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CONCEPTIONS OF BETWEEN RATIOS AND WITHIN RATIOS

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Gulseren Karagoz Akar

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The thesis of Gulseren Karagoz Akar was reviewed and approved* by the following:

Martin A. Simon
Professor of Education
Thesis Advisor
Chair of Committee

Glendon W. Blume
Professor of Education

Tadanobu Watanabe
Associate Professor of Education

Mark Levi
Professor of Mathematics

Patrick W. Shannon
Professor of Education
Head of the Graduate Program

*Signatures are on file in the Graduate School

ABSTRACT

The purpose of this study was to investigate students' conceptions of between ratios and within ratios and make distinctions with respect to those conceptions. The study included two one and one- half hour written sessions and a follow up one- hour interview session in which prospective elementary and secondary mathematics teachers reasoned about different tasks/problems in the context of ratio.

The study showed that a student may have a fairly sophisticated ability to make sense of between ratios although not even having a per-one meaning for within ratios. The study also showed that students with an understanding of within ratios may not have meaning for the ratio representing quantities coming from the same measure space, between ratios. That is, they might not have abstracted that between ratios represent the change factor from one ratio situation to the other. These results suggest that an understanding of between ratios may develop to some degree independently from the understanding of within ratios, and vice versa

In this study, I go on to discuss implications, derived from the data, for further understanding the development of within ratios.

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Chapter 1

STATEMENT OF THE PURPOSE

The purpose of this study was to investigate students' conceptions of between-ratios and within-ratios and make distinctions with respect to those conceptions.

Background

A significant amount of research has been done with regard to the conception of ratio and proportion as well as the task variables inherent to this specific context (Karplus, 1972; Karplus & Karplus, 1972; Karplus, Pulos & Stage, 1974; Karplus & Peterson, 1974; Noelting, 1980a; Noelting, 1980b;Tourniaire & Pulos, 1985; Hart, 1988; Thompson, 1988; Simon & Blume, 1994; Thompson & Bush, 2003; Lamon, 1994; Lamon 1995, Lesh, Post, Behr; 1988; Karplus & Lamon, 1994; Lamon, 1995; Kaput & West, 1994; Watanabe & Lo, 1997, Heinz, 2000). Some of these studies focused on the informal strategies students use (Karplus & Lamon, 1994; Lamon, 1995; Kaput & West, 1994; Thompson, 1994; Thompson, 1988; Watanabe &Lo, 1997). Some studies focused on structural properties of the tasks/problems requiring proportional reasoning (Noelting, 1980a; Noelting, 1980b; Hart, 1988; Karplus, 1972; Kaput & West, 1994). Some other studies considered students' conceptions of ratio and proportion and defined what those specific conceptions entail (Heinz, 2000; Thompson, 1994; Thompson & Thompson, 1994; Thompson & Thompson, 1996; Simon & Blume, 1994).

In the literature, researchers used between-ratios and within-ratios in different ways. In this study, within-ratios means that the ratio represents the original quantities, within one state, and the quantities come from two different measure spaces. Between-ratios means that the ratio represents quantities from two situations and so the quantities come from the same measure space. For instance, in the problem, “There are 3 apples for every 5 oranges in a basket, how many apples would be in the basket, if the basket had 20 oranges?” if the students think about the problem in the ratios of 5 oranges: 20 oranges and 3 apples: x apples; then this refers to between-ratios. On the other hand, if they use the ratios of 3 apples: 5 oranges and x apples: 20 oranges, then these are within-ratios.

In the literature, prior studies focused either on very primitive notions of ratio or mostly on conceptions of within-ratios. Specifically, Simon & Blume (1994) studied prospective elementary mathematics teachers’ understandings of ratio. They introduced the conception of “ratio-as-measure” as the understanding of ratio as an intensive quantity measuring the attribute in the situation, represented by within-ratios. Similarly, the studies Thompson (1994) and Thompson and Thompson (1996) conducted included a specific case of ratio conception, speed, a within ratio. Heinz (2000) focused on the characteristics of two conceptions of ratio. The first one was the identical groups conception, entailing the understanding of ratio as a representation of two extensive quantities coming from different measure spaces, namely, within-ratios. The second was ratio-as-quantity, involving the understanding of ratio as an intensive quantity measuring the relative size, the structure of which is applicable to both within and between-ratios. Heinz (2000) also studied differences between those two conceptions and expressed the need to do more research to understand fully what constitutes a web of well developed conceptions of ratio. Identifying the characteristics of conceptions of between-ratios and within-ratios to help better understanding how those different conceptions of ratio relate and to shed light on how possible paths from one conception to the other can be studied more effectively requires more research.

This study aimed at two goals: 1) to further investigate and explicate conceptions of between and within-ratios; and 2) to make distinctions with respect to these conceptions.

Overview of Study

This study involved prospective elementary and secondary mathematics teachers. It included two, one and one half hour written sessions and one hour interviews with each participant. The written sessions provided data of students' use of different strategies and approaches to different ratio tasks/problems so that the interviews could focus on students' conceptions of between and within-ratios in greater depth. For this study, I analyzed two prospective elementary mathematics teachers' written responses and transcripts of the videotapes of interviews to explicate and make distinctions among their conceptions of ratio.

Rationale for the Study

The rationale for this study for further explicating and making distinctions with respect to conceptions of between-ratios and within-ratios is:

i) Ratio and proportion is a key concept.

ii) Conceptions of between-ratios and within-ratios need to be studied.

Ratio and Proportion Is a Key Concept

Different researchers have mentioned that the concept of ratio and proportion is not only a key concept in the learning of mathematics, but also in the learning of quantitative relations in science.

The concept of ratio and proportion is vital in mathematics because it gives rise to the development of other concepts such as relative size, probability, slope of a line, similarity and trigonometric ratios (Simon & Blume, 1994). The concept of ratio and proportion is also important in the learning of science because it constitutes the basis for the calculation of chemical equivalents, use of ideal gas laws, definitions of density, speed, acceleration, operation of simple machines and analysis of electric circuits (Karplus & Peterson, 1970).

Studies Are Needed for Explicating and Making Distinctions on Conceptions of Between-Ratios and Within-Ratios

As mentioned earlier, most of the completed research involving students' conceptions of ratio has focused either on very primitive notions of ratio or on conceptions of within-ratios. The findings of these studies need to be expanded to include focus on between-ratios and attention to the relationship between conceptions of within-ratios and between-ratios. Such work is likely to shed light on the development of a useful concept of ratio, what Heinz (2000) called "ratio as quantity."

Chapter 2

LITERATURE REVIEW

Introduction

The National Council of Teachers of Mathematics stated, “The ability to reason proportionally develops in students throughout grades 5-8. It is of such great importance that it merits whatever time and effort must be expended to assure its careful development. Students need to see many problem situations that can be modeled and then solved through proportional reasoning” (NCTM 1989, p.82). As well as NCTM (1989, 2000) researchers emphasize the importance of proportional reasoning and agree that proportional reasoning is essential in the study of algebra, geometry, biology, chemistry and physics (Karplus, 1972; Karplus, Pulos, & Stage, 1974; Lamon, 1994; Lamon 1995, Lesh, Post, Behr; 1988; Simon & Blume, 1994; Thompson & Bush, 2003; Tourniaire & Pulos, 1985; Vergnaud, 1988). Lesh, Post and Behr asserted “On the one hand, it is the capstone of children’s elementary school arithmetic; on the other hand, it is the cornerstone of all that is to follow” (Lesh, Post, & Behr; 1988, p.94). Among the content areas included in proportional reasoning are linear equations, similar figures and their area and volume relationships, speed, acceleration, gas laws, definitions of density etc. (Karplus, 1972; Karplus, Pulos, & Stage, 1974; Lamon, 1994; Lamon, 1995; Simon & Blume, 1994; Thompson & Bush, 2003; Tourniaire & Pulos, 1985; Vergnaud, 1988) The concept of ratio and proportion has been widely studied (Noelting, 1980).

The articulation of different conceptions pertinent to this particular content area requires taking into account several issues. First, the concept itself is composed of complex but related constructs, such as extensive and intensive quantities, and co-variance and invariance (Lamon, 1995; Heinz, 2000). Second, no consensus exists among researchers in the use of the terminology. Some researchers even use different names for the same concepts throughout their continued work on this domain (Lesh, Post, Behr; 1988). This causes even more difficulty in understanding the complex nature of the conceptions involved in ratio. Finally, these constructs such as extensive and intensive quantities, and co-variance and invariance help to understand and define different conceptions of ratio. Therefore, information about the definition and use of ratio and rate together with the constructs involved in ratio understanding in the existing research literature is compulsory.

In sum, for this study, such definitions of the terms and the constructs are helpful in three ways: a) in the articulation of different conceptions of ratio; b) in framing how the findings of the researchers integrate; and c) in framing both the analysis and the results of this study in the existing research.

Therefore, in the following sections, different constructs and definitions of the terminology, different strategies students use, task variables, and different conceptions in the context of ratio are provided because of their importance in situating this study in the field.

Components Necessary for Different Conceptions of Ratio

In this section, I provide information on the necessary components and constructs entailed in different conceptions of ratio.

Quantity

The meaning of quantity is approached with different emphasis by different researchers (Heinz, 2000). Schwartz's approach is more of a formal way than the approach of Thompson (Thompson, 1994). The approach in both cases is important, because the former provides information about the way advanced students think about quantity and the latter provides information about how regular students reason quantitatively about situations. In other words, Thompson's emphasis is on a particular student's approach to quantity (Thompson, 1994). For this study, Thompson's approach becomes essential because of its emphasis on regular students' conceptualizations. On the other hand, knowing how an advanced student would reason about quantity and also being able to use that information to examine how particular student(s) reason quantitatively about the situations are two aspects of the same inquiry (Heinz, 2000).

Schwartz (1988) stated that quantities in mathematics are constructed by two actions-operations, namely counting and measuring, which will be discussed later. Counting is applied to discrete sets and measuring is applied to continuous sets. All quantities created by those actions have referents. The operations done on quantities might be used to generate new quantities and these new quantities might have or might not have new referents. *Referent- preserving composition* occurs when the new quantity derived from those operations does not have a new referent; that is, when the new quantity has the same referent as the quantities from which it is derived. This is seen in addition and subtraction. For instance, one can add 2 apples and 3 apples with the resulting sum of 5 apples. *Referent- transforming composition* occurs when a new quantity has a different referent from the quantities from which it is derived.

Multiplication and division are the two-referent transforming compositions because the quantities derived from those operations have different referents from the original quantities (Schwartz, 1988). For instance, 2 apples in each bag multiplied by the number of 3 bags results in 6 apples, different from apples in each bag and bags. Also, the number of 12 apples divided by the number of 3 people results in 4 apples per person.

What Thompson (1994) claimed is not different from what Schwartz (1988) claimed in the following way: Both researchers agree on the mathematical thinking of the compositions mentioned above. In other words, the actions/operations of mind applied to quantities to create new quantities are agreed upon by these researchers. However, as I would like to discuss more in the following paragraph, Thompson put emphasis more on particular students' perspectives rather than on an advanced way of creation of new quantities. I believe both Thompson's and Schwartz's contributions are important; however, since this study will focus on students' conceptions of ratio, it is particularly important that I mention how Thompson gave insight into students' perspectives on the notion of quantity.

Thompson (1994) viewed quantities as conceptual entities, which are schematic in nature and exist in students' conceptions of situations. They are schematic in the sense that there is an object with a quality. The numerical value of the object with a quality could be measured by a process within an appropriate unit or dimension. Quantities exist in students' conceptions of situations in the sense that the student thinks about the quality of the object in such a way that necessitates her/him to measure that quality. Thompson (1994) claimed that operations by which the quantities are created are unitizing and segmenting, which will be discussed later. Thompson further argued that the schematic approach to the notion of quantity helps the researcher/teacher to reason how a particular student thinks about quantitative situations (Thompson, 1994).

The importance of this can be shown as follows: In a series of teaching experiments Thompson conducted with two elementary school students that students have not approached speed as a production of a referent-transforming composition, instead those students have thought about speed as the production of referent-preserving composition. Thompson’s emphasis was that a particular student might have different notions of quantity from those of the advanced students and teachers/researchers. The existence of the two types of “compositions” necessitates thinking about two different kinds of quantities: extensive quantity and intensive quantity. Extensive quantities are the result of referent-preserving composition and intensive quantities are the result of referent-transforming compositions (Schwartz, 1988). Also, Lesh etal. claimed that extensive quantities express how much of a quantity is associated with a given object (Lesh, Post, Behr,1988). To the contrary, intensive quantity is a relationship between quantities. It is the descriptor of the quality of the objects and is not measured or counted directly. That is, intensive quantity does not give any information about the number or amount of the extensive quantities (not always necessary such as in the particular ratio situations) from which it is derived. For instance, “5 apples per 2 bags” does not give any information about the number of apples or bags. It is generally distinguished by the unit measures containing the word “per” (Schwartz, 1988). Lesh etal. also claimed that intensive quantities, relationships between one quantity and one unit of another quantity, are recognizable by the “per” in their unit labels (Lesh, Post, & Behr, 1988).

Schwartz made a distinction between two uses of intensive quantities (Schwartz, 1988). Other researchers also paid attention to this distinction when they provide definitions of ratio and rate (Kaput & West, 1994; Lamon, 1995; Thompson, 1994). The distinction is that, when the intensive quantities are applied to the homogenous referents, then the relationship embraces all parts of the referent entity. If the referent is not homogenous then the intensive quantity is a local property of the referent (Schwartz, 1988). In other words, “5 apples per 2 bags” might mean that this relationship holds for all the parts of the referent entity because the problem based on this property asks for instance “how many apples per 8 bags?” On the other hand, this relationship might be about a specific 2-bag situation that contains 5 apples.

Schwartz also claimed that there is a special kind of intensive quantity called *unit measure conversion factors*. This special intensive quantity is used to convert one unit of measure such as meter to another unit of measure such as miles. Unit measure conversion factors are also called “scalars” since they do not change the nature of the referent but only the numerical characteristics of its measure (Schwartz, 1988). This special kind of intensive quantity will be discussed later in much more detail because of its particular importance for this study.

Having discussed the notion of quantity, the discussion of how it is generated is what follows.

Quantitative Operations and Numerical Operations

Thompson provided information about quantitative operations, such as unitizing, segmentation, matching and subdividing (Thompson, 1994). He asserted that quantitative operations and numerical operations are crucial in understanding the nature of quantitative reasoning. Quantitative operations are the non-numerical operations from which quantities are derived. Numerical operations are the operations with which quantities are evaluated. Since the quantitative operations in the multiplicative world is the focus of this study, I would like to continue with how some researchers view ratio as a multiplicative relationship resulting from specific quantitative operations and what these specific operations are.

Thompson stated that “the quantitative operation of comparing two quantities multiplicatively originates in matching and subdividing with the goal of sharing” (Thompson, 1994,p.186). He further claimed that “ratio is the result of comparing two quantities multiplicatively” (Thompson, 1994, p. 190). As stated above, Thompson (1994) claimed that unitizing applies in discrete cases and segmentation applies in continuous cases.

The operation Thompson mentioned, namely subdividing, coincides with the operation Lamon mentioned in her study, namely, partitioning. She claimed that partitioning, an operation that generates quantity, is an intuitive strategy based on fair sharing. She also mentioned another operation, unitizing, as Thompson would agree. She claimed “unitizing is a cognitive process for conceptualizing the amount of a given commodity or share before, during and after the sharing process.” (Lamon, 1996, p.171). Lamon claimed “ratio is a composite unit, formed by comparing units that are themselves the result of multiple compositions of units” (Lamon, 1995, p. 169). The similarity between the definitions and the uses of quantitative operations are as follows: Both researchers claimed that with the goal of sharing/partitioning/ subdividing the extensive quantities at hand, new quantities are created. The terms used by these researchers as subdividing/ sharing/partitioning seem like the same operations as breaking down. Thompson (1994) used unitizing and building-up in the discrete cases, whereas Lamon (1996) used this operation in both discrete and continuous cases. Thompson would use segmentation, which is measuring, for continuous cases. As discussed earlier, Schwartz mentioned two operations, counting and measuring, in the construction of quantity (Schwartz, 1988). I believe, counting is compatible with building-up (unitizing), and measuring is compatible with segmenting. Similarly, unitizing applies to both discrete and continuous contexts (Kaput & West, 1994) in the sense that by counting one first establishes 1 as a unit, then by counting again, one forms 3-ones. The process of unitizing now applies that those 3-ones also could be thought of as 1-three. Therefore, when Lamon claimed that re-unitization brings about composite units construct, because one applies unitization again and again 1-[3-ones], 2-[3-ones], 3- [3-ones]. This also coincides with the idea that unitizing in discrete contexts results in groups (Kaput & West, 1994). Unitizing occurs in continuous contexts in the sense that, based on measuring, one forms segments and the issue of how much of that segment is the notion of unitizing. That is, when one segment is established, they come together based on measuring again and how much amount of that segment is the notion of unitizing. Now, the question is the construction of what researchers call a measure unit, an intensive quantity such as 4 apples per bag (Behr, Harel, Post, & Lesh, 1992).

One of the activities, which is not mentioned so far, matching, plays an important role in the construction of intensive quantities. I believe matching is the ongoing correspondence between the elements of the ratio. What I mean is that the quantities, which are segmented in the continuous case or are unitized in the discrete case, and are being compared in such a way that they are related to each other, keeping the sameness with the elements of the original ratio. Based on a general example, we can see how those different uses of terms which are unitizing, segmenting, breaking down and matching come together to form rate.

The example is as follows: the ratio a/b is understood as the intensive quantity within a problem context in which a/b is found to be equal to c/d. 1) Two extensive quantities, a and b, are related in the sense that for every unit of one quantity, a, there is a corresponding unit of another quantity, b. This process is matching. 2) One of the extensive quantities, a, is used within the same measure space to reach another extensive quantity, c. The process of reaching the other extensive quantity is unitizing (or segmenting depending on the discreteness of continuity).This process is held simultaneously for the other quantity, b and d, too. In other words, for every element of the first extensive quantity, a, there will be some amount of the other extensive quantity, b, so when one of the extensive quantities, a, is added to the original one, the other extensive quantity, b, will be added to. 3) This process will result in an anticipation that every pair of quantities has the same collection as the original one. Though this anticipation is not multiplicative in nature, it is the preservation of the original corresponding pair. 4) When the given quantity, c, is reached with repeated addition of the same amounts simultaneously, there is a group of numbers of both extensive quantities such as a.2a.3a, …c and b,2b,3b, …d. The unitized (or segmented) groups are coordinated in such a way that the extensive quantities in both parts of the ratio are corresponded [a with b, 2a with 2b, …c with d]. 4) Each part of these “sub-groups” is conceptualized as proportional amounts of the whole [breaking down]. The two subgroup quantities are conceptualized as the proportional amounts of the whole [breaking down]. 5) The sub-grouped quantities are anticipated to be bi-directional and covarying quantities as parts of the ratio.

Co-variance / Invariance

One of the core components for understanding the concept of ratio is the notion of covariance and invariance (Lamon, 1995; Noelting, 1980b; Simon & Blume, 1994b). Noelting stated that co-variation of parts is multiplicative in nature because each variation of a part inside the whole changes the relation of the co-varying part to the whole (Noelting, 1980). That is, when one of the parts of the whole changes x amount, the other part of the whole changes the same percentage.

In addition, covariance means that two different quantities are held in the mind simultaneously in such a way that "it entails coupling the two quantities, so that, in one's understanding, a multiplicative object is formed of the two. As a multiplicative object, one tracks either quantities' value with the immediate, explicit, and persistent realization that, at every moment, the other quantity also has a value" (Saldanha &Thompson, 1998). The simultaneous change in the parts of the ratio, taking into consideration of the multiplicative relationship, means covariance. This is an important part of understanding ratio because the quantity of interest, the fixed multiplicative relationship between the elements of the ratio, gets preserved. This is invariance. This creates awareness of the fact that any change in one part of the ratio will cause a simultaneous multiplicative change in the other part of the ratio to keep the invariance of the ratio.

Within- Ratios / Between-Ratios

In the literature, there is also some discussion about within and between- ratios although there is no consensus on the way they are used. Two examples of these different approaches are provided below to clarify the way I will use these terms in this study. With-in and between-ratios are important for this study because this study is aimed at characterizing conceptions of within and between- ratios. That is, students with different conceptions of ratio tend to use one of these ways or both depending on their conception. Two examples of different approaches are provided below.

