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OPTIMAL STRATEGIES USING FINANCIAL OPTIONS IN AIRLINE BOOKING

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ABSTRACT

Maximizing airline profit is challenging as several costs, such as fuel or manpower, have been increasing. Fierce competition has forced airlines to drastically reconsider their economic policy. Traditionally, revenue management techniques such as overbooking or dynamic pricing have been utilized to improve profit of the airline industry. Recently, the use of financial options theory offers a different approach to airline revenue maximization.

The purpose of this thesis is to apply financial option theory in order to maximize the expected revenue of the airline per flight. An option-based model to maximize the airline revenue under uncertainty and a numerical search method are presented. The demand distribution and the random walk of a ticket price cause uncertainty in the booking process. The proposed model assumes that the evaluation periods are discrete and uses the Cross-Ross-Rubinstein model (1979) to price the options. The complexity of the expected revenue expression does not allow for the utilization of calculus to maximize the airline's profit. Therefore, a numerical search method is used to determine the optimal values for the decisions variables, which are the initial number of call and put options and the respective strike prices, for a given distribution of demand. A numerical example is presented and sensitivity analysis is performed to test the behavior of the model when changing input parameters (skewness of demand distribution and up and down moves of the ticket price). It is found that changing the skewness of the distribution has a significant effect on the expected revenue. Moreover, we show that the parameters

of the random walk (the drift and its variance) directly affect the utilization of options, because these have a direct effect on the probability that the ticket price increases. Our analysis demonstrates the importance of accurate forecasting of demand and the precise estimation of the random walk parameters.

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CHAPTER 1

INTRODUCTION AND LITERATURE REVIEW

1.1 Introduction

Revenue management is the integrated control of capacity and price. Companies selling perishable goods or services often face the problem of selling a fixed capacity of a product over a finite horizon. If the market is characterized by customers willing to pay different prices for the product, it is often possible to target different customer segments by the use of product differentiation. Together with distribution, it represents one of the core competencies of modern airline operations. Revenue management involves utilizing passenger demand forecasting and fare mix optimization techniques to maximize profit for an airline. An airline typically offers tickets for many origin-destination itineraries in various fare classes. These fare classes not only include first, business or economy class, which are "physically" different, but also fare classes for which the difference in price can be explained by cancellation options, etc. Therefore seats on a flight are products targeting different passenger segments for different fares. A seat is no longer profitable once the aircraft has taken off, tickets are perishable products. Therefore, when considering the airline industry, revenue management can be defined as the practice of managing the booking requests with an objective of increasing airline revenue.

Airline revenue management is focused on the seat inventory control problem. This last problem concerns the allocation of finite seat inventory to the demand that occurs over time before the flight departs, with the objective of maximizing the revenue of the airline by finding the right combination of passengers - fare class on the flight. In order to decide whether or not to accept a booking request, the opportunity cost of losing the seats taken up by the booking or charging a higher price later has to be evaluated and compared to the revenue generated by accepting the booking request. Solution methods for the seat inventory control problem are concerned with approximating these opportunity costs and incorporating them in a booking control policy such that expected revenue is maximized.

Airline revenue management has also directed attention to demand forecasting. Demand forecasting is the activity of estimating the quantity of a ticket that consumers will purchase. It involves techniques including both informal methods, such as educated guesses, and quantitative methods, such as the use of historical sales data or current data from test markets. Demand forecasting may be used in making pricing decisions and in assessing future capacity requirements and these are of critical importance because airline booking control policies use demand forecasts to determine the optimal booking control strategy. Poor demand estimates result in a booking control strategy that will perform badly.

Another issue is the consideration of no-shows, cancellations and denied-boarding. In order to prevent a flight from taking off with empty seats ("empty" designates a seat that has been cancelled prior the departure, including the passengers that did not show up), airline revenue management has used overbooking. Thus, airlines routinely overbook flights to compensate for no-shows: people who reschedule or opt not to fly. An empty seat on a plane means a loss of revenue. Historical data show a lot of last-minute cancellations or rescheduling (for example, business men are known to reschedule as meetings happen to last longer). If everyone shows up, the overbooking will cause an over-sale. The airline then asks for volunteers to give up their seats or may refuse boarding to certain passengers, in exchange for a compensation, known as denied boarding costs (which may include an additional free ticket or an upgrade in a later flight). Despite the denied barding costs, overbooking is still more profitable for the airline than the flight taking off with empty seats. Some airlines, such as JetBlue Airways, do not overbook as a policy that provides incentive, and consequently avoid customer dissatisfaction. These airlines target a different customer segment, mostly tourists, with non-refundable tickets, and so most passengers show up. By and large, only economy class deals with overbooking while higher classes (business and first) do not, allowing the airline to upgrade some privileged passengers (such as "faithful" very frequent flyers) to otherwise unused seats.

Price is obviously a key variable when computing the revenues of an airline. Price differentiation is the starting point of any revenue management policy. Pricing is the manual or automatic process of applying prices to purchase and sales orders, based on factors such as the date of ticket purchase, the number of tickets purchased, the place where the tickets are bought (on the internet, through a travel agent), the cancellation options of the tickets and the refund possibility (fully or partially). Pricing leads to price differentiation, the foundation of revenue management. Price discrimination requires market segmentation in order to target different passenger types (last minute travelers, passengers who book their tickets months in advance, passengers who have flexible flying dates) and some means to discourage discount customers from becoming resellers and, by extension, competitors. This usually entails using one or more means of preventing any resale, keeping the different price groups separate, making price comparisons difficult, or restricting pricing information. The airline set up the rate fence, which is the boundary that separates segments. Pricing is thus a very common topic of research when considering airline revenue management; it is also one of the most complicated ones because of the diversity of parameters to take into account.

The contribution of revenue management is very difficult to evaluate, yet different revenue management techniques are said, according to Smith et al. (1992), to have increased American Airlines revenues by $ 500 million annually from 1989 to 1992. In 2004. Using similar techniques, Delta in 2004 announced an additional revenue of $ 300 million per year.

Economic conditions have been worsening since 2001, with emphasized effect on airline industry. Significant economic pressures from record fuel prices and intense competition, particularly from discount carriers and global airlines, have fundamentally transformed the airline industry in recent decades. This new environment has resulted in diminished profits, restructuring, more than 150,000 lost jobs and financial losses of over $ 29 billion among U.S. network carriers since 2001. Airlines have thus merged to become stronger, as Delta and Northwest did in 2008 or Air France and KLM did in 2004. The recent sharp increase of oil prices has challenged the airlines and revenue management has become a key tool to maintain profitability.

Effective revenue management can save airlines hundreds of millions of dollars a year. According to Klophaus and Polt (2006), for Lufthansa German Airline, 4.9 millions passengers in 2005 were declared as no-shows or cancellations, equivalent to 12,500 full Boeing 747s. Yet revenue management, or, more precisely overbooking, allowed

Lufthansa to carry more than 570,000 additional passengers, leading to a revenue increase of $ 105,000,000 (denied boarding cost included), as compared to the revenue without revenue management.

1.2 Literature Review

Revenue management applied to Airline industry has been widely studied in literature, our review presents an overview of these studies. It also displays the recent ideas introduced in revenue management, that have led to the model, which will be presented.

1.2.1 Techniques used in Airline Booking Process

The seat inventory control problem in airline revenue management concerns the allocation of a finite seat inventory to the demand that occurs over time. In order to decide whether or not to accept a booking request, the airline must compare the opportunity cost of losing the seat taken up by the booking or charging a higher fare later with the revenue generated by accepting the booking request. Solving the seat inventory control problem consists in approximating these opportunity costs and incorporating them in a booking control policy such that expected future revenues are maximized. The airline seat allocation problem confronts two major challenges: lack of an accurate demand forecast and the difficulty in solving large-scale dynamic programming problems, as the computation can become very complex and lengthy.

In single-leg seat inventory, booking control policies for various flight legs are made independently of each other. There are two categories of single-leg solution methods: static and dynamic. In the static model, booking limits per each period and each class are defined in the beginning of the airline booking process. Therefore, the static model does not consider the current booking status during the process. On the other hand, in the dynamic model, the booking limits are updated through the airline booking process, according to the actual status of booking. Dynamic models are therefore more accurate than static ones but more difficult to compute.

1.2.1.1 Static Solution method

Littlewood (1972) was the first to develop an Expected Marginal Seat Revenue (EMSR) approach to find an approximation to an optimal policy for the single-leg, double-fare problem. Thus, his EMSR is the basis of most research in airline revenue management. He proposes that an airline should continue to reduce the protection level for class-1 seats as long as the fare for class-2 (discount) seats satisfies:

$$f_2 \geq f_1 \Pr[d_1 > p_1]$$

where f_i is the fare (or average revenue) of the ith class (class-1 is the full-fare class and class-2 is the discount-fare class), d_1 is the full-fare demand, p_1 is the full-fare protection level and Pr[.] denotes probability. The airline therefore accepts the immediate return from selling an additional discount seat as long as the discount revenue equals or exceeds the expected full-fare revenue from the seat. Belobaba (1987a, 1987b, 1989) enhances Littlewood's work by developing an EMSR approach for a multi-fare problem. The heuristic proposed by Belobaba (EMSR-a and EMSR-b) are the ones mostly used in practice. Brumelle and McGill (1993) define optimality conditions for the airline seat allocation problem when multiple fare classes are booked into a common seating pool in the aircraft (amongst the same cabin) and when the following assumptions are made:

(1) they consider a single-leg flight

(2) the demands for different fare classes are stochastically independent

(3) the reservation requests arrive sequentially in order of increasing fare level, that is low fare booking requests come in before high fare booking requests

(4) cancellations, no-shows and overbooking are not considered

(5) inside the Economy class (which includes several fare classes), any fare class can be booked into seats not taken by lower fare class (this approach is called “nested classes”).

Van Ryzin and McGill (2000) present a Robbins-Monro (1951) stochastic approximation scheme that exploits simulation in order to solve the single-leg problem and they incorporate forecasting as an integral part of the solution, which is a distinct advantage. The approach uses only historical observations of the relative frequencies of certain seat-filling events to guide direct adjustments of the seat protection levels in accordance with the optimality conditions of Brumelle and McGill (1993). Stochastic approximation theory is used to prove the convergence of this adaptive algorithm to the optimal protection levels. The simulation study compares the revenue performance of this adaptive approach to a more traditional method that combines a censored forecasting method with a common seat allocation heuristic (EMSR-b).

1.2.1.2 Dynamic Solution Method and Bid-Price Controls

For overbooking policies and bid-price controls, we refer to Lee and Hersh (1993), Feng and Gallego (1995) and Talluri and van Ryzin (1999). Bid-price controls consist in setting threshold (or bid) prices for seats on flight legs; then each ticket for a specific itinerary and fare class is sold only if the offered fare exceeds the sum of the threshold prices of the flight legs needed to supply the specific ticket. For example, Lee and Hersh (1993) use discrete time dynamic programming in order to develop optimal rules for the single-leg problem when demand in each fare class is modeled as a stochastic process. Feng and Gallego (1995) derive optimal threshold rules when demand in each fare class is modeled as a continuous time stochastic process. In general, the incremental control approach implements a maximum number of additional reservations to be taken, based on the reservations already taken for that flight and the historical patterns for the flight. Several airlines use a level control approach in which reservations are accepted until the total number of reservations exceeds specified or authorization levels.

