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GENERALIZED S.V.D. REDUCED ORDER OBSERVERS FOR NOISY LINEAR AND NONLINEAR SYSTEMS

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Abstract

The use of observers for improvement of output feedback process control is important for achieving increased accuracy and processing efficiency, especially for large multivariate process systems. Reduced-order observers are particularly advantageous for reducing the computational complexity of estimating state variables.

For chemical processes that can be modeled as a linear time-invariant continuous system with stochastic elements of white Gaussian noise accounting for both process disturbances and sensor inaccuracies. Noise contributions to multiple sensors are characterized and scaled before being filtered by a soft sensor. The soft sensor design here is based on a best linear unbiased estimation (BLUE) of output measurements using generalized singular value decomposition (GSVD) of the coefficient matrices of measured variables and noise respectively. The resulting state estimates are obtained through a reduced-order Kalman-Bucy observer superstructure. Using chemical process examples of a biochemical continuous stirred tank reactor (CSTR) and the simplified Tennessee Eastman model, this design is shown to outperform the ordinary Kalman filter design with noisy measurements that deviate from a Gauss-Markov model.

A nonlinear observer design method is also proposed for the reduced order observation of nonlinear systems in the presence of sensor and process noise. Supernumerary sensors to the measured states are assumed to be available, and state variables unavailable for observation by measurement are estimated with the proposed observer structure that requires lower computation than full order observers. By modeling output measurements as a generalized linear combination of observable states and measurement noise, this method combines generalized singular value decomposition (GSVD) static estimation of noisy output measurement and reduced order observer theory for estimating unmeasured state variables in nonlinear systems. Using chemical process examples of a biochemical continuous stirred tank reactor (CSTR) and an iso-thermal reactor, this design is shown as a viable low computation alternative to full-order observation, with potential economic advantage in model predictive control (MPC) applications.

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Chapter 1
INTRODUCTION

Observers are computational models that can be employed to estimate and report the value of state variables of a process system. The state estimation results of observer models establish an information bridge between decision making and true process system dynamics with the system output measurements obtained via sensors. Observers modify output measurements with information known about the process system dynamics to obtain estimations of the true state of a process system or plant. The resulting state estimates from observer models are therefore useful for future decision making, such as driving feedback control action for desired chemical process results.

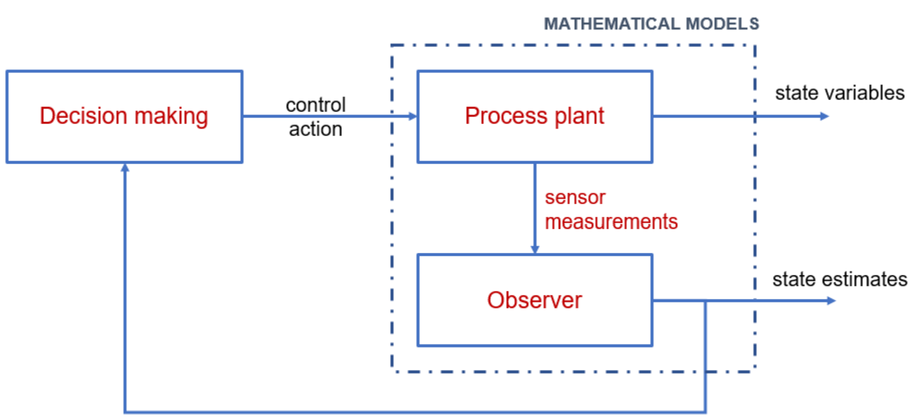


Figure 1.1. Process System Model Computation Structure

Generally, observer state variable detection is more reliable than sensor measurements alone by design. The observer design is based on the the development of approximate true process system dynamics by minimization of estimation error. The estimates obtained must approach the true states of the process system as much as possible.

While the employment of observers generally improve accuracy and efficiency in plant operations, they can also amplify errors from modeling and sensor measurement inaccuracies. These errors can result in model instability, which can translate to instability in the feedback system or increased costs of robust control. Model inaccuracies are difficult to eliminate completely, but they can be minimized by representing the systems as close to real process behavior as possible.

In reality, process systems are usually nonlinear in nature but sometimes, they can be approximated closely with linearized models because nonlinear systems come with new layers of complexity in comparison to linear systems [1]. For even more accurate estimation of process state variables, nonlinear observers can be developed to handle the peculiarities of nonlinear phenomena such as bifurcation, oscillation and chaos, etc. without linearization.

1.1 Model accuracy

For industrial and most practical applications of observer models, it is important to have an accurate process dynamics representation in the plant model for accurate state estimations. Since most practical process systems are in fact nonlinear in nature, the development of observers from nonlinear plant models is important for reducing modeling inaccuracies.

The nonlinear model representation of system dynamics can be presented with the set of differential equations of the form

$$\begin{aligned} \frac{dx}{dt} &= \mathcal{F}(x, u, t) \\ y &= \mathcal{H}(x, u, t) \end{aligned} \tag{1.1}$$

for the system state variables, x which we desire to estimate with a suitable observer, time, t, the system disturbance, u which can be used to control the evolution of the process plant, and the output measurements of the process, y.

A linear approximation of (1.1) is easily derived as

$$\begin{aligned} \dot{x} &= Ax(t) + Bu(t) \\ y &= Cx(t) + Du(t) \end{aligned} \tag{1.2}$$

where the constant matrices A, B, C and D are obtained by linearization of (1.1) around a chosen operation point of reference variables x_0 and u_0.

$$A = \left.\frac{\partial \mathcal{F}(x,u,t)}{\partial x}\right|_{x=x_0} \qquad B = \left.\frac{\partial \mathcal{F}(x,u,t)}{\partial u}\right|_{u=u_0}$$

$$C = \left.\frac{\partial \mathcal{H}(x,u,t)}{\partial x}\right|_{x=x_0} \qquad D = \left.\frac{\partial \mathcal{H}(x,u,t)}{\partial u}\right|_{u=u_0}$$

This linearization of system dynamics can be a source of modeling inaccuracy transferred to an observer model especially around operating conditions where the system exhibits non-linear behavior.

1.2 Full-order observers

Full order observers are a category of observer models designed to match the system size, in order to estimate all independent state variables of the process plant dynamically from available output measurements. These deterministic dynamic observer models were first introduced for estimation of state variables of linear time invariant (LTI) systems by D. Luenberger [2, 3].

1.2.1 Luenberger observer

For a linear time-invariant system, i.e. with system state variables independent of time given by

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{1.3}$$

where $x \in \mathbb{R}^n$ is the state variable vector, $u \in \mathbb{R}^l$ is the vector of input variables, $y \in \mathbb{R}^m$ is the output measurement vector, with constant matrices A, B, C, of appropriate dimensions and $D = 0$.

Luenberger defined the estimate of the full state variable as $\hat{x} \in \mathbb{R}^n$, so that the resulting error of estimation or divergence of state estimates, $\hat{x}$ from the true states x can be evaluated by :

$$\varepsilon = x - \hat{x}$$

The observer model mimics the dynamics of (1.3) with system output measurement as the model input,

$$\dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}u + Ly$$

The resulting error dynamics

$$\begin{aligned}
\dot{\varepsilon} &= \dot{x} - \dot{\hat{x}} \\
&= Ax + Bu - (\hat{A}\hat{x} + \hat{B}u + Ly) \\
&= Ax + Bu - \hat{A}(x - \varepsilon) - \hat{B}u - LCx \\
&= \hat{A}\varepsilon + (A - LC - \hat{A})x + (B - \hat{B})u
\end{aligned}$$

The estimation of the process state variables $\hat{x}$, with a zero dynamic estimation error results in a solution with

$$\begin{aligned}
\hat{B} &= B \\
\hat{A} &= A - LC
\end{aligned}$$

Therefore,

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}) \tag{1.4}$$

where the gain matrix, L, is chosen to obtain exponentially decaying error dynamics by making the matrix $\hat{A} = A - LC$, *Hurwitz*.

1.2.1.1 Nonlinear Luenberger Observer

The Luenberger design is applied to nonlinear systems, by comparing the model of (1.1) to (1.3) to give the formulation:

$$\begin{aligned} \dot{x}(t) &= \mathcal{F}(x,u,t) = f(x(t)) + g(x(t),u(t)) \\ y(t) &= \mathcal{H}(x,u,t) = h(x(t)) \end{aligned} \tag{1.5}$$

where f, g and h are separable nonlinear expressions of x. The resulting observer model is

$$\dot{\hat{x}} = f(\hat{x}(t)) + g(\hat{x}(t),u(t)) + L(\hat{x})(y - h(\hat{x}(t)) \tag{1.6}$$

with the observation gain matrix, $L(\hat{x})$, derived similar to (1.4) by extended linearization of f and h.

$$A(\hat{x}) = \left.\frac{\partial f(x(t))}{\partial x}\right|_{x=\hat{x}} \quad \text{and} \quad C(\hat{x}) = \left.\frac{\partial h(x(t))}{\partial x}\right|_{x=\hat{x}}$$

1.2.2 Kalman filter

The design of the Luenberger observer was based on the development of a gain matrix to converge estimation error to zero asymptotically. However, the optimum form of the full order observer model is considered to be the Kalman filter illustrated years earlier by R. E. Kalman [4], where he considered the contributions of process and measurement uncertainties, and adopted statistical methods to design a gain matrix for minimizing error covariance matrices [5].

This problem was addressed by R. E. Kalman and R. S. Bucy in 1961 by modeling the linear time-invariant continuous system with stochastic elements of white Gaussian noise accounting for both process disturbances, and sensor inaccuracies respectively [5] with

$$\begin{aligned} \dot{x} &= Ax + Bu + Fv \\ y &= Cx + w \end{aligned} \tag{1.7}$$

where $x \in \mathbb{R}^n$ is the state variable vector, $u \in \mathbb{R}^l$ is the vector of input variables, $y \in \mathbb{R}^m$ is the output measurement vector, v and w are white Gaussian noise, with constant matrices A, B, C, and F of appropriate dimensions.

The Kalman filter performs the full order dynamic estimation of state variables by

$$\dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x})$$

with error dynamics

$$\begin{aligned}\dot{\varepsilon} = \dot{x} - \dot{\hat{x}} &= Ax + Fv - A\hat{x} - K(y - C\hat{x}) \\ &= (A - KC)\varepsilon + Fv - Kw\end{aligned}$$

having the cumulative noise contribution: $\xi = Fv - Kw$

$$\dot{\varepsilon} = (A - KC)\varepsilon + \xi$$

The optimum observation gain matrix, $\hat{K}$ was obtained by Kalman and Bucy with the minimization of noise variance in error dynamics. For uncorrelated v and w, this solution is

$$\hat{K} = \hat{P}C^T W^{-1}$$

from the optimized covariance error dynamics given by the Ricatti equation

$$\dot{\hat{P}} = A\hat{P} + \hat{P}A^T - \hat{P}C^T W^{-1} C^T \hat{P} + FVF^T \tag{1.8}$$

where the variance matrices of white noise contributions have the expected values:

$$E\{v(t)v(\tau)^T\} = V(t)\Delta(t - \tau)$$

$$E\{w(t)w(\tau)^T\} = W(t)\Delta(t - \tau)$$

Therefore, the resulting optimal observer is

$$\dot{\hat{x}} = A\hat{x} + Bu + \hat{K}(y - C\hat{x}) \tag{1.9}$$

1.2.2.1 Extended Kalman Filter

Similar to the Luenberger formulation of (1.6), the design of Kalman can be extended to nonlinear systems by application of the extended separation principle and extended linearization.

$$\begin{aligned}\dot{x}(t) &= \mathcal{F}(x,u,t) = f(x(t)) + g(x(t),u(t)) + v(t)\\ y(t) &= \mathcal{H}(x,u,t) = h(x(t)) + w(t)\end{aligned} \tag{1.10}$$

where f, g and h are separable nonlinear expressions of x. The resulting observer model is

$$\dot{\hat{x}} = f(\hat{x}(t)) + g(\hat{x}(t), u(t)) + \hat{K}(\hat{x})(y - h(\hat{x}(t)) \tag{1.11}$$

with the observation gain matrix, $\hat{K}(\hat{x})$ derived similar to (1.9) by extended linearization of f and h.

$$A(\hat{x}) = \left.\frac{\partial f(x(t))}{\partial x}\right|_{x=\hat{x}} \quad \text{and} \quad C(\hat{x}) = \left.\frac{\partial h(x(t))}{\partial x}\right|_{x=\hat{x}}$$

The extended Kalman filter (EKF) provides only a near-optimal observation of state variables because of its reliance on linear approximation of system dynamics, $f(x(t))$ by extended linearization.

