{|\eta_1|^2} - \frac{\eta_2^\perp}{|\eta_2|^2}\right| \\ &\leq \frac{|\eta_1 - \eta_2|}{|\eta_1|^2} + |\eta_2|\frac{\left||\eta_2|^2 - |\eta_1|^2\right|}{|\eta_1|^2|\eta_2|^2} \\ &\leq c_T^{-2}|\eta_1 - \eta_2| + \frac{1}{|\eta_1|^2|\eta_2|}\left||\eta_2| - |\eta_1|\right|(|\eta_1| + |\eta_2|) \lesssim |\eta_1 - \eta_2| \\ &= |(x - \widetilde{z}(t)) - (x - z(t))| = |\widetilde{z}(t) - z(t)| \lesssim \nu t \qquad \text{(by the estimate (5.14)).} \end{aligned}$$

Hence

$$|V_1(\eta_1,\eta_2)| \lesssim \nu t. \tag{5.48}$$

Now for $V_2(\eta_1,\eta_2)$, note that we shall only consider $x \in \mathrm{supp}(\widetilde{\omega}^E(t))$, in which we get (5.47). In this case we get

$$|V_2(\eta_1,\eta_2)| \leq |\eta_2|^{-1}e^{-|\eta_2|^2/4\nu t} + |\eta_1|^{-1}e^{-|\eta_1|^2/4\nu t} \leq 2c_T^{-1}e^{-c_T^2/4\nu t} \lesssim \nu t. \tag{5.49}$$

Combining (5.46), (5.48) and (5.49), we get the desired inequality (5.44). The bound for the first term is complete. This concludes the proof. □

**Lemma 5.23.** *There holds*

$$\|f_2(t)\|_{L^4\cap L^{4/3}} \lesssim \|\bar{w}_1(t)\|_{L^4\cap L^{4/3}}$$

*Proof.* We have

$$\|f_2(t)\|_{L^4\cap L^{4/3}} = \|\bar{v}_1(t)\cdot\nabla\widetilde{\omega}^E(t)\|_{L^4\cap L^{4/3}} \leq \|\bar{v}_1(t)\|_{L^\infty}\|\nabla\widetilde{\omega}^E(t)\|_{L^4\cap L^{4/3}} \lesssim \|\bar{w}_1(t)\|_{L^4\cap L^{4/3}}$$

by Corollary 5.3 and Lemma 5.26. The proof is complete. □

**Lemma 5.24.** *There holds*

$$\|f_3(t)\|_{L^4\cap L^{4/3}} \lesssim t^{3/2}\left(\|\bar{w}_2(t)\|_{L^2_p} + \|\nabla \bar{w}_2(t)\|_{L^2_p}\right).$$

*Proof.* We recall that

$$f_3(x,t) = \frac{\sqrt{t}}{\nu}\bar{v}_2(\xi,t)\cdot\nabla\widetilde{\omega}^E(t,x), \qquad \xi = \frac{x-\widetilde{z}(t)}{\sqrt{\nu t}}.$$

We shall only consider $x \in \operatorname{supp}(\widetilde{\omega}^E(t))$. Since $\widetilde{\omega}^E(t)$ is supported away from $\widetilde{z}(t)$, there exists $d_T > 0$ such that

$$|x-\widetilde{z}(t)| \geq d_T \qquad \text{for} \quad x \in \operatorname{supp}(\widetilde{\omega}^E(t)). \tag{5.50}$$

Since $\int_{\mathbb{R}^2}\bar{w}_2(\xi,t)d\xi = 0$, by Lemma (5.26), we get

$$\begin{aligned}\|(1+|\xi|^2)\bar{v}_2(t)\|_{L^\infty} &\lesssim \|(1+|\xi|^2)\bar{w}_2(t)\|_{L^4} + \|(1+|\xi|^2)\bar{w}_2(t)\|_{L^{4/3}} \\ &\lesssim \|\bar{w}_2(t)\|_{L^2_p} + \|\nabla\bar{w}_2(t)\|_{L^2_p}.\end{aligned}$$

This implies that, for $x$ in the support of $\widetilde{\omega}^E(t)$, we get

$$|\bar{v}_2(t,\xi)| \lesssim \frac{1}{1+|\xi|^2}\left(\|\bar{w}_2(t)\|_{L^2_p} + \|\nabla\bar{w}_2(t)\|_{L^2_p}\right) \lesssim (\nu t)\left(\|\bar{w}_2(t)\|_{L^2_p} + \|\nabla\bar{w}_2(t)\|_{L^2_p}\right).$$