These terms are related to the two measure spaces used in the context of the problem (Lamon, 1993). For instance, consider the problem, “There are 3 apples for every 4 oranges in a basket, how many apples would be if there are 16 oranges in the basket?” If the student thinks about the problem in the ratios 4 oranges: 16 oranges and 3 apples: x apples, then this is called within ratios. On the other hand, if they use the 3 apples: 4 oranges and x apples: 16 oranges, then this is called between- ratios. Lamon declared “A student is said to be using a “within” strategy or a scalar method when s/he equates two internal or within-measure space ratios and uses the sameness of scalar operators to determine the missing term.” (Lamon, 1994, p. 95). She further claimed “We say that a student is using a ‘between’ strategy or a functional method when s/he equates two external or between-measure space ratios or relies on the functional relationship between the measure spaces to find the missing term” (Lamon, 1994, p. 96).

On the other hand, Noelting used a mixture of orange juice and water in his study and uses them as ordered pairs (the number of glasses of orange juice, the number of glasses of water). He claimed that in a proportion a/b=c/d, ‘within- state’ ratio corresponds to the ratio between orange juice and water in each drink (the ratio of a to b). On the other hand, "between-state" ratio corresponds to the ratio between the number of glasses of orange juice (or water) between each drink (the ratios of a to c or b to d) (Noelting, 1980b). Although the example above is about mixtures, Noelting emphasized that he would use the within-state ratios for the quantities in the original ratio, so a within-one system refers to quantities coming from two different measure spaces, and then he would use the between-state ratios for the quantities coming from two different ratios, so it would refer to the quantities coming from the same measure space.

In this study, therefore, whenever I refer to a ratio "a/b" as in the case above provided by Noelting (1980b), I will use within-ratios to emphasize that ratio represents the original quantities, within one state, and they come from two different measure spaces. Whenever I refer to a ratio "a/c" as in the case above, I will use between- ratios to emphasize that ratio represents quantities between two situations and they come from the same measure space.

Definitions of Ratio and Schemes Related to Conceptions of Ratio

Based on the information above regarding the quantitative operations and related terms in the context of ratio, in this section I focus on the definitions of ratio and rate provided by the field and necessary schemes involved in understanding of ratio conceptions.

Definitions of Ratio and Rate

Lesh et al. defined ratio as binary relations, which involve extensive, intensive or scalar types of ordered pairs of quantities (Lesh, Post, & Behr,1988). Hart claimed "ratio is a statement of the numerical relationship between two entities" (Hart, 1988, p. 198).

Likewise, Ohlsson defined ratio as a numerical relationship between two quantities; the relationship is how much there is one quantity in relation to another quantity (Ohlsson, 1988).

Lesh and colloquies defined rate as intensive quantities, which are recognizable by the “per” in their unit labels such as 5 miles-per-hour (Lesh, Post, & Behr,1988). Also, Ohlsson defined rate as the ratio between the quantity and a period of time. In other words it is the simplest numerical representation of change over time (Ohlsson, 1988).

Also, Vergnaud provided information on two types of ratios “functional ratios and scalar ratios” (Vergnaud, 1994). He claimed that these ratios are conceptually different because the former represents the relationship between two different quantities in different measure spaces. The latter one, on the other hand, is the quotient of quantities of the same kind and is a “scalar” because it has no dimension (Vergnaud, 1988).

Lamon claimed “ratio is a composite unit, formed by comparing units that are themselves the result of multiple compositions of units” (Lamon, 1995, p. 169). She further asserts for instance “ nine is a composite unit…a 1 nine-unit, a unit of units. Likewise, two is a composite unit…a single two-unit. A new level of complexity is reached if we consider the ratio itself as a unit…9/2 is a new unit formed by relating the 9 and the 2” (Lamon, 1993, p. 135). She neither articulated how 9 and 2 are related nor articulated how that relationship is acquired. She seems to be claiming that ratio is a composite unit resulting from the idea that the amount of one of the extensive quantities is itself a composite unit, and the amount of the other extensive quantity is also a composite unit. In this case, the multiplicative relationship between these two quantities brings about the new unit, which is ratio. Similarly, although Behr and colloquies did not articulate how the formation of an intensive quantity is established, they claim " from a composite unit such as (4-unit), (0 0 0 0), we can conceptualize another type of unit of intensive quantity" (Behr, Harel, Post,& Lesh, 1993, p.18).

Kaput & West (1994) referred to the terms particular intensive quantity as *particular ratio* and rate intensive quantity as *rate ratio*. They claimed that particular ratio is conceived as a description of a particular case. Consider the example "3 pounds for 4 dollars", they claim that if the multiplicative relationship between these quantities is assumed to be only about the particular purchase of 3 pounds for 4 dollars, then they cal it as particular ratio. On the other hand, rate ratio is conceived as the intensive quantity as indicating a quality of the situation without referring to the particulars of the given problem or situation at hand. In other words, if the example "3 pounds per 4 dollars" refers to the purchase of any sized quantity of whatever is bought then this would refer to the rate intensive quantity, rate ratio.

They also provided some rationale on how particular ratios and rate ratios would be viewed in mathematical terms. "Karplus et al. (1983) described proportions as standing for linear functional relationships. They viewed an intensive quantity (he referred to such as *rates* and *ratios*) as the multiplier m in the linear function y=mx" (Karplus et al., 1983, cited in Kaput & West, 1994, p. 240). In that sense, the particular ratio of "3 pounds per 4 dollars" would refer to a particular substitution of x and y values, meaning a particular purchase; on the other hand, the rate-ratio would be the linear function defined between two measures (Kaput & West, 1994) ---the function that determines the relationship between the two quantities. In addition, they argue that experience with the particular ratio situations as a mental operation relating a particular pair of quantities and realizing the homogeneity, the idea that all of the particular cases are governed by the same quality of interest, the attribute in the situation, must yield to the conceptualization of rate intensive quantity, rate-ratio (Kaput &West, 1994).

Thompson defined ratio as “the result of comparing two quantities multiplicatively” (Thompson, 1994; p. 190). He mentioned ratio as being the expressions of specific situations and rate as being the constant ratio, which is reflectively abstracted from ratio, through generalized situations (Thompson, 1994). He claimed that if one conceives the compared quantities in their independent, static nature then the person conceptualized ratio; however, “ as soon as one reconceives the situation as being that the ratio applies generally outside the phenomenal bounds in which it was originally conceived, then one has generalized that ratio to a rate” (Thompson, 1994, p. 192). That is, understanding of ratio within generalized situations means both that the student understands that the ratio is the expression of the multiplicative relationship for all cases representing the same attribute and also can use the abstracted idea in different contexts. Thomson (1994) claimed

> …rate is (from my point of view) a linear function that can be instantiated with the value of an approximately conceived structure. To say that an object travels at 50 miles/hour quantifies the objects’ motion, but it says nothing about a distance traveled nor about a duration traveled at that speed (Schwartz, 1988, cited in Thompson, 1994). However, conceiving speed of travel in relation to an amount of time traveled produces a specific value for the distance traveled.” (p. 192)

Schwartz, on the other hand, mentioned intensive quantities and gives emphasis to the local property and homogeneity of the referent. He claimed that intensive quantity, the relationship between quantities, could be a local property of the referent or could hold for all parts of the referent because of the homogeneity of the referent (Schwartz, 1988).

For instance, when considering 5 apples per 2 persons or 10 apples per 4 persons, if we use this expression as a particular sharing situation, then it is the particular multiplicative relationship between the number of apples and the number of persons, that is called "ratio." On the other hand, if this relationship is used to define any sharing situation of apples to people, then this is "rate."

Simon & Blume defined ratio as the quantity that is "the appropriate measure of a given attribute" (Simon & Blume, 1994). They claimed that ratio expresses a relationship, which itself is a quantity that measures a particular attribute of a situation. They paid attention to aspects of ratio; the need to distinguish between multiplicative and additive situations and ratio as a measure of an attribute.

Multiplication as Repeated Addition and "Times as Large"

Two multiplication schemes that students might be using when approaching different tasks/problems in the context of ratio include: multiplication as repeated addition and multiplication as "times as large." Thompson & Saldanha (2003) claimed "conceptualized multiplication is not the same as repeated addition" (p. 103). The reason is that repeated addition is based on replications of the identified units and counting them. Therefore this is additive in nature (Heinz, 2000). In repeated addition students pay attention to what is repeated. However, for understanding multiplication, multiplicatively students pay attention to the product, the result of having multiplied, as being made up of identical copies of some other quantity (Thompson & Saldanha, 2003). Therefore, the second interpretation implies thinking of a quantity being "times as large" as another quantity. In that sense, thinking about the product (a*b), becomes *a* times as large as *b* and *b* times as large as *a* (Thompson & Saldanha, 2003). These researchers also pointed to two meanings of part- whole relationship such that understanding multiplication, multiplicatively, necessitates understanding fractions as entailing a proportion. They claimed if one thinks of a fraction such as 3/5 as "3 out of 5 parts" then it is additive in nature. However, if one thinks of 3/5 as 3 is 3/5th of 5, then it means thinking of 5 as 5/3 times as large as 3. Therefore, this implies the understanding of the product of (a*b) being in reciprocal relationships to *a* and to *b.* That is, *b* becomes *1/a* times as large as (a*b), and *a* becomes *1/b* times as large as (a*b) (Thompson & Saldanha, 2003).

Informal Strategies Students Use When Solving Problems in the Context of Ratio

The strategies that will be focus of attention in this section are the building-up strategy, the abbreviated build- up strategy and unit-factor approaches. While defining these strategies, I will also include other researchers' use of different strategies whenever there is need to mention them. I believe this will provide insight into different strategies used by several researchers.

Building-up

One of the correct strategies on which researchers have consensus is the building-up strategy (Kaput & West, 1994; Lamon, 1993; Lamon, 1994; Thompson, 1994; Tourniaire & Pulos, 1985). Lamon claimed that the building-up strategy constitutes forming successive multiples of some other units, thus forming units of units (Lamon, 1993). This strategy is based on additive reasoning although in the strategy students use multiplication. In other words, the repeated addition version of multiplication is used in this strategy (Kaput & West, 1994). According to Lamon, students who use the building-up strategy solve the problem "Ellen, Jim, and Steve bought three helium-filled balloons and paid $2.00 for all three. They decided to go back to the store and buy enough balloons for everyone in their class. How much did they pay for 24 balloons? (Lamon, 1993, p.145) as follows:

The student would make a table and say:

2	4	6	8	10	12	14	16	($ paid)
3	6	9	12	15	18	21	24	(ballons)

So the student would argue that s/he would count by 2 at the top and 3 at the bottom <u>at the same time</u> until s/he reaches to the desired number of balloons, which are 24 in this case. Heinz claimed that the building-up strategy does not necessitate the invariance between and within ratios. The reason is that this strategy might be based solely on the conception that the final situation will be the copy of the original situation, the identical copies of the original situation, whenever the final situation is made up of equal, identical groups like the original one (Heinz, 2000). Kaput and West provided insight into what the use of this strategy's underlying conceptualization entails (Kaput & West, 1994). The use of this strategy requires one to know that there are two quantities that make up the quality of interest. Also, there is a correspondence between the two quantities in the situation such that the association between the two quantities creates the particular quality of interest. Finally, the replications--the copies-- of the coordinated original quantities are held until the targeted quantity is reached. In other words, the building-up strategy is based on the use of a repeated addition version of multiplication, which is the replication of the quantity for a given number of times. Therefore, as long as the original quantity is replicated for a given number of times the desired result is reached

(Heinz, 2000). Some researchers claimed that building- up processes might be a precursor to the understanding of rate conception (Heinz, 2000; Kaput & West, 1994; Thompson, 1994). Thompson states

> If a child is trying to find, say, how many apples there are in a basket where the ratio of pears to apples is 3:4 and there are twenty-four pears, and the child thinks "three pears to four apples, six pears to eight apples,…., twenty-four pears to thirty-two apples" (a succession of equal ratios), then this provides an occasion for the child to abstract the relationship "three apples for every four pears" (an iterable ratio relating collections of apples and pears as the amounts of either might vary), and eventually "there will be ¾ of an apple or part thereof for every pear or proportional part thereof" (an accumulation of apples and pears that carries the image that the values of both can vary, but only in constant ratio to the other). The former conception--accumulations made by iterating a ratio-- I call an *internalized* ratio, whereas the latter conception--total accumulations in constant ratio--I call an *interiorized* ratio, or a *rate* (p. 193, italics are original).

Abbreviated Build-Up

Kaput and West (1994) provided the field with another strategy students use, abbreviated build-up strategy. They claimed that the abbreviated build-up strategy is beyond the building-up strategy yet it is additive at heart. It is more sophisticated than the building-up strategy because students who use abbreviated build-up strategy know that the factor by which they will multiply the first quantity in the ratio is the same factor by which they will multiply the second quantity (Kaput & West, 1994). The explanation follows: Consider the problem "Ellen, Jim, and Steve bought three helium-filled balloons and paid $2.00 for all three. They decided to go back to the store and buy enough balloons for everyone in their class. How much did they pay for 24 balloons?" (Lamon, 1993, p.145) The use of the abbreviated build-up strategy implies to divide 24 by 3 and get 8 as the factor by which $2 dollars will be multiplied. Although students who use the abbreviated build-up strategy recognize the multiplicative relationship between ratios, still the result of division of 24 by 3 is understood as the number of groups that the original combination of a 3 to 2 ratio will be used to come to the targeted amount of balloons. Heinz claimed that the use of abbreviated build- up strategy does not necessitate the understanding of that there is an invariant relationship between and within ratios. The reason is that although quotitive division is used between the quantities of between ratios, the conceptualization underlying this strategy is based on understanding the quotient obtained from qutotitive division as the number of known-quantity increments (Kaput & West, 1994); in other words, the quotient tells how many times 3 balloons need to be repeated and 2 dollars need to be repeated (Heinz, 2000). I would like to re-emphasize one important feature of the abbreviated build-up strategy that it is based upon realizing that there is a multiplicative structure between ratios yet that multiplicative structure relies upon iterated addition. This coincides with what Schwartz referred to as "scalar ratios" (Vergnaud, 1983, cited in Schwartz, 1988) as an example of a specific type of intensive quantity. Schwartz argued that this type of intensive quantity needs to be given particular attention because of its characteristics: a) it does not change the nature of the referent of the quantity on which it operates so it is dimensionless b) it changes the magnitude of its measure.

Also, Vergnaud provided information on two types of ratios "functional ratios and scalar ratios" (Vergnaud, 1994). He claimed that these ratios are conceptually different because the former represents the relationship between two different quantities in different measure spaces. The latter one, on the other hand, is the quotient of quantities of the same kind and is a "scalar" because it has no dimension (Vergnaud, 1988). Further, he claimed that “scalar ratios” rely upon iterated addition from which the multiplicative structure is developed (Vergnaud, 1994).

Unit-Factor Approaches [Per-One]

Kaput & West provided the field with another strategy, unit factor approach (Kaput & West, 1994). Unit formation requires one to use a two-stage process, which is the division of paired extensive quantities and multiplication of the factor by the given quantity (Kaput & West, 1994). They claimed that this approach is different from the building-up strategy in the sense that students recognize the use of division to find the unit factor and then multiply this factor by the given quantity (Kaput & West, 1994). The conceptualization process involved in the unit-factor approach includes the construction of a correspondence between the quantities in the original ratio, and once the association of those quantities in the original ratio is made then the question becomes what quantity will act as a divisor and dividend; which ends up at deciding that the known quantity will be the divisor so that the result of the division can be used as the unit factor. They also claimed that unit-factor approach is embedded in the use of partitive division scheme, unlike the abbreviated build-up strategy which is embedded in the quotitive division scheme (Kaput & West, 1994). The unit factor approach is compatible with what Heinz claimed as a “per-one strategy” (Heinz, 2000). The terms these two researchers use might be different but the strategies they mention are similar in nature. Heinz used the term “per -one” for the first part of the single-unit strategy---finding the cost of one item in the problem above---. She claimed that a per-one quantity does not require one to reason about intensive quantities, rather the student could be associating the amount of one quantity with the one unit of another quantity. This means that the student could be using the idea as an association of two extensive quantities (Heinz, 2000). Therefore, the student, while finding the cost of one balloon, could be associating one balloon with the amount of money. In addition, after finding the cost of one balloon, the multiplication of the found value (the cost of one balloon) with the desired number of items indicates that the identical groups of the cost of one balloon are calculated. That is, the student replicates the cost of the one balloon until the desired number of balloons is reached. Hence, this strategy does not require the understanding of proportionality. It is based on copying identical elements until the desired number is reached.

Also, this strategy seems to be compatible with what Lamon (1993) presented as the single-unit strategy. She claimed that the single- unit strategy is " a two step process in which the cost of one item is computed and then multiplied by the desired number of items" (Lamon, 1993, p.146). Any student who would solve the problem; "Ellen, Jim, and Steve bought three helium-filled balloons and paid $2.00 for all three. They decided to go back to the store and buy enough balloons for everyone in their class. How much did they pay for 24 balloons? (Lamon, 1993, p. 145), would reason according to this strategy as follows. S/he would find the cost of one balloon by dividing 2 by 3 and then s/he would multiply this amount with 24 to find how many dollars he would spend.

Lamon (1993) claimed that this strategy is a lower level strategy than the building-up strategy because the students are not required to use composite units in applying the single-unit strategy. Interestingly, Kaput and West (1994) said that the unit-factor approach is more sophisticated than the building-up strategy, unlike Lamon. In addition, they asserted that the unit formation process should precede the rate ratio concept because “particular ratios” are used to come to the conceptualization of rate ratios.

Heinz shed light on this dilemma in her study. Although the nature of the unit-factor approach and the single-unit strategy is the same, she asserts that the former might entail three different conceptualizations, which also include the latter (Heinz, 2000). That is, first, the unit- factor approach (per-one) requires one to find out how much of a quantity corresponds to one unit of another quantity. Second, the given two extensive quantities are split into equal parts, which “do not necessarily result in one of the quantities being equal to one” (Heinz, 2000, p. 42). Third, the invariance of the value of ratio holds even when the components of the ratio are divided by the same quantity because ratio itself quantifies the invariance of the quality of interest (Heinz, 2000). That is, for the first and the second explanations, it seems that per-one and per-n are the focus. The third explanation already is beyond unit-factor approaches; however, unit-factor approaches might be a pre-cursor to the understanding of rate conception (Kaput & West, 1994) which agrees with what Heinz claimed as a third explanation, that the student knows that the ratio determines a quality of an attribute and the ratio is the multiplicative relationship between the two quantities that requires one to think about the invariance in all parts of the ratio (Heinz, 2000).

Another informal strategy provided in the literature is the ratio-unit/ build-up method (Lo & Watanabe, 1997). Lo and Watanabe claimed that any student who could find an “x elements for y elements” relationship and employ this relationship as a countable unit in a whole-number sense, the strategy used is called as ratio-unit/building-up strategy. This strategy involves the notion of homogeneity in proportion context. The intention of the user is to even out the elements of given quantities. That is, the student seeks for the least composite units to associate with the least composite unit of the other extensive quantity. This process involves the partitioning of the quantities. Then, the implicit notion that a relationship exists between the x elements and y elements of the given quantities and that this relationship needs to be preserved between the certain subsets of the given quantities becomes the heart of the strategy. (Lo & Watanabe, 1997). For instance, when given the 24 candies for 18 children and asked for the number of candies for 21 children, the student solves the problem as follows: S/he divides 24 and 18 by 6 and knowing that the relationship between 4 and 3 is preserved, s/he adds 4 more candies to 24. This is similar to per-n strategy that Heinz (2000) provided in the following way: That is, the student can use the partitive division scheme to deal out the 24 candies among 18 people. However, since the student is aware of the fact that 24 to 18 is a multiple group of 4 to3 [breaking down the given quantities to find the equivalent group of which the 24 to 18 is a multiple], s/he does not have to break it into a per-one ratio. Knowing that the quantities, which are still extensive, can be used to create another identical group, the student adds the 3 to 18 and therefore 4 to 24 to find out the number of candies for 21 people.

Task Variables

This study is focused on students' conceptions of within and between- ratios and making distinctions among them. Thus, it is important to situate what the research findings are so that it could be helpful for the data collection and preparation stage.

Numerical Features

Noelting examined the structural features of items/ problems (Noelting, 1980b). He argued that according to the different stages of structural complexity of the items (problems) used in his study, only certain problems were meaningful to students (Noelting, 1980a). That is, depending on the numbers included in the different parts of ratio, students could reason or could not reason through. For instance, when the numbers included in both parts were integer multiples of each other, then it showed some level of sophistication lower than ratios with non-integer multiples. Kaput and West (1994) pointed out to the same issue that when the numbers included in the parts of the ratio are integer multiples, then problems are easier for students to solve than when the numbers included in the parts of the ratio are non-integer multiples of each other.