1.2.3 Other Full-order Observers

Other successful results for full order observation include the derivation of geometric observers, Lyapunov-based observers [6–11] for extended linearizable nonlinear systems. The Moving horizon estimators (MHE) in particular derive an optimized nonlinear estimation of the state variables via the direct solution of the optimal observation problem for a finite time window; the noise distribution function shape is implicitly assumed to be known. However, the advantage of the MHE observer simplicity comes at the cost of increased computations. [12, 13]

1.3 Importance of observer models

Following the introduction of the full order observers of Luenberger [2] and Kalman [4], observer models have been well established for numerous applications. These applications span a wide range of modeled systems especially important to industrial process automation.

1. **Navigation** : Flight control for airplanes, missiles, and even space transportation systems are examples of processes that rely on the refining of information gathered by sensors for accurate guidance of system dynamics. Observers provide the refining computation of distance and velocity measurements for improving trajectory dynamics estimation.

2. **Fault detection** : Observer models' action on sensor measurements can be useful for the derivation of summary statistics of process results to show deviations from expectations or desired results that may have otherwise been indiscernible from sensor measurements alone.

3. **Forecasting** : In meteorology and financial markets, prediction models are built on extremely nonlinear dynamics. Observer models are applied to historical data for refining the prediction models.

4. **Feedback controller design** : In industrial manufacturing, observer models are especially relevant for improvement of process control, and achieving increased accuracy and processing efficiency. Accurate observer computations result in economic automatic control especially for complex multivariate process systems like those found in manufacturing plants with multiple products, and bio-molecular synthesis with multiple pathways and by-products.

1.4 Objective

Full order observer models can grow in complexity with the introduction of uncertainties in system and measurement models as well as with increased system size, becoming expensive to design and compute. Economic alternatives for full order state estimations have been found in the reducing of observation model size with the design of reduced order observers and functional observers [14–18] for linear systems.

The successful deployment of linear observers in the presence of noise and uncertainty, has resulted in research focused on the development of observers for nonlinear systems that go beyond linearizing the process behavior at an operating point and introducing locally stable linear observers. Because nonlinear systems generally come with new layers of complexity in comparison to linear systems, nonlinear observer models need to tackle the peculiar challenges of stability, bifurcation, chaos etc. [1].

The goal of this work is to investigate the use of a reduced order dynamic observer model modified with a static observer (soft sensor) design based on the generalized singular value decomposition (GSVD) of noisy output measurements. Drawing from the statistical background of the optimum observer of Kalman, the analogy of principal component analysis (PCA) and singular value decomposition (SVD) is used to filter noisy output measurements optimally, and combined with reduced system dynamics to estimate process state variables. The target observer design filters out both process and sensor noises while maintaining a relatively low computation demand.

The thesis is structured as follows; section 2 introduces the definitions of reduced order observer theory for linear and nonlinear systems, section 3 contains literature review of static estimation of noisy output measurements, and section 4 introduces the definition of GSVD and its application to static estimation. The proposed observer design procedure is formulated in section 5 for noisy output measurements, and applied to numerical examples for linear and nonlinear cases in section 6.

Chapter 2
REDUCED ORDER OBSERVERS

Since observer models grow in complexity with increasing system size, full state observation can be expensive to design and compute. The alternative approach for estimating state variables involves improving state estimations from output measurements. While more accurate state measurements may be achieved by employing multiple sensors to measure the system output, this can also raise operation costs significantly and complicate operations.

Economic alternatives for state estimations rely on the reduction the observation model size with the design of reduced order observers, which make it possible to estimate all desired state variables. With reduced order observers, system monitoring requires a limited number of sensors that are combined with dynamic estimations of only the unmeasured variables. This reduces the required observation system model order and numerical computations, making reduced order observers a considerably more economic option.

This same principles have been applied for the design of reduced order observers for nonlinear systems by extended linearization of system dynamics [19–21]. The use of observer models for improvement of process control is very useful for achieving accuracy and processing efficiency [22, 23], especially for complex multivariate process systems like those found in manufacturing plants with multiple products, and biomolecular synthesis with multiple pathways and by-products.

2.1 Linear Reduced Order Observer

2.1.1 Functional Observers

Functional observer models were developed for the estimation of deterministic LTI systems by the reduction of the system order by transformation to an equivalent system. [14–18] The LTI system is described by

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \\ z &= Lx \end{aligned} \tag{2.1}$$

where the state variables of x that can be directly measured by $y \in \mathbb{R}^k$, while the rest of the state variables that are not directly measurable are to be estimated with $z \in \mathbb{R}^{n-k}$.

The unmeasured states, z are estimated with an observer designed with a similar dynamics to (2.1), with

$$\begin{aligned} \dot{s} &= Ns + Jy + Hu \\ \hat{z} &= s \end{aligned} \tag{2.2}$$

and the estimation error: $\varepsilon = z - \hat{z}$. Therefore, the error dynamics is

$$\begin{aligned} \dot{\varepsilon} &= L\dot{x} - \dot{s} \\ &= L(Ax + Bu) - Ns - Jy - Hu \\ &= LAx + LBu - NLx + N\varepsilon - JCx - Hu \\ &= N\varepsilon + (LA - NL - JC)x + (LB - H)u \end{aligned}$$

The estimation of the process state variables $\hat{z}$, with a zero dynamic estimation error results in a solution with

$$\begin{aligned} LB &= H \\ LA - NL - JC &= 0 \end{aligned}$$

Therefore,

$$\dot{\hat{z}} = N\hat{z} + Hu + Jy \tag{2.3}$$

where the gain matrix, J is chosen to obtain exponentially decaying error dynamics by making the matrix$N = LAL^{\dagger} - JCL^{\dagger}$, to be *Hurwitz*. The resulting functional observer model requires accurate output measurements with y, as the reduced states of z alone are evaluated with the observer.

2.1.2 Reduced Order Kalman-Bucy

The reduced order Kalman-Bucy is a dynamic observer modeled after the optimum observer - *the Kalman filter* - for estimation of unmeasured states from output measurement by minimizing the error covariance matrix of the measured system. The reduced order Kalman-Bucy observer is designed for a dynamic process with noise-free partial measurements modeled by

$$\begin{aligned} \dot{x} &= Ax + Bu + Fv \\ y &= Cx \\ z &= \dot{y} - CBu \end{aligned} \tag{2.4}$$

where the state variables of x that can be directly estimated from y are grouped in vector x_k, while the rest of the state variables that are not directly measurable are grouped in vector x_{n-k}. Without loss of generality it is assumed $x_k \in \mathbb{R}^k$ and $x_{n-k} \in \mathbb{R}^{n-k}$.

The unmeasured states, x_{n-k} are estimated with an optimized observer gain, $\hat{K}$ obtained from the solution of the resulting algebraic Ricatti equation of the optimum error covariance matrix of (2.4) given by

$$\dot{\hat{P}} = \bar{A}\hat{P} + \hat{P}\bar{A}^T - \hat{P}A^TC^T(CFVF^TC^T)^{-1}CA\hat{P} + F\bar{V}F^T$$

where $\hat{P}$ denotes the optimum error covariance matrix, and V represents the process noise covariance matrix. [5]

$$C\dot{\hat{P}} = C\bar{A}\hat{P} + C\hat{P}\bar{A}^T - C\hat{P}A^TC^T(CFVF^TC^T)^{-1}CA\hat{P} + CF\bar{V}F^T$$

$$\bar{A} = A - FVF^TC^T(CFVF^TC^T)^{-1}CA$$

$$\bar{V} = V - VF^TC^T(CFVF^TC^T)^{-1}CFV$$

$$C\bar{V} = CV - CVF^TC^T(CFVF^TC^T)^{-1}CFV = \emptyset$$

$$C\bar{A} = CA - CFVF^TC^T(CFVF^TC^T)^{-1}CA = \emptyset$$

$$C\dot{\hat{P}} = C\hat{P}\bar{A}^T - C\hat{P}A^TC^T(CFVF^TC^T)^{-1}CA\hat{P}$$

The observer gain $\hat{K}$ optimality is not guaranteed but the steady-state solution of $C\dot{\hat{P}} = 0$, gives the solution:

$$C\hat{K}= C(\hat{P}A^T + FVF^T)C^T(CFVF^TC^T)^{-1} = I \tag{2.5}$$

The reduced order observer design for the system in (2.4) is given by

$$\begin{aligned}\dot{\hat{x}} &= A\hat{x} + Bu + \hat{K}(z - CA\hat{x})\\ &= (I - \hat{K}C)A\hat{x} + Bu + \hat{K}z\\ &= (I - \hat{K}C)(A\hat{x} + Bu) + \hat{K}\dot{y}\end{aligned}$$

From the estimation error term, $\varepsilon = \hat{x} - \hat{K}y$, the error dynamics are obtained as

$$\dot{\varepsilon} = \dot{\hat{x}} - \hat{K}\dot{y} = (I - \hat{K}C)(A\hat{x} + Bu)$$

By identifying a similarity transform, T, that establishes

$$I - \hat{K}C = T^{-1}\begin{bmatrix}\emptyset & \emptyset\\ \emptyset & I_{n-k}\end{bmatrix}T$$

where I_{n-k} is the identity matrix of dimension $n-k$,and taking $\gamma = T\varepsilon$, and $\varepsilon = T^{-1}\gamma$, the following dynamics are obtained

$$\dot{\varepsilon} = T^{-1}\begin{bmatrix}\emptyset & \emptyset\\ \emptyset & I_{n-k}\end{bmatrix}T[A(\varepsilon + \hat{K}y) + Bu]$$

$$\dot{\gamma} = T\dot{\varepsilon} = \begin{bmatrix} \emptyset & \emptyset \\ \emptyset & I_{n-k} \end{bmatrix} T[A(\varepsilon + \hat{K}y) + Bu]$$

Segmenting the dynamics according to $\gamma = \begin{bmatrix} \gamma_k \\ \gamma_{n-k} \end{bmatrix}$, $T = \begin{bmatrix} T_k & T_{n-k} \end{bmatrix}$, where $T_k \in \mathbb{C}^{n \times k}$, $T_{n-k} \in \mathbb{C}^{n \times n-k}$ and $\tilde{A} = TAT^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, we obtain

$$\dot{\gamma} = \begin{bmatrix} \dot{\gamma}_k \\ \dot{\gamma}_{n-k} \end{bmatrix} = \begin{bmatrix} \emptyset & \emptyset \\ \emptyset & I_{n-k} \end{bmatrix} \left\{ \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \gamma_k + y \\ \gamma_{n-k} \end{bmatrix} + TBu \right\}$$

giving

$$\dot{\gamma}_k = \emptyset$$

$$\dot{\gamma}_{n-k} = A_{21}(\gamma_k + y) + A_{22}\gamma_{n-k} + T_{n-k}Bu$$

The solution to $\dot{\gamma}_k = \emptyset$ is $\gamma_k = 0$, making

$$\dot{\gamma}_{n-k} = A_{21}y + A_{22}\gamma_{n-k} + T_{n-k}Bu$$

The resulting state estimate, $\hat{x} = \hat{K}y + \varepsilon = \hat{K}y + T^{-1}\gamma$, when using $T^{-1} = \begin{bmatrix} L_k & L_{n-k} \end{bmatrix}$, with $L_k \in \mathbb{C}^{n \times k}$, $L_{n-k} \in \mathbb{C}^{n \times n-k}$, and $\gamma_k = 0$, becomes

$$\hat{x} = \hat{K}y + L_{n-k}\gamma_{n-k} \tag{2.6}$$

The Kalman-Bucy reduced order observer developed with the principles of the Kalman filter also assumes a noise-free measurement in y. This design is therefore limited in its capability for handling noisy output measurements.

