SHAPE CHANGE AND STRUCTURAL PERFORMANCE OF CABLE-ACTUATED CYLINDRICAL TENSEGRITIES

A Thesis in

Aerospace Engineering

by

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Abstract

Large space structures are often desirable for increasing performance of those missions requiring high resolution or high power. As scientific needs demand larger and larger structures, the need for high efficiency, deployable structures that can satisfy the mass and volume constraints of launch vehicles also grows. Tensegrity structures offer a promising, lightweight, packaging-efficient solution to many deployable boom applications.

This thesis has two primary focuses. The first is an analysis of the range of shapes a class-2 triplex tensegrity boom may be able to achieve via actuation of select cables. The second is a design analysis of these triplex booms, to develop a better understanding of how different configuration parameters influence overall structural performance.

Tensegrity structures are self-equilibrated structures composed of compression-carrying struts and tension-carrying cables. These members are connected at nodes, which act as frictionless ball joints. The tensegrities studied in this thesis are class-2 cylindrical triplex booms, meaning that there are three struts per bay, the nodes on the top or bottom face of a straight bay lie on a circle, and at most two struts touch at a single node for multi-bay structures (improving stiffness and maintaining symmetry between adjacent bays).

Review of existing studies of tensegrity analysis methodology helps to understand the shapes a configuration may achieve. Extensive research has been done on different methods for finding equilibrated shapes for a given tensegrity—even for a triplex specifically—but little has been done to characterize the entire range of achievable shapes. This thesis develops a method for understanding the entire range of shapes that may be achieved with a triplex boom, when only the 3 supporting cables are actuated by changing their lengths.
Building on the shape change analysis, this thesis then addresses how different configuration parameters influence structural properties and performance. For a single bay, the ratio of bay height (related to strut length) to the circumscribing radius (related to face cable lengths) and the member cross-section radius are evaluated as two key design parameters studied for their effects on structural properties. For a boom of a fixed height, the number of bays are also varied as a design parameter to understand the resulting effects on structural properties. The range of achievable shapes, bending stiffness (tip deflection under load), packaging efficiency, and mass are the structural properties addressed. An example case using the Candarm2 to define design goals was addressed to show how the developed methods might be used to design a boom for a specific mission.

In general, increasing the ratio of height to circumscribing radius increases the range of achievable shapes and decreases packaging efficiency. Increasing member radius was found to decrease the range of achievable shapes, decrease tip deflection and decrease packaging efficiency. Increasing the number of bays was found to have varying effects on range of achievable shapes and packaging efficiency depending on the ratio of overall boom height to circumscribing radius. In most cases (with sufficiently high ratio of overall height to circumscribing radius), increasing the number of bays increases the range of achievable shapes and increases packaging efficiency. Increasing the number of bays (regardless of the ratio of height to circumscribing radius) was found to increase tip deflection and increase mass.

Finally, this thesis considers experimental validation of the models developed to predict shape-changing performance. A review of existing methods of actuation, control and metrology was conducted, with special attention to potential utility for these types of triplex booms. An experimental bay was constructed then actuated by manually adjusting cable lengths, while a
motion tracking system was used to determine the evolving shape. This experiment generally verified the variety of shapes that can be achieved by a triplex via actuation of just 3 cables, and that the methods developed can predict those shapes. Furthermore, it demonstrated the accuracy of the methods developed to predict member interference along a shape-change path.

A large range of achievable shapes combined with good packaging efficiency and structural performance suggests that tensegrity structures merit continuing consideration for future deployable boom applications.
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### Symbols

- $\theta$: twist angle
- $f_{ji}$: force from member $j$ imparted on member $i$
- $lij$: direction cosines between members $i$ and $j$
- $q$: force density
- $l_m$: length of member $m$
- $C$: connectivity matrix
- $Q$: Diagonal square matrix of force densities
- $D$: force density matrix
- $A$: equilibrium matrix
- $q$: force density vector
- $\alpha$: azimuth
- $\beta$: elevation
- $r_h$: slant height
- $h$: single bay height
- $r$: circumscribing radius of a bay
- $h/r$: ratio of height to circumscribing radius (one bay)
- $\beta_c$: critical elevation
- $n$: normal vector
- $a$: axis of rotation
- $\varphi$: angle of rotation
- $R$: rotation matrix
- $H$: total boom height
- $H/r$: ratio of total boom height to circumscribing radius
- $R_{mn}$: position vector of member $m$ in position $n$
- $K$: global stiffness matrix
- $\Delta U$: tip deflection vector
- $P$: external load vector
- $F$: internal load vector from pre-stress
- $E$: Modulus elasticity
- $\varepsilon$: Packaging efficiency
- $V$: volume
- $N$: number of bays
- $S_d$: strut diameter
- $r_p$: package radius
- $Ca$: cable length (face cables)
- $M$: mass
- $\rho$: density
- $EI$: bending stiffness
- $\delta$: tip deflection
- $L$: length

### Acronyms

- FEM: Finite element method(s)
- LFEM: Linear finite element method(s)
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Chapter 1

Introduction

In the drive for space exploration and discovery, scientists and engineers face countless challenges unique to the trying environment of space itself. Escaping the grasp of Earth’s gravity and atmosphere defines the majority of limitations for extra-terrestrial vehicles and instruments. These limitations typically take the form of weight and volume. In order to meet launch requirements without sacrificing the quality of a science payload, structures and instrumentation in launch vehicles are constantly being refined to reduce launch weight and volume.

Deployable structures are a natural solution to meeting volume limitations. By stowing with high packaging efficiency, deployable structures allow instruments requiring large dimensions to be stowed in a small space during launch, and then to expand in orbit. This type of structure is often used for applications where instruments need to be separated to reduce interference. Deployable structures are also used for things like solar sails, telescopes, and communication reflectors, to name only a few examples. Due to the deployment mechanisms, some deployable structures pay a penalty in structural stiffness to meet the necessary mass and volume constraints.

Tensegrity structures show great promise for excellent packaging efficiency and low mass without greatly sacrificing structural stiffness. A tensegrity is a pre-stressed, self-equilibrated structure composed of struts and cables connected at frictionless ball joints. The composition of
struts and cables enables consideration of actively controlled shape change, especially through cable actuation. Many deployable structures currently in use in space applications are designed to be deployed only once. This may be done with tensegrities as well, but an actively controlled deployment holds potential not only for increased deployment accuracy, but also improves upon the types of potential applications of the structure. A structural boom which can change its shape even after deployment may be used to maintain shape accuracy, correcting for things like thermal expansion or hysteresis in applications requiring high precision. Such a structure may also be used for applications which requires redirection of only part of a payload—like pointing in communication or observation satellites, for example—without changing the orientation of the entire satellite.

This chapter introduces to related existing deployable structure technologies which served as a motivation for the investigation of tensegrity structures. Just as important as the deployment itself is an understanding of the post-deployment applications. The tensegrities studied in this thesis are explored as potentially deployable structures, and examined for those design characteristics which make them most suited for use in for space applications.

1.1 Deployable space-craft structures

Deployable structures are essential to the development of large-scale structures in the space industry. Whether it be booms (deploying in one direction) or multi-dimensional structures, the ability to stow compactly is critical launch. Tensegrity structures—due to their composition of struts and cables—provide an excellent potential solution for the demanding constraints of launch.

There are many different types of deployable structures, each with associated applications. Solar panels, antennae, and gravity gradient stabilization booms are just a few examples of
deployable structures and their uses in space applications [1-3]. Narrowing the scope to only consider deployable booms still leaves many different varieties. Thin-walled tubular booms work like the common carpenter’s tape; they are flattened in the deployed shape, and the stored strain energy when stowed (due to their natural curvature) can be used to drive deployment [4-6]. Telescopic booms are comprised of concentric cylinders, which typically deploy with the use of a motor-driven lead-screw [7, 8]. Coilable booms are lattice trusses composed of continuous rods and bracing cables, which typically involve buckled members in the stowed configuration for very high packaging efficiencies [9, 10].

Similar to coilable booms, tensegrity structures are comprised of rods and cables. Tensegrity structures are a relatively new technology in the deployables field, but have been extensively researched for their potential uses. Furuya offered one of the first conceptual studies of tensegrities as deployable structures [11]. Knight looked at tensegrity structures—focusing in particular on their stability—for deployable antennae applications [55]. Tibert and Pellegrino considered tensegrities using tape-spring hinges in the struts for deployment [12], and Sultan and Skelton examined tensegrity deployment through cable actuation [13]. Yildiz looks at transforming a tensegrity from class-1 for stowing to class-2 once deployed by actuating cable lengths [19]. Cable actuation for deployment might be re-used after being deployed to allow the structure to continue changing length. Cable actuation, if used for deployment, could potentially also be used once deployed to further change the shape of the structure.

1.2 Articulated spacecraft Structures

Articulated structures—where the structure may change shape either by member elongation or translation—have many potential uses in space applications. Robotic arms are among
the more common examples of articulated structures. The Canadarm, for example, was used for 30 years in numerous missions on NASA’s space shuttle, and the Canadarm2 experienced similar success in mission utility on the International Space Station [14].

Articulated booms could prove useful in other space applications as well. Thermal heating in the extreme environment of space is an issue addressed in any spacecraft—and is particularly important when very long structures are involved. In cases of uneven heating, as when the sun is strong on one side of a spacecraft while the other side is in shadow, temperature differences can cause thermal expansion of only some parts of the structure, causing inaccuracies in alignment and pointed devices.

Actuated structures could be used to help improve alignment of very long structures, since they might be actively controlled and adjusted for any given situation—including adjusting for thermal expansion. Furthermore, articulated structures might be used as deployable structures which can be re-stowed and deployed repeatedly. Most designs of deployable booms are intended to lock in place once deployed, stowing only for the sake of launch. This has the benefit of allowing for passive deployment, but, actuated deployment can allow for less chaotic deployment processes overall, and accommodate applications which may require repeated stowing and deploying.

Actuated tensegrity structures have been researched extensively as well. Different methods of changing the overall structural shape are achieved in different ways depending on the study and intended application. Wijdeven and Jager considered optimization of trajectories of a shape-changing tensegrity, finding feasible morphologies of a given tensegrity based on design constraints, and a desired trajectory [15]. Iscen et al. used spooled cable actuation for a tensegrity robot to allow the structure to roll—focusing on methods of controlling the robot accurately [16]. Fest et al. developed an adjustable tensegrity in which the shape is changed by adjusting the length
of telescopic struts [17]. The use of tensegrities as shape changing devices is not seen much in actual applications today. NASA Ames developed the tensegrity super ball bot, a rolling spherical tensegrity for use as a robotic lander. However, the super ball bot has only been developed to technology readiness level 2 (i.e. the research phase with only introductory physical technology builds) [18]. Nevertheless, the burgeoning field of research dedicated to tensegrity—and even shape-changing tensegrity in particular—demonstrates the potential for these structures to be used as viable aerospace structures in the future.

1.3 Objectives

This research is motivated by a need to develop a better understanding of how cylindrical tensegrity structures may be designed to meet space mission requirements. A thorough understanding of structural properties and design parameters could be used to promote the use of these structures in real world applications. Tensegrity structures offer a vast range of potential configurations, all having unique properties that may be fine-tuned to any particular mission.

This thesis has two primary focuses. The first is an analysis of the range of shapes a class-2 triplex tensegrity boom may be able to achieve via actuation of select cables. The second is a design analysis of these triplex booms, to develop a better understanding of how different configuration parameters influence overall structural performance.

With only a few cables allowed to change length, this thesis explores the breadth of the range of achievable shapes for a given cylindrical triplex tensegrity design. Furthermore, this thesis analyzes the characteristics of the structure as a whole. Though these structures are not yet widely used in current spacecraft missions, methods like those developed in this thesis may help promote
a better understanding of the potential of these structures so that they may play part in future space missions.

This thesis is organized as follows. Chapter 2 develops a general background in tensegrity structures, explaining the basic nomenclature and characteristics which provide the basis for all tensegrity work. In Chapter 3, a geometry-based method for finding all equilibrated shapes a triplex can achieve is described, as well as how this methodology can be extended to multi-bay structures. Chapter 4 describes the methods used to analyze collision checking between members along prescribed paths, bending tip deflection and packaging efficiency. Chapter 5 uses the methods developed in Chapters 3 and 4 to compare the effects of different configuration parameters—namely ratio of bay height to circumscribing radius, member radius, and number of bays—on the structural properties described in Chapter 4. Chapter 6 discusses some existing experimental designs of shape changing tensegrities, and describes the experimental structure developed for this thesis.
Chapter 2

Tensegrity Geometry and Mechanics

Tensegrity structures hold their foundation in art and architecture applications, with the first physical tensegrity being built Kenneth Snelson as an art piece [35]. Though their exact origin is somewhat controversial (some attribute their origin to Richard Buckminster Fuller, while others attribute it to Kenneth Snelson [35]), it is undeniable that their discovery caught the eye of researchers across numerous fields, including engineering. Tensegrities offer a fascinating potential solution to countless design problems, due to their nature as lightweight, reconfigurable and flexible structures.

To date, much research has been done classifying and characterizing tensegrity structures and their related properties in order to provide a useful framework for tensegrity design. Comprised of struts and cables, the classical tensegrity design has struts “floating” amidst the cables, such that no struts touch at a given node. The number of nodes is used to describe the structures, and this holds true even for less traditional cases of tensegrity structures where multiple struts connect at a node [36]. The geometric nature of tensegrities allows for countless configurations of a given tensegrity structure, all of which have unique structural properties related to the pre-stress and member materials of the system. This means that for any design problem a vast solution space of tensegrity structures may be considered, making understanding both tensegrity geometry and mechanics critical to the decision-making process in design. This chapter discusses nomenclature and geometry in greater detail that will be used throughout the whole of this thesis. Furthermore, this chapter offers an introductory look at how to handle tensegrity mechanics, as well as existing methods to determine viable tensegrity geometries for a given application.
2.1 Tensegrity geometries

Tensegrities are a unique type of structure, being composed of struts and cables connected at nodes which are modeled as frictionless ball joints. Tensegrities are also pre-stressed structures, with the members carrying uniaxial loads. Cables (sometimes called tendons), and struts hold exclusively tension and compression respectively. A given tensegrity structure of \( n \)-nodes may have multiple equilibrated configurations—where configuration refers to the specific way in which struts and cables connect to nodes (aka the connectivity of the tensegrity) and the particular number and respective lengths of struts and cables.

Figure 2.1 shows 3 different tensegrity configurations, each containing 12 nodes for 6 struts. One category of configuration is spherical (Fig. 2.1 (a))—where the tensegrity’s nodes all lie on one of 2 concentric spheres. Spherical tensegrities are the typical geometry studied for applications in moving robotics and lander applications [18, 20, 21]. Another category of tensegrity is cylindrical (Fig. 2.1 (a) and 2.1 (b)). Cylindrical tensegrities are categorized by their topmost and bottommost faces being contained in parallel planes, where the nodes of the top and bottom faces lie on parallel plane circumscribing circles of the same radius. In the case of Fig. 2.1 (b), the tensegrity is a single bay 6-plex, where 6-plex denotes how many struts are in a bay (and consequently how many nodes are on a given face). Fig. 2.1 (c), on the other hand, shows 2 stacked 3-plex bays. Again, 3-plex refers to 3 struts for a single bay. The bays are offset in being stacked such that no struts are touching each other at a given node.

This case of multi-bay tensegrity is referred to as class-1, where the 1 denotes a single strut present at any one node. Spherical and cylindrical tensegrities are only 2 (albeit large) categories of tensegrity. Fig. 2.1 illustrates just how different a tensegrity of a given number of nodes and struts may be, with each of the 3 depicted tensegrities all having 6 struts and 12 unique nodes. The
geometries depicted have the minimum number of cables for equilibrium, with Fig 2.1 (a) and 2.1 (c) both having 24 cables, while 2.1 (b) has 36 cables.

Figure 2.1: Different configurations for a tensegrity with 6 struts and 12 nodes. Struts are in red and cables are in blue, nodes are numbered. Left (a) is an icosahedron; a spherical tensegrity. Center (b) is a single bay 6-plex cylindrical tensegrity. Right (c) is a 2-bay, class-1, 3-plex tensegrity, formed by stacking 2 3-plex cylindrical tensegrity bays.

Consequently, the different configurations have different structural properties that are important to consider when designing a tensegrity. Spherical tensegrities, for example, are appealing for lander applications, as their symmetry reduces the need for reorientation equipment, the same feature which enables the structure to easily adapt for locomotion \([18, 20, 27]\). Cylindrical tensegrities are appealing as alternative structures for boom applications \([19, 35, 36]\). The tensegrities considered for this thesis are cylindrical triplexes. In multi-bay cases, the triplex will be class-2 (denoting 2 nodes meeting at a single joint). Class-2 tensegrities offer greater structural stiffness in comparison to class-1. Conversely, class-1 structures tend to offer greater packaging efficiency. Research has been done investigating stowing tensegrities as class-1, and re-orienting into class-2 in the deployment process, thus offering high packaging efficiency in the deployment process and then re-orienting into a stiffer configuration \([19]\). The properties investigated in this dissertation, however, are obtained only considering the structure post-deployment. A class-2
triplex (3-plex) offers the simplest geometry with higher stiffness (as compared to class-1), and may be treated as a baseline case of cylindrical tensegrities.

2.2.1 Nomenclature

To better understand the literature on tensegrities, it is necessary to discuss some tensegrity-specific nomenclature. All tensegrities investigated herein are 3-plex, though the following nomenclature may be used to describe any type of cylindrical tensegrity.

A configuration of a tensegrity describes the number of nodes, the connectivity of members in relation to those nodes, and the member lengths. In most work, a cylindrical tensegrity configuration is defined in its straight axis or parallel face shape—shape referring to a particular set of nodal locations. In the straight shape, the centers of the top and bottom faces (defined by the cables in blue in Fig. 2.2 (a)) are aligned vertically, and the planes containing these faces are parallel and horizontal. The distance between the centers of the top and bottom faces is called the slant height, regardless of the configuration’s shape. Struts, the compression members, are denoted by red lines, while cables are denoted by blue and green lines. The face cables are shown in blue, and the top and bottom face nodes are connected to each other by the supporting cables, shown in green.

The twist angle is also labeled in Fig. 2.2 (b), and is defined as the angle between adjacent nodes in the straight shape when viewing the structure top-down. Twist angle for the general cylindrical $n$-plex is found as

$$
\theta = \pi \left( \frac{1}{2} - \frac{j}{n} \right)
$$

(2.1)
where \( j = 1 \) and \( n \) is the number of nodes in a face. Thus, in the case of the triplex, \( n = 3 \) and \( \theta = \pi/6 \) [22]. The configuration depicted in Fig. 2.2 shows the minimum number of cables for a cylindrical triplex. It is possible to add additional cables (in the case of the triplex, these would be additional supporting cables), as shown in Fig. 2.3 [19, 23]. These additional cables allow for additional equilibrated shapes, including straight shapes for a range of twist angles. The addition of these cables is not considered in future chapters, but the same processes discussed could readily be extended to consider such.

