NUMERICAL CONVERGENCE ANALYSIS OF A
MULTIPLE ORDER ACOUSTIC STREAMING PROBLEM
AND OTHER PHYSICAL AND NUMERICAL CONSIDERATIONS
OF THAT PROBLEM

A Thesis in
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Abstract

Two formulations of the acoustic streaming problem are developed including a Eulerian approach and an arbitrary Lagrangian Eulerian approach. Additionally, a temperature inclusion for the Eulerian case is further outlined. In the Eulerian formulation the commonly studied Surface Acoustic Wave boundary condition is implemented using a numerical approximation and related to a bulk acoustic velocity. In the arbitrary Lagrangian Eulerian formulation the statement of the boundary displacement is direct in relation to the first order solution.

Both formulations are tested using a numerical convergence scheme under the same study parameters in an attempt to compare the accuracy under the same conditions. Additional physical solutions are provided using physically meaningful parameters as defined in previous literature to validate the successful formulation of the acoustic streaming problem. Further conclusions are made based on the results of the component studies.
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Chapter 1
Introduction

1.1 Introduction

While the study of acoustic streaming finds its origins in the 19th century, when it was first denoted by Lord Rayleigh [18], it is only recently that interest in the topic has escalated. Various initiatives are under way to develop microacoustic devices capable of manipulating micro-scale fluids to open new opportunities in a number of medical and industrial applications. Some suggested uses include the improvement of micro-analytical tools such as capillary electrophoresis [10] and in developing means of biological cell capture for medical devices [14]. In fact microacoustofluidics has become an integral part of the Lab on a Chip initiative [5], which hopes to build the framework for the fabrication of devices at an unprecedentedly small scale capably of housing an array of analysis tools. The long term implications of this work could mean a vast collection of new scientific tools, as well as real-world products with the potential to increase the availability and effectiveness of medical testing.

Various works exist that focus on the formal and numerical analysis of the acoustic streaming problem [1, 13, 16]. These analyses have centered around the resulting pressure and velocity fields induced within Surface Acoustic Wave (SAW) devices. Some of these formulation, such as those in [13] and [16] appear to be competing. Also, extended formulations of the problem exist that allow for the inclusion of other variables like temperature [1].

In this study, a framework for the comparison in quality of these formulations is outlined and multiple solutions are derived and compared. The author hopes
this outline can stand as a marker for quality analysis within the growing body
of working surrounding the microacoustic problem. A lack of a robust analysis
system is evident in the common literature surrounding this mathematics, leading
to a lack of common quality standards among studies. Additionally, creating a
quality framework for such studies allows opportunities to identify sources of error
within layered numerical problems. In defining and testing this analysis system,
this study also stands as an outline of multiple growing approaches to modeling
and evaluating studies within microacoustofluidic systems.

1.1.1 The Acoustic Streaming Problem

At the microscopic scale, fluid within the confines of a narrow ($< 1 \times 10^{-5}$ m)
passage or chamber is dominated by viscous effects that make its motion very hard.
In order to incite motion or cause mixing at this scale, induced acoustic waves
are commonly used. Most notably this is achieved via SAW induced upon one or
more surfaces of an acoustic domain via piezoelectric devices. These SAWs can
induce highly laminar flow patterns in the full domain. These flow patterns can
then move and induce stresses upon objects trapped within. Figure 1.1 shows a
common approach to the problem: The three boundary conditions depicted here
are the common conditions applied to a microfluidic domain. Fixed wall boundaries,
denoted by $\Omega_f$ represent a fixed surface that is common of applications like cell
manipulation and channel mixing [5]. This boundary allows for no motion or flux
and can be represent in the domain by a zero flow boundary layer. The Surface
Acoustic Wave Boundary, $\Omega_d$, is the surface experiencing acoustic waves, which
are driven by an external source, in this case an electric transducer. As will be
expressed later in this analysis, there are multiple methodologies for accounting for
this vibration in the context of a numerical model. This boundary is usually only
required on a single surface in a SAW device in order to successfully produce a
desired response within the domain. There are multiple forms that SAW actuation
can take on a single surface, as well as other types of actuation, such as side to side
actuation [1]. An additional boundary condition is possible, which is a free surface
boundary allowing for the motion of domain walls. Often held in place by relatively
high surface tensions, these boundaries are for fluctuating, but zero-flux boundaries
under the effects of piezoelectric actuation.
The acoustic streaming problem is actually a sub-problem of a larger multiple-order acoustic problem. The zeroth order solution to the problem represents the non-induced fluid conditions within the domain. This problem is trivial, with the exception of naturally forming fluid domains. In this case, the fluid domain is actually dependent on a number of factors, such as surface boundary conditions, like hydrophobia and surface tension based on fluid properties. The first-order solution in the acoustic problem depicts the primary motion of the fluid. It is a very fast and high magnitude motion, but follows a repeated flow pattern, not allowing for the long term motion of particles. The second order of the problem is the acoustic streaming sub-problem and the primary interest of this study. This solution represents a non-oscillatory result of the induced flow, allowing for the creation of devices to move particles within a micro structure.
1.1.2 The Eulerian Acoustic Streaming Problem from Köster (2007)

Many studies of the acoustic streaming problem [13] utilize a fully Eulerian approach. This approach begins with a statement of the compressible Navier-Stokes equations and then proceeds to create sub-problem orders by assuming that the solution is the superposition of components of different order, where the latter is defined in terms of a nondimensional smallness parameter that represents a defining quantity for the problem. For example, if the motion is excited by a SAW, the boundary conditions are dictated by the amplitude of displacement of the substrate. In this case, a meaningful smallness parameter can be defined as the ratio of the displacement amplitude and the characteristic size of the device. Once a smallness parameter is defined, the zeroth order solution represents the stationary equilibrium solution about which the oscillatory motion induced by the excitation takes place. The first order solution is then represented by the rapid, quasi-elastic, motion of the fluid under excitation. The second order represents a motion that is observed at a time-scale that is significantly slower than that of the first order motion and that is induced by the nonlinear constitutive response of the fluid. This motion is the same as that which is often referred to as Stokes drift [13]. For a fixed domain, the zeroth order motion is typically one of static equilibrium, and therefore trivial.

Multiple problems arise from the use of a fully Eulerian approach to the acoustic streaming problem. The boundary conditions of the problem must be approximated by developing a relationship between the walls displacement field and the velocity field of the fluid in contact with that wall. Such an approximation allows for the entrance of error into the results. Furthermore, experimental analyses of acoustic streaming utilize particle tracking in order to create results. A comparison with simulated Eulerian results proves difficult and requires a second approximation relating Eulerian results to a single particle Lagrangian output that can be compared to experiments [16].
1.1.3 The Arbitrary Lagrangian-Eulerian Formulation from Nama et al. (2017)

Another approach to the acoustic streaming problem utilizes an ALE perspective. The ALE formulation contains some of the same developments from a purely Eulerian approach. The acoustic streaming problem is still subdivided into a first and second order. Where it is still assumed that the zeroth order problem is not necessary for a fixed domain. However, the ALE formulation dictates that there are three necessary domains of the problem, one a reference domain, another the domain after a given deformation and the final an arbitrary intermediate deformed configuration. All the operators of the first and second order problem are manipulated to be set in the intermediate deformed configuration. This deformed configuration is that of the slow-time solution, which disregards the fast rapid fluctuations of fluid excitation.

An ALE approach circumvents the need to approximate a relationship between fluid velocity and domain wall displacement, as the solution of displacement in the ALE approach is exactly related to the domain wall displacement. Additionally, the results of an ALE study do not need to be processed to relate to physical experiments, the results can already express single particle motion, that would similarly be shown by tracers in an experiment.

