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VALUE DISTRIBUTION OF AUTOMORPHIC L-FUNCTIONS

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Abstract

Significant attention has been given to study various moments of the Riemann zeta function, ζ, its logarithm and their generalizations However, not much is known about the moments of $\frac{\zeta'}{\zeta}$. and the logarithmic derivative of more general L-functions. For π, a cuspidal automorphic representation of $GL_d(\mathbb{A}_{\mathbb{Q}})$, there is an associated L-function, $L(s,\pi)$. We study the value distribution of its logarithmic derivative on the 1-line, $\frac{L'}{L}(1+it,\pi)$. We are able to prove that for $t \in [T, 2T]$, in some sense, $\frac{L'}{L}(1+it,\pi)$ has "almost" normal distribution with mean 0 and variance $\sqrt{\frac{\log(y(T))}{2y(T)}}$. An essential ingredient of the proof is the fact that our function of interest can be approximated by Dirichlet polynomial with coefficients supported on prime powers. We prove similar results for $\frac{L'}{L}(1+it,\pi\times\overline{\pi})$ and $\log(L(1+it,\pi))$.

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Dedication

For my grandfather Bogdan.

Chapter 1
Introduction

Understanding the behavior of the ζ-function and other L-functions plays a major role in modern mathematics. Two out of seven Millennium Prize Problems pose a question about the analytic behavior of L-functions. One of them, called the Riemann Hypothesis, asks whether all non-trivial zeros of the zeta function lie on the 1/2-line. It has far-reaching implications across many branches of mathematics. The other, known as the Birch and Swinnerton-Dyer Conjecture, asks about the connection between the order of vanishing of $L(1/2, E)$, an L-function attached to an elliptic curve E, and the rank of E.

A rigorous understanding of the ζ-function, and other L-functions, proved to be a difficult problem. To gain more insight into their nature, it is natural to study various moments of these functions. As an example consider the following. If we know that for every $k \in \mathbb{N}$

$$\int_0^T \left|\zeta\left(\frac{1}{2} + it\right)\right|^k \ll_\epsilon T^{1+\epsilon},$$

then for every large x there is a prime number between x and $x + x^{1/2+\epsilon}$. Although both questions are still open, we can see how knowledge of mean values sheds light onto the distribution of primes.

Furthermore, some applications of the Riemann Hypothesis seem to go beyond mathematics. In [VBPK99], Berry and Keating suggest a connection between the zeros of the ζ-function and certain phenomena in quantum mechanics.

The Riemann zeta function exhibits some unusual properties. One of them is the fact that ζ takes on every value, with one possible exception, in $\mathbb{C}$ infinitely many times. Another remarkable property, proven by Voronin, is known as the zeta universality. He showed that in the strip $1/2 < \Re(s) < 1$, any non-vanishing holomorphic function can be approximated arbitrarily well by the ζ-function.

There are many instances when it is possible to compute all the moments. However, they often do not correspond to moments of any well-known distributions. In this thesis, we investigate the behavior of the logarithm and logarithmic derivative of a general L-function on the 1-line. We exploit the fact that we can control the influence of potential nontrivial zeros off of the 1/2-line. Suitable Dirichlet polynomial approximations to the functions of interest are supported on a sparse set of prime powers. All these facts put together make the computation of all the moments feasible. In addition, we introduce a novel idea of including a short initial segment which causes the moments to be recognizable as the ones of a Gaussian distribution. In view of this, the distribution functions are somewhat "close" to the Gaussian distribution

1.1 Integral moments of the ζ-function on the $\frac{1}{2}$-line

The knowledge of Mean Values can greatly deepen our knowledge of a particular function. It is often difficult to understand the behavior of a function, but if such a function is studied together with a family of closely related functions, we often get better insight into the behavior of our function of interest. A great example is the following equivalence [Tit86]

$$M_k := \int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k} dt \ll_{k,\epsilon} T^{1+\epsilon} \;\; \forall k \in \mathbb{N}, \;\; \forall \epsilon > 0 \Leftrightarrow \zeta(\frac{1}{2}+it) \ll_{\epsilon} t^{\epsilon} \;\; \forall\, \epsilon > 0.$$

In this case we can think of $\zeta(\frac{1}{2}+it), t \in [0,T]$ as a continuous family of ζ-functions with respect to a parameter t which gives information about the point wise bound on the $\frac{1}{2}$-line.

1.1.1 Conditional results

To get some kind of feeling for the possible size of $\int_0^T |\zeta(\frac{1}{2}+it)|^{2k} dt$ we start this section with a brief summary of results conditional on the Riemann Hypothesis.

It turns out that knowledge of the location of the zeros of the ζ-function gives us better information about the size of M_k for a wide range of k. Both Ramachandra [Ram78] and Heath-Brown [HB81] independently showed that assuming the Riemann Hypothesis we have

$$\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k} dt \gg_k T\log(T)^{k^2} \quad \forall\ k>0.$$

Also, very recently Radziwill and Soundararajan obtained more precise results. In [RS96], assuming the Riemann Hypothesis, they established

$$M_k(T) \geq e^{-30k^4} T\log(T)^{k^2} \quad \forall\ k>1.$$

Let $L(s,\pi)$ be a primitive L-function, defined in Section 5 of this chapter. Under mild assumption on the size of local parameters $\alpha_\pi(j,p)$, Akbary and Fodden in [AF12] were able to prove that, for rational $k \geq 0$,

$$\int_1^T \left|L\left(\frac{1}{2}+it,\pi\right)\right|^{2k} dt \gg_{k,\pi} T\log(T)^{k^2}.$$

To complete the picture, in 2008 Soundararajan [KS00b] managed to show that for every $k>0$ and every $\epsilon>0$ we have

$$\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k} dt \ll_{k,\epsilon} T\log(T)^{k^2+\epsilon}$$

and five years later Harper, in [JH13], improved Soundararajan's result by removing the ϵ. From these we really see that we should have

$$\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k} dt \sim C_k T(\log(T))^{k^2}.$$

Furthermore, Conrey and Gonek [CG01] conjectured that for some constant g_k

$$C_k = \frac{a_k g_k}{\Gamma(k^2+1)}$$

where

$$a_k = \prod_p \left(\left(1 - \frac{1}{p}\right)^{k^2} \sum_{r=1}^{\infty} \frac{d_k^2(p^r)}{p^r} \right)$$

and in 2000 Keating and Snaith [KS00b], using random matrix theory, conjectured that the constant g_k ought to look like

$$g_k = (k^2)! \prod_{j=0}^{k-1} \frac{j!}{(j+k)!}.$$

1.1.2 Unconditional results

Now that we have some intuitive feeling regarding the possible size of $M_k(T)$, we are going to talk about the current state of knowledge of Mean Value Theorems with no assumption of the Riemann Hypothesis, or any other conjectures. As of now, the problem has been completely solved for $k = 1$ and $k = 2$. For $k = 1$ we have

Theorem 1.1.1. *As* $T \to \infty$

$$\int_0^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^2 dt = T\log(T) + (2\gamma - 1 - \log(2\pi))T + O\left(T^{\frac{1}{2}} \log(T)\right).$$

The asymptotics were first established by Hardy and Littlewood [HL16]. Later Ingham in [Ing27] proved the above theorem as well as the following

Theorem 1.1.2. *As* $T \to \infty$

$$\int_0^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^4 dt = \frac{1}{2\pi^2} T(\log(T))^4 + O(T(\log(T))^3).$$

We have some partial results in a very limited range for $k > 2$. These can be found in Ivić [Ivi03] which we state below. For fixed $k \geq 2$ define $M(k)$ to be the infimum of all M such that

$$\int_1^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^{2k} dt \ll T^{M+\epsilon}$$

for all $\epsilon > 0$. Then we have the following result

Theorem 1.1.3. *If $k \geq 2$ is a fixed number, then*

$$M(k) = \begin{cases} 1 + \frac{k-2}{4} & \text{if } 2 \leq k \leq 6, \\ 2 + \frac{3(k-6)}{11} & \text{if } 6 \leq k \leq \frac{89}{13}, \\ 1 + \frac{35(k-3)}{108} & \text{if } k \geq \frac{89}{13}. \end{cases}$$

It turns out that our knowledge of lower bounds for $\int_0^T |\zeta(\frac{1}{2}+it)|^{2k} dt$ is much deeper than the one for upper bounds. Ramachandra [Ram80] showed

$$\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k} dt \gg_k T(\log(T))^{k^2} \text{ where } \ k \in \mathbb{N}.$$

and Heath-Brown [HB81] extended this result for $k \in \mathbb{Q}^+$, which is consistent with what we expect to be the truth namely $\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k} dt \sim C_k T(\log(T))^{k^2}$.

In [Sou09] Soundararajan introduced novel ideas towards understanding

$$\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k} dt.$$

As mentioned above, assuming the Riemann Hypothesis, he was able to prove

$$\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k} dt \ll T(\log(T))^{k^2+\epsilon}.$$

1.2 Moments of $\log(\zeta(s))$

The problem of computing moments of both real and imaginary parts of

$$\log(\zeta(1/2+it))$$

turned out to be more accessible. The result of Selberg [Sel44] says that, unconditionally, we have

$$\int_0^T \left|\Im \log\left(\zeta\left(\frac{1}{2}+it\right)\right)\right|^{2k} dt \sim \frac{(2k)!}{k!(2)^{2k}} T(\log\log(T))^k$$

for all positive integers k. Further, as noted in notes at the end of chapter 14 of Montgomery and Vaughan [MV07], it can be shown that for k odd we have

$$\int_0^T \left(\Im \log\left(\zeta\left(\frac{1}{2}+it\right)\right)\right)^k dt = o(T(\log\log(T))^{\frac{k}{2}}).$$

With this and Selberg's result for even moments, one can deduce that the underlying distribution is the Gaussian one

$$\lim_{T\to\infty}\frac{1}{T}meas\left\{t\in[0,T] : \frac{\Im\left(\log\left(\zeta\left(\frac{1}{2}+it\right)\right)\right)}{\sqrt{\frac{1}{2}\log\log(T))}} \leq c\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{c} e^{-\frac{t^2}{2}}dt.$$

The distribution function of both real and imaginary part of $\log(\zeta(\sigma_0+it))$, where $1/2<\sigma_0\leq 1$ is fixed, was studied by Bohr and Jessen in [BJ30] and [BJ32]. They showed that given a rectangle $R\subset\mathbb{C}$, with sides parallel to the real and imaginary axes, there exists a real, nowhere negative, continuous function $F(u+iv)$ such that

$$\lim_{T\to\infty}\frac{1}{2T}meas\left\{t\in[-T,T]|\log\left(\zeta\left(\sigma_0+it\right)\right)\in R\right\} = \int\int_R F(u+iv)dudv.$$

This result shows that the behavior of the ζ-function on the half line is quite different from the behavior anywhere else on the complex plane. In Chapter 5 we establish the following theorem for a general class of L-functions. Below we specialize the main result to the case of the ζ-function

Theorem 1.2.1. *Assume Hypothesis H1, let $T\geq 1$, $k\in\mathbb{N}$. $y=y(T)$ be such that $y(T)\to\infty$ as $T\to\infty$ and $2<y<T^{\epsilon}$, for all $\epsilon>0$,*

$$h(t) = \frac{\log\left(\zeta\left(1+it\right)\right)-\sum_{n\leq y}\frac{\Lambda(n)}{\log(n)n^{1+it}}}{\sqrt{\frac{1}{2\log(y)y}}}.$$

Then

$$\lim_{T\to\infty}\frac{1}{T}meas\left\{t\in[T,2T]|\Re\left(h(t)\right)\leq c\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{c} e^{-\frac{x^2}{2}}dx,$$

$$\lim_{T\to\infty}\frac{1}{T}meas\left\{t\in[T,2T]|\Im\left(f(t)\right)\leq c\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{c} e^{-\frac{x^2}{2}}dx.$$

Here, by subtracting the relatively short initial segment $\sum \frac{\Lambda(n)}{\log(n)n^{1+it}}$ we see that, on the 1-line, the L function is asymptotically normally distributed. In some sense, the function $F(u+iv)$ of Bohr and Jessen is really "close" to the kernel of the Gaussian distribution

1.3 Discrete Moments of L-functions

Not all mean values are integral in nature. Suppose $L(s)$ is an L-function associated to some interesting arithmetic or geometric object. After appropriate normalization, we can suppose that all the non-trivial zeros lie in the critical strip $0 < \sigma < 1$. Quite often such a function vanishes at special values which are related to certain properties of our arithmetic or geometric object. For example, let E be an elliptic curve. The Birch and Swinnerton-Dyer Conjecture asserts a connection between the order of vanishing of $L(1/2, E)$, an L-function attached to E, and the rank of E. As an example [Sou09] considers a twist of an elliptic curve E by a quadratic character modulo d, χ_d. The associated L-function, $L(s, E_d)$, may vanish at the central point $s = \frac{1}{2}$. If one has a continuous lower bound for the moments $\sum'_{|d|\leq X} L(\frac{1}{2}, E_d)^k$ (here $\sum'$ means that we are putting a restriction on the sum. For details consult [RlS15].) then by letting $k \to 0$ we can deduce how often does $L(E_d, \frac{1}{2})$ vanishes as we vary d. Radziwiłł and Soundararajan [RlS15] established the following sharp upper bound

Theorem 1.3.1. *Let $0 \leq k \leq 1$ be a real number. For all large X we have*

$$\sum_{|d|\leq X}{}' L\left(\frac{1}{2}, E_d\right)^k \leq C(k,E)X(\log(X))^{\frac{k(k-1)}{2}},$$

for a positive constant $C(k,E)$.

This compares well with what is expected form a conjecture that Keating and Snaith have made in [KS00a]. Namely, for all $k \geq 0$ there exists a non zero constant $C_0(k,E)$ such that

$$\sum_{|d|\leq X}{}' L\left(\frac{1}{2}, E_d\right)^k \sim C_0(k,E)X(\log(X))^{\frac{k(k-1)}{2}}.$$

Another interesting example of discrete mean values was investigated by Elliott

[Ell71]. By varying the prime modulus p and corresponding Dirichlet characters χ_p; he proves the asymptotic value distribution of $\sum_{q\leq l} \chi_p(q) q^{-s}$ in the following sense.

Lemma 1.3.1. *For each positive integer* m

$$\lim_{Q\to\infty} M_Q^{-1} \sum_{p\leq Q} \sum_{\chi_p \neq \chi_0} \left(\Re \sum_{q\leq l} \frac{\chi_p(q)}{q^s} \right)^m = \mu_m,$$

exists and is finite. Moreover

$$\mu_m = 0 \quad \textit{for } m \textit{ odd}$$

$$0 \leq \mu_m \leq 2^{-m/2} \zeta(2\sigma) m! \quad \textit{for } m \textit{ even.}$$

Due to the unknown nature of μ_m for odd m, we don't fully understand the limiting behavior of the above function.

1.4 Zero Free Region

Establishing the zero-free region for L-functions started with the work of Hadamard and de la Vallée Poussin who independently proved that $\zeta(1+it) \neq 0$. However, it was not until 1899 when de la Vallée Poussin [dlVP00], by construction an auxiliary function, established the "standard" zero free region

Theorem 1.4.1. *There exist* $c > 0$ *such that* $\zeta(s) \neq 0$ *in the region* $\sigma \geq 1 - c/\log(|t|+3)$.

In proving the above theorem, the core idea was to construct an auxiliary function. Amazingly this idea carries over to more general L-functions as can be seen in the proof of Theorem 5.10 in Iwaniec and Kowalski's book [IK04]

Theorem 1.4.2. *(Theorem 5.10 in [IK04]) Let* $L(f,s)$ *be an L-function of degree* d *(for the definition look at Chapter 5 of [IK04]) such that the Rankin-Selberg convolutions* $L(f\otimes f, s)$ *and* $L(f \otimes \overline{f}, s)$ *exist, and the latter has a simple pole at* $s=1$ *while the former is entire in* $f \neq \overline{f}$. *Suppose that at the ramified primes* $|\alpha_j(p)|^2 \leq p/2$. *Then there exists an absolute constant* $c > 0$ *such that* $L(f,s)$ *has* *no zeros in the region*

$$\sigma \geq 1 - \frac{c}{d^4 \log(\mathfrak{q}(f)(|t|+3))}$$

except possibly for one simple real zero $\beta_f < 1$, in which case f is self-dual.

As can be seen from the hypothesis of the theorem, for $L(s,\pi)$ we require the existence of the Rankin-Selberg convolution L-function $L(s,\pi\times\overline{\pi})$. Since the existence of $L(s,(\pi\times\overline{\pi})\times(\overline{\pi\times\overline{\pi}}))$ is not known, we can't use the above theorem to deduce a zero free region of $L(s,\pi\times\overline{\pi})$ of "standard" type. We will present some literature that shows progress towards developing a zero-free region of "standard" type for the Rankin-Selberg L-function (defined in section 1.5 of this chapter).

In the appendix to Lapid's work [Lap13] Brumley extended his result of [Bru06] to $L(s,\pi\times\overline{\pi})$ where π is any cuspidal automorphic representation of $GL_d(\mathbb{A}_F)$(here, F is a number field). In particular he showed there exists $c>0$ such that $L(s,\pi\times\overline{\pi})\neq 0$ in the region $\sigma \geq 1 - c\left(\mathfrak{q}(\pi\times\overline{\pi})^2\mathfrak{q}(1+it,\pi\times\overline{\pi})\right)^{-(7/8-5/(8d)+\epsilon}$. This is considered to be a "narrow" zero-free region because it is a polynomial in terms of the conductor. By putting some restriction on π, Humphries and Brumley [HP18] established the following zero free region for an even more general family of L-functions

Theorem 1.4.3. *For a number field F, let π be a unitary cuspidal automorphic representation of $GL_n(\mathbb{A}_F)$ that is tempered at every nonarchimedean place outside a set of Dirichlet density zero. Then there exists an absolute constant c_π dependent on π (and hence also on n and F) such that $L(s,\pi\times\overline{\pi})$ has no zeroes in the region*

$$\sigma \geq 1 - \frac{c_\pi}{d^4 \log((|t|+3))}$$

with $|t| \geq 1$.

In Chapter 2 we prove Theorem 2.1.2 which gives a "standard" type of zero-free region $\sigma \geq 1 - c(d^2 \log(\mathfrak{q}(\pi)(|t|+5)))^{-1}$. However, we cannot exclude the existence of an exceptional real zero lying close to the point $s=1$.

1.5 Review of Probability Theory

We give some definitions and theorems from the general theory of distributions following Petrov [Pet75] and Billingsley [Bil95]

Definition 1.5.1. *[Pet75] Let $F(x)$ be a non decreasing, left-continuous function satisfying*

$$\lim_{x\to-\infty} F(x) = 0 \quad \textit{and} \quad \lim_{x\to\infty} F(x) = 1.$$

Then $F(x)$ is a distribution function of some random variable.

Definition 1.5.2. *[Pet75] Let $F(x)$ be a distribution function. We define moments of order k, where k is a positive real number, of the distribution function $F(x)$ as*

$$\alpha_k = \int_{-\infty}^{\infty} x^k dF(x)$$

and the central moments of order k as

$$\mu_k = \int_{-\infty}^{\infty} (x-\alpha_1)^k dF(x).$$

Theorem 1.5.1. *[Bil95] Let $F(x)$ be a distribution function having finite moments $\alpha_k = \int_{-\infty}^{\infty} x^k dF(x)$ of all orders. If the power series $\sum_k \alpha_k r^k/k!$ has a positive radius of convergence, then $F(x)$ is the only distribution function with the moments $\alpha_1, \alpha_2, ...$*

Theorem 1.5.2. *[Bil95] Suppose that the distribution function $F(x)$ is uniquely determined by its moments. If $\{F_n(x)\}$ is a sequence of distribution functions whose moments of every order $k = 1, 2, 3, ...$ converge to corresponding moments of $F(x)$ then $F_n(x)$ converges to $F(x)$.*

Let us recall some facts about the Gaussian distribution [Pet75]. The probability density function is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right),$$

where μ is the mean and σ is the standard deviation of the distribution. The central moments of a Gaussian distribution are given by

$$\mu_{2k} = \frac{(2k)!\sigma^{2k}}{k!2^k}, \qquad \mu_{2k+1} = 0.$$

As noted in Example 30.1 of [Bil95], since $\mu_k \leq k!$, by Theorem 1.5.1 the Gaussian distribution is uniquely determined by its moments.

1.6 Definitions

We will follow the setup as in Rudnick and Sarnak's seminal paper "Zeros of principal L-functions and random matrix theory" [RS96]. (For an in-depth introduction to the theory of automorphic L-functions I recommend the books by Goldfeld and Hundley [GH11a], [GH11b].)

Definition 1.6.1. *Let π be an unitary automorphic irreducible cuspidal representation of $GL_d(\mathbb{A}_{\mathbb{Q}})$. By Flath [Fla79] we have the following restricted tensor product decomposition $\pi = \otimes'_p \pi_p$ where π_p is an admissible, irreducible representation of $GL_d(\mathbb{Q}_p)$. The primitive L-function is defined as a product over local factors*

$$L(s,\pi) := \prod_{p<\infty} L(s,\pi_p). \tag{1.1}$$

For our purposes the local factors are best described in terms of Langlands parameters as in [RS96]

$$L(s,\pi_p) = \begin{cases} \prod_{j=1}^{d} \left(1 - \alpha_\pi(j,p)\, p^{-s}\right)^{-1} & \text{if } \pi_p \text{ is unramified,} \\ P_p(p^{-s})^{-1} & \text{if } \pi_p \text{ is ramified,} \end{cases}$$

where $P_p(x)$ is a polynomial such that $\deg(P_p) \le d$ and $P(0) = 1$. Jacquet and Shalika [JS81b] showed that $L(s,\pi)$ converges absolutely for $\sigma > 1$. Let

$$L_\infty(s,\pi) = \prod_{j=1}^{d} \pi^{-\frac{s+\mu_\pi(j)}{2}} \Gamma\left(\frac{s+\mu_\pi(j)}{2}\right).$$

Then there exists a positive integer, $q(\pi)$, called the conductor of π, and a unimodular complex number $\epsilon(\pi)$, called the root number, such that the completed L-function defined as

$$\Lambda(s,\pi) := q(\pi)^{s/2} L_\infty(s,\pi) L(s,\pi)$$

is entire (except when $L(s,\pi) = \zeta(s)$) and satisfies the functional equation [GJ72]

$$\Lambda(s,\pi) = \epsilon(\pi)\Lambda(1-s,\overline{\pi}).$$

Various bounds and conjectures about $L(s,\pi)$ are stated in terms of an analytic conductor. We follow the definition as in [IK04]. Define

$$\mathfrak{q}_\infty(s) := \prod_{j=1}^{d} \left(|s + \mu_\pi(j)| + 3\right)$$

then the analytic conductor, $\mathfrak{q}(s,\pi)$, is

$$\mathfrak{q}(s,\pi) := q(\pi)\mathfrak{q}_\infty(s).$$

As in [IK04], we have

$$\mathfrak{q}(\pi) = \mathfrak{q}(0,\pi).$$

We have the following bound for the local parameters. The proof can be found in the appendix of Rudnick and Sarnak's paper [RS96].