**Figure 2.2:** Left (a) is a labeled single bay triplex in isometric view. Nodes are labeled with arrows and numbers 1 - 6. Right (b) is the same triplex viewed from top down, with the labeled \( \theta \) being the twist angle.

**Figure 2.3:** A triplex with additional supporting cable connections
2.2 Tensegrity mechanics

Tensegrity configurations are chosen such that the structure will be in a state of equilibrium for a desired shape. Given the structure’s unique composition of struts and cables, this only includes cases where the struts carry purely compression and cables carry purely tension. Cases where members carry no force, or where members carry the wrong type of force (i.e. struts in tension or cables in compression) typically amount to the structure collapsing. Of course, in reality a strut might carry tension or a cable might go slack without physical issue, but such a state is considered outside the definition of a tensegrity structure.

It is important to understand how to know if a given tensegrity configuration and shape is in equilibrium. The methods used are derived from the method of joints, like that used to analyze a truss structure. Figure 2.4 shows a free-body diagram of the force balance at node $i$. The forces from members $ij$ and $ik$ (of lengths $l_{ij}$ and $l_{ik}$ respectively) on node $i$ are represented as $f_{j,i}$ and $f_{k,i}$. The nodal coordinates of each node can be written as $(x_i,y_i,z_i)$, $(x_j,y_j,z_j)$ and $(x_k,y_k,z_k)$ for nodes $i$, $j$, and $k$ respectively. External forces will be considered later in Chapter 4, here they will be assumed to be 0. In this regard, the equilibrium equations discussed can be considered self-equilibrium equations.

Elements connecting nodes, like element $ij$ and $ik$, can either be cables or struts, where the forces act in axial compression or axial tension for cables or struts respectively at node $i$. Representing force $f_{j,i}$ in its 3 components as $f_{j,i,x}$, $f_{j,i,y}$, and $f_{j,i,z}$, with $f_{k,i,z}$ written in the same manner, we can write the equilibrium as:

$$\frac{\Delta x_{ij}}{l_{ij}} f_{j,i,x} + \frac{\Delta x_{ij}}{l_{tk}} f_{k,i,x} = 0 \quad (2.2 \text{ a})$$
Figure 2.4: Diagram of force balance at node i

\[
\frac{\Delta y_{ij}}{l_{ij}} f_{j,i,y} + \frac{\Delta y_{ik}}{l_{ik}} f_{k,i,y} = 0 \quad (2.2 \text{ b})
\]

\[
\frac{\Delta z_{ij}}{l_{ij}} f_{j,i,z} + \frac{\Delta z_{ik}}{l_{ik}} f_{k,i,z} = 0 \quad (2.2 \text{ c})
\]

Where the \( \Delta x_{ij} / l_{ik} \) terms are direction cosines and \( \Delta x_{ij} = x_i - x_j \). The length terms can be found as:

\[
l_{ij} = \sqrt{(\Delta x_{ij})^2 + (\Delta y_{ij})^2 + (\Delta z_{ij})^2} \quad (2.3 \text{ a})
\]

\[
l_{ik} = \sqrt{(\Delta x_{ik})^2 + (\Delta y_{ik})^2 + (\Delta z_{ik})^2} \quad (2.3 \text{ b})
\]

Inserting equations (2.3) into (2.2) would result in a set of equations which are no longer linear. Instead of sorting through nonlineairities, the equations can be re-written in terms of force densities (i.e. force per unit length), \( q \), such that equation (2.2) becomes:

\[
(\Delta x_{ij}) q_{j,i,x} + (\Delta x_{ik}) q_{k,i,x} = 0 \quad (2.4 \text{ a})
\]
\[
(\Delta y_{ij})q_{ji,y} + (\Delta y_{ik})q_{ki,y} = 0 \quad (2.4b)
\]
\[
(\Delta z_{ij})q_{ji,z} + (\Delta z_{ik})q_{ki,z} = 0 \quad (2.4c)
\]

With:

\[
q_{m,x} = \frac{f_{m,x}}{l_m} \quad (2.5)
\]

The subscript \( m \) indicates any member \( m \). Equation (2.5) can be written for \( y \) and \( z \) coordinates in the same way.

In this way the net force equations have been linearized. Linearizing the equations in this way is used in force-density methods [24-26]. Note that these equations are simply a force balance with the assumption that there are no external forces acting on the system. In a case with external forces, the 0’s on the right-hand side of Equations (2.2) and (2.4) would be substituted with the respective net components of the external forces in the \( x-y-z \) directions respectively.

With the potentially large number of nodes and members which must be considered, matrix representation and analysis is expedient. Equation (2.4) can be re-written to readily allow for matrix representation as:

\[
(q_{ji,x} + q_{ki,x})x_i - (q_{ji,x})x_j - (q_{ki,x})x_k = 0
\]
\[
(q_{ji,y} + q_{ki,y})y_i - (q_{ji,y})y_j - (q_{ki,y})y_k = 0
\]
\[
(q_{ji,z} + q_{ki,z})z_i - (q_{ji,z})z_j - (q_{ki,z})z_k = 0
\]

To further simplify the matrix representation, let \( C \) represent the connectivity matrix. The connectivity matrix describes the member connections in the tensegrity. The matrix is \( m \) by \( n \), where there are \( m \) members and \( n \) nodes. Thus, each row represents a member and each column a node. Since members are defined by their connection between 2 nodes, a row representing one
member will be +1 in column \( i \), -1 in column \( j \), and 0 elsewhere, denoting its connection between nodes \( i \) and \( j \).

With this connectivity matrix, the equilibrium equations for the entire tensegrity are compactly written as:

\[
\begin{align*}
C^T QCx &= 0 \quad (2.7 \text{ a}) \\
C^T QCy &= 0 \quad (2.7 \text{ b}) \\
C^T QCz &= 0 \quad (2.7 \text{ c})
\end{align*}
\]

Where \( Q \) is a diagonal square matrix of the force densities (\( Q=\text{diag}(q) \)). The vectors \( x, y, z \) and \( q \) are of length \( m \) (for \( m \) members) and represent the nodal coordinates \((x, y, z)\) and force densities \((q)\) for each member of the tensegrity. Equation (2.7) can be re-written for greater ease of implementation depending on how the structure’s self-equilibrium will be imposed. With the introduction of a force-density matrix, \( D \), Equation (2.6) may be written as:

\[
D[x \ y \ z] = [0 \ 0 \ 0] \quad (2.8)
\]

Where the force density matrix is defined as \( D=C^T QC \). Similarly, an equilibrium matrix, \( A \), is substituted in Equation (2.7), yielding:

\[
Aq = 0 \quad (2.9)
\]

Where the equilibrium matrix is defined as:

\[
A = \begin{bmatrix}
C^T \text{diag}(Cx) \\
C^T \text{diag}(Cy) \\
C^T \text{diag}(Cz)
\end{bmatrix} \quad (2.10)
\]

Equations (2.8) and (2.9) are equivalent versions of the same self-equilibrium equations. Equation (2.8) is written in terms of nodal coordinates, while Equation (2.9) is written in terms of the force densities [19].
In addition to satisfying the self-equilibrium equations with these matrix formulations, certain rank conditions must be met to ensure the structure generated is of the correct dimension, and that a force density vector can be found. To be of the correct dimension, the following must be satisfied:

\[ n - r_D \geq d + 1 \]  \hspace{1cm} (2.11)

Where \( n \) is the number of nodes, \( r_D \) is the rank of the force-density matrix, \( D \), and \( d \) is the dimension of the problem (in future chapters, \( d \) is always taken as 3). This condition ensures that \( d \) (or more) particular solutions can be obtained. The second condition can be written as:

\[ m - r_A \geq 1 \]  \hspace{1cm} (2.12)

Where \( m \) is the number of members and \( r_A \) is the rank of the equilibrium matrix, \( A \). This condition ensures the tensegrity is pre-stressed in an equilibrated state, and that at least one force-density vector exists, reflecting at least one non-trivial solution to the self-equilibrium equations (Equation (2.9)) [19, 24, 31].

These equations, in combination with the rank conditions, are used to ensure a given tensegrity structure is indeed in a self-equilibrated state, and can be found using the preceding equations. Different methods will prefer different forms of the equation, however, these self-equilibrium equations and conditions form the foundation of tensegrity structure design.

2.2.1 Tensegrity design methods: form-finding and force-finding

As with most structures, typically a tensegrity design problem is meant to fill a specific volume or length while meeting required structural specifications, meaning there are given dimensions which the structure is allowed to fill while maintaining specified properties (i.e.
stiffness, mass etc.). Since a tensegrity structure is composed of struts and cables connecting nodes, the nodes define the space which the structure takes up, but there are numerous ways in which struts and cables may connect those nodes for any given volume—especially as the number of nodes increases. For a single volume of space, a tensegrity can have any number of nodes, with a wide variety of different connectivity of the members for a given number of nodes; both the number of nodes as well as the specific connectivity of members influence the structural properties of the structure. The tensegrity design problem can quickly become highly complicated—increasing the nodes increases the number of members needed to maintain equilibrium, but also influences the different types of possible connectivity and potentially structural characteristics.

A vast number of studies have been conducted looking at different ways of optimizing tensegrity structures. Pietroni et al. explore tensegrity designs optimized to fit a specific shape-volume [28]. Masic et al. look at the optimization of tensegrity structures in terms of mass-to-stiffness ratio [29]. Ashwear et al. explore tensegrity designs optimized for high stiffness with a high, separated lowest natural frequency [30]. Since tensegrity configurations can change dramatically for any given volume or shape, the optimization of their configurations for varying design parameters is a subject constantly explored in research. These studies seek to answer the questions: where should the nodes be located in the space, how many should there be, and what is the proper member connectivity? Though the details of the methods used to answer these questions vary, the solutions general fall under one of two methods: form-finding or force-finding.

Form-finding methods seek solutions to the self-equilibrium equations assuming the connectivity is known—i.e. how the members connect to nodes is known, but the nodal locations are not specified. The solution to form-finding methods are nodal coordinates (yielding a geometric shape satisfying the specified connectivity) and a corresponding force-density vector.
which satisfies the self-equilibrium equations. There is much literature exploring different form-finding methods, including analytic methods, dynamic relaxation methods, force-density methods and more. McCray and Crane look at the different shapes of a 4-plex tensegrity with springs as some of the cable members, using preliminary member lengths to find the nearest equilibrated form [56]. Tibert and Pellegrino offer a review of multiple different form finding methods [32], and Pietroni et al., Masic et al. and Ashwear et al. also provide examples of uses of form-finding methods [28-30].

Alternatively, force-finding methods assume the connectivity as well as the nodal coordinates are known, and solves for the force densities (if any) which satisfy the equilibrium equations. As with form-finding, there are many different approaches to using the force-finding approach. For example, Xu and Luo use a simulated annealing algorithm as a numerical optimization method of force-finding [33], and Tran and Lee offer an alternative force-finding numerical approach used in application to a tensegrity grid [26]. Force-finding methods are derivative of form-finding methods, but with additional input information to guide the methodology (i.e. including nodal coordinates along with connectivity). In particular, force-finding methods often have a lot of overlap with force density form-finding methodologies.

Force density methods make use of the force density vector, where the force density for a given member is defined in Equation (2.5). Harichandran and Sreevalli compare various force density methods and discuss respective advantages and disadvantages of these methods [34]. This dissertation focuses on a numerical approach to describe the achievable shapes of a tensegrity with prescribed connectivity and lengths of select members. This approach shares theory with both form-finding and force-finding methods, though most directly implementing aspects of the force density method. Typically form-finding and force-finding methods alike consider a problem
solved when a single tensegrity shape is found satisfying the method’s conditions. The focus of this dissertation, however, is to understand the variety of shapes a single type of tensegrity may achieve by changing some cable lengths. In this way, the input parameters vary from those described in the aforementioned methods. Despite the differences of the problem definition, the methods described in subsequent chapters pulls greatly in theory from the form finding and force finding methods described in this section.
Chapter 3

Geometric Shape Finding

The geometry for a given tensegrity structure may be as simple or as complex as the design constraints allow, resulting in a many geometric parameters to explore for any given problem. Indeed, even just considering cylindrical tensegrities, multiple connectivities may be developed for certain values of n-plex, and the value of n-plex itself is a design consideration as well. Even further, for a given n-plex and given connectivity, the member lengths are also design variables; an aspect further complicated when members are not all held at fixed lengths. The problem of form-finding for a given application has been—and continues to be—a large area of research in tensegrity (e.g. [28-32]).

There are many different approaches to solve for the possible shapes a tensegrity may assume. Most existing methods seek shapes satisfying certain input requirements, and consider the problem solved once a single shape is found. These methods may be used to find more than one shape in certain cases ([37-40]), however, there has been very little work done exploring the limits of the range of shapes a single tensegrity configuration can achieve. Some work has been done exploring the range of shapes a tensegrity with varying cable lengths might achieve for a particular design, such as Herder and Guest’s spring-elongation based triplex [41]. It is desirable, however, to have a method of finding this shape range that might be extended to a variety of geometries and physical structure parameters (such as structure height and member diameters).

To analyze the entire range of physically achievable shapes, both the geometric constraints of the configuration, as well as the physical constraints of the tensegrity structure are considered.
This chapter focuses first on the geometry of the structure, detailing a novel method for understanding all geometrically allowable shapes a given tensegrity may achieve. The geometrically viable set of solutions is then refined to only consider solutions which meet the physical restrictions of a tensegrity, i.e. requiring cables to be in tension and struts in compression. The remaining data points thus make up a solution set of all the possible shapes a particular triplex configuration can achieve.

3.1 A single-bay cylindrical triplex

The cylindrical triplex offers the simplest possible geometry for a cylindrical tensegrity, and was chosen as the focus of this thesis for that reason. More specifically, the method developed in this chapter was applied to a single bay, 12-member triplex. Figure 3.1 depicts this type of tensegrity. The strut and face cables are fixed in length, thus only allowing the supporting cables to change their lengths, with no restrictions placed on the lengths of these 3 supporting cables. By fixing the lengths of the 6 face cables and 3 struts, the range of geometrically possible shapes is limited greatly.

![Figure 3.1: A cylindrical single bay triplex. The struts and top and bottom face cables are fixed in length while the supporting cables are allowed to be any length.](image)
Cylindrical tensegrities are inherently geometric in nature, thus making geometry-based methods an intuitive approach for determining possible configurations. By fixing the lengths of particular members of the structure, the geometric solution space of the given tensegrity is also restricted. As aforementioned, the struts, as well as the top and bottom face cables are all kept fixed in length. With these restrictions, the bottom face cables can be held fixed in space, such that all movements of top face are relative to the location of the bottom face of the triplex. By fixing the bottom face cables in space, the bottom 3 nodes of the triplex are also fixed in space, defining the first set of end nodes for each of the struts. The geometric problem thus becomes one solving for the second end node location of all the struts—i.e. the location of the remaining 3 nodes.

When considering possible configurations, the end of the strut attached to the base node is fixed in space, however, the strut is free to rotate in any direction. The location of the end node of a given strut is thus constrained only by the length of the strut itself (and the base node location). The possible positions in space of the strut’s end node can achieve is shown as the surface of a sphere whose radius is the length of the strut and center is the base node (Fig. 3.2 (a): in red is the positioned first strut, with the green sphere indicating the surface on which the end node must be located). Such a surface can be found for all 3 struts (Fig 3.2 (a) shows the other 2 strut spheres in light blue and light yellow), such that the end node of any strut must be located on the surface of its respective sphere surface.

Once the first strut position is fixed, the spheres defined by the other 2 struts are considered. The remaining struts can be positioned with their end nodes anywhere on their respective sphere surfaces; however, the lengths of the top face cables are fixed. This means that the ends of the
remaining struts must also lie within reach of the cables from the end node of the first strut. For simplicity, all the cables which are fixed in length are assumed to have the same length. With this assumption, a new sphere can be defined whose center is the end node of the first strut, and whose radius is defined by the length of the cable (Fig. 3.2 (b) shows the cable’s sphere in dark blue). This third sphere will intersect in either a point or a circle with each of the 2 spheres defined by the other strut lengths (in some cases there might be no intersections, indicating no geometric solutions for that strut position).

In order to fix the second strut location, consider the intersection of the sphere defined by the second strut, and the sphere defined by the cable about the first strut’s end node (light green sphere and dark blue sphere respectively in Fig. 3.2 (b)). These two spheres typically intersect in a circle (black in Fig. 3.2 (b)). The second strut end node may be located anywhere along this circle. The end node of this second strut will be iteratively moved around this circle for the currently ‘fixed’ location of the first strut.

For a given position of the end node of the second strut, the location of the final strut can be determined. This final strut must be a cable length away from both of the end nodes of the 2 now-fixed struts, as well as a strut’s length away from the remaining base node (bright green, dark blue and pale blue spheres respectively in Fig. 3.2 (c)). This results in 3 spheres which typically intersect with each other as circles (highlighted in Fig. 3.2 (c) in black and shown alone in Fig. 3.2 (d) in black, blue and green). In most cases, these 3 circles will have 2 points of intersection (the black points in Fig. 3.2 (d)). These intersection points are the possible end node locations of the final strut. Figure 3.2 (e) and (f) show the two geometrically possible configurations from these points of intersection.
Figure 3.2 outlines the methodology used to find two geometrically viable configurations. The total solution space, however, contains far more shapes than this. Figure 3.2 (b) highlights a
circle of potential end node locations for the second strut. To generate sub figures (c)-(f) in Fig. 3.2, one possible end node location of the second strut was chosen and held fixed in space. However, the end node of the strut could actually geometrically reach any location on the highlighted circle. Thus, to expand on the geometrically viable solution space, the end node of the second strut must move through locations on the circle.

Figure 3.3 shows some sample results from such iterations. The strut shown in pink is the first strut held fixed throughout iterations. The second strut has its end-node moving around the highlighted circle, which is shown in light blue in Fig. 3.3. In each case shown, 2 solutions were found for the 3rd strut, only one of which is displayed in Fig. 3.3. It is possible for combinations of first and second strut end node locations to result in either only 1 geometrically viable solution, or no geometrically viable solutions, but such results are not shown here. Figure 3.3 left to right shows second-strut end-node locations moving counter-clockwise around the circle.