In addition, Kaput & West asserted that the following numerical features of the problems promote the use of the building-up strategy. First, students are more likely to recognize the invariant multiplicative relationship between the quantities if the ratio is given in its reduced form. This also helps them to increment the quantities in the ratio. Second, if the numbers are familiar to students such as numbers in the 9*9 table and if the quantities evenly divide one another, then it is more likely for students to recognize that the total quantity (s) as a result of incrementing. This also enables students to recognize that the total quantity is made up of *increasing multiples* of the quantities in the given ratio (Kaput & West, 1994). Conversely, the use of numbers whose magnitudes are relatively close to each other in making up the quantities of ratio could facilitate the use of additive reasoning (Heinz, 2000; Kaput & West, 1994). For instance, Heinz presented in her study that most of the students used additive thinking while working on the ratio pairs of 3 lemons to 2 limes and 4 lemons to 3 limes (Heinz, 2000). Heinz claimed that the task was a non-trivial task for the students since the numbers involved in the parts of these ratios were not integer multiples of each other, students would be more likely to use additive reasoning. This is compatible with what Karplus and colloquies stated. They claimed that the inclusion of the equivalent ratios is easier for students to deal with than the inclusion of unequal ratios (Karplus, Pulos & Stage, 1983b cited in Heinz 2000).

Similarly, Heinz (2000) claimed that when students solve problems accessing the identical groups conception, then they are able to do so to some extent. Yet, when they face the non integer multiples of the identical groups, they cannot reason further. For example, in her study students could reason about a problem involving working hours per job. They could use the ratio of 5/12 to find out the total number of working hours in the following way: They could reason that it would take the two friends to do 5/12 of the job in one hour. Then they incremented this amount one more time and concluded that 2 hours would be required to finish the 10/12 of the job. However, when they faced the 2/12 of the job left, since 2/12 was not an integer multiple of 5/12th, they could not go further and concluded that the job would take more than 2 hours. Kaput & West claimed that in the case of the building-up approach, if the numbers involved require students to deal with halving, for instance, then it might be possible to reason through the problem. However, when the numbers, as in the case of Heinz's study, represent unequal ratios then it is harder for students to reason though them with the building-up approach (Kaput & West, 1994)

Semantic Features and the Context Domain

Kaput & West argued that the use of “for each” and “for every” makes the homogeneity of the situation more explicit in the problems and therefore students are more likely to use building-up strategy (Kaput & West 1994). That is, the association between the extensive quantities in the ratio becomes easier to detect for the students. Associating those extensive quantities, students carry out the identical group formation until they reach the desired amount. In addition, they claimed that when the problems are given as containment situations such as the ratio of the number of forks to the number of spoons on a table, students are likely to work with a building-up strategy (Kaput & West 1994). The association between the two extensive quantities of the containment makes it possible to use a building-up strategy (Kaput &West, 1994).

Lamon mentioned stretcher/shrinker situations being the most difficult for students to apply multiplicative thinking. Actually, the problems of this type such as similarity problems or geometry tasks requiring the use of ratio and proportion are the ones that students generally have used additive thinking instead of multiplicative thinking as in well-chunked measures, speed, price, etc. (Lamon, 1993; Kaput & West, 1994; Heinz, 2000). Similarly, mixture problems are some of the most difficult ones for students to grasp as proportional situations (Kaput & West, 1994; Heinz, 2000). This is because the nature of the problems requires students to be able to think about what is the quality of an attribute to measure. In other words, mixture problems require students to understand that ratio of the parts to parts or parts to whole in the mixture represent the concentration quality of the mixture. In addition, they argued that when the problems are examples of part-whole situations, then they are also easier to solve (Kaput & West, 1994).In their study, they used the example of part-whole in the context of the ratio of number of girls to the number of boys in a class. They claimed that since the problem involves “for every x number of girls there are y number of boys”, the inclusion of “for every” statement makes the homogeneity of issue more explicit . This enables students to realize the invariance of the ratio in the situation more easily. Also, Lamon stated that students, even the proportional reasoners, have used the building-up strategy in reasoning through part-part-whole situations (Lamon, 1993). The example that Kaput and West used is the same type of example Lamon uses in her study [the ratio of number of girls to number of boys in the class]. Students can think of the ratio of the number of girls to the number of boys using the incrementation of the extensive quantities until they reach the amount they are given in the problem.

Two Different Stages of Mathematical Conception

Before articulating the different conceptions of ratio, mentioning the two stages in learning a new mathematical conception is necessary. Those stages become important while analyzing data on students’ approaches to different tasks/problems in the context of ratio for two reasons: First, mentioning conceptions of ratio, it refers to the understanding of the concept at an abstracted level, an anticipatory stage. Second, some of the tasks provided to students demand an abstracted level of understandings, at the anticipatory stage, such that students, without the specific abstracted level of understanding tasks/problems demand, cannot call upon the necessary knowledge to approach the tasks/problems.

Tzur & Simon (2004) mentioned two different stages of abstraction in learning a mathematical conception: participatory stage and anticipatory stage. Participatory stage refers to the understanding level such that the student has access to the knowledge only if participating in the context of the activity through which the knowledge is developed. These researchers asserted “In the context of the activity means either the learners are engaged in the activity or are somehow (e.g. chance, social interaction) prompted to use or think about the activity” (p. 15). Anticipatory stage refers to an understanding level such that the student is able to call upon the knowledge without being in the context of the activity through which it is developed. In that sense, students at an anticipatory stage of understanding, of any mathematics topic, in this case ratio, can call on the necessary activities to approach any task/problem that demands the necessary knowledge. Similarly, if the students do not possess the anticipatory stage understanding of a specific concept that the tasks/problems demand, then the student cannot call on the necessary knowledge to approach to the tasks.

Different Conceptions of Ratio

In the research literature, some of the conceptions of ratio have already been provided (Heinz, 2000; Simon & Blume, 1994; Thompson, 1994). The conceptions are identical groups conception, ratio-as-measure and ratio-as-quantity (Heinz, 2000; Kaput & West, 1994, Behr, Harel, Post, & Lesh, 1993).

Identical Groups Conception

This conception of ratio is based on a *collection of sets* of the extensive quantities in ratio or *breaking down* the ratio into equal parts. Students with this conception think of ratio as an expression that represents the association between two extensive quantities (Heinz, 2000). That is, the ratio is made up of two extensive quantities. Particularly, the quality of interest is the result of the association between the extensive quantities and ratio, constituting two extensive quantities determines the quality of interest (Heinz, 2000).

The process that underlies the *collection of sets* is the addition or at most multiplication as repeated addition. The strategy students use for collection of sets using repeated addition is “building-up” (Heinz, 2000). Since students realize that there is a quality of interest in the situation which is represented by the association between the two extensive quantities, they utilize the repeated addition until they reach their goals. They do so because students understand that the association between the extensive quantities is kept the same since the initial extensive quantities given in the ratio is repeated together until one of the known quantities is reached (Heinz, 2000). Thus, they know that since the association is kept the same, so is the quality of interest which is the result of the association. This understanding enables them to use equivalence of ratios in such a way that the same number of groups between extensive quantities belonging to the same domain acts upon the other extensive quantity in the ratio. Also, students might use the abbreviated build-up strategy since they understand that they need to use the same number of groups for each of the extensive quantities in the ratio. In other words, they understand that an equal number of groups of both extensive quantities guarantee the preservation of the quality of interest.

The process that underlies *breaking down* is partitive division within the elements of a given ratio. The reason is that students with identical groups conception understand that ratio when partitioned into equal sets preserves the quality of interest. Yet, what they do not understand is that they do not realize that a multiplicative relationship exists between the elements of the ratio, and that multiplicative relationship can be measured by ratio in such a way that the resultant quantity represents the size of one of the extensive quantities relative to the size of the other quantity in the ratio. On the other hand, association of the extensive quantities is done by partitive division in such a way that elements of the first extensive quantity are distributed over the elements of the second extensive quantity. The strategy students use for breaking down the quantities using partitive division is called per-one. In this process, ratio expresses how much of one extensive quantity associates with one unit or n units of the other extensive quantity (Heinz, 2000).

Based on these discussions, division makes sense [students with identical groups conception think of using division] in two situations: 1) If they have something at hand to compare, and 2) if they can find "integer groups" of the extensive quantities at hand. That is, for the first case, they might divide to discover which set of the extensive quantities is a representative of “more or less of a quality of interest.” In this case students utilize partitive division, therefore, the per-one strategy. In the second case, they might want to know how many groups of the extensive quantities in the given ratio need to be repeated, as in abbreviated build- up.

The goal of the students in identical groups conception is to keep the structure of the situation the same. In other words, when the situation is given as, “for every x amount of something there is y amount of another thing”, the goal of the students is to be able to observe the same structure until they accomplish the goal of the problem. The similarity between per-one [per n] strategy and building-up strategy as an implication of identical groups conception is that both strategies serve the same goal, which is to keep the structure, therefore, the quality of interest, the same. The difference is that the ultimate goal in building-up is to make twins, triples…n tiples of the same structure; whereas, the goal in per-one strategy is to make smaller groups of the given ratio. On the other hand, when the situation necessitates both partitioning and building-up in pursuing the goal, where students seek that repeated structure, then students with identical groups conception call on both strategies.

Ratio-as-Measure

Simon & Blume (1994) defined ratio-as-measure as the quantification of a given attribute. They claimed that ratio expresses a relationship, which itself is a quantity that measures a particular attribute of a situation. They paid attention to aspects of ratio, the need to distinguish between multiplicative and additive situations and ratio as a measure of an attribute.

In pursuing division as an operation to find the relationship between the quantities, the goal of a student with ratio-as-measure is to determine the relative size of one amount to the other. Students know a necessity exits in the situation to find out the relative size of the quantities because the quantities in the situation call for an indirect measure of an attribute such as lemoniness, squareness, density, sweetness, etc. For instance, when considering a situation where two people's ages are compared, the measure is direct because the interest is a difference between two extensive quantities. Contrary to the fact that the attribute, age, can be represented in absolute amounts, sweetness can not be expressed by the absolute values of sugar and water in the situation. The sweetness calls for an indirect measure. That measure, called as ratio, determines the quality of interest in the situation by relating those two quantities as relative to one another rather than in absoluteness.

The relativeness comes from the fact that both quantities covary simultaneously on two levels. First, whenever one of the quantities reaches a point, the other quantity reaches another point in such a way that the specific relationship between them can help to determine the value of one of the quantities when the other quantity's position [value] is known in the situation. Second, the covariation occurs within the extensive quantities that make up the ratio. That is, whenever the value of one of the quantities in the ratio is changed the other quantity needs to be changed in such a way that what remains the same is the multiplicative relationship between the extensive quantities as well as the measure of the quality of interest.

Identical Groups Conception Versus Ratio-as-Measure

The difference between identical groups conception and ratio-as-measure is that in the former, ratio is not an intensive quantity (Heinz, 2000). In other words, although students find, for instance "x per-one [per n]" or they keep the association between the extensive quantities so that the quality of interest will remain the same, the ratio is still made up of two extensive quantities (Heinz, 2000). When the ratio is used as an "intensive quantity," as in the case of ratio-as-measure, then the expression "x per-one" is used as an expression that reveals the relationship between x and y. However, in identical groups conception, ratio seems to be assumed as a quantity, an extensive one; on the other hand, in ratio-as-measure conception, ratio is assumed to be the multiplicative relationship between two quantities. In particular, as in ratio-as-measure, in identical groups conception of ratio, students recognize a quality of interest which is preserved, invariant. Unlike in identical groups-conception of ratio, in ratio-as-measure understanding, students realize that they can measure the invariant quality of interest by an invariant quantity, ratio, which is the multiplicative relationship between the extensive quantities.

Ratio-as-Quantity

In this understanding of ratio, as in ratio-as-measure, students realize that ratio is an intensive quantity that measures a relationship between quantities (Heinz, 2000). In particular, students recognize a relationship between two quantities, and they can use ratio as a measure of that relationship. Students realize that the nature of the relationship is a multiplicative one (Heinz, 2000). That is, they can measure one of the quantities in the ratio in terms of the other quantity. They know that the result of the measurement is an entity that represents the extent of the change in one of the quantities (Heinz, 2000). In particular, the ratio, the result of the measurement, the measure of the multiplicative relationship acts on one of the quantities and transforms it into larger or smaller quantity of the same type. Heinz claimed that ratio-as-quantity indicates for students how much of a change, how much of a transformation occurs (Heinz, 2000).

Students understand that the quantities involved in the situation have a relationship, a fixed relationship, and ratio is an appropriate way to measure that multiplicative relationship. Also, and most importantly, they understand that ratio is the quantity that represents the amount of change in size of one of the quantities such that the ratio showing the change of size in one of the quantities relative to the size of the other quantity can act on the other extensive quantity because of the covariance between extensive quantities in the ratio. If considering the problem, “How much does one need to pay for 30 limes if every 3 limes cost $2.5 dollars?” the student realizes: a) A multiplicative relationship exists between the 3 limes and 30 limes such that the measure of that relationship determines the rate at which the number of 3 limes increases. b) This rate of change in the number of the limes can act on $2.5 dollars since both of the quantities [lime and dollar] covary. c) The number, ratio, also determines the rate of change in the amount of $2.5. This means that this person is utilizing the ratio, 10, in the problem situation as the quantity by which the degree of change in the situation has been determined. The solution will probably be constructed as 2.5*(30/3) since the students with this understanding already know that the ratio, 30/3, is an intensive quantity, and they already know that the proportional amount that 3 is of 30 is the same as the proportional amount 2.5 is of the unknown quantity. Therefore, the solution comes from 30/3=x/2.5, from which x=2.5 * (30/3) is determined (Karplus et.al., 1983, cited in Heinz, 2000).

However, the distinction should be made between the students with identical groups conception who can also determine that 30 is 10 times larger than 3. As stated earlier, students with identical groups conception are not yet aware of an invariant multiplicative relationship between the quantities involved in the situation. That is, they realize that quantities such as 3 limes and $2.5 dollars covary; however, they do so because of the association between them. In other words, the quality of interest needs to be preserved in the situation and the way to do it is to repeat those quantities together until the goal is reached. Therefore, these students might recognize that 30 is 10 times larger than 3, but they do not realize that this relationship is an invariant relationship, and they do not yet realize that 10 is the measure of the invariant relationship between those quantities.

What Research Has Contributed

Researchers have studied the task variables in the context of ratio and proportion (Harel & Behr, 1989; Kaput & West, 1994, Noelting, 1985a; Noelting, 1985b), informal strategies student use (Karplus & Lamon, 1994; Kaput & West, 1994; Lamon, 1995; Thompson, 1994; Thompson, 1988; Watanabe &Lo, 1997), and different conceptions of ratio and proportion (Heinz, 2000; Thompson, 1994; Thompson &Thompson, 1996). The studies Thompson conducted only included discussions of students' conception of a specific example of ratio, speed. That is, they included well established understandings of a specific concept of within ratios, such as ratio-as-measure. The study Heinz (2000) conducted characterized the understandings of identical groups conception and ratio-as-quantity. Although, Heinz studied conceptions of ratio regarding both between- ratios and within- ratios, she expressed the need to do more research in terms of understanding fully what constitutes a web of well developed conception of ratio such that identifying the characteristics of conceptions of ratio might help to better understand how those different conceptions of ratio are related and possible paths from one conception to the other can be studied more effectively (Heinz, 2000). She raised issues in terms of further explicating the nature of different conceptions of ratio:

> Students' identical groups, part-whole, and partitive division conceptions enable them to solve some types of ratio tasks. I propose that there is some overlap in the development of those conceptions in the sense that as students develop increasingly sophisticated notions of one of those conceptions, those understandings can contribute to the development of other conceptions (Heinz, 2000, p. 136).

Thus, the question becomes what are conceptions of between and within- ratios such that knowing the particular characteristics of those conceptions might in turn shed light on the use of partitive and quotitive division schemes, part whole relationships and per-one in different conceptions of ratio? In addition, Heinz (2000) claimed the existence of different ratio and proportion tasks that cannot be solved by specific conceptions of ratio and proportion. Also, Lamon (1993) stated that, some of the students who were proportional reasoners solved different ratio and proportion tasks/problems using different conceptions. So, another question arises regarding the investigation of how students' different understandings play out in different problems/tasks in the context of ratio, and how those expressions of understandings help to make distinctions among those conceptions.

In sum, the aforementioned research on ratio suggests a need to do further research on what constitutes different pieces of a web of a well developed conception of ratio and how those conceptions differ from each other. Therefore, research is needed to further explicate and make distinctions among conceptions of between-ratios and within- ratios.

Key Aspects of the Current State of the Field

As stated several times, a great amount of research has been dedicated to children's and adults' conceptions of ratio. This current study was conducted based on the research findings presented in this chapter. The following list provides a list of what aspects of existing research was considered through the data collection and analysis of this study. Research on the key constructs of the concept of ratio includes:

1. Identification of quantitative reasoning and numerical reasoning.
2. Different multiplicative schemes.

3. Students' informal strategies prior to instruction.
4. Task variables, such as numerical features of the problems, semantic features of the problems and the context variables.
5. Learning mathematics, shedding light on the articulation of the understandings inherent to the specific content areas.
6. Conceptions of students' during instruction, revealing, to some extent, what different conceptions of ratio are used by students and revealing the differences among those conceptions.

Purpose of the Study

Existing research indicated a need to study different conceptions of ratio and to make distinctions among them. The purpose of this study was to investigate and explicate conceptions of between- ratios and within-ratios and make distinctions with respect to those conceptions.

Chapter 3

METHODOLOGY FOR THE STUDY

Overall Method

This study focused on students' conceptions of between-ratios and within-ratios and distinctions among them. Students received several problems/tasks to produce answers and justify their answers to the problems/tasks. That is, students were expected to write their solution processes using paper and pencil and justify those processes in two, one and one half hour sessions. Their answers to the problems/tasks were analyzed through their justifications and their use of different strategies. The problems/tasks were chosen in such a way that solution processes were expected to provide a way to elicit different conceptions of between and within-ratios. Then, one hour clinical interviews were conducted to further elucidate students' conceptions of between-ratios and within-ratios, underlying students' approaches to problems/tasks in the context of ratio.

The theoretical framework and conceptual framework that was the basis of this study, the justification of the method of the study, participants of the study and data sources and data analysis are presented in the following paragraphs.

Theoretical Framework

Radical constructivism is a theory of epistemology that emphasizes:

> … there can *a priori* be no such thing as mathematical structure existing apart from an individual's constructed knowledge, nor can we talk meaningfully of a problem‘s structure apart from the understandings of a problem solver. Furthermore, there is ---again *a priori*--- no way of knowing that a problem (or a mathematical concept) has the same structure for different individuals"(Goldin, 1990, p.39).

That is, being a living subject, mathematics is the outcome of human activity (Steffe & Thompson, 2000). Also, one can never have direct access to anyone else's constructions; rather one can only create models of knowledge of others (Goldin, 1990). Similarly, "It is the product of reflection-whereas reflection, as such, is not observable, its products may be inferred from observable responses" (Glasersfeld, 1896, p. 10).

The researcher, who maintains a radical constructivist theory of epistemology, understands that no two persons have the same mathematical structures. Only through the observations of reasoning of those people on different tasks/problems can an observer interpret the knowledge of those people. Thus, the researcher poses tasks/problems to observe students' mathematical behavior and also knows that for students to make sense of those tasks/problems, they need to have access to them. The reason is that without having the necessary knowledge to reason about tasks/problems, the student will not know what to do. That is:

> ….Operative knowledge, therefore, is not associative retrieval of a particular answer but rather knowledge of what to do in order to produce an answer" (Glaseresfeld, 1986, p.11).

Conceptual Framework

Previous research on conceptions of ratio (Heinz, 2000; Thompson, 1994; Simon & Blume, 1994), informal and formal strategies students use (Heinz, 2000; Kaput & West, 1994; Lamon 1995; Karplus, Pulos & Stage, 1983b cited in Heinz 2000), constructs such as intensive -extensive quantities, invariance and covariance, etc. (Schwartz ,1988; Thomson, 1994; Noelting, 1980b; Lamon, 1995) and also task variables (Kaput & West, 1994; Lamon, 1993) already provided insight into distinctions among students' different reasoning patterns (Heinz, 2000). While preparing and analyzing tasks/problems both in the problem solving written sessions and the interviews and also, the probing questions in the interviews, the influence of previous research results had consideration.

