ROBUST-ADAPTIVE ACTIVE VIBRATION CONTROL OF ALLOY AND FLEXIBLE MATRIX COMPOSITE ROTORCRAFT DRIVELINES VIA MAGNETIC BEARINGS: THEORY AND EXPERIMENT

A Thesis in
Mechanical Engineering
by
Hans A. DeSmidt

© 2005 Hans A. DeSmidt

Submitted in Partial Fulfillment of the Requirements for the Degree of

Doctor of Philosophy

May 2005
The thesis of Hans A. DeSmidt was reviewed and approved* by the following:

Kon-Well Wang  
William E. Diefenderfer Chaired Professor in Mechanical Engineering  
Thesis Co-Advisor  
Co-Chair of Committee

Edward C. Smith  
Professor of Aerospace Engineering  
Thesis Co-Advisor  
Co-Chair of Committee

Charles E. Bakis  
Professor of Engineering Science and Mechanics

Christopher D. Rahn  
Professor of Mechanical Engineering

A. Scott Lewis  
Dual Research Associate of Applied Research Lab and Mechanical Engineering

Richard C. Benson  
Professor of Mechanical Engineering  
Head of the Department of Mechanical Engineering

* Signatures are on file in the Graduate School.
ABSTRACT

This thesis explores the use of Active Magnetic Bearing (AMB) technology and newly emerging Flexible Matrix Composite (FMC) materials to advance the state-of-the-art of rotorcraft and other high performance driveline systems. Specifically, two actively controlled tailrotor driveline configurations are explored. The first driveline configuration (Configuration I) consists of a multi-segment alloy driveline connected by Non-Constant-Velocity (NCV) flexible couplings and mounted on non-contact AMB devices. The second configuration (Configuration II) consists of a single piece, rigidly coupled, FMC shaft supported by AMBs. For each driveline configuration, a novel hybrid robust-adaptive vibration control strategy is theoretically developed and experimentally validated based on the specific driveline characteristics and uncertainties. In the case of Configuration I, the control strategy is based on a hybrid design consisting of a PID feedback controller augmented with a slowly adapting, Multi-Harmonic Adaptive Vibration Control (MHAVC) input. Here, the control is developed to ensure robustness with respect to the driveline operating conditions e.g. driveline misalignment, load-torque, shaft speed and shaft imbalance. The analysis shows that the hybrid PID/MHAVC control strategy achieves multi-harmonic suppression of the imbalance, misalignment and load-torque induced driveline vibration over a range of operating conditions. Furthermore, the control law developed for Configuration II is based on a hybrid robust $H_\infty$ feedback/Synchronous Adaptive Vibration Control (SAVC) strategy. Here, the effects of temperature dependent FMC material properties, rotating-frame damping and shaft imbalance are considered in the control design. The analysis shows that the hybrid $H_\infty$/SAVC control strategy guarantees stability, convergence and imbalance vibration suppression under the conditions of bounded temperature deviations and unknown imbalance. Finally, the robustness and vibration suppression performance of both new AMB driveline configurations is experimentally confirmed using a frequency-scaled AMB driveline testrig specifically developed for this research.
G.1 Configuration I Experimental Results ........................................................... 243
  G.1.1 Configuration I Time-Domain Results .................................................. 244
  G.1.2 Configuration I Frequency-Domain Results ......................................... 252
G.2 Configuration II Experimental Results ...................................................... 257
  G.2.1 Configuration II Time-Domain Results ............................................... 257
  G.2.2 Configuration II Frequency-Domain Results ........................................ 261
LIST OF FIGURES

Fig. 1.1: Conventional supercritical tailrotor driveline and fuselage ......................... 2
Fig. 1.2: Common NCV coupling, 4-bolt metal disk coupling ..................................... 6
Fig. 1.3: Boeing AH-64 coupled tailrotor driveline-fuselage mode. ......................... 12
Fig. 1.4: 8-Pole attractive radial magnetic bearing .................................................. 15
Fig. 1.5: AMB-shaft digital control system .............................................................. 17
Fig. 1.6: AMB-Tailrotor driveline full state feedback optimal controller ................. 20
Fig. 1.7: Closed-loop hanger bearing loads and gap displacements in forward-flight .......................................................... 21
Fig. 1.8: Conventional and new actively controlled driveline configurations ......... 24
Fig. 2.1: Segmented, multi-coupling, driveline on flexible fuselage structure .......... 28
Fig. 2.2: Fixed-frame and rotating frame elastic deflections of shaft cross-section ... 29
Fig. 2.3: Pth NCV coupling connecting statically misaligned, flexible shafts ........... 31
Fig. 2.4: NCV Coupling: U-Joint with two Euler angles \( \alpha \) and \( \beta \) ......................... 32
Fig. 2.5: \( \{b\} \) in terms of \( i-1^{th} \) shaft kinematic variables and coupling Euler angles .... 33
Fig. 2.6: \( \{b\} \) in terms of \( i^{th} \) shaft kinematics variables and static misalignments ...... 34
Fig. 2.7: Driveline static misalignment due to aerodynamic load; side-view .......... 36
Fig. 2.8: Driveline static misalignment due to aerodynamic load; top-view .......... 37
Fig. 2.9: Aerodynamic loads on tailrotor driveline-fuselage structure .................... 45
Fig. 2.10: Cross-section of \( i^{th} \) shaft with mass imbalance ................................. 47
Fig. 2.11: Shaft geometric imperfection: initially bent shaft.

Fig. 2.12: $j^{th}$ beam-rod-torsion finite element of the $i^{th}$ structure.

Fig. 2.13: 8-pole radial AMB with bias and control currents.

Fig. 3.1: Segmented driveline connected by U-joint couplings.

Fig. 3.2: Nominal system whirl stability behavior.

Fig. 3.3: Misalignment-shaft speed instability region, $\tau = 0.0$.

Fig. 3.4: Misalignment-shaft speed instability region, $\tau = 0.5 \tau_{\text{max}}$.

Fig. 3.5: Effect of damping on misalignment induced.

Fig. 3.6: Effect of damping on torque induced bending-bending instability.

Fig. 3.7: Effect of damping, torque and misalignment on $1^{st}$ torsion-bending instability
\[ \square c_d = 0.0, \square c_d = 0.04 \text{ (a) } \tau = 0.0; \text{ (b) } \tau = 0.25 \tau_{\text{max}}; \text{ (c) } \tau = 0. \]
\[ 5 \tau_{\text{max}}; \text{ (d) } \tau = \tau_{\text{max}}. \]

Fig. 3.8: Effect of damping, torque and misalignment on $2^{nd}$ torsion-bending instability.
\[ \square c_d = 0.01, \square c_d = 0.2 \text{ (a) } \tau = 0.0; \text{ (b) } \tau = 0.25 \tau_{\text{max}}; \text{ (c) } \tau = 0. \]
\[ 5 \tau_{\text{max}}; \text{ (d) } \tau = \tau_{\text{max}}. \]

Fig. 4.1: Configuration I: segmented driveline with AMBs and NCV couplings.

Fig. 4.2: Driveline with hybrid PID/Multi-Harmonic Adaptive Vibration Control.

Fig. 4.3: Robust stability index, $S_{\text{PID}}$ vs. $k_p$ and $k_d$ with $k_l = 1000$ Amp/m-sec.

Fig. 4.4: MHAVC Convergence Metric vs. shaft speed for $w_{\text{eff}} = 0$ and $w_{\text{eff}} = 1000$;
\[ (a) [T_L = 0.5 T_{L\text{max}}, \theta = 4^\circ]; \text{ (b) } [T_L = T_{L\text{max}}, \theta = 4^\circ]. \]

Fig. 4.5: $\theta - \Omega_0$ MHAVC convergence regions for various $w_{\text{eff}}$ with $0 \leq T_L \leq T_{L\text{max}}$.

Fig. 4.6: MHAVC Robust Convergence Metric vs. MHAVC control effort weighting.

Fig. 4.7: PID-MHAVC/AMB-driveline converged response; worst-case vibration.

Fig. 4.8: PID-MHAVC/AMB-driveline response; worst-case control current.

Fig. 4.9: PID-MHAVC/AMB-driveline response; RMS shaft vibration.
Fig. 4.10: PID-MHAVC/AMB-driveline response; RMS control current. .......... 107
Fig. 4.11: PID-MHAVC/AMB-driveline vibration harmonics at AMB1.............. 108
Fig. 4.12: PID-MHAVC/AMB-driveline control current harmonics at AMB1. ....... 109
Fig. 4.13: Peak vibration amplitude at the AH-64 shaft operating speed. ............ 110
Fig. 4.14: Peak AMB control current at the AH-64 shaft operating speed............. 111
Fig. 4.15: PID-MHAVC controlled AMB-driveline system. ............................... 112
Fig. 4.16: Real-time Harmonic Fourier Coefficient calculator block.................. 112
Fig. 4.17: Multi-Harmonic Adaptive Vibration Control synthesis block. ............... 113
Fig. 4.18: PID-MHAVC controlled AMB-driveline shaft vibration response. ...... 114
Fig. 4.19: PID-MHAVC controlled AMB-driveline control current...................... 115
Fig. 4.20: Shaft vibration magnitude and phase at AMB1. ................................. 116
Fig. 4.21: Control current magnitude and phase at AMB1............................... 116
Fig. 5.1: Configuration II: one-piece FMC driveline with AMBs and rigid couplings................................................................. 122
Fig. 5.2: Hybrid feedback/SAVC controlled AMB-FMC driveline system. ............ 128
Fig. 5.3: LFT representation of feedback controlled AMB-FMC driveline. ............ 132
Fig. 5.4: RMS vibration and control currents vs. shaft speed, system at $T_n=85^{\circ}$ .. 135
Fig. 5.5: Effect of AMB location on transmission-zero critical speeds.................. 136
Fig. 5.6: Deviation temperature robustness margin vs. shaft speed for several values of SAVC control effort penalty weighting, $w_{eff}$ .................................. 137
Fig. 5.7: $J_{FB}$ and $J_{FBAVC}$ vs. shaft speed for two levels of temperature uncertainty; $[\delta T = 0]$ and $[0 \leq \delta T \leq 100^{\circ}F]$, with $w_{eff} 6.0x10^{-8}$...................................................... 139
Fig. 5.8: $J_{FB}$ and $J_{FBAVC}$ vs. shaft speed for two levels of temperature uncertainty; $[\delta T = 0]$ and $[0 \leq \delta T \leq 100^{\circ}F]$, with $w_{eff} 2.6x10^{-7}$.............................................. 139
Fig. 6.1: Photo of tailrotor driveline-fuselage testrig in Conventional
Configuration......................................................................................... 144
Fig. 6.2: Photo of tailrotor driveline-fuselage testrig in Configuration I ...................... 145
Fig. 6.3: Testrig diagram in Conventional Configuration ........................................... 145
Fig. 6.4: Viscous damper assembly for Conventional Configuration, 1 of 2 .......... 146
Fig. 6.5: Fiber-optic displacement probe sensor pair, 1 of 2 ............................... 147
Fig. 6.6: End-load assembly ................................................................................ 148
Fig. 6.7: Adjustable static driveline misalignment condition ............................. 150
Fig. 6.8: Shaker attached to foundation-beam for external excitations .......... 151
Fig. 6.9: Testrig 8-pole radial AMB, AMB-rotor and backup-bearing .............. 152
Fig. 6.10: Photo of testrig driveline in Configuration I ........................................ 153
Fig. 6.11: FMC shaft connected with rigid coupling (Configuration II) ............ 154
Fig. 7.1: Testrig setup for experimental validation of driveline analytical model ... 157
Fig. 7.2: Foundation input to shaft displacement FRF of non-rotating driveline ... 159
Fig. 7.3: U-Joint coupling “C” with 2° misalignment (for spinup Cases 3 & 4) .... 160
Fig. 7.4: Load-torque and shaft speed vs. time (spinup experiment Cases 1 - 4) 161
Fig. 7.5: Shaft displacement vs. time (spinup experiment Cases 1 & 4) ............. 162
Fig. 7.6: Spectrogram of shaft response, spinup experiment Case 4 ............... 163
Fig. 7.7: Shaft displacement at frequency 2Ω for spinup Cases 1-4, FEM & Exp .... 164
Fig. 7.8: Foundation acceleration at frequency 2Ω for spinup Cases 1-4, FEM & Exp ................................................ 164
Fig. 7.9: Experimentally implemented hybrid feedback/harmonic adaptive controller .............................................................................................................. 166
Fig. 7.10: Open-Loop Testrig AMB/driveline system .......................................... 167
Fig. 7.11: Digital PID controller for Configuration I and II feedback control loop 167
Fig. 7.12: Testrig Configuration I closed-loop control-path transfer functions .... 171
Fig. 7.13: Harmonic adaptive vibration control ..................................................... 172
Fig. 7.14: Harmonic Fourier Coefficient calculator block .......................... 172
Fig. 7.15: Harmonic AVC synthesis block .............................................. 174
Fig. 7.16: Schematic time-domain response of feedback/AVC controlled system. .... 175
Fig. 7.17: HFC estimation error vs. AVC update time for several signal-to-noise ratios ................................................................. 177
Fig. 7.18: Testrig setup for Configuration I experiment ................................ 179
Fig. 7.19: RMS vibration metric for Configuration I experiment ..................... 180
Fig. 7.20: RMS control current metric for Configuration I experiment .......... 181
Fig. 7.21: Shaft response, Configuration I experiment Case 6 ......................... 182
Fig. 7.22: Control currents, Configuration I experiment Case 6 ...................... 182
Fig. 7.23: Shaft response spectra, Configuration I experiment Case 6 .......... 183
Fig. 7.24: Testrig setup for Configuration II experiment .............................. 184
Fig. 7.25: RMS vibration metric for Configuration II experiment .................. 187
Fig. 7.26: RMS control current metric for Configuration II experiment .......... 187
Fig. 7.27: Shaft response, Configuration II experiment Case 1 ...................... 188
Fig. 7.28: Shaft response spectra, Configuration II experiment Case 1 .......... 189
Fig. 7.29: Driveline load-torque in quasi-steady operating environment .......... 192
Fig. 8.1: Misaligned FMC driveline with conventional single actuation-plane AMB .......................................................... 197
Fig. 8.2: Misaligned FMC driveline with proposed dual actuation-plane AMB .... 198
Fig. F.1: FMC shaft with in-plane force resultants ....................................... 240
Fig. F.2: FMC laminate ........................................................................ 241
Fig. G.1: Configuration I shaft vibration response, Case 1 ......................... 244
Fig. G.2: Configuration I control current, Case 1 ..................................... 245
Fig. G.3: Configuration I shaft vibration response, Case 2. ........................................245
Fig. G.4: Configuration I control current, Case 2 ......................................................246
Fig. G.5: Configuration I shaft vibration response, Case 3. ........................................246
Fig. G.6: Configuration I control current, Case 3 ......................................................247
Fig. G.7: Configuration I shaft vibration response, Case 4. ........................................247
Fig. G.8: Configuration I control current, Case 4 ......................................................248
Fig. G.9: Configuration I shaft vibration response, Case 5. ........................................248
Fig. G.10: Configuration I control current, Case 5 ......................................................249
Fig. G.11: Configuration I shaft vibration response, Case 6. ......................................249
Fig. G.12: Configuration I control current, Case 6 ......................................................250
Fig. G.13: Configuration I shaft vibration response, Case 7. ......................................250
Fig. G.14: Configuration I control current, Case 7 ......................................................251
Fig. G.15: Configuration I shaft vibration response, Case 8. ......................................251
Fig. G.16: Configuration I control current, Case 8 ......................................................252
Fig. G.17: Configuration I shaft vibration spectrum, Case 1 .......................................253
Fig. G.18: Configuration I shaft vibration spectrum, Case 2 .......................................253
Fig. G.19: Configuration I shaft vibration spectrum, Case 3 .......................................254
Fig. G.20: Configuration I shaft vibration spectrum, Case 4 .......................................254
Fig. G.21: Configuration I shaft vibration spectrum, Case 5 .......................................255
Fig. G.22: Configuration I shaft vibration spectrum, Case 6 .......................................255
Fig. G.23: Configuration I shaft vibration spectrum, Case 7 .......................................256
Fig. G.24: Configuration I shaft vibration spectrum, Case 8 .......................................256
Fig. G.25: Configuration II shaft vibration response, Case 1 .....................................258
Fig. G.26: Configuration II control current, Case 1 ................................................................. 258
Fig. G.27: Configuration II shaft vibration response, Case 2 .................................................. 259
Fig. G.28: Configuration II control current, Case 2 ................................................................. 259
Fig. G.29: Configuration II shaft vibration response, Case 3 .................................................. 260
Fig. G.30: Configuration II control current, Case 3 ................................................................. 260
Fig. G.31: Configuration II shaft vibration spectrum, Case 1 ................................................. 261
Fig. G.32: Configuration II shaft vibration spectrum, Case 2 ................................................. 262
Fig. G.33: Configuration II shaft vibration spectrum, Case 3 ................................................. 263
LIST OF TABLES

Table 4.1: AH-64 Driveline Parameters ............................................................... 85
Table 4.2: AMB Parameters .................................................................................. 86
Table 4.3: AH-64 Misalignment Influence Coefficients ....................................... 87
Table 4.4: Operating Condition Uncertainty Bounds ............................................ 96
Table 4.5: Robust PID Feedback Design ............................................................... 98
Table 4.6: Nominal PID/AMB Driveline Closed-Loop Natural Frequencies ........... 99
Table 5.1: FMC Material Properties and Ply Configuration ................................. 124
Table 5.2: Equivalent Isotropic Properties ........................................................... 124
Table 5.3: FMC Driveline Parameters ................................................................ 126
Table 5.4: AMB Parameters ................................................................................. 126
Table 6.1: Motor and End-Load Assembly Characteristics ................................... 149
Table 6.2: Testrig AMB and AMB Rotor Parameters ......................................... 153
Table 6.3: Driveline-Fuselage Testrig Sensors ...................................................... 155
Table 6.4: AMB/Driveline-Fuselage Closed-Loop Control System I/O ............... 155
Table 7.1: Testrig Setup for Conventional Configuration ..................................... 158
Table 7.2: Conventional Configuration Spinup Test Cases .................................. 160
Table 7.3: Configuration I and II Digital PID Controller Parameters ................. 168
Table 7.4: AH-64 and Testrig Driveline Mass Properties ..................................... 170
Table 7.5: Configuration I and II AVC Parameters ............................................. 178
Table 7.6: Configuration I Testrig Setup ................................................................. 179
Table 7.7: Configuration I Experiment Test Cases ................................................. 180
Table 7.8: Configuration II Testrig Setup ............................................................... 185
Table 7.9: Configuration II Experiment Test Cases ............................................... 185
Table G.1: Configuration I Experiment Test Cases ................................................... 243
Table G.2: Configuration II Experiment Test Cases ............................................... 257
ACKNOWLEDGMENTS

I would like to thank Professors Kon-Well Wang and Edward C. Smith for their excellent guidance in writing this thesis and for their encouragement throughout the entire research project. Their knowledge and dedication have made this learning experience an invaluable step in my career. Appreciation is also extended to the members of my Ph.D. committee, Professors Charles Bakis, Christopher Rahn, and Scott Lewis, for their helpful discussions and suggestions. I also want to thank NASA research engineer, Andrew J. Provenza, from the NASA Glenn Research Center, for his technical advice and assistance on the topic of Magnetic Bearings and associated control hardware considerations. I also appreciate the assistance of fellow graduate student, Murat Ocalan, in the assembly of the Active Magnetic Bearing/Driveline Testrig facility.

The financial support for this research was granted by the U.S. Army Research Office (ARO) MURI Program, the NASA Graduate Student Research Program, and the Penn State University Weiss Graduate Fellowship Program. Additionally, I want to thank Dr. Robert Bill of the NASA Glenn Research Center for his advice on the project and Dr. Gary Anderson of the U.S. Army Research Office for his assistance in obtaining an ARO DURIP equipment grant for the experimental facility. Finally, I want to thank my wife, Mary DeSmidt, for her infinite patience and support during my graduate studies.
Chapter 1

INTRODUCTION

1.1 Supercritical Rotorcraft Drivelines

Rotorcraft drivelines, especially helicopter tailrotor drivelines, operate under a variety of conditions where they are subject to multiple excitation sources. The vibration levels experienced by the tailrotor driveline and associated components during routine operation result in high maintenance requirements which are necessary to maintain safe, reliable operation, Hartman (1973) and Dousis (1994). To increase the power-to-weight ratio for a given driveshaft design, the current trend is toward supercritical drivelines, where the shaft operating speed, $\Omega$, is above one or more of the shaft lateral bending natural frequencies, Kraus and Darlow (1987). Figure 1.1 shows the conventional, state-of-the-art, supercritical tailrotor-driveline configuration that consists of two, relatively lightweight, shaft segments connected by flexible couplings which accommodate driveline angular misalignment. The driveshaft is mounted on a relatively flexible tailboom-fuselage structure by contact hanger bearings and viscous, squeeze-film or squirrel cage shaft dampers.
Supercritical shafts are advantageous since they operate at high speeds and thus transmit a given amount of power with less torque than sub-critical designs. Lower maximum torque requirements allow for lighter-weight shafting and driveline components, which is desirable for rotorcraft applications. One main purpose of hanger bearings is to make the driveline sub-critical by raising the lateral natural frequencies above the operating speed, $\Omega$. By removing the sub-critical design constraint, supercritical designs reduce weight and complexity by allowing longer, unsupported shaft lengths with fewer shaft segments and hanger bearings, Darlow, et al. (1990).

Despite these advantages, it is well known that supercritical shafts are prone to so-called flutter or whirl instability due to shaft structural damping in the rotating frame. Zorzi and Nelson (1977) derived dissipation functions that account for viscous and hysteretic material damping of a rotating, flexible shaft. From a finite element perspective, they showed that the rotating-frame damping results in both a regular

![Diagram of Conventional Supercritical Tailrotor Driveline and Fuselage](image-url)
damping matrix and a skew-symmetric stiffness matrix that is proportional to both the material damping and the shaft speed, $\Omega$. Furthermore, Chen and Ku (1991), used a finite-element model to study the effect of shaft boundary conditions on the whirl stability of a rotor-bearing system with rotating-frame damping. These investigations concluded that whirl instability can be delayed to higher speeds by providing sufficient auxiliary fixed-frame viscous damping of the lateral shaft motion. Typically, the critical speed above which whirl instability occurs is proportional to the ratio of fixed-frame to shaft material damping. Also, it is shown that the fixed-frame lateral damping required for stability can be reduced by introducing bearing stiffness anisotropy. Because of whirl instability, the potential weight savings offered by supercritical designs are somewhat offset by the added weight of the auxiliary fixed-frame dampers needed for stability. In addition to the added weight, dampers are maintenance intensive, and their performance is difficult to predict since their characteristics vary with operating temperature and wear, Vance (1988).

Since the function of a driveline is to transmit torque, the tailrotor driveline is subjected to a range torque loads throughout the helicopter operation. Also, the tailrotor driveline undergoes axial loading that is caused by the bending deformation of the tailboom-fuselage structure on which the driveline is mounted. Many researchers have determined that axial and torque loads can have destabilizing effects on flexible shaft and rotor-bearing structures.

Bolotin (1963) derived the virtual work expression that accounts for the transverse stiffness effect of the axial load. It was determined that static compressive axial loads below the critical buckling load decrease the shaft bending natural
frequencies, and above the critical load divergence instability occurred. Eshleman and Eubanks (1969) derived the partial differential equations of an elastic rotating shaft subjected to constant axial torque. Here they showed that the load-torque generates transverse bending moments that reduce the shaft bending natural frequencies. They showed that this effect is more significant for shafts with smaller slenderness ratios. Zorzi and Nelson (1979) derived a variational work expression for the axial load-torque effect on the transverse bending. Using a finite element approach, they showed that the load-torque results in an additional non-symmetric torque buckling stiffness matrix. Cohen and Porat (1984) showed that, under certain conditions, load-torque caused flutter instability. They also showed that this instability can be suppressed with sufficient lateral damping.

Furthermore, Yim, et al. (1986) showed that torque-induced instability of a flexible shaft was highly dependent on shaft boundary conditions, rotation speed, gyroscopic effects and torque to damping ratio. In general, over-hung rotors were more prone to load-torque instabilities, and the stability of asymmetric shaft modes were much more affected than symmetric modes. Also, Yim, et al. (1986) and Lee and Yun (1996) found that load-torque destabilized backward whirl modes in positive work systems (torque with same sense as $\Omega$) and destabilized forward whirl modes in negative work systems (torque with opposite sense as $\Omega$). Finally, Chen and Chen (1995) studied the effects of both conservative and non-conservative torque and axial loads on a cantilevered rotor-disk structure. Here the interaction between the disk gyroscopic moments, torque-load and axial load caused both flutter and divergence instability depending on the shaft speed.
Still other researches investigated periodically time-varying axial and torque loads. Chen and Ku (1990) studied a uniform elastic shaft subjected to a periodic axial load. They found that bending instability occurred when the axial load variation frequency was near twice a transverse bending natural frequency. Finally, Khader (1997) used Floquet theory to study the parametric instability zones of a cantilever shaft-disk assembly subjected to periodic follower torque and axial loads. It was discovered that the shape of the parametric insatiably zones was affected by the non-dimensional axial load to non-dimensional torque load ratio. They also determined that rotational damping was more effective in suppressing instability than translational damping. Finally, like in the constant torque case, positive and negative torque loads destabilized backward and forward gyroscopic whirl modes, respectively.

In addition to instability, shaft imbalance, due to run-out, bent shafts and density variation, is a major source of shaft vibration at the rotation speed, $\Omega$. Since the imbalance force increases with the square of rotation speed, high-speed supercritical shafts are very sensitive to rotational imbalance. Therefore, precise and frequent shaft balancing is required to reduce and keep the magnitude of the imbalance vibration within acceptable levels, Kraus and Darlow (1987) and Darlow, et al. (1990). Typically this is accomplished by attaching eccentric masses to the shaft in an attempt to cancel the imbalance. However, in the case of flexible shafts with many modes, it is very difficult to achieve perfect balancing since the imbalance distribution is inherently unknown.
1.2 Non-Constant Velocity Couplings

Another significant but less understood and sometimes overlooked source of driveline vibration and instability are the flexible couplings, Kirk, et al. (1984). Many types of flexible couplings such as U-Joints (Hooke’s Joints), metal disk, and gear-type couplings, have non-constant velocity (NCV) kinematics, Xu and Marangoni (1990). With these couplings, angular misalignment causes rotational speed variation between the coupled shaft segments. A common NCV used coupling in rotorcraft drivelines is the so-called 4-bolt disk coupling shown in Figure 1.2, Gibbons (1980).

![Flexible Disk Pack](image)

Fig. 1.2: Common NCV coupling, 4-bolt metal disk coupling.

This type of coupling accommodates angular misalignment by a flexible metal disk-pack attached alternately with bolts to opposite flanges. Since each bolt pair on the flange acts like a lateral pivot axis, the kinematics is a function of the number of bolt-pairs and their relative orientation, Dewell and Mitchell (1984). In the 4-bolt case there are effectively two 90° pivot axes, thus the 4-bolt disk coupling is kinematically equivalent to a standard U-Joint coupling but with an additional rotational spring stiffness about the pivot axes, Mancuso (1986). Equation 1.1 shows a functional representation of how the output shaft speed, \( \Omega_{i+1} \), varies periodically with the input shaft speed, \( \Omega_i \), and misalignment angle, \( \theta(t) \).
\( \vartheta(t) \), Dewell and Mitchell (1984). Where, in general, \( \vartheta(t) \), is some, nominal, static angle plus a dynamic angle due to shaft lateral bending vibration.

\[
\Omega_{i+1} = \Omega_i + f_0(\Omega_i, \vartheta, \dot{\vartheta}) + f_c(\Omega_i, \vartheta, \dot{\vartheta}) \cos(2\Omega_i t) + f_s(\Omega_i, \vartheta, \dot{\vartheta}) \sin(2\Omega_i t)
\]

(1.1)

Here \( f_0 \), \( f_c \) and \( f_s \) are the scalar speed-misalignment variation functions which are functions of the input speed, and the misalignment.

In addition to speed variation, when torque is transmitted through U-Joint or disk-type couplings in the presence of misalignment, periodic lateral bending moments are generated on the shaft coupling flanges with frequency \( 2\Omega \). Specifically, load-torque in the presence of static misalignment produces periodic moment forcing terms, and load-torque in the presence of dynamic misalignment produces periodic parametric terms, which affect stability. Thus, from a modeling perspective, the stability and response of a driveline system with NCV couplings can be described by a set of linear periodic time-varying equations.

Many researchers have investigated the stability and response of rotor-shaft systems that involve a single U-Joint coupling. Iwatsubo and Saigo (1984) studied the effect of a constant follower load-torque on the lateral stability of a nominally aligned rigid rotor-disk mounted on a compliant bearing and driven through a single U-Joint. They derived the parametric and forcing terms which account for the transverse moments induced by torque transmitted through the U-Joint. It was determined that constant load-torque induced parametric instabilities for shaft speeds near the sum-type combinations of the system transverse natural frequencies. Mazzei, et al. (1999) considered the effect of lateral shaft flexibility on the stability of a misaligned shaft driven by a single U-Joint and subjected to a constant follower load-torque. They found that constant load-torque
caused both flutter instability and parametric instability for shaft speeds near sum-type combinations of the shaft bending natural frequencies.

Asokanthan and Hwang (1996) and Asokanthan and Wang (1996) studied the stability of two torsionally flexible, misaligned shafts coupled by a U-Joint. In their analyses the shafts were driving an inertia load and the lateral shaft orientations were fixed, hence only torsional dynamics were considered. They concluded that shaft speed variation due to angular misalignment caused parametric instabilities near principle and sum-type combinations of the torsional natural frequencies. It was also found that the addition of viscous torsional damping had a stabilizing effect for principle instability zones, but destabilized the sum-type combination instability zones.

DeSmidt, et al. (2002) considered load-torque, load-inertia and misalignment angle on the stability of a shaft-disk assembly supported on a compliant bearing/damper and driven with a single U-Joint. In this analysis, both torsional and lateral shaft flexibility were considered and it was discovered that load-inertia together with misalignment caused periodic inertia coupling of the torsion and lateral modes. This torsion-lateral interaction was described by periodic inertia coupling matrices, which caused torsion-lateral parametric instability for shaft speeds near the torsion-lateral sum combinations. It was also shown that misalignment had a stabilizing effect on the load-torque induced-flutter instability near the torsional-lateral difference combination frequencies. Furthermore, it was found that sign of the load-torque determined which torsion-lateral combination frequencies were affected. Finally, they showed that the torsion-lateral instabilities could be stabilized with sufficient lateral viscous damping.
Xu, et al., Part I (1994) and Xu, et al., Part II (1994) analytically and experimentally studied the lateral vibration response of two unbalanced and misaligned shafts connected by a single U-Joint and driving a torsional inertia. The misalignment was modeled as purely static, hence the potentially destabilizing parametric terms were neglected. The misalignment at the U-Joint caused speed variation of the inertia load which generated a dynamic torque. When transmitted through the U-Joint, this dynamic torque generated transverse moments on the shaft that were periodic with frequencies $2\Omega$ and $4\Omega$ and proportional to both the misalignment and load-inertia. For certain shaft speeds, $\Omega$, it was determined that the $2\Omega$ vibration response due to the U-Joint was as significant as the imbalance vibration response.

Kato and Ota (1990) studied U-Joint frictional effects for a statically misaligned shaft driven by a single U-Joint. They considered both viscous and Coulomb friction between the U-Joint yokes and cross-piece. It was found that both viscous and coulomb friction generated harmonic lateral moments at even multiples of the shaft speed, i.e. $2\Omega$, $4\Omega$, … etc. The magnitude of the viscous friction-induced moments were proportional to the misalignment angle, while the Coulomb friction-induced moments were independent of misalignment angle. Finally, they also showed that the viscous friction-induced moments could be suppressed if the viscous friction coefficients of each yoke were equal.

Saigo, et al. (1997) studied the effect of U-Joint coulomb friction on a statically misaligned U-Joint/shaft system mounted on a flexible bearing damper. Since the torsional inertia load was small, shaft speed variation due to the U-Joint kinematics was neglected. It was found that coulomb friction induced lateral shaft vibration and, under
certain conditions, instability. They determined that lateral fixed-frame damping and bearing stiffness asymmetry were both effective in suppressing the instability and reducing the vibration amplitude. Additionally, the static misalignment angle had a stabilizing effect on the coulomb friction-induced instability.

Other researchers have investigated the vibration responses of various shaft systems that incorporate two U-Joint couplings. Both Rosenberg and Ohio (1958) and Sheu, et al. (1996) investigated the steady-state response of a misaligned, laterally flexible, torsionally rigid, shaft between two U-Joints via the harmonic balance method. Rosenberg and Ohio (1958) derived the transverse moment forcing terms for the double U-Joint/shaft system for an arbitrary single plane misalignment configuration. However, since the dynamic portion of the misalignment was neglected, parametric terms for the double U-Joint/shaft system were not developed. It was discovered that the double U-Joint moment forcing terms combined with shaft imbalance resulted in shaft vibration at odd integer multiple harmonics of the shaft operating speed, i.e. $\Omega, 3\Omega, \ldots$ etc. Sheu, et al. (1996) considered a more general misalignment configuration of the double U-Joint/shaft system that accounts for misalignment in both orthogonal planes. Both static and dynamic misalignment were considered and the transverse moment forcing and parametric terms for the laterally flexible, torsionally rigid, shaft/double U-Joint system were derived. It was shown that when the U-Joints were phased by $90^\circ$ and the input and output shafts had the same static misalignment angle, the moment forcing terms due to static misalignment were identically zero. This is the so-called parallel offset misalignment condition common in many driveline applications. However, it was also
shown that the parametric terms due to dynamic misalignment remain non-zero which resulted in $2\Omega$ harmonic lateral shaft vibration.

### 1.3 Aerodynamic Loading

Another loading and vibration source relevant to rotorcraft drivelines, which does not originate from the driveline, is steady and unsteady aerodynamic loading. One main steady aerodynamic load is the tail-rotor anti-torque force. This force is produced by the tail-rotor to counteract the main-rotor torque. Both Leishman and Bi (1989) and Norman and Yamauchi (1991) showed that in hover and forward-flight downwash from the main rotor over the tailboom produced static loading and harmonic loading at the main rotor blade-passage frequency, $\omega_{BP} = N_b \Omega_{MR}$. Where $\Omega_{MR}$ is the main rotor speed and $N_b$ is the number of blades. Furthermore, in addition to static lift forces and moments acting on the horizontal stabilizer in forward flight, Gangwani (1981) showed that main rotor tip vortices generated harmonic loads at multiples of the main rotor blade-passage frequency.

In the particular case of the Boeing AH-64 helicopter, Toossi and Callahan (1994) showed that the several natural frequencies of the tailboom structure were near multiples of the blade-passage frequency, and thus unsteady aerodynamic loads had the potential to cause significant tailboom structural vibration. Furthermore, for the case of the AH-64 tailrotor driveline, DeSmidt, et al. (1998) showed that the multi-frequency unsteady aerodynamic loads excited many coupled tailrotor driveline-fuselage modes, such as Figure 1.3.
As seen above, theses modes involved both fuselage and shaft vibration. Since the natural frequencies of several of theses modes were near multiples of the blade-passage frequency, ($\omega_{BP} \approx 18$ Hz), the unsteady aerodynamic loads caused significant, multi-frequency driveshaft vibration and loading of the hanger bearings and flexible couplings.

### 1.4 Composite Shafting

Kraus and Darlow (1987) and Darlow, et al. (1990) explored the use of light-weight Graphite/Epoxy (Gr/Ep) composite materials for supercritical helicopter power transmission shafting. They concluded that for a given power requirement, composite shafting offered significant weight savings over conventional aluminum alloy shafting. However, they also concluded that the composite shafts were typically less well balanced and inherently more sensitive to imbalance than alloy shafts. Recently, a novel approach to the design of helicopter tailrotor drivelines based on newly emerging Flexible Matrix
Composite (FMC) materials have been explored by Shin, et al. (2003) and Ocalan (2002). FMC materials have both high strain and high strength capabilities. Specifically, for a given torsional strength and stiffness, FMC shafts can be tailored to have very low transverse bending stiffness compared to conventional alloy shafts, Shan and Bakis (2002). This type of FMC shaft design was studied by Ocalan (2002) and Shin, et al. (2003) in the context of helicopter tailrotor drivelines. It was found that this low transverse stiffness allowed a single FMC shaft to accommodate the fuselage deflection and driveline misalignment without the need for multiple segments or flexible-couplings. Despite this benefit, it was found that the supercritical FMC designs were more prone to whirl instability than conventional alloy supercritical designs. The low bending stiffness resulted transverse bending natural frequencies that were lower than in the conventional case. Thus, for a given operating speed, the FMC shaft had to pass through more critical speeds and hence the design was “more” supercritical that the conventional alloy design. Also, since the FMC material damping was much larger than the aluminum alloy material damping, the FMC design required more fixed-frame damping to suppress the whirl instability. Finally, another challenge posed by the use of FMC materials in driveline applications is uncertainty and variation in the material properties. Specifically, Shan and Bakis (2003) explored and characterized the temperature and frequency dependence of the FMC material stiffness and damping properties.
1.5 Active Magnetic Bearings

In an attempt to alleviate shaft vibration in addition to reducing vibratory loading and frictional wear of driveline components, many researchers have investigated the use of active vibration control implemented via Active Magnetic Bearings (AMB). An AMB is a non-contact device that applies levitation and actuation forces to a shaft across an air-gap via a magnetic field. In theory, due to their non-contact nature, AMBs do not suffer from frictional wear or need lubrication and thus require little maintenance compared to conventional contact roller bearings, Allaire, et al. (1986). However, the full benefit of AMBs has only recently been practically and economically realizable as a result of advances in high energy product permanent magnetic materials, development of high-saturation flux ferromagnetic materials and innovations in miniaturized integrated circuit and solid state electronics, Meeks and Spencer (1990).

Magnetic bearings fall into two main categories: radial and axial magnetic bearings. As the names imply, radial magnetic bearings apply levitation and actuation forces to a shaft in the radial or transverse direction, while axial magnetic bearings do so in the axial direction. Furthermore, there are three basic magnetic bearing operating principles: electromagnetic attraction, permanent magnet repulsion, and reluctance centering. Because of their smaller size and weight along with their higher force capability, designs based on electromagnetic attraction are preferred over other approaches, Lewis (1994).

In the case of an attractive radial magnetic bearing, electro-magnetic coils surrounding the shaft generate a magnetic field that acts on a ferromagnetic rotor which produces a net magnetic force on the shaft. Figure 1.4 shows an 8-coil (8-pole), attractive
radial magnetic bearing with the rotor and driveshaft suspended in the center. Finally, in case of a power failure, each AMB has a set of backup roller bearings with a clearance slightly smaller than the air-gap distance, Meeks and Spencer (1990).

The attractive magnetic force, $F_{\text{coil}}$, generated by each coil is given by Equation 1.2

$$F_{\text{coil}} = \frac{\mu_0 A_p N_w^2 i_c^2}{h^2}$$  \hspace{1cm} (1.2)

Here $\mu_0$ is the free-space magnetic permeability, $A_p$ is coil pole face area, $N_w$ is the number of coil windings, $i_c$ is the coil current, and $h$ is air-gap distance between the rotor and the coil face, Mease (1991). Due to the inverse square attractive nature of the magnetic force, for constant coil currents the magnetic bearing acts like a negative stiffness which results in an unstable system. To counteract this and maintain stable levitation, the bearing forces on the shaft are adjusted in real-time via the coil currents using shaft position feedback information. Sinha (1990), summarized the basic stability issues related active magnetic suspension. Typically, each AMB has two collocated position sensors that measure shaft displacement relative to the stator housing in two
orthogonal directions. In addition to counteracting the negative stiffness, feedback and sometimes disturbance feed-forward information is used to actively suppress shaft vibration, so called active vibration control. One of the most basic and commonly used control algorithms is the decentralized proportional-derivative (PD) feedback law, Mease (1991). In this case, the control force of each bearing in is determined individually using only the shaft displacement and velocity information from the collocated bearing sensors. The proportional feedback acts like positive stiffness and the derivative feedback acts like damping, thus each AMB behaves like a passive bearing/damper. In other, more complex, centralized control algorithms, the bearing forces are coordinated based on all of the sensor information combined with an analytical system model and some desired performance objectives.

An AMB based active control system can be implemented using either an analog or digital approach. Successful experimental applications of analog controllers have been reported by Humphris, et al. (1986) and Nonami (1988). However, due to major advances in digital computer technology, digital signal processing, and the flexibility of software based design environments, digital controllers are now used almost exclusively. Many researchers such as Sinha, et al. (1991), Mease (1991), Lewis (1994), Suzuki (1998) and others have successfully implemented digital AMB control systems. The basic elements of an AMB digital control system are shown in Figure 1.5, which is a schematic of the closed-loop AMB-shaft system studied by Mease (1991) and Lewis (1994).
1.6 Closed-Loop Active Vibration Control of Flexible Shafts

Despite the advantages of AMBs, their application is not necessarily straightforward since the stability and performance of the closed-loop AMB-flexible shaft system is highly dependent on the control algorithm design. The synthesis of active suspension and vibration control algorithms for AMB systems remains a very challenging analytical problem which has been explored by many researchers. Without the feedback control, the AMB-flexible shaft system is typically unstable, which complicates the controller synthesis and restricts the set of feasible controller designs. In addition to the stability requirements, to avoid contacting the backup-bearings there are strict performance requirements on the relative shaft-AMB stator displacements, i.e. gap displacements. Also, the multi-input multi-output (MIMO) nature of the system further complicates the control synthesis. Typically there are two independent control inputs per AMB, i.e. the magnetic bearing forces, and two sensor outputs per AMB, i.e. the
collocated shaft displacement measurements. Usually the number of sensor outputs is much less than the number of system states, especially in the case of flexible shafts where there are theoretically an infinite number of modes. Thus, any control algorithm must be designed to use, so-called, output feedback.

Many researchers have explored proportional-integral-derivative (PID) feedback and eigenvalue assignment in the context of shaft-AMB controller designs. Nikolajsen, et al. (1979), experimented with magnetic dampers on a supercritical transmission shaft in place of the traditional squeeze-film or squirrel-cage dampers. Here, decentralized analog derivative feedback provided sufficient damping levels to pass through a critical speed. Okada, et al. (1988) explored digital PID control on an AMB system. Stanway and Burrows (1981) studied theoretical control algorithms for rotors with flexibly mounted journal bearings. Here they discussed controllability, observability and eigenvalue assignment. Salm and Schweitzer (1984) used direct output-feedback information to control shaft vibration with collocated magnetic bearings and sensors. They also discussed the use of Optimal Control theory. Nonami (1988) studied eigenvalue assignment methods for a flexible shaft-AMB system accounting for both shaft mechanical and the AMB electrical degrees-of-freedom. Here it was found that the dynamics of the electrical degrees-of-freedom, i.e. the AMB coil currents, can be considered quasi-static if the structural frequencies are well below the AMB coil-circuit cutoff frequency.

Other researchers explored sliding-mode and robust control designs. Sinha, et al. (1991) and Mease (1991) explored active vibration control of a rigid shaft mounted on two AMBs that was subjected to a mid-span rotational imbalance. Taking advantage of
the, so-called, matching conditions, they developed a full-state feedback sliding mode controller with guaranteed performance in the presence of imbalance magnitude uncertainty. Similarly, Lewis (1994) developed a sliding-mode active vibration controller for a flexible shaft-AMB system subjected to a mid-span rotational imbalance, Figure 1.5. Again, taking advantage of the matching conditions and using estimated state feedback from an observer, they developed an output-feedback sliding-mode controller with guaranteed performance in the presence of imbalance magnitude uncertainty. Here the controller and observer were based on a modally truncated shaft model accounting for the rigid body and first few flexible modes. Nonami and Takayuki (1996), explored a robust control design for a flexible rotor-AMB system using $\mu$ synthesis techniques and a modally truncated finite-element model. They compared the robust stability and performance of the $\mu$ controller to a standard $H_\infty$ design with respect to structured rotor mass uncertainty.

Other researchers investigated the effect of base motion on AMB systems. Suzuki (1998) experimentally investigated a digital PID feedback controller used in conjunction with base acceleration feedforward to suppress base-excited rotor motion. They found that PID feedback with base acceleration feedforward was more effective than standard PID control for both harmonic and random base motion. Cole, et al. (1998) explored state-space vibration control algorithms for flexible rotor-AMB system subjected to both imbalance and base-motion disturbances. Here they explored mixed objective functions to minimize response to both disturbances. Kasarda, et al. (2000) experimentally studied the effect of sinusoidal base motion on a non-rotating mass suspended on a single AMB using PD feedback to simulate stiffness and damping. Here the AMB stator was mounted
on a shaker that provided single frequency harmonic base motion. For small displacement values compared with the AMB air-gap clearance, they found that the displacement transmissibility between the stator and suspended mass was amplitude independent. However, when the displacements within the AMB were on the order of the air-gap clearance, they suggested that usual linearization procedure about a nominal gap to determine the coil control currents was inadequate.

DeSmidt, et al. (1998) explored active vibration control strategies for a tailrotor driveline-fuselage structure using magnetic actuators. Here, misalignment, load-torque and NCV coupling effects were not considered. A reduced-order finite element model of a tailrotor driveline-fuselage structure was developed and augmented with AMBs that were used as control actuators, Figure 1.6. The system was subject to multi-frequency aerodynamic loading on the fuselage and imbalance loading on the driveline.

![Diagram](image)

Fig. 1.6: AMB-Tailrotor driveline full state feedback optimal controller.
Several full-state feedback controllers based on linear quadratic Optimal Control theory were designed to minimize an object function, $J_{cost}$, of the form,

\[
J_{cost} = \int_{t_0}^{t_f} \left[ x^T Q_{\text{comp}} x + u^T R_{\text{act}} u \right] dt
\]

\[
Q_{\text{comp}} = w_b Q_{\text{bear}} + w_c Q_{\text{coup}} + w_g Q_{\text{gap}}
\]

(1.3)

Where $R_{\text{act}}$ is control effort weighing matrix, and $Q_{\text{comp}}$ is the composite state weighting matrix composed of $Q_{\text{bear}}$, $Q_{\text{coup}}$ and $Q_{\text{gap}}$ which weight the hanger bearing loads, the flexible-coupling loads and shaft displacements at the AMB locations. It was found that hanger bearing and flexible-coupling loads could be dramatically reduced in the presence of simultaneous fuselage and shaft disturbances, while maintaining shaft displacements below the air-gap clearance constraint, Figure 1.7.

