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THERMODYNAMIC FORMALISM FOR NONUNIFORMLY HYPERBOLIC DYNAMICAL SYSTEMS

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Anna Mummert

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The thesis of Anna Mummert was reviewed and approved* by the following:

Yakov Pesin
Distinguished Professor of Mathematics
Thesis Advisor
Chair of Committee

Piotr Berman
Associate Professor of Computer Science and Engineering

Svetlana Katok
Professor of Mathematics

Mark Levi
Professor of Mathematics

Arkady Tempelman
Professor of Mathematics and Statistics

Howard Weiss
Professor of Mathematics

Nigel Higson
Distinguished Professor of Mathematics
Chair of the Mathematics Department

*Signatures are on file in the Graduate School.

Abstract

This thesis examines the thermodynamic formalism of nonuniformly hyperbolic dynamical systems in two cases.

In the first part, we study the nonadditive thermodynamic formalism for the class of almost-additive sequences of potentials. We define the topological pressure $P_Z(\Phi)$ of an almost-additive sequence Φ, on a compact f-invariant set Z. We give conditions which allow us to establish a variational principle for the topological pressure. We state conditions for the existence and uniqueness of equilibrium measures. In the special case of subshifts of finite type we state conditions for the existence and uniqueness of Gibbs measures. We compare our results for almost-additive sequences to the thermodynamic formalism for additive sequences [Rue78, Sin72, Bow70], nonadditive sequences [Bar96], subadditive sequences [Fal88], and the almost-additive sequence studied by Feng and Lau [FL02, Fen04].

Second, we study the thermodynamic formalism for discontinuous potentials. We give conditions under which the topological pressure of a discontinuous potential can be defined. A corresponding variational principle is established, no additional conditions are required. This thermodynamic formalism is applied to nonuniformly hyperbolic maps f and the corresponding potentials $\varphi_t(x) = -t \log \mathrm{Jac}(df_{|E_x^u})$. Other specific examples are considered, namely countable Markov shifts [Sar99] and unimodal maps [BK98].

Contents

List of Symbols

$\mathbb{1}$	the $d \times 1$ vector of all ones 21
A	a transition matrix 3
$c_1(n)$	$c_1 : \mathbb{N} \to \mathbb{R}$, lower bound for almost-additive sequence 22, 28
$c_2(n)$	$c_2 : \mathbb{N} \to \mathbb{R}$, upper bound for almost-additive sequence 22, 28
C_1	lower bound for $c_1(n)$ 22, 28
C_2	upper bound for $c_2(n)$ 22, 28
$\mathcal{C}_n$	a cylinder set of length n 3
$\underline{CP}_Z(\varphi)$	the lower capacity pressure of φ on Z 11
$\underline{CP}_Z(\Phi)$	the lower capacity pressure of Φ on Z 17
$\underline{CP}_Z(\varphi,\mathcal{U})$	the lower capacity pressure of φ on Z with respect to the cover $\mathcal{U}$ 11
$\underline{CP}_Z(\Phi,\mathcal{U})$	the lower capacity pressure of Φ on Z with respect to the cover $\mathcal{U}$ 17
$\overline{CP}_Z(\varphi)$	the upper capacity pressure of φ on Z 11
$\overline{CP}_Z(\Phi)$	the upper capacity pressure of Φ on Z 17
$\overline{CP}_Z(\varphi,\mathcal{U})$	the upper capacity pressure of φ on Z with respect to the cover $\mathcal{U}$ 11
$\overline{CP}_Z(\Phi,\mathcal{U})$	the upper capacity pressure of Φ on Z with respect to the cover $\mathcal{U}$ 17
$\mathrm{diam}(\mathcal{U})$	the diameter of the cover $\mathcal{U}$ 4
δ_x	the probability measure supported on $\{x\}$ 4
df	the differential of f 1
E_x^s	the set of vectors in the tangent space at x which contract under iterations of the differential 1
E_x^u	the set of vectors in the tangent space at x which expand under iterations of the differential 1
f	a continuous map $f : X \to X$, the dynamical system

φ	a potential $\varphi : X \to \mathbb{R}$ 4
Φ	a sequence of potentials $\Phi = \{\varphi_n : X \to \mathbb{R}\}_{n\geq 1}$ 16
$\gamma_n(\Phi,\mathcal{U})$	$V_n(\varphi_n)$ 17
$h_\mu(f)$	the measure-theoretic (metric) entropy of f 5
$h_\mu(f,\mathcal{E})$	the measure-theoretic (metric) entropy of f with respect to the partition $\mathcal{E}$ 5
$H_\mu(\mathcal{E})$	the entropy of μ with respect to the partition $\mathcal{E}$ 5
$H_\mu(\mathcal{E}\vert\mathcal{D})$	the conditional entropy of $\mathcal{E}$ with respect to $\mathcal{D}$ 5
$\mathrm{Jac}(df)$	the Jacobian of the differential df 61, 26
$\mathcal{L}(Z)$	$\{x \in Z : V(x) \bigcap \mathcal{M}(Z,f) \neq \varnothing\}$ 4
$\mathcal{L}(\Lambda,\varphi)$	$\{x \in \Lambda : V(x) \bigcap \mathcal{M}_\varphi(\Lambda,f) \neq \varnothing\}$ 26
Λ_k	one set in the family $\{\Lambda_k\}_{k\geq 0}$ 26
$M(Z,\alpha,\varphi,\mathcal{U})$	one of three Carathéodory functions for φ 11
$\underline{M}(Z,\alpha,\varphi,\mathcal{U})$	one of three Carathéodory functions for φ 11
$\overline{M}(Z,\alpha,\varphi,\mathcal{U})$	one of three Carathéodory functions for φ 11
$M(Z,\alpha,\Phi,\mathcal{U})$	one of three Carathéodory functions for Φ 16
$\underline{M}(Z,\alpha,\Phi,\mathcal{U})$	one of three Carathéodory functions for Φ 16
$\overline{M}(Z,\alpha,\Phi,\mathcal{U})$	one of three Carathéodory functions for Φ 16
$M(Z,\alpha,\varphi,\mathcal{U},N)$	a Carathéodory function for φ (before taking the limit as N goes to infinity) 53
$\mathcal{M}(X,f)$	the set of f-invariant Borel probability measures on X 4
$\mathcal{M}_\varphi(\Lambda,f)$	the set of f-invariant Borel probability measures on Λ for which φ is integrable 26
$\mu_{x,n}$	the probability measure $\frac{1}{n}\sum_{k=0}^{n-1}\delta_{f^k x}$ 4
μ_φ	an equilibrium or Gibbs measure for the potential φ 12, 14, 15
μ_Φ	an equilibrium or Gibbs measure for the sequence of potentials Φ 21, 24
$m(\underline{U})$	the length of the string $\underline{U}$ 4
$\Omega(f)$	the nonwandering points of f 2
Ω_i	a basic set for $\Omega(f)$ 2
$P_Z(\varphi)$	the topological pressure of φ on the set Z 11
$P_Z(\Phi)$	the topological pressure of Φ on the set Z 17, 18, 22

$P_{f,Z}(\varphi)$	the topological pressure of φ on the set Z with respect to the function f (used only when necessary) 34, 30
$P_\Lambda(\varphi)$	the topological pressure of (not necessarily continuous) φ on the set Λ 26
$P_G(\varphi)$	the Gurevich pressure of φ 14
$P_Z(\varphi,\mathcal{U})$	the topological pressure of φ on the set Z with respect to the cover $\mathcal{U}$ 11
$P_Z(\Phi,\mathcal{U})$	the topological pressure of Φ on the set Z with respect to the cover $\mathcal{U}$ 17
$P_{top}(\varphi_{\|Y})$	the topological pressure of φ on Y, that is, $P_Y(\varphi)$ (used only when necessary) 15
$\mathcal{R}$	a compact smooth Riemannian manifold 1
$S_m\Phi$	an additive sequence of potentials generated by Φ 29
Σ_n	the set of two-sided sequences of n letters 3
Σ_n^+	the set of one-sided sequences of n letters 3
Σ_A	the set of allowable two-sided sequences 3
Σ_A^+	the set of allowable one-sided sequences 3
σ	the shift map 3, 14
$T_x\mathcal{R}$	the tangent manifold to $\mathcal{R}$ at $x \in \mathcal{R}$ 1
$\mathcal{U}$	a finite open cover of X 4
$\underline{U}$	a string of elements of $\mathcal{U}$ 4
$V_n(\varphi)$	the n-th variation of φ 4
$V_n(\varphi,\mathcal{U})$	the n-th variation of φ with respect to the cover $\mathcal{U}$ (used only when necessary) 45
$V(x)$	the set of weak-$*$ limits of $\mu_{x,n}$ 4
$W_m(\mathcal{U})$	the set of strings of $\mathcal{U}$ of length $m(\underline{U}) = m$ 4
$W(\mathcal{U})$	the set of all strings of $\mathcal{U}$ 4
(X,ρ)	X a compact metric space with metric ρ 4
$X(\underline{U})$	$\{x \in X : f^k x \in U_k,\ \text{for all } 0 \le k < m(\underline{U})\}$ 4
$X_f(\underline{U})$	$X(\underline{U})$ with respect to the function f (used only when necessary) 34
$Z_m(\varphi)$	the partition function for φ 13
$Z_m(Z,\varphi,\mathcal{U})$	the partition function for Z, φ, and $\mathcal{U}$ 13
$Z_m(Z,\Phi,\mathcal{U})$	the partition function for Z, Φ, and $\mathcal{U}$ 22

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Epigraph

Measure Theory
If I were to stroll along a simplex
And come to the very end
What would I find there
But my ergodic friend

Together we would travel
And find the total measure of it all
Oh the points we could see
Together just like this
For eternity

Chapter 1

Basic Notions

In this chapter the basic mathematical notions are collected. We begin with a section on hyperbolicity as it applies to this thesis. Then other preliminary definitions are given.

1.1 Hyperbolicity

A comprehensive examination of hyperbolicity as it applies to this thesis can be found in [BP02].

A dynamical system is said to be hyperbolic if at every point there exists an invariant splitting of the tangent bundle $T_xM = E_x^s \oplus E_x^u$, where E_x^s consists of those vectors that contract under iterations of the derivative of f, and E_x^u consists of those that contract under iterations of the derivative of f^{-1}. If the splitting is continuous in x and the contraction rates do not depend on x, the system is called uniformly hyperbolic, otherwise it is called nonuniformly hyperbolic.

Let $f\colon \mathcal{R} \to \mathcal{R}$ be a $C^{1+\epsilon}$ diffeomorphism of a compact smooth Riemannian manifold $\mathcal{R}$. Given $x \in \mathcal{R}$ and $v \in T_x\mathcal{R}$, define the *Lyapunov exponent* of v at x by

$$\lambda(x, v) = \limsup_{n\to\infty} \frac{\log ||df_x^n v||}{n}.$$

If x is fixed then the function $\lambda(x, \cdot)$ can achieve only finitely many distinct values $\lambda^{(1)}(x) > \cdots > \lambda^{(q(x))}(x)$. The functions $\lambda^{(i)}(x)$ and $q(x)$ are measurable and f-invariant.

Define the f-invariant set

$$\Lambda = \{x \in \mathcal{R} : \exists\, 1 \le k(x) < s(x) : \lambda^{k(x)}(x) < 0 \text{ and } \lambda^{k(x)+1}(x) > 0\}.$$

The dynamical system f is said to have *nonzero Lyapunov exponents almost-everywhere* if there exists an ergodic f-invariant Borel measure ν such that $\nu(\Lambda) = 1$. The measure ν is said to be a *hyperbolic measure* for f.

Let μ be a hyperbolic ergodic Borel f-invariant measure on $\mathcal{R}$. Thus we have that the functions $\lambda^{(i)}(x) = \lambda_\mu^{(i)}$ and $q(x) = q$ are constant μ almost everywhere, and there exists $k(= k(x))$, $1 \leq k < q$ such that

$$\lambda_\mu^{(1)} > \cdots > \lambda_\mu^{(k)} > 0 > \lambda_\mu^{(k+1)} > \cdots > \lambda_\mu^{(q)}.$$

Also, for μ almost every point $x \in \mathcal{R}$ there exist stable and unstable subspaces $E^{(s)}(x), E^{(u)}(x) \subset T_x\mathcal{R}$ such that

1. $E^{(s)}(x) \oplus E^{(u)}(x) = T_x\mathcal{R}$,
   $df_x E^{(s)}(x) = E^{(s)}(f(x))$, and $df_x E^{(u)}(x) = E^{(u)}(f(x))$,
2. For any $n \geq 0$
   $$||df_x^n v|| \leq D_1(x)\gamma^n ||v|| \text{ if } v \in E^s(x),$$
   $$||df_x^{-n} v|| \leq D_1(x)\gamma^n ||v|| \text{ if } v \in E^u(x),$$
   where $0 < \gamma < 1$ is a constant and $C_1(x) > 0$ is a measurable function,
3. $\angle(E^s(x), E^u(x)) \geq D_2(x) > 0$, where $D_2(x)$ is a measurable function and $\angle$ denotes the angle between the two subspaces, and
4. $D_1(f^n(x)) \leq D_1(x)e^{n\delta}$, $D_2(f^n(x)) \geq D_2(x)e^{-n\delta}$ for any $n \geq 0$, where $\delta > 0$ is a constant which is sufficiently small compared to $1 - \gamma$.

The regular sets are

$$\Lambda_l = \{x \in \mathcal{R} : D_1(x) \leq l \text{ and } D_2(x) \geq 1/l\}.$$

These sets Λ_l are nested and exhaust $\Lambda = \bigcup_{l \geq 1} \Lambda_l$, the set of regular points.

Theorem (Multiplicative Ergodic Theorem). If f is a C^1 diffeomorphism of a compact smooth Riemannian manifold $\mathcal{R}$, then the set of Lyapunov regular points has full measure with respect to any f-invariant Borel probability measure on $\mathcal{R}$.

For a hyperbolic measure μ, the multiplicative ergodic theorem immediately gives us that the set of Lyapunov regular points with nonzero Lyapunov exponent contains a nonuniformly hyperbolic set of full μ measure.

A diffeomorphism $f\colon X \to X$ is called *Axiom A* if the set $\Omega(f)$ of non-wandering points is hyperbolic and is the closure of the periodic points. The spectral decomposition theorem gives that $\Omega(f) = \Omega_1 \bigcup \cdots \bigcup \Omega_s$, where the basic sets Ω_i are pairwise disjoint closed sets with

1. $f(\Omega_i) = \Omega_i$ and $f_{|\Omega_i}$ is topologically transitive,

2. $\Omega_i = Z_{i,1} \bigcup \cdots \bigcup Z_{i,n_i}$, with the $Z_{i,j}$ pairwise disjoint closed sets with $f(Z_{i,j}) = Z_{i,j+1}$ and $f_{|Z_{i,j}}$ topologically mixing.

An Axiom A diffeomorphism is topologically conjugate to a one-sided subshift of finite type.

1.2 Preliminary Information

A comprehensive examination of the concepts and definitions defined here and used in this thesis can be found in [KH95].

1.2.1 Subshifts of Finite and Countable Type

The set of all two-sided (double-sided) sequences of n letters is denoted

$$\Sigma_n = \{\underline{x} = \ldots x_{-1}x_0x_1\ldots \,:\, 0 \le x_i \le n-1, \text{ for all } i \in \mathbb{Z}\},$$

and the set of all one-sided (single-sided) sequences of n letters is

$$\Sigma_n^+ = \{\underline{x} = x_0x_1\ldots \,:\, 0 \le x_i \le n-1, \text{ for all } i \in \mathbb{N}\}.$$

An $n \times n$ matrix A is a transition matrix if every entry in A is 0 or 1. We assume that the matrix is nondegenerate, that is, every row and column has at least one nonzero entry. A transition matrix gives a subset of two- (or one-) sided allowable sequences. The set of all two-sided allowable sequences is

$$\Sigma_A = \{\underline{x} = \cdots x_{-1}x_0x_1\cdots : A_{x_ix_{i+1}} = 1, \text{ for all } i \in \mathbb{Z}\},$$

and the set of one-sided allowable sequences is

$$\Sigma_A^+ = \{\underline{x} = x_0x_1\cdots : A_{x_ix_{i+1}} = 1, \text{ for all } i \in \mathbb{N}\}.$$

A cylinder set $\mathcal{C}_n$ is the set of all $\underline{x} \in \Sigma_n$ (or Σ_n, Σ_A, Σ_A^+) is the set of all allowed sequences such that the positions $-n$ to n (or the first n) are fixed.

The shift map on Σ_n, Σ_n^+, Σ_A, and Σ_A^+ is defined as $(\sigma(x))_i = x_{i+1}$. The systems (Σ_n, σ) and (Σ_n^+, σ) are called the full two- or one-sided shift, respectively. The systems (Σ_A, σ) and (Σ_A^+, σ) are called two- or one-sided subshifts of finite type. When there is no confusion, we drop the + notation for the set of one-sided sequences.

If for every i, j there exists an $N = N(i,j)$ such that $(A^N)_{ij} = 1$, then the matrix A is called irreducible, and the corresponding dynamical system is topologically transitive. If there exists an N such that for every i, j $(A^N)_{ij} = 1$, then the matrix A is called primitive, and the corresponding dynamical system is topologically mixing. If the dynamical system is topologically mixing then there exists an N such that for every $n \geq N$ and every a, b of the alphabet there exists a word of length n from a to b.

1.2.2 Covers and Sets Associated to Covers; Potentials; Variations

Let $\mathcal{U}$ be a finite open cover of a compact metric space (X, ρ), and $W_m(\mathcal{U})$ the set of all m-strings $\underline{U} = U_{i_0} \dots U_{i_m}$ of members of $\mathcal{U}$, and denote by $W(\mathcal{U}) = \bigcup_{m \geq 1} W_m(\mathcal{U})$. Let $m(\underline{U}) = m$ be the length of the string $\underline{U}$. Define the set

$$X(\underline{U}) = \{x \in X : f^k(x) \in U_{i_k},\ k = 0, \dots, m(\underline{U}) - 1\}.$$

We say that $\Gamma \subset W_m(\mathcal{U})$ covers X if $X = \bigcup_{\underline{U} \in \Gamma} X(\underline{U})$. For subshifts of finite type, if $m(\underline{U}) = n$ then $X(\underline{U})$ is a cylinder set $\mathcal{C}_n$.

Set $\operatorname{diam}(\mathcal{U})$ to be the diameter of the the collection $\mathcal{U}$, that is, the largest diameter of the sets $U \in \mathcal{U}$.

Let $\varphi \colon X \to \mathbb{R}$. The variations of φ are

$$V_n(\varphi) = \sup\{|\varphi(x) - \varphi(y)| :\ x, y \in X(\underline{U}),\ m(\underline{U}) = n\}.$$

1.2.3 Measures and Entropy

Let $\mathcal{M}(X, f)$ be the set of f-invariant ergodic Borel probability measures on X. For any subset $Z \subset X$ we denote by $\mathcal{M}(Z, f) \subset \mathcal{M}(X, f)$ the set of all f - invariant Borel probability measures on Z. If Z is compact and f-invariant, then $\mathcal{M}(Z, f) \neq \varnothing$.

For each $x \in X$ and $n \geq 0$ define a probability measure $\mu_{x,n}$ on X by

$$\mu_{x,n} = \frac{1}{n} \sum_{k=0}^{n-1} \delta_{f^k(x)},$$

where δ_x is the δ-measure supported on the point x. Denote by $V(x)$ the set of all weak-$*$ limit measures of the sequence of measures $\{\mu_{x,n}\}_{n \in \mathbb{N}}$. As X is compact, $\varnothing \neq V(x) \subset \mathcal{M}(X, f)$. Set

$$\mathcal{L}(Z) = \{x \in Z : V(x) \bigcap \mathcal{M}(Z, f) \neq \varnothing\}.$$

The set $\mathcal{L}(Z)$ is not empty when Z is compact and f-invariant; $\mathcal{L}(Z)$ may be empty if Z is either not compact or not invariant.

For dynamical system (X, f) and measure $\mu \in \mathcal{M}(X, f)$ the measure-theoretic (metric) entropy of f is denoted by $h_\mu(f)$. The measure-theoretic entropy of f with respect to the partition $\mathcal{E}$ is denoted $h_\mu(f, \mathcal{E})$. The entropy of μ with respect to the partition $\mathcal{E}$ is $H_\mu(\mathcal{E})$, and the conditional entropy of the partition $\mathcal{E}$ with respect to the partition $\mathcal{D}$ is denoted $H_\mu(\mathcal{E}|\mathcal{D})$.

A map f is called expansive if there exists an $\epsilon > 0$ so that for any points $x, y \in X$ with $\rho(f^k(x), f^k(y)) < \epsilon$ for all $k \in \mathbb{Z}$ then $x = y$.

Chapter 2

Introduction

The introduction begins with a brief overview of the area of the results of this thesis. Then a review of the classical thermodynamic formalism is given, as well as a description of more recent results relevant to this thesis. Finally, the definitions and results of this thesis are given.

2.1 Overview

The results of this thesis are in the area of thermodynamic formalism for dynamical systems. This field uses the methods of statistical mechanics to analyze the behavior of chaotic dynamical systems. A main ingredient of thermodynamic formalism is the variational principle, which is a theoretical extension of the principle that "nature minimizes free energy." This principle claims that the maximum (negative) free energy of the system associated with the potential ϕ is a topological invariant, called the topological pressure $P(\phi)$ of the potential. In other words,

$$P(\phi) = \sup \left\{ h_\mu(f) + \textstyle\int \phi \, d\mu \right\},$$

where the supremum is taken over all f-invariant Borel probability measures on X. Here $h_\mu(f)$ denotes the measure-theoretic entropy of f. The classical interpretation of topological pressure was given in the pioneering works of Ruelle [Rue78], Sinai [Sin72], and Bowen [Bow70].

