The Pennsylvania State University
The Graduate School

# MACKEY BIJECTION FOR SOME REDUCTIVE GROUPS AND CONTINUOUS FIELDS OF REDUCED GROUP $C^*$-ALGEBRAS

A Dissertation in
Mathematics
by
Angel Román-Martínez



Submitted in Partial Fulfillment
of the Requirements
for the Degree of

Doctor of Philosophy

December 2019

The dissertation of Angel Román-Martínez was reviewed and approved* by the following:

Nigel Higson
Evan Pugh University Professor of Mathematics
Dissertation Advisor, Chair of Committee

Nathanial Brown
Professor of Mathematics

Paul Baum
Evan Pugh University Professor of Mathematics

Murat Gunaydin
Professor of Physics

Eyal Subag
S. Chowla Research Assistant Professor of Mathematics
Special Member

Carina Curto
Professor of Mathematics
Co-chair of Graduate Studies in Mathematics

*Signatures are on file in the Graduate School.

# Abstract

The purpose of this dissertation is to make a further contribution to the Mackey bijection for a complex reductive group $G$ and for $SL(2, \mathbb{R})$, between the tempered dual of $G$ and the unitary dual of the associated Cartan motion group. We shall construct an embedding of the $C^*$-algebra of the motion group into the reduced $C^*$-algebra of $G$, and use it to characterize the continuous field of reduced group $C^*$-algebras that is associated to the Mackey bijection. We shall also obtain a new characterization of the Mackey bijection using the same embedding.

# Table of Contents

# Acknowledgments

Life is hard. This is especially true in graduate school. This dissertation would not have been possible without the support of so many people. First, a special thank you to my advisor, Nigel Higson, for taking a chance and taking me on. This is a wonderful project that have sparked interest. Thank you to Paul Baum, Nathanial Brown, Murat Gunaydin, and Eyal Subag for taking the time to be a part of my doctoral committee. A thank you to Svetlana Katok, Nathanial Brown, Anna Mazzucatto, Carina Curto, and Alexei Novikov, for your efforts as chair of graduate studies during my time here. I also want to thank Becky Halpenny and Allyson Borger as administrative assistants to the graduate program. I do not know how we would get by without them.

I must also extend thanks to the close friends I have made in the mathematics department. Thank you Will, Jim, Juan, Sarah, and Dom. I also want to thank the members of Club Taekwondo at Penn State. You guys kept me sane.

I have retired from the world competitive stage in Taekwondo to pursue my doctorate. The community at large have always been supportive in any endeavor one may choose. A big thank you to the community at large, especially to the old teammates and coaches back home.

Finally, and perhaps most importantly, I must thank my family, especially my parents, Angel Román-Más and Aurora Martínez, my sisters Legnamarie and Aurora Natalia Román, and my grandmother Olga Más. I would not have made it without the love and support from you. Natalia, this dissertation is for you and Dave. Of course, a big thank you to the rest of the family that would brag about me to their friends. To my grandfathers, Juan Martínez and Angel Augusto Román, I hope I made you proud.

# Dedication

This dissertation is dedicated to

Carmen Fiona Olson.

This will be your legacy. Your uncle loves you.

# Chapter 1 | Introduction

In their early days, group representation theory and $C^*$-algebra theory had a close relationship. Representation theory began earlier than the axiomization of $C^*$-algebras, but many results about representations on locally compact groups were in essence proven using representations of $C^*$-algebras. Given a locally compact group $G$, we can define a $C^*$-algebra, called the full group $C^*$-algebra, denoted by $C^*(G)$ or $C^*_{max}(G)$. Then there is a bijective correspondence between the $*$-representations of $C^*(G)$ and unitary representations of $G$. Because of this, any question we may ask about unitary representations of $G$ amounts to asking a question about $C^*(G)$. This correspondence also gave rise to many good examples of $C^*$-algebras in the first decades of $C^*$-algebra theory. See [Ros94].

Later on, however, the two subject began to diverge. For instance, when George Mackey systemized the study of unitary representations of locally compact groups through his theory of unitary induction, $C^*$-algebras played little to no role. And the methods adopted by Harish-Chandra in his study of the tempered unitary representations of reductive group were even further removed from $C^*$-algebra theory. At the same time, the study of $C^*$-algebras began to take off on its own, following its own directions, independent of group representation theory. See [Ros94].

This dissertation is about a relatively recent effort to bring back together $C^*$-algebra theory and the work of Mackey and Harish-Chandra in group representation theory. This is by no means the only project that aimed at bringing back together $C^*$-algebra theory and group representation theory. For other examples, see [Ros94].

George Mackey systemized the study of unitary representations of locally compact groups by constructing unitary representations of $G$ from representations of smaller closed subgroups of $G$. The process is known as *unitary induction*, and we will discuss this in some detail in Chapter 3. His *Imprimitivity Theorem* then gave conditions that determine when a given representation of $G$ is actually induced from a representation of a subgroup. Collectively, these methods became known as the *Mackey Machine*.

For example, consider the semidirect product $SO(3) \ltimes \mathbb{R}^3$, with product operation given by

$$(k, \nu) \cdot (k', \nu') = (kk', (k')^{-1}\nu + \nu').$$

The Mackey machine tells us that

$$\widehat{SO(3) \ltimes \mathbb{R}^3} \cong \{(l, r) | r \geq 0, l \in \mathbb{Z}, \text{ and if } r = 0, \text{ then } l \geq 0\}. \quad (1.0.0.1)$$

The individual irreducible unitary representations are constructed by unitary induction as follows. If $\nu : \mathbb{R}^3 \to \mathbb{R}$ is a linear functional, then $\exp(i\nu)$ is a unitary character on $\mathbb{R}^3$. If $\pi$ is an irreducible representation of the isotropy group $SO(3)_\nu$, then the product $\pi \otimes \exp(i\nu)$ is a representation of the closed subgroup $SO(3)_\nu \ltimes \mathbb{R}^3$. This we then unitarily induce to obtain an irreducible unitary representation of $SO(3) \ltimes \mathbb{R}^3$. The Mackey machinery tells all irreducible unitary representations are obtained in this way.

If $\nu \neq 0$ then the isotropy group is $SO(3)_\nu \cong SO(2)$, and the dual of $SO(2)$ is parametrized by $l \in \mathbb{Z}$. If $\nu = 0$, then the isotropy group is $SO(3)_\nu = SO(3)$, whose dual can be parametrized by nonnegative integers $l \geq 0$. We attach the label $r = \|\nu\|$ to the induced representation above, and after considering the equivalences among the induced representations, we get (1.0.0.1).

Harish-Chandra is famous for his work on "tempered" unitary representations of real reductive groups such as $SL(2, \mathbb{R})$ or $SL(2, \mathbb{C})$, and many more. From the point of view of $C^*$-algebra theory, a unitary representation is tempered if the $C^*$-algebra representation

$$\pi : C^*(G) \to B(H)$$

associated to $\pi$ factors through the "reduced" $C^*$-algebra, which is the quotient of $C^*(G)$ by the kernel of the regular representation

$$\lambda : C^*(G) \to B(L^2(G)).$$

See Section 4.2. We shall write

$$C_r^*(G) = C^*(G)/\ker(\lambda).$$

The irreducible tempered unitary representations are the irreducible unitary representations that are required to decompose the regular representation into irreducible representations. In a bit more detail, it was shown in the early days of $C^*$-algebra theory by I. Segal (see [Seg50]) that there is a unique measure $\mu$ on the unitary dual such that

$$\|f\|^2_{L^2(G)} = \int_{\hat{G}} \|\pi(f)\|^2_{\text{H-S}} \, d\mu(\pi) \tag{1.0.0.2}$$

for all smooth and compactly supported functions $f$ on $G$, where

$$\pi(f) = \int_G f(g)\pi(g) \, dg,$$

and where $\| \ \|_{\text{H-S}}$ denotes the Hilbert-Schmid operator norm. We call the measure $\mu$ the *Plancherel measure* and the formula (1.0.0.2) the *Plancherel formula*. The tempered unitary dual is the support of $\mu$. See [Wal92, Vol 2, Ch 13].

Harish-Chandra explicitly constructed enough of the unitary dual to support $\mu$ (meaning that the complement of the set he constructed had measure zero) and he explicitly determined $\mu$ on this set.

By way of example, consider $PGL(2, \mathbb{C})$. The Fourier inversion formula (which is equivalent to the Plancherel formula) is given as

$$f(1) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{Trace}(\pi_{2n,i\nu}(f))(4n^2 + \nu^2) \, d\nu$$

for $f \in C_c^\infty(\mathrm{PGL}(2,\mathbb{C}))$, where $\pi_{2n,i\nu}$ is an irreducible “principal series” representation of $\mathrm{PGL}(2,\mathbb{C})$ (a type of unitarily induced representation). See for example [Kna86, Equation 2.24]. We can see from this that the Plancherel measure is $d\mu(\pi_{2n,\nu}) = (4n^2+\nu^2)\ d\nu$. Let us do another example to show that not all cases are as simple as the one we just did (and not all of them involve only unitarily induced representations). Consider $\mathrm{SL}(2,\mathbb{R})$. Then the Fourier inversion formula is given as

$$f(1) = \int_{-\infty}^{\infty} \mathrm{Trace}(\pi_{+,i\nu}(f))\frac{\nu}{2}\tanh\left(\frac{\pi\nu}{2}\right)\ d\nu + \int_{-\infty}^{\infty} \mathrm{Trace}(\pi_{-,i\nu}(f))\frac{\nu}{2}\coth\left(\frac{\pi\nu}{2}\right)\ d\nu \\ + \sum_{n=2}^{\infty} 4(n-1)\,\mathrm{Trace}(D_n^+(f)) + \mathrm{Trace}(D_n^-(f)) \qquad \text{( [Kna86, Equation 2.25])}$$

with $f \in C_c^\infty(\mathrm{SL}(2,\mathbb{R}))$, where $\pi_{\pm,\nu}$ are the principal series representations and $D_n^\pm$ are the discrete series representations. See also [Var89, Chapter 6].

In 1975, George Mackey made an observation, motivated by his work on induced representation and his interest in physics. In the following passage he explains his idea in the case of $G = \mathrm{SL}(2,\mathbb{R})$:

> ... the physical interpretation suggests that there ought to exist a “natural” one-to-one correspondence between almost all of the irreducible unitary representations of $E$ [the group $\widetilde{SO(3)} \ltimes \mathbb{R}^3$ where $\widetilde{SO(3)}$ is the universal covering group of $SO(3)$] and almost all of the irreducible unitary representations of $\mathrm{SL}(2,\mathbb{C})$– in spite of rather different algebraic structures of these groups.

Let us consider instead the closely related example of $G = \mathrm{PGL}(2,\mathbb{C})$ (That is, the group $\mathrm{GL}(2,\mathbb{C})/(\mathbb{C}^\times \cdot \mathrm{Id})$), as above. Let $G_0 = SO(3) \ltimes \mathbb{R}^3$, the semi-direct product. This is called the *Cartan motion group* for $G$. Recall that the semi-direct product carries the following composition: $(k,\nu)\cdot(k',\nu') = (kk', (k')^{-1}\nu + \nu')$. Thus $G_0$ is the group of orientation-preserving isometries of a three-dimensional Euclidean space. On the other hand, $G$ is the group of orientation-preserving isometries of the complete, simply connected three-dimensional Riemannian space with a constant negative curvature, that is, a hyperbolic space $\mathbb{H}^3$. It should be obvious that the algebraic structure of $G$ and $G_0$ are very different.

The Hilbert space of states for a quantum mechanical particle is a unitary representation of $G_0$. But $\mathbb{H}^3$ is a plausible model for physical space. Mackey made the observation that if we replace the usual Euclidean space by the hyperbolic space, and work out the quantum mechanics of a single free particle, we get a unitary representation of G. It is from this that Mackey made the suggestion that there is a close relation between the unitary representations of $G_0$ and G. He proceeded to explain a one-to-one correspondence "almost everywhere" by mathematical means: there is a coincidence between the parameters that describe irreducible representations of various subgroups, that are then induced up to each group. In the example we are considering, the dual $\widehat{PGL(2, \mathbb{C})}$ is parametrized by pairs $(2l, r)$ where $l$ is a non-negative integer, $r \in \mathbb{R}$, and if $l = 0$ then $r \geq 0$. Then the Mackey bijection is given as

$$(l, r) \mapsto (2l, r)$$
$$(-l, r) \mapsto (2l, -r),$$

where $l$ and $r$ are non-negative. (Actually, Mackey stopped short of considering the case where $r = 0$, which is partly why he spoke of a correspondence almost everywhere.)

While explaining a correspondence for the complex semisimple groups seemed relatively easy, Mackey went further and attempted to develop a correspondence for $SL(n, \mathbb{R})$, which was significantly more involved and difficult . For $n \geq 3$, this was developed by induction, and he had hoped that for the general real group, a similar process could be developed. However, the discrete series presented severe difficulties.

Mackey's exposition was rather tentative in several respects, and in particular it was never made explicit exactly which class of irreducible unitary representations of G one ought to consider in the correspondence. There are various series of irreducible representations (for instance the so-called complementary series) that did not feature at all in Mackey's analysis. Perhaps because of these problems, Mackey's work did not attract much immediate attention.

After a long pause, Mackey's proposal began to be examined in detail over the past dozen years, first for complex groups in [Hig08] and ultimately for all real groups in breakthrough work of Afgoustidis [Afg18]. The current *Mackey bijection* is a certain one-to-one correspondence between the irreducible tempered representations of a real reductive group and the irreducible unitary representations of its Cartan motion group.

In [Hig08], Higson constructed in detail a Mackey bijection for complex semisimple groups. He showed that there is a bijection between the irreducible unitary representations of $G_0$ and the irreducible *tempered* representations of G. Usually this does not encompass all of the irreducible unitary representations. He then placed the reduced $C^*$-algebras of the two groups into a continuous field of $C^*$-algebras using the so-called deformation to the normal cone of $G_0$ into G (already considered in [BCH94]).

Alexandre Afgoustidis expanded on the work started by Higson and obtained generalizations to real groups in [Afg18] and [Afg19]. As recently as 2019, in [AA19], he showed that the Mackey bijection (which he called the Mackey-Higson bijection) is continuous from the space of irreducible unitary representation of $G_0$ to the space of irreducible tempered representation of G with respect to the Fell topology. He made a point to show that the inverse mapping is never continuous.

A principal goal of this dissertation is to study the continuous field $\{C^*_r(G_t)\}_{t\in\mathbb{R}}$ in more detail, and indeed to characterize it up to isomorphism when G is complex and in some other cases. This is the main subject of Chapter 5.

To explain the method, it is helpful to start with a toy model case, in which G is a semidirect product group

$$G = K \ltimes V$$

associated to the action of a compact group K on a real, finite-dimensional vector space V (this is not a reductive group, of course). Here the deformation to the normal cone associated to the embedding of K into G gives a smooth family of groups $\{G_t\}$ that is isomorphic to the constant family of groups with fiber G. However it is not *equal* to the constant family; to obtain an isomophism to the constant family we must use the family of *rescaling morphisms*

$$\alpha_t \colon G_t \longrightarrow G_t$$

$$\alpha_t(k, \nu) = (k, t\nu)$$

for $t \neq 0$. The family $\{\alpha_t\}_{t\neq 0}$ extends in a unique way to an isomorphism from the constant family of groups with fiber $G$ into the deformation to the normal cone family.

Similar rescaling morphisms of $G$ do not exist on a real reductive group except in trivial cases. But one might ask whether they nonetheless exist on the reduced group $C^*$-algebra?

The reason that one might guess that rescaling morphisms exist on at the $C^*$-algebra level is that the structure of the $C^*$-algebra is very closely related to the structure of the tempered dual of $G$, and there is a natural rescaling operation on the tempered dual. Indeed the tempered dual is parametrized by a combination of discrete and continuous parameters, with the latter belonging to vector spaces, or quotients of vectors spaces by finite group actions. So the contunuous parameters may be rescaled in the obvious way. Morever this rescaling operation plays a central role in the Mackey bijection.

Our first main result is that rescaling morphisms for connected complex reductive groups do indeed exist at the $C^*$-algebra level:

**Theorem.** *Let $G$ be a connected complex reductive group. There is a one-parameter group of automorphisms*

$$\alpha_t \colon C^*_r(G) \longrightarrow C^*_r(G) \qquad (t > 0)$$

*that implements the rescaling action on the tempered dual of $G$ in the Mackey bijection.*

See Section 5.1. Our second main result is that, at the level of continuous fields, there is a (unique) extension to $t=0$:

**Theorem.** *Let $G$ be a connected complex reductive group and let $\{f_t\}$ be a continuous section of the continuous field of $C^*$-algebras associated to the deformation to the normal cone construction for the inclusion of a maximal compact subgroup into $G$. Then the limit $\lim_{t\to 0} \alpha_t(f_t)$ exists in $C^*_r(G)$, and the formula*

$$\alpha(f_0) = \lim_{t\to 0} \alpha_t(f_t)$$

*defines an embedding of* $C^*$*-algebras*

$$\alpha\colon C^*_r(G_0) \hookrightarrow C^*_r(G).$$

See Theorems 5.2.0.1 and 5.3.1.1 for the precise statements (we have omitted here some details related to the Haar measures on the groups $G_t$, which vary with $t$).

Now, given an inclusion of $C^*$-algebras $B \to A$, there is a simple and obvious way to construct a continuous field of $C^*$-algebras with fibers

$$A_t = \begin{cases} A & t \neq 0 \\ B & t = 0, \end{cases}$$

namely we take as continuous sections all the continuous functions from $\mathbb{R}$ to $A$ whose values at $t=0$ lie in $B$. Let us call this the *mapping cone* continuous field associated to the inclusion. Using this construction, we are able to characterize the continuous field associated to the deformation to the normal cone, as follows:

**Theorem.** *Let* $G$ *be a connected complex reductive group. The continuous field of* $C^*$*-algebras* $\{\, C^*_r(G_t) \,\}_{t\in\mathbb{R}}$ *associated to the deformation to the normal cone construction is isomorphic to the mapping cone field for the embedding*

$$\alpha\colon C^*_r(G_0) \longrightarrow C^*_r(G).$$

*Indeed the morphism*

$$\{f_t\} \longmapsto \{\alpha_t(f_t)\}$$

*is a bijection from continuous sections of the deformation to the normal cone field to continuous sections of the mapping cone field.*

See Section 5.3.2 for further details (including the proper treatment of $\alpha_t$ when $t$ is negative).

To summarize, one might say that the continuous field is nothing more or less than the morphism

$$\alpha\colon C^*_r(G_0) \longrightarrow C^*_r(G).$$

What does this tell us about the Mackey bijection? Each tempered irreducible representation of G corresponds to an irreducible representation

$$\pi\colon C^*_r(G) \longrightarrow \mathcal{B}(H_\pi)$$

(in fact the range is the ideal of compact operators, $\mathcal{K}(H_\pi)$, but that is not relevant here). The composition of this representation of $C^*_r(G)$ with the embedding $\alpha$ above is not necessarily irreducible, so composition with $\alpha$ does not directly determine a map from the tempered dual of G to the unitary dual of $G_0$. However generically the restriction *is* irreducible, and it turns out that this enough to determine a unique Mackey bijection:

**Theorem.** *Let* G *be a connected complex reductive group. There is a unique bijection* $\mu$ *from the tempered unitary dual of* G *to the unitary dual of* $G_0$ *with the property that for every* $\pi \in \widehat{G}_r$*,* $\mu(\pi)$ *is a subrepresentation of* $\pi \circ \alpha$*.*

Chapter 2 reviews some structure theory about reductive Lie groups, as well as integral formulas. All of this is well-known and found in many graduate textbooks, but it is fundamentally important to our results, so we cover them here. Chapter 3 covers important facts about group representation theory and provides details about the Mackey bijection. Chapter 4 covers the full (or maximum) and reduced $C^*$-algebras and their structures in what we may think of them as a "Fourier transform" picture. We also cover the continuous fields of $C^*$-algebras and their properties. Chapter 5 then explains and proves the new results just presented above. Chapter 6 presentas the new results for $SL(2,\mathbb{R})$.

This is a joint work with Nigel Higson (see [HR19]).

# Chapter 2 Structure Theory of Reductive Groups

In this chapter we provide a review of basic results for reductive groups, which are necessary in proving the main results, including its definition, important properties, and especially, the structure theory. All results will be provided without proofs. For proofs, see [Kna86] and [Kna02].

## 2.1 Reductive Groups

There are quite a few different definitions for real reductive Lie groups. The important point is that all definitions lead to the structure theory that we need. Along the way, we will also define semisimple groups. These definitions are taken from [CSM95] and [CCH16]. See also [Wal88] and [Hum75].

Let $F$ be a field, either $\mathbb{R}$ or $\mathbb{C}$, We let $M_n(F)$ be the set of $n \times n$ matrices over $K$ and let $GL(n, \mathbb{C})$ be the set of invertible matrices. Then we can take a finite set $S$ of polynomial functions on $M_n(F)$. *A linear algebraic group* $G$ is a closed subgroup of $GL(n, F)$, defined by the zeros of the polynomials in $S$.

$SL(n, F)$ is an $F$-linear algebraic group; it is defined by $\det(g) = 1$ or $\det(g) - 1 = 0$ for a $g \in SL(n, F)$. The group $SU(2) \subset SL(2, \mathbb{C})$ is a linear algebraic group. Beside $\det(g) = 1$, the defining polynomials are the components of $gg^* = 1$, where $g^*$ is the conjugate transpose of $g$. It should be obvious that the equation given is a polynomial equation. $GL(n, F)$ itself is also an algebraic group.

Let $G$ be a linear algebraic group. Let $G^u$ be the subset of $G$ consisting of unipotent matrices (that is, $g - I$ is nilpotent, that is $(g - I)^n = 0$ for some $n$). If $G = G^u$, then $G$ is *unipotent*. We define the *unipotent radical* $R_u(G)$ to be the unique maximal, closed, connected, unipotent, normal subgroup of $G$. If $R_u(G)$ is trivial, then we say that $G$ is *reductive*. We can find an isomorphic group to the reductive group that is stable under conjugate transpose. In this dissertation, we will define a *complex reductive group* to be a connected, reductive group (in the above algebraic group sense) that is closed under conjugate transpose (this latter condition can always be arranged).

Let $S$ be a set of polynomials in $M_n(\mathbb{C})$ that are real valued on $M_n(\mathbb{R})$. Let $G_\mathbb{C}$ be the complex reductive group defined by $S$ in $GL(n, \mathbb{C})$. The subgroup $G_\mathbb{R} = G_\mathbb{C} \cap GL(n, \mathbb{R})$ is the group of *real points* of $G_\mathbb{C}$. A group $G_\mathbb{R}$ is called a *real reductive group* if $G_\mathbb{C}$ is a complex reductive group.

### 2.1.1 Semisimple Groups

Let $G^1$ be the subgroup of $G$ generated by the commutators $[g, h] = ghg^{-1}h^{-1}$ for $g, h \in G$. Then we define recursively $G^{j+1}$ to be the subgroups of $G^j$ generated by the commutators $[g, h]$ for $g, h \in G^j$. We say that $G$ is *solvable* if this series reaches the trivial group in a finite finite number of steps (for some $j$, $G^j = \{e\}$). Define the *solvable radical* of $G$ by $R(G)$ to be the unique maximal, closed, connected, solvable, normal subgroup of $G$. If $R(G)$ is trivial, then we say that $G$ is *semisimple*. All semisimple groups are reductive.

The general linear groups $GL(n, F)$ are reductive, but not semisimple. The special linear groups $SL(n, k)$ are semisimple, and thus also reductive. this dissertation will consider complex reductive group with finite center, where we can use $SL(n, \mathbb{C})$ as a running example (especially $n = 2$), and one case of real reductive group: $SL(2, \mathbb{R})$.

## 2.2 Lie Algebra

Reductive groups have special structures that we will take full advantage of here in this thesis. To explain such structures, we look at the Lie algebras of these groups.

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Recall that for a given $X \in \mathfrak{g}$, the linear map $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ is defined by $(\mathrm{ad}\, X)Y = [X, Y]$. This is a linear operator on $\mathfrak{g}$. We can then define the *Killing form* of $\mathfrak{g}$ by

$$B(X, Y) = \mathrm{Trace}(\mathrm{ad}\, X \;\; \mathrm{ad}\, Y).$$

This is a symmetric bilinear form that is invariant in the sense that $B([X, Y], Z) = B(X, [Y, Z])$ or $B((\mathrm{ad}\, X)Y, Z) = -B(Y, (\mathrm{ad}\, X), Z)$.

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Define $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}]$ where, this time, we are using the Lie bracket. We then define recursively $\mathfrak{g}^{j+1} = [\mathfrak{g}^j, \mathfrak{g}^j]$. We say that $\mathfrak{g}$ is *solvable* if $\mathfrak{g}^j = 0$ for some $j$. We then define the *radical* of $\mathfrak{g}$, denoted $\mathrm{rad}(\mathfrak{g})$, to be the maximal solvable ideal of $\mathfrak{g}$. We say that $\mathfrak{g}$ is *simple* if $\mathfrak{g}$ is nonabelian and has no proper nonzero ideals. We also say that $\mathfrak{g}$ is *semisimple* if $\mathrm{rad}\, \mathfrak{g} = 0$. Every simple Lie algebra is semisimple. There is an equivalent definition for semisimple: $\mathfrak{g}$ is semisimple if and only if $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_m$ where each $\mathfrak{g}_i$ are ideals in which are simple Lie algebras (see [Kna02, Theorem 1.51]). This decomposition is unique and the only ideals in $\mathfrak{g}$ are the sums of the various simple ideals. The direct sum here means $[\mathfrak{g}_i, \mathfrak{g}_k] = 0$.

**Theorem 2.2.0.1** ( [Kna02, Theorem 1.42]). *The Lie algebra $\mathfrak{g}$ is semisimple if and only if the Killing form for $\mathfrak{g}$ is non-degenerate.*

A Lie group is *semisimple* if its Lie algebra is semisimple. Note that this definition coincides somewhat with the above definition from the linear algebraic point of view.

**Theorem 2.2.0.2** ( [Kna02, Proposition 7.9]). *Let $G$ be an analytic subgroup of real or complex matrices whose Lie algebra $\mathfrak{g}$ is semisimple. Then $G$ has finite center and is a closed linear group.*

A Lie algebra $\mathfrak{g}$ is *reductive* if for each ideal *a* in $\mathfrak{g}$, there is an ideal $\mathfrak{b}$ in $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$. We do not, however define a reductive Lie group based on this. But, the Lie algebra of a reductive group as defined above is a reductive Lie algebra.

**Proposition 2.2.0.3** ( [Kna02] and [Kna86]). *If* $\mathfrak{g}$ *is reductive, then* $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus Z_{\mathfrak{g}}$ *with* $[\mathfrak{g}, \mathfrak{g}]$ *semisimple and* $Z_{\mathfrak{g}}$ *is the center of* $\mathfrak{g}$*, and thus, abelian.*

## 2.3 Cartan Decomposition and Cartan Motion Group

Let $G$ be a reductive group as in Section 2.1 and $\mathfrak{g}$ its reductive Lie algebra. We define a *Cartan involution*

$$\phi : \mathfrak{g} \to \mathfrak{g}$$

given by $\phi X = -X^*$, where $X^*$ is the conjugate transpose of $X$ (any involution such that, for a Killing form $B$, $-B(X, \theta Y)$ is positive definite. Any two Cartan involutions are conjugate by $\mathrm{Int}\,\mathfrak{g}$. So we will stick to this definition). Since $\phi$ is an involution, that is $\phi^2 = 1$, it only has two eigenvalues: $+1$ and $-1$. Let $\mathfrak{k}$ and $\mathfrak{p}$ be the eigenspaces of $+1$ and $-1$, respectively. Then we have the *Cartan Decomposition*

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

where the elements in $\mathfrak{k}$ are skew-Hermitian and the elements in $\mathfrak{p}$ are Hermitian. We have the following relations

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k} \quad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p} \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}.$$

So we see that $\mathfrak{k}$ is a subalgebra of $\mathfrak{g}$, but $\mathfrak{p}$ is not. We can say more about $\mathfrak{K}$. Let $K := G \cup U(n)$ for some natural number $n$ so that $K$ is a maximal compact subgroup of $G$. Then the Lie algebra of $K$ is $\mathfrak{k}$. We have that $\mathfrak{k}$ and $\mathfrak{p}$ are orthogonal under the Killing form $B$, and $B$ is positive definite on $\mathfrak{p}$ and negative definite on $\mathfrak{k}$.

**Example** Take $G = SL(n, \mathbb{C})$. Then $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ is the set of $n \times n$ complex matrices such that their traces vanish. By the Cartan decomposition we get $\mathfrak{k} = \mathfrak{su}(n)$ which is the set of all $n \times n$ complex matrices $X$ such that $\mathrm{Trace}\, X = 0$ and

$X^* + X = 0$. Then we get that $K = SU(n)$, the special unitary group. That is, for $g \in SU(n)$, we have $\det g = 1$ and $g^* g = 1$. For $G = SL(2, \mathbb{C})$, we have

$$\begin{aligned}
G = SL(2,\mathbb{C}) &= \left\{ \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \middle| \alpha\delta - \beta\gamma = 1 \right\}, \\
\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C}) &= \left\{ \begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix} \middle| \alpha, \beta, \gamma \in \mathbb{C} \right\}, \\
\mathfrak{k} = \mathfrak{su}(2) &= \left\{ \begin{bmatrix} ia & \beta \\ -\overline{\beta} & -ia \end{bmatrix} \middle| a \in \mathbb{R}, \beta \in \mathbb{C} \right\}, \\
K = SU(2) &= \left\{ \begin{bmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{bmatrix} \middle| |\alpha|^2 + |\beta|^2 = 1 \right\}, \\
\text{and} \qquad \mathfrak{p} &= \left\{ \begin{bmatrix} a & \beta \\ \overline{\beta} & -a \end{bmatrix} \middle| a \in \mathbb{R}, \beta \in \mathbb{C} \right\}.
\end{aligned}$$

Let us also list down the real case:

$$\begin{aligned}
G = SL(2,\mathbb{R}) &= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| ad - bc = 1 \right\}, \\
\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R}) &= \left\{ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \middle| a, b, c \in \mathbb{R} \right\}, \\
\mathfrak{k} = \mathfrak{so}(2) &= \left\{ \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} \middle| b \in \mathbb{R} \right\}, \\
K = SO(2) &= \left\{ \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \middle| \theta \in \mathbb{R} \right\}, \\
\text{and} \qquad \mathfrak{p} &= \left\{ \begin{bmatrix} a & b \\ b & -a \end{bmatrix} \middle| a, b \in \mathbb{R} \right\}.
\end{aligned}$$

**Proposition 2.3.0.1** ( [Kna86, Proposition 1.2]). *The map* $K \times \mathfrak{p} \to G$ *given by multiplication* $(k, X) \mapsto k \exp(X)$ *is a diffeomorphism.*

Of course, $K \subset G$ and $\exp(\mathfrak{p}) \subset G$, so we have $G = K \exp(\mathfrak{p})$, which we call the *Cartan decomposition* of $G$.

**Definition.** We define the *Cartan motion group* to be the semidirect product

$$G_0 = K \ltimes \mathfrak{g}/\mathfrak{k}$$

with multiplication defined by

$$(k, X)\cdot(k', X') = (kk', \mathrm{Ad}_{(k')^{-1}}(X) + X').$$

*Remark.* The Cartan motion group is not reductive and the structure of $G$ and $G_0$ are quite different.