![Figure 1.7: Closed-loop hanger bearing loads and gap displacements in forward-flight.](image-url)
Additionally, it was found that the control dramatically reduced the normally large hanger bearing loads encountered during spin-up as the shaft speed passes through the natural frequencies to the supercritical operating speed. Based on the desired vibration reduction and corresponding control effort, it was found that the magnetic actuator size, weight and power requirements were reasonable in the context of a rotorcraft setting.

In most real rotor-shaft and driveline systems, the exact location and distribution of the rotational imbalance is unknown. Thus, in state-space terminology, the input distribution matrix, \(B_d\), associated with the imbalance is unknown and cannot be used in control synthesis as is done in many theoretical control designs. To address this problem Cole, et al. (2000) employed a standard linear quadratic Gaussian (LQG) approach coupled with a performance enhancing, adaptive feed-forward loop. In this semi-adaptive technique, developed by Tay and Moore (1991), an estimated state feedback controller, such as LQG, stabilizes the open-loop unstable shaft-AMB system, while an adaptive feedforward, designed not affect the stability, adapts to cancel the unknown disturbances.

1.7 Thesis Objectives

As cited above, many researchers have shown that new materials such as FMC and active control technology such as AMBs potentially offer many performance enhancing and maintenance saving benefits to power transmission systems. This is especially the case in rotorcraft drivelines where reducing weight, vibration and maintenance intervals are all active research goals. To advance the state-of-the-art of rotorcraft and other high-performance drivelines, this thesis explores two new actively
controlled supercritical tailrotor driveline configurations utilizing AMB and FMC technologies. Figure 1.8 shows a conventional supercritical tailrotor driveline configuration along with the two new actively configurations which will be explored in this thesis.

In new Configuration I, the conventional contact hanger-bearing and viscous shaft dampers are replaced with non-contact AMB devices. This configuration eliminates both frictional wear and the high maintenance requirements of conventional hanger-bearings and dampers. Also, with proper control law design, AMBs offer active vibration suppression of driveshaft vibration due to such sources as; imbalance, NCV coupling effects, and fuselage-aerodynamic loads. Active suppression of imbalance loads also reduces the need for shaft balancing, which is maintenance intensive. Further, new Configuration II, in addition to replacing the conventional hanger-bearing and viscous dampers with AMBs, utilizes a single piece rigidly coupled FMC shaft to replace the conventional segmented shaft and flexible-couplings. Since it has been demonstrated that an FMC shaft with rigid couplings can safely accommodate driveline misalignments due to fuselage-deflection, see Ocalan (2002) and Shin, et al. (2003), the need for the segmented shafts and flexible-couplings can be eliminated. In addition to reducing driveline complexity and maintenance requirements, removing the flexible-couplings eliminates undesirable NCV coupling effects which are significant a source of vibration and instability.
The overall objective of this thesis is to theoretically develop and experimentally evaluate active vibration control algorithms for the two actively controlled AMB tailrotor driveline-fuselage Configurations I and II. Despite the large body of previous research in the areas of rotordynamics, AMBs, and active vibration control, several research issues still need to be addressed in order to successfully develop closed-loop control algorithms for Configurations I and II.

In Configuration I, to guarantee stability and performance, the Configuration I active control algorithm must account for the effects of the NCV couplings, which have
not been considered in previous shaft/driveline active control investigations. Previous research on non-actively controlled rotor-shaft systems has found that NCV couplings caused both vibration and instability. However, since the previous stability investigations only considered single NCV coupling/shaft configurations and neglected rotating-frame damping effects, the previous research is not directly applicable to the multi-segment/NCV coupling, supercritical, rotorcraft driveline considered in this research. Thus, in order to gain insight into the design of the active control algorithm for the multi-segment/NCV driveline in Configuration I, a detailed stability analysis of the conventionally configured driveline (i.e. passive bearings and dampers and no AMB) is conducted.

It has been shown that the tailboom and horizontal stabilizer are subjected to relatively large static aerodynamic loads which cause tailboom deflection. Since the driveline is mounted on the flexible tailboom structure, the driveline misalignment is inherently a function of the aerodynamic loads which vary with the flight condition. Furthermore, since the tailrotor anti-torque force varies with flight condition and pilot input, the driveline load-torque also varies through the flight envelope. In this research, the NCV forcing and parametric terms, which are functions of misalignment and load-torque, are considered as a form of structured model uncertainty under which the Configuration I control algorithm must be stable and have acceptable performance.

In Configuration II, the NCV terms are not an issue since only rigid couplings are involved. However, other issues related to the FMC shaft material, such as stiffness and damping temperature sensitivity and high internal damping must be considered in the design of the Configuration II control algorithm.
Furthermore, in both Configurations I and II, unlike previous studies, a priori knowledge of the shaft imbalance distribution cannot be utilized in the control design since the exact nature of the imbalance distribution is typically unknown and difficult to measure. Thus, the control must have robust vibration suppression performance characteristics in order to maintain the shaft vibrations below the AMB airgap clearance to prevent rotor/backup-bearing touchdown. Finally, the performance of both actively controlled configurations will be investigated analytically and experimentally for a variety of operating conditions utilizing a scaled AMB-supercritical tailrotor driveline-fuselage testrig developed for this research.
2.1 Introduction

To facilitate analysis of the dynamics and stability and to establish a baseline for comparing the two actively controlled driveline configurations introduced in Chapter 1, a comprehensive analytical model of a conventional supercritical rotorcraft driveline has been developed. Specifically, a finite element model of the segmented driveline-fuselage structure shown in Figure 1.1 is derived.

2.2 System Description

Equations-of-motion are derived for the segmented driveline-fuselage structure given in Figure 2.1. The driveline consists of a fixed input-shaft, a fixed output-shaft, and two flexible, intermediate shaft segments, \( n_{\text{seg}} = 2 \). The shafts are connected with non-constant velocity (NCV) flexible-couplings, A, B and C, which are relatively phased
about the shaft rotation axis by 90° as is typically the case, Mancuso (1986). Specifically, the flexible-coupling phase angles are $\psi_A = 0°$, $\psi_B = 90°$ and $\psi_C = 0°$. The intermediate shafts are mounted on the fuselage by roller bearings and lateral viscous dampers and the driveline rotation axis is offset from the fuselage centerline by a distance $h_s$. Furthermore, $L_{d_i}$ and $L_{d_j} \ [i = 1,2...n_{seg}]$ and $L_{c_i} \ [i = 1,2...n_{seg}+1]$ denote the axial locations of the bearings, dampers, and flexible couplings along driveline as measured from the global origin O.

![Diagram](image)

**Fig. 2.1:** Segmented, multi-coupling, driveline on flexible fuselage structure.

The fuselage structure is modeled as a uniform, rectangular cross-section, cantilevered beam, length $L_f$, with a rigidly attached, lumped tail inertia on the free-end. The lumped tail inertia, with mass $m_t$ and rotational inertia matrix, $\mathbf{I}_{m_t}$, accounts for the inertia properties of the vertical mast, horizontal-stabilizer, intermediate gearbox, tailrotor gearbox, and rear wheel-gear assembly. The fuselage elastic displacements are measured from the inertia-fixed coordinate frame, $\{n\} = [n_1, n_2, n_3]$, located at the global origin, O,
where \( n_1 \) is aligned with the fuselage-beam neutral axis. Specifically, on the domain \( 0 \leq x \leq L_f \), the transverse fuselage deflections in the \( n_2 \) and \( n_3 \) directions are \( v_f(x,t) \) and \( w_f(x,t) \), the axial fuselage deflection in the \( n_1 \) direction is \( u_f(x,t) \), and the fuselage elastic windup angle about the \( n_1 \) axis is \( \hat{\phi}_f(x,t) \).

The intermediate shafts, \( i =1 \) and \( 2 \), are modeled as axially rotating beams with lengths \( L_i \), and with uniform circular cross-section. Figure 2.2 shows the fixed-frame and rotating-frame elastic deflections of the \( i^{th} \) shaft at axial location, \( x \).

![Diagram of fixed-frame and rotating-frame elastic deflections of shaft cross-section.](image)

**Fig. 2.2:** Fixed-frame and rotating frame elastic deflections of shaft cross-section.

Here, the two coordinate frames, \( \{ m^i \} = \{ m_1^i, m_2^i, m_3^i \} \) and \( \{ r^i \} = \{ r_1^i, r_2^i, r_3^i \} \), have directions \( m_1^i \) and \( r_1^i \) aligned with the undeformed \( i^{th} \) shaft neutral-axis and have origins at \( x = 0 \) on the \( i^{th} \) shaft. The \( \{ m^i \} \) frame is inertially-fixed and the \( \{ r^i \} \) frame follows the \( i^{th} \) shaft rigid body rotation, \( \varphi_i(t) \), about the undeformed neutral axis. On the domain, \( 0 \leq x \leq L_i \), \( v_i(x,t) \) and \( w_i(x,t) \) measure the \( i^{th} \) shaft transverse deflection and \( u_i(x,t) \) is the \( i^{th} \) shaft axial deflection relative to the fixed-frame \( \{ m^i \} \). Furthermore, relative to the rotating-frame \( \{ r^i \} \), \( v_\eta(x,t) \), \( w_\eta(x,t) \) and \( u_\eta(x,t) \) are the physical elastic deflections of the
shaft material in the transverse and axial directions. Over the domain $0 \leq x \leq L_i$, the elastic windup of the $i^{th}$ shaft about the cross-section-fixed axis $d_1$, is $\hat{\phi}_i(x,t)$. In the fixed-frame coordinate case, both the $i^{th}$ shaft rigid-body rotation angle, $\bar{\phi}_i(t)$ and the elastic windup angle, $\hat{\phi}_i(x,t)$, are measured about the $d_1$ axis. Thus, the total rotation angle of the $i^{th}$ shaft at cross-section location $x$ is,

$$\phi_i(x,t) = \bar{\phi}_i(t) + \hat{\phi}_i(x,t) \quad 0 < x < L_i$$  \hspace{1cm} (2.1)$$

In the case of small transverse deflections, the rigid-body rotation angles, $\bar{\phi}_i(t)$ and $\varphi_i(t)$, used in each rotation sequence are approximately equal, so it is assumed that $\varphi_i(t) = \bar{\phi}_i(t)$.

In general, the $i^{th}$ shaft rigid-body rotation angle, $\bar{\phi}_i(t)$, is a function of the input shaft rotation angle, $\phi_0$, and the NCV coupling kinematics. The input-shaft is driven at a constant speed, $\Omega_0$, thus $\phi_0 = \Omega_0 t$. Furthermore, the output-shaft drives an inertia load, $J_L$, and a resistive torque load, $T_L$, which are equivalent to the tailrotor inertia and torque loads as seen through the intermediate gearbox. The output-shaft rotation angle, $\phi_L(t)$, and the corresponding output-shaft rotation speed, $\Omega_L(t)$, are functions of the input-shaft speed, $\Omega_0$, and the NCV coupling kinematics derived in the next section.

2.3 Flexible Shaft/NCV Coupling Kinematics

As discussed in Section 1.2, many types of flexible-couplings used in driveline applications have, so-called, non-constant velocity (NCV) kinematics. This means that the rotational speeds of two shafts connected by an NCV coupling deviate from each
other as a function of the relative angular misalignment between the shafts. In this section, the misalignment-shaft speed kinematic relations are derived for the nominally misaligned, flexible, shaft segments, $i$ and $i-1$, which are connected by the $P$th NCV coupling with phase angle $\psi_P$. See Figure 2.3.

Fig. 2.3: $P$th NCV coupling connecting statically misaligned, flexible shafts.

The total effective misalignment is the sum of the nominal static misalignments, $\theta_{v,P}$ and $\theta_{w,P}$, plus the net dynamic misalignments of the shafts at the $P$th coupling due to bending. The net dynamic misalignments at the $P$th coupling are defined in terms of the elastic slopes as $v_P' = v_{i,P}' - v_{i-1,P}'$ and $w_P' = w_{i,P}' - w_{i-1,P}'$ where,

$$
\begin{align*}
  v_{i-1,P}' &\equiv v_{i-1}'(x,t)
  &\bigg|_{x=L_{i-1}},
  v_{i,P}' &\equiv v_i'(x,t)
  &\bigg|_{x=0},
  w_{i-1,P}' &\equiv w_{i-1}'(x,t)
  &\bigg|_{x=L_{i-1}},
  w_{i,P}' &\equiv w_i'(x,t)
  &\bigg|_{x=0}
\end{align*}
$$

(2.2)

Here, the “’” indicates differentiation with respect to the axial coordinate, $x$. The goal of this kinematic analysis is to obtain an expression for the rigid rotation coordinate of the $i^{th}$ shaft, $\phi_i(t)$, in terms of the $i-1^{th}$ shaft rigid an elastic rotation coordinates, $\phi_{i-1}(t)$ and $\phi_{i-1}(x,t)$, the static and dynamic misalignments, and the coupling phase angle, $\psi_P$.

This analysis is carried out for Universal Joint (U-Joint) couplings, which are kinematically equivalent to the 4-bolt disk couplings commonly used in many rotorcraft.
driveline applications. Both types of couplings have two successive rotation axes from the input-yoke to the output-yoke described by the Euler angles $\alpha$ and $\beta$. See Figure 2.4

![Diagram of NCV Coupling: U-Joint with two Euler angles $\alpha$ and $\beta$.](image)

Fig. 2.4: NCV Coupling: U-Joint with two Euler angles $\alpha$ and $\beta$.

The input-yoke of the $P^{th}$ coupling, described by the frame $\{a\}$=$[a_1, a_2, a_3]$, is attached to the $i-1^{th}$ shaft at location $x = L_{i-1}$. The output-yoke of the $P^{th}$ coupling, described by the frame $\{b\}$=$[b_1, b_2, b_3]$, is attached to the $i^{th}$ shaft at location $x = 0$. The input-yoke rotation angle, $\phi_{i-1,P}(t)$, due to rigid-body rotation and elastic windup of the $i-1^{th}$ shaft is

$$\phi_{i-1,P}(t) = \phi_{i-1}(t) + \phi_{i-1}(x,t)\bigg|_{x=L_{i-1}}$$ (2.3)

Similarly, the $P^{th}$ coupling output-yoke rotation angle, $\phi_{i,P}(t)$, due to rigid-body rotation and elastic windup of the $i^{th}$ shaft is

$$\phi_{i,P}(t) = \phi_{i}(t) + \phi_{i}(x,t)\bigg|_{x=0}$$ (2.4)

Figure 2.5 shows the rotation sequence from the $\{m^{i+1}\}$ frame to the $\{b\}$ frame in terms of the elastic slopes, $v_{i-1,P}$ and $w_{i-1,P}$, the input-yoke rotation angle, $\phi_{i-1,P}(t)$, and the Euler angles, $\alpha$ and $\beta$. 
Assuming small elastic slopes, this transformation from \( \{m^{i-1}\} \) to \( \{b\} \) is written

\[
\begin{bmatrix}
    b_1 \\
    b_2 \\
    b_3
\end{bmatrix} =
\begin{bmatrix}
    \cos \alpha \cos \beta & \sin \beta & -\sin \alpha \cos \beta & \cos(\phi_{i-1,P} + \psi_P) \\
    -\sin \beta \cos \alpha & \cos \beta & \sin \alpha & \sin(\phi_{i-1,P} + \psi_P) \\
    \sin \alpha & 0 & \cos \alpha & \cos(\phi_{i-1,P} + \psi_P)
\end{bmatrix}
\begin{bmatrix}
    1 \\
    0 \\
    0 \\
    \cos(\phi_{i-1,P} + \psi_P) \\
    \sin(\phi_{i-1,P} + \psi_P) \\
    \sin(\phi_{i-1,P} + \psi_P) \cos(\phi_{i-1,P} + \psi_P)
\end{bmatrix}
\begin{bmatrix}
    v'_{i-1,P} \\
    w'_{i-1,P} \\
    v'_{i-1,P} \frac{v'^2}{2} + w'^2_{i-1,P} \\
    -v'_{i-1,P} \\
    -w'_{i-1,P} \\
    \frac{1}{2} v'_{i-1,P} w'_{i-1,P}
\end{bmatrix}
\begin{bmatrix}
    1 & 0 \\
    0 & 1 \\
    0 & 0 \\
    \frac{1}{2} v'_{i-1,P} w'_{i-1,P} & 1 - \frac{1}{2} w'^2_{i-1,P}
\end{bmatrix}
\begin{bmatrix}
    m_1^{i-1} \\
    m_2^{i-1} \\
    m_3^{i-1}
\end{bmatrix}
\]

To eliminate \( \alpha \) and \( \beta \) and find an expression for \( \phi_i(t) \), a second coordinate rotation sequence from \( \{m^{i-1}\} \) to \( \{b\} \), shown in Figure 2.6, is written in terms of the static misalignments, \( \theta_v,P \) and \( \theta_w,P \), the elastic slopes, \( v'_{i,P} \) and \( w'_{i,P} \), and the output-yoke rotation angle, \( \phi_{i,P}(t) \).

Fig. 2.5: \( \{b\} \) in terms of \( i^{th} \) shaft kinematic variables and coupling Euler angles.
Assuming small misalignments, this second transformation from \( \{m^{i-1}\} \) to \( \{b\} \) is

\[
\begin{bmatrix}
\mathbf{b}_1 \\
\mathbf{b}_2 \\
\mathbf{b}_3
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 \\
0 & \cos(\phi_{i,p} + \psi_{p}) & \sin(\phi_{i,p} + \psi_{p}) \\
0 & -\sin(\phi_{i,p} + \psi_{p}) & \cos(\phi_{i,p} + \psi_{p})
\end{bmatrix}
\begin{bmatrix}
1 & -\frac{1}{2}(v'_{i,p}^2 + w'_{i,p}^2) & v'_{i,p} \\ -v'_{i,p} & 1 & -\frac{1}{2}v'_{i,p}^2 \\ -w'_{i,p} & -\frac{1}{2}v'_{i,p}w_{i,p} & 1 + \frac{1}{2}w'_{i,p}^2
\end{bmatrix}
\begin{bmatrix}
\mathbf{m}_2^{i-1} \\
\mathbf{m}_3^{i-1} \\
\mathbf{m}_4^{i-1}
\end{bmatrix}
\]

By equating the two expressions for the output-yoke frame, \( \{b\} \), in Equations (2.5) and (2.6), the expression for the Euler angles, \( \alpha \) and \( \beta \), in (2.7), and the expression for the \( i^{th} \) shaft rigid-body rotation coordinate, \( \tilde{\phi}_i(t) \), in (2.8), are obtained.

\[
\begin{bmatrix}
\alpha \\
\beta
\end{bmatrix} =
\begin{bmatrix}
\sin(\phi_{i-1,p} + \psi_{p}) - \cos(\phi_{i-1,p} + \psi_{p}) \\
\cos(\phi_{i-1,p} + \psi_{p}) & \sin(\phi_{i-1,p} + \psi_{p})
\end{bmatrix}
\begin{bmatrix}
v'_{i,p} + \theta_{v,p} \\
w'_{i,p} + \theta_{w,p}
\end{bmatrix}
\]

\[
\tilde{\phi}_i(t) = \tilde{\phi}_i(t) + \frac{1}{4}\sin(2(\phi_{i-1,p} + \psi_{p})(v'_{i,p} + \theta_{v,p})^2 - (w'_{i,p} + \theta_{w,p})^2)
\]

\[
-\frac{1}{2}\cos(2(\phi_{i-1,p} + \psi_{p})(v'_{i,p} + \theta_{v,p})(w'_{i,p} + \theta_{w,p}) + \frac{1}{2}(\theta_{v,p} + v'_{i,p})(\theta_{w,p} - w'_{i,p}) - \frac{1}{2}(\theta_{w,p} + w'_{i,p})(\theta_{v,p} - v'_{i,p})
\]
By applying Equations (2.1) - (2.4) and (2.7) successively, expressions for the shaft rotation coordinates of a multi-coupling driveline with an arbitrary static misalignment configuration can be obtained.

2.4 Misaligned Tailrotor Driveline Kinematics

Depending on the flight condition, the tailboom structure is subjected to multiple sources of aerodynamic loading. Since the steady portion of the aerodynamic loading is typically much larger than the unsteady part, Leishman and Bi (1989), the impinging aerodynamic loads tend to cause a net static deflection and misalignment of the tailboom-driveline structure. Therefore, a statically misaligned operating condition of the tailboom-driveline structure is considered. See Figure 2.7 and Figure 2.8. Here, the steady distributed rotor downwash load is \( f_{DW}(x) \), the steady horizontal-stabilizer lift and drag forces are \( F_{Lift} \) and \( F_{Drag} \), the steady horizontal-stabilizer pitch and roll moments are \( M_{Pitch} \) and \( M_{Roll} \), and the steady tailrotor anti-torque force is \( F_{TR} \). These loads are computed based on work done by Gangwani (1981), Leishman and Bi (1989) and Norman and Yamauchi (1991). These steady loads produce static misalignments at the flexible couplings \( \delta_3, \xi_3, \) and \( \gamma_3 \) in the \( n_1-n_2 \) plane and \( \delta_2, \xi_2, \) and \( \gamma_2 \) in the \( n_1-n_3 \) plane. Furthermore, the overall driveline misalignment is characterized by the \( n_1-n_2 \) and \( n_1-n_3 \) plane angles, \( \theta_v \) and \( \theta_w \), which are linearly related to the flexible coupling misalignments via (2.9).
\[
\begin{align*}
\delta_3 &= c_{Av} \theta_v, \quad \xi_3 = c_{Bv} \theta_v, \quad \gamma_3 = c_{Cv} \theta_v \\
\delta_2 &= c_{Aw} \theta_w, \quad \xi_2 = c_{Bw} \theta_w, \quad \gamma_2 = c_{Cw} \theta_w 
\end{align*}
\] (2.9)

Where the influence coefficients \(c_{Av}, c_{Aw}, c_{Bv}, c_{Bw}, c_{Cv},\) and \(c_{Cw},\) computed from the statically misaligned equilibrium condition, describe how the overall driveline misalignment is distributed among the individual driveline flexible couplings A, B and C.

Fig. 2.7: Driveline static misalignment due to aerodynamic load; side-view.
Given the input-shaft rotation angle, $\phi_0(t) = \Omega_0 t$, the static misalignments, the $n_1$-$n_2$ plane and $\delta_2$, $\xi_2$, and $\gamma_2$ in the $n_1$-$n_3$ plane, the $n_1$-$n_2$ plane and $\delta_2$, $\xi_2$, and $\gamma_2$ in the $n_1$-$n_3$ plane, $\xi_2$, $\xi_3$, $\gamma_2$, $\gamma_3$, and the coupling phase angles $\psi_A = 0^\circ$, $\psi_B = 90^\circ$, $\psi_C = 0^\circ$, expressions for intermediate and output-shaft rotation angles, $\phi_1(x,t)$, $\phi_2(x,t)$ and $\phi_3(x,t)$, are derived by successive application of Equations (2.1) - (2.4) and (2.7) starting at coupling A and proceeding through couplings B and C. See Equations (2.10) - (2.12).

$$
\phi_1(x,t) = \phi_1(x,t) + \phi_1(x,t), \quad \text{with} \quad \phi_1(t) = \Omega_0 t + \Theta_0^0 + \Theta_1^0 \sin(2\Omega_0 t) + \Theta_1^0 \cos(2\Omega_0 t)
$$

with

$$
\begin{align*}
\Theta_0^0 &= \frac{1}{2} \left[ (\delta_3 + v_{1,A}) \delta_2 - (\delta_2 + w_{1,A}) \delta_3 \right] \\
\Theta_1^0 &= \frac{1}{4} \left[ (\delta_3 + v_{1,A})^2 - (\delta_2 + w_{1,A})^2 \right] \\
\Theta_1^0 &= -\frac{1}{2} (\delta_3 + v_{1,A}) (\delta_2 + w_{1,A})
\end{align*}
$$

Fig. 2.8: Driveline static misalignment due to aerodynamic load; top-view.

* Deflections Exaggerated
\[ \phi_2(x,t) = \tilde{\phi}_2(x,t) + \tilde{\phi}_2(t), \quad \text{with} \quad \tilde{\phi}_2(t) = \Omega_0 t + \Theta_2^0 + \Theta_2^2 \sin(2\Omega_0 t) + \Theta_2^c \cos(2\Omega_0 t) \]

with

\[
\Theta_2^0 = \frac{1}{4} \left[ \left( \delta_3 + v_{1,A} \right) \delta_2 - \left( \delta_2 + w_{1,A} \right) \delta_3 + \left( \xi_3 + v_{2,B} \right) \xi_2 - \left( \xi_2 + w_{2,B} \right) \xi_3 - \left( \xi_1 + v_{1,B} \right) \right] \\
+ \frac{1}{8} \delta_3 \delta_2 \left[ \left( v_B + \xi_3 \right)^2 - \left( w_B + \xi_2 \right)^2 \right] - \left( \delta_3^2 - \delta_2^2 \right) v_B + \xi_3 \left( w_B + \xi_2 \right) \right]
\]

\[
\Theta_2^2 = \frac{1}{4} \left[ \left( v_{1,A} + \delta_3 \right)^2 - \left( w_{1,A} + \delta_2 \right)^2 - \left( \delta^2 \right) - \delta_B \right] \left( v_B + \xi_3 \right) \left( w_B + \xi_2 \right) - 4 \delta_B \left( v_B + \xi_3 \right) \left( w_B + \xi_2 \right) \right]
\]

\[
\Theta_2^c = \frac{1}{4} \left[ \left( v_{1,A} + \delta_3 \right) \left( w_{1,A} + \delta_2 \right) + \left( \delta^2 \right) - \delta_B \right] \left( v_B + \xi_3 \right) \left( w_B + \xi_2 \right) - 4 \delta_B \left( v_B + \xi_3 \right) \left( w_B + \xi_2 \right) \right]\]

\[ \phi_L(t) = \tilde{\phi}_{2,C} + \Omega_0 t + \Theta_2^0 + \Theta_2^c \sin(2\Omega_0 t) + \Theta_2^c \cos(2\Omega_0 t) \]

with

\[
\Theta_2^L = \frac{1}{4} \left[ \left( v_{1,A} + \delta_3 \right) \left( w_{1,A} + \delta_2 \right) - \left( \delta^2 \right) - \delta_B \right] \left( v_B + \xi_3 \right) \left( w_B + \xi_2 \right) - 4 \delta_B \left( v_B + \xi_3 \right) \left( w_B + \xi_2 \right) \right]
\]

\[
\Theta_2^c = \frac{1}{4} \left[ \left( v_{1,A} + \delta_3 \right) \left( w_{1,A} + \delta_2 \right) + \left( \delta^2 \right) - \delta_B \right] \left( v_B + \xi_3 \right) \left( w_B + \xi_2 \right) - 4 \delta_B \left( v_B + \xi_3 \right) \left( w_B + \xi_2 \right) \right]\]

\[ \phi_L(t) = \tilde{\phi}_{2,C} + \Omega_0 t + \Theta_2^0 + \Theta_2^c \sin(2\Omega_0 t) + \Theta_2^c \cos(2\Omega_0 t) \]

With the small dynamic misalignments due to the elastic slopes of the intermediate shafts at the couplings defined as

\[
v_{1,A}(t) = v_{1}(x,t) \bigg|_{x=0}, \quad w_{1,A}(t) = w_{1}(x,t) \bigg|_{x=0}, \quad v_{1,B}(t) = v_{1}(x,t) \bigg|_{x=L_1}, \quad w_{1,B}(t) = w_{1}(x,t) \bigg|_{x=L_1}, \quad v_{2,B}(t) = v_{2}(x,t) \bigg|_{x=0}, \quad w_{2,B}(t) = w_{2}(x,t) \bigg|_{x=L_1}, \quad v_{2,C}(t) = v_{2}(x,t) \bigg|_{x=L_2}, \quad w_{2,C}(t) = w_{2}(x,t) \bigg|_{x=L_2}, \quad v_B(t) = v_{2}(x,t) - v_{1}(x,t) \bigg|_{x=0} \quad \text{and} \quad w_B(t) = w_{2}(x,t) - w_{1}(x,t) \bigg|_{x=L_1} \]
and the elastic windup angles at couplings B and C are

$$\hat{\phi}_{\text{B}}(t) = \phi_{sB} + \dot{\phi}_1(x,t) \bigg|_{x=L_1} \quad \text{and} \quad \hat{\phi}_{\text{C}}(t) = \phi_{sC} + \dot{\phi}_2(x,t) \bigg|_{x=L_2}$$

(2.14)

Where $\phi_{sB}$ and $\phi_{sC}$ are the static portions of the elastic windup angles induced by the nominal static operating load torque, $T_L$. The static misalignments and elastic twists are assumed to be $O(1e^{-1}) = 10^\circ$ and the elastic slopes are assumed to be $O(1e^{-3}) = 0.1^\circ$.

Noting that $\Theta_i^0$, $\Theta_i^x$ and $\Theta_i^y$ ($i = 1, 2, L$), are implicit functions of time, differentiating the expressions for the rotation angles with respect to time yields the intermediate and output-shaft rotation speed speeds. $\Omega_1 = \dot{\phi}_1(x,t)$, $\Omega_2 = \dot{\phi}_2(x,t)$ and $\Omega_L = \dot{\phi}_L(t)$. Where the “···” indicates time differentiation.

### 2.5 Energy and Dissipation Functions

The equations-of-motion of the segmented driveline-fuselage structure in Figure 2.1 are derived using energy methods. The kinetic energy of both intermediate shafts is

$$T_s = \frac{1}{2} \sum_{i=1}^{2} \int_0^{L_i} \left[ m_s (\ddot{u}_i^2 + \ddot{v}_i^2 + \ddot{w}_i^2) + I_m (\dot{v}_i^2 + \dot{w}_i^2) + J_m (\Omega_i^2 + \Omega_i^y \dot{v}_i^2) + J_m (\Omega_i^x \dot{w}_i^2 - \dot{v}_i \dot{w}_i^y) \right] dx$$

(2.15)

Where $m_s$ is shaft mass per-unit-length, and $I_m$ and $J_m$ are shaft cross-sectional transverse and polar mass moments of inertia. The kinetic energy of the equivalent tail-rotor inertia load, $J_L$, on the output-shaft is

$$T_{J_L} = \frac{J_L}{2} \Omega_L^2$$

(2.16)
In this derivation it is assumed that $J_L \gg J_m L_i$ thus the NCV coupling-induced speed variation kinematics of the intermediate shafts is neglected. Therefore,

$$\phi_i(t) \approx \Omega_0 t \quad \& \quad \Omega_j(t) \approx \Omega_0 \quad \text{(for } i = 1, 2) \quad (2.17)$$

is assumed. However, the kinematic expression for the output-shaft rotation angle, $\phi_L(t)$, in Equation (2.12), and the corresponding output-shaft speed, $\Omega_L = \dot{\phi}_L(t)$, are fully utilized in the derivation. The fuselage-beam kinetic energy is

$$T_f = \frac{1}{2} \int_0^{L_f} \left[ m_f (\dot{u}_f^2 + \dot{v}_f^2 + \dot{w}_f^2) + I_{m11} \dot{\phi}_f^2 + I_{m33} \dot{\psi}_f^2 + I_{m22} \dot{\psi}_f^2 \right] dx \quad (2.18)$$

Where, $m_f$ is fuselage mass per-unit-length and $I_{m11}$, $I_{m22}$ and $I_{m33}$ are fuselage transverse and polar cross-sectional mass moments of inertia. Finally, the kinetic energy of the rigidly attached lumped tail inertia is

$$T_t = \frac{m_t}{2} \left( N_{v_{E/O}} + N_{\omega^T \times R_{ET}} \right) \left( N_{v_{E/O}} + N_{\omega^T \times R_{ET}} \right)^T \frac{1}{2} \{ N_{\omega^T} \}_{t} \cdot \{ I_{m} \}_{t} \cdot \{ N_{\omega^T} \}_{t} \quad (2.19)$$

with,

$$N_{v_{E/O}} = \dot{u}_f (L_f, t) \mathbf{n}_1 + \dot{v}_f (L_f, t) \mathbf{n}_2 + \dot{w}_f (L_f, t) \mathbf{n}_3$$

$$R_{ET} = h_{11} \mathbf{e}_1 + h_{12} \mathbf{e}_2 + h_{12} \mathbf{e}_2$$

$$N_{\omega^T} = N_{\omega^T} = \dot{\phi}_f (L_f, t) \mathbf{t}_1 - \dot{\psi}_f (L_f, t) \mathbf{t}_2 + \dot{\psi}_f (L_f, t) \mathbf{t}_3 \quad (2.20)$$

and with,

$$\{ N_{\omega^T} \}_{t} = \begin{bmatrix} \dot{\phi}_f (L_f, t) \\ \dot{\psi}_f (L_f, t) \\ \dot{\psi}_f (L_f, t) \end{bmatrix} \quad \text{and} \quad \{ I_{m} \}_{t} = \begin{bmatrix} I_{m11} & -I_{m12} & I_{m13} \\ -I_{m12} & I_{m22} & -I_{m23} \\ I_{m13} & -I_{m23} & I_{m33} \end{bmatrix} \quad (2.21)$$

Here $m_t$ is the tail mass, $N_{v_{E/O}}$ is the velocity vector of the attachment point, E, on the fuselage-beam, $R_{ET}$ is the position vector from E to the mass-center, T, of the tail and $N_{\omega^T}$ is the angular velocity vector of the fuselage cross-section body-fixed frame, $\{ \mathbf{e} \} = \{ \mathbf{e} \}$.
[e1, e2, e3], at E. Furthermore, Imt, is the matrix of moments and products of inertia for the tail about the tail body-fixed frame, \{t\} = [t1, t2, t3], at T and NTT = \( N^T \omega^T \) is the angular velocity vector of the \{t\} frame. Finally, since the tail is rigidly attached at E, NTT = NTE.

The shaft strain energy, \( V_s \), and corresponding Rayleigh dissipation function, \( D_s \), in terms of shaft material displacements and velocities in the rotating-frame are

\[
V_s = \frac{1}{2} \sum_{i=1}^{2} \int_0^{L_i} \left[ E_s A_s \dot{u}_i'^2 + E_s I_s \left( \dot{v}_i'^2 + \dot{w}_i'^2 \right) + G_s J_s \dot{\phi}_i'^2 \right] dx \tag{2.22}
\]

and

\[
D_s = \frac{\xi_s}{2} \sum_{i=1}^{2} \int_0^{L_i} \left[ E_s A_s \ddot{u}_i'^2 + E_s I_s \left( \ddot{v}_i'^2 + \ddot{w}_i'^2 \right) + G_s J_s \ddot{\phi}_i'^2 \right] dx \tag{2.23}
\]

where \( E_s \) and \( G_s \) are shaft material elastic and shear moduli, \( A_s \) is shaft cross-sectional area, \( I_s \) and \( J_s \) are shaft cross-sectional transverse and polar area moments of inertia, and \( \xi_s \) is the shaft material viscous damping coefficient. Since rotating-frame deflections and velocities are related to the fixed-frame deflections and velocities by

\[
\begin{bmatrix}
\dot{v}_i \\
\dot{w}_i
\end{bmatrix} = \begin{bmatrix}
\cos \phi_i & \sin \phi_i \\
-\sin \phi_i & \cos \phi_i
\end{bmatrix} \begin{bmatrix}
\dot{v}_i \\
\dot{w}_i
\end{bmatrix} \quad \text{and} \quad
\ddot{v}_i = \begin{bmatrix}
\cos \phi_i & \sin \phi_i \\
-\sin \phi_i & \cos \phi_i
\end{bmatrix} \ddot{v}_i + \Omega_i \begin{bmatrix}
-\sin \phi_i & \cos \phi_i \\
\cos \phi_i & -\sin \phi_i
\end{bmatrix} \dot{v}_i
\]

where \( \Omega_i = \dot{\phi}_i(t) \), the shaft strain energy and dissipation function are rewritten as

\[
V_s = \frac{1}{2} \sum_{i=1}^{2} \int_0^{L_i} \left[ E_s A_s \ddot{u}_i'^2 + E_s I_s \left( \ddot{v}_i'^2 + \ddot{w}_i'^2 \right) + G_s J_s \ddot{\phi}_i'^2 \right] dx \tag{2.25}
\]

and

\[
D_s = \frac{\xi_s}{2} \sum_{i=1}^{2} \int_0^{L_i} \left[ E_s A_s \ddot{u}_i'^2 + E_s I_s \left( \ddot{v}_i'^2 + \ddot{w}_i'^2 + 2\Omega_i [\ddot{v}_i' \dot{v}_i' - \ddot{w}_i' \dot{w}_i'] + \Omega_i^2 [\ddot{v}_i'^2 + \ddot{w}_i'^2] \right) + G_s J_s \ddot{\phi}_i'^2 \right] dx \tag{2.26}
\]
This expression for the rotating-frame dissipation function, $D_s$, is similar to the expression derived in Zorzi and Nelson (1977). The roller bearings, which support the shafts on the foundation-beam, are modeled as discrete springs in both transverse directions. Thus, the bearing strain energy and dissipation function are written

$$V_b = \sum_{i=1}^{2} \frac{1}{2} \left[ k_{v_{bi}} \left| v_{i_{rel}}^{rel} (t) \right|^2 + k_{w_{bi}} \left| w_{i_{rel}}^{rel} (t) \right|^2 \right]$$  \hspace{1cm} (2.27)

and

$$D_b = \sum_{i=1}^{2} \frac{\xi_{ib}^2}{2} \left[ k_{v_{bi}} \left| v_{i_{rel}}^{rel} (t) \right|^2 + k_{w_{bi}} \left| w_{i_{rel}}^{rel} (t) \right|^2 \right]$$  \hspace{1cm} (2.28)

Here, $k_{v_{bi}}$ and $k_{w_{bi}}$ are the $i^{th}$ bearing stiffness coefficients and $v_{i_{rel}}^{rel}$ and $w_{i_{rel}}^{rel}$ are the relative transverse displacements between the shaft and bearing the mounting point on the fuselage, taking into account offset distance, $h_s$, between shaft and fuselage neutral axes.

$$v_{i_{rel}}^{rel} (t) = v_f (L_{bi}, t) - v_i (x_{bi}, t)$$

$$w_{i_{rel}}^{rel} (t) = [w_f (L_{bi}, t) + h_s \dot{\phi}_f (L_{bi}, t)] - w_i (x_{bi}, t)$$  \hspace{1cm} (2.29)

Furthermore, the flexible coupling strain-energy and dissipation functions are

$$V_c = \sum_{i=1}^{3} \frac{1}{2} \left[ k_{ae_{ci}} \left| u_{e_{rel}}^{rel} (t) \right|^2 + k_{te_{ci}} \left( \left[ \dot{v}_{e_{rel}}^{rel} (t) \right]^2 + \left[ \dot{w}_{e_{rel}}^{rel} (t) \right]^2 \right) \right]$$  \hspace{1cm} (2.30)

and

$$D_c = \sum_{i=1}^{3} \frac{\xi_{ec}}{2} \left[ k_{ae_{ci}} \left| u_{e_{rel}}^{rel} (t) \right|^2 + k_{te_{ci}} \left( \left[ \dot{v}_{e_{rel}}^{rel} (t) \right]^2 + \left[ \dot{w}_{e_{rel}}^{rel} (t) \right]^2 \right) \right]$$  \hspace{1cm} (2.31)
Where $kt_{ci}$ and $ka_{ci}$ are the flexible-coupling transverse and axial stiffness values, $\zeta_s$ is the coupling material viscous damping coefficient, and $v_{ci}^{rel}$, $w_{ci}^{rel}$ and $u_{ci}^{rel}$ are the relative displacements between both sides of the $i^{th}$ coupling.

$$
\begin{align*}
v_{c1}^{rel}(t) &= v_f(L_{c1},t) - v_1(0,t), \quad w_{c1}^{rel}(t) = \left[w_f(L_{c1},t) + h_s\dot{\phi}_f(L_{c1},t)\right] - w_1(0,t) \\
u_{c1}^{rel}(t) &= u_f(L_{c1},t) - h_s v_f(L_{c1},t) - u_1(0,t) \\
v_{c2}^{rel}(t) &= v_2(0,t) - v_1(L_4,t), \quad w_{c2}^{rel}(t) = w_2(0,t) - w_1(L_4,t) \\
u_{c2}^{rel}(t) &= u_2(0,t) - u_1(L_4,t) \\
v_{c3}^{rel}(t) &= v_f(L_{c3},t) - v_2(L_2,t), \quad w_{c3}^{rel}(t) = \left[w_f(L_{c3},t) + h_s\dot{\phi}_f(L_{c3},t)\right] - w_2(L_2,t) \\
u_{c3}^{rel}(t) &= u_f(L_{c3},t) - h_s v_f(L_{c3},t) - u_2(L_2,t)
\end{align*}
$$

The dissipation function of the lateral shaft dampers is

$$
D_d = \sum_{i=1}^{2} \left[ c_{d_1} v_{d_1}^{rel}(t))^2 + c_{w_2} w_{d_2}^{rel}(t))^2 \right] 
$$

(2.33)

Where $c_{d_1}$ and $c_{w_2}$ are $i^{th}$ damper viscous damping coefficients, and $\dot{v}_{d_1}^{rel}$ and $\dot{w}_{d_2}^{rel}$ are relative velocities between the shaft and the damper mounting point on the fuselage.

$$
\begin{align*}
\dot{v}_{d_1}^{rel}(t) &= \dot{v}_f(L_{d_1},t) - \dot{v}_i(x_{d_1},t) \\
\dot{w}_{d_2}^{rel}(t) &= \left[\dot{w}_f(L_{d_1},t) + h_s\dot{\phi}_f(L_{d_1},t)\right] - \dot{w}_i(x_{d_1},t)
\end{align*}
$$

(2.34)

The strain energy and dissipation function of the fuselage-beam are written

$$
V_f = \frac{1}{2} \int_0^{L_f} \left[ E_f A_f u_f^2 + E_f I_{f33} v_f^2 + E_f I_{f22} w_f^2 + G_f J_{f33} \dot{\phi}_f^2 \right] dx
$$

(2.35)

and

$$
D_f = \frac{\xi_v}{2} \int_0^{L_f} \left[ E_f A_f \dot{u}_f^2 + E_f I_{f33} \dot{v}_f^2 + E_f I_{f22} \dot{w}_f^2 + G_f J_{f33} \ddot{\phi}_f^2 \right] dx
$$

(2.36)
Where $E_f$ and $G_f$ are fuselage material elastic and shear moduli, $\xi_f$ is the foundation material viscous damping coefficient, $A_f$ is fuselage cross-section area. $I_{f33}$ and $I_{f22}$ are fuselage cross-section area moments of inertia about $n_3$ and $n_2$ and $J_{fcs}$ is the effective polar area moment of inertia for the fuselage rectangular cross-section, Beer and Johnston (1992).

### 2.6 Loading Conditions and Virtual Work Expressions

As mentioned in Chapter 1, depending on flight conditions, the tailrotor driveline-fuselage structure is subjected to multiple aerodynamic loads. These loads have been thoroughly explored both experimentally and analytically by Gangwani (1981), Leishman and Bi (1989) and Norman and Yamauchi (1991). In this section the relevant characteristics and components of these and other loads is outlined.

Figure 2.9 is a schematic representation of the aerodynamic loads relevant to the tailrotor driveline-fuselage structure.
The horizontal-stabilizer lift and drag forces are $F_{\text{Lift}}$ and $F_{\text{Drag}}$, and the horizontal-stabilizer pitch and roll moments are $M_{\text{Pitch}}$ and $M_{\text{Roll}}$. For a helicopter in forward flight, the steady part of these forces and moments are due to standard airfoil lift and drag effects. Following Gangwani (1981), these forces and moments are described by,

$$
F_{\text{Lift}}(t) = C_L \frac{1}{2} \rho_{\text{air}} V_F^2 A_{hs}, \quad F_{\text{Drag}}(t) = C_D \frac{1}{2} \rho_{\text{air}} V_F^2 A_{hs},
$$

$$
M_{\text{Pitch}}(t) = C_P \frac{1}{2} \rho_{\text{air}} V_F^2 A_{hs} L_c, \quad M_{\text{Roll}}(t) = C_R \frac{1}{2} \rho_{\text{air}} V_F^2 A_{hs} L_c
$$

(2.37)

Where $C_L$, $C_D$, $C_P$, and $C_R$ are the lift, drag, pitch and roll coefficients, $\rho_{\text{air}}$ is air-density, $V_F$ is helicopter forward speed, and $A_{hs}$ and $L_c$ are the horizontal stabilizer surface area and chord length. According to Leishman and Bi (1989), in hover and in forward flight, another major source of steady loading on the tailboom structure is main-rotor downwash. The downwash loads, $f_{SW}$ and $f_{DW}$, act in both the horizontal and vertical directions and are distributed along the length of tailboom. These loads are described as

![Diagram of aerodynamic loads on tailrotor driveline-fuselage structure.](image)
\[ f_{DW}(x) = C_{P_{top}} \frac{1}{2} \rho_{air} V_{tip}^2 W_{top}, \quad f_{SW}(x) = C_{P_{side}} \frac{1}{2} \rho_{air} V_{tip}^2 W_{side} \]  

(2.38)

Where \( C_{P_{top}}(x) \) and \( C_{P_{side}}(x) \) are distributed pressure coefficients along the top and sides of the tailboom beam, \( W_{top}(x) \) and \( W_{side}(x) \) are the planform widths of the tailboom top and side, and \( V_{tip} \) is main-rotor tip speed. Both Leishman and Bi (1989) and Norman and Yamauchi (1991) measured these pressure coefficients on scaled helicopter models in wind tunnel experiments where it was shown that the pressure coefficients, \( C_{P_{top}} \) and \( C_{P_{side}} \), mainly depend on axial location, \( x \), blade-loading coefficient \( C_f/\delta \), forward speed advance ratio \( \mu \), and the main-rotor tilt angle, \( \alpha_s \). Finally, another significant source of steady aerodynamic loading is the tail-rotor anti-torque force, \( F_{TR} \), which counteracts the main-rotor torque \( T_{MR} \).

\[ F_{TR} = \frac{T_{MR}}{L_H} \]  

(2.39)

Here, \( L_H \) is the distance from the main-rotor hub to the tail-rotor hub. The virtual work expressions for the above aerodynamic loads are written as

\[
\delta W_{HS} = F_{LH} \delta [v_f(L_f, t)] + F_{Drag} \delta [u_f(L_f, t)] + M_{Roll} \delta [\hat{\phi}_f(L_f, t)] + M_{Pitch} \delta [\dot{\phi}_f(L_f, t)]
\]

\[
\delta W_{TB} = \int_0^{L_f} \left[ f_{DW}(x) \delta [v_f(x, t)] + f_{SW}(x) \delta [w_f(x, t)] \right] dx
\]

\[
\delta W_{TR} = F_{TR} \delta [w_f(L_f, t) + w'_f(L_f, t)] d_{TR_1} + \dot{\phi}_f(L_f, t) d_{TR_2}
\]

\[
\delta W_{aero} = \delta W_{HS} + \delta W_{TB} + \delta W_{TR}
\]

(2.40)

Another significant excitation source inherent to all drivelines is rotational imbalance. Rotation imbalance occurs when the mass center of the shaft cross-section is not coincident with the rotation axis. This is due to density variations and shaft geometric
imperfections that act like distributed mass imbalance, $m_{imb}^i$, at radial and angular locations, $r_{imb}^i$ and $\phi_{imb}^i$, on the shaft cross-section.

