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STUDIES OF PARTITION FUNCTIONS WITH CONDITIONS ON PARTS AND PARITY

A Dissertation in
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by
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Abstract

This dissertation explores four topics in partition theory, with main themes on parts and parities conditions. The first topic studies many properties of the $\overline{\mathcal{EO}}$-partitions by Andrews, especially the results analogous to the work of Atkin and Swinnerton-Dyer. The second topic deals with the concept of Andrews' separable integer partition classes, which yields alternate proofs for many partition identities including little Göllnitz identities. The third topic gives infinite families of congruences for partition functions arising from Ramanujan's mock theta functions. Some other related identities analogue to Euler's pentagonal number theorem are also proved. Finally, the fourth topic presents a combinatorial proof for an overpartition identity, derived from the truncated version of a certain theta series identity.

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Chapter 1
Introduction

The problem regarding the number of ways to represent a positive integer as a sum of positive integers can be dated back to Leibniz. It was the breakthrough work of Euler, who discovered the generating function of the integer partition function, that started the intensive research on this additive number theory problem. Studies on this particular problem have been done extensively in the past centuries, both analytically and combinatorially. There are also studies on variants of the partition function, where some conditions on the properties of partitions (for examples, number of parts, parities, etc.) were further imposed.

The main theme of this dissertation is focused on properties of several variants of the partition function, in which most of them have certain restrictions on the parity of parts. Many results are new, while some present alternate proofs which may provide new insight to further generalizations. We will give an overview for each topic in this chapter, and discuss the topic in detail in the following chapters. We also collect some definitions and propositions at the end of this chapter, which will be used several times throughout the work.

1.1 $\overline{\mathcal{EO}}$−partitions

Let us define the partition function $p(n)$ to be the number of ways to write a nonnegative integer n as a sum of positive integers, regardless of the order of summation. One of the most important results in the theory of partition is the divisibility properties of $p(n)$ discovered by Ramanujan:

$$p(5n+4) \equiv 0 \pmod 5 \tag{1.1.1}$$
$$p(7n+5) \equiv 0 \pmod 7 \tag{1.1.2}$$
$$p(11n+6) \equiv 0 \pmod{11}. \tag{1.1.3}$$

In 2018, George E. Andrews studied two subsets of the integer partitions called $\mathcal{EO}$-partitions, which are integer partitions whose even parts are smaller than odd parts, and $\overline{\mathcal{EO}}$-partitions, in which further condition that only the largest even part can appear an odd number of times was imposed. Parts of the work resulted in the divisibility property of $\overline{\mathcal{EO}}$-partitions similar to Ramanujan's. We will investigate further divisibility properties and offer results analogous to the work of Atkin and Swinnerton-Dyer on the partition function $p(n)$. For example, the congruence for the 5-dissection of the generating function for $\overline{\mathcal{EO}}$-partition function.

1.2 Separable Integer Partition Classes

In an upcoming paper, George E. Andrews considered classes of integer partitions with the idea of "basis" for partitions, and called such classes separable integer partition classes (or SIP classes). The idea provided new insights on many famous partition theorems, which may also lead to more generalizations of related identities. We will give new proofs for several partition identities by using an SIP classes method, including both little Göllnitz identities and Göllnitz-Gordon identity.

1.3 Partition functions associated with Mock Theta Functions

In 2015, Andrews et al. found several combinatorial interpretations arise from many of the Ramanujan's mock theta functions. More precisely, two of the Ramanujan's mock theta functions that were considered are

$$\omega(q) := \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q;q^2)_{n+1}^2} \tag{1.3.1}$$

$$\nu(q) := \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(-q;q^2)_{n+1}}. \tag{1.3.2}$$

The two functions give rise to two variants of the function $p(n)$, namely, $p_\omega(n)$ which counts the number of partitions of n whose odd parts are smaller than twice the smallest part, and $p_\nu(n)$ which counts the number of similar partitions as in $p_\omega(n)$, but with the further condition that each part must be distinct. Some properties and congruences for both functions were discovered. We will provide new infinite families of congruences for $p_\omega(n)$. This is joint work with George Andrews, James Sellers, and Ae Ja Yee. Furthermore, we will provide some identities related to the generating functions for both $p_\omega(n)$ and $p_\nu(n)$, and also generalize some of the previous work on results analogous to Euler's pentagonal number theorem.

1.4 Truncated Theta Series Theorem

In 2013, Victor J. W. Guo and Jiang Zeng derived the truncated version of the following well-known theta series identity:

$$\frac{(q;q)_\infty}{(-q;q)_\infty} = 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2} \tag{1.4.1}$$

and also gave a combinatorial interpretation of such result in terms of inequalities involving the number of overpartitions. Later, in 2018, George E. Andrews and Mircea Merca improved the results by finding the correct additional term, which change the inequalities into a combinatorial identity. Afterward, two combinatorial proofs were created. We will provide one of the proofs which involves eliminating half of the total number of overpartitions in consideration. This is joint work with Cristina Ballantine, Mircea Merca, and Ae Ja Yee.

1.5 Collection of definitions and propositions

We collect here the definitions and well-known propositions from [2], [21], which will be referenced throughout this work.

Definition 1.5.1. *A* ***partition*** *of a positive integer n is a non-increasing finite sequence of positive integers*

$$\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k),$$

such that $\sum_{i=1}^{k} \lambda_i = n$. A partition can also be written as $\lambda_1 + \lambda_2 + \cdots + \lambda_k$.

While we formally define partition as a non-increasing sequence of positive integers, we may resort to non-decreasing convention without much hassle.

Definition 1.5.2. *The* ***q-Pochhammer symbol*** *is defined as follows:*

$$\begin{aligned}(a;q)_n &:= (1-a)(1-aq)\cdots(1-aq^{n-1}),\\ (a;q)_\infty &:= \lim_{n\to\infty}(a;q)_n = \prod_{i=1}^{\infty}(1-aq^{i-1}).\end{aligned}$$

We also defined the following shorthand notations:

$$\begin{aligned}(a_1, a_2, \cdots, a_k; q)_n &:= (a_1;q)_n(a_2;q)_n\cdots(a_k;q)_n,\\ (a_1, a_2, \cdots, a_k; q)_\infty &:= (a_1;q)_\infty(a_2;q)_\infty\cdots(a_k;q)_\infty,\\ [a;q]_n &:= (a, a^{-1}q; q)_n,\\ [a;q]_\infty &:= (a, a^{-1}q; q)_\infty,\\ [a_1, a_2, \cdots, a_k; q]_n &:= [a_1;q]_n[a_2;q]_n\cdots[a_k;q]_n,\\ [a_1, a_2, \cdots, a_k; q]_\infty &:= [a_1;q]_\infty[a_2;q]_\infty\cdots[a_k;q]_\infty.\end{aligned}$$

Definition 1.5.3. *The* ***q-binomial coefficient****, or* ***Gaussian polynomial*** *for non-negative integers a, b and positive integer k is*

$$\begin{bmatrix} a \\ b \end{bmatrix}_k := \begin{cases} \frac{(q^k;q^k)_a}{(q^k;q^k)_b(q^k;q^k)_{a-b}}, & \text{if} \quad 0 \le b \le a,\\ 0, & \text{otherwise.}\end{cases}$$

Definition 1.5.4. *For a given partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$, there is a corresponding graphical representation $\mathcal{G}_\lambda$ called **Ferrers diagram** (or Young diagram), which is defined to be the set of k rows of points, such that the i^{th} row contains λ_i points.*

Another common way to represent Ferrers diagram is by using square grid instead of points, which we will mainly use in our work.

Figure 1.1: Ferrers diagrams for the partition "$5+5+4+2+2+2+1$".

Definition 1.5.5. *An **overpartition** of a positive integer n, is a partition of n where the last occurrence (or equivalently, the first occurrence) may be overlined.*

For example, "$5+5+\overline{4}+2+2+\overline{2}+1$" is an overpartition of 21. We can use the Ferrers diagram to represent an overpartition, in which each of the rows corresponding to the overlined parts will have its rightmost square colored black.

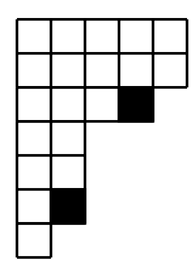
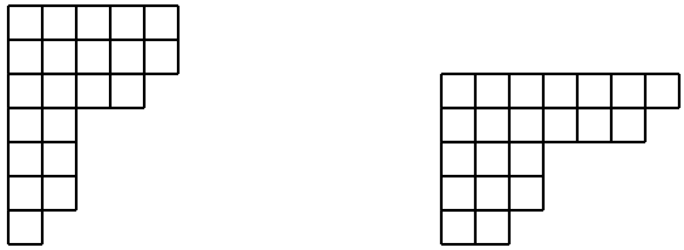


Figure 1.2: Ferrers diagram for the overpartition "$5+5+\overline{4}+2+2+\overline{2}+1$".

Definition 1.5.6. *For a given partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$, define a new partition called the **conjugate** of λ by $\lambda' = (\lambda'_1, \lambda'_2, \cdots, \lambda'_m)$, where λ'_i is the number of parts of λ that are greater or equal to i.*

Graphically, we can obtain the Ferrers diagram for the conjugate λ' of λ, by rewriting the rows of the Ferrers diagram for λ as columns and vice versa.



$5+5+4+2+2+2+1$ $\qquad$ $7+6+3+3+2$

Figure 1.3: Ferrers diagrams for a pair of conjugates.

Proposition 1.5.7 (Euler)**.**

$$(-q;q)_\infty = \frac{1}{(q;q^2)_\infty}. \tag{1.5.1}$$

Proposition 1.5.8 (q-binomial theorem)**.** *For complex numbers* $|q|, |z| < 1$,

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n = \frac{(az;q)_\infty}{(z;q)_\infty}. \tag{1.5.2}$$

Proposition 1.5.9 (Euler)**.** *For complex numbers q and z such that* $|q| < 1$,

$$\sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n} = \frac{1}{(z;q)_\infty}. \tag{1.5.3}$$

And if $|z| < 1$,

$$\sum_{n=0}^{\infty} \frac{(-z)^n q^{\frac{n(n-1)}{2}}}{(q;q)_n} = (z;q)_\infty. \tag{1.5.4}$$

Proposition 1.5.10 (Pentagonal number theorem)**.** *For complex numbers* $|q| < 1$,

$$(q;q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}}. \tag{1.5.5}$$

Proposition 1.5.11 (Jacobi triple product theorem)**.** *For complex numbers* $z \neq 0$ *and* $|q| < 1$,

$$\sum_{n=-\infty}^{\infty} z^n q^{\frac{n(n-1)}{2}} = (q;q)_\infty(-z;q)_\infty(-\frac{q}{z};q)_\infty. \tag{1.5.6}$$

Chapter 2
$\overline{\mathcal{EO}}$-partitions

2.1 Introduction

One aspect of the study of integer partitions is the divisibility condition. Ramanujan [34, 35] was the first to discover and prove the following fact. Let $p(n)$ denote the number of partitions of a positive integer n. Then

$$p(5n+4) \equiv 0 \pmod{5}, \tag{2.1.1}$$

$$p(7n+5) \equiv 0 \pmod{7}, \tag{2.1.2}$$

$$p(11n+6) \equiv 0 \pmod{11}. \tag{2.1.3}$$

Because of the divisibility behavior, some speculated that there should be certain "statistics" that would categorize partitions for such integers into equally numerous classes. Dyson [22] introduced these ideas for congruence mod 5 and 7, as the rank of a partition defined as follows: for a partition λ, let $l(\lambda)$ denote the largest part of λ and $\#(\lambda)$ the number of parts of λ. Then

$$rank(\lambda) = l(\lambda) - \#(\lambda).$$

For example, $p(4) = 5$ whose partitions are $4, 3+1, 2+2, 2+1+1, 1+1+1+1+1$. We can see that

$$rank(\text{“}4\text{”}) = 4 - 1 = 3,$$

$$rank(\text{“}3+1\text{”}) = 3 - 2 = 1,$$

$$rank(\text{"}2+2\text{"}) = 2-2 = 0,$$
$$rank(\text{"}2+1+1\text{"}) = 2-3 = -1,$$
$$rank(\text{"}1+1+1+1\text{"}) = 1-4 = -3.$$

Note that the rank for each of the partitions create a complete residue system modulo 5. Let $N(m,k,n)$ be the number of partitions of n with rank congruent to m modulo k. Dyson conjectured that

$$N(i,5,5n+4) = \frac{1}{5}p(5n+4) \qquad 0 \le i \le 4, \tag{2.1.4}$$
$$N(i,7,7n+5) = \frac{1}{7}p(7n+5) \qquad 0 \le i \le 6, \tag{2.1.5}$$

which were proved later by Atkin and Swinnerton-Dyer [15]. In fact, they also came up with congruences for the dissection of the generating function for the partition function. In particular, for modulo 5 we have the following result: let us define

$$P_5(a) = \begin{cases} [q^{5a}; q^{25}]_\infty, & a \neq 0, \\ (q^{25}; q^{25})_\infty, & a = 0. \end{cases}$$

Theorem 2.1.1. *Let*

$$\sum_{n=0}^{\infty} p(n)q^n = \sum_{n=0}^{4} \Phi_n(q^5)q^n.$$

Then

$$\Phi_0(q^5) \equiv \frac{P_5(0)P_5(2)}{P_5(1)^2} \pmod 5, \tag{2.1.6}$$
$$\Phi_1(q^5) \equiv \frac{P_5(0)}{P_5(1)} \pmod 5, \tag{2.1.7}$$
$$\Phi_2(q^5) \equiv 2\frac{P_5(0)}{P_5(2)} \pmod 5, \tag{2.1.8}$$
$$\Phi_3(q^5) \equiv 3\frac{P_5(0)P_5(1)}{P_5(2)^2} \pmod 5, \tag{2.1.9}$$
$$\Phi_4(q^5) \equiv 0 \pmod 5. \tag{2.1.10}$$

Observe that (2.1.10) implies the congruence in (2.1.1). However, the rank idea does not carry on beyond modulo 7. Nevertheless, Dyson had hoped for alternative statistics that would explain the case of modulo 13, naming it as a "crank" of the partition. It was until 1988 that the crank of the partition had been found by Andrews and Garvan [11].

Let $\omega(\lambda)$ denote the number of 1's in λ, and $\mu(\lambda)$ the number of parts in λ larger than $\omega(\lambda)$. Then the crank of the partition λ is

$$crank(\lambda) = \begin{cases} l(\lambda), & \omega(\lambda) = 0, \\ \mu(\lambda) - \omega(\lambda), & \omega(\lambda) > 0. \end{cases}$$

If we let $M(m, k, n)$ be the number of partitions of n with crank congruent to m modulo k, Andrews and Garvan proved that

$$\begin{aligned} M(i, 5, 5n+4) &= \frac{1}{5} p(5n+4) \qquad 0 \le i \le 4, \\ M(i, 7, 7n+5) &= \frac{1}{7} p(7n+5) \qquad 0 \le i \le 6, \\ M(i, 11, 11n+6) &= \frac{1}{11} p(11n+6) \quad 0 \le i \le 10. \end{aligned}$$

While the rank function may not equally distribute the partitions of $5n + k$ for other values of k, Atkin and Swinnerton-Dyer found some other relations between certain classes of rank modulo 5:

Theorem 2.1.2.

$$N(1, 5, 5n+1) = N(2, 5, 5n+1), \tag{2.1.11}$$

$$N(0, 5, 5n+2) = N(2, 5, 5n+2), \tag{2.1.12}$$

$$N(0, 5, 5n+4) = N(1, 5, 5n+4) = N(2, 5, 5n+4). \tag{2.1.13}$$

It can be seen that (2.1.4) follows from (2.1.13) and elementary property of ranks via conjugation, i.e.,

$$N(m,k,n) = N(k-m,k,n).$$

Furthermore, they also found the generating function for the difference between classes of rank modulo 5. For convenience, we will use the following notation as appeared in their paper.

$$\begin{aligned} r_a(d) = r_a(d,k) &:= \sum_{n=0}^{\infty} N(a,k,kn+d)q^n, \\ r_{ab}(d) = r_{ab}(d,k) &:= r_a(d) - r_b(d), \\ \Sigma(a,b) &:= \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\frac{75n(n+1)}{2}+5b}}{1-q^{25n+5a}}. \end{aligned}$$

Theorem 2.1.3. *For $k=5$, we have*

$$r_{12}(0) = q\frac{\Sigma(1,0)}{P_5(0)}, \tag{2.1.14}$$

$$r_{02}(0) + 2r_{12}(0) = \frac{P_5(0)P_5(2)}{P_5(1)^2} - 1, \tag{2.1.15}$$

$$r_{02}(1) = \frac{P_5(0)}{P_5(1)}, \tag{2.1.16}$$

$$r_{12}(1) = 0, \tag{2.1.17}$$

$$r_{02}(2) = 0, \tag{2.1.18}$$

$$r_{12}(2) = \frac{P_5(0)}{P_5(2)}, \tag{2.1.19}$$

$$r_{02}(3) = -q\frac{\Sigma(2,0)}{P_5(0)}, \tag{2.1.20}$$

$$r_{01}(3) + r_{02}(3) = \frac{P_5(0)P_5(1)}{P_5(2)^2}, \tag{2.1.21}$$

$$r_{02}(4) = 0, \tag{2.1.22}$$

$$r_{12}(4) = 0. \tag{2.1.23}$$

It is clear that (2.1.17) implies (2.1.11), (2.1.18) implies (2.1.12), and (2.1.22)-

(2.1.23) implies (2.1.13) respectively.

Surprisingly, these results are actually equivalent to one of the results already discovered by Ramanujan. In his Ph.D. thesis, Garvan [24] showed that the following 5-dissection due to Ramanujan [9, Entry 2.1.2] implied the results of Atkin and Swinnerton-Dyer.