By the Cartan decomposition, we have $\mathfrak{p} \cong \mathfrak{g}/\mathfrak{k}$. So we can write the Cartan motion group as

$$G_0 = K \ltimes \mathfrak{p}$$

and the action of $K$ on $\mathfrak{p}$ is compatible.

## 2.4 Iwasawa Decomposition

Consider the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$.

**Lemma 2.4.0.1 ( [Kna86, Theorem 5.13]).** Let $G$ be a reductive Lie group. If $\mathfrak{a}$ and $\mathfrak{a}'$ are two maximal abelian subspaces of $\mathfrak{p}$, then there is a member $k \in K$ such that $\mathrm{Ad}(k)\mathfrak{a} = \mathfrak{a}'$. Hence $\mathfrak{p} = \bigcup_{k \in K} \mathrm{Ad}(k)\mathfrak{a}$.

This tells us that all maximal abelian subspace are conjugate to one another. This allows us to choose a single $\mathfrak{a}$ without any loss of generality for what follows.

Let $\lambda$ be a linear functional on $\mathfrak{a}$. Now let

$$\mathfrak{g}_\lambda = \{X \in \mathfrak{g} | [H, X] = \lambda(H)X \text{ for all } H \in \mathfrak{a}\}.$$

If $\lambda$ and $\mathfrak{g}_\lambda$ are nonzero, then we say that $\lambda$ is a *restricted root* of $\mathfrak{g}$ and $\mathfrak{g}_\lambda$ is the *restricted root space*. We denote these by $\Delta(\mathfrak{g} : \mathfrak{a})$, or by $\Delta$ if there is no ambiguity. From here on out we will refer to the restricted roots simply as *roots*. Let $V$ be the vector space spanned by $\Delta$. It is possible to define a notion of *positivity* on $V$. We denote a positive element $\phi$ in $V$ by $\phi > 0$. Then

1. for $\phi \in V$, it is either $\phi > 0$ or $-\phi > 0$ or $\phi = 0$,

2. If $\phi > 0$, $\psi > 0$, and $c$ is positive scalar, then $\phi + \psi > 0$, and $c\phi > 0$.

We say $\phi > \psi$ if $\phi - \psi > 0$. One way to define a notion of positivity is through lexicographic ordering: Let $(\phi_1, \dots, \phi_k)$ be an ordered basis for $V$. Then $\phi = \sum_{i=1}^{k} a_i \phi_i$ is positive if $a_1 = \dots = a_l = 0$ and $a_{l+1} > 0$ for some $l \geq 0$. We denote the set of positive roots by $\Delta^+$ (or $\Delta^+(\mathfrak{g} : \mathfrak{a})$ if it is necessary). There are only finitely many possible $\Delta^+$. A root $\lambda$ is *simple* if $\lambda > 0$ and it does not decompose as $\lambda = \lambda_1 + \lambda_2$ where $\lambda_1, \lambda_2 > 0$. We denote the set of simple roots by $\Pi$.

Before we continue on to the Iwasawa decomposition, we lay out some root systems as examples.

**Example** Let $G = SL(n, \mathbb{C})$ so that $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, the set of traceless $n \times n$ matrices. The maximal abelian subgroup $\mathfrak{a}$ of $\mathfrak{g}$, is the set of diagonal matrices such that the sum of the diagonal is zero:

$$\mathfrak{a} = \left\{ \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{bmatrix} = \mathrm{diag}(a_1, \dots, a_n) \middle| a_1 \cdot \ldots \cdot a_n = 0 \right\}.$$

Let $e_j$ be a linear functional on $\mathfrak{a}$ defined by $e_j(\mathrm{diag}(a_1, \dots, a_n) = a_j$. Then we have the set of roots

$$\Delta = \{e_i - e_j | i \neq j\}.$$

The positive roots are (using lexicographic ordering)

$$\Delta^+ = \{e_i - e_j | i < j\}.$$

Then the simple roots are

$$\Pi = \{e_1 - e_2, e_2 - e_3, \dots, e_{n-1} - e_n\}.$$

We continue with the Iwasawa decomposition. Let

$$\mathfrak{n} = \sum_{\lambda \in \Delta^+} \mathfrak{g}_\lambda.$$

Then $\mathfrak{n}$ is a nilpotent Lie algebra of $\mathfrak{g}$.

**Proposition 2.4.0.2** ( [Kna86, Proposition 5.10])**.** *Let $\mathfrak{k}$, $\mathfrak{a}$, and $\mathfrak{n}$ be as defined above. Then $\mathfrak{g}$ is a vector-space direct sum $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$, $\mathfrak{a}$ is abelian, $\mathfrak{n}$ is nilpotent, $\mathfrak{a} \oplus \mathfrak{n}$ is a solvable Lie subalgebra, and $[\mathfrak{a} \oplus \mathfrak{n}, \mathfrak{a} \oplus \mathfrak{n}]$.*

This direct sum is called the *Iwasawa decomposition of the Lie algebra*. Note that the choice of $\mathfrak{n}$ depends on our choice of positivity of the root system. Any two choices are conjugate by a member of $N_K(\mathfrak{a})$ and this member is unique up to a member of $Z_K(\mathfrak{a})$. We define the *Weyl group* by $W = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$. We have a decomposition of the group as well.

**Theorem 2.4.0.3** ( [Kna86, Theorem 5.12])**.** *Let $G$ be a semisimple Lie group with $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ the Iwasawa decomposition of the Lie algebra $\mathfrak{g}$ of $G$. Let $A$ and $N$ be connected Lie subgroups of $G$ with Lie algebras $\mathfrak{a}$ and $\mathfrak{n}$ respectively. Then the multiplication map $K \times A \times N \to G$ given by $(k, a, n) \mapsto kan$ is a diffeomorphism.*

The decomposition of the group is simply called the *Iwasawa decomposition*. Let us show $G = SL(2, \mathbb{C})$ as an example. We have already shown that $K = SU(2)$ and $\mathfrak{k} = \mathfrak{su}(2)$. We just explained that $\mathfrak{a}$ is the set of diagonal matrices such that the sum of the diagonals is 0. So in the two dimensional case, we get

$$\mathfrak{a} = \left\{ \begin{bmatrix} t & \\ & -t \end{bmatrix} \middle| t \in \mathbb{R} \right\}.$$

The

This means that

$$A = \left\{ \begin{bmatrix} e^t & \\ & e^{-t} \end{bmatrix} \middle| t \in \mathbb{R} \right\}.$$

So $A$ is the set of all diagonal matrices such that the product of the diagonal is 1 and the diagonal entries are positive. Now the nilpotent matrix $\mathfrak{n}$ is given as

$$\mathfrak{n} = \left\{ \begin{bmatrix} 0 & z \\ 0 & 0 \end{bmatrix} \middle| z \in \mathbb{C} \right\}.$$

It is easy to see that

$$N = \left\{ \begin{bmatrix} 1 & z \\ & 1 \end{bmatrix} \middle| z \in \mathbb{C} \right\}$$

is the analytic subgroup with Lie algebra $\mathfrak{n}$. This tells us the matrix $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ with $\alpha\delta - \beta\gamma = 1$ can be written as

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} \xi & \eta \\ \overline{\eta} & \overline{\xi} \end{bmatrix} \begin{bmatrix} e^t & \\ & e^{-t} \end{bmatrix} \begin{bmatrix} 1 & z \\ & 1 \end{bmatrix}$$

with $|\xi|^2 + |\eta|^2 = 1$, $t \in \mathbb{R}$, and $z \in \mathbb{C}$.

## 2.5 Parabolic Subgroups

We follow up on the previous section: let $G$ be a reductive group with the Iwasawa decomposition $G = KAN$. Let $M = Z_K(\mathfrak{a}) = Z_K(A)$. Then $Q = MAN$ is a *minimal parabolic subgroup*. Now let $\mathfrak{m}$ be the Lie algebra of $M$, that is, $\mathfrak{m} = Z_{\mathfrak{k}}(\mathfrak{a})$. A *minimal parabolic subalgebra* $\mathfrak{q}$ is a subalgebra of $\mathfrak{g}$ that is conjugate to $\mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$.

A *(general) parabolic subalgebra* $\mathfrak{s}$ is a subalgebra of $\mathfrak{g}$ that contains a minimal parabolic subalgebra. A parabolic subalgebra has a *Langlands decomposition* similar to the one for the minimal parabolic subalgebra, which we will show below. Given such a decomposition $\mathfrak{s} = \mathfrak{m}_S \oplus \mathfrak{a}_S \oplus \mathfrak{n}_S$, we can define $A_S$ and $N_S$ to be analytic subgroups with Lie algebras $\mathfrak{a}_S$ and $\mathfrak{n}_S$ respectively. Let $M_S^0$ be the analytic subgroup with Lie algebra $\mathfrak{m}_S$. Then define $M_S = Z_K(\mathfrak{a})M_S^0$. Then $S := M_S A_S N_S$ is a *parabolic subgroup* with Lie algebra $\mathfrak{s}$ and contains a minimal parabolic subgroup. It should be noted that the Lie algebra of $M_S$ is $\mathfrak{m}_S$.

**Example** Let $G = SL(n, \mathbb{R})$. Then $MAN$ is the group of upper triangular matrices. All minimal parabolic subgroups are conjugate to this (of course, $MAN$ itself is a minimal parabolic subgroup). Then in general, a parabolic subgroup containing $MAN$ has the form of block upper triangular matrices. In $n = 4$, the minimal parabolic subgroups are conjugate to

$$\begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix}.$$

Then other parabolic subgroups containing this are $G$ and other groups of matrices with one of the following forms

$$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ 0 & 0 & 0 & * \end{bmatrix} \qquad \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix} \qquad \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix}.$$

The five parabolic subgroups shown above are not conjugates.

Now the parabolic subgroup $S$ has a decomposition $S = M_S A_S N_S$ similar to $MAN$ as shown above called the *Langlands decomposition*. However, we will not construct this decomposition for a general parabolic subgroup. Instead, we are only interested in the *cuspidal* parabolic subgroup, which we will define soon.

### 2.5.1 Cartan Subalgebra

Let $\mathfrak{g}$ be a complex reductive Lie algebra. A *Cartan subalgebra* $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$ such that it is maximal abelian and $\operatorname{ad}_{\mathfrak{g}} \mathfrak{h}$ is simultaneously diagonalizable. See [Kna02, Corollary 2.13]. We can also speak of $\theta$-stable Cartan subalgebra, where $\theta$ is the Cartan involution. In this case, we may write $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ where $\mathfrak{t} \subset \mathfrak{k}$ and $\mathfrak{a} \in \mathfrak{p}$.

**Proposition 2.5.1.1** ( [Kna02, Proposition 6.59]). *Any Cartan subalgebra* $\mathfrak{h}$ *of* $\mathfrak{g}$ *is* *conjugate via* $\mathrm{Int}\,\mathfrak{g}$ *to a* $\theta$*-stable Cartan subalgebra.*

This allows us to focus on just $\theta$-stable Cartan subalgebras. In fact, from now on, when we say Cartan subalgebra, we mean the $\theta$-stable Cartan subalgebra. It is also true that all Cartan subalgebra have the same dimension. Thus we define the *rank* of $G$ (and of $\mathfrak{g}$) to be the dimension of its Cartan subalgebra. The Cartan subalgebra $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ is said to be *maximally compact* if the dimension of $\mathfrak{t}$ is as large as possible. We also say that it is *maximally noncompact* if the dimension of $\mathfrak{a}$ is as large as possible. Maximally compact Cartan subalgebra and maximally noncompact Cartan subalgebra are the extreme ends of possible Cartan subalgebras.

Let $\mathfrak{a}$ be the maximal abelian subalgebra of $\mathfrak{p}$, just as in the Iwasawa decomposition. Then let $\mathfrak{b}$ be the maximal abelian subspace of $\mathfrak{m}$ (where $\mathfrak{m} = Z_{\mathfrak{k}}(\mathfrak{a})$). Then $\mathfrak{h} = \mathfrak{b} \oplus \mathfrak{a}$ is a Cartan subalgebra that is maximally noncompact. Now let $\mathfrak{t}$ be a maximal abelian subspace of $\mathfrak{k}$. Then define $\mathfrak{h} = Z_{\mathfrak{g}}(\mathfrak{t})$. This is a Cartan subalgebra that is maximally compact.

**Proposition 2.5.1.2** ( [Kna02, Proposition 6.61])**.** *Maximally compact (*$\theta$*-stable) Cartan subalgebras are conjugate via* $K$*. Maximally noncompact Cartan subalgebras are conjugate via* $K$*.*

**Lemma 2.5.1.3** ( [Kna02, Lemma 6.62])**.** If $\mathfrak{h}$ and $\mathfrak{h}'$ are Cartan subalgebras such that $\mathfrak{h} \cap \mathfrak{p} = \mathfrak{h}' \cap \mathfrak{p}$, then $\mathfrak{h}$ and $\mathfrak{h}'$ are conjugate via $K$.

**Proposition 2.5.1.4** ( [Kna86, Theorem 5.22])**.** *There are only finitely many nonconjugate (*$\theta$*-stable) Cartan subalgebras.*

**Proposition 2.5.1.5** ( [Kna86, Theorem 5.22(e)])**.** *All Cartan subalgebras of a complex subalgebra* $\mathfrak{g}$ *are conjugate, that is, there is only one Cartan subalgebra.*

In $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, up to conjugacy, the only Cartan subalgebra of $\mathfrak{g}$ is the diagonal subalgebra. This has the property that $\dim(\mathfrak{h} \cap \mathfrak{k}) = \dim(\mathfrak{h} \cap \mathfrak{p}) = n - 1$.

**Example** In $\mathfrak{g} = \mathfrak{sl}(4, \mathbb{R})$, The following are nonconjugate Cartan subalgebras

$$\mathfrak{h}_0 = \begin{bmatrix} s_1 & & & \\ & s_2 & & \\ & & s_3 & \\ & & & s_4 \end{bmatrix} \quad \mathfrak{h}_1 = \begin{bmatrix} t & \theta & & \\ -\theta & t & & \\ & & s_1 & \\ & & & s_2 \end{bmatrix} \quad \mathfrak{h}_2 = \begin{bmatrix} t_1 & \theta_1 & & \\ -\theta_1 & t_1 & & \\ & & t_2 & \theta_2 \\ & & -\theta_2 & t_2 \end{bmatrix}$$

where the traces are zero. Here $\dim(\mathfrak{h}_j \cap \mathfrak{k}) = j$ and $\dim(\mathfrak{h}_j \cap \mathfrak{p}) = 4 - j - 1$. So $\mathfrak{h}_0$ is maximally noncompact, and $\mathfrak{h}_2$ is maximally compact.

### 2.5.2 Cuspidal Parabolic Subalgebra

The construction is taken from [Kna86, Section V.5]. Also see [Kna02, Sections V.7 and VII.7].

Here, we show the Langlands decomposition of a cuspidal parabolic subalgebra. Let $\mathfrak{s} = \mathfrak{m}_S \oplus \mathfrak{a}_S \oplus \mathfrak{n}_S$ be the Langlands decomposition of a parabolic subalgebra. Then $\mathfrak{s}$ is *cuspidal* if $\mathfrak{m}_S$ has a compact Cartan subalgebra, call it $\mathfrak{t}_S$. Then $\mathfrak{h}_S = \mathfrak{t}_S \oplus \mathfrak{a}_S$ is a Cartan subalgebra of $\mathfrak{g}$. Instead of trying to decompose a general parabolic subalgebra $\mathfrak{s}$ and figuring out which $\mathfrak{m}_S$ has a compact Cartan subalgebra, we will begin with a $\theta$-stable Cartan subalgebra $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ and construct a cuspidal parabolic subalgebra from this.

Let $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ be a $\theta$-stable Cartan subalgebra. Then, for a linear functional $\alpha$ on $\mathfrak{h}$, We can define with $\mathfrak{t} = \mathfrak{h} \cap \mathfrak{k}$ and $\mathfrak{a} = \mathfrak{h} \cap \mathfrak{p}$

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} | [H, X] = \alpha(H)X, \text{ for all } H \in \mathfrak{h}\}.$$

Then $\alpha$ is a *root* in $\Delta(\mathfrak{g}, \mathfrak{h})$ if $\alpha \neq 0$ and $\mathfrak{g}_\alpha \neq 0$. We can also take a root system $\Delta(\mathfrak{g}, \mathfrak{a})$ which can be identified with $\Delta(\mathfrak{g}, \mathfrak{h})|_\mathfrak{a}$. Now let $F = \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}) | \alpha|_\mathfrak{a} = 0\}$ be the set of roots that vanishes on $\mathfrak{a}$. So we can extend roots in $\Delta(\mathfrak{g}, \mathfrak{a})$ to roots in $\Delta(\mathfrak{g}, \mathfrak{h})$ by $F$. We can define a lexicographic ordering on $\Delta(\mathfrak{g}, \mathfrak{h})$ by lexicographic ordering on $\mathfrak{a}$, then on $\mathfrak{t}$. In this way, the positive roots in $\Delta^+(\mathfrak{g}, \mathfrak{h})$ restricted to $\mathfrak{a}$ are also positive roots in $\Delta^+(\mathfrak{g}, \mathfrak{a})$, that is, $\Delta^+(\mathfrak{g}, \mathfrak{h})|_\mathfrak{a}$ is identified with $\Delta^+(\mathfrak{g}, \mathfrak{a})$ (this way, we can take the usual ordering in $\Delta(\mathfrak{g}, \mathfrak{a})$).

Now we can define

$$\mathfrak{m} = \mathfrak{t} \oplus \bigoplus_{\alpha \in F} \mathfrak{g}_\alpha \qquad \mathfrak{n} = \bigoplus_{\substack{\alpha \in \Delta^+(\mathfrak{g},\mathfrak{h}) \\ \alpha|_\mathfrak{a} \neq 0}} \mathfrak{g}_\alpha.$$

Then $\mathfrak{s} := \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ is a cuspidal parabolic subalgebra and the decomposition is called the *Langlands decomposition*, where $\mathfrak{m}$ is a reductive group and it has a compact Cartan subalgebra $\mathfrak{t} = \mathfrak{h} \cap \mathfrak{m}$.

A practical way of constructing the Langlands decomposition of a cuspidal parabolic subgroup is the following: start with a Cartan subalgebra $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ with $\mathfrak{t} = \mathfrak{h} \cap \mathfrak{k}$ and $\mathfrak{a} = \mathfrak{h} \cap \mathfrak{p}$. Then we let $\mathfrak{m}$ be the orthocomplement of $\mathfrak{a}$ in $Z_\mathfrak{g}(\mathfrak{a})$ with respect to the inner product $\langle X, Y \rangle = -\operatorname{Trace}(X(-\overline{Y}^{tr}))$. Define $\mathfrak{n} = \bigoplus_{\alpha \in \Delta^+(\mathfrak{g},\mathfrak{a})} \mathfrak{g}_\alpha$. Then the Langlands decomposition of the cuspidal parabolic subalgebra $\mathfrak{s}$ is $\mathfrak{s} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$.

*Remark.* Not all parabolic subgroups are cuspidal.

**Example** Recall the Cartan subalgebra for $\mathfrak{g} = \mathfrak{sl}(4, \mathbb{R})$: $\mathfrak{h}_0$, $\mathfrak{h}_1$, and $\mathfrak{h}_2$. The following are cuspidal parabolic subalgebras

$$\begin{bmatrix} * & * & * & * \\ & * & * & * \\ & & * & * \\ & & & * \end{bmatrix} \quad \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & & * \end{bmatrix} \quad \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & * & * \end{bmatrix}$$

constructed from $\mathfrak{h}_0$, $\mathfrak{h}_1$, and $\mathfrak{h}_2$, respectively. Note that the other two parabolic subalgebras are not cuspidal. See the example on page 19.

**Example** Let us illustrate the constructions of cuspidal parabolic subgroups for $G = SL(2, \mathbb{R})$. Recall that $\mathfrak{sl}(2, \mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \middle| a, b, c \in \mathbb{R} \right\}$. By the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, we get $\mathfrak{k} = \mathfrak{so}(n) = \left\{ \begin{bmatrix} & u \\ -u & \end{bmatrix} \middle| u \in \mathbb{R} \right\}$ and

$\mathfrak{p} = \left\{ \begin{bmatrix} t & u \\ u & -t \end{bmatrix} \middle| t, u \in \mathbb{R} \right\}$. From this, we have two $\theta$-stable Cartan subalgebra

$$\mathfrak{h}_0 = \left\{ \begin{bmatrix} s & \\ & -s \end{bmatrix} \middle| s \in \mathbb{R} \right\} \qquad \mathfrak{h}_1 = \left\{ \begin{bmatrix} 0 & u \\ -u & 0 \end{bmatrix} \middle| u \in \mathbb{R} \right\}.$$

Let us find the two cuspidal parabolic subalgebras. Let

$$\mathfrak{a}_0 = \mathfrak{h}_0 \cap \mathfrak{p} = \left\{ \begin{bmatrix} s & \\ & -s \end{bmatrix} \middle| s \in \mathbb{R} \right\} = \mathfrak{h}_0$$

and $\mathfrak{t}_0 = \mathfrak{h}_0 \cap \mathfrak{k} = \{0\}$. From this, we get that

$$\mathfrak{n}_0 = \left\{ \begin{bmatrix} 0 & z \\ & 0 \end{bmatrix} \middle| z \in \mathbb{R} \right\}.$$

Now we want to find $\mathfrak{m}_0$, which is the orthocomplement of $\mathfrak{a}_0$ in $Z_{\mathfrak{g}}(\mathfrak{a}_0)$. First, note that

$$Z_{\mathfrak{g}}(\mathfrak{a}_0) = \left\{ \begin{bmatrix} t & \\ & -t \end{bmatrix} \middle| t \in \mathbb{R} \right\},$$

so then we can clearly see that $\mathfrak{m}_0 = \{0\}$. Having found $\mathfrak{m}_0$, $\mathfrak{a}_0$, and $\mathfrak{n}_0$, we can construct the cuspidal parabolic subgroup. In this case,

$$A = \left\{ \begin{bmatrix} e^t & \\ & e^{-t} \end{bmatrix} \middle| t \in \mathbb{R} \right\}$$

and

$$N = \left\{ \begin{bmatrix} 1 & z \\ & 1 \end{bmatrix} \middle| z \in \mathbb{R} \right\}.$$

Now, $M^0 = \{\mathrm{Id}\}$. So then

$$M_0 = Z_K(\mathfrak{a}) M^0 = Z_K(\mathfrak{a}) = \left\{ \begin{bmatrix} \pm 1 & \\ & \pm 1 \end{bmatrix} \right\}.$$

Thus, the cuspidal parabolic subgroup is

$$S_0 = M_0A_0N_0 = \left\{ \begin{bmatrix} x & z \\ & x^{-1} \end{bmatrix} \middle| x \in \mathbb{R} - \{0\}, z \in \mathbb{R} \right\},$$

which is the minimal parabolic subgroup.

Now, for the other parabolic subgroup, constructed from $\mathfrak{h}_1$, we get $\mathfrak{a}_1 = \mathfrak{h}_1 \cap \mathfrak{p} = \{0\}$ and $\mathfrak{t}_\circ = \mathfrak{h}_1 \cap \mathfrak{k} = \left\{ \begin{bmatrix} & s \\ -s & \end{bmatrix} \middle| s \in \mathbb{R} \right\}$. Note then that $Z_{\mathfrak{g}}(\mathfrak{a}_1) = \mathfrak{g}$, so we get that $\mathfrak{m}_1 = \mathfrak{g}$. It is also clear that $\mathfrak{n}_1 = \{0\}$. Thus, the cuspidal parabolic subalgebra is $\mathfrak{s}_1 = \mathfrak{m}_1 \oplus \mathfrak{a}_1 \oplus \mathfrak{n}_1 = \mathfrak{g}$. It should then be obvious that the parabolic subgroup is $S_1 = SL(2, \mathbb{R})$.

## 2.6 Haar Measure and Integration Formulas

In this section, we review Haar measures and some integration formulas. They will be used extensively in Chapter 4 and in Chapter 5.

Let $G$ be a Lie group. There exists a Borel measure $\eta$ that is invariant under the left translation action, that is, for all $x \in G$ and all Borel sets $E$, $\eta(xE) = \eta(E)$ (for Lie groups this can be constructed from left-invariant differential forms on $G$). We then call $\eta$ the *(left) Haar measure* on $G$. There are two important properties about the Haar measure: if $K$ is a compact subgroup of $G$, then $\eta(K) < \infty$, and if $f \in C_c(G)$, then

$$\int_G f(xg)\, d\eta(g) = \int_G f(g)\, d\eta(g)$$

for all $x \in G$.

**Theorem 2.6.0.1** ( [Kna02, Theorem 8.23])**.** *If $G$ is a Lie group, then any two left Haar measure on $G$ are proportional.*

We can also define a right Haar measure $\eta_r$ (building it from right invariant smooth sections), and there is a function $\Delta : G \to \mathbb{R}^+$ defined by

$$\eta_r(x) = \Delta(g)\eta(x)$$

called the *modular* function.

**Proposition 2.6.0.2** ( [Kna02, Proposition 8.27]). *If* $G$ *is a Lie group, then* $\Delta(g) = |\det \mathrm{Ad}_g|$.

We say that the Lie group $G$ is *unimodular* if $\Delta(g) = 1$ for all $g \in G$, that is the left and right Haar measure are equal.

**Proposition 2.6.0.3** ( [Kna02, Corollary 8.31]). *The following Lie groups are unimodular:*

1. *abelian Lie groups*
2. *compact Lie groups*
3. *semisimple Lie groups*
4. *reductive Lie groups*
5. *nilpotent Lie groups.*

From now on we will make the following notations $dx := d\eta(x)$ and $d_r x := d\eta_r(x)$ for left and right Haar measure, respectively. We will also take the convention that whenever we say "Haar measure" we mean the left Haar measure, unless otherwise stated.

**Proposition 2.6.0.4** ( [Kna86, Proposition 5.26]). *Let* $G$ *be Lie group, and let* $S$ *and* $T$ *be closed subgroups such that* $S \cap T$ *is compact and such that the set of product* $ST$ *exhaust* $G$ *except possibly for a set of Haar measure 0. Then the Haar measures on* $G$*,* $S$*, and* $T$ *can be normalized so that*

$$\int_G f(x)\ dx = \int_{S\times T} f(st)\frac{\Delta_T(t)}{\Delta_G(t)}\ ds\ dt$$

*for all* $f \in C_c(G)$.

**Corollary 2.6.0.5** ( [Kna86, Consequence 1 following proposition 5.26]). *Let* $G$*,* $S$*, and* $T$ *be as in the previous proposition. Then*

$$dx = ds\ d_r t.$$

As a consequence we get

$$d(man) = dm\ da\ dn.$$

Recall that $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+(\mathfrak{g},\mathfrak{a})} \dim(\mathfrak{g}_\alpha)\alpha$. We have that

$$\Delta_{MAN}(man) = e^{2\rho \log a}.$$

and thus, we get that $d_r(man) = e^{2\rho\log a}\ dm\ da\ dn$. We also have $d(an) = da\ dn$ and since $\Delta_{AN}(an) = e^{2\rho\log a}$ we also get $d_r(an) = e^{2\rho\log a}\ da\ dn$. For details about this, see [Kna02, Section VIII.4] or [Kna86, Section V.6]. The next two corollaries are also explored in the references just mentioned.

**Corollary 2.6.0.6.** *If* $G = KAN$ *is the Iwasawa decomposition of a reductive group, then the Haar measure of* $G$, $K$, $A$, *and* $N$ *can be normalized so that*

$$dx = dk\ d_r(an) = e^{2\rho\log a}\ dk\ da\ dn.$$

**Corollary 2.6.0.7.** *If* $G$ *is a reductive group and* $MAN$ *is a parabolic subgroup, so that* $G = KMAN$, *then the Haar measure on* $G$, $MAN$, $M$, $A$, *and* $N$ *can be normalized so that*

$$dx = dk\ d_r(man) = e^{2\rho\log a}\ dk\ dm\ da\ dn.$$

Along with the Haar measure defined above, we would also like to define a Haar measure for the Cartan motion group that is compatible with the ones above. In particular, we have needs to define some measures on Lie algebras. If we consider the abelian group $A$ and the nilpotent group $N$ from the Iwasawa decomposition of the group $G$, then we know that $A \cong \mathfrak{a}$ as smooth manifolds by the exponential mapping $\exp : \mathfrak{a} \to A$. Since $\mathfrak{a}$ is a vector space, so we can define a measure $dX$ that is left and right invariant under translation. Now, the exponential mapping preserves the group operation. Then we can define a Haar measure on $A$ compatible with the measure on $\mathfrak{a}$. By Theorem 2.6.0.1 we can then take $dX = da$, that is,

$$\int_{\mathfrak{a}} f(\exp(X))\ dX = \int_A f(a)\ da$$

for $f \in C_c(A)$. By [CG90, Theorem 1.2.10], the mapping $\exp : N \to \mathfrak{n}$ maps Lebesque measure to a left invariant Haar measure, so we define a measure $dY$ on $\mathfrak{n}$ by

$$\int_{\mathfrak{n}} f(\exp(Y))\ dY = \int_N f(n)\ dn$$

for $f \in C_c(N)$.

Recall that the Cartan motion group $G_0 = K \ltimes \mathfrak{p}$ is, as a set, the Cartesian product of $K$ and $\mathfrak{p}$. We can view $G_0$ as having the product topology. Now, note that $\mathfrak{p}$ is a finite-dimensional vector space so it will naturally have an invariant measure, compatible with the ones defined for $\mathfrak{a}$ and $\mathfrak{n}$ (since $\mathfrak{p} \cong \mathfrak{a} \oplus \mathfrak{n}$). We already know that $K$ has a Haar measure. Thus, we can define the measure on $G_0$ by a product measure. So we have

$$\int_{G_0} f(k, X)\ d(k, X) = \int_K \int_{\mathfrak{p}} f(k, X)\ dk\ dX.$$

We will also need the following lemma:

**Lemma 2.6.0.8.** If $G = KAN$ is an Iwasawa decomposition, then the Haar measure of $G$, $\mathfrak{a}$, and $\mathfrak{n}$ can be normalized so that

$$dx = e^{2\rho(X)} dk\ dX\ dY$$

*Proof.* Recall the Cartan decomposition $G = K \times \mathfrak{p}$ and the Iwasawa decomposition $\mathfrak{g} \cong \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ where $\mathfrak{k}$ is the Lie algebra of $K$, $\mathfrak{a}$ is an abelian Lie algebra, and $\mathfrak{n}$ is a nilpotent Lie algebra. Since $\mathfrak{k} \cong K$ via exponential mapping (and $K$ is compact), we see that $\mathfrak{a} \oplus \mathfrak{n} \cong \mathfrak{p}$ (in general, they are not the same set). We can conclude that $K \times \mathfrak{a} \times \mathfrak{n} \cong G$ by the following diffeomorphism $(k, X, Y) \mapsto k \exp(X) \exp(Y)$.

From Corollary 2.6.0.6, we have $dx = e^{2\rho(\log a)} dk\ da\ dn$ under the Iwasawa decomposition. Also, $da = dX$ for $dn = dY$ where $a \in A$, $X \in \mathfrak{a}$, $n \in N$, and $Y \in \mathfrak{n}$. So we have $dx = e^{2\rho(X)} dk\ dX\ dY$. $\square$

This will allow us to find a relationship between the Haar measures for a reductive group $G$ and its Cartan motion group.