Particularly, the research on conceptions of ratio provides insight into how students at particular levels reason (Heinz, 2000). For instance, the identical groups conception implies ratio as a representation of two extensive quantities, the use of building-up strategy, per-one strategy [equal sharing] and abbreviated build-up strategy [use of quotitive division], and also equivalence of fractions (Heinz, 2000). Ratio-as-measure implies ratio as an intensive quantity (Simon & Blume, 1994), and the use of quotitive division and partitive division. Ratio-as-quantity implies ratio as intensive quantities such that the attribute in the situation is the multiplicative structure (Heinz, 2000). I considered these different levels of reasoning in ratio as provided by different researchers and created a chart showing the characteristics of those conceptions. This allowed me to focus on the distinctions among those conceptions. This then allowed me to pay attention to the characteristics of tasks/problems for both explicating students' understandings of between and within-ratios and also distinctions among them. Also, for the analysis, those conceptions, the constructs and the strategies helped me to determine what understandings students held, what connections students had and the distinctions among their reasoning.

Similarly, research on task variables such as numerical and semantic features and context domain (Kaput &West, 1994; Lamon, 1993) helped to determine the characteristics of tasks/problems so that the study would elicit the conceptions of between and within-ratios regardless of the tasks variables. Knowing the task variables, I could find and create tasks/problems that would allow making distinctions among their understandings. For instance, knowledge on numerical task variables allowed me to use non-integer multiplies and some difficult numbers in some other tasks to distinguish different levels of reasoning in between-ratios and within-ratios. Similarly, the knowledge of semantic features allowed me not to use "for every" or "for each" phrases that are expected to trigger the use of building-up strategy; and also allowed me to include part-whole structure in problems to observe how students use their knowledge of part-whole in reasoning in the context of ratio. Also, knowing how task variables affect students' reasoning helped to make claims about their understandings during the analysis.

Furthermore, studies point to different understanding levels of ratio, differentiating internalized ratio [particular ratios] from interiorized ratio [rate ratio] and possible paths that might provide necessary reasoning patterns [processes through building-up] to come to the rate understanding (Heinz, 2000; Kaput & West, 1994; Thompson, 1994). When analyzing the data, I looked if there were examples that would elicit the kind of reasoning that would promote rate understanding based on the processes involved in building- up.

In addition, research results on students' ways of reasoning on different tasks/problems (Heinz, 2000; Laomn, 1993; Karplus et al. 1983a) also informed both the preparation and the analysis of this study. I paid attention to the fact that different tasks/ problems might have different ways of solutions depending on the same conception of ratio or different conceptions of ratio. The same problem might be accessible to different conceptions of ratio and proportion or it might be accessible to only one type of conception of ratio. This occurred because the problem solving process assumes as

Kieren & Pirie emphasized, “ That a current state of mathematical knowing transcendently elaborates previous states, and integrates or entails these states in the sense that they can be called into current knowing actions” (Kieren & Pirie, 1991, p. 81). That is, “the mathematical knowledge of a child at a particular point in time is a transcending whole which non-destructively integrates previous states or levels of knowing and in particular can call these previous states in problem solving. “ (p. 80).

Finally, previous studies provided insight into the distinctions among different stages of understanding [participatory/anticipatory] suggesting that students at particular understanding levels might not have access to the tasks/problems that other students might have (Tzur & Simon, 2004). I used those different stages of understanding to analyze and make sense of some part of this study. Thus, previous research and the radical constructivism served well for the goal of this study: to focus on students’ current reasoning on conceptions of between and within-ratios and distinctions among them.

Written Arguments as a Way of Data Collection

As stated earlier, students were given a number of problems/tasks in the context of ratio and were asked to provide their solution processes and explanations for these problems/tasks in two, one and one half hour long problem solving written sessions. The goal for doing written sessions was to gain insight into what strategies students use, what tasks/problems they could and could not reason about and what justifications they provided that would help to elicit their understandings which underlie their approaches. Written work served as a foundation for examining students’ understandings of between and within-ratios in a greater depth during the interview. It also allowed for selection of students to interview.

In order to obtain as comprehensive a picture as possible of students' conceptions of ratio, I attempted to assemble a set of problems that would allow me to distinguish different conceptions at every level of conceptual sophistication. I created a chart based on the literature specifying known subconcepts. Again using the literature, I selected problems that could be helpful in distinguishing each subconcept. In a few cases, I created problems.

I included five ratio problems in the first problem solving written session, and five additional ratio problems in the second problem solving written session [See appendix for the whole set of problems in the written problem solving sessions]. First problem solving session included three problems for which ratios were not appropriate, and the second problem solving session included five problems for which ratios were not appropriate, so students had to choose when to use ratio. Also, they would not necessarily think they were solving ratio problems so that they would not get smart at solving them. Students used paper and pencil to solve these problems and calculators were available to students who had a need. The problems asked students to explain and justify their reasoning processes during the problem solving written sessions.

Clinical Interviews as a Source of Data Collection

The students who participated in the written problem solving subsequently participated in hour- long task-based interviews. The interviews included both the tasks from the written sessions and also some additional problems [see appendix 3 for the additional interview questions]. The interview sessions included the student and me, as researcher. I videotaped the sessions since I wanted to have verbal [data] justifications of students. Students used pencil and paper and also calculators, if they needed, so that I would have another source of data for the analysis. I asked students the problems requiring more sophisticated understandings at the beginning of the interview, and then I asked them less sophisticated ones so that they would not get smart at solving those problems.

Conducting task-based interviews permits observation and drawing conclusions about students' current mathematical knowledge. The focus of attention is on student reasoning (Goldin, 1998; Goldin, 2000; Ackerman, 1995; Steffe, 1991; Skemp, 1987; Clement, 2000). Interview settings include generally, one researcher, as problem poser and questioner, and one student, as problem solver. Interviews are videotaped and/or audio taped (Goldin, 2000; Goldin, 1998). In conducting task-based interviews, researchers aim at understanding students' internal mathematical constructions and problem solving processes. The focus is on the rationale behind the solutions students provide rather than on detecting correct or incorrect solutions (Goldin, 2000; Goldin, 1998). Thus, task-based videotaped interviews provide data on students' current conceptions through inferences drawn from their problem solving processes and nonstandard solution methods (Goldin, 2000; Clement, 2000).

The focus of attention was students' reasoning on several tasks/problems, not whether those students were able to solve tasks/problems correctly or not, in the context of ratio. The focus of attention was how they reasoned and were able to justify their solution processes, and how they were making connections among different sub-constructs such as invariant-covariant relationships, quotitive division, partitive division, part whole, per- one, etc. while providing their solutions.

I used some of the tasks/problems from the written sessions also during the interviews to investigate students' understandings in greater depth. Also, I determined what additional problems to use and probing questions to ask based on the specifics of their written problem responses. For instance, some students used part-whole relationship to solve some of the problems; thus, I decided to ask them what that part-whole represented in the problem and how they made sense of it. Similarly, some students made comments about between-ratios yet they did not make sense of the result of division of within-ratios, then I decided to ask them how they made sense of between-ratios and how they could not make sense of the within-ratios. I expected that this would provide me with insight into how those students use their division schemes in ratio situations and how they make sense of the quotient, the ratio. Also, some students, in spite of directions asking for not solving the problem but commenting on the solution provided to them, configured the proportions and solved the problems. In those cases, I decided to ask them how they made sense of the equations they configured or if they could reason about the problem without creating proportions. In addition, I asked students two additional problems; the first one was the Ice-Cream Problem. I included that problem because I realized that I needed to include really difficult numbers in a missing value problem for which the given solution was a ratio. I wanted to see if students were or were not able to reason through the problem solution regardless of the difficulty of the numbers. The second problem was the Mosaic Problem, in which the question from the written sessions requested as many solutions as possible. Generally students established a proportion and solved it. Selecting from different students' solutions, I chose one of the students' solution and provided that solution to other students and asked them if they were to make sense of that solution.

Participants of the Study

In this study, twelve prospective elementary mathematics teachers and three prospective secondary mathematics teachers participated in the written problem solving sessions, and fourteen of them participated in the interviews. One of the prospective elementary mathematics teachers withdrew from the study. For this study, I analyzed two prospective elementary mathematics teachers’ understanding of ratio. I used a combination of elementary and secondary prospective teachers, because I expected that they would demonstrate concepts that ranged from incorrect additive strategies to sophisticated concepts of ratio. Students participated on a voluntary basis and were paid five dollars per hour for the written problem solving sessions and $ 7.5 per hour for the interview sessions. I recruited the students in their mathematics methods classes. They were told that the study focused on distinctions among students ‘conceptions using particular mathematics problems.

Data Sources

The data sources for this study were students’ written responses to the tasks/problems in the written problem solving sessions, transcripts and videotapes of the interviews, and written artifacts from the interviews.

Data Analysis

I began the data analysis after I gathered data on problem solving written sessions. I analyzed each student's responses to the tasks/problems in the context of ratio. The goal of the analysis of the data from the written sessions was to gain insight into what kinds of strategies and solution processes students used, how they justified their solutions and what problems they had or did not have access to. The information gathered through their use of strategies, solution processes and justifications were important: Previous research already provided information regarding what kinds of formal and informal strategies students use and their different conceptions of ratio. Knowing what research already contributed, I analyzed each student's response in the written sessions line by line with the goal of examining what underlying conceptions of ratio they might be revealing through their strategies, processes and justifications. Based on their work on particular problems, I formulated hypotheses on the students' conceptions. I also recorded questions about their work and conceptions that would require further information. This analysis provided me with the insight to determine what questions to ask during the interviews. That is, I was able to decide what problems/tasks I did not have to pursue on during the interviews of each student and what additional probing questions I needed to ask for some of those tasks/problems. These indications helped gain more insight into what reasons those students used while using those strategies, processes and justifications, and how they made connections among those.

Then, I conducted one hour videotaped interviews with each student. After the interviews, I transferred the data to DVD's and listened to all of them. When I listened to all of those interviews, I had the goal of determining what conceptions those students might be reasoning from and what interesting reasoning patterns they seem to be presenting. That way, I could determine that six of those students applied additive reasoning in the context of ratio. Thus, I decided not to include them since this study focused on conceptions of between and within-ratios. Similarly, two of the students did not provide data of their conceptions, rather they used mostly procedures such as cross multiplication, and when probed, they kept saying that was the way they learned how to solve those problems/tasks. Elimination of those students in this study was due to a lack of evidence that would be informative in terms of students' conceptions of between and within-ratios. The last six students provided rich data in terms of their reasoning and conceptions of ratio.

I analyzed the data from all six. Three of those students seemed to have a robust understanding of ratio. That is, they knew both that the result of division within-ratios represented the multiplicative relationship between two quantities of the ratio and that number applied to other situations with the same quality of interest. They also knew that between-ratios represented the change factor, the multiplicative change from one situation to the other. Thus, those students already had understandings of both between-ratios and within-ratios and data from their work did not present an opportunity for new research insights into between and within- ratio conceptions. A fourth student demonstrated less competence. However, similar to the first three, I did not find the data to provide a basis for new insights. The data from Mark and Rita, both separately and in juxtaposition, provided the best opportunity to advance the goals of the study.

I analyzed the data line by line. While analyzing the data, I had the goal of characterizing the conceptions of between-ratios and within-ratios, what limitations and affordances those understandings held and what processes those students seemed to be using. So, I examined their reasoning on each task and highlighted the important pieces serving the goals I had. Then, I went back to the written sessions and looked at the data that I did not pursue during the interviews. Based on the hypotheses that I drew from the analysis of the transcripts of the interviews, I checked if the hypotheses still seemed plausible according to the responses in the written sessions. Then, I wrote narratives on those students' accounts both based on the data from the interviews and the written sessions. Before, during and after writing the narratives, I did three things: First, I shared the conjectures and claims of the analysis with other researchers to allow them to challenge my conjectures and /or to affirm their reasonableness. Second, I kept going back to the previous research findings to examine how the analysis of this study related to previous research results. Third, I looked for disconfirming evidence in both written sessions and the interview. For example, after getting reactions from other researchers to one of my initial hypotheses about Rita's understandings, I became aware that I did not have a kind of data needed for that hypothesis. Therefore, I modified the hypothesis. In another example, after going back to the literature about how students at higher levels of understanding might approach tasks using less sophisticated approaches, I had to consider an alternative interpretation of data from interviews with Mark. Therefore, I suggested two interpretations: either he lacked understanding or he elected to use a more primitive, but valid approach.

I did not analyze students' solutions to the distracters because analysis of those problems was not important to the goal of this study. Also, I did not include the analysis of the Squareness Problem, the Blue Jeans Problem, and the Ski-Slope Problem for this study, since data from those problems did not seem to further strengthen the analysis. My final step was to compare conceptions of the two students, allowing me to pay attention to aspects of the data that I had not concentrated on and to make further distinctions.

Chapter 4

EXPLICATING CONCENPTIONS OF BETWEEN-RATIOS AND WITHIN-RATIOS

Rita's Understanding of Between-Ratios

In the written session, Rita did not have much comment on option "a" for the Hair-Color-2 Problem. So, during the interview I asked her to comment on that option first: The following data is important because the data shows that Rita is able to realize and use the multiplicative relationship between-ratios.

Hair Color-2 Problem: Kelly likes to color her hair with a mixture of red and brown dye. When her hair was shorter; she used 15 grams of red and 17 grams of brown dye. Now, her hair is longer. She knows that the original mixture will not be enough to color all of her hair. She intends to add 7 grams of red dye to the original amount of red dye and some amount of brown dye to the original amount of brown dye. How much brown dye does she need to make sure that she has the original color?
In a group of seventh graders, the following two solutions were given:

a) The amount of brown dye Kelly needs to add to the original amount is 17*(22/15).
b) The amount of brown dye Kelly needs to add to the original amount is 22*(17/15).

Explain how each answer fits or does not fit the story? Depending on the choice of your answer, explain what (22/15) represents and/or what (17/15) represents

For the above problem, I asked Rita to explain what 22/15 represents and how she is allowed to multiply that by 17. For 22/15, Rita had already commented on dividing 22 by 15 and getting a number and then multiplying that with 17. When asked again to comment on what 22/15 represents, she offered:

> Rita: This, the representative do you want me to tell actually what it represents? It tells us I think that the result tells you what the new amount all right I am getting confused. It is telling us how it has changed from the first solution to the second solution. [Line 70]
> R: So it is like this one if that is 1 point say 5, then yeah almost, so this would tell you how much red the first one

> Rita: this is the how much you would, you would need all of the 15 and then half of 15 which would be almost 7 you know what I mean it is not like the exact decimal but to then get your new amount and to do that in the same proportion of the mixture you would have to then if this is true take the 17 to make and take one whole 17 and add another half of that to make sure that it is the exact same mixture [Line 72]…Yes, because that is the proportionate, they have to be equal to get the same color dye to multiply those [Line 84]

The excerpt above shows that Rita knows that 22/15 is the increase in one color and that the other color needs to have the same multiplicative increase. In other words, she knows that the ratio 22/15 represents the amount by which the first quantity in the original ratio has increased to reach the new quantity, so she can use her quotitive division scheme to determine the needed increase so that more of the same combination of 15 to 17 can be made. That is, she knows that 1 whole 15 is included in 22, and then 7 is left, and she thinks of that 7 as a part of 15. She also knows that the combination of 15 to 17 needs to be kept the same; that is how she knows that the other ingredient in the color, 17, is under the same change. The change in the amounts of the ratio of 15 to 17, as making more of the same combination, needs to be one whole and almost half for both quantities, so that the color is the same.

Later, when asked why she would use multiplication, she claimed:

> R: Why is multiplication there?
> Rita: Because you want to like when we think about decimal numbers and it is like a whole of parts I think, so we want to make sure that we are accounting for the 15 that we already had and we want to also add a part of that 15 which would be the decimal then so if you multiply it is just it is simplified way to do that instead of adding subtracting all that kind of [inaudible] terms [Line 76]…It [Student refers to 22/15] tells you how to increase whatever number you are working with originally [Line 78]

Again, the excerpt shows that multiplication for Rita means how much more of the old amount makes up the new amount. When she does not have a whole amount of the original quantity, then she can think of the rest of the amount, say 7, in terms of the parts of the original amount 15, [7/15]. That is, Rita has already abstracted the idea that the result of division between measures tells how many times the quantities in the original ratio situation makes up the quantities in the new situation.

The following problem, Recipe-1 Problem, was given to Rita during the written session and she solved it setting up the proportions. In the interview, I asked her if she could solve it using some other way:

Recipe-1 Problem: A recipe for salad dressing asks for 9 tablespoons of oil and 4 tablespoons of vinegar. If we want to make a batch that includes 7 tablespoons of vinegar, how much oil will we need to be consistent with the recipe? How do you know that the result makes sense? You can use diagrams, shapes, graphs etc in explaining your answer"

The following excerpt is important because Rita thinks of division between same measures. She claimed:

> Rita: Yeah, okay the original was oil and vinegar and then if we do 9 oil was the original and when we are moving to anything like the difference is 5 so then if we are going to do 7 then I guess how may times 4 goes into 7 would have to be not quite 2 and then also multiply that number by 5the 5 so then is then the new difference but the same proportion of both mixtures [Line 276].

The excerpt shows that Rita knows that dividing 7 by 4, using the quotitive division scheme, she will determine with which number to multiply those original quantities. She understands that not only the original quantities, but also their differences will be multiplied by the same number. How Rita is able to think about the division and why it is crucial is revealed in the following excerpts.

Then, I asked Rita how the number she had was helping with the difference and she provided the following: The excerpt is important because it shows how Rita thinks of the result of division between the same measures.

> Rita: It tells, because in the original mixture there was only 4 tablespoons of vinegar and there would be if this is not quite 2 then it would mean that this 1 point whatever is how much more we have what has become of the original which is now the new so it is 1 point or 1 and whatever fraction of it was before it is an addition amount which is what the fraction accounts for but what we we cant just thrown into 5 because that is not going to remain constant because we don't have 4 we have 7 now so we need to make sure that the amount by which you increase the 4 should also be the same percentage or amount or proportion that you would create then with oil [Line 278].

The above excerpt shows the amount of increase from 4 to 7 is the percentage or proportion for Rita. How that is the case is revealed in the following excerpt.

Then I gave Rita an approximate number to talk about since she kept saying 1 point something and closer to 2, and I asked her to tell what that number represented:

> R: Okay, because what does number tell you? [I gave her 1.9 as an approximate number] [Line 289]
> Rita: That number tells you how much, compared to this original this would be this new mixture if we had 1.9 here it would be a hundred and 90 percent of the whole mixture, meaning the whole mixture and plus another part and that is exactly we need to do the same for both so that it is the same mixture [Line 290].

The excerpt shows two important points: First, Rita is able to make sense of the result of division between the same measures such that the result tells how many groups of 4 are in 7. Thus, this shows that Rita is able to think about the multiplicative relationship between the same measures and for her "a between- ratio" represents the multiplicative change form one situation to the other. That is, the multiplicative change tells how much the first quantity in the original ratio needs to be increased to hit the new quantity. Second, for Rita, the result of division is the percentage by which both of the ingredients in the recipe are increased. The meaning of percentages is that percentage tells how much whole and part of the original mixture constitutes the new mixture. In other words, she not only knows the result of division, 1.9, is the amount by which each ingredient is increased, but also it is the amount by which the whole mixture is increased because each ingredient coming together constitutes the whole mixture. Again, she justifies that this is the way she will measure the quality of interest in the situation such that the same percentage, the same amount of increase, guarantees the preservation of the quality if interest.

Also, at the end of the interview, I asked her the Ice-Cream Problem, and she pursued finding the quotient obtained from dividing the quantities of between-ratios.

Ice-Cream Problem: In an ice-cream factory 12.7 liters of milk and 10.5 pounds of sugar is needed for one gallon of ice-cream. They want to make as much ice-cream as possible with 11.4 lbs of sugar. How much milk should they use to create a batch that tastes the same as their usual recipe?
Kelly solved the problem above and found the answer as 13.7 liters of milk. She then claimed that the ratio of 12.7 to 10.5 is the same as the ratio of 13.7 to 11.4. Then, she wrote:

12.7/10.5= 13.7/11.4

> Rita: What she did here she put the original milk over the original sugar try to finding out the new milk to the new sugar and because she is trying to use 11.4 pounds of sugar she is then she then took just I did before in the last couple of ones, if you divide 11.4 by original 10.5 you find out by how much the original has increased and then we can take by how much that is keep it consistent with whatever the number is of milk that we need to find out and again multiply that original number by that same multiplies to keep it consistent [Line 327].