Theorem 2.1.4. *Let* $z = e^{2\pi i/5}$, *then*

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(zq;q)_n(z^{-1}q;q)_n} = A(q^5) + (z + z^{-1} - 2)\phi(q^5) + qB(q^5) + (z + z^{-1})q^2C(q^5) - (z + z^{-1})q^3 \left[D(q^5) - (z^2 + z^{-2} - 2)\frac{\psi(q^5)}{q^5} \right],$$

where

$$\begin{aligned}
A(q) &:= \frac{[q^2;q^5]_\infty (q^5;q^5)_\infty}{[q;q^5]_\infty^2}, \\
B(q) &:= \frac{(q^5;q^5)_\infty}{[q;q^5]_\infty}, \\
C(q) &:= \frac{(q^5;q^5)_\infty}{[q^2;q^5]_\infty}, \\
D(q) &:= \frac{[q;q^5]_\infty (q^5;q^5)_\infty}{[q^2;q^5]_\infty^2}, \\
\phi(q) &:= -1 + \sum_{n=0}^{\infty} \frac{q^{5n^2}}{(q;q^5)_{n+1}(q^4;q^5)_n}, \\
\frac{\psi(q)}{q} &:= \sum_{n=0}^{\infty} \frac{q^{5n^2-1}}{(q^2;q^5)_{n+1}(q^3;q^5)_n}.
\end{aligned}$$

In [5], Andrews studied the classes of partitions whose even parts are smaller than odd parts if odd parts exist. Let $\mathcal{EO}(n)$ be the number of such partitions for n. For example, $\mathcal{EO}(8) = 12$, whose partitions are $8, 7+1, 6+2, 5+3, 5+1+1+1, 4+4, 4+2+2, 3+3+2, 3+3+1+1, 3+1+1+1+1+1, 2+2+2+2, 1+1+1+1+1+1+1+1$.

If we denote $\mathrm{eo}(q)$ as its generating function, it can be shown that

$$\mathrm{eo}(q) := \sum_{n=0}^{\infty} \mathcal{EO}(n) q^n = \frac{1}{(1-q)(q^2;q^2)_\infty}.$$

Andrews also considered the certain subset of partitions counted by $\mathcal{EO}(n)$, with further restrictions that only the largest even part will appear an odd number of times (if there is no even part, we treat such cases by noting the part 0 appears exactly once). Denote the number of such partition for n by $\overline{\mathcal{EO}}(n)$. For example, $\overline{\mathcal{EO}}(8) = 5$, whose partitions are $8, 4+2+2, 3+3+2, 3+3+1+1, 1+1+1+1+1+1+1+1$. By the parity restriction, it is easy to see that

$$\overline{\mathcal{EO}}(2n+1) = 0.$$

The generating function for $\overline{\mathcal{EO}}(n)$, denoted by $\overline{\mathrm{eo}}(q)$ can be derived as

$$\overline{\mathrm{eo}}(q) := \sum_{n=0}^{\infty} \overline{\mathcal{EO}}(n) q^n = \frac{(q^4;q^4)_\infty}{(q^2;q^4)_\infty^2}.$$

Interestingly, there is also a divisibility property for $\overline{\mathrm{eo}}(q)$, For example,

$$\mathcal{EO}(n) \equiv \begin{cases} 1 \pmod 2 & \text{if } n = 4n(3n \pm 1), \\ 0 \pmod 2 & \text{otherwise.} \end{cases} \tag{2.1.24}$$

Another example is that there is a similar congruence to Ramanujan's in modulo 5:

$$\overline{\mathcal{EO}}(10n+8) \equiv 0 \pmod 5. \tag{2.1.25}$$

In Section 2.3, we will provide two proofs of the congruence in (2.1.24). We note in passing that the second proof of (2.1.24) was due to James Sellers via personal communication. Also, (2.1.25) was proved in [5]. In fact, in Section 2.4, we will prove the analogue of Atkin-Swinnerton-Dyer congruence in modulo 5 for the 5-dissection of $\overline{\mathrm{eo}}(q)$.

The above divisibility property for modulo 5 can be explained by the statistics called even-odd crank. Let $le(\lambda)$ denote the largest even part of λ and $\#o(\lambda)$ the number of odd parts of λ, then

$$eoc(\lambda) = le(\lambda) - \#o(\lambda).$$

If we let $\overline{EO}_5(m,n)$ be the number of partitions of n with even-odd crank congruent to m modulo 5, then it can be shown that indeed

$$\overline{EO}_5(i, 10n+8) = \frac{1}{5}\overline{\mathcal{EO}}(10n+8) \quad 0 \le i \le 4.$$

We will show in Section 2.5 that there are also some other relations between certain classes of even-odd crank modulo 5:

Theorem 2.1.5.

$$\begin{aligned}
\overline{EO}_5(1, 10n) &= \overline{EO}_5(2, 10n), \\
\overline{EO}_5(0, 10n+2) &= \overline{EO}_5(1, 10n+2), \\
\overline{EO}_5(0, 10n+4) &= \overline{EO}_5(2, 10n+4), \\
\overline{EO}_5(1, 10n+4) &= \overline{EO}_5(2, 10n+6), \\
\overline{EO}_5(0, 10n+8) = \overline{EO}_5(1, 10n+8) &= \overline{EO}_5(2, 10n+8).
\end{aligned}$$

As for the generating function of $\overline{\mathcal{EO}}(n)$ with consideration on the even-odd crank, let $\overline{\mathcal{EO}}(m,n)$ be the number of partitions counted by $\overline{\mathcal{EO}}(n)$ with even-odd crank equal to m. Then

$$\overline{\mathrm{eo}}(z,q) := \sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} \overline{\mathcal{EO}}(m,n) z^m q^n = \frac{(q^4;q^4)_\infty}{(z^2q^2;q^4)_\infty (z^{-2}q^2;q^4)_\infty}.$$

From the fact that $\overline{\mathcal{EO}}(m, 2n+1) = 0$, we also have that

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} \overline{\mathcal{EO}}(m,2n) z^m q^{2n} = \frac{(q^4;q^4)_\infty}{(z^2q^2;q^4)_\infty (z^{-2}q^2;q^4)_\infty}.$$

Thus

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} \overline{\mathcal{EO}}(m,2n) z^m q^{n} = \frac{(q^2;q^2)_\infty}{(z^2q;q^2)_\infty (z^{-2}q;q^2)_\infty}. \tag{2.1.26}$$

In Section 2.5, we will investigate the 5-dissection for the infinite products above. Also, we obtain an analogue of Theorem 2.1.4 for $\overline{\text{eo}}(z,q)$.

2.2 Preliminaries

The following series transformation is needed [25, p.36, eq 2.7.1].

Proposition 2.2.1. *If* $|aq/bcd| < 1$, *then*

$$\sum_{k=0}^{\infty} \frac{1-aq^{2k}}{1-a} \frac{(a,b,c,d;q)_k}{(\frac{aq}{b},\frac{aq}{c},\frac{aq}{d},q;q)_k} \left(\frac{aq}{bcd}\right)^k = \frac{\left(aq,\frac{aq}{bc},\frac{aq}{bd},\frac{aq}{cd};q\right)_\infty}{\left(\frac{aq}{b},\frac{aq}{c},\frac{aq}{d},\frac{aq}{bcd};q\right)_\infty}.$$

Next, we define the following functions

Definition 2.2.2.

$$\begin{aligned}
A(z,\zeta,q) &:= \sum_{n=-\infty}^{\infty} \frac{(-1)^n \zeta^n q^{n^2+n}}{1-zq^{2n+1}}, \\
A_5(a,b) &:= A(q^{5a}, q^{5b}, q^{25}), \\
P(a) &:= \begin{cases} [q^{5a}; q^{50}]_\infty, & a \neq 0, \\ (q^{50}; q^{50})_\infty, & a = 0. \end{cases} \\
\overline{P}(a) &:= \begin{cases} [q^{10a}; q^{100}]_\infty, & a \neq 0, \\ (q^{100}; q^{100})_\infty, & a = 0. \end{cases}
\end{aligned}$$

It can be shown that

$$A_5(a,b) = -q^{-5b} A_5(10-a, 10-b)$$

by means of change of variable on the index of the sum as $n \mapsto n-1$. Also, $P(a)$ becomes $\overline{P}(a)$ under the change of variable $q \to q^2$. And for $1 \leq i \leq 4$

$$\begin{aligned}
P(i) &= P(10-i), \\
\overline{P}(i) &= \overline{P}(10-i).
\end{aligned}$$

Note that $A(z,\zeta,q)$ is related to the case $k=1$ of the level $2k-1$ Appell function:

$$\Sigma^{(2k-1)}(z,q) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(2k-1)n(n+1)/2}}{1-zq^n}.$$

From the original proof of Theorem 2.1.4 in [9, p.19, Lemma 2.3.2], the identity involving level 3 Appell function is used in order to transform the terms of $\Sigma^3(z,q)$ into infinite products. We require a similar identity for our function $A_5(a,b)$. It was shown in [19] that the generalization of such identity can be derived for any $k>1$. However, the analogue for the case $k=1$ was actually found by Lewis [33, (11)]:

Proposition 2.2.3.

$$\frac{[z,\zeta^2;q]_\infty (q;q)_\infty^2}{[z\zeta,\zeta,z\zeta^{-1};q]_\infty} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n \zeta^{-n} q^{n(n+1)/2}}{1-z\zeta^{-1}q^n} + \zeta \sum_{n=-\infty}^{\infty} \frac{(-1)^n \zeta^{n} q^{n(n+1)/2}}{1-z\zeta q^n}.$$

Corollary 2.2.4.

$$A_5(a-b,-b) + q^{5b} A_5(a+b,b) = \frac{P(a+5)P(2b)P(0)^2}{P(a+b+5)P(b)P(a-b+5)}.$$

Proof. Taking $q \to q^2$ and $z \to zq$ in Proposition 2.2.3 yields

$$\frac{[zq,\zeta^2;q^2]_\infty (q^2;q^2)_\infty^2}{[z\zeta q,\zeta,z\zeta^{-1}q;q^2]_\infty} = A(z\zeta^{-1},\zeta^{-1},q) + \zeta A(z\zeta,\zeta,q).$$

After taking $q \to q^{25}$, $z = q^{5a}$, $\zeta = q^{5b}$, the identity follows. □

Corollary 2.2.5.

$$A_5(a,0) = \frac{P(0)^2}{P(a+5)}.$$

Proof. Consider the limiting case of Proposition 2.2.3 with $\zeta \to 1$.

$$\frac{(q;q)_\infty^2}{[z;q]_\infty} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{1-zq^n}$$

The identity then follows from similar change of variables ($q \to q^2$, $z \to zq$ and then $q \to q^{25}$, $z = q^{5a}$). □

The next two propositions from Ramanujan are also needed.

Proposition 2.2.6.

$$1 + q^5 \frac{P(1)P(2)}{P(3)P(4)} = \frac{P(2)^2 P(5)}{P(1)P(4)^2},$$
$$1 - q^5 \frac{P(1)P(2)}{P(3)P(4)} = \frac{P(1)P(4)P(5)}{P(2)P(3)^2}.$$

Proof. This follows from the results by Ramanujan [36] or Cooper [20]:

$$1 + q\frac{[q, q^2; q^{10}]_\infty}{[q^3, q^4; q^{10}]_\infty} = \frac{[q^2, q^2, q^5; q^{10}]_\infty}{[q, q^4, q^4; q^{10}]_\infty},$$
$$1 - q\frac{[q, q^2; q^{10}]_\infty}{[q^3, q^4; q^{10}]_\infty} = \frac{[q, q^4, q^5; q^{10}]_\infty}{[q^2, q^3, q^3; q^{10}]_\infty}$$

with the change of variable $q \to q^5$. □

Proposition 2.2.7.

$$(q^2; q^2)_\infty = P(0) \left[\frac{P(4)}{P(2)} - q^2 - q^4 \frac{P(2)}{P(4)} \right].$$

Proof. This is simply the identity from Ramanujan [29]:

$$(q; q)_\infty = (q^{25}; q^{25})_\infty \left[\frac{[q^{10}; q^{25}]_\infty}{[q^5; q^{25}]_\infty} - q - q^2 \frac{[q^5; q^{25}]_\infty}{[q^{10}; q^{25}]_\infty} \right]$$

with the change of variable $q \to q^2$. □

Finally, we define the following functions for difference between even-odd crank

Definition 2.2.8.

$$\overline{r}_a(d) := \sum_{n=0}^{\infty} \overline{EO}_5(a, 10n + d) q^{2n},$$
$$\overline{r}_{ab}(d) := \overline{r}_a(d) - \overline{r}_b(d).$$

2.3 Congruence modulo 2

We now prove the congruence for $\overline{\mathcal{EO}}(n)$ modulo 2 in (2.1.24). The second proof was due to James Sellers via personal communication.

First proof. We have that

$$\begin{aligned}
\overline{eo}(q) &= \frac{(q^4;q^4)_\infty}{(q^2;q^4)^2_\infty} \\
&= \frac{(q^4;q^8)_\infty(q^8;q^8)_\infty}{(q^2;q^4)^2_\infty} - (q^8;q^8)_\infty + (q^8;q^8)_\infty \\
&= (q^8;q^8)_\infty\left[\frac{(q^4;q^8)_\infty - (q^2;q^4)^2_\infty}{(q^2;q^4)^2_\infty}\right] + (q^8;q^8)_\infty \\
&= (q^8;q^8)_\infty\left[\frac{(-q^2;q^4)_\infty - (q^2;q^4)_\infty}{(q^2;q^4)_\infty}\right] + \sum_{n=-\infty}^{\infty}(-1)^n q^{4n(3n-1)},
\end{aligned}$$

where the last equation comes from (1.5.5). The result follows from the fact that all the coefficients of $(-q^2;q^4)_\infty - (q^2;q^4)_\infty$ are all even. □

Second proof. Note that for any integer $x, (1+x)^2 \equiv 1+x^2 \pmod 2$. Thus,

$$\begin{aligned}
\overline{eo}(q) &= \frac{(q^4;q^4)_\infty}{(q^2;q^4)^2_\infty} \\
&= (q^4;q^4)_\infty \frac{(q^4;q^4)^2_\infty}{(q^2;q^2)^2_\infty} \\
&= \frac{(q^4;q^4)^3_\infty}{(q^2;q^2)^2_\infty} \\
&= \frac{(q^4;q^4)^2_\infty (q^4;q^4)_\infty}{(q^2;q^2)^2_\infty} \\
&\equiv \frac{(q^8;q^8)^2_\infty (q^4;q^4)_\infty}{(q^4;q^4)_\infty} \pmod 2 \\
&= (q^8;q^8),
\end{aligned}$$

and the theorem follows from (1.5.5). □

2.4 Congruence for 5-dissection

Theorem 2.4.1. *Let*

$$\overline{eo}(q) = \sum_{k=0}^{4} \Psi_k(q^{10}) q^{2k}.$$

Then we have that in modulo 5,

$$\begin{aligned}
\Psi_0(q^{10}) &\equiv \frac{\overline{P}(0)}{\overline{P}(1)} && \pmod 5,\\
\Psi_1(q^{10}) &\equiv 2\frac{\overline{P}(0)\overline{P}(4)}{\overline{P}(2)\overline{P}(3)} && \pmod 5,\\
\Psi_2(q^{10}) &\equiv 2\frac{\overline{P}(0)\overline{P}(2)}{\overline{P}(1)\overline{P}(4)} && \pmod 5,\\
\Psi_3(q^{10}) &\equiv 4\frac{\overline{P}(0)}{\overline{P}(3)} && \pmod 5,\\
\Psi_4(q^{10}) &\equiv 0 && \pmod 5.
\end{aligned}$$

Proof. From the generating function of the $\overline{\mathcal{EO}}-$partition:

$$\sum_{n=0}^{\infty} \overline{\mathcal{EO}}(n) q^n = \frac{(q^4;q^4)_\infty}{(q^2;q^4)_\infty^2} = (-q^2;q^2)_\infty^3 (q^2;q^2)_\infty$$

In [31], Hirschhorn and Sellers provided the congruence modulo 5 for the following infinite product:

$$(-q;q)_\infty^3 (q;q)_\infty \equiv \frac{(q^2;q^2)_\infty^3 (q;q)_\infty^3}{(q^5;q^5)_\infty} \pmod 5.$$

Also, it can be shown that [30],

$$(q;q)_\infty^3 \equiv A(q^5) - 3qB(q^5) \pmod 5,$$

where

$$A(q) = (q^2, q^3, q^5; q^5)_\infty, \qquad B(q) = (q, q^4, q^5; q^5)_\infty.$$

Thus, we obtain in modulo 5

$$\begin{aligned}
(-q;q)_\infty^3(q;q)_\infty \equiv \frac{1}{(q^5;q^5)_\infty}\Big[&A(q^5)A(q^{10}) - 3qB(q^5)A(q^{10}) - 3q^2A(q^5)B(q^{10}) \\
&+ 9q^3B(q^5)B(q^{10})\Big] \pmod 5 \\
\equiv\ &\Psi_0(q^5) + q\Psi_1(q^5) + q^2\Psi_2(q^5) + q^3\Psi_3(q^5) + q^4\Psi_4(q^5) \pmod 5.
\end{aligned}$$

Now, for each of the function $\Psi_k(q^5)$ in modulo 5, we have that

$$\begin{aligned}
\Psi_0(q^5) &\equiv \frac{(q^{10},q^{15},q^{25},q^{35},q^{40},q^{50};q^{50})_\infty(q^{20},q^{30},q^{50};q^{50})_\infty}{(q^5;q^5)_\infty} \pmod 5 \\
&= \frac{P(0)}{P(1)}, \\
\Psi_1(q^5) &\equiv 2\frac{(q^{5},q^{20},q^{25},q^{30},q^{45},q^{50};q^{50})_\infty(q^{20},q^{30},q^{50};q^{50})_\infty}{(q^5;q^5)_\infty} \pmod 5 \\
&= 2\frac{P(0)P(4)}{P(2)P(3)}, \\
\Psi_2(q^5) &\equiv 2\frac{(q^{10},q^{15},q^{25},q^{35},q^{40},q^{50};q^{50})_\infty(q^{10},q^{40},q^{50};q^{50})_\infty}{(q^5;q^5)_\infty} \pmod 5 \\
&= 2\frac{P(0)P(2)}{P(1)P(4)}, \\
\Psi_3(q^5) &\equiv 4\frac{(q^{5},q^{20},q^{25},q^{30},q^{45},q^{50};q^{50})_\infty(q^{10},q^{40},q^{50};q^{50})_\infty}{(q^5;q^5)_\infty} \pmod 5 \\
&= 4\frac{P(0)}{P(3)},
\end{aligned}$$

and $$\Psi_4(q^5) \equiv 0 \pmod 5.$$

The theorem then follows from the change of variable $q \to q^2$. □

2.5 Difference between even-odd crank

First, we rewrite the generating function in (2.1.26) into Appell function.