The distributed imbalance force, $\vec{f}_{imb}^i$, is given as

\[
\vec{f}_{imb}^i(x,t) = f_{v_{imb}}^i(x,t)m_2^i + f_{w_{imb}}^i(x,t)m_3^i
\]

with

\[
f_{v_{imb}}^i(x,t) = f_{imb}^i(x)\cos[\phi^i + \phi_{imb}^i(x)]
\]

and

\[
f_{w_{imb}}^i(x,t) = f_{imb}^i(x)\sin[\phi^i + \phi_{imb}^i(x)]
\]

For an initially bent shaft, with deformed shape $v_{iro}^i(x)$ and $w_{iro}^i(x)$, the imbalance parameters are given by

\[
m_{imb}^i(x) = m_s, \quad r_{imb}^i(x) = \sqrt{v_{ro}^i(x)^2 + w_{ro}^i(x)^2}, \quad \phi_{imb}^i(x) = \tan^{-1}\left[\frac{w_{ro}^i(x)}{v_{ro}^i(x)}\right]
\]

Since imbalance is typically specified by eccentricity, the following cross-sectional shaft eccentricity, $e_{cci}$, is defined as (2.43).
\[ e_{cc_i} = n_{imb}^i \left[ \frac{m_{imb}^i}{m_s + m_{imb}^i} \right] \]  

(2.43)

Which measures the cross-sectional mass center offset from the shaft geometric center.

The virtual work expression for the shaft rotational imbalance force is

\[ \sum \int \delta W_{imb} = \sum_{i=1}^{2} \int_{0}^{L_i} \left[ f_{vimb}^i(x,t) \delta [v_i(x,t)] + f_{wimb}^i(x,t) \delta [w_i(x,t)] \right] dx \]  

(2.44)

Fig. 2.11: Shaft geometric imperfection: initially bent shaft.

The virtual work expression for the static axial load, \( P_a \), is given as

\[ \delta W_{Pa} = P_a \sum_{i=1}^{2} \int_{0}^{L_i} \left[ v_i^i \delta v_i^i + w_i^i \delta w_i^i \right] dx - P_a \delta u_2 \bigg|_{x=L_2} \]  

(2.45)

Furthermore, to account for the tail-rotor torque-load, the virtual work expression for the resistive torque-load, \( T_L \), is given by

\[ \delta W_{TL} = T_L \sum_{i=1}^{2} \int_{0}^{L_i} \left[ v_i^i \delta v_i^i - w_i^i \delta w_i^i \right] dx - T_L \delta \phi_L \]  

(2.46)

Here \( T_L \) is the effective torque-load as seen through the intermediate gearbox. The first term of this expression accounts for the so-called axial torque effect on the intermediate
shafts, derived in Zorzi and Nelson (1979), and the second term accounts for the torque load on the output-shaft.

Finally, to account for effects of gravity, $g$, in the $n_1$-$n_2$ plane, (2.47) gives the virtual work expression for the gravity loading.

$$
\delta W_{\text{grav}} = \sum_{i=1}^{2} \int_{0}^{L_i} \left[ m_x g \delta [v_i(x,t)] \right] dx - \int_{0}^{L_f} \left[ m_y g \delta [v_f(x,t)] \right] dx
$$

(2.47)

### 2.7 Finite Element Formulation and Equations-of-Motion

In this section, the finite-element-method together with Lagrange’s equations are employed to obtain equations-of-motion for the tailrotor driveline-fuselage structure. The two intermediate shafts, ($i = 1, 2$), and the fuselage beam, ($i = f$), are discretized into $N_{el}^i$ two-node beam-rod-torsion elements each with element length $L_{el}^i = L_i / N_{el}^i$. Figure 2.12 shows the $j^{th}$ element of the $i^{th}$ structure with two 6-degree-of-freedom (DOF) nodes. Equation (2.48) shows the corresponding nodal and elemental displacement vectors

$$
q_{\text{node},i}^j = [v_{i,j} \theta_{3_{i,j}} \omega_{i,j} u_{i,j} \theta_{4_{i,j}} \theta_{5_{i,j}}]^T \quad \text{and} \quad q_{el,i}^j = \begin{bmatrix} q_{\text{node},i}^j \\ q_{\text{node},j+1}^j \end{bmatrix}
$$

(2.48)

Finally, the number of nodes in the $i^{th}$ assembled structure is related to the number of elements by

$$
N_{node}^i = N_{el}^i + 1
$$

(2.49)
The displacements on the domain of the $i^{th}$ structure at the $j^{th}$ element are related to the elemental displacement vector, $q_{elj}^i$, by

$$v_i(x,t) = N_v(x_{el}, L_{el}) q_{elj}^i (t), \quad w_i(x,t) = N_w(x_{el}, L_{el}) q_{elj}^i (t)$$

$$v'_i(x,t) = N_v'(x_{el}, L_{el}) q_{elj}^i (t), \quad w'_i(x,t) = N_w'(x_{el}, L_{el}) q_{elj}^i (t)$$

$$u_i(x,t) = N_u(x_{el}, L_{el}) q_{elj}^i (t), \quad \hat{\phi}_i(x,t) = N_\phi(x_{el}, L_{el}) q_{elj}^i (t)$$

(2.50)

where $x_{el} = x - (j-1)L_{el}^i$, and with shape functions $N_v, N_w, N_u$ and $N_\phi$ defined as

$$N_v = \begin{bmatrix} -3 \frac{x_{el}^2}{L_{el}^2} + 2 \frac{x_{el}^3}{L_{el}^3}, x_{el} - 2 \frac{x_{el}^2}{L_{el}^2}, 0, 0, 0, 0, 3 \frac{x_{el}^2}{L_{el}^2} - 2 \frac{x_{el}^3}{L_{el}^3}, \frac{x_{el}^3}{L_{el}^3}, \frac{x_{el}^2}{L_{el}^2}, 0, 0, 0 \end{bmatrix}$$

$$N_w = \begin{bmatrix} 0, 0, 1 - 3 \frac{x_{el}^2}{L_{el}^2} + 2 \frac{x_{el}^3}{L_{el}^3}, -x_{el} + 2 \frac{x_{el}^2}{L_{el}^2}, \frac{x_{el}^3}{L_{el}^3}, \frac{x_{el}^2}{L_{el}^2}, 0, 0, 0, 3 \frac{x_{el}^2}{L_{el}^2} - 2 \frac{x_{el}^3}{L_{el}^3}, \frac{x_{el}^3}{L_{el}^3}, \frac{x_{el}^2}{L_{el}^2}, 0, 0, 0 \end{bmatrix}$$

(2.51)

$$N_u = \begin{bmatrix} 0, 0, 0, 0, 1 - \frac{x_{el}}{L_{el}}, 0, 0, 0, \frac{x_{el}}{L_{el}} \end{bmatrix} \quad N_\phi = \begin{bmatrix} 0, 0, 0, 0, 0, 0, 0, 0, 1 - \frac{x_{el}}{L_{el}}, 0, 0, 0, 0, \frac{x_{el}}{L_{el}} \end{bmatrix}$$

Here $N_v$ and $N_w$ are based on cubic polynomial interpolation functions and $N_u$ and $N_\phi$ are based on linear interpolation functions, Meirovitch, (1980). After substituting the displacements in Equation (2.50) into the previously derived energy, dissipation and virtual work expressions and defining the global displacement vector, $q \in \mathbb{R}^{Ndof \times 1}$, as
the unconstrained equations-of-motions, are obtained using Lagrange’s Equations as

\[
\frac{d}{dt} \left[ \begin{array}{c} \dot{T}_f + T_i + T_s + T_{J_L} \\ \dot{V}_f + V_s + V_e + V_b \end{array} \right] + \frac{\partial (T_f + T_i + T_s + T_{J_L})}{\partial q} + \frac{\partial (V_f + V_s + V_e + V_b)}{\partial q} \\
\frac{\partial (D_f + D_s + D_c + D_b + D_d)}{\partial q} = Q_{F_a} + Q_{T_L} + Q_{imb} + Q_{aero} + Q_{grav}
\]

The undamped gyroscopic portion of the system with zero load-torque, zero axial-load, zero static misalignment and no external loading or excitation is

\[
\frac{d}{dt} \left[ \begin{array}{c} \dot{T}_f + T_i + T_s \\
\end{array} \right] + \frac{\partial (T_f + T_i + T_s)}{\partial q} + \frac{\partial (V_f + V_s + V_e + V_b)}{\partial q} = M\ddot{q} + G(\Omega_q)\dot{q} + Kq
\]

Here M, G, and K are the nominal mass, gyroscopic, and elastic stiffness matrices. See Appendix A for the elemental matrices. Since output-shaft speed, \(\Omega_L\), is a function of the input-shaft speed, \(\Omega_0\), the static misalignments, \(\delta_2, \delta_3, \xi_2, \xi_3, \gamma_2, \gamma_3\), and the dynamic misalignments, which are functions of \(q\) and \(\dot{q}\), the inertia-load terms are,

\[
\frac{d}{dt} \left[ \begin{array}{c} \dot{T}_{J_L} \\
\end{array} \right] + \frac{\partial T_{J_L}}{\partial q} = J_L\dot{\Omega}_L + J_L\Omega_L \left( \frac{d}{dt} \frac{\partial \Omega_L}{\partial q} + \frac{\partial \Omega_L}{\partial q} \right) = M_A(t)\ddot{q} + C_A(t)\dot{q} + K_A(t)q - F_A(t) + O(1e^{-6})
\]

Here the linear inertia-load terms are composed of the periodic inertia, damping, stiffness and forcing terms, corresponding to \(M_A(t) = M_A(t + T_p)\), \(C_A(t) = C_A(t + T_p)\), \(K_A(t) = K_A(t + T_p)\) and \(F_A(t) = F_A(t + T_p)\) respectively with period \(T_p = \frac{\pi}{\Omega_0}\). These periodic coupling matrices and forcing terms are functions of the static misalignment angles, input shaft speed, and inertia-load, \(J_L\). See Appendix B.

The dissipation terms of the system are
Here, $C_{sd}$ is the structural damping matrix, $K_{rd}$ is the skew-symmetric stiffness matrix due to shaft material damping in the rotating frame, and $C_{aux}$ is the auxiliary damping matrix due to the viscous shaft dampers.

The non-conservative work and applied loading described by the virtual work results in the generalized forces on the right-hand side of Equation (2.53). The axial-load, $P_a$, results in the generalized force, $Q_{pa}$ with

$$\partial W_{pa} = P_a \sum_{i=1}^{2} \int_{0}^{L_i} \left[ v_i' \delta \omega_i' - w_i' \delta \gamma_i' \right] dx - P_a \delta u_i \bigg|_{x=L_2} = \dot{Q}_{pa}^T \delta q \Rightarrow Q_{pa} = -K_{p} q + F_p$$

(2.57)

Here $K_p$ is that shaft transverse buckling stiffness matrix and $F_p$ is the forcing term affecting the shaft axial degrees-of-freedom. The load-torque, $T_L$, results in the generalized force, $Q_{TL}$, which is broken down into two components $Q_{1,TL}$ and $Q_{2,TL}.$

$$\partial W_{TL} = T_L \sum_{i=1}^{2} \int_{0}^{L_i} \left[ v_i' \delta \omega_i' - w_i' \delta \gamma_i' \right] dx - T_L \delta \phi_L = \dot{Q}_{1,TL}^T \delta q = \left[ \dot{Q}_{1,TL}^T + \dot{Q}_{2,TL}^T \right] \delta q$$

with

$$\dot{Q}_{1,TL}^T \delta q = T_L \sum_{i=1}^{2} \int_{0}^{L_i} \left[ v_i' \delta \omega_i' - w_i' \delta \gamma_i' \right] dx \Rightarrow Q_{1,TL} = -K_{T} q$$

$$\dot{Q}_{2,TL}^T \delta q = -T_L \delta \phi_L \text{ with } \delta \phi_L = \left[ \frac{\partial \phi_L}{\partial q} \right]^T \delta q \Rightarrow Q_{2,TL} = -T_L \frac{\partial \phi_L}{\partial q} = -K_{T}(t) q + F_{T}(t)$$

(2.58)

Here $K_T$ is the non-symmetric torque buckling matrix, and $K_{T}(t) = K_{T}(t + T_p)$ and $F_{T}(t) = F_{T}(t + T_p)$ are the periodic stiffness and forcing terms resulting from the interaction between the load-torque and the NCV couplings. $K_{T}(t)$ and $F_{T}(t)$ are functions of the
load-torque and the static misalignment angles. See Appendix B. The virtual work from the imbalance loading results in the single frequency harmonic loading \( F_{imb}(t) \).

\[
\delta W_{imb} = Q_{imb}^T \delta \dot{q} \\
Q_{imb}(t) = F_{imb} + F_{imb_{c1}} \sin(\Omega_0 t) + F_{imb_{c2}} \cos(\Omega_0 t) \tag{2.59}
\]

Additionally, effects of gravity result in the static loading vector \( F_{grav} \) given by

\[
\delta W_{grav} = Q_{grav}^T \delta \dot{q} \quad \text{with} \quad Q_{grav} = F_{grav} \tag{2.60}
\]

Furthermore, the virtual work from the steady aerodynamic loading results in the static loading vector \( F_{aero} \).

\[
\delta W_{aero} = Q_{aero}^T \delta \dot{q} \quad \text{with} \quad Q_{aero} = F_{aero} \tag{2.61}
\]

Finally, the equations-of-motion in Equation (2.53) can be written as the set of \( N_{dof} \) linear, periodically time-varying, ordinary differential equations.

\[
\begin{align*}
\left[ M + M_{\Delta}(t) \right] \ddot{q} + \left[ C_{sd} + C_{aux} + G(\Omega_0) + C_{\Delta}(t) \right] \dot{q} \\
\cdots + \left[ K + K_{rd}(\Omega_0) + K_T(T_L) + K_P(P_a) + K_{\Delta}(t) + K_{\Gamma}(t) \right] \ddot{q} = F_p + F_{\Delta}(t) + F_{\Gamma}(t) + F_{imb}(t) + F_{grav}
\end{align*} \tag{2.62}
\]

Where \( \dot{q} \) describes the system dynamics about the statically misaligned equilibrium condition, \( \ddot{q} \), determined by the steady aerodynamic loading \( F_{aero} \).

\[
q(t) = \ddot{q} + \dot{q}(t) \quad \text{and} \quad \ddot{q} = K^{-1} F_{aero} \tag{2.63}
\]

With the flexible coupling misalignment influence coefficients defined in (2.9) determined from (2.64).
Where \( C_\delta \) and \( C_\theta \) are observation matrices into the global degree of freedom vector.

### 2.8 Modal Reduction

In order to reduce the order of the system, a truncated modal transformation based on the first \( N_{\text{mode}} < N_{\text{dof}} \) eigenvector-modes of the system is performed as

\[
\begin{bmatrix}
\delta_2 \\
\xi_2 \\
\gamma_2 \\
\delta_3 \\
\xi_3 \\
\gamma_3
\end{bmatrix} = C_\delta \bar{q}, \quad \begin{bmatrix}
\theta_v \\
\theta_w
\end{bmatrix} = C_\theta \bar{q}
\]

(2.64)

Here \( \omega_{n_i} \) and \( V_i \) are \( i^{th} \) the natural frequency and corresponding normal modeshape of the undamped, non-gyroscopic system. Also, \( \Phi \in \mathbb{R}^{N_{\text{dof}} \times N_{\text{mode}}} \) is the modal transformation matrix and \( \eta \in \mathbb{R}^{N_{\text{mode}} \times 1} \) is the modal coordinate vector. After performing the modal reduction, the misaligned, supercritical, segmented tailrotor driveshaft-fuselage system with NCV couplings is described by the following set of \( N_{\text{mode}} \) linear, periodically time-varying, ordinary differential equations.

\[
\begin{bmatrix}
M + M_{\Delta}(t) & \mathcal{C}_s + \mathcal{C}_a + \mathcal{G}(\Omega(t)) + \mathcal{C}_\Delta(t) \\
\mathcal{C}_s & \mathcal{C}(\Omega(t)) + \mathcal{G}_F(T_L) + \mathcal{K}_{\Delta}(t) + \mathcal{K}_F(t)
\end{bmatrix}\dot{\eta} = \mathcal{F}_p + \mathcal{F}_\Delta(t) + \mathcal{F}_F(t) + \mathcal{F}_{\text{imb}}(t) + \mathcal{F}_{\text{grav}}
\]

(2.66)

With
\[ M = \Phi^T M \Phi, \quad M_\Delta(t) = \Phi^T M_\Delta(t) \Phi \]
\[ C_{sd} = \Phi^T C_{sd} \Phi, \quad C_{aux} = \Phi^T C_{aux} \Phi, \]
\[ G = \Phi^T G \Phi, \quad G_\Delta(t) = \Phi^T G_\Delta(t) \Phi \]
\[ K = \Phi^T K \Phi, \quad K_{rd} = \Phi^T K_{rd} \Phi, \quad K_T = \Phi^T K_T \Phi \]
\[ K_P = \Phi^T K_P \Phi, \quad K_\Delta(t) = \Phi^T K_\Delta(t) \Phi, \quad K_T(t) = \Phi^T K_T(t) \Phi \]
\[ F_P = \Phi^T F_P, \quad F_\Delta(t) = \Phi^T F_\Delta(t), \quad F_T(t) = \Phi^T F_T(t), \quad F_{imb}(t) = \Phi^T F_{imb}(t), \quad F_{grav} = \Phi^T F_{grav} \]

(2.67)

2.9 Magnetic Bearings

In this section, the force-current-displacement relations are derived for the magnetic bearings discussed in Section 1.5. Figure 2.13 is a cross-section of an 8-pole radial Active Magnetic Bearing (AMB). Here, the coils in each quadrant I, II, III, and IV are wired in series and have currents \( i_{\text{coil}_1}, i_{\text{coil}_II}, i_{\text{coil}_III}\) and \( i_{\text{coil}_IV} \).

![Fig. 2.13: 8-pole radial AMB with bias and control currents.](image-url)
To produce the net actuation force, $F_{b2}$, in the $b_2$ direction, quadrants I and III are operated together and, likewise, quadrants II and IV operate together to generate the actuation force $F_{b3}$, in the $b_3$ direction. In particular, the coil currents are operated about a bias current level, $i_{bias}$, and the coil currents are given as

$$i_{coil_{I}} = i_{bias} + i_2, \quad i_{coil_{III}} = i_{bias} - i_2$$
$$i_{coil_{II}} = i_{bias} + i_3, \quad i_{coil_{IV}} = i_{bias} - i_3$$

(2.68)

Here $i_2$ and $i_3$ are the AMB control currents. Based on the force-current-displacement relation for an individual coil, (1.2), the net actuation forces are

$$F_{b2} = G_b \left[ \frac{(i_{bias} + i_2)^2}{(h_{gap} - v_b)^2} - \frac{(i_{bias} - i_2)^2}{(h_{gap} + v_b)^2} \right]$$
$$F_{b3} = G_b \left[ \frac{(i_{bias} + i_3)^2}{(h_{gap} - w_b)^2} - \frac{(i_{bias} - i_3)^2}{(h_{gap} + w_b)^2} \right]$$

(2.69)

Where $v_b$ and $w_b$ are the shaft displacements relative to the AMB stator and $h_{gap}$ is the nominal rotor-stator airgap distance. Also, the AMB force constant $G_b$ is

$$G_b \equiv \cos \left( \frac{\pi}{4} \right) \mu_0 A_p N_w^2$$

(2.70)

Where $\mu_0$ is the magnetic permeability of free-space, $A_p$ is the pole face area and $N_w$ is the number of coil windings. After linearizing about the bias current and the nominal airgap, the linearized force-current-displacement relations for the $k^{th}$ AMB become

$$\begin{bmatrix} F_{b2,k} \\ F_{b3,k} \end{bmatrix} = \begin{bmatrix} k_x & 0 \\ 0 & k_x \end{bmatrix} \begin{bmatrix} v_{b,k} \\ w_{b,k} \end{bmatrix} + \begin{bmatrix} k_i & 0 \\ 0 & k_i \end{bmatrix} \begin{bmatrix} i_{2,k} \\ i_{3,k} \end{bmatrix}$$

(2.71)

with the AMB force-position and force-current gains, $k_x$ and $k_i$ given as

$$k_x = \frac{4G_b i_{bias}^2}{h_{gap}^3} \quad \text{and} \quad k_i = \frac{4G_b i_{bias}^2}{h_{gap}^2}$$

(2.72)
Here, $k_x$, acts like a negative stiffness term and is due to the inverse square attractive nature of the magnetic forces. In order to achieve stable levitation of the driveline, this negative stiffness effect must be counteracted by adjusting the control currents, $i_2$ and $i_3$, with an appropriate feedback control. See Chapters 4 and 5. Using (2.71) and (2.72) the generalized force representation, $F_{AMB} \in \mathbb{R}^{N_{dof} \times 1}$, for all of the AMBs on the structure can be expressed as

$$F_{AMB} = -K_{AMB} \vec{y} + Q_{AMB}u$$

(2.73)

Where $K_{AMB} \in \mathbb{R}^{N_{dof} \times N_{dof}}$ and $Q_{AMB} \in \mathbb{R}^{N_{dof} \times 2N_{AMB}}$ are the AMB negative stiffness and control current-force input matrices in global coordinates. Furthermore the control current input vector $u \in \mathbb{R}^{2N_{AMB} \times 1}$ is given as

$$u = \begin{bmatrix} i_{2,1} & i_{3,1} & \cdots & i_{2,N_{AMB}} & i_{3,N_{AMB}} \end{bmatrix}^T$$

(2.74)

Finally, in modal coordinates, the AMB matrices become

$$\overline{K}_{AMB} = \Phi^T K_{AMB} \Phi \text{ and } \overline{Q}_{AMB} = \Phi^T Q_{AMB}$$

(2.75)
Chapter 3

STABILITY ANALYSIS OF A SEGMENTED SUPERCritical DRIVELINE WITH NCV COUPLINGS SUBJECT TO MISALIGNMENT AND TORQUE

3.1 Introduction

To gain insight and develop design guidelines for the Configuration I active vibration control, a non-dimensional stability analysis of a conventional segmented driveline is performed. As discussed in Chapter 1, many researchers have studied the stability of shafts operating at supercritical speeds, where it has been shown that internal (rotating-frame) damping tends to cause whirl instability depending on the amount of external (fixed-frame) damping present. However, the effect of non-constant velocity (NCV) couplings has not been included in these analyses. Some researchers explored the stability of single U-joint/shaft-disk systems, where it has been shown that misalignment and load-torque generate periodic parametric terms that cause instability near certain shaft speeds. While the results were interesting, the single U-joint system does not really resemble a typical driveline. Furthermore, previous analyses of double U-joint/shaft systems only considered the periodic moment forcing terms, but neglected the potentially
destabilizing periodic parametric terms. Hence, the stability behavior of a segmented driveline involving multiple NCV couplings has not been studied. Since most drivelines consist of two or more flexible couplings, depending on the number of segments, it is important to understand the effect of misalignment and load-torque on the stability of such multi-coupling/shaft systems. Furthermore, with the trend toward supercritical drivelines, the interaction between different instability regions due to misalignment, load-torque and the rotating-frame damping-induced whirl instability must be assessed.

3.2 Driveline Non-Dimensional Equations-of-Motion

Equations of motion are derived for the segmented driveline system shown in Figure 3.1. This is very similar to the conventional tailrotor driveline system discussed in Chapter 2; however, to reduce complexity and obtain a non-dimensional formulation, the driveline is assumed to be mounted on a rigid foundation with a rigid hanger-bearing. Neglecting fuselage flexibility is a mild assumption since the fuselage structure is typically much larger and stiffer than the driveline. Also, the couplings are assumed to be rigid in the transverse direction, thus giving the intermediate shafts pinned-pinned boundary conditions in transverse directions. Furthermore, the static misalignment configuration considered in this chapter is less general in Chapter 2. The system consists of a fixed input-shaft, a fixed output-shaft, and two flexible intermediate shafts with lengths $L$. The shafts are connected by U-joint couplings A, B and C that are nominally phased by $90^\circ$, as is typically the case, Mancuso (1986). Specifically, the U-joint phase angles are, $\psi_A = 0^\circ$, $\psi_B = 90^\circ$ and $\psi_C = 0^\circ$. The intermediate shafts are flexible in bending
and torsion and are supported on a rigid bearing. The two mid-span dampers, with damping coefficient $C_d$, provide auxiliary viscous lateral damping to counteract destabilizing effects of shaft internal damping.

![Diagram of driveline components](image)

Fig. 3.1: Segmented driveline connected by U-joint couplings.

Here $\{m\} = [m_1, m_2, m_3]$, a fixed-frame aligned with intermediate shafts with $m_3$ normal to the page. The transverse displacements measured from the $\{m\}$ frame are defined as

$$v(x,t) = \begin{cases} v_1(x,t) & 0 \leq x \leq L \\ v_2(x-L,t) & L \leq x \leq 2L \end{cases}, \quad w(x,t) = \begin{cases} w_1(x,t) & 0 \leq x \leq L \\ w_2(x-L,t) & L \leq x \leq 2L \end{cases}$$ (3.1)

Where $v(x,t)$ and $w(x,t)$ measure deflection in the $m_2$ and $m_3$ directions. The total shaft rotation angle, $\phi(x,t)$, and elastic windup angle, $\dot{\phi}(x,t)$, are defined as

$$\phi(x,t) = \begin{cases} \phi_1(x,t) = \Phi(t) + \bar{\phi}_1(x,t) & 0 \leq x \leq L \\ \phi_2(x-L,t) = \Phi(t) + \bar{\phi}_2(x-L,t) & L \leq x \leq 2L \end{cases}, \quad \dot{\phi}(x,t) = \begin{cases} \dot{\phi}_1(x,t) & 0 \leq x \leq L \\ \dot{\phi}_2(x-L,t) & L \leq x \leq 2L \end{cases}$$ (3.2)

where $\Phi(t)$ are rigid body rotations and $\bar{\phi}_1(x,t)$ are elastic twist deformations of the intermediate shafts. It is assumed that input-shaft is driven at a constant speed, $\Omega_0$, and the output-shaft drives both a torsional inertial-load, $J_L$, and a resistive torque-load, $T_L$, with rotational speed $\Omega_L(t)$. Furthermore, the driveline is subjected to the static
misalignments, $\delta$ and $\gamma$, at the input-shaft and output-shafts in the $\mathbf{m}_1$-$\mathbf{m}_2$ plane. In terms of the driveline static misalignments defined in Chapter 2, the misalignment angles for this configuration are $[\delta_2 = \delta, \delta_3 = 0, \xi_2 = 0, \xi_3 = 0, \gamma_2 = \gamma, \gamma_3 = 0]$. Substituting these misalignments into Equation (2.11) gives the output-shaft rotation angle, $\phi_L(t)$

$$
\phi_L(t) = \hat{\phi}_{2,C} + \phi_0 + \frac{1}{2} \left[ \delta v'_{1,A} + v'_{1,B} w'_{2,B} - v'_{2,B} w'_{1,B} + \gamma v'_{2,C} \right] + \frac{\delta^2}{8} \left( v'_{2,C} (\gamma - w'_{1,C}) + v'_{B} w'_{B} \right) \\
\quad \ldots + \frac{1}{4} \sin(2\phi_0) \left[ v'_{1,A}^2 - (\delta + w'_{1,A})^2 - 4 \hat{\phi}_{1,B} v'_{B} w'_{B} - (1 - 2 \hat{\phi}_{1,B}^2) (v'_{B}^2 - w'_{B}^2) \right] \\
\quad \ldots - \frac{1}{4} \sin(2\phi_0) \left[ 4 \hat{\phi}_{2,C} v'_{2,C} (\gamma - w'_{2,C}) - (1 - 2 \hat{\phi}_{2,C}^2) (v'_{2,C}^2 - (\gamma - w'_{2,C})^2) \right] \\
\quad \ldots - \frac{1}{2} \cos(2\phi_0) \left[ (\delta + w'_{1,A}) v'_{1,A} - (1 - 2 \hat{\phi}_{1,B}^2) v'_{B} w'_{B} + \hat{\phi}_{1,B} (v'_{B}^2 - w'_{B}^2) \right] \\
\quad \ldots + \frac{1}{2} \cos(2\phi_0) \left[ (1 - 2 \hat{\phi}_{2,C}^2) v'_{2,C} (\gamma - w'_{2,C}) + \hat{\phi}_{2,C} v'_{B} (v'_{B}^2 - (\gamma - w'_{2,C})^2) \right] 
$$

(3.3)

With the small dynamic misalignments due to the elastic slopes of the intermediate shafts at the couplings defined as

$$
\begin{align*}
\left. v'_{1,A}(t) = v'_1(x,t) \right|_{x=0}, & \quad \left. w'_{1,A}(t) = w'_1(x,t) \right|_{x=0} \\
\left. v'_{1,B}(t) = v'_1(x,t) \right|_{x=L}, & \quad \left. w'_{1,B}(t) = w'_1(x,t) \right|_{x=L} \\
\left. v'_{2,B}(t) = v'_2(x,t) \right|_{x=0}, & \quad \left. w'_{2,B}(t) = w'_2(x,t) \right|_{x=0} \quad \text{and} \quad \left. v'_{B}(t) = v'_{2,B}(t) - v'_{1,B}(t) \right|_{x=L} \\
\left. v'_{2,C}(t) = v'_2(x,t) \right|_{x=L}, & \quad \left. w'_{2,C}(t) = w'_2(x,t) \right|_{x=L} \quad \text{and} \quad \left. w'_{B}(t) = w'_{2,B}(t) - w'_{1,B}(t) \right|_{x=L}
\end{align*}
$$

(3.4)

and the shaft elastic windup angles at the couplings B and C defined as

$$
\hat{\phi}_{1,B}(t) = \phi_{sB} + \phi_{B}(x,t) \big|_{x=L} \quad \text{and} \quad \hat{\phi}_{2,C}(t) = \phi_{sC} + \phi_{C}(x,t) \big|_{x=L}
$$

(3.5)

Where $\phi_{sB}$ and $\phi_{sC}$ are the static portions of the elastic windup due to the load torque, $T_L$, which are

$$
\phi_{sB} = \frac{T_L L}{G_s J_s} \quad \text{and} \quad \phi_{sC} = \phi_{sB} + \frac{T_L L}{G_s J_s}
$$

(3.6)

Furthermore, the time derivative of (3.3) yields the output-shaft speed, $\Omega_L(t) = \dot{\phi}_L(t)$.

The total kinetic energy of the driveline is
\[ T = \int_0^{2L} \left[ \frac{m_s}{2} (\dot{v}^2 + \dot{w}^2) + \frac{I_{m_s}}{2} (\dot{\omega}^2 + \dot{\omega}_m^2) + \frac{J_m}{2} (\dot{\Omega} + \Omega [w \dot{v} - v \dot{w}]) \right] \, dx + \frac{J_L}{2} \Omega_L^2 \]  

(3.7)

Since it is assumed that \( J_{m_s} L \ll J_L \), the effect of the intermediate shaft speed variation on the equations-of-motion is negligible compared to the effect of the output-shaft speed variation. Hence the full expressions for \( \phi_L \) and \( \Omega_L \) are used in the derivation but \( \phi \) and \( \Omega \) are approximated as

\[ \phi(x,t) \approx \Omega_0 t + \hat{\phi}(x,t) \quad \text{and} \quad \Omega(x,t) \approx \Omega_0 + \dot{\phi}(x,t) \]  

(3.8)

The total driveline strain energy is

\[ V = \int_0^{2L} \left[ \frac{E_s I_s}{2} (v^* + w^* x) + \frac{G_s I_s}{2} \dot{\phi}^2 \right] \, dx \]  

(3.9)

The total driveline Rayleigh dissipation function is

\[ D = \xi \int_0^{2L} \left[ \frac{E_s I_s}{2} (v^* + w^* x) + 2 \Omega_0 [v^* \dot{w} - w^* \dot{v}] + \Omega_0^2 [v^* + w^* x] + \frac{G_s I_s}{2} \dot{\phi}^2 \right] \, dx \]  

\[ \cdots + \frac{C_d}{2} \left( \dot{v}^* + \dot{w}^* x \right) \bigg|_{x=L/2} + \frac{C_d}{2} \left( \dot{v}^* + \dot{w}^* \right) \bigg|_{x=L+L/2} \]  

(3.10)

Here \( D \) accounts for both the rotating-frame shaft viscous material damping and the fixed-frame auxiliary dampers with damping coefficient \( C_d \).

To obtain equations-of-motion, the shaft transverse deflections, \( v(x,t) \) and \( w(x,t) \), and the shaft elastic twist angle, \( \dot{\phi}(x,t) \), are written in terms of the following modal expansion,

\[ v(x,t) = \Phi_v(x) \eta(t), \quad w(x,t) = \Phi_w(x) \eta(t), \quad \dot{\phi}(x,t) = \Phi_{\phi}(x) \eta(t) \]  

(3.11)

Where \( \eta(t) \) is the \( n \times 1 \) column vector of modal coordinates and \( \Phi_v(x) \), \( \Phi_w(x) \) and \( \Phi_{\phi}(x) \) are the corresponding row vectors of assumed mode shapes.
Here $\Phi_\psi(x)$ and $\Phi_w(x)$ contain the first two pinned-pinned bending mode shapes for both intermediate shafts, and $\Phi_\phi(x)$ is composed of the first fixed-free twisting mode shape. After substitution of the modal expansions (3.11) and (3.12), the functional representation of the output-shaft speed and rotation angle becomes

$$
\phi_L = \phi_L(t, \Omega_0, \delta, \gamma, \eta) \quad \text{and} \quad \Omega_L = \Omega_L(t, \Omega_0, \delta, \gamma, \eta, \dot{\eta})
$$

The virtual work, $\delta W_{T_L}$, and corresponding generalized force vector, $Q_{T_L}$, due to the output-shaft resistive torque load, $T_L$, are

$$
\delta W_{T_L} = -T_L \delta \phi_L = Q_{T_L}^T \delta \eta, \quad \text{with} \quad \delta \phi_L = \left[ \frac{\partial \phi_L}{\partial \eta} \right]^T \delta \eta \quad \text{thus} \quad Q_{T_L} = -T_L \frac{\partial \phi_L}{\partial \eta}
$$

After substituting the modal expansions into the energy and dissipation expressions in (3.7), (3.9) and (3.10), the equations-of-motion obtained via Lagrange’s equations are

$$
\begin{aligned}
\frac{d}{dt} \left[ \frac{\partial T}{\partial \eta} \right] - \frac{\partial T}{\partial \eta} \dot{\eta} + \frac{\partial V}{\partial \eta} - \frac{\partial D}{\partial \eta} + Q_{T_L} &= 0 \\
M \ddot{\eta} + [G + C_{sd} + C_{aux}] \dot{\eta} + [K + K_{rd}] \eta + J_L \frac{\partial \Omega_L}{\partial \eta} + J_L \frac{\partial \Omega_L}{\partial \eta} + J_L \frac{\partial \Omega_L}{\partial \eta} + T_L \frac{\partial \phi_L}{\partial \eta} &= 0
\end{aligned}
$$

**Nominal System**

**Misalignment & Torque NCV Terms**

Where the nominal system matrices are
\[ \mathbf{M} = \int_0^{2L} \left[ m_s \left( \Phi_v^T \Phi_v + \Phi_w^T \Phi_w \right) + I_{m_s} \left( \Phi_v^T \Phi_v' + \Phi_w^T \Phi_w' \right) + J_{m_s} \left( \Phi_{\phi}^T \Phi_{\phi} \right) \right] \, dx \]

\[ \mathbf{G} = J_{m_s} \Omega_0 \int_0^{2L} \left[ \Phi_v^T \Phi_v' - \Phi_w^T \Phi_w' \right] \, dx \]

\[ \mathbf{C}_{sd} = \xi_s \int_0^{2L} \left[ E_s I_s \left( \Phi_v^T \Phi_v' + \Phi_w^T \Phi_w' \right) + G_s J_s \left( \Phi_{\phi}^T \Phi_{\phi} \right) \right] \, dx \]

\[ \mathbf{C}_{aux} = c_d \left[ \left( \Phi_v^T \Phi_v + \Phi_w^T \Phi_w \right) \right]_{x=\frac{L}{2}} + \left[ \left( \Phi_v^T \Phi_v + \Phi_w^T \Phi_w \right) \right]_{x=\frac{3L}{2}} \]

\[ \mathbf{K} = \int_0^{2L} \left[ E_s I_s \left( \Phi_v^T \Phi_v' + \Phi_w^T \Phi_w' \right) + G_s J_s \left( \Phi_{\phi}^T \Phi_{\phi} \right) \right] \, dx \]

\[ \mathbf{K}_{rd} = \xi_r E_s I_s \Omega_0 \int_0^{2L} \left[ \Phi_v^T \Phi_v' - \Phi_w^T \Phi_w' \right] \, dx \]

\( \mathbf{M}, \mathbf{G}, \text{ and } \mathbf{K} \) are shaft inertia, gyroscopic and elastic stiffness matrices, \( \mathbf{C}_{sd} \) and \( \mathbf{C}_{aux} \) are the shaft structural and auxiliary damping matrices, and \( \mathbf{K}_{rd} \) is the skew-symmetric stiffness matrix due to the rotating-frame damping. The NCV terms due to misalignment and load-torque are obtained by substituting the expressions for \( \phi_L \) and \( \Omega_L \) into (3.15) and taking the necessary partial and time derivatives. After this procedure it becomes apparent that

\[ \frac{d}{dt} \left[ \frac{\partial \Omega_L}{\partial \eta} \right] - \frac{\partial \Omega_L}{\partial \eta} = 0 \quad (3.17) \]

After dropping higher-order non-linear terms, the terms due to misalignment become

\[ J_L \dot{\Omega}_L \frac{\partial \Omega_L}{\partial \eta} = \left[ \mathbf{M}_{\Delta_0} + \mathbf{M}_{\Delta_2} \sin(2\Omega_0 t) + \mathbf{M}_{\Delta_{2c}} \cos(2\Omega_0 t) \right] \dot{\eta} \]

\[ \cdots + \left[ \mathbf{C}_{\Delta_0} + \mathbf{C}_{\Delta_2} \sin(2\Omega_0 t) + \mathbf{C}_{\Delta_{2c}} \cos(2\Omega_0 t) \right] \dot{\eta} \]

\[ \cdots + \left[ \mathbf{K}_{\Delta_0} + \mathbf{K}_{\Delta_2} \sin(2\Omega_0 t) + \mathbf{K}_{\Delta_{2c}} \cos(2\Omega_0 t) \right] \dot{\eta} \]

\[ \cdots - \left[ \mathbf{F}_{\Delta_0} + \mathbf{F}_{\Delta_2} \sin(2\Omega_0 t) + \mathbf{F}_{\Delta_{2c}} \cos(2\Omega_0 t) \right] \]

and the terms due to load-torque become
\[ T_L \frac{\partial \phi_L}{\partial \eta} = \left[ K_{\Gamma_0} + K_{\Gamma_2} \cos(2\Omega_0 t) + K_{\Gamma_2} \cos(2\Omega_0 t) \right] \eta \]

After substituting (3.17), (3.18) and (3.19) into (3.15), the equations-of-motion become a set of \( n \) second-order linear periodically time-varying equations, see Equation (3.20).

\[
\begin{align*}
\left[ \bar{M} + \bar{M}_{\lambda_0} + \bar{M}_{\lambda_2} \cos(2f_0 \tilde{t}) + \bar{M}_{\lambda_s} \sin(2f_0 \tilde{t}) \right] \ddot{\bar{\eta}} \\
\cdots + \left[ \bar{G} + \bar{C}_{sd} + \bar{C}_{aux} + \bar{C}_{\Lambda_0} + \bar{C}_{\lambda_2} \cos(2f_0 \tilde{t}) + \bar{C}_{\lambda_s} \sin(2f_0 \tilde{t}) \right] \dot{\bar{\eta}} \\
\cdots + \left[ \bar{K} + \bar{K}_{rd} + \bar{K}_{\Lambda_0} + \bar{K}_{\lambda_2} \cos(2f_0 \tilde{t}) + \bar{K}_{\lambda_s} \sin(2f_0 \tilde{t}) \right] \bar{\eta} \\
\cdots = \bar{F}_{\lambda_0} + \bar{F}_{\Gamma_0} + \left( \bar{F}_{\lambda_2} + \bar{F}_{\Gamma_2} \right) \cos(2f_0 \tilde{t}) + \left( \bar{F}_{\lambda_s} + \bar{F}_{\Gamma_s} \right) \sin(2f_0 \tilde{t})
\end{align*}
\] (3.20)

and the nominal equations-of-motion are

\[
\bar{M} \ddot{\bar{\eta}} + \left[ \bar{G} + \bar{C}_{sd} + \bar{C}_{aux} \right] \dot{\bar{\eta}} + \left[ \bar{K} + \bar{K}_{rd} \right] \bar{\eta} = 0
\] (3.21)

Here the equations have been non-dimensionalized with respect to the first pinned-pinned bending natural frequency, \( \Omega_{ND} \), and the intermediate shaft segment length, \( L \). The “ \(*\)” operator indicates differentiation with respect to non-dimensional (N.D.) time, \( \tilde{t} \). The relevant N.D. parameters for a solid circular cross-section shaft are shown in Equations (3.22) – (3.25).

\[
\Omega_{ND} = \sqrt{\frac{E \pi^4}{m \pi^4}} \Rightarrow \tilde{t} = \pi \Omega_{ND}, \quad \bar{x} = \frac{x}{L}, \quad \bar{v} = \frac{v}{L}, \quad \bar{w} = \frac{w}{L}, \quad \varepsilon = \frac{d}{L}
\] (3.22)

Where the N.D. shaft displacements are \( \bar{v}, \bar{w} \) and the N.D. axial coordinates is \( \bar{x} \). Also, the shaft slenderness ratio based on shaft diameter, \( d \), is \( \varepsilon \).

\[
\mu = \frac{J_L}{m \pi^4}, \quad \tau = \frac{T_L}{m \pi^4 \Omega_{ND}}, \quad \tau_{\max} = \frac{\sigma_{\text{shear}}}{2 \pi^4 (1+\nu)}, \quad k_\phi = \frac{1}{2 \pi^4 (1+\nu)}, \quad \phi_s = \phi_s \frac{\varepsilon}{k_\phi}
\] (3.23)

Also, \( \mu \) and \( \tau \) are the N.D. load-inertia and load-torque parameters and \( \tau_{\max} \) is the maximum load-torque parameter based on the material shear yield strain, \( \varepsilon_{\text{shear}} \), and
Poisson’s ratio, \( \nu \). The N.D. torsion stiffness is, \( k_\phi \), and the static windup angle of the driveline at coupling C due to \( \tau \), is \( \phi_s \).

\[
f_0 = \frac{\Omega_0}{\Omega_{ND}}, \quad f_1 = \sqrt{\frac{1}{1 + \frac{\nu^2 \pi^2}{16}}} \approx 1, \quad f_2 = 4 \sqrt{\frac{1}{1 + \frac{\nu^2 \pi^2}{4}}} \approx 4, \quad f_\phi = \sqrt{\frac{k_\phi}{(\mu + \frac{\nu^2}{12})}} \approx \sqrt{\frac{k_\phi}{\mu}}
\]

(3.24)

The N.D. input-shaft speed is \( f_0 \), the first two N.D. bending natural frequencies are \( f_1 \) and \( f_2 \), and the first N.D. torsion natural frequency is \( f_\phi \).

\[
\zeta_1 = \frac{\xi_1 \Omega_{ND}}{2f_1}, \quad \zeta_2 = \frac{\xi_2 \Omega_{ND}}{2f_2}, \quad \zeta_\phi = \frac{\xi_\phi \Omega_{ND}}{2f_\phi}, \quad c_d = \frac{C_d}{m_s L \Omega_{ND}}
\]

(3.25)

Furthermore, \( \zeta_1 \), \( \zeta_2 \) and \( \zeta_\phi \) are the modal damping ratios and \( c_d \) is the N.D. auxiliary damping coefficient. Finally, the driveline equations-of-motion in (3.20) are valid under the parameter constraints shown in (3.26).

\[
\varepsilon \ll 1, \quad Jm_s L \ll J_L \Rightarrow f_\phi \ll \frac{2}{\varepsilon \pi^2} \sqrt{\frac{3}{5(1+\nu)}}, \quad \tau < \tau_{\text{max}}
\]

(3.26)

The N.D. nominal system matrices are \( \bar{M}, \bar{G}, \bar{C}_{sd}, \bar{C}_{aux}, \bar{K} \) and \( \bar{K}_{sd} \), and the remaining matrices and forcing terms are the NCV terms due to misalignment and load-torque. The inertia matrices, \( \bar{M}_{\lambda_0}, \bar{M}_{\lambda_{s2}} \) and \( \bar{M}_{\lambda_{c2}} \), the damping matrices, \( \bar{C}_{\lambda_0}, \bar{C}_{\lambda_{s2}} \) and \( \bar{C}_{\lambda_{c2}} \), the stiffness matrices, \( \bar{K}_{\lambda_0}, \bar{K}_{\lambda_{s2}} \) and \( \bar{K}_{\lambda_{c2}} \), and forcing terms, \( \bar{F}_{\lambda_0}, \bar{F}_{\lambda_{s2}} \) and \( \bar{F}_{\lambda_{c2}} \), are functions of the static misalignment angles, \( \delta \) and \( \gamma \), the inertia-load parameter, \( \mu \), and the static windup angle \( \phi_s \). Furthermore, the stiffness matrices, \( \bar{K}_{\tau_0}, \bar{K}_{\tau_{s2}} \) and \( \bar{K}_{\tau_{c2}} \), and forcing terms, \( \bar{F}_{\tau_0}, \bar{F}_{\tau_{s2}} \) and \( \bar{F}_{\tau_{c2}} \), are functions of the load-torque parameter, \( \tau \), and the static windup angle \( \phi_s \).
3.3 Stability Analysis

In the following sub-sections, the stability behavior of Equations (3.20) and (3.21) is analyzed. To establish a baseline for studying the stability of the full system, (3.20), the effect of shaft internal and external damping on the stability of the nominal system, (3.21), is first examined. Next, the stability of the full system is explored including the effects of internal and external damping, driveline misalignment, and load-torque.

