CENTERED COPRIME ARRAY PERFORMANCE IN THE
SHALLOW WATER ENVIRONMENT

A Dissertation in
Acoustics
by
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Submitted in Partial Fulfillment
of the Requirements
for the Degree of

Doctor of Philosophy

May 2020
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Abstract

Coprime arrays are sparse arrays that use two colinear uniformly spaced arrays to sense signals with performance comparable to a more densely spaced array of similar dimensions. Families of processors using the coprime array depend on its ability to approximate a full signal covariance matrix under certain common conditions with a smaller number of sensors than a traditional uniform linear array. This behavior is considered for a coprime hydrophone array in a shallow water environment. We show that certain assumptions fundamental to coprime array processing, including wide sense stationarity, are invalid in such an environment and quantify the effects. By examining environmental assumptions like high signal coherence and isotropic noise it is found that some metrics of performance are reduced by the sparseness of the coprime array while others are not. In the process, new theory regarding coprime arrays is developed, addressing aspects of the array that are relevant under the more realistic conditions of this work. A novel coprime array design is developed, termed the centered coprime array, that guarantees a lower bound on the array’s fully augmentable range that scales linearly with physical aperture. This design also enables a new subarray processing method, appropriate for short range beamforming, that is not possible with other sparse array geometries. Finally, numerical simulation is shown to verify performance predictions of the centered coprime array in a number of realistic operating scenarios. Ultimately, it is shown that performance of the coprime array in realistic operating scenarios is directly related to the choice of coprime factors, and thus allows for a smooth engineering tradeoff between cost and performance that can be optimized for a given application.
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\[ d \] Fundamental spacing of an array, in meters. One half-wavelength is often chosen.

\[ \lambda \] Acoustic wavelength, in meters.

\[ \omega \] Angular frequency, in radians per second.

\[ c \] Sound speed, in meters per second.

\[ \vec{k} \] Acoustic wavenumber vector, in inverse meters.

\[ L \] Array length, in units of \( d \).

\[ \theta \] Angle with respect to the array axis, in radians, measured from broadside.

\[ u \] Transformed angle, unitless. \( u = \sin \theta \).

\[ K \] Total number of elements in an array.

\[ M \] and \[ N \] Coprime array undersampling factors. Integer values chosen to be coprime.

\[ m \] and \[ n \] Subarray indices of \( M \) and \( N \) spaced arrays.

\[ \vec{u} \] Source location vector, in meters.

\[ \vec{x}_k \] Location vector of the \( k \)th array sensor.

\[ \vec{y} \] Spatial lag, in meters.

\[ c \] Sound speed, in meters per second.

\[ s_k(t) \] Output signal at the \( k \)th array sensor.
\( z_k(t) \) Noise measured at the \( k \)th array sensor.
\( A_k \) Signal attenuation at the \( k \)th array sensor.
\( \tau_k \) Time delay to the \( k \)th array sensor.
\( v(t) \) Measured array response.
\( \sigma^2 \) Power (or Variance) of the subscripted value.
\( I(\vec{u}) \) Source intensity function at location \( \vec{u} \).
\( r(\vec{x}_1, \vec{x}_2) \) Cross-correlation function between sensors at locations \( i \) and \( j \).
\( R \) Correlation function.
\( \mathbf{R} \) Correlation matrix.
\( \hat{\mathbf{R}} \) Estimated correlation matrix.
\( \mathbf{w} \) Array steering vector.
\( k_0 \) Wavenumber, in inverse meters.
\( \gamma_\vec{y} \) Spectral weight or lag \( \vec{y} \).
\( C_d \) Difference coarray set.
\( X \) Array aperture function.
\( c_d(\vec{y}) \) Difference coarray weighting function of lag \( \vec{y} \).
\( w_k \) Aperture shading function.
\( \phi \) Elevation angle, in radians.
\( \Omega \) Solid angle.
\( |N(\phi, \theta)|^2 \) Directional noise intensity at angles \( \phi \) and \( \theta \).
\( b(\phi, \theta) \) Beampattern at angles \( \phi \) and \( \theta \).
\( \log(\cdot) \) Logarithm base \( e \), unless otherwise noted, as with \( \log_{10}(\cdot) \).
\( \forall \) For all.
\( A(\vec{x}) \) Signal attenuation at location \( \vec{x} \).
$\Psi_l(z)$  Vertical mode shape function of order $l$ at depth $z$.

$C_L$  Wave correlation length, in meters.

$P$  Probability of element failure.
Acknowledgments

There are many people that I would like to thank for making this dissertation possible. Thank you Lee Culver for serving as my initial advisor and welcoming me to Penn State, acoustics, and the world of underwater acoustics research. Thank you Dave Swanson for stepping in to help me make the final push to complete this dissertation and the constant words of encouragement and advice that have helped me see things through to the end. Thank you Tom Hilands for recommending the tools needed for simulations, and for understanding what a student like myself needs to stay motivated. Thank you to the other members of my committee, Charles Holland, Ram Narayanan and Karl Reichard, without whose feedback this work would be far weaker.

I want to thank my former professors at the University of Arizona for giving me the background needed for success in graduate school, especially Michael Gehm for encouraging me to pursue graduate studies and Robin Strickland for guiding me on a path towards acoustics and giving me a strong background in signal processing that has served me well and will continue to in the future. Thank you to the professors of the Graduate Program in Acoustics for your devotion to education that has truly shaped my scholarly development.

I would like to thank all of my fellow graduate students in acoustics and especially my labmates throughout the years: Brett Bissinger, Alex Sell, Daniel Parks, and James Esplin. It has been a real pleasure meeting and getting to know so many classes of students as passionate about acoustics as myself, and it will be hard going back to the real world where our science is considered niche.

Thank you to all of the members of the administration that helped me navigate the confusing and sometimes frustrating rules and procedures. A special thanks to Karen Thal and program chair Vic Sparrow for always having the best interests of their students at heart and being willing to put in the extra effort needed to bring them success. Thank you to my friends and family that have supported me along the way. I know it has been hard on my parents and sisters to have me live so far away. Priscilla Brownlow has been incredibly patient waiting for me to finish school. Without the love and support of all of you, this dissertation wouldn’t exist.
Finally, this work would not have been possible without the financial support of the Applied Research Laboratory Eric Walker Graduate Assistantship. Additional funding was provided in fellowships and awards from the Acoustical Society of America, National Defense Industrial Association, and the Simowitz family. Simulation tools used in this work were developed by Bob Goddard of the University of Washington.
Chapter 1

Background and Motivation

In the development of new ideas to address a specific problem, assumptions are often applied to better frame the problem and simplify the understanding of these new ideas. Yet, when taken into practice, these assumptions must be reconsidered, and often modified, to account for the realities of the physical world. It is in the spirit of this latter maxim that we consider the recently suggested coprime array in the context of shallow water acoustics. The coprime array is a newly developed sensor array geometry which promises an increased number of degrees of freedom given the same number of sensors as a comparable uniform linear array [2]. In the development of this design, and the proof of its increased number of degrees of freedom, several simplifying assumptions were made. In application to a real world scenario, however, such as remote passive sensing in the underwater acoustic environment, these assumptions must be reexamined and the impact of any deviations quantified.

Our goal, in this dissertation, is to quantify and predict the performance of the coprime array in a realistic scenario. To do this, we first examine the performance of the coprime array in conditions consistent with the design’s underlying assumptions. We then consider realistic deviations from these assumptions, derived from existing literature and detailed acoustic simulations of a realistic environment. To predict performance in these circumstances, we develop new theory regarding coprime arrays, addressing aspects of the design that are immaterial under the original assumptions but have a direct impact on the performance under more realistic ocean conditions. From this new theory, improvements on existing coprime array designs (even under optimal conditions) become apparent and are described. This includes the centered coprime array, a novel approach to coprime array design that furnishes a lower bound on the fully augmentable range of the array that scales linearly with physical array aperture. Finally, we perform numerical simulations to verify our performance predictions and compare the performance of this new array geometry against existing geometries, such as the uniform linear
Figure 1.1. A uniform linear array of length $L$, with an element spacing of $d$.

array and the minimum redundancy array, quantifying coprime array performance in previously unexamined environmental conditions.

1.1 Sensor Arrays

The work presented in this dissertation will primarily address the topic of passive (receive-only) sensor array design. A sensor array is any group of sensors working in concert, and such arrays can be found in many domains, including electromagnetic and acoustic sensors. Compared to a single sensor, sensor arrays offer a number of performance benefits; the most important of these being increased directivity. While a single physical sensor has a directivity pattern set by its design, the outputs of multiple sensors in an array can be combined to generate multiple directivity patterns. Even if the individual sensors are omni-directional (that is, their directivity pattern is unity in all look directions), an array can be designed with a highly directional pattern. As such, arrays perform many functions including direction-of-arrival estimation, spatial filtering, and spatial spectrum estimation. A direct comparison can be made between the function of an array in the spatial domain and that of time sampling in the time domain, underlining its importance in understanding any signals that vary in space.

The most basic array design is the Uniform Linear Array (ULA). ULA’s consist of sensors spaced evenly and colinearly, as shown in Fig. 1.1. The spacing between adjacent elements determines the minimum wavelength that the array can process without ambiguity such that $d = \lambda/2$. This distance is often called the spatial Nyquist rate, and is commensurate with the Nyquist frequency in time sampling. When measuring wavelengths shorter than the array is designed for, spatial aliasing can occur in the form of grating lobes, as shown in Fig. 1.2. Therefore, for applications where the entire range of angles is necessary, ULA’s must adhere to the spatial Nyquist rate.

The primary functions of arrays are to isolate sources in an environment from each other and reduce the presence of noise in a measurement. To this end, two factors important to ULA design are the array aperture and the total number of sensors. According to the Rayleigh resolution criterion [3], two sources are just
resolved when the peak of the response generated by one source coincides with the null of the other. In this way, resolution directly relates to the beamwidth of an array, which is itself a function of the array geometry and shading. For the simple case of a discrete uniform linear array, the location of the first null in the beampattern is found at \( \sin(\theta) = \lambda/L \), where \( L \) is the aperture of the array [4], giving a direct relationship between an array’s resolution and aperture.

The actual usable aperture of the array can be limited by environmental factors, including the signal coherence length, which will be covered in more detail later. The total number of sensors in the array also contributes to the array’s performance by reducing noise, often quantified with a value called array gain. Under the assumption of noise on individual channels being independent and identically distributed, the array gain (in dB) is given by Eq. 1.1 where \( K \) is the number of sensors [5]. This value changes when the assumption of independence is violated, as will be covered later.

\[
\text{Array Gain} = 10 \log_{10}(K)
\] (1.1)
1.1.1 Sparse Arrays

While the Uniform Linear Array (ULA) is a very common design, there is a large class of non-uniform array designs that are also of theoretical and practical interest. Among these is the sparse array, which is any array whose average sensor spacing is greater than that dictated by the spatial Nyquist rate of \( d = \lambda / 2 \). The benefits of such arrays are two-fold. First, by decreasing the number of sensor elements along the length of the array, the material cost of a sparse array can be significantly lower than a ULA of the same aperture. Second, the reduction in the number of received channels can simplify data processing by reducing the amount of data collected and stored. However, as stated previously for a uniformly spaced array with sensor spacing greater than the spatial Nyquist rate, spatial aliasing can occur. To overcome this limitation, a number of non-uniform sampling schemes have been developed that minimize or completely eliminate spatial ambiguity resulting from the reduced number of sensors.

There are multiple methods for designing sparse arrays, but the one we will focus on utilizes the concept of the coarray [6]. For many mathematical operations on the array, including spatial spectrum estimation, calculation relies only on the spatial correlation function between spatial locations. Under certain common conditions, this correlation function is only a function of distance between locations, also known as spatial lag. The coarray represents the set of spatial lags that can be directly measured by an array, and by extension, the support of the correlation function as measured by the array. As such, considerable attention has been paid to array designs that generate favorable coarrays while minimizing the number of physical elements. Among the designs inspired by such methods are Minimum Redundancy Arrays (MRAs), Minimum Hole Arrays (MHAs or Golomb arrays), nested arrays, and coprime arrays [2, 7, 8, 9].

1.1.2 Coprime Arrays

One sparse array design of considerable contemporary interest is the coprime array. This array, first described by Pal and Vaidyanathan [2], consists of two colinear undersampled uniform linear arrays working in concert. The first array is undersampled by some integer factor \( M \) and the second by a factor \( N \), chosen to be coprime. An example of the coprime array design is shown in Fig. 1.3. While each of the two component arrays that make up the coprime array are undersampled, and thus generate grating lobes, the full array made up of elements from each coprime array does not possess such ambiguities, as shown in Fig. 1.4. Since its introduction, a wealth of literature regarding coprime sampling has been published, considering its application for Direction-of-Arrival (DOA) estimation [9], synthetic aperture sonar [10], compressive sensing [11, 12, 13, 14], modal analysis [15], and control systems [16, 17]. Extensions of the coprime array concept to new array geometries have also been explored, including coprime arrays with extended
Figure 1.3. A coprime array made up of two component uniform linear arrays undersampled by factors of $M = 3$ and $N = 4$, respectively. The minimum element separation is $d = \lambda/2$.

Figure 1.4. The beampattern of a coprime array with $M = 3$ and $N = 4$ shown with the beampatterns of the component arrays. The coprime array has an unambiguous main lobe, while the component arrays possess grating lobes.

Considering the existence of other sparse array geometries, one might question why the coprime array, in particular, has garnered so much attention in recent years. While one reason may simply be that the array geometry is new, its design as a pair of uniform arrays lends itself well to the repurposing of existing uniform linear array designs and processing methods. Indeed, work has been done demonstrating the application of uniform array shading to the outputs of the component apertures [18, 19], generalized coprime arrays [20], coprime arrays with component arrays spatially displaced [21], and multidimensional coprime arrays [22].
arrays [19]. Moreover, when compared to random arrays, or arrays generated through exhaustive searches (such as the minimum redundancy array [7]), the features of coprime arrays can be more easily predicted because of their deterministic design. Still, a number of open questions regarding coprime arrays exist, several of which are directly applicable to their application in the shallow water environment.

Since the introduction of coprime arrays, a number of questions have been raised regarding their performance and potential applications, many of which are still open. In 2012, the Office of Naval Research issued a Basic Research Challenge on “Co-Prime Sensor Array Signal Processing,” highlighting the following areas of interest: two-dimensional arrays, active systems and waveform design, random element spacing errors, imaging systems, multipath environments, and efficient Fourier transform algorithms [23]. Of these, the questions of system performance in the presence of element spacing errors and sensing in the multipath environment are of direct interest for shallow water acoustics. The former issue is relevant for practical line arrays, both moored and towed, which often consist of elements placed along a flexible line that can bend and deform during deployment and use. The issue of multipath is of interest for vertical arrays, as the shallow water environment supports a large number of reflecting paths interacting with both the sea surface and bottom, and horizontal arrays must also account for scatterers that generate coherent interference, not accounted for in the theoretical framework underpinning coprime arrays.

In addition to these direct technical questions regarding the performance and design of coprime arrays, questions of the practical utility of coprime arrays when compared to other sparse arrays designs have been noticed in the discussions following conference presentations on the topic. With that in mind, an increasingly central question in the study of coprime arrays has become “What, if any, benefits do coprime arrays offer when compared to existing sparse array geometries, such as minimum redundancy and random arrays?” While fully answering this question is outside the scope of this dissertation, it will still be addressed in passing, particularly when the potential benefits of lag redundancy are considered, and the potential for subarray processing to be used with coprime arrays is demonstrated.

1.2 Underwater acoustics

Oceans cover a large percentage of the Earth’s surface, and are of great importance for economic, ecological, and strategic purposes. As a result, sensing and exploring the underwater environment is of great value. While electromagnetic radiation at a wide range of wavelengths penetrate the atmosphere and can be measured at long distances in air, such radiation quickly absorbs in water and only penetrates a few hundred meters. For the purposes of long range sensing and communications, especially considering the sheer size of the world’s oceans, this distance is insufficient. Acoustic waves, in contrast, can propagate for very long distances in
the ocean, potentially thousands of kilometers under appropriate conditions [24]. As such, acoustics is the preferred sensing modality in the ocean, and considerable research over the last century has been dedicated to advancing the scientific understanding of how sound waves travel through various ocean environments.

1.2.1 Shallow water acoustics

The shallow water environment is at once the most important and the most difficult in underwater acoustics. Defined broadly as water of less than 200 m depth (as one might find on the continental shelf), shallow water houses a large proportion of the ocean’s life (especially commercially important fish species), and hosts the majority of non-shipping related human activity. But acoustic sensing in this environment is complicated by a number of factors that can make modeling difficult and deleteriously impact array performance. These effects include acoustic multipath, internal waves, surface and bottom interactions, volume scattering, and noise [5]. These effects lead to reduced signal coherence in shallow water [25], and low signal-to-noise ratios compared to the deep ocean.

While understanding and overcoming the effects of the shallow water environment are important, little attention has been given to how these effects might directly impact the performance of coprime arrays and other sparse array designs. Specifically, because the shallow water environment may violate the assumption of spatial stationarity, key to the development of coarray theory, sparse array performance will differ from theoretical predictions in a practical setting. Understanding, predicting, and quantifying these deviations is the primary contribution of this work.

1.3 Dissertation Overview

This dissertation attempts to integrate two separate, but related fields of study: sparse array design and shallow water acoustics. As such, a large amount of background material must be covered before new results can be presented.

We begin in Chapter 2 with a brief overview of acoustic arrays and array signal processing. We examine time-domain delay-sum beamforming, and related frequency-domain techniques that make direct use of the sample covariance matrix. This leads naturally to a discussion of coarrays, one of the theoretical bases motivating the design of sparse arrays. A short overview of sparse arrays follows, highlighting designs of practical and historical interest. Finally, we consider array performance metrics that are relevant to both uniform and sparse array designs that will later be used to quantify the performance of the coprime array in a variety of conditions. We specifically consider lag redundancy, aperture utilization, resolution, sidelobe level, and array gain.
In Chapter 3, the current state of the coprime literature is summarized. We define the concept of a coprime array and show a number of features common to the many designs that have been created as part of the coprime array family. An overview of these array designs follows, covering extensions, generalizations, and modifications of the original coprime array concept. Data gathered by the coprime array is then subject to various sorts of processing. We break these forms of processing into two major categories, multiplicative and covariance-based, and describe the uses and advantages of each. Finally, we identify the underlying assumptions that coprime arrays rely on to guarantee performance that may not hold in a realistic operating scenario.

In Chapter 4, we consider lag redundancies in the coprime array, and investigate the effects of such redundancies on sparse array performance. We also develop the centered coprime array, a novel approach to coprime array design that places certain guarantees on the array’s usable aperture, and thus ensures a certain level of resolution that cannot be guaranteed by traditional coprime array designs. We further show that this design lends itself to a subarray processing scheme that is applicable to the coprime array, but is not generally applicable to sparse arrays.

In Chapter 5, we consider the shallow water environment, and what impact the realities of this environment has on the underlying assumptions of the coprime array. We then describe a number of test cases that exemplify these different effects, and predict the performance of the coprime array under these test cases. Simulated results are presented demonstrating these test cases, and further quantify how array performance (measured by our chosen set of metrics) is impacted by channel features like signal coherence length and noise directionality.

Finally, we conclude in Chapter 6 with a brief discussion section, summarizing the primary research results of this dissertation, and suggesting future work on coprime arrays that could answer questions still left open.
Chapter 2

Acoustic Arrays

The measurement of waves is an essential task for both electromagnetic and acoustic radiation. Each sensing modality has a specialized sensor associated with it. Radar uses antennae, optics relies on Charge Coupled Devices (CCDs), and aerospace uses microphones. For the purposes of underwater acoustics, hydrophones are the sensor of choice. Hydrophones come in many varieties, but the most common are made of piezoelectric materials. These materials generate a differential voltage when subjected to pressure, allowing for the direct measurement of pressure waves through transduction. The hydrophone’s response to sounds coming from different directions is known as its directivity, and the directivity of a single sensor is a function of that sensor’s size, with respect to a given wavelength. We will primarily focus on low frequency waves (less than 1 kHz), and at this frequency range, most individual hydrophones are omnidirectional, i.e., they respond equally to sound waves impinging from any direction.

Multiple sensors working in concert are known as an array, and such arrays find wide application in every sensing modality. In underwater acoustics, hydrophone arrays of one, two, or three dimensions are all used for different purposes. We will primarily consider one-dimensional arrays (linear arrays), consisting of hydrophone elements connected by a flexible line, which may be towed behind a vessel, moored to static anchors, or otherwise suspended in the water column. Often, the output of each hydrophone is acquired on a separate channel, which can then be processed together.