The major goal of thermodynamic formalism is to generate equilibrium measures, those which achieve the supremum in the variational principle. Each equilibrium measure corresponds to a possible observable state of the system. The existence of two equilibrium measures implies that the system has a phase transition, which is an abrupt change in the the state of the system due to small continuous changes in system parameters. In particular, a system with a unique equilibrium measure has no phase transitions. Certain natural invariant measures can be generated as equilibrium measures. Examples include Sinai–Ruelle–Bowen (SRB) measures and absolutely continuous measures.

The classical thermodynamic formalism corresponds to dynamical systems which are uniformly hyperbolic, such as Anosov systems and, more generally, Axiom A systems. In the classical case it has been shown that if the potential is Hölder continuous then there exists a unique equilibrium measure.

The main goal of this thesis is to extend the thermodynamic formalism to dynamical systems possessing weaker forms of hyperbolicity, such as nonuniformly hyperbolic dynamical systems.

Little progress has been made towards establishing a thermodynamic formalism for nonuniformly hyperbolic dynamical systems. One obstacle is that traditionally the potential was required to be continuous, but many natural potential functions for nonuniformly hyperbolic dynamical systems are discontinuous. For example, SRB measures correspond to the potential $\phi(x) = -\log \operatorname{Jac}(df_{|E^u_x})$, which may be measurable but not continuous for dynamical systems such as unimodal maps, Hénon attractors, and geodesic flows on nonpositively curved manifolds. Thus discontinuous potentials require a new concept of topological pressure and a new setup for the variational principle, and new methods are needed to establish the existence and uniqueness of equilibrium measures.

The results in this thesis include two different approaches toward establishing a thermodynamic formalism for nonuniformly hyperbolic dynamical systems. Each approach requires certain assumptions on the potential function(s). One method is based on the nonadditive topological pressure for sequences of potential functions first examined by Barreira [Bar96]. The second method uses the existence of a nested family of subsets so that on the closure of each subset the potential is continuous.

In Section 2.2 we present known results in thermodynamic formalism which pertain to the results of this thesis. We then give the definitions and state the results of this thesis for both cases considered: the case of almost-additive sequences of potentials (Section 2.3) and the case of discontinuous potentials (Section 2.4). The proofs of results for the case of almost-additive sequences, as well as several examples, are given in Chapter 3. Chapter 4 contains the proofs of results for discontinuous potentials and examples. Common notation used throughout this thesis and a review of hyperbolicity are given in Chapter 1.

2.2 Known Results of Thermodynamic Formalism

Statistical mechanics examines physical systems with a large number of interacting particles. Physics tells us that the behavior of the system is governed by a Hamiltonian, which depends on the energy of the system. Unfortunately, due to the large number of particles, it is unfeasible to even attempt to solve the corresponding equations of motion. To study a system with such a large number of particles, one must not consider the motion of a single particle, but instead consider the fraction of particles exhibiting a particular property.

Mathematically, a probability distribution on the phase space of the system is required. The phase space is the set of all possible positions and velocities of all the particles. Then the fraction of particles exhibiting a certain behavior is the probability of that event.

The events that we are interested in are functions defined on the phase space. These functions are so called "thermodynamical" quantities, for example, temperature, pressure, and entropy. Because the complexity and number of interactions between the particles is so high, these thermodynamic quantities can be thought of as random variables (measurable functions on the phase space).

The systems which are studied exhibit thermodynamic behavior, meaning that the system will tend to an equilibrium state as time goes to infinity. An equilibrium state of a systems consists of one or more macroscopically homogeneous regions, called phases. Each phase is determined by its values of the thermodynamic quantities. Experimental evidence shows that these thermodynamic functions almost coincide with their means with respect to a suitable distribution.

The goal of statistical mechanics is to construct an appropriate distribution on the phase space for which the thermodynamic functions almost coincide with their mean.

Phase Space with One Site

Suppose that a physical system has one site with possible states $\{1, \dots, n\}$ and corresponding energies $E_1, \dots E_n$. This systems is put into contact with a heat source of temperature T, and heat is exchanged. The energy of the system is not fixed, and so can change. In equilibrium, the probability that the system is in state j is given by the *Gibbs distribution*

$$P_j = \frac{\exp(-\beta E_j)}{\sum_{i=1}^{n} \exp(-\beta E_i)} \ ,$$

where $\beta = 1/kT$, k a physical constant. The Gibbs distribution has the property that states with higher energy are less likely to occur.

The free energy of a system is the average energy of the system minus a constant times the entropy of the system,

$$\sum_{i=1}^{n} (-\beta E_i) P_i - (\beta T) \sum_{i=1}^{n} (-P_i \log P_i).$$

Using Calculus one can show that the Gibbs distribution maximizes the (negative) free energy, that is, "nature minimizes free energy."

One-Dimensional Lattice

Now consider a one-dimensional lattice with infinitely many sites. Each site has possible states $\{1, \ldots, n\}$. A configuration of the system is the assignment of one state to each lattice site. In other words, a configuration is any element of the two-sided sequence space of n elements, Σ_n.

Let $\underline{x} \in \Sigma_n$. The (potential) energy contribution due to $x_0 \in \{1, \ldots, n\}$ being in the zeroth place is

$$\phi(\underline{x}) = \phi_0(x_0) + \frac{1}{2} \sum_{j \neq 0} \phi_1(j; x_j, x_0),$$

where $\phi_0(k)$ is the energy due to the presence of state k, and ϕ_1 is the energy of the interaction between states. The energy of x_1 being in the first place is then $\phi(\sigma \underline{x})$, where σ is the shift map on the space of sequences Σ_n.

For a system with a finite number of sites, say $-m$ to m, let the energy of configuration $\underline{x}$ be E_m. The corresponding Gibbs distribution μ_m is proportional to $\exp(-\beta E_m(x_{-m} \cdots x_m))$. If the limit of these Gibbs distributions existed, then it would be natural to call the limit measure the Gibbs measure for Σ_n. If $V_k(\phi) \leq c\alpha^k$, then the limit of the energies for finite systems, $\lim_{m \to \infty} E_m$, will be up to a constant given by

$$\sum_{j=-\infty}^{\infty} \phi(\sigma^j \underline{x}).$$

Thus using $\sum_{j=-m}^{m} \phi(\sigma^j \underline{x})$ instead of E_m changes the limit measure only up to a constant.

For the one-dimensional lattice, the following general theorem holds for the existence of Gibbs measures.

Theorem (Bowen [Bow70]). Assume that $\varphi\colon \Sigma_n \to \mathbb{R}$ is continuous and there exist $c > 0$, $\alpha \in (0,1)$ so that $V_k(\varphi) \le c\alpha^k$ for every k. Then there is a unique shift-invariant measure μ on Σ_n for which one can find constants $M > 0$ and P such that

$$\frac{1}{M} \le \frac{\mu(\mathcal{C}_m)}{\exp(-Pm + \sum_{j=0}^{m-1} \varphi(\sigma^j \underline{x}))} \le M$$

for every $\underline{x} \in \mathcal{C}_m$ and $\mathcal{C}_m \in \Sigma_n$.

The measure μ is typically written as μ_φ and is called the *Gibbs measure* of φ. For physical systems, $\varphi = -\beta\phi$. The Gibbs measure assigns lower probability to those states with higher energy. The function $\varphi\colon \Sigma_n \to \mathbb{R}$ is called the potential, signifying its physical role as the potential energy of the system.

2.2.1 Continuous Potentials

Let (X, ρ) be a compact metric space, $f\colon X \to X$ a continuous map, and $\varphi\colon X \to \mathbb{R}$ a continuous function. Fix a finite open cover $\mathcal{U}$ of X. Consider the following collection of subsets of X:

$$\mathcal{F} = \mathcal{F}(\mathcal{U}) = \{X(\underline{U}) : \underline{U} \in W(\mathcal{U})\}$$

and the following three set functions $\xi, \eta, \psi : W(\mathcal{U}) \to \mathbb{R}$ defined as

$$\begin{aligned}
\xi(\underline{U}) &= \exp\left(\sup_{x \in X(\underline{U})} \sum_{k=0}^{m(\underline{U})-1} \varphi(f^k(x)) \right), \\
\eta(\underline{U}) &= \exp(-m(\underline{U})), \\
\psi(\underline{U}) &= m(\underline{U})^{-1}.
\end{aligned}$$

The set $W(\mathcal{U})$, the collection of subsets $\mathcal{F}$, and the three functions ξ, η, ψ generate a Carathéodory dimension structure $\tau = (W(\mathcal{U}), \mathcal{F}, \xi, \eta, \psi)$ on X. (See Pesin [Pes97] for a review of dimension theory.)

The theory of Carathéodory dimension structures implies that for any set $Z \subset X$ (Z not necessarily compact or f-invariant) and real number α

there are three Carathéodory functions M, $\underline{M}$, and $\overline{M}$. For every $Z \subset X$, we have

$$M(Z, \alpha, \varphi, \mathcal{U}) = \lim_{n\to\infty} \inf_{\Gamma} \sum_{\underline{U}\in\Gamma} \exp\left(-\alpha m(\underline{U}) + \sup_{x\in X(\underline{U})} \sum_{k=0}^{m(\underline{U})-1} \varphi(f^k(x)) \right),$$

where the infimum is taken over all $\Gamma \subset \bigcup_{k\geq n} W_k(\mathcal{U})$ that cover Z. Similarly, define

$$\underline{M}(Z, \alpha, \varphi, \mathcal{U}) = \liminf_{n\to\infty} \inf_{\Gamma} \sum_{\underline{U}\in\Gamma} \exp\left(-\alpha n + \sup_{x\in X(\underline{U})} \sum_{k=0}^{n-1} \varphi(f^k(x)) \right),$$
$$\overline{M}(Z, \alpha, \varphi, \mathcal{U}) = \limsup_{n\to\infty} \inf_{\Gamma} \sum_{\underline{U}\in\Gamma} \exp\left(-\alpha n + \sup_{x\in X(\underline{U})} \sum_{k=0}^{n-1} \varphi(f^k(x)) \right),$$

where the infima are each taken over all $\Gamma \subset W_n(\mathcal{U})$ that cover Z. In each of M, $\underline{M}$, and $\overline{M}$, if $X(\underline{U}) = \varnothing$ then set $\sup_{x\in X(\underline{U})} \varphi_{m(\underline{U})}(x) = -\infty$.

The general theory of Carathéodory dimension characteristics gives that each of the three equations above jumps from $+\infty$ to 0 at a unique critical value, which is possibly $-\infty$ or $+\infty$. Define the critical values as

$$P_Z(\varphi, \mathcal{U}) = \inf\{\alpha : M(Z, \alpha, \varphi, \mathcal{U}) = 0\} = \sup\{\alpha : M(Z, \alpha, \varphi, \mathcal{U}) = +\infty\},$$
$$\underline{CP}_Z(\varphi, \mathcal{U}) = \inf\{\alpha : \underline{M}(Z, \alpha, \varphi, \mathcal{U}) = 0\} = \sup\{\alpha : \underline{M}(Z, \alpha, \varphi, \mathcal{U}) = +\infty\},$$
$$\overline{CP}_Z(\varphi, \mathcal{U}) = \inf\{\alpha : \overline{M}(Z, \alpha, \varphi, \mathcal{U}) = 0\} = \sup\{\alpha : \overline{M}(Z, \alpha, \varphi, \mathcal{U}) = +\infty\}.$$

Theorem (Pesin [Pes97]). The following limits exist:

$$P_Z(\varphi) = \lim_{\operatorname{diam}(\mathcal{U})\to 0} P_Z(\varphi, \mathcal{U}),$$
$$\underline{CP}_Z(\varphi) = \lim_{\operatorname{diam}(\mathcal{U})\to 0} \underline{CP}_Z(\varphi, \mathcal{U}),$$
$$\overline{CP}_Z(\varphi) = \lim_{\operatorname{diam}(\mathcal{U})\to 0} \overline{CP}_Z(\varphi, \mathcal{U}).$$

The *topological pressure* of a continuous potential φ on any set $Z \subset X$ is given by $P_Z(\varphi)$. The values $\underline{CP}_Z(\varphi)$ and $\overline{CP}_Z(\varphi)$ are the *lower* and *upper capacity pressure* of φ on Z, respectively.

If Z is f-invariant, then $\underline{CP}_Z(\varphi) = \overline{CP}_Z(\varphi)$. If Z is both compact and f-invariant, then $P_Z(\varphi) = \underline{CP}_Z(\varphi) = \overline{CP}_Z(\varphi)$.

The topological pressure has the following variational principle.

Theorem (Pesin [Pes97]). Let (X, ρ) be a compact metric space, and $f\colon X \to X$ continuous. Assume that $\varphi\colon X \to \mathbb{R}$ is continuous. Let $Z \subset X$ be any set with $\mathcal{L}(Z) \neq \varnothing$. Then

$$P_{\mathcal{L}(Z)}(\varphi) = \sup\left\{h_\mu(f) + \int_Z \varphi d\mu : \mu \in \mathcal{M}(Z, f)\right\}.$$

A measure $\mu_\varphi \in \mathcal{M}(X, f)$ is an *equilibrium measure* for φ if the equation holds:

$$h_{\mu_\varphi}(f) + \int \varphi\, d\mu_\varphi = \sup\left\{h_\mu(f) + \int \varphi\, d\mu : \mu \in \mathcal{M}(X, f)\right\}.$$

Let f be expansive. Then the map $\mu \mapsto h_\mu(f) + \int \varphi d\mu$ is upper semicontinuous. Since an upper semicontinuous map achieves its supremum on a compact space, the following result for the existence of equilibrium measures holds.

Theorem (Pesin [Pes97]). Let (X, ρ) be a compact metric space, and let $f\colon X \to X$ be continuous. Assume that $\varphi\colon X \to \mathbb{R}$ be continuous. Assume that f is an expansive homeomorphism of X and that the set $\mathcal{M}(Z, f)$ is closed in $\mathcal{M}(X, f)$ in the weak-$*$ topology. Then for any continuous function φ there exists an equilibrium measure.

The existence of equilibrium measures is a property of the dynamical system f and does not require any assumptions on the potential φ. Properties of φ become important for showing uniqueness of equilibrium measures.

Let (X, f) be topologically conjugate to a mixing subshift of finite type. Then there is a unique equilibrium measure for each Hölder continuous φ. In particular, basic sets of Axiom A diffeomorphisms have a unique equilibrium measure corresponding to each Hölder continuous φ.

Theorem (Bowen [Bow70]). Let Ω_s be a basic set for an Axiom A diffeomorphism f and let $\varphi : \Omega_s \to \mathbb{R}$ be Hölder continuous. Then φ has a unique equilibrium measure μ_φ.

Compact Invariant Sets and Continuous Potentials

The expression for the topological pressure can be simplified for compact f-invariant sets.

Let (X, ρ) be a compact metric space, $f\colon X \to X$ a continuous map, and $\varphi\colon X \to \mathbb{R}$ a continuous functions. Fix a finite open cover $\mathcal{U}$ of X. Let

$Z \subset X$ be a compact f-invariant set. Define the partition function

$$Z_m(Z, \varphi, \mathcal{U}) = \inf_{\Gamma} \sum_{\underline{U} \in \Gamma} \exp \sup_{x \in X(\underline{U})} \sum_{k=0}^{m(\underline{U})-1} \varphi(f^k(x)),$$

where the infimum is taken over all $\Gamma \subset W_m(\mathcal{U})$ covering Z. If $X(\underline{U}) = \varnothing$, then set $\sum_{k=0}^{m(\underline{U})-1} \varphi(f^k x) = -\infty$. The *topological pressure* of φ on Z is

$$P_Z(\varphi) = \lim_{\mathrm{diam}(\mathcal{U}) \to 0} \lim_{m \to \infty} \frac{1}{m} \log Z_m(Z, \varphi, \mathcal{U}).$$

For a compact f-invariant set Z we have $\mathcal{L}(Z) = Z$. Thus the *variational principle*

$$P_Z(\varphi) = \sup \left\{ h_\mu(f) + \int_Z \varphi \, d\mu : \mu \in \mathcal{M}(Z, f) \right\}$$

holds.

Subshifts of Finite Type and Continuous Potentials

Subshifts of finite type are of interest in thermodynamic formalism since Gibbs measures exist for one- or two-sided subshifts of finite type, as well as for the full two-sided sequence space (the one-dimensional lattice). For subshifts of finite type the value of P in the definition of Gibbs measure is the topological pressure of the potential φ.

The expression for the topological pressure can be simplified for subshifts of finite type. Let (Σ_A, σ) be a subshift of finite type, and $\varphi \colon \Sigma_A \to \mathbb{R}$ continuous. The partition function simplifies to

$$Z_m(\varphi) = \sum_{\mathcal{C}_m \in \Sigma_A} \exp \left(\sup_{x \in \mathcal{C}_m} \sum_{k=0}^{m-1} \varphi(\sigma^k x) \right).$$

The *topological pressure* of φ on Σ_A is

$$P_{\Sigma_A}(\varphi) = \lim_{m \to \infty} \frac{1}{m} \log Z_m(\varphi).$$

The topological pressure can also be computed using a sum over the periodic points, that is with partition function

$$Z'_m(\varphi) = \sum_{x \,:\, \sigma^n x = x} \exp \left(\sum_{k=0}^{m-1} \varphi(\sigma^k x) \right)$$

replacing $Z_m(\varphi)$.

A measure $\mu \in \mathcal{M}(\Sigma_A, \sigma)$ is called a *Gibbs measure* for φ if there exist constants $M > 0$ and P such that

$$\frac{1}{M} \leq \frac{\mu(\mathcal{C}_n)}{\exp\left(-Pn + \sum_{k=0}^{n-1} \varphi(\sigma^k x)\right)} \leq M,$$

for every n, cylinder set $\mathcal{C}_n$, and $x \in \mathcal{C}_n$.

Theorem (Bowen [Bow70]). Let (Σ_A, σ) be a mixing subshift of finite type. Let $\varphi\colon \Sigma_A \to \mathbb{R}$ be continuous with $V_k(\varphi) < c\alpha^k$, for some $c > 0$ and $\alpha \in (0,1)$. Then there exists a unique Gibbs measure $\mu_\varphi \in \mathcal{M}(\Sigma_A, \sigma)$. Moreover, the Gibbs measure μ_φ is the unique equilibrium measure for φ.

Countable Markov Shifts

The thermodynamic formalism for subshifts of finite type is useful since many dynamical systems are topologically conjugate to a subshift of finite type. However, many dynamical systems are not topologically conjugate to a subshift of finite type and the results do not extend. There are dynamical systems which are conjugate to countable Markov shifts. Thus the thermodynamic formalism for countable Markov shifts has been studied.

Let S be a countable set and let $A = (a_{ij})_{S \times S}$ be the transition matrix. Let M be the set

$$M = \{x = x_0 x_1 \cdots \in S^{\mathbb{N}} : a_{x_i x_{i+1}} = 1 \text{ for all } i \geq 0\}.$$

Define the shift map $\sigma\colon M \to M$ as $(\sigma(x))_i = x_{i+1}$. Assume that (M, σ) is topologically mixing.

Let $\varphi\colon M \to \mathbb{R}$. The function φ has *summable variations* if the inequality holds

$$\sum_{n=1}^{\infty} V_n(\varphi) < \infty.$$

In [Sar99], summable variations has the sum beginning with $n = 2$; by recoding the Markov shift, one can assume that the sum begins with $n = 1$.

If $\varphi\colon M \to \mathbb{R}$ has summable variations then the *Gurevich pressure* of φ is

$$P_G(\varphi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{x\,:\,\sigma^n x = x} \exp\left(\sum_{k=0}^{n-1} \varphi(\sigma^k(x))\right) 1_{[a]}(x),$$

where $1_{[a]}$ is the indicator function on the cylinder set $[a]$; the limit exists and is independent of $a \in S$. The above definition of the Gurevich pressure was given in [Sar99]. The first pressure for countable Markov chains was given in [GS98].

The following connection between the Gurevich pressure and the topological pressure holds.

$$
\begin{aligned}
P_G(\varphi) &= \sup\{P_{top}(\varphi_{|Y}) : Y \subset M \text{ a topologically mixing finite Markov shift}\} \\
&= \sup\{P_{top}(\varphi_{|K}) : K \subset M \text{ compact and } T^{-1}K = K\}.
\end{aligned}
$$

The following variational principle holds for the Gurevich pressure.

Theorem (Sarig [Sar99]). Assume that (M, σ) is a topologically mixing countable Markov shift. Let $\varphi \colon M \to \mathbb{R}$ have summable variations and assume $\sup \varphi < \infty$. Then

$$
P_G(\varphi) = \sup\left\{ h_\mu(\sigma) + \int \varphi \, d\mu : \mu \in \mathcal{M}(X, \sigma) \text{ and } -\int \varphi \, d\mu < \infty \right\}.
$$

Using a slight modification of the proof, one can show the variational principle with the requirement that φ is integrable, without the requirement that $\sup \varphi < \infty$.

A measure $\mu \in \mathcal{M}(M, \sigma)$ is called a *Gibbs measure* for φ if there exists $B > 0$ and P such that

$$
\frac{1}{B} \le \frac{\mu(\mathcal{C}_n)}{\exp(-nP + \sum_{k=0}^{n-1} f^k(x))} \le B
$$

for every n-cylinder set $\mathcal{C}_n$ and every $x \in \mathcal{C}_n$.

The countable Markov shift M satisfies the *BIP property* if there are $b_1, \ldots, b_N \in S$ such that for all $a \in S$ there exist i, j such that $t_{b_i a} t_{a b_j} = 1$.

Theorem (Sarig [Sar99]). Let (M, σ) be a topologically mixing countable Markov shift and let φ have summable variations and finite Gurevich pressure, that is, $P_G(\varphi) < \infty$. Then a Hölder continuous φ has an invariant Gibbs measure if and only if M has the BIP property.

For subshifts of finite type, Gibbs and equilibrium measures coincide. In the countable Markov shift case, Gibbs and equilibrium measures are not guaranteed to be the same.

Theorem (Sarig [Sar99]). If μ_φ is the Gibbs measure for Hölder continuous potential φ and $h_{\mu_\varphi}(f) < \infty$, then μ_φ is the also unique equilibrium measure for φ.