# Chapter 3 | Representation Theory of Reductive Groups

In this chapter, we cover certain aspects of representation theory on real reductive groups, which will lead us to a discussion of the Mackey bijection. In particular, we will be looking at certain kinds of irreducible unitary representations called tempered representations. We will rely heavily on the structure theory discussed in the previous chapter. To get us started, we recall a few definitions and facts.

## 3.1 Definitions

Since we will be discussing Hilbert spaces, we will take the convention that the inner product is linear in the second argument and conjugate-linear in the first.

A *representation* of a topological group $G$ on a Banach space $V$ is a homomorphism $\pi : G \to GL(V)$ where $GL(V)$ is the group of bounded invertible operators whose inverse are also bounded, and where the action of $g \in G$ on $V$ is strongly continuous, that is $g \mapsto \pi(g)v$ is continuous for every $v \in V$. We call $V$ the *representation space*. An *invariant subspace* $U \subset V$ is a subspace such that $\pi(g)U \subset U$ for every $g \in G$. The representation $\pi$ is *irreducible* if there are no invariant subspace other than $0$ and $V$. We say that the representation $\pi$ is *unitary* if its representation space is a Hilbert space and $\langle \pi(g)u, \pi(g)v \rangle = \langle u, v \rangle$ for all $g \in G$ and all $u, v \in V$. Finally, a *matrix coefficient* of $\pi$ is a function $g \mapsto \langle u, \pi(g)v \rangle$ for some $u, v \in V$.

Recall that we have defined a left Haar measure $dg$ on $G$. We can then define the Hilbert space $L^2(G)$, the space of square-integrable functions on $G$ under the left Haar measure. Define $\lambda : G \to U(L^2(G))$ by $\lambda(g)f(x) = f(g^{-1}x)$ where $g, x \in G$ and $f \in L^2(G)$. This is called the *left regular representation* and is unitary (thus the reason for using the notation $U$, for a unitary invertible operator on $L^2(G)$). This is not irreducible, however. Along with the left regular representation, we also have the *right regular representation* $R$ of $G$ on $L^2(G)$, which is defined by $R(g)f(x) = f(xg)$, for $g, x \in G$ and $f \in L^2(G)$.

Let $\pi$ and $\pi'$ be two representations of $G$ with representation spaces $V$ and $V'$ respectively. An *intertwining operator* $J : V \to V'$ is an operator such that $J\pi(g) = \pi'(g)J$. We say that $\pi$ and $\pi'$ are *equivalent* if there is a bounded invertible intertwining operator from one representation space to the other. If the intertwining operator is unitary, then we say that the representations are *unitarily equivalent*. We denote by $\hat{G}$ the set of all equivalence classes of irreducible unitary representation of $G$. This set does have a topology which will be explained in the next chapter.

Given a representation $\pi$ of $G$ on a representation space $V$ and a compact subgroup $K$ of $G$, we say that $v \in V$ is $K$-*finite* if $\{\pi(k)v | k \in K\}$ span a finite-dimensional vector space. Thus we can say K-finite matrix coefficient to refer to a matrix coefficient using some K-finite vectors in $V$.

The following definition can be found in [Kna86, Section IX.3]. Suppose that $G$ has a compact center (as in the case of $SL(2, \mathbb{R})$, but this excludes $GL(n, \mathbb{R})$). An irreducible unitary representation $\pi$ of $G$ is in the *discrete series* of $G$ if the equivalent statements are satisfied:

1. some K-finite matrix coefficient of $\pi$ is in $L^2(G)$.

2. all of the matrix coefficients of $\pi$ are in $L^2(G)$.

3. $\pi$ is equivalent to a direct summand of the right regular representation of $G$ on $L^2(G)$.

Not all reductive groups have a discrete series representation. In fact, there is an important condition (which we will make use of) for a group to have a discrete series; it must have a compact Cartan subgroup. That is, the Lie algebra $\mathfrak{g}$ must have a compact Cartan subalgebra. This is the same as saying that the rank of $G$ must be equal to the rank of $K$. See [Kna86, Theorem 12.20]. As we have seen before, this is not always true. A complex reductive group does not have any discrete series representation. But $SL(2, \mathbb{R})$ does have a discrete series representation. It should also be clear that by the Peter-Weyl theorem, that any irreducible representation of a compact group is of the discrete series.

A unitary representation $\pi$ is a *tempered representation* of $G$ if all of its $K$-finite matrix coefficients are in $L^{2+\epsilon}(G)$ for every $\epsilon > 0$. The notion of tempered representation is a bit difficult to wrap our head around, but we will have an easier (equivalent) definition once we introduce the reduced $C^*$-algebras of $G$ in the next chapter (see section 4.2). In fact, it is this $C^*$-algebra definition that will serve us the best. We introduce it now while we are on the topic of representation theory. It is obvious that the representations in the discrete series are also tempered.

## 3.2 Representation of Compact Groups

### 3.2.1 Peter-Weyl Theorem

If we have a compact group $K$, then every representation of $K$ is unitary and the Peter-Weyl theorem states that

1. An irreducible unitary representation is finite-dimensional (the dimension of a representation is the dimension of the representation space),

2. Any representation $\pi$ of $K$ decomposes into an orthogonal direct sum of irreducible unitary representations of $K$,

3. The span of all matrix coefficients of all irreducible unitary representation is dense in $L^2(K)$.

See [Kna86, Theorem 1.12]. The Peter-Weyl theorem allows us to find an orthonormal basis for $L^2(K)$. We will often write

$$L^2(K) \cong \oplus_{\tau \in \hat{K}} V_\tau \otimes V_\tau^*,$$

where $\hat{K}$ is the set of equivalence classes of irreducible unitary representations of $K$, $V_\tau$ is the representation space of $\tau$ and $V_\tau^*$ is the dual to $V_\tau$, The isomorphism is built from the mapping

$$v \otimes \phi \mapsto \left[k \mapsto \left(\frac{\dim(\tau)}{\mathrm{vol}(K)}\right)^{1/2} \phi(\tau(k)^{-1}v)\right],$$

where $v \otimes \phi \in V_\tau \otimes V_\tau^*$ and $k \in K$. Any irreducible representation of a compact group is of the discrete series, and thus it is also a tempered representation.

### 3.2.2 Highest Weight Theorem

Let $K$ be a connected compact Lie group. Let $\pi$ be an irreducible representation of $K$ on $V$ which is finite-dimensional. It is a standard result that $\pi$ is a smooth function from $K$ to $GL(V)$ (see [Kna86]). From $\pi$ we can define a representation of $\mathfrak{k}$, the Lie algebra of $K$ by its differential

$$d\pi(X)v = \lim_{t \to 0} \frac{\pi(\exp tX)v - v}{t}$$

where $X \in \mathfrak{k}$ and $v \in V$. Now we let $T$ be a maximal torus (maximal connected abelian subgroup) in $K$. We can then let $\mathfrak{t}$ be the Lie algebra of $T$; $\mathfrak{t}$ is in fact a Cartan subalgebra. Let $\lambda$ be a linear functional on $\mathfrak{t}^{\mathbb{C}} := \mathfrak{t} \oplus i\mathfrak{t}$. Then define

$$V_\lambda = \{v \in V | d\pi(H)v = \lambda(H)v, \text{ for all } H \in \mathfrak{t}^{\mathbb{C}}\}.$$

If $V_\lambda \neq 0$, then $\lambda$ is a *weight* and is real-valued on $i \cdot \mathfrak{t}$. In this case, $V_\lambda$ is called the *weight space*. There are only finitely many weight spaces and there is an orthogonal decomposition $V = \oplus V_\lambda$.

Recall that we can put an ordering on $\mathfrak{t}^*$ (such as lexicographic ordering) that gives us the set of positive roots $\Delta^+(\mathfrak{k}, \mathfrak{t})$. Also note that the roots and weights are both linear functionals on $\mathfrak{t}$. From this ordering, and the fact that there are only finitely many weight spaces, a representation has a (unique) *highest weight*. A weight is *dominant* if $\frac{2\langle\lambda,\alpha\rangle}{|\alpha|^2} \geq 0$ for $\alpha \in \Delta^+(\mathfrak{k}, \mathfrak{t})$. We also say that a linear functional

$\lambda$ on $\mathfrak{t}^{\mathbb{C}}$ is *analytically integral* if there is a character $\xi_\lambda$ of $T$ with $\xi_\lambda(\exp H) = e^{\lambda(H)}$ for all $H \in \mathfrak{t}$. Such analytically integral linear functionals are real-valued on $i\mathfrak{t}$. All weights are analytically integral. The following theorem is taken from [Kna86] (see also [Vog85]).

Let $W(K,T) = N_K(T)/Z_K(T)$, which is a Weyl group and is finite.

**Theorem 3.2.2.1** (Cartan-Weyl theorem). *The set of equivalence classes of irreducible representations of $K$ is in one-to-one correspondence with analytically integral and dominant linear functional on $\mathfrak{t}^{\mathbb{C}}$. This correspondence between a (equivalence class of) representation $\pi$ and an analytically integral and dominant linear functional $\lambda$ is that $\lambda$ is the highest weight of $\pi$.*

**Proposition 3.2.2.2.** *Suppose that $\lambda$ is a linear functional on $i\mathfrak{t}$ and a positive root system is given. Then $\lambda$ is conjugate via $W$ to a dominant linear functional.*

Consider a character $\chi$ on $T$. Then there is inherently an analytically integral linear functional $\lambda'$ such that $\chi(\exp X) = e^{\lambda'(X)}$ for $X \in \mathfrak{t}$. Now $\lambda'$ is conjugate to a dominant linear functional via an element $w \in W(K,T)$, call it $\lambda$ (therefore $\lambda = w\lambda'$ or $\lambda(X) = \lambda'(\mathrm{Ad}_w(X))$). Thus $\chi(\mathrm{Ad}_w(\exp X)) = \chi(\exp(\mathrm{Ad}_w(X))) = e^{\lambda(\mathrm{Ad}_w(X))} = e^{\lambda'(X)}$. So we can define a conjugate character $\chi' = w\chi$ which has an dominant analytically integral linear functional. These results lead to

**Corollary 3.2.2.3.** *There is a one-to-one correspondence between $\hat{K}$ and $\hat{T}/W(K,T)$.*

## 3.3 Unitarily Induced Representations

Let $H$ be a closed subgroup of a locally compact group $G$, $\sigma$ a unitary representation of $H$ with representation space $V_\sigma$. Let $\eta$ be a quasi-invariant Borel measure on $G/H$ (quasi-invariant means that for a set Borel set $E \subset G/H$, $\eta(E) = 0$ if and only if $\eta(gE) = 0$ for all $g \in G$). The space $\mathrm{Ind}_H^G V_\sigma$ is the closure of the set of functions $f : G \to V_\sigma$ such that

1. $f$ is a Borel function,

2. for $h \in H$, $g \in G$ we have

$$f(gh) = \left(\frac{\Delta_H(h)}{\Delta_G(h)}\right)^{-1/2} \sigma(h)^{-1} f(g),$$

3. and the norm is given by

$$\|f\|^2 = \int_{G/H} |f(x)|^2 \, d\eta(x).$$

We define $\mathrm{Ind}_H^G \sigma(g) f(x) = f(g^{-1}x)$ for $g \in G$ and $f \in \mathrm{Ind}_H^G V_\sigma$. We call $\mathrm{Ind}_H^G \sigma$ the *unitary induced representation* of G induced from σ. The coefficient $\left(\frac{\Delta_H(h)}{\Delta_G(h)}\right)^{-1/2}$ makes $\mathrm{Ind}_G^H \sigma$ a unitary representation (without it, the action would not be unitary). See [Mac76, Section 3.2], [Var89, Section 3.3], and [Kna86, Section VII.2] for more details.

### 3.3.1 Cartan Motion Group

We recall the definition of the Cartan motion group $G_0$ associated to G (see Section 2.3). By the Cartan decomposition, we have $\mathfrak{p} \cong \mathfrak{g}/\mathfrak{k}$. So we let the *Cartan motion group* be the semidirect product $G_0 = K \ltimes \mathfrak{p}$. with multiplication defined by

$$(k, X) \times (k', X') = (kk', \mathrm{Ad}_{(k')^{-1}}(X) + X').$$

We repeat the following remark:

*Remark.* The Cartan motion group is not reductive and the structure of G and $G_0$ are quite different.

In the rest of this section, we will describe all the irreducible unitary representations on $G_0$, up to unitary equivalence.

Let χ be a unitary character on $\mathfrak{p}$, that is, a continuous group homomorphism $\chi : \mathfrak{p} \to U(\mathbb{C})$ where $U(V)$ is the group of unitary operator on the vector space V. Define the *isotropy subgroup associated to* χ as

$$K_\chi = \{k \in K | \chi(\mathrm{Ad}_k(X)) = \chi(X) \text{ for all } X \in \mathfrak{p}\}. \quad (3.3.1.1)$$

Since $K_\chi$ is a closed subgroup of $K$, it is also compact. So now let $\tau_\chi$ be an irreducible unitary representation of $K_\chi$ on a finite-dimensional vector space $W$. We can define a representation of $K_\chi \ltimes \mathfrak{p}$ on the representation space $W$ by $\tau_\chi \otimes \chi : (k, X) \mapsto \chi(X)\tau_\chi(k)$. Now we induce the representation $\tau_\chi \otimes \chi$ up to $G_0$. So let $\phi_{\chi,\tau_\chi} = \mathrm{Ind}_{K_\chi \ltimes \mathfrak{p}}^{G_0} \tau_\chi \otimes \chi$. Mackey was able to show the following (see [Mac76], [Hig08] or [Hig11]):

**Theorem 3.3.1.1.** *The induced representation $\phi_{\chi,\tau_\chi}$ is an irreducible unitary representation of $G_0$ and each irreducible unitary representation of $G_0$ is unitarily equivalent to one such induced representation.*

So far, in the previous paragraph, we have exhibited all irreducible unitary representation of $G_0$ and how to construct them. Now we describe their equivalence classes to complete the classification.

**Proposition 3.3.1.2** (See [Hig08, Section 2.1] or [Mac76, Chapter 3])**.** *two induced representations $\phi_{\chi,\tau_\chi}$ and $\phi_{\chi',\tau'_{\chi'}}$ are unitarily equivalent if and only if there is a $K \in K$ such that $\chi'(X) = \chi(\mathrm{Ad}_k(X))$ and $\tau'_{\chi'} \cong \tau_\chi \circ \mathrm{Ad}_k$.*

This tells us that each element of an equivalence class of some irreducible unitary representation are obtained by conjugacy in $K$. Thus we can conclude

$$\hat{G}_0 \cong \{(\tau_\chi, \chi) | \chi \in \hat{\mathfrak{p}}, \tau_\chi \in \hat{K}_\chi\}/K$$

where we are denoting the space of (equivalence classes of) unitary characters of $\mathfrak{p}$ by $\hat{\mathfrak{p}}$.

### 3.3.2 Principal Series

Now we go back to $G$ itself. Recall the Iwasawa decomposition $G = KAN$ associated to a system of positive roots $\Delta^+(\mathfrak{g}, \mathfrak{a})$ (to define $N$), and a cuspidal parabolic subgroup $S$ of $G$ with the Langlands decomposition $S = MAN$. Let $\sigma$ be an irreducible unitary representation of $M$ on representation space $V_\sigma$. Let $\mathfrak{a}^*$ denote the space of (real) linear functional on $\mathfrak{a}$. Then $i\nu$ is a linear functional with its image in $i\mathbb{R}$. Then, since $\exp \mathfrak{a} = A$, we can define a representation of $A$ by $a \mapsto e^{i\nu(\log a)}$

(the representation space of $\nu$ is one-dimensional). We will denote this by $e^{i\nu}$. We can then define a representation $\sigma \otimes e^{i\nu} \otimes 1$ of $MAN$ where $1$ denotes the trivial representation of $N$, thus $\sigma \otimes e^{i\nu} \otimes 1$ is unitary. Now define $L^2(G)^{\sigma,\nu}$ to be the completion of

$$\{f \in C(G, V^\sigma) | f(gman) = e^{-(\rho+i\nu)(\log a)} \sigma(m)^{-1} f(g)\},$$

under the norm $\|f\|^2 = \int_K |f(k)|^2 \, dk$ where $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+(\mathfrak{g},\mathfrak{a})} \dim(\alpha)\alpha$. The dimension of a root refers to the dimension of its restricted root space. Note that since G is unimodular, we have $\Delta_G = 1$, and $\Delta_{MAN}(man) = e^{2\rho(\log a)}$, which explains the extra factors found in the representation space (see the discussion at the beginning of this section). The representation $\pi_{\sigma,\nu} = \operatorname{Ind}_{MAN}^G \sigma \otimes e^{i\nu} \otimes 1$ on the induced representation space $L^2(G)^{\sigma,\nu}$ is

$$\pi_{\sigma,\nu}(g) f(x) = f(g^{-1}\nu).$$

$L^2(G)^{\sigma,\nu}$ can be described a bit differently using the equivalent *compact picture*. Define $L^2(K)^\sigma$ to be the completion of

$$\{f \in C(K, V^\sigma) | f(km) = \sigma(m)^{-1} f(k), k \in K, m \in K \cap M\}$$

under the same norm $\|f\|^2 = \int_K |f(k)|^2 \, dk$. By the Iwasawa decomposition $G = KAN$, we see that $L^2(G)^{\sigma,\nu}$ and $L^2(K)^\sigma$ are unitarily equivalent by restriction of functions. Now let $g \in G$ and decompose $g$ under $KAN$, as

$$g = \kappa(g) e^{H(g)} n.$$

The action of $\pi_{\sigma,\nu}$ on $L^2(K)^\sigma$ is

$$\pi_{\sigma,\nu}(g) f(k) = e^{-(\rho+i\nu)H(g^{-1}k)} f(\kappa(g^{-1}k)).$$

Note that the representation space $L^2(K)^\sigma$ is independent of $\nu$. As a consequence of general facts about unitary induction, $\pi_{\sigma,\nu}$ is unitary. See also [Kna86, Section VII.2].

While most parabolic inductions are irreducible, we may find some that are not.

Suppose that $Q = MAN$ is the minimal parabolic subgroup. Then the representation $\pi_{\sigma,\nu}$ of $G$ induced from $Q$ is called the *(unitary) principal series* if $\nu$ is imaginary on $\mathfrak{a}$. It is called *nonunitary principal series* for a general $\nu$. We will not have use for the general nonunitary principal series. Note that $M$ is compact, thus, any irreducible representation of $M$ is finite-dimensional, unitary, and of the discrete series, therefore it is also a tempered representation.

**Proposition 3.3.2.1** ( [Kna86, Proposition 7.1]). *If $G$ is complex semisimple, and $Q = MAN$ is minimal parabolic subgroup, then, the induced representation $\pi_{\sigma,\nu}$ of $G$ induced from $Q$ is irreducible for any $\sigma$ irreducible and $\nu \in \mathfrak{a}^*$.*

The irreducibility of the principal series in full generality is due to Wallach [Wal71]. Since in this dissertation, we only care about tempered representations, so we need to note the following proposition:

**Proposition 3.3.2.2** ( [Kna86, Section VII.11]). *If $\sigma$ is an irreducible tempered representation of $M$ and $\nu$ is imaginary on $\mathfrak{a}$, then $\pi_{\sigma,\nu}$ is a tempered representation.*

Note that the complex group $G$ itself does not have a compact Cartan subgroup, thus it does not have a discrete series. Recall from previous chapter that all Cartan subgroups of a complex semisimple group $G$ are conjugate, so there is only one nonequivalent cuspidal parabolic subgroup for a complex semisimple group, namely the minimal parabolic subgroup. The principal series comprises all the irreducible tempered representation of $G$.

## 3.4 Intertwining Operators

We still need to address the question of equivalence classes among the principal series. The question of equivalence classes amounts to a question of intertwining operators, which is a deep theory and perhaps beyond the scope of this dissertation, In fact, the difficulty of this concept is what contributes to one of the open question in our work, and is the reason that we will focus on the complex semisimple group and $SL(2, \mathbb{R})$. For more details about the general intertwining operator see [Kna86] and [KS80].

Let us consider the principal series. Let $W$ be the Weyl group of $G$, that is $W = W(\mathfrak{a} : G) = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$, which is a finite group. Two principal series representations $\pi_{\sigma,\nu}$ and $\pi_{\sigma',\nu'}$ are equivalent if there is an intertwining operator

$$J : L^2(K)^{\sigma,\nu} \to L^2(K)^{\sigma',\nu'}$$

such that

$$J\pi_{\sigma,\nu} = \pi_{\sigma',\nu'}J.$$

This only occurs if there is a $w \in W$ (that is, $w \in N_K(\mathfrak{a})$, and the fact here depends only on the equivalence class of $w$) such that $\sigma' \cong w\sigma$ and $\nu' = w\nu$. (see [Kna86]).

The actual construction of intertwining operators is very difficult. Here is a very short sketch. Let us consider the nonunitary principal series $\pi_{\sigma,\nu}$ (which includes the case of unitary principal series) where $\nu$ is a general complex linear functional, but we are still only concerned with minimal parabolic subgroup. We can formally define an intertwining operator $J_{w,\sigma,\nu}$ by

$$J_{w,\sigma,\nu}f(x) = \int_{\overline{N}\cap w^{-1}Nw} f(xw\overline{n})\ d\overline{n} \qquad (3.4.0.1)$$

where $f \in L^2(G)^{\sigma,\nu}$ and $\overline{N}$ is the set of the inverse conjugate transpose of elements of $N$ (i.e. if we were to consider $N$ to be the set of upper triangular matrices with diagonal entries 1, then $\overline{N}$ is the set of the lower triangular matrices with entries 1). Unfortunately, the integral in (4.5.2.1) is not always convergent. We express some conditions for this integral to exist. See [Kna86].

**Proposition 3.4.0.1** ( [Kna86, Proposition 7.8]). *Let $Q = MAN$ be a minimal parabolic subgroup and $\Delta^+(\mathfrak{g}, \mathfrak{a})$ the set of positive roots determined by (or determining) $N$. Let $w \in N_K(\mathfrak{a})$ be an element representing a coset in W. For a given linear functional $\nu$ on $\mathfrak{a} \oplus i\mathfrak{a}$, suppose that $\langle \mathrm{Re}\,\nu, \beta\rangle > 0$ for every $\beta \in \Delta^+(\mathfrak{g} : \mathfrak{a})$ for which $w\beta < 0$. Then the integral in (4.5.2.1) converges for all $f$ and $x \in G$, and thus, $J_{w,\sigma,\nu}$ is an intertwining operator.*

**Proposition 3.4.0.2** ( [KS80, Theorem 4.2]). *With $J_{w,\sigma,\nu}$ defined for $\langle \mathrm{Re}\,\nu, \beta\rangle > 0$ for $\beta \in \Delta^+(\mathfrak{g}, \mathfrak{a})$ such that $w\beta < 0$, there is a meromorphic continuation to all of $\nu$. $J_{w,\sigma,\nu}$ is* *holomorphic at* $\nu$, *except if*

$$\frac{2\langle \nu, \beta \rangle}{|\beta|^2} \in \mathbb{Z}.$$

We can conclude that if $\nu$ is imaginary, then $\frac{2\langle \nu, \beta \rangle}{|\beta|^2}$ is not an integer. Thus (4.5.2.1) exists and intertwines $\pi_{\sigma,\nu}$ and $\pi_{w\sigma,w\nu}$ for $\nu$ imaginary. We can conclude

**Proposition 3.4.0.3.** *If* $\sigma$ *and* $\sigma'$ *are unitary and* $\nu$ *and* $\nu'$ *are imaginary, then* $\pi_{\sigma,\nu}$ *and* $\pi_{\sigma'\nu'}$ *are equivalent if and only if there is a* $w \in N_K(\mathfrak{a})$ *so that* $[w] \in W$ *such that* $\sigma' \cong w\sigma$ *and* $\nu' = w\nu$.

We can parametrize the principal series of G by $(\hat{M} \times \hat{A})/W$ where $\hat{M}$ is the set of equivalence classes of irreducible unitary representation of M, $\hat{A}$ is the set of unitary character of A, and W is the Weyl group $N_K(\mathfrak{a})/Z_K(\mathfrak{a})$.

## 3.5 Mackey Bijection for Complex Groups

In the case where G is a complex reductive group, the principal series representations encompasses all the irreducible tempered unitary representations. Let us denote $\hat{G}_r$ to be the set of equivalence classes of irreducible tempered representation (the subscript r comes from the fact that this set is equivalent to reduced dual of G, to be better defined in the next chapter). Then from above we have

$$\hat{G}_r \cong (\hat{M} \times \hat{A})/W.$$

Recall that

$$\hat{G}_0 \cong \{(\tau_\chi, \chi) | \chi \in \hat{\mathfrak{p}}, \tau_\chi \in \hat{K}_\chi\}/K.$$

The Mackey bijection relates the two parametrizations of the two duals.

### 3.5.1 Construction of Mackey Bijection

In this section we focus on the complex semisimple group, which is taken from [Hig08]. We let $G = KAN$ be the Iwasawa decomposition and $M = Z_K(\mathfrak{a})$ with $Q = MAN$ the minimal parabolic subgroup as usual. It should be noted that M is a maximal torus in $K$ (consider how in the complex case, there is only one Cartan subalgebra up to conjugacy).

**Lemma 3.5.1.1** ( [Kna86, Theorem 5.13]). Let $\mathfrak{p}$ and $\mathfrak{a}$ be defined as usual ($\mathfrak{a}$ is maximal abelian subalgebra of $\mathfrak{p}$). Then $\mathfrak{p} = \bigcup_{k \in K} \mathrm{Ad}_k(\mathfrak{p})$. Furthermore, any two maximal abelian subalgebra $\mathfrak{a}$ and $\mathfrak{a}'$ are conjugate by an element of $K$.

Let $\mathfrak{a}^{\perp}$ be the orthogonal complement of $\mathfrak{a}$ in $\mathfrak{p}$. Then every character of $\mathfrak{p}$ is conjugate by an element in $K$ to a character that vanishes on $\mathfrak{a}^{\perp}$, thus can be viewed as a character of $\mathfrak{a}$, which is unique up to conjugacy by a Weyl group element in $W$. To see this, consider that the set of characters of $\mathfrak{p}$ is in one-to-one correspondence with $\mathfrak{p}$. So we can consider a character $\chi_X$ on $\mathfrak{p}$ with $X \in \mathfrak{p}$. Now conjugacy of $\chi_X$ by $k \in K$ means $k\chi_X = \chi_{\mathrm{Ad}_k(X)}$. We can also consider $k\chi_X(Y) = \chi(\mathrm{Ad}_{k^{-1}}(Y))$. From the above lemma, we see that we can find a $k \in K$ that will conjugate $X \in \mathfrak{p}$ to an element $Y \in \mathfrak{a}$. Thus $k\chi_X = \chi_Y$, which vanishes on $\mathfrak{a}^{\perp}$. Each $K$-orbit of a character $\chi$ on $\mathfrak{p}$ has a one-to-one correspondence with a $W$-orbit of a character on $\mathfrak{a}$. Thus

$$\hat{\mathfrak{p}}/K \cong \hat{\mathfrak{a}}/W.$$

Now we can make the following statement: $\phi_{\chi,\tau_\chi}$ and $\phi'_{\chi',\tau_{\chi'}}$ are unitarily equivalent if and only if there is a $[w] \in W$ such that $\chi'(X) = \chi(\mathrm{Ad}_w(X))$ and $\tau'_{\chi'} = \tau_\chi \circ \mathrm{Ad}_w$ for any $w$ in $[w]$ (this is well-defined), thus

$$\hat{G}_0 \cong \{(\tau_\chi, \chi) | \chi \in \hat{\mathfrak{a}}, \tau_\chi \in \hat{K}_\chi\}/W.$$

**Lemma 3.5.1.2** ( [Hig08, Lemma 2.2]). Let $G$ be a connected semisimple group. Then if $\chi$ is a unitary character of $\mathfrak{p}$, then the isotropy group $K_\chi$ is connected.

The connectedness will allow us to use the highest weight theorem. It is clear that $M$ is contained in $K_\chi$. Indeed, given $m \in M$, we know that $\mathrm{Ad}_m(X) = X$ for any $X \in \mathfrak{a}$. It stands to reason that $\chi(\mathrm{Ad}_m(X)) = \chi(X)$ for all $X \in \mathfrak{a}$. Furthermore, since $M$ is maximal abelian in $K$, it must also be maximal abelian in $K_\chi$. Let $W_\chi = N_{K_\chi}(M)/M$ be the Weyl group of $K_\chi$. Then by Corollary 3.2.2.3, $\hat{K}_\chi$ is in one-to-one correspondence with $\hat{M}/W_\chi$.

Let us take a closer look at $K_\chi$ and $W_\chi$. Suppose that $\chi$ is trivial. Then it should be obvious that $K_\chi = K$. Now suppose that $\chi$ is not trivial. Then by elementary linear algebra we can see that $K_\chi = M$. In this case, $W_\chi$ is trivial, So $\widehat{K}_\chi \cong \widehat{M}$. For the moment, we shall ignore the case where $\chi$ is trivial. Then we see that

$$\widehat{G}_0 \cong (\widehat{\mathfrak{a}} \times \widehat{M})/W.$$

For G, we will ignore the case where the character $\nu$ on $A$ is trivial. We also have the fact that, since $\exp(\mathfrak{a}) = A$, we get $\widehat{\mathfrak{a}} \cong \widehat{A}$, the set of characters on $A$. Then, we have the following bijection. For $\chi \in \mathfrak{a}$ and $\sigma \in \widehat{M}$ we map $(\chi, \sigma) \mapsto (\nu, \tau_\sigma)$ where $\nu = \exp(\chi)$ and $\tau_\sigma$ is the corresponding representation of $K_\chi$ with $\sigma$ as its highest weight. In summary, we have

$$\widehat{G}_0 \cong (\widehat{\mathfrak{a}} \times \widehat{M})/W \cong (\widehat{A} \times \widehat{M})/W \cong \widehat{G}_r,$$

which is what we are calling the *Mackey bijection*. The equivalence classes of irreducible tempered unitary representations of G and equivalence classes of irreducible unitary representations of $G_0$ are in bijection based on their parametrization. Theorem 5.4.2.1 gives a new characterization this bijection.

## 3.6 Mackey Bijection for $SL(2, \mathbb{R})$

Now, let $G = SL(2, \mathbb{R})$. Let us describe the two principal series representations. Here we get $M = \{\pm I\}$, so there are only two unitary character of M: $\sigma_+$, the trivial character, and $\sigma_-$, defined by $\sigma_-(\pm I) = \pm 1$. The irreducible character $\nu$ of $\mathfrak{a}$ is defined by $\nu(\mathrm{diag}(t, -t)) = i\nu t$, where we may consider $\nu$ as a real number (since $\widehat{\mathfrak{a}} \cong \mathbb{R}$). Then we have representations of the principal series $\pi_{+,\nu}$ and $\pi_{-,\nu}$, induced in the same way as defined earlier for the complex case. $\pi_{+,\nu}$ is irreducible and unitary, and $\pi_{-,\nu}$ is irreducible and unitary except where $\nu = 0$. $\pi_{-,0}$ decomposes into $D_1 \oplus D_{-1}$, where $D_1$ and $D_{-1}$ are called the *limits of discrete series*. Each of them are irreducible and unitary but they are not equivalent to each other. The only equivalences among principal series are between $\pi_{+,\nu}$ and $\pi_{+,-\nu}$, and between $\pi_{-,\nu}$ and $\pi_{-,-\nu}$.