In this chapter, we consider acoustic array processing and design. We begin by considering beamforming techniques, which combine the outputs of the sensors in an array to estimate the far field intensity pattern. We describe both time and frequency domain techniques, the latter of which inspires the concept of the coarray. Utilizing this coarray concept, we explore the field of sparse array design. Several such designs will be discussed, allowing for a better understanding of the context of coprime arrays, to be described in Chapter 3.
2.1 Beamforming

In environments with multiple sources distributed in space, it is often desirable to selectively listen to each source individually. Doing so reduces noise and interference, and improves estimation of signal characteristics, such as source strength. While this goal can be achieved by physically rotating a single directional sensor, sensor arrays offer a more flexible approach through beamforming. Beamforming is any processing method that combines outputs of multiple sensor elements to spatially filter a sound field incident on an array. By combining the outputs of multiple directions, the far field source function can be estimated, giving the locations and intensities of all sources in the far field. In this section, we consider two methods of beamforming. Time-domain beamforming uses delays on the outputs of each element of an array to compensate for the difference in arrival times for a plane wave coming from a given angle. Frequency-domain beamforming takes advantage of the Fourier transform relationship between the source function and the incident sound field, directly measuring the spatial spectrum and performing an inverse transform to yield the source function. After introducing these techniques, we further examine frequency-domain beamforming and its relationship to coarrays, which directly inspires sparse array designs that more efficiently sample the spatial spectrum, enabling frequency-domain beamforming with a reduced number of elements.

2.1.1 Time-Domain Beamforming

Time-domain beamforming, also known as delay-sum beamforming, is a straightforward approach to beamforming that acts directly on the time-domain outputs measured at each sensor location. The prototypical problem is shown in Fig. 2.1, a source at some location \( \vec{u} \) generates sound that is sensed at an array with \( K \) sensors whose element locations are \( \vec{x}_k \). If \( s(t) \) is the signal generated by the source, then the output measured in a non-dispersive isovelocity medium at each receiver location is given by Eq. 2.1, where \( c \) is the sound speed, \( A_k \) is a scalar representing signal attenuation, and \( z_k(t) \) is independent noise measured at sensor \( k \) with power \( \sigma_{z_k}^2 \). We then define \( t_0 = (\vec{u} - \vec{x}_0)/c \) as the propagation delay to the first sensor, and \( \tau_k \) as the propagation delay between arrival at the first sensor and sensor \( k \) of the array.

\[
s_k(t) = A_k s \left( t + \frac{\vec{u} - \vec{x}_k}{c} \right) + z_k(t) = A_k s(t + t_0 + \tau_k) + z_k(t) \quad (2.1)
\]

To perform time-domain beamforming, the output of each sensor is delayed to compensate for the propagation delay between each sensor. Without loss of generality, we let the position of the first sensor \( \vec{x}_0 \) be the origin. If the source is sufficiently distant, such that incoming waves can be approximated as plane
waves, and the angle between the array axis and the source vector is \( \theta \), then the inter-element delay is simply
\[
\tau_k = \frac{x_k}{c} \sin \theta. \tag{2.2}
\]
If we let \( A_k = A \sqrt{\kappa} \), then the response \( v(t) \) measured as the sum of the delayed outputs becomes
\[
v(t) = kA s(t + t_0) + \sum_{k=0}^{N} z_k(t - \tau_k) = kA s(t + t_0) + z(t), \tag{2.3}
\]
Where \( z(t) \) is the total noise, with power \( \sigma_z^2 = \sum_{k=0}^{N} \sigma_{z_k}^2 \).
To estimate the far-field source pattern using time-domain beamforming, delay outputs for multiple look directions might be applied, and the mean square power measured for the associated response. Such an estimate is shown in Fig. 2.2 for a signal of unity intensity at broadside \( (\theta = 0) \) for a twelve element uniform array with sensor spacing \( d = \lambda/2 \). The beam response to such a source is called the beam pattern of the array.

### 2.1.2 Frequency-Domain Beamforming

A common alternative to time-domain beamforming is frequency-domain beamforming, also called phased array beamforming. This method relies on the Fourier transform relationship between far field source intensity and incident sound field, using the measurements at each array element to estimate the spatial spectrum,
which is then inverted. We follow the development of Hoctor and Kassam [6] which parallels the work of Blackman and Tukey [26].

Consider a narrowband system operating at some angular frequency $\omega_0$. We define the source amplitude function

$$S(\vec{u}, t) = s(\vec{u}) \exp(j\omega_0 t),$$

(2.4)

Where $s(\vec{u})$ is the complex amplitude at location $\vec{u}$. The received amplitude at location $\vec{x}$ is then given by

$$v(\vec{x}) = \int s(\vec{u}) \exp(jk_0 \vec{u} \cdot \vec{x}) d\vec{u}. $$

(2.5)

We then consider the cross-correlation function of the amplitude between locations $\vec{x}_1$ and $\vec{x}_2$, defined as

$$r(\vec{x}_1, \vec{x}_2) = \mathbb{E}[v(\vec{x}_1)v^*(\vec{x}_2)].$$

(2.6)

For a source distribution such that all sources are uncorrelated, this reduces to the following:

$$r(\vec{x}_1, \vec{x}_2) = \int I(\vec{u}) \exp(jk_0 \vec{u} \cdot (\vec{x}_1 - \vec{x}_2)) d\vec{u},$$

(2.7)
Where $I(\bar{u})$ is the far-field intensity at location $\bar{u}$. The cross-correlation is a function only of spatial lag $\gamma = \bar{x}_1 - \bar{x}_2$, which allows us to express the correlation function and source intensity function as

$$R(\gamma) = \int I(\bar{u}) \exp(jk_0 \bar{u} \cdot \gamma) d\bar{u}. \quad (2.8)$$

Eq. 2.8 establishes the cross-correlation and source intensity functions as a Fourier transform pair. As a result, measurement of the cross-correlation function allows for an estimate of the source intensity function through an inverse Fourier transform operation. For a discretized set of weighted lag measurements, this may be expressed by Eq. 2.9, which corresponds to the Blackman-Tukey method of spectral estimation.

$$\hat{I}(\bar{u}) = \sum_{\gamma \in C_d} \gamma_y R(\gamma) \exp(-jk_0 \bar{u} \cdot \gamma) \quad (2.9)$$

Where $\gamma_y$ is the spectral weight of lag $\gamma$. A connection can be drawn here between time-domain and frequency domain beamforming, as the phase shift term in the exponential corresponds to a time delay given by the travel time between the source at some location $\bar{u}$ and a receiver at location $\gamma$. While $\gamma$ does not necessarily correspond to a physical sensor location (rather it is a spatial lag that might be measured at any physical location), it is sometimes described as a virtual sensor location, corresponding to the same lag being measured from the origin. In this way, the Blackman-Tukey spectral estimation method shown in Eq. 2.9 is analogous to a weighted time-domain beamforming operation carried out on an array of virtual sensors at locations given by the set of lags measured by the array (also known as the coarray).

An alternative formulation uses the cross-correlation matrix to similar effect. This matrix is defined by

$$R_{mn} = \mathbb{E}[v(\bar{x}_m)v^*(\bar{x}_n)], \quad (2.10)$$

Where $\bar{x}_m$ and $\bar{x}_n$ are the locations of the $m$th and $n$th sensors, respectively. For a uniformly spaced array, the far field intensity is estimated by the Bartlett beamformer given in Eq. 2.11, where $w$ is the steering vector associated with location $\bar{u}$.

$$\hat{I}(\bar{u}) = w^H R w \quad (2.11)$$

The output of this beamformer is identical to that of the beamformer given in Eq. 2.9, for a filled coarray weighted by a Bartlett window, defined by

$$\gamma_y = \begin{cases} 
L + 1 - y & : y \leq L \\
0 & : y > L
\end{cases}, \quad (2.12)$$
Where $L$ is the length of the array. The Bartlett window will be used in future sections to directly compare performance of a coprime array to a uniform array.

### 2.1.2.1 Coarrays

In examining the application of the frequency-domain beamforming techniques described above to sparse arrays, it is useful to understand the concept of the coarray, described by Hoctor and Kassam [6]. According to this, “the coarray of a given aperture may be defined as the set of all vector spacings between points in the aperture.” In other words, the coarray of an array is the set of all physically realized lags in the array. The “difference coarray,” as it were, is then formally defined as follows:

$$C_d = \{ \bar{y} | \bar{y} = \bar{x}_1 - \bar{x}_2, \text{ for } \bar{x}_1, \bar{x}_2 \in X \},$$

(2.13)

where $C_d$ is the difference coarray and $X$ is the array aperture function. It should be noted that Hoctor and Kassam also define the “sum coarray,” which may be applied for coherent imaging, as with active sonar, but for the purposes of this work we will only consider the difference coarray and refer to it simply as the coarray.

The coarray itself is valuable in that it determines the set of points on which the data covariance function can be directly estimated. As shown previously, the data covariance function can be used for beamforming, so in practice the coarray is a determining factor of beamforming performance. Hoctor and Kassam describe this in terms of the possible point spread functions that can be generated given a particular array aperture. The point spread function as described in optics is equivalent to the beam pattern used in sonar. This naturally leads to an equivalence relation that all arrays that possess the same coarray are capable of generating the same set of beam patterns given appropriate coarray weightings. Coarray weightings are the spatial frequency domain equivalent of array shading, and Hoctor and Kassam’s paper shows that all traditional shading functions have an equivalent coarray weighting that will generate the same beam pattern when applied. The Bartlett window, given in Eq. 2.12 is an example of such a weighting, which corresponds to uniform shading on a uniformly spaced linear array. Similar spectral windows can be constructed corresponding to Dolph-Chebyshev or any other shading.

For the purposes of sparse array design, four features of the coarray are of particular interest: the smallest non-zero element, the largest element, the total number of elements, and the range of contiguous elements. The smallest element in the coarray determines the smallest wavelength the array can process unambiguously. This minimum element spacing must be less than or equal to $\lambda/2$, which is the same restriction as that placed on uniform linear arrays. The largest element in the coarray determines the array’s resolution, as this is the maximum aperture.
size of the array. The total number of unique elements in the coarray gives the array’s degrees-of-freedom, and is also the total number of sources that can be determined using processing methods like compressive sensing. Finally, the range of contiguous elements describes the set of elements in the coarray which includes no holes (missing coarray elements). Considering that the coarray of a uniform linear array is contiguous along its entire length, the range of contiguous elements dictates the largest uniform linear array to which any given array is functionally equivalent (in terms of generable beam patterns). This range is also known as the “fully augmentable range” of the array, a term derived from covariance matrix augmentation [27].

While the coarray is not a complete description of an array’s performance (notably, one cannot determine array gain given the coarray alone), it is still a valuable tool for understanding and comparing different array geometries, and one which will be referred back to throughout this paper.

2.1.2.2 Minimum Variance Distortionless Response Beamforming

An alternative to the Blackman-Tukey method presented above and the related technique of the Bartlett beamformer, is the Minimum Variance Distortionless Response (MVDR) beamformer [28]. In this method, the correlation matrix ($R$) is inverted, prior to the application of the steering vector, and the resulting output is then inverted to give the beamformer output. Eq. 2.14 gives the mathematical expression of this process.

$$
\hat{I}(\vec{w}) = \frac{1}{w^H R^{-1}w}
$$

(2.14)

The name Minimum Variance Distortionless Response refers to the process giving a minimum variance unbiased estimate of the spatial field [29]. MVDR possesses characteristics of a super-resolution beamformer. With a very narrow beamwidth and an ability to resolve closely spaced sources, the method has distinct advantages over the conventional Bartlett beamforming approach. An example of the narrower beampattern afforded by MVDR can be seen in Fig. 2.3. However, this beampattern can be misleading as the narrowness of the beam does not result in a comensurate increase in noise rejection, nor an ability to overcome the Vandermonde problem associated with coherent sources. Rather, the narrow beam associated with MVDR improves the array’s capacity for direction finding and resolution of spatially compact incoherent sources [30].

This improved performance, however, comes at the cost of a computationally intensive full-rank matrix inversion operation. This is further complicated for sparse arrays, or arrays in the presence of coherent multipath, as the covariance matrix may not be invertible [31]. MVDR may still be applied, however, either through projection onto the set of invertible matrices, or through a positive definite Toeplitz completion operation [32]. In order to avoid these operations, and to
Figure 2.3. The beampattern of a uniform linear array for Bartlett beamforming and super-resolution MVDR beamforming. MVDR generates a much tighter peak, at the cost of a computationally expensive matrix inversion step.

make performance analysis computationally tractable, we will not consider MVDR processing on the coprime array, though we recognize that the fully augmentable range of the coarray is relevant to the aforementioned covariance matrix completion operations.

2.1.2.3 Compressive Sensing

One processing approach of recent interest is compressive sensing. Compressive sensing relies on underlying sparseness in the signal to identify distinct signals in noise [33, 34]. The sparse signal representation demands of compressive sensing can be equated to the assumption of compact sources in shallow water acoustics (which will be discussed in Section 3.3.3), and quite a bit of work has been done in applying compressive sensing techniques to coprime arrays [11, 12, 13, 35, 36]. Compressive sensing can be difficult to compare to traditional beamforming techniques, as the output of a compressive sensing system is not a beam response. Instead, algorithms such as Orthogonal Matching Pursuit (OMP) or LASSO gives source locations and powers for a number of sources either chosen a priori (as with OMP) or determined in process (as with LASSO). There are also Bayesian compressive sensing algorithms that allow for quantification of uncertainty, in addition to estimates of source location and power [37]. A key point on these processes is that they do not necessarily require that coarray elements be contiguous over a
range, and the measurement matrices that they rely on can be determined numerically through simulation, giving them a great deal of practical flexibility.

While we will not directly consider compressive sensing in this dissertation; it is acknowledged here because of the wealth of literature applying compressive sensing techniques to the coprime array.

2.2 Sparse Arrays

A sparse array is a sensor array with a lower density of elements than dictated by the spatial Nyquist rate \(d = \lambda/2\). While a uniformly undersampled array under this limit will experience spatial aliasing in the form of grating lobes, non-uniform sampling below this rate does not necessarily incur the same effect. A number of sparse array designs exist that handle this non-uniform sampling differently. Coprime arrays are one of the many possible sparse array geometries, and in this section we will discuss several of the alternatives that exist. These array designs will appear again in later chapters where they will be compared with coprime arrays using a number of performance metrics.

2.2.1 Minimum Redundancy Arrays

Minimum redundancy linear arrays were developed by Alan T. Moffet for applications in radio astronomy [7]. A minimum redundancy array is, by definition, any array which possesses the maximum fully augmentable range for a given number of elements. In Moffet’s paper on the topic, he shows a number of array configurations that can be deemed “Minimally Redundant” based on work by mathematicians J. Arsac, R.N. Bracewell, and J. Leech. The first examples given are of so-called “perfect arrays,” which possess no redundancies, shown in Fig. 2.4. Beyond four elements, however, such non-redundant configurations do not exist.

Moffet describes two types of minimally redundant arrays, which he terms “restricted” and “general.” An example of each of these array configurations is shown in Fig. 2.5, each with five elements. Restricted arrays have a physical aperture equal to their fully augmentable range, and as a result are hole-free up to the total length of the array. General arrays, on the other hand, allow a larger physical aperture, and thus generate coarrays that have holes. Moffet points out that there are advantages to each of these types of array. The restricted case benefits from a smaller physical aperture, which can reduce cost and may simply be dictated by necessity because of space constraints. The general case can result in a larger fully augmentable range, which has obvious benefits, but may give a number of non-contiguous measurements in the coarray that cannot be exploited by all processing frameworks.

Minimum redundancy arrays have proven to be very attractive, because (by definition) they optimally cover the coarray for a given number of elements. How-
Figure 2.4. The four perfect arrays. These minimum-redundancy arrays possess zero redundant lags, each lag is represented exactly once up to the full array length. Beyond four elements no perfect array configurations exist.

Figure 2.5. Two minimum redundancy array configurations for five elements. (a) The restricted case, where the fully augmentable range is equal to the physical aperture. (b) The general case, which possesses missing lags in its coarray.

ever, there are a number of practical considerations that limit their use. As discussed in Moffet’s paper, there is no programatic way to generate a new minimally redundant array configuration for a given number of elements or to expand an existing array configuration through the addition of elements. Thus, new array configurations can only be found through brute force search of all possible array configurations. Moreover, searching for such configurations is a computationally expensive process that has been proven to be NP-hard. Luckily, many such configurations have already been determined, in the form of Golomb rulers, and once such a ruler has been determined that design can be used for any array with that number of elements. At the time of writing, the highest order Golomb ruler known is 27, with a total length of 553 (as determined by distributed.net in February 2014) [38]. As will be shown later, this aperture far exceeds the maximum usable aperture dictated by channel coherence, even in deep water.
2.2.2 Random Arrays

The theory of random arrays was developed by Yuen Tze Lo in a series of papers produced starting in 1962. These arrays consist of a number of elements placed randomly according to some distribution over a given aperture. Despite their probabilistic nature, however, random arrays have been shown to have a number of important properties that can be predicted by theory. Main beam width, sidelobe level, and array gain can all be determined using the number of elements and the total aperture of the array [39, 40], and grating lobes are not a concern because of the aperiodic structure of the array. Furthermore, by using Monte Carlo methods to generate and analyze a large number of random arrays, designers can specifically select arrays that have favorable qualities even within a given set of parameters.

2.2.3 Nested Arrays

Nested arrays are another form of sparse array. These arrays consist of two distinct sections: a section of high element-density, and a section of low element-density. The high element-density section has $N_1$ elements with a spacing $d_1$, where $d_1$ is chosen according to the spatial Nyquist rate. The low element-density section has $N_2$ elements with a spacing of $d_2 = (N_1 + 1)d_1$ [9]. An example of this array configuration can be seen in Fig 2.6. This array is closely related to the coprime array, and has even been described as a limiting case of the generalized coprime array [20]. The nested array is notable because it possesses a full coarray up to the array aperture with a smaller number of elements than a comparable uniform linear array. However, because the high element-density section is sampled at the Nyquist rate, this section can suffer from a high amount of mutual coupling when compared to a coprime array with the same number of elements [41].

2.3 Performance metrics

Array performance is quantified using a number of standard metrics. In this section we will detail several of these metrics, which will later be used to compare the
Figure 2.7. Array geometry specific metrics for a prototype coprime array with M=3 and N=4. Lag redundancy describes all lag measurements beyond the first, aperture is both the physical extent of the array and the largest lag present in the array, and the fully augmentable range is the range of lags over which the array is hole free.

performance of the coprime array and other array designs, both under conditions consistent with the underlying assumptions in Sec. 3.3 and in the presence of shallow water effects that may alter or invalidate those assumptions. These metrics will be broken into two main categories: array geometry specific metrics and beampattern metrics. Array geometry specific metrics include redundancy and aperture utilization, which determine an array’s efficiency at sampling the spatial domain and are entirely dependent on the array’s geometry. Beampattern metrics, on the other hand, are a direct measure of the array’s practical performance, and may be subject to change when environmental conditions limit or otherwise impact the array’s performance.

2.3.1 Redundancy

Lag redundancy is a fundamental concept in sparse array design, and is a primary metric of a design’s efficacy. A lag redundancy exists when a given spacing between elements exists more than once in the array. Formally, the coarray as defined in Eq. 2.13 can be extended to give the weighting of each lag as follows:
where \( w_k \in \{0, 1\} \) is a weighting determined by the array’s aperture function. Any element of this lag weighting function such that \( c_d(y) > 1 \) is a redundant lag [42]. Fig. 2.7 shows the redundancies in the coarray weighting function of an example coprime array.

Under conditions of spatial stationarity, lag redundancy is undesirable because additional lag measurements do not increase the size of an array’s coarray, and thus do not allow for the direct synthesis of any additional array configurations through spectral weighting. Lag redundancies increase the cost of the array, through additional measurements, without improving the array’s performance. Additionally, when used for conventional beamforming, without spectral weighting applied, lag redundancies actually increase the peak sidelobe level of the array’s beampattern [43], directly reducing the array’s performance.

While these harmful effects of lag redundancy have been well examined, there are potential benefits to lag redundancy. For example, if correlation measurements for a given lag are noisy, then lag redundancy can reduce the error associated with that lag. In this way, redundancy directly improves array gain, which may be valuable for certain applications. Additionally, lag redundancy can be used to reduce the impact of coherent noise, though the process of averaging lag measurements can generate nondefinite covariance matrices [31]. Moreover, spectral weighting can eliminate the performance loss associated with lag redundancy, and is an inexpensive operation for any processing chain which includes the synthesis of the array covariance matrix.