Theorem 1.6.1. *For all primes $p < \infty$ and all $1 \leq j \leq d$ we have*

$$|\alpha_\pi(j,p)| \leq p^{1/2-1/(d^2+1)}, \qquad \Re\left(\mu_\pi(j)\right) > -\frac{1}{2}.$$

Our primary object of study is the logarithmic derivative of $L(s,\pi)$ namely

$$-\frac{L'}{L}(s,\pi) = \sum_{n=2}^{\infty} \frac{\Lambda_\pi(n)}{n^s}, \tag{1.2}$$

where

$$\Lambda_\pi(n) := \Lambda(n)a_\pi(n) = \begin{cases} \Lambda\left(p^k\right) \sum_{j=1}^{d} \alpha_\pi(j,p)^k & \text{if } n = p^k \\ 0 & \text{otherwise.} \end{cases} \tag{1.3}$$

While studying mean values of (1.2) on the $1-line$ we work with terms involving $\Lambda(n)^2|a_\pi(n)|^2$. Unfortunately, there is no natural L-function associated to such coefficient. However, coefficients coming from a Rankin-Selberg convolution will provide a sufficient substitute. In addition, using its analytic properties we obtain essential information about the average growth of $|\Lambda_\pi(n)|^2$, a strong zero-free region for $L(s,\pi)$ and an absolute convergence of the sum (1.1) for $\sigma > 1$. Further, in chapter 4 we will establish a value distribution of $L'/L(1+it, \pi \times \overline{\pi})$. This motivates the following definitions and theorems about a Rankin-Selberg L-functions.

As in the case of $L(s,\pi)$, the Rankin-Selberg L-functions is defined in terms of its local factors. Let $\overline{\pi} = \otimes_p' \overline{\pi_p}$ be the dual of π. Then

$$L(s,\pi\times\overline{\pi}) := \prod_{p<\infty} L(s,\pi_p\times\overline{\pi}_p), \tag{1.4}$$

where

$$L(s,\pi_p\times\overline{\pi}_p) = \begin{cases} \prod_{i,j}\left(1-\alpha_\pi(i,p)\,\overline{\alpha_\pi(j,p)}p^{-s}\right)^{-1} & \text{if } \pi_p \text{ is unramified} \\ P_p(p^{-s})^{-1} & \text{if } \pi_p \text{ is ramified.} \end{cases} \tag{1.5}$$

This time $P_p(x)$ is a polynomial such that $\deg(P_p)\le d^2$ and $P(0)=1$. $L(s,\pi\times\overline{\pi})$ converges absolutely for $\Re(s)>1$ and has a simple pole at $s=1$ [JS81a]. Let

$$L_\infty(s,\pi\times\overline{\pi}) = \prod_{i,j} \pi^{-\frac{s}{2}(\mu_{\pi\times\overline{\pi}}(i,j))}\Gamma\left(\frac{s}{2}+\frac{\mu_{\pi\times\overline{\pi}}(i,j))}{2}\right).$$

There exists a positive integer $q(\pi\times\overline{\pi})$ and a unimodular complex number $\epsilon(\pi\times\overline{\pi})$ such that the completed L-function

$$\Lambda(s,\pi\times\overline{\pi}) = q(\pi\times\overline{\pi})^{s/2}L_\infty(s,\pi\times\overline{\pi})L(s,\pi\times\overline{\pi})$$

has a meromorphic continuation to the whole complex plane, exhibiting simple poles at $s=0,1$ and satisfies the functional equation [JPSS83]

$$\Lambda(s,\pi\times\overline{\pi}) = \epsilon(\pi\times\overline{\pi})\Lambda(1-s,\pi\times\overline{\pi}).$$

We note that $\Lambda(s,\pi\times\overline{\pi})$ has no poles for $\Re(s)>1$ and $L(s,\pi\times\overline{\pi})$ has no zeros to the right of the 1-line. It follows that $\Re(\mu_{\pi\times\overline{\pi}}(i,j))>-1$. Further by work of Jacquet, Piatetskii-Shapiro and Shalika [JPSS83] $L(s,\pi\times\pi)$ exists. As remarked in [IK04], this implies the following bound for the analytic conductor

$$\mathfrak{q}\left(s,\pi\times\overline{\pi}\right) \le \mathfrak{q}(\pi)^{2d}(|s|+3)^{d^2}. \tag{1.6}$$

1.7 Main Results and Remarks

Statement of Hypotheses H1, H2, H3, H4 are stated at the beginning of Chapters 3 and 4. In Chapters 3, 4 and 5 we establish the main contribution of this thesis

Theorem 1.7.1. *Assume Hypothesis H1 and let $T \geq 1$, $k \in \mathbb{N}$. There exists $c_\pi > 0$ such that if $2 < y = y(T) < \frac{\exp\left((\log(T))^2\right)}{(\log(T))^2}$ and $y(T) \to \infty$ as $T \to \infty$, then*

$$\int_T^{2T} \left| -\frac{L'}{L}(1+it,\pi) - \sum_{n \leq y} \frac{\Lambda_\pi(n)}{n^{1+it}} \right|^{2k} dt = Tk! \left(\frac{\log(y)}{y} \right)^k (1+o(1)),$$

$$\int_T^{2T} \left| \log\left(L(1+it,\pi)\right) - \sum_{n \leq y} \frac{\Lambda_\pi(n)}{\log(n) n^{1+it}} \right|^{2k} dt = Tk! \left(\frac{1}{y\log(y)} \right)^k (1+o(1)).$$

If we further assume Hypothesis H2, H3, H4, then

$$\int_T^{2T} \left| -\frac{L'}{L}(1+it,\pi\times\overline{\pi}) - \sum_{n \leq y} \frac{\Lambda_{\pi\times\overline{\pi}}(n)}{n^{1+it}} \right|^{2k} dt = Tk! \left(\frac{c_0\log(y)}{y} \right)^k (1+o(1)).$$

Computing moments of L-functions prove to be a notoriously difficult problem. Here we would like to highlight the underlying principles that make the computation of all integral moments of $\frac{L'}{L}(1+it,\pi)$ possible. Similar principles apply to the case of $\frac{L'}{L}(1+it,\pi\times\overline{\pi})$. There are two major aspects that make it possible to compute all integral moments of our function of interest. First, (in Lemma 2.3.) we use the zero-free region to show that $\frac{L'}{L}(1+it,\pi) \approx \sum_{n\leq x} \frac{\Lambda_\pi(n)}{n^{1+it}}$. This is a relatively long Dirichlet polynomial. Secondly, the coefficients $\Lambda_\pi(n)$ are supported on prime powers. The analysis shows that the main term comes from the primes alone, and the rest contributes to the error term. In order to get control of the error term, we need to assume the boundedness of a certain sum which involves $\Lambda_\pi\left(p^k\right)$ where $k \geq 2$. That is where Hypothesis H1 comes in.

To motivate subtraction of the initial segment note that we expect the main contribution to come from diagonal terms of form

$$\int_T^{2T} \left| \sum_{y<n\leq x} \frac{\Lambda_\pi(n)}{n^{\sigma+it}} \right|^{2k} dt = \int_T^{2T} \left| \sum_{y^k<n\leq x^k} \frac{A_{k,\pi}(n)}{n^{\sigma+it}} \right|^2 dt \approx T \sum_{y^k<n\leq x^k} \frac{|A_{k,\pi}(n)|^2}{n^{2\sigma}}. \quad (1.7)$$

Let $y = 2$. From the analytic properties of $L(s, \pi \times \overline{\pi})$ and assumption of Hypothesis H1 we can see that the right hand side of (1.7) converges for $\sigma > \frac{1}{2}$. The series gives rise to an arithmetical constant that depends on π and k. These constants do not correspond to moments of well-known distributions. By subtracting an initial segment of length $y = y(T)$ we end up studying the tail of that convergent series. We also have control of the size of the error term. Most importantly the main term becomes asymptotic to

$$Tk! \left(\sum_{y<p} \frac{|\Lambda_\pi(p)|^2}{p^2} \right)^k \sim Tk! \left(\frac{\log(y)}{y} \right)^k ,$$

which is independent of π and the constant, that depends on k, is easy to understand. Unfortunately there is no natural L-function attached to $|\Lambda_\pi(n)|^2 = \Lambda(n)^2 |a_\pi(n)|^2$. We use Hypothesis H1 to replace those coefficients by $\log(n)\Lambda(n)|a_\pi(n)|^2$. The latter coefficients appear in the Dirichlet series expansion of $\left(\frac{L'}{L}(s, \pi \times \overline{\pi})\right)'$ and thus are easier to study.

The behavior of $\frac{L'}{L}(s, \pi \times \overline{\pi})$ is more mysterious. In order to understand it, we have to develop a little bit of theory. In particular in Chapter 2 we prove a zero-free region for $L(s, \pi \times \overline{\pi})$ from which we deduce a form of the prime number theorem with a classical error term as well as a suitable approximation to $\frac{L'}{L}(s, \pi \times \overline{\pi})$ by a Dirichlet polynomial. Unfortunately, the coefficients that involve $\Lambda(n)|a_\pi(n)|^4$ do not correspond to any known L-functions. Studying them is much more difficult. That is why we will make assumptions about their average growth (Hypothesis H2, H3, H4).

1.8 Future work

With more work it should be possible to refine the results in Theorem 1.7.1 and extract more than just one term in the main term. For example, we expect

$$\int_T^{2T} \left| -\frac{L'}{L}(1 + it, \pi) - \sum_{n \le y} \frac{\Lambda_\pi(n)}{n^{1+it}} \right|^{2k} dt = TP_k \left(\frac{\log(y)}{y} \right) (1 + o(1)) ,$$

where $P_k(x)$ is some polynomial of degree k.

The most natural extension of our work would be to fix $1/2 < \sigma < 1$ and work out the moments of $\frac{L'}{L}(\sigma + it, \pi)$. Seeing that our path of integration may cross through potential zeros of $L(s, \pi)$ we would require subtraction of said involved terms. For example, in the case of the Riemann zeta function, we have the following expansion

$$-\frac{\zeta'}{\zeta}(s) = \sum_{n\geq 1} \frac{\Lambda(n)}{n^s} e^{-\left(\frac{n}{x}\right)^2} + \frac{1}{2}\sum_{\rho} \Gamma\left(\frac{\rho - s}{2}\right) x^{\rho - s} + \frac{1}{2}\sum_{r\geq 1} \Gamma\left(-r - \frac{s}{2}\right) x^{-2r-s}$$
$$+ \sum_{r\geq 1} \frac{\zeta'}{\zeta}(-2r + s) x^{-2r} \frac{(-1)^r}{r!} - \frac{1}{2}\Gamma\left(\frac{1-s}{2}\right) x^{1-s}.$$

By subtracting terms that are influenced by potential nontrivial zeros on the σ-line we would expect the following

$$\int_T^{2T} \left| -\frac{\zeta'}{\zeta}(\sigma + it) - \sum_{n\leq y} \frac{\Lambda(n)}{n^{\sigma+it}} - \sum_{\rho\in\mathcal{R}} \frac{1}{\rho - \sigma - it} \right|^{2k} dt$$
$$= Tk! \left(\frac{\log(y)}{y^{2\sigma-1}(2\sigma - 1)} \right)^k (1 + o(1)),$$

where $\mathcal{R}$ is some region close to the σ-line.

Lastly, using results from Chapter 5 and observing that

$$L(1 + it, \pi) = \exp\left(\left(\log\left(L(1 + it, \pi)\right) - \sum_{n\leq y} \frac{\Lambda_\pi(n)}{\log(n) n^{1+it}} \right) + \sum_{n\leq y} \frac{\Lambda_\pi(n)}{\log(n) n^{1+it}} \right)$$

$$= \exp\left(\left(\log\left(L(1 + it, \pi)\right) - \sum_{n\leq y} \frac{\Lambda_\pi(n)}{\log(n) n^{1+it}} \right) \right) \prod_{p\leq P} L(s, \pi_p),$$

we may be able to compute the moments of $L(s, \pi)$ itself, on the 1-line, and shed light on its value distribution.

Chapter 2 Analytic Properties of L-functions

2.1 Zero-free Region of L-functions

In order to obtain a good approximation for $\frac{L'}{L}(s,\pi)$ we need some information about zeros of $L\left(s,\pi\right)$ near the 1-line. The following is a restatement of Theorem 5.42 in [IK04].

Theorem 2.1.1. *[IK04] Let $L(s,\pi)$ be a primitive L-function attached to a cuspidal automorphic representation of $GL_d(\mathbb{A}_{\mathbb{Q}})$. Then there exist $c>0$ such that*

$$L(s,\pi)\neq 0 \quad \textit{in the region} \quad \sigma \geq 1-\frac{c}{d^4\log(\mathfrak{q}(\pi)(|t|+3))}. \tag{2.1}$$

except for possibly one simple real zero $\beta<1$. For such β to exist, π is necessarily self dual i.e. $\pi=\overline{\pi}$.

Theorem 2.1.2. *Let $L(s,\pi\times\overline{\pi})$ be the Rankin-Selberg L-function where π is an irreducible cuspidal automorphic representation of $GL_d(\mathbb{A}_{\mathbb{Q}})$. Then there exist $c>0$ such that*

$$L(\pi\times\overline{\pi},s)\neq 0 \quad \textit{in the region} \quad \sigma \geq 1-\frac{c}{d^2\log\left(\mathfrak{q}(\pi)(|t|+5)\right)}.$$

With an exception of at most one real zero β located in

$$1-\frac{c}{d^2\log\left(\mathfrak{q}(\pi)5\right)}\leq\beta<1$$

where $\mathfrak{q}(\pi) = \mathfrak{q}(0, \pi)$.

Proof. By Proposition 5.7 in [IK04] we have the following expression valid for $1/2 \le \sigma \le 2$

$$-\frac{L'}{L}(s, \pi \times \overline{\pi}) = \frac{1}{s} + \frac{1}{s-1} - \sum_{|s+\mu_{\pi\times\overline{\pi}}(i,j)|<1} \frac{1}{s+\mu_{\pi\times\overline{\pi}}(i,j)} - \sum_{|s-\rho|<1} \frac{1}{s-\rho} + O\left(\log(\mathfrak{q}(s, \pi \times \overline{\pi})\right).$$

In our application $\sigma > 1$ so $-\Re((s-\rho)^{-1}) = -\frac{\sigma-\beta}{|s-\rho|^2} < 0$. From the analytic properties of $\Lambda(s, \pi \times \overline{\pi})$ we have $\Re(\mu_{\pi\times\overline{\pi}}(i,j)) > -1$. With this we can see that for $\sigma > 1$

$$-\Re((s+\mu_{\pi\times\overline{\pi}}(i,j))^{-1}) = -\frac{\sigma + \Re(\mu_{\pi\times\overline{\pi}}(i,j))}{|s+\mu_{\pi\times\overline{\pi}}(i,j)|^2} < 0.$$

Also, by equation (1.6) the error term can be further bounded $\log(\mathfrak{q}(s, \pi \times \overline{\pi})) \ll \log(\mathfrak{q}(\pi)^{d^2}(|s|+3)^{d^2})$. With this remark and $s = 1 + \delta + i\gamma_0$ we have following inequalities

$$-\Re\left(\frac{L'}{L}(1+\delta, \pi \times \overline{\pi})\right) \le \frac{1}{\delta} + c_1 \log(\mathfrak{q}(\pi)^{d^2}(|1+\delta|+3)^{d^2}),$$

$$-\Re\left(\frac{L'}{L}(1+\delta+i\gamma_0, \pi \times \overline{\pi})\right) \le \frac{\delta}{\delta^2+\gamma_0^2} - \frac{1}{1+\delta-\beta_0} + c_1 d^2 \log(\mathfrak{q}(\pi)(|1+\delta+i\gamma_0|+3)),$$

$$-\Re\left(\frac{L'}{L}(1+\delta+i2\gamma_0, \pi \times \overline{\pi})\right) \le \frac{\delta}{\delta^2+4\gamma_0^2} + c_1 \log(\mathfrak{q}(\pi)^{d^2}(|1+\delta+i2\gamma_0|+3)^{d^2}).$$

With these, we have

$$0 \le \Re\left(-3\frac{L'}{L}(1+\delta, \pi \times \overline{\pi}) - 4\frac{L'}{L}(1+\delta+i\gamma_0, \pi \times \overline{\pi})\right.$$

$$-\frac{L'}{L}(1+\delta+i2\gamma_0,\pi\times\overline{\pi})\Big)\leq\frac{3}{\delta}+\frac{5\delta}{\delta^2+\gamma_0^2}-\frac{4}{1+\delta-\beta_0}+c_2d^2\log\left(\mathfrak{q}(\pi)(|\gamma_0|+5)\right)$$

If $\beta_0=1$ then necessarily $\gamma_0\neq 0$ and as $\delta\to 0^+$ we obtain a contradiction. Suppose that $\beta_0<1$, set $\delta=6(1-\beta_0)$ and suppose $|\gamma_0|\geq 20(1-\beta_0)$. Then

$$0\leq\frac{1}{1-\beta_0}\left(\frac{1}{2}+\frac{30}{436}-\frac{4}{7}\right)+c_2\log\left(\mathfrak{q}(\pi)^{d^2}(|\gamma_0|+5)^{d^2}\right)$$

$$=-\frac{2}{763(1-\beta_0)}+c_2\log\left(\mathfrak{q}(\pi)^{d^2}(|\gamma_0|+5)^{d^2}\right).$$

Therefore

$$1-\beta_0\geq\frac{2}{763c_2\log\left(\mathfrak{q}(\pi)^{d^2}(|\gamma_0|+5)^{d^2}\right)}.$$

Now suppose $0<|\gamma_0|\leq 20(1-\beta_0)$

$$0\leq-\Re\left(\frac{L'}{L}(1+\delta,\pi\times\overline{\pi})\right)\leq\frac{1}{\delta}-\Re\left(\frac{1}{1+\delta-(\beta_0+i\gamma_0)}\right)$$
$$-Re\left(\frac{1}{1+\delta-(\beta_0-i\gamma_0)}\right)+c_1\log\left(\mathfrak{q}(\pi)^{d^2}(|\gamma_0|+5)^{d^2}\right)$$

$$\leq\frac{1}{\delta}-\frac{2(1+\delta-\beta_0)}{(1+\delta-\beta_0)^2+400(1-\beta_0)^2}+c_1\log\left(\mathfrak{q}(\pi)^{d^2}(|\gamma_0|+5)^{d^2}\right).$$

Let $\delta=21(1-\beta_0)$

$$0\leq\frac{1}{1-\beta_0}\left(\frac{1}{21}-\frac{44}{884}\right)+c_1\log\left(\mathfrak{q}(\pi)^{d^2}(|\gamma_0|+5)^{d^2}\right)$$

$$=-\frac{10}{4641(1-\beta_0)}+c_1\log\left(\mathfrak{q}(\pi)^{d^2}(|\gamma_0|+5)^{d^2}\right).$$

Therefore

$$1-\beta_0\geq\frac{10}{4641c_1\log\left(\mathfrak{q}(\pi)^{d^2}(|\gamma_0|+5)^{d^2}\right)}.$$

To finish the argument, suppose $\gamma_0=0$ and $\beta_0\leq\beta_1<1$ are two real zeros then

$$0 \leq -\Re\left(\frac{L'}{L}(1+\delta, \pi \times \overline{\pi})\right) \leq \frac{1}{\delta} - \frac{1}{1+\delta-\beta_0} - \frac{1}{1+\delta-\beta_1} + c_1 \log\left(\mathfrak{q}(\pi)^{d^2} 5^{d^2}\right)$$

$$\leq \frac{1}{\delta} - \frac{2}{1+\delta-\beta_0} + c_1 \log\left(\mathfrak{q}(\pi)^{d^2} 5^{d^2}\right).$$

Again, by setting $\delta = 2(1-\beta_0)$ we can deduce

$$0 \leq \frac{1}{1-\beta_0}\left(\frac{1}{2} - \frac{2}{3}\right) + c_1 \log\left(\mathfrak{q}(\pi)^{d^2} 5^{d^2}\right) = -\frac{1}{6(1-\beta_0)} + c_1 \log\left(\mathfrak{q}(\pi)^{d^2} 5^{d^2}\right),$$

$$\beta_0 \leq 1 - \frac{c}{\log\left(\mathfrak{q}(\pi)^{d^2} 5^{d^2}\right)}.$$

Thus there is at most one real zero in the region $\sigma \geq 1 - \frac{c}{\log\left(\mathfrak{q}(\pi)^{d^2} 5^{d^2}\right)}$. □

2.2 Growth of L-functions Near the 1-line

From Theorem 2.1.1 we can deduce the following bound

Lemma 2.2.1. *Given a primitive automorphic L-function $L(s, \pi)$, let $T \geq 1$, $\frac{T}{2} \leq t \leq \frac{5T}{2}$, $x \geq 2$ and $1 - \frac{c}{2d^4 \log(\mathfrak{q}(\pi)(3T+3))} \leq \sigma \leq 1 + \frac{1}{\log(x)}$. Then in this region we have*

$$\frac{L'}{L}(\sigma + it, \pi) \ll \frac{d^5}{c}(\log(\mathfrak{q}(\pi)(3T+3)))^2.$$

Proof. Recall that $\Re(\mu_\pi(j)) > -1/2$ for all $1 \leq j \leq d$ and note that, in our region of interest, we avoid contribution from a possible exceptional zero. Thus using Proposition 5.7 of [IK04] we have

$$-\frac{L'}{L}(s, \pi) = \frac{1}{s} + \frac{1}{s-1} - \sum_{|s+\mu_\pi(j))| \leq 1} \frac{1}{s+\mu_\pi(j)} - \sum_{|s-\rho|<1} \frac{1}{s-\rho} + O(\log(\mathfrak{q}(s, \pi)))$$

$$\ll d + \sum_{|s-\rho|<1} \frac{1}{|s-\rho|} + c_d \log\left(\mathfrak{q}(\pi)(|t|+4)\right).$$

Again, by Propostion 5.7 of [IK04], the sum over nontrivial zeros has at most $\log(\mathfrak{q}(it,\pi)) \ll d\log(\mathfrak{q}(\pi)(|t|+3))$ terms. By our assumption, each summand is bounded by

$$|\sigma-\rho|^{-1} \ll \left(1-\frac{c}{2d^4(\log(\mathfrak{q}(\pi)(3T+3))}-\left(1-\frac{c}{d^4(\log(\mathfrak{q}(\pi)(|t|+3))}\right)\right)^{-1}$$

$$=\frac{2d^4}{c}\frac{\log(\mathfrak{q}(\pi)(|t|+3))\log(\mathfrak{q}(\pi)(3T+3))}{2\log(\mathfrak{q}(\pi)(3T+3))-\log(\mathfrak{q}(\pi)(|t|+3))}$$

$$=\frac{2d^4}{c}\log(\mathfrak{q}(\pi)(|t|+3))\left(\frac{1}{2-\frac{\log(\mathfrak{q}(\pi)(|t|+3))}{\log(\mathfrak{q}(\pi)(3T+3))}}\right) \ll \frac{2d^4}{c}\log(\mathfrak{q}(\pi)(|t|+3)).$$

□

Lemma 2.2.2. *Given a Rankin-Selberg L-function* $L(s,\pi\times\overline{\pi})$, $s=\sigma+it$, *let* $T\geq 1$, $-T\leq t\leq T$, $x\geq 2$ *and* $1-\frac{c}{2d^2(\log(\mathfrak{q}(\pi)(T+5))}\leq\sigma\leq 1+\frac{1}{\log(x)}$. *Then in this region we have*

$$\frac{L'}{L}(s,\pi\times\overline{\pi}) \ll \frac{1}{|s-1|}+\frac{1}{|s-\beta|}+\sum_{|s+\mu_{\pi\times\overline{\pi}}(i,j)|<1}\frac{1}{|s+\mu_{\pi\times\overline{\pi}}(i,j)|} + \frac{d^4}{c}\left(\log(\mathfrak{q}(\pi)(|t|+5))\right)^2, \quad (2.2)$$

where β *is an exceptional real zero located* $1-\frac{c}{d^2\log(\mathfrak{q}(\pi)5)}\leq\beta<1$.