2.2.2 Nonadditive Sequences of Continuous Potentials

To study the dimension of Cantor-like sets with complicated geometric structure, Barreira introduced the nonadditive thermodynamic formalism [Bar96]. This is the study of the thermodynamic formalism of nonadditive sequences of potentials. A sequence $\Phi = \{\varphi_n : X \to \mathbb{R}\}$ is *additive* if $\varphi_n(x) = \sum_{k=0}^{n-1} \varphi(f^k x)$, otherwise Φ is *nonadditive*. The thermodynamic formalism for a single potential function is an additive thermodynamic formalism. (See Section 2.2.1.)

The relation between dimension theory and thermodynamic formalism was first illuminated by work of Bowen. He showed that the Hausdorff dimension of quasicircles is the unique root of the equation $P(s\varphi) = 0$ [Bow79]. Bowen's equation holds for more general situations. In every case, however, the description of the set must be regular enough to be encoded by a single function φ. For more general constructions, one function φ will not code the description of the set. Under more general requirements a sequence of functions, $\Phi = \{\varphi_n\}_{n=1}^{\infty}$ can be used to describe a set. Barreira defined the topological pressure for a sequence of functions Φ under a mild condition. Then he showed Bowen-type equations holds, which yield estimates on the dimension of the sets coded by Φ as the unique roots of $P(s\Phi) = 0$, $\underline{CP}(s\Phi)$, and $\overline{CP}(s\Phi)$ [Bar96].

Let X be a compact metric space, and $f\colon X \to X$ a continuous map. Consider a sequence of functions $\Phi = \{\varphi_n : X \to \mathbb{R}\}_{n=0}^{\infty}$. For every $Z \subset X$ define

$$M(Z, \alpha, \Phi, \mathcal{U}) = \lim_{n\to\infty} \inf_{\Gamma} \sum_{\underline{U}\in\Gamma} \exp\left(-\alpha m(\underline{U}) + \sup_{x\in X(\underline{U})} \varphi_{m(\underline{U})}(x)\right),$$

where the infimum is taken over all $\Gamma \subset \bigcup_{k\geq n} W_k(\mathcal{U})$ that cover Z. Similarly, define

$$\underline{M}(Z, \alpha, \Phi, \mathcal{U}) = \liminf_{n\to\infty} \inf_{\Gamma} \sum_{\underline{U}\in\Gamma} \exp\left(-\alpha n + \sup_{x\in X(\underline{U})} \varphi_n(x)\right),$$

$$\overline{M}(Z, \alpha, \Phi, \mathcal{U}) = \limsup_{n\to\infty} \inf_{\Gamma} \sum_{\underline{U}\in\Gamma} \exp\left(-\alpha n + \sup_{x\in X(\underline{U})} \varphi_n(x)\right),$$

where the infima are each taken over all $\Gamma \subset W_n(\mathcal{U})$ that cover Z. In each of M, $\underline{M}$, and $\overline{M}$, if $X(\underline{U}) = \varnothing$, then set $\sup_{x\in X(\underline{U})} \varphi_n(x) = -\infty$.

The general theory of Carathéodory dimension characteristics implies that each of the three equations above jumps from $+\infty$ to 0 at a unique critical value, which may be $-\infty$ or $+\infty$. Define the critical values of M, $\underline{M}$, and $\overline{M}$ as

$$\begin{aligned}
P_Z(\Phi,\mathcal{U}) &= \inf\{\alpha : M(Z,\alpha,\Phi,\mathcal{U}) = 0\} = \sup\{\alpha : M(Z,\alpha,\Phi,\mathcal{U}) = +\infty\},\\
\underline{CP}_Z(\Phi,\mathcal{U}) &= \inf\{\alpha : \underline{M}(Z,\alpha,\Phi,\mathcal{U}) = 0\} = \sup\{\alpha : \underline{M}(Z,\alpha,\Phi,\mathcal{U}) = +\infty\},\\
\overline{CP}_Z(\Phi,\mathcal{U}) &= \inf\{\alpha : \overline{M}(Z,\alpha,\Phi,\mathcal{U}) = 0\} = \sup\{\alpha : \overline{M}(Z,\alpha,\Phi,\mathcal{U}) = +\infty\}.
\end{aligned}$$

Set $\gamma_n(\Phi,\mathcal{U}) = V_n(\varphi_n)$. Thus

$$\gamma_n(\Phi,\mathcal{U}) = \sup\{|\varphi_n(x) - \varphi_n(y)| : x, y \in X(\underline{U}),\ m(\underline{U}) = n\}.$$

Theorem (Barreira [Bar96]). Let Φ be a sequence of functions satisfying

$$\lim_{\operatorname{diam}(\mathcal{U})\to 0} \limsup_{n\to\infty} \frac{\gamma_n(\Phi,\mathcal{U})}{n} = 0. \tag{2.2.1}$$

The following limits exist

$$\begin{aligned}
P_Z(\Phi) &= \lim_{\operatorname{diam}(\mathcal{U})\to 0} P_Z(\Phi,\mathcal{U}),\\
\underline{CP}_Z(\Phi) &= \lim_{\operatorname{diam}(\mathcal{U})\to 0} \underline{CP}_Z(\Phi,\mathcal{U}),\\
\overline{CP}_Z(\Phi) &= \lim_{\operatorname{diam}(\mathcal{U})\to 0} \overline{CP}_Z(\Phi,\mathcal{U}).
\end{aligned}$$

The *topological pressure* of a sequence Φ satisfying (2.2.1) on any set $Z \subset X$ is given by $P_Z(\Phi)$. The values $\underline{CP}_Z(\Phi)$ and $\overline{CP}_Z(\Phi)$ are the *lower* and *upper capacity pressure* of Φ on Z, respectively. The following variational principle holds.

Theorem (Barreira [Bar96]). Assume that Φ is a sequence of continuous functions satisfying (2.2.1), and there exists a continuous function $\psi\colon X \to \mathbb{R}$ such that

$$\varphi_{n+1} - \varphi_n \circ f \to \psi$$

uniformly on Z as $n \to \infty$. Then

$$P_{\mathcal{L}(Z)}(\Phi) = \sup\left\{h_\mu(f) + \int_Z \psi d\mu : \mu \in \mathcal{M}(Z,f)\right\}.$$

Subadditive Sequences of Potentials on Mixing Repellers

The first study of the nonadditive thermodynamic formalism was conducted by Falconer [Fal88]. He was interested in thermodynamic formalism because of its relation to the study of multifractal formalism for nonconformal dynamical systems.

For a dynamical system (X, f), any subset $Z \subset X$ is said to have *multifractal structure* if there exists a decomposition of Z in the following sense. Let $h : X \to \mathbb{R}$. Then let $Z_a = \{z \in Z : h(z) = a\}$, and $\hat{Z}$ be the set of $z \in Z$ where h is undefined. The *multifractal decomposition* is

$$Z = \hat{Z} \cup \left(\bigcup_a Z_a \right).$$

As the name implies, sets Z_a are often fractals.

Before the work of Falconer, the thermodynamic formalism for fractals, and in particular a Bowen's equation for fractals, was shown only for conformal maps. Repellers of conformal maps can be encoded by a single function; thus their dimension can be studied. Mixing repellers of certain nonconformal maps can be encoded by a subadditive sequence. A sequence $\Phi = \{\varphi : X \to \mathbb{R}\}_{n=0}^{\infty}$ is *subadditive* if $\varphi_{m+n}(x) \le \varphi_m(x) + \varphi_n(f^n x)$ for every $x \in X$. Falconer was interested in a Bowen-type equation for nonconformal mixing repellers, and so gave a thermodynamic formalism for subadditive sequences on mixing repellers.

Let $f : \mathcal{R} \to \mathcal{R}$ be a $C^{1+\epsilon}$ expanding map of a Riemannian manifold. Let $J \subset \mathcal{R}$ be an f-invariant compact set. Such a J is called a mixing repeller for f. A mixing repeller has a Markov partition, which allows one to define a topological conjugacy between (J, f) and a mixing subshift of finite type (Σ_A, σ).

Consider (Σ_A, σ), where Σ_A is a mixing subshift of finite type with transition matrix A. Let $\Phi = \{\varphi_n : \Sigma_A \to \mathbb{R}\}_{n=0}^{\infty}$ be a subadditive sequence of functions. Assume that the following conditions hold:

1. a uniform bound: $|(1/n)\varphi_n(x)| \le M$,
2. a Lipschitz condition: $|(1/n)\varphi_n(x) - (1/n)\varphi_n(y)| \le a|x - y|$, and
3. bounded variation: there exists a constant b independent of n such that $V_n(\varphi_n) \le b$.

The *topological pressure* of Φ on Σ_A is

$$P_{\Sigma_A}(\Phi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{C_n} \exp \sup_{x \in C_n} \varphi_n(x).$$

Set $\mathcal{M} = \bigcup_{N \ge 1} \mathcal{M}(J, f^N)$.

Theorem (Falconer [Fal88]). The following variational principle holds:

$$P_{\Sigma_A}(\Phi) = \sup \left\{ h_\mu(f) + \lim_{n\to\infty} \frac{1}{n} \int \varphi_n \, d\mu : \mu \in \mathcal{M} \right\}.$$

The variational principle for subadditive sequences is established using the equality

$$P_{\Sigma_A}(\Phi) = \lim_{N\to\infty} \frac{1}{N} P_{f^N,\Sigma_A}(\varphi_N),$$

and the fact that each $P_{f^N,\Sigma_A}(\varphi_N)$ has a variational principle.

Sequences of Potentials Generated by Iterated Function Systems with Overlaps

Our main motivation for studying the thermodynamic formalism for almost-additive sequences (Section 2.3 and Chapter 3) comes from the work of Feng and Lau [FL02, Fen04]. They created a thermodynamic formalism for a specific nonadditive sequence of functions generated from the study of iterated function systems with overlaps. The major contribution of Feng and Lau was the construction of a unique Gibbs (and equilibrium) measure for the sequence that they studied.

A family of contractive maps $\{S_j\}_{j=1}^m$ on $\mathbb{R}^d$ is an *iterated function system*, or an IFS. An IFS generates an invariant compact subset $K = \bigcup_{j=1}^m S_j K$. If each image under S_j has disjoint interior from the rest, then the IFS is said to be without overlaps, otherwise it is said to be with overlap.

If a set of probability weights $\{a_j\}_{j=1}^m$ is associated to the system, then there is an invariant measure on K given by

$$\mu = \sum_{j=1}^m a_j \mu \circ S_j^{-1}. \tag{2.2.2}$$

Such a measure is called *self-similar*. There has been much study into these self-similar measures. For any set of weights there exists a unique self-similar Borel regular probability measure μ_K on the set K. The measure μ_F is known to be either singular or absolutely continuous with respect to the Lebesgue measure.

A *similitude* is a contraction map of the form $f(x) = \rho x + b$, where ρ is the contraction rate. For two similitudes with the same contraction rate $\rho < \frac{1}{2}$, the set K is a Cantor set. Thus the self-similar measure is singular. For two similitudes with the same contraction rate $\rho > \frac{1}{2}$, the question of whether the self-similar measure is singular or absolutely continuous has not been fully answered. One result for $\rho > \frac{1}{2}$ gives a necessary condition for singularity. A number s is a Pisot number if it is an algebraic integer and all of its conjugates have modulus less than one; for example, the golden ratio $(\sqrt{5}+1)/2$ is a Pisot number. When ρ is the reciprocal of a Pisot number, then the corresponding measure is singular.

Strichartz, Taylor, and Zhang [STZ95] give a method for overcoming the difficulty of IFS with overlaps by "splitting" intervals which overlap into intervals which do not. The splitting of the overlapping intervals produces a splitting of the corresponding measure; the measure is split in proportion to the length of the split subintervals. A new IFS without overlaps can be generated which produces the same compact invariant set K, and the self-similar measure associated with the split is exactly the self-similar measure of the new IFS. The corresponding equation (2.2.2) for the self-similar measure of the new IFS is a set of matrix equations.

We offer an example to clarify the splitting procedure, the new IFS, and the set of matrix equations. Consider the IFS on $[0,1]$ generated by the two similitudes $S_1(x) = \rho x$ and $S_2(x) = \rho x + (1-\rho)$, where $\rho = (\sqrt{5}-1)/2$ and weights on each map in the self-similar measure are $a_i = \frac{1}{2}$, $i = 1, 2$. As $\rho > 1/2$, the images $S_1([0,1])$ and $S_2([0,1])$ overlap. Consider the new IFS generated by compositions of S_1, S_2 given by the similitudes

$$\begin{aligned} T_0(x) =& S_1 S_1(x) = \rho x, \\ T_1(x) =& S_1 S_2 S_2(x) = S_2 S_1 S_1(x) = \rho^3 x + \rho^2, \\ T_2(x) =& S_2 S_2(x) = \rho^2 x + \rho. \end{aligned}$$

The images $T_0([0,1]) = [0, \rho^2]$, $T_1([0,1]) = [\rho^2, \rho]$, and $T_2([0,1]) = [\rho, 1]$ are intervals with disjoint interiors. The self-similar equation (2.2.2) gives three matrix equations. For $A \subset [0,1]$,

$$\begin{bmatrix} \mu(T_0 T_i A) \\ \mu(T_1 T_i A) \\ \mu(T_2 T_i A) \end{bmatrix} = M_i \begin{bmatrix} \mu(T_0 A) \\ \mu(T_1 A) \\ \mu(T_2 A) \end{bmatrix}, \quad i = 0, 1, 2,$$

where

$$M_0 = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ \frac{1}{8} & \frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 \end{bmatrix}, \quad M_1 = \begin{bmatrix} 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{4} & \frac{1}{8} \\ 0 & 0 & \frac{1}{4} \end{bmatrix}.$$

Thus the study of self-similar measures for IFS with overlaps becomes the study of IFS without overlaps and with the self-similar equation (2.2.2) a set of matrix equations. To study the multifractal formalism for such IFS, Feng and Lau created a thermodynamic formalism for a certain sequence of potential functions involving matrices.

Consider (Σ_A, σ), where Σ_A is a mixing subshift of finite type with primitive transition matrix A. Let M be a Hölder continuous function on Σ_A taking values in the set of all positive $d \times d$ matrices. Define the multiplicative norm $||M|| = 1^t M 1$, where 1 is the $d \times 1$ vector of all ones. Consider the sequence $\Phi = \{\varphi_n : \Sigma_A \to \mathbb{R}\}_{n=1}^{\infty}$ where

$$\varphi_n(x) = \log ||M(x) \dots M(\sigma^{n-1}(x))||$$

The *topological pressure* of Φ on Σ_A is

$$P_{\Sigma_A}(\Phi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\mathcal{C}_n \in \Sigma_A} \sup_{x \in \mathcal{C}_n} ||M(X)M(\sigma x) \dots M(\sigma^{n-1}(x))||.$$

The topological pressure has a corresponding variational principle.

Theorem (Feng [Fen04]). Assume that M is a continuous function taking values in the nonnegative matrices. Then

$$P_{\Sigma_A}(\Phi) = \sup \left\{ h_\mu(\sigma) + \lim_{n\to\infty} \frac{1}{n} \int \varphi_n \, d\mu : \mu \in \mathcal{M}(\Sigma_A, \sigma) \right\}.$$

A measure $\mu_\Phi \in \mathcal{M}(\Sigma_A, \sigma)$ is an *equilibrium measure* for Φ if

$$h_{\mu_\Phi}(\sigma) + \lim_{n\to\infty} \frac{1}{n} \int \varphi_n \, d\mu_\Phi = \sup \left\{ h_\mu(\sigma) + \lim_{n\to\infty} \frac{1}{n} \int \varphi_n \, d\mu : \mu \in \mathcal{M}(\Sigma_A, \sigma) \right\}.$$

A measure $\nu \in \mathcal{M}(\Sigma_A, \sigma)$ is a *Gibbs measure* for Φ if there exist B, P such that

$$\frac{1}{B} \le \frac{\nu(\mathcal{C}_n)}{\exp(-nP + \varphi_n(x))} \le B.$$

Theorem (Feng and Lau [FL02]). Assume that M is Hölder continuous and takes values in the positive matrices. Then there is a unique Gibbs measure $\mu_\Phi \in \mathcal{M}(\Sigma_A, \sigma)$ for the sequence Φ. Furthermore, the measure μ_Φ is the unique equilibrium measure for Φ.

2.3 New Results for Almost-Additive Sequences of Continuous Potentials

The results on the thermodynamic formalism for almost-additive sequences of continuous potentials can be found in [Mum06a].

Feng and Lau developed a thermodynamic formalism for a sequence of functions on a subshift of finite type in the context of multifractal formalism associated to iterated function systems with overlaps ([FL02, Fen04]). These sequences of functions are not additive and fall into the category of almost-additive sequences. Conditions are given under which the thermodynamic formalism of Feng and Lau can be generalized to the class of almost-additive sequences on an arbitrary compact metric space, with respect to a continuous map f.

Let (X, ρ) be a compact metric space, $f\colon X \to X$ a continuous measure preserving transformation on X, and $\Phi = \{\varphi_n\colon X \to \mathbb{R}\}_{n=1}^{\infty}$ a sequence of continuous functions.

The sequence Φ is called *almost-additive* if there exist two functions $c_1, c_2 : \mathbb{N} \to \mathbb{R}$ such that for every $m, n \geq 1$, and $x \in X$

1. $c_1(n) \leq c_2(n)$,
2. there exist constants C_1, C_2 such that $C_1 \leq c_1(n)$ and $c_2(n) \leq C_2$, and
3. $c_1(n+m) + \varphi_n(x) + \varphi_m(f^n(x)) \leq \varphi_{n+m}(x)$
$\leq \varphi_n(x) + \varphi_m(f^n(x)) + c_2(n+m).$

In Section 3.2, we examine some properties of almost-additive sequences. We note that an almost-additive sequence may not be close to any additive sequence (see Examples 3.6.2 and 3.6.3 in Section 3.6).

Several examples of almost-additive sequences are given in Section 3.6. The thermodynamic formalism of the example sequences has been studied (see Ruelle [Rue78], Sinai [Sin72], Bowen [Bow70], Barreira [Bar96], Falconer [Fal88], Feng and Lau [FL02], [Fen04]). We compare these previous results to our thermodynamic formalism.

Topological Pressure

Set
$$Z_n(\Phi, Z, \mathcal{U}) = \inf_{\Gamma} \sum_{\underline{U} \in \Gamma} \exp \sup_{x \in X(\underline{U})} \varphi_n(x),$$
where the infimum is taken over all $\Gamma \subset W_n(\mathcal{U})$ covering Z.

The *topological pressure* of an almost-additive sequence Φ, on a compact f-invariant set $Z \subset X$, is given by
$$P_Z(\Phi) = \lim_{\operatorname{diam}(\mathcal{U}) \to 0} \lim_{n \to \infty} \frac{1}{n} Z_n(\Phi, Z, \mathcal{U}).$$

If the almost-additive sequence of functions Φ satisfies (2.2.1), that is

$$\lim_{\operatorname{diam}(\mathcal{U})\to 0} \limsup_{n\to\infty} \frac{\gamma_n(\Phi,\mathcal{U})}{n} = 0\,,$$

then the limits in the definition of the topological pressure exist. In Section 3.3, the topological pressure for almost-additive sequences on compact invariant sets is shown to be the nonadditive topological pressure.

Variational Principle

The variational principle is studied in Section 3.4. The sets Z under consideration are compact and f-invariant; thus without loss of generality we assume that $X = Z$ and write $P(\Phi) = P_Z(\Phi)$. The following general estimate of the topological pressure is obtained for any almost-additive sequence of functions.

Theorem. Suppose that Φ is an almost-additive sequence of functions satisfying property (2.2.1). Then

$$P(\Phi) \geq \sup\left\{ h_\mu(f) + \lim_{n\to\infty} \frac{1}{n} \int \varphi_n \, d\mu : \mu \in \mathcal{M}(X,f) \right\}.$$

To obtain the inverse inequality, and in particular the variational principle, the sequence of functions must satisfy an additional condition.

Theorem. Suppose that Φ is an almost-additive sequence of functions satisfying property (2.2.1), and the functions $c_1(n)$ and $c_2(n)$ satisfy

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} c_1(n-k) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} c_2(n-k). \tag{2.3.1}$$

Then

$$P(\Phi) = \sup\left\{ h_\mu(f) + \lim_{n\to\infty} \frac{1}{n} \int \varphi_n \, d\mu : \mu \in \mathcal{M}(X,f) \right\}.$$

Condition (2.3.1) is a strong requirement on the sequence Φ. It is satisfied by some almost-additive sequences but not by the sequence studied by Feng and Lau in [FL02] and [Fen04]. For these sequences we provide another condition which guarantees the variational principle holds. This second requirement is a Hölder-type requirement and it guarantees that the variational principle holds when the system is a subshift of finite type.

Theorem. Suppose that (Σ_A, σ) is a mixing subshift of finite type and Φ is an almost-additive sequence satisfying (2.2.1). Also, suppose that there exists γ such that for every $n \geq 1$ and $x \in X(\underline{U})$,

$$\varphi_n(x) \leq \sup_{z \in X(\underline{U})} \varphi_n(z) \leq \gamma + \varphi_n(x). \tag{2.3.2}$$

Then

$$P(\Phi) = \sup \left\{ h_\mu(\sigma) + \lim_{n\to\infty} \frac{1}{n} \int \varphi_n \, d\mu : \mu \in \mathcal{M}(\Sigma_A, \sigma) \right\}.$$

For an additive sequence, if φ_1 is Hölder continuous, then condition (2.3.2) holds. For almost-additive sequences, that are not additive, condition (2.3.2) holds if there are positive constants b and $\alpha \in (0, 1)$ such that $\gamma_n(\Phi, \mathcal{U}) \leq b\alpha^n$ for every n. We note that this last condition is satisfied by sequences for which each φ_n is α-Hölder continuous with common constant C, as required for the sequences studied by Feng and Lau [FL02, Fen04].

Remark. Recent work of Barreira [Bar06] implies that condition (2.3.1) is not required for the variational principle to hold.