Thus we get

$$\|f_3(t)\|_{L^4\cap L^{4/3}} \lesssim \frac{\sqrt{t}}{\nu}\|\bar{v}_2(\xi,t)\cdot\nabla\widetilde{\omega}^E(t,x)\|_{L^4\cap L^{4/3}} \lesssim t^{3/2}\left(\|\bar{w}_2(t)\|_{L^2_p} + \|\nabla\bar{w}_2(t)\|_{L^2_p}\right).$$

The proof is complete. □

We conclude this subsection by proving the Proposition 5.21

*Proof of Proposition 5.21.* The proof follows as a direct consequence of the previous lemmas for $f_i, \quad i \in \{1,2,3\}$ in this subsection. □

### 5.4.3 Apriori estimates for the remainder

In this section, we give a proof for our main Theorem 5.20, stated at the beginning of this subsection. We recall from Section 5.4.1 that $\bar{w}_1$ solves the heat transport equation

$$\partial_t\bar{w}_1 + u^\nu\cdot\nabla\bar{w}_1 - \nu\Delta\bar{w}_1 = f(x,t).$$

A standard $L^4 \cap L^{4/3}$ estimate for the heat transport equation yields

$$\begin{aligned}\frac{d}{dt}\left(\|\bar{w}_1(t)\|_{L^4\cap L^{4/3}}\right) &\lesssim \|f(t)\|_{L^4\cap L^{4/3}}\\ &\lesssim \|\bar{w}_1(t)\|_{L^4\cap L^{4/3}} + t^{3/2}\left(\|\bar{w}_2(t)\|_{L^2_p} + \|\nabla\bar{w}_2(t)\|_{L^2_p}\right) + \sqrt{\nu},\end{aligned}$$

using Proposition 5.21. Now applying Gronwall lemma for the above inequality, we have

$$\begin{aligned}\|\bar{w}_1(t)\|_{L^4\cap L^{4/3}} &\lesssim \int_0^t \left(s^{3/2}(\|\bar{w}_2(t)\|_{L^2_p} + \|\nabla\bar{w}_2(t)\|_{L^2_p}) + \sqrt{\nu}\right) ds\\ &\lesssim \int_0^t s^{3/2}(\|\bar{w}_2(t)\|_{L^2_p} + \|\nabla\bar{w}_2(t)\|_{L^2_p}) ds + \nu^{1/2} t.\end{aligned} \tag{5.51}$$

The proof is complete.

## 5.5 Proof of the inviscid limit

In this section, we conclude the proof for inviscid limit, using the apriori estimates obtained from the previous sections. Let us first prove the following proposition, before proving our main theorem, stated in the first part of this paper.

**Proposition 5.25.** *There exists a time $T > 0$, independent of the viscosity $\nu$, such that*

$$\sup_{0\le t\le T}\left(\|\bar{w}_2(t)\|_{L^2_p} + \|\bar{w}_1(t)\|_{L^4\cap L^{4/3}}\right) \lesssim 1,$$

*uniformly in $\nu$.*

*Proof.* First, we recall the following estimates for $\|\bar{w}_2(t)\|_{L^2_p}$ and $\|\bar{w}_1(t)\|_{L^4\cap L^{4/3}}$ proven in Propositions 5.10 and 5.20.

$$\begin{aligned}&\frac{d}{dt}\|\bar{w}_2(t)\|^2_{L^2_p} + \frac{\kappa}{t}(\|(1+|\xi|)\bar{w}_2(t)\|^2_{L^2_p} + \|\nabla\bar{w}_2(t)\|^2_{L^2_p}) \lesssim \|\bar{w}_2(t)\|^5_{L^2_p} + \nu\|\bar{v}_1(t)\|^4_{L^\infty} + t^{-2}\|\bar{v}_1(t)\|^2_{L^\infty}\\ &\|\bar{w}_1(t)\|_{L^4\cap L^{4/3}} \lesssim \int_0^t s^{3/2}\left(\|\bar{w}_2(t)\|_{L^2_p} + \|\nabla\bar{w}_2(t)\|_{L^2_p}\right) ds + \nu^{1/2} t.\end{aligned} \tag{5.52}$$

Let

$$\mathcal{G}(t) = \|\bar{w}_2(t)\|^2_{L^2_p} + \int_0^t s^{-1}(\|\bar{w}_2(s)\|^2_{L^2_p} + \|\nabla\bar{w}_2(s)\|^2_{L^2_p}) ds.$$

From the inequality (5.52), it is straight-forward that

$$\|\bar{w}_1(t)\|_{L^4\cap L^{4/3}} \lesssim t^{5/2}\mathcal{G}(t)^{1/2} + \nu^{1/2} t. \tag{5.53}$$