Proof. By Proposition 5.7 of [IK04] we have

$$-\frac{L'}{L}(s,\pi\times\overline{\pi}) = \frac{1}{s}+\frac{1}{s-1}-\sum_{|s+\mu_{\pi\times\overline{\pi}}(i,j)|\leq 1}\frac{1}{s+\mu_{\pi\times\overline{\pi}}(i,j)}-\sum_{|s-\rho|<1}\frac{1}{s-\rho} + O(\log(\mathfrak{q}(s,\pi\times\overline{\pi})))$$

$$\ll \frac{1}{|s-1|} + \sum_{|s+\mu_{\pi\times\overline{\pi}}(i,j)|\le 1} \frac{1}{|s+\mu_{\pi\times\overline{\pi}}(i,j)|} + \frac{1}{|s-\beta_1|} + \sum_{|s-\rho|<1} \frac{1}{|s-\rho|} + c_d \log\left(\mathfrak{q}(\pi)(|t|+4)\right).$$

Also, by Propostion 5.7 of [IK04], the sum over nontrivial zeros has at most $\log(\mathfrak{q}(it, \pi\times\overline{\pi})) \ll d^2 \log\left(\mathfrak{q}(\pi)(|t|+3)\right)$ terms. By our assumption, each summand is bounded by

$$|\sigma-\rho|^{-1} \ll \left(1 - \frac{c}{2d^2(\log\left(\mathfrak{q}(\pi)(T+5)\right)} - \left(1 - \frac{c}{d^2(\log\left(\mathfrak{q}(\pi)(|t|+5)\right)}\right)\right)^{-1}$$

$$= \frac{2d^2}{c} \frac{\log\left(\mathfrak{q}(\pi)(|t|+5)\right)\log\left(\mathfrak{q}(\pi)(T+5)\right)}{2\log\left(\mathfrak{q}(\pi)(T+5)\right) - \log\left(\mathfrak{q}(\pi)(|t|+5)\right)}$$

$$= \frac{2d^2}{c}\log\left(\mathfrak{q}(\pi)(|t|+5)\right)\left(\frac{1}{2-\frac{\log(\mathfrak{q}(\pi)(|t|+5))}{\log(\mathfrak{q}(\pi)(T+5))}}\right) \ll \frac{2d^2}{c}\log\left(\mathfrak{q}(\pi)(|t|+5)\right).$$

This completes the proof. □

Lemma 2.2.3. *Given a primitive automorphic L-function $L(s,\pi)$, let $T \geq 1$, $\frac{T}{2} \leq t \leq \frac{5T}{2}$, $x \geq 2$ and $1 - \frac{c}{2d^4 \log(\mathfrak{q}(\pi)(3T+3))} \leq \sigma \leq 1 + \frac{1}{\log(x)}$, where $c > 0$. Then in this region we have*

$$\log\left(L(s,\pi)\right) \ll_\pi \log(\mathfrak{q}(\pi)(3T+3)).$$

Proof. By Proposition 5.7 in [IK04] the following is valid for any $w = u + iv$ in the strip $-1/2 \leq u \leq 2$

$$-\frac{L'}{L}(w,\pi) = \frac{1}{w} + \frac{1}{w-1} - \sum_{|w+\mu_\pi(j))|\le 1} \frac{1}{w+\mu_\pi(j)} - \sum_{|w-\rho|<1} \frac{1}{w-\rho} + O(\log(\mathfrak{q}(w,\pi))).$$

Let $s = \sigma + it$ be as in our hypothesis. If we let the path of integration to be a line segment joining s and $2+it$ then

$$\log\left(L(s,\pi)\right) = \int_s^{2+it} -\frac{L'}{L}\left(w,\pi\right)dw + \log\left(L\left(2+it,\pi\right)\right)$$

$$\ll \int_{\sigma}^{2} \frac{d}{t} + \sum_{|u+it-\rho|<1} \frac{1}{|u+it-\rho|} + \log(\mathfrak{q}(u+it,\pi))du + \sum_{n\geq 2} \frac{|\Lambda_\pi(n)|}{\log(n)n^2}$$

$$\ll_\pi \log(\mathfrak{q}(\pi)(3T+3)). \tag{2.3}$$

The last inequality follows the fact that by Theorem 2.1.1 the sum over the zeros is empty.

□

2.3 Prime Number Theorem

Theorem 2.3.1. *Let $x \geq 3$. Then there exist constants $\epsilon, c_\pi > 0$ such that*

$$\sum_{n\leq x} \Lambda(n)|a_\pi(n)|^2(x-n) = \frac{x^2}{2} - \frac{x^{\beta+1}}{\beta(\beta+1)} + \sum_{\mu_{\pi\times\overline{\pi}}(i,j)}{}' \frac{x^{-\mu_{\pi\times\overline{\pi}}(i,j)+1}}{\mu_{\pi\times\overline{\pi}}(i,j)(1-\mu_{\pi\times\overline{\pi}}(i,j))} + O_\pi\left(x^2\exp\left(-c_\pi\sqrt{\log(x)}\right)\right).$$

Here $\sum'_{\mu_{\pi\times\overline{\pi}}(i,j)}$ is a sum with restriction that only $\mu_{\pi\times\overline{\pi}}(i,j)$ such that $1-\epsilon < \Re(\mu_{\pi\times\overline{\pi}}(i,j)) < 1$ appear.

Proof. With $\sigma_0 = 1 + \frac{1}{\log(x)}$, we have

$$\sum_{n\leq x} \Lambda(n)|a_\pi(n)|^2(x-n) = \frac{-1}{2\pi i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty} \frac{L'}{L}(w,\pi\times\overline{\pi})\frac{x^{w+1}}{w(w+1)}dw.$$

Let $2 \leq T \leq x$ be a free parameter at our disposal. Choose $T_1 \in [T, T+1]$ such that

$$|T_1 + \Im(\mu_{\pi\times\overline{\pi}}(i,j))| \gg \frac{1}{d^2}.$$

Such a T_1 exists because there are at most d^2 parameters $\mu_{\pi\times\overline{\pi}}(i,j)$. Since $\Re(\mu_{\pi\times\overline{\pi}}(i,j)) > -1$, by Lemma 2.2.2 we may estimate the infinite part of our integral in the following way

$$\frac{-1}{2\pi i}\int_{\sigma_0\pm iT_1}^{\sigma_0\pm i\infty}\frac{L'}{L}\left(w,\pi\times\overline{\pi}\right)\frac{x^{w+1}}{w(w+1)}dw$$

$$\ll_\pi \int_{\pm T_1}^{\pm\infty}\left(\log(x)+(\log(\mathfrak{q}(\pi)(|t|+5)))^2\right)\frac{x^{\sigma_0+1}}{t^2}dt \ll_\pi \frac{x^2(\log(x))^2}{T}.$$

The strategy now is to move the path of integration into the critical strip. Because of the unknown nature of local parameters at infinity $\mu_{\pi\times\overline{\pi}}(i,j)$, bounding the integral in the critical strip is a little bit technical. First of all we know that $\Re(\mu_{\pi\times\overline{\pi}}(i,j)) > -1$. Thus by looking at equation (2.2) there may be some poles coming from the factor at infinity for $\sigma < 1$. Consider the following interval

$$\mathcal{A} := \left[1-\frac{c}{2d^2\log(\mathfrak{q}(\pi)(T+5))}, 1-\frac{c}{4d^2\log(\mathfrak{q}(\pi)(T+5))}\right].$$

There is a possibility that some $\Re(\mu_{\pi\times\overline{\pi}}(i,j))$, and an exceptional zero β lie in the above interval. However, there are at most d^2 parameters $\mu_{\pi\times\overline{\pi}}(i,j)$ and at most one exceptional zero. Therefore there exist $\sigma_1\in\mathcal{A}$ such that for all i,j we have

$$|\sigma_1+\Re\left(\mu_{\pi\times\overline{\pi}}(i,j)\right)| \gg \frac{c}{4d^2(d^2+1)\log(\mathfrak{q}(\pi)(T+5))}$$

and

$$|\sigma_1-\beta| \gg \frac{c}{4d^2(d^2+1)\log(\mathfrak{q}(\pi)(T+5))}.$$

Now consider the following contour $\mathscr{C}$ with vertices σ_0-iT_1, σ_0+iT_1, σ_1+iT_1, σ_1-iT_1. The contour contains a simple pole at w=1 and possible simple poles coming from $\mu_{\pi\times\overline{\pi}}(i,j)$ and the exceptional zero β. Therefore

$$\frac{-1}{2\pi i}\int_{\mathscr{C}}\frac{L'}{L}\left(w,\pi\times\overline{\pi}\right)\frac{x^{w+1}}{w(w+1)}dw$$

$$=\frac{x^2}{2}-\frac{x^{\beta+1}}{\beta(\beta+1)}-\sideset{}{'}\sum_{\mu_{\pi\times\overline{\pi}}(i,j)}\frac{x^{-\mu_{\pi\times\overline{\pi}}(i,j)+1}}{-\mu_{\pi\times\overline{\pi}}(i,j)(1-\mu_{\pi\times\overline{\pi}}(i,j))}.$$

The horizontal paths contribute at most

$$\frac{-1}{2\pi i}\int_{\sigma_1\pm iT_1}^{\sigma_0\pm iT_1}\frac{L'}{L}\left(w,\pi\times\overline{\pi}\right)\frac{x^{w+1}}{w(w+1)}dw \ll_\pi \int_{\sigma_1}^{\sigma_0}\frac{\log(\mathfrak{q}(\pi)(T+5))^2x^{\sigma+1}}{|(\sigma\pm iT_1)(1+\sigma\pm iT_1)|}d\sigma$$

$$\ll_\pi \frac{\log(\mathfrak{q}(\pi)(T+5))^2 x^2}{T^2}.$$

By our choice of σ_1 and T_1 and the bound established in Lemma 2.2.2, the vertical path inside the critical strip can be bounded in the following way;

$$\frac{-1}{2\pi}\int_{\sigma_1-iT}^{\sigma_1+iT}\frac{L'}{L}(w,\pi\times\overline{\pi})\frac{x^{w+1}}{w(w+1)}dw$$

$$=\frac{-1}{2\pi}\int_{-T_1}^{T_1}\frac{L'}{L}(\sigma_1+it,\pi\times\overline{\pi})\frac{x^{\sigma_1+it+1}}{(\sigma_1+it)(\sigma_1+it+1)}dt$$

$$\ll_\pi \int_0^{T_1}\left|\frac{L'}{L}(\sigma_1+it,\pi\times\overline{\pi})\right|\frac{x^{\sigma_1+1}}{|\sigma_1+it||\sigma_1+it+1|}dt$$

$$\ll_\pi \int_0^{T_1}\left(\frac{1}{|\sigma_1-1+it|}+\frac{1}{|\sigma_1-\beta+it|}+\log(\mathfrak{q}(\pi)(T+5))\right.$$
$$\left.+(\log(\mathfrak{q}(\pi)(T+5)))^2\right)\frac{x^{\sigma_1+1}}{(\sigma_1)^2+t^2}dt$$

$$\ll_\pi \int_0^{T_1}\left((\log(\mathfrak{q}(\pi)(T+5)))^2\right)\frac{x^{\sigma_1+1}}{(\sigma_1)^2+t^2}dt \ll_\pi x^{\sigma_1+1}\left(\log\left(\mathfrak{q}(\pi)(T+5)\right)\right)^2.$$

Putting all the bounds together and choosing $T=\exp\left(\sqrt{c_\pi^{'}\log(x)}\right)$, where $c_\pi^{'}>0$ we get

$$\sum_{n\le x}\Lambda(n)|a_\pi(n)|^2(x-n)=\frac{x^2}{2}-\frac{x^{\beta+1}}{\beta(\beta+1)}+\sum_{\mu_{\pi\times\overline{\pi}}(i,j)}{}^{'}\frac{x^{-\mu_{\pi\times\overline{\pi}}(i,j)+1}}{\mu_{\pi\times\overline{\pi}}(i,j)(1-\mu_{\pi\times\overline{\pi}}(i,j))}$$
$$+O_\pi\left(x^2\left(\frac{(\log(x))^2}{T}+\frac{(\log(\mathfrak{q}(\pi)(T+5)))^2}{T^2}+x^{-\frac{c}{4d^2\log(\mathfrak{q}(\pi)(T+5))}}\right)\right)$$

$$=\frac{x^2}{2}-\frac{x^{\beta+1}}{\beta(\beta+1)}+\sum_{\mu_{\pi\times\overline{\pi}}(i,j)}{}^{'}\frac{x^{-\mu_{\pi\times\overline{\pi}}(i,j)+1}}{\mu_{\pi\times\overline{\pi}}(i,j)(1-\mu_{\pi\times\overline{\pi}}(i,j))}+O_\pi\left(x^2\exp\left(-c_\pi\sqrt{\log(x)}\right)\right).$$

∎

In order to unsmooth our sum, or remove the weight $x-n$ from the summand in the theorem above, we need a technical Lemma.

Lemma 2.3.1. *Let $x \geq 3$, $y = x\exp(-\eta\sqrt{\log(x)})$ for $\eta > 0$, and α a fixed complex number. Then*

$$\frac{x^{\alpha+1}}{\alpha+1} - \frac{(x-y)^{\alpha+1}}{\alpha+1} = x^{\alpha}\left(y + O_{\alpha}\left(y\exp\left(-\eta\sqrt{\log(x)}\right)\right)\right).$$

Proof.

$$\frac{x^{\alpha+1}}{\alpha+1} - \frac{(x-y)^{\alpha+1}}{\alpha+1} = \int_{x-y}^{x} u^{\alpha} du = \int_{0}^{y} (x-y+u)^{\alpha}\, du =$$

$$= x^{\alpha}\int_{0}^{y}\left(1-\left(\frac{y}{x}-\frac{u}{x}\right)\right)^{\alpha} du = x^{\alpha}\left(\int_{0}^{y} 1 + O_{\alpha}\left(\frac{y}{x}+\frac{u}{x}\right) du\right) =$$

$$= x^{\alpha}\left(y + O_{\alpha}\left(y\exp\left(-\eta\sqrt{\log(x)}\right)\right)\right).$$

∎

Theorem 2.3.2. *Let $x \geq 3$. There exist a constant $c_{\pi} > 0$ such that*

$$\psi_{\pi\times\overline{\pi}}(x) = \sum_{n\leq x}\Lambda(n)|a_{\pi}(n)|^2 = x - \frac{x^{\beta}}{\beta} + \sideset{}{'}\sum_{\mu_{\pi\times\overline{\pi}}(i,j)} \frac{x^{-\mu_{\pi\times\overline{\pi}}(i,j)}}{\mu_{\pi\times\overline{\pi}}(i,j)} + O_{\pi}\left(x\exp\left(-c_{\pi}\sqrt{\log(x)}\right)\right).$$

Proof. For convenience define the following function

$$S(x) := \sum_{n\leq x}\Lambda(n)|a_{\pi}(n)|^2(x-n).$$

By Theorem 2.3.1 there exists $c_1 > 0$ such that

$$S(x) = \frac{x^2}{2} - \frac{x^{\beta+1}}{\beta(\beta+1)} + \sideset{}{'}\sum_{\mu_{\pi\times\overline{\pi}}(i,j)} \frac{x^{-\mu_{\pi\times\overline{\pi}}(i,j)+1}}{\mu_{\pi\times\overline{\pi}}(i,j)(1-\mu_{\pi\times\overline{\pi}}(i,j))} + O_{\pi}\left(x^2\exp\left(-c_1\sqrt{\log(x)}\right)\right).$$

Let $y = x\exp(-\eta\sqrt{\log(x)})$. Then by monotonicity

$$\frac{S(x)-S(x-y)}{y} = \frac{1}{y}\int_{x-y}^{x}\psi_{\pi\times\overline{\pi}}(t)dt \le \psi_{\pi\times\overline{\pi}}(x)$$

$$\le \frac{1}{y}\int_{x}^{x+y}\psi_{\pi\times\overline{\pi}}(t)dt = \frac{S(x+y)-S(x)}{y}.$$

We may apply Lemma 2.3.1 to see that

$$S(x)-S(x-y) = \frac{x^2}{2} - \frac{(x-y)^2}{2} - \frac{x^{\beta+1}}{\beta(\beta+1)} + \frac{(x-y)^{\beta+1}}{\beta(\beta+1)}$$
$$+ \sum_{\mu_{\pi\times\overline{\pi}}(i,j)}{}^{'} \left(\frac{x^{-\mu_{\pi\times\overline{\pi}}(i,j)+1}}{\mu_{\pi\times\overline{\pi}}(i,j)(1-\mu_{\pi\times\overline{\pi}}(i,j))} - \frac{(x-y)^{-\mu_{\pi\times\overline{\pi}}(i,j)+1}}{\mu_{\pi\times\overline{\pi}}(i,j)(1-\mu_{\pi\times\overline{\pi}}(i,j))}\right)$$
$$+ O_\pi\left(x^2\exp\left(-c_2\sqrt{\log(x)}\right)\right)$$

$$= xy + \frac{y^2}{2} - \frac{x^\beta}{\beta}\left(y + O_\beta\left(y\exp\left(-\eta_1\sqrt{\log(x)}\right)\right)\right)$$
$$+ \sum_{\mu_{\pi\times\overline{\pi}}(i,j)}{}^{'} \frac{x^{-\mu_{\pi\times\overline{\pi}}(i,j)}}{\mu_{\pi\times\overline{\pi}}(i,j)}\left(y + O_\pi\left(y\exp\left(-\eta_2\sqrt{\log(x)}\right)\right)\right)$$
$$+ O_\pi\left(x^2\exp\left(-c_2\sqrt{\log(x)}\right)\right)$$

$$= xy + \frac{y^2}{2} - \frac{x^\beta y}{\beta} + \sum_{\mu_{\pi\times\overline{\pi}}(i,j)}{}^{'} \frac{x^{-\mu_{\pi\times\overline{\pi}}(i,j)}y}{\mu_{\pi\times\overline{\pi}}(i,j)} + O_\pi\left(x^2\exp\left(-c_3\sqrt{\log(x)}\right)\right).$$

Therefore

$$\frac{S(x+y)-S(x)}{y} = x + \frac{y}{2} - \frac{x^\beta}{\beta} + \sum_{\mu_{\pi\times\overline{\pi}}(i,j)}{}^{'} \frac{x^{-\mu_{\pi\times\overline{\pi}}(i,j)}}{\mu_{\pi\times\overline{\pi}}(i,j)} + O_\pi\left(x\exp\left(-c_4\sqrt{\log(x)}\right)\right).$$

By computations similar to the above one shows that

$$\frac{S(x+y)-S(x)}{y} = x + \frac{y}{2} - \frac{x^\beta}{\beta} + \sum_{\mu_{\pi\times\overline{\pi}}(i,j)}{}^{'} \frac{x^{-\mu_{\pi\times\overline{\pi}}(i,j)}}{\mu_{\pi\times\overline{\pi}}(i,j)} + O_\pi\left(x\exp\left(-c_5\sqrt{\log(x)}\right)\right).$$

This gives the desired result

$$\psi_{\pi\times\overline{\pi}}(x) = x - \frac{x^\beta}{\beta} + \sum_{\mu_{\pi\times\overline{\pi}}(i,j)}{}^{'} \frac{x^{-\mu_{\pi\times\overline{\pi}}(i,j)}}{\mu_{\pi\times\overline{\pi}}(i,j)} + O_\pi\left(x\exp\left(-c_6\sqrt{\log(x)}\right)\right).$$

□

Remark: This version of the prime number theorem for the Rankin-Selberg convolution of automorphic forms involves extra terms coming from potential exceptional zero, or poles of the gamma factor lying close to the 1-line. However, for the purposes of understanding the value distribution of $\frac{L'}{L}(1+it,\pi)$, or $\frac{L'}{L}(1+it,\pi\times\overline{\pi})$, these terms don't play a major role. Since $\beta, -\Re\left(\mu_{\pi\times\overline{\pi}}(i,j)\right) < 1$ any contribution coming from those terms will be absorbed by an error term.