Equilibrium Measures

A measure $\mu_\Phi \in \mathcal{M}(X, f)$ is an *equilibrium measure* associated with Φ if

$$h_{\mu_\Phi}(f) + \lim_{n\to\infty} \frac{1}{n} \int \varphi_n \, d\mu_\Phi = \sup \left\{ h_\mu(f) + \lim_{n\to\infty} \frac{1}{n} \int \varphi_n \, d\mu : \mu \in \mathcal{M}(X, f) \right\}.$$

The existence and uniqueness of equilibrium measures is explored in Section 3.5. The following result on existence of such measures holds without requiring the topological pressure to exist.

Theorem. Suppose that f is an expansive homeomorphism of X. Then for any almost-additive sequence of functions Φ there exists an equilibrium measure μ_Φ on X.

Corollary. Let Φ be an almost-additive sequence satisfying (2.2.1) and (2.3.1), and that f is an expansive homeomorphism of X. Then there exists a measure μ_Φ on X such that

$$P(\Phi) = h_{\mu_\Phi}(f) + \lim_{n\to\infty} \frac{1}{n} \int \varphi_n \, d\mu_\Phi. \tag{2.3.3}$$

Some authors define an equilibrium measure to be a measure μ_Φ satisfying (2.3.3).

A measure $\mu \in \mathcal{M}(\Sigma_A, f)$ on Σ_A is a *Gibbs measure* for Φ if there exist constants $A_1, A_2 > 0$ such that

$$A_1 \leq \frac{\mu(\mathcal{C}_n)}{\exp(-nP(\Phi) + \varphi_n(x))} \leq A_2$$

for any $n > 0, \mathcal{C}_n \subset \Sigma_A$ and $x \in \mathcal{C}_n$.

In Section 3.5 the following results on existence and uniqueness of Gibbs and equilibrium measures for a mixing subshift of finite type (Σ_A, σ) are obtained.

Theorem. Suppose that Φ is an almost-additive sequences of functions on a mixing subshift of finite type (Σ_A, σ) satisfying (2.2.1) and (2.3.2). Then there exists a unique Gibbs measure μ_Φ on X. Moreover, the measure μ_Φ is the unique equilibrium measure for Φ.

An Axiom A diffeomorphism $f \colon X \to X$ is expansive on its hyperbolic set. Thus there exists at least one equilibrium measure for an almost-additive sequence Φ. The following uniqueness theorem is shown in Section 3.5.

Theorem. Suppose that Ω is a basic set for an Axiom A diffeomorphism f and Φ is an almost-additive sequence of functions satisfying conditions (2.2.1) and (2.3.2). Then there exists a unique equilibrium measure μ_Φ for Φ on Ω.

In the course of proving the above theorem, the variational principle for Axiom A diffeomorphisms is established without requiring Φ to satisfy (2.3.1).

Theorem. Suppose that Ω is a basic set for an Axiom A diffeomorphism f and Φ an almost-additive sequence of functions satisfying conditions (2.2.1) and (2.3.2). Then

$$P_\Omega(\Phi) = \sup \left\{ h_\mu(\sigma) + \lim_{n\to\infty} \frac{1}{n} \int \varphi_n d\mu : \mu \in \mathcal{M}(\Omega, f) \right\}.$$

2.4 New Results for Discontinuous Potentials

The results on the thermodynamic formalism for discontinuous potentials can be found in [Mum06b].

For nonuniformly hyperbolic systems (systems with nonzero Lyapunov exponents) it is natural to consider potentials which are discontinuous. For example, the potential $\varphi(x) = -\log \operatorname{Jac}(df_{|E^u_x})$ is measurable but not continuous for many nonuniformly hyperbolic dynamical systems, such as unimodal maps, Hénon maps, and the time-one map of geodesic flows on nonpositively curved manifolds.

Many important classes of nonuniformly hyperbolic systems have a natural family of sets $\{\Lambda_l\}_{l\geq 1}$, called the regular sets, such that the potential $\varphi(x) = -\log \operatorname{Jac}(df_{|E^u_x})$ is continuous on the closure of each Λ_l, but not on $\Lambda = \bigcup_{l\geq 1} \Lambda_l$. Such a family of subsets is required for the topological pressure for discontinuous potentials.

Let (X, ρ) be a compact metric space, and $f\colon X \to X$ a continuous map. Consider an f-invariant subset $\Lambda \subset X$ possessing a nested family of subsets $\{\Lambda_l\}_{l\geq 1}$ which exhaust Λ, that is, $\Lambda_l \subset \Lambda_{l+1}$ for every $l \geq 1$ and $\Lambda = \bigcup_{l\geq 1} \Lambda_l$. Neither the Λ nor the Λ_l are required to be compact; the Λ_l are not required to be f-invariant.

Consider a measurable potential $\varphi\colon X \to \mathbb{R}$. The potential φ is *continuous with respect to the family of subsets* $\{\Lambda_l\}$ if φ is continuous on the closure of each Λ_l. The potential function φ is not (necessarily) continuous on Λ. The *topological pressure* of φ on Λ is

$$P_\Lambda(\varphi) = \sup_{l\geq 1} P_{\Lambda_l}(\varphi),$$

where $P_{\Lambda_l}(\varphi)$ is the topological pressure of φ on Λ_l (see Section 2.2.1).

The topological pressure does not depend on the choice of the family of sets $\{\Lambda_l\}$ (see Section 4.3 for details). Other properties of the topological pressure are given in Section 4.3.

Variational Principle

When the potential is not required to be continuous the set of measures considered must change; it must depend on the potential φ. We consider the supremum over f-invariant Borel probability measures μ such that φ is integrable with respect to μ. As the set Λ is not compact, the set of points considered in the variational principle must also change to depend on φ.

Set $\mathcal{M}_\varphi(\Lambda, f)$ be the set of Borel f-invariant ergodic probability measures on Λ so that φ is integrable with respect to μ. Let

$$\mathcal{L}(\Lambda, \varphi) = \{x \in \Lambda : V(x) \bigcap \mathcal{M}_\varphi(\Lambda, f) \neq \varnothing\},$$

which is Borel and f-invariant.

The following variational principle for discontinuous potentials is established in Section 4.2.

Theorem. Assume that f-invariant $\Lambda \subset X$ has a nested family of subsets $\{\Lambda_l\}_{l\geq 1}$ which exhaust Λ. Let $\varphi\colon X \to \mathbb{R}$ be continuous with respect to the family $\{\Lambda_l\}_{l\geq 1}$. Assume $\mathcal{L}(\Lambda, \varphi) \neq \varnothing$. Then

$$P_{\mathcal{L}(\Lambda,\varphi)}(\varphi) = \sup\left\{h_\mu(f) + \int_\Lambda \varphi\, d\mu : \mu \in \mathcal{M}_\varphi(\Lambda, f)\right\}.$$

Properties of the variational principle are given in Section 4.3.

Chapter 3

Almost-Additive Sequences of Continuous Potentials

In this chapter, we give a thermodynamic formalism for almost-additive sequences of potentials. The results in this chapter on the thermodynamic formalism for almost-additive sequences of continuous potentials can be found in [Mum06a].

3.1 Introduction

Let (X, ρ) be a compact metric space, $f \colon X \to X$ a continuous measure preserving transformation on X, and $\Phi = \{\varphi_n : X \to \mathbb{R}\}_{n=1}^{\infty}$ a sequence of continuous functions.

Definition 3.1.1. The sequence Φ is *almost-additive* if there exist functions $c_1, c_2 : \mathbb{N} \to \mathbb{R}$ such that for every $m, n \geq 1$, and $x \in X$

1. $c_1(n) \leq c_2(n)$,
2. there exist constants C_1, C_2 such that $C_1 \leq c_1(n)$ and $c_2(n) \leq C_2$, and
3. $$c_1(n+m) + \varphi_n(x) + \varphi_m(f^n(x)) \leq \varphi_{n+m}(x) \leq \varphi_n(x) + \varphi_m(f^n(x)) + c_2(n+m).$$

Several properties of almost-additive sequences are stated in Section 3.2. In Section 3.3 the topological pressure of an almost-additive sequence on a compact f-invariant set is given. Conditions which guarantee a variational principle corresponding to the topological pressure of an almost-additive sequence are given in Section 3.4. Equilibrium measures for almost-additive sequences are defined in Section 3.5 and conditions for existence and uniqueness of equilibrium measures are given. Several examples of almost-additive sequences are shown in Section 3.6 and results are compared.

3.2 Properties of Almost-Additive Sequences

Let Φ be an almost-additive sequence of potentials. If $c_2(n)$ is 0 for every n, then Φ is a subadditive sequence. If in addition, $c_1(n)$ is 0 for all n, then the almost-additive sequence of functions is additive.

Lemma 3.2.1. Let Φ be an almost-additive sequence of functions. Then the limit $\lim_{n\to\infty} \varphi_n/n$ exists almost everywhere; it is possibly $-\infty$ or ∞.

Proof. The sequence $(\varphi_n + C_2)/n$ is subadditive since for every m, n we have

$$\varphi_{m+n}(x) + C_2 \leq (\varphi_m(x) + C_2) + (\varphi_n(f^m(x)) + C_2).$$

By the subadditive ergodic theorem (see [Wal82]), $\lim_{n\to\infty}(\varphi_n + C_2)/n$ exists almost everywhere. As we have

$$\lim_{n\to\infty} \frac{\varphi_n}{n} = \lim_{n\to\infty} \frac{\varphi_n + c_2(n)}{n} \leq \lim_{n\to\infty} \frac{\varphi_n + C_2}{n} = \lim_{n\to\infty} \frac{\varphi_n}{n},$$

the result is shown. $\square$

The two lemmas below follow from Definition 3.1.1 part 3 and the subadditive ergodic theorem.

Lemma 3.2.2. Let Φ be an almost-additive sequence of functions. Then $\int \varphi_n \, d\mu$ is an almost-additive sequence of numbers, i.e. for $a_n = \int \varphi_n \, d\mu$ we have that

$$c_1(n+m) + a_n + a_m \leq a_{n+m} \leq a_n + a_m + c_2(n+m).$$

In particular, $\lim_{n\to\infty}(\int \varphi_n \, d\mu)/n$ exists.

Lemma 3.2.3. Let Φ be an almost-additive sequence of functions. Then for a fixed $m \geq 0$, the sequence

$$S_m\Phi = \left\{ \phi_n^m = \sum_{k=0}^{n-1} \varphi_m \circ f^k \right\}_{n=0}^{\infty}$$

is additive.

3.3 Topological Pressure

Set

$$Z_n(Z,\Phi,\mathcal{U}) = \inf_{\Gamma} \sum_{\underline{U}\in\Gamma} \exp \sup_{x\in X(\underline{U})} \varphi_n(x), \tag{3.3.1}$$

where the infimum is taken over all $\Gamma \subset W_n(\mathcal{U})$ covering Z.

Definition 3.3.1. Assume that the almost-additive sequence of functions Φ satisfies

$$\lim_{\operatorname{diam}(\mathcal{U})\to 0} \limsup_{n\to\infty} \frac{\gamma_n(\Phi,\mathcal{U})}{n} = 0\,, \tag{3.3.2}$$

The *topological pressure* of an almost-additive sequence Φ, on a compact f-invariant set $Z \subset X$, is given by

$$P_Z(\Phi) = \lim_{\operatorname{diam}(\mathcal{U})\to 0} \lim_{n\to\infty} \frac{1}{n} Z_n(Z,\Phi,\mathcal{U}).$$

When necessary, we write $P_{f,Z}(\Phi)$ for the topological pressure of Φ on Z with respect to the function f.

The following theorem shows that Definition 3.3.1 is the same as the nonadditive topological pressure for an almost-additive sequence satisfying (3.3.2) on a compact f-invariant set.

Theorem 3.3.2. Let Φ be an almost-additive sequence of functions satisfying (3.3.2) and let $Z \subset X$ be f-invariant. The following hold:

1. $CP_Z(\Phi) = \underline{CP}_Z(\Phi) = \overline{CP}_Z(\Phi) =$
$$\lim_{\operatorname{diam}(\mathcal{U})\to 0} \lim_{n\to\infty} \frac{1}{n} \log Z_n(\Phi, Z,\mathcal{U}).$$

2. If in addition Z is compact, then $P_Z(\Phi) = CP_Z(\Phi)$.

Proof. (1) Given sets $\Gamma_m \subset W_m(\mathcal{U})$ and $\Gamma_n \subset W_n(\mathcal{U})$, define the set

$$\Gamma_{m,n} = \{\underline{UV} : \underline{U} \in \Gamma_m, \underline{V} \in \Gamma_n\} \subset W_{m+n}(\mathcal{U}).$$

Since Z is f-invariant, if the two collections Γ_m and Γ_n cover Z, then their concatenation $\Gamma_{m,n}$ also covers Z. As Φ is almost-additive,

$$\sup_{x\in X(\underline{UV})} \varphi_{m+n}(x) \le \sup_{x\in X(\underline{U})} \varphi_m(x) + \sup_{x\in X(\underline{V})} \varphi_n(x) + C_2,$$

for every $\underline{UV} \in \Gamma_{m,n}$. Thus

$$Z_{m+n}(\Phi,\mathcal{U}) \leq (\exp C_2) Z_m(\Phi,\mathcal{U}) Z_n(\Phi,\mathcal{U}).$$

As $\mathcal{U}$ is a finite cover,

$$\inf_{n\geq 1} \frac{\log Z_n(\Phi,\mathcal{U})}{n} > -\infty.$$

Almost-additivity implies that the limit of $\log Z_n(\Phi,\mathcal{U})/n$ as n goes to infinity exists and is finite.

Thus, as a result of Theorem 2.2 in [Pes97] and Theorem 2.2.2,

$$\underline{CP}_Z(\Phi) = \overline{CP}_Z(\Phi) = \lim_{n\to\infty} \frac{1}{n} \log Z_n(Z,\Phi,\mathcal{U}) \,.$$

(2) For an almost-additive sequence of functions, we have $\varphi_n < \varphi_{n+1} + K$, where $K = \sup_{x\in X} |\varphi_1(x)| + \max\{|C_1|, |C_2|\} + 1$. Let $\Gamma \subset \bigcup_{n\geq 1} W_n(\mathcal{U})$ be a cover of Z. As Z is compact, we can assume that Γ is finite. Thus there exists an M such that $\Gamma \subset \bigcup_{n\leq M} W_n(\mathcal{U})$.

Set $\Gamma^n = \{\underline{U}_1 \cdots \underline{U}_n : \underline{U}_i \in \Gamma\}$ for each $n \geq 1$. As Z is f-invariant and Γ covers Z, for all n the collection Γ^n covers Z. Since Φ is almost-additive,

$$\sup_{x\in X(\underline{U}_1\cdots\underline{U}_n)} \varphi_{m(\underline{U}_1)+\cdots+m(\underline{U}_n)}(x) \leq \sum_{i=1}^{n} \sup_{x\in X(\underline{U}_i)} \varphi_{m(\underline{U}_i)}(x) + (n-1)C_2.$$

Set

$$N(\Gamma) = \sum_{U\in\Gamma} \exp\left(-\alpha m(\underline{U}) + \sup_{x\in X(U)} \varphi_{m(U)}(x)\right).$$

We have that $N(\Gamma^n) \leq [(\exp C_2)N(\Gamma)]^n$.

If $\alpha = P_Z(\Phi,\mathcal{U})$ then there exists an $m \geq 1$ and a cover $\Gamma \subset \bigcup_{n\geq m} W_n(\mathcal{U})$ of Z such that $(\exp C_2)N(\Gamma) < 1$. Set $\Gamma^\infty = \{\underline{U} : \underline{U} \in \Gamma^n$ for some $n\}$. Then

$$N(\Gamma^\infty) = \sum_{n=1}^{\infty} N(\Gamma^n) < \infty.$$

As Γ covers Z, for any $N \geq 1$ and $x \in Z$ there exists $\underline{U} \in \Gamma^\infty$ such that $x \in X(\underline{U})$ and $N \leq m(\underline{U}) \leq N + M$. Let $\Gamma^* \subset W_N(\mathcal{U})$ be the collection of strings $\underline{U}^*$ that consist of the first N elements of some string $\underline{U} \in \Gamma^\infty$. The following hold:

$$\sup \varphi_{m(\underline{U}^*)} \leq \sup \varphi_{m(\underline{U})} + KM$$

and

$$N(\Gamma^*) \leq N(\Gamma^\infty) \max\{1, \exp(\alpha M)\} \exp(KM).$$

Thus $\overline{M}(Z, \alpha, \Phi, \mathcal{U}) < \infty$, which implies that $\alpha > \overline{CP}_Z(\Phi, \mathcal{U})$. Thus

$$P_Z(\Phi, \mathcal{U}) \geq \overline{CP}_Z(\Phi, \mathcal{U}).$$

Then using part 1 above, the result is shown. $\square$

Let (Σ_A, σ) be a subshift of finite type. Let $Z \subset \Sigma_A$ be any subset. For every n there is a unique cover of Z by n-cylinders. The equations Z_n and Z_{n+l-1} (Equation (3.3.1)) satisfy the following inequality:

$$p^{l+1} \sum_{\underline{U} \in \Gamma_{l+n-1}} \exp \sup_{x \in X(\underline{U})} \varphi_n(x) \leq \sum_{\underline{V} \in \Gamma_n} \exp \sup_{x \in X(\underline{V})} \varphi_n(x) \leq \sum_{\underline{U} \in \Gamma_{l+n-1}} \exp \sup_{x \in X(\underline{U})} \varphi_n(x).$$

For an almost-additive sequence satisfying (3.3.2) this implies that

$$P_Z(\Phi, \mathcal{U}_l) = P_Z(\Phi, \mathcal{U}_1).$$

Thus

$$P_Z(\Phi) = \lim_{l \to \infty} P_Z(\Phi, \mathcal{U}_l) = P_Z(\Phi, \mathcal{U}_1) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{\underline{U} \in \Gamma} \exp \sup_{x \in X(\underline{U})} \varphi_n(x).$$

We conclude that an almost-additive sequence satisfying (3.3.2) on a subshift of finite type the topological pressure can be computed as follows.

Definition 3.3.3. Let (Σ_A, σ) be a subshift of finite type, and let

$$\Phi = \{\varphi_n \colon \Sigma_A \to \mathbb{R}\}_{n=1}^\infty$$

be an almost-additive sequence satisfying (3.3.2). The *topological pressure* of Φ on compact f-invariant set $Z \subset \Sigma_A$ is

$$P_Z(\Phi) = \lim_{n \to \infty} \frac{1}{n} Z_n(Z, \Phi, \mathcal{U}).$$

3.4 Variational Principle

For the rest of Chapter 3, we will be considering Z, a compact f-invariant subset of X; thus without loss of generality assume that $X = Z$, and write $P(\Phi) = P_Z(\Phi)$.

Theorem 3.4.1. Suppose that Φ is an almost-additive sequence of functions satisfying property (3.3.2). Then

$$P(\Phi) \geq \sup \left\{ h_\mu(f) + \lim_{n\to\infty} \frac{1}{n} \int \varphi_n \, d\mu : \mu \in \mathcal{M}(X, f) \right\}.$$

The proof of the inequality follows the line of argument in Bowen [Bow70] (see also [Pes97]). Only necessary modifications for the almost-additive case are indicated.

Let $\Gamma \in W_n(\mathcal{U})$ be a cover of X. For $\lambda > 0$ denote

$$Z(\Gamma, \lambda) = \sum_{\underline{U}\in\Gamma} \lambda^n \exp \sup_{x\in X(\underline{U})} \varphi_n(x).$$

Fix $m \geq 1$ and let $\Gamma^m = \{\underline{U}_1 \dots \underline{U}_m : \underline{U}_i \in \Gamma\}$.

Lemma 3.4.2.

$$Z(\Gamma^m, \lambda) \leq \Big[(\exp C_2) Z(\Gamma, \lambda)\Big]^m. \tag{3.4.1}$$

Proof. Almost-additivity implies

$$\sup_{x\in X(\underline{U}_1 \dots \underline{U}_m)} \varphi_{mn}(x) \leq \sup_{X(\underline{U}_1)} \varphi_n(x) + \cdots + \sup_{X(\underline{U}_m)} \varphi_n(x) + (m-1)C_2.$$

Hence

$$\begin{aligned}
&\sum_{\underline{U}_1\dots\underline{U}_m\in\Gamma^m} \lambda^{mn} \exp \sup_{x\in X(\underline{U}_1\dots\underline{U}_m)} \varphi_{mn}(x) \leq \\
&\exp\Big[(m-1)(C_2)\Big] \cdot \\
&\quad \sum_{\underline{U}_1\dots\underline{U}_m\in\Gamma^m} \left[\lambda^n \exp \sup_{x\in X(\underline{U}_1)} \varphi_n(x)\right] \dots \left[\lambda^n \exp \sup_{x\in X(\underline{U}_m)} \varphi_n(x)\right] \\
&\leq \left[(\exp C_2)\left(\sum_{\underline{U}\in\Gamma} \lambda^n \exp \sup_{x\in X(\underline{U})} \varphi_n(x)\right)\right]^m.
\end{aligned}$$

□

Lemma 3.4.3. $\lim_{n\to\infty} (\log Z_{nm}(\Phi, \mathcal{U}))/n \leq mP(\Phi, \mathcal{U})$.

Proof. Beginning with Inequality (3.4.1) of Lemma 3.4.2,

$$\lim_{n\to\infty} \frac{1}{n} \log \inf_{\Gamma\in W_n(\mathcal{U})} Z(\Gamma^m, 1) \le m \lim_{n\to\infty} \frac{1}{n} \left[\log \inf_{\Gamma\in W_n(\mathcal{U})} Z(\Gamma, 1) + C_2\right]$$
$$= m\, P(\Phi,\mathcal{U}) \qquad \square$$

Lemma 3.4.4. For any fixed $m \ge 1$ we have $C_1 + P_{f^m}(S_m\Phi) \le mP_f(\Phi)$.