**Figure 3.3:** Some geometrically viable shapes, found by iterating the second strut around the circle of intersection of the second strut’s end-node sphere and the first strut’s end node top-face cable sphere. Left to right is moving the second strut counter-clockwise around the sphere.
Procedure:

- **Step 0**: Define the fixed member lengths of the triplex (i.e. the length of the face cables and the struts) and fix the bottom face cables (and nodes) in space.
- **Step 1**: Define a sphere whose center is one node on the bottom face, and whose radius is equal to the length of the strut.
- **Step 2**: Choose one new point on the surface of this sphere, and define this point as the end node of the first strut.
  - If there are no new points, the entire solution set has been found.
- **Step 3**: Find the intersection (if any) of two new spheres. The first sphere is defined by one node of the bottom face (not used in step 2) and has radius equal to the strut length. The second sphere is defined with the center at the top face node of the first strut, and has radius equal to the length of the face cables.
- **Step 4**: Check the intersection type
  1. The intersection does not exist (this indicates the top face location of the first strut’s end node results in no geometrically viable shapes). *Return to step 2.*
  2. There is one intersection (the top face location of the first strut’s end node results in only 1 geometrically viable location for the second strut)
  3. The intersection is a circle
- **Step 5**: Fix the end-node of the top face of the second strut.
  1. If step 4 was case 2, this will be at the only intersection point found.
  2. If step 4 was case 3, choses an unused point on the intersection circle.
    - If there are no unused points, *return to step 2.*
- **Step 6**: Check for the intersection of three new spheres. The first sphere has the center at the only unused end node on the bottom face of the triplex, and has radius equal to the strut length. The other two spheres have centers at each of the defined top face end-nodes of the first 2 struts, and radii equal to the top face cable lengths. There are 2 categories of possible intersection:
  1. No intersection (indicates no geometrically viable shapes for where the first 2 struts are currently located). *Return to step 2.*
  2. One or two points of intersection: the points of intersection are the end-node location on the top face of the triplex for the final strut. Add resulting shape(s) to the solution set.
    - If step 4 resulted in case 2: *return to step 3.*
    - If step 4 resulted in case 3: move to a new point on the intersection circle from step 5. *Return to step 5.*

This procedure describes a computational process to find “all” geometrically viable configurations for this type of triplex. This method was only applied for the 3-strut triplex (and extended to multi-bay triplex towers in later chapters). Keeping the struts and face cables fixed, this method could readily be applied to cylindrical tensegrity structures of a higher n-plex. The addition of nodes,
however, increases the computational time required and was not explored. Furthermore, this method could also be extended to consider additional members having variable lengths. The addition of a single member having variable length, however, would result in one of the aforementioned spheres instead having a range of radii, thus resulting in a volume of space the related node may occupy, rather than a surface. This would greatly increase the computational time required, and though this method may be extended to include such cases, it is likely not the most efficient way to do so. Due to the time requirements, this was not explored.

Note that thus far only the geometric constraints have been considered. A more practical solution set must only include those shapes which also satisfy the equations of self-equilibrium, as explained in the following section. Furthermore, a more practical solution set must consider finite member diameters, as well as the ability of a structure to reach a given location in the solution set without member interference. A method of collision checking and the discussion of how this restricts the solution set is discussed in chapter 4.

3.1.2 Refining the solution set to only contain equilibrated shapes

Chapter 2 discussed the general formulation of the equations of self-equilibrium. Equation (2.9) in particular (shown again in Equation (3.1)) must be calculated for each of the geometrically viable shapes found. Since the connectivity is prescribed, the solution set found geometrically yields nodal coordinates. For each shape, these coordinates are included in the formation of the $A$ matrix (shown again in Equation (3.2)). For each geometrically viable shape found, a corresponding $A$ matrix is generated. Then, the equation of self-equilibrium (Equation (3.1)) is solved for the force density vector ($q$). This vector is used to check the forces in each of the
members of the triplex. As discussed in chapter 2, the only shapes of interest are those in which
the struts are in compression and cables are in tension. Thus, the solution set is refined to keep
only those shapes which are geometrically viable, and which maintain equilibrated members with
the proper member loads.

\[ Aq = 0 \] (3.1a)

\[ A = \begin{bmatrix} C^T \text{diag}(Cx) \\ C^T \text{diag}(Cy) \\ C^T \text{diag}(Cz) \end{bmatrix} \] (3.1b)

Retaining only those shapes which also satisfy equilibrium greatly reduces the size of the
solution set. Figure 3.4 shows one such solution set. Each dot in the figure corresponds to the
center of the top face for a unique shape satisfying both geometry and equilibrium. The different
colors are included purely for clarity. Figure 3.4 represents a solution surface of possible locations
of the center of the top face. The shape of this surface is unique to the prescribed lengths of the
struts and fixed cables. Non-dimensionally, this can be represented using the ratio of the parallel
slant height of the structure to the circumscribing radius \((h/r)\). The circumscribing radius is the
radius of the circle on which all nodes of the top and bottom faces lie. In the case of Fig. 3.4, this
\(h/r\) is 1, a relatively small aspect ratio.

Viewing the results for a solution surface, the centers of the top-most face of the tensegrity
provides quick insight into the general range of achievable shapes. This solution surface, however,
cannot be used on its own to generate the precise, equilibrated shape of the triplex for any given
shape. The center of the top face fixes the “translation” of the top face, but does not provide any
information about the top face orientation. Geometrically, more than one given orientation of the
top face may be possible for a given center, but only one of these orientations actually satisfies the
equations of equilibrium. Furthermore, the orientation of the top face is critical to applications
which need to know the direction the structure is “pointing” (e.g. if the structure is being use as a support for a reflector which needs to point in a certain direction). In this sense, representing the solution set in a way that includes location and orientation information as concisely as possible is beneficial.

![Figure 3.4: An isometric (left) and top-down (right) view of the solution set of geometrically viable and equilibrated shapes for a triplex with h/r ratio of 1. Each point represents the center of the top face for a unique shape (color is purely for visual clarity).](image)

### 3.1.3 Characterizing the solutions

The solution set found in the previous section represents the results of a computational approach to the problem. Rather than repeating procedure every time the results are needed, it is useful to represent the solution set in a way that any user can rapidly access and easily understand. There are many different ways to represent the solutions. On one hand, the solution set could be expressed in terms of the structure itself—perhaps describing the relative changes in length of the members themselves, member forces, or relative changes in other physical parameters. For example, Herder and Guest [41] investigate a structural design in which the cable lengths are adjusted through springs. In their study, the range of motion is explored in terms of spring elongation relative to the fixed bar lengths, clearance between bars, and angles between cables.
This thesis details an experimental realization of the focus triplex system is in Chapter 6, however, describing the solution set independently of the specifics of the structural design, such that it could be applied to any tensegrity of the same geometry as long as the lengths of the same members are held constant, makes the methodology more universally applicable.

Alternatively, the problem can be approached by treating the top-face of the triplex as a rigid body with 6 degrees of freedom, 3 corresponding to location and 3 corresponding to orientation. Its location can be described through any convenient coordinate system (Cartesian, cylindrical, etc.). Similarly, orientation can be described in a number of equivalent ways, be it through some form of rotation matrices/angles, an orientation vector, quaternions, etc. Any combination of these methods can provide an accurate description of the results. The choice of solution set representation then becomes a question of efficiency—how can the most information about the solution set be described most concisely without loss of accuracy?

The solution set described in the previous section was shown exclusively in terms of the location of the center of the top-face. This center fixes the location of the top-face, but not its orientation. To get a holistic picture, additional information is required. Furthermore, a relationship between the Cartesian x-y-z coordinates of the centers is not apparent (see Fig. 3.5), nor is a graphical representation of them in 3D space particularly useful without a large dataset giving exact coordinates. A few different methods of describing the results were considered. Ultimately, describing results in terms of cylindrical coordinates (azimuth and elevation) was chosen for its ability to provide a holistic view of the possible shapes and orientations.

The Cartesian coordinate results for the centers of the top-face from the previous section can easily be converted to spherical coordinates (i.e. into azimuth, elevation and slant height) through the following equations:
Figure 3.5: Three different projections in Cartesian coordinates of the center points from Fig. 3.4. Left (a) is top down (x-y plane), center (b) is the x-z plane and right (c) is the y-z plane.

\[ \alpha = \tan^{-1}\left( \frac{y}{x} \right) \]  

\[ \beta = \tan^{-1}\left( \frac{z}{\sqrt{x^2 + y^2}} \right) \]  

\[ r_h = \sqrt{x^2 + y^2 + z^2} \]

Where \( \alpha \) is azimuth, \( \beta \) is elevation and \( r_h \) is slant height. Figure 3.6 shows the directions in which the angles are measured. Transforming the entire data-set into azimuth and elevation of the centerline, the results are displayed in Fig. 3.7.

With the addition of radius information for each point, Fig. 3.7 is a spherical representation of the same center points plotted in Cartesian coordinates in Fig. 3.4. The azimuth and elevation of the centerline still does not provide any information about the orientation of the top face. To begin to address this, consider the plane formed by the 3 nodes of the top-face. The normal of this
plane can also be represented by the azimuth and elevation of the normal vector relative to the inertial frame. The corresponding results are shown in Figure 3.8.

**Figure 3.6:** One shape of an equilibrated triplex. Left (a) shows the line connecting the center of the top-face to the center of the bottom-face (dashed black, labeled centerline) and normal to the top face (pink arrow). Right (b) shows just the center line, with azimuth (α) and elevation (β) labeled. Note, counterclockwise from the positive x-axis is positive for azimuth, and counter clockwise from the x-y plane is positive for the elevation.

**Figure 3.7:** The azimuth and elevation of the centerline for the solution set (previously shown in x-y-z coordinates as the center points of the top-faces in Fig. 3.4)
Figure 3.8: Azimuth and Elevation of the normal vector of the top-face. Note that the azimuth and elevation are relative to the inertial frame at the origin, not the top-face itself.

With these four angles (the azimuth and elevation of the centerline and the normal of the top-face), it is possible to regenerate the complete solution set found in Fig. 3.4. This process is discussed in greater detail in the following section; however, note that doing so requires a bridge between Figures 3.7 and 3.8. A relationship between the centerline direction and the normal of the top-face is necessary in order to link the azimuth and elevations of the two. Figure 3.9 is generated for this purpose, plotting the elevation of the centerline against the elevation of the normal to the top-face for each shape in the solution set. This plot can be used to generate all the necessary information on a particular shape to find all nodal coordinates. For a particular centerline azimuth, Fig. 3.7 can be used to pick a viable centerline elevation. With the azimuth-elevation pair of the centerline, Fig. 3.9 can then be used to look up the range of viable elevations of the normal, given that elevation of the centerline. Once an elevation of the top-face normal is chosen from Fig. 3.9, Fig. 3.8 can be used to pick a viable azimuth of the top-face normal for the chosen top-face
elevation. Figures 3.7-3.9 all show the same dataset of viable shapes in terms of the center of the top face, where each point corresponds to one particular shape. The information provided using all 3 figures can be used to generate any particular shape within the solution set.

![Figure 3.9: Elevation of the centerline plotted against elevation of the top-face normal](image)

The results thus far have been expressed in terms of 4 variables; azimuth and elevation of the centerline, and azimuth and elevation of the top-face normal. Figures 3.7-3.9 could be used as look-up graphs to generate the solution set; however, it can be more useful to describe the values of the centerline and top-face normal azimuth and elevations as a range of values. The maximum elevation for both the normal and the centerline corresponds to the parallel shape, where the top-face normal is aligned with the z-axis. Thus, π/2 radians is the always the maximum elevation. The minimum elevation, however, is more difficult to write analytically.

Consider Fig. 3.7 for example. Examining only the lower-bound points of the elevation for any given azimuth, it appears that the minimum bound of elevation might be fit with a repeating concave parabola. Selecting the minimum points of one repeated section, the built-in MATLAB curve fitting tools were used to fit the minimum values of elevation to a few different types of
curves, as shown in Fig. 3.10 and Table 3.1. The different curves provide varying degrees of accuracy, as shown by the root-mean-square errors in Table 3.1.

Note using these fits as a lower-bound misses some points found from the solution set, and implies the existence of some points which were not found in the solution set. The implication of including erroneous points could have particularly negative consequences in terms of real-world applications. Shapes outside of the found data set are not in equilibrium. Thus, if this solution set is used as a guide for the extremes of shapes the triplex can take, moving the triplex to shapes with this wrong minimum elevation could result in failure. The curves chosen to fit these minima could be improved to allow for greater exclusion of unallowable solutions with the benefit of all remaining solutions being only actual solutions. This would yield a curve, repeating every 120°, which describes (with varying degrees of accuracy depending on the curve) the minimum allowable elevation as a function of azimuth.

Figure 3.10: Select equations curve-fitted to the minimum values of centerline elevation. In black is the original data set, blue are the selected minimum data to which the curves are fit.
Table 3.1: Forms of the equations from Fig. 3.10 which were fit to the minimum centerline elevations. Constants A, B, C and D are solved for using built-in MATLAB curve fitting tools, α is azimuth and β is elevation. Note the super ellipse equation was solved for β, its typical formula is of the form: $|\beta/C|^A + |\alpha/B|^A = 1$.

<table>
<thead>
<tr>
<th>Equation</th>
<th>RMSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parabolic</td>
<td>$\beta = A(\alpha - C)^2 + D$</td>
</tr>
<tr>
<td>Sine</td>
<td>$\beta = Asin(B\alpha - C) + D$</td>
</tr>
<tr>
<td>Super ellipse</td>
<td>$\beta = C \left(1 - \left</td>
</tr>
</tbody>
</table>

Describing the minimum centerline elevation as a function of azimuth offers a way to describe range of allowable elevations. The exact shape of these minimum elevations, however, are specific to the configuration parameters (i.e. $h/r$). Thus, varying parameters of the triplex configuration also changes the shape of the minimum elevation curve. Figure 3.11, for example, compares the azimuth-elevation plot of a triplex of height 2 (i.e. a height-to-circumscribing radius ratio of $h/r=2$, whereas previous plots have shown $h/r=1$) to the same plot for the triplex of $h/r=2$.

For a design analysis of these triplex structures, it is desirable to have way of describing the minimum centerline elevation which can consistently be applied regardless of the $h/r$ ratio. With this outcome in mind, it is interesting to instead look at the minimum elevation, that can be reached at any azimuth. This elevation will be referred to as the critical elevation, $\beta_c$. In this way, the solution set (for the centerline) can be given as: $-\pi \leq \alpha \leq \pi$ and $\beta_c \leq \beta \leq \pi/2$.

Restricting the solution set to only points whose elevation is greater than the critical elevation is not an unreasonable restriction considering its physical implications. A smaller elevation of the centerline corresponds to the struts angling closer to the x-y plane, as necessitated by the tensegrity geometry. As of yet, the members in the structure have not been presumed to have any finite-diameter. A more thorough discussion of finite-diameter members and their implications is pursued in Chapter 4.
Figure 3.11: Plot of the azimuth and elevation of the solution set for triplexes of h/r=2 (h/d=1, dark blue) and h/r=1 (h/d=1/2, light blue).

As a cursory investigation, however, consider Fig. 3.12. The bottom-face cables are all within the x-y plane, corresponding to member 1 in Fig. 3.12. In this case, take member 2 to be a strut, where θ denotes the angle between the two members. In generating the data set, members are assumed to have no diameter, and can be considered as only the dashed center-lines in Fig. 3.12. If the member has a diameter, however, the members actually occupy a cylindrical volume, shown by the solid rectangles. The design of the nodes can be developed such that member interference cannot happen within the node itself. Even still, it is clear from Fig. 3.12 that for small values of θ, members of finite diameters will result in interference, shown as the red region of overlap. In this sense, eliminating shapes of smaller centerline elevations (corresponding to smaller θ’s) will largely eliminate shapes where this region of overlap would likely pose a problem for physical designs.
Using the critical elevation greatly reduces the complexity of describing the solution set. With this critical elevation, any elevation of equal or larger value is viable for any azimuth of the centerline. The results for the azimuth and elevation of the top-face normal can also be modified to reflect this reduced solution set. The results of such a reduction are shown in Fig. 3.13. The relationship between elevation of the centerline and elevation of the top-face normals is also modified to reflect only the centerlines above the critical elevation.

**Figure 3.12**: A diagram of 2 members of finite diameters. For small values of \( \theta \), the region of overlap becomes an issue, resulting in member interference for physical structures.

**Figure 3.13**: Azimuth and elevation of top-face normal for shapes whose centerline elevation is above \( \beta_c \)
The solution set described in terms of critical elevation still requires a look-up chart for the azimuth and elevation of the top-face normal. The plot comparing elevation of the centerline to elevation of the top-face normal is modified for shapes above the critical elevation in Fig. 3.14.

Figure 3.14: Elevation of centerline and top-face normals for shapes whose centerline elevation are above the critical elevation

Figure 3.15 shows a cursory approach as to how the information in figure 3.14 might be provided. In blue is the same data shown in Fig. 3.14, and in black are points selected for fitting the upper and lower bounds of the data. Selection of data points was done automatically with code so that it may quickly be applied to any data-set (this is explored in greater detail in Chapter 5). The sparsity in data for shapes remaining above the critical elevation with \( h/r = l \) results in some discrepancies in selection of data points which represent (and are fitted as) the upper and lower bounds of the data. However, the selected points represent a reasonable approximation of the boundaries, as can be seen in Fig. 3.15 (these points are highlighted in black).

A linear fit of form \( y = Ax + B \) and a second order polynomial fit of form \( y = Ax^2 + Bx + C \) were fit to each of the boundaries and are plotted in Fig. 3.15. These fits were found using built-in MATLAB curve fitting tools. For the lower-bound and upper-bound fits, the linear fit resulted in
a RMS error of 0.01628 and 0.0139 respectively, the polynomial fit resulted in RMS error of 0.005814 and 0.008018 respectively. A linear fit offers greater simplicity, however, a second order polynomial fit offers greater accuracy.

Figure 3.15: Elevation of Centerline vs. Elevation of top-face normal for shapes above the critical elevation. In blue is the entirety of said dataset, in black are the points selected for the upper bound and lower bound fitting. Fit 1 refers to a fit of form $y=Ax+B$ and fit 2 refers to a fit of form $y=Ax^2+Bx+C$

Thus far the set of viable shapes has been described in terms of a critical elevation of the centerline (a value dependent on $h/r$), two equations describing the upper and lower bounds of the top-face normal elevation as a function of the centerline elevation, and a plot of azimuth vs. elevation of the top-face normal. Describing the top-face normal azimuth-elevation plot results in similar issues to the centerline azimuth-elevation in terms of fitting the lower bound. However, in the case of fitting the lower-bound of the centerline plot, errors in fitting could result in inclusion of shapes not in equilibrium. This is not necessarily the case for the top-face normal plot.

Figure 3.16 shows one repeating section, with data from the entire solution set in gray, and data for points whose centerline elevations are above the critical elevation in black. This demonstrates how straying (within a reasonable distance) outside of the solution space for selected
shapes (in black), will not typically result in unallowable shapes. In this sense, errors in fitting will likely cause discrepancies between prescribed and actual shape, but should not typically result in shapes outside of equilibrium. Thus, it is reasonable to describe this solution set with an approximate fit for the lower bound. A second order polynomial and a sine function were fit to the data with built-in MATLAB curve fitting tools, as show in Fig. 3.16, with respective RMS error of 0.02144 and 0.02293.

![Fitting lower-bound of Azimuth Elevation of top-face normals](image)

**Figure 3.16**: Azimuth vs. Elevation of the top-face normal. In gray are equilibrated shapes below the critical elevation, in black are shapes above the critical elevation. Points highlighted in green were representative of the lower-bound of points above critical elevation and used for curve-fitting.

In this section, a method of describing the equilibrated solution set was explored with the aim of most efficiently providing information that is easily accessible without loss of accuracy. Describing the azimuth and elevation of both the centerline and the top-face normal provides sufficient information for any given shape to quickly determine detailed shape information.
Examining the range of values these variables can have, the azimuths of both the centerline and the top-face normal fall in the range of \(-\pi \leq \alpha \leq \pi\). In the case of the centerline, the elevation is independent of azimuth, for all elevations satisfying \(\beta_c \leq \beta\). In the case of elevation of the top-face normal, the range of elevation satisfies: \(\beta_{\text{min}}(\alpha) \leq \beta \leq \pi/2\), where \(\beta_{\text{min}} = A\alpha^2 + B\alpha + C\) (where the polynomial fit is chosen as it was found to be more accurate). Constants \(A\), \(B\) and \(C\) are found from specific curve fits and are unique to a given \(h/r\).