Chatwin (1998) proposes two dynamic models (stationary-fares and nonstationary-fares) to deal with a multi-period overbooking problem for a single-leg flight with a single service class and use the model to calculate the optimal booking limits. Cancellations may occur at any time, including no-shows at flight departure time. At departure time, the airline may also bump passengers in excess of flight capacity and pay denied-boarding costs. Deriving conditions on the fares, refunds and on the denied-boarding penalty function (that ensure that a booking-limit policy is optimal), and estimating distributions of passenger demand for reservations and cancellations in each period, Chatwin computes the optimal booking limits. The model is applied to the discount allocation problem in which lower fare classes book prior to higher fare classes. Subramanian, Lautenbacher and Stidham (1999) develop a discrete time, finite horizon Markov decision process, and solve it by backward induction on the number of periods remaining before departure. Their model allows cancellations, overbooking and discounting; it solves for the single-leg problem. They assume a Poisson process for cancellations and equal cancellation probabilities for all classes. Biyalogorsky, Carmon, Fruchter and Gerstne (1999) propose that a strategy using overbooking with opportunistic cancellations can increase expected profits and improve allocation efficiency; their work results in deriving a new optimal rule of allocating capacity to consumers. Under their strategy, the seller can oversell capacity when high-paying consumers show up, even if capacity has already been fully booked. Then, the seller will cancel the sale to some low-paying customers while providing them with appropriate compensation. Karaesmen and van Ryzin (2004) consider an overbooking problem with multiple reservation and inventory classes, in which the multiple inventory classes may be used as substitutes to satisfy the demand of a given reservation class. They determine overbooking levels for the reservation classes, taking into account the substitution options. They model this as a two-period optimization problem: in the first period, reservations are accepted given only a probabilistic knowledge of cancellations, whereas in the second period, cancellations are realized and surviving customers are assigned to the various fare classes to maximize the net benefit of assignments. Using a stochastic gradient algorithm (Spall, 2003), they find the joint optimal overbooking levels and when numerically comparing the decisions of their model to those produced by common heuristics, they show that accounting for substitution when setting overbooking levels has a small, but still significant, impact on revenues and costs.

1.2.1.3 Recent Methods

Recently new techniques have been used to solve revenue management problems. As computers become more powerful, simulation plays a bigger role in solving airline revenue management problems, and has recently allowed "model-free" simulation approach. Gosavi, Ozkaya and Kahraman (2007) have developed a model-free simulation-based optimization model to solve the seat allocation problem. (By "model-free", these authors mean that it does not require knowledge of the structure of the stochastic system; all that is required is a numerical value of the objective function.) Their model accommodates the following realistic assumptions:

(1) random customer arrivals for booking

(2) random cancellations

(3) change in arrival rates with time

(4) concurrent (non sequential order) arrivals of passengers, i.e., arrivals do not follow any particular order such as low fare classes first, etc.

A main advantage of using a discrete-event simulator along with a numerical optimization method (these authors use a gradient-ascent technique) is that it requires only estimated numerical values of the objective function, values which can be easily provided by a discrete-event simulator. Yet, the number of simulations required per iteration grows proportionately with the number of decision variables. The model developed by Gosavi et al. (2007) captures the dynamic of cancellations and overbooking, and finally proves to produce robust solutions.

Ching, Li, Siu and Wu (2007) propose an option-based revenue management model that maximizes the travel agent's profit. Travel agents purchase a call option from their customers while selling them air tickets. The call option premium appears in the form of a discount off the regular airfare. A call option is a financial contract between a buyer and a seller. The buyer of the option has the right, but not the obligation to buy an agreed quantity of a particular commodity from the seller of the option by a certain time for a certain price, called the strike price. The seller is obligated to sell the commodity should the buyer so decide, and the buyer pays a fee (called a premium) for this right. The call option premium appears in the form of a discount off the regular airfare. Customers who are flexible regarding all aspects of their trips and have the possibility of changing their travel plans would be willing to accept the deal. In this case, in exchange for a compensation (the strike price), travel agents can call back these tickets when some other customers have an obligation to travel or purchase air tickets at much higher prices shortly before departure. Ching et al. compute the call option price according to the Black-Scholes pricing model (1973), for a discounted airfare for both American and European options.

An exhaustive literature search yields to only one paper (Akgunduz, Turkmen and Bulgak , 2007) on financial options theory applied to the airline booking system. In their work, the authors present a model where call options are used to sell cheap recallable tickets when the expected demand is higher than the capacity and put options are used to deal with travel agents to ensure a number of tickets to be sold when the expected demand is low. A put option is a contract between the seller, and the buyer of the option.

The put option allows the buyer the right but not the obligation to sell a commodity to the seller at a certain time for a certain strike price The seller has the obligation to purchase the underlying asset at that strike price, if the buyer exercises the option. Akgunduz et al. (2007) assume normal demand and overbooking with a binomial probability of no-shows and consider a multiple-fare-classes single-leg system. Using simulation, they prove for different scenarios that using financial options in airline booking is more efficient than using "regular" overbooking model, that is, it gives superior revenues. The model of Akgunduz et al. largely inspires the model described in this thesis.

A comprehensive literature review on the revenue management problem is provided by McGill and Van Ryzin (1999) and later by Chiang, Chen and Xu (2007). In addition, several books on the topic have been published recently. For instance, Robert Philips's *Pricing and Revenue Optimization* (2005) provides a comprehensive introduction of pricing and revenue management.

1.2.2 Options Analysis and Revenue Management

In recent years a new theory of pricing and operating assets has been developed when uncertainty and managerial flexibility in operating strategies are involved, this is the theory of real options (Dixit and Pindyck, 1994). Amram and Kulatilaka (1999) have studied analytical approaches applying real options. In general, financial options are used to minimize the risk exposure of a firm. Yet, there is a limited amount of literature concerning the issue of controlling risk in revenue management when the price of the underlying commodity, such as the ticket price or the capacity, fluctuates randomly over time.

Law, MacKay and Nolan (2004) use financial options to hedge rail transportation capacity. They describe a market for derivative securities (for example, futures or options) on rail car capacity within a completely deregulated rail industry. Their model is illustrated with the example of coal transportation, where they use the standard binomial option pricing model (cf. Cox, Ross and Rubinstein, 1979) to calculate the prices of European call options for transporting coal by train. The underlying asset is a unit of rail car capacity suitable for coal transportation.

Gallego and Philips (2004) define revenue management of a flexible product, a set of two or more alternatives serving the same market such that the purchase of the flexible product will be assigned to one of the alternatives by the seller at a later date. For instance, if there were three flights from A to B during the day, the customer purchasing a flexible ticket is guaranteed service by one of the three alternatives but the airline would not assign a specific flight until a later date. These flexible products present the advantage of increasing overall demand and enabling better capacity utilization at the cost of potentially cannibalizing high-fare demand for specific products. The work of Gallego and Philips (2004) derives conditions and algorithms for a single flexible product consisting of two alternatives, and a numerical example illustrates the benefits of offering flexible products. The approach is very similar to a real option-based model.

Traditionally, the main focus of the literature on revenue management concerns profit enhancement or profit maximization for industries, such as the airline industry or the car rental industry, with fluctuating commodity prices. Anderson, Davison and Rasmussen (2004) are pioneers into introducing the financial options theory in the revenue management booking system. They present a novel real options approach to revenue management that is specifically suited to the car rental business. A numerical example with actual car rental data illustrates their model, which produces minimally acceptable prices and inventory release quantities (number of cars available for rent at a given price) as a function of remaining time and available inventory. The price to rent a car is modeled as following a random walk (Enders, 1995, p. 4); with analogy with the financial options theory, Anderson et al. derive a Black-Scholes-like partial differential equation and provide an approach that includes competitive effects in revenue management settings. For commodity based service businesses, like car rentals (where customer switching costs are low), the approach is related to the well-known swing options (options that permit the holder to repeatedly exercise the right to receive greater or smaller amounts of energy; see Jaillet, Ronn and Tompaidis, 2003) used in the power industry. Modeling price as a stochastic differential equation defines then a novel approach to including these competitive effects in revenue management applications.

But as mentioned before, the utilization of financial options in the airline booking system has only been studied by Akgunduz et al. (2007). In their work, Akgunduz, Turkmen and Bulgak present a model involving two kinds of financial options: (1) call options, which the airline buys from the willing passenger who agrees to buy a therefore cheaper but recallable ticket (at a certain date in the future), and (2) put options, which the airline buys from a travel agent who accepts to purchasing an agreed-upon number of tickets at a certain date in the future. In their model, call and put options are mutually exclusive, as the airline considers call options when the expected total demand is higher than the capacity of the aircraft, and considers put options in the other case. The maturity date (time where the airline decides to exercise its option) and the strike price are decided when the airline and the passenger or travel agent agrees on the contract, i.e., the airline buys the options. As a call (put) option is a right but not an obligation, the airline can decide to recall tickets from the customer (to sell the tickets to the travel agent) up to the number of options purchased. Alternatively, the airline can also decide to let its call (put) option expire and try to sell its remaining tickets in the market. Their model incorporates no-show probabilities and denied-boarding, and they derive the expected revenue and detail a simulation methodology to explore effectiveness:

$$E[R] = \sum_{j=1}^{t}\sum_{i=1}^{n} b_{ij} f_i - n_c^e \pi_c - n_{db} c_{db} - n_p p_p - n_c p_c$$

In the above

j indexes periods $j=1,\ldots t$,

i indexes the fare class $i=1,\ldots n$,

b_{ij} is the booking limit of class i and period j,

f_i is the fare of class i (assumed to be constant over time),

n_C and n_P are respectively the initial number of call and put options purchased by the airline,

p_C and p_P are respectively the premiums (price paid by the airline per option purchased) of call and put options,

n_{db} and c_{db} are respectively the number of passengers who are denied boarding and the denied boarding cost per bumped passenger,

n_c^e is the number of call options exercised

π_c is the exercise price of call options.

1.3 Thesis Research Objectives and Contribution

The objective of this research is to develop a model with call and put options that can be used by the airlines in order to maximize their revenues. Though this work is inspired by the model of Akgunduz et al. (2007), it is different in the following:

(1) a single-fare class is studied,

(2) ticket price in the fare class is assumed to follow a random walk and has therefore an associated distribution, and

(3) a binomial options pricing model is used to derive the premiums of both types of options.