Thus, we have

$$\begin{aligned}
\mathcal{G}'(t) &= \frac{d}{dt}\|\bar{w}_2(t)\|^2_{L^2_p} + t^{-1}\left(\|\bar{w}_2(t)\|^2_{L^2_p} + \|\nabla\bar{w}_2(t)\|^2_{L^2_p}\right) \\
&\lesssim \|\bar{w}_2(t)\|^5_{L^2_p} + \nu\|\bar{v}_1(t)\|^4_{L^\infty} + t^{-2}\|\bar{v}_1(t)\|^2_{L^\infty} \qquad \text{(by (5.52))} \\
&\lesssim \mathcal{G}(t)^{5/2} + \nu\|\bar{w}_1(t)\|^4_{L^4\cap L^{4/3}} + t^{-2}\|\bar{w}_1(t)\|^2_{L^4\cap L^{4/3}} \\
&\lesssim \mathcal{G}(t)^{5/2} + \nu\left(t^{5/2}\mathcal{G}(t)^{1/2} + \nu^{1/2}t\right)^4 + t^{-2}\left(t^{5/2}\mathcal{G}(t)^{1/2} + \nu^{1/2}t\right)^2 \qquad \text{(by (5.53))} \\
&\lesssim \mathcal{G}(t)^{5/2} + \nu t^{10}\mathcal{G}(t)^2 + \nu^3 t^4 + t^3\mathcal{G}(t) + \nu.
\end{aligned}$$

By standard ODE theory, we have a time $T > 0$, which is independent of $\nu > 0$, such that $\mathcal{G}(t)$ is uniformly bounded for $t \in [0, T]$. Since $\mathcal{G}(t) \geq \|\bar{w}_2(t)\|_{L^2_p}$, the proof for $\|\bar{w}_2(t)\|_{L^2_p}$ is complete. The bound $\|\bar{w}_1(t)\|_{L^4\cap L^{4/3}} \lesssim 1$ follows from the inequality (5.53). □

We conclude this section by proving our main theorem, stated in the first part of this paper.

*Proof of Theorem 5.1.2.* We have proved that $\|\bar{w}_2(t)\|_{L^2_p}$ is uniformly bounded in $\nu$. We recall from Section 5.3 that

$$\omega^{B,\nu}(t,x) = \frac{1}{\nu t}w_2(\xi,t) = \frac{1}{\nu t}\left(G(\xi) + (\nu t)w_{2,a} + (\nu t)\bar{w}_2\right) = \frac{1}{\nu t}G(\xi) + w_{2,a} + \bar{w}_2,$$

where $G(\xi) = \frac{1}{4\pi}e^{-|\xi|^2/4}$ and $\xi = (x - \widetilde{z}(t))/\sqrt{\nu t}$. We compute

$$\begin{aligned}
\left\|\omega^{B,\nu}(t,x) - \frac{1}{4\pi\nu t}e^{-\frac{|x-\widetilde{z}(t)|^2}{4\nu t}}\right\|_{L^1_x} &= \left\|w_{2,a}(\xi,t) + \bar{w}_2(\xi,t)\right\|_{L^1_x} \\
&= \nu t\|w_{2,a}(t) + \bar{w}_2(t)\|_{L^1_\xi} \lesssim (\nu t)\left(\|w_{2,a}(t)\|_{L^2_p} + \|\bar{w}_2(t)\|_{L^2_p}\right) \\
&\lesssim (\nu t).
\end{aligned} \tag{5.54}$$

For simplicity of notations, we denote by $G_{\widetilde{z}(t)}(x)$ and $G_{z(t)}(x)$ the Gaussians $\frac{1}{4\pi\nu t}e^{-\frac{|x-\widetilde{z}(t)|^2}{4\nu t}}$ and $\frac{1}{4\pi\nu t}e^{-\frac{|x-z(t)|^2}{4\nu t}}$, respectively. Our goal now is to compare the two Gaussians in $L^1$ norm. To this end, let us denote $A = \frac{|x-\widetilde{z}(t)|^2}{4\nu t}$ and $B = \frac{|x-z(t)|^2}{4\nu t}$. We have

$$G_{\widetilde{z}(t)}(x) - G_{z(t)}(x) = e^{-A} - e^{-B} = e^{-B}\left(e^{B-A} - 1\right).$$