Proposition 2.5.1.

$$\frac{(q^2;q^2)_\infty}{(z^2q;q^2)_\infty(z^{-2}q;q^2)_\infty} = \frac{1}{(q^2;q^2)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n^2+n}}{1-z^2q^{2n+1}}$$

Proof. From Proposition (2.2.1), take $q \to q^2$ and set $a = q^2, b = z^2q, c = z^{-2}q$. Finally, take $d \to \infty$ to obtain

$$\begin{aligned}
\frac{(q^2;q^2)_\infty^2}{(z^2q;q^2)_\infty(z^{-2}q;q^2)_\infty} &= \sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2+n}(1-q^{4n+2})}{(1-z^2q^{2n+1})(1-z^{-2}q^{2n+1})} \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2+n}}{1-z^2q^{2n+1}} + \sum_{n=0}^{\infty}\frac{(-1)^n z^{-2} q^{n^2+3n+1}}{1-z^{-2}q^{2n+1}} \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2+n}}{1-z^2q^{2n+1}} - \sum_{n=0}^{\infty}\frac{(-1)^n q^{(-n-1)^2+(-n-1)}}{1-z^2q^{2(-n-1)-1}} \\
&= \sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n^2+n}}{1-z^2q^{2n+1}}
\end{aligned}$$

and the theorem follows. □

For convenience, let us define the function

$$T(b) := \sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n^2+n+b(2n+1)}}{1-q^{10n+5}}.$$

Note that $T(b) = T(5-b)$ by the change of variable $n \mapsto -n-1$. We will now 5-dissect the sum on the right hand side of Proposition 2.5.1.

Proposition 2.5.2. *Let $z = e^{2\pi i/5}$. Then*

$$\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n^2+n}}{1-z^2q^{2n+1}} = (T(0)-T(1)) + (z+z^{-1})(T(2)-T(1)).$$

Proof.

$$\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n^2+n}}{1-z^2q^{2n+1}} = \sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n^2+n}(1-q^{2n+1})(1-zq^{2n+1})(1-z^3q^{2n+1})(1-z^4q^{2n+1})}{1-q^{10n+5}}$$

$$
\begin{aligned}
&= \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+n}}{1-q^{10n+5}} \Big[(1 - q^{2n+1} + q^{4n+2} - q^{6n+3}) + z(-q^{2n+1} + q^{4n+2}) \\
&\quad + z^2(q^{4n+2} - q^{6n+3}) + z^3(-q^{2n+1} + q^{4n+2} - q^{6n+3} + q^{8n+4}) \\
&\quad + z^4(-q^{2n+1} + 2q^{4n+2} - q^{6n+3}) \Big] \\
&= (T(0) - T(1) + T(2) - T(3)) + z\,(-T(1) + T(2)) + z^2\,(T(2) - T(3)) \\
&\quad + z^3\,(-T(1) + T(2) - T(3) + T(4)) + z^4\,(-T(1) + 2T(2) - T(3)) \\
&= (T(0) - T(1)) + (z + z^{-1})(T(2) - T(1)),
\end{aligned}
$$

where we used $T(b) = T(5-b)$ in the last equation. □

Next, we will further 5-dissect the functions $T(0), T(1)$ and $T(2)$.

Proposition 2.5.3.

$$
\begin{aligned}
T(0) &= q^6 \frac{P(0)^2}{P(5)} - q^2 \frac{P(5)P(4)P(0)^2}{P(2)P(3)^2} + \frac{P(5)P(2)P(0)^2}{P(4)P(1)^2}, \\
T(1) &= -q^5 \frac{P(0)^2}{P(3)} + q \frac{P(3)P(4)P(0)^2}{P(5)P(2)P(1)} + q^4 \frac{P(2)P(0)^2}{P(4)P(1)}, \\
T(2) &= q^2 \frac{P(0)^2}{P(1)} + q^3 \frac{P(4)P(0)^2}{P(3)P(2)} - q^{11} \frac{P(1)P(2)P(0)^2}{P(5)P(4)P(3)}.
\end{aligned}
$$

Proof. We dissect each of the function $T(b)$ by setting $n = 5m + b$. By Corollary 2.2.4 and Corollary 2.2.5, we have that

$$
\begin{aligned}
T(0) &= \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+n}}{1-q^{10n+5}} \\
&= \sum_{b=0}^{4} \sum_{m=-\infty}^{\infty} \frac{(-1)^{m+b} q^{25m^2+5m+10bm+b^2+b}}{1-q^{50m+10b+5}} \\
&= \sum_{b=0}^{4} (-1)^b q^{b^2+b} \sum_{m=-\infty}^{\infty} \frac{(-1)^m q^{25m^2+(10b+5)m}}{1-q^{50m+10b+5}} \\
&= A_5(-4,-4) - q^2 A_5(-2,-2) + q^6 A_5(0,0) - q^{12} A_5(2,2) + q^{20} A_5(4,4) \\
&= q^6 A_5(0,0) - q^2 [A_5(-2,-2) + q^{10} A_5(2,2)] + [A_5(-4,-4) + q^{20} A_5(4,4)] \\
&= q^6 \frac{P(0)^2}{P(5)} - q^2 \frac{P(5)P(4)P(0)^2}{P(7)P(2)P(3)} + \frac{P(5)P(8)P(0)^2}{P(9)P(4)P(1)}
\end{aligned}
$$

$$= q^6 \frac{P(0)^2}{P(5)} - q^2 \frac{P(5)P(4)P(0)^2}{P(2)P(3)^2} + \frac{P(5)P(2)P(0)^2}{P(4)P(1)^2}.$$

Also,

$$\begin{aligned}
T(1) &= \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+3n+1}}{1-q^{10n+5}} \\
&= \sum_{b=0}^{4} \sum_{m=-\infty}^{\infty} \frac{(-1)^{m+b} q^{25m^2+15m+10bm+b^2+3b+1}}{1-q^{50m+10b+5}} \\
&= \sum_{b=0}^{4} (-1)^b q^{b^2+3b+1} \sum_{m=-\infty}^{\infty} \frac{(-1)^m q^{25m^2+(10b+15)m}}{1-q^{50m+10b+5}} \\
&= qA_5(-4,-2) - q^5 A_5(-2,0) + q^{11} A_5(0,2) - q^{19} A_5(2,4) + q^{29} A_5(4,6) \\
&= -q^5 A_5(-2,0) + q[A_5(-4,-2) + q^{10} A_5(0,2)] - q^{-1}[A_5(-6,-4) + q^{20} A_5(2,4)] \\
&= -q^5 \frac{P(0)^2}{P(3)} + q \frac{P(3)P(4)P(0)^2}{P(5)P(2)P(1)} - q^{-1} \frac{P(3)P(8)P(0)^2}{P(7)P(4)P(-1)} \\
&= -q^5 \frac{P(0)^2}{P(3)} + q \frac{P(3)P(4)P(0)^2}{P(5)P(2)P(1)} + q^4 \frac{P(8)P(0)^2}{P(4)P(9)} \\
&= -q^5 \frac{P(0)^2}{P(3)} + q \frac{P(3)P(4)P(0)^2}{P(5)P(2)P(1)} + q^4 \frac{P(2)P(0)^2}{P(4)P(1)}.
\end{aligned}$$

Finally,

$$\begin{aligned}
T(2) &= \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+5n+2}}{1-q^{10n+5}} \\
&= \sum_{b=0}^{4} \sum_{m=-\infty}^{\infty} \frac{(-1)^{m+b} q^{25m^2+25m+10bm+b^2+5b+2}}{1-q^{50m+10b+5}} \\
&= \sum_{b=0}^{4} (-1)^b q^{b^2+5b+2} \sum_{m=-\infty}^{\infty} \frac{(-1)^m q^{25m^2+(10b+25)m}}{1-q^{50m+10b+5}} \\
&= q^2 A_5(-4,0) - q^8 A_5(-2,2) + q^{16} A_5(0,4) - q^{26} A_5(2,6) + q^{38} A_5(4,8) \\
&= q^2 A_5(-4,0) - q^{-2}[A_5(-6,-2) + q^{10} A_5(-2,2)] + q^{-4}[A_5(-8,-4) + q^{20} A_5(0,4)] \\
&= q^2 \frac{P(0)^2}{P(1)} - q^{-2} \frac{P(1)P(4)P(0)^2}{P(3)P(2)P(-1)} + q^{-4} \frac{P(1)P(8)P(0)^2}{P(5)P(4)P(-3)} \\
&= q^2 \frac{P(0)^2}{P(1)} + q^3 \frac{P(4)P(0)^2}{P(3)P(2)} - q^{11} \frac{P(1)P(8)P(0)^2}{P(5)P(4)P(7)}
\end{aligned}$$

$$= q^2\frac{P(0)^2}{P(1)} + q^3\frac{P(4)P(0)^2}{P(3)P(2)} - q^{11}\frac{P(1)P(2)P(0)^2}{P(5)P(4)P(3)},$$

and we are done. □

Proposition 2.5.4.

$$\frac{T(0)-T(1)}{(q^2;q^2)_\infty} = \frac{P(0)}{P(1)} - q\frac{P(4)P(0)}{P(2)P(3)} - q^3\frac{P(0)}{P(3)}, \tag{2.5.1}$$

$$\frac{T(2)-T(1)}{(q^2;q^2)_\infty} = -q\frac{P(4)P(0)}{P(2)P(3)} + q^2\frac{P(2)P(0)}{P(1)P(4)}. \tag{2.5.2}$$

Proof. By multiplying by (2.2.7) on both of the right hand side of (2.5.1) and (2.5.2), we have that

$$\begin{aligned}
&\left[\frac{P(0)}{P(1)} - q\frac{P(4)P(0)}{P(2)P(3)} - q^3\frac{P(0)}{P(3)}\right] P(0)\left[\frac{P(4)}{P(2)} - q^2 - q^4\frac{P(2)}{P(4)}\right]\\
&= \frac{P(4)P(0)^2}{P(1)P(2)} - q\frac{P(4)^2P(0)^2}{P(2)^2P(3)} - q^2\frac{P(0)^2}{P(1)} - q^4\frac{P(2)P(0)^2}{P(1)P(4)} + 2q^5\frac{P(0)^2}{P(3)} + q^7\frac{P(2)P(0)^2}{P(3)P(4)}\\
&= \left[\frac{P(4)P(0)^2}{P(1)P(2)} + 2q^5\frac{P(0)^2}{P(3)}\right] - q\frac{P(4)^2P(0)^2}{P(2)^2P(3)} + q^2\left[-\frac{P(0)^2}{P(1)} + q^5\frac{P(2)P(0)^2}{P(3)P(4)}\right] - q^4\frac{P(2)P(0)^2}{P(1)P(4)}\\
&= B_{10}(q^5) + qB_{11}(q^5) + q^2B_{12}(q^5) + q^3B_{13}(q^5) + q^4B_{14}(q^5),
\end{aligned}$$

and also

$$\begin{aligned}
&\left[-q\frac{P(4)P(0)}{P(2)P(3)} + q^2\frac{P(2)P(0)}{P(1)P(4)}\right] P(0)\left[\frac{P(4)}{P(2)} - q^2 - q^4\frac{P(2)}{P(4)}\right]\\
&= -q\frac{P(4)^2P(0)^2}{P(2)^2P(3)} + q^2\frac{P(0)^2}{P(1)} + q^3\frac{P(4)P(0)^2}{P(2)P(3)} - q^4\frac{P(2)P(0)^2}{P(1)P(4)} + q^5\frac{P(0)^2}{P(3)} - q^6\frac{P(2)^2P(0)^2}{P(1)P(4)^2}\\
&= q^5\frac{P(0)^2}{P(3)} + q\left[-\frac{P(4)^2P(0)^2}{P(2)^2P(3)} - q^5\frac{P(2)^2P(0)^2}{P(1)P(4)^2}\right] + q^2\frac{P(0)^2}{P(1)} + q^3\frac{P(4)P(0)^2}{P(2)P(3)} - q^4\frac{P(2)P(0)^2}{P(1)P(4)}\\
&= B_{20}(q^5) + qB_{21}(q^5) + q^2B_{22}(q^5) + q^3B_{23}(q^5) + q^4B_{24}(q^5).
\end{aligned}$$

Now, from Theorem 2.5.3 we have that

$$T(0)-T(1) = \left[\frac{P(2)P(5)P(0)^2}{P(1)^2P(4)} + q^5\frac{P(0)^2}{P(3)}\right] + q\left[-\frac{P(3)P(4)P(0)^2}{P(5)P(2)P(1)} + q^5\frac{P(0)^2}{P(5)}\right]$$

$$- q^2\frac{P(4)P(5)P(0)^2}{P(2)P(3)^2} - q^4\frac{P(2)P(0)^2}{P(4)P(1)}$$
$$= C_{10}(q^5) + qC_{11}(q^5) + q^2C_{12}(q^5) + q^3C_{13}(q^5) + q^4C_{14}(q^5),$$

and also

$$T(2) - T(1) = q^5\frac{P(0)^2}{P(3)} + q\left[-\frac{P(3)P(4)P(0)^2}{P(1)P(2)P(5)} - q^{10}\frac{P(1)P(2)P(0)^2}{P(3)P(4)P(5)}\right]$$
$$+ q^2\frac{P(0)^2}{P(1)} + q^3\frac{P(4)P(0)^2}{P(2)P(3)} - q^4\frac{P(2)P(0)^2}{P(1)P(4)}$$
$$= C_{20}(q^5) + qC_{21}(q^5) + q^2C_{22}(q^5) + q^3C_{23}(q^5) + q^4C_{24}(q^5).$$

It is now left to show that $B_{10}(q^5) = C_{10}(q^5), B_{11}(q^5) = C_{11}(q^5), B_{12}(q^5) = C_{12}(q^5)$ and $B_{21}(q^5) = C_{21}(q^5)$. By using Proposition 2.2.7, we have

$$\begin{aligned} B_{10}(q^5) &= \frac{P(4)P(0)^2}{P(1)P(2)} + 2q^5\frac{P(0)^2}{P(3)} \\ &= \frac{P(4)P(0)^2}{P(1)P(2)}\left(1 + q^5\frac{P(1)P(2)}{P(3)P(4)}\right) + q^5\frac{P(0)^2}{P(3)} \\ &= \frac{P(4)P(0)^2}{P(1)P(2)}\left(\frac{P(2)^2P(5)}{P(1)P(4)^2}\right) + q^5\frac{P(0)^2}{P(3)} \\ &= \frac{P(2)P(5)P(0)^2}{P(1)^2P(4)} + q^5\frac{P(0)^2}{P(3)} \\ &= C_{10}(q^5). \end{aligned}$$

Next,

$$\begin{aligned} B_{11}(q^5) &= -\frac{P(4)^2P(0)^2}{P(2)^2P(3)} \\ &= -\frac{P(3)P(4)P(0)^2}{P(1)P(2)P(5)}\left(\frac{P(1)P(4)P(5)}{P(2)P(3)^2}\right) \\ &= -\frac{P(3)P(4)P(0)^2}{P(1)P(2)P(5)}\left(1 - q^5\frac{P(1)P(2)}{P(3)P(4)}\right) \\ &= -\frac{P(3)P(4)P(0)^2}{P(5)P(2)P(1)} + q^5\frac{P(0)^2}{P(5)} \end{aligned}$$

$$= C_{11}(q^5).$$

Also,

$$\begin{aligned}
B_{12}(q^5) &= -\frac{P(0)^2}{P(1)} + q^5 \frac{P(2)P(0)^2}{P(3)P(4)} \\
&= -\frac{P(0)^2}{P(1)} \left(1 - q^5 \frac{P(1)P(2)}{P(3)P(4)} \right) \\
&= -\frac{P(0)^2}{P(1)} \left(\frac{P(1)P(4)P(5)}{P(2)P(3)^2} \right) \\
&= -\frac{P(4)P(5)P(0)^2}{P(2)P(3)^2} \\
&= C_{12}(q^5).
\end{aligned}$$

Finally,

$$\begin{aligned}
B_{21}(q^5) &= -\frac{P(4)^2P(0)^2}{P(2)^2P(3)} - q^5 \frac{P(2)^2P(0)^2}{P(1)P(4)^2} \\
&= -\frac{P(0)^2}{P(5)} \left[\frac{P(3)P(4)}{P(1)P(2)} \left(\frac{P(1)P(4)P(5)}{P(2)P(3)} \right) + q^5 \frac{P(2)^2P(5)}{P(1)P(4)^2} \right] \\
&= -\frac{P(0)^2}{P(5)} \left[\frac{P(3)P(4)}{P(1)P(2)} \left(1 - q^5 \frac{P(1)P(2)}{P(3)P(4)} \right) + q^5 \left(1 + q^5 \frac{P(1)P(2)}{P(3)P(4)} \right) \right] \\
&= -\frac{P(0)^2}{P(5)} \left[\frac{P(3)P(4)}{P(1)P(2)} - q^5 + q^5 + q^{10} \frac{P(1)P(2)}{P(3)P(4)} \right] \\
&= -\frac{P(3)P(4)P(0)^2}{P(1)P(2)P(5)} - q^{10} \frac{P(1)P(2)P(0)^2}{P(3)P(4)P(5)} \\
&= C_{21}(q^5),
\end{aligned}$$

as desired. □

Thus we obtain a 5-dissection of $\overline{eo}(q)$ analogous to the result of Ramanujan's.

Theorem 2.5.5. *Let* $z = e^{2\pi i/5}$. *Then*

$$\frac{(q^4;q^4)_\infty}{(z^2q^2;q^4)_\infty (z^{-2}q;q^4)_\infty} = \overline{A}(q^{10}) - (z+z^{-1}+1)q^2\overline{B}(q^{10}) + (z+z^{-1})q^4\overline{C}(q^{10}) - q^6\overline{D}(q^{10}),$$

where

$$\overline{A}(q^{10}) = \frac{\overline{P}(0)}{\overline{P}(1)},$$
$$\overline{B}(q^{10}) = \frac{\overline{P}(4)\overline{P}(0)}{\overline{P}(2)\overline{P}(3)},$$
$$\overline{C}(q^{10}) = \frac{\overline{P}(2)\overline{P}(0)}{\overline{P}(1)\overline{P}(4)},$$
$$\overline{D}(q^{10}) = \frac{\overline{P}(0)}{\overline{P}(3)}.$$

Proof. This follows from equation (2.1.26), Proposition 2.5.1 to Proposition 2.5.4 and the change of variable $q \to q^2$. □

Now we relate the above equation with $\overline{r}_{ab}(d)$. The proof here follows the similar argument in [24].