As a basis for analysis, the parameters for a typical rotorcraft supercritical driveline are used in the equations. The shaft slenderness ratio is \( \varepsilon = 0.012 \), the modal damping ratios are \( \zeta_1 = \zeta_2 = \zeta_\phi = \zeta = 0.002 \), the shaft material shear yield strain is \( \varepsilon_{\text{shear}} = 2.7 \times 10^{-3} \), and Poisson’s ratio, \( \nu \), is 0.3. Finally, the inertia load parameter is, \( \mu = 0.253 \), which corresponds to a N.D. torsional natural frequency of \( f_\phi = 0.125 \). All the above parameters satisfy the constraints given in Equation (3.26). The remaining N.D. parameters, e.g. misalignment angles, \( \delta \) and \( \gamma \), load-torque, \( \tau \), external damping coefficient, \( c_d \), and shaft operating speed, \( f_0 \), will be varied to explore the stability behavior.

3.3.1 Nominal System Stability

With no misalignment and no load-torque, \((\delta = 0, \gamma = 0 \text{ and } \tau = 0)\), the driveline dynamics are described by the linear time-invariant nominal system in equation (3.21). In this case, like in Zorzi and Nelson (1977) and Chen and Ku (1991), the primary destabilizing mechanism is the rotating-frame shaft damping, \( \zeta \), which gives rise to both
the structural damping matrix, $C_{sd}$, and the destabilizing skew-symmetric stiffness matrix, $K_{rd}$. Figure 3.2 shows the whirl stability behavior in terms of the shaft speed, $f_0$, and the external to internal damping ratio.

The nominal system is stable for all subcritical operation speeds, regardless of damping. For supercritical shaft speeds, $f_0 > f_1$, external damping, $c_d$, is required for stable operation. In this analysis the external dampers are located at the mid-span of both shaft segments, which is the nodal point of the second bending mode, therefore increasing the external to internal damping ratio increases the first mode whirl-speed but has no effect on the second mode whirl-speed. This represents the realistic situation for supercritical drivelines where it may be impractical to provide external damping to all of the higher modes.

![Graph showing nominal system whirl stability behavior](image)

Fig. 3.2: Nominal system whirl stability behavior, $f_1$ 1st mode whirl-speed, $f_2$ 2nd mode whirl-speed.

The $i^{th}$ mode whirl-speed is defined as the shaft speed at which the real part of the $i^{th}$ mode eigenvalue first becomes positive and the mode becomes unstable, and overall
whirl-speed is the slowest shaft at which whirl instability occurs. Equation (3.27) shows the nominal system first and second mode whirl speeds, \( f_{w1} \) and \( f_{w2} \), and the nominal system overall whirl-speed, \( f_w \).

\[
f_{w1} = \frac{c_d}{2\zeta} + f_1, \quad f_{w2} = f_2, \quad f_w = \min(f_{w1}, f_{w2})
\]

Since, in this case, external damping does not increase the second mode whirl-speed, the largest achievable stable shaft speed range for the nominal system is \( 0 \leq |f_0| < f_2 \).

### 3.3.2 Full system stability

When misalignment and load-torque are present, the driveline dynamics are described by the linear, periodic time-varying system given by (3.20). In this analysis, the stability is determined numerically via Floquet theory by examining the eigenvalues of the Floquet Transition Matrix (FTM), see Bolotin (1963). This technique is numerically intensive, but deemed necessary to capture all the instability behavior of the system.

With Equation (3.20) recast in first order form and the forcing terms set to zero, the system is written as

\[
\begin{bmatrix}
\dot{X}_1 \\
\dot{X}_2
\end{bmatrix} = \begin{bmatrix} A_1(t) & B_1(t) \\
B_2(t) & A_2(t)\end{bmatrix} \begin{bmatrix} X_1 \\
X_2
\end{bmatrix} + \begin{bmatrix} F_1(t) \\
F_2(t)\end{bmatrix}
\]

Where \( \begin{bmatrix} A_1(t) & B_1(t) \\
B_2(t) & A_2(t)\end{bmatrix} \) is the \( 2n \times 2n \) periodic system matrix and \( \begin{bmatrix} X_1 \\
X_2\end{bmatrix} \) is the state vector. \( \bar{T} \) is the N.D. period, which is \( \bar{T} = \pi / f_0 \). Next, the FTM matrix, denoted by \( \Phi(\bar{T}) \), is generated, where

\[
\Phi(\bar{T}) = \left[ \{X_1(\bar{T})\}, \{X_2(\bar{T})\}, \ldots, \{X_{2n}(\bar{T})\} \right]
\]

and \( \left[ \{X_1(\bar{T})\}, \{X_2(\bar{T})\}, \ldots, \{X_{2n}(\bar{T})\} \right] \) are the \( 2n \) linearly independent solutions obtained by numerically integrating Equation (3.28) from 0 to \( \bar{T} \) with the following initial conditions

\[
\begin{bmatrix}
X_1(0) \\
X_2(0)
\end{bmatrix} = \begin{bmatrix} x_1(0) \\
\ldots \\
x_{2n}(0)
\end{bmatrix}
\]
The FTM matrix, $\Phi(\bar{T})$, maps the state of the system from some initial state, $X_0$, to the state at time, $\bar{t} = k\bar{T}$, such that, $X(k\bar{T}) = \Phi(\bar{T})^k X_0$, where $k$ is an integer. Thus the eigenvalues, $\lambda_i$, of $\Phi(\bar{T})$, which govern the stability of the mapping, also determine the stability of the system.

As discussed in Hsu (1963), it is expected that parametric instabilities may occur when the parameter variation frequency, which in this case is $2f_0$, is in the neighborhood of the principal, sum, and difference combination frequencies. Thus the potential shaft speed parametric instability zones are summarized as

$$f_0 = \frac{\omega_{n_i} \pm \omega_{n_j}}{2k} + \rho, \text{ for } [i,j,k=1,2,3,...]$$  \hspace{1cm} (3.31)

Where $\omega_{n_i}$ and $\omega_{n_j}$ are system natural frequencies and $\rho$ is a small frequency de-tuning parameter.

Figure 3.3 and Figure 3.4 show how the stability varies with the degree misalignment over the shaft speed range, $0 \leq |f_0| < f_2$, for several values of external damping, $c_d$, and load-torque, $\tau$. Figure 3.3 gives the results for the case of $\tau = 0$ and Figure 3.4 gives results for $\tau = 0.5 \tau_{max}$. The misalignment condition is a so-called offset misalignment, with $\delta = -\gamma$, where the input and output shafts remain parallel but are offset by some amount.
Figure 3.3 demonstrates that misalignment has both stabilizing and destabilizing effects. On one hand, for a given nominal whirl-speed, $f_w$, corresponding to a the external/internal damping ratio, $c_d/2\zeta$, small amounts of misalignment increase the effective whirl-speed by delaying the onset of instability to speeds above $f_w$. Thus, misalignment tends to stabilize the whirl instability induced by internal damping. On the other hand, despite this stabilizing effect, higher values of misalignment cause parametric instability for shaft speeds near the sum-type combinations of the system natural frequencies.

For the case with no external damping, $c_d = 0$, corresponding to $f_w = 1$ (Figure 3.3-a), it is seen that increasing the misalignment from $0^\circ$ to about $2^\circ$ delays the whirl instability from $f_0 = 1$ to about $f_0 = 1.1$. However, for misalignment greater than about $2^\circ$, bending–torsion and bending-bending parametric instability zones begin to arise near $f_0 = (f_\phi + f_1)/2$ and $f_0 = f_1$ respectively.
For cases with non-zero external damping, $c_d > 0$, small amounts of misalignment continue to provide additional stabilization of the whirl boundary beyond the amount provided by the external damping (Figure 3.3-b, d, e and f). Note, this phenomena would also be seen in Figure 3.3-c, but in this plot $c_d$ is such that $f_w = 1.75$, hence the whirl boundary is outside the plot range for this graph. As the internal damping-induced whirl instability boundary is shifted to higher shaft speeds with increasing $c_d$, more misalignment induced sum-type bending-bending and torsion-bending parametric instability zones are revealed.

**Bending-Bending Parametric Instability Zones:**

$$ f_0 = f_1 + \rho, \quad f_0 = \frac{1}{4}(f_1 + f_2) + \rho \quad \text{and} \quad f_0 = \frac{1}{2}(f_1 + f_2) + \rho \quad \text{and} \quad f_0 = f_2 + \rho $$ (3.32)

**Torsion-Bending Parametric Instability Zones:**

$$ f_0 = \frac{1}{2}(f_\phi + f_1) + \rho \quad \text{and} \quad f_0 = \frac{1}{4}(f_\phi + f_2) + \rho $$ (3.33)

As seen in Equations (3.32) and (3.33), the bending-bending combination frequencies are always supercritical, however the torsion-bending combination frequencies can be sub or supercritical depending on the torsional natural frequency $f_\phi$. In this case, since $f_\phi = 0.125$, the first torque-bending instability is subcritical, $f_0 = (f_\phi + f_1)/2 = 0.5625$ and the second torque-bending instability is supercritical, $f_0 = (f_\phi + f_2)/2 = 2.0625$. Finally, it is also apparent from Figure 3.3 that the parametric instability regions occurring at higher shaft speeds have a wider frequency width for a given misalignment.
Figure 3.4, shows the effect of non-zero load-torque, $\tau$, on the misalignment-shaft speed instability regions. Here it is seen that, similar to misalignment, load-torque has both stabilizing and destabilizing effects. On one hand, load-torque is stabilizing since it tends to enhance the misalignment induced stabilization of the whirl instability. This can be seen by comparing Figure 3.3 with Figure 3.4. This stabilization is most likely related to the asymmetric stiffness coupling matrix, $K_\phi$, which is a function of misalignment, load-inertia, load-torque and shaft speed. This is consistent with the results from Zorzi and Nelson (1977), where it was shown that bearing stiffness anisotropy delayed the whirl instability to higher speeds. In this case it is not bearing stiffness anisotropy, rather, it is an effective stiffness anisotropy created by misalignment and load-torque acting through $K_\phi$.

On the other hand, load-torque is destabilizing since it increases the size of the misalignment induced torsion-bending instability. Furthermore load-torque causes
instability at the bending-bending instability zones independent of the misalignment. This is illustrated in Figure 3.4 where the bending-bending instability zones have a finite frequency width even at zero misalignment.

3.3.3 Bending-Bending Parametric Instability

In this sub-section, since misalignment and load-torque independently cause instability at shaft speeds near the sum-type bending-bending combination frequencies, both kinds of instability are examined. Figure 3.5 and Figure 3.6 show the effect of external damping on the load-torque and misalignment induced bending-bending instability regions.

Figure 3.5 shows the misalignment induced bending-bending instability regions with $\tau = 0$ for several values of $c_d$. The size of the misalignment instability regions near $f_0 = f_1$ and $f_0 = (f_1 + f_2)/4$ are reduced by increasing the damping coefficient $c_d$ (Figure 3.5-a). However, damping is not purely stabilizing for the instability region near $f_0 = (f_1 + f_2)/2$. In this case, damping increases the threshold of destabilizing misalignment, but also increases the frequency width of the unstable region (Figure 3.5-b).
Figure 3.6 shows the load-torque induced bending-bending instability regions with $\delta = \gamma = 0^\circ$ for several values of $c_d$. Here, again external damping tends to shrink the instability region near $f_0 = f_1$ but causes spreading of the instability region near $f_0 = (f_1 + f_2)/2$.

Figure 3.5: Effect of damping on misalignment induced bending-bending instability, $\tau = 0.0$.
(a) $c_d = 0.002$, $c_d = 0.02$; (b) $c_d = 0.01$, $c_d = 0.5$;

Fig. 3.6: Effect of damping on torque induced bending-bending instability, $\delta = \gamma = 0^\circ$. (a) $c_d = 0.002$, $c_d = 0.01$, $c_d = 0.02$;
(b) $c_d = 0.01$, $c_d = 0.2$, $c_d = 0.5$;
The spreading of the misalignment and load-torque instability regions near \( f_0 = (f_1 + f_2)/2 \) could be related to the location of the external dampers. Since the dampers are at the mid-span of the shafts, external damping is provided to the first bending modes only.

### 3.3.4 Torsion-Bending Parametric Instability

In this sub-section the behavior of the misalignment induced torsion-bending instability zones is studied. Figure 3.7 and Figure 3.8 show the 1\(^{st}\) and 2\(^{nd}\) torque-bending instability zones near \( f_0 = (f_\phi + f_1)/2 \) and \( f_0 = (f_\phi + f_2)/2 \) for several values of external damping, \( c_d \), and load-torque, \( \tau \). As seen from the plots and as noted in sub-section 3.3.2, the frequency width of both torsion-bending instability zones increases with load-torque. However, load-torque alone, without misalignment, is not sufficient to cause torsion-bending instability. Figure 3.7 shows that external damping, \( c_d \), causes spreading of the 1\(^{st}\) torsion-bending instability region.
Fig. 3.7: Effect of damping, torque and misalignment on 1st torsion-bending instability. □ $c_d = 0.0$, □ $c_d = 0.04$  (a) $\tau = 0.0$; (b) $\tau = 0.25 \tau_{\text{max}}$; (c) $\tau = 0.5 \tau_{\text{max}}$; (d) $\tau = \tau_{\text{max}}$

Fig. 3.8: Effect of damping, torque and misalignment on 2nd torsion-bending instability. □ $c_d = 0.01$, □ $c_d = 0.2$  (a) $\tau = 0.0$; (b) $\tau = 0.25 \tau_{\text{max}}$; (c) $\tau = 0.5 \tau_{\text{max}}$; (d) $\tau = \tau_{\text{max}}$
Finally, since the external damper does not supply damping to either the torsion mode or the 2\textsuperscript{nd} bending mode, increasing $c_d$ has no effect on the 2\textsuperscript{nd} torsion-bending instability, see Figure 3.8.

### 3.4 Stability Analysis Summary and Conclusions

The previous sections show the effects of internal damping, external damping, misalignment, load-inertia and load-torque on the stability of a segmented shaft connected with U-Joint couplings operating at sub and supercritical speeds. The non-dimensionalized driveline model includes both shaft bending and torsion flexibility along with the kinematic effects of misalignment, coupling phase angles, and torque windup angle between the U-Joint couplings. In the equation-of-motion, internal damping results in both a damping matrix and a skew symmetric stiffness matrix, while the misalignment, load-inertia and load-torque results in constant and periodic inertia, damping, and stiffness coupling matrices, as well as forcing terms.

In the case with zero misalignment and zero load-torque, classic whirl instability occurs for shaft speeds above some supercritical operating speed that depends on the ratio of internal to external damping. Thus increasing external damping delays the onset of whirl instability to higher shaft speeds. In the case studied in this analysis, external damping can only delay whirl instability to the second critical speed, $f_w = f_2$, since the mid-span dampers do not provide damping for the second bending mode. Thus, the range shaft speed invested in the analysis is $0 \leq |f_0| < f_2$. 

When misalignment and load-torque are present, it is discovered that they both have stabilizing and destabilizing effects. For a given level of external damping, $c_d$, small amounts of misalignment, $\delta$ and $\gamma$, and load-torque, $\tau$, tend to delay the onset of whirl instability beyond the nominal critical speed $f_{crit}$. This whirl stabilization is most likely a consequence of the stiffness anisotropy created by the $K_0$ coupling matrix.

Despite the whirl stabilization, misalignment and load-torque could create shaft speed zones of parametric instability in the sum-type combination frequency regions. Specifically, misalignment causes instability in both the bending-bending and torsion-bending regions while load-torque only causes instability in the bending-bending regions. Furthermore, even though load-torque by itself does not cause instability in the torsion-bending regions, it tends to increases the frequency width of the torsion-bending instability for a given level of misalignment.

The stability results for single U-Joint systems predict both torque induced bending-bending instability and misalignment induced torsion-bending instability, however misalignment induced bending-bending instability has not been identified in previous research. This phenomena can be explained by the presence of misalignment terms in the coupling matrices that allow periodic cross-coupling between bending modes giving rise to instability in these modes.

In addition to stabilizing whirl instability, the effectiveness of using the external mid-span dampers to stabilize the parametric instability is investigated. It is shown that increasing the external damping, $c_d$, causes some parametric instability regions to shrink and others to spread. However, in both cases, increasing $c_d$ is beneficial since it increases the destabilizing torque and misalignment threshold of the parametric instability zones.
Although the stability results are for a conventionally configured driveline with no active control, the results give important insight into potential Configuration I control algorithm designs, which must be stable in the presence of the misalignment and torque induced NCV terms. In particular, since misalignment and torque NCV terms tend to cause instability when shaft speed is near bending-bending or torsion-bending sum-type combinations, a design constraint of the active control is that it should “place” the system natural frequencies to avoid this condition. For a given supercritical operating speed, assuming this constraint is satisfied together with the fact that misalignment and torque both tend to stabilize whirl instability, the maximum amount of fixed-frame damping needed for stability is the amount needed to stabilize the nominal system whirl. Thus, if the combination frequencies are avoided, the control system can be designed based on the nominal system, i.e. without the NCV terms. If the operating speed cannot be guaranteed to be outside a given combination instability zone, the control system can maintain stability by supplying sufficient damping to the particular modes involved. The damping level can be designed off-line based on given misalignment and torque bounds.
4.1 Introduction

As discussed in Chapter 1 and derived in Chapter 2, driveline misalignment and load-torque together with NCV couplings introduce linear periodic parametric terms and multiple harmonic forcing terms at integer multiples of shaft speed, $N\Omega_0$, $N = [1,2,\ldots n_h]$. Furthermore, as found in Chapter 3, misalignment and load-torque NCV effects are a significant source of instability in segmented supercritical driveline systems. Due to the significance of NCV effects on the stability and vibration response, it is clear that these effects should be accounted for in the control law design for any AMB controlled, segmented driveline system. However, NCV effects have not been considered in previous AMB-rotor vibration control investigations. To advance the state-of-the-art, the main objective of this chapter is to develop a robust active control law that can effectively suppress both the synchronous imbalance response and the multi-harmonic vibration induced by misalignment and torque NCV effects in misaligned AMB-driveline systems.
Many researchers have investigated active suppression of imbalance vibration in various AMB-rotor systems. One very successful approach for suppressing imbalance vibration is the open-loop synchronous adaptive vibration control (SAVC), Knospe, et al. (1993). In this approach, given an AMB-rotor system with a feedback controller, synchronous perturbation control inputs are generated and added to the feedback inputs. These single frequency sinusoidal control inputs are synchronized with the shaft rotation while the magnitude and phase are adjusted to minimize the imbalance response. This adjustment is performed slowly relative to the decay of the rotor’s transient response, so the process can be considered as the adaptation of a set of open-loop signals. Thus, the system is stable as long as the original closed-loop system is stable and the perturbation control inputs converge. The adaptation of the open-loop input is based on an estimate of the Fourier influence coefficient mapping, $T_{yu}$, from the control input to the response output at frequency $\Omega$. Since errors between the actual and estimated $T_{yu}$ matrix can cause the open-loop adaptation process not to converge, both the convergence and performance robustness with respect to several types of structured uncertainties in the $T_{yu}$ matrix were explored. In particular, both operating speed and static rotor stiffness uncertainty have been studied Knospe, et al. (1997).

Given the success of the SAVC approach in suppressing synchronous vibration of linear time-invariant (LTIV) plants, one focus of this chapter is to extend this control methodology to the more general case of periodically time varying (LPTV) plants with multiple harmonic excitation, e.g. the AMB-NCV-driveline system. Specifically, in this research, to ensure both stability and effective vibration suppression, a hybrid control
strategy is developed based on a decentralized PID feedback control which is augmented with a slowly updating multi-harmonic adaptive vibration control (MHAVC). The function of the PID feedback is to levitate the driveline and maintain closed-loop stability, while the MHAVC input adapts to suppress the multi-harmonic response at the shaft speed multiples, $N\Omega_0$, due to imbalance, misalignment and load-torque.

In this investigation, since misalignment and load-torque typically vary within some operating range, the NCV terms are treated as periodic time-varying model uncertainty. Thus, another focus of this chapter is to design the hybrid PID-MHAVC control system in order to ensure stability and performance robustness with respect to misalignment, load-torque and shaft imbalance uncertainties. The resulting robust PID-MHAVC controlled AMB-NCV-driveline system stability and vibration performance is investigated over a range of driveline operating conditions.

### 4.2 Segmented AMB-NCV Driveline System

Figure 4.1 is a schematic of the AMB controlled, segmented tailrotor driveline with NCV couplings. Here, the fuselage structure is assumed to be subjected to a static aerodynamic loading which produces a vertical-plane driveline misalignment, $\theta_v$, and the corresponding vertical-plane flexible coupling misalignments $\delta_3$, $\xi_3$, and $\gamma_3$. Refer to Chapter 2 for details.
Fig. 4.1: Configuration I: segmented driveline with AMBs and NCV couplings.

From the derivation in Chapter 2, the linear periodic time-varying equations-of-motion for the AMB-NCV-driveline system in modal coordinates, \( \eta \in \mathbb{R}^{N_{\text{mode}}} \) \((N_{\text{mode}} = 20)\), about the statically misaligned equilibrium condition is given by (4.1).

\[
\begin{align*}
[M + M_{\text{NCV}}(t)]\ddot{\eta} + [C + G + C_{\text{NCV}}(t)]\dot{\eta} \\
+ [K + K_{\text{rd}} + K_T + K_{\text{AMB}} + K_{\text{NCV}}(t)]\eta &= F_{\text{GRAV}} + F_{\text{NCV}}(t) + F_{\text{IMB}}(t) + Q_{\text{AMB}}u(t)
\end{align*}
\tag{4.1}
\]

Where the periodic parametric and forcing NCV terms and the synchronous shaft imbalance are given by (4.2).

\[
\begin{align*}
M_{\text{NCV}}(t) &= M_0 + M_{s2} \sin(2\Omega_0 t) + M_{c2} \cos(2\Omega_0 t) \\
C_{\text{NCV}}(t) &= C_0 + C_{s2} \sin(2\Omega_0 t) + C_{c2} \cos(2\Omega_0 t) \\
K_{\text{NCV}}(t) &= K_0 + K_{s2} \sin(2\Omega_0 t) + K_{c2} \cos(2\Omega_0 t) \\
F_{\text{NCV}}(t) &= F_0 + F_{s2} \sin(2\Omega_0 t) + F_{c2} \cos(2\Omega_0 t) \\
F_{\text{IMB}}(t) &= F_{s1} \sin(\Omega_0 t) + F_{c1} \cos(\Omega_0 t)
\end{align*}
\tag{4.2}
\]
In order to conduct a relevant numerical investigation, the model parameters are selected based on an existing supercritical driveline system. Specifically, the model parameters are chosen for the Boeing AH-64 Apache helicopter tailrotor driveline system with the passive bearings and dampers replaced with AMBs. See Table 4.1.

Table 4.1: AH-64 Driveline Parameters

<table>
<thead>
<tr>
<th>Shaft Properties</th>
<th>Operating Conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of segments, $n_{seg}$</td>
<td>Shaft Speed, $\Omega_0$</td>
</tr>
<tr>
<td>Segment length, $L_i$</td>
<td>Load Torque, $T_L$</td>
</tr>
<tr>
<td>Outer diameter, $D_i$</td>
<td>Load Inertia, $J_L$</td>
</tr>
<tr>
<td>Wall thickness, $t_s$</td>
<td></td>
</tr>
<tr>
<td>Shaft Material</td>
<td>Aluminum</td>
</tr>
<tr>
<td>Density, $\rho_s$</td>
<td>2800 kg/m$^3$</td>
</tr>
<tr>
<td>Elastic modulus, $E_s$</td>
<td>75 GPa</td>
</tr>
<tr>
<td>Shear modulus, $G_s$</td>
<td>27 GPa</td>
</tr>
<tr>
<td>Buckling torque, $T_{buck}$</td>
<td>3359 N-m</td>
</tr>
<tr>
<td>Yield torque, $T_{yield}$</td>
<td>7461 N-m</td>
</tr>
</tbody>
</table>

Furthermore, Table 4.2 summarizes the characteristics of the AMBs utilized in this analysis which were sized for the AH-64 application using a comprehensive radial magnetic bearing design code.
To investigate the effects of various amounts of driveline misalignment \( \theta \), the overall driveline misalignment angles for the vertical plane misalignment condition are (4.3)

\[
\theta_v = \theta \quad \text{and} \quad \theta_w = 0 \tag{4.3}
\]

With the flexible coupling misalignment angles given by (4.4). See Section 2.4.

\[
\begin{align*}
\delta_3 &= c_{Aw} \theta_v, \quad \xi_3 = c_{Bw} \theta_v, \quad \gamma_3 = c_{Cw} \theta_v \\
\delta_2 &= c_{Aw} \theta_w, \quad \xi_2 = c_{Bw} \theta_w, \quad \gamma_2 = c_{Cw} \theta_w \tag{4.4}
\end{align*}
\]

And where the flexible-coupling misalignment influence coefficients calculated for the AH-64 driveline-fuselage structure are given by Table 4.3.

<table>
<thead>
<tr>
<th>AMB Properties</th>
<th>AMB Location</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rotor-stator gap, ( h_{gap} )</td>
<td>AMB1, ( L_{amb1} ) 1.33 m</td>
</tr>
<tr>
<td>Backup-bearing gap, ( h_{back} )</td>
<td>AMB2, ( L_{amb2} ) 3.67 m</td>
</tr>
<tr>
<td>Saturation current, ( i_{sat} )</td>
<td>AMB3, ( L_{amb3} ) 5.34 m</td>
</tr>
<tr>
<td>Bias current, ( i_{bias} )</td>
<td>2.5 Amps</td>
</tr>
<tr>
<td>Force constant, ( G_b )</td>
<td>9.1x10^{-6} N m^2/Amp^2</td>
</tr>
<tr>
<td>Peak force capacity, ( F_{\text{max}} )</td>
<td>445 N</td>
</tr>
<tr>
<td>RMS force capacity, ( F_{\text{rms}} )</td>
<td>311 N</td>
</tr>
<tr>
<td>Control axis angle, ( \theta_b )</td>
<td>45°</td>
</tr>
<tr>
<td>Coil-Sensor offset, ( d_{sens} )</td>
<td>25.4 mm</td>
</tr>
<tr>
<td>AMB rotor length, ( L_{\text{rotor}} )</td>
<td>37.5 mm</td>
</tr>
<tr>
<td>AMB rotor mass, ( m_{\text{rotor}} )</td>
<td>0.645 kg</td>
</tr>
</tbody>
</table>
Finally, the imbalance loading is assumed to be due to a first-order sinusoidal bend or run-out distribution along each shaft segment. The resulting shaft segment imbalance load magnitude distribution is (4.5)

\[ f_{imb}(x) = m_x e_{cc} \Omega_0^2 \sin(x/L_i) \]

where \( e_{cc} \) is the eccentricity at the mid-span of each shaft segment. In this analysis \( e_{cc} \) is taken to be \( e_{cc} = 2 \times 10^{-4} \) m, which is a typical value for aluminum helicopter driveshafts, Darlow, et al. (1990). Refer to Chapter 2 for further formulation details.

### 4.3 Active Control Law Structure

Figure 4.2 is block diagram of the closed-loop AMB-driveline system developed in this investigation. \( d \in \mathbb{R}^{n_d \times 1} \) and \( z \in \mathbb{R}^{n_z \times 1} \) are disturbance input and performance output vectors. \( y \in \mathbb{R}^{n_y \times 1} \) is a vector of transverse shaft displacements measured by the AMB sensors, and \( u \in \mathbb{R}^{n_u \times 1} \) is the AMB control input vector. Since each AMB has two orthogonal axes of control and there are a total of 3 AMBs, \( n_y = n_u = 6 \).
In this hybrid control strategy, the control input, $u$, consists of two components given by

$$ u(t) = u_{PID}(t) + u_{MHAVC}(i, t) $$  \hspace{1cm} \text{(4.6)} $$

Here $u_{PID} \in \mathbb{R}^{n_u \times 1}$ is the input generated from an analog Proportional-Derivative-Integral (PID) feedback control given by

$$ u_{PID} = -K_p y - K_d \dot{y} - K_i \int y \, dt $$  \hspace{1cm} \text{(4.7)} $$

where $K_p = k_p \mathbf{I}$, $K_d = k_d \mathbf{I}$ and $K_i = k_i \mathbf{I}$ are constant diagonal matrices with the proportional, derivative and integral feedback gains $k_p$, $k_d$ and $k_i$. Furthermore, $u_{MHAVC} \in \mathbb{R}^{n_u \times 1}$ is the multi-harmonic input generated by the Multi-Harmonic-Adaptive-Vibration-Control (MHAVC) portion of the control law.

$$ u_{MHAVC}(i, t) = u_u(i) + \sum_{n=1}^{n_h} \left[ u_{u_n}(i) \sin(n\Omega_n t) + u_{v_n}(i) \cos(n\Omega_n t) \right] $$  \hspace{1cm} \text{(4.8)} $$

Fig. 4.2: Driveline with hybrid PID/Multi-Harmonic Adaptive Vibration Control.
Here $n_h$ is the number of harmonics and $u_{0}(i)$, $u_{n_{i}}(i)$ and $u_{cn}(i) \in \mathbb{R}^{n_{i} \times 1}$ for $n \in \{1,2,\ldots n_{h}\}$ are the $i$th updated Fourier coefficient vectors which are adapted to suppress steady-state vibration.

Next, since the PID feedback gains are constant, the PID controlled AMB-driveline sub-system in Figure 4.2 can be written as the LPTV state-space descriptor system (4.9).

$$
E(t) \dot{x} = A(t)x + B_{d}d(t) + B_{u}u_{MHAVC} (t) \\
y = C_{y}x \\
z = C_{z}x
$$

(4.9)

Here $x = [\eta^{T} \dot{\eta}^{T} \eta_{t}^{T}] \in \mathbb{R}^{n_{x} \times 1}$ is the closed-loop state vector, $B_{d} \in \mathbb{R}^{n_{x} \times n_{d}}$ and $B_{u} \in \mathbb{R}^{n_{x} \times n_{u}}$ are the disturbance and control input distribution matrices and $C_{z} \in \mathbb{R}^{n_{z} \times n_{x}}$ and $C_{y} \in \mathbb{R}^{n_{y} \times n_{x}}$ are performance and measured output state-space observation matrices.

Furthermore, the periodic system and derivative matrices, $A(t+T_{p}) = A(t) \in \mathbb{R}^{n_{x} \times n_{x}}$ and $E(t+T_{p}) = E(t) \in \mathbb{R}^{n_{x} \times n_{x}}$, each with period $T_{p} = \pi/\Omega_0$, are written as (4.10).

$$
E(t) = I + E_{0} + E_{s2} \sin(2\Omega_{o}t) + E_{c2} \cos(2\Omega_{o}t) \\
A(t) = A_{x} + A_{0} + A_{s2} \sin(2\Omega_{o}t) + A_{c2} \cos(2\Omega_{o}t)
$$

(4.10)

In terms of Equations (4.1), (4.2) and (4.7), the matrices in (4.9) and (4.10) are
Where $A_n$ represents the nominal PID controlled AMB-driveline system i.e. for the
operating conditions $[\Omega_0 = 0, T_L = 0, \theta = 0]$. With $P_{\text{AMB}} \in \mathbb{R}^{n_y \times \text{Nmode}}$ defined as the AMB
sensor observation matrix in modal coordinates, i.e. $\eta = P_{\text{AMB}} \eta$. Finally, the control and
disturbance matrices in (4.9) are

$$A_n = \begin{bmatrix}
0 & 0 & 0 & 0 \\
-\left(K + K_{\text{AMB}} + Q_{\text{AMB}}K_p \right) & I & 0 & 0 \\
P_{\text{AMB}} & 0 & 0 & 0 \\
C + Q_{\text{AMB}}K_d & 0 & 0 & 0 \\
\end{bmatrix}$$

$$A_0 = \begin{bmatrix}
0 & 0 & 0 & 0 \\
-\left(K_T + K_{rd} + K_0 \right) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}$$

$$A_{s2} = \begin{bmatrix}
0 & 0 & 0 & 0 \\
-\left(K_{s2} \right) & -C_{s2} & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}$$

$$A_{c2} = \begin{bmatrix}
0 & 0 & 0 & 0 \\
-\left(K_{c2} \right) & -C_{c2} & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}$$

(4.11)

$$E_0 = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}$$

$$E_{s2} = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}$$

$$E_{c2} = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}$$

(4.12)

In order to develop the MHAVC portion of the controller for the PID controlled AMB-
driveline system in (4.9), a Fourier influence coefficient mapping from the control input
to the system output must be obtained. However, since the equations in (4.9) are time-
varying, the Fourier influence coefficient mapping does not have a convenient closed-
form representation. Therefore an approximate version is obtained via a truncated harmonic balance approach.

Assuming the PID controlled AMB-driveline system in (4.9) is Bounded-Input-Bounded-Output (BIBO) stable (guaranteed through PID robust design in Section 4.4.1) and given the multi-harmonic control and disturbance inputs \( u_{\text{MHAVC}}(t) \) and \( d(t) \) in (4.8) and (4.14), the steady-state \( x \) and \( \dot{x} \) are approximated as

\[
x(t) = x_0 + \sum_{n=1}^{n_h} \left[ x_{sn} \sin(n\Omega_0 t) + x_{cn} \cos(n\Omega_0 t) \right]
\]

\[\text{(4.15)}\]

\[
\dot{x}(t) = \Omega_0 \sum_{n=1}^{n_h} \left[ x_{sn} n \cos(n\Omega_0 t) - x_{cn} n \sin(n\Omega_0 t) \right]
\]

and the steady-state outputs \( y \) and \( z \) are

\[
y(t) = y_0 + \sum_{n=1}^{n_h} \left[ y_{sn} \sin(n\Omega_0 t) + y_{cn} \cos(n\Omega_0 t) \right]
\]

\[\text{(4.16)}\]

\[
z(t) = z_0 + \sum_{n=1}^{n_h} \left[ z_{sn} \sin(n\Omega_0 t) + z_{cn} \cos(n\Omega_0 t) \right]
\]

After substituting (4.8) and (4.14)-(4.16) into (4.9) and equating like harmonics with \( n_h = 4 \), the following truncated Fourier influence coefficient representation is obtained

\[
\begin{bmatrix}
Y \\
Z
\end{bmatrix} = \begin{bmatrix}
T_{yu} & T_{yd} \\
T_{zu} & T_{zd}
\end{bmatrix}
\begin{bmatrix}
U \\
D
\end{bmatrix}
\]

\[\text{(4.17)}\]

where, \( D \in \mathbb{R}^{n_f \times n_d} \), \( U \in \mathbb{R}^{n_f \times n_u} \), \( Y \in \mathbb{R}^{n_f \times n_y} \) and \( Z \in \mathbb{R}^{n_f \times n_z} \), with \( n_f = 2n_h + 1 \), are the combined Fourier coefficient vectors.
\[ D = \begin{bmatrix} d_0^T & d_{s_1}^T & d_{s_2}^T & d_{s_3}^T & d_{s_4}^T \end{bmatrix}^T \]

\[ U = \begin{bmatrix} u_0^T & u_{s_1}^T & u_{s_2}^T & u_{s_3}^T & u_{s_4}^T \end{bmatrix}^T \]

\[ Y = \begin{bmatrix} y_0^T & y_{c_1}^T & y_{c_2}^T & y_{c_3}^T & y_{c_4}^T \end{bmatrix}^T \]

\[ Z = \begin{bmatrix} z_0^T & z_{c_1}^T & z_{c_2}^T & z_{c_3}^T & z_{c_4}^T \end{bmatrix}^T \]

(4.18)

With the influence coefficient matrices in (4.17) \( T_{yu} \in \mathbb{R}^{nfny \times nfnu} \), \( T_{yd} \in \mathbb{R}^{nfny \times nfnd} \), \( T_{zu} \in \mathbb{R}^{nfnz \times nfnu} \), \( T_{zd} \in \mathbb{R}^{nfnz \times nfnd} \) given by

\[
\begin{align*}
T_{yu} &= \begin{bmatrix} C_y & \cdots & [\Psi]\mathbb{I}^{-1} & B_u \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix} \\
T_{yd} &= \begin{bmatrix} C_y & \cdots & [\Psi]\mathbb{I}^{-1} & B_d \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix} \\
T_{zu} &= \begin{bmatrix} C_z & \cdots & B_u \\ \cdots & \cdots & \cdots \end{bmatrix} \\
T_{zd} &= \begin{bmatrix} C_z & \cdots & B_d \\ \cdots & \cdots & \cdots \end{bmatrix}
\end{align*}
\]

(4.19)

Where the matrix \( \Psi = \Psi_n(k_p, k_d, k_I) + \Delta\Psi(\Omega_0, T_L, \theta) \in \mathbb{R}^{nfnx \times nfnx} \) in terms of the system matrices in (4.9) and (4.10) is

\[
\Psi_n = \begin{bmatrix} -A_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -A_n - \Omega_0 I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \Omega_0 I & -A_n & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -A_n & -2\Omega_0 I & 0 & 0 & 0 & 0 \\ 0 & 0 & 2\Omega_0 I & -A_n & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -A_n & -3\Omega_0 I & 0 & 0 \\ 0 & 0 & 0 & 0 & 3\Omega_0 I & -A_n & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -A_n & -4\Omega_0 I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4\Omega_0 I & -A_n \end{bmatrix}
\]

(4.20)

and
Here $\Psi_n$ is the nominal system contribution and $\Delta \Psi$ is due to $\Omega_0$, $T_L$, and $\theta$ dependent terms. From the Fourier influence coefficient representation in (4.17), define the baseline response as $DT y_{PID} = 0$ and $DT z_{PID} = 0$ which is just the steady-state response of the PID controlled AMB-driveline system without any MHAVC input, (i.e. $U = 0$). In this control strategy, the MHAVC input, $U$, will be updated slowly to allow the system to reach a steady-state after each update. The steady-state response after the $i^{th}$ MHAVC update is

$$
Y_i = T_{yu} U_i + Y_{PID} \quad \text{and} \quad Z_i = T_{zd} U_i + Z_{PID}
$$

(4.22)
In order for the MHAVC to suppress driveline vibration, the MHAVC update law is obtained through least-squares minimization of the following objective function $J_i$. Refer to Appendix D for derivation details.

$$J_i = Y_i^T W Y_i + U_i^T R U_i \quad \text{with} \quad W = w_{\text{perf}} I_{n_u,n_f} \quad \text{and} \quad R = w_{\text{eff}} I_{n_u,n_f} \quad (4.23)$$

Here $W \in \mathbb{R}^{n_y \times n_y}$ and $R \in \mathbb{R}^{n_u \times n_u}$ are vibration performance and control effort weighing matrices. Where $w_{\text{perf}}$ and $w_{\text{eff}}$ (m$^2$/Amp) are scalar design variables which penalize shaft vibration and MHAVC control effort respectively. Based on least-squares minimization of $J_i$ the MHAVC update law is

$$U_{i+1} = [T_{yu}^T W T_{yu} + R]^{-1} T_{yu}^T W [T_{yu} U_i - Y_i] \quad (4.24)$$

In order to implement the MHAVC update law, the multi-harmonic transfer matrix, $T_{yu}$, must be known. However, the exact $T_{yu}$ cannot be utilized since, as shown in (4.19)-(4.21), it depends on uncertain operating conditions, $[\Omega_0, T_L, \theta]$. Therefore, some approximate version, $\hat{T}_{yu}$, must be utilized and the implemented MHAVC update law becomes.

$$U_{i+1} = [\hat{T}_{yu}^T W \hat{T}_{yu} + R]^{-1} \hat{T}_{yu}^T W [\hat{T}_{yu} U_i - Y_i] \quad (4.25)$$

Since errors between the actual $T_{yu}$ and the estimate, $\hat{T}_{yu}$, can cause (4.25) not to converge, MHAVC convergence robustness must be considered in the control design.
4.4 Robust PID-MHAVC Controller Design

Conceptually, in the hybrid PID-MHAVC control system, the function of the PID feedback is to ensure stable levitation of the AMB-driveline while the MHAVC adapts and converges to suppress the steady-state vibration. Since the MHAVC input is confined to slow adaptation rates relative to the transients of the PID controlled AMB-driveline system in (4.9), the MHAVC is essentially an adapted open-loop input which does not affect BIBO stability. Therefore, the PID and MHAVC portions of the hybrid PID-MHAVC law can be designed sequentially in two design steps. In step I, Sub-Section 4.4.1, the PID gains, \( k_p \), \( k_d \) and \( k_I \), are selected to ensure robustly stable levitation of the AMB-driveline system. In second step, Sub-Section 4.4.2, the MHAVC objective function weightings, \( w_{perf} \) and \( w_{eff} \), are designed based on convergence requirements.

From a control design point-of-view, the driveline operating conditions, \([\Omega_0, T_L, \theta]\) are assumed to be uncertain within some range. Thus, the operating condition uncertainty space, \( V_u \), is defined as

\[
V_u = \left[ \Omega_{\text{min}} \leq \Omega_0 \leq \Omega_{\text{max}}, \ T_{L_{\text{min}}} \leq T_L \leq T_{L_{\text{max}}}, \ \theta_{\text{min}} \leq \theta \leq \theta_{\text{max}} \right]
\]

and the PID-MHAVC law must be designed so that the overall system is robust with respect to these uncertainties. In particular, Table 4.4 gives the operating condition uncertainty bounds considered in this analysis.
4.4.1 Step I, Robust PID Feedback Design

In this sub-section, the PID feedback controller gains which guarantee BIBO stability of the AMB-driveline system with respect to driveline operating conditions is designed. At a given operating condition, \([\Omega_0, T_L, \theta]\), and for a given PID control design, \([k_p, k_d, k_i]\), the PID controlled AMB-driveline system (4.9) is LPTV, hence, its BIBO stability is characterized by Floquet theory. Specifically, stability is determined by evaluating the eigenvalues of the Floquet Transition Matrix of (4.9) which is

\[
\Phi(T_p) = [\{x_1(T_p)\}, \{x_2(T_p)\}, \ldots, \{x_{n_x}(T_p)\}]
\]

(4.27)

Where \([\{x_1(T_p)\}, \{x_2(T_p)\}, \ldots, \{x_{n_x}(T_p)\}]\) are the \(n_x\) linearly independent solutions at \(t = T_p\) obtained by integrating \(\dot{x} = E^{-1}(t)A(t)x\) from 0 to \(T_p\) with initial conditions \(\Phi(0) = I\).

Using this, the BIBO stability state, \(s\), for a given PID controller design and operating condition is defined as

\[
s(\Omega_0, T_L, \theta, k_p, k_d, k_i) = \begin{cases} 
0 & \text{if } \bar{\rho}[\Phi(T_p)] \geq 1 \quad \text{(unstable)} \\
1 & \text{if } \bar{\rho}[\Phi(T_p)] < 1 \quad \text{(stable)}
\end{cases}
\]

(4.28)

<table>
<thead>
<tr>
<th>Operating Condition</th>
<th>Minimum Value</th>
<th>Maximum Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Operating speed, (\Omega_0)</td>
<td>(\Omega_{\min} = 0) RPM</td>
<td>(\Omega_{\max} = 5000) RPM</td>
</tr>
<tr>
<td>Load torque, (T_L)</td>
<td>(T_{L\min} = 0) N-m</td>
<td>(T_{L\max} = 2500) N-m</td>
</tr>
<tr>
<td>Misalignment, (\theta)</td>
<td>(\theta_{\min} = 0^\circ)</td>
<td>(\theta_{\max} = 8^\circ)</td>
</tr>
</tbody>
</table>

Table 4.4: Operating Condition Uncertainty Bounds
where $\rho$ is the spectral radius. Next, as a metric for PID control design, the robust stability index, $[0 \leq S_{PID} \leq 1]$, is defined in terms of the stability state, $s$, over the operating parameter space $V_u$.

$$S_{PID}(kp, kd, kI) = \frac{1}{(\Omega_{max} - \Omega_{min})(T_{Lmax} - T_{Lmin})(\theta_{max} - \theta_{min})} \int \int \int \int s(\Omega_0, T_L, \theta, kp, kd, kI) d\Omega_0 dT_L d\theta$$  \hspace{1cm} (4.29)

For a given PID control design, $S_{PID}$ quantifies the BIBO stability robustness of (4.9) over the operating condition parameter space $V_u$. Where $S_{PID} = 1$ indicates stability over the entire $V_u$ space i.e. robust stability. Figure 4.3 shows how $S_{PID}$ varies with the active stiffness and damping gains, $kp$ and $kd$, for a given integrator gain $kI = 1000$ Amp/m-sec.

Fig. 4.3: Robust stability index, $S_{PID}$ vs. $kp$ and $kd$ with $kI = 1000$ Amp/m-sec.
This analysis shows that there is a single range of stiffness and damping gains, \( [2982 \leq k_p \leq 32320, 12 \leq k_d \leq 238] \) with \( k_I = 1000 \) Amp/m-sec, which guarantee robust BIBO stability of the AMB-NCV driveline system over the operating condition space, \( V_u \).

Furthermore, the results show that, for \( k_p < 2982 \) Amp/m, \( S_{PID} = 0 \), which indicates that the closed-loop system is unstable over the entire operating parameter space. This is a result of the negative magnetic levitation stiffness, \( K_{AMB} \), which must be counter-acted by a sufficiently large active stiffness gain, \( k_p \), for stability. To proceed with the analysis, the final robust PID controller design \( [k_p = k_p^*, k_d = k_d^* \text{ and } k_I = k_I^*] \), is selected by choosing the gains from the mid-point of the robust design region. See Table 4.5.