1.1.4 The Thermally Coupled Eulerian Formulation from Rune (2012)

Previous literature has introduced a framework for the inclusion of the heat equation in the context of the fully Eulerian acoustic streaming problem. By following this approach it will be possible to extend the reach of the earlier Eulerian analysis in predicting physical problems. Specifically, the inclusion of the coupled heat equation provides a framework for temperature solutions within a microfluidic domain. This inclusion may allow further insight into the quality of solutions of the numerically approximated acoustic streaming problem and identify error implicit in the inclusion of other factors.
1.1.5 Numerical Convergence in Acoustic Streaming Studies

It is common in engineering practice to consider a study by the residual or bulk error remaining upon convergence. However, this practice is not sufficient to understand the quality of the formulation, which can vary even within the context of a single application. For this reason, utilizing the method of manufactured solutions is increasingly common [1], [16]. This approach requires a user defined solution to the problem and a reformulation of the governing equations. The result of the numerical evaluation is then some approximated solution of the user defined solution, assuming the conditions of the study are sufficiently robust to allow for convergence. It is exactly that robustness that is being tested by this approach.

At a single element count, the solution will converge to a defined tolerance, but only by evaluating the study across multiple element accounts and comparing the result to a known solution is it possible to tell if the refinement in mesh is bettering the final result. In this new notion of convergence, the success of a formulation and numerical scheme is defined by the ability of the study to improve in solution accuracy at greater refinement. Failure to meet standards for convergence is then a clear indication that the study is ill formatted to create desired solutions.
Chapter 2  
Governing Equations

2.1 Governing Equations

2.1.1 Derivation of The Eulerian Governing Equations

The problem’s governing equations stem from the compressible Navier-Stokes system, which consists of two equations: one representing the balance of momentum and the other the balance of mass. Referring to Fig. 2.1 and letting $\Omega$ be a regular subset of the standard $d$-dimensional Euclidean point space, we state these equations in Eulerian form below:

$$\rho(\partial_t v + v \cdot \nabla v) - \nabla \cdot T - b = 0 \quad \text{in } \Omega,$$

$$\partial_t \rho + \nabla \cdot (\rho v) - q = 0 \quad \text{in } \Omega,$$

Figure 2.1. Fluid domain and boundaries.
\[ v = \partial_t u \quad \text{on } \Gamma_d. \] (2.3)

Equation (2.1) is the balance of momentum. In it \( \rho \) is the fluid mass density, \( \partial_t \) denotes differentiation with respect to time, \( v \) is the velocity field, \( T \) is the Cauchy stress tensor field, and \( b \) is the body force field. Equation (2.2) is the balance of mass, in addition to the quantities already introduced, \( q \) represents the rate of mass production, which is set to zero when simulating physical phenomena. Finally, Eq. (2.3) represents the \textit{prescribed} SAW excitation \( u \), the latter being the displacement of the boundary. The symbol ‘\( \nabla \cdot \)’ is used here to denote the divergence of the entity that follows. For a compressible linear viscous fluid, the Cauchy stress \( T \) is defined as follows:

\[ T = -p I + \mu [\nabla v + (\nabla v)^T] + (\mu_b - \frac{2}{3} \mu_s)(\nabla \cdot v) I, \] (2.4)

with

\[ p = c^2(\rho - \rho_0), \] (2.5)

where \( \rho_0 \) is the fluid’s mass density in a chosen stress-free reference state, and where \( \mu \) and \( \mu_b \) are the shear and bulk dynamics viscosities, respectively, \( c \) is the speed of sound. The quantities \( \mu_s \), \( \mu \), and \( c \) will be treated as a constants measured in said reference state. Clearly, this is an assumption: deviations from the reference state are assumed to be negligible.

The traditional approach to formulate the acoustic streaming problem is to carry out a matched asymptotic order analysis that splits the problem into sub-problems whose order is determined by way of a smallness parameter. We denote the smallness parameter by \( \epsilon \) and then represent quantities as follows (here demonstrated for the pressure as a specific example):

\[ p = p^{(0)} + \epsilon p^{(1)} + \epsilon^2 p^{(2)} + \cdots, \] (2.6)

where the number in parentheses that appears as a superscript of a quantity refers to the order of the quantity in question. For example we refer to \( p^{(1)} \) as the first order pressure. The decomposition in Eq. (2.6), which is adopted for all fields, allows us to break the original elements into sub-problems consisting of terms of equal order. The smallness parameter \( \epsilon \) is nondimensional and is defined using
some characteristic lengths and/or time scales present in the problem. In the case of microfluidic devices excited via SAWs, a convenient definition of $\epsilon$ is

$$
\epsilon = \frac{\max \|u\|}{\ell},
$$

(2.7)

that is as the ratio of the SAW’s amplitude and a characteristic length $\ell$ defining the size of the device, such as the diameter of the lumen of a micro-channel. In actual applications, $\epsilon$ is typically on the order of $1 \times 10^{-5}$. Going back to the derivation of the governing equations of the streaming problem, once the various fields in the balance equations are represented according to the strategy in Eq. (2.7), the overall problem can be split up into a cascade of problems identified with the power of the parameter $\epsilon$. One important consideration pertains to the boundary conditions.

Two boundaries exist on the domain, $\Gamma_f$ and $\Gamma_d$, such that $\Gamma_f, \Gamma_d \subseteq \Gamma$. $\Gamma_d$ is subject to the displacement $u$, as used in Eq. (2.7). The definition of $\epsilon$ is such that $u$ is specifically first order. Additionally, $u$ is expected to be a first order harmonic composed of sine and cosine components. This displacement, $u$, is stated directly on $\Gamma_d$ and will be shown to relate to a velocity in the Eulerian formulation. Elsewhere on $\Gamma$, velocity is expected to have value zero.

Adopting the definition in Eq. (2.7) for the problems discussed in this thesis, the boundary condition in Eq. (2.3) is such that the zero-th order velocity term is identically equal to zero. Hence, the zero-th order problem is one of mere static equilibrium. Neglecting gravity and in the absence of mass sources (i.e., $b = 0$ and $q = 0$), the solution to the first order problem is simply:

$$
v^{(0)} = 0, \quad p^{(0)} = 0, \quad \text{and} \quad \rho^{(0)} = \rho_0.
$$

(2.8)

Then, the first order problem is:

$$
\rho_0 \partial_t \mathbf{v}^{(1)} - \nabla \cdot \mathbf{T}^{(1)} - \mathbf{b}^{(1)} = 0 \quad \text{in } \Omega, \quad (2.9)
$$

$$
\partial_t \rho^{(1)} + \rho^{(0)} \nabla \cdot \mathbf{v}^{(1)} - q^{(1)} = 0 \quad \text{in } \Omega, \quad (2.10)
$$

$$
\mathbf{v}^{(1)} = \partial_t \mathbf{u} \quad \text{on } \Gamma_d, \quad (2.11)
$$
We note that when the field decomposition in Eq. (2.6) is substituted into the governing equations, any multiplication of first order fields gives rise to second order contributions, as is shown in Eqs. (2.12)–(2.14). These terms are essential to linking the first and second order solutions. This link implies the experimentally observed fact that the bulk acoustic vibration is a driving factor of the long-term streaming flow.

The above equations are fully time dependent. The experimental evidence suggests that the fluid particles respond to a high-frequency excitation by harmonic motion with the same frequency as that of the excitation along with a slower steady flow. From a modeling perspective, the fast harmonic motion is identified with the first order fields. Hence, for the first order problem we seek harmonic solutions. As far as the slow steady motion is concerned, this will be described by the second order fields. Hence, for the second order problem we seek steady, i.e., time independent solutions, where “time” is understood to be resolved at a scale that is coarser than that of the first order problem. Therefore, when it comes to the terms in the second order problem featuring first order fields, said terms will be re-expressed by averaging the first order solution over a period of the harmonic excitation.