Given that the aperture function of a uniform array is a rectangular function, the uniform array possesses the maximum number of redundancies for a given aperture. The resulting lag weighting function is triangular, with successively fewer redundancies for larger lags.

Coprime arrays also necessarily include lag redundancies, though far fewer than a comparable uniform linear array, as shown in Fig. 2.8. Lag redundancies exist within the uniform component arrays, and it will be shown in Sec. 4.1 that lags measured between elements of each component array also possess redundancies at semi-regular intervals. Moreover, because the component arrays are uniform, their redundancy patterns are spectrally periodic. That is, for a coprime array with coprime factors \( M \) and \( N \), coarray lags that are multiples of \( M \) or \( N \) possess larger numbers of redundancies than adjacent lags which are not multiples of those factors. For unweighted conventional beamforming, periodic redundancies gives rise to high peak sidelobe levels, when compared to the equiripple sidelobes generated from aperiodic or uniform redundancies.
2.3.2 Aperture efficiency

While correlation lag redundancy can result in increased array cost without commensurate performance improvements, missing correlation lags (termed “holes”) can result in performance degradation and limit the effective aperture of an array for certain signal processing applications. A rough tradeoff exists between redundancies and holes, with no array beyond four elements possessing a coarray that is both hole-free and non-redundant [7].

The set of contiguous lags in an array’s coarray (sometimes known as the “fully augmentable range” or the “hole-free range”) is relevant in several signal processing applications. Within this range, the coarray of a given array is identical to a uniform linear array of length equal to the array’s fully augmentable range. This allows the direct application of eigenspace techniques (like MUSIC) [32], and the completion of a Toeplitz covariance matrix without the need for interpolation or other techniques. For the purposes of algorithms like MUSIC, the fully augmentable range of the array represents the effective aperture of the array, and as shown in Sec. 2.3.3, resolution is a direct function of array aperture, with increased aperture resulting in improved resolution.

Aperture efficiency is a metric that does not appear in the literature, but can be given as the ratio of an array’s fully-augmentable range to its physical aperture, both of which are shown in Fig. 2.7. It can be described as the ratio of the array’s effective aperture to the effective aperture of a uniform linear array with the same
Figure 2.9. Beampattern metrics for a naturally weighted prototype coprime array with M=3 and N=4. Beamwidth is the distance from null-to-null of the main beam, sidelobe level is the ratio of the peak of the beam pattern to the highest sidelobe, and array gain is the ratio of the in-band signal to noise ratio before and after beamforming.

physical extent. Aperture efficiency is constrained to the interval (0, 1]. By design, a uniform array always has an aperture efficiency of 100%. A coprime array, on the other hand, will necessarily have holes [2], and thus will have an aperture efficiency less than 1. While the exact aperture efficiency of a coprime array will vary based on the array design, it will be shown in Sec. 4.3 that bounds on the aperture efficiency can be computed.

2.3.3 Resolution

Resolution is often the most important metric for an array’s performance. The resolution of an array is the smallest angle of separation between two sources such that the sources can be independently measured. Resolution can be directly related to the array beampattern through the Rayleigh criterion, which states that two sources can be distinguished when the maximum of one source’s response coincides with the first minima of the other source [3]. Thus, in this way, the resolution of an array is one half of the zero-to-zero main beamwidth, shown in Fig. 2.9. Maximum achievable beamwidth is almost exclusively a function of an array’s usable aperture, independent of other features of the coarray, which incentivizes the design of arrays with large aperture for applications where resolution is of high importance.
Unfortunately, increased array aperture cannot improve resolution to an arbitrary extent. The practical limit on coherent signal processing is the signal coherence length, which is a function of the acoustic environment and range. As such, sonar performance is optimal when the array aperture is equal to or less than the signal coherence length. For reference, the coherence length in shallow water is often limited to approximately 30 wavelengths, while the coherence length in deep water can be 100 wavelengths or more [25], the reasons for this will be covered in detail in Sec. 5.2.

2.3.4 Sidelobe level

Sidelobes are a feature of an array’s beampattern and beam response. The sidelobes are the local maxima found in sections of the beampattern away from the main beam, where no source is located. The sections of the beampattern containing sidelobes is shown in Fig. 2.9. To some extent, sidelobes are unavoidable, as all conventional discrete beamformer beampatterns will necessarily have them. However, the exact properties of the sidelobes are a direct result of the spectral weighting of the correlation function. In the absence of explicit spectral weighting, however, the array’s geometry determines the sidelobe level by way of the coarray weighting function. Both holes and redundancies in the coarray contribute to elevated sidelobe levels [42, 43], and sidelobes are particularly high when redundancies are periodic.

For imaging purposes, the peak sidelobe level determines the dynamic range. This is because signals below the peak sidelobe level of another signal are indistinguishable from those sidelobes.

2.3.5 Array gain

Oftentimes, arrays are used to amplify a signal in the presence of environmental or sensor noise. In these situations, the improvement in signal to noise ratio (SNR) becomes an incredibly important measure of an array’s performance, called array gain. For fully coherent plane wave signals, the array gain can be thought of as the ability of the array to reduce incoherent noise. Given an array beampattern $b(\phi, \theta)$ and directional noise intensity $|N(\phi, \theta)|^2$, this is often expressed as the ratio of the noise power measured at an omnidirectional sensor to the noise power at the output of beamformer as shown in Eq. 2.16 [4]. For an isotropic (spatially white) noise field, this value is the directivity index shown in Eq. 2.17, and is a measure of the array’s in-beam vs out-of-beam energy. With this in mind, sidelobe level is very important to array gain, as directivity index is directly related to the integrated sidelobe level.

\[
\text{array gain} = AG = 10 \log_{10} \left[ \frac{\int_{4\pi} |N(\phi, \theta)|^2 d\Omega}{\int_{4\pi} |N(\phi, \theta)|^2 b(\phi, \theta) d\Omega}\right]
\] (2.16)
directivity index = \( \text{DI} = 10 \log_{10} \left[ \frac{4\pi}{\int_{4\pi} b(\phi, \theta) d\Omega} \right] \) (2.17)

For fully coherent signals and fully incoherent noise, array gain is very simply calculated. Using the difference of in-phase and random phase addition, we can find that the array gain for an array with \( K \) sensors is given by \( \text{AG} = 10 \log_{10}(K) \). Of course, in practice, signals measured at a receiving array are neither fully coherent, nor are noise signals from propagating sources fully incoherent. Under this condition, we can measure the array gain directly as the difference between signal gain and noise gain for the beamformer, as shown below.

\[
\text{AG} = \frac{\text{SNR}_{\text{array}}}{\text{SNR}_{\text{element}}} = 10 \log_{10} \left( \frac{S_b}{S_e} \right) - 10 \log_{10} \left( \frac{N_b}{N_e} \right)
\] (2.18)

Where \( S_b, S_e, N_b, \) and \( N_e \) are the powers measured for the beamformed signal, the element-level signal, the beamformed noise, and element-level noise, respectively. For sparse arrays, like the coprime array, the array gain given in Eq. 2.18 is directly applicable when the covariance measurements from the array are naturally weighted. Under this processing scheme, there is a loss of array gain associated with the reduced number of independent channels. With spectral weighting, however, the array gain of a sparse array can be restored to that of a coarray equivalent uniform linear array. To illustrate this, we will consider the function of Toeplitz reconstruction and its effect on both a noise and signal covariance matrix.

Consider a fully augmentable sparse array with fewer physical elements than a coarray equivalent uniform array, as shown in Fig. 2.10. If we generate a measurement vector for this array, with zeros as placeholders for the missing elements, we can use this vector to construct a covariance matrix with missing rows and columns corresponding to the missing elements as shown in Eq. 2.19.
If we assume that the signal and noise are spatially stationary, then we can infer an important feature about the true covariance matrix: that it is Toeplitz. By utilizing this feature, we can fill the missing elements of the covariance matrix, so long as we possess at least one measurement of each diagonal (which is guaranteed by our specification of the array as a fully augmentable array). So long as our assumption of spatial stationarity holds, our new Topelitz array \( \hat{R} \) is identical to the true full covariance matrix that would be determined with a uniform array of equivalent length.

\[
\hat{R} = \text{Toep}(R) = \begin{bmatrix}
  r_0 & r_1 & r_2 & r_3 \\
  r_{-1} & r_0 & r_1 & r_2 \\
  r_{-2} & r_{-1} & r_0 & r_1 \\
  r_{-3} & r_{-2} & r_{-1} & r_0 \\
\end{bmatrix}
\] (2.20)

From here, we may now consider signal and noise gain for beamforming with this completed covariance matrix. For a plane wave signal with power \( \sigma_2^2 \) at some angle \( \theta \), the covariance matrix will have the structure given in Eq. 2.21.

\[
\hat{R}_{\text{signal}} = \sigma_2^2 \begin{pmatrix}
  1 & e^{j\pi \cos \theta} & e^{j2\pi \cos \theta} & e^{j3\pi \cos \theta} \\
  e^{-j\pi \cos \theta} & 1 & e^{j\pi \cos \theta} & e^{j2\pi \cos \theta} \\
  e^{-j2\pi \cos \theta} & e^{-j\pi \cos \theta} & 1 & e^{j\pi \cos \theta} \\
  e^{-j3\pi \cos \theta} & e^{-j2\pi \cos \theta} & e^{-j\pi \cos \theta} & 1 \\
\end{pmatrix}
\] (2.21)

Each element of the array has the same absolute value, but the phase angle is a linear function of distance from the diagonal. We may then determine a steering vector \( \mathbf{w} \), such that

\[
\mathbf{w} = 1/K' \begin{bmatrix}
  1 & e^{-j\pi \cos \theta} & e^{-j2\pi \cos \theta} & e^{-j3\pi \cos \theta} \\
\end{bmatrix}^T
\] (2.22)

This steering vector points in the true direction of the source, and thereby maximizes the beamformer output given by \( \mathbf{w}^T \hat{R}_{\text{signal}} \mathbf{w} \) for \( \sum |\mathbf{w}| = 1 \). We can then verify that the beamformer output in this look direction is given by \( \mathbf{w}^T \hat{R}_{\text{signal}} \mathbf{w} = K' \sigma_2^2 \). By a similar line of reasoning, we now consider a noise matrix whose covariance is given by Eq. \( \mathbf{R}_{\text{noise}} = \sigma_2^2 \mathbf{I} \). This covariance corresponds to spatially white noise which is independent between sensors. Under this condition, for all steering vectors \( \mathbf{w} \) such that \( \sum |\mathbf{w}| = 1 \), we find that \( \mathbf{w}^T \hat{R}_{\text{noise}} \mathbf{w} = K' \sigma_2^2 \). Given this, we find our array gain is \( K' \), the dimensionality of the covariance matrix and the number of elements in the equivalent uniform linear array, which is greater than predicted by the number of array elements.
The above array gain is a direct result of a number of assumptions about the signal and noise statistics. First, it is assumed that the signal is fully coherent along the length of the array, and is a plane wave such that the exact steering vector associated with the source is included in our set of steering vectors. Second, it is assumed that the noise is fully incoherent, such that all off-diagonal elements are identically 0. Moreover, the number of temporal snapshots is assumed large enough that the measured covariance matrix entries are identical to the true covariance entries at each measured lag. In later sections of this dissertation, those assumptions will be examined, and their effect on array gain described.
Coprime Arrays

Coprime arrays are a relatively recent sparse array design that is composed of two uniform undersampled arrays that are processed in concert [2]. Several commonalities exist between array designs in the coprime array family. First, all coprime arrays consist of two uniform arrays with interelement spacings given by $Md$ and $Nd$, where $d$ is the design spacing (often $d = \lambda/2$) and $M$ and $N$ are coprime integers, i.e. integers that share no common factor aside from unity.

The coprimality of $M$ and $N$ allows for features of the coarray to be proven. Specifically, through the use of Bézout’s Identity, which states that for any pair of integers $M$ and $N$, there exist some pair of integers $m$ and $n$ such that

$$\text{GCD}(M, N) = mM + nN. \quad (3.1)$$

Where GCD$(M, N)$ is the greatest common divisor of $M$ and $N$. For coprime $M$ and $N$, GCD$(M, N) = 1$. Using this, we may find a pair of integers $m'$ and $n'$ such $m'M - n'N = b$ for any integer $b$ (one choice being simply $m' = bm$ and $n' = -bn$). As a result, for any integer coarray spacing, there exists some pair of elements in an infinitely long coprime array that measures it. However, for a coprime array of finite length, the coarray will be a direct result of the actual element locations included in the array. With this in mind, several different coprime array designs have been developed that alter the lengths and relative positions of the two uniform arrays. A brief overview of these different designs will be given in Sec. 3.1, along with the potential applications the designs are suited for.

Data gathered by the array can be processed in a number of ways. We highlight two paradigms of coprime array processing, known as multiplicative and covariance beamforming. The former method employs two stages. The first stage beamforms the outputs of the individual component arrays to yield responses with significant ambiguity, and the second stage combines those outputs through multiplication or cross-correlation to resolve ambiguity. The latter method performs beamforming using the correlation between each element pair in the array explicitly, taking direct
advantage of the coarray properties of the coprime array.

Finally, to develop and prove the behavior of the coprime array, a number of simplifying assumptions are invoked. These include features about the signal and noise, as well as general features of the sound field measured by the array. Sec. 3.3 will detail each of these assumptions, and any connection to environmental effects that may exist. By explicitly recognizing the assumptions, the impact of deviations from the ideal case can be identified.

3.1 Coprime array designs

Since their introduction, several comprime array configurations have appeared in the literature. While all of these designs consist of two uniform component arrays with spatial sampling rates that are related in a coprime fashion, the number of elements in each subarray differ, as can the relative positions of the subarrays. In this section, we will discuss the most prominent coprime array configurations and their functional properties.

3.1.1 Prototype Coprime arrays

The coprime array, as initially described by Vaidyanathan and Pal [2], was designed to maximize the number of distinct integer elements in the coarray. The resulting array, later termed the Prototype Coprime Array [20], begins with a shared element and has a total length of \( M(N-1) \) and is pictured in Fig. 3.1. This array measures \( MN \) distinct integers with \( M + N - 1 \) elements. Though, it should be noted its coarray is not complete, as the \( MN \) distinct integers includes negative lags that are functionally equivalent to their positive counterparts for the purposes of covariance estimation. In fact, the hole free range of the coprime array is \( M + N - 1 \) [44], which is the same as a uniform array with the same number of elements. Thus, for applications that are limited by the hole-free range of the array, the prototype coprime array offers no additional aperture. Despite this fact, possible applications for prototype coprime arrays have been considered by both Weng and Djuric [45] and Vaidyanathan and Pal [46].

3.1.2 Conventional Coprime arrays

While the prototype coprime array is useful for multiplicative beamforming and creates a large number of coarray elements with a relatively small number of physical elements, the coarray of this array design did not have an extended hole-free range over a comparable uniform linear array. The hole-free range is of importance for signal processing operations (described in Section 2.1.2.2) and also gives the largest uniform linear array to which the coprime array is coarray equivalent. To achieve a filled coarray from 0 to \( MN \) (later shown to be \( MN + M - 1 \) [44])
Figure 3.1. A prototype coprime array with $M = 3$ and $N = 4$.

Figure 3.2. A conventional coprime array with $M = 3$ and $N = 4$. The extension of the $N$-spaced subarray guarantees a hole-free range of $MN + M - 1$.

lags, Vaidyanathan and Pal proposed extending one of the two arrays to twice its original length [2]. Without loss of generality, they let the array with fewer elements (the $N$-spaced array) be lengthened, resulting in an array with $2M + N - 1$ elements, accounting for the shared first element in the array. The resulting array has a total length of $N(2M - 1)$, though one half of the array is more sparsely populated. An example of this array configuration can be found in Fig. 3.2.

Termed the Conventional Coprime Array by Qin et al., this array design was used in conjunction with the MUSIC algorithm by Pal and Vaidyanathan to detect $MN$ spectral lines using only $2M + N - 1$ physical samples [41]. Since that article, a number of other authors have written papers utilizing this coprime array design [10, 11, 12, 13, 47].

While the conventional coprime array is notable for its hole-free range from 0 to $MN + M - 1$, it is worth noting that the array will, by necessity, also include additional lags in the range from $MN + M$ to $N(2M - 1)$, albeit with holes. While the use of those lags in the estimation of a covariance matrix does lead to a matrix
with undefined elements, the raw covariance estimates can be used in methods that do not require the inversion of a covariance matrix (such as compressive sensing).

3.1.2.1 Coprime Array with Displaced Subarrays (CADiS)

While the prototype and conventional coprime array designs are sparse in the sense that their average element spacing is greater than $\lambda/2$, their minimum element spacing is still $\lambda/2$. While this is often a desirable feature, there are situations where this minimum cannot be achieved, as with physical sensing elements that are themselves larger than $\lambda/2$. To cope with these situations, the Coprime Array with Displaced Subarrays (CADiS) configuration was developed by Zhang et al [36]. This array design is similar to the conventional coprime array, albeit with one of the subarrays displaced in position, as opposed to colocated.

The CADiS design is very similar to the conventional coprime array, and can be considered an extension of the concept. The CADiS array consists of two uniformly spaced colinear subarrays. The first subarray has $N$ elements with $Md$ spacing, while the second subarray has $2M - 1$ elements with $Nd$ spacing. Unlike the conventional coprime array, however, where the arrays are collocated such that the first element of each subarray is shared, the CADiS design places the first element of the $Nd$ spaced subarray at a location $Dd$ away from the last element of the $Md$ spaced subarray. Compared to the conventional coprime array, the CADiS configuration has an equal number of elements ($2M + N - 1$) but a larger minimum interelement spacing ($Md$). Moreover, the total array aperture for CADiS is $(3MN - M + 2N + D)d$, which is greater than the $(2MN - N)$ aperture of the conventional coprime array. An example of this array configuration can be found in Fig. 3.3.

While $D$ can be any value greater than $M$, Zhang et al recommend $D = M + N$ as a useful value. With $D = M + N$, the set of consecutive lags for the CADiS configuration is $MN + 2M + N - 1$ while the number of non-consecutive unique lags is $4MN + 2M - 1$ (The derivation of these results was sadly omitted from the
paper in which they were presented). It is worth noting that unlike the consecutive lags of the other coprime array designs, these lags do not extend down to zero. Thus, the range of consecutive lags does not constitute coarray equivalence with any fully sampled uniform linear array.

While the CADiS configuration does have advantages, the very large aperture can present problems for underwater acoustics. Moreover, excluding very high frequency sonar, the average sensor size for underwater acoustic arrays is much smaller than a wavelength, obviating the need for the CADiS configuration.

3.1.2.2 Generalized Coprime arrays

Generalized coprime arrays are a generalization on the form of conventional coprime arrays, and connect the concept of the conventional coprime array to nested arrays [20]. To construct a generalized coprime array, select two coprime integers $M$ and $N$. Construct two uniform linear subarrays, the first an $M$-element subarray with $Nd$ spacing, the second an $N$-element subarray with $Md/p$ spacing, where $p$ is an integer factor of $M$. In other words, follow the construction of a prototype coprime array, with the spacing in one of the arrays compressed by an integer factor $p$. A generalized coprime array with $p = 3$ is shown in Fig. 3.4.

The generalized coprime array connects directly to the conventional coprime array and nested array. The conventional coprime array is the same as a generalized coprime array with $p = 2$, while the nested coprime array is the same as a generalized coprime array with $p = M$.

Probably the most important aspect of the generalized coprime array is the hole-free range. It can be shown that the hole-free range for a generalized coprime array configuration with coprime integers $M$ and $N$ and compression factor $p$ is given by

$$ MN - (M/p)(N - 1) - 1. \tag{3.2} $$

It is worth noting that this also gives a larger hole-free range for conventional coprime arrays. If we let $p = 2$ and substitute $\hat{M} = M/p$, it can be shown that the hole-free range for a conventional coprime array is given by

$$ \hat{M}N + \hat{M} - 1, \tag{3.3} $$
Figure 3.5. An extended coprime array with $M = 3$, $N = 4$, and $c = 1$. The extended coprime array improves multiplicative beamforming performance by reducing grating lobe overlap.

which is $M - 1$ greater than the $MN$ given by Vaidyanathan and Pal. It is noted, however, that as $p$ approaches $M$, the number of consecutive lags increases. However, this also results in an array with a greater difference in element density between the section of the array containing both subarrays and the section containing only one. In order to overcome this limitation, and better utilize the full aperture of the coprime array, the extended coprime array approach may be considered.