Table 4.5: Robust PID Feedback Design

<table>
<thead>
<tr>
<th>Final Design Gains</th>
<th>Allowable Operating Condition Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stiffness Gain, ( k_p^* )</td>
<td>15000 Amp/m</td>
</tr>
<tr>
<td>Damping Gain, ( k_d^* )</td>
<td>175 Amp-sec/m</td>
</tr>
<tr>
<td>Integrator Gain, ( k_I^* )</td>
<td>1000 Amp/m-sec</td>
</tr>
</tbody>
</table>

Additionally, Table 4.6 shows the first few closed-loop natural frequencies of the nominal PID controlled AMB driveline system using the robust PID gains, \( [k_p = k_p^*, k_d = k_d^*, k_I = k_I^*] \), in the control law.
4.4.2 Step II, Robust MHAVC Design

In this sub-section, the MHAVC control effort weighting parameter, $w_{\text{eff}}$, in the objective function (4.23), is designed to guarantee robust convergence of the open-loop MHAVC input with respect to the driveline operating condition uncertainty space, $V_u$. To implement the MHAVC portion of the control, the transfer matrix estimate, $\hat{y}_u$, used to compute the MHAVC update law in (4.25) is based on the nominal PID controlled AMB-driveline system, i.e. with PID gains $[k_p = k_p^*, \ k_d = k_d^*, \ k_i = k_i^*]$ and operating conditions $[\Omega_0 = 0, \ T_L = 0, \ \theta = 0]$.

$$\hat{T}_{yu} = T_{yu} \bigg|_{\Omega_0=0, T_L=0, \theta=0} = \begin{bmatrix} C_y & \cdots & b \end{bmatrix} \begin{bmatrix} \Psi_n (k_p^*, k_d^*, k_i^*) \end{bmatrix}^{-1} \begin{bmatrix} B_u \\ \cdots \end{bmatrix}$$  \quad (4.30)

Thus, the error between $T_{yu}$ and $\hat{T}_{yu}$ only depends on the uncertain operating condition parameters $[\Omega_0, \ T_L, \ \theta]$.

<table>
<thead>
<tr>
<th>Shaft Bending Modes (Pairs)</th>
<th>Shaft Torsion Modes</th>
</tr>
</thead>
<tbody>
<tr>
<td>1\textsuperscript{st} Mode</td>
<td>86.5 Hz</td>
</tr>
<tr>
<td>2\textsuperscript{nd} Mode</td>
<td>4.1 Hz</td>
</tr>
<tr>
<td>1\textsuperscript{st} Shaft Torsion</td>
<td>133.4 Hz</td>
</tr>
<tr>
<td>2\textsuperscript{nd} Shaft Torsion</td>
<td>176.8 Hz</td>
</tr>
<tr>
<td>3\textsuperscript{rd} Mode</td>
<td>212.1 Hz</td>
</tr>
<tr>
<td>4\textsuperscript{th} Mode</td>
<td>243.7 Hz</td>
</tr>
</tbody>
</table>

Table 4.6: Nominal PID/AMB Driveline Closed-Loop Natural Frequencies
Since the PID controlled AMB-driveline system (4.9) is guaranteed BIBO stable by use of \([k_p = k_p^*, \ k_d = k_d^*, \ k_i = k_i^*]\), the convergence of the open-loop MHAVC input is governed by the following condition.

\[
\bar{\rho}[\Lambda] \geq 1 \quad \text{Divergence} \quad \text{with} \quad \Lambda = [\hat{T}_{yu}^T W \hat{T}_{yu} + R]^{-1} \hat{T}_{yu}^T W [\hat{T}_{yu} - T_{yu}] \quad (4.31)
\]

Where the MHAVC Convergence Metric, \(\bar{\rho}[\Lambda]\), is defined as the maximum eigenvalue of the matrix \(\Lambda\). Also, the converged MHAVC input and response \(U_i \rightarrow U_c\) and \(Y_i \rightarrow Y_c\) with \(i \rightarrow \infty\) are given by

\[
U_c = -[\hat{T}_{yu}^T W T_{yu} + R]^{-1} \hat{T}_{yu}^T W Y_{PID}, \quad \text{and} \quad Y_c = T_{yu} U_c + Y_{PID} \quad (4.32)
\]

Refer to Appendix E for details.

To assess the MHAVC convergence robustness with respect to the operating condition uncertainty space, \(V_u\), the MHAVC Robust Convergence Metric is defined as

\[
\sigma(w_{perf}, w_{eff}) = \sup_{V_u} \bar{\rho}\left[A(w_{perf}, w_{eff})\right] \quad (4.33)
\]

According to this definition, \(\sigma < 1\) indicates robust convergence, i.e. the MHAVC input converges at all operating conditions in \(V_u\).

To proceed with the design, the performance weighting parameter, \(w_{perf}\), is chosen as \(w_{perf} = 1\) and the MHAVC control effort weighting parameter, \(w_{eff}\), is selected to ensure robustness. Figure 4.4 shows the MHAVC Convergence Metric, \(\bar{\rho}[\Lambda]\), versus shaft speed at two different misalignment and load torque operating conditions \([T_L, \ \theta]\) for two amounts of MHAVC control effort penalty weighting \(w_{eff}\).
This figure shows that $T_L$ and $\theta$ tend to cause divergence of the MHAVC adaptation process, (4.25), at certain shaft operating speeds. Misalignment and torque tend to affect the convergence at different speeds. Comparing Figure 4.4 (a) and (b) shows the effect of two different load-torque levels ($T_L = 0.5T_{\text{max}}$, $\theta = 4^\circ$); (b) [$T_L = T_{\text{max}}$, $\theta = 4^\circ$]. In this case, misalignment, $\theta$, tends to cause divergence near $\Omega_0 = 1500$ and $\Omega_0 = 1900$ RPM while the load-torque, $T_L$, tends to cause divergence at higher speeds, $\Omega_0 > 3000$ RPM. This figure also shows that increasing the MHAVC control effort penalty weighing, $w_{\text{eff}}$, tends to reduce $\bar{\rho}[\Lambda]$ at all operating speeds $\Omega_0$, thus improving the MHAVC convergence characteristics.

To further explore the MHAVC convergence behavior, Figure 4.5 shows the effect of driveline misalignment, $\theta$, and shaft speed, $\Omega_0$, for $0 \leq T_L \leq T_{\text{max}}$ on the...
MHAVC convergence for several values of $w_{eff}$. Here the non-shaded regions are regions where $\bar{p}[\Lambda] < 1$.

Misalignment causes the MHAVC update law to diverge near various shaft speeds similar to the parametric instability regions in Chapter 3, where the system natural frequencies are now the closed-loop eigenvalues of the nominal PID controlled AMB-driveline system. By comparing the convergence regions for the three different values of $w_{eff}$ in Figure 4.5, it is clear that $w_{eff}$ has a significant impact on the convergence. This is further illustrated in Figure 4.6, which is a plot of the convergence robustness metric, $\sigma$ versus $w_{eff}$ for two different levels of operating condition uncertainty.

Fig. 4.5: $\theta - \Omega_0$ MHAVC convergence regions for various $w_{eff}$ with $0 \leq T_L \leq T_{L,\text{max}}$.
For a given level of operating condition uncertainty, convergence robustness ($\sigma < 1$) is guaranteed by selecting a sufficiently large value of $w_{eff}$, i.e. $w_{eff} > w_{eff\text{crit}}$. However, since $w_{eff}$ penalizes the MHAVC control effort, and thus limits the magnitude of the MHAVC inputs, the vibration performance with respect to $w_{eff}$ also needs to be considered.

**4.5 PID-MHAVC Performance Analysis**

In this section, the vibration performance and required control currents for the hybrid PID-MHAVC controlled AMB-driveline system are analyzed over a range of driveline operating conditions. In particular, Sub-Section 4.5.1 investigates the effect of
**4.5.1 Steady-State Performance**

To evaluate the PID-MHAVC performance, (4.32) in conjunction with (4.16) is used to compute the converged, steady-state, shaft vibration and control current response vectors \( y(t) \) and \( u(t) \). Furthermore, to assess the overall steady-state vibration and control currents, the following scalar RMS and peak vibration and control current performance metrics are defined as

\[
y_{\text{rms}} = \frac{1}{n_y} \sum_{i=1}^{n_y} \sqrt{\int_0^T y_i(t)^2 \, dt} \quad \text{and} \quad i_{\text{rms}} = \frac{1}{n_u} \sum_{i=1}^{n_u} \sqrt{\int_0^T u_i(t)^2 \, dt} \tag{4.34}
\]

and

\[
y_{\text{max}} = \sup_{0<t<T} \left[ \sup_{1<i<n_y} y_i(t) \right] \quad \text{and} \quad i_{\text{max}} = \sup_{0<t<T} \left[ \sup_{1<i<n_u} u_i(t) \right] \tag{4.35}
\]

where \( T = \frac{2\pi}{\Omega_0} \) is the fundamental period of the response.

Using the expressions in (4.35), Figures 4.7 and 4.8 show the worst-case shaft vibration response and worst-case AMB control currents over several operating condition ranges, \( V_u \), versus the MHAVC control effort weighting design parameter \( w_{\text{eff}} \).
Fig. 4.7: PID-MHAVC/AMB-driveline converged response; worst-case vibration.

Fig. 4.8: PID-MHAVC/AMB-driveline response; worst-case control current.

For lower values of $w_{eff}$, near the minimum value required for convergence, $w_{eff \text{crit}}$,

Figure 4.7 shows that the MHAVC input is very effective at suppressing shaft vibration.
over the entire range of operating conditions, $V_u$. In particular, the shaft vibration is maintained well below the backup-bearing air-gap clearance threshold. Since no knowledge of the imbalance, misalignment or load-torque is utilized in the implementation of the control, the hybrid PID-MHAVC control strategy developed here achieves robust performance. Furthermore, Figure 4.8 shows that the required control currents are near or below the bias current level, and thus the control is feasible with the magnetic bearings considered in this analysis. Finally, as expected, when $w_{eff}$ is increased, Figures 4.7 and 4.8 show that the vibration and current responses approach the PID baseline values.

Figures 4.9 and 4.10 show the RMS shaft vibration and current, $y_{rms}$ and $i_{rms}$, at three different operating conditions and three different control settings versus the shaft speed. For the case with just synchronous imbalance excitation at frequency $\Omega_0$ and no misalignment or load-torque, i.e. $[e_{cc} > 0, T_L=0 \text{ N-m}, \theta=0^\circ]$, there are no resonance peaks in the response. This is because the closed-loop natural frequencies, $\omega_n$, of the nominal PID controlled AMB-driveline system with $[k_p = k_{p*}, k_d = k_{d*}, k_i = k_{i*}]$, are placed above the maximum shaft speed i.e. $\omega_n > \Omega_0$. However, for cases with $T_L > 0$ and $\theta > 0$, resonances are excited by the $2\Omega_0$ and $4\Omega_0$ super-synchronous NCV excitations. In this case, the peak vibration amplitudes occur for shaft speeds near $\frac{1}{4}$ and $\frac{1}{2}$ multiples of the closed-loop natural frequencies, i.e. for $\Omega_0 = \omega_n/4$ and $\Omega_0 = \omega_n/2$. Despite this, Figure 4.9 shows that the PID-MHAVC achieves significant vibrations suppression across the shaft speed range.
Fig. 4.9: PID-MHAVC/AMB-driveline response; RMS shaft vibration.

Fig. 4.10: PID-MHAVC/AMB-driveline response; RMS control current.
Figure 4.10 shows that the control current magnitudes required by the PID-MHAVC only exceed the PID baseline values near resonance. Thus, for shaft speeds operating away from resonance, PID-MHAVC achieves significant vibration suppression with negligible increases in control current magnitude.

To examine the contributions of the individual vibration and control current response harmonics at frequencies $\Omega_0$, $2\Omega_0$, $3\Omega_0$ and $4\Omega_0$, Figures 4.11 and 4.12 give the baseline PID and PID-MHAVC controlled steady-state shaft vibration and control current response harmonic amplitudes versus shaft speed, $\Omega_0$, for a particular $e_{cc}$ and $[T_L, \theta]$ operating condition at AMB1.

![Diagram showing vibration and control current response harmonics](image)

Controller Parameters:
- PID
- PID-MHAVC with $w_{eff} = 1 \times 10^4 \, \mu m^2/Amp$
- PID Gains $[k_p^* = k_d^*, k_i^* = k_d^*]$

Operating Conditions:
- $e_{cc} = 200 \, \mu m$
- $T_L = T_{L \, max}$
- $\theta = 6.0^\circ$

Fig. 4.11: PID-MHAVC/AMB-driveline vibration harmonics at AMB1.
Here, the synchronous portion of the response at $\Omega_0$ is due to the shaft imbalance and the super-synchronous response at $2\Omega_0$, $3\Omega_0$ and $4\Omega_0$ is due to the misalignment and load-torque NCV effects. This figure also demonstrates that the PID-MH AVC effectively suppresses the vibration at each of the harmonics.

Figure 4.12 shows that the PID-MH AVC only requires more control current than the baseline PID controlled system near certain critical speeds.

Finally, Figures 4.13 and 4.14 show the peak vibration amplitude, $y_{\text{max}}$, and peak control current, $i_{\text{max}}$, versus misalignment at $\Omega_0=4815$ RPM, which is the actual shaft operating speed of the AH-64 Apache tailrotor driveline. Since $\Omega_0=4815$ RPM is not at
resonance, the PID-MHAVC achieves vibration suppression without increasing the control currents above the PID baseline values. This is due to the fact that, away from resonance, the MHAVC input only changes the phase of the control input to cancel the response and thus, does not require any additional control current. This phenomena is further explored in the time-domain simulation, section 4.5.2, and is experimentally verified in Chapter 7.

Fig. 4.13: Peak vibration amplitude at the AH-64 shaft operating speed.
This section describes the signal-processing required to implement the PID-MHAVC control law and investigates the time-domain response of the PID-MHAVC controlled AMB-NCV-driveline system. The closed-loop system is implemented and simulated under Matlab Simulink™. The block diagram of the overall system is given by Figure 4.15.

Fig. 4.14: Peak AMB control current at the AH-64 shaft operating speed.
The MHAVC is implemented with 3 sub-blocks. The first sub-block, is the deconvolving Harmonic Fourier Coefficient (HFC) calculator, shown in Figure 4.16.
Here $T_s$ is the sampling period and $T_{update}$ is the MHAVC update period. In this case, the highest system natural frequency is 243 Hz, so the sampling frequency $f_s$ is selected as $f_s = 1$ kHz ($T_s = 0.001$ seconds). In order to allow the system to reach steady-state after each control update, $U_{i+1}$, the update period is selected to be $T_{update} = 1.0$ seconds. After the control update, $U_{i+1}$, is computed from the MHAVC update law in the second sub-block, the result is passed to the MHAVC synthesis sub-block which re-convolves $U_{i+1}$ to produce the time-domain input signal $u_{MHAVC}$. See Figure 4.17.

![Fig. 4.17: Multi-Harmonic Adaptive Vibration Control synthesis block.](image)

Figure 4.18 and Figure 4.19 show the shaft vibration response and the control current input from the time-domain simulation at the AH-64 operating speed, $\Omega_0 = 4815$ RPM, with the AH-64 nominal operating load-torque, $T_L = 488$ N-m, and subjected to a $\theta = 6^\circ$ misalignment and shaft imbalance, $e_{cc} = 2 \times 10^{-4}$ m. For the first 5 seconds of the simulation, the system is operating under just the robust PID control i.e. [$k_p = k_p^*, k_d = k_d^*, k_i = k_i^*$]. After 5 seconds, the MHAVC portion of the control is activated with $w_{perf} = 1$ and $w_{eff} = 1 \times 10^4 \mu m^2$/Amp.
The time domain simulations confirm the stability and convergence robustness of the design and also validates the design methodology. In particular, it confirms the validity of using of the truncated harmonic balance method to approximate the multi-harmonic transfer matrix $T_{\text{yu}}$ (4.19) of the LPTV driveline system in the convergence analysis (4.31).

Also, the simulation shows the performance robustness of the PID-MHAVC controlled system with respect to the misalignment and load torque NCV effect as well as imbalance. This is the case, since, the control is able to suppress the multi-frequency driveline vibration without knowledge of the imbalance, misalignment or load torque operating conditions.

Fig. 4.18: PID-MHAVC controlled AMB-driveline shaft vibration response.
Finally in order to more thoroughly understand the effect of the MHAVC input on the AMB driveline response, the magnitude and phase of the vibration and control current harmonics versus time are shown in Figures 4.20 and 4.21. Here, the magnitude and phase values are computed from the HFC vectors $Y_i$ and $U_i$ as computed by the real-time HFC calculator block. When the MHAVC portion of the control is activated, it adapts the magnitude and phase of the overall control input $u(t) = u_{PID} + u_{MHAVC}$ to achieve vibration suppression.

Fig. 4.19: PID-MHAVC controlled AMB-driveline control current.
Fig. 4.20: Shaft vibration magnitude and phase at AMB1.

Fig. 4.21: Control current magnitude and phase at AMB1.
4.6 Configuration I Control Law Summary and Conclusions

While previous research, together with the new findings in Chapter 3, have found that NCV couplings greatly impact driveline stability and cause significant multi-harmonic excitation, this issue has yet to be addressed in the context of active magnetic bearing control of driveshafts. Due to the nature of the NCV terms, traditional robust control strategies, such as $H\infty$, $\mu$-synthesis, and Sliding Mode control, are not adequate to guarantee both robust stability and robust vibration suppression performance. Specifically, the $H\infty$ control approach, Zhou, et al. (1996), is inadequate because it does not account for the structure of the uncertain NCV terms in the control synthesis. Since the NCV driveline system has multiple uncertain parameters, not accounting for the uncertainty structure could lead to unnecessary conservatism in the control design resulting in poor closed-loop vibration performance. Furthermore, since the uncertainty is time-varying, $\mu$-synthesis techniques, Nonami and Takayuki (1996), cannot be utilized. Finally, robust Sliding Mode control techniques, Utkin (1977) and Sinha and Miller (1995), cannot be effectively utilized since the NCV terms do not satisfy a matching condition.

To overcome these challenges, this chapter develops a robust-adaptive active control law based on a hybrid PID feedback/Multi-Harmonic Adaptive Vibration Control (PID-MHAVC) approach. This control strategy effectively suppresses both the synchronous imbalance vibration and the super-synchronous vibrations induced by misalignment and load-torque NCV effects in AMB-driveline systems.
Advancing from the open-loop SAVC developed for LTIV systems in Knospe, et al. (1993), a multi-harmonic version (MHAVC) is developed for an AMB-NCV driveline (an LPTV system). By utilizing a harmonic balance approach, a multi-harmonic Fourier coefficient transfer matrix representation, $T_{uv}$, is determined for the AMB-NCV driveline system. This representation enables the MHAVC update law to be formulated based on least-squares minimization of an objective function and allows a convergence robustness analysis to be developed.

Using these results, the robust PID-MHAVC controller was designed for an AMB-NCV driveline based on an AH-64 helicopter tailrotor driveline system. In the control design, the shaft imbalance, $e_{cc}$, and the operating conditions, $\Omega_0$, $T_L$ and $\theta$, were considered to be uncertain within a given range and thus the periodic NCV terms were treated as modeling uncertainty. Robust BIBO stability was ensured by designing the PID feedback control gains and robust convergence of the MHAVC input was ensured by adjusting a control effort weighting parameter in the MHAVC objective function.

Finally, the performance of the PID-MHAVC controlled AMB driveline system is analyzed over a range of operating conditions. The results of the performance analysis show that the PID-MHAVC control effectively suppresses the multi-harmonic vibration due to shaft imbalance and the NCV effects. Since good attenuation is achieved without using explicit knowledge of the misalignment, load-torque or imbalance, the control is deemed to have robust performance. Additionally, since the required control currents are on the order of the AMB bias currents, the PID-MHAVC law developed in this chapter is
feasible for the AH-64 tailrotor driveline application with the current magnetic bearings considered in this investigation.
Chapter 5

CONFIGURATION II: ROBUST ADAPTIVE VIBRATION CONTROL OF FLEXIBLE MATRIX COMPOSITE DRIVELINES VIA MAGNETIC BEARINGS

5.1 Introduction

Recently, a novel approach to the design of helicopter tailrotor drivelines based on newly emerging Flexible Matrix Composite (FMC) materials has been explored by, Shan and Bakis (2002), Shan and Bakis (2003), Ocalan (2002) and Shin, et al. (2003). It has been found that shafts constructed of such FMC materials can provide many benefits over conventional segmented metal alloy driveshafts. In particular, Shin, et al. (2003) found that, by proper tailoring of the FMC ply thickness and fiber orientations, FMC tailrotor driveshaft designs with high torsion stiffness and low lateral-bending stiffness can be achieved with less-weight than conventional alloy shaft designs. Due to the high strain capability of FMC materials, see Shan and Bakis (2002), it was demonstrated by Ocalan (2002) and Shin, et al. (2003) that such FMC shaft designs can safely accommodate tailrotor driveline angular misalignments and eliminate the need for
flexible couplings, which are a significant source of driveline vibration and maintenance requirements.

Despite these benefits, FMC shafts have been found to have significantly more structural damping compared with conventional alloy or rigid matrix composite shafts, Shan and Bakis (2002). Consequently, FMC shafts are more prone to both self-heating, and rotating-frame damping induced whirl instability, Shin, et al. (2003). Furthermore, another challenge posed by FMC shafts is the temperature sensitivity of the FMC material properties. In Shan and Bakis (2003), it was demonstrated that the FMC ply elastic moduli and damping loss-factors were temperature dependent. The temperature sensitivity poses several challenges for the design of stable, low vibration FMC tailrotor driveline systems. In particular, the stiffness and damping temperature sensitivity together with the stiffness and damping dependent self-heating, could allow for coupled temperature/vibration dynamics which has the potential to cause limit-cycle oscillations or instability. Furthermore, the stiffness variation due to ambient temperature fluctuations could also lead to instability or high vibration levels by shifting the system natural frequencies close to or below the driveline operating speed.

To address these issues and advance the state-of-the-art, the objective of this chapter is to explore the use of non-contact Active Magnetic Bearings (AMB) technology together with active vibration control techniques in order to guarantee stable, low vibration operation of a single piece FMC tailrotor driveline. In this chapter, a robust-adaptive vibration control strategy based on a hybrid $H_\infty$/SAVC approach is developed
which accounts for the main source of vibration (shaft imbalance) and uncertainty (temperature dependent stiffness and damping) in FMC driveline systems.

5.2 AMB-FMC Driveline System

Figure 5.1 depicts the AMB-FMC flexible driveline system investigated in this research. The system consists of a single piece FMC shaft, length $L_s$, which is driven at constant rotational speed, $\Omega$. The shaft is coupled to a fixed input-shaft and fixed output-shaft via rigid couplings and is supported by three radial AMBs at locations $L_{amb1}$, $L_{amb2}$ and $L_{amb3}$. Here, $v(x,t)$ and $w(x,t)$ are the transverse shaft deflections.
Unlike the driveline configurations explored in the previous chapters, which have flexible couplings with NCV kinematics, the current FMC driveline system has rigid couplings with constant velocity kinematics. Thus, misalignment and load-torque do not explicitly affect the driveline dynamics or stability since the NCV terms are zero.

The FMC shaft considered in this investigation is composed of 4 ply layers which are assembled in symmetric layup configuration. The ply orientations are described as 

\[[+\theta_1/-\theta_1/-\theta_2/+\theta_2]\], where \(\theta_1\) and \(\theta_2\) are the fiber-orientation angles relative the shaft axis.

Based on the layup angles and the ply material properties, the equivalent isotropic elastic and shear moduli, \(E\) and \(G\), of the assembled FMC shaft are obtained using Classical Lamination Theory, Jones (1975). See Appendix F. Since the FMC material properties are temperature dependent, the equivalent isotropic shaft material parameters are

\[
E = E(T), \quad G = G(T) \quad \text{and} \quad \xi = \xi(T) \quad (5.1)
\]

Where \(E\) and \(G\) are elastic and shear moduli, \(\xi\) is an equivalent viscous damping parameter and \(T\) is the shaft temperature. Also, the nominal shaft material properties are defined as

\[
E_n = E(T_n), \quad G_n = G(T_n) \quad \text{and} \quad \xi_n = \xi(T_n) \quad (5.2)
\]

with nominal temperature, \(T_n = 85^\circ\text{F}\). Finally, the shaft material property temperature dependence is linearly approximated about the nominal values as

\[
E(T) \approx (1 + \delta_T \delta_k)E_n, \quad G(T) \approx (1 + \delta_T \delta_r)G_n
\]

\[
\xi(T)E(T) \approx (1 + \delta_T \delta_c)\xi_nE_n \quad (5.3)
\]

Where \(\delta_k\), \(\delta_r\), and \(\delta_c\), are bending stiffness, torsion stiffness and damping temperature sensitivities and \(\delta_T = T - T_n\) is the driveline deviation temperature about \(T_n\) due to the...
ambient temperature operating conditions. See Tables 5.1 and 5.2 for the FMC shaft material properties considered in this investigation.

Table 5.1: FMC Material Properties and Ply Configuration

<table>
<thead>
<tr>
<th>FMC Ply Material Properties a</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Density, ( \rho_s )</td>
<td>1650 kg/m³</td>
</tr>
<tr>
<td>Longitudinal modulus, ( E_l )</td>
<td>115 GPa</td>
</tr>
<tr>
<td>Transverse modulus, ( E_t )</td>
<td>0.275 GPa</td>
</tr>
<tr>
<td>Shear modulus, ( G_{lt} )</td>
<td>0.250 GPa</td>
</tr>
<tr>
<td>Longitudinal loss-factor, ( \eta_l )</td>
<td>0.0011</td>
</tr>
<tr>
<td>Transverse loss-factor, ( \eta_t )</td>
<td>0.080</td>
</tr>
<tr>
<td>Shear loss-factor, ( \eta_{lt} )</td>
<td>0.085</td>
</tr>
<tr>
<td>Poisson’s Ratio, ( \nu_{lt} )</td>
<td>0.38</td>
</tr>
</tbody>
</table>

FMC Shaft Ply Configuration

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of ply layers, ( n_l )</td>
<td>4</td>
</tr>
<tr>
<td>Ply layer thickness, ( t_l )</td>
<td>0.772 mm</td>
</tr>
<tr>
<td>Fiber orientations, ( \theta_l )</td>
<td>[+45°/-45°/-90°/+90°]</td>
</tr>
</tbody>
</table>

a All values at nominal temperature, \( T_n = 85°F \), Shan and Bakis (2003)

Table 5.2: Equivalent Isotropic Properties

<table>
<thead>
<tr>
<th>FMC Shaft Equivalent Isotropic Properties</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Density, ( \rho_s )</td>
<td>1650 kg/m³</td>
</tr>
<tr>
<td>Elastic modulus (nominal), ( E_n )</td>
<td>11.84 GPa</td>
</tr>
<tr>
<td>Shear modulus (nominal), ( G_m )</td>
<td>14.51 GPa</td>
</tr>
<tr>
<td>Equivalent viscous damping a, ( \xi_n )</td>
<td>5.85 x10⁻⁵</td>
</tr>
</tbody>
</table>

Temperature Sensitivity Parameters

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Bending stiffness sensitivity, ( \delta_k )</td>
<td>-0.0625 % / °F</td>
</tr>
<tr>
<td>Torsion stiffness sensitivity, ( \delta_{\tau} )</td>
<td>-2.0x10⁻³ % / °F</td>
</tr>
<tr>
<td>Damping sensitivity, ( \delta_c )</td>
<td>0.125 % / °F</td>
</tr>
</tbody>
</table>

a Viscous damping based on loss-factor at 10 Hz

The AMBs are modeled based on the linearized force-current-displacement relations about some bias current level, \( i_{bias} \). The AMB-FMC driveline equations-of-motion are obtained via Finite Element Method (FEM) and the model order is reduced by
performing a truncated modal transformation retaining the first ten ($N_{\text{mode}} = 10$) undamped, non-gyroscopic, mass-normalized modes. The resulting equations-of-motion in terms of the truncated modal coordinate vector $\eta(t) \in \mathbb{R}^{N_{\text{mode}}}$ is

$$
M \ddot{\eta} + [(1 + \delta T \delta_c)C + G(\Omega)]\dot{\eta} + [(1 + \delta T \delta_k)K + (1 + \delta T \delta_e)K_{rd}(\Omega) + K_{AMB}]\eta \\
= Q_{\text{IMB}}[\sin \Omega t \quad \cos \Omega t]^T + Q_{\text{AMB}}u(t)
$$

(5.4)

Where “ $\cdot$ ” indicates time differentiation and the matrices $M$, $C$, $G$ and $K$ are the nominal inertia, structural damping, gyroscopic and elastic stiffness matrices with $M = I$ (identity matrix). $K_{rd}$ is the skew-symmetric rotating-frame damping-stiffness matrix, Zorzi and Nelson (1977), and $K_{AMB}$, is the negative AMB stiffness matrix for the bias current level $i_{\text{bias}}$. Finally, $Q_{\text{IMB}} \in \mathbb{R}^{N_{\text{mode}}^2}$ describes the synchronous shaft imbalance excitation and $Q_{\text{AMB}} \in \mathbb{R}^{N_{\text{mode}}^2 \times 1}$ is the AMB force-current input distribution matrix, with control current input vector $u(t) \in \mathbb{R}^{n_u \times 1}$.

In this investigation, the AMB-FMC driveline is sized to replace the conventional segmented tailrotor driveline of the Boeing AH-64 Apache helicopter. In particular, the overall driveline length, $L_s$, and the shaft outer diameter, $D_s$, is kept the same as the original AH-64 driveline. Furthermore, the FMC shaft wall thickness, $t_s$, is selected such that the torsion stiffness of FMC driveline also matches that of the original AH-64 driveline. Tables 5.3 and 5.4 summarize the driveline and AMB parameters considered in this analysis. Here, the AMBs are the same design as the ones utilized in Configuration I (Chapter 4). However, due to the lower bending stiffness of the FMC shaft, a lower bias current setting, $i_{\text{bias}} = 1.5$ Amps, is selected compared with $i_{\text{bias}} = 2.5$ Amps, in the alloy shaft case.
To proceed with development of the active control, the AMB-FMC driveline system (5.4) is first recast into state-space form.
\[ \dot{x} = \left[ A_n + \delta r A_T \right] x + B_d d(t) + B_u u(t) \]
\[ z = C_x x \]
\[ y = C_y x \]

(5.5)

With state vector \( x = [\eta^T \ \eta^T] \in \mathbb{R}^{n_x \times 1} \), the nominal system matrix, \( A_n \), and deviation temperature-induced perturbation matrix, \( A_T \), are written as

\[ A_n = \begin{bmatrix} 0 & I \\ -(K + K_{rd} + K_{AMB}) & -(C + G) \end{bmatrix} \quad A_T = \begin{bmatrix} 0 & 0 \\ -(\delta r K + \delta r K_{rd}) & -\delta r C \end{bmatrix} \]

(5.6)

Also, \( d(t) = [\sin \Omega t \ \cos \Omega t]^T \) is the synchronous disturbance input due to imbalance and \( u \in \mathbb{R}^{n_u \times 1} \) is the AMB control input with corresponding input distribution matrices \( B_d \in \mathbb{R}^{n_d \times n_d} \) and \( B_u \in \mathbb{R}^{n_u \times n_u} \) written as

\[ B_d = \begin{bmatrix} 0 \\ Q_{\text{AMB}} \end{bmatrix} \quad \text{and} \quad B_u = \begin{bmatrix} 0 \\ Q_{\text{AMB}} \end{bmatrix} \]

(5.7)

Additionally, \( y \in \mathbb{R}^{n_y \times 1} \) is the output vector of shaft displacements measured by the AMB sensors and \( z \in \mathbb{R}^{n_z \times 1} \) is the performance output vector. Each of the three AMBs has two independent control axes and two corresponding displacement sensors, thus \( n_u = n_y = 6 \).

Figure 5.2 is a block diagram of the closed-loop AMB-FMC driveline system.
In this hybrid control strategy, the control input, \( u \), has two components given by (5.8).

\[
    u(t) = u_{FB}(t) + u_{SAVC}(i,t)
\]  

(5.8)

Here, \( u_{FB} \in \mathbb{R}^{n \times 1} \) is generated from an output feedback control and is designed to stabilize the supercritical shaft whirl instabilities and to ensure BIBO stable levitation of the AMB-FMC driveline system, Eq. (5.5). Furthermore, the open-loop adaptive portion of the control, \( u_{SAVC} \in \mathbb{R}^{n \times 1} \), is

\[
    u_{SAVC}(i,t) = U_i(\sin \Omega t + j \cos \Omega t)
\]  

(5.9)

Where \( U_i \in \mathbb{R}^{n \times 1} \) is the \( i^{th} \) updated complex Fourier coefficient control input that is adapted to suppress steady-state synchronous imbalance response.

In this investigation, a full-order dynamic output feedback controller, \( C_{FB}(s) \), is considered, thus

Fig. 5.2: Hybrid feedback/SAVC controlled AMB-FMC driveline system.
\[ C_{FB}(s) := \begin{cases} \dot{x}_c &= A_c x_c + B_c y \\ u_{FB} &= C_c x_c + D_c y \end{cases} \]  

(5.10)

Where \( x_c \in \mathbb{R}^{n_x \times 1} \) is the controller state and \( n_c = n_x \) is the controller order. In this case, since the first 10 modes are considered, \( n_x = 20 \). From (5.5) and (5.10), the feedback controlled driveline system is

\[
\begin{align*}
\dot{x}_{cl} &= A_{cl} x_{cl} + B_{dcl} d(t) + B_{ucl} u_{SAVC}(t) \\
z &= C_{zcl} x_{cl} \\
y &= C_{ycl} x_{cl}
\end{align*}
\]

(5.11)

with augmented state vector \( x_{cl} = [x^T \ x_c^T]^T \) and

\[
\begin{align*}
A_{cl} &= A_{ncl} + \delta_T A_{Tcl} \\
A_{ncl} &= \begin{bmatrix} A_n + B_u D_c C_y & B_u C_c \\ B_c C_y & A_c \end{bmatrix}, A_{Tcl} = \begin{bmatrix} A_T & 0 \\ 0 & 0 \end{bmatrix} \\
C_{ycl} &= [C_y \ 0] \\
C_{zcl} &= [C_z \ 0] \\
B_{dcl} &= \begin{bmatrix} B_d \\ 0 \end{bmatrix}, B_{ucl} = \begin{bmatrix} B_u \\ 0 \end{bmatrix}
\end{align*}
\]

(5.12)

Assuming the feedback controlled AMB-FMC driveline system in (5.11) is BIBO stable, the steady-state response after the \( i \)th SAVC control input, \( U_i \), is

\[ Y_i = T_{yu} U_i + Y_{FB} \]  

(5.13)

Where \( T_{yu} \) is the synchronous transfer matrix from the control input to the measured output 5.14

\[ T_{yu} = C_{ycl} \left[j \Omega I - A_{cl} \right]^{-1} B_{ucl} \]  

(5.14)

and \( Y_{FB} = T_{yu} D \) is the steady-state imbalance response of the feedback controlled driveline system without SAVC input.
The SAVC update law is obtained via least-squares minimization of the following objective function $J_i$

$$J_i = Y_i^*Y_i + U_i^*RU_i \quad \text{with} \quad R = w_{\text{eff}}I$$ (5.15)

where “*” indicates complex conjugate transpose and $w_{\text{eff}}$ (m$^2$/Amp), weights the SAVC control effort. Based on least-squares minimization of $J_i$, the SAVC update law is

$$U_{i+1} = [T_{yu}^*T_{yu} + R]^{-1}T_{yu}^*[T_{yu}U_i - Y_i]$$ (5.16)

Since the actual transfer matrix, $T_{yu}$, is a function of the temperature uncertainty parameter, $\delta_T$, the nominal transfer matrix

$$\hat{T}_{yu} = T_{yu} \bigg|_{\delta_T=0} = C_{ynl} [j\Omega I - A_{ncl}]^{-1}B_{ncl}$$ (5.17)

is utilized instead. Thus, the implemented SAVC update law is

$$U_{i+1} = [\hat{T}_{yu}^*\hat{T}_{yu} + R]^{-1}\hat{T}_{yu}^*[\hat{T}_{yu}U_i - Y_i]$$ (5.18)

If the SAVC input is updated slowly relative to the settling-time of the feedback controlled driveline system in (5.11), then the overall Hybrid Feedback/SAVC controlled driveline system is stable if and only if the system in (5.11) is stable and if the SAVC adaptation process in (5.18) converges. In particular, the SAVC converges if and only if the following condition is satisfied.

$$\overline{\rho}[\hat{T}_{yu}^*\hat{T}_{yu} + R]^{-1}\hat{T}_{yu}^*[\hat{T}_{yu} - T_{yu}] < 1$$ (5.19)

Where $\overline{\rho}[ ]$ is the spectral radius and the converged input and response, $U_{AVC}$ and $Y_{FBAVC}$,

$$U_{AVC} = -[\hat{T}_{yu}^*T_{yu} + R]^{-1}\hat{T}_{yu}^*Y_{FB}$$
$$Y_{FBAVC} = T_{yu}U_{AVC} + Y_{FB}$$ (5.20)
Since errors between the actual transfer matrix, \( T_{yu} \), and the nominal transfer matrix, \( \hat{T}_{yu} \), could cause the SAVC input not to converge, the convergence robustness must be considered in the control design.

5.4 Control Synthesis

Conceptually, in the hybrid Feedback-SAVC control system, the function of the feedback control in (5.10) is to ensure BIBO stable levitation of the AMB-FMC-driveline while the SAVC adapts and converges to suppress the steady-state imbalance vibration. Since the slowly adapted SAVC input does not affect BIBO stability, the feedback controller and the SAVC design and analyses can be conducted sequentially.

Since the control system must be robust with respect to temperature deviations, \( \delta_T \), about the nominal temperature, \( \bar{T} \) is considered as a bounded uncertainty parameter in the control design. Thus, the feedback controlled AMB-FMC driveline system is rewritten as a linear fractional transformation (LFT) about the nominal feedback controlled system, \( G(s) \), with deviation temperature uncertainty block \( \Delta = \delta_T I \).
With nominal open-loop AMB-FMC driveline plant

\[
P(s) := \begin{bmatrix}
\dot{x} = A_n x + B_p p(t) + B_d d(t) + B_u u(t) \\
q = C_q x \\
z = C_z x \\
y = C_y x
\end{bmatrix}
\]

and with temperature dependent uncertainty matrix in (5.5) written as

\[
\delta_T A = B_p \Delta C_q \quad \text{with} \quad \Delta = \delta_I
\]

Where \( B_p \) and \( C_q \) describe the uncertainty structure.

To ensure that the feedback controlled closed-loop driveline system in (5.11) is robustly stable with respect to temperature deviations, \( \delta_T \), the feedback portion of the control, \( C_{FB}(s) \), is synthesized using a robust \( H_\infty \) design approach.

In particular, \( C_{FB}(s) \), is synthesized by minimizing the \( H_\infty \) norm of the closed-loop transfer function \( G_{pq}(s) \), over the set of stabilizing controllers, Zhou, et al. (1996). Since
\[ \| \Delta \| = | \gamma |, \text{ when } \| G_{wp}(s) \|_\infty < \gamma, \text{ BIBO stability of the feedback controlled driveline system is guaranteed for all bounded temperatures deviations satisfying} \]
\[ | \delta_t | \leq \delta_{\text{stab}} = 1/ \gamma \]  
(5.23)

where \( \delta_{\text{stab}} \) is the deviation temperature robust stability margin.

In this investigation, the \( H_\infty \) feedback controller is computed using the MATLAB \( \text{LMI Control Toolbox}\) software package. For the AMB-FMC driveline system studied in this research, a feedback controller, \( C_{FB}(s) \), which achieves \( \gamma = 0.0034 \) \( (\delta_{\text{stab}} = 294^\circ F) \) is synthesized. Since \( \delta_{\text{stab}} = 294^\circ F \) is well above the expected shaft temperature deviations, this is considered a robust design and is utilized for the feedback portion of the hybrid feedback/SAVC law in the subsequent analysis.

To analyze the convergence robustness of the SAVC portion of the control, the convergence criteria in (5.19) is iteratively solved using a bisection algorithm to determine the deviation temperature robust convergence margin, \( \delta_{\text{conv}} \), for a range of shaft speeds. Finally, the overall deviation temperature robustness margin \( \delta_{\text{max}} \) for a given shaft speed, \( \Omega \), is defined and computed as
\[ \delta_{\text{max}} = \min[\delta_{\text{stab}}, \delta_{\text{conv}}] \]  
(5.24)

To assess the vibration performance, the following worst-case steady-state vibration indices are defined for a given deviation temperature bound, \( \Delta T \)
\[ J_{FB} = \sup_{0 \leq |\delta_T| \leq \Delta T} \left[ \frac{1}{n_y} \sqrt{Y_{FB} * Y_{FB}} \right], \quad J_{FBAVC} = \sup_{0 \leq |\delta_T| \leq \Delta T} \left[ \frac{1}{n_y} \sqrt{Y_{FBAVC} * Y_{FBAVC}} \right] \]  
(5.25)
Where $\Delta T \leq \delta_{T_{\text{max}}}$ and $0 \leq |\delta_T| \leq \Delta T$. Here $J_{FB}$ measures the imbalance vibration response of the robust $H_\infty$ feedback controlled driveline system and $J_{FBAVC}$ measures the driveline vibration under the hybrid $H_\infty$ feedback/SAVC law.

5.5 Nominal System Closed-Loop Analysis

In this section, the closed-loop performance of the $H_\infty$/SAVC controlled AMB-FMC driveline at the nominal temperature, $T_n=85^\circ F$, is investigated assuming no temperature deviations or uncertainty, i.e. $\delta_T=0$. In particular, Figure 5.4 shows the RMS vibration and control currents of the AMB-FMC driveline under $H_\infty$ and $H_\infty$/SAVC control with two amounts of SAVC control effort penalty weighting $w_{\text{eff}}$. Here, the $H_\infty$/SAVC achieves significant vibration suppression compared with the $H_\infty$ baseline except near two critical shaft speeds, $\omega_{c_1} \approx 2800$ and $\omega_{c_2} \approx 5500$ RPM. As expected, the case with the lowest SAVC penalty weighting, $w_{\text{eff}}$, achieves the best vibration suppression. Furthermore, except near the two critical speeds, $\omega_{c_1}$ and $\omega_{c_2}$, the required control currents are near or below the AMB bias currents. Also, similar to the behavior demonstrated in Chapter 4, the hybrid feedback/AVC law can achieve vibration suppression with less current than the baseline feedback control except near certain critical speeds.
Near the operating speeds, $\omega_{c1}$ and $\omega_{c2}$, the SAVC requires excessive currents to suppress the vibration and thus, due to AMB current saturation limitations, the imbalance vibration cannot as effectively be reduced at these speeds. Physically, this phenomena is due to transmission zeros in the control path which are introduced by the $H_\infty$ feedback controller $C_{FB}(s)$. See Figure 5.3. The control path zeros attenuate the effectiveness of the SAVC portion of the control at frequencies corresponding to the imaginary part of the transmission zeros. This phenomena was not encountered with the PID-MHAVC law developed in Chapter 4 since the PID feedback does not introduce any transmission zeros or any internal model dynamics.

Fig. 5.4: RMS vibration and control currents vs. shaft speed, system at $T_n=85^\circ$.
The effect of the transmission-zero critical speeds is not necessarily a serious issue, since most drivelines, such as helicopter drivelines, typically operate at a single speed $\Omega$. Thus, unless $\Omega$ corresponds with one of the transmission-zero speeds, the hybrid $H_\infty$/SAVC law can be used to achieve effective vibration suppression. However, one way to address this issue is through proper selection of the AMB locations along the driveline. It is found that the transmission zeros are very sensitive to the AMB locations, and thus the transmission-zero critical speeds can be shifted away from a given operating speed by proper AMB placement. This is demonstrated in Figure 5.5, where the $H_\infty$/SAVC controlled AMB-FMC driveline RMS vibration and control currents for two sets of AMB locations are shown.

Fig. 5.5: Effect of AMB location on transmission-zero critical speeds.
5.6 Closed-Loop Analysis with Shaft Temperature Deviations

In this section the effect of temperature uncertainty, $\delta_T$, is considered. Using (5.24) and (5.25), the system robustness and imbalance vibration performance is analyzed over a range of shaft speeds, $\Omega = [0 - 6000]$ RPM. This speed range includes both sub and supercritical operation.

Figure 5.6 shows how $\delta_{T_{\text{max}}}$ varies with $\Omega$ for several values of the SAVC control effort weighting parameter, $w_{\text{eff}}$.

![Graph showing deviation temperature robustness margin vs. shaft speed for several values of SAVC control effort penalty weighting, $w_{\text{eff}}$.](image)

Fig. 5.6: Deviation temperature robustness margin vs. shaft speed for several values of SAVC control effort penalty weighting, $w_{\text{eff}}$.

With no SAVC effort penalty (i.e. $w_{\text{eff}} = 0$), stability and convergence is guaranteed for temperature deviations of approximately 100°F about the nominal temperature, $T_n = 85°F$, over most of the RPM range. However, for operation speeds near
the transmission-zero critical speeds \(\omega_{c1} \approx 2800\) and \(\omega_{c2} \approx 5500\) RPM, the temperature robustness becomes significantly less. By penalizing the SAVC input (i.e. selecting \(w_{eff} > 0\)) the robustness margin near, \(\omega_{c1}\) and \(\omega_{c2}\), can be increased. In particular, as seen in Figure 5.6, if \(w_{eff}\) is chosen to be \(w_{eff} \geq w_{eff}^* = 5.5 \times 10^{-8}\) m²/Amp, the system can tolerate temperature deviations up to 100°F and still remain stable and converge across the examined RPM range. That is, \(w_{eff} = w_{eff}^* = 5.5 \times 10^{-8}\) m²/Amp guarantees \(\delta_{T_{max}} > 100\)°F over the speed range \(\Omega = [0 - 6000]\) RPM.

Considering \(w_{eff} = w_{eff}^* = 5.5 \times 10^{-8}\) m²/Amp to be the lowest acceptable value for the design, the vibration performance for two values of \(w_{eff} \geq w_{eff}^*\) is examined in Figures 5.7 and 5.8. Note that no knowledge of the shaft imbalance is utilized in the control synthesis and vibration is suppression is achieved in the presence of the uncertainty \(\delta_T\).

In each plot, the worst-case vibration performance indices, \(J_{FB}\) and \(J_{FBAVC}\), are shown for the nominal system and for a 100°F deviation temperature uncertainty bound. By comparing the two performance indices, \(J_{FB}\) and \(J_{FBAVC}\), and by considering \(J_{FB}\) as the baseline performance, the effectiveness of the SAVC can be assessed.
Fig. 5.7: $J_{FB}$ and $J_{FBAVC}$ vs. shaft speed for two levels of temperature uncertainty; $[\delta_T = 0]$ and $[0 \leq \delta_T \leq 100^\circ F]$, with $w_{eff} = 6.0 \times 10^{-8}$.