In this way, even with no knowledge of tensegrity structures, the solution space (in terms of azimuth and elevation of the centerline and top-face normal) could be generated with the described ranges of these 4 variables. Furthermore, this information is sufficient to define allowable shapes in complete detail.

The following section describes how, given these four variables, the nodal coordinates of the reconfigured structure can be obtained. With the nodal coordinates of any equilibrated shape, the solution set can be described in whatever format may be most useful for a given application.

### 3.2 Re-generating the solution set with minimal given information

Having defined the solution space in terms of the ranges of four angles, it is useful to understand how to go from these angles back to the nodal coordinates. With the nodal coordinates, the solution set may be represented in any way which is most useful.

The previous section discussed how the four angles of interest are inter-related, so this procedure will begin by assuming the four values of interest are known and viable. Thus, centerline azimuth and elevation \((\alpha_1 \text{ and } \beta_1)\) and the top-face normal azimuth and elevation \((\alpha_2 \text{ and } \beta_2)\) are known. Furthermore, the bottom-face node locations are known, as is the connectivity, and the fixed member lengths.
Starting with the azimuth and elevation of the centerline, $\alpha_1$ and $\beta_1$ effectively represent a vector, in the direction of which the top-face center must lie. This can be visualized as a line, described by the following equation:

$$
\begin{align*}
\{x, y, z\} &= \begin{pmatrix}
t \cos(\alpha_1) \cos(\beta_1) \\
t \sin(\alpha_1) \cos(\beta_1) \\
\text{t} \sin(\beta_1)
\end{pmatrix}
\end{align*}
$$

(3.4)

Where $t$ is a parameterized variable and can take on any value. Without considering member constraints, the center of the top-face could be located anywhere along this line.

The values of $\alpha_2$ and $\beta_2$ can be used to define the unit normal ($n$) of the top-face with the following equation:

$$
\begin{align*}
n &= \begin{pmatrix}
\cos(\alpha_2) \cos(\beta_2) \\
\sin(\alpha_2) \cos(\beta_2) \\
\sin(\beta_2)
\end{pmatrix}
\end{align*}
$$

(3.4)

This normal can be used to define a plane on which the top-face nodes will lie. Thus far, only the normal is found to define the plane, which fixes the orientation of the plane. Since this plane contains the top-face nodes, it also contains the center of the top-face, and thus the plane location can be fixed with the centerline. Figure 3.17 shows a visualization of the centerline orientation, and an arbitrary location of the top-face plane. The plane can “move” along the centerline and still satisfy the normal vector prescribed.
**Figure 3.17:** A visualization of the centerline and a particular location of the top-face normal plane.

**Figure 3.18:** A visualization of member length requirements for a particular top-face node. The normal plane containing all top-face nodes is shown in green (for some arbitrary location), the sphere whose surface must contain the top-face node connecting to node 1 as a strut is shown and yellow. The intersection of the strut sphere and the normal plane is shown in red. The intersection of the sphere containing all nodes of radius l (not shown) and the strut sphere (yellow) is in blue.
To fix the plane in place (and consequently solve for the remaining nodal positions), consider the member constraints. From a given bottom-face node, the corresponding top-face end node must be located on a possible plane a strut length away. Figure 3.18 shows the sphere defined by bottom-face node 1 in yellow and its intersection with a plane at some position $t$ along the centerline. There is one such sphere for all 3 of the bottom-face nodes. This position $t$, by definition of the centerline, corresponds to a potential center point of the top-face. Since each node of the top-face is equidistant from the center, the distance from the center to a given node (i.e. the circumscribing radius) is also fixed, and can be found using Equation (3.5).

$$r = \frac{Ca}{\sqrt{3}}$$  \hspace{1cm} (3.5)

Equation (3.5) describes $r$ as the distance from the top-face center to a top-face node, with $Ca$ being the cable length. The factor of $1/\sqrt{3}$ comes from the geometry of an equilateral triangle. The current center of the top face is defined by the $t$ value on the centerline, and thus can also be used as the center of a sphere of radius $r$. Based on the defined connectivity of the triplex, the end node for a given strut must be on the surface of the sphere of radius equal to the strut length and the sphere of radius $r$. Each of these spheres intersect with the plane containing the top nodes, and each other.

The intersection of a sphere and a plane, or sphere and a sphere (if there is any intersection), is either a point or a circle. Most cases are a circle, and the circles of intersection are shown in Fig. 3.18 (the red circle is the intersection of the strut sphere and the normal plane, the blue circle is the intersection of the sphere of radius $r$ (sphere not depicted) and the strut sphere). For some values of $t$, the resulting circles will intersect with each other. These intersections represent potential end node locations of the top-face. For each of the 3 bottom-face nodes, there can be 1, 2 or no intersections depending on the $t$ value. No intersections for any one of the nodes indicates
a wrong value of $t$. For the case of 1 or 2 intersections for all the nodes, all combinations of nodes need to be checked against the required member lengths. Thus, to find a shape for a particular set of $\alpha_1, \beta_1, \alpha_2$ and $\beta_2$, only the correct value of $t$ needs to be solved for based on the prescribed member lengths. This can be done using the following procedure.

**Procedure:**

- **Step 0:** Define the bottom-face by fixing the first 3 nodes in space based on prescribed cable lengths
- **Step 1:** Define 3 spheres, each with a center at one of the 3 bottom nodes and all of radius equal to the strut length
- **Step 2:** Define the normal of the plane containing the top-face nodes based on $\alpha_2$, $\beta_2$, and Equation (3.5)
- **Step 3:** Define the centerline based on the current values of $\alpha_1, \beta_1$ and Equation (3.4), in terms of $t$
- **Step 4:** Define the start value of $t$ by solving Equation (3.4) where $x=y=z=0$ (the center of the bottom-face)
- **Step 5:** Choose a sufficiently small increment $dt$, and add this to the start value of $t$
- **Step 6:** Define the current potential center of the top-face as the point corresponding to the current value of $t$
- **Step 7:** Fix the normal plane containing the top-face nodes in space with the potential center. Use this same point to define a sphere with this point as the center and of radius $r$
- **Step 8:** Check for intersections between the normal plane and the 3 spheres of radius equal to the strut length
  - If there is no intersection for any one of the 3 spheres: *Return to step 5*
- **Step 9:** Check for the intersection between the sphere of radius $r$ and the 3 spheres of radius equal to the strut length
  - If there is no intersection for any one of the 3 spheres: *Return to step 5*
- **Step 10:** Check for intersections between the points/circles of intersection found in step 8 and the points/circles of intersection found in step 9 corresponding to the same top-face nodes (i.e. only check the intersection found for node 1 in step 8 to the intersection found for node 1 in step 9)
  - If there is no intersection for any of the 3 potential mutual intersections: *Return to step 5*
  - If there is at least 1 intersection for each step 8 and step 9 strut sphere (i.e. a point or a circle of points for each of the 3 bottom nodes in step 8 and 9), then the mutual intersections (i.e. points from step 8 intersecting points from step 9, points from step 8 intersecting a point on the circle from step 9 [or vice versa], or an intersection of the circle from step 8 and the circle from step 9) are potential locations of the top-face nodes.
• **Step 11**: Check the distances between all possible combinations of the potential top-face nodes against each other and ensure they are mutually a cable length away from each other, as well as a strut length away from the bottom-face node.
  - If there is only 1 potential point for each top-face node, then these are the final top-face nodal coordinates
  - If there is more than 1 potential point for any of the top-face nodes, then checking the distances for every combination of top-face nodes will reveal which point is the correct point for each as the final top-face nodal coordinate

This procedure iteratively solves for a value of $t$ that yields the correct nodal coordinates for a given azimuth elevation of the centerline and top-face normal. To regenerate the entire solution set, the four angles may be iterated through their respective ranges, using the above procedure to generate the nodal coordinates for each combination of the 4 angles. This particular procedure does not require any checking of equilibrium. Describing the solution in terms of these 4 angles in effect over-constrains the problem, such that any coordinates found satisfying the member lengths by default of being within the prescribed angle ranges, also satisfies equilibrium. This method thus demonstrates the utility of using these four angles to describe any solution set found for the cylindrical triplex structures explored.

### 3.3 Extension to multi-bay triplex structures

The methods discussed thus far have all pertained to a single-bay triplex. This type of triplex can be stacked to create a multi-bay structure. There are multiple ways to stack the single bays, resulting in different structural properties for the overall structural tower. The traditional definition of a tensegrity aligns itself with a class-1 tower, where no struts are touching other struts. Class-1 booms offer the potential for greater packaging efficiency; however, they are not as stiff. Increasing the class (i.e. how many struts touch each other at a given node) increases the overall stiffness of the boom [19]. For a triplex, there are multiple ways to orient the stacking of the bays
when creating a boom. In the case of class-2, the top face of the first bay becomes the bottom face of the second bay, such that nodes 4-5-6 of the bottom bay are congruent with nodes 1-2-3 of the second bay, thus making the structure have 9 nodes overall (as opposed to the 12 nodes in a class-1 2-bay tower).

In all the single bay triplexes depicted thus far, the twist angle between node 1 and node 4 is 30° in the counter-clockwise direction. In effect, this means that the bottom face may be translated up by the slant height and rotated 30° counter-clockwise to get the top face. When a second bay is added, the bottom face of the second bay is shared with the top face of the first bay. The angle of twist for the second bay is still required to be 30° for the straight configuration, but this second bay can be rotated in the clockwise or the counterclockwise direction independent of the first bay. A twist angle that is 30° in the counterclockwise direction, like the first bay, leads to the top-most face bay having an offset of 60° with the bottom-most face. In this case, it takes 4 stacked bays, all rotating by 30° in the same (counterclockwise) direction, to return back to the bottom-most orientation. In this sense, every 4 bays in a multi-bay class-2 triplex boom will be a repeating geometry.

Alternatively, if the second bay is rotated 30° clockwise, in the opposite direction of the first bay, then the top-most face will return to the same orientation as the bottom-most bay, thus having every 2 bays be a repeating structure for a multi-bay boom. These two directions of stacking are shown in Fig. 3.19.

The idea of creating the faces of bays through translation and rotation of the bottom-most face can be extended to any number of faces, such that a boom of any number of bays may be created through simple translations and rotations of the bottom-most bay. In this way, all the methods thus far for a single bay may be compounded to show the results for any number of bays.
To extend the results, however, the direction of rotation is important to maintain, as a twist angle in the clockwise direction will have different results than a twist angle in the counterclockwise direction. That is, the results discussed in section 3.1 are specific to a single bay triplex with a $30^\circ$ counterclockwise twist. The results for the same triplex but with a clockwise twist would have different orientation. For ensuing discussion, all results were obtained considering only a counterclockwise twist, simply because these results were already found from section 3.1, and then can be directly applied to additional bays. The methods discussed could be extended to any combination of clockwise and counterclockwise twists, however, by regenerating the results for section 3.1 with a clockwise twist instead and applying the correct twist directions to the respective bays.
**Figure 3.19:** Two different ways of stacking 2 bays in class-2 configurations. Left (a & b) have both bays in the counterclockwise direction (a is isometric view and b is top-down). Right (c & d) is the first bay in counterclockwise direction and second bay in clockwise direction (c is isometric and d is top-down).

To demonstrate how the results can be extended for any number of bays, consider first a single 2-bay shape. To demonstrate generality, assume that the shapes of the first bay and second bay are independent of each other, as show in Fig. 3.20.

**Figure 3.20:** The individual bay shapes, and those bays stacked to make a class-2 tower of 2 bays.

To stack these two bays in order to create a 2-bay, class-2 triplex tower, a simple rotation and translation is applied to the second bay. To stack the bays in this way, the normal of the top face of bay 1 and the normal of the bottom face of bay 2 must be aligned, and the centers and nodal locations of the bay 1 top face and bay 2 bottom face must also be the same. Creating a rotation matrix that will align the normal of the bottom face of bay 2 to the normal of the top face of bay 1 will yield a matrix that can be applied to all nodes of bay 2 for proper orientation. This rotation matrix can be formed by finding the axis of rotation (Equation (3.6)) and angle of rotation (Equation (3.7)), and is described by Equation (3.8).

\[ \mathbf{a} = \frac{\mathbf{n}_2 \times \mathbf{n}_1}{||\mathbf{n}_2 \times \mathbf{n}_1||} \]  \hspace{1cm} (3.6)

In Equation (3.7), \( \mathbf{a} \) is the axis of rotation, \( \mathbf{n}_1 \) is the top-face normal of bay 1 and \( \mathbf{n}_2 \) is the bottom-face normal of bay 2.
\[ \phi = \cos^{-1}(\mathbf{n}_2 \cdot \mathbf{n}_1) \]  

(3.7)

In Equation (3.8) \( \phi \) is the angle of rotation about the axis of rotation, \( \mathbf{a} \), to get from \( \mathbf{n}_2 \) to \( \mathbf{n}_1 \). The rotation matrix, then, is defined as follows:

\[
\mathbf{R} = \begin{bmatrix}
    a_x^2(1 - c) + c & a_xa_y(1 - c) - a_zs & a_xa_z(1 - c) + a_ys \\
    a_xa_y(1 - c) + a_zs & a_y^2(1 - c) + c & a_ya_z(1 - c) - a_xs \\
    a_xa_z(1 - c) - a_ys & a_ya_z(1 - c) + a_xs & a_z^2(1 - c) + c
\end{bmatrix}
\]  

(3.8)

Where \( c \) is \( \cos(\phi) \), \( s \) is \( \sin(\phi) \), and \( a_x \), \( a_y \) and \( a_z \) are the x, y and z components of \( \mathbf{a} \) respectively. An additional translation, as defined by Equation (3.9), can be added to the rotated nodal coordinates to translate the second bay on top of the first.

\[
\mathbf{T} = \begin{bmatrix}
    c_{1,x} - c_{2,x} \\
    c_{1,y} - c_{2,y} \\
    c_{1,z} - c_{2,z}
\end{bmatrix}
\]  

(3.9)

Where the center of the top face of bay 1 is defined by \( (c_{1,x}, c_{1,y}, c_{1,z}) \) and the center of the bottom face of node 2 (before translation) is defined by \( (c_{2,x}, c_{2}, c_{2,z}) \). Since the rotation matrix, \( \mathbf{R} \), only aligns the normals, an additional rotation must be applied to the second bay to align the nodes such that it is a class-2 tower. This rotation can be applied by using Equations (3.6)-(3.8), but substituting a vector pointing from the (now shared) center of the top face of bay 1, to node 4 of bay 1, and a vector pointing from the center of the bottom face of bay 2, to node 1 of bay 2. This new rotation matrix can be applied to the rotated and translated bay-2 nodes, effectively aligning node 1 of bay 2 to node 4 of bay 1, and thus creating a class-2 tower.

The results for a single bay serve as a baseline for a tower of any number of bays. For one of the equilibrated positions of bay 1, the second bay may independently take on any of the equilibrated positions found for bay 1, relative to the top of bay 1 (see Fig. 3.21 (b)). This can be done for each equilibrated position found for the first bay. In this way, if \( n \) discrete solutions were numerically found for a single bay, then a tower of \( N \) bays will have \( n^N \) discrete solutions. Of
course, not all solutions need to be displayed to get a good representation of the continuous data set. Since the results were found computationally, it is relatively straightforward to extend the numerical analysis to select a representative amount of data from a single bay and use that for each additional bay, or else to find all solutions and only work with a select number of the final solutions as a representative set.

When considering multi-bay structures, the ratio of interest is the overall height (i.e. distance from the bottom-most face center to the top-most face center when all bays are in the parallel shape), denoted $H$, to the circumscribing radius, $r$. Figure 3.22 shows the results for $H/r=8$ for a 2-bay tower. Clearly, the results become much more complicated with the addition of bays. For a single bay, the results form a single surface of solutions for the top-face centers. With the addition of bays, the solution set becomes a cloud rather than a surface (see Fig. 3.22 (a)), due to the bays being allowed to have shapes independent of each other. This essentially means that for a given azimuth-elevation pair for the centerline, there may be more than one shape the tower can take, each with its own corresponding azimuth-elevation of the top-face normal.

**Figure 3.21**: A visualization of how the single bay results can be translated to a second bay. Left (a) is all the single bay results for a triplex of $h/r=4$. Right (b) shows a 2-bay triplex, each bay of $h/r=4$, ($H/r=8$) with bay 1 in an arbitrary position, and the points in blue being the relative locations of the second bay top-face center that can be reached with the first bay in the shown shape. The second bay plotted in the 2-bay tower is in the parallel shape.
The results shown in Fig. 3.22 allow each bay to take on any possible equilibrated shape. The data set can be restricted, as discussed in earlier sections, to only include shapes where the centerline elevation for each bay is above its critical elevation. This means that each bay individually is only allowed to take on shapes with a centerline elevation above the critical elevation. Figure 3.23 shows how implementing this restriction influences the results.
Figure 3.23: Results for shapes whose centerline elevation for each bay is above the critical elevation. Left (a) shows centers of the top-most face, and right (b) shows the azimuth and elevation of the centerlines to the top-most face.

The results can be found using the same methods for 4 bays and 8 bays as well. For comparison, 4 different configurations for a structure of a net height ratio of $H/r=8$ were considered: a single bay of $h=8$, two bays each of $h=4$, four bays each of $h=2$ and 8 bays each of $h=1$. The results are shown for comparison in Figures 3.24 and 3.25. These figures clarify how the addition of bays greatly increase the range of potential achievable shapes. In Fig. 3.24, the azimuth and elevation of the centerlines shown are with no restrictions on the allowed shapes, and show how by 4 bays, any azimuth-elevation pairing is possible. Restricting the data set to only allow bays to individually have an elevation above the critical elevation results in Fig. 3.25. This shows that by 8 bays, any azimuth-elevation combination of the centerline is possible.

Figure 3.24: Azimuth and elevation of centerline (from bottom-most face to top-most face) of multiple configurations, all of $H/r=8$. No restrictions are placed on the allowed shapes for this figure. One bay is of $h=8$, two bays each of $h=4$, four bays each of $h=2$ and 8 bays each of $h=1$.

The methods described here can be applied to any combination of number of bays and individual bay heights. The results for multi-bay towers are directly derivative from the results of
a single bay within that tower, thus making the design of the unit bays critical to understanding the overall tower behavior. In general, it is clear that the addition of bays greatly increases the range of achievable shapes which are achievable.