The principal series are not the only tempered representation of G. The limits of discrete series are also tempered, and $G = SL(2, \mathbb{R})$ has representations of the discrete series which are parametrized by $n \in \mathbb{Z}$ with $|n| \geq 2$. We shall denote the discrete series representations by $D_n$ with $n \geq 2$ and $n \leq -2$. So the equivalence classes of irreducible tempered representations of G can be described by

$$\hat{G}_r \cong \{D_n | n \in \mathbb{Z} - \{0\}\} \cup \{\pi_{+,\nu} | \nu \in \mathbb{R}^+ \cup \{0\}\} \cup \{\pi_{-,\nu} | \nu \in \mathbb{R}^+\}.$$

Now let us explore the irreducible unitary representation of the Cartan motion group of G. In this case, however, $K_\chi$ is not neccessarily connected. For more information about a general real reductive group with representations of the discrete series (that is, it has a compact Cartan subgroup, such as $SL(2, \mathbb{R})$) see [Afg18]. In that paper he uses the notion of the minimal K-type, which we will not cover here. Instead, we will only focus on the particular example. Recall that

$$\hat{G}_0 \cong \{(\chi, \tau_\chi) | \chi \in \hat{A}, \tau_\chi \in K_\chi\}/W,$$

where $W = \{I, \begin{bmatrix} & 1 \\ -1 & \end{bmatrix}\}$ and $\hat{A} \cong \mathbb{R}$. Suppose that $\chi_s \in \hat{A}$ with $s \in \mathbb{R}$. We have two cases. Suppose $\chi_s$ is non-trivial, that is, $s \neq 0$. Then $K_{\chi_s} = M = \{\pm I\}$. So for this $K_{\chi_s}$ we have two non-equivalent representations: $\sigma_+$ and $\sigma_-$. We have the following irreducible unitary representations of $G_0$: $\phi_{s,\pm} = \mathrm{Ind}_{K_{\chi_s} \ltimes \mathfrak{a}}^{G_0}(\chi_s, \sigma_\pm)$, with equivalences between $\phi_{s,\pm}$ and $\phi_{-s,\pm}$. Now consider $\chi_0$, the trivial character on $A$. Then $K_{\chi_0} = K = SO(2)$. In this case, all irreducible representations in $\hat{K}_{\chi_0}$ are of the form

$$\tau_n \left( \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \right) = e^{in\theta}$$

with $n \in \mathbb{Z}$ and there are no equivalences among them.

The previous paragraph allows us to build a bijection between $\hat{G}_0$ and $\hat{G}_r$. We write the following mapping: let $\phi_{s,\pm} \in \hat{G}_0$ with $s > 0$ and map

$$\phi_{s,\pm} \mapsto (\chi_s, \sigma_\pm) \mapsto \pi_{\pm,\nu_s} = \mathrm{Ind}_{MAN}^{G} \sigma_\pm \otimes \nu_s.$$

Here, since $\chi_s \in \hat{A}$ as well as $\nu_s \in \hat{A}$, we consider $\chi_s = \nu_s$ (we are just mapping the notations). On the other hand, let $\phi_{0,\tau_n} \in \hat{G}_0$ (with $n \in \mathbb{Z} - \{0\}$) defined from $\chi_0 \in \hat{A}$, the trivial representation. Then we can map

$$\phi_{0,\tau_n} \mapsto (\chi_0, \tau_n) \mapsto D_n.$$

Finally, define

$$\phi_{0,\tau_0} \mapsto (\chi_0, \tau_0) \mapsto \pi_{+,0}.$$

These mappings together define a bijection, and thus, we have

$$\hat{G}_0 \cong \hat{G}_r$$

for the case of $G = SL(2, \mathbb{R})$.

In [AA19], the authors show that the Mackey bijection is, in fact, continuous, but cannot be a homeomorphism. $\hat{G}_r$ and $\hat{G}_0$ requires us to use $C^*$-algebras in order to define a topology (see Chapter 4). It is then from this that a notion of continuity makes sense.

# Chapter 4 | Reduced $C^*$-Algebras and Continuous Fields

$C^*$-algebras have their own representation theory, which we will explore in this chapter. We will define the full and reduced $C^*$-algebras and indicate how they relate to group representation theory. Once we have accomplished this, we will then construct a continuous field of $C^*$-algebras that will play a central role in our analysis of the Mackey bijection.

## 4.1 Full $C^*$-Algebras of Lie Groups

While the Cartan motion group is not reductive, it is unimodular (left and right Haar measure are equal), so in this dissertation we will focus only on the unimodular case, since reductive groups are also unimodular (see Proposition 2.6.0.3). Let $G$ be a unimodular Lie group. Given a unitary representation $\pi$ of $G$ on a representation Hilbert space $V$, we can extend it to a representation of the convolution algebra of integrable functions $f \in L^1(G)$. We define $\pi(f)$ as a weak integral

$$\pi(f)v = \int_G f(g)\pi(g)v \, dg$$

for $v \in V$. Thus $\pi(f)v$ is the unique vector such that

$$\langle u, \pi(f)v \rangle = \int_G f(g) \langle u, \pi(g)v \rangle \, dg$$

for $u \in V$. See [Dix77, Chapter 13]. A few of the immediate properties are

1. $\|\pi(f)\|_\pi \leq \|f\|_1$,
2. $\pi(f)^* = \pi(f^*)$, with $f^*(x) = \overline{f(x^{-1})}$ (since G is unimodular),
3. $\pi(f * f') = \pi(f)\pi(f')$ where $f * f'(x) = \int_G f(xy^{-1})f'(y) \, dy = \int_G f(y)f'(x^{-1}y) \, dy$ is the convolution.

Define a (new) norm on $L^1(G)$ by

$$\|f\|_{\text{full}} = \sup_{\pi \in \hat{G}} \|\pi(f)\|_\pi.$$

By property 1, this is well-defined and finite. Then we define a $C^*$-algebra by taking the closure of $L^1(G)$ with respect to the norm $\| \, \|_{\text{full}}$, and denote it by $C^*(G)$.This is called the *full* $C^*$*-algebra* of $G$.

We can define representations of $C^*$-algebras. A $*$-homomorphism $\phi : A \to B(V)$ from a $C^*$-algebra to the set of bounded operators on a Hilbert space $V$ is called a *representation* of $A$ on $V$. We say that a representation $\phi$ of $A$ on $V$ is *non-degenerate* if $\overline{\phi[A]V} = V$ where

$$\phi[A]V := \{\phi(a)v | a \in A, v \in V\}.$$

Everything that we have defined for representations of topological groups, we can also define on representations of $C^*$-algebras. We have notions of irreducible representations, intertwining operators between two representation spaces, and equivalence of representations. We can define $\hat{A}$ to be the set of equivalence classes of irreducible representations of $A$. See [Dix77].

**Proposition 4.1.0.1.** *There is a one-to-one correspondence between unitary representations of* $G$ *and non-degenerate representations of* $C^*(G)$.

*Sketch of Proof.* The idea behind the correspondence is explained by Rosenberg in his article [Ros94], which we will paraphrase here. As we saw above, given a unitary representation $\pi$ of $G$ on $V$, we can define a representation $\phi_\pi$ on $C^*(G)$ by defining on $f \in L^1(G)$

$$\phi_\pi(f)v = \int_G f(g)\pi(g)v \, dg$$

with $v \in V$ and extending $\phi_\pi$ to $C^*(G)$.

On the other hand, Let $\phi$ be a representation of $C^*(G)$ on $V$. In general, $G$ does not fit inside of $C^*(G)$. But $G$ does fit inside of the multiplier algebra of $C^*(G)$, denote it by $M(C^*(G))$. We can canonically extend $\phi$ to a representation $\tilde{\phi}$ of $M(C^*(G))$ on the same representation space. Then we can restrict $\tilde{\phi}$ to $G$, and we get a representation $\pi_\phi = \tilde{\phi}|_G$ of $G$ on $V$. □

As Rosenberg remarks in [Ros94],

> ... essentially all questions one can ask about the unitary representation theory of $G$ are equivalent to questions about the structure of $C^*(G)$.

It is with this in mind that we shall tackle the Mackey bijection phenomenon by taking the $C^*$-algebra approach.

**Corollary 4.1.0.2.** $\widehat{C^*(G)} \cong \hat{G}$. □

Given a $C^*$-algebra $A$, we can define a topology on $\hat{A}$, called the *Fell topology*. Take an irreducible representation $\pi : A \to B(H)$. Then $\ker \pi$ is an ideal of $A$. An ideal $J$ in $A$ is *primitive* if $J = \ker \pi$ for some irreducible representation $\pi$ of $A$. The Jacobson topology on the set of all primitive ideals $\mathrm{Prim}(A)$ of $A$ is defined as follows: the closed sets of primitive ideals are given by general ideals I:

$$F_I = \{J \in \mathrm{Prim}(A) | I \subset J\}.$$

Define a mapping $\pi \mapsto \ker \pi$. This mapping $\hat{A} \to \mathrm{Prim}(A)$ is a canonical surjection. The Fell topology on $\hat{A}$ is then the inverse image of the Jacobson topology of the defined mapping. This means that we have a topology on $\widehat{C^*(G)}$. By the one-to-one correspondence between $\widehat{C^*(G)}$ and $\hat{G}$, we can impose a topology on $\hat{G}$ as well. We will not have much use for these notions of topology in this dissertation, however, Afgoustidis and Aubert showed in [AA19] that the Mackey bijection $\hat{G} \to \hat{G}_0$ is continuous, with respect to the topology defined here. See [Dix77].

## 4.2 Reduced $C^*$-Algebras of Lie Groups

Recall that the left regular representation $\lambda$ of $G$ on the Hilbert space $L^2(G)$ is defined by $\lambda(g)f(x) = f(g^{-1}x)$. Let $h \in L^1(G)$. Then we can define

$$\lambda(h)f(x) = h * f(x) = \int_G h(g)\lambda(g)f(x)\ dg = \int_G h(g)f(g^{-1}x)\ dg,$$

where $h * f$ is the *convolution* of $h$ and $f$. Note that the image of $\lambda$ is in $B(L^2(G))$, which has a norm, the bounded operator norm, which we will denote by $\| \|_B$. We define the *reduced* $C^*$*-algebra* of $G$, denoted $C^*_r(G)$, to be the completion of $L^1(G)$ (we can also use $C_c^\infty(G)$) in the norm

$$\|f\|_{C^*_r(G)} = \|\lambda(f)\|_B,$$

where $f \in L^1(G)$.

Now we describe the notion of weak containment and a new and equivalent definition for tempered representation. Given a representation $\pi$ of a $C^*$-algebra $A$, and a set $S$ of representations of the same $A$, we say that $\pi$ is *weakly contained* in $S$ if

$$\bigcap_{\sigma \in S} \ker \sigma \subset \ker \pi.$$

Likewise, we will say that a representation $\pi \in \hat{G}$ (which is the same as to say $\pi \in C^*(G)$), is *weakly contained* in $\lambda$ if

$$\ker \lambda \subset \ker \pi.$$

Recall that $\ker \lambda$ is an ideal of $C^*(G)$. By the first isomorphism theorem and the fact that images of $C^*$-algebra homomorphism are closed, we have

$$C_r^*(G) \cong C^*(G)/\ker\lambda.$$

We take a look at the quotient itself. We use the result from [Dix77, 2.11.2], which states

**Proposition 4.2.0.1.** *Let* $A$ *be a* $C^*$*-algebra with a closed two-sided ideal* $I$*. Let*

$$\hat{A}_I = \{\pi \in \hat{A} | \pi(x) = 0, \textit{ for all } x \in A\}.$$

*For each* $\pi \in \hat{A}_I$*, let* $\pi'$ *be the quotient representation of* $A/I$ *(letting* $\rho$ *be the canonical mapping from* $A$ *onto* $A/I$*, we are defining* $\pi'$ *to be the unique representation such that* $\pi' \circ \rho = \pi$*). Then the mapping* $\pi \mapsto \pi'$ *is a homeomorphism from* $\hat{A}_I$ *onto* $\widehat{A/I}$*.*

Consider a representation $\sigma' \in C^*(\widehat{G)/\ker}\lambda$. By the proposition, this has a unique correspondence with a representation $\sigma \in \widehat{C^*(G)}$ such that $\sigma(f) = 0$ for all $f \in \ker\lambda$. This means that $\ker\lambda \subset \ker\sigma$. On the other hand, consider a representation $\sigma \in \widehat{C^*(G)}$ such that $\ker\lambda \subset \ker\sigma$, thus we have that $\sigma(f) = 0$ for all $f \in \ker\lambda$. By the proposition above, there is a unique corresponding representation $\sigma' \in C^*(\widehat{G)/\ker}\lambda$. Thus we can give yet another definition: a representation $\sigma \in \widehat{C^*(G)}$ is *weakly contained* in $\lambda$ if $\sigma$ has a canonical correspondence with a representation $\sigma' \in \widehat{C_r^*(G)}$.

Recall that $\hat{G} \cong \widehat{C^*(G)}$. We will then define $\hat{G}_r$ to be the closed subset of $\hat{G}$ characterized by the following: $\sigma \in \hat{G}$ is also in $\hat{G}_r$ if the corresponding $\sigma \in \widehat{C^*(G)}$ has a corresponding representation $\sigma' \in \widehat{C_r^*(G)}$ (that is, $\ker\lambda \subset \ker\sigma$). In other words, $\hat{G}_r$ is the set of equivalence classes of irreducible representation such that, when viewed as a representation of $C^*(G)$, is weakly contained in $\lambda$. We call $\hat{G}_r$ the *reduced dual* of $G$. In [CHH88], it is shown that reduced dual and irreducible tempered dual are equivalent (See the definitions in Section 3.1.

We then have that for a complex reductive group $G$, $\hat{G}_r$ consist only of the principal series. If $G = SL(2, \mathbb{R})$, then $\hat{G}_r$ are the principal series and the discrete series.

## 4.3 Deformation Space

There are a few ways to define the following deformation space. For example, see [HSSH18], [Hig08] and [Hig10]. We will closely follow the definition in [Hig08] and in [Hig10]. See also [HR19, Section 2]. Let $M$ be a smooth submanifold of a smooth manifold $V$, both without boundary. We define a set

$$\mathbb{N}_V M := N_V M \times \{0\} \sqcup V \times \mathbb{R}^\times$$

where $N_V M$ is the normal bundle to $M$, that is, $N_V M = TV/TM$.

**Theorem 4.3.0.1.** *There is a unique topology and a unique smooth structure on* $\mathbb{N}_V M$ *such that*

(i) *The natural map* $N_V M \to \mathbb{R}$ *is smooth.*

(ii) *If* $f$ *is a smooth function on* $G$*, then the function*

$$\begin{cases} (g,t) \mapsto f(g) & t \neq 0 \\ (X_k, 0) \mapsto f(k) \end{cases}$$

*is smooth on* $\mathbb{N}_V M$*.*

(iii) *If* $f$ *is a smooth function on* $G$*, and if* $f$ *vanishes on* $M$*, then the function* $\delta f$ *defined by*

$$\delta f : \begin{cases} (g,t) \mapsto t^{-1} f(g) & t \neq 0 \\ (X_k, 0) \mapsto X_k(f) & t = 0 \end{cases}$$

*is smooth on* $\mathbb{N}_V M$*.*

(iv) *At every point, local coordinates can be selected from functions of the above types.*

*Remark* ( [Hig08, Remark 4.2]). The deformation space is taken from algebraic geometry where its counterpart is the *deformation to the normal cone*. See [Ful84, Chapter 5].

To illustrate (iv), let $(x_1, \ldots, x_k, y_1, \ldots, y_l)$ be a coordinate chart on V such that $(x_1, \ldots, x_k)$ is a coordinate chart on M and $y_j$ vanishes on M, then

$$(x_1, \ldots, x_k, \delta y_1, \ldots, \delta y_l, t)$$

is a coordinate chart on the deformation space $\mathbb{N}_V M$. In other words, the coordinate gives a diffeomorphism from a neighborhood of $\mathbb{N}_V M$ to an open subset of $\mathbb{R}^{k+l} \times \mathbb{R}$. See [Hig10].

We can provide a simple example to illustrate a deformation space. Let M be a smooth manifold and and let V be a vector bundle over M. Then we can view M as the zero section in V. Thus, M is a closed submanifold of V. The normal bundle $N_V M$ can then be identified with V itself. Then the mapping $\mathbb{N}_M V \to V \times \mathbb{R}$ defined by

$$\begin{aligned} (p, X_p, 0) &\mapsto (X_p, 0) && t = 0 \\ (p, X_p, t) &\mapsto (t^{-1} X_p, t) && t \neq 0 \end{aligned}$$

is a diffeomorphism from $\mathbb{N}_M V$ to $V \times \mathbb{R}$.

### 4.3.1 Deformation Space of a Reductive Group

Let G be a reductive Lie group. We will show that the deformation space of G over K is

$$\mathbb{N}_G K = ((K \ltimes \mathfrak{p}) \times \{0\}) \ \sqcup \ (G \times \mathbb{R}^\times).$$

Recall that there is a diffeomorphism $K \times \mathfrak{p} \cong G$ defined by $(k, X) \mapsto k \exp(X)$. So the mapping $\Phi : \mathbb{N}_G K \to G \times \mathbb{R}$ defined by

$$\begin{aligned} (k \exp(X), t) &\mapsto (k \exp(t^{-1} X), t) && \text{if } t \neq 0 \\ (k, X, 0) &\mapsto (k \exp(X), 0) && \text{if } t = 0 \end{aligned}$$

is a diffeomorphism. We now have that $\mathbb{N}_G K$ is a smooth bundle over $\mathbb{R}$. We show that $N_G K$ is a Lie group. We already know that it is a smooth manifold. The commuting diagram of multiplication operations

$$\begin{array}{ccc} K \times K & \longrightarrow & K \\ \downarrow & & \downarrow \\ G \times G & \longrightarrow & G \end{array}$$

induces a diagram

$$\begin{array}{ccc} TK \times TK & \longrightarrow & TK \\ \downarrow & & \downarrow \\ TG \times TG & \longrightarrow & TG, \end{array}$$

and we obtain from this a multiplication operation

$$N_G K \times N_G K \longrightarrow N_G K.$$

It makes the normal bundle into a Lie group. If we trivialize the tangent bundles on $G$ and $K$ by left translations, then we obtain an identification of bundles and Lie groups

$$N_G K \cong K \ltimes \mathfrak{g}/\mathfrak{k}$$

where on the right is the semidirect product group associated to the adjoint action.

We then define $G_t$ as the fiber over $t \in \mathbb{R}$. As a Lie group, $G_t = G$ for $t \neq 0$ and $G_0$ is the Cartan motion group. However, we shall want to equip the groups $G_t$ with a varying family of Haar measures. Choose a (left) Haar measure $dg$ on $G$. Recall that $\mathfrak{p} \cong \mathfrak{a} \times \mathfrak{n}$. There is a one to one correspondence between $C_c^\infty(\mathbb{N}_K G)$ and $C_c^\infty(K \times \mathfrak{a} \times \mathfrak{n} \times \mathbb{R})$ that is defined as follows. Let $f \in C_c^\infty(\mathbb{N}_K G)$. We define $\tilde{f} \in C_c^\infty(K \times \mathfrak{a} \times \mathfrak{n} \times \mathbb{R})$ as

$$\tilde{f}(k, X, Y, t) = \begin{cases} f(k \exp(tX) \exp(tY), t) & \text{if } t \neq 0 \\ f(k, X, Y, 0) & \text{if } t = 0. \end{cases} \tag{4.3.1.1}$$

We want to define a Haar measure $dg_t$ on $G_t$ so that it varies smoothly with $t$. We define

$$\int_{G_t} f_t(g)\ dg_t = \int_K \int_{\mathfrak{a}} \int_{\mathfrak{n}} \tilde{f}(k, X, Y, t)\ e^{2t\rho(X)}\ dk\ dX\ dY. \tag{4.3.1.2}$$

It is clear from Lemma 2.6.0.8 that the measure on the right hand side of (4.3.1.2) varies smoothly with t and is a Haar measure, even when $t = 0$. Then we have

$$\begin{aligned}\int_K \int_{\mathfrak{a}} \int_{\mathfrak{n}} \tilde{f}(k, X, Y, t)\ e^{2t\rho(X)}\ dk\ dX\ dY \\ &= \int_K \int_{\mathfrak{a}} \int_{\mathfrak{n}} f(k \exp(tX) \exp(tY), t)\ e^{2\rho(tX)}\ dk\ dX\ dY \\ &= \int_K \int_{\mathfrak{a}} \int_{\mathfrak{n}} f(k \exp(X) \exp(Y), t)\ t^{-\dim(\mathfrak{p})} e^{2\rho(X)}\ dk\ dX\ dY.\end{aligned}$$

So the Haar measure on $G_t$ ($t \neq 0$) is $|t|^{-\dim \mathfrak{p}}$ times the Haar measure on $G_1$ (identifying $G_1$ with G). At $G_0$, by the right-hand side of (4.3.1.2), the Haar measure is just the usual Haar measure on the Cartan motion group. In summary, if $\{f_t\}$ is a smooth section of $\{C_r^*(G_t)\}$, then for each $t \neq 0$, we have

$$\int_{G_t} f_t(g)\ dg_t = \int_K \int_{\mathfrak{a}} \int_{\mathfrak{n}} f(k \exp(tX) \exp(tY)) e^{2\rho(tX)}\ dk\ dX\ dY. \tag{4.3.1.3}$$

## 4.4 Continuous Fields of $C^*$-Algebras

The definitions and theory which we follow in this section are found in [Dix77, Chapter 10]. Let T be a topological space. Let $\{A_t\}_{t\in T}$ be a family of $C^*$-algebras indexed by T. We consider a *section* or a *vector field* $\{f_t\} \in \prod_{t\in T} A_t$ so that $f_t \in A_t$. We can make the following definitions: given $\{f_t\}, \{h_t\} \in \prod_{t\in T} A_t$ and a scalar $\lambda$, then

$$\{(f+h)_t\} = \{f_t + h_t\} \qquad \{(\lambda f)_t\} = \{\lambda f_t\}$$
$$\{(fh)_t\} = \{f_t h_t\} \qquad \{f_t\}^* = \{f_t^*\}.$$

Note we have not said anything about the norm on a section. Let $\Gamma \subset \prod_{t\in T} A_t$ be a family of sections with the following properties:

1. $\Gamma$ is closed under the operations defined above,
2. for every $t \in T$, the set $\{f_t | \{f_t\} \in \Gamma\}$ is dense in $A_t$,

3. for every $\{f_t\} \in \Gamma$, the function $t \mapsto \|f_t\|_t$ is continuous (where $\| \|_t$ is the norm on $A_t$),

4. given $\{h_t\} \in \prod_{t \in T} A_t$, if , at each point $t \in T$ and for every $\epsilon > 0$, there is an $\{f_t\} \in \Gamma$ such that for every $s$ in a neighborhood of $t$ we have $\|f_s - h_s\|_s < \epsilon$, then $\{h_t\} \in \Gamma$.

**Definition.** We call the pair $(A_t, \Gamma)$ a *continuous field* of $C^*$-algebras over $T$, and the elements of $\Gamma$ are the *continuous sections* of $(A_t, \Gamma)$.

We will provide a simple, but important, example. Let $T$ be a topological space and let $A$ be a $C^*$-algebra. Let $\Gamma$ be the set of continuous functions $f : T \to A$. We can define $A_t = A$, and thus $(A_t, \Gamma)$ is a continuous field of $C^*$-algebras over $T$. $(A_t, \Gamma)$ is called a *constant field*.

Let $(A_t, \Gamma)$ and $(A'_t, \Gamma')$ be two continuous fields of $C^*$-agebras over the same topological space $T$. Let $\Phi = \{\Phi_t\}_{t \in T}$ be a family of $C^*$ isomorphism from $A_t$ onto $A'_t$ ($\Phi_t : A_t \to A'_t$) If $\Phi$ maps $\Gamma$ to $\Gamma'$ bijectively then we say that $\Phi$ is an *isomorphism* of continuous fields from $(A_t, \Gamma)$ onto $(A'_t, \Gamma')$.

Recall that a subset of a topological vector space $V$ is a *total set* if its linear span is dense in $V$. Given a continuous field of $C^*$-algebras $(A_t, \Gamma)$ over a topological space $T$, a subset $\Lambda$ of $\Gamma$ is *total* if for every $t \in T$, the set $\{f_t | \{f_t\} \in \Gamma\}$ is total in $A_t$.

**Proposition 4.4.0.1** ( [Dix77, Proposition 10.2.3]). *Let $\{A_t\}_{t \in T}$ be a family of $C^*$-algebras over a topological space $T$, and a subset $\Lambda \subset \prod_{t \in T} A_t$ that satisfies the first three properties of continuous sections (replacing $\Gamma$ with $\Lambda$). Then there is a unique subset $\Gamma \subset \prod_{t \in T} A_t$ that contains $\Lambda$ and satisfies all four properties of continuous sets.*

In particular, $\{x_t\} \in \Gamma$ if and only if for every $t_0 \in T$ and every $\epsilon > 0$, there is an $\{x'\}_t$ in the span of $\Lambda$ such that

$$\|x_t - x'_t\| < \epsilon$$

in a neighborhood of $t_0$. See [Dix77, Proposition 10.2.2] for details. In this case, we call $\Lambda$ a *generating family* for $\Gamma$. See for example Section 5.2.

**Proposition 4.4.0.2** ( [Dix77, Proposition 10.2.4]). *Let* $(A_t, \Gamma)$ *and* $(A'_t, \Gamma')$ *be two continuous field of* $C^*$*-algebras over the same topological space* $T$. *For each* $t \in T$, *let* $\Phi_t : A_t \to A'_t$ *be a* $C^*$*-algebra isomorphism. Then* $\Phi = \{\Phi_t\}_{t\in T}$ *is an isomorphism of continuous fields from* $(A_t, \Gamma)$ *onto* $(B_t, \Gamma')$ *if and only if* $\Phi[\Lambda] \subset \Gamma'$ *.*

*Remark.* Throughout the dissertation, we will denote a continuous field of $C^*$-algebras simply by $\{A_t\}_{t\in T}$, or, when there is no ambiguity, by $\{A_t\}$. We will understand $\{f_t\} \in \{A_t\}_{t\in T}$ to be a continuous section.

### 4.4.1 Continuous Fields of Reduced $C^*$-Algebras of a Reductive Group

Let $\{C^*_r(G_t)\}_{t\in\mathbb{R}}$ be the family of $C^*$-algebras associated to the fibers $G_t$ of $\mathbb{N}_K G$ (each of which is a reductive group except at $G_0$, which is the Cartan motion group). For each $C^*_r(G_t)$ we use the Haar measure $dg_t$, and thus, the norm on this $C^*$-algebra is dependent on the Haar measure. Denote $\| \|_t := \| \|_{C^*_r(G_t)}$. We wish to define a space of continuous sections of $\{C^*_r(G_t)\}$.

Let $f \in C^\infty_c(\mathbb{N}_K G)$. Define $\{f_t\}$ to be a section on $\{C^*_r(G_t)\}$ by $f_t(g) = f(g, t)$ for $t \neq 0$ and $f_t(k, X) = f(k, X, 0)$ for $t = 0$. Then $\{f_t\}$ is a continuous section of $\{C^*_r(G_t)\}$ by the following lemma:

**Lemma 4.4.1.1** (See [Hig08, Lemma 6.13]). If $f$ is a smooth and compactly supported function on $\mathbb{N}_K G$, then the function

$$t \longmapsto \|f_t\|_t$$

is continuous.

Thus, by Proposition 4.4.0.1, $\{C^*_r(G_t)\}$ becomes a continuous field of $C^*$-algebras with continuous sections generated by elements of $C^\infty_c(\mathbb{N}_K G)$.

### 4.4.2 Multiplier Algebras

We want to look at certain elements of the multiplier algebra of $C^*_r(G_t)$ for each $t \in \mathbb{R}$. Take $\phi \in C^\infty(K)$. Then we can define an action on $C^\infty_c(G_t)$ on the left and right by convolution as follows: let $f_t \in C_c^\infty(G_t)$ and

$$\phi * f_t(g) = \int_K \phi(k) f_t(k^{-1}g)\ dk \tag{4.4.2.1}$$

and

$$f_t * \phi(g) = \int_K f_t(gk^{-1})\phi(k)\ dk, \tag{4.4.2.2}$$

which are elements of $C_c^\infty(G_t)$. Now, we have $\lambda(f_t * \phi) = \lambda(f_t)\lambda(\phi)$ and $\lambda(\phi * f_t) = \lambda(\phi)\lambda(f_t)$. Indeed, given $h \in L^2(G_t)$, we have

$$\begin{aligned}\lambda(f_t * \phi)h &= \int_{G_t} f_t * \phi(g)\lambda(g)h\ dg_t \\ &= \int_{G_t}\int_K f_t(gk^{-1})\phi(k)\ dk\ \lambda(g)h\ dg_t.\end{aligned}$$

Make the following change of variables: $g := gk^{-1}$. Then

$$\begin{aligned}&= \int_{G_t}\int_K f_t(g)\phi(k)\lambda(gk)h\ dk\ dg_t \\ &= \int_{G_t}\int_K f_t(g)\phi(k)\lambda(g)\lambda(k)h\ dk\ dg_t \\ &= \int_{G_t} f_t(g)\lambda(g)\int_K \phi(k)\lambda(k)h\ dk\ dg_t \\ &= \int_{G_t} f_t(g)\lambda(g)(\lambda(\phi)h)\ dg_t \\ &= \lambda(f_t)\lambda(\phi)h.\end{aligned}$$

By these actions, we have $C^*$-morphism from $C_r^*(K)$ into $M(C_r^*(G_t))$ for all $t \in \mathbb{R}$, including $t = 0$. This will be pertinent to the proofs in Chapter 5.

## 4.5 Structure of the Reduced $C^*$-Algebras of Some Lie Groups

In this section we review the structure of the reduced $C^*$-algebra of reductive groups and their motion groups as shown in [Hig08] and [CCH16]. Because of the mapping used to exhibit the structure, we will call these description the *Fourier transform pictures* of the reduced group $C^*$-algberas.

### 4.5.1 Structure of the Reduced $C^*$-Algebra of a Cartan Motion Group

We consider the Cartan motion group $G_0 = K \ltimes \mathfrak{p}$. We consider $\mathfrak{p}$, a vector space, as an abelian group. Let $\nu \in \mathfrak{p}^*$, the dual space of $\mathfrak{p}$. The function $\exp(i\nu)$ is a unitary character on the additive group $\mathfrak{p}$, and so we may form the unitarily induced representation

$$\pi_\nu = \operatorname{Ind}_{\mathfrak{s}}^{K \ltimes \mathfrak{p}} exp(i\nu)$$

of the motion group $G_0$. By definition, its Hilbert space is the completion of the space of smooth functions $\psi \colon G_0 \to \mathbb{C}$ such that

$$\psi(k, X) = \psi(k) \exp(-i\nu(X)) \qquad (\forall (k, X) \in G_0)$$

in the norm induced from the inner product

$$\langle \phi, \psi \rangle = \int_K \overline{\phi(k)} \psi(k) \, dk.$$

The action of $G_0$ is by left translation.

Of course the Hilbert space identifies with $L^2(K)$. Under this identification the subgroup $K \subseteq G_0$ acts by left translation, whereas an element $X \in \mathfrak{p}$ acts by pointwise multiplication by the function $k \mapsto \exp(i\nu(k^{-1} \cdot X))$.