We have

$$
\begin{aligned}
B - A &= (4\nu t)^{-1}\left(|x - z(t)|^2 - |x - \widetilde{z}(t)|^2\right) = (4\nu t)^{-1}\left(2x\cdot(\widetilde{z}(t) - z(t)) + |z(t)|^2 - |\widetilde{z}(t)|^2\right)\\
&\lesssim (4\nu t)^{-1}\left(|x||\widetilde{z}(t) - z(t)| + |\widetilde{z}(t) - z(t)|\right)\\
&\lesssim |x| + 1 \qquad (\text{since} \quad |\widetilde{z}(t) - z(t)| \lesssim \nu t)\\
&\leq |x - z(t)| + |z(t)| + 1 \lesssim \frac{|x - z(t)|}{\sqrt{\nu t}} + 1.
\end{aligned}
$$

Here we used the standard fact of the vortex-wave system that $|z(t)| \lesssim 1$ for any fixed interval of time. For, one can see that $|z(t)| \leq |z_0| + \int_0^t |v^E(z(s), s)|ds \leq |z_0| + t\|v^E\|_{L^\infty}$. Hence we get

$$
|G_{\widetilde{z}(t)}(x) - G_{z(t)}(x)| \lesssim e^{-\frac{|x-z(t)|^2}{4\nu t} + M_T\frac{|x-z(t)|}{\sqrt{\nu t}}} \qquad \text{for some} \quad M_T > 0. \tag{5.55}
$$

Integrating both sides of the inequality (5.55) in $x \in \mathbb{R}^2$, we have

$$
\|G_{z(t)} - G_{\widetilde{z}(t)}\|_{L^1_x} \lesssim \int_{\mathbb{R}^2} e^{-\frac{|x-z(t)|^2}{4\nu t} + M_T\frac{|x-z(t)|}{\sqrt{\nu t}}}dx.
$$

Making the change of variables $y = \frac{x - z(t)}{\sqrt{\nu t}}$ in the above integral, we thus obtain

$$
\|G_{z(t)} - G_{\widetilde{z}(t)}\|_{L^1_x} \lesssim \nu t. \tag{5.56}
$$

Combining the inequalities (5.54) and (5.56), we get

$$
\left\|\omega^{B,\nu}(t,x) - \frac{1}{4\pi\nu t}e^{-\frac{|x-z(t)|^2}{4\nu t}}\right\|_{L^1_x} \lesssim \nu t.
$$

The inequality $\|\omega^{E,\nu}(t) - \omega^E(t)\|_{L^4\cap L^{4/3}} \lesssim \nu$ follows directly from the expansion (5.42), the inequality (5.53) and the uniform bound of $\mathcal{G}(t)$. The proof is complete. □

## 5.6 Appendix

In this section, we collect several useful lemmas used in this section.

**Lemma 5.26** (Elliptic estimates)**.** *Let $v = K \star \omega$ be the velocity vector field obtained from the vorticity $\omega$ on $\mathbb{R}^2$. Define the norm $\|\cdot\|_{L^4\cap L^{4/3}} = \|\cdot\|_{L^4} + \|\cdot\|_{L^{4/3}}$. There hold the following inequalities*

$$
\|v\|_{L^\infty} \lesssim \|\omega\|_{L^4\cap L^{4/3}}, \qquad \|v\|_{L^\infty} \lesssim \|\omega\|_{L^1\cap L^\infty}.
$$

*Moreover, if $\int_{\mathbb{R}^2}\omega(x)dx = 0$, then*

$$
\|(1 + |x|^2)v\|_{L^\infty} \lesssim \|(1 + |x|^2)\omega\|_{L^4\cap L^{4/3}}.
$$

*Proof.* From the Biot-Savart law which relates $v$ and $\omega$, we have

$$
\begin{aligned}
|v(x)| &\lesssim \int_{\mathbb{R}^2} \frac{|\omega(y)|}{|x-y|}dy = \left(\int_{|x-y|\le R} + \int_{|x-y|\ge R}\right) \frac{|\omega(y)|}{|x-y|}dy \\
&\lesssim \left(\int_{|x-y|\le R} |x-y|^{-4/3}dy\right)^{3/4} \|\omega\|_{L^4} + \left(\int_{|x-y|\ge R} |x-y|^{-4}dy\right)^{1/4} \|\omega\|_{L^{4/3}} \\
&\lesssim R^{1/2}\|\omega\|_{L^4} + R^{-1/2}\|\omega\|_{L^{4/3}}.
\end{aligned} \tag{5.57}
$$

Thus choosing $R = \frac{\|\omega\|_{L^{4/3}}}{\|\omega\|_{L^4}}$, we have $\|v\|_{L^\infty} \lesssim \|\omega\|_{L^{4/3}}^{1/2}\|\omega\|_{L^4}^{1/2}$, which gives the first inequality. As for the second, we use $\|\omega\|_{L^p} \le \|\omega\|_{L^1}^{1/p}\|\omega\|_{L^\infty}^{1-1/p}$.