3.1.3 Extended Coprime arrays

Extended coprime arrays were originally suggested by Adhikari et al in the context of multiplicative beamforming [18, 19]. The development of the extended coprime array begins by considering multiplicative beamforming on a prototype coprime array. The beam pattern generated here is found by multiplying the response of the first subarray with the complex conjugate of the response of the second subarray. It is shown that for the prototype coprime array, the sidelobe level of the resulting beam pattern is greater than that of a uniform linear array because of overlap between the grating lobes that appear in each subarray’s beam pattern. Noting that the grating lobes are images of the main lobe and that main lobe width is a function of array aperture, the proposed solution was to extend both arrays by some factor to reduce (and ultimately eliminate) grating lobe overlap. The resulting array design consists of two uniform linear subarrays, the first with $[M(c+1)]$ $Nd$-spaced elements and the second with $[N(c+1)]$ $Md$-spaced elements. Essentially, this is the same as extending both subarrays of the prototype coprime array by an extension factor ($c$). An extended coprime array with $c = 1$ can be seen in Fig. 3.5.

While extended coprime arrays have been examined for multiplicative beamforming processing, specifically determining optimal extension factors for different array shadings, these arrays have additional properties that make them attractive for covariance-based processing as well. Unlike other coprime array designs that increase the aperture, for example, the extended coprime array maintains the same
element density along the entire array. Extended coprime arrays will be the primary array design considered in this paper, and a number of other features of this array geometry will be given, including features of its coarray for covariance based processing.

3.1.4 Multidimensional Lattice Arrays

In addition to the one-dimensional linear arrays previously described, a multidimensional extension of the coprime array concept was developed by Vaidyanathan and Pal [22, 48]. Coprime multidimensional lattice arrays are based on a pair of integer matrices (M and N) which are chosen to be commuting and coprime. These matrices define a pair of multidimensional subarrays with elements placed along non-rectangular lattice points. That is, the element locations are given by M\(\vec{n}_1\) and N\(\vec{n}_2\) where \(\vec{n}_1\) and \(\vec{n}_2\) are integer vectors. Despite the change from one dimension to multiple, a number of similarities exist between one dimensional coprime arrays and the higher dimensional generalization.

For example, if \(\vec{n}_1\) and \(\vec{n}_2\) are restricted to the fundamental parallelepipeds (FPD) of N and M, respectively, then the resulting coarray will have \(\text{det} MN\) distinct elements in the symmetrical parallelepipeds (SPD) of MN [22]. This is directly analogous to the element range and coarray of the prototype coprime array. Likewise, if we instead restrict \(\vec{n}_1\) to SPD(N) or FPD(2N), then the resulting coarray includes all integer vectors in FPD(MN). This design is analogous to the conventional coprime array, as it results in a filled coarray. Furthermore, if M and N are chosen to be an adjugate pair, the coarray tiling is rectangular, and if M and N are skew-circulant, the resulting array approximates a symmetric diamond.

The present research is primarily interested in the design and function of linear coprime arrays, thus, no further discussion will be given to the properties of multidimensional lattice arrays. Those interested in further information on this topic are directed to the source papers by Vaidyanathan and Pal.

3.2 Processing paradigms

Signal processing performed on element-level data from a coprime array can be performed in multiple ways. In this section we consider two paradigms for processing data from a coprime array to generate a far field beam response. This response is often used for direction of arrival estimation, to determine the location and signal strength of sources in the far field of the array. The first method considered is termed “Multiplicative Beamforming,” and results from a two step process. The first step generates the beam response for each of the component uniform arrays individually, and the second step cross-correlates each of these responses to yield a final response for the full array. The second method, termed “Covariance Beamforming,” directly computes the correlation function measured between each
pair of elements in the full array, which can then be used as an input to several types of beamformers, including a traditional frequency-domain beamformer. This can also include the use of a sample covariance matrix, but as the coprime array is sparse, the covariance matrix for the coprime array will necessarily have gaps when compared against a uniform array. Methods for augmenting the covariance matrix to fill these holes will also be discussed.

### 3.2.1 Multiplicative beamforming

One method for processing data from a coprime array is to perform a process we will call “Multiplicative Beamforming.” In this method, the two component arrays are beamformed separately, and the resulting complex beam responses are then cross-correlated together, resulting in the elimination of grating lobes. A version of this method, using DFT filter banks, was discussed in one of the earliest papers on coprime arrays [2], and is also the impetus behind the extended coprime array design [18]. Additionally, while it does not explicitly use the correlation step, a search-free method developed by Weng and Djuric [45] works on a similar principle to multiplicative beamforming, considering the interesections between estimates from each of the two component arrays to determine direction-of-arrival [45]. While multiplicative beamforming may perform differently when compared to covariance based techniques, the use of traditional uniform beamforming processes can be valuable for certain applications, especially when such processing can be handled in hardware. Even beyond those advantages, however, multiplicative beamforming’s operation is illustrative of general properties of coprime arrays, and can help to show how two ambiguous beampatterns can be combined to form a single unambiguous one.

It has been demonstrated that for stationary signals, multiplicative beamforming is equivalent to covariance based techniques with application of an appropriate windowing function [49].

Mathematically, multiplicative beamforming follows Eq. 3.4.

\[
\begin{align*}
\text{rmultiplicative}(\theta) &= \sqrt{r_1(\theta)r_2^*(\theta)} \\
\end{align*}
\]  

Where \(r_1(\theta)\) and \(r_2(\theta)\) are the beam responses for the first and second arrays, respectively. The beampatterns for each of the component arrays possess grating lobes that are evenly spaced in the angular space given by \(u = \sin(\theta)\). Because the undersampling factors are coprime, the grating lobes are not coincident, though they may overlap, as shown in Fig. 3.6. The locations where grating lobes overlap can act as the peak sidelobes in the full array multiplicative beampattern. Thus, to minimize or eliminate grating lobe overlap, array extension can be used. To further reduce sidelobe levels, array shading can be used, though any increase in beamwidth this causes should be compensated for with additional aperture.
extension. It is worth noting that the natural beampattern generated by multiplicative beamforming is very different from the natural beampattern generated by the Blackman-Tukey method shown in Fig. 1.4, as a result of differences in the effective spectral weighting of the output.

Multiplicative beamforming has been experimentally verified by Bush and Xi-ang [50] using a set of electret microphones and a temporal windowing operation to emulate a free-field condition. Not only did this experiment verify the beampattern of the coprime array, but it also proved that the grating lobe cancellation of the coprime array at the design frequency $f_c$ applies at all lower frequencies as well, thereby defining the full operating frequency range of the coprime array. This paper also demonstrated a key flaw of multiplicative beamforming with coprime arrays, namely that a coherent source located too near the grating lobes of one of the subarrays can generate a number of image sources. This effect was mitigated when wideband signals were utilized.

3.2.2 Covariance beamforming

An alternative method of beamforming, and the one that will be used in this dissertation, is “Covariance Beamforming.” Covariance beamforming directly calculates the sample cross-correlation function between sensors in the array, and uses
those values to perform beamforming. The Blackman-Tukey spectral estimation technique shown in 2.1.2 is an example of this form of beamforming, as are the majority of implementations of compressive sensing and MUSIC to the coprime array found in the literature. Methods that use the sample covariance matrix, such as the Bartlett and Minimum-Variance Distortionless Response beamformers, can also be considered covariance beamformers.

While it is often assumed that covariance measurements for a given lag are identical under the condition of wide sense stationarity (see Sec. 3.3.4), in practice these measurements may vary. To reduce the effect of this noise, multiple measurements of the covariance for a given lag may be averaged together. The benefits of this averaging will depend both on the number of measurements and the degree of correlation between those elements. The number of unique measurements in a given array is determined by the number of lag redundancies in the array, while the correlation will depend on higher-order statistics of the spatial field, specifically the second-order correlation. When the second-order correlation is zero, lag averaging yields Eq. 3.5.

\[
\hat{R}(\tilde{y}) = \frac{1}{c_d(\tilde{y})} \sum_{y=x_1-x_2} v(\tilde{x}_1)v^*(\tilde{x}_2) = R(\tilde{y}) + \frac{z_y}{\sqrt{c_d(\tilde{y})}}
\]  

(3.5)

Where \(z_y\) is gaussian random noise associated with a lag of \(\tilde{y}\). Notably, when coarray weighting is applied in the Blackman-Tukey method of beamforming, the associated noise is also increased. As a result, while spectral weighting can synthesize beampatterns, it cannot replicate the noise reduction associated with redundant lag measurements. However, the magnitude of noise varies significantly with lag, often decreasing with increasing lag. In the ideal case of spatially uncorrelated or perfectly isotropic noise (see Sec. 3.3.2) and large snapshot length, noise power approaches zero for all non-zero lags.

A similar process to lag averaging for the covariance matrix is called “Covariance Augmentation.” This name reflects that the sample covariance matrix for a sparse array will have missing rows and columns corresponding to missing elements in the array. In the data covariance matrix, the off-diagonal elements of the array correspond to covariance measurements of a given lag, with the distance from the main diagonal giving the lag measured. To perform covariance augmentation, one must first assume the form that the full covariance matrix should take. For a wide sense stationary sound field, the matrix will be both positive definite and Toeplitz.

One way to obtain a Toeplitz matrix consistent with the measured covariance entries is through lag redundancy averaging. This averaging follows precisely the form of Eq. 3.5, with the resulting measurement being used to generate a Toeplitz covariance matrix \(\hat{R}\). While this matrix will suffice for the Bartlett beamformer, it may not be suitable for processors that require a matrix inversion (like MVDR). This is because the redundancy averaging process does not necessarily yield a positive definite matrix [31]. In order to overcome this, one can perform eigenvalue
decomposition on the Toeplitz matrix $\hat{R}$, setting any negative eigenvalues of the matrix to either zero or the value of the smallest non-negative eigenvalue. This effectively projects the matrix into the space of positive definite matrices, yielding an invertible matrix suitable for processors like MVDR. Alternative algorithms may directly yield such a positive-definite Toeplitz matrix, though the physical implications of these methods may not be as clear as the above redundancy averaging method [32].

3.3 Underlying assumptions

A number of underlying assumptions simplify the physical models that coprime arrays are built on. To better understand how the coprime array might behave in practice, however, those assumptions must be examined. In this section we will consider four assumptions regarding both the underlying signal and its propagation through space that are fundamental to the development of the coprime array. We will discuss how these assumptions may fail in practical application, and how such deviations might impact the performance of the coprime array.

3.3.1 Signal coherence

Coherence describes the extent to which phase relationships between two measurements separated in time, space, or both are stable. Two measurements are perfectly coherent when their phase relationship is fixed, and perfectly incoherent when their phase relationship is uniformly random. There are two common assumptions regarding coherence for sources in array processing. The first is that measurements of a given source are perfectly coherent at all locations and times. Under this assumption, the cross correlation between measurement locations will have a fixed amplitude and a phase angle that corresponds to the travel time between those measurement locations. The second assumption is that different sources in the environment will be completely uncorrelated with each other.

In practice, signal coherence is limited both in time and space. The ocean channel possesses random inhomogeneities that vary in both space and time, which can result in random travel-time variation from a source to a receiving array. The extent of this variation depends on a large number of factors, and can be expressed as a spatial and temporal coherence function. Alternatively, coherence can be expressed simply by coherence time and coherence length, which give the time and length scales beyond which a signal is approximately incoherent with itself. Rules of thumb for coherence lengths in the ocean exist and differ greatly between deep water (where coherence lengths are often on the order of 50 wavelengths) and shallow water (which are often less than 30 wavelengths) [25]. Environmental parameters that impact coherence length will be considered in Sec. 5.2.
The second assumption, that sound sources in the ocean are mutually uncorrelated, is often reliable, though the extent to which this is true depends greatly on the system bandwidth and the measurement time. While two truly monochromatic sources at the same frequency will (by definition) be perfectly coherent, in practice, all sources experience some degree of phase variation that will result in incoherence over a long enough period of time. Even reflections of a source from discrete scatterers in the environment can be incoherent with the direct path of the source with sufficient environmental inhomogeneity. In the absence of such effects, however, coherent sources can lead to interference patterns along the array, resulting in the covariance matrix taking on a Vandermode form from which the sources cannot be individually resolved [30].

### 3.3.2 Uncorrelated noise

With any real measurement, noise is likely to be present, and the statistics of that noise can be important to understanding an array’s performance in beamforming operations. There are many possible models for noise, but one of the most common is that of uncorrelated Gaussian noise. In this model, the noise covariance matrix can be modeled as a diagonal matrix, and is often further simplified with the assumption that the sensor noises are identically distributed, making the covariance matrix a scalar multiple of the identity matrix $\sigma_n^2 I$. This model is very common in radar signal processing, where electrical noise is of much greater importance than the small amount of propagating noise in the environment.

In underwater acoustics, a number of other models are used. The isotropic noise model treats the noise arriving at each sensor as a superposition of random plane waves coming equally from all directions in a solid sphere around the array [51]. The resulting correlation function follows Eq. 3.6, which (as shown in Fig. 3.7) is zero for integer multiples of $\lambda/2$.

$$R(\vec{y}) = \frac{\sin(k|\vec{y}|)}{k|\vec{y}|}$$ (3.6)

Given that $d = \lambda/2$ is the conventional array spacing, and sparse arrays are often limited to a grid with this spacing, the implication is that narrowband noise near the design frequency of the array is uncorrelated. However, for anisotropic noise (i.e. noise with non-uniform directionality), noise correlation between sensors is non-trivial and is a direct function of the sensor spacing and directional noise statistics [52].

The underwater environment contains both correlated and uncorrelated sources of noise, and the level of correlation can also depend on array orientation and look direction. Modal propagation in shallow water, in particular, leads to high noise gain at angles associated with the channel modes for long-range propagating noise [53], the same angles on which any long-range signal propagates. As a result, the
Figure 3.7. The spatial correlation function of isotropic noise as a function measurement lag. For lags that are multiples of $\lambda/2$, sensor noise is uncorrelated.

assumption of zero correlation for non-zero lags is flawed. This non-zero correlation can lead to reduced array gain, the extent of this reduction depending both on the second-order correlation and fourth-order correlation that reflects the correlation between the spatial correlation measurements. This loss in array gain occurs for both fully-populated and sparse arrays, but array sparsity incurs an additional penalty when the fourth-order correlation is below unity, which will be shown in Ch. 5.

### 3.3.3 Compact sources

One common assumption in array signal processing is that sources are spatially compact and can be approximated as point sources. This assumption arises when the distance from the source to the receiver is large, both compared to the wavelengths of interest and the physical extent of the source and array. While long range compared to the wavelength ensures that only propagating waves are measured along the array, long range compared to the size of the source and the array ensures that waves impinging on the array are approximately plane waves. Under this latter condition, the apparent source angle, as measured normal to the array axis, is constant along the length of a linear array.

In the situation that sources are not compact, a number of effects can be observed. Of particular note for coprime arrays is that sources with a finite physical
extent can cause widening in the beam response, which can cause overlap of grating lobes for multiplicative beamforming operations. For sources near the array, wavefront curvature can reduce beamforming performance (an effect which can be mitigated through subarray processing, to be discussed in Sec. 4.4). For most examples throughout this work, however, source ranges will be selected to remain consistent with the assumption of compact sources.

### 3.3.4 Wide Sense Stationarity

In order to apply the coarray concept to arrays, spatial Wide Sense Stationarity (WSS) is often assumed. For a process to be WSS it must have a constant mean, and a covariance function that depends only on the spatial lag. Mathematically, these properties are expressed as

\[
E[v(\bar{x})] = E[v(\bar{x} + \bar{y})] \forall \bar{y}
\]

(3.7)

\[
r(\bar{x}_1, \bar{x}_2) = E[v(\bar{x}_1)v^*(\bar{x}_2)] = E[v(\bar{x}_1 - \bar{x}_2)v^*(0)]
\]

(3.8)

Physically, WSS implies that the covariance between sensors is invariant upon translation, so long as the relative distance and orientation of the sensors is maintained. The value of WSS can be seen in the coarray, as the WSS assumption allows for the simplification of the generalized cross-correlation function to the function shown in Eq. 2.8 that depends only on spatial lag. It is as a direct result of this condition that repeated measurements of the same lag can be considered “redundant.” WSS can also be interpreted as the condition that the spatial spectrum does not change in space. Under this interpretation, it becomes apparent that any environmental features that may impact the spatial spectrum can also result in a loss of stationarity.

Failures of other assumptions, as with incoherent sources, inhomogeneous noise, and wave-front curvature, can all be expressed as losses of Wide Sense Stationarity. As a result, while WSS is a key assumption for the development of coarray theory, deviations from it will be considered in the context of those other conditions.
Chapter 4

Centered Coprime Arrays

Given the relationship between coprime arrays and uniform arrays, coprime arrays possess features not found in other sparse arrays. Specifically, regularity in lag redundancies can be exploited, both to predict performance and to enable sub-array processing techniques often reserved for uniform arrays. This allows the coprime array to better handle non-stationary fields, like those generated by a nearby source.

While a number of array designs exist in the coprime array family, our interest is in a novel design: the Centered Coprime Array (CCA). This array design follows directly from the concept of the extended coprime array, where both of the component arrays are built to exceed the prototypical length of $MNd$, where $M$ and $N$ are coprime numbers and $d$ is the unit spacing at the design frequency of the array. The Centered Coprime Array is positioned such that the physical center of the array is precisely $MN/4$ spatial units away from the nearest shared elements in the two arrays. As will be shown in this chapter, this choice of element positions ensures that the centered coprime array’s resulting coarray will be hole-free up to a lower bound that can be derived analytically.

In this chapter, we define lag redundancy and consider its effects on the performance metrics defined in Sec. 2.3. We show that lag redundancy can reduce noise energy in non-zero lags, allow identification of spatial interference from coherent scatterers, and allow measurement of wavefront curvature from short range sources. We also show that in the absence of spectral weighting, lag redundancies can increase sidelobe levels. Next, we consider lag redundancy in coprime arrays and show the manner of lag repetition along the length of the array. Using this as a theoretical basis, bounds on the hole-free range of an extended coprime array are given and refined through deliberate positioning of the array’s center, resulting in the Centered Coprime Array configuration. Finally, we consider the coarrays of subsections of the Centered Coprime Array, and show that (properly chosen) these subsections have identical coarray weighting functions, lending them to use...
for subarray-based techniques that are traditionally reserved for uniform linear arrays.

4.1 Lag Redundancy

When pairs of elements in an array measure a given spatial lag more than one time, that lag is said to be redundant. For a wide sense stationary signal, such redundant lags do not contribute additional information for the estimation of the correlation function, and thus represent a cost in both elements and processing with no associated benefit. Uniform linear arrays possess a large number of redundant lags. In fact, of all discrete array designs of a given minimum element spacing and length, the uniform linear array contains the maximum number of redundancies. Sparse arrays, on the other hand, are designed to possess a smaller number of lag redundancies through non-uniform spacing along the length of the array.

However, when the sound field does not fit the ideal condition of wide sense stationarity, additional measurements of the same lag can offer some benefits. In this section we will consider the performance effects of lag redundancies on an array. It will be shown that lag redundancies can reduce measurement noise and help to identify standing interference patterns associated with a multipath fading channel. Finally, the impact of redundancies on sidelobe levels on the array will be considered, including effective sidelobe levels caused by noisy covariance measurements.

4.1.1 Error reduction

In addition to its directional characteristics, one of the primary functions of an acoustic array is to reduce the effects of noise on a given measurement. In a general sense, this is achievable because of differences in the spatial and temporal autocorrelation of the noise field and the signal sound field. Often, the noise is less correlated (or uncorrelated) in time and/or space, while the signal is highly correlated in both. Other times, the noise may be highly correlated, but originating from a different location such that beam steering can reject the noise. In either case, it is the differences in autocorrelation that allow the signal to be isolated.

Relative noise reduction does not occur in a single sample. Rather, it is the combination of multiple samples that allow this reduction to occur. Using the Blackman-Tukey method from Eq. 2.9, we can consider the individual spatial correlation measurements as a mean of samples taken at $T$ instants in time at each lag redundancy where the measurement occurs in the array. We may then consider the measurement as the sum of a signal and noise, each with their own spatial and temporal correlation functions. This yields the following equation:
\[
\hat{R}(\tilde{y}) = \frac{1}{k_{\tilde{y}}} \sum_{x_n = x_m = \tilde{y}} \left( \frac{1}{T} \sum_{T} (v(x_n^* + z(x_n^*))(v^*(x_m^*) + z^*(x_m^*)) \right) \tag{4.1}
\]

Where \(k_{\tilde{y}}\) is the number of redundancies of lag \(\tilde{y}\). Expanding the product yields four terms in the sum, representing the auto-correlation of the signal, the noise, and the cross-correlations between them. Under the common assumptions that the signal and noise are independent and the noise is uncorrelated with itself in space and time, we yield the following expression for the sample correlation as a function of redundancies and snapshots.

\[
\hat{R}(\tilde{y}) = R(\tilde{y}) + Z(\tilde{y}) \tag{4.2}
\]

Where \(Z(\tilde{y})\) is the snapshot noise at lag \(\tilde{y}\). This noise term is a sum of the product of normal random variables. However, by the central limit theorem, it can be shown that this noise term is approximately normal with zero mean and a variance related to the signal and noise powers (\(\sigma^2_v\) and \(\sigma^2_z\), respectively) by Eq. 4.3.

\[
\sigma^2_Z = \frac{2\sigma_v\sigma_z + \sigma^2_z}{Tk_{\tilde{y}}} \tag{4.3}
\]

As can be seen, the covariance noise power drops at a rate of \(1/Tk_{\tilde{y}}\). Thus, as the number of snapshots and redundancies increases, the error in the correlation measurement drops. In the limit where \(T\) goes to infinity, snapshot error is reduced to 0, and redundancies beyond the first are irrelevant. This is consistent with the law of large numbers, in that the mean value of the measurements converge to the theoretical true value of the correlation. However, when snapshots are limited, it can be shown that redundancies effectively multiply the number of snapshots for the purposes of noise reduction.