Although this problem, presented within-ratios, Rita showed her preference for using between-ratios. Rita showed that the multiplicative relationship between-ratios guarantee the same quality of interest between more than one ratio situation. To keep the quality of interest the same, she needs to multiply both quantities in the original ratio by the same number so that the increase from one situation to the other is the same for both quantities.

So far the data has shown that Rita has abstracted the notion that a between ratio represents the multiplicative increase or decrease from one situation to the other. She understands that a between ratio tells how much the first quantity in the ratio needs to be increased or decreased to reach the targeted quantity. She knows that the multiplicative relationship between the quantities of the same measures from one ratio situation to the other brings about more of the original combination of the ratio. Therefore, she is able to deal with problems with divisibility failure.

Rita's Understanding of Within-Ratios

In this section, in light of the data, I show that Rita's understanding of within-ratios is limited and I compare her reasoning to an understanding of within-ratios multiplicatively.

In the written sessions, Rita thought about the Mixture Problem, and she chose "b" as an option, and disregarded option "a" as being plausible as a solution to the problem and made comments about the "c" option, which appears later. During the interview I asked her to comment on a follow up question so that her way of thinking about ratio would be much more explicit.

Mixture Problem: A group of fifth graders will mix two types of juice mixtures and will put them into jars so they can serve them to the kindergarten students during the week. In the first jar, they put 36 grams of lemon juice and 32 grams of lime juice. In the second jar, they put 20 grams of lemon juice and 16 grams of lime juice. To label the jars, the fifth graders wanted to use one number to accurately represent the lemon-lime flavor in that jar. They considered the following three ideas for labeling the jars:

a) There is 4 more lemons in the first mixture so put 4 on the first label. There are 5 more lemons in the second mixture, so put 5 on the second label.
b) The ratio of 36/32 for the first type of mixture and the ratio of 20/15 for the second type of mixture.
c) 1.125 (the result of dividing 36 by 32) and 1.33 (the result of dividing 20 by 15).

Which idea(s) would accurately indicate the lemon-lime flavor? Why? Provide an explanation for the reason (s) that each of these options would or would not indicate the mixture's lemon-lime flavor.

When I asked Rita if she would choose 9 to 8 ratio and 4 to 3 ratio for the mixtures respectively instead of the original ratios given in option "b" and why, she said "because it is a simplified ratio like if you divide 36 by 4 and you are going to get 9 and if you divide 32 by 4 you would get 8 so in that way even lesser amount it is that is the most simplified you can need" [Line 146]. Then I asked her to reason without using the idea that she can divide 36 and 32 by 4 and get 9 and 8 respectively. I asked her how she would know that the ratio of 36 to 32 would be the same as the ratio of 9 to 8. She claimed the following excerpt below, which is important because it shows how Rita makes sense of ratio in the mixture problem.

> Rita: I guess for every so many parts on top there should be then so many other different parts like with the lemon and lime so you need to know [inaudible minute 39] how much you are making of it how many in relation to one another so there should always be 1 less of the second I guess with this would be lime always be 1 less lime use for every I am just using numbers for every 8 limes use you would need another lemon but then you could duplicate that again and say oohh I need if I am going to use 16 of limes then I would need an 18 , because it would be the same, it would be two , 2 of those 2 of

> the 9 over 8 ratio mixtures and you would have to account for then because it was 2 mixtures so you would have to have 2 less limes [Line 154].

The above excerpt shows that Rita knows that the within ratio, 9 lemons to 8 limes, is made up of two different quantities associated to each other, such that, the association creates a unique relationship between those quantities. That is, she thinks of the quantities in ratio as parts associated to each other in such a way that they differ from each other by 1 part in each specific combination. When Rita thinks of doubling, she thinks of making more of the same mixture. This means that she thinks that the association between 9 and 8 makes a combination, and when she makes more of the same mixture she makes identical copies of the same combination. Similarly, when she thinks of the combination of the 9 to 8 ratio as a lesser amount of the combinations of 36 to 32, through her quotitive division scheme, she thinks of breaking those larger amounts into more groups identical to each other, such as four groups of the combination of 9 to 8. This means that she thinks of those quantities as two related amounts rather than the ratio indicating the relationship between those quantities, i.e. ratio as an intensive quantity. She also knows that when she makes more or less amounts of the same combination, the difference between the quantities of 9 lemons and 8 limes needs to be under the same change since each combination differs from each other by 1 part and that association between those two quantities creates the particular taste.

One point is worth emphasizing: In this problem, Rita is able to think about the quantities in the original within ratio in such a way that those quantities make up a group that defines a particular quality of interest. Those two quantities that make up the particular group are thought as parts. Her underlying conception seems to be an identical groups conception. With caution, though, this does not mean that she is not capable of thinking about those quantities in a multiplicative way, as an intensive quantity. More evidence arises in the future as to whether she can or cannot make sense of a within ratio as an intensive quantity.

For the Mixture Problem during the written sessions, Rita commented on option "c", which appears below as well. When I asked her about the same option during the interview, Rita's reasoning is important in terms of showing that she does not have meaning for the result of division within-ratios.

During the interview, Rita claimed the following for option "c":

> Rita : I don't think that will tell you anything because again it is not whole to part and it is part to part and that does not really tell us anything about the mixture in itself because if you are trying to figure out how much of the entire mixture it is kind of just for lemon and or just lime and so if you divide like how much there is lemon by how much lime you are not you don't have a whole to measure that against you know what I am saying it is like you are dividing into each other instead of like it should be divided from the whole and I don't think that is a useful number [Line 174].

The excerpt shows, that for Rita, the result of the division of the parts in the original ratio by each other does not tell anything about the mixture. The only ratios that are meaningful to her for one mixture are part-to-whole ratios.

Also, in the written part Rita claimed the following for option "c":

> " Idea C is confusing with fractions being converted to decimals. What do we do w/ that decimal when we need to make more of that mixture?"

What this shows is that Rita does not know that 1.125 or 1.33 represents all the mixtures of the same taste, but also, she does not know that 1.125 is the amount of lemon for every lime. Had she known this, she would have not said that she would not know what to do with that number if she had to remake more of the same mixture.

Also, the "b" option of the Hair Color-2 Problem, gave a window on Rita's understanding of within-ratios.

Hair Color-2 Problem: Kelly likes to color her hair with a mixture of red and brown dye. When her hair was shorter; she used 15 grams of red and 17 grams of brown dye. Now, her hair is longer. She knows that the original mixture will not be enough to color all of her hair. She intends to add 7 grams of red dye to the original amount of red dye and some amount of brown dye to the original amount of brown dye. How much brown dye does she need to make sure that she has the original color?
In a group of seventh graders, the following two solutions were given:

c) The amount of brown dye Kelly needs to add to the original amount is 17*(22/15).
d) The amount of brown dye Kelly needs to add to the original amount is 22*(17/15).

Explain how each answer fits or does not fit the story? Depending on the choice of your answer, explain what (22/15) represents and/or what (17/15) represents

> Rita: I don't think that because 17 over 15 I don't think is a useful number I don't think that like although I I don't know I just was kind of work out in my head and trying to force it to work but I don't think that if you divide that it will give you a number that you should use for anything [Line 102]… I think that it is like a ratio 17 to 15, but I don't think that it will give you any good because that is not necessarily finding out how much concentration of what color is in it it just figures comparison [Line 104].

For Rita, the within- ratio 17/15 has no meaning, and she does not understand that the result of division of 17 by 15 represents all the mixtures of the same color.

The following excerpt shows that for two mixtures to taste the same or not, Rita does not have any other way than using the proportions between the same measures.

Ice-cream Problem: In an ice-cream factory 12.7 liters of milk and 10.5 pounds of sugar is needed for one gallon of ice-cream. They want to make as much ice-cream as possible with 11.4 lbs of sugar. How much milk should they use to create a batch that tastes the same as their usual recipe?

Kelly solved the problem above and found the answer as 13.7 liters of milk. She then claimed that the ratio of 12.7 to 10.5 is the same as the ratio of 13.7 to 11.4. Then, she wrote:

12.7/10.5= 13.7/11.4

Rita already explained how she would know that Kelly was right. She claimed that: "…she then took just I did before in the last couple of ones, if you divide 11.4 by original 10.5 you find out by how much the original has increased and then we can take by how much that is keep it consistent with whatever the number is of milk that we need to find out and again multiply that original number by that same multiplies to keep it consistent" [Line 327].Then, when I asked if there was any way that she would know other than what she was saying that 12.7 /10.5 and 13/7/11.4 are the same ratios, she claimed:

> R: All right, what about this, you are talking here, but what about this 12. 7 and 10.5 and then 13. 7 to 11. 4 you can say anything about these things, [referring to 12.7/10.5 and 13.7/11.4, the researcher wrote those ratios separately on the paper] instead of using these can you talk about like it [Line 334].
> Rita: Was a whole [Line 335].
> R: Yeah, what about this is keeping the same here, or why do I know that these [Researcher referring to 12.7/10.5 and 13.7/11.4] are the same [Line 336].
> Rita: Without doing part to part I guess if you were to [Line 337].
> R: Yeah without doing sugar to sugar milk to milk, if just you are given this [12.7/10.5] and you are given that [13.7/11.4], how would you know that this might be the same or is the same as this one [Line 338].
> **Rita: If, it is all going to back part to part for me I don't know anything to do any other way** [Line 339].
> R: Can you, you can say part to part thinking [Line 340].
> Rita: How do I know that this is the same as this, both both parts should be multiplied and increased or whatever by the same amount so [Line 341].

The excerpt shows that for Rita to know if two mixtures are the same or not is only through the use of multiplying both parts by the same amount so that the percentages are kept the same, other than that she does not have any other way to show if those mixtures are the same or not. That is, she cannot use the division within-ratios to quantify the taste of the ice-cream and to show that those ice-cream mixtures are the same.

Thus, the data above shows that Rita is not able to make sense of ratio situations multiplicatively in terms of two quantities from different measures spaces. In other words, she does not use partitive division to share the amount of the first quantity in the amount of the second quantity, as in per-one; nor does she know that the within -ratios represents the size of one quantity relative to the size of the other quantity. Thus, she lacks the components for understanding of within-ratios multiplicatively [i.e. the quotient obtained from dividing the first quantity in the ratio by the second quantity is the multiplicative relationship between those quantities and quantifies the quality of interest]

Rita's Understanding of Part Whole Relationship in the Context of Ratio

In this section, I present data on what Rita understands about part whole relationship in ratio situations.

In the Flour-Oil-Sugar Problem, Rita revealed information of how she thinks of part- whole relationships and how she uses percentages as a way to justify whether two mixtures have the same taste or not. The following excerpts are important because Rita, when asked to show if two or more ratio situations have the same quality of interest or not, thinks of part-whole relationships and decimal notation, percentages, of part whole. Also, her way of justifying whether two or more mixtures have the same quality of interest is important because her reasoning provides insight into how she makes sense of ratio situations.

Flour-Oil-Sugar Problem: A mixture of 46 cups is made from flour, oil, and sugar with the proportions of 2.1, 1.3, and 2.5 respectively. How much oil is used for this mixture? Explain your answer.
For this problem, in the written part Rita had written the following:

> "2.1 + 1.3 + 2.5=5.9
>
> 46* $\frac{2.1}{5.9}$ =16.37 cups of flour
>
> 46* $\frac{1.3}{5.9}$ = 10.14 cups of oil
>
> 46* $\frac{2.5}{5.9}$ = 9.49 cups of sugar should be used according to the total mixture amt. & the proportions of each ingridient

During the interview I asked Rita to explain what $\frac{2.1}{5.9}$ represented in her written solution:

> Rita: I think that this number [Student refers to] $\frac{2.1}{5.9}$ tells you the proportion of how much of ones flour was in the total like part over whole would be how much of the how much of the new mixture should remain does not matter what volume you are making of it [Line 180].
> Then later, … That [Student refers to $\frac{2.1}{5.9}$] is a percentage of how much of that item is in the whole so for example like in my cake I would like if is it is like 20 percent if I like divided this and it was like 0.2 I would know that it would be 20 percent of the 5, the original cake [minute 54. 44] from the 2.1 cups of whatever the cake is made and then we know that from the from that original proportion from the original cake we need to then make a cake that is I guess larger in I don't know my head that needs to be 46 cups instead of the original small 5.9 [Line 210].

Both excerpts above show that Rita thinks of $\frac{2.1}{5.9}$ as the representation of the relationship between one of the quantities and the whole of which that quantity is a part. The relationship is such that Rita knows that the decimal notation of $\frac{2.1}{5.9}$ indicates what portion the first quantity, the amount of flour given in the numerator, is of the whole mixture; the amount is given in the denominator. Rita knows that the fraction representation, $\frac{2.1}{5.9}$, is a measure of the portion the first quantity is of the whole mixture. This is a measure for her because she knows that as long as the relationship between the quantities that make up the original mixture and the whole original mixture is kept the same, the quality of interest, the taste of the mixture is kept the same. Therefore, she thinks of percentages, the decimal notation of part to whole, as a measure of the quality of interest because percentage or fraction is a way to evaluate the quality of interest.

Then, later, when I asked how the percentages were kept the same, Rita explained:

> Rita: Because we are keeping the like the we are multiplying by the same amount each ingredients would be multiplied by the same amount so that there are new amount s of that ingredient but in the same proportion as in the form [Line 244].
> Then later ...I know that that would preserve the percentages because we are just changing the amount we are not changing we don't want to change the composition, and so again like This 5.9 is still made up of these three things, but then if I am changing the amount again it should be made up of those three things in the same amount just a large amount of each one [Line 268].

The above excerpt shows that Rita knows that multiplication makes the composition enlarge by the same amount into a greater mixture. She knows that multiplication preserves the composition of the quantities in the original mixture to the greater mixture since it increases all the original quantities by the same amount. This then results in the fact that the original whole is increased by the same amount into a greater mixture. Therefore, she understands the invariant relationship among different mixtures as a fraction of the whole. That is, she understands that the fraction of the whole, the original mixture, is the same as the fraction of any whole. She can use her understanding of equivalent fractions as representing the same fractional part. The use of equivalence of fractions makes sense to her because equivalent fractions allow her to take care of two things: a large mixture or a smaller mixture [amounts change] and also the same fractional part [the part-whole relationship stays the same]. Particularly, she knows that if she increases or decreases the amounts in the original ratio, made up of two parts, she knows that the part whole relationship will stay the same. Also, if she changes the part whole relationship, she knows that the part to part relationship changes too. Therefore, Rita knows, not only that percentages being the same from one mixture to the other keeps the quality of interest the same, but also that she can justify how that is possible.

In the written session, Rita also solved the Hair-Color-1 Problem, the following way: This is important because it shows again how Rita makes uses of percentages and what percentages mean for her when the percentages are not the same from one mixture to the other.

> Hair Color-1 Problem: Kelly likes to color her hair with a mixture of red and brown dyes. When her hair was shorter, she used 4 grams of red and 5 grams of brown dye. Now, her hair is longer. She knows that the original mixture of 4 grams of red and 5 grams of brown will not be enough to color all of her hair. So she thinks that if she adds 1 gram of each color to the original mixture, she will preserve the color. Will new mixture be the same color as the original? Explain your reasoning.

She wrote on her paper, see Figure 1,(arrows hers).

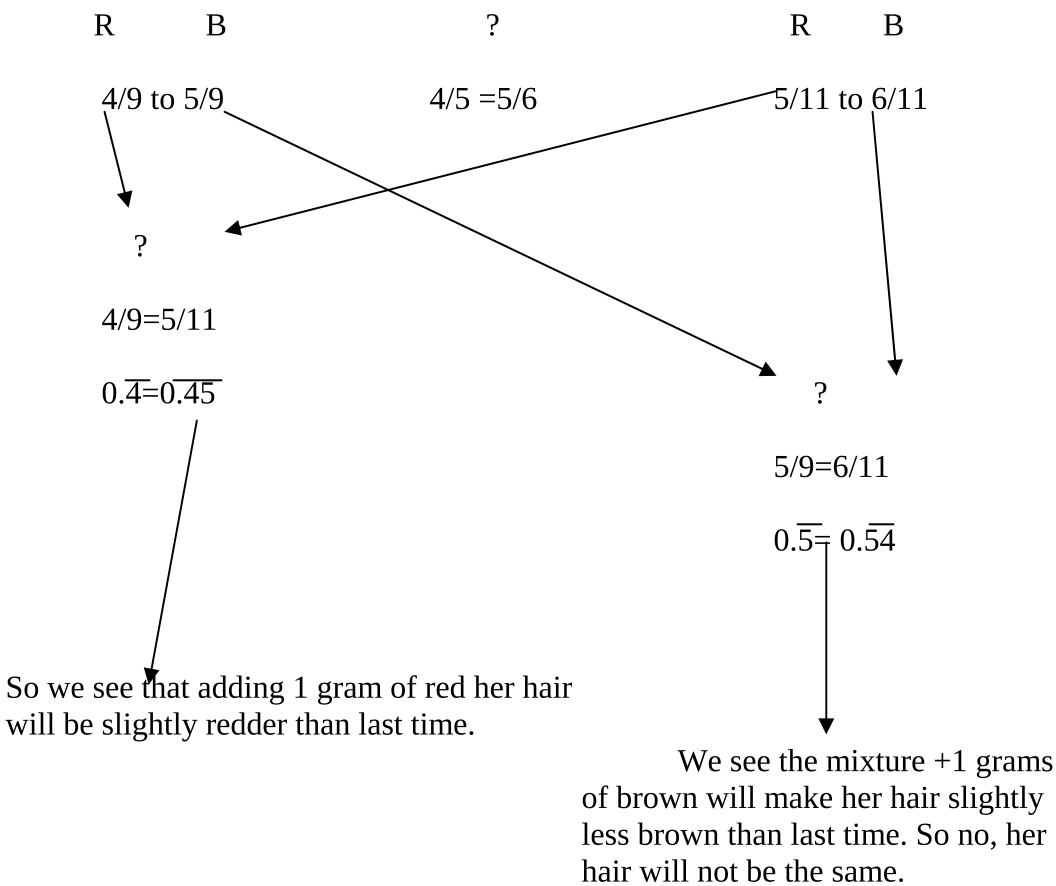

Figure1. Rita’s Reasoning on Hair-Color-1 Problem

This also shows that for Rita to evaluate if the two mixtures are the same or not, she will make the original ratio of 4/5 into another ratio such as a new ratio, 4/9 which represents the portion that the first quantity in the ratio is of the whole. The decimals not being the same means that the mixtures are different since they represent different portion sizes between the first quantity and the whole. In other words, she knows that when part-whole relationship is not the same, the part-to-part relationship that creates a particular quality of interest is not the same either.

Summary of Rita's Conceptions

In sum, the data shows that, given a ratio situation, Rita has *only* the multiplicative relationship between the quantities coming from the same measure space. On the other hand, she cannot reason about within-ratios multiplicatively. Also, Rita has abstracted the invariant relationship among different ratio situations as fraction of the whole.

Particularly, Rita's understanding of the multiplicative relationship between-ratios indicates that through her quotitive division scheme, she has abstracted the notion that a between ratio represents the multiplicative increase or decrease from one situation to another. Thus, she is able to deal with problems with divisibility failure. On the other hand, Rita's understanding of within ratio shows the characteristics of identical groups conception such that within ratio represents two extensive quantities that are associated to create a particular quality of interest. Since Rita is thinking of the association between the quantities in the original ratio by a difference of so many parts, her considering the quantities in the original ratio, the quantities within ratio, is shaped in such a way that she realizes that the multiplicative relationship between-ratios needs to apply to the difference between the quantities in the original ratio in each new situation to keep the quality of interest. Nevertheless, Rita is not able to think about the result of division of the quantities within- ratio. That is, Rita is not able to make sense of ratio situations multiplicatively in terms of two quantities coming from different measures spaces. In other words, she does not use partitive division to share the amount of first quantity in the amount of the second quantity, as in per-one; nor does she know the within ratio represents the size of one quantity relative to the size of the other quantity. Finally, as seen in several examples, she can think about the invariance of a situation in terms of part-whole relationships. For instance, she is able to think about the result of division of the first quantity by the whole [within the original ratio situation] and also the result of division between-ratios as percentages, representing how much whole and part of the original mixture makes up the new mixture; and, that is the invariant relationship, multiplicative relationship, between not only two mixtures but also the quantities of the same measure space from different ratio situations.