To proceed as stated, and assuming that the time averaged result is independent of time as it would be in a steady solution, a time averaging operation is required and it is defined as follows: given a time dependent field \( \phi \), its time average over a period is denoted by \( \langle \phi \rangle \) and is computed as

\[
\langle \phi \rangle = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} \phi \, dt, \tag{2.15}
\]

where \( \omega \) is the frequency of the acoustic excitation.
2.1.2 Statement of the Eulerian Acoustic Problem

As already mentioned, the first order problem is often referred to as the *acoustic subproblem*. For these problem we seek harmonic solutions of the type

\[
\mathbf{v}^{(1)}(x, t) = \mathbf{v}_c^{(1)}(x) \cos(\omega t) + \mathbf{v}_s^{(1)}(x) \sin(\omega t),
\]

and

\[
\rho^{(1)}(x, t) = \rho_c^{(1)}(x) \cos(\omega t) + \rho_s^{(1)}(x) \sin(\omega t),
\]

where the spatial domains of these functions is \( \Omega \) and the subscripts ‘\( c \)’ and ‘\( s \)’ denote the ‘cosine’ and ‘sine’ components of the harmonic response, respectively.

As the stress response is linear in \( \rho \) and \( \mathbf{v} \), the assumed solution form is directly translated into the following form for the stress:

\[
T^{(1)}(x, t) = T_c^{(1)}(x) \cos(\omega t) + T_s^{(1)}(x) \sin(\omega t)
\]

with

\[
T_{c,s}^{(1)} = -c^2 \rho_{c,s}^{(1)} \mathbf{l} + \mu \left[ \nabla \mathbf{v}_{c,s}^{(1)} + \left( \nabla \mathbf{v}_{c,s}^{(1)} \right)^T \right] + \left( \mu_b - \frac{2}{3} \mu_s \right) \left( \nabla \cdot \mathbf{v}_{c,s}^{(1)} \right) \mathbf{l}.
\]

Substituting the assumed form of the solution into Eqs. (2.9)–(2.11) we obtain the set of equations that are actually solved when we talk about the first order problem (acoustic) problem:

\[
\rho_0 \omega \mathbf{v}_s^{(1)} - \nabla \cdot T_c^{(1)} - b_c^{(1)} = 0 \quad \text{in } \Omega,
\]

\[
-\rho_0 \omega \mathbf{v}_c^{(1)} - \nabla \cdot T_s^{(1)} - b_s^{(1)} = 0 \quad \text{in } \Omega,
\]

\[
\omega \rho_s^{(1)} + \rho_0 \nabla \cdot \mathbf{v}_c^{(1)} - q_c^{(1)} = 0 \quad \text{in } \Omega,
\]

\[
\omega \rho_c^{(1)} + \rho_0 \nabla \cdot \mathbf{v}_s^{(1)} - q_s^{(1)} = 0 \quad \text{in } \Omega,
\]

\[
\mathbf{v}_c^{(1)} = \omega \mathbf{u}_s \quad \text{on } \Gamma_d,
\]

\[
\mathbf{v}_s^{(1)} = -\omega \mathbf{u}_c \quad \text{on } \Gamma_d.
\]

It is important to note that the boundary value problem (BVP) in Eqs. (2.20)–(2.25) is *time independent* as the time dependence of the full first order fields (as opposed to their cosine and sine components) has been assumed to be harmonic.

From a mathematical perspective, so long as the excitation frequency \( \omega \) is not a resonant frequency of the system, the above BVP has features that are very similar
to a linear elastostatics problem.

### 2.1.3 Statement of the Full Eulerian Streaming Problem

As already mentioned, the streaming phenomenon is registered at a temporal scale that is slower than that of the acoustic excitation. Therefore, Eqs. (2.12)–(2.14) are not solved as such. Rather, they are time averaged over a period of excitation. Furthermore, supported by experiments, is it assumed that the average motion is steady. With this in mind, and with a slight abuse of notation, we will refer to $v^{(2)}$, $\rho^{(2)}$, and $p^{(2)}$ as the time averaged second order fields, i.e.,

\[
\langle v^{(2)} \rangle \rightarrow v^{(2)}, \quad \langle \rho^{(2)} \rangle \rightarrow \rho^{(2)}, \quad \text{and} \quad \langle p^{(2)} \rangle \rightarrow p^{(2)}.
\] (2.26)

With the above convention the equations that actually define the second order problem are as follows:

\[
-\nabla \cdot T^{(2)} - b^{(2)} = \langle -\rho^{(1)} \frac{\partial v^{(1)}}{\partial t} - \rho_0 (\nabla v^{(1)})^T v^{(1)} \rangle \quad \text{in} \quad \Omega,
\] (2.27)

\[
\rho_0 \nabla \cdot v^{(2)} - q^{(2)} = \langle -\nabla \cdot (\rho^{(1)} v^{(1)}) \rangle \quad \text{in} \quad \Omega,
\] (2.28)

\[
v^{(2)} = -\langle u \cdot \nabla v^{(1)} \rangle \quad \text{on} \quad \Gamma_d.
\] (2.29)

Before making explicit the form of $T^{(2)}$ and of the terms in angle brackets, it is important to note that the BVP in Eqs. (2.27)–(2.29) has a feature that makes it fundamentally different from the first order problem. Specifically, we observe that Eq. (2.28) does not contain the unknown $\rho^{(2)}$. Formally, this implies that this equation must be viewed as a kinematic constraint equation on the field $v^{(2)}$ that is exactly analogous to the incompressibility constraint of the classical Stokes flow problem (cf. [11]). The immediate consequence of this problem is that the relationship between stress and density can no longer be described by a constitutive relation. That is, the stress tensor $T^{(2)}$ must have the form

\[
T^{(2)} = -p^{(2)} I + \mu [\nabla v^{(2)} + (\nabla v^{(1)})^T],
\] (2.30)

where, as in the Stokes problem, $p^{(2)}$ must be interpreted as a multiplier responsible for the enforcement of the kinematic constraint in Eq. (2.28) rather than the physical pressure. A consequence of this fact is that, from a rigorous viewpoint, the
second order density field results are undetermined in the above formulation. The fact that \( p^{(2)} \) is governed by a kinematic constraint has important ramifications for the numerical solution of the problem in that the function spaces selected for the solution of the problem cannot be chosen in arbitrary ways [11].

An additional consequence of the form of Eq. (2.28) is not only that \( p^{(2)} \) is dictated by kinematic (as opposed to physical) conditions, but that the solution for \( p^{(2)} \) when the boundary conditions are of pure velocity type (as they are for the problem of interest) is not unique. Again, this is a well-known fact in the solution of the Stokes and Navier-Stokes problems [4, 11]. In order to obtain a problem that admits a unique solution, the field \( p^{(2)} \) must be constrained by a single scalar equation. Typically, one requires that the average of \( p^{(2)} \) over the solution’s domain be equal to zero. However, this condition can make the numerical solution of the problem difficult and therefore a different condition has been used herein as discussed in the result section of this thesis.

Going back to Eqs. (2.27)–(2.29), because of the harmonic form for the first order fields, the terms in angle brackets can be given explicit forms. In fact, given any two scalar-, vector-, or tensor-valued fields of the form \( \phi(x, t) = \phi_c(x) \cos(\omega t) + \phi_s(x) \cos(\omega t) \) and \( \psi(x, t) = \psi_c(x) \cos(\omega t) + \psi_s(x) \cos(\omega t) \), and given a meaningful product, denoted here by \( \diamond \), we have

\[
\langle \phi \diamond \psi \rangle = \frac{1}{2} (\phi_c \diamond \psi_c + \phi_s \diamond \psi_s).
\] (2.31)

It is well known that in an Eulerian formulation the stream lines of the field \( v^{(2)} \) do not represent particle trajectories in the streaming motion [6,15,16]. However, it is often the case, especially in comparing theoretical results to experimental measurements that one needs to determine the velocity field that, when steady, has streamlines representing particle trajectories. This field is referred to as the Lagrangian mean velocity [6] and is typically denoted by \( v^L \). In order to determine this field from \( v^{(2)} \) one must employ a correction referred to as Stokes drift. Following [6], and referring the to assumed form of the first order velocity field, we consider a field \( \zeta \) such that \( \partial_t \zeta = v^{(1)} \):

\[
\zeta(x, t) = \frac{1}{\omega} \left[ v_c^{(1)}(x) \sin(\omega t) - v_s^{(1)}(x) \cos(\omega t) \right].
\] (2.32)