![Figure 3.25: Plots of azimuth vs elevation of the centerline (from bottom-most face to top-most face) above critical elevation. All configurations have an \( \frac{H}{r}=8 \). One bay has \( h=8 \), two bays each have \( h=4 \), four bays each have \( h=2 \) and 8 bays each have \( h=1 \).](image)

The results thus far have not considered physical design parameters, such as member interference or structural properties. When considering the limitations of the range of achievable shapes, these physical parameters have great influence—especially member interference. Furthermore, structural properties—such as bending or twisting stiffness—are critical parameters, often driving design selections in real world applications. All of these properties are important to consider when understanding the design of a tensegrity structure. The next chapter focuses on the influence of such parameters on the range of achievable shapes, and how these parameters vary for a different number of bays of the same \( \frac{H}{r} \) ratio.
Chapter 4

Structural Properties of Cable-Actuated Tensegrities

Understanding and specifying a structure’s properties is a critical aspect of mission design. Geometry, materials, and dimension all influence structural properties, such as stiffness, packaging efficiency, and range of achievable shapes. Chapter 3 discussed the methods which can be used to provide an understanding of what shapes can be achieved for a given tensegrity configuration. These shapes consider geometry and equilibrium based on that geometry, but they do not consider any physical properties of structure itself. While all the shapes found may conceptually be possible, they do consider the fact that members have finite diameters, nor the need to start from one position and move members continuously to reach a different shape. Furthermore, when physical properties—such as member material and radius—are assigned to the structure, properties such as structural strength and packaging efficiency may also be considered.

This chapter discusses the methods used to analyze these properties. Chapter 5 will expand upon these methods, and use them to gain insight into how different configuration parameters—such as $h/r$, member radius and number of bays—may be used to influence overall structural properties like tip deflection under bending load and packaging efficiency.
4.1 Member interference

While exploring the design potential of a cable-actuated triplex, Chapter 3 established the range of geometrically viable and equilibrated shapes possible for a specific configuration. This solution set was obtained by neglecting member diameters, and did not consider a starting shape. In effect, this means that each shape found was considered individually, and was only considered problematic (even if geometrically possible and in equilibrium) if the members intersected exactly (i.e. the line connecting the nodes of one member directly passed through the line connecting the nodes of another member).

In reality, each member will have a finite diameter, and cannot pass through another member. This means there is a minimum non-zero, finite distance—related to member radii—which the members must maintain between each other. Furthermore, the solution set discussed thus far takes each shape as its own, unique shape. In effect, this means that it is assumed the structure can reach that position regardless of its initial shape. This is not necessarily true when finite member radii and the starting shape are included. Depending on the member radii and the starting shape, certain other shapes may not be reachable without requiring members to touch or pass through each other, or the structure violating equilibrium in the process. For practical applications which would find an actuated structure useful, it is thus necessary to consider the limitations imposed on the solution set by considering member interference.

In most applications, the struts and cables can be approximated as long, thin cylinders. In this way, the centerline through these cylinders is the same line connecting nodes that was used in previous chapters. The outer edge of the member is some radius away from that centerline, perpendicular to the line, at any given point on the member. Thus, for members to touch or
interfere, the centerlines of the members must be separated by a distance of the sum of each members’ radius (or less).

To illustrate the methods used to check for member interference, consider first two members, denoted member 1 and member 2. To check for member interference between one starting position (A) and one ending position (B), the center line of each member is represented parametrically as a line as follows:

\[
R_m^n(\delta) = \frac{R_{mb}^n + R_{ma}^n}{2} + \frac{R_{mb}^n - R_{ma}^n}{2} \delta \quad \text{with} \quad -1 \leq \delta \leq +1 \tag{4.1}
\]

Where \( R \) denotes a position vector to a given point on the member, subscript \( m \) denotes member 1 or 2, and superscript \( n \) denotes position A or B. Subscripts \( a \) and \( b \) denote the 2 nodes which define each member (see Fig. 4.1). Equation (4.1) can thus be used to describe 4 distinct lines: member 1 in position A and position B, and member 2 in position A and position B, where any point on any of these lines can be found with a unique value of \( \delta \) between -1 (corresponding to node \( a \)) and 1 (corresponding to node \( b \)).

![Figure 4.1: Labeling of member 1 (red and pink) and member 2 (blue and light blue) for member interference checking. The labeling numbers refer to member 1 or 2, lowercase a or b refers to node a or b on the respective members, and capital A or B refers to the respective member in position A or position B.](image-url)
For small movements between positions A and B, each point on a member may be assumed to move linearly between its positions from A to B (see Fig. 4.2). In this sense, an additional line may be considered for each point (corresponding to $\delta$) on each member. These lines may be parametrically written using the results from Equation (4.1) as follows:

$$R_m(\delta, \varepsilon) = \frac{R_m^B + R_m^A}{2} + \frac{R_m^B - R_m^A}{2} \varepsilon \quad \text{with} \quad -1 \leq \varepsilon \leq +1 \quad (4.2)$$

Where $R_m$ denotes the position vector of the point $\delta$ on member $m$. The variable $\varepsilon$ denotes the position of point $\delta$ along its linear path between positions A (corresponding to $\varepsilon=-1$) and position B (corresponding to $\varepsilon=1$). Variables $R_m^A$ and $R_m^B$ are found from Equation (4.1) for any $\delta$ of interest on member $m$ in positions A and B respectively.

![Figure 4.2: Visual representation of the movement of members 1 and 2 from position A to position B. From red to pink is member 1 moving from A to B. From blue to light blue is member 2 moving from A to B. Each line in between the respective colors represents a single $\varepsilon$ value between A ($\varepsilon=-1$) and B ($\varepsilon=1$).](image)

To check for collisions between positions A and B, the minimum $\varepsilon$ is incrementally changed by $\Delta\varepsilon$ from -1 to 1. For each value of $\varepsilon$, the current line defining members 1 or 2 can be written using Equation (4.2) as follows:

$$R_m(\delta) = \frac{R_m^{aN} + R_m^{bN}}{2} + \frac{R_m^{bN} - R_m^{aN}}{2} \delta \quad (4.3)$$
Figure 4.2 shows 25 values of $\varepsilon$ between -1 and 1, representing the linear paths moved through by the 2 members. The lines shown are the same as in Fig. 4.1, and are defined by points chosen purely for visual clarity. Figure 4.2 demonstrates how representing the movement of a member from A to B in this way sweeps out a surface that the member travels through. The surface swept out by these member movements may intersect with each other, as they do in Fig 4.2. This does not necessarily mean member interference has occurred. For member interference to occur, the member lines must intersect for the same $\varepsilon$ value. To account for this, the value of $\varepsilon$ is incrementally changed by $\Delta\varepsilon$ simultaneously for each of the members. For a given value of $\varepsilon$, the members are represented as lines described by Equation (4.2). With 2 distinct equations of lines, one for each member, the minimum distance between any 2 members can be found for a given value of $\varepsilon$. If this minimum distance is less than the sum of both member’s radii, then a collision has occurred.

There are a few parameters to consider when numerically implementing this method of checking for member collision. To start, it is important that the step size $\Delta\varepsilon$ is small enough to ensure that a possible collision will not be missed. Since the parameter defining the presence of a collision is the sum of the radii of the two members being considered, $\Delta\varepsilon$ should be smaller than the sum of these radii. A $\Delta\varepsilon$ of at least $\frac{1}{2}$ the sum of the member radii is taken as a reasonable maximum step size. Smaller step sizes can only help, as this is not the only source of numerical error. However, minimizing computational time is also important, as a single tensegrity bay has 12 members to check against multiple other members. If each member is checked against all 11 other members, this is 77 calculations for a single step value of $\varepsilon$. Thus, especially for larger spacing between positions A and B, it is desirable to minimize the time required to check between two positions within reason. Since this method of collision checking assumes linear movements
between two positions, small differences in position between A and B are desirable for best accuracy.

With this in mind, the movements discussed in this thesis which are checked for collisions are small enough that $\Delta \varepsilon$ is chosen such that there are at least 100 steps between position A and B to ensure robust results, as this is still relatively time efficient, but more accurate than a $\Delta \varepsilon$ of $\frac{1}{2}$ the sum of the member radii.

Another point of consideration is which members to check against each other. For members sharing a node, the minimum distance between the representative lines will always be at the node, since by necessity they intersect there. In reality, the design of the node itself will account for this to some extent, with the restriction on movement being specific to the type of node. Furthermore, strut members may be tapered near the nodes, to further extend the range of motion. Figure 4.3 shows how the region of overlap can be reduced by tapering the members near the nodes. Restrictions on movement based on collisions near a node are thus heavily dependent on the design of the nodes and the members themselves. Furthermore, it is more straightforward to think of collisions here as function of the angle between the members. In this sense, for members sharing a node, a collision occurs for a minimum angle related directly to the member thickness. This could be implemented in lieu of the described method looking at centerlines for members sharing a node. This thesis only checked for collisions between members which did not share nodes. Reducing the number of members to check also helps to greatly reduce computation times.
Figure 4.3: A visualization of how tapering members can reduce region of overlap. Left shows two cylindrical members with the region of overlap in red. Right shows the same members, but tapered near the node in blue. This tapering gets rid of a region of overlap altogether for the shown $\theta$.

As a final consideration, note that the aforementioned method assumes linear movements between position A and position B. In movements along any path prescribed in this research, each individual A and B step is dictated by the density of the solution set found by the methods described in Chapter 3. Since generating the solution set is the most computationally intensive part of the process, the density of the set is assumed to be great enough such that the linear assumption is accurate.

The degree of accuracy is at the discretion of the designer when it comes to physical implementation. Though rods are typically inextensible, the cables have a bit of room for error in shape. Cables may be designed to have varying degrees of extensibility based on material properties and the degree to which they are pre-stressed (i.e. exactly how much pre-tension they carry). Consideration of varying tension levels in the cables, as well as material properties, may allow for greater errors without loss of equilibrium in the structure.

Furthermore, considering the data set for the location of the center of the top face, the average distance between a single point in the baseline single bay case and any of its adjacent data set positions is $\sim 2\%$ of the circumscribing diameter. The maximum distance a single node moves from adjacent positions is $\sim 4.4\%$ of the circumscribing diameter, and 99% of the data set has a
maximum distance from adjacent points below 3% of the circumscribing diameter. Movements between adjacent shapes along any path prescribed by movements of the top-face center will not deviate greatly from this average movement, and this step size has been found to be sufficiently small for a linear assumption to provide reasonable accuracy. Inaccuracies in the linear assumption appear (most directly) as requiring members to change lengths. The greatest change in length for members which are assumed to be fixed in length is 0.007% of the circumscribing diameter for movements between adjacent shapes. The average change in length between adjacent shapes is 0.0009% of the circumscribing diameter.

Though the shapes between which the members move will vary in separation, potentially outside these adjacent shapes, any deviation from these values will be minimal. Thus, these values prove sufficiently small for the linear assumption to be reasonable.

4.1.1 Radial Path restrictions on achievable shapes

Accounting for members having a finite diameter leads to the question: how does member interference restrict achievable shapes? For a given starting shape, member collisions may occur when trying to move from one position to another. A reasonable starting shape to assume is the parallel shape.

From this straight shape, paths are considered in straight lines moving radially outward. These paths are followed by the center of the topmost face, and “straight” is defined with regards to the x-y plane. The center of the top-most face follows these paths in the x-y plane, with the z value being dictated by finding the point on the solution surface with the minimum distance from the path in x-y. Figure 4.4 shows a top-down view of the solution set of top-face centers for a single bay of $h/r=1$, with pink lines showing a few such radial paths. Figure 4.5 highlights just one of
these paths, showing the projection of this radial path onto the solution surface, and highlighting in green the points selected in following this path.

Figure 4.4: Top-down view of all equilibrated top-face centers for h/r=1 in blue, lines in pink are radial paths.

The method described in the previous section is applied along these paths, starting with the straight configuration as position A, and the next adjacent shape corresponding to the next top-face center along the radial line as position B. Moving along this path, the initial position B becomes the new position A, with the next point along the path becoming the new position B, and so on until the edge of the dataset for the current path is reached.
Figure 4.5: A top down (left) and isometric (right) view of the top-face centers of equilibrated shapes for h/r=1. The line shown in pink is an example radial path, with the selected top-face centers following that path highlighted on the solution surface in green.

Considering finite member radii, the solution set in terms of azimuth-elevation of the centerline may be modified to only include shapes which may be reached through these types of paths from the straight shape. Including member radius effects, the critical elevation for reachable shapes increases, with greater radii causing greater restrictions. Figure 4.6 shows the original azimuth-elevation plot for a triplex of h/r=1 without member radii consideration, overlaid with the azimuth-elevation plot for a ratio of member radius to circumscribing radius (m/r) of 0.01 (i.e. member radius is 1% of circumscribing radius). Figure 4.7 shows how changing this ratio through multiple values reduces the critical elevation for the case of h/r=1.

In these cases, all member radii are considered equal to each other. The case with cables having a smaller diameter was tested (cable diameters equal to ½ strut diameters), but it was found to have no influence on path restrictions for the single bay cases. This means that all member collisions for single bay cases in these types of paths come from strut-strut interference. Thus, for simplicity, all member diameters are kept equal to each other in these calculations.
Figure 4.6: Azimuth vs elevation for a single bay triplex of h/r=1. In dark blue are the centers of all equilibrated shapes unrestricted by finite member radii. Overlaid in light blue is the same plot, but excluding points which would require member interference to reach from the straight configuration if members have a radius equal to 1% of the circumscribing radius.

Figure 4.7: Critical elevation (considering potential member interference) as a function of member radius (plotted as a percent of circumscribing radius)
This analysis can be extended to multi-bay cases as well. However, selecting points that follow these radial paths is not as straightforward for multi-bay cases. For a single x-y values along the line, there may be multiple z values in the solution set within an acceptable tolerance range. There are a few different ways this might be addressed.

For instance, the solution set could be constrained by only the bottom bay to change shape, with additional bays kept straight. The resulting solution set for 2 bays, each of \( h/r=1 \) is shown in Fig. 4.8 (a). This approach results in a solution surface, as for a single bay, rather than a solution cloud when bays are allowed to move independently of each other. This might be a reasonable assumption to make for 2 bays, but becomes impractical for realistic applications of a large number of bays, or even 2 bays of large \( h/r \) values. Controlling the movement of the top-most center by movements of the bottom-most bay would require extreme precision of the bottom-bay shapes for any movements which aren’t particularly course.

An alternative way to approach the multi-bay case would be to prescribe that the bays have the same shape as each other. This is shown for 2 bays of \( h/r=1 \) in Fig. 4.8 (b). This approach also results in a solution surface, rather than a cloud. This type of movement, however, is perhaps more practical in that all the bays having the same shape results in some symmetry of cable lengths between bays, which could also result in a reduction of required actuators. Reducing the number of actuators reduces the number of moving parts—something always desirable in terms of reliability—and reduces the controls that must be considered. However, the possible shapes achievable are undeniably reduced by restricting the bay shapes to all be the same.

A solution \textit{surface} is desirable in terms of ease of prescribing a path, as it allows the path to be prescribed as an x-y projection, and will result in only one shape being closest to the path. If it is more desirable, however, to have a large variety of achievable shapes, it is necessary to consider bays moving independently of one another. This is shown in Fig. 4.8 (c) for 2 bays of \( h/r=1 \). The optimal way of describing the position of a multi-bay structure will likely vary depending on the application. A multi-bay shape changing structure might be used as a robotic
arm, for example, where the curvature of the ‘inside’ of the tower may be a more desirable descriptor of location. In cases of pointing, azimuth-elevation may be the optimal descriptor, and in these cases, it is not unreasonable to assume the radius related to that azimuth-elevation is insignificant. Prescribing paths by x-y projections is similar to prescribing azimuth-elevation (and could readily be translated to azimuth-elevation if desired). Since this is the most straight-forward way of prescribing a path for a single bay, and offers some numerical simplicity, this was chosen as the method of focus when considering bays moving independently of each other.

Radial paths for independently moving bays may have multiple z values (corresponding to multiple shapes) that satisfy the path constraints. Consider checking for member interference along the same radial paths, but for a 2-bay case. For these radial movements, all shapes within a tolerance of the current prescribed position along that path were considered and compared. Each shape in that tolerance was checked against the previous shape. If only one shape could be achieved without member interference, that shape became the next position. If multiple shapes were found without member interference, then the shape requiring the least movement of the nodes was given priority as the next shape.

As with the single bay case, the path is chosen by moving along the solution set found for the centers of the topmost face. When generating this solution set for multiple bays, the single bay results are translated to additional bays. Thus, the solution set checked for multi-bay cases may be reduced by only considering values from the single bay results that are above the critical elevation associated with the member radius of interest. This reduced data set must then also be checked for interference between bays. The results of member radii influence on the critical elevation for independent bay movements found by only considering the single bay shapes without member interference are shown in Fig. 4.9 for two bays, each of $h/r=1$. Member interference between
Figure 4.8: Different ways to allow for a 2-bay structure (each bay h/r=1) to change shape. Top row (a) shows when the bottom bay may change, second bay is held straight. Middle row (b) shows both bays moving in the same way simultaneously. Bottom row (c) shows each bay moving independently.
directly adjacent bays (outside of within a single bay itself) would come from, by necessity of a class-2 boom, members sharing a node. Checking for collisions between adjacent bays could be done with the same method described in section 4.1 by considering angles between members, and is not investigated in this thesis.

Extension of this method of collision checking to account for interference between bays when there are more than 2 bays in the boom is also considered. When checking for interference between non-adjacent bays for a large number of bays, the process becomes more complicated and numerically intensive. Specifying the path only in $x$-$y$ is less practical for a large number of bays, as many shapes with different $z$ values could satisfy the $x$-$y$ position. Furthermore, the likelihood of finding multiple shapes for a given center location (even including specifying $z$) is greater for higher number of bays—meaning there are multiple redundant shapes for one end position, some of which may experience interference along the path while others may not.

![Figure 4.9: Critical elevation vs. member radius (plotted as a percent of the circumscribing radius) for a 2-bay boom, each bay having $h/r=1$.](image)
Path planning as such was not considered extensively. Instead, a single path is considered for an 8-bay case here as an example. The methods could be extended to any specified path for any number of bays.

Interference between bays in this 8-bay case may be considered by methods very similar to those of the 2-bay case. A coarse check is first implemented to expedite the process. First, the centerline of the 2 bays being considered are checked for their minimum distance relative to each other. These are checked the same way as members are checked against other members in the single bay case, but the minimum allowable distance is taken as the 2 times the circumscribing radius. If any point on the centerline of the two bays being checked is found to be less than the circumscribing diameter away from each other, then the current bay positions are run through a finer check, where individual members of one bay are checked against the members of the other bay. Adjacent bays, as mentioned, would always find interference between members sharing nodes (and, in the case of the coarse check, at their shared center in the intermediate face). A small angle check between adjacent bays/node-sharing members could be implemented alongside the method described here to check as well. Adjacent bay collisions are not considered in this thesis otherwise.

Figure 4.10 shows the cartesian plot of the topmost face centers of the 8-bay case, where each bay has \( h/r=4 \) (making the net height 32). The member radii were taken as 3% of the circumscribing radius. Highlighted in green is the path chosen. The path is specified in spherical coordinates, prescribing the azimuth and elevation of the center. The radius of points on the path are always greater than the radius of points in the solution set, so only the largest values of radius of center points are considered. For a given point along the path, there are multiple points within the dataset very close to each other that might be considered. The density of the dataset generated
will influence how many points are found with their some tolerance of the desired azimuth-elevation and an acceptable radius.