2.2 Nonlinear Reduced Order Theory

Even with all the noted developments in estimation over the years, there is still difficulty in estimation of nonlinear dynamics with noisy output measurement by reduced order observers without the limitation of noise-free measurement assumptions.

For nonlinear systems with plant models of the form in (1.1) with process disturbance and measurement disturbance that can be represented with

$$\begin{aligned} \dot{x} &= \mathcal{F}(x, u, v) \\ y &= \mathcal{H}(x, w) \end{aligned} \tag{2.7}$$

where $x \in \mathbb{R}^n$ is the state variable vector, $u \in \mathbb{R}^l$ is the vector of input variables, $y \in \mathbb{R}^m$ is the output measurement vector, v and w are random noise representations of system and measurement disturbances respectively.

A nonlinear reduced order observer can be designed for estimation of the unmeasured state variables of the process, if the following assumptions hold.

The system under consideration (2.7) satisfies the following conditions:

1. The system is observable.

2. The system is process disturbance affine.

3. Functions $\mathcal{F}$ & $\mathcal{H}$ belong in $\mathcal{C}_1$.

4. The system dynamics can be similarly partitioned into measured and unmeasured states, such that the state variables of $x \in \mathbb{R}^n$ that can be obtained from output measurements are grouped as $x_1 \in \mathbb{R}^{n-p}$, and unmeasured variables are grouped as $x_2 \in \mathbb{R}^p$.

$$\begin{aligned} \dot{x}_1 &= \mathcal{F}_1(x, u, v) \\ \dot{x}_2 &= \mathcal{F}_2(x, u, v) \\ y &= \mathcal{H}(x, w) \end{aligned} \tag{2.8}$$

For a system described by (2.8), estimation of unmeasured states, $\hat{x}_2$ is extracted from original system dynamics and output measurements [24], by the expression

$$\hat{x}_2 = \Phi(\hat{x}_1) + z \tag{2.9}$$

where $\Phi(\hat{x}_1)$ is an arbitrary function of the measured states and z is an arbitrary parameter of unknown dynamics. The resulting estimation error defined as $\varepsilon = x_2 - \hat{x}_2$, will have the dynamics

$$\dot{\varepsilon} = \dot{x}_2 - \dot{\hat{x}}_2$$

$$
\begin{aligned}
\dot{\varepsilon} &= \mathcal{F}_2(x,u,v) - \dot{\hat{x}}_2 \\
&= \mathcal{F}_2(x,u,v) - \dot{\Phi}(\hat{x}_1) - \dot{z}
\end{aligned}
$$

With an exponentially decaying error dynamics, as $\dot{\varepsilon} \to 0$, we have

$$
\dot{z} = \Psi(x,u) = \mathcal{F}_2(x,u,v) - \dot{\Phi}(\hat{x}_1)
$$

where

$$
\dot{\Phi}(\hat{x}_1) = \frac{\partial \Phi(\hat{x}_1)}{\partial t} = \frac{\partial \Phi(\hat{x}_1)}{\partial x_1}\frac{\partial x_1}{\partial t}
$$

$$
\dot{z} = \mathcal{F}_2(x,u,v) - \frac{\partial \Phi(\hat{x}_1)}{\partial x_1}\mathcal{F}_1(x,u,v) \tag{2.10}
$$

After establishing the dynamics of z with (2.10), the unmeasured states are obtained from (2.9) with the transformation $\Phi(\hat{x}_1)$. Therefore the resulting full state estimates can be obtained for $\hat{x}_1$ from noisy output measurements by static observation, and $\hat{x}_2$ from the reduced order observer.

Chapter 3
OUTPUT MEASUREMENT STATE ESTIMATION

Advancements of the reduced order observer model design have since evolved into the development of functional transformations for minimizing observer order, such as functional observer of section 2. These advancements, while further reducing observation costs and model complexities, do not address the limitations of the reduced order Kalman-Bucy observer with noisy measurements.

A static observer model is desired for the optimization of sensor measurements over noise contributions for state variable estimation. This can be achieved by the reduction of noisy process measurements to the principal component of the state vector for minimal observation error with a target linear transformation of output measurements:

$$\hat{x} = Gy \tag{3.1}$$

Direct state variable estimates, $\hat{x}$ from output measurements in (3.1) that can be used to guide process observation is obtained using linear algebra and statistical analysis tools for optimal estimation and robustness.

3.1 Noise-free output measurement

First, we consider a simplified version of output measurement that is noise-free. The resulting least squares estimation problem of linear measurements is given by:

$$y = Cx$$

with measurement error, $\varepsilon = Cx - y$. $C \in \mathbb{R}^{m \times n}$, since $y \in \mathbb{R}^m$ is a linear function of state variables, $x \in \mathbb{R}^n$.

3.1.1 Least squares estimation

Its estimate, $\hat{x} = \arg \min_{\varepsilon^2}\{y = f(x)\}$, is obtained by the operation of $\min \|Cx - y\|^2$ that gives the elegant solution:

$$C^T C x = C^T y$$

$$\hat{x} = (C^T C)^{-1} C^T y = Gy \tag{3.2}$$

For a case with non-singular $C \in \mathbb{R}^{m \times m}$, this solution simplifies to $\hat{x} = C^{-1}y$, with G equivalent to the ordinary inverse of C. However, for the more general case of $C \in \mathbb{R}^{m \times n}$, $m \neq n$ which is a lot more typical with real processes, this ordinary inverse doesn't exist.

Our goal is to obtain G that closely approximates the state variables from sensor measurements. For example if $m \geq n$ like in processes with overdetermined sensor measurements such as large scale plants which may require repeated output measurements for improved accuracy of the measurements used for driving operation and control.

3.1.2 Moore-Penrose pseudoinverse

E. H. Moore (1920) and R. Penrose (1955) [25] both discovered a unique inverse for any matrix $C \in \mathbb{R}^{m \times n}$ with $m \neq n$ known popularly as *Moore-Penrose pseudoinverse*. The pseudoinverse of C, $C^\dagger$ satisfies the following equation:

$$C = CC^\dagger C \tag{3.3}$$

where $C^\dagger \in \mathbb{R}^{n \times m}$. $C^\dagger$ can be used in the absence of ordinary inverse C^{-1}, so that state variable estimates are obtained by

$$\hat{x} = C^\dagger y$$

3.1.3 Singular value decomposition

The pseudoinverse solution can be generalized for $m \neq n$ cases with *Definition* 1. [26]

Definition 1. SVD *provides a generalized diagonalization for rectangular matrices such as $C \in \mathbb{R}^{m \times n}$ with $m \neq n$. This operation decomposes the matrix as:*

$$C = U\Sigma V^T$$

$U =$ left orthogonal matrix $\in \mathbb{R}^{m \times m}, V =$ right orthogonal matrix $\in \mathbb{R}^{n \times n}$, and Σ is a real positive definite diagonal matrix $\in \mathbb{R}^{m \times n}$.

From the defintion, if $m \geq n$, then $\Sigma \in \mathbb{R}^{m \times n}$ with diagonals $(\sigma_1, ..., \sigma_n)$ is such that: $(\sigma_1 \geq \sigma_2 \geq, ..., \geq \sigma_n > 0)$.

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_n \\ 0 & 0 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} \sigma \\ \emptyset \end{bmatrix}$$

and a pseudoinverse for every matrix $C \in \mathbb{R}^{m \times n}$ can be computed as

$$C^\dagger = (V\Sigma^T U^T U \Sigma V^T)^{-1} V \Sigma^T U^T = V(\Sigma^T \Sigma)^{-1} \Sigma^T U^T$$

$$C^\dagger = V \begin{bmatrix} \sigma^{-1} & \emptyset \end{bmatrix} U^T = V \Sigma^\dagger U^T \tag{3.4}$$

When $m < n$,

$$C^\dagger = V \begin{bmatrix} \sigma^{-1} \\ \emptyset \end{bmatrix} U^T = V \Sigma^\dagger U^T$$

Therefore, there exists a unique pseudoinverse $C^\dagger$, for every matrix $C \in \mathbb{R}^{m \times n}$ for all $m \neq n$, given by the inverse of the singular value decomposition of C.

Remark 1. *From the definition it is trivial to observe that the pseudoinverse is also robust to rank deficient matrices C. As such, the proposed work can also be employed for processes where measurements overlap (one of the many reasons for rank deficient C).*

3.2 Noisy output measurement

This noise-free measurement assumption is however far removed from real expectations, and in many cases may not be sufficient for accurate estimation of state variables.

For noisy output measurements, the Gauss-Markov model of (1.7), is insufficient to address the limitations of the reduced order Kalman-Bucy with singular or colored noise. The singular noise covariance matrix limitation of the reduced order Kalman-Bucy were considered by M. F. Hutton [27], with a recommendation of partial noise-free output measurements assumption. Similarly, measurement with colored noise can be either approximated as noise-free, or lumped with process dynamics to form meta-states [5, 28].

3.2.1 Generalized Linear Model

For the complete analysis of noisy output measurements, a generalized linear model is proposed for the accommodation of measurement disturbances of varying noise profiles with

$$y = Cx + Ew \tag{3.5}$$

where y is the output measurement vector, x is the state variable vector, and w is white Gaussian noise representations of output measurement disturbances. The introduction of matrix E allows the incorporation of noise correlation for dealing with the singular and colored noise limitations of Gauss-Markov noisy measurement model of R. E. Kalman. The covariance matrix of the noise contributions, $W = EE^T$.

3.2.2 Thikonov Regularized Least Squares Estimation

A method for computing the noise-free approximation of output measurement is the Thikohov regularization [29] method which solves the constrained least squares problem of $\min_x \|Cx - y\|^2$, subject to bounded noise contributions in $d = Ew$ with $\min_d (Lx - d)^2$.

$$\hat{x} = \arg\min_x \{\|Cx - y\|^2 + \|Lx\|^2\}$$

where $L = \mu I_n$ is a symmetric regularization matrix chosen arbitrarily to minimize noise contributions with μ^2 in the objective function.

$$\hat{x} = (C^T C + \mu^2 I)^{-1} C^T y \tag{3.6}$$

Note. Evaluating μ^2 in (3.6) is integral to obtaining the best estimate, $\hat{x}$, however this process requires the knowledge of the bound on noise contribution to error, d which is unknown. The best method for evaluating $\lambda = \mu^2$ is from the solution of the general cross-validation (GCV) equation:

$$\min_{\lambda} V(\lambda) = \frac{\|C\hat{x} - y\|^2}{[tr(I - C(\lambda))]^2} \tag{3.7}$$

where $C^{\dagger}(\lambda) = (C^T C + \mu^2 I)^{-1} C^T$ and approximating feasible noise regularization bounds using the largest and smallest singular values of C before interpolating for a minimum μ^2 satisfying (3.6). The solution to (3.7) is iterative and computationally expensive, especially with increasing system size.

3.2.3 Best Linear Unbiased Estimation

The best linear unbiased estimator (BLUE) of state variables is based on an effective minimization of noise contributions, $\underset{x,w}{min} \|w\|^2$ subject to $y = Cx + Ew$ and has the solution

$$\hat{x} = \min_{x} (Cx - y)^T W^{-1} (Cx - y) \tag{3.8}$$

where $W = EE^T$. The best linear unbiased estimation of noisy linear output measurements is advantageous over other regularization methods, because it is non-iterative and saves on computation. This solution however requires the existence of a non-singular W, limiting its applications [30]. The static estimates of observable state variables may be obtained by a generalized singular value decomposition (GSVD) of noisy output measurements, even with singular W.

Chapter 4
GSVD STATE ESTIMATION

The application of GSVD for estimating unmeasured state variables is of interest because of the challenges from unobserved state variables encountered in model predictive control (MPC). They can be useful for achieving efficient and economic feedback control design for nonlinear systems without the approximations of system dynamics by extended linearization like the EKF or iterative estimation like moving-horizon estimators.