Theorem 2.5.6.

$$\overline{r}_{01}(0) = \frac{\overline{P}(0)}{\overline{P}(1)},$$
$$\overline{r}_{12}(0) = 0,$$
$$\overline{r}_{01}(2) = 0,$$
$$\overline{r}_{12}(2) = -\frac{\overline{P}(4)\overline{P}(0)}{\overline{P}(2)\overline{P}(3)},$$
$$\overline{r}_{01}(4) = -\frac{\overline{P}(2)\overline{P}(0)}{\overline{P}(1)\overline{P}(4)},$$
$$\overline{r}_{02}(4) = 0,$$
$$\overline{r}_{01}(6) = -\frac{\overline{P}(0)}{\overline{P}(3)},$$
$$\overline{r}_{12}(6) = 0,$$
$$\overline{r}_{01}(8) = 0,$$
$$\overline{r}_{12}(8) = 0.$$

Proof. We write Theorem 2.5.5 as

$$\sum_{k=0}^{4} z^k \sum_{n=0}^{\infty} \overline{EO}_5(k,2n)q^{2n} = \overline{A}(q^{10}) - (z+z^{-1}+1)q^2\overline{B}(q^{10}) + (z+z^{-1})q^4\overline{C}(q^{10}) - q^6\overline{D}(q^{10})$$

Let $\overline{A}(q^{10}) = \sum_{n=0}^{\infty} a_n q^{10n}$. By collecting powers of q congruent to 0 modulo 10, we can see that

$$\sum_{k=0}^{4} z^k \sum_{n=0}^{\infty} \overline{EO}_5(k,10n)q^{10n} = \sum_{n=0}^{\infty} a_n q^{10n}.$$

For the coefficient of q^{10n}, we have

$$\sum_{k=0}^{4} z^k \overline{EO}_5(k,10n) = a_n,$$

i.e.,

$$(\overline{EO}_5(0,10n) - a_n) + \overline{EO}_5(1,10n)z + \overline{EO}_5(2,10n)z^2 + \overline{EO}_5(3,10n)z^3 + \overline{EO}_5(4,10n)z^4 = 0.$$

Since $1+z+z^2+z^3+z^4$ is the minimal polynomial of z, we have that

$$\overline{EO}_5(0,10n) - a_n = \overline{EO}_5(1,10n) = \overline{EO}_5(2,10n) = \overline{EO}_5(3,10n) = \overline{EO}_5(4,10n).$$

In particular,

$$\overline{r}_{01}(0) = \sum_{n=0}^{\infty} (\overline{EO}_5(0,10n) - \overline{EO}_5(1,10n))q^{10n} = \sum_{n=0}^{\infty} a_n q^{10n} = \overline{A}(q^{10}),$$

and

$$\overline{r}_{12}(0) = \sum_{n=0}^{\infty} (\overline{EO}_5(1,10n) - \overline{EO}_5(2,10n))q^{10n} = 0.$$

The remaining results follow similarly. □

With this, Theorem 2.1.5 follows directly from the theorem above as a corollary, i.e.,

$$\overline{EO}_5(1, 10n) = \overline{EO}_5(2, 10n),$$
$$\overline{EO}_5(0, 10n+2) = \overline{EO}_5(1, 10n+2),$$
$$\overline{EO}_5(0, 10n+4) = \overline{EO}_5(2, 10n+4),$$
$$\overline{EO}_5(1, 10n+4) = \overline{EO}_5(2, 10n+6),$$
$$\overline{EO}_5(0, 10n+8) = \overline{EO}_5(1, 10n+8) = \overline{EO}_5(2, 10n+8).$$

Chapter 3
Separable Integer Partition Classes

3.1 Introduction

In [7], Andrews introduced the notion of separable integer partition classes as follows.

Definition 3.1.1. *A* ***separable integer partition class (SIP)*** $\mathcal{P}$ ***with modulus*** k*, is a subset of all the integer partitions which contains a subset* $\mathcal{B} \subset \mathcal{P}$ *that satisfies the following conditions:*

1. *For an integer* $n \geq 1$*, the number of partitions in* $\mathcal{B}$ *with* n *parts is finite.*

2. *Every partition in* $\mathcal{P}$ *can be written uniquely as*

$$(b_1 + \pi_1) + (b_2 + \pi_2) + \cdots + (b_n + \pi_n), \tag{*}$$

   where $0 < b_1 \leq b_2 \leq \cdots \leq b_n$ *creates a partition in* $\mathcal{B}$ *and* $0 \leq \pi_1 \leq \cdots \leq \pi_n$ *creates a partition whose parts are divisible by* k*.*

3. *All partitions written in the form of* (*) *must be in* $\mathcal{P}$*.*

The set $\mathcal{B}$ *is called the* **basis** *of* $\mathcal{P}$*.*

For example, consider the set of all integer partitions (denoted by $\mathcal{P}_{\mathbb{N}}$). It turns out that this is an SIP class of modulus 1, with the basis as

$$\mathcal{B}_{\mathbb{N}} = \left\{ \underbrace{1 + 1 + \cdots + 1}_{n} \middle| n \in \mathbb{N} \right\},$$

this is because for any partition with n parts, say $\pi_1 + \pi_2 + \cdots + \pi_n$ in $\mathcal{P}_{\mathbb{N}}$, we can write it as

$$(1 + (\pi_1 - 1)) + (1 + (\pi_2 - 1)) + \cdots + (1 + (\pi_n - 1)).$$

Another example is the set of integer partitions with distinct parts (denoted by $\mathcal{P}_{\mathcal{D}}$). This is also an SIP class of modulo 1, with the basis as

$$\mathcal{B}_{\mathcal{D}} = \left\{ \underbrace{1 + 2 + \cdots + n}_{n} \middle| n \in \mathbb{N} \right\}.$$

Again, this is because for any partition $\pi_1 + \pi_2 + \cdots + \pi_n$ with $0 < \pi_1 < \pi_2 < \cdots < \pi_n$, we can write it as

$$(1 + (\pi_1 - 1)) + (2 + (\pi_2 - 2)) + \cdots + (n + (\pi_n - n)).$$

and note that $(\pi_1 - 1) + (\pi_2 - 2) + \cdots + (\pi_n - n)$ creates an ordinary partition with parts less than or equal to n

From the above examples, note that for a fixed positive integer $n \geq 1$, there is only one element of length n in the basis for each case. In general, it is possible for an SIP class to have a basis which has more than one element of length n. Nevertheless, it can be seen that

Theorem 3.1.2. *For an SIP class $\mathcal{P}$ of modulus k with the basis $\mathcal{B}$, let $B(n; q)$ be the generating function for partitions in $\mathcal{B}$ with n parts. Then the generating function for partitions in $\mathcal{P}$ is*

$$P(q) = \sum_{n=0}^{\infty} \frac{B(n; q)}{(q^k; q^k)_n}.$$

For example, if we consider the SIP class $\mathcal{P}_{\mathbb{N}}$, then we can see that $B(n; q) = q^{1+1+\cdots+1} = q^n$. Therefore, we obtain the following identity

$$\sum_{n=0}^{\infty} \frac{q^n}{(q; q)_n} = \frac{1}{(q; q)_\infty}. \tag{3.1.1}$$

And if we consider the SIP class $\mathcal{P}_{\mathcal{D}}$, then $B(n; q) = q^{1+2+\cdots+n} = q^{\frac{n(n+1)}{2}}$. This yields the identity

$$\sum_{n=0}^{\infty} \frac{q^{\frac{n(n+1)}{2}}}{(q;q)_n} = (-q;q)_\infty. \tag{3.1.2}$$

Note that (3.1.1) and (3.1.2) were due to Euler [2].

We will investigate the application of SIP classes on several sets of $\mathcal{P}$, and give alternate proofs of certain partition identities. In Section 3.2, we will give another proof for the generating functions of $\mathcal{EO}-$partitions and $\mathcal{OE}-$partitions. With the help of the basis theorem in Section 3.3, we will give alternate proofs for both little Göllnitz identities and the second Göllnitz-Gordon identity. Finally, in Section 3.5, we will prove a special case from an infinite family of identities similar to Rogers-Ramanujan identity.

3.2 $\mathcal{EO}-$partitions and $\mathcal{OE}-$partitions

In [6], Andrews considered all the possibilities of restriction on integer partitions with regards to the separation of parities. In particular, there were the case of partitions whose even parts are smaller than the odd parts if odd parts exist (this is the $\mathcal{EO}-$partition in Chapter 2) and the partitions with at least one odd part, and all odd parts are smaller than the even parts if even parts exist (here, we name this the $\mathcal{OE}-$partition). Let $\mathcal{OE}(n)$ be the number of the partitions for n in the latter case. It is easy to derive the generating function for both $\mathcal{EO}(n)$ and $\mathcal{OE}(n)$ as

$$\mathrm{eo}(q) := \sum_{n=0}^{\infty} \mathcal{EO}(n)q^n = \sum_{n=0}^{\infty} \frac{q^{2n}}{(q^2;q^2)_n(q^{2n+1};q^2)_\infty},$$
$$\mathrm{oe}(q) := \sum_{n=0}^{\infty} \mathcal{OE}(n)q^n = \sum_{n=0}^{\infty} \frac{q^{2n+1}}{(q;q^2)_{n+1}(q^{2n+2};q^2)_\infty}.$$

In fact, both generating functions can be simplified further into the following expressions:

Theorem 3.2.1.

$$eo(q) = \frac{1}{(1-q)(q^2;q^2)_\infty}, \tag{3.2.1}$$

$$oe(q) = \frac{1}{1-q}\left(\frac{1}{(q;q^2)_\infty} - \frac{1}{(q^2;q^2)_\infty}\right). \tag{3.2.2}$$

Proof. Considering the $\mathcal{EO}$-partition, this is actually an SIP class of modulo 2, in which we can come up with the basis $\mathcal{B}$ as

$$\mathcal{B} = \left\{ \underbrace{1+\cdots+1}_{n} \middle| n \in \mathbb{N} \right\} \cup \left\{ \underbrace{2+\cdots+2}_{k} + \underbrace{3+\cdots+3}_{n-k} \middle| n \in \mathbb{N},\ 0 < k \le n \right\}.$$

Here, the set on the left takes care of partitions without an even part, and the right one for those with nonzero even parts. Hence we can write the generating function as

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{q^n + \sum_{k=1}^{n} q^{2k+3(n-k)}}{(q^2;q^2)_n} &= \sum_{n=0}^{\infty} \frac{q^n \left(1 + \sum_{k=0}^{n-1} q^{n+k}\right)}{(q^2;q^2)_n} \\
&= \sum_{n=0}^{\infty} \frac{q^n \left(\frac{1-q+q^n-q^{2n}}{1-q}\right)}{(q^2;q^2)_n} \\
&= \frac{1}{1-q}\left[\sum_{n=0}^{\infty} \frac{q^{2n}}{(q^2;q^2)_n} + (1-q)\sum_{n=0}^{\infty} \frac{q^n}{(q^2;q^2)_n} - \sum_{n=0}^{\infty} \frac{q^{3n}}{(q^2;q^2)_n}\right] \\
&= \frac{1}{1-q}\left[\frac{1}{(q^2;q^2)_\infty} + \frac{1-q}{(q;q^2)_\infty} - \frac{1}{(q^3;q^2)_\infty}\right] \\
&= \frac{1}{(1-q)(q^2;q^2)_\infty},
\end{aligned}$$

where the penultimate step follows from the q-binomial theorem in (1.5.2).

The same idea can be applied to $\mathcal{OE}$-partition, which is also an SIP class of modulo 2 with the following basis

$$\mathcal{B} = \left\{ \underbrace{1+\cdots+1}_{k} + \underbrace{2+\cdots+2}_{n-k} \middle| n \in \mathbb{N},\ 0 < k \le n \right\}.$$

So we can write the generating function as

$$
\begin{aligned}
\sum_{n=0}^{\infty} \frac{\sum_{k=1}^{n} q^{k+2(n-k)}}{(q^2;q^2)_n} &= \sum_{n=0}^{\infty} \frac{\sum_{k=1}^{n} q^{2n-k}}{(q^2;q^2)_n} \\
&= \sum_{n=0}^{\infty} \frac{q^n \sum_{k=0}^{n-1} q^k}{(q^2;q^2)_n} \\
&= \sum_{n=0}^{\infty} \frac{q^n \left(\frac{1-q^n}{1-q}\right)}{(q^2;q^2)_n} \\
&= \frac{1}{1-q}\left[\sum_{n=0}^{\infty} \frac{q^n}{(q^2;q^2)_n} - \sum_{n=0}^{\infty} \frac{q^{2n}}{(q^2;q^2)_n}\right] \\
&= \frac{1}{1-q}\left[\frac{1}{(q;q^2)_\infty} - \frac{1}{(q^2;q^2)_\infty}\right]
\end{aligned}
$$

Again, the last step follows from the q-binomial theorem in (1.5.2). □

We note that (3.2.1) was proved in [5, p. 435, eq. (2.1)], and (3.2.2) was proved in [6, p. 243], where both proofs used alternate approaches compared to our proof here.

3.3 Existence of basis in some special SIP classes

The two examples above presented the cases where each of the basis is simple to describe. However, this may not be the case for other kinds of SIP classes, especially with those of higher modulus $k > 1$. Nevertheless, it might still be possible to get around the problem of explicitly presenting the basis $\mathcal{B}$ by finding the recurrence relation of $B(n,q)$. This can be done by using the condition that follows from the uniqueness of representation in the definition of SIP class. It is still required that we know such basis exists for our set $\mathcal{P}$. The following theorem, due to Andrews [7], gives a large type of such sets $\mathcal{P}$ with the conditions on the difference between parts, that are guaranteed to have a basis $\mathcal{B}$.

Theorem 3.3.1. *Let* $\{c_1, \cdots, c_k\}$ *be a set of positive integers where* $c_r \equiv r \pmod{k}$, *and* $\{d_1, \cdots, d_k\}$ *be a set of nonnegative integers. Let* $\mathcal{P}$ *be the set of all integers* *partitions of the form*

$$b_1 + b_2 + \cdots + b_j,$$

where $0 < b_1 \leq b_2 \leq \cdots \leq b_j$ *with the condition that for* $1 \leq r \leq k$ *and each* b_i*:*

1. *if* $b_i \equiv c_r \pmod{k}$, *then* $b_i \geq c_r$,

2. *if* $i > 1$, *then* $b_i - b_{i-1} \geq d_r$.

Then $\mathcal{P}$ *is an SIP class with modulus* k*. And the basis* $\mathcal{B}$ *is the set of all those partitions*

$$\beta_1 + \beta_2 + \cdots + \beta_j,$$

such that if $\beta_1 \equiv r \pmod{k}$, *then* $\beta_1 = c_r$, *and for* $2 \leq i \leq j$, *if* $\beta_i \equiv r \pmod{k}$, *then*

$$d_r \leq \beta_i - \beta_{i-1} < d_r + k.$$

3.4 Göllnitz & Gordon

Göllnitz [26, eq. (2.22, 2.24)] found the following partition identities.

Theorem 3.4.1 (First little Göllnitz identity)**.** *The number of partitions of* n *such that the difference between parts is at least 2 and in which no consecutive odd numbers appear as parts, is equal to the number of partitions of* n *into parts congruent to 1, 5 or 6 (mod 8).*

Theorem 3.4.2 (Second little Göllnitz identity)**.** *The number of partitions of* n *such that each part is larger than 1, the difference between parts is at least 2 and in which no consecutive odd numbers appear as parts, is equal to the number of partitions of* n *into parts congruent to 2, 3 or 7 (mod 8).*

Also, the following variations are also found by Göllnitz [26, pp. 162-163, Satz 2.1 and 2.2] and Gordon [27, p. 741, Thms. 2 and 3.]

Theorem 3.4.3 (First Göllnitz-Gordon identity)**.** *The number of partitions of* n *such that the difference between parts is at least 2 and in which no consecutive even numbers appear as parts, is equal to the number of partitions of* n *into parts congruent to 1, 4 or 7 (mod 8).*

Theorem 3.4.4 (Second Göllnitz-Gordon identity)**.** *The number of partitions of n such that each part is larger than 2, the difference between parts is at least 2 and in which no consecutive even numbers appear as parts, is equal to the number of partitions of n into parts congruent to 3, 4 or 5 (mod 8).*

We can rewrite both little Göllnitz identities in terms of the generating function as

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}(-q^{-1};q^2)_n}{(q^2;q^2)_n} = \frac{1}{(q,q^5,q^6;q^8)_\infty}, \tag{3.4.1}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}(-q;q^2)_n}{(q^2;q^2)_n} = \frac{1}{(q^2,q^3,q^7;q^8)_\infty}, \tag{3.4.2}$$

and both Göllnitz-Gordon identities as

$$\sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(q^2;q^2)_n} = \frac{1}{(q,q^4,q^7;q^8)_\infty},$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+2n}(-q;q^2)_n}{(q^2;q^2)_n} = \frac{1}{(q^3,q^4,q^5;q^8)_\infty}.$$

It is worth mentioning that (3.4.1) and (3.4.2) are special cases of the Lebesgue identity [32]

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}(a;q)_n}{(q;q)_n} = (-q;q)_\infty(aq;q^2)_\infty,$$

where we set $a \to -q^{-1}, q \to q^2$ for (3.4.1) and $a \to -q, q \to q^2$ for (3.4.2). Andrews [7] had proved the first Göllnitz-Gordon identity by means of SIP classes. We will now prove both of the little Göllnitz identities and the second Göllnitz-Gordon identity in similar fashion.