The unitary representation $\pi_\nu$ of $G_0$ integrates to a $C^*$-algebra representation

$$\pi_\nu \colon C^*_r(G_0) \longrightarrow \mathcal{K}(L^2(K)),$$

and the Riemann-Lebesgue lemma for the ordinary Fourier transform implies the following result:

**Lemma 4.5.1.1** (See [Hig08, Section 3.1])**.** There is a morphism $C^*$-algebras

$$\pi\colon C_r^*(G_0) \longrightarrow C_0(\mathfrak{p}^*, \mathcal{K}(L^2(K)))$$

such that

$$\pi(f)(\nu) = \pi_\nu(f) \qquad (\forall \nu \in \mathfrak{p}^*). \qquad \square$$

There is a natural action of $K$ on $\sigma$: we define $k.\sigma(X) = \sigma(\mathrm{Ad}_k(X))$. Now consider the $C^*$-algebra $\mathcal{K}(L^2(K))$ of compact operator on $L^2(K)$. We can also define an action of $K$ on $\mathcal{K}(L^2(K))$. Let $T \in \mathcal{K}(L^2(K))$. Then we define $k.T(F) = T(\rho(k)F\rho(k)^{-1})$ where $F \in L^2(K)$ and $\rho$ is the right regular representation. We can then define the following $C^*$-algebra, $C_0(\mathfrak{p}^*, \mathcal{K}(L^2(K)))^K$ consisting of continuous functions $f$, that vanishes at infinity, from $\mathfrak{p}^*$ to $\mathcal{K}(L^2(K))$ with the added property that $f(k.\sigma) = k.f(\sigma)$ for every $k \in K$ and $\sigma \in \mathfrak{p}^*$. In this case, we say that $f$ is $K$-*equivariant*.

**Theorem 4.5.1.2** (See for example [Hig08, Thm 3.2])**.** *The morphism* $\pi$ *in Lemma* 4.5.1.1 *induces an isomorphism*

$$C_r^*(G_0) \xrightarrow{\cong} C_0(\mathfrak{p}^*, \mathcal{K}(L^2(K)))^K. \qquad \square$$

*Remark.* The representations $\pi_\nu$ are not irreducible. See subsection 3.3.1.

### 4.5.2 Structure of the Reduced $C^*$-Algebra of a Reductive Group

We have just described the structure of a reduced $C^*$-algebra of a Cartan motion group (the same description applies to any semidirect product $K \ltimes V$ where $K$ is a compact group and $V$ is abelian). In this section, we will describe the structure of the reduced $C^*$-algebra of a complex reductive group and of $SL(2, \mathbb{R})$. In general, the reduced $C^*$-algebras of reductive groups are very complicated. For a more complete structure theory of a real reductive group, see [CCH16].

Let G be a real reductive group with, and let $S = MAN$ be a cuspidal parabolic subgroup with a Langlands decomposition. Given $\sigma \in \widehat{M}$ and $\nu \in \mathfrak{a}^*$ (so that $e^{i\nu} \in \widehat{A}$), let $H_{\sigma,\nu}$ be the representation space of $\sigma \otimes e^{i\nu}$ and $H_\sigma$, the representation space of $\sigma \otimes$ (trivial). We begin with a theorem

**Theorem 4.5.2.1** ( [CCH16, Theorem 4.7]). *The $C^*$-algebra of a real reductive group G acts by compact operators in any irreducible unitary representation of G.*

This tells us that for any $f \in C^*(G)$, $\pi_{S,\sigma,\nu}(f) = \operatorname{Ind}_S^G \sigma \otimes e^{i\nu} \otimes 1$ is a compact operator on $\operatorname{Ind}_S^G H_{\sigma,\nu}$. Recall that $\lambda[C^*(G)] = C_r^*(G) \subset B(L^2(G))$. In particular, $\lambda(f)$ (with $f \in C^*(G)$) will act on an element $\phi \in \operatorname{Ind}_S^G H_{\sigma,\nu}$ by

$$\lambda(f)\phi(x) = \int_G f(g)\phi(g^{-1}x)\, dg = \int_G f(g)\pi_{S,\sigma,\nu}(g)\phi(x)\, dg = \pi_{S,\sigma,\nu}(f)\phi(x).$$

So $C_r^*(G)$ acts as compact operators on $\operatorname{Ind}_S^G H_{\sigma,\nu}$. From now on, when we say $f \in C_r^*(G)$, we really mean an element $\lambda(f)$ with $f \in C^*(G)$. Letting $f \in C_r^*(G)$, $\pi_{S,\sigma,\nu}(f)$ will then make sense.

**Theorem 4.5.2.2** (See for example [CCH16, Corollary 4.13]). *Let G be a real reductive group, and let $\sigma \in \widehat{M}$. There is a $C^*$-algebra morphism*

$$\pi_{S,\sigma} : C_r^*(G) \to C_0(\mathfrak{a}^*, \mathcal{K}(\operatorname{Ind}_S^G H_\sigma)) \tag{4.5.2.1}$$

*such that $\pi_{S,\sigma}(f)(\nu) = \pi_{S,\sigma,\nu}(f)$ for all $\nu \in \mathfrak{a}^*$.*

Let $S_1 = M_1A_1N_1$ and $S_2 = M_2A_2N_2$ be parabolic subgroups and $\sigma_1 \in \widehat{M}_1$ and $\sigma_2 \in \widehat{M}_2$. Then $(S_1, \sigma_1)$ and $(S_2, \sigma_2)$ are *associate* if there is an element of G that conjugates $M_1A_1$ to $M_2A_2$ and conjugates $\sigma_1$ to a representation unitarily equivalent to $\sigma_2$. See [CCH16, Definition 5.2]. Recall that $W_S = N_K(\mathfrak{a})/M$ is the Weyl group. We can define a finite group $W_{S,\sigma} = \{w \in N_K(\mathfrak{a}) | \operatorname{Ad}_w^* \sigma \cong \sigma\}/M$. For each $w \in W_{S,\sigma}$, we define $U_{w,\sigma,\nu} : \operatorname{Ind}_P^G H_{\sigma,\nu} \to \operatorname{Ind}_P^G H_{w,\operatorname{Ad}_w^* \nu}$ to be the unitary operator that intertwines $\pi_{\sigma,\nu}$ and $\pi_{\sigma,\operatorname{Ad}_w^* \nu}$. To see that such an operator exists, see [KS80, Section I.6 and I.8] and [Kna86, Chapter VII and Section XIV.6]. In section 3.4, we considered unitary intertwining operator for principal series, that is, the induced representation from a minimal parabolic subgroup. Then we define

$$C_0(\mathfrak{a}^*, \mathcal{K}(\operatorname{Ind}_S^G H_\sigma))^{W_\sigma}$$

to be the $C^*$-algebra of all functions $F : \hat{A} \to \mathcal{K}(\operatorname{Ind}_S^G H_\sigma)$ such that

$$U_{w,\sigma,\nu}F(\nu) = F(\operatorname{Ad}_w^* \nu)U_{w,\sigma,\nu},$$

for all $w \in W_\sigma$. The main theorem about the reduced $C^*$-algebra of a reductive group is:

**Theorem 4.5.2.3.** *Let $G$ be a real reductive group. Then there is a $C^*$-algebra isomorphism*

$$C_r^*(G) \cong \bigoplus_{[(S,\sigma)]} C_0(\mathfrak{a}^*, \mathcal{K}(\operatorname{Ind}_S^G H_\sigma))^{W_\sigma},$$

*where the direct sum is taken over the associate classes of a pair $(S, \sigma)$ where $S = M_S A_S N_S$ is a cuspidal parabolic subgroup and $\sigma \in \hat{M}_S$.*

We shall call this theorem the *Fourier structure theorem*. There is an extensive sketch of the proof in the appendix.

If $G$ is a complex reductive group $G$, there is only one cuspidal parabolic subgroup to consider for our associate classes: the minimal parabolic subgroup $Q = MAN$. Then the mapping given in (4.5.2.1) is the principal series representation. By Proposition 3.4.0.3, given two nonequivalent unitary representation $\sigma, \sigma' \in \hat{M}$, $\pi_\sigma$ and $\pi_{\sigma'}$ are unitarily equivalent if and only if there is a $w \in W$ (the only Weyl group we need to consider of the complex case) such that $\sigma' \cong w\sigma$. The collection of associate classes over which we take the direct sum in Theorem 4.5.2.3 becomes $\hat{M}/W$. The representation space $\operatorname{Ind}_S^G H_\sigma$ is just $L^2(K)^\sigma$. So we get

$$C_r^*(G) \cong \bigoplus_{\sigma \in \hat{M}/W} C_0(\mathfrak{a}^*, \mathcal{K}(L^2(K)^\sigma))^{W_\sigma}. \tag{4.5.2.2}$$

For $G = SL(2, \mathbb{R})$, there are two non-conjugate cuspidal parabolic subgroups: the minimal parabolic subgroup (the group of upper triangular matrices) and $SL(2, \mathbb{R})$ itself. In the case of the minimal parabolic subgroup, the morphisms

(4.5.2.1) are the principal series representations $\pi_{+,\nu}$ and $\pi_{-,\nu}$. The Weyl group is just $W = \{I, w\} \cong \mathbb{Z}/2\mathbb{Z}$ where $w = \begin{bmatrix} & 1 \\ -1 & \end{bmatrix}$. Recall that $\pi_{+,\nu} \cong \pi_{+,w\nu}$ and $\pi_{-,\nu} \cong \pi_{-,w\nu}$, in fact, $w\nu = -\nu$. On the other hand, considering $SL(2,\mathbb{R})$ as a cuspidal parabolic subgroup, $\mathfrak{a}$ is trivial, and so we get the representations of the discrete series, $\pi_n$ where $n \in \mathbb{Z} - \{-1, 0, 1\}$. There are no equivalences among these representations. So we have

$$C_r^*(G) \cong \bigoplus_{\substack{n \in \mathbb{N} \\ n \notin \{-1,0,1\}}} \mathcal{K}(H_n) \oplus C_0(\mathfrak{a}^*, \mathcal{K}(L^2(K)^+))^{\mathbb{Z}/2\mathbb{Z}} \oplus C_0(\mathfrak{a}^*, \mathcal{K}(L^2(K)^-))^{\mathbb{Z}/2\mathbb{Z}} \quad (4.5.2.3)$$

where $H_n$ is the representation space of representation of the discrete series $\pi_n$ and $L^2(K)^{\pm}$ is the (compact picture) representation space of the principal series representation $\sigma = \pm$ respectively.

### 4.5.3 Positive Weyl Chamber

We will make a modification to the isomorphism in (4.5.2.2) in preparation for our work on the main results of this dissertation in Chapter 5. Let G be a complex reductive group with the Iwasawa decomposition KAN and the parabolic subgroup $Q = MAN$. Let $\sigma \in \hat{M}$. There is a linear functional $\lambda_\sigma : \mathfrak{m} \to i\mathbb{R}$ such that

$$\sigma(\exp(X)) = e^{\lambda_\sigma(X)}.$$

We can extend $\lambda_\sigma$ to a linear functional $\lambda_\sigma : \mathfrak{k} \to i\mathbb{R}$ by writing $\mathfrak{k} = \mathfrak{m} \oplus \mathfrak{m}^\perp$ using any K-invariant inner product, and defining $\lambda_\sigma$ to be zero on $\mathfrak{m}^\perp$. Then we can identify $W_\sigma$ with the Weyl group of the compact isotropy group $K_\sigma := \{k \in K | \lambda_\sigma(\mathrm{Ad}_k(X)) = \lambda(X),\ \text{for all } X \in \mathfrak{k}\}$. In fact, $W_\sigma$ is a Weyl group. It is also true that $M \subset K_\lambda$. Now, $\mathfrak{m}$ is a Cartan subalgebra of $\mathfrak{k}_\lambda$, the Lie algebra of $K_\lambda$. Given a positive root system $\Delta_\sigma^+ := \Delta(\mathfrak{k}_\sigma : \mathfrak{m})$ (ordering and positivity have been chosen), we can define a *positive Weyl chamber* for $\sigma$ by

$$\mathfrak{m}_{\sigma,+} = \{X \in \mathfrak{m} | \alpha(X) > 0,\ \text{for all } \alpha \in \Delta_\sigma^+\}.$$

Then we have that $\mathfrak{m}_{\sigma,+}$ is a connected open component (in fact, an open convex cone) in $\mathfrak{m}$. All elements in $\mathfrak{m}$ are conjugate to an element in $\mathfrak{m}_{\sigma,+}$ by an element in $W_\sigma$ and for a given orbit $[X] = \{\mathrm{Ad}_w X | w \in W_\sigma\}$, there is one and only one $Y \in \mathfrak{m}_{\sigma,+}$ such that $Y \in [X]$ (see [Kir08, Chapter 7] or [Kna86, Sections IV.4 and IV.9]). Thus, $\mathfrak{m}_{\sigma,+}$ is a fundamental domain of $\mathfrak{m}/W_\sigma$.

Since G is a complex group, we have that $\mathfrak{a} = i\mathfrak{m}$. We can also define a positive Weyl chamber for $\sigma$ in $\mathfrak{a}$ by $\mathfrak{a}_{\sigma,+} := i\mathfrak{m}_{\sigma,+}$. Recall that there is an isomorphism between $\mathfrak{a}$ and $\mathfrak{a}^*$ given by the following: for every $\lambda \in \mathfrak{a}^*$, there is an $H_\lambda \in \mathfrak{a}$ such that

$$\lambda(X) = \langle H_\lambda, X \rangle \qquad \forall X \in \mathfrak{a}$$

where the inner product is K-invariant. We can define a *positive Weyl chamber* for $\sigma$

$$\mathfrak{a}^*_{\sigma,+} = \{\lambda \in \mathfrak{a}^* | H_\lambda \in \mathfrak{a}_{\sigma,+}\}.$$

Now $\mathfrak{a}^*_{\sigma,+}$ is an open convex cone of $\mathfrak{a}^*$ and it is a fundamental domain for $\mathfrak{a}^*/W_\sigma$. Now we can restrict the domain of the Fourier transform picture in Theorem 4.5.2.3 to get

**Theorem 4.5.3.1.** *Let* G *be a complex reductive group. The representations of* G *in the unitary principal series induce a* $C^*$*-algebra isomorphism*

$$\oplus_{\sigma \in \widehat{M}_+} \pi_\sigma \colon C^*_r(G) \xrightarrow{\cong} \bigoplus_{\sigma \in \widehat{M}_+} C_0\Big(\mathfrak{a}^*_{+,\sigma}, \mathcal{K}(L^2(K)^\sigma)\Big),$$

*where* $\widehat{M}_+ \subseteq \widehat{M}$ *is a positive Weyl chamber for the action of* $W$.

# Chapter 5
# Main Results for the Case of Complex Reductive Groups

In this chapter, we shall prove our main results, focusing on the complex case. First, we will show that we can construct an embedding

$$\alpha : C_r^*(G_0) \hookrightarrow C_r^*(G)$$

using the continuous fields of $C^*$-algebras that we have defined in the previous chapter. Then we will characterize this continuous field by showing that it is isomorphic to a simpler continuous field, where we no longer have to scale the Haar measure. Finally, with the new embedding map $\alpha$ we will give a new characterization to the Mackey Bijection. This is joint work with Nigel Higson reported in [HR19] and some passages below are taken verbatim from [HR19].

In the previous chapter, we have been considering the fibers of the deformation continuous field of reduced $C^*$-algebras as being distinct from one another. But in order to develop our result we will actually have need to compare them, that is we shall need to put them on the same "playing field." Recall that we chose a Haar measure for G. Then for $t \neq 0$, $C_r^*(G_t)$ has a norm and product that is dependent on the Haar measure $dg_t = |t|^{-\dim \mathfrak{p}} dg$ (it acts on $L^2(G, dg_t)$). First for $t \neq 0$ the left translation action of $G_t \cong G$ on itself integrates to a $C^*$-algebra representation

$$\lambda_t \colon C_r^*(G_t) \longrightarrow \mathfrak{B}(L^2(G, dg)),$$

defined by

$$\begin{aligned}\lambda_t(f_t)\phi &= \int_{G_t} f_t(g)\lambda(g)\phi \ dg_t \\ &= \int_G g_t(g)\lambda(g)\phi \ |t|^{-\dim(\mathfrak{p})} \ dg \\ &= |t|^{-\dim(\mathfrak{p})}\lambda(f_t)\phi\end{aligned}$$

where $f_t \in C_r^*(G_t)$ and $\phi \in L^2(K, dg)$. The image of $\lambda_t$ is independent of $t$, and is the image of the regular representation of $C_r^*(G)$ itself. So we obtain a family of $*$-isomorphisms

$$\lambda_t \colon C_r^*(G_t) \xrightarrow{\cong} C_r^*(G) \tag{5.0.0.1}$$

above, where here we view $C_r^*(G)$ as the concrete $C^*$-algebra of operators on $L^2(G, dg)$ generated by the left regular representation. These are given by the formulas

$$f_t \longmapsto \left[g \mapsto |t|^{-\dim(\mathfrak{p})} f_t(g)\right]$$

for $f_t \in C_c^\infty(G_t)$.

## 5.1 Scaling Automorphism

Let $G$ be a connected complex reductive group. In this section we shall construct a one-parameter group of automorphisms

$$\alpha_t \colon C_r^*(G) \longrightarrow C_r^*(G).$$

The automorphisms will be parametrized by the multiplicative group of positive real numbers, rather than the usual additive group of real numbers, and so the group law is $\alpha_{t_1} \circ \alpha_{t_2} = \alpha_{t_1 t_2}$.

### 5.1.1 Scaling Automorphism for Positive t

Using the Fourier transform picture for $C_r^*(G)$, (see 4.5.3.1), the construction is extremely simple. Let $\sigma \in \widehat{M}_+$. Define, for $t > 0$, an automorphism

$$\alpha_{\sigma,t} : C_0\Big(\mathfrak{a}^*_{+,\sigma}, \mathcal{K}(L^2(K)^\sigma)\Big) \longrightarrow C_0\Big(\mathfrak{a}^*_{+,\sigma}, \mathcal{K}(L^2(K)^\sigma)\Big)$$

by

$$\alpha_{\sigma,t}(f)(\nu) = f(t^{-1}\nu). \tag{5.1.1.1}$$

The individual one-parameter groups $\alpha_{\sigma,t}$ may be combined by direct sum into a one-parameter group of automorphisms

$$\oplus_\sigma \alpha_{\sigma,t} \colon \bigoplus_{\sigma\in\widehat{M}_+} C_0\Big(\mathfrak{a}^*_{+,\sigma}, \mathcal{K}(L^2(K)^\sigma)\Big) \longrightarrow \bigoplus_{\sigma\in\widehat{M}_+} C_0\Big(\mathfrak{a}^*_{+,\sigma}, \mathcal{K}(L^2(K)^\sigma)\Big),$$

and then we define automorphisms $\alpha_t$ of $C_r^*(G)$ by means of the commuting diagram

$$\begin{array}{ccc} C_r^*(G) & \xrightarrow{\quad \alpha_t \quad} & C_r^*(G) \\ {\scriptstyle \oplus_\sigma \pi_\sigma} \downarrow \cong & & \cong \downarrow {\scriptstyle \oplus_\sigma \pi_\sigma} \\ \bigoplus_{\sigma\in\widehat{M}_+} C_0\Big(\mathfrak{a}^*_{+,\sigma}, \mathcal{K}(L^2(K)^\sigma)\Big) & \xrightarrow[\oplus_\sigma \alpha_{\sigma,t}]{} & \bigoplus_{\sigma\in\widehat{M}_+} C_0\Big(\mathfrak{a}^*_{+,\sigma}, \mathcal{K}(L^2(K)^\sigma)\Big). \end{array} \tag{5.1.1.2}$$

It is clear that $\{\alpha_t\}$ is a one-parameter group of automorphisms. Indeed, let $t_1$ and $t_2$ be multiplicative real numbers. Then for each $\sigma \in \hat{M}_+$ and $F \in C_0(\mathfrak{a}^*_{+,\sigma}, \mathcal{K}(L^2(K)^\sigma)\Big)$ ,

$$\alpha_{\sigma,t_1} \circ \alpha_{\sigma,t_2}(F)(\nu) = \alpha_{\sigma,t_2}(F)(t_1^{-1}\nu) = F(t_2^{-1}t_1^{-1}\nu) = \alpha_{\sigma,t_1t_2}(F)(\nu),$$

and thus, it follows that

$$(\oplus\alpha_{\sigma,t_1}) \circ (\oplus\alpha_{\sigma,t_2}) = \oplus\alpha_{\sigma,t_1t_2}.$$

Finally, we can conclude

$$\alpha_{t_1} \circ \alpha_{t_2} = \Big[(\oplus\pi_\sigma)^{-1} \circ (\oplus\alpha_{\sigma,t_1}) \circ (\oplus\pi_\sigma)\Big] \circ \Big[(\oplus\pi_\sigma)^{-1} \circ (\oplus\alpha_{\sigma,2_1}) \circ (\oplus\pi_\sigma)\Big]$$

$$
\begin{aligned}
&= \Big[(\oplus\pi_\sigma)^{-1} \circ (\oplus\alpha_{\sigma,t_1}) \circ (\oplus\alpha_{\sigma,t_2}) \circ (\oplus\pi_\sigma)\Big] \\
&= \Big[(\oplus\pi_\sigma)^{-1} \circ (\oplus\alpha_{\sigma,t_1t_2}) \circ (\oplus\pi_\sigma)\Big] \\
&= \alpha_{t_1t_2}.
\end{aligned}
$$

### 5.1.2 Scaling Automorphisms for Negative t

As we shall soon see, the key property of the rescaling automorphism $\alpha_t$, which is immediate from its definition, is that if $\sigma\in\widehat{M}_+$ and if $\nu\in\mathfrak{a}^*_{\sigma,+}$, then

$$
\pi_{\sigma,\nu}(\alpha_t(f)) = \pi_{\sigma,t^{-1}\nu}(f) \tag{5.1.2.1}
$$

for all $f\in C^*_r(G)$. We shall want to extend this to negative $t$, and to this end we define automorphisms

$$
\alpha_t\colon C^*_r(G) \longrightarrow C^*_r(G)
$$

for $t<0$ as follows.

First we define $\mathfrak{a}^*_{-,\sigma}$ to be the negative of the Weyl chamber $\mathfrak{a}^*_{+,\sigma}$. This is simply another Weyl chamber for $W_\sigma$, and so all the constructions that we made in Section 4.5.3 using $\mathfrak{a}^*_{+,\sigma}$ can be repeated for $\mathfrak{a}^*_{-,\sigma}$. In particular there is an isomorphism of $C^*$-algebras

$$
\oplus_{\sigma\in\widehat{M}_+}\pi_\sigma\colon C^*_r(G) \xrightarrow{\cong} \bigoplus_{\sigma\in\widehat{M}_+} C_0\Big(\mathfrak{a}^*_{-,\sigma}, \mathcal{K}(L^2(K)^\sigma)\Big),
$$

We now define $\alpha_t\colon C^*_r(G) \to C^*_r(G)$ for $t<0$ by means of the commuting diagram

$$
\begin{array}{ccc}
C^*_r(G) & \xrightarrow{\quad\alpha_t\quad} & C^*_r(G) \\
{\scriptstyle \oplus_\sigma\pi_\sigma}\big\downarrow{\scriptstyle\cong} & & {\scriptstyle\cong}\big\downarrow{\scriptstyle \oplus_\sigma\pi_\sigma} \\
\bigoplus_{\sigma\in\widehat{M}_+} C_0\Big(\mathfrak{a}^*_{-,\sigma}, \mathcal{K}(L^2(K)^\sigma)\Big) & \xrightarrow[\oplus_\sigma\alpha_{\sigma,t}]{} & \bigoplus_{\sigma\in\widehat{M}_+} C_0\Big(\mathfrak{a}^*_{+,\sigma}, \mathcal{K}(L^2(K)^\sigma)\Big).
\end{array}
$$

where $\alpha_{\sigma,t}(h)(\nu) = h(t^{-1}\nu)$. The key property (5.1.2.1) now holds for all $t\neq0$, for all $\nu \in \mathfrak{a}^*_{+,\sigma}$ and all $f \in C^*_r(G)$.

## 5.2 Limit Formula

The main result of this section links the scaling automorphisms

$$\alpha_t \colon C^*_r(G) \to C^*_r(G)$$

from (5.1.1.2) with the regular representations

$$\lambda_t \colon C^*_r(G_t) \to C^*_r(G)$$

from (5.0.0.1) as follows:

**Theorem 5.2.0.1.** *If* $\{f_t\}$ *is any continuous section of the continuous field* $\{C^*_r(G_t)\}$*, then the limit*

$$\lim_{t\to 0} \alpha_t(\lambda_t(f_t))$$

*exists in* $C^*_r(G)$*.*

*Remark.* Of course we exclude the value $t=0$ in forming the limit.

We shall proving the theorem by carrying out an explicit computation with a suitable collection of continuous sections. To this end, recall that a collection $\mathcal{F}$ of continuous sections of the continuous field of $C^*$-algebras $\{C^*_r(G_t)\}$ is a *generating family* if for every continuous section $s$, every $\varepsilon > 0$ and every $t_0 \in \mathbb{R}$ there is some element $f \in \mathcal{F}$ and a neighborhood $U$ of $t_0 \in \mathbb{R}$ such that

$$t \in U \quad \Rightarrow \quad \|f_t - s_t\|_{C^*_r(G_t)} < \varepsilon.$$

See Proposition 4.4.0.1.

**Example** An obvious example to this is the set of all the continuous sections $\{f_t\}$ that are defined from $f \in C^\infty_c(\mathbb{N}_K G)$ with $f_t(g) = f(g,t)$, by the very definition. Another example is the set of continuous sections $\{f_t\}$ defined by $f \in C^\infty_0(\mathbb{N}_K G)$ with $f_t(g) = f(g,t)$, that is, the set of continuous sections $\{f_t\}$ in $\{C^*_r(G_t)\}$ that vanishes at infinity. In fact, this set forms a $C^*$-algebra, with the norm defined by $\|\{f_t\}\| = \sup_{t\in\mathbb{R}} \|f_t\|_{C^*_r(G_t)}$.

**Lemma 5.2.0.2.** If the limit in Theorem 5.2.0.1 exists for a generating family of continuous sections of $\{C^*_r(G_t)\}$, then it exists for all continuous sections of $\{C^*_r(G_t)\}$.

*Proof.* Let $\{f_t\}$ be a continuous section of $\{C^*_r(G_t)\}$ and let $\mathcal{F}$ be some set of generating functions for continuous sections of $\{C^*_r(G_t)\}$ such that the theorem is true. Let $\epsilon > 0$ be given. Then there is a continuous section $s \in \mathcal{F}$ and a $\delta_1 > 0$ such that

$$|t| < \delta_1 \quad \Rightarrow \quad \|f_t - s_t\| < \epsilon/2.$$

Recall that $C^*$-isomorphisms are isometric, so we have

$$|t| < \delta_1 \quad \Rightarrow \quad \|\alpha_t(\lambda_t(f_t)) - \alpha_t(\lambda_t(s_t))\| < \epsilon/2.$$

By the hypothesis, $F = \lim_{t\to 0} \alpha_t(\lambda_t(s_t))$ exists. So there is a $\delta_2 > 0$ such that

$$|t| < \delta_2 \quad \Rightarrow \quad \|F - \alpha_t(\lambda_t(s_t))\| < \epsilon/2.$$

Let $\delta > 0$ be the smaller of $\delta_1$ and $\delta_2$. Then

$$\begin{aligned}\|F - \alpha_t(\lambda_t(f_t))\| &\leq \|F - \lambda_t(\alpha_t(s_t))\| + \|\lambda_t(\alpha_t(s_t)) - \lambda_t(\alpha_t(f_t))\| \\ &< \epsilon/2 + \epsilon/2 = \epsilon,\end{aligned}$$

proving the lemma. □

### 5.2.1 Generating Family of Continuous $K$-Finite Sections

**Definition.** If $K$ acts continuously on a complex vector space $X$, then a vector $x \in X$ is said to be $K$-*finite* if the linear span of the orbit of $x$ under the action of $K$ is finite-dimensional and the action on this finite-dimensional space is continuous, or equivalently if $x$ lies in the image under the natural map

$$\bigoplus_{\tau\in\widehat{K}} V_\tau \otimes_{\mathbb{C}} \mathrm{Hom}_K(V_\tau, X) \longrightarrow X \tag{5.2.1.1}$$

of the span of finitely many summands $V_\tau \otimes_{\mathbb{C}} \mathrm{Hom}_K(V_\tau, X)$ (here $V_\tau$ is the representation space for a representative of $\tau \in \widehat{K}$).

**Definition.** We shall call the minimal set of $\tau \in \widehat{K}$ contributing to the image of $x$ in 5.2.1.1 the $K$-*isotypical support* of $x \in X$. This natural mapping is defined by $v \otimes \phi \mapsto \phi(v)$, where $v \in V_\tau$ and $\phi \in \mathrm{Hom}_K(V_\tau, X)$, and the $K$ action is given by $k(v \otimes \phi) = (\tau(k)) \otimes \phi$ where $k \in K$.

By the Peter-Weyl theorem, we get

$$C_r^*(K) \cong \bigoplus_{\tau \in \hat{K}} \mathrm{End}(V_\tau). \tag{5.2.1.2}$$

Indeed, by Peter-Weyl theorem we know that

$$L^2(K) \cong \bigoplus_{\tau \in \hat{K}} V_\tau \otimes V_\tau^*$$

and given $f \in C^\infty(K)$, $\lambda(f)$ acts on $L^2(K)$ and we have

$$\lambda(f)(v \otimes \phi)(k) = (\tau(f)v) \otimes \phi(k).$$

Moreover, given $\epsilon > 0$, by (5.2.1.2), for a given $f \in C_r^*(K)$, we can find a $u \in C_r^*(K)$, such that $\tau(u)$ is nonzero for only finitely many $\tau$, and $\|f - u\| < \epsilon$.

**Definition.** We will say that $f_t \in C_r^*(G_t)$ is *right* $K$-*finite* if there is a $u \in C_r^*(K)$ with $\tau(u)$ nonzero for only finitely many $\tau \in \hat{K}$ and a $f_t' \in C_r^*(G_t)$ such that $f_t = f_t' * u$. Similarly we can define the same thing for *left* $K$-*finite* if we can find $f_t' \in C_r^*(G_t)$ and $u \in C_r^*(K)$ with $\tau(u) \neq 0$ for only finitely many $\tau \in \hat{K}$ such that $f_t' = u * f_t$.

**Lemma 5.2.1.1.** Let $f \in C_c^\infty(\mathbb{N}_K G)$. Given $\epsilon > 0$, there is an $h \in C^*(K)$ such that

$$\sup_{t \neq 0} \|h * f_t - f_t\|_{C_r^*(G_t)} < \epsilon$$
$$\sup_{t \neq 0} \|f_t * h - f_t\|_{C_r^*(G_t)} < \epsilon.$$

We need the following lemma:

**Lemma 5.2.1.2.** Let $f \in C_c^\infty(\mathbb{N}_K G)$, and $f_t(g) = f(g,t)$. There is a positive number $Q < \infty$ such that, for every $t \in \mathbb{R}$,

$$\mathrm{vol}_t(\mathrm{supp}(f_t)) \leq Q$$

where $\mathrm{vol}_t(H)$ is the volume of a subset $H_t$ of $G_t$, defined by the Haar measure $dg_t$.