4.1 Generalized Singular Value Decomposition

Definition 2. GSVD *For every matrix pair (A, B) with equal number of columns, there exists a generalized SVD from the factorized decomposition by orthogonalization and SVD of the composite matrix, with:*
1. Orthogonalization of (A, B):

$$\begin{bmatrix} A \\ B \end{bmatrix} = Q\,R = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} R$$

where Q is an orthogonal matrix, and R is a positive definite upper triangular matrix, and (Q_1, Q_2) are of equal size as (A, B) respectively.
2. SVD of the orthogonal matrix, Q:

$$Q_1 = U\Sigma_1 Z^T \qquad\qquad Q_2 = V\Sigma_2 Z^T$$

$$\Sigma_1 = \begin{bmatrix} c_1 & 0 & \cdots & 0 \\ 0 & c_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_n \\ 0 & 0 & \cdots & 0 \end{bmatrix} \quad \Sigma_2 = \begin{bmatrix} s_1 & 0 & \cdots & 0 \\ 0 & s_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & s_n \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

and the standard SVD of the composite matrix [26, 31, 32] gives singular values,

$$\Sigma = diag.\{c_1/s_1, c_2/s_2, \cdots, c_n/s_n\}$$

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} U & \emptyset \\ \emptyset & V \end{bmatrix} \begin{bmatrix} \Sigma_1 \\ \Sigma_2 \end{bmatrix} X^T$$

where $X = R^T Z$.

The GSVD of a matrix pair (A,B) as shown by Van Loan [31] is given by

$$A = U\Sigma_1 X^T$$

$$B = V\Sigma_2 X^T$$

The matrix X is chosen to impose a mapping constraint on the pair, such that

$$\Sigma_1^T \Sigma_1 + \Sigma_2^T \Sigma_2 = I$$

The matrices U, V are orthogonal while X is a square invertible matrix.

4.2 GSVD static observer

The estimation of state variables from noisy output measurements can be achieved strictly based on the generalized singular values of C^T and E^T given by Σ_1 and Σ_2 respectively, with the state and noise principal components factorized by the constraint $\Sigma_1^T \Sigma_1 + \Sigma_2^T \Sigma_2 = I$ such that noise contributions are weighted into Σ_1 through an invertible matrix X. GSVD gives a constrained rotation of the two principal components of state variables, x, and noise, w, that make up the output measurements, y into a single plane.

The resulting decomposition from y projections, is obtained as

$$\begin{aligned} C &= X\Sigma_1^T U^T \\ E &= X\Sigma_2^T V^T \end{aligned} \tag{4.1}$$

and output measurement becomes:

$$y = X\Sigma_1^T U^T x + X\Sigma_2^T V^T w$$

$$X^{-1}y = \Sigma_1^T U^T x + \Sigma_2^T V^T w$$

Multiplying by Σ_1 gives

$$\Sigma_1 X^{-1}y = \Sigma_1\Sigma_1^T U^T x + \Sigma_1\Sigma_2^T V^T w$$

$$U^T x = (\Sigma_1\Sigma_1^T)^{-1}\Sigma_1 X^{-1}y - (\Sigma_1\Sigma_1^T)^{-1}\Sigma_1\Sigma_2^T V^T w$$

This equation can be further reduced by multiplying with $V^{\#}$, a permutation of the orthogonal matrix V so that $V^{\#}\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} V^T = \emptyset$.

$$\hat{x} = U(\Sigma_1\Sigma_1^T)^{-1}\Sigma_1 X^{-1}y$$

$$\hat{x} = U\Sigma_1^{\dagger} X^{-1}y \tag{4.2}$$

Proposition 1. *All noisy output measurements described by the generalized linear model of* (3.5) *can be confidently approximated by* (3.1) *using GSVD linear transformation obtained in* (4.2)*.*

$$\hat{x} = Gy = U\Sigma_1^{\dagger} X^{-1}y.$$

Proof. Consider the general solution of the generalized linear model from (3.8) given as

$$C^T W^{-1} C x = C^T W^{-1} y$$

with noise covariance matrix, $W = EE^T$ [30]. Using *Definition* 2 :

$$U\Sigma_1 X^T (X\Sigma_2^T \Sigma_2 X^T)^{-1} X\Sigma_1^T U^T x = U\Sigma_1 X^T (X\Sigma_2^T \Sigma_2 X^T)^{-1} y$$

$$U\Sigma_1(\Sigma_2^T\Sigma_2)^{-1}\Sigma_1^TU^Tx = U\Sigma_1X^T(X\Sigma_2^T\Sigma_2X^T)^{-1}y$$

$$(\Sigma_2^T\Sigma_2)^{-1}\Sigma_1^TU^Tx = X^T(X\Sigma_2^T\Sigma_2X^T)^{-1}y$$

$$\Sigma_1^TU^Tx = (\Sigma_2^T\Sigma_2)X^T(X\Sigma_2^T\Sigma_2X^T)^{-1}y$$

$$\Sigma_1^TU^Tx = X^{-1}y$$

$$\hat{x} = U\Sigma_1^\dagger X^{-1}y$$

□

Remark 2. *Equation* (4.2) *therefore gives a noniterative solution of the constrained least squares problem of noisy output measurements. This is the "best linear unbiased estimate" of state variables, and the solution exists for all dimensions of noise spectral density, W.*

4.2.1 GSVD and the Pseudoinverse Relationship

The pseudoinverse and SVD inverse of C can be shown to be resulting limits of the GSVD inverse.

Consider a case of only white noise contributions to measurement, so that, $E = I$, i.e. it is an identity matrix. The state variable estimation for this scenario reduces to the standard singular value decomposition of C from:

$$\begin{bmatrix} C \\ I \end{bmatrix} = \begin{bmatrix} U & \emptyset \\ \emptyset & V \end{bmatrix} \begin{bmatrix} \Sigma_1 \\ \Sigma_2 \end{bmatrix} X^T$$

Therefore:

$$X^T = \Sigma_2^{-1}V^T$$

$$C = U(\Sigma_1\Sigma_2^{-1})V^T \equiv U\Sigma V^T$$

The pseudoinverse solution is in fact a limiting case of the GSVD approximation [26,31]. The constraint on the GSVD becomes trivial, because $W = I$, and the BLUE solution results in a pseudoinverse solution of (3.1):

$$\hat{x} = (C^TC)^{-1}C^Ty = C^\dagger y$$

4.2.2 GSVD static observer properties

This method is suitable for the desired linear transformation of y by G, because of its inherent linearity in the stretching of state measurement along its optimized singular values and rotating onto the common principal components of the state variables and noise, i.e. the columns of X^T.

Conditions:

(a) Linearity - the output measurement has to be a linear combination of state variables and disturbances for the GSVD observer to be feasible.

(b) Observability - the measurement coefficient matrix, C must satisfy $rank(C) = n$.

Static observer approach:

1. Identify C and E.
2. Perform GSVD to compute U, Σ_1 and X. Derive the static observer $G = U(\Sigma_1\Sigma_1^T)^{-1}\Sigma_1 X^{-1}$.
3. The GSVD static estimator is $\hat{x} = Gy$.

Remark 3. *The GSVD static estimator is robust and stable for all column dimensions of noise correlation E when having the same number of rows as C. It's advantage over BLUE for singular $W = EE^T$ is non-trivial, and particularly relevant for system output with combinations of noisy and noise-free measurements.*

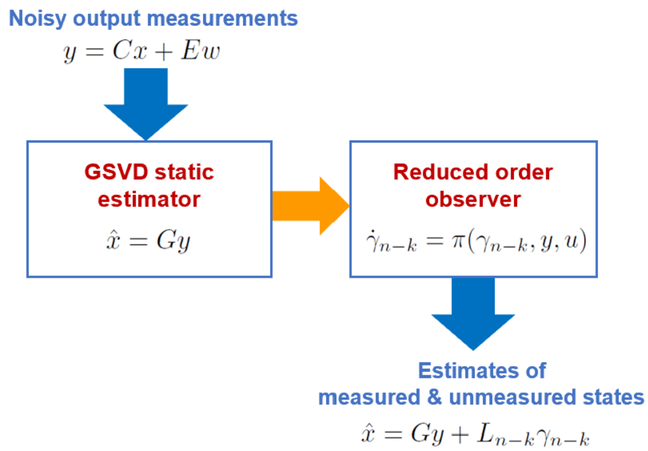


Figure 4.1. GSVD reduced observer design flow diagram

A significant limitation of static observers is condition 2, that is that $rank(C) = n$. In this section this condition is relaxed to the system being observable, *i.e.* the measurement coefficient matrix, C must satisfy $rank(A, C) = n$. The question becomes how to optimally identify the static component of the observer. The application of GSVD to the estimation of noisy output measurements is now extended to reduced order observation of unmeasured state variables similar to the reduced order Kalman-Bucy observer as depicted in Figure 4.1.

4.3 Modified Reduced Order Kalman-Bucy

Proposition 2. *Similar to the design procedure outlined in section 2, the reduced order Kalman-Bucy observer gain can be modified for the continuous time-invariant system with output measurement given by*

$$y = Cx + Ew$$

for all dimensions of spectral density matrix of noise, $W = EE^T$ with a GSVD static observer with $\hat{K} = G$ from (2.5)*, such that*

$$C\hat{K} = I$$

Proof. Consider the GSVD of the output measurements given by (4.1) and (4.2):

$$C = X\Sigma_1^T U^T$$
$$G = U\Sigma_1^\dagger X^{-1}$$

The solution for optimized observer gain is given by

$$CG = X\Sigma_1^T U^T U\Sigma_1^\dagger X^{-1} = I$$

□

Remark 4. *The resulting estimation presents an alternative to the perfect measurement assumptions of other reduced order observer models.*

For real output measurement obtained with $y = Cx + Ew$ as defined above, and row rank$(C) = k \neq m$ i.e. in case of repeated state variables in measurement, with $n > m \geq k$, an estimate of unmeasured states can be obtained from a GSVD reduced order observer designed in the following steps:

1. Identify matrices A, B, C and E.

2. Perform GSVD on (C, E) to find U, Σ_1 and X. Compute $G = U(\Sigma_1\Sigma_1^T)^{-1}\Sigma_1 X^{-1}$.

3. Obtain an invertible transformation matrix $T \in \mathbb{R}^{n\times n}$ such that:

$$I - GC = T^{-1}\begin{bmatrix} \emptyset & \emptyset \\ \emptyset & I_{n-k} \end{bmatrix} T$$

4. Compute matrices A_{21}, A_{22}, T_{n-k}, L_{n-k} as shown in the previous section.

5. Derive the reduced-order dynamic observer to estimate the state x

$$\begin{aligned} \dot{\gamma}_{n-k} &= A_{21}y + A_{22}\gamma_{n-k} + T_{n-k}Bu \\ \hat{x} &= Gy + L_{n-k}\gamma_{n-k} \end{aligned} \tag{4.3}$$

with initial condition $\gamma_{n-k}(0) = L_{n-k}^\dagger(x(0) - Gy(0))$

The GSVD approach can be applied for the evaluation of approximate noise-free measurement that can be processed in reduced order observer models for continuous process variable estimations.

4.4 GSVD Reduced Order Nonlinear Observer

For nonlinear state estimation, the application of GSVD will be extended to systems with uncertainty using functional dynamics of detectable states for estimating unmeasured state variables for plant models of the form:

$$\begin{aligned} \dot{x} &= f(x) + g(x)u + G(x)v(t) \\ y &= h(x) + E(x)w(t) \end{aligned} \tag{4.4}$$

where $x \in \mathbb{R}^n$ is the state variable vector, $u \in \mathbb{R}^l$ is the vector of input variables, $y \in \mathbb{R}^m$ is the output measurement vector, $v(t)$ and $w(t)$ are white Gaussian noise representations of system and measurement disturbances respectively. The functions $f(x)$, $g(x), h(x), G(x)$ and $E(x)$ are assumed to be sufficiently smooth (i.e., they belong in $\mathcal{C}_1$). The system output measurements y is described by a disturbance affine nonlinear model

$$\mathcal{H}(x, w) = h(x) + E(x)w(t)$$

and the relative degree between output y and states x when $w = 0$, defined as $r_i, \forall i = 1, \ldots, n$ is finite, with a number of r_i being zero. This implies that a subset of the states can be algebraically identified by the measurements.