Fig. 5.8: $J_{FB}$ and $J_{FBAVC}$ vs. shaft speed for two levels of temperature uncertainty; $[\delta_T = 0]$ and $[0 \leq \delta_T \leq 100^\circ F]$, with $w_{eff} = 2.6 \times 10^{-7}$.
Figure 5.7 demonstrates that, for lower values of control penalty, i.e. for $w_{eff}$ near $w_{eff}^*$, the SAVC input adapts and converges to achieve significant imbalance vibration suppression for operating speeds away from the transmission-zeros critical speeds $\omega_{c1}$ and $\omega_{c2}$. Thus, the hybrid $H_\infty$ feedback/SAVC law achieves robust performance for $\Omega$ away from $\omega_{c1}$ and $\omega_{c2}$. However, for speeds near $\omega_{c1}$ and $\omega_{c2}$, the SAVC input is less effective. In fact, the presence of the uncertainty, $\delta_T$, causes the SAVC to increase the vibration above the feedback controlled baseline imbalance response, (see “a” and “b” in Figure 5.7). Since $\delta_T$ alters the FMC shaft stiffness and damping, $\delta_T$ shifts the transmission-zeros critical frequencies, $\omega_{c1}$ and $\omega_{c2}$ about their nominal values. This frequency shifting is reflected in the plots of the worst-case case performance indices by the presence of the peaks “a” and “b” about $\omega_{c1}$ and “d” and “c” about $\omega_{c2}$ in Figure 5.7. The separation bandwidth between “a” and “b” and between “c” and “d” increases with the deviation temperature uncertainty $\delta_T$.

Finally, comparing Figures 5.7 and 5.8 demonstrates the effect of increasing $w_{eff}$ on the SAVC vibration suppression performance and robustness. Increasing $w_{eff}$ improves performance robustness for shaft speeds near the critical speeds, $\omega_{c1}$ and $\omega_{c2}$. However, since increasing $w_{eff}$ limits the SAVC control input magnitudes, it reduces the maximum achievable vibration suppression. Therefore, larger values of $w_{eff}$ are only beneficial for shaft speeds near the transmission-zero critical speeds.
5.7 Configuration II Active Control Summary and Conclusions

In this chapter a hybrid $H_{\infty}$ feedback/SAVC control law is developed for the AMB-FMC tailrotor driveline system (Configuration II) considering effects of temperature dependent FMC material properties, rotating-frame damping and shaft imbalance. The hybrid $H_{\infty}$ feedback/SAVC control strategy is conceptually similar to the hybrid PID-feedback/MHAVC law developed for the AMB-NCV driveline system (Configuration I) in Chapter 4. In each configuration, the feedback control ensures BIBO stable levitation of the driveline throughout a range of operating conditions (robust stability), while the open-loop control inputs (SAVC or MHAVC) are adapted to suppress the vibrations.

Since the high strain capability of FMC shafting enables driveline misalignment to be accommodated without flexible couplings, the AMB-FMC driveline configuration explored in this chapter has several advantages over both the Conventional Configuration and Configuration I drivelines. Compared with the previous configurations, Configuration II reduces the number of driveline components by replacing the multi-segment driveshaft and associated flexible couplings with a single-piece, rigidly coupled, FMC shaft. By reducing the number of driveline components, the overall driveline maintenance “turn-around” times could be improved. Furthermore, since rigid couplings have constant velocity shaft-speed kinematics as opposed to NCV kinematics, replacing the flexible couplings with rigid couplings has the advantage of directly eliminating the misalignment and load-torque induced super-synchronous driveline vibrations. As a result, the primary source of vibration in the AMB-FMC driveline system is only due to
the synchronous imbalance excitation. To suppress this vibration, the adaptive portion of the AMB-FMC driveline control law utilizes a synchronous adaptive vibration control (SAVC) approach instead of the multi-harmonic adaptive vibration control (MHAVC) approach required for the Configuration I AMB-NCV driveline in Chapter 4. Furthermore, unlike Configuration I, the absence of the NCV effects in Configuration II enables robust $H_\infty$ control theory to be utilized in the synthesis of the feedback portion of the control law. Therefore, the control strategy developed for the AMB-FMC driveline is a hybrid $H_\infty$ feedback/SAVC control.

The results of the investigation and analysis show that the $H_\infty$ feedback/SAVC hybrid control strategy can guarantee stability, convergence and imbalance vibration suppression of the AMB-FMC driveline under the conditions of bounded shaft temperature deviations and unknown shaft imbalance. Furthermore, the controller is effective at both sub and supercritical operating speeds. Specifically, if the weighting on SAVC effort is low, the control is very effective and robust for suppressing imbalance vibration at shaft speeds away from closed-loop system transmission-zeros. Additionally, for shaft speeds operating near the closed-loop system zeros, the temperature robustness and imbalance vibration suppression performance can be greatly improved by moderately increasing the SAVC control effort penalty weighting. Finally, by adjusting the AMB locations, the transmission-zero critical speeds can be effectively shifted to allow low vibration operation at a particular desired shaft operating speed.
Chapter 6

AMB/SUPERCritical Tailrotor Driveline-Fuselage Testrig
Design and Experimental Setup

6.1 Introduction

In order to experimentally validate the comprehensive tailrotor driveline-fuselage analytical model and to experimentally evaluate the closed-loop performance of the Configuration I and II control laws, an AMB/supercritical driveline-fuselage testrig has been designed and constructed. The testrig is designed to investigate a variety of different driveline configurations including the conventional configuration and the two actively controlled AMB/driveline Configurations I and II.

6.2 Driveline Testrig Overview

Figures 6.1 and 6.2 are photographs of the driveline testrig in both the conventional configuration and in the actively controlled configuration I.
Furthermore, for clarity, Figure 6.3 is a schematic drawing of the driveline testrig in the Conventional Configuration. The Conventional Configuration of the testrig is based on the Boeing AH-64 supercritical tailrotor driveline, whose parameters are utilized throughout this research. In particular, the testrig driveline natural frequencies and first few foundation-beam natural frequencies have been designed to match the full-scale AH-64 driveline and tailboom-fuselage natural frequencies.
Fig. 6.2: Photo of tailrotor driveline-fuselage testrig in Configuration I.

Fig. 6.3: Testrig diagram in Conventional Configuration.
The testrig is designed to investigate the driveline vibration and loads under various foundation excitations, misalignment conditions and torque loads relevant to the real helicopter tailrotor driveline system.

In the conventional configuration, the testrig main driveshaft (or intermediate shafts according to Chapter 2) consists of two 32” long shaft segments connected with U-joint couplings and supported on a flexible foundation beam by a contact roller bearing and two viscous dampers. The shafts are stainless-steel and have a 3/8” diameter with a solid circular cross-section.

Figure 6.4 shows the design of the viscous lateral shaft dampers. Here, the lateral shaft motion is transferred into the two piston-dampers via the grooved non-rotating bearing-damper ring assembly.

![Viscous damper assembly diagram](image)

Fig. 6.4: Viscous damper assembly for Conventional Configuration, 1 of 2.
Furthermore, the relative displacements between the shaft and foundation beam in both the horizontal and vertical planes are measured at two shaft locations with fiber optic displacement probes. See Figure 6.5.

On the input-side, the input-shaft connected at U-joint “A” is driven with a 3-phase AC motor rated at 2 hp at 1750 RPM with a maximum operating speed of 3600 RPM. In order to investigate the effect static misalignment on the driveline dynamics, both the motor and input-shaft are mounted on a misalignable base plate. This allows a controlled amount of input-shaft static misalignment, $\theta_A$, at U-joint coupling “A” to be selected without the need for large static forces to generate static misalignment of the foundation structure as with the real tailrotor driveline-fuselage system.

Fig. 6.5: Fiber-optic displacement probe sensor pair, 1 of 2.
On the output-side, connected at U-joint “C”, the driveline drives the end-load assembly, shown in Figure 6.6, whose main function is to simulate the effect of the tailrotor torsional inertia and torque loads. The torsional load-inertia is scaled based on matching the first driveline torsional natural frequency of the AH-64 and load-torque is scaled based on matching the torsional yield stress factor-of-safety. The inertia load is generated, in part, by two flywheels and the resistive torque load is selected by adjusting the two friction-based magnetically actuated brakes. Additionally, the total driveline load-torque and output-shaft speed are measured by the combined torque and RPM sensor mounted in the end-load assembly, protected by a torque-overload clutch coupling. Finally, like on the input-side, the end-load assembly is mounted on a misalignable base plate which allows a set amount of static output-shaft misalignment, $\theta_C$, at U-joint “C” to be selected without the need for large static forces.

![End-load assembly diagram](image)

**Fig. 6.6**: End-load assembly.
The driveline static misalignment angles $\theta_A$ and $\theta_C$ can be selected independently to allow a variety static misalignment conditions to be studied. Figure 6.7 shows the testrig with static misalignment at both the input and the output sides.

<table>
<thead>
<tr>
<th>Motor</th>
<th>End-Load Assembly</th>
</tr>
</thead>
<tbody>
<tr>
<td>Torque Load</td>
<td>Type</td>
</tr>
<tr>
<td>Two adjustable disk-friction brakes</td>
<td>Effective Torque Range</td>
</tr>
<tr>
<td>0 to 10 N-m</td>
<td>Inertia Load</td>
</tr>
<tr>
<td>Speed Range</td>
<td>Type</td>
</tr>
<tr>
<td>0 to 3600 RPM</td>
<td>Two flywheels coupled through 1:1.5 ratio gearbox</td>
</tr>
<tr>
<td>Max Power</td>
<td>Effective Polar Inertia</td>
</tr>
<tr>
<td>2 Hp</td>
<td>0.078 kg·sec$^2$</td>
</tr>
<tr>
<td>Max Torque</td>
<td></td>
</tr>
<tr>
<td>12 N-m</td>
<td></td>
</tr>
</tbody>
</table>
The foundation structure, which is analogous to the tailboom-fuselage, is a steel I-beam (W5x19 cross-section) pinned at one end and mounted on a coil-spring on the free end. Additionally, the end-load assembly is bolted to the free-end of the foundation beam. This is analogous to the effect of the lumped empennage mass and inertia on the helicopter tailboom. The spring stiffness was selected based on matching the first tailboom-fuselage vertical bending natural frequency of the AH-64. The tailboom-fuselage aerodynamic loading and other disturbances originating from the fuselage, are represented by the 50 Lbf capacity shaker at the tip of the foundation beam. The shaker input force and corresponding structural response at the shaker attachment point is measured via an impedance head mounted between the foundation beam and the shaker. Finally, 4 tri-axis accelerometers measure the vibration response of the foundation beam. See Figure 6.8.

Fig. 6.7: Adjustable static driveline misalignment condition.
6.3 Active Magnetic Bearings

The magnetic bearings selected for implementing the actively controlled AMB/driveline Configurations I and II on the testrig are 8-pole radial magnetic bearings manufactured by Revolve Inc. See Figure 6.9 for a photograph of the AMB and rotor in disassembled form.
Each AMB is equipped with a pair of inductive displacement sensors which measure shaft position within the AMB. Furthermore, to prevent rotor-stator contacts, each AMB is equipped with a radially clearanced backup-bearing. There are a total of 4 AMBs along with the necessary power amplifiers and control hardware available in the laboratory. See Table 6.2 for a summary of the relevant testrig AMB parameters.
Table 6.2: Testrig AMB and AMB Rotor Parameters

<table>
<thead>
<tr>
<th>AMB Parameters</th>
<th>AMB Rotor Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>Magnetic gap</td>
<td>Outer Diameter</td>
</tr>
<tr>
<td>525 µm</td>
<td>34.29 mm</td>
</tr>
<tr>
<td>Backup-bearing gap</td>
<td>Lamination ID</td>
</tr>
<tr>
<td>225 µm</td>
<td>19.53 mm</td>
</tr>
<tr>
<td>Max force</td>
<td>Mass</td>
</tr>
<tr>
<td>76 N</td>
<td>0.281 kg</td>
</tr>
<tr>
<td>Bandwidth</td>
<td>Length</td>
</tr>
<tr>
<td>550 Hz</td>
<td>41.22 mm</td>
</tr>
<tr>
<td>Saturation current</td>
<td>Transverse moment of Inertia</td>
</tr>
<tr>
<td>3.0 Amp</td>
<td>6.5 x10^{-5} kg-m^2</td>
</tr>
<tr>
<td>Force constant</td>
<td>Polar moment of Inertia</td>
</tr>
<tr>
<td>2.104x10^6 N-µm^2/Amp^2</td>
<td>4.3 x10^{-5} kg-m^2</td>
</tr>
<tr>
<td>Number of coils</td>
<td>Stator-sensor offset</td>
</tr>
<tr>
<td>8</td>
<td>21.5 mm</td>
</tr>
<tr>
<td>AMB axis angle</td>
<td>Steel Shaft</td>
</tr>
<tr>
<td>45°</td>
<td>U-Joint “B”</td>
</tr>
<tr>
<td>Stator-sensor offset</td>
<td>AMB 2</td>
</tr>
</tbody>
</table>

Fig. 6.10: Photo of testrig driveline in Configuration I.
6.4 Flexible Matrix Composite Shaft

Finally, the Flexible Matrix Composite (FMC) shafts, which are used in the experimental investigation of Configuration II, have been fabricated based on ongoing research by Professor C. E. Bakis in the Penn State ESM Composites Laboratory. See Shan and Bakis (2002). Figure 6.11 is a photo of the FMC shaft connected to the end-load assembly via a rigid coupling.

![FMC shaft connected with rigid coupling (Configuration II).](image)

Fig. 6.11: FMC shaft connected with rigid coupling (Configuration II).

6.5 System Inputs and Outputs

In the Conventional Configuration and in the actively controlled Configurations I and II, all measurements are simultaneously recorded on a PC running a LabVIEW™, based multi-channel data acquisition system capable of 10 kHz sampling per channel.
The type and number of sensors used for the experimental measurements of the driveline-fuselage testrig are summarized in Table 6.3.

Table 6.3: Driveline-Fuselage Testrig Sensors

<table>
<thead>
<tr>
<th>Sensors</th>
<th>Outputs</th>
<th>QTY</th>
<th>Total Outputs</th>
</tr>
</thead>
<tbody>
<tr>
<td>Accelerometer</td>
<td>2</td>
<td>4</td>
<td>8</td>
</tr>
<tr>
<td>Impedance Head</td>
<td>2</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>Optical Displacement Probe</td>
<td>1</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>Two-Axis Load Cell*</td>
<td>2</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>RPM Sensor</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Torque Sensor</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

* only in Conventional Configuration

20

In addition to the measurements recorded by the above sensors and data acquisition system, the closed-loop AMB control system is implemented for the two actively controlled configurations (Configuration I and II). The active vibration control algorithms are implemented digitally on a PC with a dSPACE™ real-time controller board. The basic inputs and outputs of the closed-loop AMB/driveline-fuselage vibration control system are summarized in Table 6.4.

Table 6.4: AMB/Driveline-Fuselage Closed-Loop Control System I/O

<table>
<thead>
<tr>
<th>Control System I/O</th>
<th>Inputs</th>
<th>Outputs</th>
<th>QTY</th>
</tr>
</thead>
<tbody>
<tr>
<td>8-Pole Radial AMB</td>
<td>N/A</td>
<td>N/A</td>
<td>“</td>
</tr>
<tr>
<td>AMB shaft disp. sensors</td>
<td>2</td>
<td>2</td>
<td>“</td>
</tr>
<tr>
<td>AMB coil current. sensors</td>
<td>N/A</td>
<td>2</td>
<td>“</td>
</tr>
</tbody>
</table>

up to 4

“ “

“ “
Chapter 7

EXPERIMENTAL INVESTIGATION

7.1 Introduction

The first objective of the experimental investigation is to validate the comprehensive tailrotor driveline-fuselage analytical model developed in Chapter 2. The investigation will compare analytical predictions with experimental measurements of the conventionally configured driveline-fuselage structure under various operating conditions and shaft speeds. In particular, the experimental investigation will focus on validation of the misalignment and torque-induced periodic parametric and forcing terms due to NCV couplings. The second objective is to implement and evaluate the stability, performance, and robustness of the closed-loop vibration control laws developed in this research for the actively controlled Configurations I and II.
7.2 NCV-Driveline Model Validation

The driveline model evaluation and validation is performed on the driveline in the Conventional Configuration and is conducted in two main phases. Figure 7.1 shows the schematic of the experimental setup used for driveline model validation. Furthermore the Testrig driveline setup details are summarized in Table 7.1.

![Fig. 7.1: Testrig setup for experimental validation of driveline analytical model.](image)

In the first phase, the driveline-foundation structural model is validated without any rotor-dynamic effects i.e. for $\Omega = 0$ RPM. In the second phase, effects of misalignment and load-torque on the Testrig driveline dynamics are experimentally investigated over a range of shaft speeds and then compares with analytical predictions.
Table 7.1: Testrig Setup for Conventional Configuration

<table>
<thead>
<tr>
<th>Driveshaft</th>
<th>Bearings &amp; Dampers</th>
</tr>
</thead>
<tbody>
<tr>
<td>Material</td>
<td>303 Stainless-Steel</td>
</tr>
<tr>
<td>Outer diameter</td>
<td>9.525 mm</td>
</tr>
<tr>
<td>Wall thickness</td>
<td>Solid</td>
</tr>
<tr>
<td>Number of segments</td>
<td>2</td>
</tr>
<tr>
<td>Segment length</td>
<td>$L_{s1} = 0.816$ m</td>
</tr>
<tr>
<td></td>
<td>$L_{s2} = 0.816$ m</td>
</tr>
<tr>
<td>Coupling type</td>
<td>U-Joints (A, B &amp; C)</td>
</tr>
<tr>
<td>Passive dampers</td>
<td>2</td>
</tr>
<tr>
<td>Passive bearings</td>
<td>1</td>
</tr>
<tr>
<td>Magnetic bearings</td>
<td>None</td>
</tr>
</tbody>
</table>

Figure 7.2 gives results of the first phase of the model validation. This figure shows a comparison between an experimentally determined FRF and the predicted results from the Finite Element Model developed in Chapter 2 (using the Testrig model parameters). This figure shows an FRF from the shaker input to shaft displacement of the non-rotating driveline generated from a [0 - 130 Hz] swept-sine shaker input. The shaft displacement is measured from one of the non-contact fiber-optic probes, and the shaker input force is measured via a load-cell at the shaker-foundation attachment point.
As seen in Figure 7.2, the FEM analytical predictions for the first few natural modes are in reasonable agreement with the experimentally measured values.

To validate the effects of misalignment and load-torque, several driveline spinup tests are performed for several sets of misalignment and load-torque operating conditions. See Table 7.2 for a summary of the operating conditions and see Figure 7.3 for a photo of the driveline under misalignment.
Table 7.2: Conventional Configuration Spinup Test Cases

<table>
<thead>
<tr>
<th>Spinup Test $\Omega =$ 0 to 1440 RPM</th>
<th>Driveline Misalignment Condition</th>
<th>Load-Torque Brake Setting</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case 1</td>
<td>Horizontal-Plane: $\delta_2 = 0^\circ$, $\xi_2 = 0^\circ$, $\gamma_2 = 1^\circ$&lt;br&gt;Vertical-Plane: $\delta_3 = 0^\circ$, $\xi_3 = 0^\circ$, $\gamma_3 = 0^\circ$</td>
<td>$T_L = 3$ N-m</td>
</tr>
<tr>
<td>Case 2</td>
<td>Horizontal-Plane: $\delta_2 = 0^\circ$, $\xi_2 = 0^\circ$, $\gamma_2 = 1^\circ$&lt;br&gt;Vertical-Plane: $\delta_3 = 0^\circ$, $\xi_3 = 0^\circ$, $\gamma_3 = 0^\circ$</td>
<td>$T_L = 5$ N-m</td>
</tr>
<tr>
<td>Case 3</td>
<td>Horizontal-Plane: $\delta_2 = 0^\circ$, $\xi_2 = 0^\circ$, $\gamma_2 = 2^\circ$&lt;br&gt;Vertical-Plane: $\delta_3 = 0^\circ$, $\xi_3 = 0^\circ$, $\gamma_3 = 0^\circ$</td>
<td>$T_L = 3$ N-m</td>
</tr>
<tr>
<td>Case 4</td>
<td>Horizontal-Plane: $\delta_2 = 0^\circ$, $\xi_2 = 0^\circ$, $\gamma_2 = 2^\circ$&lt;br&gt;Vertical-Plane: $\delta_3 = 0^\circ$, $\xi_3 = 0^\circ$, $\gamma_3 = 0^\circ$</td>
<td>$T_L = 5$ N-m</td>
</tr>
</tbody>
</table>

Fig. 7.3: U-Joint coupling “C” with $2^\circ$ misalignment (for spinup Cases 3 & 4).
Figure 7.4 shows the load-torque and the shaft speed time histories for the conventional configuration spinup test cases. Here, the load-torque has both static and shaft speed dependent portions. The static portion of the load-torque is due to the brake torque setting, and, it is hypothesized that, the speed dependent portion is due to gearbox oil viscosity effects. In order to ensure a more accurate model validation, this measured load-torque with the shaft speed dependence is incorporated into the time-domain spinup simulations.

![Figure 7.4: Load-torque and shaft speed vs. time (spinup experiment Cases 1 - 4).](image)

Figure 7.5 shows the shaft vibration response for two different spinup test cases as measured by one of the fiber-optic displacement probes. Further, Figure 7.6 shows a time-frequency spectrogram of the shaft response for spinup Case 4.
As expected, due to the LPTV nature of the equations-of-motion derived in Chapter 2 and from the harmonic balance analysis conducted in Chapter 4, the response spectrum is concentrated at integer multiples of the shaft speed, $N\Omega$. In particular, the shaft imbalance contributes to the synchronous portion of the response at the shaft speed $\Omega$, while the misalignment and load-torque cause super-synchronous vibration at the $2\Omega$, $3\Omega$ and $4\Omega$ harmonics. The remaining response bands at non-integer multiples of $\Omega$ are due to gearbox meshing excitations originating from the end-load assembly.

Fig. 7.5: Shaft displacement vs. time (spinup experiment Cases 1 & 4).
To investigate the accuracy of the NCV-driveline analytical model with respect to misalignment and load-torque effects, time-domain simulations for spinup Cases 1-4 are performed using the model and compared with experimental results. The results of this comparison are shown in Figures 7.7 and 7.8. Figure 7.7 shows the $2\Omega$ displacement response amplitudes for spinup Cases 1-4 for a point on the shaft, and Figure 7.7 shows the $2\Omega$ acceleration response amplitudes for a point on the foundation beam.
Case #
1) Mis. = 1°, $T_{L_{max}} = 3.0$ N-m
2) Mis. = 1°, $T_{L_{max}} = 5.0$ N-m
3) Mis. = 2°, $T_{L_{max}} = 3.0$ N-m
4) Mis. = 2°, $T_{L_{max}} = 5.0$ N-m

Fig. 7.7: Shaft displacement at frequency $2\Omega$ for spinup Cases 1-4, FEM & Exp.

Case #
1) Mis. = 1°, $T_{L_{max}} = 3.0$ N-m
2) Mis. = 1°, $T_{L_{max}} = 5.0$ N-m
3) Mis. = 2°, $T_{L_{max}} = 3.0$ N-m
4) Mis. = 2°, $T_{L_{max}} = 5.0$ N-m

Fig. 7.8: Foundation acceleration at frequency $2\Omega$ for spinup Cases 1-4, FEM & Exp.
These comparisons indicate that the comprehensive driveline analytical model developed in Chapter 2 reasonably predicts effects of imbalance, misalignment and load-torque on the driveline dynamics. This validation of the analytical model reinforces both the novel misalignment induced stability/instability behavior discovered in Chapter 3 and the Configurations I and II closed-loop results and conclusion presented in Chapters 4 and 5.

7.3 Active Controller Experimental Implementation

To experimentally implement the actively controlled driveline Configurations I and II on the Testrig, a combination of digital active controller hardware is utilized. Both the Configuration I and II hybrid control strategies are combinations of a real-time feedback control loop and a slowly updating harmonic adaptive vibration control (AVC). Since the feedback and the adaptive control loops have different sampling rate requirements, they are implemented on separate hardware and then combined and injected into the AMB-driveline system as the hybrid control law, $u = u_{FB} + u_{AVC}$. The feedback portion of the Configuration I and II controllers are implemented on a Revolve™ multi-channel MB350T digital PID controller, while the harmonic adaptive control (MHAVC for Config. I and SAVC for Config. II) is implemented on a dSPACE™ model 1103 digital controller in conjunction with MATLAB™ Simulink® and Real-Time Workshop®. See Figure 7.9 for a block diagram of the overall closed-loop AMB-driveline Testrig system. Since the feedback portion of the control is a digital PID
controller with low-pass filtering and lead-lag compensation, the feedback portions of
the Configurations I and II controllers are not identical to the ones developed in Chapters
4 and 5. Despite these hardware limitations, as long as the digital feedback controlled
AMB-Driveline subsystem is BIBO stable, the performance and convergence robustness
of harmonic adaptive vibration control strategies (SAVC and MHAVC) can be evaluated
and assessed experimentally.

Fig. 7.9: Experimentally implemented hybrid feedback/harmonic adaptive controller.

Figure 7.10 shows the basic elements of the open-loop Testrig AMB/driveline system.
7.3.1 PID Feedback Control Implementation

Figure 7.11 gives the overall structure of the digital PID feedback controller utilized in the experimental implementation.

To determine the PID feedback control parameters for the Configurations I and II Testrig drivelines that achieve robustly stable levitation, the stability for a range of controller parameters is calculated using the driveline analytical model. Using the stability analysis
results as a guide, various sets of PID controller parameters from the predicted sub-set of stabilizing control parameters are tested experimentally. The results of this analysis/trial-and-error process are summarized in Table 7.3, which shows the sets of digital PID controller parameters that experimentally achieve stable closed-loop levitation of the Configuration I and II Testrig drivelines. These values are utilized in the subsequent experimental investigation of Configuration I and II.

<table>
<thead>
<tr>
<th>Digital PID Controller Parameters</th>
<th>Configuration I</th>
<th>Configuration II</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sampling Period, $T_s$</td>
<td>1/10000 sec</td>
<td>1/10000 sec</td>
</tr>
<tr>
<td>Bias current, $i_{bias}$</td>
<td>1.0 Amps</td>
<td>0.55 Amps</td>
</tr>
<tr>
<td>Active Stiffness Gain, $k_p$</td>
<td>5000 Amps/m</td>
<td>4000 Amps/m</td>
</tr>
<tr>
<td>Active Damping Gain $k_d$</td>
<td>4.0 Amps-sec/m</td>
<td>2.5 Amps-sec/m</td>
</tr>
<tr>
<td>Integrator Gain, $k_i$</td>
<td>1000 Amps/m-sec</td>
<td>1000 Amps/m-sec</td>
</tr>
<tr>
<td>Lead-lag filter pole, $p_{LL}$</td>
<td>200 Hz</td>
<td>280 Hz</td>
</tr>
<tr>
<td>Lead-lag filter zero, $z_{LL}$</td>
<td>100 Hz</td>
<td>170 Hz</td>
</tr>
<tr>
<td>Low-pass filter cutoff, $\omega_{LP}$</td>
<td>800 Hz</td>
<td>800 Hz</td>
</tr>
<tr>
<td>Low-pass filter damping ratio, $\zeta_{LP}$</td>
<td>0.707</td>
<td>0.707</td>
</tr>
</tbody>
</table>

In Configuration I, it is found that three AMBs ($n_{amb} = 3$) are sufficient to achieve stable levitation, while in Configuration II, four AMBs ($n_{amb} = 4$) are required due to the increased lateral flexibility of the FMC shafting.

Both the Testrig stability analysis and experimental results show that the levitation stability of Configuration II (AMB-FMC driveline) is much more sensitive to the PID controller parameters compared with Configuration I (AMB-NCV alloy
driveline). Consequently, for the same bias current level, $i_{bias}$, the region of stabilizing PID control gains is much larger for Testrig Configuration I compared with Testrig Configuration II. For Configuration I, an acceptable PID controller design was achieved with $i_{bias}=1.0$ Amp. However, an acceptable PID controller design for Configuration II could only be achieved for a reduced bias current level, $i_{bias}=0.55$ Amp.

According to (2.69) – (2.72), the maximum AMB actuation force, $F_{max}$, is

$$F_{max} = 4G_b \frac{i_{bias}^2}{h_{gap}}$$

(7.1)

and the negative stiffness due to magnetic levitation is

$$k_x = 4G_b \frac{i_{bias}^2}{h_{gap}^3}$$

(7.2)

Thus, the reduced bias current level in Configuration II translates into less negative stiffness but also means less available actuation authority.

Through analytical investigation, it is concluded that the difficulties encountered in achieving stable levitation of the AMB-FMC Testrig driveline at higher bias current levels ($i_{bias} > 0.6$ Amp) is a result of the relatively large AMB rotor-to-shaft mass ratio, $\mu_r = M_{rotor} / M_{shaft}$, of the Configuration II Testrig driveline. See Table 7.4.
The relatively large AMB rotor mass ratio, $\mu_r = 0.89$, of Testrig Configuration II represents an extreme case and is not true in general of AMB-FMC drivelines. In particular, as the driveline scale increases, $\mu_r$ tends to decrease (e.g. $\mu_r = 0.054$ for the full scale AH-64 Configuration II driveline). Thus, the Configuration II Testrig driveline levitation difficulties encountered at higher bias current levels are an artifact of the particular testrig scaling and not a general problem inherent to AMB-FMC driveline systems.

Prior to driveline operation, the closed-loop transfer functions of the non-rotating ($\Omega = 0$ RPM) magnetically levitated Configration I and II driveline systems are characterized. Specifically, by injecting white noise disturbances into the AMB control inputs, the control-path transfer functions, $G_{nn}(j\omega)$, from the AMB control inputs, $u$, to the measured shaft displacements, $y$, are characterized.

$$G_{nn}(j\omega) = \frac{Y_n(j\omega)}{U_m(j\omega)} \quad (7.3)$$
Where indices $n$ and $m$ range from $[1, \ldots, 2n_{amb}]$. Figure 7.12 shows a comparison between the analytically and experimentally determined control-path transfer functions for the Configuration I Testrig driveline.

![Graph showing transfer functions for AMB1, AMB2, and AMB3](image)

**Fig. 7.12: Testrig Configuration I closed-loop control-path transfer functions.**

7.3.2 Harmonic Adaptive Control Implementation

The adaptive portions of the control in Configuration I and II are implemented on a dSPACE™ digital controller board which has 20 analog input channels and 8 analog output channels. Specifically, Figure 7.13 is a block diagram of the harmonic adaptive vibration control (AVC), which is composed of three main sub-blocks.
The first sub-block, shown in Figure 7.14, is the Harmonic Fourier Coefficient (HFC) calculator.

This block computes the steady-state response HFC vector, $Y_i$, after each AVC update. Here, $Y_i$ is

$$Y_i = [y_{s1}^T \quad y_{c1}^T \quad \ldots \quad y_{sn}^T \quad y_{cn}^T]^T$$  \hspace{1cm} (7.4)$$

which corresponds to the steady-state shaft vibration response, $y(t)$.

$$y(t) = y_0 + \sum_{n=1}^{nh} \left[ y_{sn} \sin(n\Omega_0 t) + y_{cn} \cos(n\Omega_0 t) \right]$$  \hspace{1cm} (7.5)$$

Due to the presence of measurement noise, $e(t)$, the measured shaft response is
\[
\dot{y}(t) = y(t) + e(t)
\] (7.6)

Based on the measurements \( \dot{y}(t) \), the estimate of the response HFC vector,

\[
\hat{Y} = [\hat{y}_{s1}^T \hat{y}_{c1}^T \ldots \hat{y}_{s_nh}^T \hat{y}_{c_nh}^T]^T
\] (7.7)

is computed in real-time at each desired shaft speed harmonic, \( m\Omega \), via a moving average convolution scheme given by

\[
\hat{y}_{sm} = \frac{2}{T_{conv}} \int_{0}^{T_{conv}} \dot{y}(t)\sin(m\Omega t)dt
\]  
\[m = 1, \ldots, n_h\]  
(7.8)

\[
\hat{y}_{cm} = \frac{2}{T_{conv}} \int_{0}^{T_{conv}} \dot{y}(t)\cos(m\Omega t)dt
\]  
(7.9)

Where \( T_{conv} \) is the convolution window history length, see Figure 7.16. By selecting \( T_{conv} = N_p T_p \), where \( T_p = 2\pi/\Omega \) is the shaft rotation period and \( N_p \) is some integer number of revolutions, the resulting estimated sin and cosine Fourier coefficient vectors, \( \hat{y}_{sm} \) and \( \hat{y}_{cm} \), of the \( m^{th} \) response harmonic are

\[
\hat{y}_{sm} = y_{sm} + \frac{1}{N_p T_p} \int_{0}^{N_p T_p} 2e(t)\sin(m\Omega t)dt
\]  
\[m = 1, \ldots, n_h\]  
(7.9)

\[
\hat{y}_{cm} = y_{cm} + \frac{1}{N_p T_p} \int_{0}^{N_p T_p} 2e(t)\cos(m\Omega t)dt
\]

Assuming the measurement noise, \( e(t) \), is stationary with

\[
E[e(t)] = 0, \, \text{var}[e(t)] = \sigma_e^2
\] (7.10)

where \( E[ \cdot ] \) is the expectation operator and \( \text{var}[ \cdot ] \) is the variance, the resulting estimated Fourier coefficient vectors have a Gaussian distribution with the following statistics.
Using the estimated response HFC vector, $\hat{Y}_i$, together with the AVC update law (defined in Chapter 4), the second sub-block updates the AVC control input HFC vector, $U_{i+1}$. Finally, the AVC control input, $U_{i+1}$, is transformed into the time-domain via the convolutions in the Harmonic AVC Synthesis block. See Figure 7.15,

Refer to Appendix G for complete Simulink block diagrams of the controller.

Figure 7.16 is a schematic representation of the time-domain response of a system with slowly updating harmonic AVC inputs. Here $T_s$ is the real-time sampling period, which is selected to satisfy the Nyquist sampling criteria i.e. $1/T_s >> 2 f_{bw}$ where $f_{bw}$ is the shaft response bandwidth in Hz. Furthermore, the AVC update period, $T_{update}$, is chosen such that $T_{update} = T_{decay} + T_{conv}$ where $T_{decay}$ is the transient setting-time of the PID controlled AMB-driveline sub-system. This ensures that system reaches a new steady-
state after each AVC control update and allows the AVC control to be considered as an open-loop input. Additionally, waiting for a period $T_{\text{decay}}$ after the control update reduces errors in the estimated steady-state response HFC vector, $\hat{Y}_i$, due to transients.

$$T_{\text{decay}} = \sup_{1 \leq n \leq N_{\text{model}}} \left[ -\frac{1}{\omega_n \zeta_n} \ln \left[ 0.05 \sqrt{1 - \zeta_n^2} \right] \right]$$ (7.12)

Where $\omega_n$ and $\zeta_n$ are the natural frequency and damping ratio of the $n^{\text{th}}$ mode.

In order to determine an appropriate convolution time, $T_{\text{conv}} = N_p T_p$, (more precisely to determine $N_p$), which ensures accurate estimates of $Y_i$, the selection is based on the following signal-to-noise (SNR) ratios.

$$\text{SNR}_{sm} = \frac{Y_{sm}}{\sigma_e} \quad \text{SNR}_{cm} = \frac{Y_{cm}}{\sigma_e} \quad \text{with} \quad [m = 1, \ldots, n_h]$$ (7.13)

Furthermore, based on equations (7.9) – (7.11) and (7.13) the HFC estimation errors are
\[ \Delta_{s_m} = \frac{\sqrt{\text{var}[\hat{y}_{s_m} - y_{s_m}]} }{y_{s_m}} \sqrt{\frac{2}{N_p T_p}} \sqrt{\frac{2}{\text{SNR}_{s_m}}} \sqrt{\frac{2}{T_{\text{conv}}}} \quad \text{with } [m = 1, \ldots, n_h] \quad (7.14) \]

\[ \Delta_{c_m} = \frac{\sqrt{\text{var}[\hat{y}_{c_m} - y_{c_m}]} }{y_{c_m}} \sqrt{\frac{2}{N_p T_p}} \sqrt{\frac{2}{\text{SNR}_{c_m}}} \sqrt{\frac{2}{T_{\text{conv}}}} \]

and the worst-case estimation error over the harmonics is

\[ \Delta_{\text{HFC}} \equiv \max \left[ \sup_{1 \leq m \leq n_h} [\Delta_{s_m}], \sup_{1 \leq m \leq n_h} [\Delta_{c_m}] \right] \quad (7.15) \]

Thus, for a good design, \( T_{\text{conv}} \) should be selected such that \( \Delta_{\text{HFC}} \ll 1 \) which indicates a small estimation error. The statistical analysis in (7.11) shows that, for any given measurement noise level \( \sigma_e^2 \), the variance of the HFC estimate can be made arbitrarily small by selecting a sufficiently large convolution period \( T_{\text{conv}} \) (sufficiently large \( N_p \)).

Since \( T_{\text{update}} = T_{\text{decay}} + T_{\text{conv}} \), Figure 7.17 shows the worst case HFC estimation error, \( \Delta_{\text{HFC}} \), versus the AVC update period for several signal-to-noise ratios.
In Configuration I, the adaptive control attempts to suppress both the synchronous imbalance response and the super-synchronous harmonic NCV effects (MHAVC). In Configuration II, since the NCV effects are eliminated by the use of rigid couplings, only the synchronous response will be suppressed (SAVC). Table 7.5 summarizes the parameters utilized in the experimental implementation of the AVC control in each configuration.

Fig. 7.17: HFC estimation error vs. AVC update time for several signal-to-noise ratios.
Furthermore, the update law dependents on the HFC mapping from $U_i$ to $Y_i$. The actual HFC mapping, $y_{uT}$, depends on the driveline operating conditions. However, as shown with the robust convergence and performance analyses in Chapters 4 and 5, the AVC update law can be successfully approximated by using the HFC map of the levitated non-rotating system $\hat{y}_{uT}$. To implement Configurations I and II, $\hat{y}_{uT}$ is estimated offline with batch least-squares system identification of the non-rotating, levitated, PID feedback controlled driveline. Specifically, a set of $p >> 2n_u n_h$ randomly generated HFC input vectors, $U_i$, is injected into the system and the corresponding response HFC vector, $Y_i$, is stored. Finally, the estimate, $\hat{y}_{uT}$, is obtained via (7.16).

$$\hat{y}_{uT} = [Y_1 \cdots Y_p][U_1 \cdots U_p]^T \begin{bmatrix} [U_1 \cdots U_p][U_1 \cdots U_p]^T \end{bmatrix}^{-1}$$  \hspace{1cm} (7.16)$$

To gauge the vibration performance and the control effort, the following root mean square (RMS) metrics, $y_{rms}$ and $i_{rms}$, are defined for the entire system.

$$y_{rms} = \frac{1}{n_y} \sum_{i=1}^{n_y} \frac{1}{N_{pT_p}} \int_0^{N_{pT_p}} y_i^2 \, dt$$ \hspace{1cm} \text{and} \hspace{1cm} i_{rms} = \frac{1}{n_u} \sum_{i=1}^{n_u} \frac{1}{N_{pT_p}} \int_0^{N_{pT_p}} u_i^2 \, dt$$  \hspace{1cm} (7.17)$$

**Table 7.5: Configuration I and II AVC Parameters**

<table>
<thead>
<tr>
<th>Harmonic Adaptive Vibration Controller Parameters</th>
<th>Configuration I (MHAVC)</th>
<th>Configuration II (SAVC)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of Harmonics, $n_h$</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>Harmonics to be suppressed</td>
<td>$\Omega, 2\Omega, 3\Omega, 4\Omega$</td>
<td>$\Omega$</td>
</tr>
<tr>
<td>Update Period, $T_{update}$</td>
<td>3 sec</td>
<td>3 sec</td>
</tr>
<tr>
<td>Convolution Period, $T_{conv}$</td>
<td>1 sec</td>
<td>1 sec</td>
</tr>
</tbody>
</table>


7.4 Experimental Results: Configuration I with PID-MHACVC Control

Figure 7.18 and Table 7.6 show the experimental setup used for the Configuration I experimental investigation.

Fig. 7.18: Testrig setup for Configuration I experiment.

Table 7.6: Configuration I Testrig Setup

<table>
<thead>
<tr>
<th>Driveshaft</th>
<th>Bearings &amp; Dampers</th>
</tr>
</thead>
<tbody>
<tr>
<td>Material</td>
<td>303 Stainless-Steel Passive dampers none</td>
</tr>
<tr>
<td>Outer diameter</td>
<td>9.525 mm Passive bearings none</td>
</tr>
<tr>
<td>Wall thickness</td>
<td>Solid Magnetic bearings 3 AMB</td>
</tr>
<tr>
<td>Number of segments</td>
<td>2 AMB location</td>
</tr>
<tr>
<td>Segment length</td>
<td>$L_{s1} = 0.816$ m $L_{s2} = 0.816$ m</td>
</tr>
<tr>
<td>Couplings</td>
<td>3 U-Joints</td>
</tr>
</tbody>
</table>

Table 7.7 shows the different test cases which were conducted in Configuration I.
Table 7.7: Configuration I Experiment Test Cases

<table>
<thead>
<tr>
<th>Case Number</th>
<th>Shaft Speed</th>
<th>Load-Torque</th>
<th>Misalignment (All Cases)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Low Torque Cases</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Case 1</td>
<td>$\Omega = 10.3 \text{ Hz}$</td>
<td>$T_L = 1.76 \text{ N-m}$</td>
<td>Horizontal-Plane: $\delta_2 = 0^o$, $\xi_2 = 0^o$, $\gamma_2 = 3^o$</td>
</tr>
<tr>
<td>Case 2</td>
<td>$\Omega = 16.3 \text{ Hz}$</td>
<td>$T_L = 2.09 \text{ N-m}$</td>
<td>Vertical-Plane: $\delta_3 = 0^o$, $\xi_3 = 0^o$, $\gamma_3 = 0^o$</td>
</tr>
<tr>
<td>Case 3</td>
<td>$\Omega = 22.2 \text{ Hz}$</td>
<td>$T_L = 2.64 \text{ N-m}$</td>
<td></td>
</tr>
<tr>
<td>Case 4</td>
<td>$\Omega = 30.1 \text{ Hz}$</td>
<td>$T_L = 3.04 \text{ N-m}$</td>
<td></td>
</tr>
<tr>
<td>High Torque Cases</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Case 5</td>
<td>$\Omega = 10.0 \text{ Hz}$</td>
<td>$T_L = 5.03 \text{ N-m}$</td>
<td></td>
</tr>
<tr>
<td>Case 6</td>
<td>$\Omega = 16.0 \text{ Hz}$</td>
<td>$T_L = 5.16 \text{ N-m}$</td>
<td></td>
</tr>
<tr>
<td>Case 7</td>
<td>$\Omega = 22.5 \text{ Hz}$</td>
<td>$T_L = 5.21 \text{ N-m}$</td>
<td></td>
</tr>
<tr>
<td>Case 8</td>
<td>$\Omega = 30.3 \text{ Hz}$</td>
<td>$T_L = 5.27 \text{ N-m}$</td>
<td></td>
</tr>
</tbody>
</table>

The resulting RMS vibration and control current metrics for the Configuration I test cases under PID and PID-MHAVC control are summarized in Figures 7.19 and 7.20.

![Fig. 7.19: RMS vibration metric for Configuration I experiment.](image-url)
The time-domain response for each of the experiment test cases is given in Appendix G. In particular, Figures 7.21 and 7.22 give the vibration and control current responses for Case 6, where the results are similar to the remain test cases.
Fig. 7.21: Shaft response, Configuration I experiment Case 6.

Fig. 7.22: Control currents, Configuration I experiment Case 6.
Finally, Figure 7.23 shows the spectrum of a typical Configuration II response.

The experimental results for Configuration I demonstrate that PID-MHAVC control strategy achieves significant vibration suppression of imbalance, misalignment and load-torque induced vibrations. Since vibration suppression was achieved for all test cases without knowledge of misalignment, load-torque or imbalance utilized in the control law, the robustness claims made in Chapter 4 are experimentally confirmed. Additionally, by examining the control currents in Figures 7.20 and 7.22, the experiment results show that the PID-MHAVC control requires less control current than the PID baseline control.
This reduction in required control current is in agreement with the analytical predictions made in Chapter 4.

7.5 Experimental Results: Configuration II with PID-SAVC Control

Figure 7.24 and Table 7.8 show the experimental setup used for the Configuration II experimental investigation.

Fig. 7.24: Testrig setup for Configuration II experiment.
Table 7.8: Configuration II Testrig Setup

<table>
<thead>
<tr>
<th>Driveshaft</th>
<th>Bearings &amp; Dampers</th>
</tr>
</thead>
<tbody>
<tr>
<td>material</td>
<td>FMC(^a) with Ply layer angles [45°/-45°/-90°/90°]</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>outer diameter</td>
<td>9.525 mm</td>
</tr>
<tr>
<td>wall thickness</td>
<td>0.75 mm</td>
</tr>
<tr>
<td>Number of segments</td>
<td>2</td>
</tr>
<tr>
<td>Segment length</td>
<td>(L_{s1} = 0.819) m (L_{s2} = 0.851) m</td>
</tr>
<tr>
<td>Couplings</td>
<td>3 Rigid (Clamp-Type)</td>
</tr>
</tbody>
</table>

Table 7.9 gives the different test cases which were conducted in the Configuration II experimental investigation.

Table 7.9: Configuration II Experiment Test Cases

<table>
<thead>
<tr>
<th>Case Number</th>
<th>Shaft Speed</th>
<th>Load-Torque</th>
<th>Misalignment (All Cases)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case 1</td>
<td>(\Omega = 10.0) Hz</td>
<td>(T_L = 0) N-m</td>
<td>Horizontal-Plane: (\delta_2 = 0^\circ, \xi_2 = 0^\circ, \gamma_2 = 0^\circ)</td>
</tr>
<tr>
<td>Case 2</td>
<td>(\Omega = 15.0) Hz</td>
<td>(T_L = 0) N-m</td>
<td>Vertical-Plane: (\delta_3 = 0^\circ, \xi_3 = 0^\circ, \gamma_3 = 0^\circ)</td>
</tr>
<tr>
<td>Case 3</td>
<td>(\Omega = 25.0) Hz</td>
<td>(T_L = 0) N-m</td>
<td></td>
</tr>
</tbody>
</table>

Here, the Configuration II experiment is conducted at three different operating speeds i.e. Cases 1–3. However, due to several limitations of the testrig hardware in Configuration II, no driveline misalignment \((\delta_2 = \xi_2 = \gamma_2 = \delta_3 = \xi_3 = \gamma_3 = 0^\circ)\) and no load-torque \((T_L = 0\)
N-m) is applied during the tests. Unlike Configuration I with flexible-couplings to accommodate the misalignments, in Configuration II misalignments produce shaft curvature. Due to the relatively large AMB length - shaft length ratio that occurs at the testrig scale, shaft curvature within the AMB causes internal AMB rotor/stator clearance problems. of this issue. Therefore, only the aligned FMC driveline is explored in the Configuration II experiment test cases. Furthermore, due to the low stiffness of the FMC shaft material, the clamp-type rigid couplings used in the Configuration II Testrig drivelining did not achieve sufficient clamping force to allow transmission of the load-torque without slippage. Therefore, the Configuration II experiment was conducted without any applied load torque. Refer to Chapter 8 for more discussion and possible future research related to these issues.