This work provides some important features, as this type of model maximizes the expected revenue by finding the optimal values of six decision variables: the initial numbers of purchased call and put options, the premiums of these options (paid by the airline) as well as the optimal strike prices for both types of options. But in this model, the options are priced according to the binomial option pricing model (BOPM), resulting in reducing the number of decision variables from six to four (initial numbers of options purchased and strike prices). (BOPM is an options valuation method developed by Cox, Ross, and Rubinstein in 1979. It is an iterative procedure, allowing for the specification of nodes, or points in time, during the time span between the valuation date and the option's expiration date. The model reduces the likelihood of price changes, removes the possibility for arbitrage, attempt to profit by exploiting price differences, assumes a perfectly efficient market, and shortens the duration of the option. Thus, it is a mathematical valuation of the option at each point in time specified.) Moreover, introducing uncertainty for the ticket price at each period makes the model more realistic. A single-leg single-fare class flight is studied, yet different prices associated with different probabilities, according to the assumptions of the random walk model, are computable during one period; introducing several classes would only make computations more complex as demand between fare classes are assumed independent. So, a methodology is proposed to compute the local optima of the decisions variables in order to maximize the expected revenue of the airline. A numerical example is given to explain the methodology and sensitivity analysis of the example is conducted to examine the effect on the optimal solution to variations in the parameters of the random walk and the skewness of the demand distribution.

1.4 Overview of Thesis

The research presented in this thesis is categorized according to the following:

- Chapter 2 provides a description of the model presented in this thesis and derives the objective function. It also derives the binomial option pricing of the call and put options utilized in the model described earlier.

- Chapter 3 provides a numerical methodology to solve the model presented in Chapter 2. A numerical example is used to illustrate the method presented. Sensitivity analysis is performed to demonstrate how changing some of the parameter values affects the objective function.

- Future work and concluding remarks are addressed in Chapter 4.

CHAPTER 2

MODEL FORMULATION

This chapter presents a model for computing the expected revenue of an airline when using call and put options. These financial options are used to hedge risk due to demand the airline is experiencing when booking tickets. After describing the model, the price of the option used is computed using the binomial options pricing model.

2.1 Model

2.1.1 Description of the model

The time interval between the beginning of airline booking and the flight departure is divided in J periods (of length h). A single-class, single-leg flight is considered. During the first period, the airline sells tickets to customers and travel agents with call and put options. The options expire at the beginning of period J, at time T, which is called "expiration or maturity date". From the first through the penultimate period, the airline sells tickets "regularly", i.e. without options, at a price S_j depending on the market conditions for period $j = 1,\ldots, J\text{-}1$. The price S_j of tickets at period j is a random walk. The price of tickets at the beginning of period 1 is denoted by S_0. The model uses European options, i.e. the options cannot be exercised before the expiration date.

Description of the model with call options

At the beginning of the first period, n_{0C} tickets are sold with call options, which means that these n_{0C} tickets sold during the first period can be recalled by the airline if needed at the expiration date T of the option. The airline pays the option price (which is also called option premium) c_S to the passenger, and hence the passenger buying a ticket with a call option pays only S_0 - c_S (discounted price) to the airline. A call option is a contract, which gives the airline the right to recall the ticket at time T. When buying the tickets sold with a call option, the customer agrees to sell the tickets back to the airline at price K if the airline decides to exercise its right to recall the ticket. This price K is called strike price or exercise price, and is decided upfront when the contract is settled.

If the airfare at time T (maturity date) is greater than the exercise price K, then the airline will recall the tickets and resell them at the market price S_J, and hence the net cash flow at the beginning of period 1 (time 0) to the airline for this ticket which is recalled is $S_0 - c_S + B_r(J)\ (S_J - K)$, where $B_r(J)$ is the discount factor of J periods. It is assumed that the cash flow occurs at the end of period J. If S_J is smaller than the exercise price K, then the airline will not exercise the option and will not recall the tickets. The loss to the airline is only the premium paid for the lowest fare tickets (c_S), resulting in a net cash flow of S_0 - c_S at the beginning of the first period. The two possible net profits at time 0 from one ticket hedged with a call option are displayed in Figure 2.1.

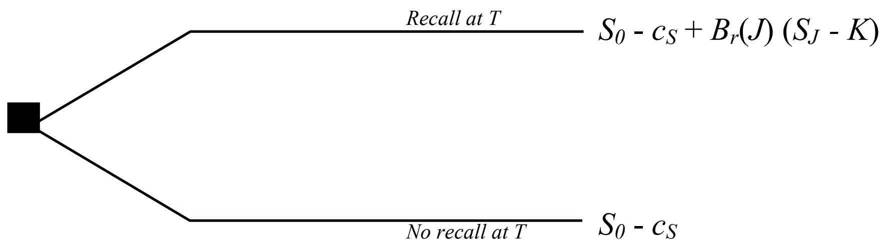


Figure 2.1: Possible net profit from one ticket with call option

Description of the model with put options

At the beginning of the first period (beginning of the airline booking process), the airline purchases n_{0P} put options from one (several) travel agent(s) at price p_S per ticket, which is called the premium of put option. As having one or several travel agents does not change the following, they will be referred as one entity. By hedging n_{0P} tickets with put option, the airline pays a price of p_S per ticket to the travel agent, which gives the airline the right, at time T, to sell up to n_{0P} tickets to this travel agent at a price L per ticket. This price L is called “exercise or strike price” and is decided upfront when the airline buys the put options from the travel agent.

If $S_J < L$ at time T, then the airline will exercise its right to sell the ticket to the travel agent at the strike price L, resulting in a net cash flow of $B_r(J)\ L$ - p_S. It is assumed that the cash flow occurs at the end of period J. If $S_J > L$, then the airline will not exercise the right to sell the ticket to the travel agent and will sell the ticket in the market at price S_J, resulting in a cash flow of $B_r(J)\ S_J$ - p_S. The two possible net profits from one ticket hedged with a put option are displayed in Figure 2.2.

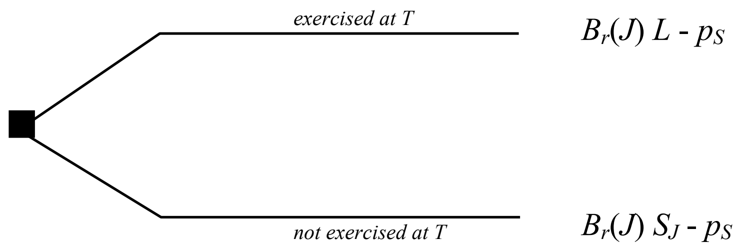


Figure 2.2: Possible net profit from one ticket with put option

From the above description of the model, it follows that if the airline expects the total demand to be greater than the capacity of the aircraft, it will purchase call options on some low-fare tickets, which could be recalled and sold at a higher price. On the other hand, if the airline expects the total demand to be lower than the capacity of the aircraft, it will purchase put options to ensure the sale of some remaining tickets. By definition, the call and put options are mutually exclusive.

2.1.2 Assumptions

- The airline determines the number of call and put options to be purchased, respectively n_{0C} and n_{0P}.
- The probability of a passenger not showing up, s, is assumed to be constant, and is called the probability of no-shows.
- A single class flight is considered in this research.
- Demand is a random variable modeled by a discrete distribution. Demands between periods are independent.

- Fare per class evolves according to changes following a random walk; an analogy can be made between stock price and ticket price. The distribution of the price is detailed in section 2.4.

- The binomial option pricing model is used to price the call and put option premiums, as described in section 2.4.

- Cash flows are assumed to occur at the end of each period.

- Overbooking is allowed.

- The model uses European options, that is the options are exercised only at maturity date, if exercised.

2.1.3 Variables

To build the model of airline booking using financial options, it is essential to define the following variables:

- S_0 is the price paid for each ticket per customer at the beginning of the airline booking process.

- S_j is the random variable characterizing the price paid for each ticket per customer during period j of the booking process, j=1, …, J. Specifically, S_J is the expected price per unit of ticket during the last period J, so this is also the price per unit of ticket at time T.

- d_j is the demand during period j, j=1,…, J and follows the discrete distribution $f_{D_j}(d_j)$.

- s is the probability of a no-show.

- n_j is the number of tickets sold by the airline during period j, j=1,… , J-1, at price S_j. As introduced by Akgunduz et al. (2007), an authorization level, a_j, is computed to limit the number of tickets sold per period j, that is,

  for j =1,… , $(J - 1)$, $\quad n_j = \min(d_j, a_j)$, [2.1]

  where for j =1,… , $(J - 1)$, $\quad a_j = (1 + s)\, \mathrm{E}[d_j]$

  and s is the probability of a no-show.

- n_J^C is the number of tickets sold by the airline during the last period at price S_J in the call option model.

- n_J^P is the number of tickets sold by the airline during the last period at price S_J in the put option model.

- n_J is the number of tickets sold during the last period: $n_J = n_J^C + n_J^P$.

- c_S is the premium of one call option.

- p_S is the premium of one put option.

- K is the strike price associated with a call option to compensate the customer from whom the ticket is recalled at time T.

- L is the strike price associated with a put option, at which the airline sells the tickets to the travel agent at time T.

- n_{0C} is the number of tickets sold with a call option to the customers by the airline at the beginning of the airline booking process (t=0).

- n_{0P} is the number of units of put options purchased from the agent by the airline at the beginning of the airline booking process (t=0).

- m is the number of units of tickets sold in the market from the first period until the end of the penultimate period (*j = 1,..., J-1*) of the airline booking process:

$$m = \sum_{j=1}^{J-1} n_j \ . \qquad [2.2]$$

- C is the capacity of the plane (number of seats available).

- T is the expiration or maturity date of the option.

- n_C^e is the number of tickets recalled from the customers.

- n_P^e is the number of tickets under put options sold to the travel agent.

- n_{db} is the number of passengers who are denied boarding.

- c_{db} is the cost of denying boarding to a passenger.

- $B_r(j)$ is the discount factor between time the initiation of the booking process and period j, where $B_r(j) = e^{-rjh}$ and r is the nominal interest rate (continuous compounding).

- μ and σ are the random walk parameters

- u and d are the binomial parameters

In order to formulate the model more easily, the two mutually exclusive types of options are presented separately to give birth to two sub-models described in the two following sections. The two derived models will then be gathered into a single one, as it will be shown in section 2.5.

2.2 Call Options Modeling

In this section, it is assumed that at the beginning of the booking process (time 0) the total expected demand is higher than the capacity of the aircraft; only call options are considered.