Assuming that the system is evolving in the neighborhood of a known point x_0, the generalized linear model of output measurements in (2.8) with x_1 measurable states is of the form

$$y = Cx_1 + Ew$$

with

$$C = \left. \frac{\partial h(x_1)}{\partial x_1} \right|_{x=x_0}$$

and

$$E = E(x)|_{x=x_0}$$

The estimation of x_1 is achieved with the generalized singular values of C^T and E^T given by Σ_1 and Σ_2 respectively, with the constraint: $\Sigma_1^T\Sigma_1 + \Sigma_2^T\Sigma_2 = I$. The resulting decomposition from y projections, is obtained with

$$\hat{x}_1 = U(\Sigma_1\Sigma_1^T)^{-1}\Sigma_1 X^{-1} y$$

$$\hat{x}_1 = U\Sigma_1^\dagger X^{-1} y. \tag{4.5}$$

Therefore we can obtain state estimates of x_1 directly from output measurement by (4.5) with the GSVD gain matrix,

$$\hat{K} = U\Sigma_1^\dagger X^{-1}.$$

The gain matrix is calculated offline at the initial state of the system x_0, however it can be updated as the system evolves if necessary, for example when the system evolves away from the neighborhood of x_0. This is achieved by updating $C(\hat{x}_1) = \left.\frac{\partial h(x_1)}{\partial x_1}\right|_{x=\hat{x}}$ and $E(\hat{x})$ based on the current system estimate and performing the GSVD calculations on-line. A secondary advantage of the static observation part is that GSVD is an algebraic non-iterative procedure, implying reduced computational overhead.

Consider the nonlinear system described by (2.8). From reduced order observer theory, the unmeasured state variables can be estimated with a linear transformation of observed states in (2.9) and (2.10).

$$\Phi(\hat{x}_1) = T\hat{x}_1$$

Therefore

$$\hat{x}_2 = T\hat{K}y + z \tag{4.6}$$

The resulting reduced order dynamics is obtained as

$$\dot{z} = \Psi(\hat{x}_1, z, u)$$

$$\dot{z} = \mathcal{F}_2(\hat{x}_1, z, u) - T\hat{K}\frac{\partial h(\hat{x}_1)}{\partial x_1}\mathcal{F}_1(\hat{x}_1, z, u)$$

$$\dot{z} = \mathcal{F}_2(\hat{x}_1, z, u) - T\hat{K}\Pi(\hat{x}_1, z, u) \tag{4.7}$$

with $z = \hat{x}_2 - T\hat{x}_1$, and matrix T is chosen for stable dynamics in (4.7) from the *Hurwitz* solution of

$$\frac{\partial \Psi}{\partial x_2} = \frac{\partial \mathcal{F}_2}{\partial x_2} - T\hat{K}\frac{\partial \Pi}{\partial x_2} \tag{4.8}$$

where

$$\Pi = \frac{\partial h(\hat{x}_1)}{\partial x_1}\mathcal{F}_1(\hat{x}_1, z, u).$$

The function $\dfrac{\partial \Pi}{\partial x_2}$ doesn't contain noise v as an argument, since the system is disturbance input affine.

Algorithm 1: GSVD nonlinear reduced-order observer design steps

---

Input: An estimate of the state operating point x_0

Recast output measurement system in form of (2.8)

Identify measured and unmeasured states x_1 and x_2 in the output model.

$C,\, E \leftarrow \left.\dfrac{\partial h(x_1)}{\partial x_1}\right|_{x=x_0}$ and $E(x)|_{x=x_0}$ respectively

Gain matrix $\hat{K} \xleftarrow{\text{GSVD}} (C^T,\, E^T)$

Observable state estimates $\hat{x}_1 \xleftarrow{(4.2)} y$

Input: Desired (stable) dynamic behavior $\left(\dfrac{\partial \Psi}{\partial x_2}\right)_{set}$

Obtain $T \xleftarrow{(4.8)} \left(\dfrac{\partial \Psi}{\partial x_2}\right)_{set}$

Integrate reduced state dynamics $z \xleftarrow{(4.7)} \dot{z}$

Unmeasured states $\hat{x}_2 \xleftarrow{(4.6)} y, z$

Output: State estimates $\hat{x}_1$ and $\hat{x}_2$

Chapter 5
APPLICATIONS AND DISCUSSION

5.1 Introduction

The results of the applications of the reduced-order observers designed with GSVD modifications in section 4 is illustrated for linear and nonlinear system examples below.

5.2 Linear System Examples

5.2.1 Biochemical CSTR

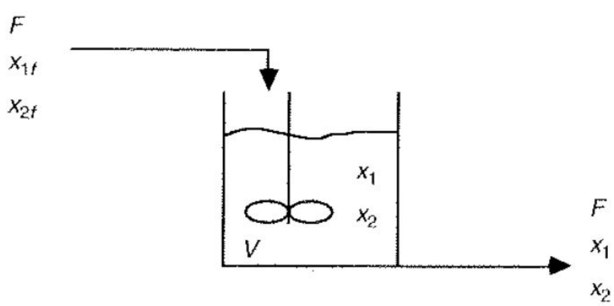


Figure 5.1. A simple bioreactor

The biochemical CSTR example [33] depicted in Figure 5.1, was used to illustrate the performance of the GSVD reduced order Kalman-Bucy observer for process variable estimation from repeated output measurements of a biochemical process modeled with:

$$\begin{aligned} \dot{x}_1 &= (\mu - d)x_1 \\ \dot{x}_2 &= d(x_{2f} - x_2) - \frac{\mu x_1}{\mathrm{Y}} \end{aligned} \tag{5.1}$$

where:

$$\begin{aligned} x_1 &= \text{biomass (cell) concentration} = \text{mass of cells/volume} \\ x_2 &= \text{substrate concentration} = \text{mass of substrate/volume} \\ d &= \text{dilution rate} = \frac{F}{V} = \text{volumetric flowrate/reactor volume} \\ x_f &= \text{substrate feed concentration} \end{aligned}$$

$$\mu = \frac{\mu_{max} x_2}{k_m + x_2 + k_1 x_2^2}$$

Table 5.1. Biochemical CSTR process parameter values

$x_{2fs} = 4.0\ g/L$	$\mu_{max} = 0.53\ hr^{-1}$
$d_s = 0.3\ hr^{-1}$	$k_m = 0.12\ g/L$
$Y = 0.4$	$k_1 = 0.4545\ L/g$

Given the reactor parameter values in Table 5.1 the system is observed to exhibit two notable steady states of operation that are highlighted below:

$$\text{Stable: } x_0 = \begin{pmatrix} 1.5302 \\ 0.1746 \end{pmatrix} \quad \text{Unstable: } x_0 = \begin{pmatrix} 0.9951 \\ 1.5122 \end{pmatrix}$$

After linearizing the nonlinear process model of (5.1) in the neighborhood of the above identified operating conditions of equilibrium point $x_0 = \begin{pmatrix} x_{1s} \\ x_{2s} \end{pmatrix}$, we obtain

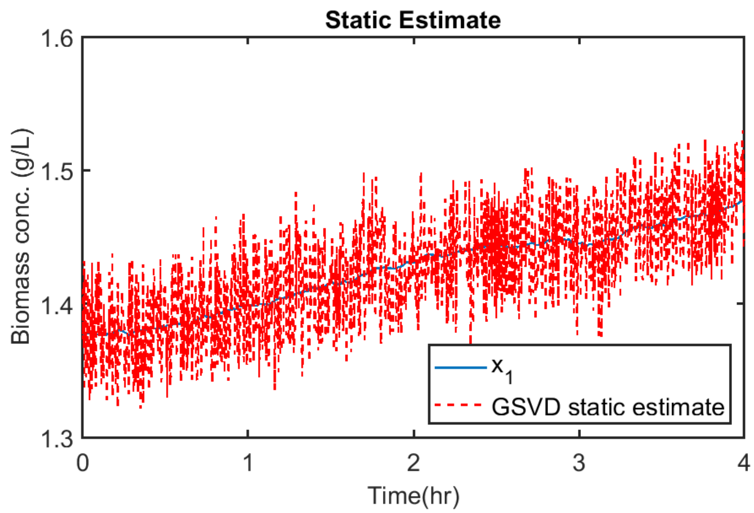
$$A = \begin{bmatrix} \frac{\mu_{max} x_{2s}}{k_m + x_{2s} + k_1 x_{2s}^2} - d & \frac{\mu_{max} x_{1s}(k_m - k_1 x_{2s}^2)}{(k_m + x_{2s} + k_1 x_{2s}^2)^2} \\ \frac{-\mu_{max} x_{2s}}{Y(k_m + x_{2s} + k_1 x_{2s}^2)} & -d - \frac{\mu_{max} x_{1s}(k_m - k_1 x_{2s}^2)}{Y(k_m + x_{2s} + k_1 x_{2s}^2)^2} \end{bmatrix}$$

$$B = \begin{bmatrix} -x_{1s} \\ x_{2fs} - x_{2s} \end{bmatrix}$$

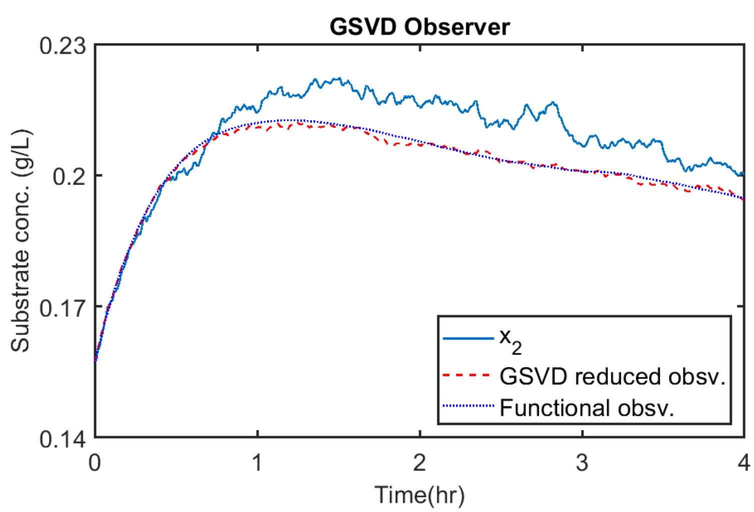
For repeated output measurements of biomass concentration, x_1 with measurement noise, we use

$$C = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \quad E = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$$

The results in Figures 5.2 and 5.3 present a comparison of the GSVD modified reduced order Kalman-Bucy results for x_2 estimation from repeated x_1 measurements with other established observer models.