Proof of Theorem 3.4.1. According to Theorem 3.3.1, this is an SIP of modulus 2 with $\{c_1, c_2\} = \{1, 2\}$ and $\{d_1, d_2\} = \{3, 2\}$. If we set $b_g(n, h)$ to be the generating function for its basis with n parts and the largest part equal to h, we have that

$$b_g(1,h) = \begin{cases} q, & \text{if } h = 1, \\ q^2, & \text{if } h = 2, \\ 0, & \text{otherwise}, \end{cases} \tag{3.4.3}$$

and for $n > 1$

$$b_g(n,h) = \begin{cases} q^h \left[b_g(n-1,h-4) + b_g(n-1,h-3) \right], & \text{if } h \text{ is odd}, \\ q^h \left[b_g(n-1,h-3) + b_g(n-1,h-2) \right], & \text{if } h \text{ is even}. \end{cases} \tag{3.4.4}$$

By (3.4.3) and (3.4.4), we have that for $n = 2$,

$$b_g(2,h) = \begin{cases} q^5 + q^6, & \text{if } h = 4, \\ q^6 + q^7, & \text{if } h = 5, \\ 0, & \text{otherwise}. \end{cases} \tag{3.4.5}$$

From (3.4.4), it can be seen that,

$$b_g(n, 2h-1) = q b_g(n, 2h-2). \tag{3.4.6}$$

With (3.4.4) and (3.4.6), we have that

$$\begin{aligned} b_g(n, 2n+2h-2) &= q^{2n+2h-2} \left[b_g(n-1, 2n+2h-5) + b_g(n-1, 2n+2h-4) \right] \\ &= q^{2n+2h-2} \left[b_g(n-1, 2n+2h-4) + q b_g(n-1, 2n+2h-6) \right]. \end{aligned} \tag{3.4.7}$$

It turns out that

$$q^{n^2+n+h^2-2}(1+q) \begin{bmatrix} n-2 \\ h-1 \end{bmatrix}_2,$$

also satisfy the equation (3.4.7) when $n > 2$, and satisfy (3.4.5) when $n = 2$. This can be verified with the aid of the following identities for q-binomial coefficients [2, p.

35, eq. (3.3.4)]:

$$\begin{bmatrix} a \\ b \end{bmatrix}_n = \begin{bmatrix} a-1 \\ b-1 \end{bmatrix}_n + q^{nb} \begin{bmatrix} a-1 \\ b \end{bmatrix}_n . \tag{3.4.8}$$

Thus, the generating function for the first little Göllnitz identity is

$$\begin{aligned} g(q) &= \sum_{n=0}^{\infty} \frac{\sum_{j=0}^{\infty} b_g(n,j)}{(q^2;q^2)_n} \\ &= 1 + \frac{q+q^2}{1-q^2} + \sum_{n=2}^{\infty}\sum_{j=1}^{\infty} \frac{(1+q)^2 q^{n^2+n+j^2-2} \begin{bmatrix} n-2 \\ j-1 \end{bmatrix}_2}{(q^2;q^2)_n} \\ &= 1 + \frac{q^2(1+q^{-1})}{1-q^2} + (1+q)^2 \sum_{n=2}^{\infty} \frac{q^{n^2+n-1}}{(q^2;q^2)_n} \sum_{j=0}^{n-2} q^{j^2+2j} \begin{bmatrix} n-2 \\ j \end{bmatrix}_2 \\ &= 1 + \frac{q^2(1+q^{-1})}{1-q^2} + (1+q^{-1})(1+q) \sum_{n=2}^{\infty} \frac{q^{n^2+n}(-q^3;q^2)_{n-2}}{(q^2;q^2)_n} \\ &= \sum_{n=0}^{\infty} \frac{q^{n^2+n}(-q^{-1};q^2)_n}{(q^2;q^2)_n}, \end{aligned} \tag{3.4.9}$$

where the penultimate step follows from finite version of Euler's theorem [37] (cf. [2, p. 36, eq. 3.3.6])

Alternatively, we can rewrite (3.4.9) as

$$\begin{aligned} g(q) &= 1 + \frac{q+q^2}{1-q^2} + (1+q^{-1}) \sum_{n=2}^{\infty} \frac{q^{n^2+n}}{(q^2;q^2)_n} \left(\sum_{j=0}^{n-2} q^{j^2+2j} \begin{bmatrix} n-2 \\ j \end{bmatrix}_2 + \sum_{j=1}^{n-1} q^{j^2} \begin{bmatrix} n-2 \\ j-1 \end{bmatrix}_2 \right) \\ &= 1 + (1+q^{-1}) \sum_{n=1}^{\infty} \frac{q^{n^2+n}}{(q^2;q^2)_n} \sum_{j=0}^{n-1} q^{j^2} \begin{bmatrix} n-1 \\ j \end{bmatrix}_2 \\ &= 1 + \sum_{n=1}^{\infty} \frac{q^{n^2+n}}{(q^2;q^2)_n} \left(\sum_{j=0}^{n-1} q^{j^2} \begin{bmatrix} n-1 \\ j \end{bmatrix}_2 + \sum_{j=1}^{n} q^{j^2-2j} \begin{bmatrix} n-1 \\ j-1 \end{bmatrix}_2 \right) \\ &= \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q^2;q^2)_n} \sum_{j=0}^{n} q^{j^2-2j} \begin{bmatrix} n \\ j \end{bmatrix}_2 \\ &= \sum_{j=0}^{\infty} \frac{q^{2j^2-j}}{(q^2;q^2)_j} \sum_{n=0}^{\infty} \frac{q^{n^2+(2j+1)n}}{(q^2;q^2)_n} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=0}^{\infty} \frac{q^{2j^2-j}}{(q^2;q^2)_j}(-q^{2j+2};q^2)_\infty \qquad \text{(by (1.5.4))} \\
&= (-q^2;q^2)_\infty \sum_{j=0}^{\infty} \frac{q^{2j^2-j}}{(q^4;q^4)_j} \\
&= (-q^2;q^2)_\infty(-q;q^4)_\infty \qquad \text{(by (1.5.4))} \\
&= \frac{(q^2;q^8)_\infty}{(q^2,q;q^4)_\infty} \\
&= \frac{1}{(q,q^5,q^6;q^8)_\infty},
\end{aligned}
$$

and we are done. □

Proof of Theorem 3.4.2. For this case, the partition is an SIP of modulus 2 with $\{c_1, c_2\} = \{3, 2\}$ and $\{d_1, d_2\} = \{3, 2\}$. The only difference for the recurrence relations compared to the previous one is the initial conditions, i.e., if we set $b_{g'}(n, h)$ to be the generating function for its basis with n parts and largest part equal to h, we have that

$$
b_{g'}(1,h) = \begin{cases} q^2, & \text{if } h = 2, \\ q^3, & \text{if } h = 3, \\ 0, & \text{otherwise,} \end{cases} \tag{3.4.10}
$$

and for $n > 1$

$$
b_{g'}(n,h) = \begin{cases} q^h\left[b_{g'}(n-1,h-4) + b_{g'}(n-1,h-3)\right], & \text{if } h \text{ is odd,} \\ q^h\left[b_{g'}(n-1,h-3) + b_{g'}(n-1,h-2)\right], & \text{if } h \text{ is even.} \end{cases}
$$

This again yields

$$
b_{g'}(n,2h+1) = qb_{g'}(n,2h), \tag{3.4.11}
$$

and

$$b_{g'}(n,2n+2h) = q^{2n+2h}\left[b_{g'}(n-1,2n+2h-2) + qb_{g'}(n-1,2n+2h-4)\right]. \tag{3.4.12}$$

In this case, it can be shown that

$$q^{n^2+n+h^2+2h}\begin{bmatrix} n-1 \\ h \end{bmatrix}_2,$$

satisfies both (3.4.10) and (3.4.12). Thus, the generating function for the second little Göllnitz identity is

$$\begin{aligned} g'(q) &= \sum_{n=0}^{\infty} \frac{\sum_{j=0}^{\infty} b_{g'}(n,j)}{(q^2;q^2)_n} \\ &= 1 + \sum_{n=1}^{\infty}\sum_{j=0}^{\infty} \frac{(1+q)q^{n^2+n+j^2+2j}\begin{bmatrix} n-1 \\ j \end{bmatrix}_2}{(q^2;q^2)_n} \qquad (3.4.13) \\ &= 1 + (1+q)\sum_{n=1}^{\infty} \frac{q^{n^2+n}}{(q^2;q^2)_n} \sum_{j=0}^{n-1} q^{j^2+2j}\begin{bmatrix} n-1 \\ j \end{bmatrix}_2 \\ &= 1 + (1+q)\sum_{n=1}^{\infty} \frac{q^{n^2+n}(-q^3;q^2)_{n-1}}{(q^2;q^2)_n} \quad \text{(by [37] (cf. [2, p. 36, eq. 3.3.6]))} \\ &= \sum_{n=0}^{\infty} \frac{q^{n^2+n}(-q;q^2)_n}{(q^2;q^2)_n}. \end{aligned}$$

Alternatively, (3.4.13) can be written as

$$\begin{aligned} g'(q) &= 1 + \sum_{n=1}^{\infty} \frac{q^{n^2+n}}{(q^2;q^2)_n}\left(\sum_{j=0}^{n-1} q^{j^2+2j}\begin{bmatrix} n-1 \\ j \end{bmatrix}_2 + \sum_{j=1}^{n} q^{j^2}\begin{bmatrix} n-1 \\ j-1 \end{bmatrix}_2\right) \\ &= \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q^2;q^2)_n}\sum_{j=0}^{n} q^{j^2}\begin{bmatrix} n \\ j \end{bmatrix}_2 \\ &= \sum_{j=0}^{\infty} \frac{q^{2j^2+j}}{(q^2;q^2)_j}\sum_{n=0}^{\infty} \frac{q^{n^2+(2j+1)n}}{(q^2;q^2)_n} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=0}^{\infty} \frac{q^{2j^2+j}}{(q^2;q^2)_j}(-q^{2j+2};q^2)_\infty \qquad \text{(by (1.5.4))} \\
&= (-q^2;q^2)_\infty \sum_{j=0}^{\infty} \frac{q^{2j^2+j}}{(q^4;q^4)_j} \\
&= (-q^2;q^2)_\infty(-q^3;q^4)_\infty \qquad \text{(by (1.5.4))} \\
&= \frac{(q^6;q^8)_\infty}{(q^2,q^3;q^4)_\infty} \\
&= \frac{1}{(q^2,q^3,q^7;q^8)_\infty},
\end{aligned}
$$

and we are done ☐

Proof of Theorem 3.4.4. For this case, the partition is an SIP of modulus 2 with $\{c_1, c_2\} = \{3, 4\}$ and $\{d_1, d_2\} = \{2, 3\}$. If we set $b_{G'}(n, h)$ to be the generating function for its basis with n parts and largest part equal to h, we have that

$$
b_{G'}(1,h) = \begin{cases} q^3, & \text{if } h = 3, \\ q^4, & \text{if } h = 4, \\ 0, & \text{otherwise,} \end{cases} \tag{3.4.14}
$$

and for $n > 1$

$$
b_{G'}(n,h) = \begin{cases} q^h \left[b_{G'}(n-1,h-2) + b_{G'}(n-1,h-3)\right], & \text{if } h \text{ is odd,} \\ q^h \left[b_{G'}(n-1,h-3) + b_{G'}(n-1,h-4)\right], & \text{if } h \text{ is even.} \end{cases} \tag{3.4.15}
$$

Just like the first Gollnitz-Gordon identity, (3.4.15) yields

$$
b_{G'}(n,2h) = q b_{G'}(n,2h-1),
$$

and

$$
b_{G'}(n,2n+2h-1) = q^{2n+2h-1}\left[b_{G'}(n-1,2n+2h-3) + q b_{G'}(n-1,2n+2h-5)\right]. \tag{3.4.16}
$$

As for this case, it can be shown that

$$q^{n^2+2n+h^2-1}\begin{bmatrix} n-1 \\ h-1 \end{bmatrix}_2,$$

also satisfies both (3.4.14) and (3.4.16). Thus, the generating function for the second Gollnitz-Gordon identity is

$$\begin{aligned}
G'(q) &= \sum_{n=0}^{\infty}\frac{\sum_{j=0}^{\infty} b_{G'}(n,j)}{(q^2;q^2)_n} \\
&= 1+\frac{q^3+q^4}{1-q^2}+\sum_{n=2}^{\infty}\sum_{j=1}^{\infty}\frac{(1+q)q^{n^2+2n+j^2-1}\begin{bmatrix} n-1 \\ j-1 \end{bmatrix}_2}{(q^2;q^2)_n} \qquad (3.4.17) \\
&= 1+\frac{q^3(1+q)}{1-q^2}+(1+q)\sum_{n=2}^{\infty}\frac{q^{n^2+2n}}{(q^2;q^2)_n}\sum_{j=0}^{n-1}q^{j^2+2j}\begin{bmatrix} n-1 \\ j \end{bmatrix}_2 \\
&= 1+\frac{q^3(1+q)}{1-q^2}+(1+q)\sum_{n=2}^{\infty}\frac{q^{n^2+2n}(-q^3;q^2)_{n-1}}{(q^2;q^2)_n} \text{(by [37] (cf. [2, p. 36, eq. 3.3.6]))} \\
&= \sum_{n=0}^{\infty}\frac{q^{n^2+2n}(-q;q^2)_n}{(q^2;q^2)_n}.
\end{aligned}$$

We can rewrite (3.4.17) as

$$\begin{aligned}
G'(q) &= 1+\frac{q^3+q^4}{1-q^2}+\sum_{n=2}^{\infty}\frac{q^{n^2+2n}}{(q^2;q^2)_n}\left(\sum_{j=0}^{n-1}q^{j^2+2j}\begin{bmatrix} n-1 \\ j \end{bmatrix}_2+\sum_{j=1}^{n}q^{j^2}\begin{bmatrix} n-1 \\ j-1 \end{bmatrix}_2\right) \\
&= \sum_{n=0}^{\infty}\frac{q^{n^2+2n}}{(q^2;q^2)_n}\sum_{j=0}^{n}q^{j^2}\begin{bmatrix} n \\ j \end{bmatrix}_2 \\
&= \sum_{j=0}^{\infty}\frac{q^{2j^2+2j}}{(q^2;q^2)_j}\sum_{n=0}^{\infty}\frac{q^{n^2+(2j+2)n}}{(q^2;q^2)_n} \\
&= \sum_{j=0}^{\infty}\frac{q^{2j^2+2j}}{(q^2;q^2)_j}(-q^{2j+3};q^2)_\infty \qquad \text{(by (1.5.4))} \\
&= (-q;q^2)_\infty\sum_{j=0}^{\infty}\frac{q^{2j^2+2j}}{(-q;-q)_{2j+1}}.
\end{aligned}$$

Now consider

$$
\begin{aligned}
G'(-q^2) &= (q^2;q^4)_\infty \sum_{j=0}^{\infty} \frac{q^{4j^2+4j}}{(q^2;q^2)_{2j+1}} \\
&= \frac{(q^2;q^4)_\infty}{q} \sum_{j=0}^{\infty} \frac{q^{(2j+1)^2}}{(q^2;q^2)_{2j+1}} \\
&= \frac{(q^2;q^4)_\infty}{2q} \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^2;q^2)_n}(1-(-1)^n) \\
&= \frac{(q^2;q^4)_\infty}{2q} \Big((-q;q^2)_\infty - (q;q^2)_\infty\Big) \qquad \text{(by (1.5.4))} \\
&= \frac{(q^2;q^4)_\infty}{2q(q^4;q^4)_\infty} \left(\sum_{n=-\infty}^{\infty} q^{2n^2-n} - \sum_{n=-\infty}^{\infty} (-1)^n q^{2n^2-n}\right) \qquad \text{(by (1.5.6))} \\
&= \frac{(q^2;q^4)_\infty}{(q^4;q^4)_\infty} \sum_{n=-\infty}^{\infty} q^{8n^2-6n} \\
&= \frac{(q^2;q^4)_\infty}{(q^4;q^4)_\infty} (-q^{14},-q^2,q^{16};q^{16})_\infty. \qquad \text{(by (1.5.6))}
\end{aligned}
$$

Thus

$$
\begin{aligned}
G'(q) &= \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}(q^7,q,q^8;q^8)_\infty \\
&= \frac{(q^2;q^4)_\infty}{(q;q)_\infty}(q^7,q,q^8;q^8)_\infty \\
&= \frac{1}{(q^3,q^4,q^5;q^8)_\infty},
\end{aligned}
$$

as desired. □

3.5 A family of Rogers-Ramanujan-Type identities

In [3], Andrews discovered the following theorem while searching for Rogers-Ramanujan-Type identities by computer:

Theorem 3.5.1. *For $k \geq 2$, let $\mathfrak{A}_k(n)$ denote the number of partitions of n into parts congruent to $0, -1 \pmod{k}$ but not congruent to $k(k-1) \pmod{k^2}$. Let $\mathfrak{B}_k(n)$ denote the number of partitions of n of the form $n = b_1 + \cdots + b_s$, where $b_i - b_{i+1} \geq$*

$j(k-1)$ (assume $b_{s+1}=0$) with j the least nonnegative residue of $-b_i \pmod{k}$. Then $\mathfrak{A}_k(n)=\mathfrak{B}_k(n)$ for all n.

The proof of this theorem followed from the similar method in [1]. We will prove this theorem for the case $k=2$ using the idea of SIP classes.