As discussed in [6], the field \( \zeta \) represents a displacement field that, to the first
order, can be viewed as providing the actual location of a particle with nominal position \( \mathbf{x} \) as \( \mathbf{x} + \mathbf{\zeta} \). Then, it can be shown that the Lagrangian mean velocity of a particle with nominal position \( \mathbf{x} \) is

\[
\mathbf{v}^L = \mathbf{v}^{(2)} + \langle \mathbf{\zeta} \cdot \nabla \mathbf{v}^{(1)} \rangle.
\]

(2.33)

As \( \mathbf{v}^{(1)} \) is computed in the solution of the first order problem, and \( \mathbf{\zeta} \) is a straightforward consequence of such solution, we see that in the Eulerian formulation of the problem the Lagrangian velocity can only be obtained by post processing the solution of the second order problem along with that of the first order problem. The formulation presented in the following section is one in which \( \mathbf{v}^L \) is computed directly.

### 2.1.4 Governing Equations in an Arbitrary Lagrangian–Eulerian Framework

In an ALE formulation the domain of the functions appearing in the governing equations is chosen so as to serve a convenient purpose [8]. The change of domain does not alter the physics of the problem as it is a mere change of variables granted, of course, that the change of variables in question is smooth with smooth inverse.

The rationale behind the adoption of an ALE scheme for streaming problems is the form of the boundary conditions discussed in the previous section. Specifically, we recall that the domain occupied by the fluid is time dependent because the portion of the walls of the domain that are in contact with the harmonically excited substrate move as SAWs propagate. The boundary conditions discussed in the previous section are approximate because they pertain to a computational domain that is (i) time independent and (ii) prescribed by way of a truncated Taylor series expansion. By adopting an ALE formulation, the governing equations can be written over a truly time independent domain which is that occupied by the fluid before excitation. By doing so, the statement of the problem’s boundary conditions can be made exact. The change of domain can be prescribed so as to have a convenient physical meaning. The ALE scheme followed in this thesis is that demonstrated in [16] and is summarized in this section. Although unusual in fluid mechanics, the map between the physical time dependent domain and the time independent computational domain is chosen to be the displacement of
the fluid particles from the reference configuration $\Omega$ which is now understood to be the domain occupied by the fluid at rest before the onset of the harmonic excitation. Once this choice is made, the formulation still makes use of a matched asymptotics methodology except for the fact that it makes more the mathematical representation of the time scale separation that must exist in order to derive meaningful time-averaged equations for the streaming motion.

Following [16, 20], two time scales are defined: a fast time scale within which time is denoted by $t$, and a slow time scale within which time is denoted by $T$. For the specific problem at hand, it has been shown is [16, 20] that the appropriate scaling between the slow and fast time scale is as follows:

$$T = \epsilon^2 t,$$  \hspace{1cm} (2.34)

where $\epsilon$ is the same smallness parameter introduced earlier. As is typically seen in homogenization theory (cf. [2, 12, 20]), the explicit introduction of scales (whether spatial or temporal) must be reflected in how derivatives are computed across the scales. This is because it is understood that a function $\phi(t)$ will be viewed as being a function of the type $\hat{\phi}(t, T)$, i.e., depending on time in different ways based on the time scale of observation. With this in mind, in the case of Eq. (2.34), differentiation with respect to time will be carried out according to the following operation:

$$\partial_t \to \partial_T + \epsilon^2 \partial_t$$  \hspace{1cm} (2.35)

Next, let $\Omega_0$ denote the time-independent domain occupied by the fluid before excitation. Also, let $\Omega_t$ denote the time dependent domain occupied by the fluid at time $t$. Let $X$ denote a material particle in $\Omega_0$. The motion causes a particle $X$ to occupy a position $y$ in $\Omega_t$. At the slow time scale, the particle appears to occupy a position $x$ which is the time-average of $y$. The acoustic subproblem is concerned with what happens at the fast time scale, whereas the streaming problem is interested in determining the time averaged motion represented by $x$. To this end, following [16], we introduce a domain $\Omega_T$, which is the image of $\Omega_0$ under the time-averaged motion. Correspondingly then, the overall motion is decomposed in the following fashion:

$$y = x(X, T) + \xi(x, t) \text{ with } \quad x = X + u^{(o)}(X, T),$$  \hspace{1cm} (2.36)
where $u^{(0)}(X, T)$ is the displacement field of the time averaged motion and $\xi(x, t)$ is the displacement of a particle from its time averaged position to its actual position on the fast time scale. With this in mind, the domain $\Omega_T$ is the image of $\Omega_0$ under the motion that has $u^{(0)}(X, T)$ as its displacement. $\Omega_T$ is therefore time dependent. However, on the slow time scale the walls of the fluid domain do not move. Therefore, $u^{(0)}(X, T) = 0$ everywhere on $\Gamma_0$, the boundary of $\Omega_0$. This in turn implies that the boundary of $\Omega_T$ coincides with the boundary of $\Omega_0$. Furthermore, displacement excitation on the fast scale caused by SAWs must coincide with $\xi(x, t)$.

The main assumption concerning $\xi$ that is done in [16] is that this is a harmonic function with period equal to that of the excitation. Based on this assumption, Nama et al. proceeds to use a matched asymptotics approach and time averaging to deduce the governing equations for the field $\xi$ and the velocity field of the motion on the slow time scale. Here, we report the necessary kinematics to derive these equations.

We denote by $F$ the deformation gradient of the overall motion. Hence, from Eq. (2.36), we have

$$F = F^{(0)}F_\xi, \quad F^{(0)} = I + \nabla X u^{(0)}, \quad F_\xi = I + \nabla x \xi,$$

where the notation ‘$\nabla_z$’ represents the gradient operator relative to the position variable ‘$z$’. Equation (2.37) is simply a statement of the chain rule of calculus (cf. [9]). For convenience we denote by $J_\xi$ the determinant of $F_\xi$. Using a well-known representation formula for the determinant (based on the Cayley-Hamilton theorem, cf. [9]), we can then write

$$J_\xi = \det(F_\xi) = 1 + \mathcal{I}_1(\nabla_x \xi) + \mathcal{I}_2(\nabla_x \xi) + \mathcal{I}_3(\nabla_x \xi),$$

where, given any second order tensor $A$, $\mathcal{I}_j(A) (j = 1, \ldots, 3)$ are the principal invariants of $A$, i.e.,

$$\mathcal{I}_1(A) = \text{tr} A, \quad \mathcal{I}_2(A) = \frac{1}{2} [(\text{tr} A)^2 - \text{tr}(A^2)], \quad \mathcal{I}_3(A) = \det A.$$

Using Eqs. (2.35), (2.36), and (2.37), one can show that the material velocity field
is given by
\[ \mathbf{v} = \partial_t \xi + \epsilon^2 \mathbf{F}_\xi \partial_T \mathbf{u}^{(0)}, \quad (2.40) \]
where we view \( \mathbf{v} \) as a field with \( \Omega_t \) as its spatial domain. As \( \xi \) is the response to the harmonic excitation and is therefore of order \( \epsilon \), the result in Eq. (2.40) allows for an easy identification of the acoustic and streaming components of the velocity. For convenience, we set
\[ \mathbf{v}_\xi = \partial_t \xi, \quad (2.41) \]
and we view this as a field with \( \Omega_T \) as its spatial domain. Also, given a quantity \( \phi(y, t) \), we will define \( \phi^*(x, t) \) by its remapping onto the \( \Omega_T \) domain, i.e.,
\[ \phi^*(x, t) = \phi(y, t)|_{y = x + \xi(x, t)}. \quad (2.42) \]