It should be noted that snapshot noise is distinct from the steady state noise contribution, which is the noise contribution that remains in the limit where \(T\) approaches infinity. Notably, while the steady state noise for uncorrelated noise is approximately 0 for all non-zero lags, snapshot noise appears at all lags.

### 4.1.2 Interference pattern identification

One of the defining characteristics of shallow water acoustic propagation is the presence of multipath. As the name suggests, multipath occurs when multiple propagation paths exist between the source and receiver, resulting in the total measured pressure being a sum of time-shifted copies of the signal. For narrowband signals, these time shifts translate directly to phase shifts, resulting in an interference pattern along the array. As such, the shallow water environment exhibits the qualities of a fading channel, with amplitude and phase varying as a
function both of time and space. In this section, we will show that lag redundancies can help to identify and compensate for the presence of an interference pattern, particularly when said redundancies are spread across the length of the array.

We begin by modeling the fading channel as a channel whose transfer function varies randomly in time and space following some distribution. One common model is a zero mean bivariate distribution, resulting in what’s called a Rayleigh fading channel [54]. Importantly, these channel statistics are correlated in time and space, and the scale of this correlation greatly impacts how an array functions in the channel. Generally, fading is categorized as either slow or fast, depending on whether the degree of fading varies over a single measurement period or not. The justification for the use of Rayleigh fading will be given in Sec. 5.1.1, as will an examination of the channel parameters that determine the length scales of the variation.

One limiting case of channel fading is known as slow fading. Slow fading occurs when the rate of change in the channel statistics is such that they are approximately uniform over the measurement. In the context of array processing, this means that the spatial variation of the channel statistics is greater than the array length. In this situation, array redundancy does not significantly impact performance, as each redundant lag experiences approximately the same level of amplitude fading.

Fast fading, on the other hand, occurs when the correlation length of the channel fading statistics is equal to or less than the length of the array. In this case, different elements along the array experience different signal levels. As such, redundant lags may measure different values based on their position along the array’s length. Notably, if the signal-to-noise ratio of certain elements drops below an acceptable threshold, a redundant lag from another position along the array may allow for the reconstruction of the signal. This is very similar to the telecommunications method of time diversity [54], wherein data is transmitted multiple times to cope with signal fading.

Reframing channel fading not merely as a complication to be compensated for but as something to identify and measure, lag redundancies offer some means to this end. In high signal-to-noise environments, amplitude fading can be directly measured by observing channel outputs, but correlation measurements between sensors can help to reject noise. Plotting the amplitude of correlation measurements for the same lag at different points along an array can be one way to identify amplitude fading. An example of this sort of measurement is shown in Fig. 4.1, measuring the interference pattern along an array in the presence of five randomly placed plane wave interferers. As can be seen, the interference fringes on the pattern manifest as a spatially varying pattern in the correlation measurement. When the spacing between redundancies is less than one half the spatial bandwidth of the interference pattern, then said pattern can be measured. As will be shown in this chapter, certain redundancies of the coprime array repeat at regular intervals
Figure 4.1. Correlation amplitude of $L - 1$ lag measurements each spaced $d/2$ apart for a uniform array in the presence of a single strong signal with multiple scatterers placed randomly in the channel. The variation in received amplitude is consistent with channel fading.

of the coprime factors $M$ and $N$, and thus these values can be selected based on the relevant length scales of the interference pattern.

Another potential manner of interference identification creates a virtual array of sensors generated by correlating each sensor with a subsequent neighbor or by performing this same process on each component array to form a virtual coprime array. The advantages of either of these methods is beyond the scope of this work, but there is potential merit in considering such channel estimation as some form of higher-order spectrum estimation on the channel’s amplitude effects.

4.1.3 Sidelobe Level

An important figure of merit for array performance is the peak sidelobe level. While the integrated sidelobe level determines the average white noise gain for a point source, the peak sidelobe level determines the dynamic range for imaging applications, as well as the potential for masking by interferers. Through the process
of array shading or spectral weighting, sidelobe level can be controlled. However, lag redundancies also have an effect as they determine the natural spectral weighting of the array. Not only are the total number of redundancies important, but also the ratios between lags with multiple redundancies.

The output of a Blackman-Tukey spectral estimate follows the form of Eq. 2.9, which is a sum of complex exponentials. As such, the values $\gamma_{\hat{y}}$ and $R(\hat{y})$ give the weights and phases of cosines being added together. As shown in Fig. 4.2, for a compact source at a given location the sum of these exponentials will have a distinct peak, as the phases of the various cosines all line up at that location. Outside of that region, however, the different phases cause the individual sinusoidal contributions to cancel each other out to some extent. Additionally, the magnitude of these components are determined by the spectral weighting or (in the absence of such weighting) the number of redundancies. Because the sinusoidal functions are periodic, the peaks of sinusoids whose spatial frequencies share a common factor will line up at points away from the source location, resulting in local maxima in the side lobe level. It has been observed that periodic lag redundancies increase peak side lobe level, which is a direct result of amplifying these reinforcing sinusoids [42].

Because of their impact on sidelobe level, the pattern of redundancies in the coprime array are of interest. We will consider both the number of redundancies, and the relationship between lags with a high number of redundancies. Unfortu-
nately, as will be shown in Sec. 4.1, the coprime array possesses a large number of redundancies that are spectrally periodic at each of the component array frequencies, $M$ and $N$. As a result, the sidelobe levels of the coprime array will be higher than another array with a comparable number of coarray redundancies but a more even or random spectral pattern.

The high number of redundant lags related by integers multiples in the coprime array allows us to estimate the naturally weighted sidelobe level for a narrowband signal \textit{a priori}. As the peak sidelobe level is the ratio of the peak level to the maximum level in the sidelobes, we may yield the following approximation.

\[ \text{Peak Sidelobe Level} = 10 \log_{10} \left( \frac{\sum_{\bar{g}} k_{\bar{g}}}{X + \sqrt{Y}} \right) \]  

Where $\sum_{\bar{g}} k_{\bar{g}}$ is the sum of all redundancies in the array, $X$ is the highest sum of periodically related redundancies in the array, and $Y$ represents the mean square value of the energy remaining in non-periodic lags. For any array, the total number of redundancies is given by $K(K - 1)$, where $K$ is the total number of elements in the array. For an extended coprime array we know that $K$ is less than or equal to $(N + M)\alpha + 1$, where $N$ and $M$ are the array’s coprime factors and $\alpha$ is the extension coefficient of the array.

Likewise, the number of periodic redundancies can also be determined by means of the single subarray lags, which will be derived in Sec. 4.2.1. With this we can determine that the redundancies that are multiples of lag $M$ will have the highest multiplicity, with a total sidelobe amplitude of $N\alpha(N\alpha + 1)$. This can be further justified, as the highest point in the coprime sidelobes is coincident with the grating lobes of the component array with the largest number of elements, namely the $M$-spaced component array.

The remaining term in Eq. 4.4, however, requires full knowledge of the array redundancies. This can be found using the values determined in Sec. 4.1, yielding a rather large expression. However, a useful approximation can be found, by neglecting the term for the non-periodic redundancies and approximating $x(x + 1) \approx x^2$. This simplified form reduces to

\[ \text{Peak Sidelobe Level} \approx 20 \log_{10} \left( 1 + \frac{M}{N} \right) \]  

While simple, this equation holds closely to the true sidelobe levels for coprime arrays, as shown in Fig. 4.3. This approximation is most appropriate for prime $M$, large values of $N$, and large $\alpha$, but is still useful even outside of those restrictions. More importantly, we can determine two engineering recommendations from this. First, sidelobe levels will often (but not always) tend towards the range of 3-6 dB. Second, better values are yielded when the coprime factors $M$ and $N$ are larger and closer in value to each other. Even at its best, however, the naturally weighted sidelobe level for a coprime array is higher than that of a uniform linear array.
Compared to a uniform linear array, the sidelobe level for a coprime array is relatively poor. However, the process of spectral weighting can be used to fix this problem and yield any beampattern that could be realized on a ULA. While this method is very effective with coherent fields there are tradeoffs in the presence of noise. To illustrate this, we will consider two limiting cases: when the coarray measurements are perfectly correlated and uncorrelated.

Suppose some lag, \( \bar{\gamma} \), is measured \( k_{\bar{\gamma}} \) times, averaged, and weighted by a value \( \gamma_{\bar{\gamma}} \), where \( \gamma \) is the desired weighting function. Using Eq. 2.9 we find that the output depends on the sample cross correlation value \( \hat{R}(\bar{\gamma}) \). When the lag measurements are perfectly correlated \( \hat{R}(\bar{\gamma}) = k_{\bar{\gamma}} R(\bar{\gamma}) / k_{\bar{\gamma}} = R(\bar{\gamma}) \), but when these measurements are uncorrelated \( \hat{R}(\bar{\gamma}) = \sqrt{k_{\bar{\gamma}}} R(\bar{\gamma}) / k_{\bar{\gamma}} = R(\bar{\gamma}) / \sqrt{k_{\bar{\gamma}}} \). Thus, for coherent lag measurements, the intensity estimate depends only on the weighting function \( \gamma \), but for incoherent lag measurements we have an effective weighting given by \( \hat{\gamma}_{\bar{\gamma}} = \gamma_{\bar{\gamma}} / \sqrt{k_{\bar{\gamma}}} \).

While the weighting function can be any function, it is often chosen to correspond to some traditional windowing function on a uniform array. In the case of the Bartlett window, given by Eq. 2.12, it can be shown that \( \gamma_{\bar{\gamma}} \geq k_{\bar{\gamma}} \) for any sparse array design. As such, the spectral weighting function results in a relative increase in the measured spatial spectrum of the noise, particularly for lags with low multiplicity within the array. The can be further related to the case of partially incoherent signals, which will be considered in Sec. 5.2.
4.2 Lag redundancy in coprime arrays

Despite the importance of lag redundancy to array performance in non-stationary fields and the naturally-weighted beampattern of the array, lag redundancies in the coprime array have not been given much consideration in the literature. In their introductory paper, Vaidyanathan and Pal make oblique reference to the presence of redundancies by stating that the coprime array measures both positive and negative versions of specific lags, and that these lags are functionally equivalent for the purposes of covariance estimation [2]. BouDaher et al. gave some consideration to coarray redundancies and plotted the coarray weighting function for a conventional coprime array configuration, though this was not contextualized in terms of array performance or natural beampatterns [44]. Moreover, the physical locations of these redundant lag measurements have not been previously quantified or considered. Instead, the trend in the literature has been to consider only the coarray, as opposed to the coarray weighting function.

One exception to this trend can be found in a paper by Raza et al [55], where lag redundancies are considered in order to eliminate shared elements in a conventional coprime array. While the proofs in this work follow a similar format to those to follow, the application to conventional coprime arrays differs from the extended coprime arrays considered here, and the application to eliminate redundant portions of the array differs from our goal of quantifying such redundancy to leverage performance advantages.

In this section we consider lag redundancies in an extended coprime array. To do this, we break the set of lags into two sets. The first set consists of lags that are independently measured by each of the component arrays, the redundancies of which are numerous, evenly distributed in space, and consistent with the coarray weighting functions of uniform linear arrays. The second set consists of lags that are measured by pairs of unshared elements in each of the component arrays, which we term “colags.” We show that this second set of lags forms a disjoint set from the first, and that the repetitions of these lags are less numerous and are semi-regularly spaced in the physical domain. Not only will these results be shown to have performance implications for the coprime array, as discussed in Sec. 4.1, but they will also be used to motivate and prove the design of the Centered Coprime Array.

4.2.1 Single component array lag redundancy

The coprime array is made of two uniform component arrays, each of which is undersampled by some integer factor. Naturally, the full array will contain all of the lags measured by each of the component array elements alone, in addition to those lags that are measured by pairs of elements taken from each of the component array. In this section we consider only the lags that are measured by each of the
component arrays. We show that the component arrays only share lags that are multiples of the product \( MN \), and the resulting coarray weightings are similar to that of a uniform linear array.

As with any uniform array, each of the two component arrays measures a set of lags that are integer multiples of the spacing of that array. That is, the \( M \)-spaced array measures all lags \( mM \) and the \( N \)-spaced array measures all lags \( nN \), where \( m \) and \( n \) are integers less than \( L/M \) and \( L/N \), respectively. Measurements of these lags repeat at even intervals given by the array spacing. That is, each lag measured in the \( M \)-spaced array repeats every \( M \) units and each lag measured in the \( N \)-spaced array repeats every \( N \) units. Moreover, the lags measured by each component array are exclusive to that array. That is, any lag which is measured by one component array that is not a multiple of \( MN \) is not measured by the other component array. The proof of each of these properties follows.

**Lemma 1.** Each lag measured in a component array repeats at even intervals of the component array spacing.

**Proof.** Let \( y \) be a lag measured in array \( M \).

Then there exist integer element indices \( m_1 \) and \( m_2 \) such that \( y = m_1M - m_2M = (m_1 - m_2)M \).

The location of this measurement, \( x \), is defined as the center point between the two sensor locations: \( x = (m_1M + m_2M)/2 = (m_1 + m_2)M/2 \).

Consider the lag measured between points \((m_1 + 1)M\) and \((m_2 + 1)M\). Because the array \( M \) contains all integer multiples of \( M \), the points \((m_1 + 1)\) and \((m_2 + 1)\) are also elements in the array.

The lag, \( y_0 \), between these two points is \( y_0 = ((m_1 + 1) - (m_2 + 1))M = (m_1 - m_2)M = y \).

Thus, the lag measured is a repetition of \( y \).

The location of this measurement, \( x_0 \), is given by \( x_0 = (m_1 + 1 + m_2 + 1)M/2 = (m_1 + m_2)M/2 + M = x + M \).

Therefore, given a measurement of a lag \( y \) at position \( x \), there is a repetition of that lag exactly \( M \) spaces away.

The same argument can be applied to array \( N \). \(\square\)

**Lemma 2.** Lags measured in one component array that are not integer multiples of \( MN \) are not measured in the other component array.

**Proof.** Let \( y \) be a non-zero lag measured in array \( M \) which is not a multiple of \( MN \).

Then there exist integer element indices \( m_1 \) and \( m_2 \) such that \( y = m_1M - m_2M = (m_1 - m_2)M = mM \), where \( m \) is an integer.

Thus, \( y \) is an integer multiple of \( M \).

Suppose \( y \) is also measured in array \( N \).
Then there must exist some integer element indices \( n_1 \) and \( n_2 \) such that 
\[ y = (n_1 - n_2)N = nN, \]
where \( n \) is an integer.

Thus, \( y \) is also an integer multiple of \( N \).

If \( y = mM \) and \( y = nN \), then \( mM = nN \) and \( m/n = N/M \). However, because \( M \) and \( N \) are coprime, the fraction \( N/M \) is irreducible. Thus, \( m/n = N/M \) only if \( m = aN \) and \( n = aM \) for some integer \( a \).

Thus, \( y = mM = aNM \), which is a multiple of \( MN \), contradicting the earlier assumption.

Therefore, \( y \) must not be measured in array \( N \).

The regular repetition of lag measurements results in coarray weighting functions similar to that of a uniform linear array. If \( L_m \) and \( L_n \) are the lengths of the \( M \)-spaced component array and the \( N \)-spaced component array, respectively, then the coarray weightings for the lags measured by the component arrays are given by

\[
\gamma_y = \begin{cases} 
(L_m - y)/M + 1 & : y = mM \leq L_m \\
(L_n - y)/N + 1 & : y = nN \leq L_n \\
(L_m - y)/M + (L_n - y)/N - \lfloor L/MN \rfloor + 2 & : y = aMN \leq L 
\end{cases} \tag{4.6}
\]

Fig. 4.4 shows an example of this weighting function for a coprime array with \( M = 3 \) and \( N = 4 \). While this set does cover a considerable number of lags, the component arrays processed separately in this manner have a significant number of holes. As will be shown in the next section, many of the missing lags are measured by pairs of elements taken from each of the component arrays, termed colags, albeit with fewer redundant measurements.

### 4.2.2 Colag redundancy

In addition to the lags measured by each of the component arrays individually, there are also lags that are measured by pairs of elements taken from each of the two arrays and take the form \( y = mM - nN \) where \( m \) and \( n \) are integer values. We define the term “colags” to describe this set of lags. Colags are important, because they represent the additional measurements gained by processing the arrays in concert. Additionally, because of Euclid’s theorem, there exist values of \( m \) and \( n \) that yield any integer lag \( y \) [2]. However, this includes values like \( m = N \) and \( n = 0 \) that would be considered shared elements between the two arrays. We may, therefore, exclude those shared elements and consider only lags between unshared elements in each of the arrays.

**Definition 1.** A colag is a lag measured between one element of the \( M \)-spaced array and one element of the \( N \)-spaced array such that neither element is shared. The set of colags in an array is given by \( C_c = \{y | y = \pm (mM - nN), \text{ for } mM, nN \in A \setminus \{aMN \text{ for } a \in \mathbb{Z}\} \} \), where \( A \) is the full array aperture function.
Figure 4.4. Coarray weighting function for the lags measured by the individual component arrays of a coprime array of length $L = 30$ with $M = 3$ and $N = 4$. The individual component arrays display a triangular weighting function consistent with a uniform linear array, with each contributing measurements for lags that are multiples of $MN$.

An important feature of the set of colags is that they form a disjoint set from the set of lags measured by each individual component array, described above.

**Lemma 3.** Lags measured between unshared elements of each component array are not measured in either component array individually.

**Proof.** Let $y$ be a colag such that $y = mM - nN$, where $mM$ and $nN$ are unshared elements of the composite array.

Suppose that the lag $y$ is measured in one of the component arrays.

If $y$ is measured in the M array, then there exist integer element indices $m_1$ and $m_2$ such that $y = m_1M - m_2M$.

This implies that $mM - nN = m_1M - m_2M$.

Rearranging the above gives, $nN = (m - (m_1 - m_2))M = m'M$, where $m'$ is an integer.

Because $N$ and $M$ are coprime, this is true if and only if $n$ is a multiple of $M$. However, this yields a contradiction because $nN$ is an unshared element of the two arrays. So $y$ is not measured in the M array.

The same argument can be used to show that $y$ is not measured in the N array. Therefore, colags are not measured in either component array individually. □
While lags within the component arrays repeat with a regular physical spacing determined by the individual array’s spacing, colag spacing is only semi-regular, with two lag measurements, each complementary to each other, within any given $MN$ length. This can be shown by combining the results of two lemmas. The first lemma shows by construction the location of a non-complementary colag at a location $MN$ away from any given measurement, and also shows by contradiction that this is the nearest non-complementary lag. The second lemma shows by construction the location of a complementary lag equidistant any point $AMN/2$ where $A$ is an integer. For the purposes of correlation estimation, complementary lags are functionally equivalent [6].

Lemma 4. As shown in Fig. 4.5, given the location of a colag, that colag must also be measured exactly $MN$ spaces away. This is the nearest non-complementary colag.

Proof. Let $y$ be a colag such that $y = mM - nN$. The location of $y$ is given by $\bar{x} = (mM + nN)/2$.

Consider the elements $(m + N)M$ and $(n + M)N$. Since $M$ and $N$ are both integers, these are both elements in the respective arrays $M$ and $N$.

The lag between these elements is $(m + N)M - (n + M)N = mM - nN + NM - NM = mM - nN = y$. The position of these lags is $\bar{x_0} = ((m + N)M + (n + M)N)/2 = (mM + nN + 2MN)/2 = \bar{x} + MN$.

Therefore, the lag between these two elements is $y$, and the position is $MN$ spaces away.