Proof. We can see that the generating function for $\mathfrak{A}_2(n)$ is

$$\sum_{n=0}^{\infty} \mathfrak{A}_2(n) q^n = \frac{1}{(q, q^3, q^4; q^4)_\infty}.$$

As for $\mathfrak{B}_2(n)$, the difference conditions can be written as

$$b_i - b_{i+1} \geq \begin{cases} 0, & \text{if } b_i = 2t, \\ 1, & \text{if } b_i = 2t+1. \end{cases}$$

This is an SIP of modulus 2 with $\{c_1, c_2\} = \{1, 2\}$ and $\{d_1, d_2\} = \{1, 0\}$. If we set $b_2(n,h)$ to be the generating function for its basis with n parts and largest part equal to h, we have that

$$b_2(1,h) = \begin{cases} q, & \text{if } h = 1, \\ q^2, & \text{if } h = 2, \\ 0, & \text{otherwise,} \end{cases} \tag{3.5.1}$$

and for $n > 1$

$$b_2(n,h) = \begin{cases} q^h \left[b_2(n-1,h) + b_2(n-1,h-1) \right], & \text{if } h = 2t, \\ q^h \left[b_2(n-1,h-1) + b_2(n-1,h-2) \right], & \text{if } h = 2t+1. \end{cases} \tag{3.5.2}$$

Thus for $n = 2$, we have that

$$b_2(2,h) = \begin{cases} q^3 + q^4, & \text{if } h = 2, \\ q^4 + q^5, & \text{if } h = 3, \\ 0, & \text{otherwise.} \end{cases} \tag{3.5.3}$$

From (3.5.2), it can be seen that,

$$b_2(n,2h+1) = qb_2(n,2h). \tag{3.5.4}$$

With (3.5.2) and (3.5.4), we have that

$$\begin{aligned} b_2(n,2h) &= q^{2h}\left[b_2(n-1,2h)+b_2(n-1,2h-1)\right] \\ &= q^{2h}\left[b_2(n-1,2h)+qb_2(n-1,2h-2)\right]. \end{aligned} \tag{3.5.5}$$

It turns out that

$$q^{2n+h^2-2}(1+q)\begin{bmatrix} n-2 \\ h-1 \end{bmatrix}_2,$$

also satisfy the equation (3.5.5) when $n>2$, and satisfy (3.5.3) when $n=2$. Thus, the generating function for $b_2(n)$ is

$$\begin{aligned} \sum_{n=0}^{\infty}\mathfrak{B}_2(n)q^n &= \sum_{n=0}^{\infty}\frac{\sum_{j=0}^{\infty}b_g(n,j)}{(q^2;q^2)_n} \\ &= 1+\frac{q+q^2}{1-q^2}+\sum_{n=2}^{\infty}\sum_{j=1}^{\infty}\frac{(1+q)^2q^{2n+j^2-2}\begin{bmatrix} n-2 \\ j-1 \end{bmatrix}_2}{(q^2;q^2)_n} \\ &= 1+\frac{q^2(1+q^{-1})}{1-q^2}+(1+q)^2\sum_{n=2}^{\infty}\frac{q^{2n-1}}{(q^2;q^2)_n}\sum_{j=0}^{n-2}q^{j^2+2j}\begin{bmatrix} n-2 \\ j \end{bmatrix}_2 \\ &= 1+\frac{q^2(1+q^{-1})}{1-q^2}+(1+q^{-1})(1+q)\sum_{n=2}^{\infty}\frac{q^{2n}(-q^3;q^2)_{n-2}}{(q^2;q^2)_n} \\ &= \sum_{n=0}^{\infty}\frac{q^{2n}(-q^{-1};q^2)_n}{(q^2;q^2)_n}. \end{aligned} \tag{3.5.6}$$

Alternatively, we can rewrite (3.5.6) as

$$\begin{aligned} \sum_{n=0}^{\infty}\mathfrak{B}_2(n)q^n &= 1+\frac{q+q^2}{1-q^2}+(1+q^{-1})\sum_{n=2}^{\infty}\frac{q^{2n}}{(q^2;q^2)_n}\left(\sum_{j=0}^{n-2}q^{j^2+2j}\begin{bmatrix} n-2 \\ j \end{bmatrix}_2+\sum_{j=1}^{n-1}q^{j^2}\begin{bmatrix} n-2 \\ j-1 \end{bmatrix}_2\right) \\ &= 1+(1+q^{-1})\sum_{n=1}^{\infty}\frac{q^{2n}}{(q^2;q^2)_n}\sum_{j=0}^{n-1}q^{j^2}\begin{bmatrix} n-1 \\ j \end{bmatrix}_2 \end{aligned}$$

$$
\begin{aligned}
&= 1 + \sum_{n=1}^{\infty} \frac{q^{2n}}{(q^2;q^2)_n} \left(\sum_{j=0}^{n-1} q^{j^2} \begin{bmatrix} n-1 \\ j \end{bmatrix}_2 + \sum_{j=1}^{n} q^{j^2-2j} \begin{bmatrix} n-1 \\ j-1 \end{bmatrix}_2 \right) \\
&= \sum_{n=0}^{\infty} \frac{q^{2n}}{(q^2;q^2)_n} \sum_{j=0}^{n} q^{j^2-2j} \begin{bmatrix} n \\ j \end{bmatrix}_2 \\
&= \sum_{j=0}^{\infty} \frac{q^{j^2}}{(q^2;q^2)_j} \sum_{n=0}^{\infty} \frac{q^{2n}}{(q^2;q^2)_n} \\
&= (-q;q^2)_\infty \cdot \frac{1}{(q^2;q^2)_\infty} \\
&= \frac{1}{(q,q^3,q^4;q^4)_\infty} \\
&= \sum_{n=0}^{\infty} \mathfrak{A}_2(n) q^n,
\end{aligned}
$$

which proves the theorem. □

Chapter 4
Partitions associated with Mock Theta Functions

4.1 Introduction

Regarded as one of the most mysterious objects in Mathematics, mock theta functions are the list of functions in $q-$series with particular sets of special properties discovered by Ramanujan. The first appearance of such functions was recorded along with other results in his last letter to Hardy in 1920 [18, p. 220], in which he merely stated several identities involving mock theta functions, without rigorously defining what they are. Later in 1976, Andrews found other examples of mock theta functions and their identities hidden inside the sheaf of Ramanujan's manuscript, the so-called "Lost Notebook" [4, pp. 5-8]. Since then, the properties of mock theta functions have been an intense research topic.

One of the interesting features of mock theta functions is that many of such functions also admit combinatorial interpretations. For example, consider the following third order mock theta functions [36, 39],

$$\omega(q) := \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q;q^2)_{n+1}^2},$$
$$\nu(q) := \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(-q;q^2)_{n+1}}.$$

Also, let us define $p_\omega(n)$ and $p_\nu(n)$ by

$$\sum_{n=1}^{\infty} p_\omega(n)q^n := \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)(q^{n+1};q)_n(q^{2n+2};q^2)_\infty},$$
$$\sum_{n=0}^{\infty} p_\nu(n)q^n := \sum_{n=0}^{\infty} q^n(-q^{n+1};q)_n(-q^{2n+2};q^2)_\infty.$$

From the definition, we can see that $p_\omega(n)$ counts the number of partitions of n where each odd part is less than twice the smallest part, while $p_\nu(n)$ also counts the number partitions of n where each odd part is less than twice the smallest part, with an additional requirement that each part must be distinct. It has been shown in [10] that

$$\sum_{n=1}^{\infty} p_\omega(n)q^n = q\omega(q),$$
$$\sum_{n=0}^{\infty} p_\nu(n)q^n = \nu(-q).$$

There is also a connection between $\overline{\mathcal{EO}}(n)$ in Chapter 2 and $p_\nu(n)$ found in [5]:

$$p_\nu(2n) = \overline{\mathcal{EO}}(2n).$$

In other words, $\overline{eo}(q)$ is basically the even part of the series $\nu(-q)$.

Studies on the divisibility properties of partitions derived from both $\omega(q)$ and $\nu(q)$ were initiated by the work of Garthwaite and Penniston [23], who found infinitely many congruences of Ramanujan-type related to the coefficients of $\omega(q)$. The first explicit congruences of $p_\omega(n)$ were due to Waldherr [38] with suggested computation done by Lovejoy. In section 4.3 we will prove the following infinite family of congruences of $p_\omega(n)$ in modulo of powers of 2. This is joint work with George Andrews, James Sellers, and Ae Ja Yee [14].

Theorem 4.1.1. *For nonnegative integers n and k,*

$$p_\omega\left(2^{2k+3}n + \frac{11\cdot 2^{2k}+1}{3}\right) \equiv 0 \pmod 4,$$

$$p_\omega\left(2^{2k+3}n+\frac{17\cdot 2^{2k}+1}{3}\right)\equiv 0 \pmod 8,$$
$$p_\omega\left(2^{2k+4}n+\frac{38\cdot 2^{2k}+1}{3}\right)\equiv 0 \pmod 4.$$

In [10], there was also a consideration for a partition counted by $p_\omega(n)$, with a milder restriction that only the smallest part cannot be repeated. This resulted in the following identity

$$\sum_{n=1}^{\infty}\frac{q^n}{(q^{n+1};q)_n(q^{2n+1};q^2)_\infty}=-1+(-q;q)_\infty.$$

Motivated by the identity above, in Section 4.4 we will prove the following variant:

Theorem 4.1.2.

$$\sum_{n=1}^{\infty}\frac{q^n}{(-q^{n+1};q)_n(-q^{2n+1};q^2)_\infty}=\sum_{n=1}^{\infty}\left[(-q)^{\frac{n(3n+1)}{2}}-(-q)^{\frac{n(3n-1)}{2}}\right].$$

There were also further variants of the generating functions of $p_\omega(n)$ and $p_\nu(n)$ in [10], which yielded results analogous to Euler's pentagonal number theorem.

$$\sum_{n=1}^{\infty}\frac{q^n}{(-q^n;q)_{n+1}(-q^{2n+2};q^2)_\infty}=\sum_{j=0}^{\infty}(-1)^jq^{6j^2+4j+1}(1+q^{4j+2}), \tag{4.1.1}$$
$$\sum_{n=0}^{\infty}q^n(q^{n+1};q)_n(q^{2n+2};q^2)_\infty=\sum_{j=0}^{\infty}(-1)^jq^{j(3j+2)}(1+q^{2j+1}). \tag{4.1.2}$$

In Section 4.5, we will prove generalizations for both of the variants:

Theorem 4.1.3. *For a nonnegative integer k,*

$$\sum_{n=1}^{\infty}\frac{q^n(q^{2n};q^2)_k}{(-q^n;q)_{n+2k+1}(-q^{2n+2k+2};q^2)_\infty}=\sum_{j=0}^{\infty}(q^2;q^4)_jq^{2(k+1)j+1}.$$

Theorem 4.1.4. *For a nonnegative integer k,*

$$\sum_{n=0}^{\infty}q^n(q^{n+1};q)_{n+2k}(q^{2n+2k+2};q^2)_\infty=\sum_{j=0}^{\infty}(q;q^2)_jq^{(2k+1)j}.$$

Note that both theorems are reduced to (4.1.1) and (4.1.2) when $k = 0$, and by using Entry 9.5.2 from [8, p. 238]

$$\sum_{n=0}^{\infty}(q;q^2)_n q^n = \sum_{n=0}^{\infty}(-1)^n q^{n(3n+2)}(1+q^{2n+1}). \tag{4.1.3}$$

We will also give alternate representations for some small values of k in Section 4.6.

4.2 Preliminaries

First, we introduce other kinds of functions called Ramanujan theta functions.

Definition 4.2.1.

$$\phi(q) := \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2 (q^2;q^2)_\infty,$$
$$\psi(q) := \sum_{n=-\infty}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}.$$

Note that we can rewrite both functions as

$$\phi(q) = \frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^2 (q^4;q^4)_\infty^2}, \tag{4.2.1}$$
$$\psi(q) = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty}. \tag{4.2.2}$$

The following relations between $\phi(q)$ and $\psi(q)$ are from Entry 25 (i), (ii), and Entry 25 (v), (vi) in [17] respectively.

Proposition 4.2.2.

$$\phi(q) = \phi(q^4) + 2q\psi(q^8),$$
$$\phi(q)^2 = \phi(q^2)^2 + 4q\psi(q^4)^2.$$

Also, consider a formula from [39, p. 63]:

$$f(q^8) + 2q\omega(q) + 2q^3\omega(-q^4) = \frac{\phi(q)\phi(q^2)^2}{(q^4;q^4)_\infty^2} =: F(q), \tag{4.2.3}$$

where $f(q)$ is another third order mock theta function:

$$f(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2}. \tag{4.2.4}$$

Lemma 4.2.3. *The 4-dissection of $F(q)$ is*

$$F(q) = \sum_{n=0}^{3} F_n(q^4) q^n,$$

where

$$\begin{aligned} F_0(q) &= \frac{\phi(q)^3}{(q;q)_\infty^2}, \\ F_1(q) &= \frac{2\phi(q)^2\psi(q^2)}{(q;q)_\infty^2}, \\ F_2(q) &= \frac{4\phi(q)\psi(q^2)^2}{(q;q)_\infty^2}, \\ F_3(q) &= \frac{8\psi(q^2)^3}{(q;q)_\infty^2}. \end{aligned}$$

Proof. By Proposition 4.2.2,

$$\begin{aligned} F(q) &= \frac{\phi(q)\phi(q^2)^2}{(q^4;q^4)_\infty^2} \\ &= \frac{\left(\phi(q^4) + 2q\psi(q^8)\right)\left(\phi(q^4)^2 + 4q^2\psi(q^8)^2\right)}{(q^4;q^4)_\infty^2} \\ &= \frac{\phi(q^4)^3 + 2q\psi(q^8)\phi(q^4)^2 + 4q^2\phi(q^4)\psi(q^8)^2 + 8q^3\psi(q^8)^3}{(q^4;q^4)_\infty^2}, \end{aligned}$$

and the result follows. □

Lemma 4.2.4.

$$1 + \sum_{n=1}^{\infty} (-q;q)_{n-1} q^n = (-q;q)_\infty.$$

Proof. This follows from that fact that both sides are the generating functions for partitions with distinct parts, where the sum on the left hand side represent cases with the largest part as n. □

The next two lemmas follow simply from induction and applications of the binomial theorem.

Lemma 4.2.5. *For any positive integer n,*

$$(1+x)^{2^n} \equiv (1+x^2)^{2^{n-1}} \pmod{2^n}.$$

Lemma 4.2.6. *For any prime p,*

$$(1+x)^p \equiv (1+x^p) \pmod{p}.$$

The following Rogers-Fine identity [8, Eq. 9.1.1, p. 223] is also needed.

Proposition 4.2.7.

$$\sum_{n=0}^{\infty} \frac{(\alpha;q)_n}{(\beta;q)_n}\tau^n = \sum_{n=0}^{\infty} \frac{(\alpha;q)_n(\alpha\tau q/\beta;q)_n\beta^n\tau^n q^{n^2-n}(1-\alpha\tau q^{2n})}{(\beta;q)_n(\tau;q)_{n+1}}.$$

4.3 Congruences of $p_\omega(n)$

First, we have the following cases of Theorem 4.1.1 when $k=0$:

Theorem 4.3.1. *For any nonnegative integers n,*

$$p_\omega(8n+4) \equiv 0 \pmod{4}, \tag{4.3.1}$$

$$p_\omega(8n+6) \equiv 0 \pmod{8}, \tag{4.3.2}$$

$$p_\omega(16n+13) \equiv 0 \pmod{4}. \tag{4.3.3}$$

Proof. From the left hand side (4.2.3), it can be seen that only the term $2q\omega(q)$ creates q^n where n is not congruent to $0, 3$ or 7 modulo 8. Now consider the terms for q^{8n+4} and q^{16n+3} in $F(q)$. We will show that the coefficients for such terms are congruent to 0 modulo 8, which will imply (4.3.1) and (4.3.3).

For $q^{8n+4} = q^{4(2n+1)}$, it is sufficient to show that the odd powers of q in $F_0(q)$ vanish modulo 8. By Lemma 4.2.3, Lemma 4.2.5, and (4.2.1),

$$\begin{aligned}
F_0(q) &= \frac{\phi(q)^3}{(q;q)_\infty^2} \\
&= \frac{(q^2;q^2)_\infty^{15}}{(q;q)_\infty^8 (q^4;q^4)_\infty^6} \\
&\equiv \frac{(q^2;q^2)_\infty^{15}}{(q^2;q^2)_\infty^4 (q^4;q^4)_\infty^6} \pmod 8 \\
&= \frac{(q^2;q^2)_\infty^{11}}{(q^4;q^4)_\infty^6}.
\end{aligned}$$

Thus $F_0(q)$ is and even function in modulo 8, which implies the first congruence (4.3.1).

As for $q^{16n+13} = q^{4(4n+3)+1}$, it is sufficient to show that the powers of the form q^{4n+3} in $\frac{1}{2}F_1(q)$ vanish modulo 4. By Lemma 4.2.3, Lemma 4.2.5, (4.2.1), and (4.2.2),

$$\begin{aligned}
\frac{1}{2}F_1(q) &= \frac{\phi(q)^2\psi(q^2)}{(q;q)_\infty^2} \\
&= \frac{(q^2;q^2)_\infty^{9}}{(q;q)_\infty^6 (q^4;q^4)_\infty^2} \\
&\equiv \frac{(q^2;q^2)_\infty^{9}}{(q;q)_\infty^2 (q^2;q^2)_\infty^2 (q^4;q^4)_\infty^2} \pmod 4 \\
&= \frac{(q^2;q^2)_\infty^{7}}{(q;q)_\infty^2 (q^4;q^4)_\infty^2} \\
&= \phi(q)(q^2;q^2)_\infty^2.
\end{aligned}$$

Now, by Lemma 4.2.5,

$$(q^2;q^2)_\infty^2 \equiv (q^4;q^4)_\infty \pmod 2.$$

So we can write $(q^2;q^2)_\infty^2$ as

$$(q^2;q^2)_\infty^2 = (q^4;q^4)_\infty + 2q^2X(q^4),$$

for some function $X(q^4)$. Hence, by Proposition 4.2.2,

$$\begin{aligned}\frac{1}{2}F_1(q) &\equiv (\phi(q^4)+2q\psi(q^8))((q^4;q^4)_\infty + 2q^2X(q^4)) && \pmod 4,\\ &\equiv \phi(q^4)(q^4;q^4)_\infty + 2q\psi(q^8)(q^4;q^4)_\infty + 2q^2\phi(q^4)X(q^4) && \pmod 4.\end{aligned}$$

Thus, the coefficients of q^{4n+3} in $\frac{1}{2}F_1(q)$ vanish modulo 4. This implies the third congruence (4.3.3).

Finally, consider the powers $q^{8n+6} = q^{4(2n+1)+2}$ in $F(q)$. We will show that in $\frac{1}{4}F_2(q)$, all odd powers will have the coefficients congruent to 0 modulo 4, which will imply (4.3.2). Again, by Lemma 4.2.3, Lemma 4.2.5, (4.2.1), and (4.2.2),

$$\begin{aligned}\frac{1}{4}F_2(q) &= \frac{\phi(q)\psi(q^2)^2}{(q;q)_\infty^2}\\ &= \frac{(q^2;q^2)_\infty^3(q^4;q^4)_\infty^2}{(q;q)_\infty^4}\\ &\equiv \frac{(q^2;q^2)_\infty^3(q^4;q^4)_\infty^2}{(q^2;q^2)_\infty^2} \pmod 4\\ &= (q^2;q^2)_\infty(q^4;q^4)_\infty^2,\end{aligned}$$

which implies the second congruence (4.3.2). □

We now prove the cases of any positive integer k by induction.