Mark's Understanding of Ratio

Each section related to Mark's concept of ratio is labeled either "Beginning of Interview" or "End of Interview". Beginning of Interview refers to before Mark reasons about the Mosaic Problem, and End of Interview refers to Mark's reasoning about the Mosaic Problem and after.

Beginning of Interview: Mark's Understanding of Within-ratios

In the written sessions, the Hair Color-2 Problem was given to Mark. He had solved the problem instead of explaining how the answers fit the story and what the ratios represented. At the beginning of the interview, I asked Mark to account for the solutions provided to him.

Hair Color-2 Problem: Kelly likes to color her hair with a mixture of red and brown dye. When her hair was shorter, she used 15 grams of red and 17 grams of brown dye. Now, her hair is longer. She knows that the original mixture will not be enough to color all of her hair. She intends to add 7 grams of red dye to the original amount of red dye and some amount of brown dye to the original amount of brown dye. How much brown dye does she need to make sure that she has the original color?

In a group of seventh graders, the following two solutions were given:

a) The amount of brown dye Kelly needs to add to the original amount is 17*(22/15).

b) The amount of brown dye Kelly needs to add to the original amount is 22* (17/15).

Explain how each answer fits or does not fit the story? Depending on the choice of your answer, explain what (22/15) represents and/or what (17/15) represents.

Mark explained what 17/15 ratio represents and he said "So, for every 15 grams of red dye, she needs 17 grams of brown dye" [Line 57]. Then, I asked him to comment on that more:

> R: Can you show to me you can put into mathematical terms or, is there a way you can show me for every 15 grams of red there will be 17 grams of brown dye [Line 75]?
>
> ……
>
> M: For every gram of red dye it is going to be 1.15 for brown [Line 78].

The excerpt shows that Mark knows that the ratio 17/15 represents the relationship between red and brown dye, that is, he interprets that ratio as "for every 15 grams of red, there would be 17 grams of brown." He also offered that the result of the division tells how many grams of red dye are needed for 1 gram of brown dye.

Then I asked him to show 1.15, the result of dividing 17 by 15, using any kind of graphs, diagrams or shapes, he drew two bars and said:

> M: This for every 1 gram of red there is going to be 1 a little bit more brown [He made two bars, one of them is for R and the other is for B and Brown is bigger than the Red one] [Line 80]

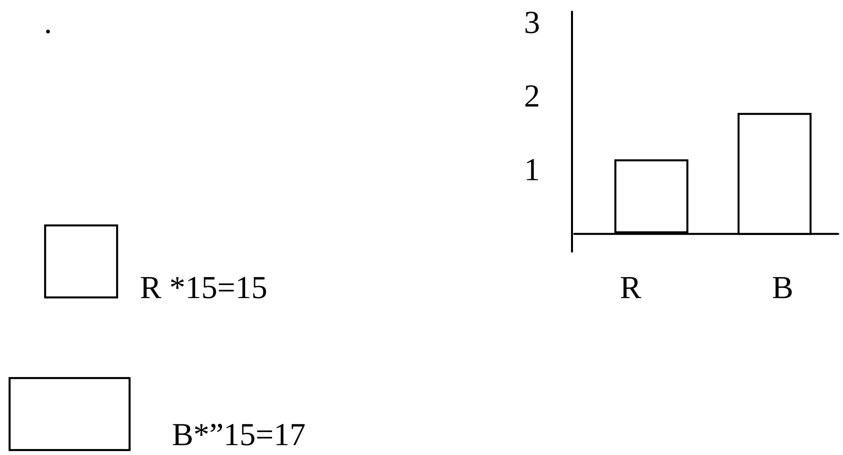

Figure 2. Mark’s Reasoning on Within-Ratios at the Beginning of the Interview.

The excerpt shows that Mark knows that 17 grams of brown and 15 grams of red are related to each other such that when he partitioned 17 grams of red into 15 equal parts, the result would be “for every gram of red, there will be 1.15 grams of brown” and he can represent the result of division as two different bars because he is thinking that the result of division, 1.15, represents two quantities: 1 gram or red and 1.15 grams of brown. In this, Mark is engaged in approaching the Hair Color-2 Problem as a per-one strategy [unit-factor approach].

The way Mark is thinking does not mean that Mark is *only* capable of thinking about ratio as representing two separate quantities. That is, it is important to restate that students at higher- level understandings might choose to use less sophisticated understandings to communicate their thinking. So, the fact that Mark’s way of reasoning on Hair Color-2 Problem resembles unit- factor approach [with a graphic representation of two extensive quantities] (two bars) does not indicate that he is only capable of thinking about within-ratios as two separate quantities.

Then he commented on what he wrote above as R*15=15 and B*15=17.

> M: Right, yeah, and then there is 15 of those whatever you need for red, and then 15 of those will be what you need for brown, or if for brown that is bigger, so you need 15 here and 17 of there [Line 86].

The excerpt also emphasizes the fact that Mark's use of partitioned values, 1 gram of red and 1.15 grams of brown, seem to be based on a covariation of two quantities and how the1 gram of red and 1.15 grams of brown ratio has the same quality of interest as the original quantities. In other words, the excerpt shows that Mark knows that the two original quantities of 15 grams of red dye and 17 grams of brown dye can be partitioned into 15 equal parts such that the result of the partition is 1 gram for red and 1.15 grams for brown. He knows that he can accumulate those 1 grams to make up the original 15 and at the same time he has to accumulate 15 portions of 1.15 grams of brown because of the specific combination they make up, i.e. the original color. Thus he knows that "1.15 grams of brown dye and 1 gram of red dye" has the same color as "17 grams of brown dye and 15 grams of red dye." Another important point is that, Mark is able to think about how many times both quantities need to be iterated. In other words, he knows that since 1 gram of red dye becomes 15 grams of red dye at the end and because15 is 15 times bigger than 1, and he knows that he needs to iterate both quantities at the same time; he knows that the other quantity needs to be iterated the same number of times.

Also, in another problem, the Mixture Problem, Mark showed that he was thinking about the ratios in terms of per-one [unit factor]. For that problem Mark had chosen both "b" and "c" options during the written sessions.

Mixture problem: Problem: “A group of fifth graders will mix two types of juice mixtures and will put them into jars so they can serve them to the kindergarten students during the week.

In the first jar, they put 36 grams of lemon juice and 32 grams of lime juice. In the second jar, they put 20 grams of lemon juice and 16 grams of lime juice. To label the jars, the fifth graders wanted to use one number to accurately represent the lemon-lime flavor in that jar. They considered the following three ideas for labeling the jars:

a) There is 4 more lemons in the first mixture so put 4 on the first label. There are 5 more lemons in the second mixture, so put 5 on the second label.

b) the ratio of 36/32 for the first type of mixture and the ratio of 20/15 for the second type of mixture.

c) 1.125 (the result of dividing 36 by 32) and 1.33 (the result of dividing 20 by 15).

Which idea(s) would accurately indicate the lemon-lime flavor? Why? Provide an explanation for the reason (s) that each of these options would or would not indicate the mixture’s lemon-lime flavor.”

Mark was asked to comment on the c option for the problem:

> R: Okay, how about those, you said also c, first of all how 36 divided 32 and 1.125 are the same, in the context of the problem, in terms of ratio, not that I can divide that the result is that because I already gave you that [Line 139].
> M: That is also another ratio, because that is like if you take 36 to 32, 1.125 means for every 1 gram, 1 ounce or whatever of lime juice there will be 1.125 lemon juice in the mixture [Line 140]
> Then later... 1 and 1.125 so whenever you have like 2, 2 times 1.125, 3 will be 3 times 1.125, four whatever and when you get to 32 then it will be 32 times 1.125 [Line 148].

The excerpt above shows that in the Mixture Problem, Mark thinks of the result of division within measures as two quantities such that the result is “for so many parts of one of the quantities, there is 1 part of another quantity,” and he would be able to iterate those quantities. The way Mark reasons the iteration is that 2- times or 3-times the combination of “1 gram of lime to 1.125 grams of lemon” represents a different number of groups of that combination. That is, Mark knows that he can create different volumes of the same mixture using the result of division and iterating it more times.

Beginning of Interview: Mark's Understanding of Between-Ratios

The following data shows Mark's limitation on the Hair Color-2 Problem at the beginning of the interview.

I asked Mark to reason about option "a". He tried to establish the proportions to solve the problem like he did in the written session. Then, I asked him to think about the specific parts of the question, such as the meaning of the ratio, 22/15.

Hair Color-2 Problem: Kelly likes to color her hair with a mixture of red and brown dye. When her hair was shorter, she used 15 grams of red and 17 grams of brown dye. Now, her hair is longer. She knows that the original mixture will not be enough to color all of her hair. She intends to add 7 grams of red dye to the original amount of red dye and some amount of brown dye to the original amount of brown dye. How much brown dye does she need to make sure that she has the original color?
In a group of seventh graders, the following two solutions were given:

e) The amount of brown dye Kelly needs to add to the original amount is 17*(22/15).
f) The amount of brown dye Kelly needs to add to the original amount is 22* (17/15).

Explain how each answer fits or does not fit the story? Depending on the choice of your answer, explain what (22/15) represents and/or what (17/15) represents.

The importance of the excerpt below is that, Mark shows that he does not know the meaning of the ratio [22/15].

> R: You are doing again setting up the proportion, okay I see, that is fine too, first of all, if you can comment on that number, what that 22 divided 15 represent [Line 69]?
> M: 22 divided by 15 will give you the number, I don't know that one, I know for b, like 17 over 15 will give you brown to red, the ratio of brown to red [Line 70].

So, the above excerpt shows that although Mark is able to think through within-ratios flexibly, he is not able to make sense of “a between ratio.” He does not have a way to think about the ratio [22/15] and he cannot think about the result of division 22 by 15.

Beginning of Interview: Mark’s Use of Part Whole Relationship in the Context of Ratio

For the Mixture Problem, I asked Mark to comment on whether he could use the 9 to 8 ratio instead of the 36 to 32 ratio.

Mixture problem: Problem: “A group of fifth graders will mix two types of juice mixtures and will put them into jars so they can serve them to the kindergarten students during the week.

In the first jar, they put 36 grams of lemon juice and 32 grams of lime juice. In the second jar, they put 20 grams of lemon juice and 16 grams of lime juice. To label the jars, the fifth graders wanted to use one number to accurately represent the lemon-lime flavor in that jar. They considered the following three ideas for labeling the jars:

a) There is 4 more lemons in the first mixture so put 4 on the first label. There are 5 more lemons in the second mixture, so put 5 on the second label.

b) the ratio of 36/32 for the first type of mixture and the ratio of 20/15 for the second type of mixture.

c) 1.125 (the result of dividing 36 by 32) and 1.33 (the result of dividing 20 by 15).

Which idea(s) would accurately indicate the lemon-lime flavor? Why? Provide an explanation for the reason (s) that each of these options would or would not indicate the mixture’s lemon-lime flavor.”

He said “It will be the same because it will be the same ratio… 36 to 32, 36 to 32 is the same as 9 to 8… you divide it by 4” [Lines 124, 126, 128]. The fact that Mark realizes that 9 to 8 and 36 to 32 is the same ratio indicates that Mark is thinking about those ratios representing the same mixture. This might mean two things: First, Mark might be aware of the fact that the invariant relationship between those different mixtures is a ratio within the particular problem. That is, he might be aware of the invariance for just the mixture problem such that different representations of 9 to 8, 18 to 16 etc. [within the boundaries of 9 to 8 and 36 to 32] refer to the same mixture. Secondly, Mark might be aware of the fact that the invariance relationship, ratio, is the representation of all the mixtures outside the boundaries of the mixture problem. In other words, he might know that ratio, the quotient within-ratios, is the multiplicative relationship between the quantities from two different measure spaces, and it is the quantification of the taste.

When I asked if he could explain how that was possible, he said the following which shows that Mark relates the invariance of within-ratios to the invariance of the percentages or part-whole relationships to show that those ratios are the same. Thus, the part to whole ratio is a way to evaluate that two mixtures have the same concentration:

> R: How do you know that you can divide by 4, not that 36 is divisible by 4 and 32 is divisible by 4, I am not interested in that, how do you know hat you are allowed to divide by 4, what is it the same between them [Line 129].
> M: The percentages [Line 130].
> R: How do you know that percentages is the same [Line 131]
> M: Because if you make this one it will be the same thing, 9 over 17 and 8 over 17, it should be the same [checks with the calculator] yeah, [Line 132].
> R: How do you know like [Line 133].
> M: I mean it is divisible by 4 but also I just think that by the ratio then 36 to 32 will be the same as like 27 to 24 , 18 to 16 and 9 to 8 are all the same because it is a just form of simplifying , because they are all the same it will be always be 53 percent and 47 percent [Line 134]. [Mark divided the first quantities with the whole amount of mixture in the calculator]

The above excerpt shows that Mark knows that the ratio of 36 to 32 is the same as 27 to 24, 18 to 16 and 9 to 8, which means that whenever he reduces the combination of 9 lemons to 8 limes from the original 36 lemons to 32 limes, the relationship, for so many parts of lemon there is this many parts of lime, does not change, so the taste does not change. When he claims that “Because they are all the same it will be always 53 percent and 47 percent," he knows that part-whole relationship is invariant from one situation to the other.

In sum, so far, Mark shows that he is able to make sense of ratio, within-ratios, as the representation of the relationship between two quantities coming from different measure spaces. Mark is able to use his partitive division scheme to equally share the first quantity in the ratio among each unit of the second quantity. He knows that the result of division of the first quantity by the second quantity in within-ratios shows the relationship "for each part of one quantity, there are so many parts of another quantity." Also, he is able to increase the quantities obtained from the result of division, per-one quantities, until he reaches the original quantities in the ratio situation. In that sense, the strategy, so far, that Mark uses to make sense of the tasks/problems resembles per-one strategy [unit-factor approach]. Also, Mark realizes the invariance at least among different within-ratios in the given situation; that is, given simplified ratio representation, he knows that that represents the same mixture as the original ratio. In addition, Mark realizes that the part-whole relationship is the invariant relationship among different situations such that the decimal notation indicates what portion the first quantity is of the whole. On the other hand, Mark does not have any meaning for between-ratios. In other words, he cannot think about the result of division between-ratios.

End of Interview: Mark's Understanding of Between-Ratios: Case of the Mosaic Problem

Later in the interview, and in contrast to the Hair Color-2 Problem, Mark was able to reason about between-ratios in the Mosaic Problem.

The Mosaic Problem: Two sisters, Shari and Melissa, decided to create a mosaic design on their table top from broken pieces of tile. They worked for sometime, and then they had a break. When they had the break they realized that, it took 16 minutes to finish an area of 40 cm square. If they worked at the same rate, how much of the table top could they finish in 36 minutes?

Can you provide as many solutions as to this problem? Explain how your solutions make sense in terms of the problem context.

Somebody solved the problem as follows:
They divided 36 by 16, 36/16=2.25, and multiplied that by 40, so 2.25*40=90

Can you account for the solution? What does 2.25 represent? Why does it make sense to multiply it with 40?"

Mark was asked to comment on the meaning of 2.25, and he said:

> M: It is is just minutes 2.25 represent how much longer they worked, not how much longer, 2.25 will be, they work that much harder, it is , they are both in minutes they worked for 2.25 times as long, there you go, so using use that way 36 to 16 means they worked, so there is 16 minutes, one time they worked there is 2.25 times as long as they worked , so it means like 2.25 longer so therefore if they did it 40 originally , they did it 40 times that , because they are working at the same rate will cover that much work for [Line 188].

The above shows that Mark knows that the result of the division of the ratio 36/16 represents the change in the amount of time. He knows that to keep the quality of interest "rate" the same, he has to increase the other quantity, which is the work done in so many minutes, with the same amount, too. This means that Mark understands that the amount by which one set of the quantities of the same type is increased; the other set of quantities of the same type has to be increased by the same amount, too.

Demands of the Mosaic Problem as Compared to the Hair Color-2 Problem

The Mosaic Problem was set as 36/16=2.25 and 2.25 *40=90 square cm. Thus, that was the whole solution, and it was given to Mark. In other words, this is a missing value problem and the solution is given. When one has the understanding of quotitive division and also knows that one needs to multiply or divide the quantities in the ratio with the same number, then one can make sense of the solution. Thus, the fact that the whole solution was given, and Mark already knew that 36 and 16 were coming from the same units, minutes, he used quotitive division. That is, he was able to call on the activity of how many times 16 goes into 36, since the quantities in the question were coming from the same measure space. Quotitive division in the context of ratio made sense to Mark because he knew that both quantities in the ratio need to be multiplied with or divided by the same number. In other words, Mark knew that when he multiplied the original quantities in the ratio with the same number or when divided by the same number he was maintaining the relationship between the quantities.

On the other hand, Mark did not have a way to think about the ratio [22/15] in the Hair Color-2 Problem prior to his work on the Mosaic Problem. One way to understand this is that he did not recognize 17*22/15 as a solution to a missing value problem. He did not interpret the expression as saying divide 15 into 22 to get the number by which 17 must be enlarged. This suggests that he did not have an abstract understanding of the ratio 22/15, an understanding of that ratio as the change factor. Only when he was considering the solution of a missing value problem (the Mosaic Problem) was he able to think about the meaning of the between- ratios.

Tzur & Simon (2004) in their postulation of *participatory* and *anticipatory* stages of understanding provided a way to make sense of Mark's different responses to the two problems. At the participatory stage of understanding, the student needs to be able to be thinking about or be involved in the activity through which the understanding developed. For Mark, he was able to solve the Mosaic Problem because it cued him to his activity of solving missing- value problems by multiplying each quantity by the same number. At the anticipatory stage, the student no longer requires a focus on the original activity. He is able to call on the understanding when needed. Thus, for Mark, the Hair Color-2 required an anticipatory level of understanding, because it did not cue him for the activity of solving missing-value problems by multiplying each quantity by the same number.

End of Interview: Mark's Understanding of Between-ratios: Case of the Hair Color-2 Problem

After he engaged in the Mosaic Problem, Mark wanted to go back to the Hair Color- 2.

> M: Yeah. So the 22 over 15 is how much times more red dye they put in so they need 17 by the same, how much more times there is, so 17 times however much more times they have for red will be how much you need for brown dye…[Line 276].

The excerpt above shows that Mark now recognizes that 22/15 is the change factor. This means that, because of the Mosaic Problem, Mark was able to recognize 17*22/15 as a solution to a missing value problem. He, then, was able to think about 22/15 as how many times 15 go into 22. Since he knew already that he needs to multiply both quantities in the ratio with the same number, he was able to make sense that 17 needs to be enlarged by the same number, too.

Either Mark's understanding advanced, as a result of his work on the Mosaic Problem, from a participatory to an anticipatory understanding, or the Mosaic Problem cued him for thinking about the activity of solving missing value problems by multiplying each quantity by the same number, thus allowing Mark to solve the Hair Color-2 Problem with only a participatory understanding.

End of the Interview: Mark's Reasoning on the Mosaic Problem

This section provides data on Mark's reasoning on the Mosaic Problem with out analysis. Then, in the following sections, I will present the analysis. The reason for providing the whole data is to show his original line of reasoning and the way the interview progressed based on probing.

For the Mosaic Problem, I asked Mark to solve the problem in a different way than he did in the written sessions, that is, without setting up proportions.

The Mosaic Problem: Two sisters, Shari and Melissa, decided to create a mosaic design on their table top from broken pieces of tile. They worked for sometime, and then they had a break. When they had the break they realized that, it took 16 minutes to finish an area of 40 cm square. If they worked at the same rate, how much of the table top could they finish in 36 minutes?
Can you provide as many solutions as to this problem? Explain how your solutions make sense in terms of the problem context.

> R: …All right, this was for you, first of all you solved it using proportions, can you solve it without using proportions, if you cannot that is fine too [Line 175]
> M: without the proportion I don't know, it would take them all right if it took them 16 minutes to do 40 cm that means they are working at a rate of 2.5 cm per minute because yeah 40 cm over 16 minutes it will be 2.5 cm per minute so it is the same rate, how much it is going to be for 36 minutes, if you are going to work 2. 5 cm per minutes, that is just 2.5 times 36, it will tell you how much of mosaic which is [Line 176].
> R: Why you are allowed to multiply 2.5 by 36?