Chapter 3 Value Distribution of $\frac{L'}{L}(1+it,\pi)$

3.1 Approximation of $\frac{L'}{L}(1+it,\pi)$

Hypothesis H 1. *[RS96] For any fixed $k \geq 2$,*

$$\sum_{p} \frac{(\log(p))^2 |a_\pi(p^k)|^2}{p^k} < \infty.$$

Where $a_\pi(p^k)$ is defined as in equation (1.3)

Lemma 3.1.1. *Let $x = \exp((\log(T))^2)$ and $T \geq 1$. Suppose that $T \leq t \leq 2T$. Then there is a positive constant c_π such that*

$$-\frac{L'}{L}(1+it,\pi) = \sum_{n\leq x} \frac{\Lambda_\pi(n)}{n^{1+it}} + O_\pi\left(\frac{1}{T^{c_\pi}}\right).$$

Proof. Let $W = \frac{T}{2}$ and $\theta = \frac{1}{\log(x)}$. By Theorem 5.3 and Corollary 5.3 of Montgomery and Vaughan [MV07] (where $4^{\sigma_0} + x^{\sigma_0}$ should be $4^{\sigma_0}x^{\sigma_0}$) we have

$$\sum_{n\leq x} \frac{\Lambda_\pi(n)}{n^{1+it}} = \frac{1}{2\pi i}\int_{\theta-iW}^{\theta+iW} -\frac{L'}{L}(1+it+w,\pi)\frac{x^w}{w}dw + R,$$

where

$$R \ll \sum_{\substack{x/2<n<2x \\ n\neq x}} \frac{|\Lambda_\pi(n)|}{n} \min\left(1, \frac{x}{W|x-n|}\right) + \frac{(4x)^{\frac{1}{\log(x)}}}{W} \sum_{n=1}^{\infty} \frac{|\Lambda_\pi(n)|}{n^{1+\frac{1}{\log(x)}}}.$$

By Theorem 2.1.1 and Lemma 2.2.1 there exists a positive constant c such that whenever $\frac{T}{2} \leq t \leq \frac{5T}{2}$ and

$$1 - \frac{c}{2d^4 \log(\mathfrak{q}(\pi)(3T+3))} \leq \sigma \leq 1 + \frac{1}{\log(x)},$$

we have $L(\sigma+it,\pi) \neq 0$ and

$$\frac{L'}{L}(\sigma+it,\pi) \ll d^5(\log(\mathfrak{q}(\pi)(3T+3)))^2.$$

Let

$$\phi = -\frac{c}{2d^4 \log(\mathfrak{q}(\pi)(3T+3))}.$$

Application of the Cauchy residue theorem to the rectangle with corners at $\theta - iW$, $\theta + iW$, $\phi + iW$, $\phi - iW$, which we will call $\mathscr{C}$, yields the following;

$$\frac{1}{2\pi i}\int_{\mathscr{C}} -\frac{L'}{L}(1+it+w,\pi)\frac{x^w}{w}dw = -\frac{L'}{L}(1+it,\pi).$$

The horizontal paths of the above integral contribute

$$\int_{\phi\pm iW}^{\theta\pm iW} -\frac{L'}{L}(1+it+w,\pi)\frac{x^w}{w}dw \ll \int_{\phi}^{\theta} d^5(\log(\mathfrak{q}(\pi)(3T+3)))^2 \frac{x^\sigma}{|\sigma \pm iW|}d\sigma$$

$$\ll_\pi \frac{(\log(\mathfrak{q}(\pi)(3T+3)))^2}{W}\int_{\phi}^{\theta} x^\sigma d\sigma \ll_\pi \frac{(\log(\mathfrak{q}(\pi)(3T+3)))^2}{T},$$

and the vertical line $\Re w = \phi$ contributes

$$\int_{\phi-iW}^{\phi+iW} -\frac{L'}{L}(1+it+w,\pi)\frac{x^w}{w}dw \ll d^5(\log(\mathfrak{q}(\pi)(3T+3)))^2 x^\phi \int_0^W \frac{1}{|\phi+iv|}dv$$

$$\ll_\pi (\log(\mathfrak{q}(\pi)(3T+3)))^2 x^\phi \left(1+\log\left(\frac{W}{|\phi|}\right)\right) \ll_\pi (\log(\mathfrak{q}(\pi)(3T+3)))^3 \exp(\phi \log(x))$$

$$\ll (\log(\mathfrak{q}(\pi)(3T+3)))^3 \exp\left(-c'\left(\frac{(\log(T))^2}{2d^4\log(\mathfrak{q}(f)(3T+3))}\right)\right) \ll_\pi \frac{1}{T^{c_\pi}},$$

where c_π is some positive constant that depends on π. Now we proceed to work out an appropriate bound for the error term R arising form Perron's formula.We deal with the finite sum first.

$$\sum_{\substack{x/2<n<2x \\ n\neq x}} \frac{|\Lambda_\pi(n)|}{n}\min\left(1,\frac{x}{W|x-n|}\right) = \frac{x}{W}\sum_{\frac{x}{2}<n<x\left(1-\frac{1}{W}\right)} \frac{|\Lambda_\pi(n)|}{n(x-n)}$$

$$+\sum_{x\left(1-\frac{1}{W}\right)<n\leq x\left(1+\frac{1}{W}\right)} \frac{|\Lambda_\pi(n)|}{n} + \frac{x}{W}\sum_{x\left(1+\frac{1}{W}\right)<n<2x} \frac{|\Lambda_\pi(n)|}{n(n-x)} = A_1(x)+A_2(x)+A_3(x).$$

By the Cauchy-Schwartz inequality together with Theorem 2.3.2, we have

$$A_1(x) \ll \frac{x}{W}\left(\sum_{\frac{x}{2}<n<x\left(1-\frac{1}{W}\right)} |\Lambda_\pi(n)|^2\right)^{\frac{1}{2}}\left(\sum_{\frac{x}{2}<n<x\left(1-\frac{1}{W}\right)} \frac{1}{n^2(x-n)^2}\right)^{\frac{1}{2}}$$

$$\ll \frac{x^{\frac{3}{2}}\log(x)^{\frac{1}{2}}}{W}\left(\int_{\frac{x}{2}}^{x\left(1-\frac{1}{W}\right)} \frac{1}{u^2(x-u)^2}du\right)^{\frac{1}{2}} = \frac{\log(x)^{\frac{1}{2}}}{W}\left(\int_{\frac{1}{2}}^{1-\frac{1}{W}} \frac{1}{u^2(1-u)^2}du\right)^{\frac{1}{2}}$$

$$\ll \frac{\log(x)^{\frac{1}{2}}}{W}\left(\int_{\frac{1}{2}}^{1-\frac{1}{W}} \frac{1}{(1-u)^2}\right)^{\frac{1}{2}} \ll \left(\frac{\log(x)}{W}\right)^{\frac{1}{2}} \ll \frac{1}{T^{c_\pi}}.$$

The term $A_3(x)$ can be treated in the same way as $A_1(x)$. The treatment yields the same bound. Before we work with $A_2(x)$, note that

$$\sum_{n\leq x}\frac{1}{n^2}=\zeta(2)-\frac{1}{x}+O\left(\frac{1}{x^2}\right).$$

Together with the Cauchy-Schwartz inequality and Theorem 2.3.2, we have

$$A_2(x)\ll\left(\sum_{x\left(1-\frac{1}{W}\right)<n<x\left(1+\frac{1}{W}\right)}|\Lambda_\pi(n)|^2\right)^{\frac{1}{2}}\left(\sum_{x\left(1-\frac{1}{W}\right)<n<x\left(1+\frac{1}{W}\right)}\frac{1}{n^2}\right)^{\frac{1}{2}}$$

$$\ll\left(\log(x)\sum_{x\left(1-\frac{1}{W}\right)<n<x\left(1+\frac{1}{W}\right)}\Lambda(n)|a_\pi(n)|^2\right)^{\frac{1}{2}}\left(\frac{1}{x}\left(\frac{1}{1-\frac{1}{W}}-\frac{1}{1+\frac{1}{W}}\right)+\frac{1}{x^2}\right)^{\frac{1}{2}}$$

$$\ll_\pi\left(\log(x)\left(\frac{x}{W}+x\exp\left(-c\sqrt{\log(x)}\right)\right)\right)^{\frac{1}{2}}\left(\frac{1}{xW}+\frac{1}{x^2}\right)^{\frac{1}{2}}$$

$$\ll_\pi\left((\log(T))^2\left(\frac{1}{W^2}+\frac{1}{WT^c}\right)\right)^{\frac{1}{2}}\ll_\pi\frac{1}{T^{c_\pi}}.$$

Lastly, to finish bounding the error term R, we deal with the Dirichlet series

$$\frac{(4x)^{\frac{1}{\log(x)}}}{W}\sum_{n=1}^{\infty}\frac{|\Lambda_f(n)|}{n^{1+\frac{1}{\log(x)}}}\ll\frac{(4x)^{\frac{1}{\log(x)}}}{W}\left(\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{1+\frac{1}{\log(x)}}}\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty}\frac{\Lambda(n)\,|a_\pi(n)|^2}{n^{1+\frac{1}{\log(x)}}}\right)^{\frac{1}{2}}$$

$$\ll\frac{1}{T}\left(\frac{\zeta'}{\zeta}\left(1+\frac{1}{\log(x)}\right)\right)^{\frac{1}{2}}\left(\frac{L'}{L}\left(1+\frac{1}{\log(x)},\pi\times\overline{\pi}\right)\right)^{\frac{1}{2}}\ll\frac{\log(x)}{T}\ll\frac{(\log(T))^2}{T}.$$

The last bound follows from the fact that both $-\frac{\zeta'}{\zeta}(\sigma)$ and $-\frac{L'}{L}(\sigma,\pi\times\overline{\pi})$ are $\sim\frac{1}{\sigma-1}$ as $\sigma\to1^+$. Putting all bounds together we obtain our result.

□

The following lemma lies at the heart of the uniformity of the distribution.

Lemma 3.1.2. *Let $\Lambda_\pi(n)$ be defined as in (1.3) and assume Hypothesis H1. Then*

$$\sum_{y<n} \frac{|\Lambda_\pi(n)|^2}{n^2} \sim \frac{\log(y)}{y}, \tag{3.1}$$

$$\sum_{\substack{y<n \\ n\neq p}} \frac{|\Lambda_\pi(n)|^2}{n^2} \ll \frac{1}{y}. \tag{3.2}$$

Proof. We apply Theorem 2.3.2

$$\sum_{y<n} \frac{\log(n)\Lambda(n)|a_\pi(n)|^2}{n^2} = \int_y^\infty \frac{\log(u)}{u^2} d\psi_{\pi\times\overline{\pi}}(u)$$

$$= \frac{\log(u)\psi_{\pi\times\overline{\pi}}(u)}{u^2}\Bigg|_y^\infty + \int_y^\infty \psi_{\pi\times\overline{\pi}}(u)\left(\frac{2\log(u)-1}{u^3}\right)du$$

$$= -\frac{\log(y)}{y} + O\left(\frac{\log(y)}{y}\exp\left(-c\sqrt{\log(y)}\right)\right) + \int_y^\infty \frac{2\log(u)-1}{u^2}du$$
$$+ O\left(\int_y^\infty \frac{\exp\left(-c\sqrt{\log(u)}\right)\log(u)}{u^2}du\right)$$

$$= \frac{\log(y)}{y} + \frac{1}{y} + O\left(\frac{\log(y)}{y}\exp\left(-c\sqrt{\log(y)}\right)\right).$$

Hence

$$\sum_{y<n} \frac{\log(n)\Lambda(n)|a_\pi(n)|^2}{n^2} = \frac{\log(y)}{y} + o\left(\frac{\log(y)}{y}\right). \tag{3.3}$$

As mentioned in [RS96] (equation 2.26 on p 281) if we use $|\alpha_\pi(j,p)| \leq p^{\frac{1}{2}-\frac{1}{d^2+1}}$ and assume Hypothesis H1, then

$$B(u) = \sum_{n\leq x} \frac{\log(n)\Lambda(n)|a_\pi(n)|^2 - |\Lambda_\pi(n)|^2}{n} \ll 1.$$

With this we immediately see

$$\sum_{y<n} \frac{\log(n)\Lambda(n)|a_\pi(n)|^2 - |\Lambda_\pi(n)|^2}{n^2} = \int_y^\infty \frac{1}{u} dB(u) = \frac{B(u)}{u}\bigg|_y^\infty + \int_y^\infty \frac{B(u)}{u^2} du \ll \frac{1}{y}. \tag{3.4}$$

Combining equations (3.3) and (3.4) we establish (3.1). From $|\alpha_\pi(j,p)| \leq p^{\frac{1}{2}-\frac{1}{d^2+1}}$ and Hypothesis H1 one can also deduce

$$\sum_{n \neq p} \frac{|\Lambda_\pi(n)|^2}{n} \ll 1.$$

Thus, the second statement follows from crude bound

$$\sum_{\substack{y<n \\ n\neq p}} \frac{|\Lambda_\pi(n)|^2}{n^2} \ll \frac{1}{y} \sum_{\substack{y<n \\ n\neq p}} \frac{|\Lambda_\pi(n)|^2}{n} \ll \frac{1}{y}.$$

□

3.2 Mean Values of Dirichlet Polynomials

In this section we prove a Lemma which is key in obtaing all the moments of $\frac{L'}{L}(1+it,\pi)$. First we state a crucial Lemma which can be found in Tsang's thesis [Tsa84]. This Lemma is a multidimensional version of Corollary 3 found in Montgomery and Vaughan's work [MV74].

Lemma 3.2.1. *(Tsang) Let $\{a_n\}$ and $\{b_n\}$ be two sequences of complex numbers. For any $T,\ H \in \mathbb{R}$ and any $m,\ k \geq 0$, we have*

$$\int_T^{T+H} \left(\sum_n a_n n^{-it}\right)^m \left(\overline{\sum_n b_n n^{-it}}\right)^k dt$$

$$= H \sum_n A_n \overline{B_n} + O\left(\left(\sum_n n\,|A_n|^2\right)^{\frac{1}{2}} \left(\sum_n n\,|B_n|^2\right)^{\frac{1}{2}}\right)$$

where

$$A_n = \sum_{n_1 \ldots n_m = n} a_{n_1} \ldots a_{n_m} \quad \textit{and} \quad B_n = \sum_{n_1 \ldots n_k = n} b_{n_1} \ldots b_{n_k} \quad \textit{where} \quad n = 1, 2, 3, \ldots$$

Lemma 3.2.2. *Assume Hypothesis H1, let* $T \geq 0$ *and* $2 < y < \frac{x}{\log(x)}$*, then*

$$\int_T^{2T} \left(\sum_{y<n\leq x} \frac{\Lambda_\pi(n)}{n^{1+it}} \right)^k \left(\sum_{y<n\leq x} \frac{\overline{\Lambda_\pi(n)}}{n^{1-it}} \right)^l dt$$

$$= \begin{cases} Tk! \left(\frac{\log(y)}{y}\right)^k + o\left(Tk!\left(\frac{\log(y)}{y}\right)^k\right) + O_k\left(\log(x)^{2k}\right), & \text{if } k = l \\ O_{k,l}\left(T\frac{\log(y)^{\frac{k+l}{2}-1}}{y^{\frac{k+l}{2}}} + \log(x)^{k+l}\right) & \text{if } k \neq l. \end{cases}$$

Proof. To simplify presentation of the proof we define

$$A\left(n_1, \ldots, n_k\right) := \frac{\Lambda_\pi\left(n_1\right) \ldots \Lambda_\pi\left(n_k\right)}{n_1 \ldots n_k}$$

By Lemma 3.2.1 we have

$$\int_T^{2T} \left(\sum_{y<n\leq x} \frac{\Lambda_\pi(n)}{n^{1+it}} \right)^k \left(\sum_{y<n\leq x} \frac{\overline{\Lambda_\pi(n)}}{n^{1-it}} \right)^l dt =$$

$$= T \sum_n \left(\sum_{\substack{y<n_1\leq x, \ldots, y<n_k\leq x \\ n_1\ldots n_k = n}} A\left(n_1, \ldots, n_k\right) \right) \left(\sum_{\substack{y<n_1\leq x, \ldots, y<n_l\leq x \\ m_1\ldots m_l = n}} \overline{A\left(m_1, \ldots, m_l\right)} \right)$$

$$+O\left(\left(\sum_n n \left| \sum_{\substack{n_1\ldots n_k = n \\ y<n_i\leq x}} A\left(n_1, \ldots, n_k\right) \right|^2 \right)^{\frac{1}{2}} \left(\sum_n n \left| \sum_{\substack{n_1\ldots n_l = n \\ y<n_i\leq x}} \overline{A\left(m_1, \ldots, m_l\right)} \right|^2 \right)^{\frac{1}{2}} \right).$$

Lemma 3.9 on page 298 of [RS96] states that given $1 \leq k \leq l$

$$\sum_{\substack{n_1\ldots n_k = m_1 \ldots m_l \\ n_i, m_j \leq x}} \frac{\Lambda_\pi(n_1)\ldots\Lambda_\pi(n_k)\overline{\Lambda_\pi(m_1)}\ldots\overline{\Lambda_\pi(m_l)}}{n_1\ldots n_k} = \begin{cases} O(\log(x)^{2k}) & \text{if } k = l \\ O(\log(x)^{2k-2}) & \text{if } k < l. \end{cases}$$

Sums in the error term can be bounded in the following way

$$\sum_n n \left| \sum_{\substack{y<n_1\leq x, \ldots, y<n_k\leq x \\ n_1\ldots n_k = n}} A\left(n_1, \ldots, n_k\right) \right|^2$$

$$= \sum_{\substack{n_1 \dots n_k = m_1 \dots m_k \\ y < n_i, m_j \le x}} \frac{\Lambda_\pi(n_1) \dots \Lambda_\pi(n_k) \overline{\Lambda_\pi(m_1)} \dots \overline{\Lambda_\pi(m_k)}}{n_1 \dots n_k} \ll \log(x)^{2k} + \log(y)^{2k}.$$

Thus we have

$$\int_T^{2T} \left(\sum_{y < n \le x} \frac{\Lambda_\pi(n)}{n^{1+it}} \right)^k \left(\sum_{y < n \le x} \frac{\overline{\Lambda_\pi(n)}}{n^{1-it}} \right)^l dt$$

$$= T \sum_n \left(\sum_{\substack{n_1 \dots n_k = n \\ y < n_i \le x}} A(n_1, \dots, n_k) \right) \left(\sum_{\substack{m_1 \dots m_l = n \\ y < m_i \le x}} \overline{A(m_1, \dots, m_l)} \right) + O\left(\log(x)^{k+l}\right).$$

In dealing with the main term there are two cases that we need to consider. First we look at the case when $k = l$

$$T \sum_n \left(\sum_{\substack{n_1 \dots n_k = n \\ y < n_i \le x}} A(n_1, \dots, n_k) \right) \left(\sum_{\substack{m_1 \dots m_k = n \\ y < m_i \le x}} \overline{A(m_1, \dots, m_l)} \right).$$

Recall that $\Lambda_\pi(n) = \Lambda(n) a_\pi(n)$ thus the sum above is supported on prime powers. We split the above into two sums according to n being square free, and non-square free. First, if n is square free then by the constraints put on the sums we see that each $n_i = p_i$ is a distinct prime. Therefore above sum restricted to square free n is

$$T \sum_{\substack{n \\ squarefree}} \sum_{\substack{p_1 \dots p_k = n \\ y < p_i \le x \\ p_i \ne p_j}} \sum_{\substack{q_1 \dots q_k = n \\ y < q_i \le x \\ q_i \ne q_j}} A(p_1, \dots, p_k) \overline{A(q_1, \dots, q_k)}$$

We note that $p_1 \dots p_k = q_1 \dots q_k$ and there are $k!$ ways to permute $q_i's$. Now we add terms where primes are repeated and the above becomes

$$= k! T \sum_{y < p_1 \le x} \dots \sum_{\substack{y < p_k \le x \\ p_i \ne p_j}} |A(p_1, \dots, p_k)|^2 = Tk! \sum_{y < p_1 \le x} \dots \sum_{y < p_k \le x} |A(p_1, \dots, p_k)|^2$$

$$+ O_k\left(T \sum_{y<p_1\le x} \dots \sum_{\substack{y<p_k\le x \\ p_i=p_j \\ \text{for some } i\neq j}} |A(p_1,\dots,p_k)|^2\right) = Tk!\left(\sum_{y<p\le x}\frac{|\Lambda_\pi(p)|^2}{p^2}\right)^k$$

$$+ O_k\left(T\sum_{y<p_1\le x}\dots\sum_{y<p_{k-2}\le x}|A(p_1,\dots,p_{k-2})|^2\sum_{y<p\le x}\frac{|\Lambda_\pi(p)|^4}{p^4}\right)$$

$$= Tk!\left(\sum_{y<p}\frac{|\Lambda_\pi(p)|^2}{p^2} + O\left(\sum_{x<p}\frac{|\Lambda_\pi(p)|^2}{p^2}\right)\right)^k$$

$$+ O_k\left(T\left(\sum_{y<p\le x}\frac{|\Lambda_\pi(p)|^2}{p^2}\right)^{k-2}\left(\sum_{y<p\le x}\frac{|\Lambda_\pi(p)|^2\log(p)^2 p^{1-\frac{2}{d^2+1}}}{p^4}\right)\right)$$

$$= Tk!\left(\sum_{y<p}\frac{|\Lambda_\pi(p)|^2}{p^2} + O_k\left(\frac{\log(x)}{x}\right)\right)^k + O_k\left(T\left(\frac{\log(y)}{y}\right)^{k-2}\frac{\log(y)}{y^2}\right)$$

$$= Tk!\left(\sum_{y<p}\frac{|\Lambda_\pi(p)|^2}{p^2}\right)^k + O_k\left(T\left(\frac{\log(y)}{y}\right)^{k-1}\frac{\log(x)}{x} + T\frac{\log(y)^{k-1}}{y^k}\right)$$

$$= Tk!\left(\frac{\log(y)}{y}\right)^k + o\left(Tk!\left(\frac{\log(y)}{y}\right)^k\right) + O_k\left(T\frac{\log(y)^{k-1}}{y^k}\right).$$

To justify last step we used the assumption that $y < \frac{x}{\log(x)}$. Now we consider sum involving terms when n is not square free. The method is essentially the same as the method used to establish Lemma 3.9 of [RS96]. As in the proof of Lemma 3.9, suppose $n = r_1^{\gamma_1}\dots r_h^{\gamma_h}$ has h distinct prime factors where $1 \le h \le k$. Then

$$T\sum_{\substack{n \\ \text{not square} \\ \text{free}}}\sum_{\substack{p_1^{\alpha_1}\dots p_k^{\alpha_k}=n \\ y<p_i^{\alpha_i}\le x}}\sum_{\substack{q_1^{\beta_1}\dots q_k^{\beta_k}=n \\ y<q_i^{\beta_i}\le x}} A\left(p_1^{\alpha_1},\dots,p_k^{\alpha_k}\right)\overline{A\left(q_1^{\beta_1},\dots,q_k^{\beta_k}\right)}$$

$$= T \sum_{h=1}^{k-1} \sum_{\substack{y^k < n \le x^k \\ \text{not square} \\ \text{free,} \\ \text{n has h distinct} \\ \text{prime factors}}} \sum_{\substack{p_1^{\alpha_1} \ldots p_k^{\alpha_k} = n \\ y < p_i^{\alpha_i} \le x \\ \exists i \neq j \;\; s.t. \;\; p_i = p_j}} \sum_{\substack{q_1^{\beta_1} \ldots q_k^{\beta_k} = n \\ y < q_i^{\beta_i} \le x}} A\left(p_1^{\alpha_1}, \ldots, p_k^{\alpha_k}\right) \overline{A\left(q_1^{\beta_1}, \ldots, q_k^{\beta_k}\right)}$$

$$+ \sum_{\substack{y^k < n \le x^k \\ \text{not square} \\ \text{free,} \\ \text{n has k distinct} \\ \text{prime factors}}} \sum_{\substack{p_1^{\alpha_1} \ldots p_k^{\alpha_k} = n \\ y < p_i^{\alpha_i} \le x}} \sum_{\substack{q_1^{\beta_1} \ldots q_k^{\beta_k} = n \\ y < q_i^{\beta_i} \le x}} A\left(p_1^{\alpha_1}, \ldots, p_k^{\alpha_k}\right) \overline{A\left(q_1^{\beta_1}, \ldots, q_k^{\beta_k}\right)} = I + II.$$

To bound II note that since n is not square free, there exists $1 \le i \le k$ such that $\alpha_i \ge 2$. After rearrangement, II is bounded by

$$\ll_k T \sum_{\substack{n \text{ not} \\ \text{square} \\ \text{free}}} \sum_{\substack{p_1^{\alpha_1} \ldots p_k^{\alpha_k} = n \\ y < p_i^{\alpha_i} \le x \\ p_i' s \text{ distinct} \\ \alpha_1 \ge 2}} \left|A\left(p_1^{\alpha_1}, \ldots, p_k^{\alpha_k}\right)\right|^2 \ll_k T \sum_{\substack{y < p_1^{\alpha_1} \le x \\ \alpha_1 \ge 2}} \cdots \sum_{y < p_k^{\alpha_k} \le x} \left|A\left(p_1^{\alpha_1}, \ldots, p_k^{\alpha_k}\right)\right|^2$$

$$\ll_k T \left(\sum_{\substack{y < n \le x \\ n \neq p}} \frac{|\Lambda_\pi(n)|^2}{n^2} \right) \left(\sum_{y < n \le x} \frac{|\Lambda_\pi(n)|^2}{n^2} \right)^{k-1} \ll_k T \frac{\log(y)^{k-1}}{y^k}.$$

We will make use of Lemma 3.8 of [RS96] which, in our notation, states

Lemma 3.2.3. *Assume $k + l \ge 3$ and $\alpha_i, \beta_j \ge 1$. Then*

$$\sum_p \sum_{\alpha_1 + \ldots + \alpha_k = \beta_1 + \ldots + \beta_l} \left| \frac{\Lambda_\pi(p^{\alpha_1}) \ldots \Lambda_\pi(p^{\alpha_k}) \overline{\Lambda_\pi(p^{\beta_1})} \ldots \overline{\Lambda_\pi(p^{\beta_l})}}{p^{\alpha_1 + \ldots + \alpha_k}} \right|$$

$$= \sum_p \sum_{\max(k,l) \le m} \frac{L_k(p,m) L_l(p,m)}{p^m} < \infty,$$

where

$$L_k(p,m) := \sum_{\alpha_1+\ldots+\alpha_k=m} |\Lambda_\pi(p^{\alpha_1})\ldots\Lambda_\pi(p^{\alpha_k})|.$$