In this example, the closest 5 points to the specified path point were all considered. If any of the 5 points were a distance greater than a specified tolerance (in this case, 0.2, chosen based on observed data density) away from the nearest point, they were excluded from consideration: this is to account for cases where data sparsity essentially results in false positives for nearby points. All the points selected this way are highlighted in the Fig. 4.10 in pink. If any of these points could be reached from the previous shape without member interference, then the boom was allowed to continue along the path. If interference occurred for all points, then the path ends. If more than one of the 5 points considered do not have member interference, then the point closest to the specified path point is chosen.

Figure 4.10: Some points of the solution set found for 8 bays allowed to move independently (topmost centers plotted in dark blue). The green points dictate a path for the center to follow in terms of azimuth-elevation of the centerline, the pink points are points from the solution set considered along the green path
Figure 4.11 shows a few shapes as the boom moves along the specified path. No member collisions were found along this path. If a collision had been found, the path would have ended when a single point along the path had all shapes (among the nearest 5 shapes) causing member interference. Even in this had been case, it may still be possible to continue along the path if different exact shapes had been chosen in previous path points. Furthermore, if the start and end location are all that are required, then the path does not need to be “linear”, which could enable many possible paths to consider. Path planning was not considered in greater detail for this thesis, but the methods for member interference checking described in this section could be applied to any specified path.

As an example, Fig. 4.12 shows a particular case where, from visual inspection, it is clear member interference would occur between non adjacent bays. The first case of member interference found with the methods developed in this chapter are marked in Fig. 4.12 (d). This serves as a visual confirmation of the success of this methodology for checking for member interference.

Member interference checking with the methods described in this section were shown for cases of 1 and 2 bays, each of $h/r=1$, single path of 8 bays with each $h/r=4$, and a single position with known interference for a 16-bay boom, each with $h/r=4$. Changing values of $h/r$ and varying the number of bays is saved for consideration in Chapter 5 as part of a design analysis of cylindrical triplex towers.
Figure 4.11: Select steps along the prescribed path for an 8-bay boom. Each bay has $h/r = 4$ with $r=1$. 
Figure 4.12: A 16 bay boom (each bay h/r=4) where there known interference between non-adjacent bays. Top two (a and b) show two different perspectives of the overall boom. Bottom 2 (c and d) zoom in on the region of interference, with point of collision found by the methodology developed in this chapter marked in blue stars.
4.2 Structural stiffness

When considering how best to design a structure, the required stiffness of the structure is often a property of great importance. There are different ways to account for structural stiffness. Twisting, bending, and buckling are all different types of loading with different results depending on structural design. These properties have been evaluated by researchers in the past [19, 42-45]. Yildiz conducts an in-depth optimization analysis of triplex booms, offering insight on tensegrity structures stiffness with different design parameters in consideration [19].

This work focuses on bending tip deflection and twist deflection as a way of considering the effects of the various design configuration parameters on structural stiffness. To evaluate the effects of bending and twisting, a linear finite element method is used. This method is based on the linear portion of the method described by Yildiz [19]. Each member of the triplex is treated as an element, with the member nodes acting as the element nodes. Yildiz [19] discusses the basis of the typical finite element method approach in some detail. The governing equation for this method is described by Equation (4.4).

\[ K \Delta U = P - F \]  \hspace{1cm} (4.4)

Where \( K \) is the global stiffness matrix, \( \Delta U \) is the tip deflection, \( P \) is the external loading, and \( F \) is the internal load vector from pre-stress. Inclusion of the pre-stress \( F \) is considered outside the scope of linear analysis by some definitions, but is essential for tensegrity analysis. Not that the linear assumptions made with this method only hold for deflections which are small compared to the overall dimensions of structure. In particular, non-linear methods may include significant phenomena particular to tensegrity, such as cables going slack [19]. This linear method is applied as a simpler method of understanding the general structural characteristics of tensegrity booms.
Figure 4.13: A visual of the repeating unit, 2 bays, of a class-2 triplex with reinforcing cables. Used for validation of numerical finite element methods used to compare against the results of Yildiz [19].

This numerical method was checked against the results of Yildiz [19] for validation. An overall structure height of 14 m is used, with a circumscribing radius of 254 mm. The struts were modeled as hollow, with an outer diameter of 30 mm and an inner diameter as dictated by Table 4.1 [19]. The triplex shapes feature additional reinforcing cables (which were included for this comparison, but not included otherwise in this thesis), and each consecutive bay is rotated in the opposite direction, such that every 2 bays are a repeating unit (Fig. 4.13). The angle of rotation in the x-y plane of adjacent faces is referred to as the twist angle, and is also shown in table 4.1. Cable diameters and the pre-stress are also shown in Table 4.1. A load of 20 N was applied in the x direction at each of the top-most nodes, while the 3 bottom-most nodes are held fixed in x-y-z. The struts were taken as Kevlar 49 resin-impregnated strands (with a modulus of elasticity, $E=124$...
GPa), and cables were taken as Unidirectional Mitsubishi K13C2U UHN /epoxy ($E=536$ GPa) [19].

Figure 4.14 shows the resulting comparison for bending tip deflection against the number of bays. Since the method used by Yildiz is nonlinear, some deviation in results would not be surprising, especially as the deflections get larger. Figure 4.15 shows the difference in results with respect to Yildiz’ values. The linear assumptions used are evidently less valid for larger deflections, like those found for larger numbers of bays. However, even for these larger errors, the difference in deflection values between these two methods does not exceed 0.27% of the overall height. This error is deemed sufficiently small for the analysis to follow. This numerical linear FEM is used to evaluate both bending and twist deflections.

**Figure 4.14**: A comparison of the results from the linear finite element method used against the nonlinear FEM used by Yildiz [19].
For both the bending and twist deflections evaluated, the boundary conditions are always implemented such that the bottom-most 3 nodes are fixed. The loading in each case is always applied to the top-most face, with an equal load $P$ applied at each of the top 3 nodes. The triplex top-faces repeat every 4 bays with the class-2 stacking methods previous discussed, so structures with any number bays which are not a factor of 4 will have top-faces at a different orientation. The loading is applied in the same way to any top face for bending, such that the direction of the external force vector, $P$, is constant, and it is simply translated to be applied wherever the top-face nodes are located. For twisting, the direction of the external force vector is always perpendicular to the line connecting the center of the top-most face to the node where the load is applied, and thus the exact orientation of the applied load will depend on the orientation of the topmost face.
**Table 4.1:** Values used for generating Fig. 4.10 and 4.11. All radii are in mm. For column 2, radius of face cables refers to the bottom-most and top-most face cables of the overall boom. Adapted from Yildiz [19].

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4.2.1 Bending Tip Deflection

When considering bending tip deflection, the external forces are loaded as shown in Fig. 4.16. The materials are held the same as in the previous section, with the struts taken as solid members of radius 30 mm, while the cables have a radius of 10 mm. To simulate pure bending, the loads are applied at each of the top 3 nodes in the same direction with respect to the x axis. In Fig. 4.16, the external load is applied in the x direction. Tip deflection for bending is calculated as the net movement of the center of the top-most face. Varying the angle with respect to the x-direction that these loads are applied in was found to have no effect on net tip deflection, indicating isotropic bending stiffness.

![Figure 4.16](image)

*Figure 4.16: A visualization of how the external loading is applied to simulate pure bending. The triangle shown is a representative case of the top-most face of the structure.*

Aside from material properties and structural geometry, the pre-stress factor of the tensegrity will influence the bending tip deflection. Holding all else constant, the pre-stress factor is varied from 0.01 N/mm to 1000 N/mm for a single triplex bay. This deflection analysis is
repeated for multiple values of $h/r$ and shown in Fig. 4.17. For low values of $h/r$, changing the pre-stress coefficient has very little effect on bending tip deflection. However, for higher values of $h/r$, increasing the pre-stress can decrease the bending tip deflection. Furthermore, comparing (a) to (b) in Fig. 4.17, it can be seen that this trend is true regardless of the applied load—higher external loads increase tip deflection values overall, but the trend of the effects of pre-stress remain the same.

**Figure 4.17**: Effects of increasing pre-stress coefficient against the bending tip deflection, where tip deflection is plotted as a percent of the bay height. Top (a) shows the results for an applied load of 20 N, while bottom (b) shows the results for 5000 N.
4.2.2 Twist Deflection

The same linear FEM numerical analysis was implemented to simulate a pure twisting load on these triplex booms. The materials and member radii are held the same as in the bending case. To simulate twisting, loads are applied perpendicular to the vector connecting the center of the top face to the node at which the load is applied, as shown in Fig. 4.18. The loads may be applied in a clockwise or counterclockwise orientation. When addressing the effects of twisting, there are two main things to consider. First, and the primary consideration, is the deflection angle. This angle is calculated as the angle between the vector pointing from the face center to a given node before twist is applied, and the same vector to the same node after twist is applied. This angle will be the same for any chosen top-face node due to the rotational axisymmetric properties of the booms. Another aspect to consider is the deflection in z caused by extension-twist coupling. The twisting motion will typically cause the structure to wind or unwind, depending on the direction of the applied loads.

![Pure Twist](image)

*Figure 4.18: A visualization of how loads are applied to simulate pure twist. Shown is counterclockwise twisting. External load vectors may be applied in the opposite direction than is shown to apply clockwise twist.*
Figure 4.19 shows the effects of increasing pre-stress coefficient for a given external load, $P$. Changing the value of $h/r$ for a single bay was found to have no effect on the deflection angle. Furthermore, the magnitude of the deflection angle is also unaffected by changing the twist from counterclockwise to clockwise rotation. The exact shape of the pre-stress coefficient vs deflection angle curve does change with respect to the magnitude of the applied load, however, the general trend of increasing pre-stress coefficient causing a decreased deflection angle remains the same.

![Figure 4.19: A plot of the pre-stress coefficient versus deflection angle. This plot is unaffected by h/r.](image)

When considering the deflection in the z direction, the direction of the applied twist will influence whether the structure “winds” or “unwinds”. Winding denotes a negative z deflection, while unwinding denotes a positive z deflection. Figure 4.20 shows that, in general, increasing the pre-stress coefficient decreases the overall z deflection, as would be expected. Figure 4.20 also shows the effect of twisting in the clockwise vs counterclockwise direction. Figure 4.20 shows that changing the value of $h/r$ changes the magnitude of the effects of pre-stress. In general, the trend of increasing pre-stress decreasing deflection remains the same regardless of applied loads. Furthermore, Fig. 4.20 (a) shows the typical triplex considered for this paper, where the top nodes are rotated 30 degrees counterclockwise from the bottom nodes. Figure 4.20 (b) and (c) shows that...
for this case, twisting in the clockwise direction thus results in winding, while counterclockwise results in unwinding. Figure 4.20 (d) shows a triplex of the same connectivity, but with nodes rotated in the clockwise direction instead. Figure 4.20 (e) and (f) show that for this orientation, clockwise twist results in unwinding and counterclockwise twist results in winding. This indicated that the geometry of the structure, dictated by the direction of the rotation of nodes between faces, is directly coupled with the effects of the direction of twisting.

![Figure 4.20](image)

**Figure 4.20**: Effects of pre-stress coefficient on z tip deflection. Left column (a, b, c) shows the typical triplex with clockwise and counterclockwise twist. Right column (d, e, f) shows a triplex with top nodes rotated in the opposite direction with clockwise and counterclockwise twist.
4.2.3 Effects of curved shapes on structural stiffness

Applications involving dramatic structural shape changes are likely analogous to robotic arm applications. In such cases, typically the base of the structure will be fixed, while the tip will carry some sort of loading. To consider the effects of different shapes with such loading conditions, a single bay triplex may first be considered. For a given bay height, each shape is considered with the same loading conditions—20 N in the x-direction, with the bottom 3 nodes completely fixed. The same materials and member radii are used as for the bending and twist deflection cases. A pre-stress coefficient of 0.05 N/mm is used.

For a single bay triplex with a height of 8 m and circumscribing radius of 1 m, Fig. 4.21 shows how different shapes influence the tip deflection for the applied load. Figure 4.21 shows all the possible centers of the top face, corresponding to different shapes this triplex can achieve. The figure is color coded to correspond to different values of total tip deflection. This figure shows how certain shapes relative to the starting straight shape will result in more or less tip deflection depending on centerline azimuth. Such results could be used to help specify shape limitations based on maximum allowable deformations for a given application.

This analysis may similarly be applied to multi-bay structures. Visualization of multi-bay structures, however, is more complicated, as multi-bay structures may have multiple possible shapes for a single azimuth-elevation pair of the centerline. Figure 4.22 shows 5 shapes, each of $H/r=8$ (where $H$ refers to total height), with the same azimuth-elevation of the centerline. An elevation of 0.65 radians was chosen, since this is approximately the value of the critical elevation for a single bay of $h/r=8$ with the given member radii. Increasing the number of bays can only increase the range of attainable elevations, so this value works for the 2-bay boom as well. An
azimuth of 2.35 radians was chosen to be near a zone of more extreme deflections, as found in Fig. 4.21.

Figure 4.21: Centers of all possible equilibrated shapes for a triplex of h/r=8. The tip deflection is plotted as the color. Top (a) and middle (b) show the centers in cartesian coordinates, while bottom (c) shows the centerline azimuth-elevation of these points.
Figure 4.22: Five different triplex shapes, all of net height 8 m with circumscribing radius of 1 m, each with the same centerline azimuth-elevation of 2.35 rad and 0.65 rad respectively. Leftmost is a single bay, while the right 4 are 2 bay booms.
Figure 4.21 indicates that the tip deflection for a given load will depend on the exact shape of the structure. This is true for the multi-bay case as well. Deflection is further dependent on the combination of shapes between the two bays. Figure 4.23 shows the resulting tip deflections corresponding to the shapes from Fig. 4.22. The effects of additional bays for a fixed overall height is explored in greater detail in the next chapter, but generally speaking increasing the number of bays is found to increase bending tip deflections.

Figure 4.23 shows that even for a single, 2-bay structure with a given centerline azimuth and elevation, the exact shapes may vary and have different resulting tip deflections. Such analysis may be useful in applications, as understanding the tip deflection for a given azimuth-elevation would allow for prescribing a shape based on minimizing (or otherwise specifying) the tip deflection. This analysis could be applied to any number of bays, for any combination of azimuth and elevation, as necessary to meet mission requirements.

![Figure 4.23: Resulting tip deflections of a single bay (red) and 4 different 2 bay shapes (blue) for the same centerline azimuth-elevation. (a)-(e) correspond to (a)-(e) of Fig. 4.21.](image)
4.3 Packaging efficiency

When considering deployable booms, packaging efficiency is a key structural property. There are many different ways to consider stowing deployable tensegrity structures. Yildiz discusses packaging efficiency with the struts stored vertically and packed in a cylinder, noting that cables could be stored within hollow struts and other empty volume within the cylinder [19]. This method of packing offers the smallest possible packaged volume without changing strut lengths, but would require complicated deployment methods to transform between the stowed and deployed state. Tibert studies a case where the struts have tape-spring hinges in the middle of their members, thus stowing in a smaller cylinder by bending at the hinges and deploying naturally as they are pulled out by a screw pulling at the top-most face [32]. The packed state, in this case, maintains some of the final geometry, albeit with struts bent to increase efficiency.

For this thesis, all struts are considered solid members in that they cannot change length or bend. The cables, however, are assumed to have a mechanism which allows them to change length. This mechanism could be used both for deployment as well as shape changing after deployment. With this in mind, a packaged state is proposed as allowing the top face cables to extend to let the struts lay flat on top of the face cables. A top down view of a single bay stacked this way is shown in Fig. 4.24. Additional bays would be stacked in the same way, such that a second bay would share the extended face cables, and the top face of the second bay would be rotated to overlap in x-y with the bottommost face. Thus for 2 bays, the intermediate face has extended cable lengths, but the bottommost and topmost faces have fixed cable lengths. The struts would be overlapped in x-y, such that the top down view of the packed state would look the same as shown in Fig. 4.24 regardless of the number of bays. With this type of packing, some external assistance would be
required to create the required rotation between faces and restore the extended intermediate cables, but these transformations could likely be done with a fairly straightforward method.

![Figure 4.24: A top down view of what a packaged triplex boom would look like. In blue are face cables, with the dashed-line bottom face cables being underneath the struts in red. The top face cables in solid blue are allowed to extend length for packing.](image)

When evaluating packaging efficiency, Equation (4.5) shows how efficiency is defined here.

\[
ε = 1 - \left( \frac{V_{\text{stowed}}}{V_{\text{deployed}}} \right)
\]  

[4.5]

Where \(ε\) denotes the packaging efficiency and \(V\) refers to the minimum cylindrical volume the structure can fit within. The volume of the stowed cylinder is defined by Equation (4.6).

\[
V_{\text{stowed}} = \pi (r_p)^2 S_d N
\]  

[4.6]

Where \(r_p\) is the package radius (dependent on strut length), \(S_d\) is the strut diameter, and \(N\) is the number of bays. Calculating the volume this way assumes the construction of the triplex allows
for the struts and cables to overlap at nodes. This might be done with insets carved out of the struts. Alternatively, the struts might be allowed to be offset from the cables, such that the side of the strut touches the side of the cable instead of the centers of each overlapping. This would increase the volume from that described in Equation (4.6) marginally. For this thesis, Equation (4.6) is taken as a sufficient approximation.

A detailed look at how different configuration parameters influence packaging efficiency is reserved for Chapter 5. In general, based on this definition of packaging efficiency, increasing strut length will result in worse packaging efficiency, as will increasing member diameter.
Chapter 5

Design Analysis of Cylindrical Triplex Tensegrities

Design analysis of structures is an important aspect of any spacecraft mission. In order to optimize a structure, an extensive understanding of mission requirements is critical, as is a thorough understanding of the different aspects of the structure which are subject to design specification, and how these aspects changes influence structural performance. Tensegrity structures are not yet found in any space platforms. This is in part due to the desire to reduce risk—launching a new technology, even if it promises better, more efficient performance, inherently has more risk than using a long-standing product which has been used successfully in the past. Tensegrity structures for space applications is a burgeoning research field, and the continued enhancement of understanding the strengths and weaknesses of these structures can only advance the technological readiness of tensegrities for future missions.

Previous chapters have explored various methods which can be used to best understand the structural properties of tensegrities. In this chapter, these methods will be used to study how different design parameters of the considered triplex booms influence the overall characteristics of the structure. Then, as an example design case, these findings are used to see how a tensegrity design compares to the Candarm2.
5.1 Single bay design parameters

The triplex tensegrities considered in this thesis are exclusively stacked as class-2 booms. This stacking offers symmetry between adjacent bays which allows the results of a single bay to serve as a basis for additional bays. For a single bay, the main design parameters to consider are the ratio of bay height to circumscribing radius, $h/r$, and member radius. In the case of $h/r$, the circumscribing radius is always held fixed, such that larger values of $h/r$ always correspond to taller bays. This can be varied in tandem with varying member radius to study influence of member radius on structural properties.

Figure 5.1 shows the influence of member radius and the value of $h/r$ on the critical elevation. Increasing the value of $h/r$ decreases the critical elevation, thus increasing the range of achievable shapes. This figure also shows the effects of member radius on achievable shapes. As one might expect, thicker members create greater restrictions on the shapes that can be reached from the starting straight shape.