With the above definitions in place, the balance of momentum and mass rewritten over the domain \( \Omega_T \) taken on the following form:
\[ J_\xi \rho^* \left[ \partial_t \mathbf{v}^* + \nabla x \mathbf{v}^* \mathbf{F}_{\xi}^{-1} (\mathbf{v}^* - \mathbf{v}_\xi) \right] - \nabla \cdot J_\xi \mathbf{T}^* \mathbf{F}_{\xi}^{-T} - J_\xi \rho^* \mathbf{b}^* = \mathbf{0} \quad \text{in } \Omega_T, \quad (2.43) \]
\[ \frac{\partial \rho^*}{\partial t} + \mathbf{F}_{\xi}^{-T} \rho \nabla \cdot (\mathbf{v}^* - \mathbf{v}_\xi) + \rho^* \mathbf{F}_{\xi}^{-T} : \nabla \mathbf{v}^* - q^* = \mathbf{0} \quad \text{in } \Omega_T. \quad (2.44) \]

Before proceeding further, to provide compact forms of the balance of momentum for the acoustic and streaming sub-problems generated by the matched asymptotic expansion, it is convenient to introduce the Second Piola Kirchoff stress tensor induced by the ALE map relating the domains \( \Omega_t \) and \( \Omega_T \):
\[ \mathbf{P}^* = J_\xi \mathbf{T}^* \mathbf{F}_{\xi}^{-T}. \quad (2.45) \]

### 2.1.5 Statement of the Arbitrary Lagrangian-Eulerian Governing Equations

As already mentioned, the ALE formulation presented in [16] is predicted on choosing \( \Omega_T \) as the domain of the various problems that arise when employing the order decomposition shown in Eq. (2.6). This said, one of the immediate results is that
\[ \rho^*^{(0)} = \rho_0 \quad \text{in } \Omega_T. \quad (2.46) \]
With this result in hand, applying Eq. (2.38), one can show that [16]:

\[ \rho^{* (1)} + \rho_0 \nabla_x \cdot \xi = 0 \quad \text{in } \Omega_T. \] (2.47)

This result implies that that \(\rho^{* (1)}\) can be directly computed once the harmonic displacement \(\xi\) is known. As it turns out, \(\xi\) is the only unknown that appears in the first order balance of momentum equation. Again, following Eq. (2.38) and Eq. (2.6), the balance of momentum yields the following in the first order equation:

\[ \rho_0 \partial_t \xi - \nabla_x \cdot P^{* (1)} = 0 \quad \text{in } \Omega_T, \] (2.48)

\[ \xi = u \quad \text{on } \partial \Omega_T, \] (2.49)

where \(\partial \Omega_T\) is the boundary of \(\Omega_T\), \(u\) is still the displacement induced by the SAW excitation, and where the Piola-Kirchoff stress tensor \(P^{* (1)}\) is found to be:

\[ P^{* (1)} = \left( c_0^2 \rho_0 + \mu_0 \partial_t \right) (\nabla_x \cdot \xi) I + \mu \partial_t [\nabla_x \xi + (\nabla_x \xi)^T]. \] (2.50)

The first order acoustic problem has now been reduced to a single equation with only one numerically meaningful solution, \(\xi\).

Going back to the expressions of the balance of mass and momentum, the terms involving second order quantities are

\[ \partial_t \rho^{* (2)} + \rho_0 \nabla_x \cdot v^{* (2)} + \rho^{* (1)} \nabla_x \cdot v^{* (1)} - \rho_0 (\nabla_x \xi)^T : \nabla_x v^{* (1)} = 0 \quad \text{in } \Omega_t \] (2.51)

and

\[ \rho_0 \partial_t v^{* (2)} + \rho^{* (1)} \partial_t v^{* (1)} + \rho_0 (\nabla_x \cdot \xi) \partial_t v^{* (1)} - \nabla_x P^{* (2)} = 0 \quad \text{in } \Omega_t. \] (2.52)

We now recall that differentiation with respect time is carried out as shown in Eq. (2.35). This fact, along with a careful order analysis (cf. [16]), yields the following remarkable simplification:

\[ \nabla_x \cdot v^{* (2)} = 0 \quad \text{and} \quad \nabla_x \cdot P^{* (2)} = 0 \quad \text{in } \Omega_T. \] (2.53)

Similarly to what has been

As we had seen in the case of Eq. (2.28), again we have that the first of Eqs. (2.53) has the form of a kinematic constraint. In turn, this implies that the
second order stress is not entirely governed by a constitutive response function but must include a hydrostatic component that has a multiplier determined via the enforcement of the kinematic constraint in question. Therefore, the second order Piola-Kirchhoff stress tensor is given by

\[
P^{\star(2)} = -q \mathbf{l} - \frac{1}{2} \epsilon_0^2 \rho_0 [((\nabla_x \cdot \xi)^2 + (\nabla_x \xi)^T \cdot \nabla_x \xi) + \mu [\nabla_x \mathbf{v}^{(2)} + (\nabla_x \mathbf{v}^{(2)})^T]}
- \mu [\nabla_x \mathbf{v}^{(1)} \cdot \nabla_x \xi + (\nabla_x \xi)^T (\nabla_x \mathbf{v}^{(1)})^T] - \mu_b [\nabla_x \mathbf{v}^{(1)} \cdot \nabla_x \mathbf{v}^{(1)}]
+ \left\{ \epsilon_0^2 \rho_0 (\nabla_x \cdot \xi) I + \mu (\nabla_x \mathbf{v}^{(1)} + (\nabla_x \mathbf{v}^{(1)})^T]
+ \mu_b (\nabla_x \cdot \mathbf{v}^{(1)}) I \right\} [(\nabla_x \cdot \xi) I - (\nabla_x \xi)^T]
\]

(2.54)

\(P^{\star(2)}\) now satisfactorily ties the first order solution to the second.

As was done in the Eulerian approach, time averaging is now enforced on the second order equations. To avoid cumbersome repetitions, and with a slight abuse of notation, we will refer to \(P^{\star(2)}\) as its time average, i.e.,

\[
\langle P^{\star(2)} \rangle \rightarrow P^{\star(2)}.
\]

(2.55)

As far as the time average of \(\mathbf{v}^{\star(2)}\), Nama et al. [16] showed that it coincides with the material velocity of the zero-order motion. That is, \(\langle \mathbf{v}^{\star(2)} \rangle\) described the velocity of a particle that moves following the average position of particles. In fluid mechanics, this notion of velocity average is known as the Lagrangian mean velocity [6] and, as already done in eq. (2.33), we denote it by \(\mathbf{v}^L\):

\[
\mathbf{v}^L = \langle \mathbf{v}^{\star(2)} \rangle.
\]

(2.56)

With the adoption of the above conventions, the acoustic streaming problem formulated within the specified ALE framework is therefore

\[
\nabla_x \cdot P^{\star(2)} = 0 \quad \text{in } \Omega_T,
\]

(2.57)

\[
\nabla_x \cdot \mathbf{v}_L = 0 \quad \text{in } \Omega_T,
\]

(2.58)

\[
\mathbf{v}_L = 0 \quad \text{on } \partial \Omega_T,
\]

(2.59)

where the boundary condition in Eq. (2.59) can be considered to be exact.

By contrast with the Eulerian formulation, we see that the current formulation
delivers the Lagrangian mean velocity as the solution of the second order problem.