The number of tickets recalled from the customers is n_C^e, if the airline decides to recall the tickets at maturity date T at price K per ticket. If the strike price K is smaller than the market price S_J, then the airline recalls tickets at price K and sells them directly in the market during the last period at price S_J per ticket. The number of tickets recalled n_C^e, at the expiration date T, depends on the number of tickets the airline has sold until time T $\left(m = \sum_{j=1}^{J-1} n_j \right)$, and the expected demand in period J ($E[d_J]$). As n_C^e cannot exceed the initial number of tickets hedged with call options (n_{0C}), it follows that:

$$n_C^e = \min\left\{ m + E\left[d_j\right] - C, \quad n_{0C} \right\} \qquad [2.3]$$

It is assumed in [2.3] that the sum of the number of tickets sold until period *J-1* and the expected demand in period J would be greater than or equal to the capacity of the aircraft, that is,

$$m + E[d_J] - C \geq 0 \quad . \qquad [2.4]$$

If m + E[d_J] – C < 0, that is if the airline expects to have its flight depart with empty seats, the airline will not recall any of the n_{0C} tickets ($n_C^e = 0$). Therefore as the number of tickets recalled must be non-negative, it can be defined as following:

$$n_C^e = \max\left(\min\{m + E[d_J] - C, \quad n_{0C}\}, \quad 0\right) \quad . \qquad [2.5]$$

There are two possible scenarios regarding the number of tickets sold in the last period J, n_J^C, at price S_J per ticket::

(i) If $C \leq m + E\left[d_J\right]$, that is, if the sum of the number of tickets sold until the end of period J-1 and the expected demand in period J is greater than or equal to the aircraft capacity, then the airline recalls n_C^e tickets hedged with call options (n_C^e was defined in [2.5]) that are sold in the market during the last period. Moreover, the airline is also selling at price S_J the number of remaining seats, which is C - $n_{0C} - m$. Yet, no more tickets than demanded can be sold, so the number of tickets sold during the last period at price S_J is:

$$n_J^C = \min\left\{C - n_{0C} - m + n_C^e, \quad d_J\right\} \quad . \qquad [2.6]$$

(ii) If $m + E\left[d_J\right] < C$, that is if the airline expects the sum of the tickets sold until the end of period J-1 and the expected demand of tickets during period J to be smaller than the aircraft capacity, then the company will not recall the tickets (from [2.5], $n_C^e = 0$), and the number of tickets sold during the last period is the minimum of the demand during period J, d_J, and the remaining seats (C - $n_{0C} - m$), that is,

$$n_J^C = \min\{ C - n_{0C} - m, d_J \} \quad . \qquad [2.7]$$

Combining [2.6] and [2.7], the expression for the number of tickets sold during the last period, n_J^C, is derived, as follows:

$$n_J^C = \min\{d_J, \quad C - n_{0C} - m + n_C^e\} \quad . \qquad [2.8]$$

Therefore, in this last period J, the net cash flow for the airline is the sum of the inflow generated by the sale of the n_J tickets at price S_J during period J and the cost of recalling n_C^e tickets at price K at time T:

$$C_J = n_J^C S_J - K n_C^e \quad . \qquad [2.9]$$

Moreover, without considering any type of options, the expected net income earned from the first period until the end of the penultimate period R_0 is

$$E[R_0] = E\left[\sum_{j=1}^{J-1} B_r(j) n_j S_j\right] \quad . \qquad [2.10]$$

$$E[R_0] = \sum_{j=1}^{J-1} \left(B_r(j) \ E\left[n_j S_j\right]\right) \quad . \qquad [2.11]$$

The objective function is to maximize the expected profit to the airline. Considering only call options and including the net cash flow at time 0 (n_{0C} (S_0 - c_S)) and during period J (defined in [2.9]), the objective function is therefore defined by

$$E[R] = n_{0C} S_0 + E[R_0] - n_{0C} c_S + B_r(J) \ E[n_J^C S_J - K n_C^e], \qquad [2.12]$$

which is equal to

$$E[R] = n_{0C} S_0 + E[R_0] - n_{0C} c_S + B_r(J) E[n_J^C S_J] - B_r(J) K \, E[n_C^e] \ . \qquad [2.13]$$

2.3 Put Options Modeling

In this section, it is assumed that at the beginning of the booking process (time 0) the total expected demand is smaller than the capacity of the aircraft; only put options are considered and call options are not used.

The number of tickets under a put option sold to the travel agent at time T is n_P^e, as the airline has bought n_{0P} put options from the travel agent at the beginning of the airline booking process. It is important to remind the reader that the airline will only decide to sell n_P^e tickets to the travel agent at maturity date T at price L, if the strike price L is greater than the price of the market S_J.

If, at time T, the sum of the tickets sold until the end of the penultimate period and the expected demand of period J is less than the capacity of the aircraft (m + E[d_J] < C), then the airline will require the travel agent to buy n_P^e tickets hedged with put options. A first expression of n_P^e can thus be derived, as follows:

$$n_P^e = C - \left(m + E\left[d_J\right]\right) \quad . \qquad [2.14]$$

Yet, the airline cannot force the travel agent to buy more than n_{0P} tickets at price L. Therefore [2.14] becomes:

$$n_P^e = \min\left\{C - \left(m + E\left[d_J\right]\right), \quad n_{0P}\right\} \quad . \qquad [2.15]$$

If $C - m$ - E[d_J] < 0, that is, if at time T the expected total number of tickets sold at the end of period J is greater than the capacity of the aircraft, then the airline will not exercise its option to sell any of the n_{0P} tickets to the travel agent:

$$n_P^e = 0 \,. \qquad [2.16]$$

Now combining the two outcomes ($C - m$ - E[d_J] < 0 and $C - m$ - E[d_J] ≥ 0) [2.15] and [2.16] leads to derive a single expression of n_P^e , as follows :

$$n_P^e = \max\left(0, \quad \min\left\{C - \left(m + E[d_J]\right), \quad n_{0P}\right\}\right). \qquad [2.17]$$

There are two possible scenarios regarding the number, n_J^P, of tickets sold in the last period J at price S_J per ticket:

(i) If, at time $T,$ the sum of the tickets sold until the end of period J - 1 and the expected demand of period J is less than the capacity of the aircraft (m + E[d_J]< C), then the airline will sell n_P^e tickets to the travel agent at the exercise price L. After selling these n_P^e to the travel agent, the airline sells the remaining empty seats ($C - m$ - n_P^e) in the market at price S_J , during the last period. Yet the airline cannot sell more tickets at price S_J than the demand d_J during period J ($\min\left\{d_J, \quad C - m - n_P^e\right\}$). Hence, the number of tickets directly sold in the market is

$$n_J^P = \min\left\{d_J, \quad C - m - n_P^e\right\} \quad . \qquad [2.18]$$

(ii) If $C \leq m$ + E[d_J], that is, if the sum of tickets sold until the end of period J - 1 and the expected demand in period J is greater than or equal to the capacity of the aircraft, then the company will not exercise its put options ($n_P^e = 0$ from [2.16]), but will sell the remaining tickets in the market, and the number of tickets sold during the last period is

$$n_J^P = \max\{0, \min\{ d_J, C - m\}\} . \qquad [2.19]$$

Combining the two scenarios ((i) and (ii)), the definition of n_J^P is as follows:

$$n_J^P = \max\left(0, \quad \min\left\{C - m - n_P^e, \quad d_J\right\}\right). \qquad [2.20]$$

The utilization of the maximum function is due to the fact that n_J^P is a positive variable: if there are no remaining seats after selling the n_P^e tickets to the travel agent, that is if $C - m - n_P^e \leq 0$, then the airline will not sell any ticket to the customers at price S_J during last period J ($n_J^P = 0$).

Therefore, in this last period J, the net cash flow for the airline is

$$C_J = n_J^P S_J + n_P^e L \qquad . \qquad [2.21]$$

The objective is to maximize the profit to the airline. Only considering put options, and including the net cash flow at time 0 ($- n_{0P}\, p_S$) and during period J (see [2.21]), the objective function is defined by:

$$E[R] = (E[R_0] + B_r(J)\, E[n_J^P S_J]) - n_{0P}\, p_S + B_r(J)\, L\, E[n_P^e] \ , \qquad [2.22]$$

where E[R_0] is the expected revenue from period 1 up to and including period *J-1* and was given in [2.11].

2.4 Overbooking and Denied Boarding Costs

The authorization levels defined in [2.1] allow underbooking and overbooking, as the total number of tickets sold can differ from the capacity of the aircraft. The options introduced in this model hedge the risk of underbooking or overbooking a flight, because

(i) By buying put options, the airline reduces the risk of underbooking as it can ensure the sale of some of its tickets.

(ii) By buying call options, the airline controls the overbooking.

Yet, because of the uncertainty of demand, underbooking and overbooking are still possible. Underbooking costs the airline revenue, but it is not a negative cash flow. On the other hand, the cost per seat overbooked is the extra compensation, c_{db}, the airline pays to the dissatisfied customer who is bumped. The number of denied boardings, n_{db}, is the difference between the number of passengers showing up for boarding the flight and the capacity of the aircraft. Therefore the total cost of overbooking is the cost of denying boarding to n_{db} passengers.

In order to define the expression of the total number of tickets sold during the booking process, with call options or put options, we examine the two models separately.

Call option model

In the call option model described in section 2.2, the total number of seats sold from the beginning of the first period until the beginning of the last period is $m + n_{0C}$. During the last period, the airline sells n_J^C tickets at price S_J . Therefore, during the total length of the booking process, from time 0 until the end of period J, the total number of tickets sold is

$$N_C = n_{0C} + m + n_J^C \quad . \qquad [2.23]$$

Put option model

In the put option model described in section 2.3, the total number of seats sold from the beginning of the first period until the beginning of the last period is m. During the last period, the airline sells n_J^P tickets at price S_J and n_P^e tickets at price L. Therefore, during the total length of the booking process, from time 0 until the end of period J, the total number of tickets sold is:

$$N_P = m + n_P^e + n_J^P \quad . \qquad [2.24]$$

General denied boarding

The two models are mutually exclusive, because

- call options are used if, at the beginning of the booking process, the airline expects the total demand to be larger than the capacity of the flight.
- put options are used if, at the beginning of the booking process, the airline expects the total demand to be smaller than the capacity of the flight.

The number of tickets sold during period J at price S_J is denoted by n_J and can be derived by $n_J = n_J^C + n_J^P$, when considering a single expression.

The total number of tickets sold during the entire booking process can be defined with a unique expression, as follows:

$$N = n_{0C} + m + n_P{}^e + n_J \quad . \qquad [2.25]$$

If the total number of passengers showing up for boarding happens to be larger than the capacity of the aircraft, then denied boarding costs are involved. As the proportion of no-shows is s, the number of passengers with tickets who show up for a given flight is $(1-s)N$ where N is defined in [2.25]. Hence the number of passenger who are denied boarding, n_{db}, is as follows:

$$n_{db} = \max\{0, \quad (1-s)N - C\} \quad . \qquad [2.26]$$

The cost of denying a passenger to board the aircraft is c_{db}; thus, the total expected cost of denied boarding for the airline is

$$C_{db} = c_{db} E[n_{db}] \, , \qquad [2.27]$$

where n_{db} is was defined in [2.26].

2.5 Objective Function

The two scenarios described in sections 2.2 and 2.3 are mutually exclusive; therefore, when eliminating the redundancy between [2.13] and [2.20], one single objective function can be derived. The total denied boarding cost for the airline, derived in [2.27], must be introduced, and discounted from the Jth period.