To show that an exact lag repetition cannot be closer than $MN$, suppose there exists a repetition of lag $y$ at position $\bar{x}'$ such that $|\bar{x} - \bar{x}'| < MN$ and $\bar{x} \neq \bar{x}'$.

Then $y = mM - nN = m'N - n'N$, for some $m' \neq m$ and $n' \neq n$.

Rearranging this yields $(m - m')/(n - n') = N/M$.

Because $M$ and $N$ are coprime, this is only true if $m - m' = aN$ and $n - n' = aM$ for some integer $a$.

This yields $\bar{x}' = (m'M + n'N)/2 = ((m + aN)M + (n + aM)N)/2 = (mM + nN)/2 + aMN = \bar{x} + aMN$.

Thus, $\bar{x}'$ cannot be less than $MN$ spaces from $\bar{x}$. \qed
Lemma 5. As shown in Fig. 4.6, for every colag and integer $A$ there exists a complementary repetition of that colag equidistant the point $AMN/2$.

Proof. Let $y$ be a colag such that $y = mM - nN$.

For any integer $A$, the distance from the lag position to the point $AMN/2$ is given by $|\bar{x} - AMN/2| = |(mM + nN)/2 - AMN/2|$.

Consider a point reflected about the point $AMN/2$, called $\bar{x}^*$, such that $\bar{x}^* - AMN/2 = -(\bar{x} - AMN/2)$.

We may rearrange this to yield $\bar{x}^* = AMN - \bar{x} = AMN - (mM + nN)/2 = ((AN - m)M + (AM - n)N)/2$.

This corresponds to a pair of elements in the composite array, $(AN - m)$ in the $M$ array and $(AM - n)$ in the $N$ array. The lag of this pair of elements is $y^* = (AN - m)M - (AM - n)N = -mM + nN = nN - mM = -y$.

Thus, given a colag at location $\bar{x}$, there exists a complementary repetition of that colag equidistant any point $AMN/2$. $\blacksquare$

Treating a colag and its complement as functionally identical for the purposes of correlation estimation, we can determine the rate of lag repetition for colags. Unlike lags measured in the component arrays, which repeat on a regular interval given by their element spacing, colags are distributed semi-regularly with two repetitions appearing every $MN$ spaces. This can be intuitively understood by recognizing that $MN$ is the fundamental repetition rate of the full coprime array. Moreover, the sections ranging from one shared element to the next are symmetrical. As a result, each colag is centered exactly once in any $MN/2$ length section spanning from a shared element. This theorem can be formally proven using the previous two lemmas.

Theorem 6. As shown in Fig. 4.7, given an infinite array, exactly one repetition of each colag exists in the range from $AMN/2$ to $(A+1)MN/2$ for any integer $A$. 

Figure 4.6. Any lag ($y$) must have a complementary lag ($-y$) that appears equidistant from each point halfway between shared elements.
Figure 4.7. Every measured lag (or its complement) must be centered in each $MN/2$ region, bounded on one side by a shared array element.

Proof. Let $y$ be a colag such that $y = mM - nN$ at location $\bar{x} = (mM + nN)/2$.

Then, as shown in Lemma 4, this lag must repeat every $MN$ spaces. Thus, for every integer $a$, there exists a repetition of lag $y$ at location $\bar{x}' = \bar{x} + aMN$.

Therefore, there exists a repetition of $y$ in the range:

$$(A + 1)MN/2 > \bar{x}' \geq (A - 1)MN/2.$$  

This can be broken into two cases.

In the trivial case, $(A + 1)MN/2 > \bar{x}' \geq AMN/2$, which is the desired range and a colag exists in this range.

Alternatively, if $AMN/2 > \bar{x}' \geq (A - 1)MN/2$, then the given colag is outside of the desired range. By Lemma 5, however, there exists a repetition of $y$ at lag $\bar{x}'^*$ such that $\bar{x}'^*$ is equidistant from the point $AMN/2$. This repetition must therefore be in the range:

$$(A + 1)MN/2 \geq \bar{x}'^* > AMN/2.$$  

Thus, $\bar{x}'^*$ is in the desired range. Therefore, there exists at least one repetition of colag $y$ in the range of $AMN/2$ to $(A + 1)MN/2$.

Exclusivity of this measurement can be derived by considering that $\bar{x}'$ and $\bar{x}'^*$ actually measure lags $y$ and $-y$, respectively, and making use of the exclusivity shown in Lemma 4.

For finite length arrays, the existence of a lag in each $MN/2$ length section depends on the extent of the array to either side of that subsection. This results in a “stair-step” pattern for the redundancies in the coarray. The exact number of redundancies for each colag should be determined directly from the design, but the coarray weighting can be determined a priori within one. That is, for an array of length $L$, with coprime values $M$ and $N$, the number of repetitions of colag $y$ will be in the set

$$\gamma_y \in \{\lfloor2(L - y)/MN\rfloor, \lceil2(L - y)/MN\rceil\}. \quad (4.7)$$

Fig. 4.8 shows an example of the coarray weighting function including colags. The stai-step like function can be clearly seen, as can the relatively low number of colag measurements, in comparison to nearby component array lags. This lack of
Figure 4.8. Coarray weighting function, including colags, for an array of length $L = 30$ with $M = 3$ and $N = 4$. Colag weighting follows a stairstep-like function, bounded by the number of $MN/2$ length sections capable of measuring the given lag (shown with a dashed line).

smoothness contributes to the large sidelobes of the coprime array’s natural beam pattern.

4.2.3 Redundancy in other sparse arrays

While coprime arrays have less redundancies than a comparable uniform linear array, they do still possess many more redundancies than some other sparse array designs. Fig. 4.9(a) shows the coarray weights of a minimum hole array, also known as a Golomb ruler, of length 34 [56]. This array possesses no redundancies besides the necessary redundancies at the zero lag, though the hole-free range of the array is much smaller than a comparable coprime array. Similarly, Fig. 4.9(b) shows the coarray weightings for a restricted minimum redundancy array of length 36 [7]. Not only does this array have far fewer total non-zero redundancies than the coprime array shown in Fig. 4.8 (18 compared to 182), but it is also hole-free for the full length of the array, giving the minimum redundancy array an aperture efficiency of 100%.

For perfectly wide sense stationary signals, the reduced redundancies of these sparse array designs offer a reduction of the number of elements with little or no loss in performance. However, when wide sense stationarity is violated, the loss of
redundancies can result in performance losses, as discussed in Sec. 4.1.

4.3 Hole-free range

When considering the coarray of a given design, the hole-free range is an essential figure of merit. This value determines the largest uniform linear array to which a given sparse array is coarray equivalent, and thus capable of generating the same set of beam patterns for wide sense stationary sound fields. For the coprime array, the exact hole-free range of the array can be determined directly by exhaustively searching the lags in the array. However, a lower bound on the hole-free range can also be determined analytically, given a specific choice for the array’s physical center.

Consider a coprime array of length \( L \), centered about the origin such that a shared element in the array exists at either \( MN/4 \) or \(-MN/4\). We will call this a Centered Coprime Array.

**Theorem 7.** For a centered coprime array of length \( L > MN/2 - 1 \), the hole-free range will be bounded from below by \( L - MN/2 + 1 \). This bound is illustrated in Fig. 4.10.
Figure 4.10. An array centered on an appropriate $MN/2$ region must measure every lag up to $L - MN/2 + 1$.

Proof. Let $y$ be a colag such that $y \leq L - MN/2 + 1$.

If the lag is at position $\pi$, then the elements are positioned at $x_- = \pi - y/2$ and $x_+ = \pi + y/2$.

The array is limited to the range $[-L/2, L/2]$, which are the bounds on $x_-$ and $x_+$, respectively.

Thus, for the largest possible lag of $L - MN/2 + 1$, $\pi \geq -L/2 + y/2 = -MN/4 + 1/2$ and $\pi \leq L/2 - y/2 = MN/4 - 1/2$.

Thus, a repetition of lag $y$ exists if $\pi$ is in the range $[-MN/4+1/2, MN/4-1/2]$.

Moreover, we note that $\pi$, is an average between two elements in the array and must take the form $\pi = \pm MN/4 + z/2$ where $z$ is an integer. Additionally, because the points $-MN/4$ and $MN/4$ are reflection points about which the infinite version of each component array is symmetrical, it can be concluded that any lags centered on those points cannot be colags. Thus extending the range to $[-MN/4, MN/4]$.

Noting that this is a contiguous region of length $MN/2$, that either begins or ends with a shared element, as shown in Theorem 6, a repetition must exist in this range.

Therefore, the lag $y \leq L - MN/2 + 1$ must exist in this array. 

A direct corollary of this handles arrays with a shared first element. To do this, we consider the section of the array that is morphologically identical to a Centered Coprime Array, and apply the above theorem to that section.

Corollary 7.1. For any even integer $B$ greater than or equal to 2, a coprime array with a shared first element will be fully augmentable up to lag $BMN/2 + 1$ if the total length of the array is greater than or equal to $(B + 1)MN/2$.

Proof. Let the first element of the array be defined as position 0.

The point $(B + 1)MN/4$ is $MN/4$ units away from the points $BMN/4$ and $BMN/4 + MN/2$. 

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If \( B \) is a multiple of 4, then \( BMN/4 \) is a multiple of \( MN \) and is thus a shared element of the array. Otherwise, \( BMN/4 + MN/2 \) is a multiple of \( MN \) and is thus a shared element.

Thus, the point \((B + 1)MN/4\) is exactly \( MN/4 \) units away from a shared element in the array.

Truncating the array to length \( L = (B + 1)MN/2 \), we see that the array is now in the same form as the previously described centered coprime array.

Thus, by application of Theorem 7, the hole-free range of the array is bounded from below by \( L - MN/2 + 1 = BMN/2 + 1 \).

The usefulness of this lower bound depends on two properties. First, the lower bound should be tight for certain array designs. That is, there should exist some array designs for which the hole-free range is equal to the lower bound. Additionally, the bound should not apply to extended coprime arrays that are not centered coprime arrays. That is, there should exist extended coprime array designs that begin with a shared element with hole-free ranges beneath the lower bound for centered coprime arrays. Fig. 4.11 shows the hole-free ranges, determined directly from the coarray, of both centered and un-centered coprime arrays as a function of array length. The lower bound for the array length for a centered coprime array is also shown, and at certain points is equal to the true hole-free range of the array. Thus, the lower bound is tight for certain array configurations. Moreover, the un-centered coprime array is not consistently above the lower bound for the centered coprime array. Thus, this lower bound is a feature unique to the centered coprime array design. However, certain un-centered array designs possess hole-free ranges greater than that of an equal-length centered coprime array. As such, there is still value in performing a search of array configurations to determine the appropriate starting location and length of a coprime array to maximize the hole-free range.

Hole-free ranges have been considered for coprime array designs in the past, and can be compared to the centered coprime array. For the prototype coprime array of length \( MN - N + 1 \), discussed in Sec. 3.1.1, the hole free range is \( M + N - 1 \) [44]. In comparison, for a centered coprime array of the same length, the lower bound is \( MN/2 - N + 2 \). When \( N = M + 1 \), the centered coprime array possesses a larger hole-free range when \( M > 5 \). The conventional coprime array, with a total length of \( 2MN - N + 1 \), has a hole-free range up to \( MN + M - 1 \) [44]. An equal length centered coprime array has a considerably larger hole-free range of \( 3MN/2 + N \) (bearing in mind that \( M \) is chosen to be less than \( N \) in the conventional coprime array design).

### 4.4 Centered virtual subarrays

Up to this point, we have focused only on the problem of direction of arrival estimation on a distant source, such that the array length is much less than the
range to the source. For both spherically and cylindrically spreading waves, like those found in the shallow water ocean, the wave fronts at such long distances are approximately planar. However, when the distance from the array to the source is on the order of the array length, the impinging wavefronts will be curved, as shown in Fig. 4.12. In this section, we will consider one potential method of handling this situation on a coprime array, which we will call Centered Virtual Subarray (CVS) processing.

Subarray processing is a technique which has traditionally been used with uniform linear arrays. In subarray processing, subsets of the sensors in the array are processed separately and the resulting outputs of those subarrays can be combined to yield estimates of a source’s bearing and range. Conceptually, this process is very simple as any subsection of a uniform linear array is, itself, a uniform linear array with the same minimum spacing as the full array. Also, because these subarrays have a smaller aperture than the full array, the plane wave approximation holds for nearer source locations (though limitations do still apply).

While subarray processing is a straightforward technique on uniform arrays, it is not necessarily so for sparse arrays. Because of the non-uniform spacing of elements in a sparse array, not all subsections of the array will have the same coarray, and thus their beam patterns may vary significantly. A clear example of this can be seen in Fig. 4.13, which shows that subsections of a Golomb Array (as

Figure 4.11. The hole-free ranges of a centered and un-centered coprime array configurations for different array lengths ($L$) given $M = 3$ and $N = 4$. The dashed line shows the lower bound $L - MN/2 + 1$ on the hole-free range of a centered coprime array.
A source generates curved wavefronts for an array whose length $L$ is on the order of the source distance $r_s$. Apparent source angle along the array varies from the true source bearing $\theta$.

Subsections of a Golomb Array do not possess a full complement of lags, with only one section possessing the $\lambda/2$ spacing. This spacing is only present in one section of the array. This results in the remaining subsections suffering source angle ambiguity.

While extended coprime arrays are not uniform, they do still possess some level of regularity, both in element spacing and the positions of lag repetitions. This regularity can be exploited for the purposes of subarray processing by properly selecting overlapping subsections, each with the same coarray.

To develop a set of Centered Virtual Subarrays, we first begin with a centered coprime array with a length no less than $5MN/2$. We then consider each section of
Figure 4.14. A coprime array of length $5MN/2$ with the position of a Centered Virtual Subarray marked. This subarray uses lags centered in the $MN/2$ range made up of elements within the subarray length $L_s = 3MN/2$.

the array of length $MN/2$ beginning or ending with a shared element as the center sections of our potential subarrays. From here, we must select an appropriate length for the subarrays ($L_s$), keeping in mind that the hole-free ranges of the resulting subarrays are subject to the same limits as any other centered coprime array. An example Centered Virtual Subarray can be seen in Fig. 4.14. While these overlapping subarrays can be processed as is, we propose an additional step to reduce overlap of lag measurements. Specifically, the covariance estimate for each subarray is calculated only using those lags centered within the $MN/2$ range that forms the center of the subarray.

The above formulation has a number of desirable properties. First, it can be seen that the number of lag redundancies for each subarray are identical, resulting in identical natural beampatterns. Moreover, the locations of lag measurements for any given lag between two adjacent subarrays will be mirror images of each other. Additionally, by restricting the range of lag measurements used for covariance estimation, while allowing the use of elements over a larger range, some of the benefits of a large aperture array for reducing beamwidth can be maintained while minimizing the effective aperture with respect to the wavefront curvature.

To test the behavior of the coprime subarray we used a simple free-field simulation. We considered a 500 Hz source at a distance of 100 m from the center of a 64.5 m array with minimum spacing $\lambda/2 = 1.5$ m. For the sake of comparison, we considered both a 43-element uniform array and a coprime array with $M = 3$ and $N = 4$ (22-elements). The coprime array had a length of $7MN/2$, which was broken into five overlapping subarrays of length $L_s = 3MN/2$.

The benefit of the reduced aperture can be seen in Fig. 4.15 which shows the beam response for a full coprime array, compared the the response for a single subarray. As can be seen, the beam response for the subarray is centered near the true source location, albeit with a wide main lobe.

Moreover, multiple subarrays can be used in tandem. As shown in Fig. 4.16, each subarray yields a slightly different bearing estimate, as the wavefront curvature dictates at different points along the array. By combining these measurements, through the use of a Kalman filter for example [57], the true source location can
Figure 4.15. Beam responses of a uniform array, coprime array, and a single virtual subarray to a source at angle $\theta = \pi / 6$ and distance $r_s = 100$ m. Wavefront curvature has resulted in poor localization for both of the full arrays, but not for the subarray.

be triangulated. Additionally, such triangulation can also give a source range in addition to a bearing estimation.

Subarrays can also be used for the related problem of bearing estimation using a curved array. For a towed array during a turn, for example, the array shape may be perturbed, resulting in a performance loss on the array [58]. While the effects of such perturbations can be reversed through knowledge of the true element location, subarrays offer another method of compensation so long as the radius of curvature of the array is greater than $MN/2$, the length over which each subarray’s lags are centered.

Ultimately, centered virtual subarrays are a flexible processing tool with a history of use for uniform arrays that can also be used on coprime arrays, despite their sparsity. However, it should be noted that many of the advantages of subarray beamforming, such as the ability to determine range and handle nearby sources, can also be achieved with near-field beamforming techniques using the full array.
Figure 4.16. Beam responses of centered virtual subarrays to a source at angle $\theta = \pi / 6$ and distance $r_s = 100$ m. The bearing estimate of each subarray can be combined to triangulate a source bearing and range.
Chapter 5

Shallow Water Environments

The shallow water environment presents a number of complications for acoustic propagation not present in the basic models used to develop coarray theory. Rather than a stationary isotropic medium of infinite extent with uncorrelated noise on each channel, the shallow water ocean has an acoustic medium with temporal and spatial variation, two very important boundaries in the ocean surface and bottom, and propagating noise with non-isotropic components. In this section we develop a model for how this shallow water environment changes the amplitude, phase, and noise characteristics of a received signal, and describe our method for simulating the channel’s effect on a narrow-band signal to test the coprime array’s efficacy in this environment. These models will be directly related to resolution, array gain, and sidelobe level as appropriate.

We begin by examining how the ocean acts as a two dimensional waveguide, and how modal propagation results in spatially varying transmission loss. Next, we examine how random variation in the environment and scattering from the ocean surface and bottom results in reduced signal coherence through random phase variation. Finally, we consider how spatially varying propagating noise results in noise that shows marked dependence between channels.

These effects, which characterize the primary difficulties of processing signals measured in shallow water, can be simulated using a combination of methods. Specifically, propagation simulation using a parabolic equation model generates a signal with transmission loss characteristics consistent with a shallow water waveguide and noise with a spatially dependent spectrum, while a Gaussian random walk applied to the phase of the resulting signal is used to account for the loss of coherence shown through empirical measurements of shallow water channels. This chapter will discuss the details of these simulations, including the parameters chosen for our test environment, which will be used to measure the performance of a centered coprime array.

To examine the effects of the shallow water environment, we must first extend
the basic signal model described in Eq. 2.5. For simplicity, we will consider a “frozen” environment that does not change over the integration time of our measurement, with no side information for real-time channel estimation. Accounting for the addition of channel effects and noise yields a measured signal function given by Eq. 5.1.

$$\tilde{V}(\bar{x}, t) = A(\bar{x})v(\bar{x}) \exp(j(\omega_0 t + \phi(\bar{x}, t))) + z(\bar{x}, t) \quad (5.1)$$

Where $A(\bar{x})$ is the channel-induced amplitude modulation resulting from Transmission Loss (TL), $\phi(\bar{x}, t)$ is channel-induced phase variation, and $z(\bar{x}, t)$ is an additive noise term, representing the sum of both measurement noise and propagating noise within the environment. Using this measured signal in the Blackman-Tukey spectral estimation of intensity, we can determine the impact of channel effects as they relate to the statistics of the channel-induced variables. By substitution of Eq. 5.1 into Eq. 2.6, we yield the cross-correlation of a received signal including channel effects, given by Eq. 5.2.

$$\tilde{r}(\bar{x}_1, \bar{x}_2) = A(\bar{x}_1)A(\bar{x}_2)(r(\bar{x}_1, \bar{x}_2) \ast \exp(j(\phi(\bar{x}_1, t) - \phi(\bar{x}_2, t)))) + A(\bar{x}_1)E[v(\bar{x}_1)z^*(\bar{x}_2)] + A(\bar{x}_2)E[z(\bar{x}_1)v^*(\bar{x}_2)] + E[z(\bar{x}_1)z^*(\bar{x}_2)] \quad (5.2)$$

Neglecting the second and third terms of the equation, which depend on the cross correlation of the signal and noise and will tend towards zero for sufficient measurement time, yields:

$$\tilde{r}(\bar{x}_1, \bar{x}_2) = A(\bar{x}_1)A(\bar{x}_2)(r(\bar{x}_1, \bar{x}_2) \ast \exp(j(\phi(\bar{x}_1, t) - \phi(\bar{x}_2, t)))) + E[z(\bar{x}_1)z^*(\bar{x}_2)]$$

$$= r(\bar{x}_1, \bar{x}_2)r_{env}(\bar{x}_1, \bar{x}_2) + E[z(\bar{x}_1)z^*(\bar{x}_2)] \quad (5.3)$$

Examination of this equation reveals the salient statistics of the channel parameters. Namely, the amplitude of the transmission loss directly affects the magnitude of the measured correlation and the resulting signal-to-noise ratio, the difference in channel induced random phase (and thus the joint statistics of said phase) introduces a phase shift to the measured correlation, and the autocorrelation of the noise field affects the resulting signal-to-noise ratio. As such, our examination of these variables will focus on the magnitude of the transmission loss variation (also known as the Scintillation Index or fading strength), the joint statistics of phase variation (also known as the phase coherence), and the spatial correlation of the noise field. These environmental deviations can be expressed with the environmental correlation function $r_{env}$.