2.1.6 The Eulerian Coupled Heat Transfer Equation

In the context of the fully Eulerian problem, it is possible to add the heat equation as another coupled conservation equation, following a scheme implemented in [1]. In this approach the density of the first order acoustic problem is described in terms of both pressure and temperature. This leads to the inclusion of the temperature variable in not only the heat equation, but in the conservation of mass. Denoting the temperature by $T$, and using the symbol $\Sigma$ for the Cauchy stress, the full depiction of the Eulerian first order problem then becomes [1]:

\begin{align}
\rho_0 \partial_t \mathbf{v}^{(1)} - \nabla \cdot \Sigma^{(1)} - \mathbf{b}^{(1)} &= 0 \quad \text{in } \Omega, \\
\partial_t \rho^{(1)} + \rho_0 \nabla \cdot \mathbf{v}^{(1)} - \alpha T_0 \frac{\partial T^{(1)}}{\partial t} - q^{(1)} &= 0 \quad \text{in } \Omega, \\
\partial_t T^{(1)} - \frac{k_{th}}{\rho_0 C_p} \Delta T^{(1)} + \frac{\alpha T_0}{\rho_0 C_p} \partial_t p^{(1)} - h^{(1)} &= 0 \quad \text{in } \Omega,
\end{align}

where we have introduced five new parameters, $\gamma$, a parameter denoting the thermodynamic identity, $\alpha$, the thermal expansion coefficient, $k$, the isentropic compressibility, $k_{th}$, the thermal conductivity, $C_p$, the specific heat capacity for a given constant pressure in the working fluid, as well as a new source term $h$, the heat equation source term. We note that $T_0$ is the reference temperature corresponding to the state of fluid at which $\rho_0$ and $c_0$ are measured. In this new context of the fully Eulerian problem we will continue to assume the approximated boundary conditions as previously stated:

\begin{equation}
\mathbf{v}^{(1)} = \partial_t \mathbf{u} \quad \text{on } \Gamma_d,
\end{equation}

In the case of an assumed isentropic condition, the equations reduce to the following form [1]:

\begin{align}
\rho_0 \partial_t \mathbf{v}^{(1)} - \nabla \cdot \Sigma^{(1)} - \mathbf{b}^{(1)} &= 0 \quad \text{in } \Omega, \\
\partial_t \rho^{(1)} + \rho_0 \nabla \cdot \mathbf{v}^{(1)} - q^{(1)} &= 0 \quad \text{in } \Omega, \\
\partial_t T^{(1)} + \frac{\alpha T_0}{\rho_0 C_p} \partial_t p^{(1)} - h^{(1)} &= 0 \quad \text{in } \Omega.
\end{align}
Chapter 3

Numerical Implementation

3.1 FE Formulation and The Method of Manufactured Solutions

The weak form of each of the problems was obtained using a standard Galerkin formulation as can be found in many textbooks [11] and will not be repeated here.

COMSOL Multiphysics® [7] was used to compute solutions. Here COMSOL Multiphysics® was used not as a FE package but as an integrated programming environment and FE library. Specifically, all formulations were implemented through the ‘Weak Form PDE’ interface. The derivation of the weak forms of the boundary value problems (BVPs) considered and their translation in the appropriate syntax for COMSOL Multiphysics® was done in Wolfram Mathematica [19].

In order to allow for a direct comparison between different formulations we use the method of manufactured solutions (MMS) [17]. This method has become a de facto “industry standard” in numerical computing and allows us to easily compute the error of the numerical solution relative to that of an exact solution of arbitrary complexity.

For a given BVP, the MMS consists in arbitrarily choosing a function (or functions) that serve as the exact solution of the BVP at hand. This function, referred to as the manufactured solution, is substituted in the strong form of the governing equations and of the boundary conditions to determine the source terms and boundary data that one would have to provide to a numerical code to determine the numerical solution. The key element of the method is that the computer code to test is not given the chosen exact solution, but only the data obtained as stated.
If the code is free of errors, it will deliver a numerical solution that can then be compared with the selected manufactured solution. Therefore, the MMS can be used to assess (i) the correctness of a code, and (ii) the order of convergence of the numerical method implemented in the code.

3.2 Numerical Implementation of the ALE and Eulerian Acoustic Problems

3.2.1 Preliminary Considerations

In the preceding Chapter we presented different formulation and it is not entirely obvious how to organize a proper comparison of their solutions. For the first order problem, it turns out that the fields $\zeta$ (cf. (2.32)) and the field $\xi$ can be considered to be the same field [6, 16]. Therefore, selecting a manufactured solution for $\xi$ in the ALE formulation delivers a corresponding manufactured solution for $v^{(1)}$ in the Eulerian formulation. In turn, the latter can be used to determine the corresponding manufactured solution for the Eulerian first order density function using the (Eulerian) mass balance equation for the first order problem. In practice, the field that is most desirable to obtain is the Lagrangian mean velocity field. Hence, we chose a manufactured solution for $v_L$ and then, we determined the corresponding manufactured solution for the Eulerian field $v^{(2)}$ using Eq. (2.33). As far as the second order pressure field for the Eulerian and the ALE formulations are concerned, these fields are multiplier fields that must enforce a kinematic constraint. Therefore, there is no criterion that can be used to select a physically meaningful comparison. Hence, out of convenience, we selected the same function for the two fields.

3.2.2 The Manufactured Pressure and Velocity Solutions

To analyze convergence the studies of the Fully Eulerian and ALE formulations of the problem will be tested against a pre-determined solution of the first and second order field variables. The fields were selected based on the benefit of being able to compare them to previous literature [13]. Keeping in mind the discussion in the
previous section, the manufactured solutions used in this thesis are

\[
\xi_x = x_0 \left[ \cos(\omega t) \sin \left( 2\pi \frac{x + y}{\ell} \right) + \sin(\omega t) \sin \left( 2\pi \frac{x - y}{\ell} \right) \right], \quad (3.1)
\]

\[
\xi_y = x_0 \left[ \cos(\omega t) \cos \left( 2\pi \frac{x + y}{\ell} \right) + \sin(\omega t) \cos \left( 2\pi \frac{x - y}{\ell} \right) \right], \quad (3.2)
\]

\[
\xi_z = x_0 \left[ \cos(\omega t) \sin \left( 2\pi \frac{x + y}{\ell} \right) + \sin(\omega t) \sin \left( 2\pi \frac{x + y}{\ell} \right) \right], \quad (3.3)
\]

\[
v^{(1)} = \partial_t \xi, \quad (3.4)
\]

\[
\rho^{(1)} = \frac{\rho_0}{\omega} \nabla \cdot v^{(1)}, \quad (3.5)
\]

\[
v^L_x = v_0 \cos \left( 2\pi \frac{x + y}{\ell} \right), \quad (3.6)
\]

\[
v^L_y = v_0 \sin \left( 2\pi \frac{x + y}{\ell} \right), \quad (3.7)
\]

\[
v^L_z = v_0 \sin \left( 2\pi \frac{x + y}{\ell} \right), \quad (3.8)
\]

\[
v^{(2)} = v^L - \langle \xi \cdot \nabla v^{(1)} \rangle, \quad (3.9)
\]

\[
p^{(2)} = p_0 \sin \left( 2\pi \frac{x + y}{\ell} \right). \quad (3.10)
\]

In order to analyze the impact of the coupled heat equation, it is necessary to introduce a solution for first order temperature. The manufactured solution for \( T^{(1)} \) is stated below:

\[
T^{(1)} = T_0 \sin \left( \frac{xy}{\ell^2} \right), \quad (3.11)
\]

where \( T_0 \) is a parameter lending the relevant units to the temperature solution, having a value of one as with previous solutions.

For the calculation of a physical acoustic streaming problem, it is realistic to introduce an associated wall temperature boundary condition or some other heat function, owing to the non thermal isolation of the system. This analysis includes a fixed wall temperature as used in previous literature [1].

\[
T = T_0 \quad \text{on } \Gamma_d. \quad (3.12)
\]

This condition is not required in order to evaluate the solution using the method of manufactured solutions.
3.2.3 Actuation in the Physical Solution

The physical solution of the Eulerian problem will follow relevant actuation as presented in previous literature [1, 16]. That actuation has two forms, the first being a side actuation on two wall surfaces as stated below:

\[ U_x = d_0 \quad \text{on} \quad \Gamma_{d1}, \quad (3.13) \]
\[ U_x = -d_0 \quad \text{on} \quad \Gamma_{d2}, \quad (3.14) \]

The second actuation is a traditional SAW interface and is actuated on one surface as follows:

\[ U_x = 0 \quad \text{on} \quad \Gamma_d, \quad (3.15) \]
\[ U_y = 0 \quad \text{on} \quad \Gamma_d, \quad (3.16) \]
\[ U_z = d_0 \sin\left(\frac{2\pi x}{L_1}\right) \quad \text{on} \quad \Gamma_d, \quad (3.17) \]

where \( L_1 \) is a parameter that references the actuation length of the domain, \( L_1 = 250 \mu m \). This actuation maintains the acoustic form expected by the problem and uses a scale on the same magnitude as a piezoelectric SAW boundary. The form of this actuation is also adapted from [1].