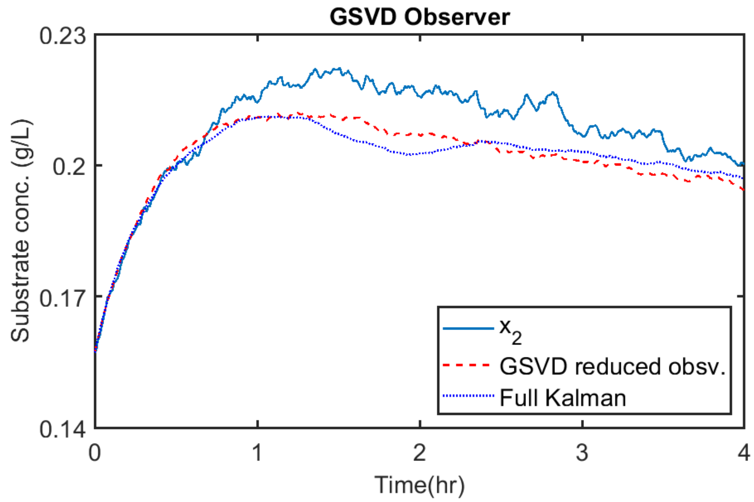


(a) Measurement of biomass concentration



(b) Functional observer estimation of substrate concentration

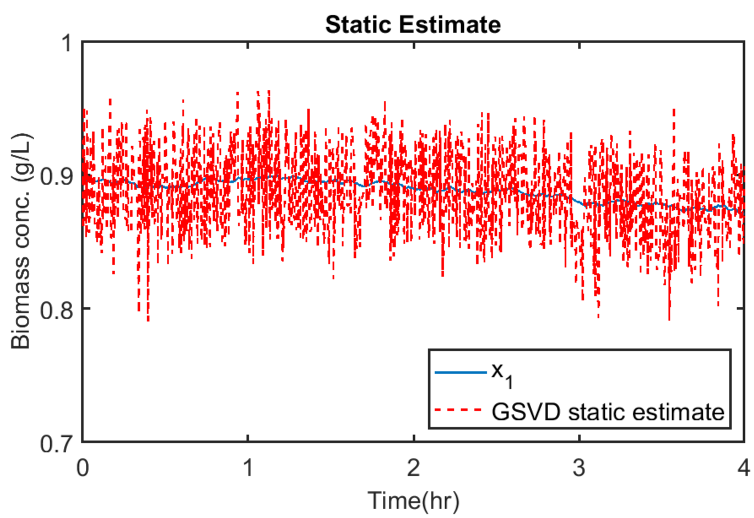
The GSVD observer is shown to outperform its reduced order counterpart, the functional observer, especially around the unstable steady state reference where the



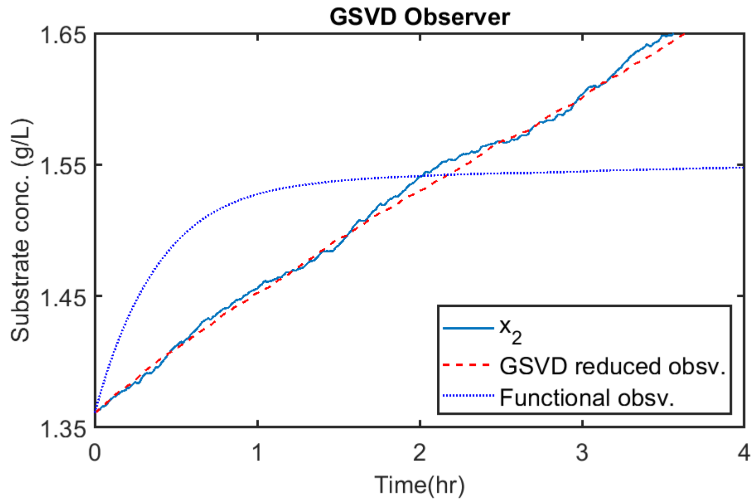
(c) Optimal observer estimation of substrate concentration

Figure 5.2. Comparison of GSVD reduced observer result to a functional observer and the full order Kalman observer when the process evolves around the stable steady state

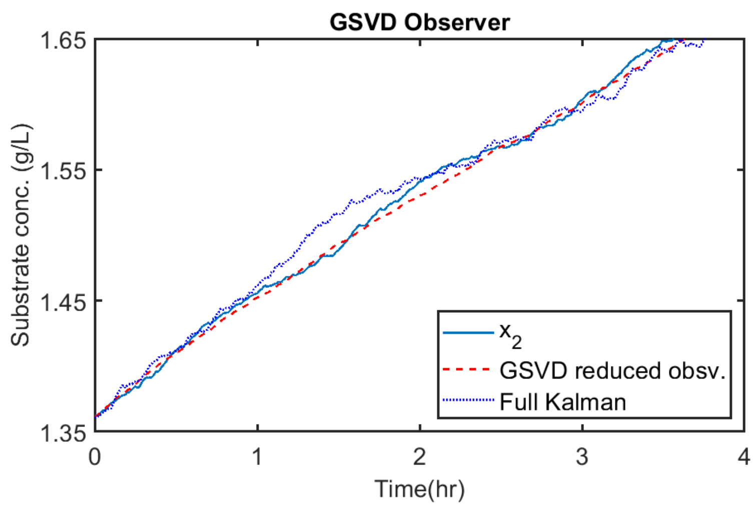
process nonlinearity is more pronounced. Another advantage of the GSVD observer over the functional observer model lies in its deviation from the assumption of perfect measurement characteristic of its counterparts.



(a) Measurement of biomass concentration



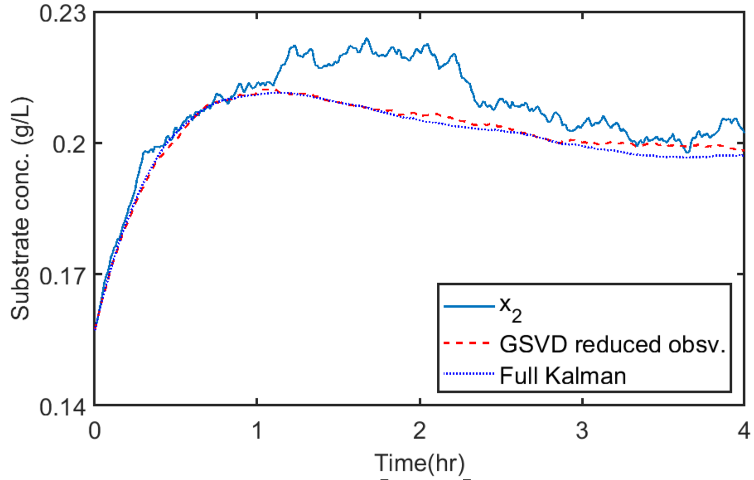
(b) Functional observer estimation of substrate concentration



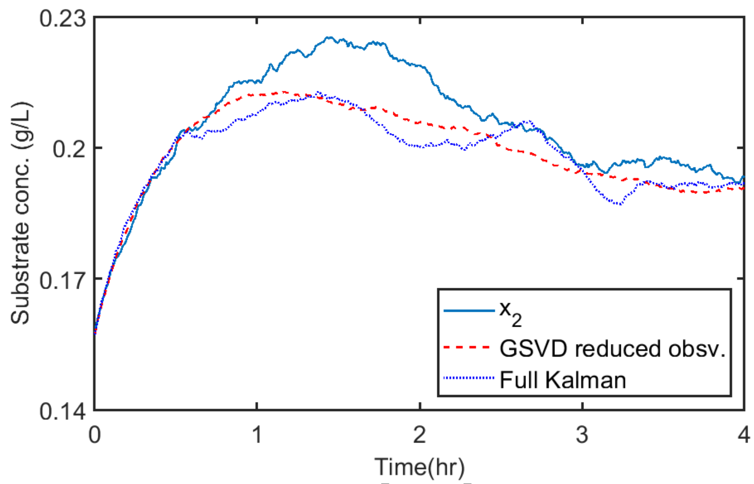
(c) Optimal observer estimation of substrate concentration

Figure 5.3. Comparison of GSVD reduced observer result to a functional observer and the full order Kalman observer when the process evolves around the unstable steady state

The resulting best linear state unbiased estimates of x_2 from x_1 measurements is approximate to the optimal observer estimates in both Figure 5.2 and Figure 5.3 with reduced computation demands. An average estimation time of 0.0043 *sec* was observed in comparison to 0.0084 *sec* for the Kalman filter in 100 runs with MATLAB© [34].



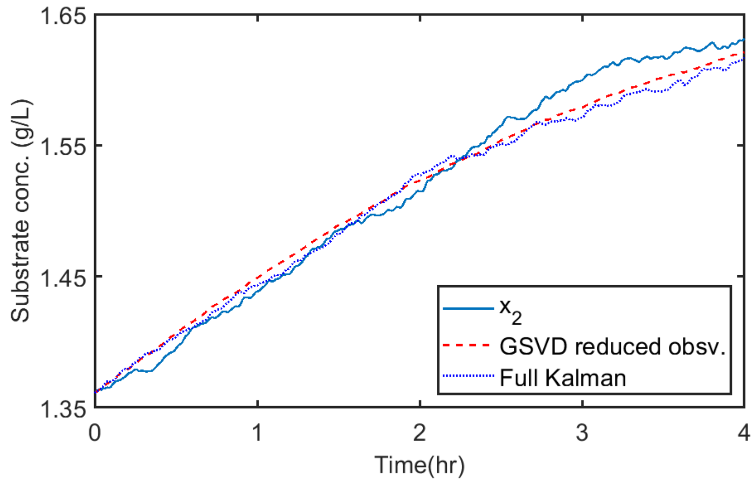
(a) $E = \begin{bmatrix} 5 & 0 \\ -3 & 1 \end{bmatrix}$



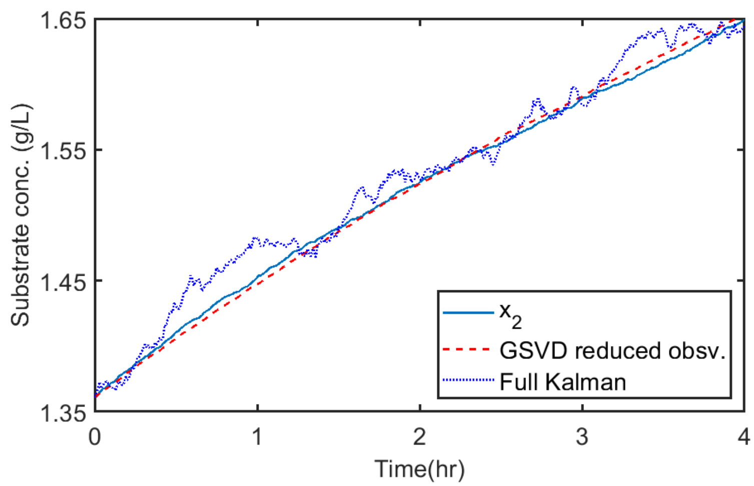
(b) $E = \begin{bmatrix} 25 & 2 \\ -3 & 1 \end{bmatrix}$

Figure 5.4. Comparison of GSVD reduced observer result to optimal filter with increasing noise correlations around the stable steady state

The advantage of the GSVD reduction becomes even more pronounced in more correlated measurement noise cases depicted in Figure 5.4 and Figure 5.5 with increasing correlations in noise contributions to output measurement, i.e. deviations from Gauss Markov white noise measurement model assumption. The GSVD reduced observer outperforms the Kalman filter with better convergence at state variable



(a) $E = \begin{bmatrix} 5 & 0 \\ -3 & 1 \end{bmatrix}$



(b) $E = \begin{bmatrix} 25 & 2 \\ -3 & 1 \end{bmatrix}$

Figure 5.5. Comparison of GSVD reduced observer result to optimal filter with increasing noise correlations around the unstable steady state

true values. Note that an average estimation time of 0.0154 *sec* was observed in comparison to 0.0290 *sec* for the Kalman filter in 100 runs with MATLAB© [34], this observation implies that the reduced observer is twice computationally faster to full order Kalman filter for the illustrated application.

This advantage is particularly important because of the error contributions of multiple sensors in large multivariate systems, that can lead to growing stochastic contributions exhibited as colored noise in output measurements. Colored noise may be observed in process measurements, when y begins to deviate from the Gauss-Markov model of white noise stochastic linear approximation of (1.7) due to varying inaccuracies of sensors and measurements that may not be completely independent. Therefore, white Gaussian noise assumption of the Kalman filter can be limiting in the evaluation of real measurements, and the proposed observer can avoid this problem.

5.2.2 Simplified Tennessee Eastman Process

The Tennessee Eastman process (TEP) model, first described by J. J. Downs and E. F. Vogel [35], is a multiple-input, multiple-output (MIMO) continuous process of a 2-phase reactor, with flash separation and stripping product recovery. It exemplifies real a chemical process and was designed as tool for comparative studies.

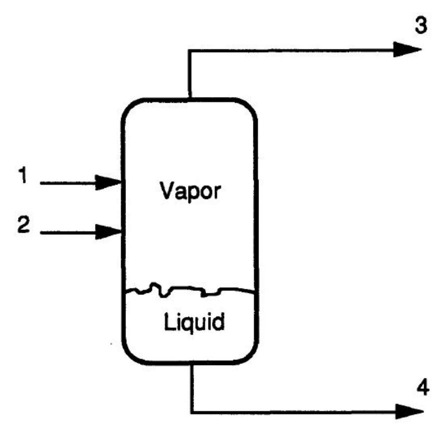


Figure 5.6. Simplified TE process diagram

The simplified TEP has multiple state variables and outputs, and is employed here to further illustrate the performance of the GSVD reduced order observer without relying on multiple output measurements for single variable estimation. TEP was simplified and linearized to state-space formulation by N. L. Ricker [36] as follows:

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

with state variables, $x \in \mathbb{R}^8$, manipulated variables, $u \in \mathbb{R}^4$, and measured output, $y \in \mathbb{R}^{10}$ for the chemical reaction:

$$A + C \rightarrow D$$

with inert B fed in stream 2 and reacted products collected from stream 3. A schematic of the chemical process is shown in Figure 5.6, and the state variables and measured variables listed in Table 5.2 and Table 5.3 respectively.