With the channel correlation function, and its dependence on environmental parameters, we can find the effect of the shallow water environment on the array beam response as given by the Blackman-Tukey spectral estimation method in Eq. 2.9. Substituting the estimate of cross-correlation given in Eq. 5.3 yields:
The shallow water channel introduces significant amplitude modulation to signals that it carries. This modulation, referred to as “Transmission Loss” (TL), results from multipath propagation and the related phenomenon of modal propagation. In this section, we introduce the features of the shallow water environment that
lead to spatially varying TL, including multipath induced by surface and bottom reflections, horizontal multipath from bearing dependent channel features, and mode-shape variation induced by non-uniformity in the soundspeed profile. Taking the physics of the environment into account, we develop statistics for the variation of TL in the channel, as well as the vertical and horizontal length scales that dictate how TL varies in space. We then consider the parabolic equation model, and discuss its use in simulating TL for a shallow water environment with a typical downward refracting soundspeed profile. Finally, we apply our model of TL to the coprime array and examine its effect on sidelobe level, array gain, and resolution.

While the reason we consider the distribution of the TL may be easily understood, the interest in the length scales of this variation may require some justification. The vertical and horizontal length scales of the TL function are important for two reasons. The first is when measuring the wavenumber spectrum in a channel is of interest, as is often the case for vertical arrays. In this case, ensuring that the minimum spacing of the array is sufficient to measure the full range of modes is important. Another matter of interest is “signal dropout,” which occurs when a receiver is placed in a deep null in the TL pattern. Considering Eq. 5.3, it can be seen that a null in the TL for some location will reduce the amplitude of the resulting covariance measurements which use that sensor. The length scale of the variation dictates the range over which we can expect such nulls to occur. When the length scale of variation is large compared to the array aperture (sometimes called ”slow fading”), TL over the array becomes a constant such that $A(\tilde{x}) = A_{array}$, while a length scale that is short compared to the array aperture (”fast fading”) will result in TL between element locations that are approximately independently identically distributed Rayleigh random variables. This information, in combination with a noise threshold, can be used to predict the likelihood of signal dropout for a given sensor, as well as the likelihood that signal dropout at one measurement location will affect nearby elements.

5.1.1 Waveguide Effects on Transmission Loss

The impact of the ocean surface and bottom are of great importance in littoral waters. There are two conceptual approaches to this problem common in the literature: modal propagation and ray propagation. Each has its own strengths, and they can be expressed as extensions of one another [5]. However, for our purposes, we will focus on modal propagation for the qualitative understanding of the problem it gives. Modal propagation treats the shallow water ocean as a 2-dimensional waveguide with the water column as the medium and the bottom and surface as distinct boundaries. The orthogonal (or near orthogonal) eigenmodes which satisfy these boundaries determine the vertical wavenumber spectra supported by the environment, and those (potentially complex) wavenumbers influence the propagation of those modes.
For low frequencies, with wavelengths on the order of the channel depth, the modal propagation model is particularly useful. Consider the Green’s function for an outgoing wave from a point, decomposed as the sum of products of the Hankel’s function and vertical modes, selected to satisfy the pressure release boundary condition of the surface and the complex boundary condition of the ocean bottom [5], as shown in Eq. 5.9.

$$\Psi(r, z) = \frac{j}{4} \sum_{l} \psi_l(z_0) \psi_l(z) H_0^{(1)}(\xi_l r)$$ (5.9)

The vertical mode shapes, $\psi_l(z)$, can be determined analytically for simple environments, but can also be found through numerical means, particularly in cases with a vertically dependent sound speed profile. The coefficient $\xi_l$ is found as the solution to the Sturm-Liouville problem, and act as the complex wavenumber for a given mode, determining both its phase speed and attenuation. In general, a signal generated in the ocean will excite multiple modes, and differences in phase and group speed for those modes results in spatially varying interference between modes.

An example of TL variation for a shallow water waveguide can be seen in Fig. 5.1 from [1]. The figure in question was generated through simulation of a 112 Hz point source with a modal propagation model, and exhibits the horizontal and vertical variation predicted by that model. Notably, at ranges beyond 3 km, the horizontal variation is on the ranges of hundreds of meters while the vertical variation is on the order of a wavelength ($\approx 14$ m).

The interference between modes is the primary source of TL variation in the shallow water channel, and while such interference is ostensibly deterministic for a given environmental realization, uncertainties about the ocean depth, bottom composition, and other relevant ocean parameters make predicting the TL difficult. A comparison between simulated and experimental TL can be seen in Fig. 5.2 from [1]. Notably, while the presence of nulls and peaks in the TL pattern is realized, the exact location and values of these variations are not.

### 5.1.2 Other Effects

While waveguide effects are of great importance for determining TL, other features of the shallow water environment can also be of interest.

Horizontal multipath, where a signal is either reflected or refracted such that sound from the source reaches the receiver from multiple bearings, can change the TL field. Much like modal interference, these horizontal multipaths can interfere with each other.

Soundspeed profile is also of importance, as the soundspeed will alter the shapes of modes supported by the ocean waveguide (though this can be accounted for through numerical modeling). Specific types of soundspeed profiles, however, can
Figure 5.1. Simulated transmission loss pattern for a 112 Hz point source in the environment of the SwellEx-96 experiment [1]. The horizontal scale of variation is greater than the vertical, particularly at large distances from the source.

have notable influence on the wavenumber spectrum. Downward refracting profiles result in higher attenuation from bottom loss effects when compared to a constant sound speed profile. This can reduce the range of energetic modes, and result in larger variation scales.

While all of these can be included in more detailed treatments of the shallow water ocean, they will not be given individual attention in this work.

5.1.3 Propagation Simulation

Propagation simulations are one way to generate realistic TL fields. Such simulations offer complete control over the parameters of the simulated experiment that is not available in a real ocean environment. However, these simulations come with a cost in accuracy, as all computational models entail some amount of approximation to the true solution and limits on the accuracy of those solutions. A number of propagation models exist, each with their own advantages and disadvan-
Figure 5.2. Simulated transmission loss patterns of the SwellEx-96 experiment along a given track compared against experimentally measured transmission loss along the same track [1]. While differences in the patterns are evident, statistical similarities give insight into the behavior of the waveguide.

Parabolic equation models are based on solutions to the eponymous parabolic equation shown in Eq. 5.10, which is the result of a paraxial (small angle) approximation of the 2-D Helmholtz equation [59]. The parabolic equation includes a number of restrictions when compared to a full wave equation solution. Modern implementations of the parabolic equation model, such as RAM, can relax these limitations to the point that the models are appropriate. For example, by using a Padé series approximation for the quantity $\sqrt{1 + q}$, the usable range of the paraxial approximation can be expanded beyond 75° [60].

$$2i\omega_0 \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} + k_0^2(n^2 - 1)\psi = 0 \quad (5.10)$$

The simulated environment used was a 100 m isovelocity channel with a sandy
bottom. For the purposes of this dissertation, the most notable limitation of the modeling is that spatial variation in the water column and temporal variation in both the water column and the sea surface are ignored. This results in a significantly overestimated spatial coherence along the array, which must be compensated for by an additional step in the modeling to simulate the travel time variation induced by these processes. However, this model serves as a good first step in the simulation chain.

To augment the transmission loss variation, a Gaussian random walk is applied to the level from element to element. The scale of the variation can then be changed as needed to approximate the effect of different environments.

5.1.4 Transmission Loss and Sidelobe level

As discussed in Sec. 2.3.4, sidelobes are those sections of the beam response beyond the main lobe, whose height is determined by the spectral weighting of the array. In the same way that array shading alters sidelobe characteristics through adjusting the gain applied to elements along the array, transmission loss imposed on the array by the environment also alters the sidelobe level.

Consider the effective spectral weight given in Eq. 5.5. The effective weight for any given lag depends on the sum of the product of random vectors determined by the environment. Transmission Loss (TL) directly relates to the amplitude of those random vectors, and it can be seen that the fundamental length scale of the TL variation determines the correlation of the amplitudes.

Fig. 5.3 shows the peak sidelobe level for a coprime array and a 13-element MRA, with and without covariance augmentation, for simulated data as a function of the normalized variation scale. Given a spatially-correlated random TL field, the normalized variation scale is defined as the distance at which the covariance of the TL at two points drops to $1/e$, normalized by the acoustic wavelength. To better illustrate the effects of transmission loss, the data is generated such that the amplitude is Rayleigh distributed and the phase coherence is infinite across the array. As can be seen in the figure, when fading is slow the augmented arrays approach the performance of the uniform linear array of the same length. However, fast fading across the array causes an increase in peak sidelobe level. The coprime array, with its greater number of lag repetitions, suffers a less severe loss of performance than the minimally redundant array, and shows less variation between environmental realizations for the same variation scale.

It can also be seen that while transmission loss variation does degrade performance for the augmented array, there is still a performance gain in peak sidelobe level when compared to the same array without covariance augmentation.

Fig. 5.4 shows the peak sidelobe levels for three sets of coprime array factors. The arrays with larger factors (corresponding with fewer elements), show less robustness to short variation scales.
A quantitative description of the relationship sidelobe level and transmission loss length will be pursued with the addition of signal incoherence, because of the importance of random phase in this development.

### 5.1.5 Transmission Loss and Array gain

Based on the changes to the sidelobe level resulting from transmission loss, one might expect a significant change in array gain as an effect of transmission loss variation. As discussed in Sec. 2.3.5, array gain is closely related to the integrated sidelobe level of the beam pattern, which measures the ratio of energy in and out of the main beam. Fig. 5.5 shows the integrated sidelobe level as a function of variation scale for an incoming wave on an MRA and Fig. 5.6 shows the same for a selection of coprime array factors.

We note, however, that while the beam pattern is, by definition, the response of an array to a unit power source, the very application of random variation to this source may not qualify for the purposes of Eq. 2.16. Thus, in comparison we will consider the more general definition for array gain as expressed in Eq. 2.18. For this, we need the ratio of array power to element power for both signal and noise. Considering that transmission loss is an effect on the signal, we can disregard the effect on noise. Thus, the peak received level on the array can serve as a measure.
Figure 5.4. Peak sidelobe level as a function of transmission loss length scale as simulated for three sets of coprime factors with length $L = 60$ averaged over 400 fading realizations. Denser arrays with more lag redundancies converge faster to the performance of the ULA as fading scale increases.

of array gain.

Fig. 5.7 shows the peak level, as measured in the previous examples compared against that of a single element. The figure does not show the pronounced difference that might be expected from the previous figures.

To understand this, we must first consider the distribution of the peak intensity, which may be approximated as a sum of identically distributed squared Rayleigh random variables (particularly for the small lags which make up the bulk of the measured lags). This distribution tends towards a Gamma distribution with parameters $K$ and $2\sigma^2$ where $K$ is the number of Rayleigh variables summed: $\Gamma(K, 2\sigma^2)$. As the mean of of this distribution is the product of its parameters $2K\sigma^2$, the normalized value is not a function of the number of elements. Instead, the effect of transmission loss variation along the array is only expressed in the other parameters of the peak distribution, such as variance, skewness, and kurtosis, which all depend on the parameter $K$.

5.1.6 Transmission Loss and Resolution

As discussed in Sec. 2.3.3, the primary determinant of array resolution is aperture. Thus, transmission loss reduces array resolution when effective aperture is somehow reduced. This is best understood using a “Sensor failure model”, where an element
Figure 5.5. Integrated sidelobe level as a function of transmission loss length scale as simulated for three arrays of length $L = 60$ averaged over 400 fading realizations. While the performance of the un-augmented arrays stays near $10\log_{10}(K)$, the augmented arrays approach the performance of the ULA with a greater number of elements.

is determined to have “failed” or suffered signal dropout when the SNR drops below some given threshold. Resolution is reduced when the failure of elements in an array reduces the usable aperture of the array, particularly those elements near the ends of the array. Moreover, when the number of redundancies is small, there is a risk that elements within the array contribute to unique lags and may fail, reducing the usable aperture for processing methods that rely on a fully populated coarray. We quantify this risk through the use of Single Points of Failure (SPOF’s), and find a relationship between the number of SPOFs and the coprime array design parameters $L$, $M$, and $N$.

One way to model the effect of transmission loss on array performance is to use a sensor failure model. This model approaches elements on a pass-fail basis, where an element whose SNR is too low is deemed to have failed. A major advantage of this approach is that it can be combined with other potential causes of failure, such as electrical failure or mechanical defects, to determine an overall failure rate for the array. The probability of failure for an element as a result of insufficient SNR is a function of the probability density function of the transmission loss, the average SNR, and the SNR threshold. The probability of failure is simply the complement of the detection probability, closed form solutions of which exist for common fading distributions such as Rayleigh and Nakagami [61].

For processing methods which utilize the full physical aperture of the array, the
Figure 5.6. Integrated sidelobe level as a function of transmission loss length scale as simulated for three sets of coprime factors with length $L = 60$ averaged over 400 fading realizations. Denser arrays again show a faster convergence to the uniform linear array behavior.

resolution depends on the largest lag measured in the array, between its first and last elements. When transmission loss results in failure for the extreme elements of the array, resolution is negatively impacted. The probability of resolution loss for a given environment is given by the probability of the elements on either end of the array failing. In a fast fading environment, these are independent of each other. In the case where one of these elements fails, the likelihood of additional failures within the variation scale is increased. As such the usable aperture of the array will vary on the order of the variation scale, with probability given by twice the single element signal drop-out rate.

For applications where lags beyond the hole-free range are usable, this explanation is sufficient. However, when the hole-free range determines the effective aperture (as with covariance inversion techniques), consideration must be given to elements within the array that contribute to unique lags and lags with a limited number of redundancies. Consider some lag $\gamma$ within the hole free range of the array, with $\gamma$ repetitions. This lag will be measured, so long as a single repetition of this lag remains after accounting for elements with signal dropout.

For lags with a single repetition, this is simply given as the the probability of loss for that lag is the $1 - (1 - P)^2$, where $P$ is the probability of failure for a single element. This is understood by considering that the lag will fail to be measured if either of the measuring elements fails. For lags with a larger number of redundancies, however, this becomes more complex, as elements may
Figure 5.7. Normalized peak level as a function of transmission loss length scale as simulated with an array of level $L = 60$ compared against the level measured by a single element. The small difference suggests little change in array gain arising purely from transmission loss variation.

be shared between lags, and different permutations of multi-element failure need to be considered. Development of a full theory for the reliability of sparse arrays is beyond the scope of this paper, however, some insight can be found in the concept of single points of failure in the array.

A single point of failure (SPOF) describes a part of a system that, if it fails, will cause a failure for the full system. In the context of an array, this describes an element whose lone failure will result in a reduction of effective aperture for the array. For a uniform array, there are two SPOF’s (namely the elements at the end of the array) while for a minimum redundancy array all elements are SPOF’s. For a coprime array, the number of elements that act as SPOF’s can be determined by considering the lags with redundancy of one. Solving Eq. 4.7 for the lag difference $L - y$, we find:

$$\text{(5.11)}$$

$$L - y < MN$$

This implies that elements contributing to lags greater than $L - MN$ may be SPOF’s. In order for an element to contribute to this set, it must be within $MN$ units of one end of the array. As such, the number of SPOF’s for a coprime array is bounded from above by $2(M + N) - 1$, supposing all elements near the edges
of the array are SPOF’s (which may or may not be the case). As such, coprime arrays with larger coprime factors which are more sparse possess a larger number of SPOFs for the same array length.

For the purposes of reliability, the number of SPOF’s is important, as (assuming element failure $P$ is independent between elements) the probability of any SPOF failing is given by:

$$1 - (1 - P)^{n_{sp}}$$

(5.12)

Where $n_{sp}$ is the number of single points of failure in the array. While this number is greater than that of a uniform array (with two SPOFs), the number that do exist does not scale with the array extension for a given set of coprime factors. As such, extended coprime arrays are no more susceptible to loss of aperture from the failure of a single element than a smaller array, but a tradeoff exists in that arrays with greater sparsity also possess a greater number of SPOFs. An engineering decision can be made using Eq. 5.11 and Eq. 5.12 to find an acceptable set of coprime factors. Specifically, for a given threshold of failure $P_f$ we find:

$$MN < L - \frac{\log(1 - P_f)}{\log(1 - P)}$$

(5.13)

An example employing this in design may be selecting coprime factors $M$ and $N$ for a 50 wavelength long array given an acceptable failure rate $P_f$ of 20% given an individual element failure rate $P$ of 1%. Using Eq. 5.13 we find that the coprime factor product $MN$ must be less than 27.8, thus factors $M = 4$ and $N = 5$ may be chosen to satisfy the reliability criterion, while factors $M = 5$ and $N = 6$ would be unacceptable despite otherwise fulfilling design requirements for a coprime array of this length.

### 5.2 Phase Variation

Beamforming operations work primarily through the application of phase adjustments to measurements taken along the array. As such, the way that phase is modulated by the environment is of great practical interest. Specifically, as shown in Eq. 5.2, the difference in channel induced phase between points directly affects the measured correlation for a given lag. In this section, we consider empirical measurements of phase variation and coherence length in shallow water. We then employ a Gaussian random walk in time variation applied to simulated signals taken at different locations to induce phase variations whose statistics match those found through empirical measurements. This empirical model is then used to extend the propagation modeling of the previous section.

A number of physical processes contribute to phase variation in the ocean. *Fundamentals of Shallow Water Acoustics* notes several such processes, including
internal waves, thermohaline structure, bottom roughness, and surface waves. All of these contribute to variations in wavespeed or acoustic path in such a way that travel-time is altered, and the spatially varying nature of these features leads to coherence loss between sensors with increasing distance. However, a full treatment of random medium theory is outside the scope of this paper, and interested readers are encouraged to explore *Wave Propagation and Scattering in Random Media* by Ishimaru [62].

### 5.2.1 Travel-Time Variation Modeling

While it may be possible to individually model each physical process’ effects on an array’s performance, in practice the relevant environmental parameters are not always measured with enough precision to accurately model the acoustic environment. As such, there is value in considering gross statistics of the channel which can inform array design. One such statistic is known as channel coherence, which measures the degree to which phase can be stably predicted in space and time. Because conventional beamforming operates through phase alignment of measurements on different channels, the coherence length is a key figure in predicting array performance. Notably, because of the shallow water effects already described, the coherence length in shallow water tends to be lower than that of deep water environments [25].

Coherence length considerations are already common in array design. Traditionally, the coherence length is used to determine the maximum lag usable by an array, and thus the limits on resolution. Coherence also limits the array gain achievable by an array, as additional sensors added beyond the coherence length contribute to signal power and noise power at the same rate. This property of coherence can also be used to estimate the coherence length, as the marginal gain of adding successive elements can be used to regress to an approximate value for the channel coherence length. As such, not only is this channel metric conceptually intuitive, but also practically estimable in deployment by empirical means.

For the purposes of array processing, we are interested not only in the phase variation between pairs of sensors, but also between sets of sensor pairs. Fig. 5.8 shows a diagram of the simplified problem, where two pairs of sensors, each separated by some lag $y$, are situated such that the centers of the pairs are separated by a distance $d$.

A common model for coherence is to treat the phase at a given point as the sum of a deterministic phase component and a zero-mean random component which varies in space. Consider a pair of elements separated by some distance $y$. We consider the model from Carey, where $\phi(x_1)$ and $\phi(x_2)$ are zero-mean random phase perturbations (each following a von Mises distribution) related by the following covariance function:
Figure 5.8. Sensor positioning. Two pairs of sensors, separated by some distance \((d)\).

\[ R_\phi = \exp(-y^2/C_L^2) \] (5.14)

Where \(C_L\) is the phase coherence length of the channel (this notation eschews the normalization by array length given in Carey). As such, the correlation between the two random phases decreases with increasing distance between the elements.