> M: Because it is 2.5 cm per minute, if 40 over 36 minutes was 2.5 cm per minute multiplying that 36 minutes, is technically over 1, the minutes will cancel, so 36 times 2.5 will be 90 cm because minutes will cancel [Line 178]… Because you know, well if you know that they can do 2.5 cm for every almost every minute, if you want to see how much they are doing in 36 minutes, you just take 36 times 2.5 cm…the ratio is 1 minute is 2.5 cm , so 36 minutes 90 cm [Line 184].

Then, I gave the solution to the task [36/16=2.25, 2.25*40=90] and asked Mark to explain whether the solution made sense to him or not.

> M: it is is just minutes 2.25 represent how much longer they worked, not how much longer, 2.25 will be, they work that much harder, it is , they are both in minutes they worked for 2.25 times as long, there you go, so using use that way 36 to 16 means they worked, so there is 16 minutes, one time they worked there is 2.25 times as long as they worked , so it means like 2.25 longer so therefore if they did it 40 originally , they did it 40 times that , because they are working at the same rate will cover that much work for [Line 188]

Then, when I asked Mark, he explained his reason for how he was able to multiply 2.25 with 40, Mark engaged in adding the amounts of 16 minutes and 40 square cm repeatedly to reach the result. At this moment the interview became much more interesting because of Mark's line of reasoning.

> M: They originally worked for 16, this is 16 minutes, and they worked then 40 cm, so if you know 36 over 16 is, going from 16 to 36 , would be like one, two and 2 and two five so they are staying at the same rate so it will be 40, 80 and a fourth of that" [Line 196].
> R: A fourth of that
> M: Because point 25 is one fourth, so like 16, 32 and a 4^{th} of 16 and so then 40, 80 and a fourth of it is 10 so it is
> R: Fourth of what
> M: The area
> R: The fourth of which one, like 16, 32 and then 36 and then
> M: 40, 80 and 90
> R: A fourth of what

M: fourth of the 40
R: Then you are going to add that to and you know you can get 4^{th} of 40 because
M: Because you went from 16 to 32 and 4^{th} of 16 is 4 to get 36 and then 40, 80 and 4^{th} of 10, 4^{th} of 40 is ten so then you have the 80 and then 90 [Line 206].
R: So you know that the remaining 4 is 4^{th} of 16 why you are using that.
M: Because it is 2.25
R: Why you are allowed to find 4^{th} of 16 in the context of the problem, why it makes sense
M: You can use a 4^{th} because 36 is not a multiple of 16 it is more than that so it is, it is not like 32 and than only a fraction of that so that is why you can use one 4^{th} for 16 and get 4, it was just like it was 32 minutes so 16 minutes to 32 minutes would be 40 cm to 80 cm since this is 36 it is going to be a 4^{th} a 4^{th} more, a 4^{th} as much so from 16 to 36 it will be 40 to 90, but without using ratios, **you can wrap all these** [Line 210]
R: How
M: [The student draws a graph]so it starts here at 16 over here you are at 40 so it is about there and than 36 it is about 90 , [inaudible minute 1:1: 50] rate , it is going to be the it is going to be the constant line [Line 212]
R: What does that line represent?
M: It tells you how much they will do in a time period, so instead of doing proportions, it would be like if they worked 10 minutes it is going to be about 30 around or it will be around 25 because 2.5 you can't tell from the graph but you can tell it is going to go up it is like rise over run so 2.5 over 1 right so there will be 2.5 cm square [Line 214].

Mark's Reasoning on the Mosaic Problem and After

Although I cannot claim definitely that Mark learned (i.e. advanced his understanding of within-ratios), Mark's reasoning and use of representations seems to be more sophisticated after the Mosaic Problem. Mark's description of a line graph contrasts with his earlier use of a bar graph to show the relative size of two quantities.

M: [making a graph see his writing] so it starts here at 16 over here you are at 40 so it is about there and than 36 it is about 90 , [inaudible minute 1:1: 50] rate , it is going to be the it is going to be the constant line [Line 212]

R; what does that line represent
M: it tells you how much they will do in a time period, so instead of doing proportions, it would be like if they worked 10 minutes it is going to be about 30 around or it will be around 25 because 2.5 you can't tell from the graph but you can tell it is going to go up it is like rise over run so 2.5 over 1 right so there will be 2.5 cm square [Line 214]

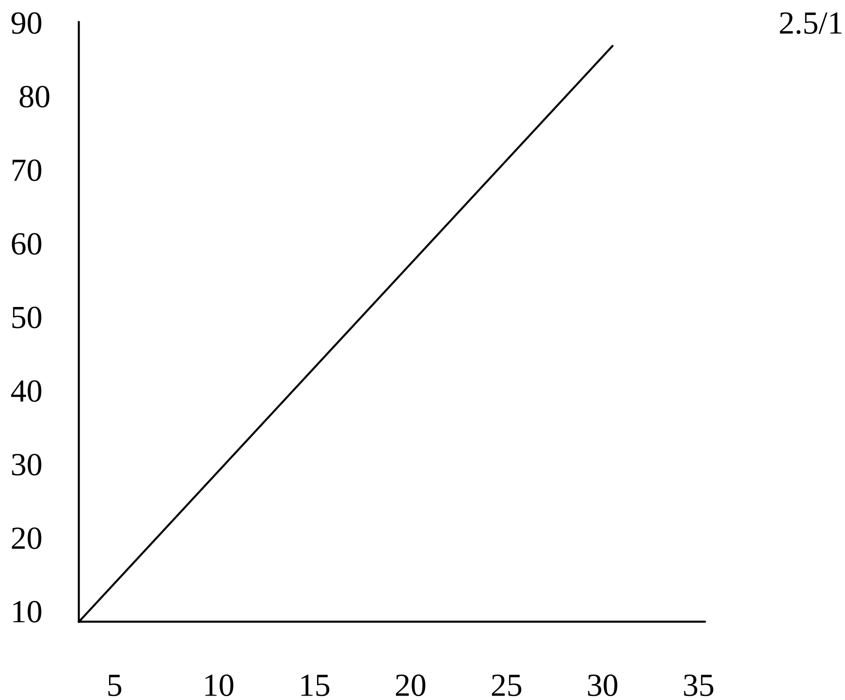

Figure 3. The Graph Mark Drew for the Mosaic Problem.

Thompson (1994) mentioned ratio as being the expressions of specific situations and rate as being the constant ratio, which is reflectedly abstracted from ratio, through generalized situations. He further claimed that "as soon as one reconceives the situation as being that the ratio applies generally outside the phenomenal bounds in which it was originally conceived, then one has generalized that ratio to a rate…rate is (from my point of view) a linear function that can be instantiated with the value of an approximately conceived structure" (p.192).

In that sense, the above excerpt shows that Mark is able to use the line representation as a representation of constant ratio which represented the invariant relationship, "2.5 square cm for 1 minute," among all the cases outside the phenomenal bounds of the Mosaic Problem.

Thompson further claims;

> …To say that an object travels at 50 miles/hour quantifies the objects' motion, but it says nothing about a distance traveled nor about a duration traveled at that speed (Schwartz, 1988, cited in Thompson, 1994). However, conceiving speed of travel in relation to an amount of time traveled produces a specific value for the distance traveled (p. 192).

In that sense, the excerpt on line 214 shows that Mark could think of 10 minutes as worth of 2.5 times as much work, which is 25 square cm. Thus, he was able to think of 10 minutes and 25 square cm of work producing a specific value for the invariant relationship of 2.5 square cm per minute among all the cases with the same rate.

Then, when I asked him to comment on the problem below, the Ice-Cream Problem, he could reason about it using within-ratios:

Ice-Cream Problem: In an ice-cream factory 12.7 liters of milk and 10.5 pounds of sugar is needed for one gallon of ice-cream. They want to make as much ice-cream as possible with 11.4 lbs of sugar. How much milk should they use to create a batch that tastes the same as their usual recipe?
Kelly solved the problem above and found the answer as 13.7 liters of milk. She then claimed that the ratio of 12.7 to 10.5 is the same as the ratio of 13.7 to 11.4. Then, she wrote:

$$12.7/10.5= 13.7/11.4$$

The following excerpt is interesting because it shows that Mark can reason about evaluation of the "quality of interest" using the result of division within-ratios.

> R: Okay, right, the question is why should I believe her that what she is saying is true [Line 215]

M: There is two ways of proportion, either way, she would the proportion but the decision as originally it takes 12.7, 12. 7 liters of milk and 10.5 pounds of sugar and you want to make as of the11.4 pounds of sugar so how much milk to use, so fraction so 12.7 liters of milk over 10.5 sugar should be the same as 13.7 liters of milk and 11.4 pounds of sugar [Line 216].
R: And the reason is
M: You don't want me to use proportions
R: Yeah, she used proportions so why it would make sense or would it make sense at al
M: It should, we will be taking 12.7 divided by 10,5 will give you a number and then that number times then, the total 12.7divided by 10.5 will give you a number [Line 220].
R: It will give you what
M: The ratio of milk to sugar
R: And that ratio will give you
M: How much liters of milk per pounds of sugar so 13.7 divided by11.4 will give you the same number if you use the same number it should be the same taste [Line 224]

According to line 224, Mark knows that ratio is the representation of the multiplicative relationship between the two quantities such that the same multiplicative relationship needs to hold for the other situation for the quality of interest to stay the same. That is, Mark used the result of division within-ratios as a way to quantify taste. Understanding within-ratios multiplicatively means knowing that the ratio is the measure of the quality of interest, the quantification of the attribute in the situation, which in this case is taste. That is, ratio is the measure of the multiplicative relationship between the quantities coming from different measure spaces.

Then, later, Mark again calculated 12.7 divided by 10.5, made a graph to communicate his thinking and claimed the following:

R: What does that line tell you, what is that line [Line 242].
M: that is, it will be 1.21 over divided by 1. for 1.21 liters of milk there is one pound of sugar [Line 243].

The above excerpt shows that the line is the relationship between the two quantities that make up the ratio, such that no matter what amounts constitute the mixture, the relationship between the quantities does not change. The invariant relationship is a multiplicative relationship such that the quotient, which is the result of division of the quantities in within-ratios, represents that. Again, if one understands within-ratios multiplicatively, the understanding extends to the fact that the multiplicative relationship represented by ratio holds for all the situations with the same quality of interest.

Comparison of Marks' Line of Reasoning on Within-Ratios at the Beginning and the End of the Interview

In this section, I contrast the characteristics of Mark's reasoning on within-ratios at the beginning of the interview and the end of the interview. Doing so highlights different levels of reasoning within the understanding of within-ratios.

At the beginning of the interview, as Mark has shown his reasoning on several tasks, Mark was able to think about ratio situations in such a way that he was able to make sense of the result of division within-ratios "for each part of one quantity, there exist so many parts of another quantity." He engaged in partitioning to come to the result of division within-ratios. In addition, what Mark used to represent the result of division within-ratios was through two different bars, which represented two different quantities in the original ratio situation. Also, both in Hair Color-2 Problem and in the Mixture Problem, Mark showed that he was able multiply or divide the amounts of original quantities in the ratio with the same number. He knew that when he multiplied or divided the original quantities with the same numbers, he kept the invariance of within-ratios.

However, at the end of the of the interview, after he engaged in the Mosaic Problem, Mark showed that he was able to think about the result of division within-ratios, in such a way that ratio is the invariant relationship between all the cases of the same quality of interest. In other words, Mark could reason that once given the result of division and one of the quantities in the ratio, he would be able to find the other quantity since a unique relationship exists between the quantities in the ratio such that the result of division tells what the size of one of the quantities is, given the size of the other quantity. Also, in both the Mosaic Problem and the Ice-Cream Problem, he began using the line representation to talk about the ratio as a representation of the multiplicative relationship between all the cases of the same quality of interest. Finally, in the Ice-Cream Problem, using the ratio, the result of division, the quotient within-ratios, Mark was able to quantify the quality of taste and justify whether two ratio situations have the same quality of interest using the quotient within-ratios.

End of Interview: Comparison of Mark's Performance before, during and after the Mosaic Problem

Mark's reasoning on the Mosaic Problem presented somewhat different levels of reasoning about within-ratios. That is, Mark used a unit-factor approach to reason about the problem, at first. Later he was able to think about the within-ratios as an intensive quantity that quantifies the quality of interest in the situation. In chapter 5, I will use this analysis to contribute to a hypothesis about developing a conception of interiorized ratio [rate-ratio] given the student's conception of internalized ratio [particular ratios].

Early in the problem, Mark reasoned as follows:

The Mosaic Problem: Two sisters, Shari and Melissa, decided to create a mosaic design on their table top from broken pieces of tile. They worked for sometime, and then they had a break. When they had the break they realized that, it took 16 minutes to finish an area of 40 cm square. If they worked at the same rate, how much of the table top could they finish in 36 minutes?

Can you provide as many solutions as to this problem? Explain how your solutions make sense in terms of the problem context.

> R: …All right, this was for you, first of all you solved it using proportions, can you solve it without using proportions, if you cannot that is fine too [Line 175]
> M: Without the proportion I don't know, it would take them all right if it took them 16 minutes to do 40 cm that means they are working at a rate of 2.5 cm per minute because yeah 40 cm over 16 minutes it will be 2.5 cm per minute so it is the same rate, how much it is going to be for 36 minutes, if you are going to work 2. 5 cm per minutes, that is just 2.5 times 36, it will tell you how much of mosaic which is.
> R: Why you are allowed to multiply 2.5 by 36?
> M: Because it is 2.5 cm per minute, if 40 over 36 minutes was 2.5 cm per minute multiplying that 36 minutes, is technically over 1, the minutes will cancel, so 36 times 2.5 will be 90 cm because minutes will cancel [Line 178]
> ….
> M: Because you know, well if you know that they can do 2.5 cm for every almost every minute, if you want to see how much they are doing in 36 minutes, you just take 36 times 2.5 cm
> M: the ratio is 1 minute is 2.5 cm , so 36 minutes 90 cm [Line 184].

The excerpt above shows that Mark knows that there is a relationship between the amount of time and how much work is done such that 40 square cm of work for 16 minutes means 1 minute for 2.5 square cm of work completed. Then, he can accumulate or he can use multiplication as an easier way of accumulation of the quantities of 1 minute and 2.5 square cm of work to come to the targeted amounts of 36 minutes and 90 square cm of work.

Then, I gave him the solution of "36/16=2.25, 2.25* 40=90". After I probed him, he started to break the problem down in the following way.

> M: They originally worked for 16 , this is 16 minutes, and they worked then 40 cm, so if you know 36 over 16 is, going from 16 to 36 , would be like one, two and 2 and two five so they are staying at the same rate so it will be 40, 80 and a fourth of that" [Line 196]
> R: A fourth of that
> M: Because point 25 is one fourth, so like 16, 32 and a 4th of 16 and so then 40, 80 and a fourth of it is 10 so it is
> R: Fourth of what
> M: The area
> R: The fourth of which one, like 16, 32 and then 36 and then
> M: 40, 80 and 90
> R: A fourth of what
> M: Fourth of the 40
> R: Then you are going to add that to and you know you can get 4th of 40 because
> M: Because you went from 16 to 32 and 4th of 16 is 4 to get 36 and then 40, 80 and 4th of 10, 4th of 40 is ten so then you have the 80 and then 90 [Line 206]
> R: So you know that the remaining 4 is 4th of 16 why you are using that
> M: Because it is 2.25
> R: Why you are allowed to find 4th of 16 in the context of the problem, why it makes sense
> M: You can use a 4th because 36 is not a multiple of 16 it is more than that so it is, it is not like 32 and than only a fraction of that so that is why you can use one 4th for 16 and get 4, it was just like it was 32 minutes so 16 minutes to 32 minutes would be 40 cm to 80 cm since this is 36 it is going to be a 4th a 4th more, a 4th as much so from 16 to 36 it will be 40 to 90, but without using ratios, **you can wrap all these** [Line 210]
> **R: How**
> M: [the student draws a graph] so it starts here at 16 over here you are at 40 so it is about there and than 36 it is about 90 , [inaudible minute 1:1: 50] rate , it is going to be the it is going to be the constant line [Line 212]
> R: What does that line represent
> M: It tells you how much they will do in a time period, so instead of doing proportions, it would be like if they worked 10 minutes it is going to be about 30 around or it will be around 25 because 2.5

> you can’t tell from the graph but you can tell it is going to go up it is like rise over run so 2.5 over 1 right so there will be 2.5 cm square [Line 214].

For the excerpt below:

> M: They originally worked for 16 , this is 16 minutes, and they worked then 40 cm, so if you know 36 over 16 is, going from 16 to 36 , would be like one, two and two and two five so they are staying at the same rate so it will be 40, 80 and a fourth of that” [Line 196]

Mark knew that he could unpack that multiplicative relationship, “2.25 as much,” and iterate the number of minutes at the same time iterating the area covered at every minute since he already knew that those quantities of “minutes” and “area” are covarying quantities [an image of succession of equal ratios]. Thinking about the fractional part was important because that helped him to focus on the fact that 4 and 10 are $1/4^{th}$ of 16 and 40, respectively, which represented the quality of interest, rate. At first, Mark is focuses on different quantities of minutes and then focuses on different quantities of area.

Then he talked about the problem again;

> M: Because you went from 16 to 32 and 4^{th} of 16 is 4 to get 36 and then 40, 80 and 4^{th} of 10, 4^{th} of 40 is ten so then you have the 80 and then 90 [Line 206].

The above excerpt shows that he was thinking about those quantities of minute and area as segmented quantities such that: 16 32 36.

40 80 90.

He focused <u>again</u> on those quantities as coordinated and accumulating at the same time [an image of an iterable ratio relating the collections of two quantities in the ratio as the amounts of either might vary].

Then, he was asked to talk about the fractional part $1/4^{th}$:

> M: You can use a 4^{th} because 36 is not a multiple of 16 it is more than that so it is, it is not like 32 and than only a fraction of that so that is why you can use one 4^{th} for 16 and get 4, it was just like it was 32 minutes so 16 minutes to 32 minutes would be 40 cm to 80 cm since this is 36 it is going to be a 4^{th} a 4^{th} more, a 4^{th} as much so from 16 to 36 it will be 40 to 90, but without using ratios, you can wrap all these [Line 210].

Mark‘s focus was both on the whole segmentation of the coordinated values and the parts of those segmented values. In other words, Mark explained how the increase in one quantity has to be coordinated with the increase in the other quantity.

At that moment, he thought of “wrapping these all”:

> R: How
> M: [the student draws a graph] so it starts here at 16 over here you are at 40 so it is about there and than 36 it is about 90 , [inaudible minute 1:1: 50] rate , it is going to be the it is going to be the constant line [Line 212]
> R: What does that line represent
> M: It tells you how much they will do in a time period, so instead of doing proportions , it would be like if they worked 10 minutes it is going to be about 30 around or it will be around 25 because 2.5 you can’t tell from the graph but you can tell it is going to go up it is like rise over run so 2.5 over 1 right so there will be 2.5 cm square [Line 214].

Mark knew that the relationship for the specific case of 10 minutes to some amount of work is the same for all cases. That relationship was the multiplicative relationship such that the amount of work would be 2.5 times as much the amount of time given. That is, coming to the realization that all the cases of the original ratio situation is represented by the unique relationship of “ 2.5 square cm for 1 minute”, he could think of 10 minutes as worth 2.5 times as much work, which is 25 square cm. Also, notice that Mark drew a line indicating that he knew that 2.5 square cm per minute represented all the cases of the same quality of interest. Whereas, before, he seemed to be reasoning about particular ratios, he now seemed to be reasoning about an interiorized ratio, a ratio-as-measure---an intensive quantity that measures the quality of interest for all the situations of the same type.

In sum, it seems that Mark’s activity with the Mosaic Problem led either to learning or to a reorienting to something previously learned. In either case, his explanations and representations were more sophisticated following his work on the Mosaic Problem.

Chapter 5

RESULTS AND IMPLICATIONS FOR FUTURE RESEARCH

Results of the Study

Understanding of Between- Ratios and Understanding of Within-Ratios

Understanding of Between- Ratios Does Not Require an Understanding of Within-Ratio.

The data on Rita's reasoning showed that she had abstracted the result of division of quantities from the same measure space, between ratios, as the multiplicative increase or decrease from one situation to another. At the same time, she showed that she did not have any meaning for the result of division of quantities coming from different measure spaces, within ratios. In other words, she did not have the understanding of within ratios as an intensive quantity.

Understanding of Between- Ratios Does Not Require an Understanding of a Per-One Approach

Although Rita understood between ratios, she was not even able to offer a per-one interpretation of the quotient given. Given the result of division of the quantities (within ratios), Rita commented that the number was useless and would not help her to make more of the same mixture. I note however, that the problems I used to assess her understanding of within ratios required a significant understanding. I did not determine if she could have reasoned with within ratios if the original quantities were in a simple ratio such as 1:2.