It remains to check the last inequality. We shall check it for $v_2$, the second component of $v$. The estimate on $v_1$ is similar. First, we check

$$
|x||v_2(x)| \lesssim \int_{\mathbb{R}^2} \frac{1}{|x-y|}|y||\omega(y)|dy. \tag{5.58}
$$

By Biot-Savart law and $\int_{\mathbb{R}^2} \omega(y)dy = 0$, we have

$$
|v_2(x)| = \frac{1}{2\pi}\left|\int_{\mathbb{R}^2} \frac{x_1-y_1}{|x-y|^2}\omega(y)dy\right| \le \frac{1}{2\pi}\int_{\mathbb{R}^2}\left|\frac{x_1-y_1}{|x-y|^2} - \frac{x_1}{|x|^2}\right||\omega(y)|dy.
$$

Now we have

$$
\frac{x_1-y_1}{|x-y|^2} - \frac{x_1}{|x|^2} = \frac{1}{|x|^2|x-y|^2}\left(|x|^2(x_1-y_1) - x_1|x-y|^2\right).
$$

It follows that $|x|^2(x_1-y_1) - x_1|x-y|^2 \le 4|x||y||x-y|$. Hence,

$$
|x|\left[\frac{x_1-y_1}{|x-y|^2} - \frac{x_1}{|x|^2}\right] \le \frac{4|y|}{|x-y|},
$$

which gives (5.58). Now multiplying both sides of (5.58) by $|x|$, we have

$$
\begin{aligned}
|x|^2|v_2(x)| &\lesssim \int_{\mathbb{R}^2} \frac{|x||y|}{|x-y|}|\omega(y)|dy \le \int_{\mathbb{R}^2} \frac{|y|+|x-y|}{|x-y|}|y||\omega(y)|dy. \\
&= \int_{\mathbb{R}^2} \frac{1}{|x-y|}|y|^2|\omega(y)|dy + \int_{\mathbb{R}^2} |y||\omega(y)|dy
\end{aligned}
$$

Let us first treat the first term in the above. Repeating the argument of (5.57) for $\omega = |y|^2|\omega(y)|$, we have

$$
\int_{\mathbb{R}^2} \frac{1}{|x-y|}|y|^2|\omega(y)|dy \lesssim \|(1+|y|^2)\omega(y)\|_{L^4\cap L^{4/3}}.
$$

For the second term, using Hölder inequality, we get

$$\int_{\mathbb{R}^2} |y||\omega(y)|dy = \int_{\mathbb{R}^2} \frac{|y|}{1+|y|^2}(1+|y|^2)|\omega(y)|dy \lesssim \|(1+|y|^2)|\omega(y)|\|_{L^{4/3}}.$$

Thus

$$|x|^2|v_2(x)| \lesssim \|(1+|x|^2)\omega\|_{L^4\cap L^{4/3}}.$$

The lemma follows. □

**Lemma 5.27.** *Let $z_1, z_2 \in \mathbb{C}$ and $\psi$ be the angle between $z_1$ and $z_2$. Assuming that $|z_1| < |z_2|$ and $\sin(\psi) \neq 0$, there holds*

$$\frac{1}{|z_1+z_2|^2} - \frac{1}{|z_2|^2} = \frac{1}{|z_2|^2}\sum_{n=1}^{\infty}(-1)^n \frac{|z_1|^n}{|z_2|^n}\frac{\sin((n+1)\psi)}{\sin(\psi)}.$$

*Proof.* Let $\frac{z_1}{z_2} = z = re^{i\psi}$. We have

$$\frac{1}{|z_1+z_2|^2} - \frac{1}{|z_2|^2} = \frac{1}{|z_2|^2}\left(\frac{1}{|1+z|^2} - 1\right).$$

Now for $|z| < 1$, we have

$$\begin{aligned}\frac{1}{|1+z|^2} &= \frac{1}{(1+z)(1+\bar{z})} = (1 - z + z^2 - \cdots)(1 - \bar{z} + \bar{z}^2 + \cdots)\\ &= 1 - (z+\bar{z}) + (z^2 + z\bar{z} + \bar{z}^2) - (z^3 + z^2\bar{z} + z\bar{z}^2 + \bar{z}^3) + \cdots.\end{aligned}$$

Now for each $n$, we have

$$z^n + z^{n-1}\bar{z} + \cdots + z\bar{z}^{n-1} + \bar{z}^n = \frac{z^{n+1} - \bar{z}^{n+1}}{z - \bar{z}} = r^n \frac{\sin((n+1)\psi)}{\sin\psi}.$$

This concludes the proof. □

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