Proof of Theorem 4.1.1. Define the sequence g_k by the following recurrence relation

$$g_k = 4g_{k-1} - 1. \tag{4.3.4}$$

Solving this yields

$$g_k = 2^{2k}g_0\frac{2^{2k}-1}{3}.$$

In particular, we have that for any nonnegative integer n,

$$g_k = \begin{cases} 2^{2k+3}n + \frac{11\cdot 2^{2k}+1}{3} & \text{if } g_0 = 8n+4, \\ 2^{2k+3}n + \frac{17\cdot 2^{2k}+1}{3} & \text{if } g_0 = 8n+6, \\ 2^{2k+4}n + \frac{38\cdot 2^{2k}+1}{3} & \text{if } g_0 = 16n+3. \end{cases}$$

Our goal is to show that $p_\omega(g_k) \equiv 0 \pmod 4$ when $g_0 = 8n+4$ or $16n+3$, and $p_\omega(g_k) \equiv 0 \pmod 8$ when $g_0 = 8n+6$.

From the definition of g_k in (4.3.4), it can be seen that for any $k > 0$, we will have $g_k \equiv 3 \pmod 4$. Thus, when we consider the left hand side of (4.2.3), the coefficients of q^{g_k} must come from the terms $2q\omega(q) + 2q^3\omega(-q^4)$. Next, note that

$$\begin{aligned} 2q\omega(q) + 2q^3\omega(-q^4) &= 2\left[q\omega(q) - q^{-1}\left(-q^4\omega(-q^4)\right)\right] \\ &= 2\left[\sum_{n=1}^{\infty} p_\omega(n)q^n - \sum_{n=1}^{\infty} p_\omega(n)(-1)^n q^{4n-1}\right], \end{aligned}$$

hence the coefficient of q^{g_k} is equal to

$$2\left(p_\omega(g_k) - (-1)^{g_{k-1}} p_\omega(g_{k-1})\right).$$

Consider the first two cases of $g_0 = 8n+4$ or $16n+3$. By the induction hypothesis, we have that $p_\omega(g_{k-1}) \equiv 0 \pmod 4$. Thus, in order to show that $p_\omega(g_k) \equiv 0 \pmod 4$, it suffices to show that the coefficient of q^{g_k} in $F(q)$ vanishes in modulo 8. This actually follows from Lemma 4.2.3 with $F_3(q)$ and from the fact that $g_k \equiv 3 \pmod 4$.

As for the case $g_0 = 8n+6$, the induction hypothesis implies that $p_\omega(g_{k-1}) \equiv 0 \pmod 8$. So to show that $p_\omega(g_k) \equiv 0 \pmod 8$, we must have that the coefficient of q^{g_k} in $F(q)$ vanishes in modulo 16. Since

$$\begin{aligned} g_k &= 4g_{k-1} - 1 \\ &= 4(g_{k-1} - 1) + 3, \end{aligned}$$

it suffices to show that the coefficient of $q^{g_{k-1}-1}$ in $F_3(q)/8$ vanishes in modulo 2. Now, by Lemma 4.2.3, Lemma 4.2.5 and Lemma 4.2.6,

$$\begin{aligned}\frac{1}{8}F_3(q) &= \frac{\psi(q^2)^3}{(q;q)_\infty^2} \\ &= \frac{(q^4;q^4)_\infty^3}{(q;q)_\infty^2(q^2;q^4)_\infty^3} \\ &= \frac{(q^4;q^4)_\infty^6}{(q;q)_\infty^2(q^2;q^2)_\infty^3} \\ &\equiv \frac{(q^4;q^4)_\infty^6}{(q^2;q^2)_\infty^4} \pmod 2 \\ &\equiv (q^4;q^4)_\infty^4 \pmod 2.\end{aligned}$$

However, $g_0 \equiv 2$ and $g_k \equiv 3$ modulo 4 for any $k > 0$. So $g_{k-1} - 1 \not\equiv 0 \pmod 4$. And the result follows. □

4.4 An identity analogous to $q\omega(q)$

We now prove Theorem 4.1.2.

Proof. First, note that

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{q^n}{(-q^{n+1};q)_n(-q^{2n+1};q^2)_\infty} &= \frac{1}{(-q;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(-q;q)_n(-q;q^2)_n q^n}{(-q;q)_{2n}} \\ &= \frac{1}{(-q;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(-q;q)_n q^n}{(-q^2;q^2)_n} \\ &= \frac{(q^2;q^2)_\infty}{(-q;q)_\infty}\sum_{n=0}^{\infty}\frac{(-q^{2n+2};q^2)_\infty}{(q^{2n+2};q^2)_\infty}\cdot\frac{q^n}{(q;q)_n} \\ &= (q;q)_\infty\sum_{n=0}^{\infty}\frac{q^n}{(q;q)_n}\sum_{m=0}^{\infty}\frac{(-1;q^2)_m q^{m(2n+2)}}{(q^2;q^2)_m} \quad \text{(by (1.5.2))} \\ &= (q;q)_\infty\sum_{m=0}^{\infty}\frac{(-1;q^2)_m q^{2m}}{(q^2;q^2)_m}\sum_{n=0}^{\infty}\frac{q^{n(2m+1)}}{(q;q)_n} \\ &= (q;q)_\infty\sum_{m=0}^{\infty}\frac{(-1;q^2)_m q^{2m}}{(q^2;q^2)_m(q^{2m+1};q)_\infty} \quad \text{(by (1.5.4))}\end{aligned}$$

$$= \sum_{m=0}^{\infty} (-1, q; q^2)_m q^{2m}$$
$$= \sum_{m=0}^{\infty} (-1; -q)_{2m} q^{2m}.$$

Thus,

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{q^n}{(-q^{n+1}; q)_n (-q^{2n+1}; q^2)_\infty} \\
&\quad = -\frac{1}{(-q; q^2)_\infty} + \sum_{m=0}^{\infty} (-1; -q)_{2m} q^{2m} \\
&\quad = -(q; -q)_\infty + 1 + 2 \sum_{m=1}^{\infty} (q; -q)_{2m-1} q^{2m} \quad \text{(by (1.5.1))} \\
&\quad = \left(-1 - \sum_{m=1}^{\infty} (q; -q)_m (-q)^m\right) + 1 + 2 \sum_{m=1}^{\infty} (q; -q)_{2m-1} q^{2m} \text{ (Lemma 4.2.4)} \\
&\quad = -\sum_{m=1}^{\infty} (q; -q)_{2m-1} q^{2m} + \sum_{m=1}^{\infty} (q; -q)_{2m-2} q^{2m-1} \\
&\qquad + 2 \sum_{m=1}^{\infty} (q; -q)_{2m-1} q^{2m} \\
&\quad = \sum_{m=1}^{\infty} (q; -q)_{2m-1} q^{2m} + \sum_{m=1}^{\infty} (q; -q)_{2m-2} q^{2m-1} \\
&\quad = \sum_{m=1}^{\infty} (q; -q)_{m-1} q^m.
\end{aligned}$$

Now, consider a generating function

$$\sum_{m=1}^{\infty} (-q; q)_{m-1} (-q)^m = \sum_{n=1}^{\infty} (pd_e(n) - pd_o(n)) q^n,$$

where $pd_e(n)$ is a number of partition of n into distinct parts with largest part even, similarly to $pd_o(n)$ but with largest part odd. By employing Franklin involution used in the proof of the pentagonal number theorem [2, Theorem 1.6, p. 10], similar argument shows that

$$pd_e(n) - pd_o(n) = \begin{cases} 1, & n = \frac{k(3k+1)}{2}, \\ -1, & n = \frac{k(3k-1)}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

So

$$\sum_{m=1}^{\infty} (-q;q)_{m-1}(-q)^m = \sum_{n=1}^{\infty} \left[q^{\frac{n(3n+1)}{2}} - q^{\frac{n(3n-1)}{2}} \right].$$

Therefore

$$\sum_{n=1}^{\infty} \frac{q^n}{(-q^{n+1};q)_n(-q^{2n+1};q^2)_\infty} = \sum_{n=1}^{\infty} \left[(-q)^{\frac{n(3n+1)}{2}} - (-q)^{\frac{n(3n-1)}{2}} \right],$$

as desired. □

4.5 Analogues of the pentagonal number theorem

The proof for both Theorem 4.1.3 and Theorem 4.1.4 follows the similar lines in [10].

Proof of Theorem 4.1.3.

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{q^n(q^{2n+2};q^2)_k}{(-q^n;q)_{n+2k+1}(-q^{2n+2k+2};q^2)_\infty} \\
&= \sum_{n=0}^{\infty} \frac{q^{n+1}(q^{2n+2};q^2)_k}{(-q^{n+1};q)_{n+2k+2}(-q^{2n+2k+4};q^2)_\infty} \\
&= \sum_{n=0}^{\infty} \frac{q^{n+1}(-q;q)_n(q^{2n+2};q^2)_k}{(-q;q)_{2n+2k+2}(-q^{2n+2k+4};q^2)_\infty} \\
&= \frac{q}{(-q^2;q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^n(-q;q)_n(-q^2;q^2)_{n+k+1}(q^{2n+2};q^2)_k}{(-q;q)_{2n+2k+2}} \\
&= \frac{q}{(-q^2;q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^n(q^2;q^2)_n(q^{2n+2};q^2)_k}{(q;q)_n(-q;q^2)_{n+k+1}} \\
&= \frac{q}{(-q^2;q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^n(q^2;q^2)_{n+k}}{(q;q)_n(-q;q^2)_{n+k+1}}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{q(q^2;q^2)_\infty}{(-q^2;q^2)_\infty(-q;q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^n(-q^{2n+2k+3};q^2)_\infty}{(q;q)_n(q^{2n+2k+2};q^2)_\infty} \\
&= q(q;q)_\infty \sum_{n=0}^{\infty} \frac{q^n}{(q;q)_n} \sum_{m=0}^{\infty} \frac{(-q;q^2)_m q^{m(2n+2k+2)}}{(q^2;q^2)_m} \quad \text{(by (1.5.2))} \\
&= q(q;q)_\infty \sum_{m=0}^{\infty} \frac{(-q;q^2)_m q^{2(k+1)m}}{(q^2;q^2)_m} \sum_{n=0}^{\infty} \frac{q^{(2m+1)n}}{(q;q)_n} \\
&= q(q;q)_\infty \sum_{m=0}^{\infty} \frac{(-q;q^2)_m q^{2(k+1)m}}{(q^2;q^2)_m} \cdot \frac{1}{(q^{2m+1};q)_\infty} \quad \text{(by (1.5.3))} \\
&= \sum_{m=0}^{\infty} \frac{(q;q)_{2m}(-q;q^2)_m q^{2(k+1)m}}{(q^2;q^2)_m} \\
&= \sum_{m=0}^{\infty} (q;q^2)_m(-q;q^2)_m q^{2(k+1)m+1} \\
&= \sum_{n=0}^{\infty} (q^2;q^4)_n q^{2(k+1)n+1},
\end{aligned}
$$

and we are done. □

Proof of Theorem 4.1.4.

$$
\begin{aligned}
&\sum_{n=0}^{\infty} q^n(q^{n+1};q)_{n+2k}(q^{2n+2k+2};q^2)_\infty \\
&\quad = \sum_{n=0}^{\infty} \frac{q^n(q;q)_{2n+2k}(q^{2n+2k+2};q^2)_\infty}{(q;q)_n} \\
&\quad = (q^2;q^2)_\infty \sum_{n=0}^{\infty} \frac{(q;q^2)_{n+k}q^n}{(q;q)_n} \\
&\quad = (q;q)_\infty \sum_{n=0}^{\infty} \frac{q^n}{(q;q)_n(q^{2n+2k+1};q^2)_\infty} \\
&\quad = (q;q)_\infty \sum_{n=0}^{\infty} \frac{q^n}{(q;q)_n} \sum_{m=0}^{\infty} \frac{q^{(2n+2k+1)m}}{(q^2;q^2)_m} \quad \text{(by (1.5.3))} \\
&\quad = (q;q)_\infty \sum_{m=0}^{\infty} \frac{q^{(2k+1)m}}{(q^2;q^2)_m} \sum_{n=0}^{\infty} \frac{q^{n(2m+1)}}{(q;q)_n} \\
&\quad = (q;q)_\infty \sum_{m=0}^{\infty} \frac{q^{(2k+1)m}}{(q^2;q^2)_m(q^{2m+1};q)_\infty} \quad \text{(by (1.5.3))} \\
&\quad = \sum_{m=0}^{\infty} \frac{(q;q)_{2m}q^{(2k+1)m}}{(q^2;q^2)_m}
\end{aligned}
$$

$$= \sum_{n=0}^{\infty} (q;q^2)_n q^{(2k+1)n},$$

as desired. □

4.6 Representations for some special cases

We have seen that (4.1.2) is a special case of Theorem 4.1.4 with $k = 0$. It turns out that there is also a nice representation for $k = 1$.

Theorem 4.6.1.

$$\sum_{n=0}^{\infty} q^n (q^{n+1};q)_{n+2}(q^{2n+4};q^2)_\infty = 1 + (1-q)\sum_{n=0}^{\infty} (-1)^n q^{3n^2+8n+3}(1+q^{2n+3}).$$

Proof. By Theorem 4.1.4 with $k = 1$, we have that

$$\sum_{n=0}^{\infty} q^n (q^{n+1};q)_{n+2}(q^{2n+4};q^2)_\infty = \sum_{n=0}^{\infty} (q;q^2)_n q^{3n}.$$

Set $q \to q^2$ and take $\alpha = q, \tau = q^3$ in Proposition 4.2.7 gives

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(q;q^2)_n q^{3n}}{(\beta;q^2)_n} &= \sum_{n=0}^{\infty} \frac{(q;q^2)_n (q^6/\beta;q^2)_n \beta^n q^{2n^2+n}(1-q^{4n+4})}{(\beta;q^2)_n (q^3;q^2)_{n+1}} \\ &= (1-q)\sum_{n=0}^{\infty} \frac{(q^6/\beta;q^2)_n \beta^n q^{2n^2+n}(1-q^{4n+4})}{(\beta;q^2)_n (1-q^{2n+1})(1-q^{2n+3})}.\end{aligned}$$

After taking $\beta \to 0$, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} (q;q^2)_n q^{3n} &= (1-q)\sum_{n=0}^{\infty} \frac{(-1)^n q^{3n^2+6n}(1-q^{4n+4})}{(1-q^{2n+1})(1-q^{2n+3})} \\ &= (1-q)\sum_{n=0}^{\infty} (-1)^n q^{3n^2+6n}\left(-1 + \frac{1}{1-q^{2n+1}} + \frac{1}{1-q^{2n+3}}\right) \\ &= (1-q)\left[\sum_{n=0}^{\infty} (-1)^{n+1} q^{3n^2+6n} + \sum_{n=0}^{\infty} \frac{(-1)^n q^{3n^2+6n}}{1-q^{2n+1}} + \sum_{n=0}^{\infty} \frac{(-1)^n q^{3n^2+6n}}{1-q^{2n+3}}\right] \\ &= 1 + (1-q)\left[\sum_{n=0}^{\infty} (-1)^{n+1} q^{3n^2+6n} + \sum_{n=1}^{\infty} \frac{(-1)^n q^{3n^2+6n}}{1-q^{2n+1}} + \sum_{n=0}^{\infty} \frac{(-1)^n q^{3n^2+6n}}{1-q^{2n+3}}\right]\end{aligned}$$

$$= 1 + (1-q)\Bigg[\sum_{n=0}^{\infty}(-1)^{n+1}q^{3n^2+6n} + \sum_{n=0}^{\infty} -\frac{(-1)^n q^{3(n+1)^2+6(n+1)}}{1-q^{2(n+1)+1}}$$

$$+ \sum_{n=0}^{\infty}\frac{(-1)^n q^{3n^2+6n}}{1-q^{2n+3}}\Bigg]$$

$$= 1 + (1-q)\left[\sum_{n=0}^{\infty}(-1)^{n+1}q^{3n^2+6n} + \sum_{n=0}^{\infty}\frac{(-1)^n q^{3n^2+6n}(1-q^{6n+9})}{1-q^{2n+3}}\right]$$

$$= 1 + (1-q)\left[\sum_{n=0}^{\infty}(-1)^{n+1}q^{3n^2+6n} + \sum_{n=0}^{\infty}(-1)^n q^{3n^2+6n}(1+q^{2n+3}+q^{4n+6})\right]$$

$$= 1 + (1-q)\sum_{n=0}^{\infty}(-1)^n q^{3n^2+8n+3}(1+q^{2n+3}),$$

as desired. □

It is also possible to get a representation for the case $k = 2$ in the form of double series.