Proof. Let $\mathcal{V} = \mathcal{U} \vee \cdots \vee f^{-m+1}\mathcal{U}$. Then $W_n(\mathcal{V})$ and $W_{mn}(\mathcal{U})$ are in one-to-one correspondence. Namely, for $\underline{U} = U_{i_0} \dots U_{i_{mn-1}}$ let $\underline{V} = V_{i_0} \dots V_{i_{m-1}}$ where $V_{i_k} = U_{i_{km}} \cap \cdots \cap f^{-m+1}U_{i_{km+m-1}}$. Then

$$X_f(\underline{U}) = X_{f^m}(\underline{V}), \tag{3.4.2}$$

since

$$\{x \in X \colon f^k \in U_k, k = 0, \dots, mn-1\} = \{x \in X \colon f^{mk} \in V_k, k = 0, \dots, n-1\}.$$

By almost-additivity and equality (3.4.2),

$$\sup_{x\in X(\underline{V})} \sum_{k=0}^{n-1} \varphi_m(f^{mk}(x)) + (n-1)C_1 \le \sup_{x\in X(\underline{U})} \varphi_{mn}(x),$$

which implies that

$$C_1 + P_{f^m}(S_m\Phi, \mathcal{V}) \le \lim_{n\to\infty} \frac{1}{n} \log \inf_{\Gamma\in W_{mn}(\mathcal{U})} \sum_{\underline{U}\in\Gamma} \exp \sup_{x\in X(\underline{U})} \varphi_{mn}.$$

The desired result follows from Lemma 3.4.3. $\square$

By a simple calculation, the following lemma may be verified.

Lemma 3.4.5. Let $\lambda > 0$. Suppose that $(\exp C_2)Z(\Gamma, \lambda) < 1$ for some Γ covering X. Then $\lambda \le \exp(-P(\Phi,\mathcal{U}))$.

The proof of Theorem 3.4.1 uses the following three lemmas.

Lemma 3.4.6 (Bowen [Bow70]). Let real numbers $a_1, a_2, \dots, a_n$ be given. Then the quantity

$$F(p_1, \dots p_n) = \sum_{i=1}^{n} -p_i \log p_i + \sum_{i=1}^{n} p_i a_i$$

has maximum value $\log \sum_{i=1}^{n} \exp a_i$ for $(p_1, \dots p_n)$ with $p_i \ge 0$ and $p_1 + \cdots + p_n = 1$, and the maximum is assumed only at $p_j = e^{a_i}(\sum_{i=0}^{n} e^{a_i})^{-1}$.

Set $H_\mu(\mathcal{E})$ to be the entropy of μ with respect to the partition $\mathcal{E}$, and $H_\mu(\mathcal{E}|\mathcal{D})$ as the conditional entropy of the partition $\mathcal{E}$ with respect to the partition $\mathcal{D}$.

Lemma 3.4.7 (Bowen [Bow70]). Let X be a compact metric space, $\mu \in \mathcal{M}(X, f)$, $\epsilon > 0$, and $\mathcal{E}$ a finite Borel partition. There is a $\delta > 0$ so that $H_\mu(\mathcal{E}|\mathcal{D}) < \epsilon$ whenever $\mathcal{D}$ is a partition with diameter less than δ.

Lemma 3.4.8 (Bowen [Bow70]). Let $\mathcal{A}$ be a finite open cover of X. For each $n > 0$ there is a Borel partition $\mathcal{D}(n)$ of X so that

1. $D \in \mathcal{D}(n)$ lies inside some member of $T^{-k}\mathcal{A}$ for each $k = 0, \dots, n-1$.
2. At most $n|\mathcal{A}|$ sets in $\mathcal{D}(n)$ can have a point in all their closures.

Lemma 3.4.9. Suppose $\mathcal{D}$ is a Borel partition of X such that each $x \in X$ is in the closure of at most M members of $\mathcal{D}$. Then

$$h_\mu(f, \mathcal{D}) + \lim_{n\to\infty} \frac{1}{n} \int \varphi_n \, d\mu \le P(\Phi) + \log M.$$

Proof. Let $\mathcal{U}$ be a finite open cover of X each member of which intersects at most M members of $\mathcal{D}$. For each $B \in \mathcal{D}_m = \mathcal{D} \vee \cdots \vee f^{-m+1}\mathcal{D}$, pick x_B with $\int_B \varphi_m d\mu \le \mu(B)\varphi_m(x_B)$. Now, using properties of entropy (see for example Katok and Hasselblatt [KH95]), and Lemma 3.4.6 we have

$$\begin{aligned}
h_\mu(f, \mathcal{D}) + \lim_{n\to\infty} \frac{1}{n} \int \varphi_n d\mu &\le \frac{1}{m} \left(H_\mu(\mathcal{D}_m) + \int \varphi_m d\mu + C_2 \right) \\
&\le \frac{1}{m} \left(\sum_{B \in D_m} \mu(B) \left(-\log \mu(B) + \varphi_m(x_B) \right) + C_2 \right) \\
&\le \frac{1}{m} \log \sum_{B \in D_m} \exp \varphi_m(x_B) + \frac{C_2}{m}. \qquad (3.4.3)
\end{aligned}$$

Fix $K > 0$. Let $\Gamma_m \subset W_m(\mathcal{U})$ be a cover of X so that

$$\sum_{\underline{U} \in \Gamma_m} \exp \sup_{x \in X(\underline{U})} \varphi_m(x) \le K + \inf_{\Gamma \in W_m(\mathcal{U})} \sum_{\underline{U} \in \Gamma} \exp \sup_{x \in X(\underline{U})} \varphi_m(x). \qquad (3.4.4)$$

For each x_B pick $\underline{U}_B \in \Gamma_m$ with $x_B \in X(\underline{U}_B)$. This map $B \to \Gamma_m$ is at most M^m to one. As $\varphi_m(x_B) \le \sup_{x \in X(\underline{U}_B)} \varphi_m(x)$, and by inequalities (3.4.3) and (3.4.4),

$$\begin{aligned}
h_\mu(f, \mathcal{D}) + \lim_{n\to\infty} \frac{1}{n} \int \varphi_n \, d\mu &\le \frac{1}{m} \log \sum_{\underline{U} \in \Gamma_m} M^m \exp \sup_{x \in X(\underline{U})} \varphi_m(x) + \frac{C_2}{m} \\
&\le \log M + \frac{1}{m} \log Z_m(\Phi, \mathcal{U}) + \frac{C_2}{m} + \frac{\log K}{m}.
\end{aligned}$$

Let $m \to \infty$ and then $\mathrm{diam}(\mathcal{U}) \to 0$ to obtain the desired inequality. $\square$

Proof of Theorem 3.4.1. Let $\mathcal{E}$ be a Borel partition and $\epsilon > 0$. By using the conclusion of Lemma 3.4.7 there exists a finite open cover $\mathcal{A}$ of X so that $H_\mu(\mathcal{E}|\mathcal{D}) \leq \epsilon$ whenever $\mathcal{D}$ is a partition every member of which is contained in some member of $\mathcal{A}$.

Fix $n > 0$. Let $\mathcal{E}_n = \mathcal{E} \vee \cdots \vee f^{-n+1}\mathcal{E}$, and $\mathcal{D}(n)$ be as in Lemma 3.4.8. Then by Lemma 3.4.9

$$\begin{aligned} h_\mu(f, \mathcal{E}_n) + \lim_{m\to\infty} \frac{1}{m} \int \varphi_m \, d\mu &\leq \frac{1}{n} \left(h_\mu(f^n, \mathcal{E}_n) + \lim_{m\to\infty} \frac{1}{m} \int \phi^n_{m,f^n} \, d\mu \right) \\ &\leq \frac{1}{n} \left(h_\mu(f^n, \mathcal{D}(n)) + \lim_{m\to\infty} \frac{1}{m} \int \phi^n_{m,f^n} \, d\mu \right) + \\ &\qquad \frac{1}{n} H_\mu(\mathcal{E}_n|\mathcal{D}(n)) \\ &\leq \frac{1}{n} \left(P_{f^n}(S_n\Phi) + \log n|\mathcal{A}| \right) + \frac{1}{n} H_\mu(\mathcal{E}_n|\mathcal{D}(n)). \end{aligned}$$

We have $H_\mu(\mathcal{E}_n|\mathcal{D}(n)) \leq \sum_{k=0}^{n-1} H_\mu(f^{-k}\mathcal{D}|\mathcal{D}(n))$. For each k, $\mathcal{D}(n)$ refines $f^{-k}\mathcal{A}$, and one has $H_\mu(f^{-k}\mathcal{E}|\mathcal{D}(n)) \leq \epsilon$ (since μ is f-invariant, $f^{-k}\mathcal{A}$ bears the same relation to $f^{-k}\mathcal{E}$ as $\mathcal{A}$ to $\mathcal{E}$). Hence, using Lemma 3.4.4,

$$h_\mu(f, \mathcal{E}) + \lim_{m\to\infty} \frac{1}{m} \int \varphi_m d\mu \leq P_f(\Phi) + \frac{\log n|\mathcal{A}|}{n} + \epsilon + \frac{C_1}{n}.$$

Letting $n \to \infty$ and then $\epsilon \to 0$ shows the desired inequality. $\square$

Theorem 3.4.10. Suppose that Φ is an almost-additive sequence of functions satisfying property (3.3.2), and the functions $c_1(n)$ and $c_2(n)$ satisfy

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} c_1(n-k) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} c_2(n-k). \tag{3.4.5}$$

Then

$$P(\Phi) = \sup \left\{ h_\mu(f) + \lim_{n\to\infty} \frac{1}{n} \int \varphi_n d\mu : \mu \in \mathcal{M}(X, f) \right\}.$$

We show that for an almost-additive sequence Φ satisfying conditions (3.3.2) and (3.4.5)

$$P(\Phi) \leq \sup \left\{ h_\mu(f) + \lim_{n\to\infty} \frac{1}{n} \int \varphi_n d\mu : \mu \in \mathcal{M}(X, f) \right\}.$$

The following lemmas are needed in the proof of the inequality.

Lemma 3.4.11 (Bowen [Bow70]). Fix a finite set E and $h \geq 0$. Let

$$R(k,h) = \left\{ \underline{a} \in E^k : H(\underline{a}) \leq h \right\}.$$

Then $\limsup_{k\to\infty} \frac{1}{k} \log |R(k,h)| \leq h$.

Lemma 3.4.12. Let $x \in X$, $\mu \in V(x)$. Fix $\delta > 0$, and $\epsilon > 0$. Then there exists $m, N \in \mathbb{N}$ for which one can find a string $\underline{U} \in W_N(\mathcal{U})$ satisfying the following:

1. $x \in X(\underline{U})$,
2. $\underline{U}$ contains a substring of length $km \geq N - m$ which, when viewed as $\underline{a} = a_0, \ldots, a_{k-1} \in (\mathcal{U}^m)^k$, satisfies $\frac{1}{m} H(\underline{a}) \leq h_\mu(f) + \delta$, and
3. $\sup_{x \in X(\underline{U})} \varphi_N(x) \leq \varphi_N(x) + \gamma_N(\Phi, \mathcal{U})$
   $\leq N \left(\lim_{n\to\infty} \frac{1}{n} \int \varphi_n d\mu + 3\epsilon \right) + \gamma_N(\Phi, \mathcal{U})$.

Proof. Parts (1) and (2) can be found in Bowen [Bow70]. Let N be large enough so that

$$\left| \frac{1}{N} \int \varphi_N \, d\mu - \lim_{n\to\infty} \frac{1}{n} \int \varphi_n \, d\mu \right| \leq \epsilon, \quad \text{and}$$
$$\left| \delta_{x,N}(\varphi_1) - \int \varphi_1 \, d\mu \right| \leq \epsilon.$$

Thus

$$\begin{aligned}
&\left| \frac{1}{N} \; \varphi_N(x) - \lim_{n\to\infty} \frac{1}{n} \int \varphi_n \, d\mu \right| \leq \\
&\quad \left| \frac{1}{N} \varphi_N(x) - \frac{1}{N} \int \varphi_N \, d\mu \right| + \left| \frac{1}{N} \int \varphi_n \, d\mu - \lim_{n\to\infty} \frac{1}{n} \int \varphi_n \, d\mu \right| \leq \\
&\quad \left| \frac{1}{N} \sum_{k=0}^{N-1} \varphi_1 \left(f^k(x) \right) - \int \varphi_1 \, d\mu \right| + \\
&\qquad \left| \frac{1}{N} \sum_{k=0}^{N-1} c_2 (N-k) - \frac{1}{N} \sum_{k=0}^{N-1} c_1 (N-k) \right| + \epsilon \leq B(N) + 2\epsilon,
\end{aligned}$$

where

$$B(n) = \left| \frac{1}{n} \sum_{k=0}^{n-1} c_2 (n-k) - \frac{1}{n} \sum_{k=0}^{n-1} c_1 (n-k) \right|.$$

As (3.4.5) is satisfied we have for N large enough that $B(N) < \epsilon$, and the result is shown. $\square$

Proof of Theorem 3.4.10. Let $\mathcal{U}$ be a finite open cover of X and $\epsilon > 0$. Cover X with countably many sets X_m, where each X_m is the set of points for which Lemma 3.4.12 holds with this m and some $\mu \in V(x)$.

Choose an ϵ-dense set $u_1, \ldots, u_r$ in the closed interval

$$\left[-\left|\left|\lim_{n\to\infty} \varphi_n/n\right|\right|_\infty, \left|\left|\lim_{n\to\infty} \varphi_n/n\right|\right|_\infty\right].$$

Cover each X_m with the sets $Y_m(u_i)$, where each $Y_m(u_i)$ is the set of $x \in X_m$ for which Lemma 3.4.12 holds for this m and some measure $\mu \in V(x)$ with $\lim_{n\to\infty}(\int \varphi_n d\mu)/n \in [u_i - \epsilon, u_i + \epsilon]$.

Next, cover each $Y_m(u)$ by $\Gamma_{m,u}$ and show that $(\exp C_2)Z(\Gamma_{m,u}, \lambda)$ can be made as small as desired. By taking the unions of such $\Gamma_{m,u}$ a cover Γ of X is obtained with $(\exp C_2)Z(\Gamma, \lambda) < 1$.

Let $S(N)$ be the number of all possible strings $\underline{U}$ satisfying Lemma 3.4.12 for some $x \in Y_m(u)$. By Lemma 3.4.12 (2), and Lemma 3.4.11,

$$\begin{aligned} S(N) &\leq |\mathcal{U}|^m \left|\left\{\underline{V} \in (\mathcal{U}^m)^k : H(\underline{V}) \leq m(h_\mu(f) + \epsilon)\right\}\right| \\ &\leq \exp\left[N\left(h_\mu(f) + \epsilon\right)\right] \end{aligned}$$

for all sufficiently large N.

For every integer N_0, the strings satisfying Lemma 3.4.12, for some $x \in Y_m(u)$ and $N \geq N_0$, cover the set $Y_m(u)$. Let $\Gamma_{m,u}$ be the collection of $\underline{U}$ showing up in the present situation for some sufficiently large $N > N_0$ fixed. One can show

$$\begin{aligned} Z(\Gamma_{m.u}, \lambda) &\leq \sum_{N=N_0}^{\infty} \lambda^N S(N) \exp\left[N\left(\lim_{n\to\infty} \frac{1}{n}\int \varphi_n \, d\mu + 3\epsilon\right) + \gamma_N(\Phi, \mathcal{U})\right] \\ &\leq \sum_{N=N_0}^{\infty} \lambda^N \exp\left[N\left(\lim_{n\to\infty} \frac{1}{n}\int \varphi_n \, d\mu + h_\mu(f) + 4\epsilon + \lim_{n\to\infty} \frac{\gamma_n(\Phi, \mathcal{U})}{n}\right)\right] \\ &\leq \sum_{N=N^0}^{\infty} \beta^N = \frac{\beta^{N_0}}{1-\beta}, \end{aligned}$$

where

$$\beta = \lambda \exp\left(c + 4\epsilon + \lim_{n\to\infty} \frac{\gamma_n(\Phi, \mathcal{U})}{n}\right) < 1 \qquad (3.4.6)$$

and

$$c = \sup_\mu \left(h_\mu(f) + \lim_{n\to\infty} \frac{1}{n}\int \varphi_n \, d\mu\right).$$

From equation (3.4.6) we see that for

$$\lambda < \exp -\left(c + 4\epsilon + \lim_{n\to\infty} \frac{\gamma_n(\Phi, \mathcal{U})}{n}\right),$$

any $Y_m(u)$ can be covered by $\Gamma_{m,u} \subset \bigcup_{m\geq 0} W_m(\mathcal{U})$ with $(\exp c_2)Z(\Gamma_{m,u}, \lambda)$ as small as desired. Taking the unions of such $\Gamma_{m,u}$ we obtain a Γ covering X with $(\exp c_2)Z(\Gamma, \lambda) < 1$. By Lemma 3.4.5, $\lambda \leq \exp(-P(\Phi, \mathcal{U}))$, meaning

$$P(\Phi, \mathcal{U}) \leq c + 4\epsilon + \lim_{n\to\infty} \frac{\gamma_n(\Phi, \mathcal{U})}{n}.$$

The result is shown by letting $\epsilon \to 0$ and $\mathrm{diam}(\mathcal{U}) \to 0$. □

The following theorem will be shown after the existence and uniqueness of Gibbs (and equilibrium) measures is shown for mixing subshifts of finite type.

Theorem 3.4.13. Suppose that (Σ_A, σ) is a mixing subshift of finite type and Φ an almost-additive sequence satisfying (3.3.2). Also, suppose that there exists γ such that for every $n \geq 1$, $x \in X(\underline{U})$,

$$\varphi_n(x) \leq \sup_{x \in X(\underline{U})} \varphi_n(x) \leq \gamma + \varphi_n(x).$$

Then

$$P(\Phi) = \sup\left\{h_\mu(\sigma) + \lim_{n\to\infty} \frac{1}{n} \int \varphi_n \, d\mu : \mu \in \mathcal{M}(\Sigma_A, \sigma)\right\}.$$

The following corollary will be shown.

Corollary 3.4.14. Suppose that Φ is an almost-additive sequence satisfying (3.3.2) and (3.4.5), and that f is an expansive homeomorphism of X. Then there exists a measure μ_Φ on X such that

$$P(\Phi) = h_{\mu_\Phi}(f) + \lim_{n\to\infty} \frac{1}{n} \int \varphi_n d\mu_\Phi. \tag{3.4.7}$$

3.5 Equilibrium Measures

Definition 3.5.1. Let Φ be an almost-additive sequence of continuous potentials. A measure $\mu_\Phi \in \mathcal{M}(X, f)$ is an *equilibrium measure* associated with Φ if

$$h_{\mu_\Phi}(f) + \lim_{n\to\infty} \frac{1}{n} \int \varphi_n \, d\mu_\Phi = \sup\left\{h_\mu(f) + \lim_{n\to\infty} \frac{1}{n} \int \varphi_n \, d\mu : \mu \in \mathcal{M}(X, f)\right\}.$$

Theorem 3.5.2. Suppose that f is an expansive homeomorphism of X. Then for any almost-additive sequence of functions Φ there exists an equilibrium measure μ_Φ on X.

Proof. We show that the function

$$\mu \mapsto h_\mu(f) + \lim_{n\to\infty} \tfrac{1}{n} \int \varphi_n d\mu$$

is upper semicontinuous. Then use the fact that an upper semicontinuous function achieve its supremum on a compact set.

The map $\mu \mapsto h_\mu(f)$ is known to be upper semicontinuous for an expansive homeomorphism, see for example Bowen [Bow70].

As a consequence of almost-additivity for the sequence $\int \varphi_n d\mu$ we have

$$\frac{1}{m}\left[\int \varphi_m \, d\mu + C_1\right] \leq \lim_{n\to\infty} \frac{1}{n} \int \varphi_n \, d\mu \leq \frac{1}{m}\left[\int \varphi_m \, d\mu + C_2\right]$$

for any $m \geq 1$ and any measure $\mu \in \mathcal{M}(X, f)$. Let $\mu_k \in \mathcal{M}(X, f)$ converge to μ in the weak-$*$ topology. We have for any measure $\nu \in M(X, f)$

$$\left| \left(\lim_{n\to\infty} \frac{1}{n} \int \varphi_n \, d\nu\right) - \frac{1}{m} \int \varphi_m \, d\nu \right| \leq \frac{\sup\{|C_1|, |C_2|\}}{m}.$$

Thus

$$\lim_{k\to\infty} \lim_{n\to\infty} \frac{1}{n} \int \varphi_n \, d\mu_k = \lim_{n\to\infty} \frac{1}{n} \int \varphi_n \, d\mu,$$

which shows that the map $\mu \mapsto (h_\mu(f) + \lim_{n\to\infty} \frac{1}{n} \int \varphi_n d\mu)$ is upper semicontinuous. □

Corollary 3.4.14 is immediate from Theorem 3.4.10 and Theorem 3.5.2.

3.5.1 Subshifts of Finite Type

Definition 3.5.3. A measure $\mu \in \mathcal{M}(\Sigma_A, f)$ on Σ_A is a *Gibbs measure* for Φ if there exist constants $A_1, A_2 > 0$ such that

$$A_1 \leq \frac{\mu(\mathcal{C}_n)}{\exp(-nP(\Phi) + \varphi_n(x))} \leq A_2 \tag{3.5.1}$$

for any $n > 0, \mathcal{C}_n \subset \Sigma_A$ and $x \in \mathcal{C}_n$.

Theorem 3.5.4. Suppose that Φ is an almost-additive sequences of functions on a mixing subshift of finite type (Σ_A, σ) satisfying (3.3.2) and

$$\varphi_n(x) \leq \sup_{x \in X(\underline{U})} \varphi_n(x) \leq \gamma + \varphi_n(x). \tag{3.5.2}$$

Then there exists a unique Gibbs measure μ_Φ on X. Moreover, the measure μ_Φ is the unique equilibrium measure for Φ.

As the system is mixing there is an integer $p > 0$ such that $A^p > 0$. This implies that for any cylinders $\mathcal{C}_n \subset \Sigma_A$ and $J_l \subset \Sigma_A$, there exists a p-cylinder $K_p \subset \Sigma_A$ such that $\mathcal{C}_n K_p J_l \subset \Sigma_A$.