By the virtue of the restrictions on the sum I, let $n = r_1^{\gamma_1}\ldots r_h^{\gamma_h}$ be the prime factorization of n. Further restrictions in the inner sum require that

$$p_1^{\alpha_1}\ldots p_k^{\alpha_k} = r_1^{\gamma_1}\ldots r_h^{\gamma_h} = q_1^{\beta_1}\ldots q_k^{\beta_k}.$$

By collection exponents which belong to the same prime factor we may express each exponent γ_g in the following way $\gamma_g = \sum_{i=1}^{m_g}\alpha_{i,g} = \sum_{i=1}^{n_g}\beta_{i,g}$.. Where $m_g, n_g \geq 1$ and $\sum_{g=1}^{h} m_g = \sum_{g=1}^{h} n_g = k$. Now I can be rewritten as follows

$$I = T\sum_{h=1}^{k-1}\prod_{g=1}^{h}\sum_{r_g}\sum_{\sum_{i\leq m_g}\alpha_{i,g}=\sum_{j\leq n_g}\beta_{i,g}} A\left(r_g^{\alpha_{1,g}},\ldots,r_g^{\alpha_{m_g,g}}\right)\overline{A\left(r_g^{\beta_{1,g}},\ldots,r_g^{\beta_{n_g,g}}\right)}$$

$$= T\sum_{h=2}^{k-1}\prod_{g=1}^{h}\sum_{r_g}\sum_{\max(m_g,n_g)\leq\gamma_g}\frac{L_{m_g}(r_g,\gamma_g)L_{n_g}(r_g,\gamma_g)}{r_g^{2\gamma_g}}.$$

Note that $n^{-1} = \prod_{g=1}^{h} r_g^{-\gamma_g} < \frac{1}{y^k}$. In order to apply Lemma 3.2.3 we split the above sum according to size of $m_g + n_g$ i.e. above is

$$= T\sum_{h=2}^{k-1}\prod_{g=1}^{h}\sum_{r_g}\sum_{\substack{\max(m_g,n_g)\leq\gamma_g \\ \exists g \text{ s.t. } m_g+n_g\ <3}}\frac{L_{m_g}(r_g,\gamma_g)L_{n_g}(r_g,\gamma_g)}{r_g^{2\gamma_g}}+$$

$$+T\sum_{h=2}^{k-1}\prod_{g=1}^{h}\sum_{r_g}\sum_{\substack{\max(m_g,n_g)\leq\gamma_g \\ m_g+n_g\ \geq 3}}\frac{L_{m_g}(r_g,\gamma_g)L_{n_g}(r_g,\gamma_g)}{r_g^{2\gamma_g}} = III + IV.$$

First, we can deal with above defined sum IV

$$IV \ll T\sum_{h=2}^{k-1}\prod_{g=1}^{h}\sum_{r_g}\sum_{\substack{\max(m_g,n_g)\leq\gamma_g \\ m_g+n_g\ \geq 3}}\frac{L_{m_g}(r_g,\gamma_g)L_{n_g}(r_g,\gamma_g)}{y^k r_g^{\gamma_g}} \ll_k \frac{T}{y^k}.$$

Before working with III we make the following observation. Since for all g we have $m_g, n_g \geq 1$ we see that whenever $m_g + n_g\ < 3$ we actually have $m_g = n_g = 1$

(note if $h=1$ then necessarily $m_g+n_g \geq 3$). Now suppose the number of $g's$ such that $m_g+n_g \; < 3$ is j and note that $1 \leq j < h$ (the case $j=h$ is impossible because than $\sum m_g = h \leq k-1$). After rearrangement we can bound III by

$$III \ll_k T \sum_{h=2}^{k-1} \prod_{g=1}^{h} \sum_{r_g} \sum_{\substack{\max(m_g,n_g)\leq\gamma_g \\ \exists g \text{ s.t. } m_g+n_g \; <3}} \frac{L_{m_g}(r_g,\gamma_g)L_{n_g}(r_g,\gamma_g)}{r_g^{2\gamma_g}}$$

$$= T \sum_{h=2}^{k-1} \sum_{j=1}^{h-1} \prod_{f=1}^{j} \sum_{r_f} \sum_{\substack{\max(m_f,n_f)\leq\gamma_f \\ m_f+n_f \; <3}} \frac{L_{m_f}(r_f,\gamma_f)L_{n_f}(r_f,\gamma_f)}{r_f^{2\gamma_f}}$$

$$\times \prod_{g=j+1}^{h} \sum_{r_g} \sum_{\substack{\max(m_g,n_g)\leq\gamma_g \\ m_g+n_g \; \geq 3}} \frac{L_{m_g}(r_g,\gamma_g)L_{n_g}(r_g,\gamma_g)}{r_g^{2\gamma_g}}$$

$$\ll_k T \sum_{h=2}^{k-1} \sum_{j=1}^{h-1} \prod_{f=1}^{j} \sum_{r_f} \sum_{1\leq\gamma_f} \frac{|\Lambda_\pi(r_f^{\gamma_f})|^2}{r_f^{2\gamma_f}} \prod_{g=j+1}^{h} \sum_{r_g} \sum_{\substack{\max(m_g,n_g)\leq\gamma_g \\ m_g+n_g \; \geq 3}} \frac{L_{m_g}(r_g,\gamma_g)L_{n_g}(r_g,\gamma_g)}{y^{k-j} r_g^{\gamma_g}}$$

$$\ll_k T \sum_{h=2}^{k-1} \sum_{j=1}^{h-1} \prod_{f=1}^{j} \left(\sum_{y<n_f\leq x} \frac{|\Lambda_\pi(n_f)|^2}{n_f^2} \right) \frac{1}{y^{k-j}} \ll_k T \sum_{h=2}^{k-1} \sum_{j=1}^{h-1} \left(\frac{\log(y)}{y} \right)^j \frac{1}{y^{k-j}}$$

$$\ll_k T \frac{(\log(y))^{k-2}}{y^k}.$$

This proves that

$$\int_T^{2T} \left| \sum_{y<n\leq x} \frac{\Lambda_\pi(n)}{n^{1+it}} \right|^{2k} dt = Tk! \left(\frac{\log(y)}{y} \right)^k + o\left(Tk! \left(\frac{\log(y)}{y} \right)^k \right)$$

$$+ O_k\left(T \frac{\log(y)^{k-1}}{y^k} \right) + O\left(\log(x)^{2k} \right).$$

To deal with case $k \neq l$ we first note that the case where n is square free is impossible by unique factorization of n and the fact that $\Lambda_\pi(n)$ is supported on prime powers, so we are left with n non square free. Also, the above argument shows

$$\sum_{\substack{n \\ \text{not square} \\ \text{free}}} \left| \sum_{\substack{p_1^{\alpha_1}\ldots p_k^{\alpha_k}=n \\ y<p_i^{\alpha_i}\leq x}} A\left(p_1^{\alpha_1},\ldots,p_k^{\alpha_k}\right) \right|^2 \ll \frac{\log(y)^{k-1}}{y^k}.$$

With this wee see

$$\int_T^{2T} \left(\sum_{y<n\leq x} \frac{\Lambda_\pi(n)}{n^{1+it}} \right)^k \left(\sum_{y<n\leq x} \frac{\overline{\Lambda_\pi(n)}}{n^{1-it}} \right)^l dt$$

$$= T \sum_{\substack{n \\ \text{not square} \\ \text{free}}} \sum_{\substack{p_1^{\alpha_1}\ldots p_k^{\alpha_k}=n \\ y<p_i^{\alpha_i}\leq x}} \sum_{\substack{q_1^{\beta_1}\ldots q_l^{\beta_l}=n \\ y<q_i^{\beta_i}\leq x}} A\left(p_1^{\alpha_1},\ldots,p_k^{\alpha_k}\right) \overline{A\left(q_1^{\beta_1},\ldots,q_l^{\beta_l}\right)} + O\left(\log(x)^{k+l}\right)$$

$$\ll T \left(\sum_{\substack{n \\ \text{not square} \\ \text{free}}} \left| \sum_{\substack{p_1^{\alpha_1}\ldots p_k^{\alpha_k}=n \\ y<p_i^{\alpha_i}\leq x}} A\left(p_1^{\alpha_1},\ldots,p_k^{\alpha_k}\right) \right|^2 \right)^{\frac{1}{2}}$$

$$\times \left(\sum_{\substack{n \\ \text{not square} \\ \text{free}}} \left| \sum_{\substack{q_1^{\beta_1}\ldots q_l^{\beta_l}=n \\ y<q_i^{\beta_i}\leq x}} \overline{A\left(q_1^{\beta_1},\ldots,q_l^{\beta_l}\right)} \right|^2 \right)^{\frac{1}{2}} + \log(x)^{k+l}$$

$$\ll T \frac{\log(y)^{\frac{k+l}{2}-1}}{y^{\frac{k+l}{2}}} + \log(x)^{k+l}.$$

□

3.3 Value Distribution of $\frac{L'}{L}(1+it,\pi)$

Theorem 3.3.1. *Assume Hypothesis H1, let $T \geq 1$, $k,l \in \mathbb{N}$. There exists $c_\pi > 0$ such that if $2 < y = y(T) < \frac{\exp\left((\log(T))^2\right)}{(\log(T))^2}$ and $y(T) \to \infty$ as $T \to \infty$, then*

$$\int_T^{2T} \left(-\frac{L'}{L}(1+it,\pi) - \sum_{n \leq y} \frac{\Lambda_\pi(n)}{n^{1+it}} \right)^k \left(-\frac{L'}{L}(1-it,\overline{\pi}) - \sum_{n \leq y} \frac{\overline{\Lambda_\pi(n)}}{n^{1-it}} \right)^l dt$$

$$= \begin{cases} Tk!\left(\frac{\log(y)}{y}\right)^k + o\left(Tk!\left(\frac{\log(y)}{y}\right)^k\right) + O_{k,\pi}\left(T^{1-c_\pi}\right), & \text{if } k = l \\ O_{k,l,\pi}\left(T\frac{\log(y)^{\frac{k+l}{2}-1}}{y^{\frac{k+l}{2}}} + T^{1-c_\pi}\right) & \text{if } k \neq l. \end{cases}$$

Proof. By Lemma 3.1.1 we have

$$-\frac{L'}{L}(1+it,\pi) = \sum_{n \leq x} \frac{\Lambda_\pi(n)}{n^{1+it}} + O_\pi\left(\frac{1}{T^{c_\pi}}\right),$$

for some $c_\pi > 0$. Now we may proceed with the computation

$$\int_T^{2T} \left(-\frac{L'}{L}(1+it,\pi) - \sum_{n \leq y} \frac{\Lambda_\pi(n)}{n^{1+it}} \right)^k \left(-\frac{L'}{L}(1-it,\overline{\pi}) - \sum_{n \leq y} \frac{\overline{\Lambda_\pi(n)}}{n^{1-it}} \right)^l dt$$

$$= \int_T^{2T} \left(\sum_{y<n \leq x} \frac{\Lambda_\pi(n)}{n^{1+it}} + O_\pi\left(\frac{1}{T^{c_\pi}}\right) \right)^k \left(\sum_{y<n \leq x} \frac{\overline{\Lambda_\pi(n)}}{n^{1-it}} + O_\pi\left(\frac{1}{T^{c_\pi}}\right) \right)^l dt$$

$$= \int_T^{2T} \left(\sum_{y<n \leq x} \frac{\Lambda_\pi(n)}{n^{1+it}} \right)^k \left(\sum_{y<n \leq x} \frac{\overline{\Lambda_\pi(n)}}{n^{1-it}} \right)^l dt$$
$$+ O_{k,l,\pi}\left(\sum_{\substack{a=0 \\ a+b\neq 0}}^{k} \sum_{b=0}^{l} \int_T^{2T} \left| \sum_{y<n \leq x} \frac{\Lambda_\pi(n)}{n^{1+it}} \right|^{k+l-a-b} \frac{1}{T^{(a+b)c_\pi}})dt \right)$$

$$= \int_T^{2T} \left(\sum_{y<n\le x} \frac{\Lambda_\pi(n)}{n^{1+it}} \right)^k \left(\sum_{y<n\le x} \frac{\overline{\Lambda_\pi(n)}}{n^{1-it}} \right)^l dt + O_{\pi,k,l}\left(T^{1-c_\pi}\right)$$

$$= \begin{cases} Tk! \left(\frac{\log(y)}{y}\right)^k + o\left(Tk!\left(\frac{\log(y)}{y}\right)^k\right) + O_{k,\pi}\left(T^{1-c_\pi} + (\log(T))^{4k}\right), & \text{if } k = l \\ O_{k,l,\pi}\left(T\frac{\log(y)^{\frac{k+l}{2}-1}}{y^{\frac{k+l}{2}}} + T^{1-c_\pi} + (\log(T))^{2(k+l)}\right) & \text{if } k \neq l. \end{cases}$$

The last equality follows from Lemma 3.2.2. □

Equipped with Theorem 3.3.1 we are ready to prove our main result.

Theorem 3.3.2. *Assume Hypothesis H1, let $T \ge 1$, $k \in \mathbb{N}$. $y = y(T)$ be such that $y(T) \to \infty$ as $T \to \infty$ and $2 < y < T^\epsilon$, for all $\epsilon > 0$. Define functions $f(t)$, $I_k(T)$ and $J_k(T)$ in the following way*

$$f(t) := \frac{-\frac{L'}{L}(1+it,\pi) - \sum_{n\le y} \frac{\Lambda_\pi(n)}{n^{1+it}}}{\sqrt{\frac{\log(y)}{2y}}},$$

$$I_k(T) := \frac{1}{T}\int_T^{2T} \left(\Re\left(f(t)\right)\right)^k dt, \qquad J_k(T) := \frac{1}{T}\int_T^{2T} \left(\Im\left(f(t)\right)\right)^k dt.$$

Then

$$\lim_{T\to\infty} I_{2k-1}(T) = \lim_{T\to\infty} J_{2k-1}(T) = 0, \qquad \lim_{T\to\infty} I_{2k}(T) = \lim_{T\to\infty} J_{2k}(T) = \frac{(2k)!}{2^k k!}.$$

Proof.

$$I_{2k-1}(T) = \frac{1}{T}\int_T^{2T} \left(\Re\left(f(t)\right)\right)^{2k-1} dt = \frac{1}{T}\int_T^{2T} \left(\frac{f(t)+\overline{f(t)}}{2}\right)^{2k-1} dt$$

$$= \frac{1}{T}\sum_{j=0}^{2k-1} \binom{2k-1}{j} \frac{1}{2^{2k-1}} \int_T^{2T} f(t)^j \overline{f(t)}^{(2k-1)-j} dt = O_{k,\pi}\left(\frac{1}{\log(y)}\right).$$

Now for the imaginary part of our function of interest

$$J_{2k-1}(T) = \frac{1}{T}\int_T^{2T} (\Im(f(t)))^{2k-1}\, dt = \frac{1}{T}\int_T^{2T} \left(\frac{f(t)-\overline{f(t)}}{2i}\right)^{2k-1} dt$$

$$= \frac{1}{T}\sum_{j=0}^{2k-1}\binom{2k-1}{j}\frac{1}{(2i)^{2k-1}}\int_T^{2T} (f(t))^j \left(-\overline{f(t)}\right)^{(2k-1)-j} dt = O_{k,\pi}\left(\frac{1}{\log(y)}\right).$$

Taking the limit as $T \to \infty$, we obtain the first part of our result. Now for the even powers, we will see that the main term comes from the middle term in the binomial expansion

$$I_{2k}(T) = \frac{1}{T}\int_T^{2T} (\Re(f(t)))^{2k}\, dt = \frac{1}{T}\int_T^{2T} \left(\frac{f(t)+\overline{f(t)}}{2}\right)^{2k} dt$$

$$= \frac{1}{T}\binom{2k}{k}\frac{1}{2^{2k}}\int_T^{2T} |f(t)|^{2k}\, dt + \frac{1}{T}\sum_{\substack{j=0\\ j\neq k}}^{2k}\binom{2k}{j}\frac{1}{2^{2k}}\int_T^{2T} f(t)^j\overline{f(t)}^{2k-j} dt$$

$$= \frac{(2k)!}{2^k k!} + o\left(\frac{(2k)!}{2^k k!}\right) + O_{k,\pi}\left(\frac{1}{\log(y)}\right).$$

Lastly, even moments of the imaginary part of $f(t)$ are

$$J_{2k}(T) = \frac{1}{T}\int_T^{2T} (\Im(f(t)))^{2k}\, dt = \frac{1}{T}\int_T^{2T} \left(\frac{f(t)-\overline{f(t)}}{2i}\right)^{2k} dt$$

$$= \frac{1}{T}\binom{2k}{k}\frac{(-1)^k}{(2i)^{2k}}\int_T^{2T} |f(t)|^{2k}\, dt + \frac{1}{T}\sum_{\substack{j=0\\ j\neq k}}^{2k}\binom{2k}{j}\frac{1}{2^{2k}}\int_T^{2T} (f(t))^j\left(-\overline{f(t)}\right)^{2k-j} dt$$

$$= \frac{(2k)!}{2^k k!} + o\left(\frac{(2k)!}{2^k k!}\right) + O_{k,\pi}\left(\frac{1}{\log(y)}\right).$$

Again, taking the limit as $T \to \infty$, we obtain the second part of our result.

□

Theorem 3.3.3. *Assume Hypothesis H1, let* $T \geq 1$, $k \in \mathbb{N}$. $y = y(T)$ *be such that* $y(T) \to \infty$ *as* $T \to \infty$ *and* $2 < y < T^{\epsilon}$, *for all* $\epsilon > 0$, *then*

$$\lim_{T\to\infty} \frac{1}{T} meas \left\{ t \in [T, 2T] | \Re \left(f(t) \right) \leq c \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{c} e^{-\frac{x^2}{2}} dx,$$

$$\lim_{T\to\infty} \frac{1}{T} meas \left\{ t \in [T, 2T] | \Im \left(f(t) \right) \leq c \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{c} e^{-\frac{x^2}{2}} dx.$$

Proof. The theorem follows by combining Theorem 3.3.2 and Theorem 1.5.2. □

Chapter 4
Value Distribution of $\frac{L'}{L}(1+it,\pi\times\overline{\pi})$

4.1 Dirichlet Polynomial Approximation of $\frac{L'}{L}(1+it,\pi\times\overline{\pi})$

In order to obtain results analogous to the ones in Chapter 3, we need to make additional assumptions. Hypothesis H2 is analogous to the Hypothesis H1 in the sense that wee need to control growth of $|\Lambda_{\pi\times\overline{\pi}}(p^k)|^2 = (\log(p))^2|a_\pi(p^k)|^4$, where $k \geq 2$. The main term will be asymptotic to $Tk!\left(\sum_{y<p}\frac{|\Lambda_{\pi\times\overline{\pi}}(p)|^2}{p^2}\right)^k$. However, we could not establish the size of that sum unconditionally thus we make an assumption about its growth via Hypothesis H3. Hypothesis H4 is necessary to establish the technical Lemma 4.2.1.

Hypothesis H 2. *For any fixed $k \geq 2$,*

$$\sum_p \frac{(\log(p))^2|a_\pi(p^k)|^4}{p^k} < \infty.$$

Hypothesis H 3. *There exists a positive constant c_0 such that as $y \to \infty$*

$$\sum_{y<p} \frac{(\log(p))^2|a_\pi(p)|^4}{p^2} \sim \frac{c_0\log(y)}{y}.$$

Hypothesis H 4. *For all primes $p < \infty$ and all $1 \leq j \leq d$ there exist $\beta > 0$ such that*

$$|\alpha_\pi(j,p)| \leq p^{\frac{1}{4}-\beta}.$$

Lemma 4.1.1. *Let* $x = \exp\left((\log(T))^2\right)$ *and* $T \geq 1$. *Suppose that* $T \leq t \leq 2T$. *Then there is a positive constant* c_π *such that*

$$-\frac{L'}{L}(1+it, \pi \times \overline{\pi}) = \sum_{n \leq x} \frac{\Lambda_{\pi\times\overline{\pi}}(n)}{n^{1+it}} + O_\pi\left(\frac{1}{T^{c_\pi}}\right),$$

where $\Lambda_{\pi\times\overline{\pi}}(n) = \Lambda(n)|a_\pi(n)|^2$.