□

*proof to Lemma 5.2.1.1.* Let $\epsilon > 0$ be given. Since $f$ is uniformly continuous, there is a neighborhood $U$ in $K$ of the identity $e \in K$ such that

$$|f(g,t) - f(kg,t)| < \epsilon$$
$$|f(g,t) - f(gk,t)| < \epsilon$$

for all $(g,t) \in \mathbb{N}_K G$, $t \neq 0$, and $k \in U$. There is a function $h : K \to \mathbb{R}$ with the properties

1. $\int_K h(k)\ dk = 1$,
2. the closure of the support of $h$ is contained in $U$,
3. and $h \geq 0$,

such that $|h * f(k) - f(k)| < \epsilon$ for any $f : K \to \mathbb{C}$. [1]

we note that

$$\begin{aligned}(h * f)(g,t) &= \int_K h(k) f(k^{-1}g,t)\ dk \\ &= \int_K h(k) f(k^{-1}g,t)\ dk - \int_K h(k) f(g,t)\ dk + \int_K h(k) f(g,t)\ dk\end{aligned}$$

which implies that

$$(h * f)(g,t) - f(g,t) = \int_K h(k)[f(k^{-1}g,t) - f(g,t)]\ dk$$

[1] The *Fejer kernel* is a familiar example of a function that essentially satisfies the properties we need. See [Rud76].

which then gives us

$$|(h * f)(g,t) - f(g,t)| \leq \int_K h(k)|f(k^{-1}g,t) - f(g,t)| \, dk.$$

Let

$$M = \sup_{\substack{(g,t)\in \mathbb{N}_K G \\ k\in \mathrm{supp}(h) \\ t\neq 0}} |f(k^{-1}g,t) - f(g,t)|,$$

then

$$|(h * f)(g,t) - f(g,t)| \leq \int_K h(k) M \, dk = M.$$

But note that $M < \epsilon$ by uniform continuity, so

$$|(h * f)(g,t) - f(g,t)| < \epsilon.$$

By Lemma 5.2.1.2, we have

$$\begin{aligned} \|(h * f_t)_t - f_t\|_{C_r^*(G_t)} &\leq \|(h * f_t) - f_t\|_{L^1(G_t)} \\ &= \int_{G_t} |h * f_t(g) - f_t(g)| \, d_t g \\ &< \epsilon \cdot \mathrm{vol}_t(\mathrm{supp}(f_t)) \\ &\leq \epsilon \cdot Q \end{aligned}$$

for any $t \in \mathbb{R} - \{0\}$. □

**Lemma 5.2.1.3.** There exists a generating family of continuous sections for the continuous field $\{C_r^*(G_t)\}$ consisting of smooth and compactly supported functions on $\mathbb{N}_G K$ that are K-finite for both the left and right translation actions of K on $\mathbb{N}_G K$.

*Proof.* Given a smooth and compactly supported complex function f on $\mathbb{N}_G K$ and $\varepsilon > 0$, by Lemma 5.2.1.1, we can find smooth functions $\phi$ and $\psi$ on K such that

$$\|f - \phi * f * \psi\| < \varepsilon,$$

where the norm is that of the $C^*$-algebra of continuous sections of $\{C_r^*(G_t)\}$ that vanish at infinity. Recall that we can find a $u, w \in C^*(K)$ such that $\tau(u)$ and $\tau(w)$ are nonzero for only finitely many $\tau \in \hat{K}$ that approximate $\phi$ and $\psi$. So we can have

$$\|f - u * f * w\| < \varepsilon.$$

The function $u * f * w$ on $\mathbb{N}_G K$ is smooth, compactly supported and left and right K-finite. The collection of all elements of this form, for all f and all $\varepsilon > 0$, is a generating family, as required.

□

**Lemma 5.2.1.4.** A representation $\tau \in \widehat{K}$ is included in the K-isotypical decomposition of $L^2(K)^\sigma$ if and only if $\sigma$ is a weight of $\tau$, and therefore $\tau$ is included in only finitely many of the spaces $L^2(K)^\sigma$, as $\sigma$ ranges over $\widehat{M}$.

*Proof.* By the Peter-Weyl theorem, the Hilbert space $L^2(K)^\sigma$ has a K-isotypical decomposition

$$L^2(K)^\sigma \cong \bigoplus_{\tau \in \widehat{K}} V_\tau \otimes (V_\tau^*)^\sigma$$

where an element $\phi \in (V_\tau^*)^\sigma$ has the property

$$m^{-1}\phi(v) = \phi(\tau(m)v) = \sigma(m)\phi(v)$$

for $m \in M$ and $v \in V_\tau$. We will show that $(V_\tau^*)^\sigma \neq \{0\}$ if and only if $\sigma$ is a weight of $\tau$. Note that

$$V_\tau \cong \mathbb{C}_{\sigma_1} \oplus \cdots \oplus \mathbb{C}_{\sigma_n},$$

where $\sigma_1, \ldots, \sigma_n$ are all the weights of $\tau$ and $\mathbb{C}_\sigma$ is the weight space of $\sigma$. Furthermore,

$$V_\tau^* \cong \mathbb{C}_{\overline{\sigma}_1} \oplus \cdots \oplus \mathbb{C}_{\overline{\sigma}_n}.$$

If $\sigma$ is a weight of $\tau$, then $\mathbb{C}_{\overline{\sigma}}$ is a constituent of $V_\tau^*$ and there is a nonzero $\phi \in \mathbb{C}_{\overline{\sigma}}$ such that $m\phi = \overline{\sigma}(m)\phi$ for all $m \in M$, and thus $\phi \in (V_\tau^*)^\sigma$.

If $(V_\tau^*)^\sigma \neq \{0\}$, then there is a nonzero $\phi \in V_\tau^*$ such that $m\phi = \overline{\sigma}(m)\phi$ for all $m \in M$. Let

$$\phi = \beta_1\phi_1 + \cdots + \beta_n\phi_n,$$

where $\phi_j$ is a basis element of $\mathbb{C}_{\overline{\sigma}_j}$. Then

$$\begin{aligned} m\phi &= \beta_1 m\phi_1 + \cdots + \beta_n \phi_n \\ &= \beta_1 \overline{\sigma}_1(m)\phi_1 + \cdots + \beta_n \overline{\sigma}_n(m)\phi_n \end{aligned}$$

and

$$\overline{\sigma}(m)\phi = \beta_1 \overline{\sigma}(m)\phi_1 + \cdots + \beta_n \overline{\sigma}(m)\phi_n.$$

So we get $\beta_j \overline{\sigma}_j(m) = \beta_j \overline{\sigma}(m)$ for all $j$ and all $m \in M$. Not all $\beta_j$ are zero, so we have $\overline{\sigma}_j(m) = \overline{\sigma}(m)$ for some $j$ and for all $m \in M$. Therefore $\sigma$ is a weight of $\tau$.

□

**Lemma 5.2.1.5.** Let $\{f_t\}$ be a right $K$-finite continuous section of $\{C^*_r(G_t)\}$. If for every $\sigma \in \widehat{M}_+$ the limit

$$\lim_{t\to 0} \pi_\sigma(\alpha_t(\lambda_t(f_t)))$$

exists in $C_0(\mathfrak{a}^*_{+,\sigma}, \mathfrak{K}(L^2(K)^\sigma))$, then the limit

$$\lim_{t\to 0} \alpha_t(\lambda_t(f_t))$$

exists in $C^*_r(G)$.

*Proof.* By definition, there is an $f_t \in C^*(G_t)$ and a $u \in C^*_r(K)$ with $\tau(u)$ nonzero for only finitely many $\tau \in \hat{K}$ (say $\tau_1, \ldots, \tau_p$) such that $f_t = f'_t * u$. Note that for $\nu \otimes \phi \in V_\tau \otimes V^*_\tau$,

$$\lambda(u)(\nu \otimes \phi) = (\tau(u)\nu) \otimes \phi.$$

$\lambda(u)$ is nonzero only on those $L^2(K)^\sigma$ that includes at least one $\tau_1, \ldots, \tau_p$. By Lemma 5.2.1.4, $\lambda(u)$ is nonzero for only finitely many $L^2(K)^\sigma$.

Therefore, if $f$ is right $K$-finite, then the element

$$\pi_\sigma(\lambda_t(f_t)) \in C_0(\mathfrak{a}^*_{+,\sigma}, \mathcal{K}(L^2(K)^\sigma)$$

is nonzero for only a finite set of $\sigma \in \widehat{M}$ that is independent of $t$. Therefore under the hypotheses of the lemma the limit

$$\lim_{t\to 0} \bigoplus_{\sigma} \pi_\sigma(\alpha_t(\lambda_t(f_t))) = \bigoplus_{\sigma} \lim_{t\to 0} \pi_\sigma(\alpha_t(\lambda_t(f_t)))$$

exists: we can commute the limit and the direct sum because only finitely many summands are nonzero. The lemma follows from the fact that $\oplus_\sigma \pi_\sigma$ is isometric.

□

### 5.2.2 Limit Formula for a Matrix Coefficient

Recall from Section 4.4.2 that $\lambda(f_t * \phi) = \lambda(f_t)\lambda(\phi)$ and $\lambda(\phi * f_t) = \lambda(\phi)\lambda(f_t)$ for $f_t \in C^*_r(G_t)$ and $\phi \in C^*_r(K)$. If we were to restrict the action above to $L^2(K)^\sigma$, then we would get

$$\pi_{\sigma,\nu}(f * \phi) = \pi_{\sigma,\nu}(f)\pi_\sigma(\phi) \quad \text{and} \quad \pi_{\sigma,\nu}(\phi * f) = \pi_\sigma(\phi)\pi_{\sigma,\nu}(f),$$

where $\pi_\sigma(\phi)$ is an element of $C^*_r(K)$, whose action is restricted to $L^2(K)^\sigma$. That is, given $h \in L^2(K)^\sigma \subset L^2(K)$, we have $\pi_\sigma(\phi)h = \lambda(\phi)h$.

**Lemma 5.2.2.1.** Let $\{f_t\}$ be a left and right K-finite continuous section of $\{C^*_r(G_t)\}$ and let $\sigma \in \widehat{M}$. If the limit

$$\lim_{t\to 0} \langle \phi, \pi_{\sigma,\nu}(\alpha_t(\lambda_t(f_t)))\psi \rangle$$

exists for every $\phi, \psi \in L^2(K)^\sigma$, uniformly in $\nu \in \mathfrak{a}^*$, then the limit

$$\lim_{t\to 0} \pi_\sigma(\alpha_t(\lambda_t(f_t)))$$

exists in $C_0(\mathfrak{a}^*_{+,\sigma}, \mathcal{K}(L^2(K)^\sigma))$.

*Proof.* Let $S \subseteq \widehat{K}$ be the union of the K-isotypical supports of $\{f_t\}$ for the left and right translation actions, and let

$$L^2(K)^\sigma_S = \operatorname{Image}\Big( \bigoplus_{\tau\in S} V_\tau \otimes \operatorname{Hom}_K(V_\tau, L^2(K)^\sigma) \longrightarrow L^2(K)^\sigma \Big).$$

This is a finite-dimensional subspace of $L^2(K)^\sigma$, and the operators $\pi_{\sigma,\nu}(f_t)$ vanish on its orthogonal complement for all $\nu$ and all t. If for any given $\nu \in \mathfrak{a}^*$ the limits

$$\lim_{t\to 0}\langle \phi, \pi_{\sigma,\nu}(\alpha_t(\lambda_t(f_t)))\psi\rangle \tag{5.2.2.1}$$

exist for all $\phi, \psi \in L^2(K)^\sigma_S$, then the limit

$$\lim_{t\to 0} \pi_{\sigma,\nu}(\alpha_t(\lambda_t(f_t)))$$

exists in $\mathfrak{K}(L^2(K)^\sigma)$. If the limits (5.2.2.1) exist uniformly in $\nu$ as $\phi$ and $\psi$ range over an orthonormal basis for $\phi, \psi \in L^2(K)^\sigma_S$, then the limit

$$\lim_{t\to 0} \pi_{\sigma}(\alpha_t(\lambda_t(f_t)))$$

exists in $C_0(\mathfrak{a}^*_{+,\sigma}, \mathfrak{K}(L^2(K)^\sigma))$, as required. □

**Lemma 5.2.2.2.** Let $\sigma\in\widehat{M}_+$, let $\nu\in\mathfrak{a}^*_{+,\sigma}$ and let $t\neq 0$. If $f_t\in C^*_r(G_t)$ is represented by a smooth and compactly supported function on $G_t$, and if $\phi, \psi\in C^\infty(K)^\sigma$, then

$$\begin{aligned}\Big\langle \phi, \pi_{\sigma,\nu}(\alpha_t(\lambda_t(f_t)))\psi\Big\rangle \\ = \int_{\mathfrak{a}}\int_{\mathfrak{n}} \Big(\phi^* {*} f_t {*} \psi\Big)\Big(\exp(-tY)\exp(-tX)\Big)e^{i\nu(X)}e^{t\rho(X)}\, dX\, dY,\end{aligned}$$

where $\phi^*(k) = \overline{\phi(k^{-1})}$.

*Proof.* It follows from the definition of the scaling automorphisms $\alpha_t$ and the morphism $\pi_\sigma$ that

$$\Big\langle \phi, \pi_{\sigma,\nu}(\alpha_t(\lambda_t(f_t)))\psi\Big\rangle = \int_G f_t(g)\Big\langle \phi, \pi_{\sigma,t^{-1}\nu}(g)\psi\Big\rangle\, |t|^{-d} dg. \tag{5.2.2.2}$$

If we insert into this formula the definition of the $L^2$-inner product, then we obtain

$$\begin{aligned}\int_G f_t(g)\langle \phi, \pi_{\sigma,t^{-1}\nu}(g)\psi\rangle\, |t|^{-d}dg \\ = \int_G f_t(g)\Big(\int_K \phi^*(k^{-1})\Big(\pi_{\sigma,t^{-1}\nu}(g)\psi\Big)(k)\, dk\Big)\, |t|^{-d}dg,\end{aligned}$$

and rearranging, and making the substitution $g := k\gamma^{-1}$ we get

$$\int_G f_t(g)\langle \phi, \pi_{\sigma,t^{-1}\nu}(g)\psi\rangle\ |t|^{-d}dg \\ = \int_K \int_G \phi^*(k^{-1}) f_t(k\gamma^{-1})\Big(\pi_{\sigma,t^{-1}\nu}(k\gamma^{-1})\psi\Big)(k)\ dk\ |t|^{-d}d\gamma. \quad (5.2.2.3)$$

Now, according to the definition of the principal series representations (see Subsection 3.3.2,

$$\Big(\pi_{\sigma,t^{-1}\nu}(k\gamma^{-1})\psi\Big)(k) = e^{-(\rho+it^{-1}\nu)H(\gamma)}\psi(\kappa(\gamma)).$$

Inserting this into the right-hand side of (5.2.2.3) we obtain

$$\int_K \int_G \phi^*(k^{-1}) f_t(k\gamma^{-1}) e^{(-\rho+it^{-1}\nu)H(\gamma)}\psi(\kappa(\gamma))\ |t|^{-d}d\gamma\ dk. \quad (5.2.2.4)$$

If we use the formula for the Haar measure on G given in Lemma 2.6.0.8, then we obtain from (5.2.2.4) the integral

$$\int_K \int_{\mathfrak{a}} \int_{\mathfrak{n}} \int_K \phi^*(k_1^{-1}) f_t\Big(k_1 \exp(-tY)\exp(-tX)k_2^{-1}\Big) \\ \times\ \phi(k_2) e^{(-\rho+it^{-1}\nu)(tX)} e^{2\rho(tX)}\ dk_1\ dX\ dY\ dk_2.$$

This is

$$\int_{\mathfrak{a}} \int_{\mathfrak{n}} \Big(\phi^* * f_t * \psi\Big)\Big(\exp(-tY)\exp(-tX)\Big) e^{i\nu(X)} e^{\rho(tX)}\ dX\ dY,$$

as required. ☐

**Lemma 5.2.2.3.** Let $\sigma \in \widehat{M}_+$ and let $\nu \in \mathfrak{a}^*_{+,\sigma}$. For any $\phi, \psi \in L^2(K)^\sigma$, we have

$$\lim_{t\to 0} \langle \phi, \pi_{\sigma,\nu}(\alpha_t(\lambda_t(f_t)))\psi\rangle = \int_{\mathfrak{a}} \int_{\mathfrak{n}} (\phi^* * f_0 * \psi)(e, X+Y) e^{i\nu(X)}\ dX\ dY.$$

The convergence is uniform in $\nu \in \mathfrak{a}^*_{+,\sigma}$.

*Proof.* This follows immediately from Lemma 5.2.2.2 above and (4.3.1.1). ☐

Now we can prove the main theorem of this section.

*Proof of Theorem 5.2.0.1.* According to Lemma 5.2.0.2, we only need verify that the limit in the statement of the theorem exists for a generating family of continuous sections, and we shall use Lemma 5.2.1.3 to work with the generating family of continuous sections $\{f_t\}$ associated to the smooth, compactly supported, left and right $K$-finite functions on $\mathbb{N}_G K$. Lemma 5.2.2.3 shows that for every $\sigma \in \widehat{M}_+$ and every $\nu \in \mathfrak{a}^*_{\sigma,+}$ the individual matrix coefficients of $\pi_{\sigma,\nu}(\alpha_t(\lambda_t(f)))$ converge to limits as $t \to 0$, uniformly in $\nu$. Lemmas 5.2.1.5 and 5.2.2.1 complete the proof. □

# 5.3 Embedding of the Reduced Group $C^*$-Algebras

## 5.3.1 The Embedding Morphism

Let $f \in C^*_r(G_0)$. Extend $f$ in any way to a continuous section $\{f_t\}$ of $\{C^*_r(G_t)\}$ and then form the limit

$$\alpha(f) = \lim_{t \to 0} \alpha_t(\lambda_t(f_t)) \tag{5.3.1.1}$$

in $C^*_r(G)$.

**Theorem 5.3.1.1.** *The formula* (5.3.1.1) *defines an embedding of* $C^*$*-algebras*

$$\alpha \colon C^*_r(G_0) \longrightarrow C^*_r(G).$$

*Proof.* Since both $\lambda_t$ and $\alpha_t$ are isometric,

$$\| \lim_{t \to 0} \alpha_t(\lambda_t(f_t)) \| = \lim_{t \to 0} \|f_t\| = \|f_0\|.$$

Moreover if $\{f'_t\}$ is a different extension of $f$ to a continuous section, then

$$\| \lim_{t \to 0} \alpha_t(\lambda_t(f_t)) - \lim_{t \to 0} \alpha_t(\lambda_t(f'_t)) \| = \| \lim_{t \to 0} \alpha_t(\lambda_t(f_t - f'_t)) \| = 0.$$

So the limit is independent of the extension, and it defines an isometric $*$-homomorphism, as required. □

### 5.3.2 Mapping Cone Fields

We begin with a very elementary construction:

**Definition.** Let $\beta : B \to A$ be an embedding of a $C^*$-algebra $B$ into a $C^*$-algebra $A$. The *mapping cone* continuous field of $C^*$-algebras over $\mathbb{R}$ associated to $\beta$ has fibers

$$\mathrm{Cone}(\beta)_t = \begin{cases} A & t \neq 0 \\ B & t = 0. \end{cases}$$

and the continuous sections are all those set-theoretic sections $\{f_t\}$ for which the function

$$t \mapsto \begin{cases} f_t & t \neq 0 \\ \beta(f_0) & t = 0 \end{cases}$$

from $\mathbb{R}$ to $A$ is norm-continuous.

We shall apply this construction to the embedding from Theorem 5.3.1.1. In particular, we have

$$\mathrm{Cone}(\alpha)_t = \begin{cases} C_r^*(G) & t \neq 0 \\ C_r^*(G_0) & t = 0. \end{cases}$$

The continuous sections $\{f_t\}$ are defined from norm-continuous functions $f : \mathbb{R} \to C_r^*(G)$ where $f(0) \in \alpha[C_r^*(G_0)]$, and thus $f_0 = \beta^{-1}(f(0))$.

**Theorem 5.3.2.1.** *The fiber isomorphisms*

$$\begin{cases} \alpha_t \circ \lambda_t \colon C_r^*(G_t) \longrightarrow C_r^*(G) & t \neq 0 \\ \mathrm{id} \colon C_r^*(G_0) \longrightarrow C_r^*(G_0) & t = 0 \end{cases}$$

*define an isomorphism of continuous fields from the deformation field $\{C_r^*(G_t)\}$ to the mapping cone field for the embedding*

$$\alpha \colon C_r^*(G_0) \longrightarrow C_r^*(G).$$

*Proof.* By Proposition 4.4.0.2 It suffices to show that for any continuous section $\{f_t\}$

of $\{C^*_r(G_t)\}$, the image section of the mapping cone field is continuous. But the image section is $\{\widehat{f}_t\}$, where

$$\widehat{f}_t = \begin{cases} \alpha_t(\lambda_t(f_t)) & t \neq 0 \\ f_0 & t = 0. \end{cases}$$

This is obviously a continuous section of the mapping cone field away from t=0, and continuity at t=0 is proved using Theorem 5.2.0.1 and the definition of $\alpha$. □

## 5.4 Characterization of the Mackey Bijection

The previous accounts of the Mackey bijection have all been organized around the concept of *minimal* $K$-*type* of an irreducible representation of $G$. In particular, in Chapter 3, we showed the Mackey bijection through the highest weight theorem for the complex case. Compare [AA19]. Here we shall give a different treatment that is organized around the embedding

$$\alpha\colon C^*_r(G_0) \longrightarrow C^*_r(G),$$

and hence around the family of rescaling automorphisms $\{\alpha_t\}$.

Recall that $\widehat{G}_0$ consists of equivalence classes of irreducible unitary representations of $G_0$. We can parametrize this as $(\widehat{M} \times \mathfrak{a}^*)/W$. We will denote the elements of $\widehat{G}_0$ by $\pi(\sigma, \nu)$, where $\sigma \in \widehat{M}$ and $\nu \in \mathfrak{a}^*$. Also recall that $\widehat{G}_r$ consists of equivalence classes of irreducible, tempered, unitary representations, that is, the unitary principal series representations $\pi_{\sigma,\nu}$. Denote the elements of $\widehat{G}_r$ by $\iota(\sigma, \nu)$.

### 5.4.1 The Principal Series as Representations of the Motion Group

**Lemma 5.4.1.1.** The composition of the principal series representation $\pi_{\sigma,\nu} = \operatorname{Ind}_Q^G \sigma \otimes e^{i\nu}$ of the connected complex reductive group $G$ with the morphism

$$\alpha\colon C^*_r(G_0) \longrightarrow C^*_r(G)$$

is the unitary representation $\operatorname{Ind}_{M\ltimes\mathfrak{p}}^{K\ltimes\mathfrak{p}} \sigma\otimes e^{i\nu}$ of the motion group $G_0$.

*Proof.* The Hilbert space of the representation $\pi = \operatorname{Ind}_{M\ltimes\mathfrak{p}}^{K\ltimes\mathfrak{p}} \sigma\otimes e^{i\nu}$ is the completion of the space of smooth functions $f\colon G_0 \to \mathbb{C}$ such that

$$f(g\cdot(m,X)) = \sigma(m)^{-1}e^{-i\nu(X)}f(g)$$

in the norm associated to the inner product

$$\langle f_1, f_2\rangle = \int_K \overline{f_1(k)}f_2(k)\ dk.$$

The action of $G_0$ is by left translation. The Hilbert space identifies with $L^2(K)^\sigma$ by restriction of functions to K, and in this realization the action of $G_0$ is

$$(\pi(k,X)\psi)(k_1) = e^{i\nu(k_1^{-1}k\cdot X)}\psi(k^{-1}k_1).$$

The matrix coefficient associated to $\phi,\psi\in L^2(K)$ and $f\in C_c^\infty(G_0)$ is therefore

$$\langle\phi,\pi(f)\psi\rangle = \int_K\int_K\int_{\mathfrak{g}/\mathfrak{k}} \overline{\phi(k_1)}f(k,X)e^{i\nu(k_1^{-1}k\cdot X)}\phi(k^{-1}k_1)\ dk\ dk_1\ dX.$$

Making the change of variables $k_2 := k_1^{-1}k$ we get

$$\langle\phi,\pi(f)\psi\rangle = \int_K\int_{\mathfrak{g}/\mathfrak{k}}\int_K \overline{\phi(k_1)}f(k_1k_2,X)e^{i\nu(k_2\cdot X)}\phi(k_2^{-1})\ dk_1\ dX\, dk_2,$$

and then the further change of variables $Z := k_2\cdot X$ gives

$$\langle\phi,\pi(f)\psi\rangle = \int_K\int_{\mathfrak{g}/\mathfrak{k}}\int_K \overline{\phi(k_1)}f(k_1k_2,k_2^{-1}\cdot Z)e^{i\nu(Z)}\phi(k_2^{-1})\ dk_1\ dZ\, dk_2.$$

Now let us insert into the integral above the formula

$$\begin{aligned}(\phi^* * f * \psi)(k,Z) &= \int_K\int_K \overline{\phi(k_1)}f(k_1\cdot(k,Z)\cdot k_2)\psi(k_2^{-1})\ dk_1\ dk_2\\ &= \int_K\int_K \overline{\phi(k_1)}f(k_1kk_2,k_2^{-1}\cdot Z)\ \psi(k_2^{-1})\ dk_1\ dk_2.\end{aligned}$$

We obtain

$$\langle \phi, \pi(f)\psi \rangle = \int_{\mathfrak{g}/\mathfrak{k}} (\phi^* * f * \psi)(e, Z) e^{i\nu(Z)}\, dZ.$$

But Lemma 5.2.2.3 shows that this is precisely $\langle \phi, \pi_{\sigma,\nu} \circ \alpha(f)\psi \rangle$, and the proof is complete. □

In the following lemma and in the next subsection we shall make use of the classifcation of irreducible representations of the compact connected group $K_\nu$ by highest weights. Rather than choose a dominant Weyl chamber for the action of the Weyl group $W_\nu$ on $\widehat{M}$, we shall associate to a given irreducible representation $\tau$ the $W_\nu$ orbit of all possible highest weights for all possible choices of dominant Weyl chamber. See Corollary 3.2.2.3. We shall use brackets, as in $[\theta]$ (with $\theta \in \hat{M}$), to denote this orbit.

**Lemma 5.4.1.2.** The composition of the principal series representation

$$\pi_{\sigma,\nu} = \operatorname{Ind}_Q^G \sigma \otimes e^{i\nu}$$

of the connected complex reductive group G with the morphism

$$\alpha\colon C^*_r(G_0) \longrightarrow C^*_r(G)$$

decomposes as a direct sum

$$\bigoplus_{[\theta] \in \widehat{M}/W_\nu} m(\sigma, \theta) \cdot \pi(\theta, \nu)$$

as a representation of $G_0$, where $m(\sigma, \theta)$ is the multiplicity with which the weight $\sigma$ occurs in the representation of $K_\nu$ with highest weight $\theta$.

*Proof.* We showed in the Lemma 5.4.1.1 that $\pi_{\sigma,\nu} \circ \alpha$ is the induced representation $\operatorname{Ind}_{M \ltimes \mathfrak{p}}^{K \ltimes \mathfrak{p}} \sigma \otimes e^{i\nu}$. Let us analyze this representation by induction in stages:

$$\operatorname{Ind}_{M \ltimes \mathfrak{p}}^{K \ltimes \mathfrak{p}} \sigma \otimes e^{i\nu} \cong \operatorname{Ind}_{K_\nu \ltimes \mathfrak{p}}^{K \ltimes \mathfrak{p}} \operatorname{Ind}_{M \ltimes \mathfrak{p}}^{K_\nu \ltimes \mathfrak{p}} \sigma \otimes e^{i\nu}$$

As in the proof of Lemma 5.4.1.1, we can realize $\operatorname{Ind}_{M \ltimes \mathfrak{p}}^{K_\nu \ltimes \mathfrak{p}} \sigma \otimes e^{i\nu}$ on the Hilbert space

$L^2(K_\nu)^\sigma$, and in this realization an element $X \in \mathfrak{p}$ acts as multiplication by the function

$$k \longmapsto e^{i\nu(k^{-1}\cdot X)} \qquad (\forall k \in K_\nu).$$

But if $k \in K_\nu$ then by definition, $\nu(k^{-1}\cdot X) = \nu(X)$. So the subgroup $\mathfrak{p}$ of $K_\nu \ltimes \mathfrak{p}$ acts on $L^2(K_\nu)^\sigma$ by the unitary character $\exp(i\nu)$. It follows that

$$\mathrm{Ind}_{M\ltimes\mathfrak{p}}^{K_\nu\ltimes\mathfrak{p}} \sigma\otimes e^{i\nu} = \bigoplus_{\tau\in\widehat{K}_\nu} m(\tau)\ \tau\otimes e^{i\nu},$$

where $m(\tau)$ is the multiplicity with which $\tau\in\widehat{K}_\nu$ occurs in $L^2(K_\nu)^\sigma$. By the Peter-Weyl theorem, $m(\tau)$ is the multiplicity with which the weight $\sigma$ occurs in $\tau$; compare the proof of Lemma 5.2.1.5. The lemma follows from this and the classification of irreducible representations of the connected group $K_\nu$ by their highest weights. □

### 5.4.2 Characterization of the Mackey Bijection

**Theorem 5.4.2.1.** *There is a unique bijection*

$$\mu\colon \widehat{G}_r \longrightarrow \widehat{G}_0$$

*such that for every $\iota \in \widehat{G}_r$, the element $\mu(\iota) \in \widehat{G}_0$ may be realized as a unitary subrepresentation of $\iota \circ \alpha$.*

*Proof.* The existence part of the theorem is handled by the Mackey bijection from [Hig08], which we reviewed in Chapter 3, which is the map

$$\mu\colon \iota(\sigma,\nu) \longmapsto \pi(\sigma,\nu).$$

Indeed by Lemma 5.4.1.2, the representation $\pi(\sigma,\nu)$ occurs within $\iota(\sigma,\nu)$ with multiplicity one.

As for uniqueness, suppose we are given any bijection $\mu$, as in the statement of the theorem. It follows from Lemma 5.4.1.2 that $\mu$ must have the form

$$\mu\colon \iota(\sigma,\nu) \longmapsto \pi(\theta,\nu)$$

for some $\theta \in \widehat{M}/W_\nu$ with $\theta \geq \sigma$. So for each fixed $\nu \in \mathfrak{a}^*$, we obtain from $\mu$ a bijection of sets

$$\mu_\nu\colon \widehat{M}\big/W_\nu \longrightarrow \widehat{M}\big/W_\nu$$

defined by

$$\mu\colon \iota(\sigma,\nu) \longmapsto \pi(\mu_\nu(\sigma),\nu).$$

We need to show that $A_\nu$ is the identity map for all $\nu$.

Now it follows from Lemma 5.4.1.2 that $\mu_\nu^{-1}$ has the property that

$$\mu_\nu\Big([\sigma]\Big) \geq [\sigma] \qquad \forall\, [\sigma] \in \widehat{M}/W_\nu,$$

and so of course the inverse bijection has the property that

$$\mu_\nu^{-1}\Big([\sigma]\Big) \leq [\sigma] \qquad \forall\, [\sigma] \in \widehat{M}/W_\nu.$$

for all $\sigma \in \widehat{M}/W_\nu$. It follows from this that $\mu_\nu^{-1}$ maps each of the finite sets

$$S_\sigma = \{\, [\theta] \in \widehat{M}/W_\nu \;:\; [\theta] \leq [\sigma] \,\}$$

into itself, and this in turn implies that $\mu_\nu^{-1}$ is the identity map, as required. $\square$

# Chapter 6 | Main Result for the Case of $SL(2, \mathbb{R})$

In this chapter we adapt the results given in Chapter 5 to the case of $G = SL(2, \mathbb{R})$. We will show that there is an embedding $C_r^*(G_0) \to C_r^*(G)$. To do this we shall construct a rescaling automorphism in the Fourier transform picture. Recall (4.5.2.3) that the Fourier transform picture is

$$C_r^*(G) \cong \left[ \bigoplus_{\substack{n \in \mathbb{N} \\ n \neq \{-1,0,1\}}} \mathcal{K}(H_n) \right] \oplus \left[ C_0(\mathfrak{a}^*, \mathcal{K}(L^2(K)^-))^{\mathbb{Z}/2\mathbb{Z}} \right] \oplus \left[ C_0(\mathfrak{a}^*, \mathcal{K}(L^2(K)^-))^{\mathbb{Z}/2\mathbb{Z}} \right]$$

where $H_n$ is the representation space of the discrete series and $L^2(K)^{\pm}$ are the compact picture representation spaces of the principal series. We will call the direct sum of the compact operators $\mathcal{K}(H_n)$ the *discrete series component* of $C_r^*(G)$ and we call the components with the continuous functions the *principal series components*. We will tackle each component separately and then bring them back together again.