Thus, the expression for the objective function is the following:

$$
\begin{aligned}
E[R] = {} & n_{0C}(S_0 - c_S) - n_{0P}\, p_S + \left(E[R_0] + B_r(J)\; E\left[\left(n_J^C + n_J^P\right)S_J\right]\right) \\
& + B_r(J)\left(-K\; E\left[n_C^e\right] + L\; E\left[n_P^e\right]\right) \\
& - B_r(J)\; c_{db} E\left[n_{db}\right]
\end{aligned}
$$

[2.28]

After factorization, [2.28] becomes:

$$
E[R] = n_{0C}(S_0 - c_S) - n_{0P}\, p_S + E[R_0] + B_r(J)\left(E\left[n_J\, S_J\right] - K\; E\left[n_C^e\right] + L\; E\left[n_P^e\right] - c_{db} E\left[n_{db}\right]\right).
$$

[2.29]

In the next section, the decision variables c_S and p_S will be shown to be functions of two other decision variables, K and L.

2.6 Binomial options pricing

In the expected revenue expression derived in [2.29], it is essential to derive not only the distribution of the ticket price but also the premiums of both call and put options, c_S and p_S, in order to reduce the number of decision variables. In order to do so, we use the binomial option pricing.

2.6.1 The Binomial Option Pricing Model (BOPM)

BOPM is an options valuation method developed by Cox et al. (1979). The model uses a "discrete-time" model of the varying price of the underlying asset over the time span between time 0 and the expiration date *T* and traces the evolution of the underlying price via a binomial tree. This option valuation method is an iterative process that works backwards through the tree to the first node (valuation date), where the calculated result is the value of the option. Thus, it is a mathematical valuation of the price of the underlying asset and of the option at each point in time.

Because of the assumption of continuously compounded returns, although the price of the ticket may go up and down, even cumulative down movements will never cause the price to be negative. Modeling the ticket price as a random walk is realistic, because over the length of the booking process, the ticket price of a same fare class is subject to variations, due to the influence of various factors.

2.6.2 Distribution of the ticket price, S_j

2.6.2.1 Evolution of the ticket price, S_j

The price of the ticket at time 0 is S_0, and S_j , j = 1,…, J is the price of the ticket during period j. Let u_j and d_j be the binomial parameters used to quantify the increase and decrease of the ticket price, S_j, over period j. In each time interval, S_j may increase in value to u_jS_j (with u_j>1) with a probability p_j or it may decrease in value to d_jS_j (with $0 \le d_j <1$) with probability 1- p_j. This process is described with Figure 2.3.

S_j

$S_{j+1} = u_j\, S_j$, with probability p_j ,

$S_{j+1} = d_j\, S_j$, with probability $(1\text{-}p_j)$.

Figure 2.3: Ticket Price Evolution

The risk-free interest rate for period j, j = 1, …, J , is r_j, and we assume that S_j behaves as described in Figure 2.3. Assuming continuous compounding, the discount factor relative to the interest rate, r_j, for one binomial step, is $B_{r_j} = e^{-r_j h}$, where h is the length of a period. After the time interval h, there are two possible outcomes that are discounted in the following tree:

S_j

$B_{r_j} u_j\, S_j$, with probability p_j,

$B_{r_j} d_j\, S_j$, with probability $(1\text{-}p_j)$.

Figure 2.4: Possible discounted revenues for one ticket

It is assumed (Cox, Ross and Rubinstein, 1979) that the price of a ticket during period j, S_j, is equal to the discounted value of S_{j+1}, and, hence,

$$S_j = B_{r_j}\left[p_j\, u_j S_j + \left(1 - p_j\right) d_j S_j\right] \quad , \tag{2.30}$$

from which p_j is obtained:

$$p_j = \frac{B_{r_j}^{-1} - d_j}{u_j - d_j} \quad . \qquad [2.31]$$

2.6.2.2 Recombinant trees

To derive the distribution of the ticket price and to compute the option premiums, u_j and d_j are now assumed to be constant, i.e., for j=1, …, J, $u_j = u$, and $d_j = d$. Moreover, it is assumed that the interest rate r_j is constant over time; hence for j=1, …, J, $r_j = r$ and $B_{r_j} = e^{-rh}$, denoted by B_r. Therefore, the parameter p is constant, $p_j = p$, and:

$$p = \frac{B_r^{-1} - d}{u - d} \qquad [2.32]$$

The number of possible nodes is now significantly reduced and the tree is said to recombine, as shown in Figure 2.5.

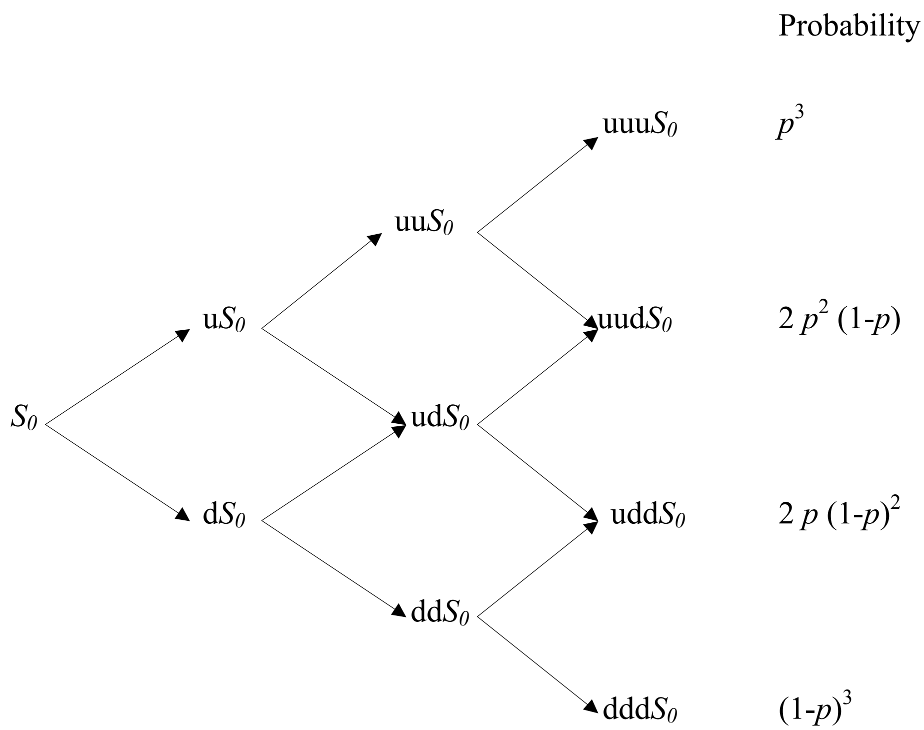


Figure 2.5: Recombinant tree

During period j, after k increases and j-k decreases, k = 0, …, j, the price of the ticket is defined as follows:

$$S_j = u^k d^{j-k} S_0 \quad . \qquad [2.33]$$

The probability that there are exactly k price increases and j - k price decreases out of j moves (periods), j = 1, …, J and k = 0, …, j, is

$$P_{j,p}(k) = \binom{j}{k} p^k (1-p)^{j-k} \quad . \qquad [2.34]$$

Hence, this is the probability that the price of the ticket, starting with the value S_0, undergoes k price increases in j periods (as defined in [2.33]):

$$P\left(S_j = u^k d^{j-k} S_0\right) = P_{j,p}(k) \quad , \qquad [2.35]$$

with $j = 1, \ldots, J$ and $k = 0, \ldots, j$.

2.6.3 European Options Pricing

In our model, we decide to use European call and put options, that is the options are exercised only at maturity date T, if exercised.

Assuming the call and put options have the same maturity date but different exercise prices (K and L), the values of the call option and put option at time T, beginning at period J, are the following:

$$c_S(J) = \max\{0, S_J - K\} \qquad [2.36]$$

$$p_S(J) = \max\{0, L - S_J\} \qquad [2.37]$$

After k price increases, the price of the ticket, S_J, during period J is $S_J = u^k d^{J-k} S_0$, as seen in [2.33]. This will occur with probability $P_{J,p}(k)$, $k = 0, \ldots, J$, as defined in [2.34].

Therefore, the expected values for the call and the put European options are

$$E[c_S(J)] = \sum_{k=0}^{J} P_{J,p}(k) \max\{0, u^k d^{J-k} S_0 - K\}, \qquad [2.38]$$

$$E[p_S(J)] = \sum_{k=0}^{J} P_{J,p}(k) \max\{0, L - u^k d^{J-k} S_0\} \quad . \qquad [2.39]$$

Hence, at time 0, with $B_r(J)$ the discount factor for J periods, the respective premiums c_S and p_S, of the European call and put options are defined as follows:

$$\begin{aligned} c_S &= B_r(J)\, E[c_S(J)] \\ &= B_r(J) \sum_{k=0}^{J} P_{J,p}(k) \max\{0, u^k d^{J-k} S_0 - K\} \end{aligned} \quad , \qquad [2.40]$$

$$\begin{aligned} p_S &= B_r(J)\, E[p_S(J)] \\ &= B_r(J) \sum_{k=0}^{J} P_{J,p}(k) \max\{0,\ L - u^k d^{J-k} S_0\} \end{aligned} \quad . \qquad [2.41]$$

So, it can be seen that for optimizing E[R] as defined in [2.29] requires only solving for K, L, n_{0C}, and n_{0P} , as two decision variables c_S and p_S can be expressed as functions of K and L.

2.6.4 Derivation of the binomial parameters u and d

The binomial parameters u and d play a key role in pricing options. For a given pair of parameters (u, d), the distribution of the ticket price and the premiums of the call and put options are known. Yet, although we assumed that u and d are constant over time until the last period, the binomial parameters u and d are difficult to estimate for the airline. This is why, from the assumption of continuously compounding of the BOPM, the drift, μ, and the standard deviation, σ, are introduced (see Cox et al. , 1979), as follows:

$$\mu T = E\left[\ln\left(\frac{S_J}{S_0}\right)\right], \qquad [2.42]$$

$$\sigma^2 T = \mathrm{var}\left(\ln\left(\frac{S_J}{S_0}\right)\right) \quad . \qquad [2.43]$$

The drift, μ, is a deterministic term expressing that the sale of the ticket should have a positive return in the long term, that is, the mean of the random walk of the prices is positive and increasing with time. As shown in Equation [2.42], the drift, μ, is the logarithmic change of the ticket price over *J* periods. The binomial parameters *u* and *d* can be expressed as functions of μ and σ. It seems easier for the airline to estimate the logarithmic change in price, using historical data, than to evaluate directly the binomial parameters *u* and *d*.

Cox et al. (1979) prove that the binomial parameters *u* and *d* are solutions of a non-linear system and solve for *u* and *d* in linear order of *h*, where *h* is the length of a period. In the solution they propose, the drift μ does not explicitly affect *u* and *d*. Yet, when deriving *u* and *d* in this case, it seems relevant to introduce the mean of the random walk, μ, which the ticket price follows. Hence, in the model presented in this thesis, the binomial parameters, *u* and *d*, as proposed by Jarrow and Rudd (1982) are used.