As shown from the analysis in Chapter 5, since there are no NCV effects in the rigidly coupled FMC driveline system, the misalignment and load-torque do not explicitly affect the Configuration II driveline dynamics (unlike Configuration I). Therefore, the Configuration II experiment test Cases 1-3 are adequate to validate and explore the converge behavior and the imbalance vibration suppression performance of the feedback/SAVC controlled AMB-FMC driveline system. Specifically, the resulting RMS vibration and control current metrics for the Configuration II test cases under PID and PID-SAVC control are summarized in Figures 7.25 and 7.26.
Fig. 7.25: RMS vibration metric for Configuration II experiment.

Fig. 7.26: RMS control current metric for Configuration II experiment.
See Appendix G for the time-domain response data from each test case. In particular, Figure 7.27 and Figure 7.28 give the vibration response and the corresponding response spectrum for a particular case.

Fig. 7.27: Shaft response, Configuration II experiment Case 1.
The Configuration II experimental results shows that the response is concentrated at the operating speed $\Omega$. This is as expected since the are no NVC effects due to the use of rigid couplings. Thus, only synchronous imbalance excitation is present in the system. The experimental results for Configuration II also demonstrate that PID-SAVC control achieves significant vibration suppression of imbalance vibration. Since this vibration suppression is achieved for all test cases without knowledge of the imbalance, the PID-SAVC successfully adapts and has robust performance characteristics. Also, similar to Configuration I, and as predicted by the analysis in Chapter 5, the Configuration II
experimental results show that the feedback/SAVC control achieves vibration suppression with less control current than the baseline feedback controlled system.

7.6 Experimental Investigation Summary and Conclusions

Utilizing the frequency-scaled driveline Testrig developed in this research, the stability and performance of the conventionally configured driveline and the actively controlled driveline configurations I and II are experimentally investigated over a range of driveline operating conditions.

In the first phase of the investigation, the experimentally measured driveline vibration is compared with analytical predictions under various misalignment, load-torque and shaft speed operating conditions. The results of this comparison show that the comprehensive driveline analytical model developed in Chapter 2 accurately predicts the multi-harmonic driveline vibration response due NCV coupling kinematics, driveline misalignment, and load-torque.

In the second phase of the investigation, the two actively controlled driveline configurations I and II are experimentally implemented and evaluated using the driveline Testrig and the AMB hardware. In Configuration I, the experimental results demonstrate that the hybrid PID-MH AVC strategy developed in this thesis successfully adapts and converges to suppress the multi-harmonic driveline vibration induced by misalignment, load-torque and imbalance. Since no knowledge of the misalignment, load-torque or imbalance is utilized in the control implementation, the experiment demonstrates the robustness of the PID-MH AVC control strategy. Furthermore, the Configuration II
experiment demonstrates the successful operation of the AMB-FMC driveline system and validates the robustness and the imbalance vibration suppression capabilities of the PID-SAVC control strategy. Additionally, the experimental results from Configurations I and II confirm the control current trends predicted by the analyses in Chapters 4 and 5. Specifically, the experiments show that the PID-MHAVC and PID-SAVC strategies (in Configurations I and II respectively) achieved vibration suppression without increasing the overall RMS control current relative to the PID feedback controlled baseline.

Finally, the experimental results (see Appendix G) and the time-domain analysis in Chapter 4 demonstrate that PID-MHAVC (and PID-SAVC) achieves approximately 95% convergence after two AVC update steps. Thus $T_{convergence} \cong 2T_{update}$. In the case of the experiment, since $T_{update} = 3$ sec, the convergence time is $T_{convergence} = 6$ sec. The AVC convergence time is an important aspect of the control performance especially in a quasi-steady environment where the driveline misalignment and load-torque operating conditions are changing frequently. Figure 7.29 is a schematic plot of driveline load-torque in a quasi-steady operating environment.
In the case of rotorcraft drivelines, such a quasi-steady operating environment occurs during maneuvering flight. In particular, Horn, et al. (2002), showed that driveline load-torque varied in a quasi-steady manner during various pull-up and pitch-over maneuvers. In order for the adaptive driveline vibration control (MHAVC and SAVC) to be beneficial during such maneuvers, the convergence time should less than the quasi-steady operation periods. Horn, et al. (2002) showed that the quasi-steady operation periods can be on the order of 10 to 15 seconds during aggressive pull-up and pitch-over maneuvers and larger for less aggressive maneuvering. Therefore, PID-MHAVC and PID-SAVC control with $T_{update} = 3$ sec ($T_{convergence} = 6$ sec) would be adequate to suppress the rotorcraft driveline vibration during maneuvering flight. Finally, based on the relation between the SNR and $T_{update}$ in (7.14) and in Figure 7.17, the AVC update and convergence times could be further reduced by ensuring a sufficient SNR is present in the control system.

Fig. 7.29: Driveline load-torque in quasi-steady operating environment.

![Driveline load-torque in quasi-steady operating environment](image-url)
Chapter 8

CONCLUSIONS AND RECOMMENDATIONS FOR FUTURE WORK

8.1 Summary and Conclusions

To advance the state-of-the-art of rotorcraft and other high-performance drivelines, this thesis explores two novel actively controlled tailrotor driveline configurations utilizing both AMB and FMC technologies. In the first configuration, Configuration I, the conventional contact hanger-bearings and viscous shaft dampers are replaced with non-contact AMB devices. In Configuration II, the conventional hanger-bearings and viscous dampers are replaced with AMBs, and the conventional segmented shaft and flexible-couplings are replaced with a rigidly coupled single-piece FMC shaft.

This research theoretically develops and experimentally validates novel hybrid robust-adaptive vibration control strategies for both of the new actively controlled driveline configurations based on their specific characteristics and uncertainties.

To facilitate analysis and development of the active control laws, a comprehensive analytical model of the driveline system was developed for both driveline
configurations. The Configuration I model consists of a multi-segment flexible driveshaft supported by magnetic bearings, and includes effects of rotating-frame damping, magnetic field negative stiffness, imbalance, non-constant velocity flexible-coupling kinematics, misalignment, load-torque and foundation flexibility. The Configuration II model is similar but with the non-constant velocity flexible-coupling kinematics removed and the FMC shaft modeled with classical lamination theory.

To experimentally validate the comprehensive tailrotor driveline-fuselage analytical model and evaluate the closed-loop performance of the control algorithms developed for Configurations I and II, a comprehensive, frequency-scaled, active magnetic bearing controlled, multi-segment driveline-foundation testrig is designed and constructed. The testrig driveline features adjustable misalignments, load-torque and load-inertia, and is mounted on a flexible foundation beam structure coupled with electromagnetic shaker. The testrig is fully instrumented with accelerometers, optical encoder, eddy current probes, torque sensor, and three digitally controlled magnetic bearings. Comparisons between analytical predictions and experimental results indicate that the comprehensive driveline analytical model developed in this research successfully predicts effects of imbalance, misalignment and load-torque on driveline dynamics.

The control strategy developed for Configuration I is based on a hybrid design consisting of a PID feedback controller augmented with a slowly adapting, Multi-Harmonic Adaptive Vibration Control (MHAVC). The MHAVC is based on a novel, multi-harmonic Fourier coefficient transfer matrix representation of the linear periodic time-varying driveline system obtained via a harmonic balance approach. From this, the
convergence condition and the relations for the converged response of the adaptive control law was formulated.

Both the analytical and experimental results for Configuration I demonstrate that the PID-MHAVC control strategy achieves significant multi-harmonic vibration suppression of the synchronous imbalance, and the super-synchronous misalignment and load-torque induced vibrations. Furthermore, since this vibration suppression was achieved without knowledge of misalignment, load-torque or shaft imbalance condition, the control strategy developed for the segmented AMB supercritical driveline system is robust. Finally, both analysis and experiment demonstrate that, except near the closed-loop natural frequencies, the PID-MHAVC achieved vibration suppression without increasing the overall control current relative to the PID controlled baseline.

The control strategy developed for the AMB-FMC tail rotor driveline system (Configuration II) is based on a hybrid $H_\infty$ feedback/Synchronous Adaptive Vibration Control (SAVC) strategy. This control considers the effects of temperature dependent FMC material properties, rotating-frame damping and shaft imbalance. The analysis shows that this hybrid control strategy can guarantee stability, convergence and imbalance vibration suppression under the conditions of bounded temperature deviations and unknown imbalance. Furthermore, the controller is effective and robust at both sub and supercritical operating speeds except at speeds near the closed-loop transmission zeros. The experimental investigation confirms the synchronous suppression abilities of the control.
The robustness and performance characteristics of the hybrid feedback/harmonic adaptive vibration control strategies developed in this research demonstrate the feasibility and benefits of employing AMB and FMC shaft technology in future rotorcraft driveline systems.

8.2 Recommendations for Future Work

One important topic for future investigations is to explore new AMB designs which expand the operating envelope of AMB-FMC drivelines and allow the high strain capabilities of FMC shaft technology to be more fully utilized. Current AMB designs are optimized for use with more conventional segmented alloy driveshafts and do not effectively tolerate large shaft curvatures. In the conventional segmented driveline case, since driveline misalignments are accommodated by flexible couplings, the shaft segment curvatures are relatively small and thus the shaft curvature tolerance of the AMB is not an issue. However, in the case of rigidly coupled FMC driveshafts, where misalignments are accommodated by shaft bending, the curvature limitations imposed by the AMBs may reduce the maximum misalignment that the AMB-FMC driveline system can tolerate. See Figure 8.1.
Conventional radial magnetic bearing designs have a single actuation plane (coil-plane) and thus are essentially point-load devices and cannot effectively control shaft curvatures within the AMB. Under conditions of high shaft curvature due to driveline misalignment, as shown in Figure 8.1, even if the nominal airgap, $h_{\text{gap}}$, is achieved and maintained in the sensor-plane by a control law, shaft/backup-bearing contact cannot be avoided due to internal AMB clearances.

One proposed solution to explore would be the design of an AMB with both force and moment actuation capability to enable increased shape control authority of the shaft.

Fig. 8.1: Misaligned FMC driveline with conventional single actuation-plane AMB.
This could be achieved with an additional coil-plane and sensor-plane within the AMBs. See Figure 8.2.

Fig. 8.2: Misaligned FMC driveline with proposed dual actuation-plane AMB.

This dual actuation plane AMB could reduce shaft curvature within the AMB and allow the AMB-FMC driveline system to operate under more severe misalignment conditions than with conventional single plane AMBs. In addition to shape control, the added moment control authority could also increase the effectiveness of any vibration control and potentially allow for fewer AMBs to be utilized in the overall control system. One obvious design challenge would be to design the dual-plane AMB with minimal weight and size increases over the single plane AMB design. Since the coils share the loading,
the size of coils in the dual-plane AMB device could be smaller than the coils of the single plane AMB. Thus, with proper coil sizing, the size and mass of the dual plane AMB could be close to the single plane AMB design. Furthermore, if the increased modal control authority of the dual plane AMB enables fewer AMBs to be utilized, this is also a potential weight saving benefit to be explored.

Another important issue raised by the Configuration II experimental investigation is related to the rigid-coupling/FMC shaft attachment method and the coupling design. Due to the low stiffness of the FMC shaft material, the conventional clamp-type rigid couplings were not able to achieve sufficient clamping force to allow transmission of the load-torque without slippage. Future investigations could explore new rigid-coupling designs and FMC/coupling attachment methods to address this issue. In particular, future investigations should focus on reversible (i.e. non-adhesive based) designs which would allow repeated assembly and disassembly of FMC driveline system.

Finally, another important aspect of AMB-FMC drivelines which should be investigated is in the area of system design. Previous researchers have conducted various design studies for passive-bearing based, FMC and rigid matrix composite drivelines based on tailoring the ply orientation angles and ply thickness, see Darlow, et al. (1990) and Shin, et al. (2003). In these studies, significant weight savings over conventional alloy shaft designs were achieved. Furthermore, in part, this thesis explores the use of AMB technology with FMC driveshafts and develops a robust-adaptive vibration control strategy for a given FMC shaft design. In order to take full advantage of the benefits offered by an AMB-FMC driveline configuration, a simultaneous FMC structural
tailoring and active control law design needs to be performed. By combining the adaptive control strategy and the corresponding robust design methodologies developed in this thesis with a comprehensive FMC shaft design, an overall optimum AMB-FMC design can be achieved. In the case of rotorcraft driveline systems, the minimum weight AMB-FMC driveline design should be explored considering driveline torsional stiffness, torsional bucking, maximum allowable misalignment, closed-loop robustness with respect to FMC temperature sensitivity and required control power.


Appendix A

FINITE ELEMENT MODEL ELEMENTAL MATRICES

A.1 Shaft Elemental Matrices

The \( i^{th} \) shaft inertia matrix of the \( j^{th} \) element for \( i=[1, 2] \) and \( j=[1, 2, \cdots, N_{el}^j] \) is

\[
M_{el,i}^j = \int_0^{L_{el}} \left[ m_s (N_{u}^T N_{u} + N_{v}^T N_{v} + N_{w}^T N_{w}) + Im_s (N_{v}^T N_{v} + N_{w}^T N_{w}) + Jm_s N_{\phi}^T N_{\phi} \right] dx \tag{A.1}
\]

The \( i^{th} \) shaft gyroscopic matrix of the \( j^{th} \) element for \( i=[1, 2] \) and \( j=[1, 2, \cdots, N_{el}^j] \) is

\[
G_{el,i}^j = \int_0^{L_{el}} m_s \Omega_i (N_{v}^T N_{v} - N_{w}^T N_{w}) dx \tag{A.2}
\]

The \( i^{th} \) shaft damping matrix of the \( j^{th} \) element for \( i=[1, 2] \) and \( j=[1, 2, \cdots, N_{el}^j] \) is

\[
C_{el,i}^j = \int_0^{L_{el}} \left[ E_s A_s N_{u}^T N_{u} + E_s I_s (N_{v}^T N_{v} + N_{w}^T N_{w}) + G_s J_s N_{\phi}^T N_{\phi} \right] dx \tag{A.3}
\]

The \( i^{th} \) shaft elastic stiffness matrix of the \( j^{th} \) element for \( i=[1, 2] \) and \( j=[1, 2, \cdots, N_{el}^j] \) is
\[
\mathbf{K}_{el,j}^i = \int_0^{L_{el}^i} \left[ E_s A_s N_u^T N_u' + E_s I_1 \left( N_v^x T N_v^x + N_w^x T N_w^x \right) + G_s J_s N_y^T N_y' \right] dx
\]  \hspace{1cm} (A.4)

The \(i^{th}\) shaft rotational-damping stiffness matrix of the \(j^{th}\) element for \(i = [1, 2]\) and \(j = [1, 2, \ldots, N_{el}^i]\) is

\[
\mathbf{K}_{rd_{el,j}}^i = \int_0^{L_{el}^i} \left( N_v^x T N_v^x - N_w^x T N_w^x \right) dx
\]  \hspace{1cm} (A.5)

The \(i^{th}\) shaft axial-buckling stiffness matrix of the \(j^{th}\) element for \(i = [1, 2]\) and \(j = [1, 2, \ldots, N_{el}^i]\) is

\[
\mathbf{K}_{P_{el,j}}^i = -P_a \int_0^{L_{el}^i} \left( N_v^x T N_v^x + N_w^x T N_w^x \right) dx
\]  \hspace{1cm} (A.6)

The \(i^{th}\) shaft torque-buckling stiffness matrix of the \(j^{th}\) element for \(i = [1, 2]\) and \(j = [1, 2, \ldots, N_{el}^i]\) is

\[
\mathbf{K}_{T_{el,j}}^i = T_1 \int_0^{L_{el}^i} \left( N_v^x T N_v^x - N_w^x T N_w^x \right) dx
\]  \hspace{1cm} (A.7)

### A.2 Fuselage-Beam Elemental Matrices

The fuselage-beam inertia matrix of the \(j^{th}\) element for \(j = [1, 2, \ldots, N_{el}^j]\) is
The fuselage-beam damping matrix of the $j^{th}$ element for $j=[1,2,\ldots,N_{el}]$ is

$$M_{el,j}^f = \int_0^{L_{el}} \left[ m_f (N_u^T N_u + N_v^T N_v + N_w^T N_w) + I m_{f33} N_v^T N_v' + I m_{f22} N_w^T N_w' + I m_{f11} N_\theta^T N_\theta \right] dx \quad (A.8)$$

The fuselage-beam elastic stiffness matrix of the $j^{th}$ element for $j=[1,2,\ldots,N_{el}]$ is

$$C_{el,j}^f = \xi \int_0^{L_{el}} \left[ E_f A_f N_u^T N_u' + E_f I_{f33} N_v^T N_v' + E_f I_{f22} N_w^T N_w' + G_f J_{fcs} N_\theta^T N_\theta \right] dx \quad (A.9)$$

The fuselage-beam elastic stiffness matrix of the $j^{th}$ element for $j=[1,2,\ldots,N_{el}]$ is

$$K_{el,j}^f = \int_0^{L_{el}} \left[ E_f A_f N_u^T N_u' + E_f I_{f33} N_v^T N_v' + E_f I_{f22} N_w^T N_w' + G_f J_{fcs} N_\theta^T N_\theta \right] dx \quad (A.10)$$
Appendix B

MISALIGNMENT-INDUCED INERTIA LOAD AND TORQUE LOAD

PERIODIC NCV COUPLING TERMS

B.1 Load-Inertia NCV Terms

The kinetic energy of the torsional inertia load, $J_L$, is

$$T_{J_L} = \frac{1}{2} J_L \Omega_L^2 \quad (B.1)$$

Time differentiation of (2.12), gives the load rotation speed, $\Omega_L = \frac{d\phi_L}{dt}$, which is a function of the static misalignments and twists, $\theta_{ABC} = [\delta_2 \; \delta_3 \; \xi_2 \; \xi_3 \; \gamma_2 \; \gamma_3 \; \phi_{sB} \; \phi_{sC}]$, and shaft elastic slopes and twists, $q_{ABC} = [v'_{1A} \; -w'_{1A} \; v'_{1B} \; -w'_{1B} \; \hat{\phi}_{1B} \; v'_{2B} \; -w'_{2B} \; v'_{2C} \; -w'_{2C} \; \hat{\phi}_{2C}]^T$, at couplings A, B and C. Functionally, $\Omega_L$ is written as

$$\Omega_L = \Omega_L(\Omega_0, \theta_{ABC}, q_{ABC}, q_{ABC}) \quad (B.2)$$

where $q_{ABC}$ is related to the global unconstrained generalized coordinate vector, $q$, via the transformation $q_{ABC} = T_{abc}q$. According to Lagrange’s Equations, the contribution of the torsional load-inertia kinetic energy to the equations-of-motion is
\[
\begin{split}
\frac{d}{dt}\left( \frac{\partial T_{LL}}{\partial q} \right) - \frac{\partial T_{IL}}{\partial q} &= J_L \dot{\Omega}_L \frac{\partial \Omega_L}{\partial q} + J_L \Omega_L \left( \frac{d}{dt} \left( \frac{\partial \Omega_L}{\partial q} \right) - \frac{\partial \Omega_L}{\partial q} \right) \\
\cdots &= T_{abc} \left[ J_L \Omega_L \left( \frac{d}{dt} \left( \frac{\partial \Omega_L}{\partial q_{ABC}} \right) - \frac{\partial \Omega_L}{\partial q_{ABC}} \right) + J_L \dot{\Omega}_L \frac{\partial \Omega_L}{\partial q_{ABC}} \right]
\end{split}
\]  
(B.3)

with

\[
\frac{d}{dt} \left( \frac{\partial \Omega_L}{\partial q_{ABC}} \right) - \frac{\partial \Omega_L}{\partial q_{ABC}} = 0 
\]  
(B.4)

and

\[
\begin{split}
J_L \dot{\Omega}_L \frac{\partial \Omega_L}{\partial q_{ABC}} &= \left[ m_{\Lambda_0} + m_{\Lambda_22} \sin(2\Omega_0 t) + m_{\Lambda_22} \cos(2\Omega_0 t) \right]_{ABC} \\
\cdots &= \left[ c_{\Lambda_0} + c_{\Lambda_22} \sin(2\Omega_0 t) + c_{\Lambda_22} \cos(2\Omega_0 t) \right]_{ABC} \\
\cdots &= \left[ k_{\Lambda_0} + k_{\Lambda_22} \sin(2\Omega_0 t) + k_{\Lambda_22} \cos(2\Omega_0 t) \right]_{ABC} \\
\cdots &= -f_{\Lambda_0} + f_{\Lambda_22} \sin(2\Omega_0 t) + f_{\Lambda_22} \cos(2\Omega_0 t) 
\end{split}
\]  
(B.5)

Where the inertia coupling matrices are given by (B.6) – (B.8).

\[
\mathbf{m}_{\Delta_0} = J_L \frac{1}{8}
\]

\[
\begin{bmatrix}
3\ddot{\xi}_2 + \dot{\xi}_3^2 & 2\ddot{\xi}_2\dot{\xi}_3 & 3\ddot{\xi}_2\dot{\xi}_3 + \ddot{\xi}_3^3 & \ddot{\xi}_3^2 + \ddot{\xi}_2\dot{\xi}_3 & 0 \\
2\ddot{\xi}_2\dot{\xi}_3 & \ddot{\xi}_3^2 + \ddot{\xi}_2\dot{\xi}_3 & \ddot{\xi}_3^3 + \ddot{\xi}_2\dot{\xi}_3 & \ddot{\xi}_3^2 + \ddot{\xi}_2\dot{\xi}_3 & 0 \\
3\ddot{\xi}_3^2 + \ddot{\xi}_2\dot{\xi}_3 & \ddot{\xi}_3^2 + \ddot{\xi}_2\dot{\xi}_3 & 2\ddot{\xi}_2^2 + \ddot{\xi}_2\dot{\xi}_3 & \ddot{\xi}_3^2 + \ddot{\xi}_2\dot{\xi}_3 & 0 \\
\ddot{\xi}_3^2 + \ddot{\xi}_2\dot{\xi}_3 & \ddot{\xi}_3^2 + \ddot{\xi}_2\dot{\xi}_3 & 2\ddot{\xi}_2^2 + \ddot{\xi}_2\dot{\xi}_3 & \ddot{\xi}_3^2 + \ddot{\xi}_2\dot{\xi}_3 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
\[
\begin{pmatrix}
\delta_2 \xi_2 - \delta_3 \xi_3 & -\delta_3 \xi_2 + 3\delta_2 \xi_3 & \delta_2 \gamma_2 - \delta_3 \gamma_3 & -\delta_3 \gamma_2 + 3\delta_2 \gamma_3 & 4\delta_2 \\
3\delta_3 \xi_2 - \delta_2 \xi_3 & -\delta_2 \xi_2 + \delta_3 \xi_3 & 3\delta_3 \gamma_2 - \delta_2 \gamma_3 & -\delta_2 \gamma_2 + 3\delta_3 \gamma_3 & 4\delta_3 \\
\xi_2^2 - \xi_3^2 & 2\xi_2 \xi_3 & \gamma_2 \xi_2 - \gamma_3 \xi_3 & 3\gamma_2 \xi_2 - \gamma_3 \xi_3 & 4\xi_2 \\
2\xi_2 \xi_3 & -\xi_2^2 + \xi_3^2 & -\gamma_3 \xi_2 + 3\gamma_2 \xi_3 & -\gamma_2 \xi_2 + 3\gamma_3 \xi_3 & 4\xi_3 \\
0 & 0 & 0 & 0 & 0 \\
3\xi_2^2 + \xi_3^2 & 2\xi_2 \xi_3 & 3\gamma_2 \xi_2 + \gamma_3 \xi_3 & \gamma_3 \xi_2 + \gamma_2 \xi_3 & 4\xi_2 \\
2\xi_2 \xi_3 & \xi_2^2 + \xi_3^2 & \gamma_3 \xi_2 + \gamma_2 \xi_3 & \gamma_2 \xi_2 + 3\gamma_3 \xi_3 & 4\xi_3 \\
3\gamma_2 \xi_2 + \gamma_3 \xi_3 & \gamma_3 \xi_2 + \gamma_2 \xi_3 & 3\gamma_2^2 + \gamma_3^2 & 2\gamma_2 \gamma_3 & 4\gamma_2 \\
\gamma_3 \xi_2 + \gamma_2 \xi_3 & \gamma_2 \xi_2 + 3\gamma_3 \xi_3 & 2\gamma_2 \gamma_3 & \gamma_2^2 + 3\gamma_3^2 & 4\gamma_3 \\
4\xi_2 & 4\xi_3 & 4\gamma_2 & 4\gamma_3 & 8
\end{pmatrix}
\]

\[
\mathbf{m}_{\Delta \alpha} = \frac{J_L}{4}
\begin{pmatrix}
2\delta_2 \delta_3 & \delta_2^2 + \delta_3^2 & \delta_3 \xi_2 + \delta_2 \xi_3 & \delta_2 \xi_2 + \delta_3 \xi_3 & 0 \\
\delta_2^2 + \delta_3^2 & 2\delta_2 \delta_3 & \delta_2 \xi_2 + \delta_3 \xi_3 & \delta_3 \xi_2 + \delta_2 \xi_3 & 0 \\
\delta_3 \xi_2 + \delta_2 \xi_3 & \delta_2 \xi_2 + \delta_3 \xi_3 & 2\xi_2 \xi_3 & \xi_2^2 + \xi_3^2 & 0 \\
\delta_2 \xi_2 + \delta_3 \xi_3 & \delta_3 \xi_2 + \delta_2 \xi_3 & \xi_2^2 + \xi_3^2 & 2\xi_2 \xi_3 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\delta_3 \xi_2 - \delta_2 \xi_3 & \delta_2 \xi_2 - \delta_3 \xi_3 & 0 & \xi_2^2 - \xi_3^2 & 0 \\
-\delta_2 \xi_2 + \delta_3 \xi_3 & -\delta_3 \xi_2 + \delta_2 \xi_3 & -\xi_2^2 + \xi_3^2 & 0 & 0 \\
\delta_3 \gamma_2 - \delta_2 \gamma_3 & \delta_2 \gamma_2 - \delta_3 \gamma_3 & -\gamma_3 \xi_2 + \gamma_2 \xi_3 & \gamma_2 \xi_2 - \gamma_3 \xi_3 & 0 \\
-\delta_2 \gamma_2 + \delta_3 \gamma_3 & -\delta_3 \gamma_2 + \delta_2 \gamma_3 & -\gamma_2 \xi_2 + \gamma_3 \xi_3 & \gamma_3 \xi_2 - \gamma_2 \xi_3 & 0 \\
2\delta_3 & 2\delta_2 & 4\phi_{\alpha \beta} \xi_2 + 2\xi_3 & 2\xi_2 - 4\phi_{\alpha \beta} \xi_2 & -4\xi_2 \xi_3 \\
\end{pmatrix}
\]
\[
\begin{bmatrix}
\delta_3 \xi_2 - \delta_2 \xi_3 & - \delta_2 \xi_2 + \delta_3 \xi_3 & \delta_3 \gamma_2 - \delta_2 \gamma_3 & - \delta_2 \gamma_2 + \delta_3 \gamma_3 & 2 \delta_3 \\
\delta_2 \xi_2 - \delta_3 \xi_3 & - \delta_3 \xi_2 + \delta_2 \xi_3 & \delta_2 \gamma_2 - \delta_3 \gamma_3 & - \delta_3 \gamma_2 + \delta_2 \gamma_3 & 2 \delta_2 \\
0 & - \xi_2^2 + \xi_3^2 & - \gamma_3 \xi_2 + \gamma_2 \xi_3 & - \gamma_2 \xi_2 + \gamma_3 \xi_3 & 4 \phi_{AB} \xi_2 + 2 \xi_3 \\
\xi_2^2 - \xi_3^2 & 0 & \gamma_2 \xi_2 - \gamma_3 \xi_3 & \gamma_3 \xi_2 - \gamma_2 \xi_3 & 2 \xi_2 - 4 \phi_{AB} \xi_2 \\
0 & 0 & 0 & 0 & 0 \\
-2 \xi_2 \xi_3 & - \xi_2^2 - \xi_3^2 & - \gamma_3 \xi_2 - \gamma_2 \xi_3 & - \gamma_2 \xi_2 - \gamma_3 \xi_3 & -4 \phi_{AB} \xi_2 - 2 \xi_3 \\
- \xi_2^2 - \xi_3^2 & 2 \xi_2 \xi_3 & - \gamma_2 \xi_2 - \gamma_3 \xi_3 & - \gamma_3 \xi_2 - \gamma_2 \xi_3 & -2 \xi_2 + 4 \phi_{AB} \xi_3 \\
- \gamma_3 \xi_2 + \gamma_2 \xi_3 & - \gamma_2 \xi_2 - \gamma_3 \xi_3 & -2 \gamma_2 \gamma_3 & - \gamma_2^2 - \gamma_3^2 & -2 \gamma_2 \gamma_3 - 4 \gamma_2 \phi_{AC} \\
- \gamma_2 \xi_2 - \gamma_3 \xi_3 & - \gamma_3 \xi_2 - \gamma_2 \xi_3 & - \gamma_2^2 - \gamma_3^2 & -2 \gamma_2 \gamma_3 & -2 \gamma_2 + 4 \gamma_3 \phi_{AC} \\
-4 \phi_{AB} \xi_2 + 2 \xi_3 & -2 \xi_2 + 4 \phi_{AB} \xi_3 & -2 \gamma_3 - 4 \gamma_2 \phi_{AC} & -2 \gamma_2 + 4 \gamma_3 \phi_{AC} & 8 \gamma_2 \gamma_3
\end{bmatrix}
\]

\[
\mathbf{m}_{\lambda \lambda} = \frac{f_L}{2}
\]

\[
\begin{bmatrix}
- \delta_2^2 & 0 & - \delta_2 \xi_2 & 0 & 0 \\
0 & \delta_3^2 & 0 & \delta_3 \xi_3 & 0 \\
- \delta_2 \xi_2 & 0 & - \xi_2^2 & 0 & 0 \\
0 & \delta_3 \xi_3 & 0 & \xi_3^2 & 0 \\
0 & 0 & 0 & 0 & 0 \\
- \delta_2 \xi_3 & 0 & - \xi_2 \xi_3 & 0 & 0 \\
0 & \delta_3 \gamma_2 & 0 & \gamma_2 \xi_3 & 0 \\
- \delta_2 \gamma_3 & 0 & - \gamma_3 \xi_2 & 0 & 0 \\
- \delta_2 & \delta_3 & - \xi_2 + 2 \phi_{AB} \xi_3 & 2 \phi_{AB} \xi_2 + \xi_3 & \xi_3^2
\end{bmatrix}
\]
The damping coupling matrices are given by (B.9) – (B.11).

\[
\begin{bmatrix}
0 & -\delta_2\xi_3 & 0 & -\delta_2\gamma_3 & -\delta_2 \\
\delta_3\xi_2 & 0 & \delta_3\gamma_2 & 0 & \delta_3 \\
0 & -\xi_2\xi_3 & 0 & -\gamma_3\xi_2 & -\xi_2 + 2\phi_3\xi_3 \\
\xi_2\xi_3 & 0 & \gamma_2\xi_3 & 0 & 2\phi_3\xi_2 + \xi_3 \\
0 & 0 & 0 & 0 & \xi_2^2 - \xi_3^2 \\
\xi_2^2 & 0 & \gamma_2\xi_2 & 0 & \xi_2 - 2\phi_3\xi_3 \\
0 & -\xi_3^2 & 0 & -\gamma_3\xi_3 & -2\phi_3\xi_2 - \xi_3 \\
\gamma_2\xi_2 & 0 & \gamma_2^2 & 0 & \gamma_2 - 2\gamma_3\phi_3 \\
0 & -\gamma_3\xi_3 & 0 & -\gamma_3^2 & -\gamma_3 - 2\gamma_2\phi_3 \\
\xi_2 - 2\phi_3\xi_3 & -2\phi_3\xi_2 - \xi_3 & \gamma_2 - 2\gamma_3\phi_3 & -\gamma_3 - 2\gamma_2\phi_3 & -2\gamma_\Delta \\
\end{bmatrix}
\]

\[
\mathbf{c}_{\Delta_0} = \frac{J_L\Omega_0}{2}
\]
\[
\begin{bmatrix}
-\delta_3 \xi_2 + \delta_2 \xi_3 & \delta_2 \xi_2 + \delta_3 \xi_3 & -\delta_3 \gamma_2 + \delta_2 \gamma_3 & \delta_2 \gamma_2 + \delta_3 \gamma_3 & 0 \\
-\delta_2 \xi_2 - \delta_3 \xi_3 & -\delta_3 \xi_2 + \delta_2 \xi_3 & -\delta_2 \gamma_2 - \delta_3 \gamma_3 & -\delta_3 \gamma_2 - \delta_2 \gamma_3 & 0 \\
0 & \xi_2^2 + \xi_3^2 & \gamma_3 \xi_2 - \gamma_2 \xi_3 & \gamma_2 \xi_2 + \gamma_3 \xi_3 & 0 \\
-\xi_2^2 - \xi_3^2 & 0 & -\gamma_2 \xi_2 - \gamma_3 \xi_3 & \gamma_3 \xi_2 - \gamma_2 \xi_3 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & -\xi_2^2 - \xi_3^2 & -\gamma_2 \xi_2 + \gamma_3 \xi_3 & -\gamma_3 \xi_2 + \gamma_2 \xi_3 & 0 \\
\xi_2^2 + \xi_3^2 & 0 & \gamma_2 \xi_2 + \gamma_3 \xi_3 & -\gamma_3 \xi_2 - \gamma_2 \xi_3 & 0 \\
\gamma_3 \xi_2 - \gamma_2 \xi_3 & -\gamma_2 \xi_2 - \gamma_3 \xi_3 & 0 & -\gamma_2^2 - \gamma_3^2 & 0 \\
\gamma_2 \xi_2 + \gamma_3 \xi_3 & \gamma_3 \xi_2 - \gamma_2 \xi_3 & \gamma_2^2 + \gamma_3^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[\mathbf{c}_{\Delta\Omega} = J_L \Omega_0\]

\[
\begin{bmatrix}
\delta_2^2 & -\delta_2 \delta_3 & \delta_2 \xi_2 & -\delta_2 \xi_3 & 0 \\
\delta_2 \delta_3 & -\delta_3^2 & \delta_3 \xi_2 & -\delta_3 \xi_3 & 0 \\
\delta_2 \xi_2 & -\delta_3 \xi_2 & \xi_2^2 & -\xi_2 \xi_3 & 0 \\
\delta_2 \xi_3 & -\delta_3 \xi_3 & \xi_2 \xi_3 & \xi_3^2 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\delta_2 \xi_2 & -\delta_3 \xi_2 & \xi_2^2 & -\xi_2 \xi_3 & 0 \\
\delta_2 \xi_3 & -\delta_3 \xi_3 & \xi_2 \xi_3 & \xi_3^2 & 0 \\
\delta_2 \gamma_2 & -\delta_3 \gamma_2 & \gamma_2^2 & -\gamma_2 \gamma_3 & 0 \\
\delta_2 \gamma_3 & -\delta_3 \gamma_3 & \gamma_2 \gamma_3 & \gamma_3^2 & 0 \\
2\delta_2 & -2\delta_3 & 2\xi_2 & -4\phi_{\gamma \xi_2} & -4\phi_{\gamma \xi_3} - 2\xi_3 & -2\xi_3^2 \\
\end{bmatrix}
\]
\[
\begin{bmatrix}
-\delta_2 \xi_2 & \delta_2 \xi_3 & -\delta_2 \gamma_2 & \delta_2 \gamma_3 & 0 \\
-\delta_3 \xi_2 & \delta_3 \xi_3 & -\delta_3 \gamma_2 & \delta_3 \gamma_3 & 0 \\
-\xi_2 & \xi_2 \xi_3 & -\gamma_2 \xi_2 & \gamma_3 \xi_2 & 0 \\
-\xi_2 \xi_3 & \xi_3 & -\gamma_2 \xi_3 & \gamma_3 \xi_3 & 0 \\
0 & 0 & 0 & 0 & 0 \\
-\xi_2 & \xi_2 \xi_3 & -\gamma_2 \xi_2 & \gamma_3 \xi_2 & 0 \\
-\xi_2 \xi_3 & \xi_3 & -\gamma_2 \xi_3 & \gamma_3 \xi_3 & 0 \\
-\gamma_2 \xi_2 & \gamma_2 \xi_3 & -\gamma_2 \gamma_3 & \gamma_2 \gamma_3 & 0 \\
-\gamma_3 \xi_2 & \gamma_3 \xi_3 & -\gamma_2 \gamma_3 & \gamma_3 \gamma_3 & 0 \\
-2\xi_2 + 4\phi_3 \xi_3 & 4\phi_3 \xi_2 + 2\xi_3 & -2\gamma_2 + 4\gamma_3 \phi \Delta & 2\gamma_3 + 4\gamma_2 \phi \Delta & 2\gamma_2 \xi_3 \\
\end{bmatrix}
\]

\[
\mathbf{c}_{\Delta_2} = J_0 \Omega_0 \begin{bmatrix}
\delta_2 \delta_3 & \delta_2 \xi_3 & \delta_2 \gamma_3 & \delta_2 \xi_3 & 0 \\
\delta_3 \xi_2 & \delta_3 \xi_3 & \delta_3 \gamma_2 & \delta_3 \gamma_3 & 0 \\
\delta_3 \xi_3 & \delta_3 \xi_3 & \xi_3 & -\xi_2 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\delta_3 \xi_2 & \delta_3 \xi_3 & \xi_3 & -\xi_2 & 0 \\
\delta_3 \xi_3 & \delta_3 \xi_3 & \xi_3 & -\xi_2 & 0 \\
\delta_3 \gamma_2 & \delta_3 \gamma_3 & \gamma_2 \xi_3 & \gamma_2 \xi_3 & 0 \\
\delta_3 \gamma_3 & \delta_3 \gamma_3 & \gamma_2 \xi_3 & \gamma_3 \xi_3 & 0 \\
2\delta_3 & 2\delta_3 & 4\phi_3 \xi_3 & 2\xi_3 & 2\xi_3 - 4\phi_3 \xi_3 & -4\xi_2 \xi_3 & 0 \\
\end{bmatrix}
\]

\( \Delta_2 \)
The stiffness coupling matrices are given by (B.12) – (B.14).
\[
\begin{array}{cccc}
-2\delta_3\xi_2 + 2\delta_2\xi_3 & 0 & 2\delta_2\xi_2 + 2\delta_3\xi_3 \\
-2\delta_2\xi_2 - 2\delta_3\xi_3 & 0 & -2\delta_3\xi_2 + 2\delta_2\xi_3 \\
2\xi_2\xi_3 - 2\delta_2\delta_3 - 2\gamma_2\gamma_3 & 0 & \delta_\Delta^2 + \gamma_\Delta^2 + \xi_\Sigma^2 + 2\xi_3^2 \\
\delta_\Delta^2 + \gamma_\Delta^2 - \xi_\Sigma^2 - 2\xi_2^2 & 0 & 2\delta_2\delta_3 + 2\gamma_2\gamma_3 - 2\xi_2\xi_3 \\
\cdots & 0 & 0 & \cdots \\
2\delta_2\delta_3 + 2\gamma_2\gamma_3 - 2\xi_2\xi_3 & 0 & -\delta_\Delta^2 - \gamma_\Delta^2 - \xi_\Sigma^2 - 2\xi_3^2 \\
-\delta_\Delta^2 - \gamma_\Delta^2 + \xi_\Sigma^2 + 2\xi_2^2 & 0 & 2\xi_2\xi_3 - 2\delta_2\delta_3 - 2\gamma_2\gamma_3 \\
2\gamma_3\xi_2 - 2\gamma_2\xi_3 & 0 & -2\gamma_2\xi_2 - 2\gamma_3\xi_3 \\
2\gamma_2\xi_2 + 2\gamma_3\xi_3 & 0 & 2\gamma_3\xi_2 - 2\gamma_2\xi_3 \\
0 & 0 & 0 & 0 \\
2\delta_3\xi_2 - 2\delta_2\xi_3 & 2\delta_2\gamma_2 + 2\delta_3\gamma_3 & 2\delta_3\gamma_2 - 2\delta_2\gamma_3 & 0 \\
2\delta_2\xi_2 + 2\delta_3\xi_3 & -2\delta_3\gamma_2 + 2\delta_2\gamma_3 & 2\delta_2\gamma_2 + 2\delta_3\gamma_3 & 0 \\
2\delta_3\delta_3 + 2\gamma_2\gamma_3 - 2\xi_2\xi_3 & 2\gamma_2\xi_2 + 2\gamma_3\xi_3 & -2\gamma_3\xi_2 + 2\gamma_2\xi_3 & 0 \\
-\delta_\Delta^2 - \gamma_\Delta^2 + \xi_\Sigma^2 + 2\xi_2^2 & 2\gamma_3\xi_2 - 2\gamma_2\xi_3 & 2\gamma_2\xi_2 + 2\gamma_3\xi_3 & 0 \\
0 & 0 & 0 & 0 \\
2\xi_2\xi_3 - 2\delta_2\delta_3 - 2\gamma_2\gamma_3 & -2\gamma_2\xi_2 - 2\gamma_3\xi_3 & 2\gamma_3\xi_2 - 2\gamma_2\xi_3 & 0 \\
\delta_\Delta^2 + \gamma_\Delta^2 - \xi_\Sigma^2 - 2\xi_2^2 & -2\gamma_3\xi_2 + 2\gamma_2\xi_3 & -2\gamma_2\xi_2 - 2\gamma_3\xi_3 & 0 \\
-2\gamma_3\xi_2 + 2\gamma_2\xi_3 & \delta_\Delta^2 - \gamma_\Delta^2 - \xi_\Sigma^2 - 2\gamma_3^2 & 2\delta_2\delta_3 + 2\gamma_2\gamma_3 - 2\xi_2\xi_3 & 0 \\
-2\gamma_2\xi_2 - 2\gamma_3\xi_3 & 2\delta_2\delta_3 + 2\gamma_2\gamma_3 - 2\xi_2\xi_3 & -\delta_\Delta^2 - \gamma_\Delta^2 + \xi_\Sigma^2 - 2\gamma_2^2 & 0 \\
0 & 0 & 0 & 0 \\
\end{array}
\]
$$k_{\Delta 4} = J_L \Omega_0^2$$

$$\begin{bmatrix}
- \delta_2 \delta_3 & - \delta_2 \xi_3 & - \delta_2 \xi_2 & 0 \\
- \delta_2 \xi_2 & - \delta_2 \delta_3 & - \delta_2 \xi_3 & 0 \\
- \delta_3 \xi_3 & - \delta_3 \xi_2 & - \xi_2 \xi_3 & - \xi_2 \xi_2 \\
- \delta_3 \xi_2 & - \delta_3 \xi_3 & - \xi_2^2 & - \xi_2 \xi_2 \\
0 & 0 & 0 & 0 \\
- \delta_3 \xi_2 & - \delta_2 \xi_3 & - \xi_2 \xi_3 & \frac{1}{2} (\delta_\Delta^2 + \gamma_\Delta^2 - \xi_2^2) - \xi_2^2 \\
- \delta_3 \xi_3 & - \delta_2 \xi_2 & \frac{1}{2} (-\delta_\Delta^2 - \gamma_\Delta^2 + \xi_2^2) - \xi_2^2 & - \xi_2 \xi_3 \\
- \delta_3 \gamma_2 & - \delta_2 \gamma_2 & - \gamma_2 \xi_3 & - \gamma_2 \xi_2 \\
- \delta_3 \gamma_3 & - \delta_2 \gamma_3 & - \gamma_3 \xi_3 & - \gamma_3 \xi_2 \\
- 2 \delta_3 & - 2 \delta_2 & - 4 \phi_{AB} \xi_2 + 2 \xi_3 & - 2 \xi_2 + 4 \phi_{AB} \xi_3 & 4 \xi_2 \xi_3 \\
\end{bmatrix}$$

\[
\begin{array}{cccc}
\delta_2 \xi_2 & \delta_2 \xi_3 & \delta_2 \gamma_2 & 0 \\
\delta_2 \xi_3 & \delta_2 \xi_2 & \delta_2 \gamma_3 & 0 \\
\xi_2 \xi_3 & \frac{1}{2} (-\delta_\Delta^2 - \gamma_\Delta^2 + \xi_2^2) + \xi_3^2 & \gamma_3 \xi_2 & \gamma_3 \xi_2 & 0 \\
0 & 0 & 0 & 0 \\
\xi_2 \xi_2 & \xi_2 & \gamma_2 \xi_2 & \gamma_2 \xi_2 & 0 \\
\xi_2^2 & \xi_2^2 & \gamma_2 \xi_2 & \gamma_2 \xi_2 & 0 \\
\gamma_2 \xi_3 & \gamma_2 \xi_2 & \gamma_2 \xi_2 & \gamma_2 \xi_2 & 0 \\
\gamma_2 \xi_3 & \gamma_2 \xi_2 & \gamma_2 \xi_2 & \gamma_2 \xi_2 & 0 \\
4 \phi_{AB} \xi_2 + 2 \xi_3 & 2 \xi_2 - 4 \phi_{AB} \xi_3 & 2 \gamma_2 + 4 \gamma_2 \phi_{BC} & 2 \gamma_2 - 4 \gamma_2 \phi_{BC} & - 4 \gamma_2 \gamma_3 \\
\end{array}
\]
The forcing terms are given by (B.15) – (B.17).
\[
\begin{align*}
\mathbf{f}_{\Delta 0} &= \frac{J_L \Omega_0^2}{4} \begin{bmatrix}
2\delta_2 (\gamma_2 \gamma_3 - \xi_2 \xi_3) + \delta_3 (\delta_\Sigma^2 + \xi_\Lambda^2 - \gamma_\Lambda^2) \\
\delta_2 (-\delta_\Sigma^2 + \xi_\Lambda^2 - \gamma_\Lambda^2) - 2\delta_3 (\gamma_2 \gamma_3 - \xi_2 \xi_3) \\
2\xi_2 (\delta_2 \delta_3 + \gamma_2 \gamma_3) + \xi_3 (\delta_\Sigma^2 - \xi_\Lambda^2 - \gamma_\Lambda^2) \\
\xi_2 (-\delta_\Sigma^2 + \xi_\Lambda^2 - \gamma_\Lambda^2) - 2\xi_3 (\delta_2 \delta_3 + \gamma_2 \gamma_3)
\end{bmatrix} \\
&\quad \begin{bmatrix}
0 \\
-2\xi_2 (\delta_2 \delta_3 + \gamma_2 \gamma_3) + \xi_3 (\delta_\Sigma^2 + \xi_\Lambda^2 + \gamma_\Lambda^2) \\
\xi_2 (\delta_\Sigma^2 - \xi_\Lambda^2 + \gamma_\Lambda^2) + 2\xi_3 (\delta_2 \delta_3 + \gamma_2 \gamma_3) \\
-2\gamma_2 (\delta_2 \delta_3 - \xi_2 \xi_3) + \gamma_3 (\delta_\Sigma^2 - \xi_\Lambda^2 - \gamma_\Lambda^2) \\
\gamma_2 (\delta_\Sigma^2 - \xi_\Lambda^2 + \gamma_\Lambda^2) + 2\gamma_3 (\delta_2 \delta_3 - \xi_2 \xi_3)
\end{bmatrix}
\end{align*}
\]
(B.15)

\[
\begin{align*}
\mathbf{f}_{\Delta a_2} &= J_L \Omega_0^2 \begin{bmatrix}
\frac{1}{2}\delta_2 (\delta_\Sigma^2 + \xi_\Lambda^2 - \gamma_\Lambda^2) \\
\frac{1}{2}\delta_3 (\delta_\Sigma^2 + \xi_\Lambda^2 - \gamma_\Lambda^2) \\
\frac{1}{2}\xi_2 (\delta_\Sigma^2 + \xi_\Lambda^2 - \gamma_\Lambda^2) \\
\frac{1}{2}\xi_3 (\delta_\Sigma^2 + \xi_\Lambda^2 - \gamma_\Lambda^2)
\end{bmatrix} \\
&\quad \begin{bmatrix}
0 \\
\frac{1}{2}\xi_2 (\delta_\Sigma^2 + \xi_\Lambda^2 - \gamma_\Lambda^2) \\
\frac{1}{2}\xi_3 (\delta_\Sigma^2 + \xi_\Lambda^2 - \gamma_\Lambda^2) \\
\frac{1}{2}\gamma_2 (\delta_\Sigma^2 + \xi_\Lambda^2 - \gamma_\Lambda^2) \\
\frac{1}{2}\gamma_3 (\delta_\Sigma^2 + \xi_\Lambda^2 - \gamma_\Lambda^2)
\end{bmatrix}
\end{align*}
\]
(B.16)

\[
\begin{align*}
\mathbf{f}_{\Delta c_2} &= J_L \Omega_0^2 \begin{bmatrix}
\delta_2 (-\delta_\Sigma \delta_3 + \xi_2 \xi_3 - \gamma_2 \gamma_3) \\
\delta_3 (-\delta_\Sigma \delta_3 + \xi_2 \xi_3 - \gamma_2 \gamma_3) \\
\xi_2 (-\delta_\Sigma \delta_3 + \xi_2 \xi_3 - \gamma_2 \gamma_3) \\
\xi_3 (-\delta_\Sigma \delta_3 + \xi_2 \xi_3 - \gamma_2 \gamma_3)
\end{bmatrix} \\
&\quad \begin{bmatrix}
0 \\
\xi_2 (-\delta_\Sigma \delta_3 + \xi_2 \xi_3 - \gamma_2 \gamma_3) \\
\xi_3 (-\delta_\Sigma \delta_3 + \xi_2 \xi_3 - \gamma_2 \gamma_3) \\
\gamma_2 (-\delta_\Sigma \delta_3 + \xi_2 \xi_3 - \gamma_2 \gamma_3) \\
\gamma_3 (-\delta_\Sigma \delta_3 + \xi_2 \xi_3 - \gamma_2 \gamma_3)
\end{bmatrix}
\end{align*}
\]
(B.17)