## 6.1 Principal Series Components

In Subsection 4.5.3 we chose some positive Weyl chambers $\mathfrak{a}^*_{\sigma,+} \subset \mathfrak{a}^*$ so that in our description of $C_r^*(G)$ we no longer need to take into account the intertwining operators. We will do similar things here and in fact will be easier. Recall that $\mathfrak{a} \cong \mathbb{R}$, so that then $\mathfrak{a}^* \cong \mathbb{R}$. Also recall that among the principal series representations, the unitary equivalences are $\pi_{\pm,\nu} \cong \pi_{\pm,-\nu}$. These equivalences are given by unitary intertwining operators

$$U_{w,\pm,\nu} : L^2(K)^{\pm} \to L^2(K)^{\mp}$$

such that $U_{w,\pm,\nu}\pi_{\pm,\nu}(g) = \pi_{\pm,-\nu}(g)U_{w,\pm,\nu}$ for all $g \in G$. Then for a function

$$F \in C_0(\mathbb{R}, \mathcal{K}(L^2(K)^{\pm}))^{\mathbb{Z}/2\mathbb{Z}}$$

we have

$$U_{w,\pm,\nu}F(\nu)U^*_{w,\pm,\nu} = F(-\nu). \tag{6.1.0.1}$$

Let $\{e_n\}_{n\in\mathbb{Z}}$ be the standard orthonormal basis for $L^2(K)$, so that given $k_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, we have $e_n(k_\theta) = e^{in\theta}$. Then $L^2(K)$ decomposes orthogonally into $L^2(K)^+$ and $L^2(K)^-$ where $L^2(K)^+$ has the orthonormal basis $\{e_{2n}\}_{n\in\mathbb{Z}}$ and $L^2(K)^-$ has the orthonormal basis $\{e_{2n+1}\}_{n\in\mathbb{Z}}$. Define

$$t_n(\nu) = \frac{\Gamma(\frac{1}{2}(n+1+i\nu))}{\Gamma(\frac{1}{2}(n+1-i\nu))}, \qquad n \in \mathbb{Z},$$

where $\Gamma$ is the classical gamma function. Then $t_n(\nu)$ are holomorphic and zero free near $\mathbb{R} \subset \mathbb{C}$, in fact, $|t_n(\nu)| = 1$ for all $\nu \in \mathbb{R}$. See [Var89, lemma 21, page 243].

**Proposition 6.1.0.1** ( [Var89, Proposition 22, page 243]). *If $\nu \in \mathbb{R}$ and $\{e_n\}$ is an orthonormal basis, then there is a unitary operator $U_\nu : L^2(K)^{\pm} \to L^2(K)^{\pm}$ such that*

$$U_\nu e_n = t_n(\nu)e_n$$

*and*

$$U_\nu^{-1}\pi_{\pm,\nu}U_\nu = \pi_{\pm,-\nu}.$$

It is clear that the unitary operator $U_\nu$ decomposes as a direct sum $U_{w,+,\nu} \oplus U_{w,-,\nu}$. Furthermore,

$$t_n(\nu) = \begin{cases} 1 & n \geq 0 \\ (-1)^n & n < 0. \end{cases}$$

See [Var89, page 246]. We can then conclude that $U_{w,+,0} = \mathrm{Id}$ and $U_{w,-,0} = W$, where

$$W(e_{2n+1}) = \begin{cases} e_{2n+1} & n \geq 0 \\ -e_{2n+1} & n < 0. \end{cases}$$

By (6.1.0.1), for $F \in C_0(\mathbb{R}, \mathcal{K}(L^2(K)^+))^{\mathbb{Z}/2\mathbb{Z}}$, we get

$$U_{w,+,0}F(0)U_{w,+,0}^* = F(0),$$

so we can rewrite the even principal series part of the Fourier transform picture as

$$C_0(\mathbb{R}, \mathcal{K}(L^2(K)^+))^{\mathbb{Z}/2\mathbb{Z}} \cong C_0([0, \infty), \mathcal{K}(L^2(K)^+)).$$

On the other hand, take $D_+ = \mathrm{span}\{e_{2n+1} | n \geq 0\}$ and $D_- = \mathrm{span}\{e_{2n+1} | n < 0\}$, so that $L^2(K)^- = D_+ \oplus D_-$. For $F \in C_0(\mathbb{R}, \mathcal{K}(L^2(K)^-))^{\mathbb{Z}/2\mathbb{Z}}$ we have

$$U_{w,-,0}F(0)U_{w,-,0} = F(0),$$

which means that $F(0)$ must be block diagonal for $D_+ \oplus D_-$. We will then write the Fourier transform picture as follows

$$C_0(\mathbb{R}, \mathcal{K}(L^2(K)^-))^{\mathbb{Z}/2\mathbb{Z}} \cong C_0^-([0, \mathbb{R}), \mathcal{K}(L^2(K)^-))$$

so that if $F \in C_0^-([0, \infty), \mathcal{K}(L^2(K)^-))$ then $F(0)$ is block diagonal under the decomposition $L^2(K)^- = D_+ \oplus D_-$. Let us take the convention that

$$C_0^+([0, \infty), \mathcal{K}(L^2(K)^+)) = C_0([0, \infty), \mathcal{K}(L^2(K)^+)).$$

*Remark.* Almost all the principal series representations $\pi_{\pm,\nu}$ are irreducible except one. The only reducible principal series is $\pi_{-,0}$ which reduces into two *limits of discrete series* $\pi_1$ and $\pi_{-1}$, each of which are irreducible. See [Kna86, Proposition 2.7]. The representation spaces of the limits of discrete series are isomorphic to $D_+$ and $D_-$. The fact that $\pi_{-,0}$ is reducible is the reason why we need a special condition on restricting the domain to some positive Weyl chamber.

Now we may proceed to construct a scaling automorphism as in Chapter 5. For $t \neq 0$, we have standard isomorphisms

$$\lambda_t : C_r^*(G_t) \xrightarrow{\cong} C_r^*(G)$$

as in (5.0.0.1), defined by

$$f_t \mapsto \left[ g \mapsto |t|^{-d} f_t(g) \right]$$

where $f_t \in C_c^\infty(G_t)$. We define the scaling automorphisms for $t > 0$

$$\alpha_{\pm,t} : C_0^\pm([0,\infty), \mathcal{K}(L^2(K)^\pm)) \to C_0^\pm([0,\infty), \mathcal{K}(L^2(K)^\pm)),$$

by

$$\alpha_{\pm,t}(f)(\nu) = f(t^{-1}\nu).$$

For $t < 0$, we can define the scaling automorphism the same way as we did in Section 5.1. We had defined the $*$-morphism

$$\pi_\pm : C_r^*(G) \to C_0^\pm([0,\infty), \mathcal{K}(L^2(K)^\pm)),$$

but we can instead define it as

$$\pi_\pm : C_r^*(G) \to C_0^\pm((-\infty,0], \mathcal{K}(L^2(K)^\pm)).$$

In this case, for $t < 0$, we will define the scaling automorphism

$$\alpha_{\pm,t} : C_0^\pm((-\infty,0], \mathcal{K}(L^2(K)^\pm)) \to C_0^\pm([0,\infty), \mathcal{K}(L^2(K)^\pm))$$

by $\alpha_{\pm,t}(f)(\nu) = f(t^{-1}\nu)$.

**Proposition 6.1.0.2.** *Let $\{f_t\}$ be a continuous section of $\{C_r^*(G_t)\}$ represented by a smooth and compactly supported function $f \in C_c^\infty(\mathbb{N}_K G)$, and let $\nu \in [0,\infty)$. For any $\phi, \psi \in L^2(K)^\pm$, the limit of the matrix coefficients*

$$\lim_{t\to 0} \langle \phi, \alpha_\pm(\pi_{\pm,\nu}(\lambda_t(f_t)))\psi \rangle$$

*exists. The convergence is uniform in* $\nu \in [0, \infty)$.

*Proof.* From the definition of the scaling automorphisms $\alpha_\pm$ and the morphisms $\pi_\pm$, we get that

$$\langle \phi, \alpha_\pm(\pi_{\pm,\nu}(\lambda_t(f_t)))\psi = \int_G f_t(g)\langle \phi, \pi_{\pm,t^{-1}\nu}(g)\psi\rangle |t|^{-\dim(\mathfrak{p})}\, dg.$$

The right hand side integral is precisely (5.2.2.2). We can then follow the proof of Lemma 5.2.2.2 to obtain

$$\begin{aligned}&\langle \phi, \alpha_\pm(\pi_{\pm,\nu}(\lambda_t(f_t)))\psi\rangle \\ &\qquad = \int_{\mathfrak{a}}\int_{\mathfrak{n}} (\phi^* * f_t * \psi)(\exp(-tY)\exp(-tX))e^{i\nu(X)}e^{\rho(tX)}\, dX\, dY.\end{aligned}$$

The limit and uniform convergence follows from Lemma 5.2.2.3. □

**Proposition 6.1.0.3.** *Let* $\{f_t\}$ *be a left and right* K*-finite continuous section of* $\{C_r^*(G_t)\}$*. If the limit*

$$\lim_{t\to 0}\langle \phi, \alpha_{\pm,t}(\pi_{\pm,\nu}(\lambda_t(f_t)))\rangle$$

*exists for every* $\phi, \psi \in L^2(K)^\pm$*, uniformly in* $\nu \in [0, \infty)$*, then the limit*

$$\lim_{t\to 0} \alpha_{\pm,t}(\pi_\pm(\lambda_t(f_t)))$$

*exists in* $C_0^\pm([0, \infty), \mathcal{K}(L^2(K)^\pm))$.

*Proof.* The proof is precisely the same as the proof for Lemma 5.2.2.1. □

## 6.2 Discrete Series Component

In this section, we define scaling automorphisms for components involving the discrete series representation spaces. For $n \in \mathbb{Z} - \{-1, 0, 1\}$ and $t \neq 0$ (here, $t$ can be positive or negative), we define the scaling automorphism

$$\alpha_{n,t} : \mathcal{K}(H_n) \to \mathcal{K}(H_n)$$

simply by

$$\alpha_{n,t}(T) = T$$

for $T \in H_n$. That is, the rescaling automorphisms are trivial on the discrete series components. Here, we want to show that the limit

$$\lim_{t \to 0} \alpha_{n,t}(\pi_n(\lambda_t(f_t)))$$

exists. We begin with a lemma involving matrix coefficients.

**Lemma 6.2.0.1.** Let $\{f_t\}$ be a continuous section of $\{f_t\}$ represented by a smooth, compactly supported function $f \in C_c^\infty(\mathbb{N}_K G)$. Let $n \in \mathbb{Z} - \{-1, 0, 1\}$. For any $v, w \in H_n$, the limit of the matrix coefficient

$$\lim_{t \to 0} \langle v, \alpha_{n,t}(\pi_n(\lambda_t(f_t)))w \rangle$$

exists.

*Proof.* By the definition of $\alpha_{n,t}$ and $\lambda_t$, we get

$$\begin{aligned} \langle v, \alpha_{n,t}(\pi_n(\lambda_t(f_t)))w \rangle &= \langle v, \pi_n(\lambda_t(f_t))w \rangle \\ &= \int_G f_t(g) \langle v, \pi_n(g)w \rangle \; |t|^{-\dim(\mathfrak{p})} \; dg. \end{aligned}$$

Using the formula for the Haar measure given in lemma 2.6.0.8 we get

$$\begin{aligned} &\int_G f_t(g) \langle v, \pi_n(g)w \rangle \; |t|^{-\dim(\mathfrak{p})} \; dg \\ &= \int_K \int_{\mathfrak{a}} \int_{\mathfrak{n}} f_t(k \exp(X) \exp(Y)) \langle v, \pi_n(k \exp(X) \exp(Y))w \rangle \; |t|^{-\dim(\mathfrak{p})} e^{2\rho(X)} \; dk \; dX \; dY \\ &= \int_K \int_{\mathfrak{a}} \int_{\mathfrak{n}} f_t(k \exp(tX) \exp(tY)) \langle v, \pi_n(k \exp(tX) \exp(tY))w \rangle \; e^{2\rho(tX)} \; dk \; dX \; dY, \end{aligned} \tag{6.2.0.1}$$

where the last equality is given by change of variables $tX := X$. Take the limit of the right hand side of (6.2.0.1) as t goes to zero and we get

$$\int_K \int_{\mathfrak{a}} \int_{\mathfrak{n}} f_0(k, X+Y)\langle v, \pi_n(k)w\rangle \; dk \; dX \; dY. \tag{6.2.0.2}$$

Since $f_0$ is smooth and compactly supported in $G_0$, we get that the function

$$F_0(k) := \int_{\mathfrak{a}} \int_{\mathfrak{n}} f_0(k, X, Y) \; dX \; dY$$

on K is smooth, and in particular bounded. Thus the integral (6.2.0.2)

$$\int_K F_0(k)\langle v, \pi_n|_K(k)w\rangle \; dk$$

is bounded. Therefore, the limit exists. □

**Lemma 6.2.0.2.** Let $n \in \mathbb{Z}$ and let $\{f_t\}$ be a continuous section of $\{C_r^*(G_t)\}$ represented by a smooth, compactly supported function $\{f_t\}$. If the limit

$$\lim_{t \to 0}\langle v, \alpha_{n,t}(\pi_n(\lambda_t(f_t)))w\rangle$$

exists for every $v, w \in H_n$, then the limit

$$\lim_{t \to 0} \alpha_{n,t}(\pi_n(\lambda_t(f_t)))$$

exists in $\mathcal{K}(H_n)$. □

Thus we can conclude that the limit

$$\lim_{t \to 0} \alpha_{n,t}(\pi_n(\lambda_t(f_t)))$$

exists.

**Proposition 6.2.0.3.** *Let $\{f_t\}$ be a left and right K-finite continuous sections of $\{C_r^*(G_t)\}$. The limit*

$$\lim_{t \to 0} \bigoplus_{\substack{n \in \mathbb{Z} \\ n \notin \{-1,0,1\}}} \alpha_{n,t}(\pi_n(\lambda_t(f_t)))$$

*exists in* $\bigoplus_{n\in\mathbb{Z}-\{-1,0,1\}}\mathcal{K}(H_n)$ *and is given by*

$$\lim_{t\to 0}\bigoplus_{\substack{n\in\mathbb{Z}\\ n\notin\{-1,0,1\}}}\alpha_{n,t}(\pi_n(\lambda_t(f_t)))=\bigoplus_{\substack{n\in\mathbb{Z}\\ n\notin\{-1,0,1\}}}\lim_{t\to 0}\alpha_{n,t}(\pi_n(\lambda_t(f_t))).$$

*Proof.* For all $n\in\mathbb{Z}-\{-1,0,1\}$, the restriction of the discrete series representations decomposes as follows, if $n\geq 1$ then

$$\pi_n|K\cong\tau_n\oplus\tau_{n+2}\oplus\tau_{n+4}\oplus\cdots,$$

and if $n\leq -1$ then

$$\pi_n|K\cong\tau_n\oplus\tau_{n-2}\oplus\tau_{n-4}\oplus\cdots.$$

For example see [Lan85, Chapter 9]. It is easy to see that the K-type $\tau_m$ is a constituent of $\pi_n|_K$ for only finitely many $n$.

By the hypothesis, there are elements $u,u'\in C_r^*(K)$ and a continuous section $\{f_t'\}$ of $\{C_r^*(G_t)\}$ such that

$$f_t=u'*f_t'*u$$

and $\tau_m(u)$ and $\tau_m(u')$ are nonzero for only finitely many $\tau_m\in\hat{K}$. We can conclude that $\pi_n|_K(u)$ and $\pi_n|_K(u')$ are nonzero for only finitely many $n\in\mathbb{Z}-\{-1,0,1\}$. Therefore $\pi_n|_K(f_t)$ is nonzero for only finitely many $n\in\mathbb{Z}-\{-1,0,1\}$.

It is then easy to see that

$$\lim_{t\to 0}\bigoplus_{\substack{n\in\mathbb{Z}\\ n\notin\{-1,0,1\}}}\alpha_{n,t}(\pi_n(\lambda_t(f_t)))=\bigoplus_{\substack{n\in\mathbb{Z}\\ n\notin\{-1,0,1\}}}\lim_{t\to 0}\alpha_{n,t}(\pi_n(\lambda_t(f_t)))\in\bigoplus_{\substack{n\in\mathbb{Z}\\ n\notin\{-1,0,1\}}}\mathcal{K}(H_n)$$

is well defined. □

*Remark.* In general, discrete series representations $\pi$ are irreducible unitary representations. We can write

$$\pi|_K\cong\bigoplus_{\tau\in\hat{K}}m_\tau\tau,$$

where $m_\tau$ is the multiplicity of the K-type $\tau$ in $\pi|_K$. See [Kna86, equation (8.5)]. By the theorem given in [Kna86, Theorem 8.1], since $\pi$ is an irreducible unitary representation, the multiplicity $m_\tau$ satisfies $m_\tau \leq \dim \tau$ for all $\tau \in \hat{K}$. Furthermore, the corollary given in [Kna86, Corollary 12.22(a)] states that if G is linear connected semisimple with rank $G = \operatorname{rank} K$, then any given K-type, $\tau$, occurs (nontrivially) in only finitely many discrete series representations. This is a special case of the uniform admissablilty theorem. See [CCH16, Theorem 5.5].

## 6.3 The Limit Formula and the Embedding Morphism

Now we define the scaling automorphisms at the level of the reduced group $C^*$-algebra. For brevity, let

$$\pi = \left[ \bigoplus_{\substack{n \in \mathbb{Z} \\ n \notin \{-1,0,1\}}} \pi_n \right] \oplus \pi_+ \oplus \pi_-$$

be the $C^*$-isomorphism from $C^*_r(G)$ to the Fourier structure picture. If $t > 0$, we define

$$\alpha_t : C^*_r(G) \to C^*_r(G)$$

by the composition

$$\alpha_t \ = \ \pi^{-1} \ \circ \ \left( \left[ \bigoplus_{\substack{n \in \mathbb{Z} \\ n \notin \{-1,0,1\}}} \alpha_{n,t} \right] \oplus \alpha_{+,t} \oplus \alpha_{-,t} \right) \ \circ \ \pi. \tag{6.3.0.1}$$

In the case that $t < 0$, let $\pi'$ be the $C^*$-isomorphism

$$\pi' : C^*(G) \longrightarrow \left[ \bigoplus_{\substack{n \in \mathbb{Z} \\ n \notin \{-1,0,1\}}} \mathcal{K}(H_n) \right] \oplus C_0((-\infty, 0], \mathcal{K}(L^2(K)^+)) \oplus C_0^-((-\infty, 0], \mathcal{K}(L^2(K)^-)),$$

and define $\alpha_t$ by the composition

$$\alpha_t = \pi^{-1} \circ \left( \left[ \bigoplus_{\substack{n\in\mathbb{Z} \\ n\notin\{-1,0,1\}}} \alpha_{n,t} \right] \oplus \alpha_{+,t} \oplus \alpha_{-,t} \right) \circ \pi'.$$

**Theorem 6.3.0.1.** *If $\{f_t\}$ is any continuous section of the continuous field $\{C_r^*(G_t)\}$, then the limit*

$$\lim_{t\to 0} \alpha_t(\lambda_t(f_t))$$

*exists in $C_r^*(G)$.*

*Proof.* By Lemma 5.2.1.3, there is a generating family of continuous sections that are represented by smooth and compactly supported functions on $\mathbb{N}_K G$ and are left and right K-finite. If $\{f_t\}$ is such a continuous section, then by Proposition 6.1.0.3, Proposition 6.2.0.3, the formula (6.3.0.1), and the fact that $\left[\bigoplus_n \pi_n\right] \oplus \pi_+ \oplus \pi_-$ is isometric, we get that

$$\lim_{t\to 0} \alpha_t(\lambda_t(f_t))$$

exists in $C_r^*(G)$. By Lemma 5.2.0.2, we conclude that the limit exists for any continuous section of $\{C_r^*(G_t)\}$. □

We can now define an embedding of the reduced group $C^*$-algebras. Let $f_0 \in C_r^*(G_0)$. Extend $f_0$ in any way to a continuous section $\{f_t\}$ of $\{C_r^*(G_t)\}$ and define

$$\alpha(f_0) := \lim_{t\to 0} \alpha_t(\lambda_t(f_t)) \tag{6.3.0.2}$$

in $C_r^*(G)$. Theorem 5.3.1.1 tells us that (6.3.0.2) is well-defined and isometric. We can construct the mapping cone continuous field as in Subsection 5.3.2. Then Theorem 5.3.2.1 gives us the following theorem

**Theorem 6.3.0.2.** *There is an isomorphism of continuous fields from the deformation continuous field for $G = SL(2,\mathbb{R})$ to the mapping cone continuous field for the embedding*

$$\alpha : C_r^*(G_0) \to C_r^*(G).$$

## 6.4 Characterization of the Mackey Bijection

In this section, we adapt the results of Section 5.4 for the case of $G = SL(2, \mathbb{R})$.

Recall in Section 3.6 that we constructed the irreducible unitary representations of $G_0$. They are

$$\phi_{\nu,\pm} = \mathrm{Ind}_{M \ltimes \mathfrak{p}}^{G_0} \sigma_\pm \otimes e^{i\nu},$$

where $\sigma_\pm \in \hat{M}$, $\nu \in \mathfrak{a}^*$,and $\nu \neq 0$, and

$$\phi_{0,\tau_n} = \mathrm{Ind}_{K \ltimes \mathfrak{p}}^{G_0} \tau_n \otimes (\text{trivial}),$$

where $\tau_n \in \hat{K}$ and $\tau \neq 0$. In particular, we have $\phi_{0,\tau_n}(k, X) = \tau_n(k)$ for $k \in K$ and $X \in \mathfrak{p}$.

### 6.4.1 Principal Series Representations

**Lemma 6.4.1.1.** The composition of the principal series $\pi_{\pm,\nu}$ of $G = SL(2, \mathbb{R})$ with the morphism

$$\alpha : C_r^*(G_0) \to C_r^*(G)$$

is the unitary representation $\mathrm{Ind}_{M \ltimes \mathfrak{p}}^{K \ltimes \mathfrak{p}} \sigma_\pm \otimes e^{i\nu}$ of the Cartan motion group $G_0$.

*Proof.* The proof is exactly the same as in the proof of Lemma 5.4.1.1 with $\sigma$ being either $\sigma_+$ or $\sigma_-$. □

Then we have

**Lemma 6.4.1.2.** If $\nu \neq 0$, then the composition of Lemma 6.4.1.1 $\pi_{\pm,\nu} \circ \alpha$ is an irreducible unitary representation of the Cartan motion group $G_0$. Furthermore,

$$\pi_{\pm,\nu} \circ \alpha = \phi_{\nu,\pm}.$$

For the case of $\nu = 0$, we have

**Lemma 6.4.1.3.** The composition of the principal series $\pi_{+,0}$ of $G = SL(2, \mathbb{R})$ with the morphism

$$\alpha : C_r^*(G_0) \to C_r^*(G)$$

decomposes as

$$\pi_{+,0} \circ \alpha = \bigoplus_{n \in \mathbb{Z}} \phi_{0,\tau_{2n}}.$$

*Proof.* By Lemma 6.4.1.1, we have $\pi_{+,0} \circ \alpha = \mathrm{Ind}_{M \ltimes \mathfrak{p}}^{G_0} \sigma_+ \otimes (\text{trivial})$. We can realize this representation on $L^2(K)^+$. Recall that

$$L^2(K)^+ \cong \bigoplus_{\tau_n \in \hat{K}} V_{\tau_n} \otimes (V_{\tau_n}^*)^+,$$

where $\phi \in (V_{\tau_n}^*)^+$ has the property that

$$\phi(\tau_n(m)v) = \sigma_+(m)\phi(v) = \phi(v),$$

for all $m \in M$ and for all $v \in V_{\tau_n}$. But,

$$\begin{aligned}
\phi(\tau_n(k_\pi)v) &= \phi(e^{in\pi}v) \\
&= e^{in\pi}\phi(v) \\
&= (-1)^n\phi(v).
\end{aligned}$$

Therefore $(V_{\tau_n}^*)^+ \neq 0$ if and only if $n$ is even. Therefore, we get

$$L^2(K)^+ \cong \bigoplus_{n \in \mathbb{Z}} V_{2n} \otimes (V_{2n}^*)^+. \tag{6.4.1.1}$$

See for example Lemma 5.2.1.4. Under the decomposition (6.4.1.1), the action of $\mathrm{Ind}_{M \ltimes \mathfrak{p}}^{G_0} \sigma_+ \otimes \text{trivial}$ becomes $\tau_{2n}$ on the component $V_{2n} \otimes (V_{2n}^*)^+$. We conclude

$$\mathrm{Ind}_{M \ltimes \mathfrak{p}}^{G_0} \sigma_+ \otimes \text{trivial} = \bigoplus_{n \in \mathbb{Z}} \phi_{0,\tau_{2n}}.$$

□

And finally,

**Lemma 6.4.1.4.** The composition of the principal series $\pi_-, 0$ of $G = SL(2, \mathbb{R})$ with the morphism

$$\alpha : C_r^*(G_0) \to C_r^*(G)$$

decomposes as

$$\pi_{-,0} = \bigoplus_{n \in \mathbb{Z}} \phi_{0,\tau_{2n+1}}.$$

Moreover, the compositions of the limits of discrete series representations $\pi_1$ and $\pi_{-1}$ with the morphism

$$\alpha : C_r^*(G_0) \to C_r^*(G)$$

decompose as

$$\pi_1 \circ \alpha = \bigoplus_{n=0}^{\infty} \phi_{0,\tau_{2n+1}}$$
$$\pi_{-1} \circ \alpha = \bigoplus_{n=0}^{\infty} \phi_{0,\tau_{-2n-1}}.$$

*Proof.* The proof follows precisely as in the proof of Lemma 6.4.1.3, this time, obtaining

$$L^2(K)^- \cong \bigoplus_{n \in \mathbb{Z}} V_{2n+1} \otimes (V_{2n+1}^*)^-. \tag{6.4.1.2}$$

Thus we get

$$\pi_{-,0} \circ \alpha = \bigoplus_{n \in \mathbb{Z}} \phi_{0,\tau_{2n+1}}.$$

Under the decomposition (6.4.1.2), $D_+$ acts nontrivially only on the components with $n$ non-negative and $D_-$ acts nontrivially only on the components with $n$ negative. So we get

$$\pi_1 \circ \alpha = \bigoplus_{n=0}^{\infty} \phi_{0,\tau_{2n+1}}$$
$$\pi_{-1} \circ \alpha = \bigoplus_{n=0}^{\infty} \phi_{0,\tau_{-2n-1}}.$$

□

### 6.4.2 Discrete Series Representations

**Lemma 6.4.2.1.** Let $n$ be an integer such that $|n| \geq 2$. The composition of the discrete series representation $\pi_n$ with the morphism

$$\alpha : C_r^*(G_0) \longrightarrow C_r^*(G)$$

decomposes as

$$\pi_n \circ \alpha(f) = \phi_{0,\tau_n}(f) \oplus \phi_{0,\tau_{n+2}}(f) \oplus \phi_{0,\tau_{n+4}} \oplus \cdots ,$$

if $n \geq 2$, and

$$\pi_n \circ \alpha(f) = \phi_{0,\tau_n}(f) \oplus \phi_{0,\tau_{n-2}}(f) \oplus \phi_{0,\tau_{n-4}} \oplus \cdots ,$$

if $n \leq -2$, where $f \in C_c^\infty(G_0)$.

*Proof.* Let

$$F_0(k) = \int_{\mathfrak{a}} \int_{\mathfrak{n}} f(k, X + Y)\ dX\ dY.$$

Also let $v, w \in H_n$, vectors in the representation space of $\pi_n$. Then by (6.2.0.2)

$$\begin{aligned} \langle v, (\pi_n \circ \alpha)(f) w \rangle &= \int_K F(k) \langle v, \pi_n|_K(k) w \rangle\ dk \\ &= \langle v, \pi_n|_K(F) w \rangle, \end{aligned}$$

so we have

$$\pi_n \circ \alpha(f) = \pi_n|_K(F).$$

Furthermore, since

$$\begin{aligned} \pi_n|_K &\cong \tau_n \oplus \tau_{n+2} \oplus \tau_{n+4} \oplus \cdots \qquad n \geq 2, \\ \pi_n|_K &\cong \tau_n \oplus \tau_{n-2} \oplus \tau_{n-4} \oplus \cdots \qquad n \leq 2, \end{aligned}$$

we get

$$\pi_n \circ \alpha(f) = \tau_n(F) \oplus \tau_{n+2}(F) \oplus \tau_{n+4} \oplus \cdots \qquad n \geq 2,$$
$$\pi_n \circ \alpha(f) = \tau_n(F) \oplus \tau_{n-2}(F) \oplus \tau_{n-4} \oplus \cdots \qquad n \leq 2.$$

On the other hand,

$$\begin{aligned}
\phi_{0,\tau_n}(f) &= \int_K \int_{\mathfrak{p}} f(k, X)\phi_{0,\tau_n}(k, X)\ dk\ dX \\
&= \int_K \int_{\mathfrak{p}} f(k, X)\tau_n(k)\ dk\ dX \\
&= \int_K F(k)\tau_n(k)\ dk \\
&= \tau_n(F).
\end{aligned}$$

We can then conclude

$$\pi_n \circ \alpha(f) = \phi_{0,\tau_n}(f) \oplus \phi_{0,\tau_{n+2}}(f) \oplus \phi_{0,\tau_{n+4}} \oplus \cdots \qquad n \geq 2,$$
$$\pi_n \circ \alpha(f) = \phi_{0,\tau_n}(f) \oplus \phi_{0,\tau_{n-2}}(f) \oplus \phi_{0,\tau_{n-4}} \oplus \cdots \qquad n \leq 2,$$

and we are done. □

### 6.4.3 Characterization of the Mackey Bijection

We recall the Mackey bijection $\mu : \hat{G}_r \to \hat{G}_0$ given in Section 3.6. It is defined by

$$\mu : \begin{cases} \pi_{\pm,\nu} \mapsto \phi_{\nu,\pm} & \nu > 0, \\ \pi_{+,0} \mapsto \phi_{0,\tau_0}, \\ D_1 \mapsto \phi_{0,\tau_1}, \\ D_{-1} \mapsto \phi_{0,\tau_{-1}}, \\ \pi_n \mapsto \phi_{0,\tau_n} & |n| \geq 2. \end{cases} \tag{6.4.3.1}$$

**Theorem 6.4.3.1.** *There is a unique bijection*

$$\mu : \hat{G}_r \to \hat{G}_0$$

*such that for every* $[\pi] \in \hat{G}_r$*, the element* $\mu([\pi]) \in \hat{G}_0$ *may be realized as a unitary subrepresentation of* $\pi \circ \alpha$*.*

*Proof.* The existence is given by (6.4.3.1). From Lemma 6.4.1.2, Lemma 6.4.1.3, Lemma 6.4.1.4, and Lemma 6.4.2.1, the bijection satisfies the necessary condition.