Member diameters are indicated non-dimensionally as a percent of the circumscribing radius in Fig. 5.1. Since the circumscribing radius is held fixed for the plotted cases, each x-tick value of the plot corresponds to the same member diameter value, regardless of $h/r$.

Also note that for some values of $h/r$, the critical elevation approaches and reaches $\pi/2$, corresponding to the straight shape. This means the member diameter is so thick compared to $h/r$ that even the straight shape exhibits member interference. These, of course, are impractical designs, but are kept in the plot to show the trend this extreme is approached. Overall, from the perspective of greater shape range being desirable, higher values of $h/r$ and lower values of member diameter are desirable.
Figure 5.1: Critical elevation vs. member radius as a percent of the circumscribing radius. Different colored lines correspond to different values of h/r.

The method of packing or stowing for launch is described in Chapter 4, and is defined again with Equations (5.1) and (5.2).

\[ \varepsilon = 1 - \frac{V_{\text{stowed}}}{V_{\text{deployed}}} \]  

With:

\[ V_{\text{stowed}} = \pi (r_p)^2 S_d N \]  

The package radius, \( r_p \), can be defined using geometry and the law of cosines with respect to the strut length and bottom face cable length by Equation (5.3).

\[ r_p = \left( \frac{Ca}{2 \cos(30^\circ)} \right)^2 + (S - Ca)^2 - 2 \left( \frac{Ca}{2 \cos(30^\circ)} \right) (S - Ca) \cos(150^\circ) \]  

Where \( Ca \) is the length of the face cables and \( S \) is the length of the struts. Clearly, the packaging efficiency is worse for longer struts. Increasing values of \( h/r \) directly increases the length of struts, and thus decrease packaging efficiency. Furthermore, increasing values of member diameter directly increase the height of the stowed volume, thus decreasing packaging efficiency. This is
reflected in Fig. 5.2. Thus, from the perspective of packaging efficiency, lower values of $h/r$ and member radius are desirable. Note however, that for some higher values of $h/r$ and member radius, the packaging efficiency as defined here may be negative, corresponding to the packaged state being larger in volume than the deployed state. This happens as the structure requires bays to become very tall and thin, since the struts then become so long that storing them vertically is in fact a smaller volume than when they are laid flat. For such cases, this method of packaging would not be recommended, and other types of stowing would need to be explored.

![Graph showing packaging efficiency vs. member radius for different values of h/r. The value of r is held fixed, and refers to deployed circumscribing radius.](image)

**Figure 5.2:** Packaging efficiency vs. member radius for different values of $h/r$. Member radius is plotted as a percent of the circumscribing radius. The value of $r$ is held fixed, and refers to deployed circumscribing radius.

When considering stiffness via bending tip deflection, the structural materials, pre-stress coefficient, and circumscribing radius are all held fixed to allow for comparison. The struts were taken as solid Kevlar 49 resin-impregnated strands ($E=124$ GPa), with a radius of 30 mm. Cables were taken as unidirectional Mitsubishi K13C2U UHN /epoxy ($E=536$ GPa) of radius 10 mm. The pre-stress coefficient was taken as 0.122 N/mm. These materials and member sizes were based on the findings of Yildiz [19]. The loading force is also held at a fixed value of 20 N. Thus, the only
parameters which are allowed to change for this comparison are the values of $h/r$ (with $r$ held fixed at 250 mm) and the member radius. For fixed member radii, increasing $h/r$ increases the tip deflection. This is to be expected, since increasing $h/r$ is increasing the total height, and thus offers a quick validation of the linear FEM used. Figure 5.3 reflects these results, showing different values of $h/r$ as different colored lines. When comparing increasing member radii, a factor is multiplied by the original member radii. Figure 5.3 shows the results non-dimensionally, with the strut radius marked as a percentage of the circumscribing radius. More material, as expected, offers more resistance to deflection, thus greater member radii decrease the bending tip deflections.

![Figure 5.3: Tip deflection vs. member radius plotted as percent of circumscribing radius for different values of $h/r$. The value of $r$ is held fixed at 250 mm.](image)

**5.2 Influence of number of bays on structural properties**

In many types of missions, the driving design parameter for a boom will often be the overall height (or length). In this way, for a given overall height, the number of bays may be varied by changing the height of individual bays to isolate and observe the effects of additional bays within a boom. Optimization of the number of bays will depend strongly on specific mission
requirements, such as a required minimum stiffness, natural frequency or maximum base footprint on the spacecraft. In general, however, the effects of additional numbers of bays may be isolated by fixing the member radii, materials, pre-stress coefficient, and ratio of total height to circumscribing radius, $H/r$, to see how the number of bays influences structural performance.

In each of the cases shown in this section, the circumscribing radius is 0.25 m and the member radii are 30 mm and 10 mm for struts and cables respectively. The materials of the members and the pre-stress are the same as in Section 5.1. In the case of critical elevation, increasing the number of bays initially decreases the critical elevation, thus increasing the range of shapes that can be achieved. When holding the net height fixed and changing the number of bays, the critical elevation for some number of bays may eventually reach $-\pi/2$ radians, indicating all azimuth-elevation combinations are possible for the centerline (from bottom-most face to top-most face). The number of bays required to reach this point (if it is reached) varies based on the individual bay heights, $h/r$ (dictated by $H/r$ and the number of bays). Accounting for members having a finite diameter can severely restrict the critical elevation, especially for small values of $h/r$ for individual bays.

Figure 5.4 shows results for the three values of $H/r$ in different colors. In general, increasing the number of bays (up to a point) will decrease the critical elevation, and higher values of $H/r$ will result in lower critical elevations being reduced more quickly. If $H/r$ is sufficiently low (like in the case of $H/r=32$ and 16) increasing the number of bays beyond a certain point results in individual bay $h/r$ small enough (relative to member diameter) to begin increasing critical elevation again. Numbers of bays beyond 16 are not considered with member collision in extensive detail. The methods developed are sufficient to analyze the optimal number of bays in terms of achievable shapes given the member radii.
Figure 5.4: Critical elevation vs number of bays for different values of $H/r$, where $r$ is held constant. Struts have a radius of 30 mm and cables a radius of 10 mm.

Figure 5.5 shows the effects of increasing the number of bays on the packaging efficiency. The optimal number of bays in terms of packaging efficiency will vary based on the value of $H/r$. As mentioned in the previous section, some configuration parameter combinations will result in a negative packaging efficiency as defined here. The exact number of bays resulting in positive packaging efficiencies, and an optimal packaging efficiency, is dependent entirely on the value of $H/r$. Generally, however, larger numbers of bays tends to result in negative packaging efficiencies. Furthermore, for very large values of $H/r$, it is possible that this method of packing cannot result in a positive packaging efficiency. As mentioned for the single bay case, these particular designs would require the use of a different method of packing.
The effects of the number of bays on the bending tip deflection can be found in Fig. 5.6. For a given total height, increasing the number of bays increases the tip deflection under a fixed bending load. The same materials, diameters, pre-stress and applied external load use in the previous section are used to generate these. Increasing the number of bays decreases $h/r$ for each bay, which in turn requires struts to have smaller angles with respect to $x$-$y$ plane, and thus are less aligned with the long axis of the boom. This in turn makes the boom more susceptible to deflection from bending loads.

**Figure 5.5:** Plots of packaging efficiency vs number of bays, where the different colored lines indicate different values of $H/r$. The value of $r$ is held fixed between these different cases. Values of $H/r$ are separated into separate plots purely for visual clarity.

**Figure 5.6:** Tip deflection plotted as a function of the number of bays. Circumscribing radius is held fixed for each case at 250 mm.
In the case of single-bay designs, the mass of the overall structure is directly tied to the materials chosen and the member radius. The materials and member radius for the results shown here are kept the same as those used for the bending tip deflection cases. They also share the same circumscribing radius as the bending tip cases, 0.25 m.

In the case of multi-bay structures, the results are slightly less straightforward. Generally speaking, the struts in a tensegrity will be much more massive than the cables. This means the mass of the overall boom is dominated by the volume of struts present. The overall mass is described by Equation (5.4).

\[ M = \rho_s V_s N_s + \rho_c V_c N_c \]  \hspace{1cm} [5.4]

Where \( M \) is overall mass, \( \rho \) is density, \( V \) is volume, \( N \) is number of bays, and the subscripts \( S \) and \( C \) denote struts and cables respectively. Increasing the number of bays will decrease the length (and thus volume) overall of the struts, but it will increase the number of struts and cables overall. The results of varying the number of bays for different values of \( H/r \) are shown in Fig. 5.7. Generally speaking, increasing the number of bays for a given total height will also increase the overall mass of the structure.

\[
\begin{align*}
\text{Figure 5.7: Plot of the total mass vs number of bays for different values of } H/r.
\end{align*}
\]
The type of analysis used to determine the relationship between the design parameters—i.e. member diameter, height to circumscribing radius, and the number of bays—can be used to optimize a triplex boom for a given application. The optimal number of bays will vary depending on the value of $H/r$, while increasing the member diameter must be weighed against conflicting optimals of decreasing tip deflection vs increasing mass. The range of achievable shapes can also be considered, though the largest amount of achievable shapes in terms of the centerline is highly dependent on the value of $H/r$.

5.3 An example case—Canadarm2

The exact design of any boom will depend greatly on the intended mission. For example, if the boom is intended to be used as a support structure that only needs small adjustments for alignment, a wide range of shape change is not critical, while properties like stiffness and packaging efficiency will still be very important. Since the study of the range of motion was a main focus for this thesis, the Canadarm2 will be used as a comparative case study.

Figure 5.8: An image of Canadarm2 taken by astronaut David Saint-Jacques on the ISS. (Credit: Canadian Space Agency/NASA) [47].
The Canadarm2 is used as a robotic arm to assist with maneuvers on the International Space Station, such as maintenance, moving equipment, and capturing visiting vehicles [46]. As a robotic arm, a wide range of reachable volume is necessary to assist in maneuvering, as well as sufficient stiffness to handle and move the objects it will be maneuvering. The design of the Canadarm2, depicted in Fig. 5.8, was likely driven primarily by these two factors. The Canadarm2 does not change any member lengths, and thus has the same “stowed” volume as when it is fully extended; in the fully folded state it simply has a different, shape than when fully extended. Detailed specifications of the Canadarm2 are difficult to find. Table 5.1 shows the specifications used for this comparison, taken from sources [46], [48] and [49]. In the case of bending stiffness, the value was found experimentally from a model meant to simulate the Canadarm2 by Lanouette et al. [49].

As a rough approximation, the extended Canadarm2 can be approximated as a cylindrical beam. The tip deflection can be found by Equation (5.5).

$$\delta_t = \frac{FL^3}{3EI} \quad [5.5]$$

<table>
<thead>
<tr>
<th>Specification</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total length</td>
<td>17.6 m</td>
</tr>
<tr>
<td>Arm radius</td>
<td>175 mm</td>
</tr>
<tr>
<td>Total mass</td>
<td>1800 kg</td>
</tr>
<tr>
<td>Material</td>
<td>19 plies of carbon high strength</td>
</tr>
<tr>
<td>carbon fiber-thermoplastic</td>
<td></td>
</tr>
<tr>
<td>Bending Stiffness (EI)</td>
<td>3.53x10^6 Nm^2</td>
</tr>
</tbody>
</table>
Where $\delta_t$ is the tip deflection, $F$ is the applied tip load, $L$ is the length, and $EI$ is the bending stiffness. With the values from table 5.1 and an applied load of $F=20$ N, the bending tip deflection of the Canadarm2 can be approximated as 0.01 m (10 mm).

Tensegrity structures differ greatly from the design of the Candarm2. To approximate the Candarm2 as a solid cylinder for tip deflection likely results in smaller tip deflections than would actually be found when the offset members and joints are accounted for. However, the Candarm2 is undeniably more like a solid cylinder than a tensegrity structure, which is more akin to a truss structure. In this sense, it is only reasonable to expect that a tensegrity structure will have less mass and better packaging efficiency, while the Canadarm2 will have greater stiffness for the same cylindrical volume. With this in mind, the methods developed throughout this thesis are applied to the design of a class-2 triplex boom, varying the number of bays and keeping the overall height fixed at 17.6 m.

Some results are shown in Figures 5.9-5.11 for a general understanding of the trends. In addition to varying the number of bays, the size of the circumscribing radius may be changed as well. The circumscribing radius is analogous to the 175 mm arm radius of the Canadarm2. In the case of packaging efficiency (Fig. 5.9), Equation (5.1) is used to define the efficiency, such that values closer to 1 are more efficient. The Candarm2 was not designed as a deployable structure, and essentially takes up the same amount of volume when fully extended vs fully folded. While packaging efficiency is important, it is difficult to compare packaging efficiencies directly. Figure 5.9 (b) shows the same values as Fig. 5.9 (a), but in terms of the stowed volume instead. For values of $r$ larger than 175 mm, the deployed volume will be larger than that of the Candarm2, but the packaging efficiency will change based on values of $r$ (generally improving for higher $r$ values). In this sense, the stowed volume might allow for some larger values of $r$ where the packaged state
takes up the same volume as the canadarm2. Figure 5.9 (b) provides insight in the potentially achievable stowed volumes.

**Figure 5.9**: Packaging efficiency of triplex booms of total height 17.6 m. Left (a) is number of bays vs efficiency, right (b) is number of bays vs stowed volume.

**Figure 5.10**: Tip deflection of triplex booms under 20 N bending load. Deflection is plotted in mm against number of bays. The overall height is fixed at 17.6 m. The two plots show the same tip deflections, but look at different values of circumscribing radius (separated for clarity).

Figure 5.10 shows the bending tip deflection as the number of bays are varied, as well as the circumscribing radius. For these calculations, the struts are taken as hollow (this improves bending stiffness as well as mass), with an outer radius of 30 mm and inner radius of 29.7 mm.
These values are chosen based roughly on the optimization work done by Yildiz [19]. The cables have a radius of 10 mm. The materials are kept the same as in previous chapters. The pre-stress coefficient is 0.122 N/m, and a load of 20 N is applied at each topmost face node, with the bottommost face nodes fixed in x-y-z. As with packaging efficiency, the overall height is fixed at 17.6 m, with the number of bays as well as the circumscribing radius being varied. The triplex booms are expected to have greater tip deflections than the cylindrical representation of the Canadarm2 for the same volume. Figure 5.10 (b) zooms in on some more promising values of circumscribing radius. Sacrificing volume for smaller tip deflections, a greater resistance to bending can be optimized by changing the value of circumscribing radius for a given number of bays.

Mass can also be calculated, using Equation (5.4). Figure 5.11 shows the results of varying the number of bays and varying the circumscribing radius on overall mass. As expected, increasing the number of bays generally increases the overall mass of the structure. However, the Canadarm2 mass is 1800 kg. Figure 5.11 demonstrates how for even the preliminary values considered, the mass in any of the cases is less than the mass of the Canadarm2.

![Figure 5.11: Mass vs. number of bays for different values of circumscribing radius. All booms have an overall height of 17.6 m.](image)
Two primary ways to optimize a triplex boom for this application were considered. Achieving the same bending resistance as the Candarm2 is impractical for comparable values of circumscribing radii. In practical applications, a tensegrity design might have to consider the use of multiple booms, or else different boom configurations (i.e. different numbers of struts per bay, different materials etc.) to achieve the same bending deflections while occupying the same volume. If bending stiffness is considered the limiting factor, more accuracy in bending tip deflection calculations would also be required, and perhaps merit the use of a nonlinear finite element method, rather than the linear one developed in this thesis. Instead, the bending case will here be minimized (rather than meeting a strict maximum deflection) when considering 2 different limiting factors. First, consider the volume as the limiting factor.

When the Candarm2 is fully extended, its volume (approximated as aligned cylinders) is 1.6933 m$^3$. This volume is better compared with the deployed volume of the triplex rather than the stowed—packaging efficiency in this case is more of a bonus. The limiting volume is that of the Canadarm2, and thus triplex booms with circumscribing radii of 175 mm or less are within the acceptable volume range. For a given value of circumscribing radius, increasing the number of bays yields a smaller per-bay height (thus decreasing $h/r$). Thus, for any value of circumscribing radius, there is a maximum number of bays which are possible without starting with the struts (of radius 30 mm) touching. The circumscribing radius was varied in 1 mm increments. The smallest value of circumscribing radius found possible was 134 mm, with a maximum of 7 bays possible. The resulting deployed volumes as a function of circumscribing radii are plotted in Fig. 5.12 (a). Figure 5.12 (b) shows the maximum number of bays as a function of viable circumscribing radii. Viable circumscribing radii are taken as those in which the straight configuration for the maximum number of bays does not require struts to touch (the number of bays are not checked beyond 40).
Note that as the number of bays approach the maximum, the restrictions on motion will be greater due to members being closer to touching in the straight configuration.

Figure 5.12: Circumscribing radii effects on deployed volume. Left (a) shows deployed volume as a function of circumscribing radius. Right (b) shows the maximum possible number of bays without the straight shape requiring members to touch.

Figure 5.13 shows the resulting tip deflections of the viable values of $H/r$. Clearly lower values of $H/r$ result in lower tip deflections. Since the height is fixed, this means the maximum value of $r$, 175 mm, is most desirable in terms of tip deflection. Figure 5.13 (b) shows the same plot, but only for $r=175$ mm. The optimal number of bays in terms of minimizing tip deflection is found for 10 bays for this ratio of $H/r$.

Figure 5.13: Tip deflection as a function of the number of bays for different values of $H/r$. $H$ is held fixed at 17.6 m. Left (a) varies $r$, with higher values of $r$ shown in the direction of the arrow (more red). Right (b) shows only the line of $H/r$ with the lowest possible tip deflection, with $r=175$ mm.
The range of achievable shapes is important to consider for robotic applications. A more detailed study characterizing the solution set of achievable shapes should be considered for such applications than can be explored in this thesis. While azimuth and elevation of the bottom-to-top face centerline is relevant for pointing based applications, robotic manipulators are chiefly concerned with the exact location and orientation of the end-point of the boom. In terms of the centerline, this would mean including the radial distance to the center of the topmost face as well (relative to the origin). This was not studied in any detail for this thesis outside of a single-bay case.

Increasing the number of bays, to a point, will increase the range of possible shapes. For a circumscribing radius of 175 mm and net height of 17.6 m, the critical elevation reaches \(-\pi/2\) for only four bays, and is not changed by increasing the number of bays to 10. In terms of the centerline, then, a large range of shapes is available with 10 bays, so this might be an optimal build for the volume-limited case.