3.2.4 Mesh Refinements

Multiple mesh refinements were created in the process of analyzing the convergence quality of the acoustic streaming problems. The goal of these mesh refinements was to select a wide enough range of points to sufficiently summarize the reduction in error with mesh refinement, while still achieving a refinement sufficient enough to reflect a realistic mesh for engineering purposes. The meshes selected are depicted in Fig. 3.1. As can be seen in Fig. 3.1 the mesh refinement follows the following rule: \( \ell/(n + 5) \), with \( n = 1, \ldots, 5 \).

3.2.5 Relevant Parameters

The following are the parameters used in all solution sets. In the case of the manufactured solution, these parameters are trivial. However, they are essential
Figure 3.1. Mesh refinements used in the convergence study. The solution’s domain is a cube of unit length: $\ell = 1$ m. From left to right, there are 5 uniform meshes consisting of cubes with side equal to $h = \ell/5$, $h = \ell/10$, $h = \ell/15$, $h = \ell/20$, and $h = \ell/25$, respectively.

to the physically meaningful solution presented later in this analysis. Table 3.1 reports the values used for the convergence study, whereas Table 3.2 reports the

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Description</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
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<td>J/(kg·K)</td>
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<td>m</td>
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<tr>
<td>$h$</td>
<td>mesh diameter</td>
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<td>m</td>
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<td>Pa</td>
</tr>
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<td>$pr_0$</td>
<td>manufactured pressure amplitude</td>
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<td>Pa</td>
</tr>
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<tr>
<td>$v_0$</td>
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<td>manufactured displacement amplitude</td>
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<td>$\mu_s$</td>
<td>shear viscosity</td>
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<td>Pa·s</td>
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</table>

values used for the physical study.

### 3.2.6 Element Discretization

The FE elements used in the all solutions where standard Lagrange polynomials. The ascribed element order for each variable is given below:
Table 3.2. Physical Problem Parameters.

<table>
<thead>
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<th>Parameters</th>
<th>Description</th>
<th>Value</th>
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<tbody>
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<td>µm</td>
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<td>µm</td>
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<td>µm</td>
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<tr>
<td>$\mu_s$</td>
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<td>$8.9 \times 10^{-7}$</td>
<td>Pa·s</td>
</tr>
</tbody>
</table>

Table 3.3. Element used in the FE calculations.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Description</th>
<th>Order</th>
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</tr>
<tr>
<td>$\rho^{(1)}$</td>
<td>Eulerian first order density</td>
<td>Linear</td>
</tr>
<tr>
<td>$v^{(2)}$</td>
<td>Eulerian second order velocity</td>
<td>Quadratic</td>
</tr>
<tr>
<td>$p^{(2)}$</td>
<td>Eulerian and ALE second order pressure</td>
<td>Linear</td>
</tr>
<tr>
<td>$\xi$</td>
<td>ALE first order displacement</td>
<td>Quadratic</td>
</tr>
<tr>
<td>$v^L$</td>
<td>ALE Lagrangian second order velocity</td>
<td>Quadratic</td>
</tr>
<tr>
<td>$T^{(1)}$</td>
<td>Eulerian Temperature</td>
<td>Linear</td>
</tr>
</tbody>
</table>

3.2.7 Error Measurements

For each scalar solution variable, $\rho^{(1)}, T^{(1)}$ and $p^{(2)}$, one measure of error will be used to evaluate convergence. That measure is the $L^2$ relative error measurement, which for a standard discretization on a quasi-uniform mesh is expected to yield a slope one greater than the element order [3]. The $L^2$ norm of the relative error is implemented as follows (using $\rho^{(1)}$ as an example):

$$
\|e(\rho^{(1)})\|_{L^2, \text{Rel}} = \left[ \int_{\Omega} (\rho_h^{(1)} - \rho_{\text{exact}}^{(1)})^2 \, dv \right]^{1/2} \left[ \int_{\Omega} (\rho_{\text{exact}}^{(1)})^2 \, dv \right]^{-1/2},
$$

(3.18)

where $\rho_h^{(1)}$ represents the numerical solution for $\rho^{(1)}$ obtained with a mesh with diameter $h$. In the case of the variables, $v^{(1)}, \xi, v^{(2)},$ and $v^L$ the theory of partial differential equation allows us to also compute the error in their gradients, which
is referred to as the error in the $H^1$ (semi-)norm. Using $\mathbf{v}^{(1)}$ as an example, the relative error in the $H^1$ norm is computed as follows:

$$
\|e(\mathbf{v}^{(1)})\|_{H^1,\text{Rel}} = \left\{ \int_{\Omega} \left[ (\nabla \mathbf{v}^{(1)}_h - \nabla \mathbf{v}^{(1)}_{\text{exact}}) : (\nabla \mathbf{v}^{(1)}_h - \nabla \mathbf{v}^{(1)}_{\text{exact}}) \, dv \right] \right\}^{1/2}
\left[ \int_{\Omega} \nabla \mathbf{v}^{(1)}_{\text{exact}} : \nabla \mathbf{v}^{(1)}_{\text{exact}} \, dv \right]^{-1/2}.
$$

(3.19)

The $L^2$ error norm is also implemented for the vector solution variables, yielding two measures of error for $\mathbf{v}^{(1)}$, $\xi$, $\mathbf{v}^{(2)}$, and $\mathbf{v}^L$. In a convergence plot these error measurements are presented as the logarithm of the relevant norm versus the logarithm of $h^{-1}$.

### 3.2.8 Expectations of the ALE and Eulerian Solutions

The expectation of this study is that the numerical boundary approximation in the Eulerian formulation of the problem will restrict the accuracy of the second order solution. In this convergence study, that should be visible in the plots of second order solution convergence. An averaged convergence rate will also be provided to consider for each case.
Chapter 4

Results

4.1 Results

4.1.1 The 3D Eulerian Solution

Solutions of the 3D Eulerian problem can be seen to have demonstrated a high level of accuracy at the greatest mesh refinement. Figure 4.1 represents \( u_x^L \), as well as its companion manufactured solution. Both are shown at the most refined mesh as defined in Figure 3.1. Figure 4.2 represents the Eulerian second order pressure solution and the manufactured solution for the same variable.

Figure 4.3, a plot of convergence for the 3D Eulerian problem was also created using these solutions, as well as the error measurements from Eq. (3.18) and Eq. (3.19). The specific data points correspond to the refinements of Fig. 3.1 and are presented on a logarithmic scale.
Figure 4.2. Approximate (left) and manufactured (right) solutions for $p^{(2)}$ using the most refined mesh as defined in Fig. 3.1.

Figure 4.3. Convergence of the Eulerian Solution

4.1.2 The 3D ALE Solution

The same select solution plots of the 3D ALE problem also present a similarly high level of accuracy in depicting the expected solution. Figure 4.4 represents $v^L_x$ and its companion manufactured solution. Both are shown at the most refined mesh as defined in Fig. 3.1. Figure 4.5 represents the second order pressure solution and the manufactured solution for the same variable of the ALE problem. Figure 4.6 demonstrates the log-scaled convergence of the ALE solutions with increased mesh refinements. This figure, in parallel with Fig. 4.3, are the best visual depiction of the difference in convergence quality between the two formulations. The discretization of the 3D problem is consistent with the 2D, pressure terms have a linear element order, while velocity terms have a quadratic element order.
Figure 4.4. The Solution and Manufactured Solution of $v_L$ along $\hat{e}_x$ using the most refined mesh as defined in Figure 3.1.