With the definitions
Finally, the load-inertia NCV coupling matrices and forcing terms in global unconstrained coordinates are

\[
\begin{align*}
\delta_\Sigma^2 &= \delta_2^2 + \delta_3^2, \\
\xi_\Sigma^2 &= \xi_2^2 + \xi_3^2, \\
\gamma_\Sigma^2 &= \gamma_2^2 + \gamma_3^2, \\
\delta_\Delta^2 &= \delta_2^2 - \delta_3^2, \\
\xi_\Delta^2 &= \xi_2^2 - \xi_3^2, \\
\gamma_\Delta^2 &= \gamma_2^2 - \gamma_3^2.
\end{align*}
\] (B.18)

\[
\mathbf{M}_\Delta(t) = \mathbf{T}_{abc}^T \left[ \mathbf{m}_{\Delta 0} + \mathbf{m}_{\Delta 2} \sin(2\Omega_0 t) + \mathbf{m}_{\Delta 4} \cos(2\Omega_0 t) \right] \mathbf{T}_{abc}
\]

\[
\mathbf{C}_\Delta(t) = \mathbf{T}_{abc}^T \left[ \mathbf{c}_{\Delta 0} + \mathbf{c}_{\Delta 2} \sin(2\Omega_0 t) + \mathbf{c}_{\Delta 4} \cos(2\Omega_0 t) \right] \mathbf{T}_{abc}
\]

\[
\mathbf{K}_\Delta(t) = \mathbf{T}_{abc}^T \left[ \mathbf{k}_{\Delta 0} + \mathbf{k}_{\Delta 2} \sin(2\Omega_0 t) + \mathbf{k}_{\Delta 4} \cos(2\Omega_0 t) \right] \mathbf{T}_{abc}
\]

\[
\mathbf{F}_\Delta(t) = \mathbf{T}_{abc}^T \left[ \mathbf{f}_{\Delta 0} + \mathbf{f}_{\Delta 2} \sin(2\Omega_0 t) + \mathbf{f}_{\Delta 4} \cos(2\Omega_0 t) \right]
\] (B.19)

**B.2 Load-Torque NCV Terms**

In (2.58), the load-torque NCV terms are given as the generalized force vector \( Q_{2,TL} \), which is

\[
Q_{2,TL} = -T_L \frac{\partial \phi_L}{\partial q}
\] (B.20)

Since the load rotation angle, \( \phi_L \), is functional represented as

\[
\phi_L = \phi_L(\phi_0, \theta_{ABC}, q_{ABC}, t)
\] (B.21)

\( Q_{2,TL} \) becomes

\[
Q_{2,TL} = -T_L \frac{\partial \phi_L}{\partial q} = -T_L T_{abc} \frac{\partial \phi_L}{\partial q_{ABC}} = -T_{abc} \left[ T_L \frac{\partial \phi_L}{\partial q_{ABC}} \right]
\] (B.22)

with (B.23)
\[ T_L \frac{\partial \phi_L}{\partial q_{ABC}} = \begin{bmatrix} k_{r_0} + k_{r_2} \sin(2\Omega q t) + k_{r_2} \cos(2\Omega q t) \end{bmatrix} q_{ABC} \]

\[ \cdots \begin{bmatrix} f_{r_0} + f_{r_2} \sin(2\Omega q t) + f_{r_2} \cos(2\Omega q t) \end{bmatrix} \]

Where the stiffness coupling matrices are given by (B.24) – (B.26).

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[ k_{r_0} = T_L \begin{bmatrix}
\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{2} & \phi_{sB} & \xi_2 & \frac{1}{2} & -\phi_{sB} & 0 & 0 & 0 \\
0 & 0 & \phi_{sB} & \frac{1}{2} & -\xi_3 & -\phi_{sB} & -\frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & \xi_2 & -\xi_3 & 0 & -\xi_2 & \phi_{sB} & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2} & -\phi_{sB} & -\xi_2 & -\frac{1}{2} & \phi_{sB} & 0 & 0 & 0 \\
0 & 0 & -\phi_{sB} & \frac{1}{2} & \xi_3 & -\phi_{sB} & \frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & -\phi_{sC} & -\gamma_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \gamma_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\gamma_2 & \gamma_3 & 0 \\
\end{bmatrix}
\]

\[ k_{r_2} = T_L \begin{bmatrix}
\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{2} & \phi_{sB} & \xi_2 & \frac{1}{2} & -\phi_{sB} & 0 & 0 & 0 \\
0 & 0 & \phi_{sB} & \frac{1}{2} & -\xi_3 & -\phi_{sB} & -\frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & \xi_2 & -\xi_3 & 0 & -\xi_2 & \phi_{sB} & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2} & -\phi_{sB} & -\xi_2 & -\frac{1}{2} & \phi_{sB} & 0 & 0 & 0 \\
0 & 0 & -\phi_{sB} & \frac{1}{2} & \xi_3 & -\phi_{sB} & \frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & -\phi_{sC} & -\gamma_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \gamma_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\gamma_2 & \gamma_3 & 0 \\
\end{bmatrix}
\]
\[ \mathbf{k}_{r_{c2}} = T_L \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\phi_{aB} & -\frac{1}{2} & \xi_3 & \phi_{aB} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \phi_{aB} & \xi_2 & \frac{1}{2} & -\phi_{aB} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \phi_{aB} & \xi_2 & \frac{1}{2} & -\phi_{aB} & 0 & 0 \\ 0 & 0 & \phi_{aB} & \xi_3 & \xi_2 & 0 & -\xi_3 & -\xi_2 & 0 \\ 0 & 0 & -\phi_{aB} & -\xi_3 & -\xi_2 & \phi_{aB} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & -\phi_{aB} & -\xi_2 & \phi_{aB} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & -\phi_{aC} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\gamma_3 & -\gamma_2 \end{bmatrix} \]  

(B.26)

And the forcing terms are given by (B.27).

\[ \mathbf{f}_{r_0} = -\frac{T_L}{2}, \quad \mathbf{f}_{r_{a2}} = \frac{T_L}{2}, \quad \text{and} \quad \mathbf{f}_{r_{c2}} = \frac{T_L}{2} \begin{bmatrix} \delta_2 \\ \delta_3 \\ \xi_2 \\ \xi_3 \\ \gamma_2 \\ \gamma_3 \\ 0 \\ \frac{1}{2} \end{bmatrix}, \quad \mathbf{f}_{r_{a2}} = \frac{T_L}{2} \begin{bmatrix} -\delta_2 \\ -\delta_3 \\ -\xi_3 - 2\phi_{aB}\xi_2 \\ -\xi_2 - 2\phi_{aB}\xi_3 \\ \xi_3 + 2\phi_{aB}\xi_2 \\ \xi_2 + 2\phi_{aB}\xi_3 \\ \xi_3 + \phi_{aC}\gamma_2 \\ \gamma_2 + 2\phi_{aC}\gamma_3 \\ \gamma_3 + \phi_{aC}\gamma_2 \end{bmatrix}, \quad \text{and} \quad \mathbf{f}_{r_{c2}} = \frac{T_L}{2} \begin{bmatrix} \delta_2 \\ \delta_3 \\ \xi_2 - 2\phi_{aB}\xi_3 \\ \xi_3 - 2\phi_{aB}\xi_2 \\ -\xi_2 + \xi_3^2 \\ -\xi_3 + \xi_2^2 \\ -\xi_2 + 2\phi_{aB}\xi_3 \\ -\xi_3 + 2\phi_{aB}\xi_2 \\ \gamma_3 + 2\phi_{aC}\gamma_3 \\ \gamma_2 + \phi_{aC}\gamma_2 \end{bmatrix} \]  

(B.27)

Finally, the load-torque NCV coupling matrices and forcing terms in global unconstrained coordinates are

\[
\mathbf{K}_r(t) = \mathbf{T}_{abc}^T \left[ \mathbf{k}_{r_0} + \mathbf{k}_{r_{a2}} \sin(2\Omega_0 t) + \mathbf{k}_{r_{c2}} \cos(2\Omega_0 t) \right] \mathbf{T}_{abc}^T
\]

\[
\mathbf{f}_r(t) = \mathbf{T}_{abc}^T \left[ \mathbf{f}_{r_0} + \mathbf{f}_{r_{a2}} \sin(2\Omega_0 t) + \mathbf{f}_{r_{c2}} \cos(2\Omega_0 t) \right] \mathbf{T}_{abc}^T
\]

(B.28)
NON-DIMENSIONAL SYSTEM MATRICES FOR CONVENTIONAL

DRIVELINE

C.1 Non-Dimensional Nominal System Matrices

The non-dimensional (N.D.) nominal mass and gyroscopic matrices, \( \bar{\mathbf{M}} \) and \( \bar{\mathbf{G}} \), are

\[
\bar{\mathbf{M}} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

\[
\bar{\mathbf{G}} = f_0 \frac{\dot{\xi}^2 \pi^2}{8}
\]

\[
\begin{bmatrix}
0 & f_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-f_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & f_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & f_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -f_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -f_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & f_2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & f_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -f_2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -f_2 & 0 \\
\end{bmatrix}
\]  \hspace{1cm} (C.1)

The N.D. structural and auxiliary damping matrix, \( \bar{\mathbf{C}}_{sd} \) and \( \bar{\mathbf{C}}_{aux} \), are

\[
\bar{\mathbf{C}}_{sd} = f_1 \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & f_1 \dot{\xi} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & f_1 \dot{\xi} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & f_1 \dot{\xi} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & f_2 \dot{\xi} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & f_2 \dot{\xi} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & f_2 \dot{\xi} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & f_2 \dot{\xi} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & f_2 \dot{\xi} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & f_2 \dot{\xi} \\
\end{bmatrix}
\]

\hspace{1cm} (C.2)

and
Finally, the N.D. elastic and rotating-frame-damping stiffness matrices, $\bar{K}$ and $\bar{K}_{rd}$, are

$$\bar{K}_{aux} = \begin{bmatrix}
    c_d & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & c_d & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & c_d & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & c_d & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & c_d & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & c_d & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & c_d & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & c_d \\
\end{bmatrix}$$ \hspace{1cm} (C.3)

and

$$\bar{K} = \begin{bmatrix}
    f_1^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & f_1^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & f_1^2 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & f_2^2 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & f_2^2 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & f_2^2 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & f_\phi^2 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & f_\phi^2 \\
\end{bmatrix}$$ \hspace{1cm} (C.4)

and

$$\bar{K}_{rd} = 2f_0 \begin{bmatrix}
    0 & f_1\xi_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
    -f_1\xi_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & f_1\xi_1 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & f_2\xi_2 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & -f_2\xi_2 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & f_2\xi_2 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & f_\phi^2 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & f_\phi^2 & 0 \\
\end{bmatrix}$$ \hspace{1cm} (C.5)

C.2 Non-Dimensional Load-Inertia NCV Terms

The N.D. inertia coupling matrices are given by (C.6) – (C.8).
$$\mathbf{M}_{\Delta_0} = \mu_\pi^2 \begin{bmatrix}
\frac{\lambda_0^2}{4} & 0 & \frac{\delta y}{4} & 0 & \frac{\lambda_0^2}{2} & 0 & \frac{\delta y}{2} & 0 & \frac{\delta}{2\pi} \\
0 & \frac{\delta y}{4} & 0 & \frac{\delta y}{4} & 0 & \frac{\delta y}{2} & 0 & \frac{\delta y}{2} & 0 \\
-\frac{\delta y}{4} & 0 & \frac{3\delta y}{4} & 0 & -\frac{\delta y}{2} & 0 & -\frac{3\delta y}{2} & 0 & -\frac{\gamma}{2\pi} \\
0 & \frac{\delta y}{4} & 0 & \frac{\gamma^2}{4} & 0 & \frac{\delta y}{2} & 0 & \frac{\gamma^2}{2} & 0 \\
\delta y & 0 & \frac{3\delta y}{2} & 0 & 3\delta^2 & 0 & \delta y & 0 & \frac{\delta}{\pi} \\
0 & \frac{\delta y}{2} & 0 & \frac{\delta y}{2} & 0 & \delta^2 & 0 & -\delta y & 0 \\
\frac{\delta y}{2} & 0 & \frac{3\delta y}{2} & 0 & \delta y & 0 & 3\gamma^2 & 0 & \frac{\gamma}{\pi} \\
0 & \frac{\delta y}{2} & 0 & \frac{\gamma^2}{2} & 0 & -\delta y & 0 & \gamma^2 & 0 \\
\frac{\delta}{\mu_\pi} & 0 & \frac{\gamma}{\mu_\pi} & 0 & 2\frac{\delta}{\mu_\pi} & 0 & \frac{2\gamma}{\mu_\pi} & 0 & 0 
\end{bmatrix} \tag{C.6}$$

$$\mathbf{M}_{\Delta_2} = \mu_\pi^2 \begin{bmatrix}
0 & -\frac{\delta^2}{2} & 0 & -\frac{\delta y}{2} & 0 & -\frac{\delta^2}{2} & 0 & -\delta y & 0 \\
-\frac{\delta^2}{2} & 0 & \frac{\delta y}{2} & 0 & -\frac{\delta^2}{2} & 0 & -\delta y & 0 & -\frac{\delta}{2\pi} \\
0 & \frac{\delta y}{2} & 0 & \frac{\gamma^2}{2} & 0 & \delta y & 0 & \gamma^2 & \frac{\gamma}{\pi} \\
-\frac{\delta y}{2} & 0 & \frac{\gamma^2}{2} & 0 & -\delta y & 0 & \gamma^2 & 0 & -\frac{\gamma}{2\pi} \\
0 & \frac{\delta y}{2} & 0 & -\frac{\gamma^2}{2} & 0 & -2\delta^2 & 0 & -2\delta y & 0 \\
-\delta^2 & 0 & \delta y & 0 & -2\delta^2 & 0 & -2\delta y & 0 & -\frac{\delta}{2\pi} \\
0 & -\delta y & 0 & -\frac{\gamma^2}{2} & 0 & -2\delta y & 0 & \gamma^2 & \frac{\gamma}{\pi} \\
\delta y & 0 & -\frac{\gamma^2}{2} & 0 & 2\delta y & 0 & 2\gamma^2 & 0 & \gamma \frac{\gamma}{\pi} \\
0 & -\frac{\delta}{\mu_\pi} & 2\frac{\gamma}{\mu_\pi} & -\frac{\gamma}{\mu_\pi} & 0 & 2\frac{\delta}{\mu_\pi} & -\frac{4\gamma}{\mu_\pi} & \frac{2\gamma}{\mu_\pi} & 0 
\end{bmatrix} \tag{C.7}$$

$$\mathbf{M}_{\Delta_4} = \mu_\pi^2 \begin{bmatrix}
-\delta^2 & 0 & 0 & 0 & -2\delta^2 & 0 & 0 & 0 & -\frac{\delta}{2\pi} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \gamma^2 & 0 & 0 & 0 & -2\gamma^2 & 0 & -\frac{\gamma}{2\pi} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \gamma^2 & 0 & 0 & 0 & -2\gamma^2 & 0 & -\frac{\gamma}{2\pi} \\
-\delta^2 & 0 & 0 & 0 & -4\delta^2 & 0 & 0 & 0 & -\frac{\delta}{2\pi} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -2\gamma^2 & 0 & 0 & 0 & -4\gamma^2 & 0 & -\frac{\gamma}{2\pi} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{\delta}{\mu_\pi} & 0 & -\frac{\gamma}{\mu_\pi} & 2\frac{\gamma}{\mu_\pi} & -\frac{2\delta}{\mu_\pi} & 0 & \frac{2\gamma}{\mu_\pi} & -\frac{4\gamma}{\mu_\pi} & \frac{\gamma^2}{\mu_\pi^2} 
\end{bmatrix} \tag{C.8}$$
The N.D. damping coupling matrices are given by (C.9) – (C.11).

\[ \mathbf{C}_{\Delta_0} = f_0 \mu \pi^2 \begin{bmatrix} 0 & \delta^2 & 0 & \delta \gamma & 0 & 2\delta^2 & 0 & -2\delta \gamma & 0 \\ -\delta^2 & 0 & -\delta \gamma & 0 & -2\delta^2 & 0 & 2\delta \gamma & 0 & 0 \\ 0 & \delta \gamma & 0 & \gamma^2 & 0 & 2\delta \gamma & 0 & -2\gamma^2 & 0 \\ -\delta \gamma & 0 & -\gamma^2 & 0 & -2\delta \gamma & 0 & 2\gamma^2 & 0 & 0 \\ 0 & 2\delta^2 & 0 & 2\delta \gamma & 0 & 4\delta^2 & 0 & -4\delta \gamma & 0 \\ -2\delta^2 & 0 & -2\delta \gamma & 0 & -4\delta^2 & 0 & 4\delta \gamma & 0 & 0 \\ 0 & -2\delta \gamma & 0 & -2\gamma^2 & 0 & -4\delta \gamma & 0 & 4\gamma^2 & 0 \\ 2\delta \gamma & 0 & 2\gamma^2 & 0 & 4\delta \gamma & 0 & -4\gamma^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]  
(C.9)

\[ \mathbf{C}_{\Delta_{\Delta_2}} = f_0 \mu \pi^2 \begin{bmatrix} 2\delta^2 & 0 & 2\delta \gamma & 0 & 4\delta^2 & 0 & -4\delta \gamma & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2\delta \gamma & 0 & -2\gamma^2 & 0 & -4\delta \gamma & 0 & 4\gamma^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4\delta^2 & 0 & 4\delta \gamma & 0 & 8\delta^2 & 0 & -8\delta \gamma & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4\delta \gamma & 0 & 4\gamma^2 & 0 & 8\delta \gamma & 0 & -8\gamma^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{4\delta}{\mu\pi} & \frac{4\gamma}{\mu\pi} & \frac{8\phi}{\mu\pi} & \frac{8\delta}{\mu\pi} & 0 & \frac{8\gamma}{\mu\pi} & 0 & 0 & \frac{16\phi}{\mu\pi} & \frac{2\gamma^2}{\mu\pi^2} \end{bmatrix} \]  
(C.10)

\[ \mathbf{C}_{\Delta_{\epsilon_2}} = f_0 \mu \pi^2 \begin{bmatrix} 0 & -2\delta^2 & 0 & -2\delta \gamma & 0 & -4\delta^2 & 0 & 4\delta \gamma & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2\delta \gamma & 0 & 2\gamma^2 & 0 & 4\delta \gamma & 0 & -4\gamma^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -4\delta^2 & 0 & -4\delta \gamma & 0 & -8\delta^2 & 0 & 8\delta \gamma & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -4\delta \gamma & 0 & -4\gamma^2 & 0 & -8\delta \gamma & 0 & 8\gamma^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{4\delta}{\mu\pi} & \frac{4\gamma}{\mu\pi} & \frac{8\phi}{\mu\pi} & 0 & \frac{8\delta}{\mu\pi} & 0 & 0 & \frac{16\phi}{\mu\pi} & \frac{8\gamma}{\mu\pi} & 0 \end{bmatrix} \]  
(C.11)

The N.D. stiffness coupling matrices are given by (C.12) – (C.14).
$$\mathbf{K}_{\delta_0} = f_0^2 \mu \pi^2$$

$$
\begin{bmatrix}
-\delta^2 & 0 & -\frac{1}{2}(\delta - \gamma)^2 & 0 \\
0 & -\delta^2 & 0 & \frac{1}{2}(\delta - \gamma)^2 \\
-\frac{1}{2}(\delta + \gamma)^2 & 0 & -\gamma^2 & 0 \\
2\gamma^2 & 0 & (\delta - \gamma)^2 & 0 \\
0 & -4\delta^2 - 2\gamma^2 & 0 & -2\delta^2 \\
-(\delta + \gamma)^2 & 0 & -2\delta^2 & 0 \\
0 & (\delta + \gamma)^2 & 0 & 2\delta^2 + 4\gamma^2 \\
0 & 0 & 0 & 0
\end{bmatrix} \ldots \tag{C.12}
$$

$$\mathbf{K}_{\delta_2} = f_0^2 \mu \pi^2$$

$$
\begin{bmatrix}
0 & 2\delta^2 & 0 & -(\delta - \gamma)^2 \\
0 & 0 & \delta^2 + \gamma^2 & 0 \\
0 & 0 & (\delta - \gamma)^2 & 0 \\
-(\delta^2 + \gamma^2) & 0 & 0 & -2\gamma^2 \\
0 & 0 & 0 & 0 \\
0 & 4\delta^2 & 0 & 2(\delta + \gamma)^2 \\
0 & 0 & 0 & 0 \\
0 & 0 & -2(\delta^2 + \gamma^2) & 0 \\
-2(\delta^2 + \gamma^2) & 0 & 0 & 4\gamma^2 \\
0 & 0 & 0 & 0 \\
0 & \frac{4\delta}{\mu \pi} & \frac{8\gamma_0}{\mu \pi} & \frac{4\gamma}{\mu \pi}
\end{bmatrix} \ldots \tag{C.13}
$$
Furthermore, the N.D. forcing terms are given as (C.15)

\[ \mathbf{\bar{F}}_{\Delta_0} = f_0^2 \mu \pi \]
\[ \mathbf{\bar{F}}_{\Delta_0} = f_0^2 \mu \pi \]
\[ \mathbf{\bar{F}}_{\Delta_0} = f_0^2 \mu \pi \]
\[ \mathbf{\bar{F}}_{\Delta_0} = f_0^2 \mu \pi \]
\[ \mathbf{\bar{F}}_{\Delta_0} = f_0^2 \mu \pi \]
\[ \mathbf{\bar{F}}_{\Delta_0} = f_0^2 \mu \pi \]
\[ \mathbf{\bar{F}}_{\Delta_0} = f_0^2 \mu \pi \]
\[ \mathbf{\bar{F}}_{\Delta_0} = f_0^2 \mu \pi \]
\[ \mathbf{\bar{F}}_{\Delta_0} = f_0^2 \mu \pi \]

C.3 Non-Dimensional Load-Torque NCV Terms

The N.D. stiffness coupling terms are given as (C.16) – (C.18).

\[ \mathbf{\bar{K}}_{\Gamma_0} = \tau \pi^2 \]

(C.14)

(C.15)
\[
\begin{bmatrix}
0 & -\phi_s & -1 & -\phi_s & 4 & 2\phi_s & -2 & -2\phi_s & 0 \\
-\phi_s & 0 & -\phi_s & 1 & 2\phi_s & -4 & -2\phi_s & 2 & 0 \\
-1 & -\phi_s & 0 & \phi_s & 2 & 2\phi_s & -4 & -6\phi_s & \frac{\gamma}{\pi} \\
-\phi_s & 1 & \phi_s & 0 & 2\phi_s & -2 & -6\phi_s & 4 & 0 \\
4 & 2\phi_s & 2 & 2\phi_s & 0 & -4\phi_s & 4 & 4\phi_s & 0 \\
2\phi_s & -4 & 2\phi_s & -2 & -4\phi_s & 0 & 4\phi_s & -4 & 0 \\
-2 & -2\phi_s & -4 & -6\phi_s & 4 & 4\phi_s & 0 & 4\phi_s & 2\frac{\gamma}{\pi} \\
-2\phi_s & 2 & -6\phi_s & 4 & 4\phi_s & -4 & 4\phi_s & 0 & 0 \\
0 & 0 & \frac{2\gamma}{\mu\pi} & 0 & 0 & 0 & \frac{4\gamma}{\mu\pi} & 0 & 0 \\
\end{bmatrix}
\]

\( K_{g2} = \epsilon\pi^2 \)  \hfill (C.17)

\[
\begin{bmatrix}
-\phi_s & 0 & -\phi_s & 1 & 2\phi_s & -4 & -2\phi_s & 2 & 0 \\
0 & \phi_s & 1 & \phi_s & -4 & -2\phi_s & 2 & 2\phi_s & 0 \\
-\phi_s & 1 & \phi_s & 0 & 2\phi_s & -2 & -6\phi_s & 4 & 0 \\
1 & \phi_s & 0 & -\phi_s & -2 & -2\phi_s & 4 & 6\phi_s & \frac{\gamma}{\pi} \\
2\phi_s & -4 & 2\phi_s & -2 & -4\phi_s & 0 & 4\phi_s & -4 & 0 \\
-4 & -2\phi_s & -2 & -2\phi_s & 0 & 4\phi_s & -4 & -4\phi_s & 0 \\
-2\phi_s & 2 & -6\phi_s & 4 & 4\phi_s & -4 & 4\phi_s & 0 & 0 \\
2 & 2\phi_s & 4 & 6\phi_s & -4 & -4\phi_s & 0 & -4\phi_s & 2\frac{\gamma}{\pi} \\
0 & 0 & 0 & \frac{2\gamma}{\mu\pi} & 0 & 0 & 0 & \frac{4\gamma}{\mu\pi} & 0 \\
\end{bmatrix}
\]

\( K_{g2} = \epsilon\pi^2 \)  \hfill (C.18)

Finally, the N.D. forcing terms are given as (C.19)

\[
\begin{bmatrix}
-\frac{\delta + \gamma}{2} \\
0 \\
-\frac{\delta - \gamma}{2} \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix}, \quad \begin{bmatrix}
\gamma\phi_s \\
\gamma\phi_s \\
\frac{\gamma\phi_s^2}{2} \\
\frac{\gamma\phi_s^2}{2} \\
\frac{\gamma\phi_s^2}{2} \\
\frac{\gamma\phi_s^2}{2} \\
\frac{\gamma^2\phi_s^2}{\mu\pi} \\
\frac{\gamma^2\phi_s^2}{\mu\pi} \\
\frac{\gamma^2\phi_s^2}{\mu\pi} \\
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
\frac{\delta + \gamma}{2} - \gamma\phi_s^2 \\
\gamma\phi_s \\
\frac{\gamma\phi_s^2}{2} \\
\frac{\gamma\phi_s^2}{2} \\
\frac{\gamma\phi_s^2}{2} \\
\frac{\gamma\phi_s^2}{2} \\
\frac{\gamma^2\phi_s^2}{\mu\pi} \\
\frac{\gamma^2\phi_s^2}{\mu\pi} \\
\frac{\gamma^2\phi_s^2}{\mu\pi} \\
\end{bmatrix}
\]

\( \bar{F}_{g0} = \epsilon\pi \)  \hfill (C.19)

\[
\begin{bmatrix}
\delta - \gamma + 2\gamma\phi_s^2 \\
-\delta - \gamma + 2\gamma\phi_s^2 \\
\frac{\gamma^2\phi_s^2}{\mu\pi} \\
\frac{\gamma^2\phi_s^2}{\mu\pi} \\
\frac{\gamma^2\phi_s^2}{\mu\pi} \\
\end{bmatrix}
\]
Appendix D

LEAST-SQUARES FORMULATION OF ADAPTIVE CONTROL UPDATE LAW

Given the transfer matrix \( T_{yu} \in \mathbb{R}^{n \times n} \) from the control input to the response output Fourier coefficient vectors, \( U \in \mathbb{R}^{n \times 1} \) and \( Y \in \mathbb{R}^{n \times 1} \), and given the baseline system response \( y_b \in \mathbb{R}^{n \times 1} \), the system steady-state response is given by

\[
Y = T_{yu} U + Y_b \tag{D.1}
\]

If the control input \( U \) is updated at each time interval \( i \), the steady state relation is written

\[
Y_i = T_{yu} U_i + Y_b \tag{D.2}
\]

Where \( Y_i \) is the steady-state response to the \( i^{th} \) control update \( U_i \). The update law for \( U_i \) can be obtained through minimization of the quadratic objective function \( J_i \) given as

\[
J_i = Y_i^T W Y_i + U_i^T R U_i \tag{D.3}
\]

Where \( W = W^T \) and \( R = R^T \) are symmetric, positive definite, weighting matrices. By using the fact that

\[
Y_{i-1} = T_{yu} U_{i-1} + Y_b \tag{D.4}
\]

\( Y_i \) can be written in terms \( U_i, U_{i-1} \) and \( Y_{i-1} \) as

\[
Y_i = T_{yu} \Delta U_i + Y_{i-1} \quad \text{with} \quad \Delta U_i = U_i - U_{i-1} \tag{D.5}
\]

Substituting (D.5) into (D.3), the objective function is expressed as
\[ J_i = \left[ T_{yu} \Delta U_i + Y_{i-1} \right]^T W \left[ T_{yu} \Delta U_i + Y_{i-1} \right] + U_i^T RU_i \]  
(D.6)

Since \( J_i \) is quadratic with \( J_i \geq 0 \), the control input \( U_i \) that minimizes \( J_i \) satisfies

\[ \frac{\partial J_i}{\partial U_i} = 0 = T_{yu}^T W T_{yu} \Delta U_i + T_{yu}^T W T_{yu} \Delta U_i + T_{yu}^T W Y_{i-1} + T_{yu}^T W T_{yu} Y_{i-1} + RU_i + R^T U_i \]  
(D.7)

Since \( W = W^T \) and \( R = R^T \) and \( \Delta U_i = U_i - U_{i-1} \) (D.7) becomes

\[ 0 = T_{yu}^T W T_{yu} (U_i - U_{i-1}) + T_{yu}^T W Y_{i-1} + RU_i \]  
(D.8)

Solving for \( U_i \) the control update law becomes

\[ U_i = \left[ T_{yu}^T W T_{yu} + R \right]^{-1} T_{yu}^T W T_{yu} U_{i-1} - Y_{i-1} \]  
(D.9)

Or, in terms of the indices \( i \) and \( i+1 \) the control update law is

\[ U_{i+1} = \left[ T_{yu}^T W T_{yu} + R \right]^{-1} T_{yu}^T W T_{yu} U_i - Y_i \]  
(D.10)
APPENDIX E

ADAPTIVE CONTROL CONVERGENCE CONDITION

Given the Fourier coefficient transfer matrix $T_{yu} \in \mathbb{R}^{n \times n}$ and the baseline system response vector $Y_b \in \mathbb{R}^{n \times 1}$, the steady-state response $Y_i \in \mathbb{R}^{n \times 1}$ to the $i^{th}$ updated control input $U_i \in \mathbb{R}^{n \times 1}$ is

$$Y_i = T_{yu}U_i + Y_b$$

(E.1)

Also, given the control update law based on an estimate of the Fourier coefficient transfer matrix $\hat{T}_{yu} \in \mathbb{R}^{n \times n}$, the control update law is

$$U_{i+1} = \left[\hat{T}_{yu} W \hat{T}_{yu} + R\right]^{-1} \hat{T}_{yu} W \left[\hat{T}_{yu} U_i - Y_i\right]$$

(E.2)

Substituting the expression for $Y_i$ in (E.1) into the update law in (E.2), the recursion relation of the control input $U_i$ is expressed as

$$U_{i+1} = PU_i + QY_b$$

(E.3)

With matrices $P$ and $Q$ defined as

$$P = \left[\hat{T}_{yu} W \hat{T}_{yu} + R\right]^{-1} \hat{T}_{yu} W \left[\hat{T}_{yu} - T_{yu}\right]$$

(E.4)

$$Q = -\left[\hat{T}_{yu} W \hat{T}_{yu} + R\right]^{-1} \hat{T}_{yu} W$$
Since $Y_b$ is a constant vector, the stability of the above discrete-time system only depends on the eigenvalues of the mapping matrix, $P$, from $U_i$ to $U_{i+1}$. Specifically, the stability condition of (E.3) is

\[
\overline{\rho}[P] < 1 \quad \text{Stable} \\
\overline{\rho}[P] \geq 1 \quad \text{Unstable}
\] (E.5)

Where $\overline{\rho}[\cdot]$ is the spectral radius. If (E.3) is stable then update law in (E.2) is said to converge, otherwise the update law it is said to diverge.

Assuming the update law (E.2) converges, then $U_i \rightarrow U_{i+1} \rightarrow U_c$ as $i \rightarrow \infty$ where $U_c$ is the converged control input vector. Substituting $U_c$ for $U_i$ and $U_{i+1}$ into (E.3) and solving for $U_c$

\[
U_c = (I - P)^{-1} Q Y_b
\] (E.6)

Furthermore substituting $U_c$ for $U_i$ in the expression for $Y_i$ in (E.1), the converged output response $Y_i \rightarrow Y_{i+1} \rightarrow Y_c$ with $i \rightarrow \infty$ is expressed as

\[
Y_c = T_{yu} (I - P)^{-1} Q Y_b + Y_b
\] (E.7)

Finally, substituting the expressions for matrices $P$ and $Q$ defined in (E.4) into (E.6) and (E.7), the converged control input and response vectors become

\[
U_c = -[\hat{T}_{yu}^T W T_{yu} + R]^{-1} \hat{T}_{yu}^T W Y_b \\
Y_c = -T_{yu} [\hat{T}_{yu}^T W T_{yu} + R]^{-1} \hat{T}_{yu}^T W Y_b + Y_b
\] (E.8)
Appendix F

FMC SHAFT LAYUP AND EQUIVALENT MATERIAL PROPERTIES

The FMC shaft considered in this investigation is based on a symmetric ply layup configuration where the fiber orientations are wound in $+\theta_i$ and $-\theta_i$ directions for each specified ply orientation angle $\theta_i$. Here $\theta_i$ is the fiber angle relative to the shaft axis. Figure F.1 is a schematic of the FMC composite shaft and Figure F.2 gives the layer locations, $z_k$, through the shaft wall thickness relative the laminate mid-plane.

Fig. F.1: FMC shaft with in-plane force resultants.
Classical Lamination Theory is used for calculating the laminate stiffness matrices from the orthotropic lamina, as given by Jones (1975). Due to the symmetry of the composite layup considered in this investigation, coupling between the bending and twist terms in the constitutive relations is neglected. Thus, the in-plane force resultants, $N_x$, $N_y$ and $N_{xy}$, on the composite laminate can be related to the mid-plane strains, $\varepsilon_x$, $\varepsilon_y$ and $\varepsilon_{xy}$, through the matrix $[A]$.

$$
\begin{bmatrix}
N_x \\
N_y \\
N_{xy}
\end{bmatrix} =
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\varepsilon_{xy}
\end{bmatrix}
$$

(F.1)

Where

$$
\begin{bmatrix}
N_x \\
N_y \\
N_{xy}
\end{bmatrix} = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} 
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\sigma_{xy}
\end{bmatrix}
\, dz
$$

(F.2)

and

$$
[A] = \sum_{k=1}^{n} [T]_k^T [Q]_k [T]_k^{-1} (z_k - z_{k-1})
$$

(F.3)
Here, \([A]\) is a function of the lamina stiffness matrices, \([Q]_k\), and their respective transformations, \([T]_k\), due to the winding angle, \(\theta_k\), given by

\[
[T]_k = \begin{bmatrix}
\cos^2 \theta_k & \sin^2 \theta_k & 2 \cos \theta_k \sin \theta_k \\
\sin^2 \theta_k & \cos^2 \theta_k & -2 \cos \theta_k \sin \theta_k \\
\cos \theta_k \sin \theta_k - \cos \theta_k \sin \theta_k & \cos^2 \theta_k - \sin^2 \theta_k & \\
\end{bmatrix}
\]  

(F.4)

Finally, using these expressions, the laminate stiffness matrix is computed as

\[
\begin{bmatrix}
\frac{1}{E^*} & -\frac{v_{xy}}{E_y} & 0 \\
-\frac{v_{xy}}{E_y} & \frac{1}{E_y} & 0 \\
0 & 0 & \frac{1}{G^*} \\
\end{bmatrix} = t_s[A]^t
\]  

(F.5)

where \(t_s\) is the shaft wall thickness and \(E^*\) and \(G^*\) are the equivalent Young’s modulus and shear modulus of the assembled FMC shaft.
Appendix G

EXPERIMENTAL RESULTS

G.1 Configuration I Experimental Results

Figures G.1 through G.24 show the experimentally measured shaft vibration response and control currents for the Configuration I test cases outlined in Table G.1.

Table G.1: Configuration I Experiment Test Cases

<table>
<thead>
<tr>
<th>Case Number</th>
<th>Shaft Speed</th>
<th>Load-Torque</th>
<th>Misalignment (All Cases)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Low Torque Cases</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Case 1</td>
<td>$\Omega = 10.3$ Hz</td>
<td>$T_L = 1.76$ N-m</td>
<td></td>
</tr>
<tr>
<td>Case 2</td>
<td>$\Omega = 16.3$ Hz</td>
<td>$T_L = 2.09$ N-m</td>
<td>$\delta_2 = 0^\circ$, $\xi_2 = 0^\circ$, $\gamma_2 = 3^\circ$</td>
</tr>
<tr>
<td>Case 3</td>
<td>$\Omega = 22.2$ Hz</td>
<td>$T_L = 2.64$ N-m</td>
<td>Vertical-Plane: $\delta_3 = 0^\circ$, $\xi_3 = 0^\circ$, $\gamma_3 = 0^\circ$</td>
</tr>
<tr>
<td>Case 4</td>
<td>$\Omega = 30.1$ Hz</td>
<td>$T_L = 3.04$ N-m</td>
<td></td>
</tr>
<tr>
<td>High Torque Cases</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Case 5</td>
<td>$\Omega = 10.0$ Hz</td>
<td>$T_L = 5.03$ N-m</td>
<td></td>
</tr>
<tr>
<td>Case 6</td>
<td>$\Omega = 16.0$ Hz</td>
<td>$T_L = 5.16$ N-m</td>
<td></td>
</tr>
</tbody>
</table>
G.1.1 Configuration I Time-Domain Results

The figures in this sub-section give the time-domain shaft vibration response and control currents for the Configuration I test cases.

Fig. G.1: Configuration I shaft vibration response, Case 1.
Fig. G.2: Configuration I control current, Case 1.

Fig. G.3: Configuration I shaft vibration response, Case 2.
Activate MHAVC

Fig. G.4: Configuration I control current, Case 2.

Fig. G.5: Configuration I shaft vibration response, Case 3.
Fig. G.6: Configuration I control current, Case 3.

Fig. G.7: Configuration I shaft vibration response, Case 4.
Fig. G.8: Configuration I control current, Case 4.

Fig. G.9: Configuration I shaft vibration response, Case 5.
Activate MHAVC

Fig. G.10: Configuration I control current, Case 5.

Activate MHAVC

Fig. G.11: Configuration I shaft vibration response, Case 6.
Fig. G.12: Configuration I control current, Case 6.

Fig. G.13: Configuration I shaft vibration response, Case 7.
Fig. G.14: Configuration I control current, Case 7.

Fig. G.15: Configuration I shaft vibration response, Case 8.
G.1.2 Configuration I Frequency-Domain Results

The figures in this sub-section give the spectrum of the steady-state shaft vibration response for each of the Configuration I experimental test cases. Here, the response spectrum at each operating condition is computed via an FFT analysis.
Fig. G.17: Configuration I shaft vibration spectrum, Case 1.

Fig. G.18: Configuration I shaft vibration spectrum, Case 2.
Fig. G.19: Configuration I shaft vibration spectrum, Case 3.

Fig. G.20: Configuration I shaft vibration spectrum, Case 4.
Fig. G.21: Configuration I shaft vibration spectrum, Case 5.

Fig. G.22: Configuration I shaft vibration spectrum, Case 6.
Fig. G.23: Configuration I shaft vibration spectrum, Case 7.

Fig. G.24: Configuration I shaft vibration spectrum, Case 8.
G.2 Configuration II Experimental Results

Figures G.25 through G.33 show the experimentally measured shaft vibration response and control currents for the Configuration II test cases outlined in Table G.2.

Table G.2: Configuration II Experiment Test Cases

<table>
<thead>
<tr>
<th>Case Number</th>
<th>Shaft Speed</th>
<th>Load-Torque</th>
<th>Misalignment (All Cases)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case 1</td>
<td>$\Omega = 10.0 \text{ Hz}$</td>
<td>$T_L = 0 \text{ N-m}$</td>
<td>Horizontal-Plane: $\delta = 0^\circ$, $\xi = 0^\circ$, $\gamma = 0^\circ$</td>
</tr>
<tr>
<td>Case 2</td>
<td>$\Omega = 15.0 \text{ Hz}$</td>
<td>$T_L = 0 \text{ N-m}$</td>
<td>Vertical-Plane: $\delta = 0^\circ$, $\xi = 0^\circ$, $\gamma = 0^\circ$</td>
</tr>
<tr>
<td>Case 3</td>
<td>$\Omega = 25.0 \text{ Hz}$</td>
<td>$T_L = 0 \text{ N-m}$</td>
<td></td>
</tr>
</tbody>
</table>

G.2.1 Configuration II Time-Domain Results

The figures in this sub-section give the time-domain shaft vibration response and control currents for the Configuration II test cases.
Fig. G.25: Configuration II shaft vibration response, Case 1.

Fig. G.26: Configuration II control current, Case 1.
Fig. G.27: Configuration II shaft vibration response, Case 2.

Fig. G.28: Configuration II control current, Case 2.
Fig. G.29: Configuration II shaft vibration response, Case 3.

Fig. G.30: Configuration II control current, Case 3.
G.2.2 Configuration II Frequency-Domain Results

The figures in this sub-section give the spectrum of the steady-state shaft vibration response for each of the Configuration II experimental test cases. Here, the response spectrum at each operating condition is computed via an FFT analysis.

Fig. G.31: Configuration II shaft vibration spectrum, Case 1.
Fig. G.32: Configuration II shaft vibration spectrum, Case 2.
Note, in Case 3 of Configuration II, the PID controlled shaft imbalance vibration response amplitudes exceeds the AMB backup-bearing clearances resulting periodic shaft/backup-bearing impacts, see Figure G.29. This non-linear impact phenomena results in a multi-harmonic response which is observed in Figure G.33 for the system under PID control. However, once the Synchronous Adaptive Vibration Control (SAVC) is activated, the synchronous portion of the response is suppressed and the contact is prevented.
Hans A. DeSmidt was born November 27, 1973 in Richmond Virginia, USA. He graduated May 1992 from Hermitage High School in Henrico County, VA, where he participated in the Virginia Junior Academy of Science and was a member of the National Honor Society. In 1996, he graduated Summa Cum Laude with a B.S. degree in Engineering Science and Mechanics from the Virginia Polytechnic Institute and State University (Virginia Tech) in Blacksburg, VA. While attending Virginia Tech, he was a member of Tau Beta Pi and Phi Eta Sigma engineering honor societies. Additionally, in 1994 and 1995, he was awarded to participate in the National Science Foundation Summer Undergraduate Research Program (NSF-SURP), where he published two undergraduate research papers on the topic composite materials. In 1996, he began his graduate studies in the Mechanical Engineering Department at The Pennsylvania State University, in University Park, PA, where he was both a NASA Graduate Student Research Program (GSRP) fellow and a Weiss Graduate Research Fellow. His research and course work was in the areas of Structural Dynamics, Vibrations, Rotordynamics, Active Control Theory, and Magnetic Bearing technology. While conducting his Ph.D. thesis investigation, he published seven conference papers and three academic journal papers on topics concerning various aspects of his research. Finally, he is both a member of the American Society of Mechanical Engineers (ASME) and the American Helicopter Society (AHS).