We use the notation $A \approx B$ to mean that there exist constants D_1, D_2 so that $D_1 A \leq B \leq D_2 A$.

The following lemma can be shown through a series of computations that follow the work of Feng and Lau [FL02, Fen04].

Lemma 3.5.5. $\sum_{\mathcal{C}_n \in \Sigma_A} \sup_{x \in \mathcal{C}_n} \exp \varphi_n(x) \approx \exp(nP(\Phi))$.

For each integer $n > 0$ let $\mathcal{B}_n$ be the σ-algebra generated by the cylinders $\mathcal{C}_n \subset \Sigma_A$. Define a sequence of probability measures $\{\nu_{n,\Phi}\}$ on $\mathcal{B}_n$ by

$$\nu_{n,\Phi}(\mathcal{C}_n) = \frac{\sup_{x \in \mathcal{C}_n} \exp \varphi_n(x)}{\sum_{\mathcal{C}_n \subset \Sigma_A} \sup_{x \in \mathcal{C}_n} \exp \varphi_n(x)}, \quad \forall \mathcal{C}_n \subset \Sigma_A.$$

There exists a subsequence $\{\nu_{n_k,\Phi}\}$ that converges in the weak-$*$ topology to a probability measure ν_Φ.

Lemma 3.5.6. For all $n > 0$, $\mathcal{C}_n \in \Sigma_A$,

$$\sup_{x \in \mathcal{C}_n} \exp \varphi_n(x) \exp(-nP(\Phi)) \approx \nu_\Phi(\mathcal{C}_n).$$

Lemma 3.5.7. There is a unique σ-invariant, ergodic Gibbs measure μ_Φ on Σ_A.

Proof. Let μ_Φ be a limit point of a subsequence of

$$\left\{ \frac{1}{m} \left(\nu_\Phi + \nu_\Phi \circ \sigma^{-1} + \cdots + \nu_\Phi \circ \sigma^{-(m-1)} \right) \right\}$$

in the weak-$*$ topology. By definition μ_Φ is a σ-invariant probability measure on Σ_A. For each $\mathcal{C}_n \subset \Sigma_A$ and $l > p$ we have that $\sigma^{-l}(\mathcal{C}_n)$ is the union of all cylinders $D_l \mathcal{C}_n \subset \Sigma_A$. Thus for every $l > p$ we have

$$\nu_\Phi \circ \sigma^{-l}(\mathcal{C}_n) \approx \sup_{x \in \mathcal{C}_n} \exp \varphi_n(x) \exp(-nP(\Phi)).$$

Sum over all $\mathcal{C}_n \subset \Sigma_A$, and divide by m. Taking the limit as m goes to infinity yields that μ_Φ satisfies equation (3.5.1).

Given any $\mathcal{C}_n$ and $D_l \subset \Sigma_A$ and any $i > n + 2p$,

$$\mu_\Phi\left(\mathcal{C}_n \bigcap \sigma^{-i}(D_l)\right) \geq C\mu_\Phi(\mathcal{C}_n)\mu_\Phi(D_l),$$

for some $C > 0$. As any Borel set can be approximated within ϵ by a finite disjoint union of cylinder sets, the above inequality holds for all Borel sets. Thus for any μ_Φ positive measure Borel sets $A, B \subset \Sigma$ there exists $n > 0$ such that $\mu_\Phi(A \bigcap \sigma^{-n}(B)) > 0$. Hence μ_Φ is ergodic.

Any two distinct ergodic measures must be singular, but property (3.5.1) shows that they must be absolutely continuous to each other. Thus the measure μ_Φ is unique. □

Lemma 3.5.8. The Gibbs measure μ_Φ is an equilibrium measure for Φ.

Proof. Since μ_Φ satisfies equation (3.5.1), for each $n \in \mathbb{N}, \mathcal{C}_n \subset \Sigma_A$, and $x \in \mathcal{C}_n$ we have $\log A_1 \leq nP(\Phi) + \log \mu_\Phi(\mathcal{C}_n) - \varphi_n(x) \leq \log A_2$. Integrate over $x \in \mathcal{C}_n$, sum over all $\mathcal{C}_n \subset \Sigma_A$, and divide by n to get

$$\frac{\log A_1}{n} \leq P(\Phi) + \frac{1}{n}\sum_{\mathcal{C}_n \subset \Sigma_A} \mu_\Phi(\mathcal{C}_n)\log \mu_\Phi(\mathcal{C}_n) - \frac{1}{n}\int \varphi_n \, d\mu_\Phi \leq \frac{\log A_2}{n}.$$

Let n go to infinity and combine with Theorem 3.4.1 to obtain the result. □

During the proof of the previous lemma the variational principle for subshifts of finite type was shown without requiring (3.4.5), which proves Theorem 3.4.13.

Lemma 3.5.9. The Gibbs measure μ_Φ is the unique equilibrium measure.

The following two lemmas are used.

Lemma 3.5.10 (Bowen [Bow70]). Let X be a compact metric space, μ a measure in $\mathcal{M}(X, f)$, and $\mathcal{D} = \{D_1, \ldots, D_n\}$ a Borel partition of X. Suppose $\{\mathcal{F}_m\}_{m=1}^\infty$ is a sequence of partitions so that

$$\operatorname{diam}(\mathcal{F}_m) = \max_{F \in \mathcal{F}_m} \operatorname{diam}(F) \to 0$$

as $m \to \infty$. Then there is a sequence of partitions $E_m = \{E_1^m, \ldots, E_n^m\}$ so that

1. Each E_i^m is a union of members of $\mathcal{F}_m$,

2. $\lim_{m\to\infty} \mu(E_i^m \triangle D_i) = 0$ for each i.

Lemma 3.5.11 (Bowen [Bow70]). Suppose $0 \leq p_i, \ldots, p_m \leq 1$, with $s = p_1 + \cdots + p_m \leq 1$ and $a_1, \ldots, a_m \in \mathbb{R}$. Then

$$\sum_{i=1}^{m} p_i(a_i - \log p_i) \leq s(\log \sum_{i=1}^{m} e^{a_i} - \log s).$$

Proof of Lemma 3.5.9. Let $\nu \in \mathcal{M}(\Sigma_A, \sigma)$ be a second equilibrium measure, i.e., $P(\Phi) = h_\nu(\sigma) + \lim_{n\to\infty} \int \varphi_n \, d\nu$.

First suppose that ν is singular with respect to the Gibbs measure μ_Φ. Then there is a Borel set B with $\sigma(B) = B$, $\mu_\Phi(B) = 0$, and $\nu(B) = 1$. Let $\mathcal{F}_m = \sigma^{-[\frac{m}{2}]+1}\mathcal{U} \vee \cdots \vee \mathcal{U}$. Then $\operatorname{diam}(\mathcal{F}_m) \to 0$ as $m \to \infty$. Applying Lemma 3.5.10 to the partition $\{B, X \setminus B\}$ one finds sets E^m which are unions of elements of $\mathcal{F}_m$ and satisfy $(\mu + \nu)(B \triangle E^m) \to 0$. As $\mu + \nu$ is σ-invariant and $\sigma^{-m+[\frac{m}{2}]}(B) = B$, one has $(\mu + \nu)(B \triangle G^m) \to 0$, where $G^m = \sigma^{-m+[\frac{m}{2}]} E^m$ is a union of members of $\sigma^{-m+1}\mathcal{U} \vee \cdots \vee \mathcal{U}$. For every ϵ there exists an m large enough so that

$$\begin{aligned} P(\Phi) =& h_\nu(\sigma) + \lim_{n\to\infty} \frac{1}{n} \int \varphi_n d\nu \\ \leq& \frac{1}{m} \left(H_\nu(\mathcal{U} \vee \cdots \vee \sigma^{-m+1}\mathcal{U}) + \int \varphi_m d\nu \right) + \epsilon. \end{aligned}$$

In other words we have that

$$mP(\Phi) \leq \sum_{B \in \mathcal{U} \vee \cdots \vee \sigma^{-m+1}\mathcal{U}} \left[-\nu(B) \log \nu(B) + \int_B \varphi_m d\nu \right].$$

As φ_m is continuous there exists a d so that for each B one can find $x_B \in B$ with $\varphi_m(x) \leq \varphi_m(x_B) + d$ on B. Thus the following equation holds:

$$mP(\Phi) \leq d + \epsilon + \sum_B \nu(B)(\varphi_m(x_B) - \log \nu(B))$$

One can show that

$$\begin{aligned} mP(\Phi) \leq& d + \epsilon + \sum_{B \in G^m} \nu(B)(\varphi_m(x_B) - \log \nu(B)) + \\ & \sum_{B \in X \setminus G^m} \nu(B)(\varphi_m(x_B) - \log \nu(B)). \end{aligned}$$

Applying Lemma 3.5.11 gives

$$mP(\Phi) - d \le \nu(G^m) \log \sum_{B \in G^m} \exp \varphi_m(x_B) + \\ \nu(X \setminus G^m) \sum_{B \in X \setminus G^m} \exp \varphi_m(x_B) + 2K,$$

where $K = \sup_{0 \le s \le 1}(-s \log s)$. Using the fact that μ_Φ is a Gibbs measure, we have that

$$\begin{aligned} -2K - d - \epsilon \le \nu(G^m) \log \sum_{B \in G^m} & \exp(\varphi_m(x_B) - mP) + \\ & \nu(X \setminus G^m) \exp(\varphi_m(x_B) - mP(\Phi)) \\ & \le \nu(G^m) \log \sum_{B \in G^m} A_1^{-1} \mu_\Phi(B) + \nu(X \setminus G^m) \log \sum_{B \in X \setminus G^m} A_1^{-1} \mu_\Phi(B) \\ & \le \log A_1^{-1} + \nu(G^m) \log \mu_\Phi(G^m) + \nu(X \setminus G^m) \log \mu(X \setminus G^m). \end{aligned}$$

Letting $\epsilon \to 0$, we have $m \to \infty$, $\nu(G^m) \to 1$, and $\mu(G^m) \to 0$, which leads to a contradiction with the above inequality.

In general, for $\nu' \in M(\Sigma_A, \sigma)$, write $\nu' = \beta\nu + (1-\beta)\mu'$, where $\beta \in [0,1]$, $\nu \in M(\Sigma_A, \sigma)$ is singular with respect to μ_Φ and $\mu' \in M(\Sigma_A, \sigma)$ is absolutely continuous with respect to μ_Φ. As ν and μ' are supported on disjoint sets,

$$\begin{aligned} & h_{\nu'}(\sigma) + \lim_{n \to \infty} \frac{1}{n} \int \varphi_n \, d\nu' \\ & \quad = \beta \left(h_\nu(\sigma) + \lim_{n \to \infty} \frac{1}{n} \int \varphi_n \, d\nu \right) + (1-\beta) \left(h_\mu(\sigma) + \lim_{n \to \infty} \frac{1}{n} \int \varphi_n \, d\mu \right). \end{aligned}$$

Suppose that ν' is an equilibrium measure for Φ. As μ_Φ is ergodic, it must be that $\beta = 0$ or $\beta = 1$. Above we showed that $h_\nu(\sigma) + \lim_{n \to \infty} \frac{1}{n} \int \varphi_n d\nu$ is not equal to the topological pressure $P(\Phi)$. Thus $\nu' = \mu'$ and write $\nu' = \frac{d\nu'}{d\mu_\Phi} \mu_\Phi$. As ν' and μ_Φ are σ-invariant, $\frac{d\nu'}{d\mu_\Phi}$ must be a constant. Since both measures are probability measures, they are equal. □

3.5.2 Axiom A Diffeomorphisms

Theorem 3.5.12. Suppose that Ω is a basic set for an Axiom A diffeomorphism f and Φ an almost-additive sequence of functions satisfying conditions (3.3.2) and (3.5.2). Then there exists a unique equilibrium measure μ_Φ for Φ.

In the course of proving Theorem 3.5.12, the variational principle will be established without requiring Φ to satisfy (3.4.5).

Corollary 3.5.13. Suppose that Ω is a basic set for an Axiom A diffeomorphism f and Φ an almost-additive sequence of functions satisfying conditions (3.3.2) and (3.5.2). Then

$$P(\Phi) = \sup\left\{h_\mu(\sigma) + \lim_{n\to\infty}\frac{1}{n}\int \varphi_n \, d\mu : \mu \in \mathcal{M}(X, f)\right\}.$$

The existence of a unique equilibrium measure for Axiom A systems is shown using a conjugacy with a mixing subshift of finite type.

Lemma 3.5.14. Let $\pi\colon \Sigma_A \to X$ be continuous and surjecative such that $\pi \circ \sigma = f \circ \pi$. Then $P_\sigma(\Phi \circ \pi) \geq P_f(\Phi)$.

Proof. Let $\mathcal{U}$ be an open cover of X, and $s \in \Sigma_A$. If $s \in \pi^{-1}(X(\underline{U}))$, then $\pi s \in X(\underline{U})$. Therefore, for fixed $n \geq 1$,

$$\sup_{x\in X(\underline{U})} \varphi_n(x) = \sup_{y\in\pi^{-1}X(\underline{U})} \varphi_n \circ \pi(y).$$

Thus $P_f(\Phi,\mathcal{U}) = P_\sigma(\Phi \circ \pi, \pi^{-1}\mathcal{U})$. We have that

$$P_\sigma(\Phi \circ \pi, \pi^{-1}\mathcal{U}) \leq P_\sigma(\Phi \circ \pi) + V_1(\Phi \circ \pi, \pi^{-1}\mathcal{U}).$$

Letting $\mathrm{diam}(\mathcal{U}) \to 0$ gives that $V_1(\Phi \circ \pi, \pi^{-1}\mathcal{U}) \to 0$. Thus the result is shown. $\square$

Proof of Theorem 3.5.12. Let $\mathcal{U}$ be a Markov partition of the basic set Ω with $\mathrm{diam}(\mathcal{U}) \leq \epsilon$, A the transition matrix for $\mathcal{U}$, and $\pi\colon \Sigma_A \to \Omega$. As in Bowen [Bow70], π is one-to-one except on a set of measure zero.

First assume that $f_{|\Omega}$ is mixing. Then $\sigma_{|\Sigma_A}$ is mixing and there exists an equilibrium measure $\mu_{\Phi\circ\pi}$. Let $\mu_\Phi = \pi{*}\mu_{\Phi\circ\pi}$, thus for every measurable set E we have $\mu_\Phi(E) = \mu_{\Phi\circ\pi}(\pi^{-1}E)$. Then μ_Φ is f-invariant. The measure spaces $(\Sigma_A, \sigma, \mu_{\Phi\circ\pi})$ and (X, f, μ_Φ) are conjugate, since π is one-to-one except on a set of $(\mu_\phi \circ \pi)$-measure zero. In particular, $h_{\mu_\Phi}(f) = h_{\mu_{\Phi\circ\pi}}(\sigma)$ and so by Lemma 3.5.14

$$\begin{aligned} P_f(\Phi) \geq h_{\mu_\Phi}(f) + \lim_{n\to\infty}\frac{1}{n}\int \varphi_n \, d\mu_\Phi &= h_{\mu_{\Phi\circ\pi}}(\sigma) + \lim_{n\to\infty}\frac{1}{n}\int (\varphi_n \circ \pi)\, d\mu_{\Phi\circ\pi} \\ &= P_\sigma(\Phi \circ \pi) \geq P_f(\Phi). \end{aligned}$$

Hence $P_\sigma(\Phi \circ \pi) = P_f(\Phi)$ and μ_Φ is an equilibrium state for Φ.

If $f_{|\Omega}$ is not mixing, then we use the spectral decomposition to show that there exists a bijection between $M(\Omega, f)$ and $M(X_1, f^m)$. Thus for the entropy we have $h_{\mu'}(f^m) = mh_\mu(f)$ and for the integrals we have

$$\lim_{n\to\infty} \frac{1}{n} \int \sum_{k=0}^{m-1} \varphi_n \circ f^k d\mu' = m \lim_{n\to\infty} \frac{1}{n} \int \varphi_n d\mu.$$

Maximizing $h_{\mu'}(f^m) + \lim_{n\to\infty} \frac{1}{n} \int \sum_{k=0}^{m-1} \varphi_n \circ f^k d\mu'$ is equivalent to maximizing $h_\mu(f) + \lim_{n\to\infty} \frac{1}{n} \int \varphi_n \, d\mu$. For Φ Hölder on Ω, $S_m\Phi$ will be Hölder on X_1 and therefore the result is shown since X_1 is a mixing basic set of f^m.

To show that the measure is unique, suppose μ is any equilibrium state of Φ and choose $\nu \in M(\Sigma_A, \sigma)$ with $\pi * \nu = \mu$. Then $h_\nu(\sigma) \geq h_\mu(f)$ and so

$$\begin{aligned} P_\sigma(\Phi \circ \pi) \geq h_\nu(\sigma) + \lim_{n\to\infty} \frac{1}{n} \int \varphi_n \circ \pi d\nu &\geq h_\mu(f) + \lim_{n\to\infty} \frac{1}{n} \int \varphi_n d\mu \\ &= P_f(\Phi) = P_\sigma(\Phi \circ \pi). \end{aligned}$$

Thus ν is an equilibrium measure for $\Phi \circ \pi$ which implies $\nu = \mu_{\Phi\circ\pi}$. Then $\mu = \pi * \mu_{\Phi\circ\pi} = \mu_\Phi$. $\square$

3.6 Examples

We give four examples of almost-additive sequences. The thermodynamic formalism of these sequences has been studied; these previous results are compared with the thermodynamic formalism for almost-additive sequences.

3.6.1 Additive Sequences of Potentials

We compare the thermodynamic formalism for almost-additive sequences to that of additive sequences [Rue78, Sin72, Bow70].

Let $\phi : X \to \mathbb{R}$ be a continuous function. The sequence

$$\varphi_n(x) = \sum_{k=0}^{n-1} \phi(f^k(x)).$$

is an additive sequence and thus is almost-additive with $c_1(n)$ and $c_2(n)$ identically equal to zero.

These sequences trivially satisfy properties (3.3.2) and (3.4.5). Thus we can apply Definition 3.3.1 to obtain the classical topological pressure of ϕ

on a compact f-invariant set Z,

$$P_Z(\phi) = \lim_{\mathrm{diam}(\mathcal{U})\to 0} \lim_{n\to\infty} \frac{1}{n} \log \inf_{\Gamma} \sum_{\underline{U}\in\Gamma} \exp \sup \sum_{k=0}^{n-1} \phi(f^k(x)),$$

where the infimum is taken over all $\Gamma \subset W_n(\mathcal{U})$ covering Z. Theorem 3.4.10 gives the classical variational principle

$$P_Z(\phi) = \sup \left\{ h_\mu(f) + \int \phi d\mu : \mu \in \mathcal{M}(Z, f). \right\}$$

If ϕ is Hölder continuous then condition (3.5.2) holds, and we recover the existence and uniqueness of Gibbs and equilibrium measures under the same conditions as in the classical case. (See Section 2.2.1.)

3.6.2 Nonadditive Sequences of Potentials

We compare the thermodynamic formalism for almost-additive sequences to that of nonadditive sequences [Bar96].

Consider a sequence satisfying (3.3.2) for which there is a continuous function $\psi \colon X \to \mathbb{R}$ so that

$$\varphi_n - \varphi_{n-1} \circ f \to \psi \tag{3.6.1}$$

uniformly on X as $n \to \infty$. Set $\gamma_n = ||\varphi_n - \varphi_{n-1} \circ f - \psi||_\infty$. Then for every $m \geq 1$ and every $x \in X$, we have that

$$-\gamma_m + \psi(x) \leq \varphi_m(x) - \varphi_{m-1}(f(x)) \leq \psi(x) + \gamma_m.$$

Fix two numbers $m, n \geq 1$. One can show the following inequality holds.

$$\begin{aligned} \sum_{k=0}^{n-1} \left[\psi\left(f^k(x)\right) - \gamma_{(m+n)-k} \right] + \varphi_m(f^n(x)) \leq \varphi_{n+m}(x) \\ \leq \sum_{k=0}^{n-1} \left[\psi\left(f^k(x)\right) + \gamma_{(m+n)-k} \right] + \varphi_m(f^n(x)). \end{aligned} \tag{3.6.2}$$

Without loss of generality set $\varphi_0 = 0$. The above inequality with $m = 0$ implies that

$$\sum_{k=0}^{n-1} \left[\psi\left(f^k(x)\right) - \gamma_{n-k} \right] \leq \varphi_n(x) \leq \sum_{k=0}^{n-1} \left[\psi\left(f^k(x)\right) + \gamma_{n-k} \right]. \tag{3.6.3}$$

Combining (3.6.2) and (3.6.3) yields

$$-\left(\sum_{k=0}^{n-1}\gamma_{(m+n)-k}+\sum_{k=0}^{n-1}\gamma_{n-k}\right)+\varphi_n(x)+\varphi_m(f^n(x))\le\varphi_{n+m}(x)$$
$$\le\varphi_n(x)+\varphi_m(f^n(x))+\left(\sum_{k=0}^{n-1}\gamma_{(m+n)-k}+\sum_{k=0}^{n-1}\gamma_{n-k}\right).$$

The above inequality shows that a sequence satisfying (3.6.1) is almost-additive if the function

$$c_2(n+m)=\sum_{k=0}^{n-1}\gamma_{(m+n)-1}+\sum_{k=0}^{n-1}\gamma_{n-k}$$

is bounded by a constant C_2 for every $n\ge 1$; since $c_1(n)=-c_2(n)$, for every n, we would also have the bound $-C_2\le c_1(n)$, for every n. This condition on the functions c_1, c_2 is satisfied if the sequence $\varphi_n-\varphi_{n-1}\circ f$ converges fast enough to the function ψ.