Theorem 4.6.2.

$$\sum_{n=0}^{\infty} q^n (q^{n+1};q)_{n+4}(q^{2n+6};q^2)_\infty = 1+\frac{q^2(1-q^{11})}{1+q} - \frac{1-q^3}{1+q}\sum_{j=0}^{5}\sum_{n=0}^{\infty}(-1)^n q^{3n^2+10n+2+(2n+5)j}.$$

Proof. Again, set $k = 2$ in Theorem 4.1.4 gives

$$\sum_{n=0}^{\infty} q^n (q^{n+1};q)_{n+4}(q^{2n+6};q^2)_\infty = \sum_{n=0}^{\infty}(q;q^2)_n q^{5n}.$$

In the Rogers-Fine identity in Proposition 4.2.7, setting $q \to q^2$ and $\alpha = q, \tau = q^5$ gives

$$\sum_{n=0}^{\infty}\frac{(q;q^2)_n q^{5n}}{(\beta;q^2)_n} = \sum_{n=0}^{\infty}\frac{(q;q^2)_n (q^8/\beta;q^2)_n \beta^n q^{2n^2+3n}(1-q^{4n+6})}{(\beta;q^2)_n (q^5;q^2)_{n+1}}$$

$$= (1-q)(1-q^3)\sum_{n=0}^{\infty}\frac{(q^8/\beta;q^2)_n \beta^n q^{2n^2+3n}(1-q^{4n+6})}{(\beta;q^2)_n (1-q^{2n+1})(1-q^{2n+3})(1-q^{2n+5})}.$$

After taking $\beta \to 0$, we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty}(q;q^2)_n q^{5n} &= (1-q)(1-q^3)\sum_{n=0}^{\infty}\frac{(-1)^n q^{3n^2+10n}(1-q^{4n+6})}{(1-q^{2n+1})(1-q^{2n+3})(1-q^{2n+5})} \\
&= \frac{(1-q)(1-q^3)}{1-q^2}\sum_{n=0}^{\infty}(-1)^n q^{3n^2+10n}\left(\frac{1}{1-q^{2n+1}}-\frac{q^2}{1-q^{2n+5}}\right) \\
&= \frac{(1-q)(1-q^3)}{1-q^2}\left[\sum_{n=0}^{\infty}\frac{(-1)^n q^{3n^2+10n}}{1-q^{2n+1}}-\sum_{n=0}^{\infty}\frac{(-1)^n q^{3n^2+10n+2}}{1-q^{2n+5}}\right] \\
&= \frac{(1-q)(1-q^3)}{1-q^2}\left[\frac{1}{1-q}-\frac{q^{13}}{1-q^3}+\sum_{n=0}^{\infty}\frac{(-1)^n q^{3(n+2)^2+10(n+2)}}{1-q^{2(n+2)+1}}\right. \\
&\quad \left.-\sum_{n=0}^{\infty}\frac{(-1)^n q^{3n^2+10n+2}}{1-q^{2n+5}}\right] \\
&= \frac{(1-q)(1-q^3)}{1-q^2}\left[\frac{1}{1-q}-\frac{q^{13}}{1-q^3}+\sum_{n=0}^{\infty}\frac{(-1)^n q^{3n^2+10n+2}(q^{12n+30}-1)}{1-q^{2n+5}}\right] \\
&= 1+\frac{q^2(1-q^{11})}{1+q}-\frac{1-q^3}{1+q}\sum_{j=0}^{5}\sum_{n=0}^{\infty}(-1)^n q^{3n^2+10n+2+(2n+5)j},
\end{aligned}$$

and the result follows. □

Chapter 5
Combinatorial proof of a truncated Theta Series Theorem

5.1 Introduction

The following theta series identity

$$\frac{(q;q)_\infty}{(-q;q)_\infty} = 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2}, \tag{5.1.1}$$

is usually attributed to Gauss [2, p. 23, eq. (2.2.13)]. Note that the reciprocal of the product on the left hand side represents the generating function for the number of overpartitions of n, i.e. [21],

$$\frac{(-q;q)_\infty}{(q;q)_\infty} = \sum_{n=0}^{\infty} \overline{p}(n) q^n. \tag{5.1.2}$$

Influenced by the work of Andrews and Merca [12], Guo and Zeng [28] proved the truncated version of the identity (5.1.1):

$$\begin{aligned}\frac{(-q;q)_\infty}{(q;q)_\infty}&\left(1 + 2\sum_{n=1}^{k}(-1)^n q^{n^2}\right)\\ &= 1 + (-1)^k \sum_{n=k+1}^{\infty} \frac{(-q;q)_k(-1;q)_{n-k}q^{(k+1)n}}{(q;q)_n} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_1.\end{aligned} \tag{5.1.3}$$

Using (5.1.2), this resulted in the following inequality :

$$(-1)^k \left(\overline{p}(n) + 2 \sum_{j=1}^{k} \overline{p}(n - j^2) \right) \geq 0. \tag{5.1.4}$$

Recently, Andrews and Merca [13] found an alternate form of the identity (5.1.3):

$$\begin{aligned} &\frac{(-q;q)_\infty}{(q;q)_\infty} \left(1 + 2 \sum_{n=1}^{k} (-1)^n q^{n^2} \right) \\ &\quad = 1 + 2(-1)^k \frac{(-q;q)_k}{(q;q)_k} \sum_{j=0}^{\infty} \frac{q^{(k+1)(j+k+1)}(-q^{j+k+2};q)_\infty}{(1-q^{j+k+1})(q^{j+k+2};q)_\infty}. \end{aligned} \tag{5.1.5}$$

With this, they interpreted the right hand side in terms of partitions generating function and obtain the following identity:

Theorem 5.1.1. *For $n, k \geq 1$,*

$$(-1)^k \left(\overline{p}(n) + 2 \sum_{j=1}^{k} \overline{p}(n - j^2) \right) = \overline{M}_k(n),$$

where $\overline{M}_k(n)$ is the number of overpartitions of n such that the first part larger than k must appear at least $k+1$ times.

An analytic proof was provided in [13]. In section 5.2, we will give one of the two combinatorial proofs for Theorem 5.1.1, which is part of the joint work with Cristina Ballantine, Mircea Merca and Ae Ja Yee [16].

5.2 Proof of Theorem 5.1.1

Proof. Since $\overline{p}(n) = \overline{M}_0(n)$, we can rewrite Theorem 5.1.1 into the following equivalent form for $n \geq 1$ and $k \geq 0$:

$$2\overline{p}(n - (k+1)^2) = \overline{M}_{k+1}(n) + \overline{M}_k(n). \tag{5.2.1}$$

We let $\overline{P}(n)$ be the set of overpartitions of n and $\overline{\mathcal{M}}_k(n)$ be the set of overpartitions of n whose first part larger than k must appear at least $k+1$ times. It is worth mentioning that $\overline{\mathcal{M}}_k(n)$ and $\overline{\mathcal{M}}_{k+1}(n)$ are not disjoint (for example, $3+3+3$ belongs to both $\overline{\mathcal{M}}_1(9)$ and $\overline{\mathcal{M}}_2(9)$).

We can interpret the left hand side of (5.2.1) that there are two copies of the set $\overline{P}(n-(k+1)^2)$. First, we will provide a bijection for one of its copies into parts of the set $\overline{\mathcal{M}}_k(n)$, which we will separate into the following two subsets:

- $\overline{\mathcal{S}}_k(n)$: the set of overpartitions from $\overline{\mathcal{M}}_k(n)$, where there are at least $k+1$ nonoverlined parts of $k+1$.
- $\overline{\mathcal{NS}}_k(n)$: the set of overpartitions from $\overline{\mathcal{M}}_k(n)$, where there are less than $k+1$ nonoverlined parts of $k+1$.

It can be seen that $\overline{\mathcal{S}}_k(n)$ and $\overline{\mathcal{NS}}_k(n)$ are disjoint, and that

$$|\overline{\mathcal{M}}_k(n)| = |\overline{\mathcal{S}}_k(n)| + |\overline{\mathcal{NS}}_k(n)|. \tag{5.2.2}$$

Define a 1-1 correspondence between the set $\overline{P}(n-(k+1)^2)$ and $\overline{\mathcal{S}}_k(n)$ as follows: For an overpartition λ in $\overline{P}(n-(k+1)^2)$, insert a blank square of size $(k+1)\times(k+1)$ into the Ferrers diagram of λ. For an overpartition μ in $\overline{\mathcal{S}}_k(n)$, delete $k+1$ nonoverlined parts of $k+1$ from the Ferrers diagram of μ.

$\lambda : \overline{5}+\overline{3}+2+2$ $\qquad$ $\mu : \overline{5}+3+3+3+\overline{3}+2+2$

Figure 5.1: 1-1 correspondence between $\overline{P}(n-3^2)$ and $\overline{\mathcal{S}}_2(n)$.

The bijection implies $|\overline{P}(n-(k+1)^2)| = |\overline{\mathcal{S}}_k(n)|$. By (5.2.1) and (5.2.2), we now only have to show that

$$|\overline{P}(n-(k+1)^2)| = |\overline{\mathcal{NS}}_k(n)| + |\overline{\mathcal{M}}_{k+1}(n)|. \tag{5.2.3}$$

Note that any overpartitions in $\overline{\mathcal{NS}}_k(n)$ will either have no parts of $k+1$, or contain exactly k nonoverlined parts of $k+1$, with one overlined part of $k+1$ (from the requirement of $\overline{\mathcal{M}}_k(n)$, the next part larger than k must appear at least $k+1$ times). Thus we can separate $\overline{\mathcal{NS}}_k(n)$ further into the following two disjoint sets:

- $\overline{\mathcal{A}}_k(n)$: the set of overpartitions of n where no parts equal to $k+1$, and the first part larger than $k+1$ appears at least $k+1$ times.
- $\overline{\mathcal{B}}_k(n)$: the set of overpartitions of n with exactly $k+1$ parts of $k+1$, and the last part of $k+1$ is overlined.

Also, we can separate $\overline{\mathcal{M}}_{k+1}(n)$ further into another two disjoint sets:

- $\overline{\mathcal{C}}_k(n)$: the set of overpartitions of n where no parts equal to $k+1$, and the first part larger than $k+1$ appears at least $k+2$ times.
- $\overline{\mathcal{D}}_k(n)$: the set of overpartitions of n where the parts $k+1$ appear at least once, and the first part larger than $k+1$ appears at least $k+2$ times.

It can be seen that

$$|(\overline{\mathcal{A}}_k(n) \cup \overline{\mathcal{B}}_k(n)) \sqcup (\overline{\mathcal{C}}_k(n) \cup \overline{\mathcal{D}}_k(n))| = |\overline{\mathcal{NS}}_k(n)| + |\overline{\mathcal{M}}_{k+1}(n)|.$$

Here, we need to emphasize that not all of these sets are piecewise disjoint. For example, $5+4+4+4+\overline{4}+2+\overline{2}$ belongs to both $\overline{\mathcal{A}}_2(25)$ and $\overline{\mathcal{C}}_2(25)$, while $4+4+4+4+3+3+\overline{3}$ belongs to both $\overline{\mathcal{B}}_2(25)$ and $\overline{\mathcal{D}}_2(25)$. Thus we have to treat overpartitions μ in $\overline{\mathcal{NS}}_k(n)$ and $\tilde{\mu}$ in $\overline{\mathcal{M}}_{k+1}(n)$ as separated objects.

We will define a bijection between the set $\overline{P}(n-(k+1)^2)$ and $(\overline{\mathcal{A}}_k(n) \cup \overline{\mathcal{B}}_k(n)) \sqcup (\overline{\mathcal{C}}_k(n) \cup \overline{\mathcal{D}}_k(n))$ by Ferrers diagram transformations. For an overpartition λ, we define certain parameters of λ as follows:

- t: the number of the parts $k+1$ in λ,
- s: the first part of λ larger than $k+1$,
- m: the number of the parts s in λ.

First, we will provide transformations from the set $\overline{P}(n-(k+1)^2)$ into $(\overline{\mathcal{A}}_k(n) \cup \overline{\mathcal{B}}_k(n)) \sqcup (\overline{\mathcal{C}}_k(n) \cup \overline{\mathcal{D}}_k(n))$. Let λ be an overpartition in $\overline{P}(n-(k+1)^2)$. Consider the following cases.

Case I: $t = 0$.

We insert a blank square of size $(k+1)\times(k+1)$ with its lower right corner colored black into λ. This yield an overpartition μ which is an element of $\mathcal{B}_k(n)$.

$\lambda : 5+\overline{5}+\overline{2}+1$ $\quad$ $\mu : 5+\overline{5}+3+3+\overline{3}+\overline{2}+1$

Figure 5.2: Case I transformation with $k = 2$.

Case II: $t \geq 1$, *and either* $m = 0$ *or* $m \geq 1$ *with* $t+k+1 < s$.

First, we insert a blank square of size $(k+1)\times(k+1)$ above the rows of length $k+1$ in λ, then conjugate all the rows of length $k+1$ in λ. This yield an overpartition μ which is an element of $\mathcal{A}_k(n)$.

$\lambda : 5+\overline{5}+\overline{3}+1$ $\quad$ $\mu : 5+\overline{5}+4+4+\overline{4}+1$

Figure 5.3: Case II transformation with $k = 2$.

Case III: *$t \geq 1, m \geq 1$ and $t + k + 1 > s$.*

First, we insert a blank square of size $(k+1) \times (k+1)$ above the rows of length $k+1$ in λ, then conjugate the top s nonoverlined rows of length $k+1$, and place them above the rows of length s in λ. This yield an overpartition $\tilde{\mu}$ which is an element of $\mathcal{D}_k(n)$.

$\lambda : 5 + \overline{5} + 3 + 3 + \overline{3} + 1$ $\qquad$ $\tilde{\mu} : 5 + 5 + 5 + 5 + \overline{5} + \overline{3} + 1$

Figure 5.4: Case III transformation with $k = 2$.

Case IV: *$t \geq 1$, $m \geq 1$ and $t + k + 1 = s$.*

We consider the following two possible cases:

(i) *s is not overlined.* First, we insert a blank square of size $(k+1) \times (k+1)$ above the rows of length $k+1$ in λ, then conjugate all the rows of length $k+1$ in λ. This yield an overpartition μ which is an element of $\mathcal{A}_k$.

(ii) *s is overlined.* First, we erase the color black on the rightmost square of s, then insert a blank square of size $(k+1) \times (k+1)$ above the rows of length $k+1$ in λ, then conjugate all the rows of length $k+1$. This yield an overpartition $\tilde{\mu}$ which is an element of $\mathcal{C}_k$.

$\lambda : 5+5+3+\overline{3}+1$ $\qquad$ $\mu : 5+5+5+5+\overline{5}+1$

$\lambda : 5+\overline{5}+3+\overline{3}+1$ $\qquad$ $\tilde{\mu} : 5+5+5+5+\overline{5}+1$

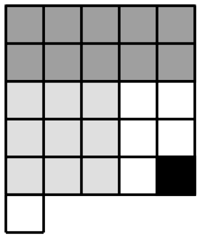
Figure 5.5: Case IV transformations with $k = 2$.

Because the elements from each $\mathcal{A}_k(n) \cup \mathcal{B}_k(n)$ and $\mathcal{C}_k(n) \cup \mathcal{D}_k(n)$ are now viewed as different partitions, the mapping as described above is injective.

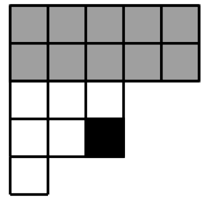
Next, we define inverse transformations from the set $(\overline{\mathcal{A}}_k(n) \cup \overline{\mathcal{B}}_k(n)) \sqcup (\overline{\mathcal{C}}_k(n) \cup \overline{\mathcal{D}}_k(n))$ into $\overline{P}(n-(k+1)^2)$. We first consider an overpartitions μ in $\overline{\mathcal{NS}}_k(n)$. This splits into two subcases.

Case 1A: $\mu \in \mathcal{A}_k(n)$. (There are no rows of length $k+1$ and $m \geq k+1$.)

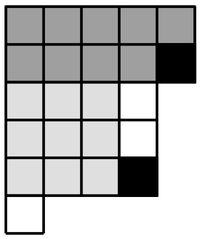
First, we remove a blank square of size $(k+1) \times (k+1)$ from the lower left corner of the rows of length s in μ, then conjugate the remaining parts. The resulted overpartition λ will be either from the Case II (when $m = k+1$) or the Case IV(i) (when $m > k+1$).



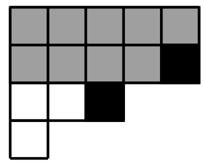
$\mu : 5 + 5 + 5 + 5 + \overline{5} + 1$



$\lambda : 5 + 5 + 3 + \overline{3} + 1$



$\mu : 5 + \overline{5} + 4 + 4 + \overline{4} + 1$

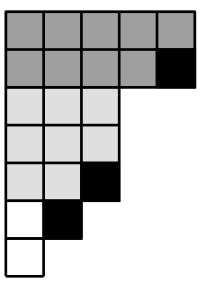


$\lambda : 5 + \overline{5} + \overline{3} + 1$

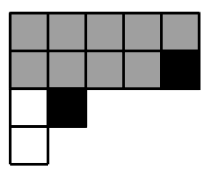
Figure 5.6: Case 1A transformations with $k = 2$.

Case 1B: $\mu \in \mathcal{B}_k(n)$. (There are exactly $k+1$ rows of length $k+1$, which is colored black at the lower right corner.)

We remove all the $k+1$ rows of length $k+1$ from μ. The resulted overpartition λ will be from Case I.



$\mu : \overline{5} + 3 + 3 + \overline{3} + \overline{2} + 1$



$\lambda : \overline{5} + \overline{2} + 1$

Figure 5.7: Case IB transformation with $k = 2$.

Next, we consider an overpartition $\tilde{\mu}$ from $\overline{\mathcal{M}}_{k+1}(n)$. Again, this splits into two subcases.

Case 2A: $\tilde{\mu} \in \mathcal{C}_k(n)$. (There are no rows of length $k+1$ and $m \geq k+2$.)

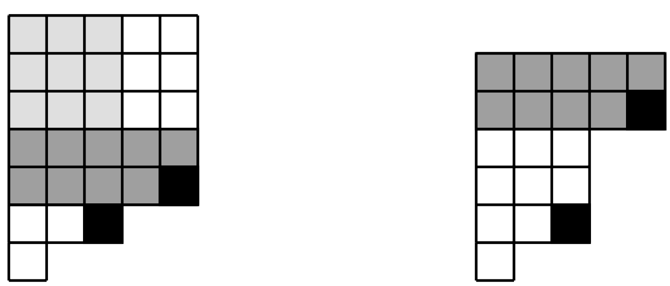
First, we remove a blank square of size $(k+1) \times (k+1)$ from the lower left corner of the rows of length s in μ, then conjugate the remaining parts. Also, fill the lower right corner of the rows of s in μ with black color. The resulted overpartition λ will be from Case IV(ii).

$\tilde{\mu} : 5+5+5+5+\overline{5}+1$ $\qquad$ $\lambda : 5+\overline{5}+3+\overline{3}+1$

Figure 5.8: Case 2A transformations with $k = 2$.

Case 2B: $\tilde{\mu} \in \mathcal{D}_k(n)$. (Rows of length $k+1$ exist and $m \geq k+2$.)

First, we remove a blank square of size $(k+1) \times (k+1)$ from the upper left corner of the rows of length s in μ, then conjugate the remaining parts, and insert them above the rows of length $k+1$. The resulted overpartition λ will be from Case III.



$\tilde{\mu} : 5+5+5+5+\overline{5}+\overline{3}+1$ $\qquad$ $\lambda : 5+\overline{5}+3+3+\overline{3}+1$

Figure 5.9: Case 2B transformation with $k = 2$.

Again, the mapping as described above is injective.

Thus we have established a bijection between the set $\overline{P}(n-(k+1)^2)$ and $(\overline{\mathcal{A}}_k(n)\cup \overline{\mathcal{B}}_k(n)) \sqcup (\overline{\mathcal{C}}_k(n) \cup \overline{\mathcal{D}}_k(n))$, which proves the equation (5.2.3). And the theorem follows. $\square$

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