The derivation of the binomial parameters u and d is presented in an intuitive form, as follows:

$$u = e^{\mu h + \sigma\sqrt{h}} \ , \qquad [2.44]$$

$$d = e^{\mu h - \sigma\sqrt{h}} \ . \qquad [2.45]$$

2.7 Final problem definition

Using the expressions [2.40] and [2.41] derived from the option pricing reduces the decision variables from $n_{0C,}$ $n_{0P,}$ K, L, c_S, and p_S to $n_{0C,}$ $n_{0P,}$ K, and L. Substituting c_S and p_S in [2.29], the expected revenue can be expressed as follows:

$$\begin{aligned} E[R] = {} & n_{0C}\left(S_0 - B_r(J)\sum_{k=0}^{J} P_{J,p}(k)\max\left\{0,\ \left(u^k d^{J-k} S_0\right) - K\right\}\right) \\ & - n_{0P}\left(B_r(J)\sum_{k=0}^{J} P_{J,p}(k)\max\left\{0,\ L - u^k d^{J-k} S_0\right\}\right) \\ & + E[R_0] + B_r(J)\left(E[n_J S_J] - K\,E\left[n_C^e\right] + L\,E\left[n_P^e\right] - c_{db}E[n_{db}]\right). \end{aligned}$$

[2.46]

The values of the binomial parameters u and d, which characterize the ticket price value and the premiums of call and put options, are defined in [2.44] ad [2.45]. With historical data, the airline can estimate σ (standard deviation of the price change per period) and μ (risk-neutral drift, i.e., average mean of the price changes over J periods).

The objective is to find the optimal values of the decision variables (namely the initial number of call and put options, n_{0C} and n_{0P}, purchased by the airline, and the respective exercise prices K and L) that maximize the expected revenue defined in [2.46], for a given initial price of the ticket S_0, and price and demand distributions. Yet, the objective function defined in Equation [2.46] is difficult to solve to obtain a closed-form solution for the optimal values of the decision variables. Thus, numerical methods have to be utilized. In this thesis, a numerical search procedure is used to obtain the optimal solution. This will be described in Chapter 3, using a numerical example.

CHAPTER 3

NUMERICAL EXAMPLE

This chapter illustrates the model of chapter 2 with a numerical example. After the method of implementation is presented, a numerical example is used to illustrate how the method would be applied. Finally, sensitivity analysis is conducted on the example to examine the objective function's sensitivity to variations in some of the input parameters.

3.1 Proposed method

3.1.1 Description

The formulation of the expected revenue is complex, because there is a distribution for the demand for each period and the ticket price changes, following a random walk model. Moreover, the numbers of options purchased or exercised have to be integers. Therefore, taking partial derivatives of the objective function, with respect of the decision variables, and setting them to zero cannot solve this problem. We recommend a numerical search methods.

The chart in Figure 3.1 presents the methodology suggested when financial options are used to compute the expected revenue of the airline.

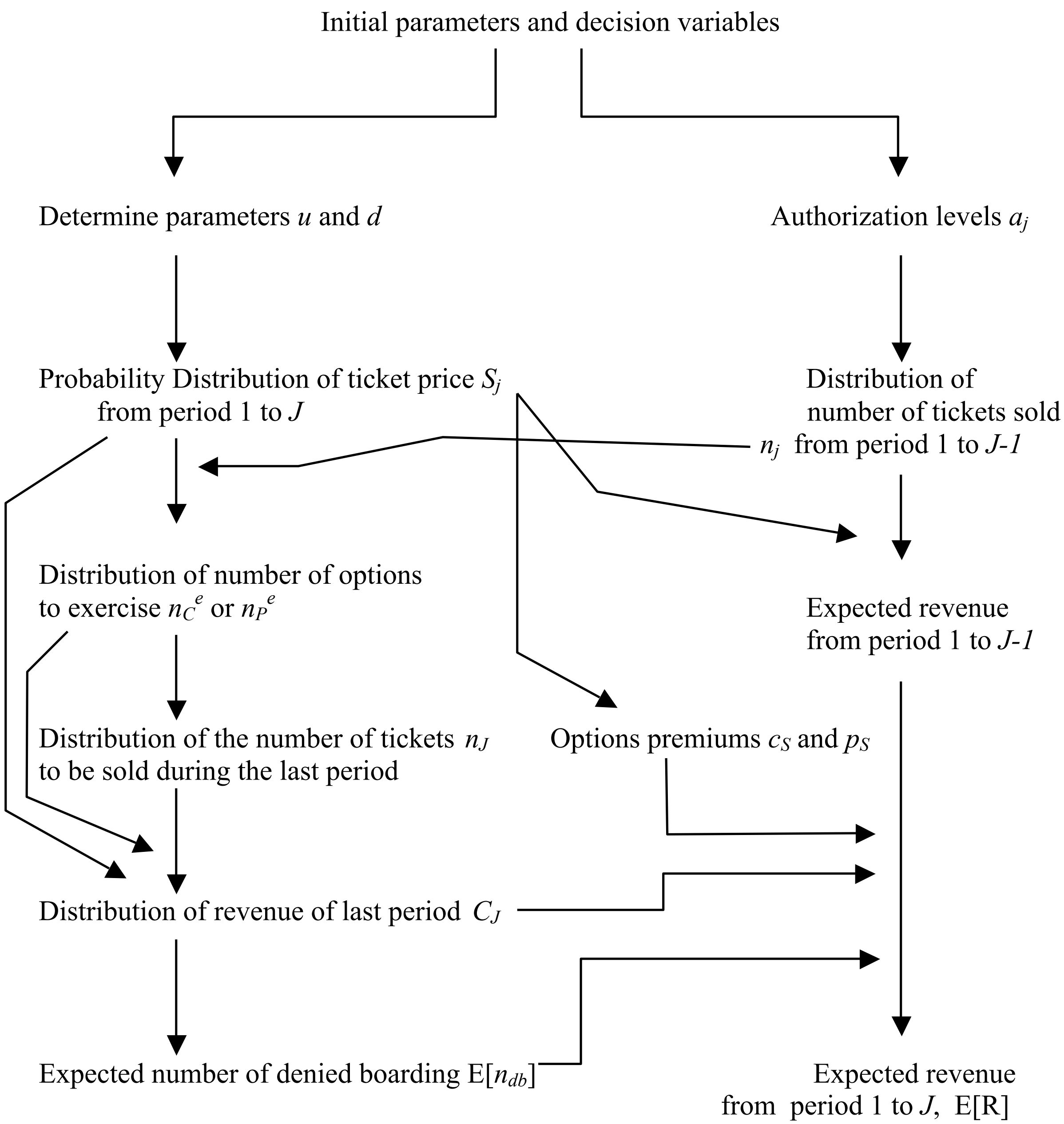


Figure 3.1: Computation of the expected revenue

Initial parameters: number of periods J, length of one period h, capacity C, demand d_j (j=1,…, J), initial price S_0, mean drift μ and its standard deviation σ, interest rate r, probability of no-show s, denied boarding cost c_{db}.

Decision variables: numbers of options initially purchased n_{0C} and n_{0P}, strike prices K and L

First, the user would forecast the distribution of demand for each period. Then the user computes the authorization levels for each period, to determine the distribution of the number of tickets sold per period. Logically, if the total expected number of tickets sold exceeds the capacity of the aircraft, call options are used; otherwise put options are used. The method can calculate the call and put option premiums for input call and put strike prices. At the same time, the distribution of the ticket price is computed for each period. Then, the user can compute the distribution of the revenue until the beginning of last period, not including the options. Also the demand of last period is considered, and, consequently, the distribution of the number of exercised options is derived at the considered strike price. Then the user can compute the distribution of the revenue of the last period, and, consequently, the expected revenue. In order to maximize the revenue, a numerical optimization method is implemented, changing not only the initial numbers of call and put options purchased, but also the strike prices at which the options will be exercised. The user selects the maximum improvement in the expected revenue to determine when to stop the search.

In the numerical example presented in this thesis, an optimization algorithm is coded in Microsoft Visual Basic for Applications. An Excel Workbook is used to enter the input parameters and display the results. The code can be found in Appendix A.

We decide that the demand distribution is identical for each period. The binomial parameters u and d are initially calculated (when the ticket price increases over a binomial step, it is multiplied by u; otherwise it decreases and it is multiplied by d). The initial number of call and put options to be utilized is set to 0. Then the expected revenue is computed. The optimization algorithm consists in changing the strike prices K and L

respectively by ΔK and ΔL and the number of options by Δn_{0C} and Δn_{0P}. For each trial (change in one of the decision variables), the expected revenue is calculated (see Figure 3.1) and it is considered improved if it results in a minimal increase (ΔE[R]) of the expected revenue, E[R], as compared to the expected revenue computed with the previous combination of the decision variables. If not, the optimization algorithm stops and the decision variables are said to be optimal.

3.1.2 Two examples with financial options

Two numerical examples are computed with and without financial options to show the efficiency of financial options. Initially, before any optimization, the numbers of call and put options purchased by the airline are set to zero, with the respective strike prices for the call and put options K = \$ 50 and L = \$ 200. Moreover, it is decided to stop the process if the improvement of the expected revenue is less than or equal to ΔE[R] = 0.01%. The incremental step in the number of options are $\Delta n_{0C} = 1$ and $\Delta n_{0P} = 1$. Concerning the exercised prices, the incremental step are $\Delta K = \Delta L$ = \$ 5. In both examples, the initial price of the ticket is S_0 = \$ 250, the demand distribution is assumed to be the same for each period and is according to Table 3.1.

Demand	30	40	50	60	70	80
Probability	0.05	0.15	0.3	0.3	0.15	0.05

Table 3.1: Demand distribution for each period

It is also decided that the exercised price, L, cannot be greater than S_0, because the travel agent would not agree to buy the tickets at a higher price than the initial ticket price. Therefore, the exercise price for put options must meet the following constraint: $L \leq \$ 250$.

The average return per change in the ticket price is $\mu = 0.11$ and the standard deviation is $\sigma = 0.167$, resulting in the following binomial parameters: $u = 1.1$ and $d = 0.95$. The input parameters are summarized in Table 3.2.

Number of periods	J=5
Length of interval time	h=1/5=0.20 time units
Denied boarding cost	$c_{db} = \$ 200$
Initial exercise price of call options	$K = \$ 50$
Initial exercise price of put options	$L = \$ 200$
Initial ticket price	$S_0 = \$ 250$
Binomial parameters	$\mu = 0.11$ $\sigma = 0.167$
Probability of no-shows	s =0.1
Risk-free interest rate	r = 5%

Table 3.2: Input parameters used to illustrate the efficiency of the model

3.1.2.1 Numerical illustration of the efficiency of call options

When the capacity of the aircraft is 200 passengers, the expected demand is greater than the capacity of the aircraft (by 50 seats), and logically call options should be used. To prove the efficiency of the call options, the expected revenue is first computed without call options, and then with the optimal number of call options found through the optimization algorithm. The results are displayed in Table 3.3.