On the other hand, there are sequences which are almost-additive but do not satisfy (3.6.1). For example, let $X=S^1, f(x)=x$. The sequence $\Phi=\{\varphi_n(x)=\sin(x+n\alpha)\}$ is almost-additive with $c_1=-2, c_2=2$, but there are α for which

$$\begin{aligned}\varphi_n(x)-\varphi_{n-1}\circ f(x)&=\sin(x+n\alpha)-\sin(x+(n-1)\alpha)\\&=k_1\cos(k_2n+k_3),\end{aligned}$$

which does not converge as $n\to\infty$, where k_i are constants depending on α.

An almost-additive sequence satisfying (3.3.2) and (3.4.5) need not satisfy (3.6.1). Thus the set of sequences admitting a variational principle as in Barreira [Bar96] (see Section 2.2.2) is not disjoint from the set of almost-additive sequences, but neither class is contained in the other.

For sequences satisfying properties (3.3.2) and (3.6.1), which are also almost-additive satisfying (3.4.5), we can apply Theorem 3.4.10 to obtain the variational principle. Since these sequences satisfy (3.6.1), the variational principle can be expressed as

$$P_Z(\Phi)=\sup\left\{h_\mu(f)+\int\psi\,d\mu:\mu\in\mathcal{M}(Z,f)\right\}.$$

For a compact f-invariant set Z, we note that $\mathcal{L}(Z)=Z$.

Suppose that f is an expansive homeomorphism of X. Theorem 3.5.2 implies that for sequences satisfying (3.6.1) which are also almost-additive there exists an equilibrium measure. Uniqueness of equilibrium measures follows from Theorem 3.5.12 for f which are Axiom A, and almost-additive sequences satisfying conditions (3.3.2) and (3.5.2).

3.6.3 Subadditive Sequences of Potentials on Mixing Repellers

We compare the thermodynamic formalism for almost-additive sequences to that of subadditive sequences [Fal88] (see also [Bar96]).

Consider $(X, f) = (\Sigma_A, \sigma)$, where Σ_A is a mixing subshift of finite type with transition matrix A. Let Φ be an almost-additive sequence such that $c_2(n) \equiv 0$. Assume that the sequence Φ is uniformly bounded, satisfies a Lipschitz condition, and has bounded variations (see Section 2.2.2).

The conditions on the sequence Φ imply that these sequences satisfy conditions (3.3.2) and (3.5.2). Thus the topological pressure of the sequence Φ on the set Z can be defined (see Definition 3.3.3). Theorem 3.4.13 gives a corresponding variational principle, and Theorem 3.5.4 gives the existence and uniqueness of a Gibbs (and equilibrium) measure.

The thermodynamic formalism for these subadditive sequences was first introduced by Falconer [Fal88] in the study of mixing repellers. They are a particular case of the nonadditive thermodynamic formalism of Barreira [Bar96] (see Section 2.2.2).

3.6.4 Sequences of Potentials Generated by Iterated Function Systems with Overlaps

We compare the thermodynamic formalism for almost-additive sequences to the sequence studied by Feng and Lau [FL02, Fen04].

Let (Σ_A, σ), where Σ_A is a mixing subshift of finite type with primitive transition matrix A. Let M be a Hölder continuous function on Σ_A taking values in the set of all positive $d \times d$ matrices. Define the matrix norm $||M|| = 1^t M 1$, where 1 is the $d \times 1$ vector of all ones. The nonadditive sequence $\Phi = \{\varphi_n : \Sigma_A \to \mathbb{R}\}_{n=1}^{\infty}$ where

$$\varphi_n(x) = \log ||M(x) \dots M(\sigma^{n-1}(x))||$$

is almost-additive with $c_1(n) = c_1 < 0$ and $c_2 = 0$. Since $|| \cdot ||$ is a multiplicative norm, we have that $||AB|| \leq ||A|| \cdot ||B||$. Thus

$$\log ||M(x) \dots M(\sigma^{n+m-1}(x))|| \leq \log||M(x) \dots M(\sigma^{n-1}(x))|| + \\ \log ||M(\sigma^n(x)) \dots M(\sigma^{n+m-1}(x))||.$$

Feng and Lau [FL02] show that there is a constant $C > 0$ such that

$$\frac{\min_{i,j} M_{i,j}(x)}{\max_{i,j} M_{i,j}(x)} > C \quad \text{for all } x \in \Sigma_A,$$

which implies that $M(x) \geq \frac{C}{d}(1 \cdot 1^t)M(x)$. Thus

$$\log ||M(x) \dots M(\sigma^{n+m-1}(x))|| \geq \log ||M(x) \dots M(\sigma^{n-1}(x))|| + \\ \log ||M(\sigma^n(x)) \dots M(\sigma^{n+m-1}(x))|| + \log \tfrac{C}{d}.$$

As $C \leq 1$, we have that $\log \frac{C}{d} < 0$.

Definition 3.3.3 gives the topological pressure

$$P_{\Sigma_A}(\Phi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\mathcal{C}_n \in \Sigma_A} \sup_{x \in \mathcal{C}_n} ||M(x)M(\sigma x) \dots M(\sigma^{n-1}(x))||.$$

Applying Theorem 3.4.13 gives the variational principle.

$$P_{\Sigma_A}(\Phi) = \sup \left\{ h_\mu(\sigma) + \lim_{n\to\infty} \frac{1}{n} \int \log ||M(y)M(\sigma y) \dots M(\sigma^{n-1}y)|| d\mu \right\}$$

where the supremum is taken over all $\mu \in \mathcal{M}(\Sigma_A, \sigma)$. The above pressure and variational principle are those given by Feng and Lau [FL02],[Fen04]. Applying Theorem 3.5.4 gives the existence of a unique Gibbs and equilibrium measure, which recovers the results of Feng and Lau (see Section 2.2.2).

Chapter 4

Discontinuous Potentials

In this chapter a thermodynamic formalism for discontinuous potentials is given. The results on the thermodynamic formalism for discontinuous potentials can be found in [Mum06b].

4.1 Introduction

Let X be a compact metric space, $f\colon X \to X$ a continuous map, and potential $\varphi\colon X \to \mathbb{R}$ be measurable but not necessarily continuous on X. Assume that there exists an f-invariant set $\Lambda \subset X$ with a nested family of subsets $\{\Lambda_l\}$ which exhaust Λ. We do not require that either Λ or the Λ_l is compact; the Λ_l are not required to be f-invariant.

Definition 4.1.1. If a potential is continuous on the closure of each Λ_l, then it is said to be *continuous with respect to the family of subsets* $\{\Lambda_l\}$.

Definition 4.1.2. Let $\varphi\colon X \to \mathbb{R}$ be continuous with respect to $\{\Lambda_l\}$. The *topological pressure* of φ on Λ is

$$P_\Lambda(\varphi) = \sup_{l \geq 1} P_{\Lambda_l}(\varphi),$$

where $P_{\Lambda_l}(\varphi)$ is the topological pressure of (continuous) φ on Λ_l.

In Section 4.3 we show that, in a sense, the topological pressure of φ on Λ does not depend on the family $\{\Lambda_l\}$. Several other properties of the topological pressure are given in the same section. A variational principle for the topological pressure is shown in Section 4.2. The topological pressure and variational principle of several examples of dynamical systems and discontinuous potentials are considered in Section 4.4.

4.2 Variational Principle

Theorem 4.2.1. Assume that f-invariant $\Lambda \subset X$ has a nested family of subsets $\{\Lambda_l\}$ which exhaust Λ. Let $\varphi\colon X \to \mathbb{R}$ be continuous with respect to the family $\{\Lambda_l\}$. Assume that $\mathcal{L}(\Lambda, \varphi) \neq \varnothing$. Then

$$P_{\mathcal{L}(\Lambda,\varphi)}(\varphi) = \sup\left\{h_\mu(f) + \int_\Lambda \varphi\, d\mu : \mu \in \mathcal{M}_\varphi(\Lambda, f)\right\}.$$

Proposition 4.2.2.

$$P_\Lambda(\varphi) \geq \sup\left\{h_\mu(f) + \int_\Lambda \varphi\, d\mu : \mu \in \mathcal{M}_\varphi(\Lambda, f)\right\}.$$

Proof. Fix $\mu \in \mathcal{M}_\varphi(\Lambda, f)$. We will show that there exists an l so that

$$P_{\Lambda_l}(\varphi) \geq h_\mu(f) + \int_\Lambda \varphi\, d\mu. \tag{4.2.1}$$

We use the following lemma in the proof of the variational principle.

Lemma 4.2.3 (see Pesin [Pes97]). For any $\epsilon > 0$ there exits δ, $0 < \delta \leq \epsilon$, a finite Borel partition $\zeta = \{Z_1, \ldots, Z_m\}$ of Λ, and a finite open cover $\mathcal{U} = \{U_1, \ldots, U_k\}$, $k \geq m$, of X such that

1. $\operatorname{diam}(U_i) \leq \epsilon$, $\operatorname{diam}(Z_j) \leq \epsilon$, $i = 1, \ldots, k$, $j = 1, \ldots, m$;
2. $\overline{U}_i \subset Z_i$, $i = 1, \ldots, m$;
3. $\mu(Z_i \setminus U_i) \leq \delta$, $i = 1, \ldots, m$ and $\mu(\bigcup_{i=m+1}^k U_i) \leq \delta$; and
4. $2\delta \log m \leq \epsilon$.

Given $y \in \Lambda$, let $t_n(y)$ denote the number of those p, $0 \leq p < n$ for which $f^p(y) \in U_i$, for some $i = m+1, \ldots, k$. Using Lemma 4.2.3 part 3 and the Birkhoff Ergodic theorem, there exists $N_1 > 0$ and a set $A_1 \subset \Lambda$ such that $\mu(A_1) \geq 1 - \delta$ and for any $y \in A_1$ and $n \geq N_1$,

$$n^{-1} t_n(y) \leq 2\delta. \tag{4.2.2}$$

As f is continuous on a compact metric space, the measure-theoretic entropy $h_\mu(f)$ is finite. Set $\zeta_n = \zeta \vee f^{-1}\zeta \vee \cdots \vee f^{-n+1}\zeta$. Using the Shannon-McMillan-Breiman theorem, there exists $N_2 > 0$ and a set $A_2 \subset \Lambda$ so that $\mu(A_2) \geq 1 - \delta$ and for any $y \in A_2$ and $n \geq N_2$,

$$\mu(Z_{\zeta_n}(y)) \leq \exp(-(h_\mu(f, \zeta) - \delta)n), \tag{4.2.3}$$

where $Z_{\zeta_n}(y)$ denotes the element of the partition ζ_n containing y and $h_\mu(f,\zeta)$ denotes the measure-theoretic entropy of the partition ζ with respect to f.

Using the Birkhoff Ergodic theorem, there exists a large $N_3 > 0$ and a set $A_3 \subset \Lambda$ so that $\mu(A_3) \geq 1-\delta$ and for any $y \in A_3$ and $n \geq N_3$,

$$\left| \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(y)) - \int_\Lambda \varphi \, d\mu \right| \leq \delta. \tag{4.2.4}$$

Set $N = \max\{N_1, N_2, N_3\}$ and $A = A_1 \bigcap A_2 \bigcap A_3$. Notice that the set A satisfies $\mu(A) \geq 1 - 3\delta$. Choose any $n \geq N$ and any

$$\lambda < h_\mu(f,\zeta) + \int_\Lambda \varphi \, d\mu - \epsilon. \tag{4.2.5}$$

Since $\Lambda = \bigcup_{l=1}^\infty \Lambda_l$ and $\Lambda_l \subset \Lambda_{l+1}$, we can choose l so that $\mu(\Lambda_l) > 1-\delta$. We have that

$$\mu(\Lambda_l \bigcap A) > 1 - 4\delta. \tag{4.2.6}$$

Set

$$M(Y,\alpha,\varphi,\mathcal{U},N) = \inf_\Gamma \sum_{\underline{U}\in\Gamma} \exp\left(-\alpha m(\mathcal{U}) + \sup_{x\in X(\underline{U})} \sum_{k=0}^{m(\underline{U})-1} \varphi(f^k x) \right),$$

where $\Gamma \subset \bigcap_{m\geq n} W_m(\mathcal{U})$ that cover Y. (See Section 2.2.1.) Choose a cover $\Gamma \subset W(\mathcal{U})$ of Λ_l with $m(\underline{U}) \geq N$, for every $\underline{U} \in \Gamma$, such that

$$\left| \sum_{\underline{U}\in\Gamma} \exp\left(-\lambda m(\underline{U}) + \sup_{x\in X(\underline{U})} \sum_{k=0}^{m(\underline{U})-1} \varphi(f^k(x)) \right) - M(\Lambda_l,\lambda,\varphi,\mathcal{U},N) \right| \leq \delta. \tag{4.2.7}$$

Let $\Gamma_p \subset \Gamma$ be the Q of strings from the cover Γ for which $m(\underline{U}) = p$ and $X(\underline{U}) \bigcap A \neq \varnothing$. Denote by P_p the cardinality of Γ_p. Set $Y_p = \bigcup_{\underline{U}\in\Gamma_p} X(\underline{U})$.

Lemma 4.2.4. We have $P_p \geq \mu(Y_p \bigcap A) \exp\left((h_\mu(f,\zeta) - \delta - 2\delta \log m)p\right)$.

Proof. Let L_p be the number of those elements Z_{ζ_p} of the partition ζ_p for which

$$Z_{\zeta_p} \bigcap Y_p \bigcap A \neq \varnothing. \tag{4.2.8}$$

The collection of such partition elements covers the set $Y_p \bigcap A$. Thus

$$\sum \mu(Z_{\zeta_p}) \geq \mu(Y_p \bigcap A), \tag{4.2.9}$$

where the sum is taken over all elements of the partition ζ_p for which condition (4.2.8) holds. Since $Z_{\zeta_p} \bigcap A_2 \neq \varnothing$ we have that

$$L_p \geq \mu(Y_p \bigcap A) \exp((h_\mu(f, \zeta) - \delta)p), \tag{4.2.10}$$

using inequalities (4.2.3) and (4.2.9). Fix a string $\underline{U} \in \Gamma_p$. Set $S_p(\underline{U})$ to be the number of those elements $Z_{\zeta_p} = Z_{i_0} \cap \cdots \cap f^{-(p-1)} Z_{i_{p-1}}$ of the partition ζ_p, for which $Z_{\zeta_p} \bigcap X(\underline{U}) \bigcap A \neq \varnothing$. Since $X(\underline{U}) \bigcap A_1 \neq \varnothing$ we have that there is only one choice for the partition element Z_{i_j} for every j except at most $2\delta p$ times when there are no more than m possibilities. The following estimate holds.

$$S_p(\underline{U}) \leq m^{2\delta p} = \exp(2\delta p \log m). \tag{4.2.11}$$

Since $P_p S_p(\underline{U}) = L_p$, the inequality follows using (4.2.10) and (4.2.11). $\square$

The following estimate holds.

$$\begin{aligned}
&\sum_{\underline{U} \in \Gamma} \exp\left(-\lambda m(\underline{U}) + \sup_{x \in X(\underline{U})} \sum_{k=0}^{m(\underline{U})-1} \varphi(f^k(x))\right) \\
&\geq \sum_{p=N}^{\infty} \sum_{\underline{U} \in \Gamma_p} \exp\left(-\lambda p + \sup_{x \in A \bigcap X(\underline{U})} \sum_{k=0}^{p-1} \varphi(f^k(x))\right) \\
&\geq \sum_{p=N}^{\infty} P_p \exp\left(\left(-\lambda + \int_\Lambda \varphi \, d\mu - \delta\right) p\right) \\
&\geq \sum_{p=N}^{\infty} \mu(Y_p \bigcap A) \exp\left(\left(h_\mu(f, \zeta) + \int_\Lambda \varphi \, d\mu - 2\delta - 2\delta \log m - \lambda\right) p\right) \\
&\geq \sum_{p=N}^{\infty} \mu(Y_p \bigcap A) \geq \mu(\Lambda_l \bigcap A) \geq 1 - 4\delta.
\end{aligned}$$

Above we used that for sufficiently small ϵ inequality (4.2.5) implies

$$h_\mu(f, \zeta) + \int_\Lambda \varphi \, d\mu - 2\delta - 2\delta \log m - \lambda > 0.$$

Thus $M(\Lambda_l, \lambda, \varphi, \mathcal{U}, N) \geq 1 - 5\delta \geq 1/2$ for ϵ, and thus δ, small enough. Hence, $P_{\Lambda_l}(\varphi, \mathcal{U}) \geq \lambda$, which implies

$$P_{\Lambda_l}(\varphi) \geq P_{\Lambda_l}(\varphi, \mathcal{U}) \geq h_\mu(f, \zeta) + \int_\Lambda \varphi \, d\mu - \epsilon.$$

Letting ϵ go to zero gives that $h_\mu(f, \zeta) \to h_\mu(f)$; thus we have (4.2.1). $\square$

Corollary 4.2.5.

$$P_{\mathcal{L}(\Lambda,\varphi)}(\varphi) \geq \sup\left\{h_\mu(f) + \int_\Lambda \varphi\, d\mu : \mu \in \mathcal{M}_\varphi(\Lambda, f)\right\}.$$

Proof. Given a measure $\mu \in \mathcal{M}_\varphi(\Lambda, f)$ denote by Z_μ the set of points in Λ defined as $Z_\mu = \{x \in \Lambda : V(x) = \{\mu\}\}$. We have that $\mu(Z_\mu) = 1$ and $Z_\mu \subset \mathcal{L}(Z, \varphi)$. Therefore, by Proposition 4.2.2,

$$P_{\mathcal{L}(\Lambda,\varphi)}(\varphi) \geq P_{Z_\mu}(\varphi) \geq h_\mu(f) + \int_\Lambda \varphi\, d\mu. \qquad \square$$

Proposition 4.2.6.

$$P_{\mathcal{L}(\Lambda,\varphi)}(\varphi) \leq \sup\left\{h_\mu(f) + \int_\Lambda \varphi\, d\mu : \mu \in \mathcal{M}_\varphi(\Lambda, f)\right\}.$$

Proof. As $\mathcal{L}(\Lambda, \varphi) = \bigcup_{l\geq 1}(\mathcal{L}(\Lambda, \varphi) \bigcap \Lambda_l)$, we have that

$$P_{\mathcal{L}(\Lambda,\varphi)}(\varphi) = \sup_{l\geq 1} P_{\mathcal{L}(\Lambda,\varphi)\bigcap \Lambda_l}(\varphi).$$

We show that for every $l \geq 1$

$$P_{\mathcal{L}(\Lambda,\varphi)\bigcap \Lambda_l}(\varphi) \leq \sup\left\{h_\mu(f) + \int_\Lambda \varphi d\mu : \mu \in \mathcal{M}_\varphi(\Lambda, f)\right\}.$$

Let E be a finite set and $\underline{a} = (a_0, \dots, a_{k-1}) \in E^k$. Define the measure $\mu_{\underline{a}}$ on E by

$$\mu_{\underline{a}}(e) = \frac{1}{k}(\text{the number of those } j \text{ for which } a_j = e).$$

Set

$$H(\underline{a}) = -\sum_{e\in E} \mu_{\underline{a}}(e) \log \mu_{\underline{a}}(e).$$

Consider the set

$$R(k, h, E) = \{\underline{a} \in E^k : H(\underline{a}) \leq h\}.$$

The following statement describes the asymptotic growth in k of the number of elements in the set $R(k, h, E)$.

Lemma 4.2.7 (Bowen [Bow70]). The following inequality holds.

$$\limsup_{k\to\infty} \frac{1}{k} \log |R(k, h, E)| \leq h.$$

Let $\mathcal{U} = \{U_1, \dots U_r\}$ be an open cover of X and $\epsilon > 0$. Set

$$\gamma_l(\mathcal{U}) = \sup\{|\varphi(x) - \varphi(y)| : x, y \in U_i \bigcap \Lambda_l \text{ for some } U_i \in \mathcal{U}\}.$$

Notice that $\gamma_l(\mathcal{U}) \to 0$ as $\operatorname{diam}(\mathcal{U}) \to 0$.

Lemma 4.2.8. Given $x \in \mathcal{L}(\Lambda, \varphi) \bigcap \Lambda_l$ and $\mu \in V(x) \bigcap \mathcal{M}_\varphi(\Lambda, f)$, there exists a number $m > 0$ such that for any $n > 0$ one can find $N > n$ and a string $\underline{U} \in S(\mathcal{U})$ with $m(\underline{U}) = N$ satisfying:

1. $x \in X(\underline{U})$;
2. $$\sup_{x \in X(\underline{U}) \bigcap \Lambda_l} \sum_{k=0}^{N-1} \varphi(f^k(x)) \leq N \left(\gamma_l(\mathcal{U}) + \int_\Lambda \varphi d\mu + \epsilon \right);$$
3. the string $\underline{U}$ contains a substring $\underline{U}'$ of length $m(\underline{U}') = km \geq N - m$ which, being written as $\underline{a} = (a_0, \dots, a_{k-1})$, satisfies the inequality
$$\frac{1}{m} H(\underline{a}) \leq h_\mu(f) + \epsilon. \tag{4.2.12}$$

Proof. Parts 1 and 3 are found in Pesin [Pes97]. Since μ_{x,n_j} converges to the measure μ and φ is integrable with respect to μ we obtain for sufficiently large N that

$$\left| \sup_{y \in X(\underline{U}) \bigcap \Lambda_l} \sum_{k=0}^{N-1} \varphi(f^k(y)) - N \int_\Lambda \varphi d\mu \right| \leq$$

$$\left| \sup_{y \in X(\underline{U}) \bigcap \Lambda_l} \sum_{k=0}^{N-1} \varphi(f^k(y)) - \sum_{k=0}^{N-1} \varphi(f^k(x)) \right| + \left| \sum_{k=0}^{N-1} \varphi(f^k(x)) - N \int_\Lambda \varphi d\mu \right| \leq$$

$$N\gamma_l(\mathcal{U}) + N\epsilon.$$ □

Given a number $m > 0$, denote by Y_m the set of points $y \in \mathcal{L}(\Lambda, \varphi) \bigcap \Lambda_l$ for which Lemma 4.2.8 holds for this m and a measure $\mu \in V(x) \bigcap \mathcal{M}_\varphi(\Lambda, f)$. We have that $\mathcal{L}(\Lambda, \varphi) \bigcap \Lambda_l = \bigcup_{m>0} Y_m$. Denote also by $Y_{m,u}$ the set of points $y \in Y_m$ for which Lemma 4.2.8 holds for some measure μ satisfying $\int_\Lambda \varphi d\mu \in [u - \epsilon, u + \epsilon]$. Set

$$c = \sup \left\{ h_\mu(f) + \int_\Lambda \varphi d\mu : \mu \in \mathcal{M}_\varphi(\Lambda, f) \right\}.$$

Note that if $x \in Y_{m,u}$ then the corresponding measure-theoretic entropy for μ and f satisfies $h_\mu(f) \le c - u + \epsilon$.