Proof. Let $W = \frac{T}{2}$ and $\theta = \frac{1}{\log(x)}$. By Theorem 5.3 and Corollary 5.3 of Montgomery and Vaughan [MV07] (where $4^{\sigma_0} + x^{\sigma_0}$ should be $4^{\sigma_0}x^{\sigma_0}$) we have

$$\sum_{n \leq x} \frac{\Lambda_{\pi\times\overline{\pi}}(n)}{n^{1+it}} = \frac{1}{2\pi i}\int_{\theta - iW}^{\theta+iW} -\frac{L'}{L}(1+it+w, \pi\times\overline{\pi})\frac{x^w}{w}dw + R,$$

where

$$R \ll \sum_{\substack{x/2<n<2x \\ n\neq x}} \frac{\Lambda_{\pi\times\overline{\pi}}(n)}{n}\min\left(1, \frac{x}{W|x-n|}\right) + \frac{(4x)^{\frac{1}{\log(x)}}}{W}\sum_{n=1}^{\infty}\frac{\Lambda_{\pi\times\overline{\pi}}(n)}{n^{1+\frac{1}{\log(x)}}}.$$

By Theorem 2.1.2 and Lemma 2.2.2 there exists a positive constant b_π such that whenever $\frac{T}{2} \leq t \leq \frac{5T}{2}$ and

$$1 - \frac{1}{2b_\pi \log(\mathfrak{q}(\pi)(3T+5))} \leq \sigma \leq 1 + \frac{1}{\log(x)}$$

we have $L(\sigma+it, \pi\times\overline{\pi}) \neq 0$ and

$$\frac{L'}{L}(\sigma+it, \pi\times\overline{\pi}) \ll_\pi (\log(\mathfrak{q}(\pi)(3T+5)))^2.$$

Let

$$\phi = -\frac{1}{2b_\pi\log(\mathfrak{q}(\pi)(3T+5))}.$$

Application of the Cauchy residue theorem to the rectangle with corners at $\theta - iW$, $\theta+iW$, $\phi+iW$, $\phi-iW$, which we will call $\mathscr{C}$, yields the following

$$\frac{1}{2\pi i}\int_{\mathscr{C}} -\frac{L'}{L}(1+it+w,\pi\times\overline{\pi})\frac{x^w}{w}dw = -\frac{L'}{L}(1+it,\pi\times\overline{\pi}).$$

The horizontal paths of the above integral contribute

$$\int_{\phi\pm iW}^{\theta\pm iW} -\frac{L'}{L}(1+it+w,\pi\times\overline{\pi})\frac{x^w}{w}dw \ll \int_{\phi}^{\theta} d^5(\log(\mathfrak{q}(\pi)(3T+3)))^2\frac{x^\sigma}{|\sigma\pm iW|}d\sigma$$

$$\ll_\pi \frac{(\log(\mathfrak{q}(\pi)(3T+3)))^2}{W}\int_{\phi}^{\sigma} x^\sigma d\sigma \ll_\pi \frac{(\log(\mathfrak{q}(\pi)(3T+3)))^2}{T},$$

and the vertical line $\Re w = \phi$ contributes

$$\int_{\phi-iW}^{\phi+iW} -\frac{L'}{L}(1+it+w,\pi\times\overline{\pi})\frac{x^w}{w}dw \ll d^5(\log(\mathfrak{q}(\pi)(3T+3)))^2x^\phi\int_0^W \frac{1}{|\phi+iv|}dv$$

$$\ll_\pi (\log(\mathfrak{q}(\pi)(3T+3)))^2x^\phi\left(1+\log\left(\frac{W}{|\phi|}\right)\right) \ll_\pi (\log(\mathfrak{q}(\pi)(3T+3)))^3\exp(\phi\log(x))$$

$$\ll_\pi (\log(\mathfrak{q}(\pi)(3T+3)))^3\exp\left(-\left(\frac{(\log(T))^2}{2b_\pi\log(\mathfrak{q}(\pi)(3T+5)}\right)\right) \ll_\pi \frac{1}{T^{c_\pi}}.$$

Now we proceed to work out an appropriate bound for the error term R arising form Perron's formula. We deal with the finite sum first.

$$\sum_{\substack{x/2<n<2x\\ n\neq x}}\frac{\Lambda_{\pi\times\overline{\pi}}(n)}{n}\min\left(1,\frac{x}{W|x-n|}\right) = \frac{x}{W}\sum_{\frac{x}{2}<n<x\left(1-\frac{1}{W}\right)}\frac{\Lambda_{\pi\times\overline{\pi}}(n)}{n(x-n)}$$

$$+\sum_{x\left(1-\frac{1}{W}\right)<n\leq x\left(1+\frac{1}{W}\right)}\frac{\Lambda_{\pi\times\overline{\pi}}(n)}{n}+\frac{x}{W}\sum_{x\left(1+\frac{1}{W}\right)<n<2x}\frac{\Lambda_{\pi\times\overline{\pi}}(n)}{n(n-x)} = A_1(x)+A_2(x)+A_3(x).$$

By Theorem 2.3.2 we have $\psi_{\pi\times\overline{\pi}}(u) = u+E(u)$ where $E(u) \ll_\pi u\exp(-a_\pi\sqrt{\log(u)})$

$$A_1(x) = \frac{x}{W}\int_{\frac{x}{2}}^{x\left(1-\frac{1}{W}\right)} \frac{1}{u(x-u)} d\psi_{\pi\times\overline{\pi}}(u)$$

$$= \frac{x}{W}\left(\int_{\frac{x}{2}}^{x\left(1-\frac{1}{W}\right)} \frac{1}{u(x-u)} du + \int_{\frac{x}{2}}^{x\left(1-\frac{1}{W}\right)} \frac{1}{u(x-u)} dE(u)\right)$$

$$= \frac{x}{W}\left(\frac{\log\left(\frac{u}{x-u}\right)}{x}\Bigg|_{\frac{x}{2}}^{x(1-\frac{1}{W})} + \frac{E(u)}{u(x-u)}\Bigg|_{\frac{x}{2}}^{x(1-\frac{1}{W})} - \int_{\frac{x}{2}}^{x\left(1-\frac{1}{W}\right)} \frac{E(u)(2u-x)}{u^2(x-u)^2} du\right)$$

$$\ll_\pi \frac{x}{W}\left(\frac{\log(W)}{x} + \frac{x\exp(-a_\pi\sqrt{\log(x)})}{x(1-\frac{1}{W})\frac{x}{W}} + \exp(-a_\pi\sqrt{\log(x)}) \times \int_{\frac{x}{2}}^{x\left(1-\frac{1}{W}\right)} \frac{(2u-x)}{u(x-u)^2} du\right)$$

$$\ll_\pi \frac{1}{T^{c_\pi}} + \frac{1}{WT^{a_\pi}}\int_{\frac{1}{2}}^{1-\frac{1}{W}} \frac{2u-1}{u(1-u)^2} du \ll_\pi \frac{1}{T^{c_\pi}}.$$

For some positive constant c_π. The term $A_3(x)$ can be treated in the same way as $A_1(x)$, the treatment yields the same bound. Now to bound $A_2(x)$

$$A_2(x) \le \int_{x(1-\frac{1}{W})}^{x(1+\frac{1}{W})} \frac{1}{u} d\psi_{\pi\times\overline{\pi}}(u) = \int_{x(1-\frac{1}{W})}^{x(1+\frac{1}{W})} \frac{1}{u} du + \int_{x(1-\frac{1}{W})}^{x(1+\frac{1}{W})} \frac{1}{u} dE(u)$$

$$= \int_{1-\frac{1}{W}}^{1+\frac{1}{W}} \frac{1}{u} du + \frac{E(u)}{u}\Bigg|_{x(1-\frac{1}{W})}^{x(1+\frac{1}{W})} + \int_{x(1-\frac{1}{W})}^{x(1+\frac{1}{W})} \frac{E(u)}{u^2} du \ll_\pi$$

$$\ll_\pi \frac{1}{W} + \exp\left(-a_\pi\sqrt{\log(x)}\right) + \frac{\exp\left(-a_\pi\sqrt{\log(x)}\right)}{W} \ll_\pi \frac{1}{T^{c_\pi}}.$$

Lastly, to finish bounding the error term R, we deal with the Dirichlet series

$$\frac{(4x)^{\frac{1}{\log(x)}}}{W}\sum_{n=1}^{\infty} \frac{\Lambda_{\pi\times\overline{\pi}}(n)}{n^{1+\frac{1}{\log(x)}}} \ll \frac{\log(x)}{T} \ll \frac{(\log(T))^2}{T} \ll \frac{1}{T^{c_\pi}}.$$

□

Lemma 4.1.2. *Assume Hypothesis H2, and that for all* $1 \le j \le d$, $|\alpha_\pi(j,p)| \le p^{1/4-\beta}$ *for some* $\beta > 0$. *Then*

$$\sum_{\substack{y<n \\ n\neq p}} \frac{(\Lambda_{\pi\times\overline{\pi}}(n))^2}{n^2} \ll \frac{1}{y}.$$

Proof. Choose an integer B such that $4B\beta > 1$

$$\sum_{\substack{y<n \\ n\neq p}} \frac{(\Lambda_{\pi\times\overline{\pi}}(n))^2}{n^2} \le \frac{1}{y}\sum_{\substack{y<n \\ n\neq p}} \frac{(\Lambda_{\pi\times\overline{\pi}}(n))^2}{n} \le \frac{1}{y}\left(\sum_{2\le k\le B}\sum_p \frac{(\Lambda_{\pi\times\overline{\pi}}(p^k))^2}{p^k}\right)$$

$$\le \frac{1}{y}\left(\sum_{2\le k\le B}\sum_p \frac{(\Lambda_{\pi\times\overline{\pi}}(p^k))^2}{p^k} + \sum_{k>B}\sum_p \frac{(\Lambda_{\pi\times\overline{\pi}}(p^k))^2}{p^k}\right)$$

$$\ll \frac{1}{y}\left(1 + \sum_{k>B}\sum_p \frac{(\Lambda(p^k))^2}{p^{4k\beta}}\right) = \frac{1}{y} + \frac{1}{y}\sum_p (\log(p))^2 \left(\frac{1}{1-\frac{1}{p^{4\beta}}} - \frac{1-\left(\frac{1}{p^{4\beta}}\right)^{B+1}}{1-\frac{1}{p^{4\beta}}}\right)$$

$$= \frac{1}{y} + \frac{1}{y}\sum_p (\log(p))^2 \frac{1}{p^{4\beta B}\left(p^{4\beta}-1\right)} \ll \frac{1}{y}.$$

□

4.2 Mean Values of Dirichlet Polynomials

Lemma 4.2.1. *Assume* $k + l \ge 3$, *Hypotheses H2, H3 and H4. Then*

$$\sum_p \sum_{\alpha_1+\ldots+\alpha_k=\beta_1+\ldots+\beta_l} \frac{\Lambda_{\pi\times\overline{\pi}}(p^{\alpha_1})\ldots\Lambda_{\pi\times\overline{\pi}}(p^{\alpha_k})\Lambda_{\pi\times\overline{\pi}}(p^{\beta_1})\ldots\Lambda_{\pi\times\overline{\pi}}(p^{\beta_l})}{p^{\alpha_1+\ldots+\alpha_k}} =$$

$$= \sum_p \sum_{\max(k,l)\le m} \frac{M_m(p,k)M_m(p,l)}{p^m} < \infty.$$

Where

$$M_m(p,k) := \sum_{\alpha_1+\ldots+\alpha_k=m} \Lambda_{\pi\times\overline{\pi}}(p^{\alpha_1})\ldots\Lambda_{\pi\times\overline{\pi}}(p^{\alpha_k}).$$

Proof. The proof follows the same line of logic as the proof of Lemma 3.8 in [RS96]. By Hypothesis H4

$$M_m(p,k) \leq \binom{m-1}{k-1}(\log(p))^k p^{2m(1/4-\beta)}.$$

Let K be such that $4K\beta > 1$. Then the terms where $m \geq K$ can be bounded by

$$\sum_p \sum_{K\leq m} \frac{\binom{m-1}{k-1}\binom{m-1}{l-1}(\log(p))^{k+l}p^{4m(1/4-\beta)}}{p^m} \ll_{k,l} \sum_p \sum_{K\leq m} \frac{m^{k+l-2}(\log(p))^{k+l}}{p^{4m\beta}}$$

$$\ll_{k,l} \sum_p \frac{(\log(p))^{k+l}}{p^{4K\beta}} \sum_{0\leq m} \frac{(m+K)^{k+l-2}}{p^{4m\beta}} \ll_{k,l} \sum_p \frac{(\log(p))^{k+l}}{p^{4K\beta}} \ll_{k,l} 1.$$

Now that our inner sum is finite it suffices to fix $\max(k,l) \leq m_0 < K$ and bound

$$\sum_p \frac{M_{m_0}(p,k)M_{m_0}(p,l)}{p^{m_0}} \leq \left(\sum_p \frac{(M_{m_0}(p,k))^2}{p^{m_0}}\right)^{1/2} \left(\sum_p \frac{(M_{m_0}(p,l))^2}{p^{m_0}}\right)^{1/2}.$$

If $l = 1$ the corresponding sum is finite by Hypothesis H2. Without loss of generality suppose $k \geq 2$ and apply Hypothesis H4 to all but one term, say that corresponding to α_1, in the inner sum

$$\sum_p \frac{(M_{m_0}(p,k))^2}{p^{m_0}}$$

$$\ll \sum_p \frac{1}{p^{m_0}} \left(\sum_{\alpha_1+\ldots+\alpha_k=m_0} \Lambda_{\pi\times\overline{\pi}}(p^{\alpha_1})\log(p)^{k-1}\left(p^{2(\alpha_2+\ldots+\alpha_k)(\frac{1}{4}-\beta)}\right)\right)^2$$

$$\ll \sum_p \left(\sum_{\alpha_1+\ldots+\alpha_k=m_0} \frac{\Lambda_{\pi\times\overline{\pi}}(p^{\alpha_1})}{p^{\frac{\alpha_1}{2}+2\beta'(m_0-\alpha_1)}} \right)^2 \ll_{k,m_0} \sum_p \left(\sum_{\alpha_1\leq m_0-1} \frac{\Lambda_{\pi\times\overline{\pi}}(p^{\alpha_1})}{p^{\frac{\alpha_1}{2}+2\beta'(m_0-\alpha_1)}} \right)^2$$

$$\ll_{k,m_0} \sum_p \sum_{\alpha\leq m_0-1} \frac{(\Lambda_{\pi\times\overline{\pi}}(p^{\alpha}))^2}{p^{\alpha+4\beta'}} \ll_{k,m_0} \sum_p \sum_{\alpha\leq m_0-1} \frac{(\log(p))^2|a_\pi(p^\alpha)|^4}{p^{\alpha+4\beta''}}$$

$$\ll_{k,m_0} \sum_p \frac{(\log(p))^2|a_\pi(p)|^4}{p^{1+4\beta''}} + 1 \ll_{k,m_0} 1.$$

Where the last inequality follows from Hypothesis H3.

□

Lemma 4.2.2. *Assume Hypotheses H2, H3, H4, let* $T \geq 0$ *and suppose* $2 < y < \frac{x}{\log(x)}$. *Then*

$$\int_T^{2T} \left(\sum_{y<n\leq x} \frac{\Lambda_{\pi\times\overline{\pi}}(n)}{n^{1+it}} \right)^k \left(\sum_{y<n\leq x} \frac{\Lambda_{\pi\times\overline{\pi}}(n)}{n^{1-it}} \right)^l dt =$$

$$= \begin{cases} Tk! \left(\frac{c_0\log(y)}{y}\right)^k + o\left(Tk!\left(\frac{c_0\log(y)}{y}\right)^k\right) + O_{k,\pi}\left(\log(x)^{2k}\right), & \textit{if } k = l \\ O_{k,l,\pi}\left(T\frac{\log(y)^{\frac{k+l}{2}-1}}{y^{\frac{k+l}{2}}} + \log(x)^{k+l}\right) & \textit{if } k \neq l. \end{cases}$$

Proof. To simplify the presentation of the proof we define

$$B(n_1,\ldots,n_k) := \frac{\Lambda_{\pi\times\overline{\pi}}(n_1)\ldots\Lambda_{\pi\times\overline{\pi}}(n_k)}{n_1\ldots n_k}.$$

We apply Lemma 3.2 of [Tsa84] and obtain

$$\int_T^{2T} \left(\sum_{y<n\leq x} \frac{\Lambda_{\pi\times\overline{\pi}}(n)}{n^{1+it}} \right)^k \left(\sum_{y<n\leq x} \frac{\Lambda_{\pi\times\overline{\pi}}(n)}{n^{1-it}} \right)^l dt$$

$$= T\sum_{n}\left(\sum_{\substack{y<n_1\le x,\dots,y<n_k\le x\\ n_1\dots n_k=n}} B\left(n_1,\dots,n_k\right)\right)\left(\sum_{\substack{y<m_1\le x,\dots,y<m_l\le x\\ m_1\dots m_l=n}} B\left(m_1,\dots,m_l\right)\right)$$

$$+O\left(\left(\sum_{n} n\left|\sum_{\substack{n_1\dots n_k=n\\ y<n_i\le x}} B\left(n_1,\dots,n_k\right)\right|^2\right)^{\frac{1}{2}}\left(\sum_{n} n\left|\sum_{\substack{m_1\dots m_l=n\\ y<m_i\le x}} B\left(m_1,\dots,m_l\right)\right|^2\right)^{\frac{1}{2}}\right).$$

In dealing with the main term there are two cases that we need to consider. First we look at the case when when $k = l$

$$T\sum_{n}\left(\sum_{\substack{n_1\dots n_k=n\\ y<n_i\le x}} B\left(n_1,\dots,n_k\right)\right)\left(\sum_{\substack{m_1\dots m_k=n\\ y<m_i\le x}} B\left(m_1,\dots,m_l\right)\right).$$

Recall that $\Lambda_{\pi\times\overline{\pi}} = \Lambda(n)|a_\pi(n)|^2$. Thus the sum above is supported on prime powers. We split the above into two sums according to n being square free, and non-square free. First, if n is square free then by the constrains put on the sums we see that each $n_i = p_i$ is a distinct prime. Therefore the sum restricted to square free n is

$$T\sum_{\substack{n\\ squarefree}}\sum_{\substack{p_1\dots p_k=n\\ y<p_i\le x\\ p_i\ne p_j}}\sum_{\substack{q_1\dots q_k=n\\ y<q_i\le x\\ q_i\ne q_j}} B\left(p_1,\dots,p_k\right)B\left(q_1,\dots,q_k\right).$$

We note that $p_1\dots p_k = q_1\dots q_k$ and there are $k!$ ways to permute $q_i's$. Now we add terms where primes are repeated and the above becomes

$$= k!T\sum_{\substack{y<p_1\le x\\ p_i\ne p_j}}\dots\sum_{y<p_k\le x}\left|B\left(p_1,\dots,p_k\right)\right|^2 = Tk!\sum_{y<p_1\le x}\dots\sum_{y<p_k\le x}\left|B\left(p_1,\dots,p_k\right)\right|^2$$

$$+ O_k \left(T \sum_{\substack{y<p_1\le x \\ p_i = p_j \\ \text{for some } i \neq j}} \cdots \sum_{y<p_k\le x} \left| B\left(p_1, \ldots, p_k\right)\right|^2 \right) = Tk! \left(\sum_{y<p\le x} \frac{|\Lambda_{\pi\times\overline{\pi}}(p)|^2}{p^2} \right)^k$$

$$+ O_k \left(T \sum_{y<p_1\le x} \cdots \sum_{y<p_{k-2}\le x} \left| B\left(p_1, \ldots, p_{k-2}\right)\right|^2 \sum_{y<p\le x} \frac{|\Lambda_{\pi\times\overline{\pi}}(p)|^4}{p^4} \right)$$

$$= Tk! \left(\sum_{y<p} \frac{|\Lambda_{\pi\times\overline{\pi}}(p)|^2}{p^2} + O\left(\sum_{x<p} \frac{|\Lambda_{\pi\times\overline{\pi}}(p)|^2}{p^2} \right) \right)^k$$

$$+ O_k \left(T \left(\sum_{y<p\le x} \frac{|\Lambda_{\pi\times\overline{\pi}}(p)|^2}{p^2} \right)^{k-2} \left(\sum_{y<p\le x} \frac{|\Lambda_{\pi\times\overline{\pi}}(p)|^2 \log(p)^2 p^{1-2\beta}}{p^4} \right) \right)$$

$$= Tk! \left(\sum_{y<p} \frac{|\Lambda_{\pi\times\overline{\pi}}(p)|^2}{p^2} + O_k\left(\frac{\log(x)}{x} \right) \right)^k + O_k \left(T \left(\frac{\log(y)}{y} \right)^{k-2} \frac{\log(y)}{y^2} \right)$$

$$= Tk! \left(\sum_{y<p} \frac{|\Lambda_{\pi\times\overline{\pi}}(p)|^2}{p^2} \right)^k + O_k \left(T \left(\frac{\log(y)}{y} \right)^{k-1} \frac{\log(x)}{x} + T \frac{\log(y)^{k-1}}{y^k} \right)$$

$$= Tk! \left(\frac{c_0 \log(y)}{y} \right)^k + o\left(Tk! \left(\frac{\log(y)}{y} \right)^k \right) + O_k \left(T \frac{\log(y)^{k-1}}{y^k} \right).$$

To justify the last step we used the assumption that $y < \frac{x}{\log(x)}$. Now we consider the sum involving terms when n is non square free. The method is essentially the same as the method used to establish Lemma 3.9 of [RS96]. As in the proof of Lemma 3.9, suppose $n = r_1^{\gamma_1} \ldots r_h^{\gamma_h}$ has h distinct prime factors where $1 \le h \le k$. Then

$$T \sum_{\substack{n \\ \text{not square} \\ \text{free}}} \sum_{\substack{p_1^{\alpha_1} \ldots p_k^{\alpha_k} = n \\ y < p_i^{\alpha_i} \le x}} \sum_{\substack{q_1^{\beta_1} \ldots q_k^{\beta_k} = n \\ y < q_i^{\beta_i} \le x}} B\left(p_1^{\alpha_1}, \ldots, p_k^{\alpha_k}\right) B\left(q_1^{\beta_1}, \ldots, q_k^{\beta_k}\right)$$

$$= T \sum_{h=1}^{k-1} \sum_{\substack{y^k < n \le x^k \\ \text{not square} \\ \text{free,} \\ \text{n has h distinct} \\ \text{prime factors}}} \sum_{\substack{p_1^{\alpha_1} \ldots p_k^{\alpha_k} = n \\ y < p_i^{\alpha_i} \le x \\ \exists i \neq j \ s.t. \ p_i = p_j}} \sum_{\substack{q_1^{\beta_1} \ldots q_k^{\beta_k} = n \\ y < q_i^{\beta_i} \le x}} B\left(p_1^{\alpha_1}, \ldots, p_k^{\alpha_k}\right) B\left(q_1^{\beta_1}, \ldots, q_k^{\beta_k}\right)$$

$$+ \sum_{\substack{y^k < n \le x^k \\ \text{not square} \\ \text{free,} \\ \text{n has k distinct} \\ \text{prime factors}}} \sum_{\substack{p_1^{\alpha_1} \ldots p_k^{\alpha_k} = n \\ y < p_i^{\alpha_i} \le x}} \sum_{\substack{q_1^{\beta_1} \ldots q_k^{\beta_k} = n \\ y < q_i^{\beta_i} \le x}} B\left(p_1^{\alpha_1}, \ldots, p_k^{\alpha_k}\right) B\left(q_1^{\beta_1}, \ldots, q_k^{\beta_k}\right) = I + II.$$

To bound II note that since n is not square free, there exists $1 \le i \le k$ such that $\alpha_i \ge 2$, after rearrangement, II is bounded by

$$II \ll_k T \sum_{\substack{n \\ \text{not square} \\ \text{free}}} \sum_{\substack{p_1^{\alpha_1} \ldots p_k^{\alpha_k} = n \\ y < p_i^{\alpha_i} \le x \\ p_i' s \ \text{distinct} \\ \alpha_1 \ge 2}} \left| B\left(p_1^{\alpha_1}, \ldots, p_k^{\alpha_k}\right) \right|^2$$

$$\ll_k T \sum_{\substack{y < p_1^{\alpha_1} \le x \\ \alpha_1 \ge 2}} \cdots \sum_{y < p_k^{\alpha_k} \le x} \left| B\left(p_1^{\alpha_1}, \ldots, p_k^{\alpha_k}\right) \right|^2$$

$$\ll_k T \left(\sum_{\substack{y < n \le x \\ n \neq p}} \frac{|\Lambda_{\pi \times \overline{\pi}}(n)|^2}{n^2} \right) \left(\sum_{y < n \le x} \frac{|\Lambda_{\pi \times \overline{\pi}}(n)|^2}{n^2} \right)^{k-1} \ll_{\pi,k} T \frac{\log(y)^{k-1}}{y^k}.$$

To bound I we proceed in exact same way as in chapter 3. Let $n = r_1^{\gamma_1} \ldots r_h^{\gamma_h}$ be the prime factorization of n. Further restrictions in the inner sum require that

$$p_1^{\alpha_1} \ldots p_k^{\alpha_k} = r_1^{\gamma_1} \ldots r_h^{\gamma_h} = q_1^{\beta_1} \ldots q_k^{\beta_k}.$$

By collection exponents which belong to the same prime factor we may express each exponent γ_g in the following way $\gamma_g = \sum_{i=1}^{m_g} \alpha_{i,g} = \sum_{i=1}^{n_g} \beta_{i,g}$.. Where $m_g, n_g \ge 1$ and $\sum_{g=1}^{h} m_g = \sum_{g=1}^{h} n_g = k$. Now we can rewrite I in the following way

$$I = T\sum_{h=1}^{k-1}\prod_{g=1}^{h}\sum_{r_g}\sum_{\sum_{i\le m_g}\alpha_{i,g}=\sum_{j\le n_g}\beta_{i,g}} B\left(r_g^{\alpha_{1,g}},\dots,r_g^{\alpha_{m_g,g}}\right) B\left(r_g^{\beta_{1,g}},\dots,r_g^{\beta_{n_g,g}}\right)$$

$$= T\sum_{h=2}^{k-1}\prod_{g=1}^{h}\sum_{r_g}\sum_{\max(m_g,n_g)\le\gamma_g}\frac{M_{m_g}(r_g,\gamma_g)M_{n_g}(r_g,\gamma_g)}{r_g^{2\gamma_g}}$$