An Understanding of Within- Ratios May Develop to Some Degree Independently from the Understanding of Between- Ratios.

Combining result #1 with data on Mark, we can consider that the development of the two types of ratio can develop somewhat independently. Mark, at the beginning of the interview, showed that he abstracted the result of division of the quantities coming from different measure spaces, within ratios, as “for every unit of one quantity, there are so many units of another quantity.” At the same time, he did not have any meaning for the ratio representing quantities coming from the same measure space, between ratios. That is, he had not abstracted that between ratios represent the change factor from one ratio situation to the other.

An Understanding of Only One of the Two Types of Ratio is Sufficient to Deal with Missing Value Problems that Pervade School Problems on Ratio.

That is, the results suggest that curriculum developers and teachers need to pay attention to different tasks/problems to both promote and assess understandings of between ratios and within ratios.

Description of Distinctions Between Levels of Abstraction in Understanding Between Ratios

The results of this study on Mark’s data showed that different levels of abstraction exist for between ratios. Mark’s data showed that for Hair Color-2 Problem, in which students were asked about the meaning of the between-ratio, 22/15, Mark could not make sense of the ratio at the beginning of the interview. However, Mark was able to make sense of the Mosaic Problem.

For the Hair Color-2 Problem, the knowledge of 22/15 as the change factor is required to make sense of the solution process given for that problem. That is, one needs to be able to realize that 22/15 is the change factor so that the solution process given for the Hair-Color 2 Problem is meaningful. The fact that Mark was not able to make sense of the Hair-Color 2 Problem showed that this problem required Mark to have an abstraction beyond his current understanding. In other words, the conception needed for the Hair Color-2 Problem was beyond Mark's current reasoning.

On the other hand, the knowledge required to make sense of the Mosaic Problem was the knowledge of multiplying both quantities with the same number to reach the targeted quantity. The fact that Mark was able to make sense of the Mosaic Problem indicates that the Mosaic Problems did not require the same level of abstraction as the Hair Color-2 Problem. An abstract understanding of between ratios requires understanding the ratio as representing the change factor, independent of the activity of looking for a common multiplier.

Development of a Conceptualization of Rate

This study showed that Mark presented somewhat different levels of reasoning about within ratios at the beginning of the interview as compared to the end of the interview. Whether or not this was an indication of learning on Mark's part or only a reorienting to something previously learned, it can be viewed as pointing to a learning progression that fits with Thompson's (1994) analysis of conceptualization of speed and the sequence of progress proposed by Heinz (2000). That is, Mark's data might be an example of a student who had the understanding of particular ratios [internalized ratio] at the beginning of the interview, and he came to an understanding of rate [interiorized ratio] by the end of the interview.

The teaching experiment Thompson (1994) conducted with a fifth grade student, JJ, suggests that JJ had the notion of speed as “every second, there is some amount of distance traveled.” The study suggested that the conceptualization process for speed entails: 1) an image of two quantities covarying; 2) “an image of two segments, one for distance and one for time, where the segments were partitioned proportionally according to units of time” (p. 200); and 3) “an image of segmented total distance in relation to a segmented total amount of time” (p. 204).

The difference between Mark and JJ is that Mark already knew that he would divide the total amount of work with the total amount of time to find out the work per minute for the Mosaic Problem. Still, what Mark went through when he was engaged in the Mosaic Problem resembles the conceptualization process JJ went through and also what Thompson (1994) stated:

> If a child is trying to find, say, how many apples there are in a basket where the ratio of pears to apples is 3:4 and there are twenty-four pears, and the child thinks “three pears to four apples, six pears to eight apples,…., twenty-four pears to thirty-two apples” (a succession of equal ratios), then this provides an occasion for the child to abstract the relationship “three apples for every four pears” (an iterable ratio relating collections of apples and pears as the amounts of either might vary), and eventually “there will be ¾ of an apple or part thereof for every pear or proportional part thereof” (an accumulation of apples and pears that carries the image that the values of both can vary, but only in constant ratio to the other). The former conception-accumulations made by iterating a ratio- I call an *internalized* ratio, whereas the latter conception-total accumulations in constant ratio-I call an *interiorized* ratio, or a *rate* (p. 193, italics are original).

Similarly, Heinz (2000) proposed that students would go through a sequence of 1) a correspondence of two quantities, 2) a measuring act on the quantities from the same measure spaces, 3) a coordination of the incremented quantities resulting from the measuring, 4) a segmentation of the quantities resulting from the coordination of the incremented quantities, 5) a coordination of the segmented quantities, 6) conceptualization of each of those segmented quantities as proportional amounts of the whole, 7) coordination of each of those segmented quantities as proportional amounts, and 8) conceptualization of the two quantities as bi-directional covarying quantities.

In that sense, Mark's data might be an example of a student who had an image of succession of equal ratios, and then had an image of an iterable ratio relating the collections of two quantities in the ratio as the amounts either might vary, and then, had an image of an accumulation of two quantities that carries the image that the values of both quantities can vary, but only in constant ratio to the other. Thus, I propose a rationale for how Mark's engagement with the building up strategy might have created an opportunity for him to have an image of a constant ratio, rate.

I will discuss how Mark's reasoning could be seen as Mark's progression from particular ratios [internalized ratios] to rate ratios [interiorized ratios] and my conjecture of how this can be explained. Mark already knew that the result of division within ratios represented "for every unit of one quantity, there exist some other units of another quantity". This can be seen as reasoning about a particular ratio, internalized ratio. He also knew that he had to multiply both quantities in the original ratio with the same number. Important to remember is that after he engaged in the Mosaic Problem, he made the following claim:

M:…You can wrap all these [Line 210]
R: How
M: [the student drew a graph] so it starts here at 16 over here you are at 40 so it is about there and than 36 it is about 90 , [inaudible minute 1:1: 50] rate , it is going to be the it is going to be the constant line [Line 212].
R: What does that line represent
M: It tells you how much they will do in a time period, so instead of doing proportions, it would be like if they worked 10 minutes it is going to be about 30 around or it will be around 25 because 2.5 you can't tell from the graph but you can tell it is going to go up it is like rise over run so 2.5 over 1 right so there will be 2.5 cm square [Line 214].

I propose the following as an explanation for how Mark came to an understanding of interiorized ratios. Mark already had gone through the accumulation of the quantities of 16 minutes and 40 square cm until he reached the targeted quantity of 36 minutes. His focus was first on the specific amounts of 16 minutes and 40 square cm, 32 minutes and 80 square cm and 36 minutes and 90 square cm. Once he made the segments of 16 minutes until 36 minutes and 40 square cm until 90 square cm, his focus was on both the whole segment and the specific amounts of that segment, such as 16 to 40, and 32 to 80 and 36 to 90. Mark knew that *each of* those pairs of quantities have a particular multiplicative relationship. Since he already knew that 16 and 40 are related to each other by 2.5 square cm for 1 minute, this made him realize that *each of* those specific quantities, 16 to 40, 32 to 80, 36 to 90, are all related to each other by the same relationship of 2.5 square cm per one minute (an image of an accumulation of two quantities that carries the image that the values of both quantities can vary, but only in constant ratio to the other). At that moment, he claimed that, "You can wrap all these," [Line 210] because he realized that the ratio is a constant ratio, invariant for all cases with the same property. His use of the line graph and explanation of it demonstrated his reasoning about an interiorized ratio or rate.

In sum, once Mark was probed to talk about how he was able to multiply the original quantities in the ratio with the same number to hit the targeted quantity, he started to use his image of covarying quantities. He knew that for every so many of the first quantity, he needed to have so many of the other quantity. He also knew that he had to match the same fractional amounts of those covarying original quantities. Accumulating the set of quantities for both of the original quantities in the ratio, he made two segments, such as 16, 32, 36 and 40, 80, 90. He knew that the corresponding quantities in the whole segment had the same quality of interest. Once he had the whole accumulated segment, his focus was on both the whole segment and the specific pieces of that segment, such as 16 to 40, and 32 to 80 and 36 to 90. He then realized that those specific quantities, and also the whole segment, are related to each other by a multiplicative relationship. That is, all the specific quantities are parts of the larger quantities and the larger quantities are multiplies of the smaller quantities. Thus, he recognized that all the specific quantities (bigger or smaller in amount) are all related to each other with the same ratio "for every unit of one quantity there exist so many units of the other quantity." Thus, he came to the interiorized rate as the constant ratio.

Implications for Future Research

1- The study suggests that the understanding of between ratios may develop to some extent independently of the understanding of within ratios. This finding begs the question of how these two somewhat independent subconcepts become a unified (abstract) concept of ratio – what Heinz (2000) refers to as “ratio as quantity.”

2- The study suggests that understanding of within ratios may develop from per-one strategies and that between ratios may develop from building- up strategies. Teaching experiments that promote and study the development of each kind of ratio would provide important next steps in verifying these hypotheses and in understanding the processes involved.

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Appendix 1

First Written Problem Solving Session

Directions:
You will be given a number of problems to work on. Please provide your thinking process for each of the problem as much as possible. You may use diagrams, shapes, graphs, etc in explaining your answer. If you use a calculator, record on the paper what you did (eg., 52 x 34).

1- A new housing subdivision offers lots of three different sizes: 185 feet by 245 feet, 75 feet by 114 feet, and 455 feet by 508 feet. If you were to view these lots from above, which would appear most square? Which would be least square? Explain your answers.

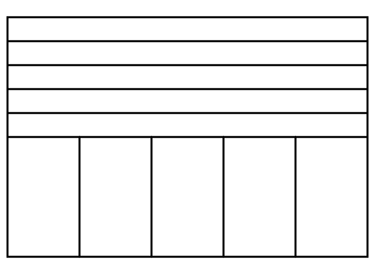
2- I have 1 2/3 cups of milk. My recipe calls for 2 ¾ cups of milk. How should I adjust the ingredients in the recipe so that I can use 1 2/3 cups of milk instead of 2 ¾ cups? (You do not need to do the calculation; just indicate what calculation you would do.)

3- Which of the following sequences is <u>most</u> likely to result from flipping a fair coin five times? Explain your reasoning.

a) HHHTT
b) TTTTT
c) THTTT
d) All four sequences are equally likely.

4- Kelly likes to color her hair with a mixture of red and brown dyes. When her hair was shorter, she used 4 grams of red and 5 grams of brown dye. Now, her hair is longer. She knows that the original mixture of 4 grams of red and 5 grams of brown will not be enough to color all of her hair. So she thinks that if she adds 1 gram of each color to the original mixture, she will preserve the color. Will the new mixture be the same color as the original? Explain your reasoning.

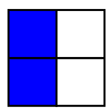
5- Amy was solving the problem of dividing a rectangular cake into tenths. She drew the diagram below. Do you agree with her that this is a correct solution? If you agree, explain why? If you disagree, explain what you think is wrong with her solution.



6- A mixture of 46 cups is made from flour, oil, and sugar with the proportions of 2.1, 1.3, and 2.5 respectively. How much oil is used for this mixture? Explain your answer.

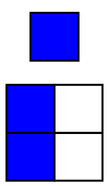
7- Two cars are traveling at the same speed. After 2 hours, Shelly is 30 miles ahead of Kristin. If they continue at the same constant speed, how far will Shelly be ahead after 4 hours? Explain your reasoning.

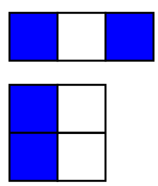
8- The owners of the Blue Jean Company wanted to expand the area designated for coloring jeans. Along with this expansion, they decided to try some new mixtures for the dye they used. The classic color mixture was two gallons of blue dye and two gallons of white dye.



Classic mixture

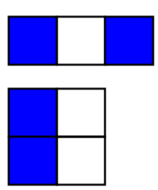
In new mixture #1 they added 1 gallon of blue dye to the classic mix and in new mixture #2 they added 2 gallons of blue dye and 1 gallon of white dye to the classic mixture.





New Mixture #1 New Mixture #2

Dale, one of the developers, objected to the idea of creating the mixtures in this way. He argued that both new mixtures would be coloring the jeans in the same way. He argued the following:



The original classic mixture has 1 gallon of blue dye for 1 gallon of white dye, and 1 blue for another white. Then, in proposed mixture 2, we add 2 blues and 1 white to the original classic mixture. Adding 2 blues and 1 white is the same as adding 1 blue because 1 blue and 1 white cancel each other out. Then, what we have is 1 blue left for the mixture 2.

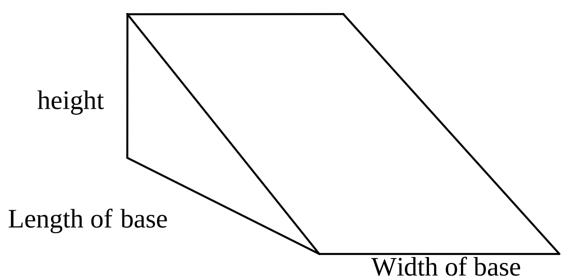
This is the same as 1 blue color we have for the mixture 1. So, in both new mixtures we start with the classic mixture and add 1 blue. So, the new mixtures we think we are creating differently are actually the same. Do you think Dale’s **argument** is valid? Why do you think it is valid or invalid? Do no computation.

Appendix 2

Second Written Problem Solving Session

Directions:

You will be given a number of problems to work on. Please provide your thinking process for each of the problem as much as possible. You can use diagrams, shapes, graphs etc in explaining your answer.



1- In Kansas, there are no mountains for skiing. An enterprising group built a series of ski ramps and covered them with plastic fiber that permitted downhill skiing. It is your job to rate them in terms of steepness. What would you use for the steepness of the ski ramps:

a) The difference between height and length of base.

b) The ratio of height to length of base

c) The difference between the height and the width of the base

d) The quotient (the result of the division of height by the length of base)

Explain why your choice is meaningful?

2- Gill played the lottery weekly the last two months. So far he has never won, but he decided to continue to play for the following reason: “Lottery is a game based on chance, sometimes you win sometimes you lose. I have already played many times and I have never won, so that increases my chances of winning now”. What is your opinion with regard to Gill’s explanation?

3- Andy is 12 years old and his brother is 15 years old. When Andy becomes twice as old as he is now, how old will his brother be? Explain your answer.

4- The following was given to a groups of seventh graders:

[Read but don't solve the problem, then comment on the students' solutions]

Kelly likes to color her hair with a mixture of red and brown dye. When her hair was shorter, she used 15 grams of red and 17 grams of brown dye. Now, her hair is longer. She knows that the original mixture will not be enough to color all of her hair. She intends to add 7 grams of red dye to the original amount of red dye and some amount of brown dye to the original amount of brown dye. How much brown dye does she need to make sure that she has the original color?

In a group of seventh graders, the following two solutions were given:

g) The amount of brown dye Kelly needs to add to the original amount is 17*(22/15).

h) The amount of brown dye Kelly needs to add to the original amount is 22*(17/15).

Explain how each answer fits or does not fit the story? Depending on the choice of your answer, explain what (22/15) represents and/or what (17/15) represents.

5- Find a fraction between $\frac{3}{5}$ and $\frac{4}{5}$? Explain your answer.

6- A group of fifth graders will mix two types of juice mixtures and will put them into jars so they can serve them to the kindergarten students during the week.

In the first jar, they put 36 grams of lemon juice and 32 grams of lime juice. In the second jar, they put 20 grams of lemon juice and 15 grams of lime juice. To label the jars, the fifth graders wanted to use one number to accurately represent the lemon-lime flavor in that jar. They considered the following three ideas for labeling the jars:

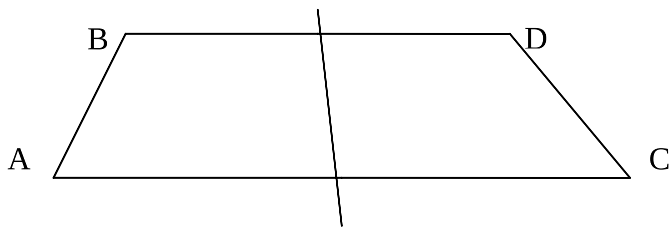
a) There is 4 more lemons than limes in the first jar. So, put 4 on the label. There is 5 more lemons than limes in the second jar. So, put 5 on the label.

b) the ratio of 36/32 for the first type of mixture and the ratio of 20/15 for the second type of mixture.

c) 1.125 (the result of dividing 36 by 32) and 1.33.. (the result of dividing 20 by 15).

Which idea(s) would accurately indicate the lemon-lime flavor? Why? Provide an explanation for the reason (s) that each of these options would or would not indicate the mixture’s lemon-lime flavor.

7- Kelly thinks that the line passing through the midpoints of the segments BD and AC in trapezoid ABCD divides the trapezoid into two equal areas. Do you agree or disagree with Kelly? If yes, why? If no, what is lacking in her reasoning?



8- A recipe for salad dressing asks for 9 tablespoons of oil and 4 tablespoons of vinegar. If we want to make a batch that includes 7 tablespoons of vinegar, how much oil will we need to be consistent with the recipe? How do you know that the result makes sense?

9- A second grade student was solving the following problem:

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  204
- 162
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She claimed that 2 away from 4 is 2, and she cannot take 6 away from zero so she needs to borrow 1 from 2 and then 6 away from 10 is 4. And since she has only 1 left then 1 way from 1 would be zero, so the result it 42.
When she was asked to show with unifix cubes, what the "1" represented when she borrowed it from 2, she showed only 1 cube.

Do you think this second grader is thinking properly or not? If yes, why is that? If no, what is wrong with her thinking?

10- Two sisters, Shari and Melissa decided to create a mosaic design on their table top from broken pieces of tile. They worked for sometime, and then they had a break. When they had the break they realized that, it took 16 minutes to finish an area of 40 cm square. If they worked at the same rate, how much of the table top could they finish in 36 minutes?

Can you provide as many solutions as to this problem? Explain how your solutions make sense in terms of the problem context.

Appendix-3

Additional Interview Questions

1- In a ice-cream factory 12.7 liters of milk and 10.5 pounds of sugar is needed for one gallons of ice-cream. The producers realized that they had 11.4 pounds of sugar for the new gallon and wanted to produce the same ice-cream with what they have. How much milk do they need to use?"

Kelly solved the problem above and found the answer as 13.7 liters of milk. She then claimed that the ratio of 12.7 to 10.5 is the same as the ratio of 13.7 to 11.4. Then, she wrote:

12.7/10.5= 13.7/11.4

Why should I believe her that this is a true statement? Can you convince me that this is a true statement?

2- Two sisters, Shari and Melissa decided to create a mosaic design on their table top from broken pieces of tile. They worked for sometime, and then they had a break. When they had the break they realized that, it took 16 minutes to finish an area of 40 cm square. If they worked at the same rate, how much of the table top could they finish in 36 minutes?

Can you provide as many solutions as to this problem? Explain how your solutions make sense in terms of the problem context.

Somebody solved the problem as follows:

They divided 36 by 16, 36/16=2.25 and then multiplied that by 40, so 2.25*40=90.

Can you account for this solution? What do 2.25 represent? Why does it make sense to multiply it with 40?

VITA

Gulseren Karagoz Akar
916 Southgate Drive, Apt# 10, State College, PA, 16801
Phone: 814-865-1544
gxk902@psu.edu

Education:

Ph.D., Mathematics Education, The Pennsylvania State University, 2007
Concentrations: Conceptualization of Mathematics Topics, Teaching for Understanding
Dissertation: Conceptions of Between Ratios and Within Ratios

M.Ed., Mathematics Education, The Pennsylvania State University, 2001
Concentrations: Learning of Mathematics
Thesis: Teaching for Understanding-The Case of Equivalence of Fractions

B.S., Mathematics Education, Middle East Technical University, Ankara, TURKEY, 1996

Presentations:

Heller, S., Karagoz Akar, G., & Sullivan, P. (2004, June). *The evolution of our mathematics resulting from student interviews*. Paper presented at the Mid-Atlantic Center for Mathematics Research Conference, State College, PA.

Karagoz Akar, G. & Watanabe, T. (2004, October). *The relationships among informal strategies students use in solving problems in proportional situations*. Poster session presented at the 26th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Toronto, Ontario, Canada.

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Papers contributed to:

Heid, M. K., Zembat, I.O., Peters, S., Sullivan, P, Portnoy, N., & Wilson, P.S. (2006, April). *Prospective secondary mathematics teachers' ways of mathematical thinking*. Paper presented at the research pre-session of the annual meeting of the National Council of Teachers of Mathematics, St. Louis, MO.