Let $G_{m,u}$ be the collection of all strings $\underline{U}$ described in Lemma 4.2.8 that correspond to all $x \in Y_{m,u}$ and all N exceeding some number N_0. It follows from Inequality (4.2.12) that for any $x \in Y_{m,u}$, the substring constructed in Lemma 4.2.8 is contained in $R(k, m(h+\epsilon), \mathcal{U}^m)$, where $h = c - u + \epsilon$. Therefore, the total number of the strings constructed in Lemma 4.2.8 does not exceed $b(N) = |\mathcal{U}|^m |R(k, m(h+\epsilon), \mathcal{U}^m)|$. By Lemma 4.2.7 we obtain that

$$\limsup_{N\to\infty} \frac{\log b(N)}{N} \le h + \epsilon. \tag{4.2.13}$$

Since the collection of strings $G_{m,u}$ covers the set $Y_{m,u}$ we conclude using Lemma 4.2.8 and (4.2.13) that

$$\begin{aligned} M'(Y_{m,u}, \lambda, \varphi, \mathcal{U}, N_0) \le &\\ &\sum_{N=N_0}^{\infty} b(N) \exp\left(-\lambda m(\underline{U}) + \sup_{x\in X(\underline{U})\bigcap \Lambda_l} \sum_{k=0}^{m(\underline{U})-1} \varphi(f^k(x))\right) \le \\ &\sum_{N=N_0}^{\infty} b(N) \exp\left(-\lambda m(\underline{U}) + N\left(\int_\Lambda \varphi d\mu + \gamma_l(\mathcal{U}) + \epsilon\right)\right). \end{aligned}$$

If N_0 is sufficiently large, we have that $b(N) \le \exp(N(h+2\epsilon))$. Hence,

$$M'(Y_{m,u}, \lambda, \varphi, \mathcal{U}, N_0) \le \frac{\beta^{N_0}}{1-\beta}, \tag{4.2.14}$$

where

$$\beta = \exp\left(-\lambda + h + \int_\Lambda \varphi d\mu + \gamma_l(\mathcal{U}) + 3\epsilon\right).$$

It follows from (4.2.14) that if $\lambda > c + \gamma_l(\mathcal{U}) + 4\epsilon$ then $m_C(Y_{m,u}, \lambda) = 0$. Hence, $\lambda \ge P_{Y_{m,u}}(\varphi, \mathcal{U})$. Set $A = |\int_\Lambda \varphi d\mu|$. Assume that points $u_1, \dots, u_r$ form an ϵ-net of the interval $[-A, A]$. Then

$$\mathcal{L}(\Lambda, \varphi) \bigcap \Lambda_l = \bigcup_{m=1}^{\infty} \bigcup_{i=1}^{r} Y_{m,u_i}.$$

We have that $\lambda \ge P_{Y_{m,u_i}}(\varphi, \mathcal{U})$ for any m and i. Therefore,

$$\lambda \ge \sup_{m,i} P_{Y_{m,u_i}}(\varphi, \mathcal{U}) = P_{\mathcal{L}(\Lambda,\varphi)\bigcap\Lambda_l}(\varphi, \mathcal{U}).$$

This implies that $c + \gamma_l(\mathcal{U}) + 4\epsilon \geq P_{\mathcal{L}(\Lambda,\varphi)\bigcap\Lambda_l}(\varphi,\mathcal{U})$. Since ϵ can be chosen arbitrarily small it follows that $c + \gamma_l(\mathcal{U}) \geq P_{\mathcal{L}(\Lambda,\varphi)\bigcap\Lambda_l}(\varphi,\mathcal{U})$. Taking the limit as $\operatorname{diam}(\mathcal{U}) \to 0$ yields $c \geq P_{\mathcal{L}(\Lambda,\varphi)\bigcap\Lambda_l}(\varphi)$ and the desired result follows. □

4.3 Properties of the Topological Pressure

The definition of the topological pressure for discontinuous potentials relies on the family of set $\{\Lambda_l\}$, however, the value of $P_\Lambda(\varphi)$ is independent of the family $\{\Lambda_l\}$ in the following sense.

Theorem 4.3.1. Assume that f-invariant $\Lambda \subset X$ has two nested families of subsets $\{\Lambda_l\}$ and $\{\Gamma_l\}$ which exhaust Λ. Let $\varphi \colon X \to \mathbb{R}$ be continuous with respect to both $\{\Lambda_l\}$ and $\{\Gamma_l\}$. Then

$$P_\Lambda(\varphi) = \sup_{l\geq 1} P_{\Lambda_l}(\varphi) = \sup_{l\geq 1} P_{\Gamma_l}(\varphi).$$

Proof. Set $P'_\Lambda(\varphi) = \sup_{l\geq 1} P_{\Lambda_i}(\varphi)$ and $P''_\Lambda(\varphi) = \sup_{l\geq 1} P_{\Gamma_i}(\varphi)$.

For every $\epsilon > 0$ there exists an n such that $P_{\Lambda_n}(\varphi) \geq P'_\Lambda(\varphi) - \epsilon$. As the Γ_l exhaust Λ, we can write $\Lambda_n = \bigcup_{m\geq 1}(\Lambda_n \bigcap \Gamma_m)$. As φ is continuous on Λ_n and each Γ_m,

$$P_{\Lambda_n}(\varphi) = \sup_{m\geq 1} P_{\Lambda_n\bigcap\Gamma_m}(\varphi) \leq \sup_{m\geq 1} P_{\Gamma_m}(\varphi) = P''_\Lambda(\varphi).$$

Thus $P''_\Lambda(\varphi) \geq P'_\Lambda(\varphi) - \epsilon$ for every ϵ. Reversing the roles of $P'_\Lambda(\varphi)$ and $P''_\Lambda(\varphi)$ gives the result. □

The topological pressure of φ on Λ satisfies many of the properties of the classical topological pressure. The theorem below follows from the definition of the topological pressure (as a supremum) and the corresponding results for continuous potentials.

The potential φ is *cohomologous* to ψ if there exists a continuous f such that $\varphi - \psi = f - f \circ T$.

Theorem 4.3.2. Assume that f-invariant Λ has a nested family of subsets $\{\Lambda_l\}$ which exhaust Λ. Let $\varphi \colon X \to \mathbb{R}$ be continuous with respect to $\{\Lambda_l\}$. Let $Z \subset \Lambda$ be any subset. The following properties hold.

1. $P_\varnothing(\varphi) \leq 0$,

2. $P_{Z_1}(\varphi) \leq P_{Z_2}(\varphi)$, if $Z_1 \subset Z_2 \subset \Lambda$,

3. $P_Z(\varphi) = \sup_{i\geq 1} P_{Z_i}(\varphi)$, where $Z = \bigcup_{i\geq 1} Z_i$,

4. If f is a homeomorphism then $P_Z(\varphi) = P_{f(Z)}(\varphi)$,

5. If $h\colon X \to X$ is a homeomorphism which commutes with f (that is $f \circ h = h \circ f$) then $P_Z(\varphi) = P_{h(Z)}(\varphi \circ h^{-1})$,

6. $|P_Z(\varphi) - P_Z(\psi)| \leq ||\varphi - \psi||$, where $||\cdot||$ denotes the supremum norm,

7. $P_Z(\varphi + c) = P_Z(\varphi) + c$,

8. $P_Z(t\varphi + (1-t)\psi) \leq tP_Z(\varphi) + (1-t)P_Z(\psi)$,

9. If φ is cohomologous to ψ, and then $P_Z(\varphi) = P_Z(\psi)$.

Assume that f-invariant subset Λ has a nested family of subsets $\{\Lambda_l\}$ which exhaust Λ. Let $\varphi\colon X \to \mathbb{R}$ be continuous with respect to $\{\Lambda_l\}$. Each function in the family $\varphi_t = t\varphi\colon X \to \mathbb{R}$ is also continuous with respect to $\{\Lambda_l\}$. Define the *pressure function* $\Psi\colon \mathbb{R} \to \mathbb{R}$ associated to φ as

$$\Psi(t) = P_{\mathcal{L}(\Lambda,\varphi_t)}(\varphi_t) = \sup\left\{h_\mu(f) - t\int_\Lambda \varphi\, d\mu : \mu \in \mathcal{M}_{\varphi_t}(\Lambda, f)\right\}.$$

The potential φ_1 is integrable with respect to a measure μ if and only if the potential φ_t is integrable with respect to the measure μ. Thus the sets of measures $\mathcal{M}_{\varphi_t}(\Lambda, f)$ are equal for every t, that is $\mathcal{M}_{\varphi_t}(\Lambda, f) = \mathcal{M}_{\varphi_1}(\Lambda, f)$, for every $t \in \mathbb{R}$. This implies that $\mathcal{L}(\Lambda, \varphi_t) = \mathcal{L}(\Lambda, \varphi_1)$ for every $t \in \mathbb{R}$. In particular, $\Psi(t)$ is well-defined for all t.

Theorem 4.3.3. Assume that f-invariant $\Lambda \subset X$ has a nested family of subsets $\{\Lambda_l\}$ which exhaust Λ. Let φ be continuous with respect to the family $\{\Lambda_l\}$. The following properties hold for the pressure function Ψ.

1. Monotone Decreasing: If $t < s$, then $\Psi(t) \geq \Psi(s)$.

2. Subadditivity: $\Psi(s + t) \leq \Psi(s) + \Psi(t)$.

Proof. For every $\mu \in \mathcal{M}_{\varphi_1}(\Lambda, f)$ we have that $-\int t\varphi\, d\mu > -s\int \varphi\, d\mu$. Thus

$$h_\mu(f) - t\int \varphi\, d\mu > h_\mu(f) - s\int \varphi\, d\mu.$$

For every $\mu \in \mathcal{M}_\varphi(\Lambda, f)$,

$$h_\mu(f) + \int (s+t)\varphi\, d\mu \leq (h_\mu(f) + \int s\varphi\, d\mu) + (h_\mu(f) + \int t\varphi\, d\mu). \qquad \square$$

Property (9) in Theorem 4.3.2 gives invariance under cohomology. The following theorem uses this invariance to show that when looking for equilibrium measures we can restrict our attention to functions which depend only on future behavior.

Theorem 4.3.4. Let $\varphi\colon X \to \mathbb{R}$ have summable variations. Then φ is homologous to $\psi\colon X \to \mathbb{R}$ with summable variations and $V_1(\varphi) < \infty$ with $\psi(x) = \psi(y)$, whenever $x_i = y_i$ for all $i \geq 0$.

Proof. For each state t pick $\underline{a}_t = \dots a_{-1} t a_1 \dots \in X$. Define $r\colon X \to X$ by $r(\underline{x}) = \underline{x}^\star$, where $\underline{x}^\star_k = x_k, k \geq 0$ and $(\underline{a}_{x_0})_k, k \leq 0$. Let

$$u(\underline{x}) = \sum_{j=0}^{\infty} (\varphi(\sigma^j \underline{x}) - \varphi(\sigma^j r(\underline{x}))).$$

We have that $\sigma^j \underline{x}$ and $\sigma^J r\underline{x}$ agree in places $-j$ to ∞. Thus we conclude $|\varphi(\sigma^j \underline{x}) - \varphi(\sigma^j(r\underline{x}))| \leq V_j(\varphi)$. Since φ has summable variations, the function u is defined and continuous.

If $x_i = y_i$ for all $|i| \leq n$, then for $j \in [0, n]$, the following hold;

$$|\varphi(\sigma^j \underline{x}) - \varphi(\sigma^j \underline{y})| \leq V_{n-j}(\varphi),$$

$$|\varphi(\sigma^j r\underline{x}) - \varphi(\sigma^j r\underline{y})| \leq V_{n-j}(\varphi).$$

Hence

$$\begin{aligned} |u(\underline{x}) - u(\underline{y})| \quad &\leq \textstyle\sum_{j=0}^{[\frac{n}{2}]} \left[|\varphi(\sigma^j \underline{x}) - \varphi(\sigma^j \underline{y})| + |\varphi(\sigma^j r\underline{x}) - \varphi(\sigma^j r\underline{y})| \right] \\ &+ \textstyle\sum_{j=[\frac{n}{2}]}^{\infty} \left[|\varphi(\sigma^j \underline{x}) - \varphi(\sigma^j \underline{y})| + |\varphi(\sigma^j r\underline{x}) - \varphi(\sigma^j r\underline{y})| \right] \\ &\leq 2 \textstyle\sum_{j=[\frac{n}{2}]}^{\infty} V_j(\varphi) < \infty. \end{aligned}$$

In conclusion, since φ has summable variations u does as well.

Hence the function $\psi(\underline{x}) = \varphi(\underline{x}) - u(\underline{x}) + u(\sigma\underline{x})$ has summable variations. Now

$$\begin{aligned} \psi(\underline{x}) &= \varphi(\underline{x}) + \sum_{j=0}^{\infty} (\varphi(\sigma^{j+1} r\underline{x}) - \varphi(\sigma^{j-1}\underline{x})) + \sum_{j=0}^{\infty} (\varphi(\sigma^{j+1}\underline{x}) - \varphi(\sigma^j r\sigma\underline{x})) \\ &= \varphi(r\underline{x}) + \sum_{j=0}^{\infty} (\varphi(\sigma^{j+1} r\underline{x}) - \varphi(\sigma^j r\sigma\underline{x})), \end{aligned}$$

which depends only on $\{x_k\}_{k=0}^{\infty}$ due to the presence of the function r. $\square$

4.4 Examples

The thermodynamic formalism for discontinuous potentials applies to several interesting examples.

4.4.1 Systems with Nonzero Lyapunov Exponents

The setup for the thermodynamic formalism for discontinuous potentials in this thesis was developed from the general theory of dynamical systems with nonzero Lyapunov exponents. Such systems have a subset Λ, which is of interest dynamically, with a natural set of nested exhaustive subsets. The concepts and definitions of systems with nonzero Lyapunov exponents can be found in [BP02].

Let $f\colon \mathcal{R} \to \mathcal{R}$ be a $C^{1+\epsilon}$ diffeomorphism of a compact smooth Riemannian manifold $\mathcal{R}$. Let μ be an ergodic Borel, f-invariant hyperbolic measure on $\mathcal{R}$. Recall that Λ_l, $l \geq 1$ is the set of regular points $x \in \mathcal{R}$,

$$\Lambda_l = \{x \in \mathcal{R} : D_1(x) \leq l \text{ and } D_2(x) \geq 1/l\}.$$

These sets are closed, $\Lambda_l \subset \Lambda_{l+1}$, and the set $\Lambda = \cup_{l\geq 1}\Lambda_l$ is f-invariant and coincides with $\mathcal{R}$ up to a set of μ measure zero.

Thus for any potential function φ which is continuous with respect to $\{\Lambda_l\}$, the topological pressure can be defined as (4.1.2), and the variational principle, Theorem 4.2.1, holds. In particular, the topological pressure can be defined for the function $\varphi_t(x) = -t \log \operatorname{Jac}(df_{|E^u_x})$, which is continuous on each Λ_l, but can be unbounded on Λ. The following variational principle holds.

Theorem 4.4.1. Let $f : \mathcal{R} \to \mathcal{R}$ be a $C^{1+\epsilon}$ diffeomorphism of a compact Riemannian manifold. Assume that there exists a hyperbolic measure ν on $\mathcal{R}$. Then

$$P_{\mathcal{L}(\Lambda,\varphi_1)}(\varphi_t) = \sup\left\{h_\mu(f) - t\int_\Lambda \log \operatorname{Jac}(df_{|E^u_x})\, d\mu : \mu \in \mathcal{M}_{\varphi_1}(\Lambda, f)\right\}.$$

4.4.2 Countable Markov Shifts

The thermodynamic formalism for discontinuous potentials is compared with that of countable Markov shifts [Gur69, Sar99, Sar03]. The thermodynamic formalism for countable Markov shifts has recently been of great interest. The existence and uniqueness of (Gibbs and) equilibrium measures has so far proven to be more complex than its subshift of finite type counterpart (see for example [Aar97, ADU93, MU01, Yur99, PS05]).

Let (M, σ) be a countable Markov shift. Assume that $\phi : M \to \mathbb{R}$ has summable variations, that is,

$$\sum_{n=1}^{\infty} V_n(\phi) < \infty.$$

The Gurevich pressure of ϕ is

$$P_G(\phi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{x\,:\,\sigma^n x = x} \exp\left(\sum_{k=0}^{n-1} \phi(\sigma^k(x))\right) 1_{[a]}(x),$$

where $1_{[a]}$ is the indicator function on the cylinder set $[a]$; the limit exists and is independent of $a \in S$.

The topological pressure for discontinuous potentials applies to countable Markov shifts as the space M is not compact. A potential function with summable variations is continuous on M, though possibly unbounded.

The following connection between the Gurevich pressure and the topological pressure holds.

$$\begin{aligned} P_G(\varphi) &= \sup\{P_{top}(\varphi_{|Y}) : Y \subset M \text{ topologically mixing finite Markov shift}\} \\ &= \sup\{P_{top}(\varphi_{|K}) : K \subset M \text{ compact }, T^{-1}K = K\} \end{aligned} \tag{4.4.1}$$

Starting with the set $\Lambda_l = \{\omega = \omega_0\omega_1\omega_2 \cdots \in M : \forall i,\, 0 \le a_i < l\}$ one can add a finite number of states and construct a topologically mixing finite Markov shift Z_l, with $\Lambda_l \subset Z_l \subset M$. (For each pair of letters $0 \le a_i, a_j < l$, one must add only the finite number of letters which form an allowable word beginning with a_i and ending with a_j.) These sets Z_l are compact f-invariant subsets of M. The family of sets $\{Z_l\}$ has a subfamily Z_{l_k} such that $Z_{l_k} \subset Z_{l_{k+1}}$. The set $\Lambda = \bigcup_{k\ge1} Z_{l_k}$ is σ-invariant and strictly contained in M. If ϕ has summable variations, then ϕ is continuous; in particular, ϕ is continuous on each Z_{l_k}. Thus the topological pressure of ϕ on Λ can be defined as $P_\Lambda(\phi) = \sup_{k\ge1} P_{Z_{l_k}}(\phi)$. Theorem 4.3.1 and Equation 4.4.1 give the following theorem.

Theorem 4.4.2. Assume M is a topologically mixing countable Markov shift. Let $\phi\colon M \to \mathbb{R}$ have summable variations. Then

$$P_\Lambda(\phi) = P_G(\phi).$$

Theorem 4.3.1 and Theorem 4.4.2 together imply that the Gurevich pressure can be computed (as in (4.1.2)) by taking the supremum over sets which are neither compact nor f-invariant, for example the sets Λ_l.

4.4.3 Unimodal Maps

The thermodynamic formalism for discontinuous potentials is applied to certain unimodal maps. Results of Bruin and Keller [BK98] are used.

Let $f\colon \mathcal{I} = [0,1] \to \mathcal{I}$ be a unimodal map with critical point p. Let $\mathcal{P} = \bigcup_{k\geq 0} f^{-k}(p)$, and let $\Lambda_l = ([0, p-1/l] \bigcup [p+1/l, 1]) \bigcap \mathcal{P}$. Then the family $\{\Lambda_l\}$ is nested and exhausts $\Lambda = \mathcal{I} \setminus \mathcal{P}$. Thus the topological pressure of any φ continuous with respect to the family $\{\Lambda_l\}$ can be defined as (4.1.2).

The function $\varphi_t(x) = -t \log|f'|$ is bounded, hence continuous, on the closure of each Λ_l. Thus the topological pressure and pressure function can be defined for φ_t for every t.

For a unimodal map f which has the following properties:

1. f is C^3 on $\mathcal{I} - \{p\}$
2. there exists an $l > 1$ and a continuous strictly positive $M\colon \mathcal{I} \to \mathbb{R}$ such that $|f'(x)| = M(x)|x-p|^{l-1}$ for every $x \in \mathcal{I}$,

work of Bruin and Keller [BK98] shows that φ_1 is integrable with respect to every measure $\mu \in \mathcal{M}(\mathcal{I}, f)$. (In [BK98], the function is also assumed to have negative Schwartzian derivative. This condition can be dropped. See [Bru95, GS05].) Thus $\mathcal{M}_{\varphi_1}(\Lambda, f) = \mathcal{M}(\Lambda, f)$, and $\mathcal{L}(\Lambda, \varphi_1) = \mathcal{L}(\Lambda)$. The following variational principle holds.

Theorem 4.4.3. Let f be a unimodal map with critical point p satisfying properties 1 and 2. Then

$$P_{\mathcal{L}(\Lambda)}(\varphi_t) = \sup\left\{ h_\mu(f) - t\int \log|f'|d\mu : \mu \in \mathcal{M}(\Lambda, f)\right\}.$$

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Vita

Anna Mummert

Born: April 7, 1978, Wellsville, New York.

Education

PhD	August 2006	The Pennsylvania State University Thesis Advisor: Yakov Pesin
BA	May 2000	State University of New York College at Geneseo *summa cum laude*

Publications

1. *Thermodynamic Formalism for Almost-additive Sequences.* Anna Mummert. Accepted for publication. Discrete and Continuous Dynamical Systems, October 2006, Volume 16, Number 2, Pages 435-454.

2. *A Variational Principle for Discontinuous Potentials.* Anna Mummert. Accepted for publication. Ergodic Theory and Dynamical Systems.

3. *Thermodynamic Formalism for Nonuniformly Hyperbolic Dynamical Systems.* PhD dissertation, August 2006.