Number of call options purchased by the airline	Expected Revenue
Without call options $n_{0C} = 0$ $K = \$\ 50$ $n_{0P} = 0$ $L = \$\ 200$	E[R] = \$ 51,437.68
With call options $n_{0C}^* = 60$ $K^* = \$\ 50$ $n_{0P}^* = 0$ $L^* = \$\ 200$	E[R]* = \$ 59,917.48

Table 3.3: Optimization and comparison for call options

Therefore, by comparison with the expected revenue without any option, using call options when the expected demand is larger than the capacity of the airplane (by 50 seats) results in an improvement of the expected revenue of 16.48% when the optimal number of call options is 60.

3.1.2.2 Numerical illustration of the efficiency of put options

When the capacity of the aircraft is 300 passengers, the expected demand is lower than the capacity of the aircraft (by 50 seats), and logically put options should be used. To prove the efficiency of the put options, the expected revenue is first computed without put options; then the optimization algorithm is run and the expected revenue is then computed with the optimal number of put options initially purchased. The results are displayed in Table 3.4.

Number of put options purchased by the airline	Expected Revenue
Without put options $n_{0C} = 0$ $K = \$\ 50$ $n_{0P} = 0$ $L = \$\ 200$	E[R] = \$ 77,497.63
With put options $n_{0C}* = 0$ $K* = \$\ 50$ $n_{0P}* = 45$ $L* = \$\ 250$	E[R]* = \$ 94,886.86

Table 3.4: Optimization and comparison for put options

Therefore, by comparison with the expected revenue without any option, using put options when the expected demand is lower than the capacity of the airplane results in an improvement of the expected revenue of 22.44% when the optimal number of put options is 45.

3.2 Sensitivity analysis

We conduct a sensitivity analysis on the example presented in section 3.1.2.2. The capacity of the aircraft is C = 300. The initial price of the ticket (at the beginning of the airline booking process) is set at S_0 = \$ 250. At the beginning of each period, the ticket price is subject to the following variations: it can go up by 10% (resulting in u=1.1) or it can go down by 5% (resulting in d=0.95).

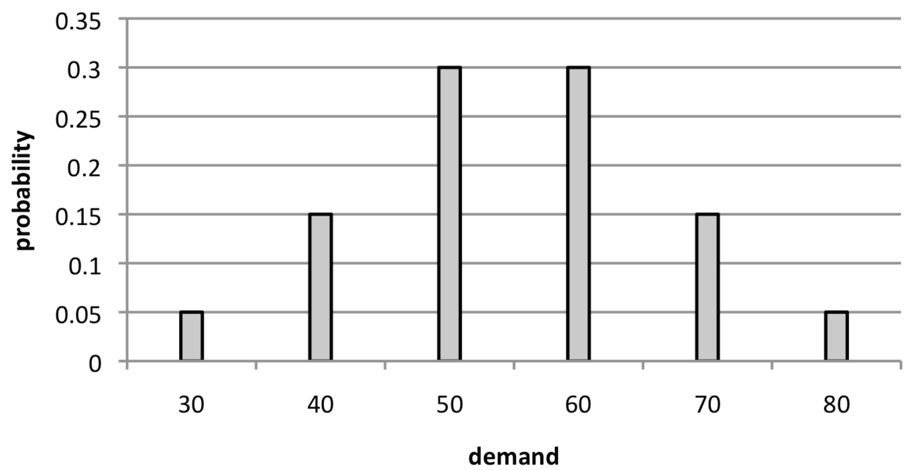
3.2.1 Sensitivity analysis conducted on the demand distribution

The sensitivity to the skewness of the demand distribution is studied. Three different scenarios are compared.

Symmetric demand

The first scenario considered is the one studied in section 3.1.2.2, where the demand, described by Table 3.1 and Figure 3.2 (see below), is symmetrical.

Figure 3.2: Symmetrical demand distribution



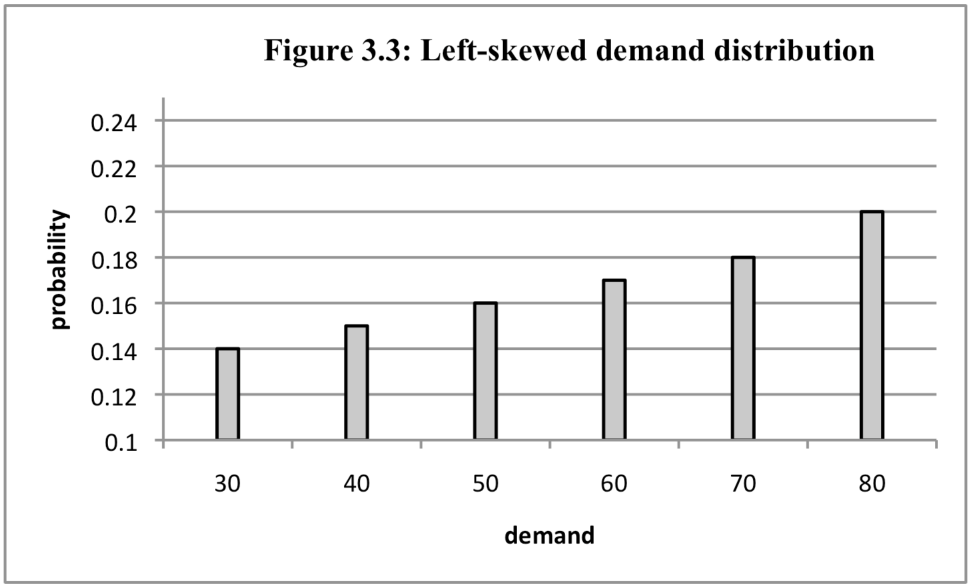
In this case, the optimal number of options to be used is 55 put options, so that the revenue is increased from \$ 77,497.63 to \$ 94,886.84 or by 22.44%.

Left-skewed demand

The second scenario observed is for a left-skewed demand, as described in Table 3.5 and Figure 3.3.

Demand	30	40	50	60	70	80
Probability	0.14	0.15	0.16	0.17	0.18	0.2

Table 3.5: Left-skewed demand distribution



The total expected demand is 285 seats, which is less than the capacity of the aircraft ($C = 300$). The results are displayed in Table 3.6.

Configuration of decision variables	Expected Revenue
Without put options $n_{0C} = 0$ $n_{0P} = 0$ $K = \$ 50$ $L = \$ 200$	E[R]= \$ 80,043.97
With put options $n_{0C}^* = 0$ $n_{0P}^* = 50$ $K^* = \$ 50$ $L^* = \$ 250$	E[R]*= \$ 94,194.17

Table 3.6: Results for left-skewed demand distribution

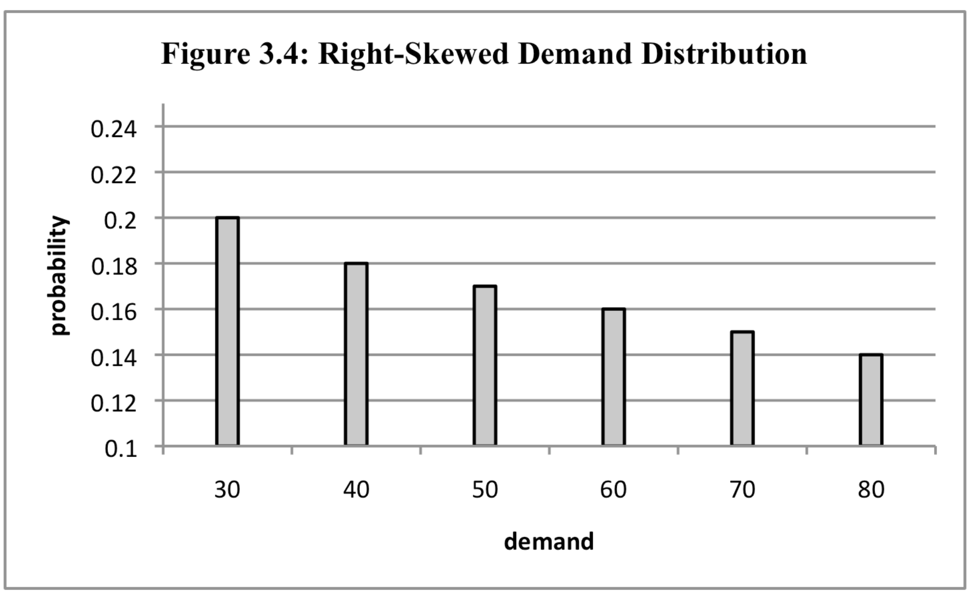
The optimal scenario results in a total increase of the expected revenue of 17.68%. The results are consistent with the model, as they do not involve any call options.

Right-skewed demand

The third scenario observed if for a right-skewed demand, as described by Table 3.7 and Figure 3.4.

Demand	30	40	50	60	70	80
Probability	0.2	0.18	0.17	0.16	0.15	0.14

Table 3.7: Right-skewed demand distribution



The total expected demand is 265 seats, which is smaller than the capacity of the aircraft. The results are displayed in Table 3.8.

Configuration of decision variables	Expected Revenue
Without put options $n_{0C} = 0$ $n_{0P} = 0$ $K = \$\ 50$ $L = \$\ 200$	E[R]= \$ 75,064.61
With put options $n_{0C}{*} = 0$ $n_{0P}{*} = 67$ $K{*} = \$\ 50$ $L{*} = \$\ 250$	E[R]*= \$ 97,944.28

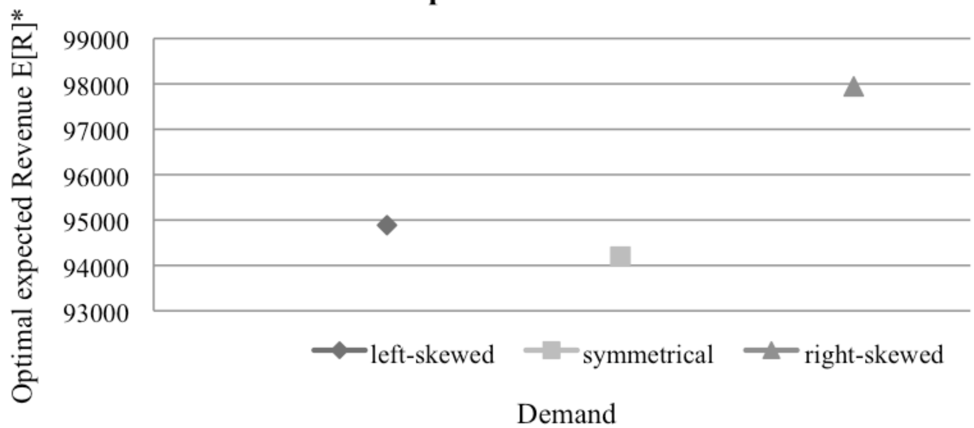
Table 3.8: Results for left-skewed demand distribution

The optimal scenario results in a total increase of expected revenue of 30.48%. The results are consistent with the model, as they do not involve any call options.