Table 5.2. State variables of simplified TE process

x_1	molar holdup of A	kmol
x_2	molar holdup of B	kmol
x_3	molar holdup of C	kmol
x_4	molar holdup of D	kmol
x_5	feed 1 valve position	%
x_6	feed 2 valve position	%
x_7	purge valve position	%
x_8	product valve position	%

Table 5.3. Output variables of simplified TE process

y_1	feed 1 flow	kmol/hr
y_2	feed 2 flow	kmol/hr
y_3	purge flow	kmol/hr
y_4	product flow	kmol/hr
y_5	pressure	kPa
y_6	liquid inventory	% of vessel vol.
y_7	A in purge	mol%
y_8	B in purge	mol%
y_9	C in purge	mol%
y_{10}	instantaneous cost	$/kmol

A noise correlation matrix, E is derived from the recorded measurement uncertainties of y_2, y_4, y_5 and y_7 at the near optimal operation of this process with a model predictive control design according to N.L. Ricker. [36]

$$E = \begin{bmatrix} 0 & -5.62 \\ 30 & -9.5 \\ 150 & 200 \\ 16 & 0 \end{bmatrix}$$

The GSVD reduced order observer design for the LTI system is performed using four measurements only as shown in Figure 5.7, and molar rates of the reactants were observed around the process steady state as shown in Figure 5.8.

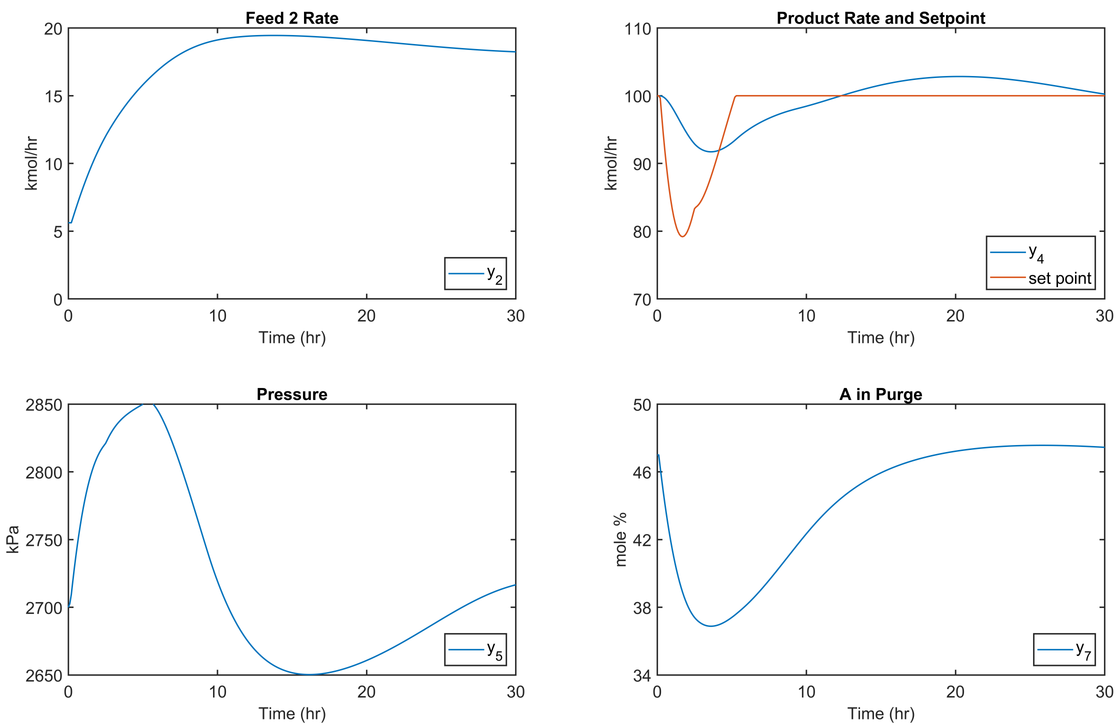
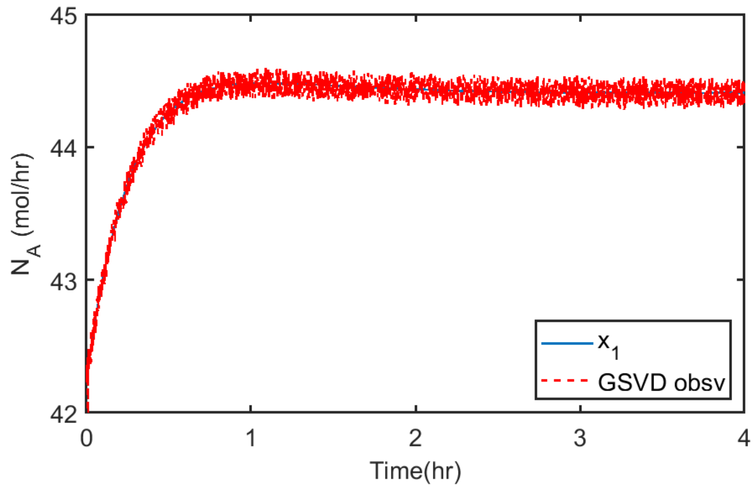
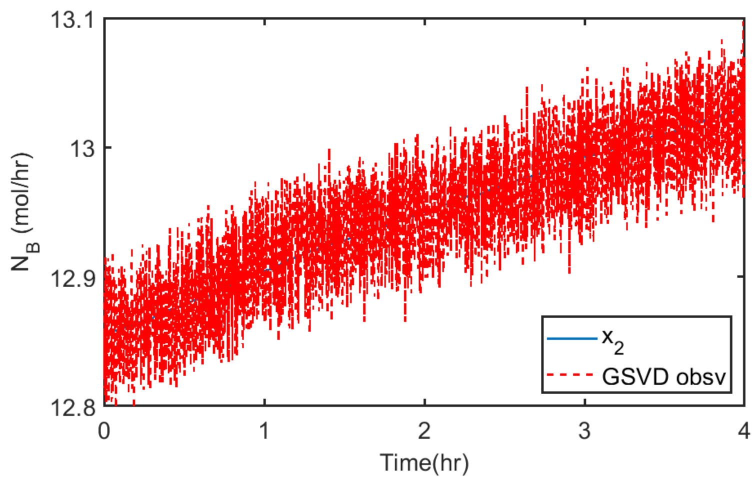


Figure 5.7. Simplified Tennessee Eastman MPC output measurement

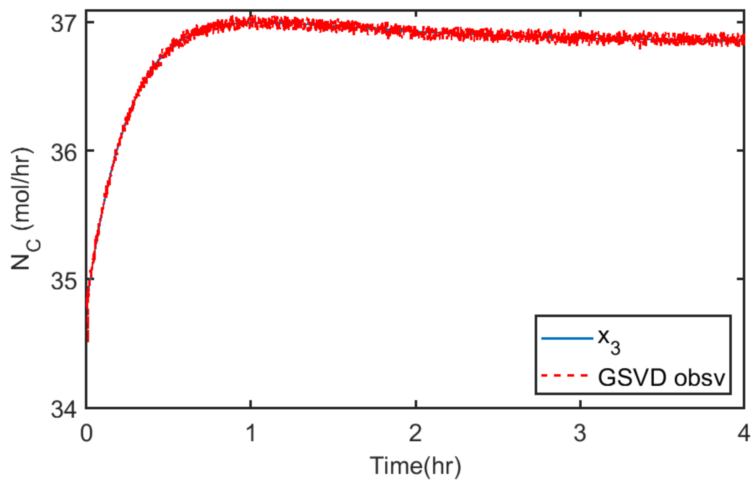
Results of the GSVD reduced order observer estimations of the molar rates of the process: x_1, x_2, x_3, x_4, is presented in Figure 5.8 from only four output measurements: y_2, y_4, y_5, y_7, instead of all $y \in \mathbb{R}^{10}$ presented by N. L. ricker. The average estimation time of $1.0 \times 10^{-4}s$ was observed in 100 runs with MATLAB© [34].



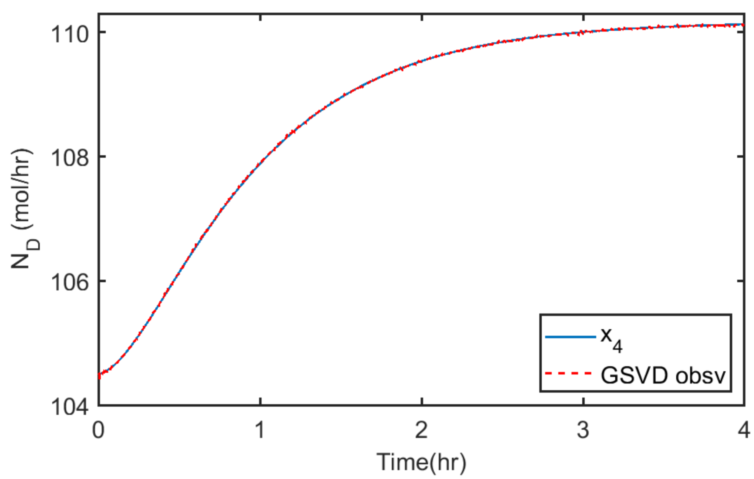
(a) Molar holdup of A



(b) Molar holdup of B



(c) Molar holdup of C



(d) Molar holdup of D

Figure 5.8. GSVD reduced observer results of unmeasured state variables

5.3 Nonlinear System Examples

For the illustration of GSVD modified observation for nonlinear systems, we compare the performance of the GSVD reduced order nonlinear observer to the extended Kalman filter.

5.3.1 Biochemical CSTR

The same biochemical reactor example of section 5.2.1 above is revisited without linearization, to illustrate the GSVD reduced order nonlinear observer estimation of substrate concentration from noisy biomass concentration measurements.

$$\dot{x}_1 = (\mu - d)x_1 + v_1(t)$$

$$\dot{x}_2 = d(x_{2f} - x_2) - \frac{\mu x_1}{Y} + v_2(t)$$

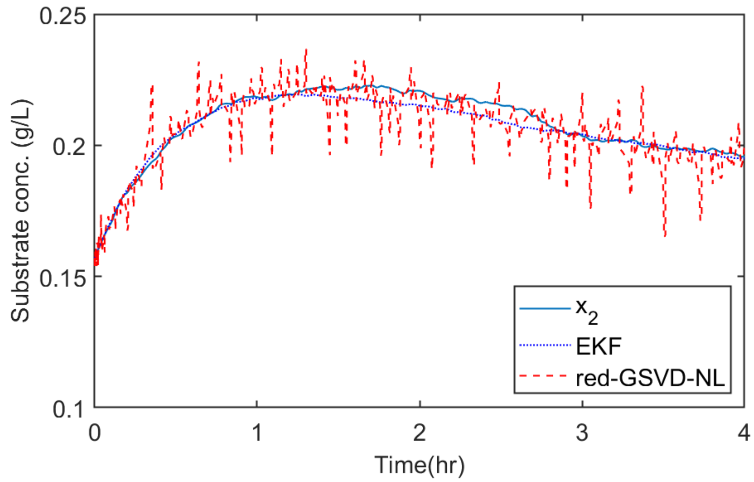
Figure 5.9 presents a comparison of the GSVD reduced order nonlinear observer results for x_2 estimation from the repeated x_1 measurements with a full order EKF.