As for uniqueness, suppose there is another bijection

$$\mu_0 : \hat{G}_r \to \hat{G}_0$$

that satisfies the stated condition of the theorem. By the given condition we must have

$$\mu_0 : \pi_{\pm,\nu} = \phi_{\nu,\pm},$$

for $\nu > 0$. We will proceed by process of elimination. Note that $\phi_{0,\tau_0}$ is contained as a subrepresentation only in $\pi_{+,0} \circ \alpha$, so we must have $\mu_0(\pi_{+,0}) = \phi_{0,\tau_0}$. Next, $\phi_{0,\tau_1}$ is contained as a subrepresentation only in $D_1 \circ \alpha$, so we have $\mu_0(D_1) = \phi_{0,\tau_1}$. Likewise, since $\phi_{0,\tau_{-1}}$ is contained only in $D_{-1} \circ \alpha$, we have $\mu_0(D_{-1}) = \phi_{0,\tau_{-1}}$. Now, $\phi_{0,\tau_2}$ is contained in both $\pi_{+,0} \circ \alpha$ and in $\pi_2 \circ \alpha$. But $\mu_0(\pi_{+,0})$ has already been defined, so we must have $\mu_0(\pi_2) = \phi_{0,\tau_2}$. Next, $\phi_{0,\tau_3}$ is contained in both $D_1 \circ \alpha$ and in $\pi_3 \circ \alpha$, but $\mu_0(D_1)$ is already defined, so we must have $\mu_0(\pi_3) = \phi_{0,\tau_3}$. We continue in this way to get $\mu_0(\pi_n) = \phi_{0,\tau_n}$ for $n \geq 2$. Similarly, we must have $\mu_0(\pi_n) = \phi_{0,\tau_n}$ for $n \leq -2$, and so we conclude $\mu_0 = \mu$.

□

# Appendix
# Sketch of Proof of Fourier Structure Theorem

In this appendix we will sketch a proof of Theorem 4.5.2.3, by taking the proof from [CCH16] which uses the language of Hilbert $C^*$-module and translating it into our context. For the full proof to the Fourier structure theorem for a real reductive group, see [CCH16].

## 1 Hilbert $C^*$-Module and the Action of the Reduced $C^*$-Algebra

Let $G$ be a real reductive group with Iwasawa decomposition $KAN$ and a parabolic subgroup $S = MAN$ with Langlands decomposition. Let $L = MA$ be its *Levi factor*. Let $\sigma \in \hat{M}$ and $\nu \in \mathfrak{a}^*$ so that $e^{i\nu} \in \hat{A}$. Then $\sigma \otimes e^{i\nu}$ is a representation of $L$ on the representation space $H_{\sigma,\nu}$. Also, $H_\sigma$ is the representation space of $\sigma \in \hat{M}$.

The $C^*$-algebra $C^*_r(G)$ acts on the left of $C^*_r(G/N)$ by convolution and $C^*_r(L)$ acts on the right, making $C^*_r(G/N)$ a Hilbert bimodule. Naturally $C^*_r(L)$ acts on $H_{\sigma,\nu}$, so we can define a new Hilbert space $C^*_r(G/N) \otimes_{C^*_r(L)} H_{\sigma,\nu}$ on which $C^*_r(G)$ will act from the left. The following theorem is given in [CCH16, Proposition 4.2], [Cla13, Corollary 3.5], and [Rie74, Theorem 5.12]:

**Theorem 1.0.1.** *Given a tempered representation $\tau$ of $L$ on $H_\tau$, there is a mapping from $C^*_r(G/N) \otimes_{C^*_r(L)} H_\tau$ onto a dense subset of $\operatorname{Ind}_S^G H_\tau$ that unitarily intertwines the action* *of* $C^*_r(G)$.

Define a $C_0(\hat{A})$-module $C_0(\hat{A}, H_\sigma)$. Note that $L = MA$ acts on $C_0(\hat{A}, H_\sigma)$, with the following rule: letting $F \in C_0(\hat{A}, H_\sigma)$, the actions are $a.F : \nu \mapsto \nu(a)F(\nu)$ and $m.F : \nu \mapsto \sigma(m)F(\nu)$. In particular, the mapping $ev_\nu$ maps $C_0(\hat{A}, H_\sigma)$ to $H_{\sigma,\nu}$. It is clear that $C^*_r(L)$ acts on $C_0(\hat{A}, H_\sigma)$, so we have the *parabolically induced Hilbert module*

$$C^*_r(G/N) \otimes_{C^*_r(L)} C_0(\hat{A}, H_\sigma),$$

upon which $C^*_r(G)$ will act. We can define

$$\operatorname{Ind}_S^G C_0(\hat{A}, h_\sigma) := \{f : G \to C_0(\hat{A}, H_\sigma) | f(gman) = \delta(man)\sigma(m)^{-1} f(g)\},$$

where $\delta$ makes the space unitary.

The above theorem applies even in the case of Hilbert modules:

**Theorem 1.0.2.** *There is a mapping* $\Lambda$ *from* $C^*_r(G/N) \otimes_{C^*_r(L)} C_0(\hat{A}, H_\sigma)$ *onto a dense subset of* $\operatorname{Ind}_S^G C_0(\hat{A}, H_\sigma)$ *unitarily intertwining the action of* $C^*_r(G)$. □

**Theorem 1.0.3** ( [CCH16, Corollary 4.12])**.** *The* $C^*$*-algebra* $C^*_r(G)$ *acts as compact operators on the Hilbert module* $C^*_r(G/N) \otimes_{C^*_r(L)} C_0(\hat{A}, H_\sigma)$.

**Corollary 1.0.4.** $C^*_r(G)$ *acts as compact operators on* $\operatorname{Ind}_P^G C_0(\hat{A}, H_\sigma)$.

*Sketch of Proof.* It should be clear that

$$\mathcal{K}(C^*_r(G/N) \otimes_{C^*_r(L)} C_0(\hat{A}, H_\sigma) \cong \mathcal{K}(\operatorname{Ind}_S^G C_0(\hat{A}, H_\sigma)).$$

Now use Theorem 1.0.1. □

Let us describe the action of $C^*_r(G)$ on $\operatorname{Ind}_S^G C_0(\hat{A}, H_\sigma)$. First, we look at the action of $G$ on $\operatorname{Ind}_S^G C_0(\hat{A}, H_\sigma)$. Let $F \in \operatorname{Ind}_S^G C_0(\hat{A}, H_\sigma)$ and $g \in G$. We will let $\pi'_\sigma$ be the representation of $G$ on $\operatorname{Ind}_S^G C_0(\hat{A}, H_\sigma)$. Then, for $x \in G$, we have

$$\pi'_\sigma(g)F(x) = F(g^{-1}x),$$

and thus for $f \in C_r^*(G)$, we get

$$\pi'_\sigma(f)F(x) = \int_G f(g)F(g^{-1}x)\ dg. \tag{1.0.1}$$

This is a compact operator on $\operatorname{Ind}_P^G C_0(\hat{A}, H_\sigma)$ and is corresponds with the action of $f$ on $C_r^*(G/N) \otimes_{C_r^*(L)} C_0(\hat{A}, H_\sigma)$ by convolution. So we have

$$\pi'_\sigma : C_r^*(G) \to \mathcal{K}(\operatorname{Ind}_S^G C_0(\hat{A}, H_\sigma)). \tag{1.0.2}$$

**Proposition 1.0.5.** $\pi_{S,\sigma}[C_r^*(G)] \cong \pi'_\sigma[C_r^*(G)]$.

The construction of this isomorphism will proceed in a few steps.

**Lemma 1.0.6.** $\operatorname{Ind}_S^G C_0(\hat{A}, H_\sigma) \cong C_0(\hat{A}, \operatorname{Ind}_S^G H_\sigma)$.

*Proof.* We define

$$\Xi : \operatorname{Ind}_S^G C_0(\hat{A}, H_\sigma) \to C_0(\hat{A}, \operatorname{Ind}_S^G H_\sigma)$$

by $\Xi(F)(\nu)(g) = F(g)(\nu)$ with $F \in \operatorname{Ind}_S^G C_0(\hat{A}, H_\sigma)$, $\nu \in \hat{A}$, and $g \in G$. This is invertible, taking $G \in C_0(\hat{A}, \operatorname{Ind}_S^G H_\sigma)$, we have $\Xi^{-1}(G)(g)(\nu) = G(\nu)(g)$. □

We can define an action of $C_r^*(G)$ on $C_0(\hat{A}, \operatorname{Ind}_S^G H_\sigma)$ so that it is compatible with $\pi'_\sigma$. We define

$$\pi''_\sigma(f) = \Xi \circ \pi'_\sigma(f) \circ \Xi^{-1} \tag{1.0.3}$$

for all $f \in C_r^*(G)$. In this way, the action becomes

$$\begin{aligned}
(\pi''_\sigma(g)V)(\nu)(x) &= \Xi(\pi'_\sigma(g)\Xi^{-1}V)(\nu)(x) \\
&= (\pi'_\sigma(g)\Xi^{-1}V)(g)(\nu) \\
&= \Xi^{-1}V(g^{-1}x)(\nu) \\
&= V(\nu)(g^{-1}x) \\
\Rightarrow (\pi''_\sigma(g)V)(\nu)(g) &= V(\nu)(g^{-1}x).
\end{aligned}$$

The action of $C_r^*(G)$ by $\pi''_\sigma$ is through compact operators on $C_0(\hat{A}, \operatorname{Ind}_S^G H_\sigma)$.

**Lemma 1.0.7.** If X is a topological space and H is a Hilbert space, then

$$\mathcal{K}(C_0(X, H)) \cong C_0(A, \mathcal{K}(H)).$$

*Sketch of Proof.* Note that $C_0(X, H) \cong H \otimes C_0(X)$. Then $\mathcal{K}(H \otimes C_0(X)) \cong \mathcal{K}(H) \otimes C_0(X)$. Finally $\mathcal{K}(H) \otimes C_0(A) \cong C_0(A, \mathcal{K}(H))$. See for example [Lan95, Chapter 4]. □

Let

$$\pi'''_\sigma : C^*_r(G) \to C_0(\hat{A}, \mathcal{K}(\operatorname{Ind}_S^G H_\sigma))$$

be defined by $\pi'''_\sigma(f) = \Theta(\pi''_\sigma(f))$. Now we describe the action of $\pi'''_\sigma$. Let $\phi \in \operatorname{Ind}_S^G H_\sigma$. Then

$$\begin{aligned}
\pi'''_\sigma(f)(\nu)\phi &= \Theta(\pi''_\sigma(f))(\nu)\phi \\
&= \pi''(f)(\phi)(\nu) \\
\Rightarrow (\pi'''_\sigma(f)(\nu)\phi)(x) &= \pi''_\sigma(f)(\phi)(\nu)(x) \\
&= \int_G f(g)\pi''_\sigma(g)\phi(\nu)(x)\ dg \\
&= \int_G f(g)\phi(\nu)(g^{-1}x)\ dg.
\end{aligned}$$

Recall that we can treat $\phi$ as a constant function in $C_0(\hat{A}, \operatorname{Ind}_S^G H_\sigma)$. Moreover, recall the mapping $ev_\nu : C_0(\hat{A}, \operatorname{Ind}_S^G H_\sigma) \to \operatorname{Ind}_S^G H_{\sigma,\nu}$. So $\phi(\nu) \in \operatorname{Ind}_S^G H_{\sigma,\nu}$. Thus we have

$$\begin{aligned}
\int_G f(g)\phi(\nu)(g^{-1}x)\ dg &= \int_G f(g)\pi_{S,\sigma,\nu}(g)\phi(\nu)(x)\ dg \\
&= \pi_{S,\sigma,\nu}(f)\phi(\nu)(x).
\end{aligned}$$

So we have $\pi'''_\sigma(f)(\nu)\phi = \pi_{S,\sigma,\nu}(f)\phi(\nu) = \pi_{S,\sigma}(f)(\nu)\phi(\nu)$, thus concluding that $\pi'''_\sigma = \pi_{S,\sigma}$. This proves Proposition 1.0.5. □

## 2 Decomposition of the Reduced Group $C^*$-Algebra

Following the notation in [CCH16], define

$$C_r^*(G)_{S,\sigma} \subset \mathcal{K}(C_r^*(G/N) \otimes_{C_r^*(L)} C_0(\hat{A}, \operatorname{Ind}_S^G H_\sigma))$$

to be the image of $C_r^*(G)$ as compact operators on $C_r^*(G/N) \otimes_{C_r^*(L)} C_0(\hat{A}, \operatorname{Ind}_S^G H_\sigma)$. Two pairs $(S_1, \sigma_1)$ and $(S_1, \sigma_2)$ are *associate* if there is an element of $G$ that conjugates the Levi factor of $S_1$ to the Levi factor of $S_2$ and conjugates $\sigma_1$ to a representation unitarily equivalent to $\sigma_2$ (see [CCH16, Definition 5.2]). Then

**Theorem 2.0.1** ( [CCH16, Proposition 5.17]). *The $C^*$-algebra homomorphism*

$$C_r^*(G) \to \bigoplus_{[S,\sigma]} C_r^*(G)_{[S,\sigma]}$$

*where $[S, \sigma]$ is the associate class of the pair $(S, \sigma)$.*

Recall from Corollary 4.15 that the actions of $C_r^*(G)$ are unitarily equivalent on $C_r^*(G/N) \otimes_{C_r^*(L)} C_0(\hat{A}, \operatorname{Ind}_S^G H_\sigma)$ and $\operatorname{Ind}_S^G C_0(\hat{A}, \operatorname{Ind}_S^G H_\sigma)$. Combine this with Proposition 1.0.5 and we get that

$$C_r^*(G)_{[S,\sigma]} \cong \pi_{S,\sigma}[C_r^*(G)] \subset C_0(\hat{A}, \mathcal{K}(\operatorname{Ind}_S^G H_\sigma)),$$

thus we can conclude

**Theorem 2.0.2.** $C_r^*(G) \cong \bigoplus_{[S,\sigma]} \pi_{S,\sigma}[C_r^*(G)]$.

## 3 Structure of a Component of the Reduced Group $C^*$-Algebra

To finish, we look at the structure of $\pi_{S,\sigma}[C_r^*(G)]$. Recall that we defined $W_\sigma = \{w \in N_K(\mathfrak{a}) | \operatorname{Ad}_w^* \sigma \cong \sigma\}/M$. We can also define $W_{\sigma,\nu} = \{w \in W_\sigma | \operatorname{Ad}_w^* \nu = \nu\}$. Now define a fixed-point algebra $C = C_0(\hat{A}, \mathcal{K}(\operatorname{Ind}_S^G H_\sigma))^{W_\sigma}$. $F \in C$ if $U_{w,\sigma,\nu} F(\nu) =$

$F(\mathrm{Ad}_w^* \nu)U_{w,\sigma,\nu}$ for all intertwining operator generated from all $w \in W_\sigma$. It is then obvious that

$$\pi_{S,\sigma}[C_r^*(G)] \subset C_0(\hat{A}, \mathcal{K}(\mathrm{Ind}_S^G H_\sigma))^{W_\sigma}.$$

We wish to show that

**Proposition 3.0.1.** $\pi_{S,\sigma}[C_r^*(G)] = C_0(\hat{A}, \mathcal{K}(\mathrm{Ind}_S^G H_\sigma))^{W_\sigma}$.

Before proving the proposition, we state a definition, some lemmas and a famous theorem:

**Theorem 3.0.2** (Harish-Chandra's Completeness Theorem)**.** *Fixing $\sigma$ and $\nu$, the span of all intertwining operator $U_{w,\sigma,\nu}$, generated from $w \in W_{\sigma,\nu}$, is exactly the full commutant of $\pi_{S,\sigma,\nu}$.*

**Lemma 3.0.3.** Let $H$ be a Hilbert space. Suppose $A \subset \mathcal{K}(H)$ and $B \subset B(H)$ are $C^*$-algebras such that $B$ is finite-dimensional. Then $M' = N$ if and only if $N' \cap \mathcal{K}(H) = M$.

*Idea of Proof.* We can consider $A, B \subset M_n(\mathbb{C})$ as $C^*$-algebras. Then it is elementary that $M' = N$ if and only if $M = N''$ which follows from a so-called double commutant theorem. In general, given $A$ and $B$ as in the hypothesis, any element in $A$ can be approximated by finite rank operators (which are equivalent to $n \times n$ matrices). On the other hand, since $B$ is finite-dimensional, it is isomorphic to $M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C})$ (see [Mur90, Theorem 6.3.8]). We then can see that, by approximation of the $n \times n$ matrices, that the conclusion is true. ☐

**Definition** ( [Dix77, Definition 11.1.1])**.** Let $A$ be a $C^*$-algebra, and $B$ a sub-$C^*$ algebra of $A$ if the following conditions are satisfied:

1. For every irreducible representation $\pi$ of $A$, $\pi|_B$ is irreducible,
2. and if $\pi$ and $\pi'$ are inequivalent irreducible representations of $A$, then $\pi|_B$ and $\pi'|_B$ are inequivalent.

**Lemma 3.0.4** ( [Dix77, Lemma 11.1.4])**.** Let $A$ be a liminal $C^*$-algebra with Hausdorff spectrum, $\hat{A}$, and $B$ a rich sub-$C^*$-algebra of $A$. Then $B = A$.

*proof to Proposition 3.0.1.* It suffices to show that $\pi_{S,\sigma}[C_r^*(G)]$ is a rich subalgebra of $C_0(\hat{A}, \mathcal{K}(\operatorname{Ind}_S^G H_\sigma))^{W_\sigma}$. A representation of $C_0(\hat{A}, \mathcal{K}(\operatorname{Ind}_S^G H_\sigma))^{W_\sigma}$ is of the form $\mathrm{ev}_\nu$ defined by $\mathrm{ev}_\nu(F) = F(\nu)$, which is a compact operator on $\operatorname{Ind}_S^G H_{\sigma,\nu}$. According to [Dix77, Proposition 5.4.13], $\mathrm{ev}_\nu$ decomposes into a direct sum of irreducible representations. Let $\mathrm{ev}_\nu = \oplus r_{\nu,i}$ so that $r_{\nu,i}$ is an irreducible representation with representation space $H_i \subset \operatorname{Ind}_S^G H_{\sigma,\nu}$. Note for any $w \in W_{\sigma,\nu}$ we have

$$U_{w,\sigma,\nu} F(\nu) = F(\nu) U_{w,\sigma,\nu} \tag{3.0.1}$$

But now suppose that $r_{\nu,i}$ is not irreducible when restricted to $\pi_{S,\sigma}[C_r^*(G)]$. Suppose we can decompose $H_i$ into $V_i \oplus W_i$ so that $r_{\nu,i}$ is invariant on $V_i$ and $W_i$ when restricted to $\pi_{S,\sigma}[C_r^*(G)]$.

Note that the Harish-Chandra Completeness Theorem tells us that

$$\operatorname{span}\{U_{w,\sigma,\nu} | w \in W_{\sigma,\nu}\} = \{\pi_{S,\sigma,\nu}(f) | f \in C_r^*(G)\}',$$

and using the lemma, we get that

$$\operatorname{span}\{U_{w,\sigma,\nu} | w \in W_{\sigma,\nu}\}' \cap \mathcal{K}(\operatorname{Ind}_S^G H_{\sigma,\nu}) = \{\pi_{S,\sigma,\nu} | f \in C_r^*(G)\}. \tag{3.0.2}$$

By (3.0.1) and (3.0.2), we must have that $F(\nu) = \pi_{S,\sigma,\nu}(f)(\nu)$ for some $f \in C_r^*(G)$. But then we would be able to decompose $r_{\nu,i}(F) = r_{\nu,i}(\pi_{S,\sigma}(f))$ into invariant operators on $V_i$ and $W_i$, which is a contradiction. We can then conclude that $r_{\nu,i}$ must be irreducible when restricted to $\pi_{S,\sigma}[C_r^*(G)]$.

As for the second part of the definition of a rich sub-$C^*$-algebra, suppose that $r_{\nu,i}$ and $r_{\nu,j}$ are inequivalent ($i \neq j$ and for now, we are considering within one $\mathrm{ev}_\nu$) representation of $C_0(\hat{A}, \mathcal{K}(\operatorname{Ind}_S^G H_\sigma))^{W_\sigma}$ on Hilbert spaces $H_i$ and $H_j$ respectively. But suppose that $r_{\nu,i}|_{C_r^*(G)}$ and $r_{\nu,j}|C_r^*(G)$ are equivalent. Then there is an invertible unitary operator $U : H_i \to H_j$ such that

$$U r_{\nu,i}(\pi_{S,\sigma}(f)) = r_{\nu,j}(\pi_{S,\sigma}(f)) U$$
$$U \pi_{S,\sigma,\nu}(f)|_{H_i} = \pi_{S,\sigma,\nu}(f)|_{H_j} U$$

for all $f \in C_r^*(G)$. Let $U_0$ be an operator on $\operatorname{Ind}_S^G H_\sigma = \oplus H_i$ so that $U_0|_{H_i} = U$ and $U_0|_{H_j} = U^*$ (qualitatively, if we restrict our attention to $H_i \oplus H_j$, we can view $U_0$ as $\begin{bmatrix} & U^* \\ U & \end{bmatrix}$). Then

$$U_0 \pi_{S,\sigma,\nu}(f) = \pi_{S,\sigma,\nu}(f) U_0$$

for all $C_r^*(G)$. By Harish-Chandra's completeness theorem, we must have that $U_0 \in \operatorname{span}\{U_{w,\sigma,\nu} | w \in W_{\sigma,\nu}\}$. It is then clear that for all

$$F \in C_0(\hat{A}, \mathcal{K}(\operatorname{Ind}_S^G H_\sigma))^{W_\sigma}$$

we have $U_0 F(\nu) = F(\nu) U_0 \Rightarrow U_0 \operatorname{ev}_\nu(F) = \operatorname{ev}_\nu(F) U$. If we restrict to $H_i$ we get $U r_{\nu,i}(F) = r_{\nu,j} U$, which is a contradiction. We conclude $r_{\nu,i}|_{C_r^*(G)}$ and $r_{\nu,j}|_{C_r^*(G)}$ are inequivalent whenever $r_{\nu,i}$ and $r_{\nu,j}$ are inequivalent.

Now suppose $\nu \neq \nu'$ and $r_{\nu,i}$ and $r_{\nu',j}$ are inequivalent but $r_{\nu,i}|_{\pi_{S,\sigma}[C_r^*(G)]}$ and $r_{\nu',j}|_{\pi_{S,\sigma}[C_r^*(G)]}$ are equivalent. Let $H_i$ be the representation space of $r_{\nu,i}$ and $H_j'$ the representation space of $r_{\nu',j}$. There is an invertible unitary transformation $U : H_i \to H_j'$ such that for all $f \in C_r^*(G)$,

$$U r_{\nu,i}(f) = r_{\nu',j}(f) U$$
$$U \pi_{S,\sigma,\nu}(f)|_{H_i} = \pi_{S,\sigma,\nu'}(f)|_{H_j'} U.$$

Let $H_0$ and $H_0'$ be so defined that $\operatorname{Ind}_S^G H_{\sigma,\nu} = H_i \oplus H_0$ and $\operatorname{Ind}_S^G H_{\sigma,\nu'} = H_j' \oplus H_0'$. Then let us extend $U$ to $U_0$ so that $U_0|_{H_i} = U$ and $U_0|_{H_0} = 0$ (in this sense, think of $U_0 : H_i \oplus H_0 \to H_j' \oplus H_0'$ as $\begin{bmatrix} U & 0 \\ 0 & 0 \end{bmatrix}$). Then it is clear that

$$U_0 \pi_{S,\sigma,\nu}(f) = \pi_{S,\sigma,\nu'}(f) U_0$$

for all $f \in C_r^*(G)$. Such $U_0$ can only exist if there is a $w \in W_\sigma$ such that $\operatorname{Ad}_w^* \nu = \nu'$. Then $\operatorname{ev}_\nu \cong \operatorname{ev}_{\nu'}$, so we can consider $r_{\nu,i}$ and $r_{\nu',j}$ to be subrepresentation of the same $\operatorname{ev}_\nu$, and follow the argument from the previous paragraph.

□

Theorem 2.0.2 and Theorem 3.0.1 proves Theorem 4.5.2.3.

# Bibliography


[AA19] A. Afgoustidis and A.-M. Aubert. Continuity of the Mackey-Higson bijection. Preprint, 2019. arXiv:1901.00144.

[Afg18] A. Afgoustidis. On the analogy between real reductive groups and Cartan motion groups. I: The Mackey-Higson bijection. Preprint, 2018. arXiv: 1510.02650.

[Afg19] A. Afgoustidis. On the analogy between real reductive groups and Cartan motion groups. III: A proof of the Connes-Kasparov isomorphism. Preprint, 2019. arXiv:1602.0889.

[CCH16] P. Clare, T. Crisp, and N. Higson. Parabolic induction and restriction via $C^*$-algebras and Hilbert $C^*$-modules. *Compos. Math.*, 152(6):1286–1318, 2016.

[CG90] Lawrence J. Corwin and Frederick P. Greenleaf. *Representations of nilpotent Lie groups and their applications. Part I*, volume 18 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1990. Basic theory and examples.

[CHH88] M. Cowling, U. Haagerup, and R. Howe. Almost $L^2$ matrix coefficients. *J. Reine Angew. Math.*, 387:97–110, 1988.

[Cla13] Pierre Clare. Hilbert modules associated to parabolically induced representations. *J. Operator Theory*, 69(2):483–509, 2013.

[CSM95] Roger Carter, Graeme Segal, and Ian Macdonald. *Lectures on Lie groups and Lie algebras*, volume 32 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1995. With a foreword by Martin Taylor.

[Dix77] J. Dixmier. *$C^*$-algebras*. North-Holland Publishing Co., Amsterdam-New York-Oxford, 1977. Translated from the French by Francis Jellett, North-Holland Mathematical Library, Vol. 15.

[Ful84] W. Fulton. *Intersection theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1984.

[Hig08] N. Higson. The Mackey analogy and K-theory. In *Group Representations, Ergodic Theory, and Mathematical Physics: A Tribute to George W. Mackey*, volume 449 of *Contemp. Math.*, pages 149–172. Amer. Math. Soc., Providence, RI, 2008.

[Hig10] Nigel Higson. The tangent groupoid and the index theorem. In *Quanta of maths*, volume 11 of *Clay Math. Proc.*, pages 241–256. Amer. Math. Soc., Providence, RI, 2010.

[Hig11] N. Higson. On the analogy between complex semisimple groups and their Cartan motion groups. In *Noncommutative geometry and global analysis*, volume 546 of *Contemp. Math.*, pages 137–170. Amer. Math. Soc., Providence, RI, 2011.

[HR19] Nigel Higson and Angel Román. The Mackey bijection for complex reductive groups and continuous fields of reduced group $C^*$-algebras. Preprint, 2019. arXiv:1908.11008.

[HSSH18] A.R. Haj Saeedi Sadegh and N. Higson. Euler-like vector fields, deformation spaces and manifolds with filtered structure. *Doc. Math.*, 23:293–325, 2018.

[Hum75] James E. Humphreys. *Linear algebraic groups*. Springer-Verlag, New York-Heidelberg, 1975. Graduate Texts in Mathematics, No. 21.

[Kir08] Alexander Kirillov, Jr. *An introduction to Lie groups and Lie algebras*, volume 113 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2008.

[Kna86] Anthony W. Knapp. *Representation Theory of Semisimple Groups: An Overview Based on Examples*. Princeton landmarks in mathematics. Princeton University Press, Princeton, New Jersey, 1986.

[Kna02] A. W. Knapp. *Lie groups beyond an introduction*, volume 140 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, second edition, 2002.

[KS80] A. W. Knapp and E. M. Stein. Intertwining operators for semisimple groups. II. *Invent. Math.*, 60(1):9–84, 1980.

[Lan85] S. Lang. $SL_2(\mathbb{R})$, volume 105 of *Graduate texts in Mathematics*. Springer-Verlag, New York, 1985.

[Lan95] E. C. Lance. *Hilbert* C*-*Modules: A toolkit for operator algebraists*, volume 210 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, 1995.

[Mac76] G. W. Mackey. *The theory of unitary group representations*. University of Chicago Press, Chicago, Ill.-London, 1976. Based on notes by James M. G. Fell and David B. Lowdenslager of lectures given at the University of Chicago, Chicago, Ill., 1955, Chicago Lectures in Mathematics.

[Mur90] Gerard J. Murphy. *C*-Algebras and Operator Theory*. Academic Press, INC., San Diego, CA, 1990.

[Rie74] Marc A. Rieffel. Induced representations of C*-algebras. *Advances in Math.*, 13:176–257, 1974.

[Ros94] Jonathan Rosenberg. C*-algebras and Mackey's theory of group representations. In C*-*algebras: 1943–1993 (San Antonio, TX, 1993)*, volume 167 of *Contemp. Math.*, pages 150–181. Amer. Math. Soc., Providence, RI, 1994.

[Rud76] Walter Rudin. *Principles of mathematical analysis*. McGraw-Hill Book Co., New York-Auckland-Düsseldorf, third edition, 1976. International Series in Pure and Applied Mathematics.

[Seg50] I. E. Segal. An extension of Plancherel's formula to separable unimodular groups. *Ann. of Math. (2)*, 52:272–292, 1950.

[Var89] V. S. Varadarajan. *An Introduction to Harmonic Analysis on Semisimple Lie Groups*, volume 16 of *Cambridge studies in advanced mathematics*. Cambridge University Press, 1989.

[Vog85] David A. Vogan, Jr. Classifying representations by lowest K-types. In *Applications of group theory in physics and mathematical physics (Chicago, 1982)*, volume 21 of *Lectures in Appl. Math.*, pages 269–288. Amer. Math. Soc., Providence, RI, 1985.

[Wal71] N. R. Wallach. Cyclic vectors and irreducibility for principal series representations. *Trans. Amer. Math. Soc.*, 158:107–113, 1971.

[Wal88] N. R. Wallach. *Real reductive groups. I*, volume 132 of *Pure and Applied Mathematics*. Academic Press Inc., Boston, MA, 1988.

[Wal92] Nolan R. Wallach. *Real reductive groups. II*, volume 132 of *Pure and Applied Mathematics*. Academic Press, Inc., Boston, MA, 1992.

# Vita

**Angel Román-Martínez**

| | |
|---|---|
| Place of birth: | Bayamon, Puerto Rico |
| Education: | MS in Mathematics, University of Puerto Rico, Rio Piedras, 2013 |
| | BS in Mathematics, University of Puerto Rico, Pio Piedras, 2009 |
| Publication: | The Mackey bijection for complex reductive groups and continuous fields of reduced group $C^*$-algebras, preprint in arXiv: 1908.11008, https://arxiv.org/abs/1908.11008, co-authored with Nigel Higson, 2019 |
| Athletic career: | Head coach of Club Taekwondo, Penn State University, 2013 to 2018 |
| | Member of the Puerto Rico national Taekwondo team, 2004 to 2013 |
| | Male athlete of the year in Taekwondo, awarded by the Puerto Rico Olympic Committee, 2004, 2005, 2008, 2009, 2011 |
| | Member of the Puerto Rico Olympic team, Taekwondo, in Beijing 2008 |
| | Gold medalist in Deaflympics, 2009 and 2013 |