Note that $n^{-1}=\prod_{g=1}^{h} r_g^{-\gamma_g}<\frac{1}{y^k}$. In order to apply Lemma 4.2.1, we split the sum according to size of m_g+n_g and obtain

$$= T\sum_{h=2}^{k-1}\prod_{g=1}^{h}\sum_{r_g}\sum_{\substack{\max(m_g,n_g)\le\gamma_g\\ \exists g \text{ s.t. } m_g+n_g\ <3}}\frac{M_{m_g}(r_g,\gamma_g)M_{n_g}(r_g,\gamma_g)}{r_g^{2\gamma_g}}$$

$$+ T\sum_{h=2}^{k-1}\prod_{g=1}^{h}\sum_{r_g}\sum_{\substack{\max(m_g,n_g)\le\gamma_g\\ m_g+n_g\ \ge 3}}\frac{M_{m_g}(r_g,\gamma_g)M_{n_g}(r_g,\gamma_g)}{r_g^{2\gamma_g}} = III + IV.$$

First, we can deal with IV. By Lemma 4.2.1.

$$IV \ll T\sum_{h=2}^{k-1}\prod_{g=1}^{h}\sum_{r_g}\sum_{\substack{\max(m_g,n_g)\le\gamma_g\\ m_g+n_g\ \ge 3}}\frac{M_{m_g}(r_g,\gamma_g)M_{n_g}(r_g,\gamma_g)}{y^k r_g^{\gamma_g}} \ll_k \frac{T}{y^k}.$$

Before working with III we make the following observation. Since for all g we have $m_g,n_g\ge 1$ we see that whenever $m_g+n_g\ <3$ we actually have $m_g=n_g=1$ (note if $h=1$ then necessarily $m_g+n_g\ge 3$). Now suppose the number of $g's$ such that $m_g+n_g\ <3$ is j and note that $1\le j<h$ (the case $j=h$ is impossible because than $\sum m_g=h\le k-1$). After rearrangement we can bound III by

$$III \ll_k T\sum_{h=2}^{k-1}\prod_{g=1}^{h}\sum_{r_g}\sum_{\substack{\max(m_g,n_g)\le\gamma_g\\ \exists g \text{ s.t. } m_g+n_g\ <3}}\frac{M_{m_g}(r_g,\gamma_g)M_{n_g}(r_g,\gamma_g)}{r_g^{2\gamma_g}}$$

$$= T\sum_{h=2}^{k-1}\sum_{j=1}^{h-1}\prod_{f=1}^{j}\sum_{r_f}\sum_{\substack{\max(m_f,n_f)\le\gamma_f\\ m_f+n_f\ <3}} \frac{M_{m_f}(r_f,\gamma_f)M_{n_f}(r_f,\gamma_f)}{r_f^{2\gamma_f}}$$

$$\times\prod_{g=j+1}^{h}\sum_{r_g}\sum_{\substack{\max(m_g,n_g)\le\gamma_g\\ m_g+n_g\ \ge 3}} \frac{M_{m_g}(r_g,\gamma_g)M_{n_g}(r_g,\gamma_g)}{r_g^{2\gamma_g}}$$

$$\ll T\sum_{h=2}^{k-1}\sum_{j=1}^{h-1}\prod_{f=1}^{j}\sum_{r_f}\sum_{1\le\gamma_f}\frac{|\Lambda_{\pi\times\overline{\pi}}(r_f^{\gamma_f})|^2}{r_f^{2\gamma_f}}\prod_{g=j+1}^{h}\sum_{r_g}\sum_{\substack{\max(m_g,n_g)\le\gamma_g\\ m_g+n_g\ \ge 3}} \frac{M_{m_g}(r_g,\gamma_g)M_{n_g}(r_g,\gamma_g)}{y^{k-j}r_g^{\gamma_g}}$$

$$\ll_k T\sum_{h=2}^{k-1}\sum_{j=1}^{h-1}\prod_{f=1}^{j}\left(\sum_{y<n_f\le x}\frac{|\Lambda_{\pi\times\overline{\pi}}(n_f)|^2}{n_f^2}\right)\frac{1}{y^{k-j}} \ll_k T\sum_{h=2}^{k-1}\sum_{j=1}^{h-1}\left(\frac{\log(y)}{y}\right)^j\frac{1}{y^{k-j}}$$

$$\ll_k T\frac{(\log(y))^{k-2}}{y^k}.$$

Following the above argument we can see that the main contribution comes from integers n which are square free. Thus suitably adjusting the above argument, we can deal with the error term in the following way

$$\sum_n n\left|\sum_{\substack{y<n_1\le x,\ldots,y<n_k\le x\\ n_1\ldots n_k=n}} B(n_1,\ldots,n_k)\right|^2 \ll_k \sum_{\substack{n\\ \text{square}\\ \text{free}}} n\left|\sum_{\substack{y<n_1\le x,\ldots,y<n_k\le x\\ n_1\ldots n_k=n}} B(n_1,\ldots,n_k)\right|^2$$

$$= k!\underbrace{\sum_{y<p_1\le x}\ldots\sum_{y<p_k\le x}}_{p_i\ne p_j} p_1\ldots p_k\left|B(p_1,\ldots,p_k)\right|^2$$

$$\ll_k \sum_{y<p_1\le x}\cdots\sum_{y<p_k\le x}\frac{(\Lambda_{\pi\times\overline{\pi}}(p_1))^2\ldots(\Lambda_{\pi\times\overline{\pi}}(p_k))^2}{p_1\ldots p_k} \ll_k \left(\sum_{y<p\le x}\frac{(\Lambda_{\pi\times\overline{\pi}}(p))^2}{p}\right)^k$$

$$\ll_k (\log(x))^{2k}.$$

Putting all bounds together we have the first part of our Lemma.

$$\int_T^{2T} \left| \sum_{y<n\le x} \frac{\Lambda_{\pi\times\overline{\pi}}(n)}{n^{1+it}} \right|^{2k} dt = Tk! \left(\frac{c_0 \log(y)}{y} \right)^k + o\left(Tk! \left(\frac{c_0 \log(y)}{y} \right)^k \right) + O_{k,\pi}\left(T \frac{\log(y)^{k-1}}{y^k} + \log(x)^{2k} \right).$$

To deal with case $k \neq l$ we first note that case where n is square free is impossible by unique factorization of n and the fact that $\Lambda_{\pi\times\overline{\pi}}(n)$ is supported on prime powers, so we are left with n non square free. Also, the above argument shows that

$$\sum_{\substack{n \\ \text{not square} \\ \text{free}}} \left| \sum_{\substack{p_1^{\alpha_1}\dots p_k^{\alpha_k}=n \\ y<p_i^{\alpha_i}\le x}} B\left(p_1^{\alpha_1}, \dots, p_k^{\alpha_k}\right) \right|^2 \ll \frac{\log(y)^{k-1}}{y^k}.$$

With this we see

$$\int_T^{2T} \left(\sum_{y<n\le x} \frac{\Lambda_{\pi\times\overline{\pi}}(n)}{n^{1+it}} \right)^k \left(\sum_{y<n\le x} \frac{\overline{\Lambda_{\pi\times\overline{\pi}}(n)}}{n^{1-it}} \right)^l dt$$

$$= T \sum_{\substack{n \\ \text{not square} \\ \text{free}}} \sum_{\substack{p_1^{\alpha_1}\dots p_k^{\alpha_k}=n \\ y<p_i^{\alpha_i}\le x}} \sum_{\substack{q_1^{\beta_1}\dots q_l^{\beta_l}=n \\ y<q_i^{\beta_i}\le x}} B\left(p_1^{\alpha_1}, \dots, p_k^{\alpha_k}\right) B\left(q_1^{\beta_1}, \dots, q_l^{\beta_l}\right) + O\left(\log(x)^{k+l}\right)$$

$$\ll T \left(\sum_{\substack{n \\ \text{not square} \\ \text{free}}} \left| \sum_{\substack{p_1^{\alpha_1}\dots p_k^{\alpha_k}=n \\ y<p_i^{\alpha_i}\le x}} B\left(p_1^{\alpha_1}, \dots, p_k^{\alpha_k}\right) \right|^2 \right)^{\frac{1}{2}}$$

$$\times \left(\sum_{\substack{n \\ \text{not square} \\ \text{free}}} \left| \sum_{\substack{q_1^{\beta_1}\dots q_l^{\beta_l}=n \\ y<q_i^{\beta_i}\le x}} B\left(q_1^{\beta_1}, \dots, q_l^{\beta_l}\right) \right|^2 \right)^{\frac{1}{2}} + \log(x)^{k+l} \ll T \frac{\log(y)^{\frac{k+l}{2}-1}}{y^{\frac{k+l}{2}}} + \log(x)^{k+l}.$$

□

4.3 Value Distribution of $\frac{L'}{L}(1+it,\pi\times\overline{\pi})$

Theorem 4.3.1. *Assume Hypotheses H2, H3, H4, let $T \geq 1$, and suppose $k,l \in \mathbb{N}$. There exists $c_\pi > 0$ such that if $2 < y = y(T) < \frac{\exp\left((\log(T))^2\right)}{(\log(T))^2}$ and $y(T) \to \infty$ as $T \to \infty$, then*

$$\int_T^{2T} \left(-\frac{L'}{L}(1+it,\pi\times\overline{\pi}) - \sum_{n\leq y} \frac{\Lambda_{\pi\times\overline{\pi}}(n)}{n^{1+it}} \right)^k \times \left(-\frac{L'}{L}(1-it,\pi\times\overline{\pi}) - \sum_{n\leq y} \frac{\Lambda_{\pi\times\overline{\pi}}(n)}{n^{1-it}} \right)^l dt$$

$$= \begin{cases} Tk!\left(\frac{c_0\log(y)}{y}\right)^k + o\left(Tk!\left(\frac{\log(y)}{y}\right)^k\right) + O_{k,\pi}\left(T^{1-c_\pi}\right), & \text{if } k = l \\ O_{k,l,\pi}\left(T\frac{\log(y)^{\frac{k+l}{2}-1}}{y^{\frac{k+l}{2}}} + T^{1-c_\pi}\right) & \text{if } k \neq l. \end{cases}$$

Proof. By Lemma 4.1.1 there exists $c_\pi > 0$ such that

$$-\frac{L'}{L}(1+it,\pi\times\overline{\pi}) = \sum_{n\leq x} \frac{\Lambda_{\pi\times\overline{\pi}}(n)}{n^{1+it}} + O_\pi\left(\frac{1}{T^{c_\pi}}\right).$$

Now we may proceed with the computation

$$\int_T^{2T} \left(-\frac{L'}{L}(1+it,\pi\times\overline{\pi}) - \sum_{n\leq y} \frac{\Lambda_{\pi\times\overline{\pi}}(n)}{n^{1+it}} \right)^k \times \left(-\frac{L'}{L}(1-it,\pi\times\overline{\pi}) - \sum_{n\leq y} \frac{\Lambda_{\pi\times\overline{\pi}}(n)}{n^{1-it}} \right)^l dt$$

$$= \int_T^{2T} \left(\sum_{y<n\leq x} \frac{\Lambda_{\pi\times\overline{\pi}}(n)}{n^{1+it}} + O_\pi\left(\frac{1}{T^{c_\pi}}\right)\right)^k \left(\sum_{y<n\leq x} \frac{\Lambda_{\pi\times\overline{\pi}}(n)}{n^{1-it}} + O_\pi\left(\frac{1}{T^{c_\pi}}\right)\right)^l dt$$

$$= \int_T^{2T} \left(\sum_{y<n\le x} \frac{\Lambda_{\pi\times\overline{\pi}}(n)}{n^{1+it}} \right)^k \left(\sum_{y<n\le x} \frac{\Lambda_{\pi\times\overline{\pi}}(n)}{n^{1-it}} \right)^l dt$$

$$+ O_{k,l,\pi} \left(\sum_{a=0}^{k} \sum_{\substack{b=0 \\ a+b\neq 0}}^{l} \int_T^{2T} \left| \sum_{y<n\le x} \frac{\Lambda_{\pi\times\overline{\pi}}(n)}{n^{1+it}} \right|^{k+l-a-b} \frac{1}{T^{(a+b)c_\pi}})dt \right)$$

$$= \int_T^{2T} \left(\sum_{y<n\le x} \frac{\Lambda_{\pi\times\overline{\pi}}(n)}{n^{1+it}} \right)^k \left(\sum_{y<n\le x} \frac{\Lambda_{\pi\times\overline{\pi}}(n)}{n^{1-it}} \right)^l dt + O_{\pi,k,l}\left(T^{1-c_\pi}\right)$$

$$= \begin{cases} Tk! \left(\frac{c_0 \log(y)}{y}\right)^k + o\left(Tk!\left(\frac{c_0\log(y)}{y}\right)^k\right) + O_{k,\pi}\left(T^{1-c_\pi} + (\log(T))^{4k}\right), & \text{if } k = l \\ O_{k,l,\pi}\left(T\frac{\log(y)^{\frac{k+l}{2}-1}}{y^{\frac{k+l}{2}}} + T^{1-c_\pi} + (\log(T))^{2(k+l)}\right) & \text{if } k \neq l. \end{cases}$$

The last equality follows from Lemma 4.2.2. □

Equipped with Theorem 4.3.1 we are ready to prove our main result.

Theorem 4.3.2. *Assume Hypotheses H2, H3, H4, let $T \geq 1$, and suppose $k \in \mathbb{N}$ and $y = y(T)$ is such that $y(T) \to \infty$ as $T \to \infty$ and $2 < y < T^\epsilon$, for all $\epsilon > 0$. Define functions $g(t)$, $M_k(T)$ and $N_k(t)$ in the following way*

$$g(t) := \frac{-\frac{L'}{L}(1+it, \pi\times\overline{\pi}) - \sum_{n\le y} \frac{\Lambda_{\pi\times\overline{\pi}}(n)}{n^{1+it}}}{\sqrt{\frac{c_0\log(y)}{2y}}},$$

$$M_k(T) := \frac{1}{T}\int_T^{2T} \left(\Re\left(g(t)\right)\right)^k dt, \qquad N_k(T) := \frac{1}{T}\int_T^{2T} \left(\Im\left(g(t)\right)\right)^k dt.$$

Then

$$\lim_{T\to\infty} M_{2k-1}(T) = \lim_{T\to\infty} N_{2k-1}(T) = 0, \qquad \lim_{T\to\infty} M_{2k}(T) = \lim_{T\to\infty} N_{2k}(T) = \frac{(2k)!}{2^k k!}.$$

Proof.

$$M_{2k-1}(T) = \frac{1}{T}\int_T^{2T} \left(\Re\left(g(t)\right)\right)^{2k-1} dt = \frac{1}{T}\int_T^{2T} \left(\frac{g(t)+\overline{g(t)}}{2}\right)^{2k-1} dt$$

$$= \frac{1}{T}\sum_{j=0}^{2k-1}\binom{2k-1}{j}\frac{1}{2^{2k-1}}\int_T^{2T} g(t)^j\overline{g(t)}^{(2k-1)-j}dt = O_{k,\pi}\left(\frac{1}{\log(y)}\right).$$

Now for the imaginary part of our function of interest

$$N_{2k-1}(T) = \frac{1}{T}\int_T^{2T} \left(\Im\left(g(t)\right)\right)^{2k-1} dt = \frac{1}{T}\int_T^{2T} \left(\frac{g(t)-\overline{g(t)}}{2i}\right)^{2k-1} dt$$

$$= \frac{1}{T}\sum_{j=0}^{2k-1}\binom{2k-1}{j}\frac{1}{(2i)^{2k-1}}\int_T^{2T} (g(t))^j\left(-\overline{g(t)}\right)^{(2k-1)-j} dt = O_{k,\pi}\left(\frac{1}{\log(y)}\right).$$

Taking the limit as $T \to \infty$, we obtain the first part of our result. Now for the even powers, we will see that the main term comes from the middle term in the binomial expansion

$$M_{2k}(T) = \frac{1}{T}\int_T^{2T} \left(\Re\left(g(t)\right)\right)^{2k} dt = \frac{1}{T}\int_T^{2T} \left(\frac{g(t)+\overline{g(t)}}{2}\right)^{2k} dt$$

$$= \frac{1}{T}\binom{2k}{k}\frac{1}{2^{2k}}\int_T^{2T} |g(t)|^{2k}\, dt + \frac{1}{T}\sum_{\substack{j=0\\ j\neq k}}^{2k}\binom{2k}{j}\frac{1}{2^{2k}}\int_T^{2T} g(t)^j\overline{g(t)}^{2k-j}dt$$

$$= \frac{(2k)!}{2^k k!} + o\left(\frac{(2k)!}{2^k k!}\right) + O_{k,\pi}\left(\frac{1}{\log(y)}\right).$$

Lastly, even moments of the imaginary part of $g(t)$ are

$$N_{2k}(T) = \frac{1}{T}\int_T^{2T} \left(\Im\left(g(t)\right)\right)^{2k} dt = \frac{1}{T}\int_T^{2T} \left(\frac{g(t)-\overline{g(t)}}{2i}\right)^{2k} dt$$

$$= \frac{1}{T}\binom{2k}{k}\frac{(-1)^k}{(2i)^{2k}}\int_T^{2T} |g(t)|^{2k}\, dt + \frac{1}{T}\sum_{\substack{j=0\\ j\neq k}}^{2k}\binom{2k}{j}\frac{1}{2^{2k}}\int_T^{2T} (g(t))^j\left(-\overline{g(t)}\right)^{2k-j} dt$$

$$= \frac{(2k)!}{2^k k!} + o\left(\frac{(2k)!}{2^k k!}\right) + O_{k,\pi}\left(\frac{1}{\log(y)}\right).$$

Again, taking the limit as $T \to \infty$, we obtain the second part of our result.

□

Theorem 4.3.3. *Assume Hypotheses H2, H3, H4, let $T \geq 1$, and suppose $k \in \mathbb{N}$ and $y = y(T)$ is such that $y(T) \to \infty$ as $T \to \infty$ and $2 < y < T^\epsilon$, for all $\epsilon > 0$, then*

$$\lim_{T\to\infty} \frac{1}{T} meas\left\{t \in [T, 2T] | \Re\left(g(t)\right) \leq c\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{c} e^{-\frac{x^2}{2}} dx,$$

$$\lim_{T\to\infty} \frac{1}{T} meas\left\{t \in [T, 2T] | \Im\left(g(t)\right) \leq c\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{c} e^{-\frac{x^2}{2}} dx.$$

Proof. The theorem follows by combining Theorem 4.3.2 and Theorem 1.5.2. □

Chapter 5 Value Distribution of $\log\left(L\left(1+it,\pi\right)\right)$

5.1 Dirichlet Polynomial Approximation of $\log\left(L\left(1+it,\pi\right)\right)$

Lemma 5.1.1. *Let $x = \exp((\log(T))^2)$ and $T \geq 1$. Suppose that $T \leq t \leq 2T$. Then there is a positive constant c_π such that*

$$\log\left(L(1+it,\pi)\right) = \sum_{n\leq x} \frac{\Lambda_\pi(n)}{\log(n)n^{1+it}} + O_\pi\left(\frac{1}{T^{c_\pi}}\right).$$

Proof. Let $W = \frac{T}{2}$ and $\theta = \frac{1}{\log(x)}$. By Theorem 5.3 and Corollary 5.3 of Montgomery and Vaughan [MV07] (where $4^{\sigma_0} + x^{\sigma_0}$ should be $(4x)^{\sigma_0}$) we have

$$\sum_{n\leq x} \frac{\Lambda_\pi(n)}{\log(n)n^{1+it}} = \frac{1}{2\pi i}\int_{\theta-iW}^{\theta+iW} \log\left(L(1+it+w,\pi)\right)\frac{x^w}{w}dw + S, \tag{5.1}$$

where

$$S \ll \sum_{\substack{x/2<n<2x\\ n\neq x}} \frac{|\Lambda_\pi(n)|}{\log(n)n}\min\left(1, \frac{x}{W|x-n|}\right) + \frac{(4x)^{\frac{1}{\log(x)}}}{W}\sum_{n=1}^{\infty}\frac{|\Lambda_\pi(n)|}{\log(n)n^{1+\frac{1}{\log(x)}}}.$$

By Theorem 2.1.1 and Lemma 2.2.3 there exists a positive constant c such that whenever $\frac{T}{2} \leq t \leq \frac{5T}{2}$ and

$$1-\frac{c}{2d^4\log(\mathfrak{q}(\pi)(3T+3))}\le\sigma\le 1+\frac{1}{\log(x)},$$

we have $L(\sigma+it,\pi)\neq 0$ and

$$\log(L(\sigma+it,\pi))\ll_\pi \log(\mathfrak{q}(\pi)(3T+3)).$$

Let

$$\phi=-\frac{c}{2d^4\log(\mathfrak{q}(\pi)(3T+3))}.$$

We apply the Cauchy residue theorem to the rectangle with corners at $\theta-iW$, $\theta+iW$, $\phi+iW$, $\phi-iW$, which we will call $\mathscr{C}$, yields the following

$$\frac{1}{2\pi i}\int_{\mathscr{C}}\log(L(1+it+w,\pi))\frac{x^w}{w}dw=\log(L(1+it,\pi)).$$

The horizontal paths of above integral contribute

$$\int_{\phi\pm iW}^{\theta\pm iW}-\frac{L'}{L}(1+it+w,\pi)\frac{x^w}{w}dw\ll_\pi\int_\phi^\theta\log(\mathfrak{q}(\pi)(3T+3))\frac{x^\sigma}{|\sigma\pm iW|}d\sigma$$

$$\ll_\pi\frac{\log(\mathfrak{q}(\pi)(3T+3))}{W}\int_\phi^\theta x^\sigma d\sigma\ll_\pi\frac{\log(\mathfrak{q}(\pi)(3T+3))}{T}.$$

The vertical line $\Re w=\phi$ contributes

$$\int_{\phi-iW}^{\phi+iW}-\frac{L'}{L}(1+it+w,\pi)\frac{x^w}{w}dw\ll_\pi\log(\mathfrak{q}(\pi)(3T+3))x^\phi\int_0^W\frac{1}{|\phi+iv|}dv$$

$$\ll_\pi\log(\mathfrak{q}(\pi)(3T+3))x^\phi\left(1+\log\left(\frac{W}{|\phi|}\right)\right)\ll_\pi(\log(\mathfrak{q}(\pi)(3T+3)))^2\exp(\phi\log(x))$$

$$\ll_\pi(\log(\mathfrak{q}(\pi)(3T+3)))^2\exp\left(-c'\left(\frac{(\log(T))^2}{2d^4\log(\mathfrak{q}(f)(3T+3)}\right)\right)\ll_\pi\frac{1}{T^{c_\pi}},$$

where c_π is some positive constant that depends on π. Now we proceed to work out an appropriate bound for the error term S arising in Perron's formula (5.1). We deal with the finite sum first. Recall in the proof of Lemma 3.1.1 we had the following bound $R \ll T^{-c_\pi}$ and since $S \leq R$ the desired bound follows.