Another way to consider designing the boom would be to let mass be the limiting factor. The Canadarm2 mass is taken as 1800 kg [46]. Figure 5.14 shows how mass varies with the circumscribing radius, from 175 mm to 2600 mm. For most cases, even with 40 bays, the boom mass does not exceed the mass of the Candarm2. The lowest number of bays found to require a mass exceeding 1800 kg is 39 bays for a value of \(r = 2325\) mm. The lowest value of \(r\) exceeding 1800 kg was found for 40 bays with \(r = 1925\) m. Thus, for the values of \(r\) considered, a circumscribing radius exceeding 2325 mm must have 38 bays or fewer, and a value of \(r\) more than 1925 mm but less than 2325 mm must have 39 bays or fewer.
With mass as the limiting factor instead of volume, finding a configuration with the same tip deflections (or less) is more feasible. Generally, increasing the circumscribing radius will improve resistance of the structure to bending tip deflections. This a larger radius is more desirable for decreasing tip deflections. In terms of range of achievable shapes, however, higher values of individual bay $h/r$ improve upon the range of shapes. These higher individual bay $h/r$ values are achieved (for a fixed overall height) with smaller circumscribing radii. Thus, for the mass-limited case, a circumscribing radius of 600 mm is chosen: the smallest circumscribing radius (found from 100mm increments, Fig. 5.10) which can match or improve upon the Candarm2 tip deflections.

The tip deflections for the case of $r=0.6$ m are, overall, lower than the case of $r=175$ mm, as would be expected from increasing the circumscribing radius. Figure 15 shows the tip deflections for the circumscribing radius of 0.6 m. This figure shows, like in the previous case, that a lower number of bays generally reduced the tip deflection the most. In this case, the minimum tip deflection is found for 5 bays. If the only restraint in stiffness, however, is to match
that of the Canadarm2, then Fig. 15 shows that any number of bays between 2 and 18 would be acceptable (all having less than 10 mm of tip deflection).

![Graph showing tip deflection vs number of bays](image)

**Figure 5.15:** Tip deflection as a function of the number of bays for the case of a net height of 17.6 m and circumscribing radius of 0.6 m

Generally speaking, increasing the number of bays (to a point) for a fixed boom height will increase the range of achievable shapes. Figure 5.16 shows the critical elevation vs number of bays for the case of $r = 0.6$ m. From this figure, it can be seen that at some point beyond 8 bays, increasing the number of bays results in each bay having a small enough $h/r$ that member interference becomes an issue, thus resulting in greater restriction on achievable shapes than lower numbers of bays. From Fig. 5.16, it is also seen that any number of bays between 4 and 8 results in a critical elevation of $-\pi/2$. In terms of the centerline from bottom-most to top-most face, then, any number of bays between 4 and 8 would allow for the maximum range of achievable shapes.
Since both stiffness and the critical elevation can be satisfied for a range of number of bays, packaging efficiency may be considered to help determine the best configuration. Figure 5.17 shows the packaging efficiency for a boom of $r=0.6$ m. Considering the restraints of tip deflection and critical elevation, packaging efficiency can be used to choose amongst the resulting range of acceptable bays, being between 4 and 8. For lower numbers of bays, Fig. 5.17 shows that packaging efficiency is improved by increasing the number of bays. Thus, a build with a circumscribing radius would be recommended to have 8 booms. This number of booms offers better structural stiffness (with a tip deflection of 7.6 mm), the maximum range of achievable shapes, as well as a positive packaging efficiency (with the stowed volume ~30% of the deployed volume).

*Figure 5.16: Critical elevation as a function of number of bays for circumscribing radius of 2.6 m and a net height of 17.6 m.*
The Candarm2 is taken as an example of how the methods developed in this thesis might be used to design a reconfigurable tensegrity boom for a specific mission. For robotic manipulators requiring high stiffness, additional research would be needed to improve the stiffness of a triplex boom—both in the FEM code itself to determine deflections more accurately, as well as consideration of alternative configurations, such as multiple booms, more struts etc. A more detailed way to characterize the range of achievable shapes would likely be required for robotic manipulators as well.

The methods developed in this thesis are sufficient to provide the set of all achievable shapes for a given triplex configuration. However, the azimuth and elevation of the centerline are prioritized here as a way of characterizing the shapes. This is insufficient to fully describe the shape range for a robotic manipulator, and more work would need to be done to characterize the solution sets if robotic arms were the intended use. With this in mind, the Candarm2 was used as an example to provide some basic guidelines for a potential triplex boom design. This analysis could be improved by extensive implementation of more sophisticated design optimization methods. The two cases evaluated here indicate that deeper consideration of different
circumscribing radii (especially in the case of mass being the limiting factor) could lead to more optimal builds, which have good stiffness as well as a large range of achievable shapes. The overall results include a few potential designs for triplex booms that could most closely meet the mission goals of Canadarm2. While additional research would need to be done to improve the analysis for a robotic arm specifically, this case study still effectively demonstrates the utility of the methods developed in this thesis for preliminary design of a triplex boom for a specific mission.
Chapter 6
Experimental Verification

In previous chapters, this thesis developed methods to address the structural performance of triplex booms, considering the effects of various design parameters. A few important practical considerations must be addressed, however, if such structures are to actually find use in space applications. For one, the shape change analysis assumes some method of actuation exists which can accommodate the cable length changes required, without addressing what this method might actually be. Furthermore, it does not address a way to accommodate non-ideal actuator performance, to understand whether or not these shapes have actually been achieved and within what degree of accuracy.

To begin to address these issues, an experimental structure was built and tested to characterize shape change along a representative prescribed path. This experiment was designed to verify that a range of shapes were possible with all but 3 members being fixed in length, as well as to verify the collision-checking methods along the prescribed path. In doing so, methods of actuation and metrology were considered, and are discussed for their potential uses in future designs.
6.1 Existing design considerations

In designing an experimental structure, two major considerations were addressed: how to actuate the cables, and how to measure the shapes. The experiment mainly intended to verify that different shapes could be achieved and test for collisions along a path. Materials selection was thus somewhat arbitrary, as the analysis methods developed to address achievable shapes and to detect collisions can account for any member radius, and do not depend on the specific material.

Actuation is an extremely important aspect of changing tensegrity shapes. Due to project budget constraints, cable lengths were changed by hand. This was, by far, the biggest source of error in the experiment.

Automating the process of actuation and metrology could greatly improve the accuracy. In autonomous designs, actuation and metrology methods are often tied together for implementation with controls systems. Much research has been done in the past addressing methods of controls for shape-changing tensegrity structures. Henrickson et al. developed a shape control method for tensegrities based on changing the tension in the cables [50]. Widjeven and Jager describe a control methodology based on prescribing and objective and constraints which allows for the length change of any of the members [15]. These methods are more model-focused, and centered on the control aspects of the problem.

Other work has been done considering the actuation method itself, in tandem with the controls. Bronfeld looked at shape change controls using Assur Trusses by changing some specified member lengths. In Bronfeld’s experimental design, a pseudo-tensegrity is constructed (with the top and bottom faces of a triplex being made of plates, to which the struts and supporting cables are separately connected). This structure is an approximation of an actual triplex, but allows for an understanding of shape change by changing struts and supporting cable lengths. The struts
are actuated cylinders, made from commercially-available linear actuators, and the cables are actuated with winches using a motor to spool cables. For metrology, linear potentiometers are used to measure linear actuator strokes and calculate strut movement, rotational potentiometers similarly measure motor movement, and load cells are mounted on the cables to measure tension. These methods of measurement are used for control methods developed by Bronfeld [51]. Böhm and Zimmermann considered a vibration-based method of tensegrity movement with only one actuator, implemented by using the force from an electromagnet to change the lengths of springs in the structure. This research used an optical laser sensor was used to measure displacements of the tips of struts as a metrology method [52]. Chen et al. used a tensegrity design with motors to spool the cables, considering both control methods as well as multiple types of motors, selecting a final motor for actuation based on experimental performance. This work also mentioned the use of motor encoders as a metrology method for the shape control, with the note that future work should include autonomous sensors and better feedback control [53]. Sychterz and Smith actuated a cylindrical tensegrity with continuous cables running through channels at nodes, with an optical motion tracking system capturing movement of end nodes. Their controls methodology used information from the feedback of strain gauges and motor control data [54].

The works discussed here represent a small selection of the experimental work that has been reported in literature. Each tensegrity build offers a unique combination of actuation and metrology methods, the choices of which are motivated by the application, funding and available resources. The most elegant system designs intrinsically integrate actuation and metrology methods with the control system to provide an efficient method of shape change. The focus of this thesis was not to design a tensegrity structure, but to outline a methodology which could be used for more detailed design in the future. Methods of actuation, control and metrology like those
discussed in this section would need to be thoroughly considered as well in the physical design of a tensegrity structure. Though project budget constraints did not allow for integrated actuation/metrology with control systems in this thesis research, a physical model was built to verify some of the range of achievable shapes and collision checking along a prescribed path, using the methodology developed in previous chapters.

6.1.1 Experimental build

Figure 6.1 shows the experimental build. The triplex was made of 3 threaded ¼ in. diameter, 1 ft. long aluminum rods, and low-stretch, no-flatten braided 3/16 in. diameter polyester rope. Eye hooks with a 1 in. diameter were screwed together at each end of the aluminum rods, acting as nodes. The top and bottom face cables were attached to the eye hooks using rope clamps to keep their lengths fixed. The supporting cables were attached to the top-face nodes with a rope clamp, then threaded through the bottom-face node eyehooks, then passed around a re-routing rod to redirect the cable to a rod where they could be spooled and clamped (at clamping rod in Fig. 6.1). The bottom face nodes were fixed in space on the base board using additional strings through the bottom-face eyehooks to secure their location. Measuring ticks separated by ¼ in. were made along the flat stretch of supporting cable between the re-routing rod and clamping rod, used with the cable measurement marker. Motion tracking markers were affixed to the top-face eyehooks, and directly adjacent to the bottom-face eyehooks. A Vicon motion tracking system was used to track these markers, with accuracy on the order of 1 mm.
A pre-prescribed spiral path (in the x-y plane for the top-face centers) was planned, shown in Fig. 6.2 (a). The required length of each supporting cable through each discrete step of the spiral path was calculated. The cable measuring markers on the excess portion of the supporting cable were used to track cable lengths. Though the motion tracking system offers good accuracy, the method of actuation left considerable room for error. First, the marker on the cable was around ¼ in. thick itself, making accurate positioning relative to that marker difficult. The cable material was chosen because of its low-stretch property. However, the cables were clamped by wrapping around a threaded rod, and then pinching the cable between the base board and a large washer. When cables were tightened and loosened for a single position, tensions in certain cables made it difficult to maintain the exact positions a given cable was set to. Furthermore, the length of each supporting cable was measured by eye, while changing 3 cable lengths individually for 53 unique
steps in the spiral path. This undoubtedly led to significant human error, causing the path to vary from the desired prescribed spiral.

Figure 6.2 (b) shows the path the structure actually followed based on the motion tracking data. Rather than compare the motion tracking data to the intended path, the motion tracking path used was instead used to dictate the path in the models. The path was checked for confirmation that the points the center moved through were in fact on the solution surface, and that member collision occurred on the path when expected.

![Figure 6.2](image)

*Figure 6.2: The planned spiral path in red (left, (a)) and the path followed in the experiment, shown as pink stars for each location of the top-face center (right (b)).*

### 6.2 Results

The primary purpose of this experiment was to confirm that a variety of shapes could be achieved by only actuating 3 cables in a triplex. Figure 6.4 visually confirms this, showing multiple shapes the experimental structure achieved. This experiment also aimed to prove the range of achievable shapes considering member collisions could be predicted with the models described in this thesis.

Figure 6.5 shows the generated solution surface of the top-face centers for a triplex corresponding to the experimental configuration. The path the triplex moved through is shown on
the solution surface in pink. In general, this path was close to the solution surface generated by the methods detailed in this thesis. The distance between each point captured by the motion tracking system and the nearest point on the solution surface was calculated. Some error arises from the sparsity of the points in the solution surface, as the truly closest possible point may not be included for every point along the path. Furthermore, the markers for the motion tracking have a finite diameter, and were attached at the top of the eyehooks, slightly above the actual node location. This generally caused the tracked location to be slightly higher than the locations on the solution surface. It also allowed their exact orientation and location relative to the center of the top face to vary slightly depending on the particular shape. With this in mind, the distances between the solution surface and the motion-captured positions are shown in Fig. 6.6. The maximum distance was found to be 0.55 in. Given the offset of the eyehooks as well as the sparsity of data to compare to, these distances are promising in terms of verifying the ability generate a solution surface.

When following the prescribed path in pink in Fig. 6.2 (b), no collisions were found in points the motion tracking system successfully captured. While the experiment was run until members visibly collided, unfortunately the motion tracking system lost the markers (due to the marker orientations varying relative to each other, since they are not on the same object). The points shown in Fig. 6.2 (b) are only those picked up by the motion tracking system, and thus only show points without collision. When the solution set that was found is prescribed to follow that path, it finds no member interference as well. Further experiments would need to be run to confidently say this experiment verifies the collision-checking methods, but the results thus far are promising.
Figure 6.3: Three different shapes of the experimental triplex structure, achieved by only changing lengths of the 3 supporting cables
Figure 6.4: The solution surface of the center of the top face for all possible shapes (blue dots), overlaid with the path followed by the top-face center in the experiment (pink).

Figure 6.5: The distance between the points tracked by the motion capture system in the experiment, and the nearest point in the solution set.
Chapter 7

Summary and Conclusions

This thesis developed a methodology for conducting design analysis of shape-changing class-2 triplex booms that use cable actuation. Chapter 1 provided an overview of deployable structures and articulated spacecraft structures. Potential applications for which a shape changing tensegrity boom might be well suited were also identified. Examples of existing research on actuated tensegrity structures were explored for their relevance to this thesis.

Chapter 2 provided a more thorough introduction to tensegrity structures. Nomenclature and physical tensegrity characteristics were described in detail, focusing on tensegrity geometries as well as mechanics. More in-depth methods of studying tensegrity mechanics were also developed, after reviewing existing research and the underlying theory for form-finding and force-finding methods.

Using this general understanding of tensegrity, Chapter 3 described the development of a computational method for defining the range of shapes achievable for a single-bay triplex actuated by the supporting cables. This method first finds all geometrically possible shapes for a single triplex bay, then refines the solution set to contain only those shapes that are in equilibrium. There are many different ways to depict the solution set generated using this method, a few of which were explored in detail. In particular, the critical elevation was identified as a defining characteristic of the range of achievable shapes for triplex bays—defined such that any elevation of the centerline above this critical elevation can be reached at any azimuth value (-π to π). Furthermore, a method of extending the single-bay results to multi-bay structures was developed, using a transformation between adjacent bays to compound the single-bay results.
Next, in Chapter 4, a methodology for understanding some structural characteristics was developed. First, a way to identify when member interference can occur along a prescribed path was considered. This member interference was found, for single-bay cases, to restrict the achievable critical elevation; this was based on the range of shapes that could be reached in radial paths from an initial straight shape. These results were then extended to multi-bay cases to observe the effect on overall critical elevation. A method to check for interference between members of non-adjacent bays was also described.

Stiffness and packaging efficiency, key characteristics of structural booms, were considered in Chapter 5. Structural stiffness was explored in terms of the bending tip deflection under load, using linear finite element models to calculate deflections. Torsional loads were also considered using linear finite element models to calculate the twist deflections. The effects of pre-stress on the deflections were also considered, finding in most cases that increasing pre-stress decreases deflection. Packaging efficiency was also addressed, packing struts flat in a triangular pattern described by a cylinder whose radius is dictated by strut length and whose overall height is dictated by member diameter and the number of bays.

Using these methods, analysis detailing the effects of different configuration parameters on structural properties and performance was conducted. In the case of a single bay, the ratio of bay height to circumscribing radius \((h/r)\) and member radius were the driving parameters. Increasing \(h/r\) was found to decrease the critical elevation (i.e. increase the range of achievable shapes) and decrease packaging efficiency. Similarly, increasing member radii was found to decrease the critical elevation, decrease tip deflection under bending, and decrease packaging efficiency.
Then, the effects of changing the number of bays for a fixed overall ratio of boom height to circumscribing radius ($H/r$) were considered. These effects tend to vary case-by-case depending on $H/r$. In terms of critical elevation, for sufficiently high values of $h/r$ for each bay (dictated by $H/r$ and the number of bays), the critical elevation is generally increased by adding more bays. Sufficiently high values of $H/r$ may eventually reach a critical elevation of $-\pi/2$, indicating the maximum range of shapes in terms of centerline tip motion are achieved. Tip deflection under bending load was generally found to increase as the number of bays were increased. Packaging efficiency was found to also depend on the specific value of $H/r$. Generally speaking, more bays are required to have a better packaging efficiency for higher values of $H/r$, while the specific optimal number of bays will vary case-by-case.

Mass was also considered, showing that increasing the number of bays increases the overall structural mass. An example design case was considered, using some specifications from the Canadarm2 to dictate the design goals of a triplex boom. This comparison recommended a build of 10 bays for a boom with the same circumscribing radius of 175 mm for a case where the volume must be less than or equal to the Canadarm2 volume. In the mass limited case (a more optimal solution), a build of 8 booms with a circumscribing radius of 600 mm will have a larger volume than the Canadarm2, but will offer better bending stiffness, lower mass, a wide range of achievable shapes and a good packaging efficiency.

Finally, some basic experiments were pursued and described in Chapter 6. First, existing actuation and metrology methods were reviewed. Methods of actuating cables were explored in particular, along with ways to measure the movement of the members and thus the overall shape of the structure. Then, an experimental structure designed for this thesis was constructed. Using a motion tracking system, a wide variety of shapes were found to be achieved by a single triplex bay
when only the supporting cables were actuated. The path followed in the experiment (based on the
points captured by the motion tracking system) found no direct collisions, which was verified
visually during the experiment.

In summary, this thesis accomplished its two main goals: first, developing an
understanding of the range of achievable shapes for a triplex boom and, second, developing
methods and conducting analyses to better understand how configuration parameters influence
structural properties and performance. The methods developed could readily be used for the design
of a triplex boom given specific mission requirements.

7.1 Future work

The research described in this thesis laid some groundwork for the design analysis of future
triplex booms. While the work completed showed successful implementation of these design
analysis tools, more work could be done in the future to improve the overall methodology and
design analysis of cylindrical tensegrity booms.

With regards to triplex booms, inclusion of small-angle collision checking between
members that share nodes would improve the accuracy of the overall range of achievable shapes
found. Member interference checking was successfully implemented for triplex booms between
members not sharing nodes, but the inclusion of members that do share nodes would result in
higher accuracy models of achievable shapes that are more directly applicable to physical designs.

The method developed for bending and twist deflection analysis could also be improved
by using non-linear finite element analysis. While the linear FEM used provided helpful insight
into general trends of the structure, non-linear FEM would more accurately capture the behavior
of tensegrity structures and should be implemented where more accurate results are required or when linear results are known to be inaccurate.

Finally, the experimental verification could be greatly improved. Higher accuracy actuation methods (e.g. with motors and controls) would greatly improve the accuracy of the experiment and allow for better and more direct comparison between the experimental results and the model predictions. Furthermore, implementation of motor-based actuation could greatly reduce the time it takes to run an experiment, and thus allow for more robust exploration of the potential solution space. Additional motion paths would allow for verification of the ability to reach all prescribed shapes found with the methods developed in this thesis, as well as more accurate verification of the collision checking methods used.
Bibliography


[51] Bronfeld, A. “Shape change algorithm for a tensegrity device,” Master’s thesis, Mechanical Engineering Dept., Tel Aviv University, Tel Aviv Israel