Extending this to the multi-sensor problem shown in Fig. 5.8, we consider the average covariance measurement across the two pairs. Momentarily neglecting both the TL and noise correlation terms in Eq. 5.3, we model this using four complex phasors of equal length with phases \(\phi_1, \phi_2, \phi_3, \text{ and } \phi_4\), that are related according to the above coherence function. This yields the model shown in Eq. 5.15.

\[ \tilde{R}_y = R_y \cos \left( \frac{\omega (\phi_1 - \phi_2) - (\phi_3 - \phi_4)}{2} \right) \exp \left( j\omega \frac{(\phi_1 - \phi_2) + (\phi_3 - \phi_4)}{2} \right) \] (5.15)

This equation exhibits both amplitude and phase variation from the cosine and complex exponential terms, respectively. The amplitude variation results from phase differences of the correlation estimates. Again, we consider the phase, now of the average covariance of the pairs:

\[ \tilde{\phi}(y, d) = \frac{1}{2}((\phi_1 - \phi_2) + (\phi_3 - \phi_4)) \] (5.16)

This term has a mean of zero and variance that is the sum of the individual phase variances and the covariance modeled by Eq. 5.14:

\[
\text{Var}(\tilde{\phi}(y, d)) = \frac{1}{4} \sum_{i,j=1}^{4} \text{Cov}(\phi_i, \phi_j) \\
= \sigma^2 + \sigma^2 \left( e^{-(d^2/C_L^2)} - e^{-y^2/C_L^2} - \frac{1}{2} e^{-(d+y)^2/C_L^2} - \frac{1}{2} e^{-(d-y)^2/C_L^2} \right)
\] (5.17)
Considering each individual term gives a physical understanding of the variance. The first term ($\sigma^2$) gives the variance when all of the sensors are mutually independent. The second term ($\sigma^2 e^{-d^2/C^2_L}$) represents the coherence between the first and third, and the second and fourth elements, giving the portion of the variance that is shared between the pairs. The third term ($-\sigma^2 e^{-y^2/C^2_L}$) represents the correlation within each pair, and has a negative sign because the difference taken between the pair reduces the variance. The fourth ($-\sigma^2 e^{-(d+y)^2/C^2_L}$) and fifth ($-\sigma^2 e^{-(d-y)^2/C^2_L}$) terms show the correlation between the inside and outside sensors, respectively. Again, the contribution of these terms to the variance is negative because of the sign difference.

We can also get considerable insight by considering the limits of the above equation. In the limit that $C_L$ is large compared to $y$ (that is, when coherence between all elements is high), we find that the total variance goes to zero.

In the limit that $d$ is small, the coherence converges to the following:

$$\text{Var}(\tilde{\phi}(y,d)) = 2\sigma^2 (1 - e^{-y^2/C^2_L})$$

Which is consistent with the two measurements being identical, and the standard deviation remaining the same.

In the limit that $d$ is large (that is, the sensor pairs are separated by a large distance), we find that:

$$\text{Var}(\tilde{\phi}(y,d)) = \sigma^2 (1 - e^{-y^2/C^2_L})$$

Which is consistent with the measurements being independent.

We may also wish to consider the case where $d = y$, which is realized in a uniform array when the center element is shared. This results in a variance given by:

$$\text{Var}(\tilde{\phi}(y,d)) = \frac{\sigma^2}{2} (1 - e^{-(2y)^2/C^2_L})$$

Which can be understood by considering that the contribution to the variance of the innermost pair sensors is completely eliminated, resulting in a measurement whose phase variance depends only on the outermost pair.

### 5.2.2 Phase Variation Simulation

Despite the importance of phase variation and the resulting signal incoherence on the performance of an array, the environmental model chosen does not yield significant signal incoherence with the chosen parameters. As such, the results of the propagation simulation detailed in the previous section must be augmented with an additional phase and level variation step.

One method that yields results consistent with Carey’s coherence model is a Gaussian random walk. In this method, phase between two individual points vary
independently following a Gaussian distribution, and the phase variation over some path is found by integrating the phase variations between adjacent points.

The above Gaussian model is useful for our purposes, as it exhibits a feature known as scale invariance. As such, the behavior over any given discretization of the model is self-similar, allowing us to choose an arbitrary spacing over which to apply the model, so long as the relevant variances are scaled properly. For the sake of simplicity, the scale is chosen to be the unit interelement spacing of \( d = \lambda/2 \), with a variance adjusted accordingly. Thus, for elements positioned according to \( d = \lambda/2 \), the phase added for successive locations is given by Eq. 5.21, where \( C_L \) is the coherence length, as expressed in Eq. 5.14, and \( \sigma \) is the phase variance for a given location.

\[
\Delta \phi = N(0,(\sigma d/C_L)^2) \tag{5.21}
\]

5.3 Noise

In its broadest definition, noise describes any measured value that is not considered to be part of the signal of interest. This includes electronic variation that originates in the processing, thermal noise or flow noise localized to a sensor, propagating noise originating in the environment, or even sounds in the environment that may be considered a “signal” for a different application. The physical mechanisms and resulting statistics depend greatly on the type of noise encountered, and thus we must take care in correctly framing the concept of noise for our purposes.

Unlike some applications where noise is primarily electrical in nature and thus inherently independent between channels, the majority of noise in the shallow water environment at the frequencies of interest is actually propagating noise in the environment. The noise statistics, particularly the correlation between sensors, depends on the spatial distribution of the noise following a Fourier transform relationship in precisely the manner used for determining the cross-correlation for sources of interest in Eq. 2.8. Thus, by modeling the spatial distribution of noise sources, the impact of noise on array performance can be examined. In this section, we examine both surface generated noise and noise from distant shipping traffic. While the former source of noise is horizontally isotropic, resulting in noise that is nearly independent between channels, the latter source is often much more compact, resulting in a highly directional noise response for the array.

5.3.1 Surface noise

One of the primary sources of noise in the ocean is the sea surface, specifically the breaking of waves at the sea surface and the vibrations of bubbles entrained in these breaking waves. If we model noise from the sea surface as being uniformly distributed on the planar boundary of the ocean, this results in a model such that
the noise distribution is uniform in azimuth and varies with angle from the horizontal. Reflections from the ocean bottom must then be accounted for, resulting in noise impinging on the array from a wide range of elevation angles. At this point, we can simplify the Fourier relationship in Eq. 2.8 based on bearing uniformity. Cox [52] provides such an equation for horizontally aligned sensors in Eq. 5.22.

\[
R_N(\vec{y}) = \int I(\vec{u}) \exp(jk_0 \vec{u} \cdot \vec{y}) d\vec{u} = \frac{1}{2} \int_0^{\pi} J_0[k_0 u \sin \phi] I(\phi) \sin \phi d\phi. \tag{5.22}
\]

Where \( J_0 \) denotes the zero-order Bessel function of the first kind (not to be confused with the spherical Bessel function of the first kind), and \( I(\phi) \) is the angular distribution of noise intensity as measured at the receiver location. Unfortunately, there is no widely applicable form for \( I(\phi) \) in shallow water, as the ambient noise directivity depends on both frequency and sea state [63]. However, if the angular distribution is taken to be uniform, this yields the familiar un-normalized sinc function, for isotropic noise as shown previously in Sec. 3.3.2.

It is at this point that we consider the specific properties of shallow water modal propagation. Rather than arriving at a continuous range of angles, some portion of the arriving noise will be found in discrete angles associated with propagating modes. For a horizontal array, the effect this will have depends on the steering angle of the array. When steered broadside, the vertically propagating direct sound from the surface may dominate, resulting in a field that still looks largely continuous. However, as the array is steered towards endfire the low angle propagating modes may dominate. As such, by the sifting property of integration, this discrete spectrum results in a received portion of the noise that is a sum of a finite set of scaled Bessel functions, as shown in Eq. 5.23.

\[
R_N(\vec{y}) = \frac{1}{2} \sum_n J_0[k_0 s \sin(\phi_n)] I(\phi_n) \sin(\phi_n) d\phi \tag{5.23}
\]

Fig. 5.9 shows the correlation as a function of lag for a range of arrival angles. Notably, while the isotropic noise model predicts zeros at multiples of a half wavelength, corresponding to the sensor grid, noise associated with discrete angles has a non-zero correlation at those same points. As a result, the correlation depends on the range of angles supported by the waveguide. However, even with discrete arrival angles, the sum of the incoming waves may approximate a sinc function, as shown with the dotted line, also in Fig. 5.9.

Regardless, it can be seen that as the envelope of either function decreases with distance, the sinc function at a rate of \( 1/y \), and the cylindrical Bessel function at a rate which approaches \( \sqrt{2/\pi y} \), for large values of \( y \). Thus, for sensor separations on the order of a half-wavelength, noise correlation drops off quickly, even if it does not vanish as is often assumed.
Figure 5.9. The noise correlation shows dependence on angle with the array, and differs from the isotropic noise case. A sum over the plotted angles, however, does approximate isotropic noise, particularly in the location of the first and second zeros.

For very low frequencies, however, such that the spacing between elements is small compared to a wavelength, noise correlation becomes a distinct concern and can contribute to a loss of array gain.

5.3.2 Shipping traffic noise

In addition to surface noise, there are also other sources of noise in the ocean. Shipping traffic, in particular, is of note in shallow water environments, as such environments are often home to ports and are otherwise the most active for all sorts of ocean vessels. For a small number of relatively close ships, an aggregate model for the noise is often not appropriate. Rather, such sources should be modeled in the same way as most signals of interest: a spatially compact semi-coherent source. For distant shipping lanes whose noise is the sum of multiple ships, however, alternative models may be employed. Specifically, this model is appropriate when considering a collection of sources whose individual angular distances (and channel-induced source spreading) render them unresolvable.

Unlike the azimuthally uniform model used in the previous section, we consider noise from a limited range of source angles. To model noise originating from a shipping lane, we will use a Gaussian distributed source, or (more precisely) a source with azimuthal directionality following a von Mises distribution (an angular analog to the Gaussian distribution). This model has been employed for other
localized noise, such as storms, and is analytically tractable [64]. Supposing the 
noise originates at some angle $\theta_n$ with respect to the array axis, with angular 
variance given by $\sigma_{\theta_n}^2$ and integrated noise power of $\sigma_{n}^2$, we may express the noise 
distribution by the following equation:

$$I(\theta, \phi) = g(\theta)I(\phi) = \exp\left(\frac{1}{\sigma_{\theta_n}^2} \cos(\theta - \theta_n)\right) I(\phi)$$  \hspace{1cm} (5.24)

Where $I(\phi)$ is the vertical directivity of the field. Again, we consider the 
relationship in Eq. 2.8, and take the Fourier transform of the above equation to 
yield a correlation function. Following the process given in Walker et al we yield 
the following integral equation:

$$R_N(y) = \frac{1}{2} \int_0^\pi I(\phi) \sin \phi I_0 \left( \sqrt{\frac{1}{\sigma_{\theta_n}^2} \cos \theta_n - iky \sin \phi} \right)^2 + \left( \frac{1}{\sigma_{\theta_n}^2} \sin \theta_n \right)^2 \, d\phi$$  \hspace{1cm} (5.25)

Where $I_0$ is the modified Bessel function of the first kind of order zero. One 
interesting thing to note with this equation is that as $\sigma_{\theta_n}$ approaches infinity, the 
avimuthal distribution approaches uniformity and Eq. 5.25 simplifies to Eq. 5.23.

One key difference between Eq. 5.25 and the result given for azimuthally uniform 
noise is that the former is complex. This results in a response that varies with look 
angle, which is to be expected given the angular dependence of the noise.

The inverse relationship between the angular variance of the noise and the noise 
correlation width is shown in Fig. 5.10 for elevation independent noise originating 
from broadside. In this case, the correlation is real, allowing us to directly 
compare against the isotropic noise case. As can be seen, for broadly distributed 
noise the correlation is well approximated by the isotropic model, but as the noise 
distribution becomes narrower the correlation broadens. Additionally, sufficiently 
small values of $\sigma_{\theta_n}$ cause the argument of the modified Bessel function to become 
real and the correlation to lose its oscillatory characteristics, a feature consistent 
with the von Mises distribution converging to the Gaussian distribution for these 
values.

### 5.3.3 Coprime array gain

With the nature of surface and shipping noise in the shallow water ocean now understood, 
the performance of coprime arrays can be examined through array gain. The mean array gain is the most interesting, as seen in Eq. 5.4 this depends 
only on the second order statistics of the signal and noise. By examination of 
Eq. 5.23 and Eq. 5.25 for surface and shipping noise, respectively, it is seen that 
even when the assumption of isotropic noise is relaxed, the noise correlation is
Figure 5.10. Normalized shipping noise correlation dependence with angular deviation $\sigma_{\theta_n}$ for $\theta_n = 0$. Noise correlation along the array increases with decreasing angular variance.

still Wide Sense Stationary. Thus, the mean array gain for an augmented coprime array is the same as that of a fully populated linear array.
Summary and Conclusion

The coprime array is an interesting array topology that takes advantage of the properties of coprime numbers to ensure an array with a nearly complete coarray using fewer sensors than a traditionally dense uniform array [2]. In this dissertation we have explored the coprime array, and its many variations, in the context of sparse array signal processing and shallow water acoustics. We have used the properties of coprime numbers to describe the spatial repetition of lag measurements in a coprime array, a specific measure of redundancy that has not received much consideration in the coprime literature. Using these repetitions the Centered Coprime Array was developed, a new design in the coprime array family. These repetitions were also exploited to recommend Coprime Subarray Processing, a processing technique that exploits the regularity of coprime arrays to subdivide samples on a sparse array in a manner usually applied to uniform ones.

Centered Coprime Arrays are based on the Extended Coprime Array design, developed by Adhikari et al. [18] where the two component arrays that form the coprime array are both extended by a constant factor. However, by adjusting the center point of the array such that it lies a distance of $MN/4$ away from the nearest shared element, the Centered Coprime Array can guarantee features about its coarray that are not guaranteed given the normal construction that begins with a shared first element. Specifically, a lower bound for the fully augmentable range of the array can be found that scales linearly with the array aperture. This bound is shown to be tight for Centered Coprime Array, and not applicable to other Extended Coprime Array designs. Depending on the constraints of the array design, this bound can be used to select an array length or set of coprime factors to meet other design requirements.

The Coprime Subarray Processing technique was also developed. This technique applies a common processing method for densely spaced uniform arrays to the more sparse coprime array. The use of subarrays for coprime arrays is justified through the pattern of repetitions in an Extended Coprime Array, and this
pattern is used to recommend a selection of subarrays that are themselves evenly spaced and possessing certain useful symmetries. Moreover, when applied to the Centered Coprime Array, the Subarray Processing technique benefits from full use of the array aperture, subarrays that are themselves Centered Coprime Arrays, and a reference subarray conveniently located at the center of the full array.

By considering the impact of lag redundancies on array performance, we have also shown that the redundant lags present in the coprime array can increase the robustness of the design to environmental effects when compared to minimally redundant array designs while still enjoying some of the benefits of sparseness when compared to uniform arrays. In other words, the coprime array strikes a balance between these two extremes of array design, tunable through the selection of coprime factors $M$ and $N$. This robustness was shown to be particularly valuable in shallow water acoustic environments, where environmental variations in Transmission Loss (TL), signal phase, and noise intensity can challenge the assumptions of Wide Sense Stationarity (WSS) that sparse arrays are based on.

The specific effects of environmental variation on sparse array performance can be neatly summarized. A reduction of redundant lags in an array does not alter the average beam pattern for an array after coarray weighting has been applied, but does increase the variance of the beam pattern about that mean value. Thus, for metrics like array gain and resolution, which are generally understood as mean performance metrics, array sparseness does not correlate with performance in a mean sense. However, for metrics like peak sidelobe level, which are themselves measures of beam response variation, the degree of sparseness in an array shows inverse correlation with its performance in the presence of shallow water environmental effects.

## 6.1 Application of results

In deploying a horizontal coprime array in shallow water, the results of this paper can help guide both the design of the array and processing of data measured on the array. Lag redundancy has been shown to have performance benefits, particularly for peak sidelobe level, and those benefits must be balanced against the cost of adding additional elements. Luckily, coprime array redundancy can be directly controlled through the selection of coprime factors $M$ and $N$, unlike Minimally Redundant Arrays (MRAs) whose conceptual design dictates a specific level of redundancy. Increasing these coprime factors results in an array with fewer redundancies (and thus lower robustness to environmental variation) and a less efficient use of the array aperture, but with the benefit of fewer elements and channels of data to process.

The steps for designing a Centered Coprime Array are, in many ways, similar to those used for designing other arrays. First, the fundamental spacing is chosen based on the frequencies of interest. Then, the array length is determined using
the channel coherence or physical limits imposed on the array. Dividing the array length by the spacing gives the number of elements needed for a fully populated uniform array, which will then be the integer array length \( L \). From there, the options for coprime factors \( M \) and \( N \) can be determined, with any pair of coefficients such that that product \( MN \) is less than \( L \). To reduce the set of eligible coprime factors, one might consider only pairs such that \( M = N - 1 \) [18], a common convention that maximizes sparsity and has been followed throughout this dissertation. The larger the coprime factors chosen, the greater the sparsity, which has been the guiding principle for selection in the literature. However, the results of this work give reason to select smaller factors as well, depending on requirements for the fully augmentable range of the array, peak sidelobe level in a given environment, or array reliability.

The Coprime Subarray Processing method of segmenting array data should be considered in the same circumstances that subarray methods are normally employed on a uniform array. This includes close range localization, as demonstrated earlier, where the reduced aperture of the subarrays can better cope with significant wavefront curvature.

Possibly the most important application of these results is to see where coprime arrays are of benefit for practical sonar systems, and where it is more appropriate to use either a uniform or a more sparse design like the MRA. Broadly speaking, snapshot limited imaging in incoherent environments, where the peak sidelobe level determines the dynamic range of the image, will always see benefits from higher redundancy. In this situation, coprime arrays with small coprime factors are preferable to MRAs, but still inferior to uniform arrays of the same length. On the other extreme, detection of a steady signal in noise where coherence levels are high and snapshots are effectively unlimited are the least sensitive to the disadvantages of sparsity even when that noise may be non-isotropic and would benefit from MRAs. It is, therefore, the transition region, where tradeoffs must be made between cost and performance that coprime arrays find themselves of greatest utility, not just because they sit between the extremes of the uniform and Minimally Redundant arrays, but because the design is flexible enough to be applied throughout the entire tradeoff region.

### 6.2 Future work

While this paper has explored many aspects of the coprime array in shallow water, significant work can still be done to better understand this family of arrays.

First, this paper primarily focused on horizontal arrays, such as those towed behind a ship or moored in a static position in the water column, but vertical arrays are also deployed in shallow water for many applications. Planar and volumetric arrays are also used in shallow water, and extension of the results presented here for linear coprime arrays might be found for the higher dimension generalization.
of coprime arrays. Even other linear coprime array designs, such as those in Ch. 3, will have patterns of redundancy (and thus shallow water performance) that differs from the Centered Coprime Array design considered here.

Another limitation for the problem statement of this work was in the estimation goal of the processor. The problem of spatial intensity estimation was the primary focus of this work. While a measure of intensity can be useful, particularly for direction of arrival estimation or acoustic imaging, it does not represent the whole range of useful data that can be gathered from an array. The problems of waveform estimation and communication in the shallow water channel were not explored in this work, but both may find use for coprime arrays with appropriate processing techniques. Waveform estimation, in particular, must take into account the bandwidth of the source, which was not given much consideration in the narrowband paradigm of this work.

Likewise, while this work focused on the Blackman-Tukey spectral estimation technique, which was chosen for its clear relationship with the coarray, other forms of processing can be applied to a coprime array. Many of these techniques, such as MUSIC and compressive sensing, have been explored by other researchers, though not necessarily in the shallow water environment. Several of these methods are matrix based methods that include inversion of matrices, and thus are sensitive to the conditioning of the array covariance matrix. Explicit examination of how redundancy and environmental variation can impact the conditioning of a matrix was not considered here, but could help guide application of these methods to this family of arrays.

6.2.1 Effects of array shape

Array shape was briefly considered in Sec. 4.1.1 as a source or random phase error for a signal that might result in a loss of signal coherence. However, array shape is a much broader problem than this basic consideration might suggest. As many underwater arrays are constructed on flexible lines, the shape of the array can change over time. For towed arrays, this motion is often a response to the towing action of the ship, water flow around the array itself, and variations in buoyancy along the array. For moored arrays, oceans currents can generate catenary that varies slowly in time. For arrays that sit on the ocean bottom, variations in the ground level and the degree to which the array becomes buried in the sediment can impact array shape.

Such position errors can have a strong impact on the performance of an array, but have not been treated explicitly in this work. Previous work has shown that circular curvature can result in increased sidelobe levels for coprime arrays under multiplicative beamforming schemes [58], but such results have not been generalized to covariance-based methods, nor have other types of array shape perturbation. That said, the features that allow for the creation of coprime subarrays
to determine wavefront curvature could also be used to determine and manage array shape curvature. Future work might employ these features of the coprime array, in addition to models estimating the array shape curvature itself, to improve the performance of beamforming on flexible linear arrays.
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