From the examination of both nonlinear observers at the stable (Figure 5.9a) and unstable (Figure 5.9b) state references of the model, the EKF gives very good results for the stable reference but loses estimation accuracy under unstable operation conditions as a consequence of extended linearization of system dynamics in the EKF design. With an offline only calculation of the gain matrix, the GSVD reduced nonlinear observer is much more robust maintaining accurate estimations of the unmeasured state variable at both stable and unstable operating conditions. This can be observed in Table 5.4 where the EKF error is four times larger than the GSVD estimator error. The error presented in the table is defined as

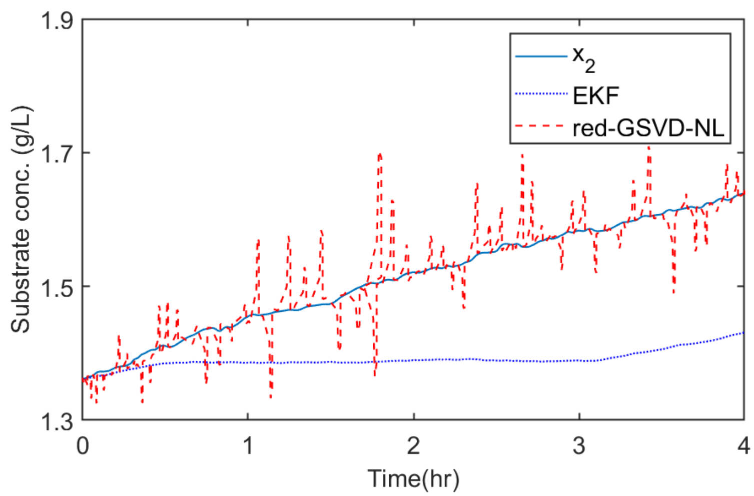
$$Error = \frac{\int_0^{t_f} \|\varepsilon\|_1 dt}{\int_0^{t_f} dt}$$

.

Another advantage of the GSVD nonlinear reduced order observer is seen in the observed average computation time in comparison to the full order EKF with



(a) Stable steady-state reference estimation of substrate concentration



(b) Unstable steady-state reference estimation of substrate concentration

Figure 5.9. Comparison of GSVD reduced order nonlinear observer with Extended Kalman filter for the biochemical reactor

MATLAB© [34]. The GSVD nonlinear observer was observed to be much faster and is presented in Table 5.4. Thus, for the specific example the proposed GSVD observer outperformed EKF in both accuracy of estimation and computational demand for a range of operating conditions, especially at unstable steady state with more pronounced nonlinear behavior.

Table 5.4. Estimation results for biochemical CSTR

Observer		Error	Average time (sec)
Stable x_0	GSVD	0.0128	0.0073
	EKF	0.0142	0.0745
Unstable x_0	GSVD	0.0348	0.0029
	EKF	0.1265	0.0398

5.3.2 Non-isothermal Reactor

A second illustration of the GSVD reduced order nonlinear observer is performed with a well mixed non-isothermal reactor example from [37] with the following endothermic reactions of A reactant species, D desired products, and byproducts U and R.

$$\begin{aligned} A &\xrightarrow{k_1} D \\ A &\xrightarrow{k_2} U \\ A &\xrightarrow{k_3} R \end{aligned} \tag{5.2}$$

The reaction system is modeled by

$$\begin{aligned} V\frac{dC_A}{dt} &= F(C_{A0} - C_A) - \sum_{i=1}^{3} k_{i0}\exp\left(\frac{-E_i}{RT}\right) C_A V \\ V\frac{dC_D}{dt} &= -FC_D + k_{10}\exp\left(\frac{-E_1}{RT}\right) C_A V \\ V\frac{dT}{dt} &= F(T_{A0} - T) + \frac{UA}{\rho c_p}(T_c - T) + \sum_{i=1}^{3} \frac{(-\Delta H_i)}{\rho c_p} k_{i0}\exp\left(\frac{-E_i}{RT}\right) C_A V \end{aligned} \tag{5.3}$$

In expressing the plant model in the description of (4.4), we have

$$\dot{x} = f(x) + g(x)u + G(x)\theta_k(t)$$

for the following identified variables:

$$x_1 = C_A - C_{As}$$

$$x_2 = C_D - C_{Ds}$$
$$x_3 = T - T_s$$
$$u = T_c - T_{cs}$$
$$\theta_1 = \Delta H_1 - \Delta H_{10}$$
$$\theta_2 = \Delta H_2 - \Delta H_{20}$$
$$\theta_3 = \Delta H_3 - \Delta H_{30}$$
$$\theta_4 = T_{A0} - T_{A0s}$$

We have :

$$f(x) = \begin{bmatrix} \frac{F}{V}(C_{A0} - C_{As} - x_1) - \sum_{i=1}^{3} \pi_i \\ \\ -\frac{F}{V}(C_{Ds} + x_2) + \pi_1 \\ \\ \frac{F}{V}(T_{A0s} - T_s - x_3) - \sum_{i=1}^{3} \frac{(-\Delta H_i)}{\rho c_p}\pi_i + \frac{UA}{\rho c_p}(T_{cs} - T_s - x_3) \end{bmatrix}$$

$$g(x) = \begin{bmatrix} 0 \\ 0 \\ \frac{UA}{\rho c_p} \end{bmatrix}, \quad G(x) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \pi_1 & \pi_2 & \pi_3 & \frac{F}{V} \end{bmatrix}$$

where

$$\pi_i = k_{i0}\exp\left(\frac{-E_i}{R(x_3 + T_s)}\right)(x_1 + C_{As})$$

For the output measurements model according to (4.4)

$$y = h(x) + E(x)w(t)$$

we have

$$h(x) = \frac{(-\Delta H_{i0})}{\rho c_p}\pi_i$$

$$E(x) = \begin{bmatrix} -\pi_1 & 0 & 0 \\ 0 & -\pi_2 & 0 \\ 0 & 0 & -\pi_3 \end{bmatrix}$$

from which we can directly estimate $\begin{bmatrix} x_1 \\ x_3 \end{bmatrix}$ by a GSVD static estimator using the following parameters:

F	$= 3$	$m^3\ hr^{-1}$
V	$= 1$	m^3
A	$= 6$	m^2
U	$= 10^3$	$kcal\ hr^{-1}\ m^{-2}\ K^{-1}$
R	$= 1.987$	$kcal\ kmol^{-1}\ K^{-1}$
c_p	$= 0.231$	$kcal\ kg^{-1}\ K^{-1}$
ρ	$= 900$	$kg\ m^{-3}$
ΔH_{10}	$= 5.4 \times 10^3$	$kcal\ kmol^{-1}$
ΔH_{20}	$= 5.1 \times 10^3$	$kcal\ kmol^{-1}$
ΔH_{30}	$= 5.0 \times 10^4$	$kcal\ kmol^{-1}$
k_{10}	$= 3.36 \times 10^6$	hr^{-1}
k_{20}	$= 7.21 \times 10^6$	hr^{-1}
k_{30}	$= 1.25 \times 10^7$	hr^{-1}
E_1	$= 8.0 \times 10^3$	$kcal\ kmol^{-1}$
E_2	$= 9.0 \times 10^3$	$kcal\ kmol^{-1}$
E_3	$= 9.5 \times 10^3$	$kcal\ kmol^{-1}$
C_{A0s}	$= 3.75$	$kmol\ m^{-3}$
T_{A0s}	$= 310$	K
C_{As}	$= 0.913$	$kmol\ m^{-3}$
C_{Ds}	$= 1.66$	$kmol\ m^{-3}$
T_s	$= 302$	K
T_{cs}	$= 320$	K

The GSVD observer gain was computed offline for this example. The resulting nonlinear reduced order estimation of concentration of desired products, C_D i.e. x_2 from output measurements with x_1 and x_3, is presented in Figure 5.10 and Table 5.5

below in comparison with the EKF. Both observers produce similar accuracy with converging estimates, but the GSVD reduced order observer achieves the same result with much less computation time. An average estimation time of 0.9862 *sec* was observed in comparison to 12.1554 *sec* for the EKF in simulations with MATLAB© [34]. Thus, for this example as well, the proposed GSVD observer outperformed EKF in both accuracy of estimation and computational demand for the range of operating conditions.

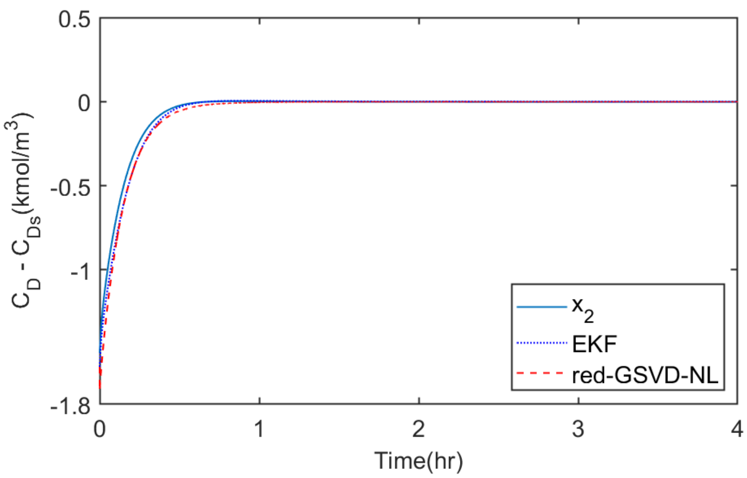


Figure 5.10. Comparison of GSVD reduced order nonlinear observer with Extended Kalman filter for the Non-isothermal chemical reactor

Table 5.5. Estimation results for non-isothermal chemical reactor

Observer	Error	Average time (sec)
GSVD	0.0013	0.9862
EKF	0.0021	12.1554

Chapter 6
CONCLUSION

6.1 Conclusions

The estimation of process system variables from noisy output measurements with GSVD for linear and nonlinear systems was investigated with the modifications of the Kalman filter.

GSVD was chosen to solve the constrained least squares problem of state estimation from noisy output measurements expressed in generalized linear model, because of its robustness for singular correlations of noise contributions. The resulting static estimates from GSVD is equivalent to the best linear unbiased estimation. The GSVD estimator approach was subsequently extended to the design of reduced order observation models.

For linear time-invariant systems, the modification of reduced order Kalman-Bucy design with GSVD static estimation was proposed. The observer was constructed for estimation of unmeasured state variables by adding reduced state variable dynamics to static estimates of noisy output measurement. The new design is simple, non-iterative and stable. The resulting reduced order observer maintains near-optimal observations in comparison to the full order Kalman filter, and outperforms other reduced order observers especially around unstable steady states, while retaining the robustness and simplicity in the derivation of the static component.

The GSVD reduced order Kalman-Bucy observer avoids the perfect measurement assumptions of functional observers and the original reduced Kalman-Bucy by extracting static estimates from a generalized linear measurement model instead of a Gauss-Markov model. This modification allows for the treatment of correlated and colored noise cases, which are more commonly encountered in industry. Because of its design accommodations for colored noise, the GSVD observer has an advantage over the Kalman filter with increasing deviations from Gauss-Markov white noise assumptions, as illustrated in the biochemical process example.

Similarly, the GSVD reduced order nonlinear design provides a reduced order alternative to the EKF for nonlinear process observation to access unmeasured state variables. This observer is simple in construction from a nonlinear system model by functional reduction of the plant model and the GSVD of output measurements. The resulting estimation of unmeasured state variables by GSVD reduced order nonlinear observer was achieved much faster than with the EKF. The GSVD reduced order nonlinear observer is stable and robust in handling process and measurement disturbances because the design eliminates the linearization of system dynamics requirement in the design of the EKF.

With its reduced computational demand and stable estimation at stable and unstable operation modes of nonlinear systems, the GSVD reduced order nonlinear observer provides an economic alternative for system estimation and feedback in model predictive control (MPC). The illustrative examples treated in chapter 5 show that the proposed GSVD observer outperformed EKF in both accuracy of estimation with less computational demand for the range of operating conditions.

Generally, computational demand and instrumentation costs can be significantly reduced using the new methods to obtain unmeasured process state variables.

6.2 Future work

The proposed future work in this area will be for the extension of GSVD state estimation to development of economic model predictive control techniques in place of high-gain observers. The computational cost savings of state estimation may translate into even more reduced costs in MPC.

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