Figure 4.5. The Solution and Manufactured Solution of Second Order Pressure

4.1.3 The Physical Solution

Figures 4.7, 4.8 and 4.9 were created by implementing the side to side physical actuation as described in Eq: (3.13). The black arrows depict the Lagrangian mean flow velocity, $v^L$, in the volume. While the depiction in [1] only depicts a 2D case, the same effect is now visible in this 3D case. Similarly, Figs. 4.10 and 4.11 were created by implementing the SAW actuation profile from Eqs. (3.15), (3.16) and (3.17). Initially the solution in Figs. 4.10 and 4.11 appear to show only a limited depiction of the same effect from Fig. 4.8. However, this reduced effect is expected and noted in [1]. Upon further consideration of the problem, the same effect becomes visible and it can be seen that flow is indeed directed along the center of the channel and then directed back along the walls in Fig. 4.12.
4.1.4 Thermally Coupled Solution

The inclusion of the heat equation in the 3D Eulerian Problem, creates three solutions in the first order, temperature, velocity and pressure. Where the magnitude of the velocity and pressure are assumed to be greater than those of the second order streaming problem. Figure 4.13 shows the temperature solution of the heat coupled Eulerian problem, as well as the manufactured solution of temperature. The same graph as in Figs. 4.3 and 4.6 was created to consider the impact of the heat equation inclusion on solution accuracy and is demonstrated in Fig. 4.14.

4.1.5 Table of Calculation Times and Degrees of Freedom

<table>
<thead>
<tr>
<th>Solution</th>
<th>Degrees of Freedom</th>
<th>Time</th>
<th>Units</th>
<th>Platform</th>
</tr>
</thead>
<tbody>
<tr>
<td>Physical SAW Actuation</td>
<td>213909</td>
<td>349</td>
<td>s</td>
<td>COMSOL</td>
</tr>
<tr>
<td>Physical Side Actuation</td>
<td>213909</td>
<td>352</td>
<td>s</td>
<td>COMSOL</td>
</tr>
<tr>
<td>Manufactured Eulerian</td>
<td>455085</td>
<td>4128</td>
<td>s</td>
<td>COMSOL</td>
</tr>
<tr>
<td>Manufactured Temperature</td>
<td>433105</td>
<td>1845</td>
<td>s</td>
<td>COMSOL</td>
</tr>
<tr>
<td>Manufactured ALE</td>
<td>415529</td>
<td>3434</td>
<td>s</td>
<td>COMSOL</td>
</tr>
</tbody>
</table>
Figure 4.7. Depiction of side to side actuation of an acoustic streaming chamber using ZY cut planes of $v^L$. 
Figure 4.8. Depiction of side to side actuation of an acoustic streaming chamber using a ZX cut plane of $v^L$. 
Figure 4.9. Depiction of side to side actuation of an acoustic streaming chamber using a ZX cut plane of $p_2$. 
Figure 4.10. Physical Solution of SAW Acoustic Streaming Showing $v_L$ Using a ZX Cut Plane
Figure 4.11. Physical Solution of SAW Acoustic Streaming Showing Second order Pressure Using a ZX Cut Plane
Figure 4.12. Physical Solution of SAW Acoustic Streaming Showing Second order Pressure Using a ZX Cut Plane

Figure 4.13. The Eulerian Solution and Manufactured Solution of Temperature at the Greatest Mesh Refinement in Figure 3.1
Figure 4.14. Convergence of the First Order Eulerian Solutions with Temperature at the Five Mesh Refinements in Figure 3.1
Chapter 5
Discussion

5.1 Comparison of Results

5.1.1 Result Quality

5.1.1.1 Averaged Convergence rates

From Figs. 4.6 and 4.3, it is possible to create averaged values of solution convergence over the mesh refinement sweep presented in Fig. 3.1. The expectations of these values is discussed in Chapter 3 and are based on previous literature [3]. Far greater improvement was demonstrated for the Lagrangian mean velocity convergence in the ALE study, than the Eulerian study. This is likely the result of the difference in approach to the SAW actuated surface, $\Gamma_d$. Eq. (2.29) in the statement of the

<table>
<thead>
<tr>
<th>Solution</th>
<th>Variable</th>
<th>Averaged Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Eulerian $L^2$</td>
<td>$u^{(1)}$</td>
<td>2.857</td>
</tr>
<tr>
<td>Eulerian $H^1$</td>
<td>$v^{(1)}$</td>
<td>1.913</td>
</tr>
<tr>
<td>Eulerian $L^2$</td>
<td>$u^L$</td>
<td>3.070</td>
</tr>
<tr>
<td>Eulerian $H^1$</td>
<td>$v^L$</td>
<td>2.199</td>
</tr>
<tr>
<td>Eulerian $L^2$</td>
<td>$p^{(2)}$</td>
<td>1.992</td>
</tr>
<tr>
<td>ALE $L^2$</td>
<td>$\xi$</td>
<td>2.936</td>
</tr>
<tr>
<td>ALE $H^1$</td>
<td>$\xi$</td>
<td>1.961</td>
</tr>
<tr>
<td>ALE $L^2$</td>
<td>$v^L$</td>
<td>3.606</td>
</tr>
<tr>
<td>ALE $H^1$</td>
<td>$v^L$</td>
<td>2.565</td>
</tr>
<tr>
<td>ALE $L^2$</td>
<td>$p^{(2)}$</td>
<td>2.053</td>
</tr>
</tbody>
</table>
Eulerian problem is the result of a Taylor series expansion, that is not utilized in the ALE formulation. Instead the ALE formulation induces the boundary condition in Eq. (2.59), which is a result of a different approach. Additionally, \( v^{(2)} \), the second order velocity in the Eulerian formulation, is not the Lagrangian mean velocity, as in the ALE formulation. Although, the solution \( v^{(2)} \) can be used to find \( v_L \), it incorporates an approximation not present in the ALE formulation. This approximation likely supplies additional error to the solution of Lagrangian mean velocity. These topics will be further explored in future works.

5.1.2 The Thermally Coupled Eulerian Solution

The inclusion of the temperature solution in the isentropic case presents little change in solution time as shown in Table 4.1 and no noticeable change in first order convergence as shown in Figure 4.14. This shows that the inclusion of a temperature solution in the acoustic streaming problem is not associated with significant computational costs or diminishing accuracy of other solution variables when incorporated in the isentropic case.

5.1.3 The Physical Solutions

The physical solutions in Figure 4.7, 4.8, 4.9, 4.10 and 4.12 presented some changes from the 2D version presented in [1]. However, the addition of new near boundaries dictating flow are likely the contributing factor. It is nonetheless possible to see the same oscillation which draws flow through the center, meeting at the origin and then back along the walls of the chamber. As expected, this flow was more apparent by side actuation, than a SAW actuation profile. For the purpose of this analysis, this actuation and the comparison with [1], were selected so as to demonstrate the ALE formulation in effect with a solution already established by an Eulerian approach. It can be expected that either formulation is functional in the context presented here.
Chapter 6

Conclusion

The formulation of two different approaches to the acoustic streaming problem were presented based on previous literature (ef. [1], [13], [16]). The intention of this analysis was too demonstrate the successful application of these approaches and too allow for a comparative analysis. To this end, multiple studies were conducted utilizing the method of manufactured solutions, as well as physical parameters and actuation as shown in Figs. 4.6, 4.3, 4.7, and 4.10. The results demonstrate the ability of either case to create numerically accurate solutions, as well as those physically expected based on previous literature. Furthermore, this application demonstrates a rigorous approach to the internal analysis that should be expected of such multiple order problems. Finally, the case can be made based on the results provided that the ALE formulation presents preferential accuracy for acoustic streaming applications.
Bibliography