$\square$

Lemma 5.1.2. *Let $\Lambda_\pi(n)$ be defined as in (1.3) and assume Hypothesis H1 then*

$$\sum_{y<n} \frac{|\Lambda_\pi(n)|^2}{(n\log(n))^2} \sim \frac{1}{y\log(y)},$$

$$\sum_{\substack{y<n\\ n\neq p}} \frac{|\Lambda_\pi(n)|^2}{(n\log(n))^2} \ll \frac{1}{y(\log(y))^2}.$$

Proof. By equations (3.3) and (3.4) we have

$$A(u) := \sum_{n<u} \frac{|\Lambda_\pi(n)|^2}{n^2} = C_\pi - \frac{\log(u)}{u} + O\left(\frac{1}{u}\right).$$

The first part of our lemma can be established using Abel's summation formula

$$\sum_{y<n} \frac{|\Lambda_\pi(n)|^2}{(n\log(n))^2} = \int_y^\infty \frac{1}{(\log(u))^2} dA(u) = -\frac{A(y)}{(\log(y))^2} - \int_y^\infty A(u)\frac{d}{du}\left(\frac{1}{(\log(u))^2}\right) du$$

$$= -\frac{C_\pi}{(\log(y))^2} + \frac{1}{y\log(y)} + O\left(\frac{1}{y(\log(y))^2}\right) + \frac{C_\pi}{(\log(y))^2} + \int_y^\infty \left(\frac{\log(u)}{u} + O\left(\frac{1}{u}\right)\right)\left(-\frac{2}{u(\log(u))^3}\right) du$$

$$= \frac{1}{y\log(y)} + O\left(\frac{1}{y(\log(y))^2}\right) - 2\int_y^\infty \frac{1}{u^2(\log(u))^2} du + O\left(\frac{1}{y(\log(y))^3}\right)$$

$$= \frac{1}{y\log(y)} + O\left(\frac{1}{y(\log(y))^2}\right) + O\left(\frac{1}{(\log(y))^2}\int_y^\infty \frac{1}{u^2} du\right) \sim \frac{1}{y\log(y)}$$

To obtain the second part of our Lemma, we apply a trivial bound on the summand via Lemma 3.1.2

$$\sum_{\substack{y<n\\ n\neq p}} \frac{|\Lambda_\pi(n)|^2}{(n\log(n))^2} \ll \frac{1}{(\log(y))^2} \sum_{\substack{y<n\\ n\neq p}} \frac{|\Lambda_\pi(n)|^2}{n^2} \ll \frac{1}{(\log(y))^2 y}.$$

□

5.2 Mean Values of Dirichlet Polynomials

Lemma 5.2.1. *Assume Hypothesis H1, let $T \geq 1$ and $2 < y < \frac{x}{\log(x)}$. Then*

$$\int_T^{2T} \left(\sum_{y<n\leq x} \frac{\Lambda_\pi(n)}{\log(n) n^{1+it}} \right)^k \left(\sum_{y<n\leq x} \frac{\overline{\Lambda_\pi(n)}}{\log(n) n^{1-it}} \right)^l dt$$

$$= \begin{cases} Tk! \left(\frac{1}{y\log(y)}\right)^k + o\left(Tk!\left(\frac{1}{y\log(y)}\right)^k\right) + O_k\left((\log\log(x))^k\right), & \text{if } k = l \\ O_{k,l}\left(T(y\log(y))^{-\frac{k+l}{2}}\log(y)^{-1} + (\log\log(x))^{\frac{k+l}{2}}\right) & \text{if } k \neq l. \end{cases}$$

Proof. As in the proof of Lemma 3.2.2, to simplify presentation, let

$$C(n_1, \ldots, n_k) := \frac{\Lambda_\pi(n_1)\ldots\Lambda_\pi(n_k)}{\log(n_1)\ldots\log(n_k) n_1\ldots n_k}.$$

By Lemma 3.2.1 we have

$$\int_T^{2T} \left(\sum_{y<n\leq x} \frac{\Lambda_\pi(n)}{\log(n) n^{1+it}} \right)^k \left(\sum_{y<n\leq x} \frac{\overline{\Lambda_\pi(n)}}{\log(n) n^{1-it}} \right)^l dt$$

$$= T\sum_n \left(\sum_{\substack{n_1\ldots n_k = n\\ y<n_i\leq x}} C(n_1,\ldots,n_k) \right) \left(\sum_{\substack{m_1\ldots m_l = n\\ y<m_i\leq x}} \overline{C(m_1,\ldots,m_l)} \right)$$
$$+ O\left(\left(\sum_n n \left| \sum_{\substack{n_1\ldots n_k = n\\ y<n_i\leq x}} C(n_1,\ldots,n_k) \right|^2 \sum_n n \left| \sum_{\substack{m_1\ldots m_l = n\\ y<m_i\leq x}} \overline{C(m_1,\ldots,m_l)} \right|^2 \right)^{\frac{1}{2}} \right).$$

In dealing with the main term there are 2 cases that we need to consider. First, we look at the case when $k = l$

$$T \sum_{n} \left(\sum_{\substack{n_1 \ldots n_k = n \\ y < n_i \leq x}} C(n_1, \ldots, n_k) \right) \left(\sum_{\substack{m_1 \ldots m_k = n \\ y < m_i \leq x}} \overline{C(m_1, \ldots, m_l)} \right).$$

Recall that $\Lambda_\pi(n) = \Lambda(n) a_\pi(n)$ thus the sum above is supported on prime powers. We split the above into two sums according to n being square-free, and not square-free. First, if n is square-free, then by the constraints put on the sums we see that each $n_i = p_i$ is a distinct prime. Therefore the said sum is

$$T \sum_{\substack{n \\ squarefree}} \sum_{\substack{p_1 \ldots p_k = n \\ y < p_i \leq x \\ p_i \neq p_j}} \sum_{\substack{q_1 \ldots q_k = n \\ y < q_i \leq x \\ q_i \neq q_j}} C(p_1, \ldots, p_k) \overline{C(q_1, \ldots, q_k)}.$$

We note that $p_1 \ldots p_k = q_1 \ldots q_k$ and there are $k!$ ways to permute $q_i' s$. Now we add terms where primes are repeated and the above becomes

$$= k! T \underbrace{\sum_{y < p_1 \leq x} \cdots \sum_{y < p_k \leq x}}_{p_i \neq p_j} |C(p_1, \ldots, p_k)|^2 = Tk! \sum_{y < p_1 \leq x} \cdots \sum_{y < p_k \leq x} |C(p_1, \ldots, p_k)|^2$$

$$+ O_k \left(T \underbrace{\sum_{y < p_1 \leq x} \cdots \sum_{y < p_k \leq x}}_{\substack{p_i = p_j \\ \text{for some } i \neq j}} |C(p_1, \ldots, p_k)|^2 \right) = Tk! \left(\sum_{y < p \leq x} \frac{|\Lambda_\pi(p)|^2}{(p \log(p))^2} \right)^k$$

$$+ O_k \left(T \sum_{y < p_1 \leq x} \cdots \sum_{y < p_{k-2} \leq x} |C(p_1, \ldots, p_{k-2})|^2 \sum_{y < p \leq x} \frac{|\Lambda_\pi(p)|^4}{(p \log(p))^4} \right)$$

$$= Tk! \left(\sum_{y < p} \frac{|\Lambda_\pi(p)|^2}{(p \log(p))^2} + O\left(\sum_{x < p} \frac{|\Lambda_\pi(p)|^2}{(p \log(p))^2} \right) \right)^k$$

$$+ O_k \left(T \left(\sum_{y < p \leq x} \frac{|\Lambda_\pi(p)|^2}{(p \log(p))^2} \right)^{k-2} \left(\sum_{y < p \leq x} \frac{|\Lambda_\pi(p)|^2 \log(p)^2 p^{1 - \frac{2}{d^2+1}}}{(p \log(p))^4} \right) \right)$$

$$= Tk!\left(\sum_{y<p}\frac{|\Lambda_\pi(p)|^2}{(p\log(p))^2}+O_k\left(\frac{1}{x\log(x)}\right)\right)^k + O_{k,\pi}\left(T\left(\frac{1}{y\log(y)}\right)^{k-2}\frac{1}{(\log(y))^3y^2}\right)$$

$$= Tk!\left(\sum_{y<p}\frac{|\Lambda_\pi(p)|^2}{(\log(p)p)^2}\right)^k + O_{k,\pi}\left(T\left(\frac{1}{\log(y)y}\right)^{k-1}\frac{1}{\log(x)x}+T\frac{1}{\log(y)^{k+1}y^k}\right)$$

$$= Tk!\left(\frac{1}{\log(y)y}\right)^k + o\left(Tk!\left(\frac{1}{\log(y)y}\right)^k\right) + O_{k,\pi}\left(T\frac{1}{\log(y)^{k+1}y^k}\right).$$

Recall that in the proof of Lemma 3.2.2 we showed

$$\sum_{\substack{n\\ \text{not square}\\ \text{free}}}\left(\sum_{\substack{p_1^{\alpha_1}\dots p_k^{\alpha_k}=n\\ y<p_i^{\alpha_i}\le x}}\left|A\left(p_1^{\alpha_1},\dots,p_k^{\alpha_k}\right)\right|\right)^2 \ll \frac{\log(y)^{k-1}}{y^k}.$$

Thus contribution from n not square-free is at most

$$T\sum_{\substack{n\\ \text{not square}\\ \text{free}}}\left(\sum_{\substack{n_1\dots n_k=n\\ y<n_i\le x}}C\left(n_1,\dots,n_k\right)\right)\left(\sum_{\substack{m_1\dots m_k=n\\ y<m_i\le x}}\overline{C\left(m_1,\dots,m_k\right)}\right)$$

$$\ll T\sum_{\substack{n\text{ not}\\ \text{square}\\ \text{free}}}\left(\sum_{\substack{n_1\dots n_k=n\\ y<n_i\le x}}\left|C\left(n_1,\dots,n_k\right)\right|\right)^2 = T\sum_{\substack{n\text{ not}\\ \text{square}\\ \text{free}}}\left(\sum_{\substack{n_1\dots n_k=n\\ y<n_i\le x}}\frac{\left|A\left(n_1,\dots,n_k\right)\right|}{\log(n_1)\dots\log(n_k)}\right)^2$$

$$\ll \frac{T}{(\log(y))^{2k}}\sum_{\substack{n\\ \text{not square}\\ \text{free}}}\left(\sum_{\substack{n_1\dots n_k=n\\ y<n_i\le x}}\left|A\left(n_1,\dots,n_k\right)\right|\right)^2 \ll \frac{T}{(\log(y))^{2k}}\frac{\log(y)^{k-1}}{y^k}$$

$$= \frac{T}{(\log(y))^{k+1} y^k}.$$

By the above argument, we can see that sums in the error term are dominated by terms where n is square-free

$$\sum_{n} n \left| \sum_{\substack{y<n_1\le x,\dots,y<n_k\le x \\ n_1\dots n_k=n}} C(n_1,\dots,n_k) \right|^2 \ll \sum_{\substack{n \\ \text{sqare free}}} n \left| \sum_{\substack{y<n_1\le x,\dots,y<n_k\le x \\ n_1\dots n_k=n}} C(n_1,\dots,n_k) \right|^2$$

$$\ll_k \sum_{y<p_1\le x} \dots \sum_{y<p_k\le x} p_1\dots p_k \left|C(p_1,\dots,p_k)\right|^2 = \left(\sum_{p\le x} \frac{|\Lambda_\pi(p)|^2}{p(\log(p))^2} \right)^k$$

$$\ll_k (\log\log(x))^k.$$

Therefore, in the case $k = l$ we obtain the first part of our lemma

$$\int_T^{2T} \left| \sum_{y<n\le x} \frac{\Lambda_\pi(n)}{\log(n) n^{1+it}} \right|^{2k} dt = Tk! \left(\frac{1}{y\log(y)} \right)^k (1+o(1)) + O_k\left((\log\log(x))^k \right).$$

The case $k \neq l$ is dealt in a similar manner as in chapters 3 and 4. Again, by unique factorization and the constraints put on our sum of interest, the case where n is square free is impossible.

$$\int_T^{2T} \left(\sum_{y<n\le x} \frac{\Lambda_\pi(n)}{\log(n) n^{1+it}} \right)^k \left(\sum_{y<n\le x} \frac{\overline{\Lambda_\pi(n)}}{\log(n) n^{1-it}} \right)^l dt$$

$$= T \sum_{\substack{n \\ \text{not square} \\ \text{free}}} \left(\sum_{\substack{n_1\dots n_k=n \\ y<n_i\le x}} C(n_1,\dots,n_k) \right) \left(\sum_{\substack{m_1\dots m_l=n \\ y<m_i\le x}} \overline{C(m_1,\dots,m_l)} \right)$$

$$+ O_k\left((\log\log(x))^{\frac{k+l}{2}} \right)$$

$$\ll_k T \left(\sum_{\substack{n \\ \text{not square} \\ \text{free}}} \left| \sum_{\substack{n_1 \ldots n_k = n \\ y < n_i \leq x}} C(n_1, \ldots, n_k) \right|^2 \right)^{\frac{1}{2}}$$

$$\times \left(\sum_{\substack{n \\ \text{not square} \\ \text{free}}} \left| \sum_{\substack{m_1 \ldots m_l = n \\ y < n_i \leq x}} C(m_1, \ldots, m_l) \right|^2 \right)^{\frac{1}{2}} + (\log\log(x))^{\frac{k+l}{2}}$$

$$\ll_k T(\log(y))^{\frac{-(k+1)}{2}} y^{\frac{-k}{2}} (\log(y))^{\frac{-(l+1)}{2}} y^{\frac{-l}{2}} + (\log\log(x))^{\frac{k+l}{2}}$$

$$= T(\log(y))^{-\frac{(k+l)}{2}-1} y^{-\frac{k+l}{2}} + (\log\log(x))^{\frac{k+l}{2}}.$$

This proves the second part of our lemma.

□

5.3 Value Distribution of $\log(L(1+it,\pi))$

Theorem 5.3.1. *Assume Hypothesis H1, let $T \geq 1$, $k, l \in \mathbb{N}$. There exists $c_\pi > 0$ such that if $2 < y = y(T) < \frac{\exp\left((\log(T))^2\right)}{(\log(T))^2}$ and $y(T) \to \infty$ as $T \to \infty$, then*

$$\int_T^{2T} \left(\log(L(1+it,\pi)) - \sum_{n \leq y} \frac{\Lambda_\pi(n)}{\log(n) n^{1+it}} \right)^k \times \left(\log(L(1-it,\overline{\pi})) - \sum_{n \leq y} \frac{\overline{\Lambda_\pi(n)}}{\log(n) n^{1-it}} \right)^l dt$$

$$= \begin{cases} Tk! \left(\frac{1)}{\log(y)y} \right)^k + o\left(Tk! \left(\frac{1}{\log(y)y} \right)^k \right) + O_{k,\pi}\left(T^{1-c_\pi}\right), & \textit{if } k = l \\ O_{k,l,\pi}\left(T(\log(y))^{-\frac{k+l}{2}-1} y^{-\frac{k+l}{2}} + T^{1-c_\pi} \right) & \textit{if } k \neq l. \end{cases}$$

Proof. By Lemma 5.1.1 we have

$$\log\left(L\left(1+it,\pi\right)\right)=\sum_{n\le x}\frac{\Lambda_\pi(n)}{\log(n)n^{1+it}}+O_\pi\left(\frac{1}{T^{c_\pi}}\right),$$

for some $c_\pi>0$. Now we may proceed with the computation

$$\int_T^{2T}\left(\log\left(L\left(1+it,\pi\right)\right)-\sum_{n\le y}\frac{\Lambda_\pi(n)}{\log(n)n^{1+it}}\right)^k \times\left(\log\left(L\left(1-it,\overline{\pi}\right)\right)-\sum_{n\le y}\frac{\overline{\Lambda_\pi(n)}}{\log(n)n^{1-it}}\right)^l dt$$

$$=\int_T^{2T}\left(\sum_{y<n\le x}\frac{\Lambda_\pi(n)}{\log(n)n^{1+it}}+O_\pi\left(\frac{1}{T^{c_\pi}}\right)\right)^k\left(\sum_{y<n\le x}\frac{\overline{\Lambda_\pi(n)}}{\log(n)n^{1-it}}+O_\pi\left(\frac{1}{T^{c_\pi}}\right)\right)^l dt$$

$$=\int_T^{2T}\left(\sum_{y<n\le x}\frac{\Lambda_\pi(n)}{\log(n)n^{1+it}}\right)^k\left(\sum_{y<n\le x}\frac{\overline{\Lambda_\pi(n)}}{\log(n)n^{1-it}}\right)^l dt + O_{k,l,\pi}\left(\sum_{\substack{a=0\\a+b\neq 0}}^{k}\sum_{b=0}^{l}\int_T^{2T}\left|\sum_{y<n\le x}\frac{\Lambda_\pi(n)}{\log(n)n^{1+it}}\right|^{k+l-a-b}\frac{1}{T^{(a+b)c_\pi}})dt\right)$$

$$=\int_T^{2T}\left(\sum_{y<n\le x}\frac{\Lambda_\pi(n)}{\log(n)n^{1+it}}\right)^k\left(\sum_{y<n\le x}\frac{\overline{\Lambda_\pi(n)}}{\log(n)n^{1-it}}\right)^l dt+O_{\pi,k,l}\left(T^{1-c_\pi}\right)$$

$$=\begin{cases}Tk!\left(\frac{1}{\log(y)y}\right)^k+o\left(Tk!\left(\frac{1}{\log(y)y}\right)^k\right)+O_{k,\pi}\left(T^{1-c_\pi}+(\log\log(T))^k\right), & \text{if } k=l\\ O_{k,l,\pi}\left(T(\log(y))^{-\frac{k+l}{2}-1}y^{-\frac{k+l}{2}}+T^{1-c_\pi}+(\log\log(T))^{\frac{k+l}{2}}\right) & \text{if } k\neq l.\end{cases}$$

The last equality follows from Lemma 5.2.1. □

Equipped with Theorem 5.3.1 we are ready to show that the moments of, appropriately modified, $\log(L(1+it,\pi))$ approach those of the Gaussian distribution.

Theorem 5.3.2. *Assume Hypothesis H1, let $T \geq 1$, $k \in \mathbb{N}$. $y = y(T)$ be such that $y(T) \to \infty$ as $T \to \infty$ and $2 < y < T^{\epsilon}$, for all $\epsilon > 0$. Define functions $h(t)$, $I_k(T)$ and $J_k(T)$ in the following way*

$$h(t) := \frac{\log\left(L\left(1+it,\pi\right)\right) - \sum_{n\leq y} \frac{\Lambda_\pi(n)}{\log(n)n^{1+it}}}{\sqrt{\frac{1}{2\log(y)y}}},$$

$$O_k(T) := \frac{1}{T}\int_T^{2T} \left(\Re\left(h(t)\right)\right)^k dt, \qquad P_k(T) := \frac{1}{T}\int_T^{2T} \left(\Im\left(h(t)\right)\right)^k dt.$$

Then

$$\lim_{T\to\infty} P_{2k-1}(T) = \lim_{T\to\infty} Q_{2k-1}(T) = 0, \qquad \lim_{T\to\infty} P_{2k}(T) = \lim_{T\to\infty} Q_{2k}(T) = \frac{(2k)!}{2^k k!}.$$

Proof.

$$P_{2k-1}(T) = \frac{1}{T}\int_T^{2T} \left(\Re\left(h(t)\right)\right)^{2k-1} dt = \frac{1}{T}\int_T^{2T} \left(\frac{h(t)+\overline{h(t)}}{2}\right)^{2k-1} dt$$

$$= \frac{1}{T}\sum_{j=0}^{2k-1}\binom{2k-1}{j}\frac{1}{2^{2k-1}}\int_T^{2T} h(t)^j\overline{h(t)}^{(2k-1)-j} dt = O_{k,\pi}\left(\frac{1}{\log(y)}\right).$$

Now for the imaginary part of our function of interest

$$Q_{2k-1}(T) = \frac{1}{T}\int_T^{2T} \left(\Im\left(h(t)\right)\right)^{2k-1} dt = \frac{1}{T}\int_T^{2T} \left(\frac{h(t)-\overline{h(t)}}{2i}\right)^{2k-1} dt$$

$$= \frac{1}{T}\sum_{j=0}^{2k-1}\binom{2k-1}{j}\frac{1}{(2i)^{2k-1}}\int_T^{2T} (h(t))^j\left(-\overline{h(t)}\right)^{(2k-1)-j} dt = O_{k,\pi}\left(\frac{1}{\log(y)}\right).$$

Taking the limit as $T \to \infty$, we obtain the first part of our result. Now for the even powers, we will see that the main term comes from the middle term in the binomial expansion

$$P_{2k}(T) = \frac{1}{T}\int_T^{2T} \left(\Re\left(h(t)\right)\right)^{2k} dt = \frac{1}{T}\int_T^{2T} \left(\frac{h(t)+\overline{h(t)}}{2}\right)^{2k} dt$$

$$= \frac{1}{T}\binom{2k}{k}\frac{1}{2^{2k}}\int_T^{2T} |h(t)|^{2k}\, dt + \frac{1}{T}\sum_{\substack{j=0\\ j\neq k}}^{2k}\binom{2k}{j}\frac{1}{2^{2k}}\int_T^{2T} h(t)^j\overline{h(t)}^{2k-j} dt$$

$$= \frac{(2k)!}{2^k k!} + o\left(\frac{(2k)!}{2^k k!}\right) + O_{k,\pi}\left(\frac{1}{\log(y)}\right).$$

Lastly, even moments of the imaginary part of $h(t)$ are

$$Q_{2k}(T) = \frac{1}{T}\int_T^{2T} \left(\Im\left(h(t)\right)\right)^{2k} dt = \frac{1}{T}\int_T^{2T} \left(\frac{h(t)-\overline{h(t)}}{2i}\right)^{2k} dt$$

$$= \frac{1}{T}\binom{2k}{k}\frac{(-1)^k}{(2i)^{2k}}\int_T^{2T} |h(t)|^{2k}\, dt + \frac{1}{T}\sum_{\substack{j=0\\ j\neq k}}^{2k}\binom{2k}{j}\frac{1}{2^{2k}}\int_T^{2T} (h(t))^j\left(-\overline{h(t)}\right)^{2k-j} dt$$

$$= \frac{(2k)!}{2^k k!} + o\left(\frac{(2k)!}{2^k k!}\right) + O_{k,\pi}\left(\frac{1}{\log(y)}\right).$$

Again, taking the limit as $T \to \infty$, we obtain the second part of our result.

□

Theorem 5.3.3. *Assume Hypothesis H1, let $T \geq 1$, $k \in \mathbb{N}$. $y = y(T)$ be such that $y(T) \to \infty$ as $T \to \infty$ and $2 < y < T^\epsilon$, for all $\epsilon > 0$, then*

$$\lim_{T\to\infty}\frac{1}{T} meas\left\{t \in [T,2T] | \Re\left(h(t)\right) \leq c\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{c} e^{-\frac{x^2}{2}} dx,$$

$$\lim_{T\to\infty}\frac{1}{T} meas\left\{t \in [T,2T] | \Im\left(h(t)\right) \leq c\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{c} e^{-\frac{x^2}{2}} dx.$$

Proof. The theorem follows by combining Theorem 5.3.2 and Theorem 1.5.2. □

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