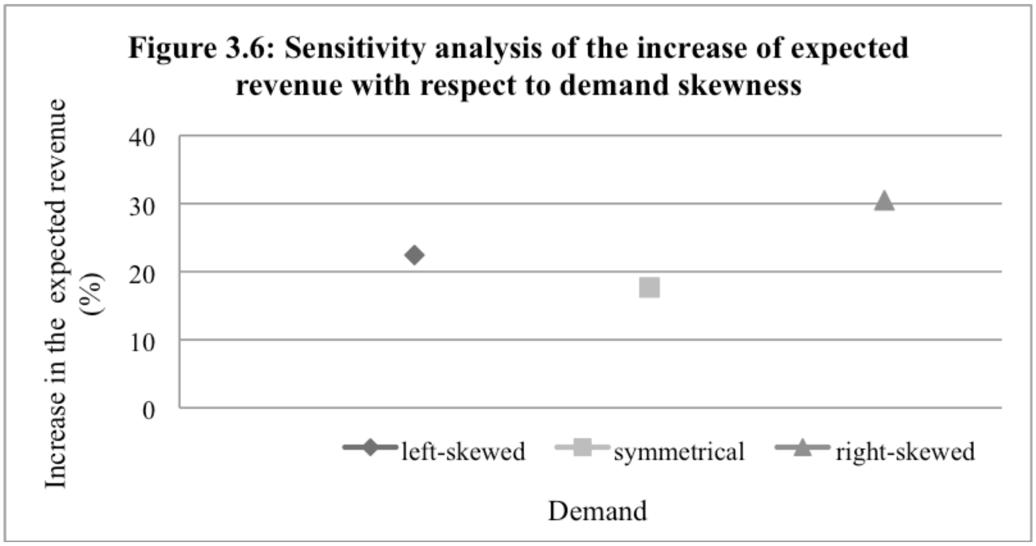
Analysis of the three cases

It may be noticed that even though the optimal value of L is \$ 250 in all three cases, it is more likely that the travel agent will negotiate a price less than \$ 250. In the sensitivity analysis conducted, the ticket price is more likely to drop, as the value of p used is 0.402, and the expected demand is lower than the capacity of the aircraft. For each distribution, the model shows an increase in expected revenue by purchasing call options at the beginning of the booking process when the total expected demand is smaller than the capacity of the aircraft. In Figure 3.5, the "optimal" expected revenues (with optimal values of the decision variables) for each demand skewness are shown.

Figure 3.5: Sensitivity analysis of the optimal expected revenue with respect to demand skewness



As shown in Figure 3.6, the sensitivity analysis to demand skewness demonstrates that the percentage increase in the expected revenue is larger when the demand is right-skewed, that is, when the probability of a low demand is higher than for a high demand. In this case, the improvement of the expected revenue using the model proposed is higher.



The reader can also notice that, because in the cases studied in this sensitivity analysis where the total expected demand is always smaller than the capacity, *C*, of the aircraft, the optimization process never used call options, which confirms the mutually exclusivity property of the two types of options.

3.2.2 Sensitivity analysis conducted on the parameters μ and σ

A sensitivity analysis is also conducted on the parameters μ and σ. Because the parameters directly influence the binomial parameters u and d (see Equations [2.44] and [2.45]), they influence not only the price probability distribution (see Equation [2.35]), but also the call and put option premiums (price at which the airline buys the options), as shown by Equations [2.40] and [2.41]. The changes in the parameters μ and σ are $\Delta\mu = 0.1$, for $0.11 \leq \mu \leq 0.41$, and $\Delta\sigma = 0.1$, for $0.167 \leq \sigma \leq 0.467$.

The results for different pairs (μ, σ) are displayed in Table 3.9. As, for each trial, no call options are involved, the optimal initial number of call options, n_{0C}*, and the optimal strike price, K*, are not shown in the table. For each case, n_{0C}* = 0, K* = \$ 50, and c_S = \$ 178.65.

μ	σ	d	u	Expected revenue E[R]* (\$)	optimal initial put n_{0P}*	optimal strike price L* (\$)	put premium p_S* (\$)	probability p
0.110	0.167	0.949	1.102	89110.74	45	250	21.178	0.271
0.110	0.267	0.907	1.152	97621.81	55	250	29.275	0.339
0.110	0.367	0.868	1.205	97150.11	55	250	39.441	0.364
0.110	0.467	0.830	1.260	95749.48	45	250	49.293	0.373
0.210	0.167	0.968	1.124	74703.16	0	200	0.000	0.142
0.210	0.267	0.926	1.175	88728.06	45	250	29.907	0.259
0.210	0.367	0.885	1.229	88478.88	45	250	38.369	0.305
0.210	0.467	0.846	1.285	88224.97	45	250	45.633	0.328
0.310	0.167	0.987	1.146	74706.59	0	200	0.000	0.017
0.310	0.267	0.944	1.199	79937.31	35	200	4.366	0.180
0.310	0.367	0.903	1.254	88355.91	45	250	38.526	0.248
0.310	0.467	0.863	1.311	88053.46	45	250	47.598	0.283
0.410	0.267	0.963	1.223	74700.57	0	200	0.000	0.103
0.410	0.367	0.921	1.279	88378.45	45	250	35.193	0.192
0.410	0.467	0.881	1.338	87985.08	45	250	47.048	0.239

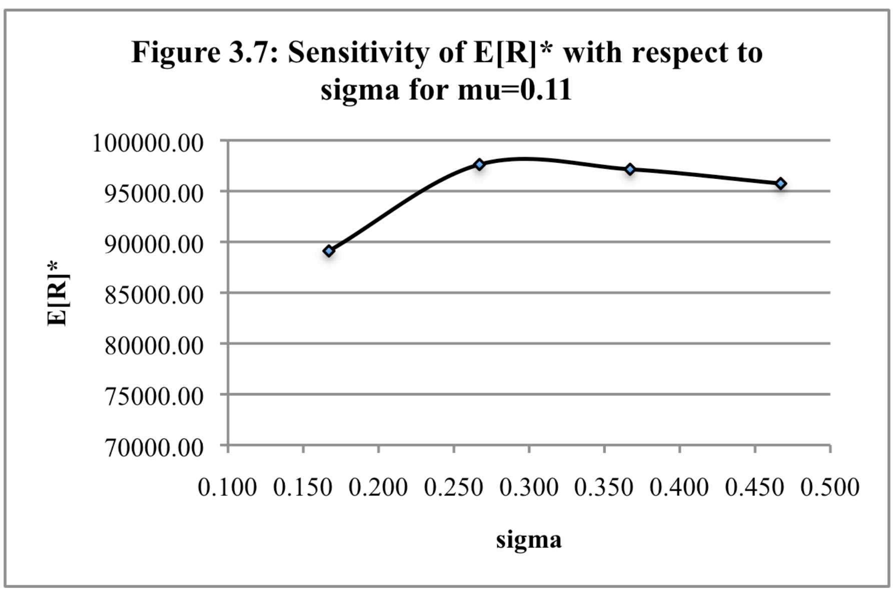
Table 3.9: Sensitivity analysis on μ and σ.

Changing μ and σ influences the probability p that the price will go up. When the probability $p \leq 0.142$, $n_{0P}^* = 0$, and the expected revenue for the airline will not improve significantly (increase of 0.01%). It is also observed that the expected revenue E[R] is an increasing function of L for $p > 0.180$, that is, when the probability that the ticket price will decrease at each period, $1 - p < 0.82$.

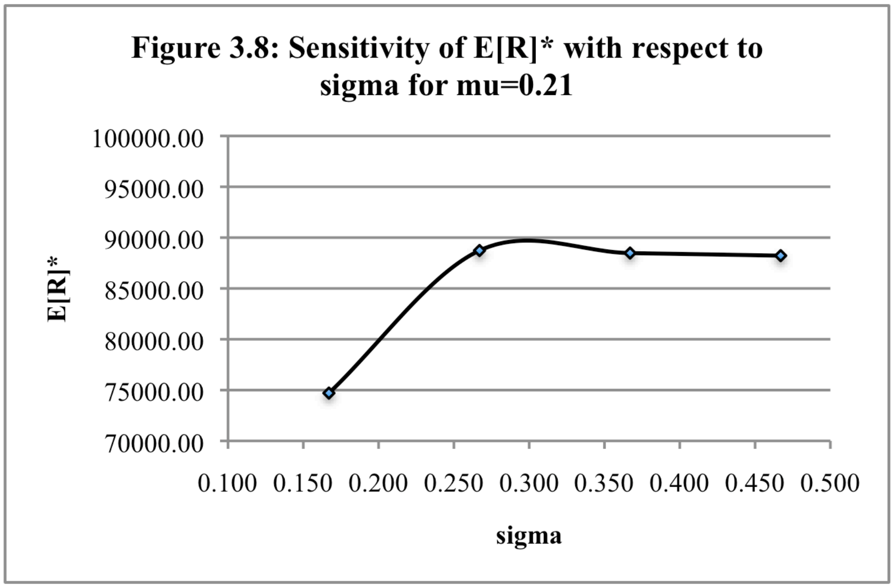
Sensitivity analysis with respect to σ

The sensitivity analysis with respect to σ is conducted for different values of μ, as follows:

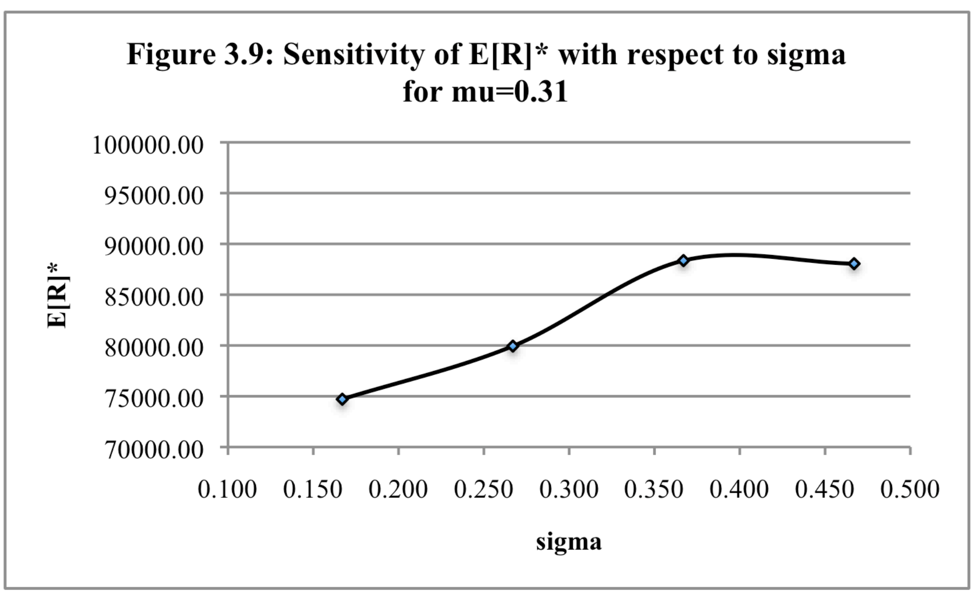
- If $\mu = 0.11$ (Figure 3.7)



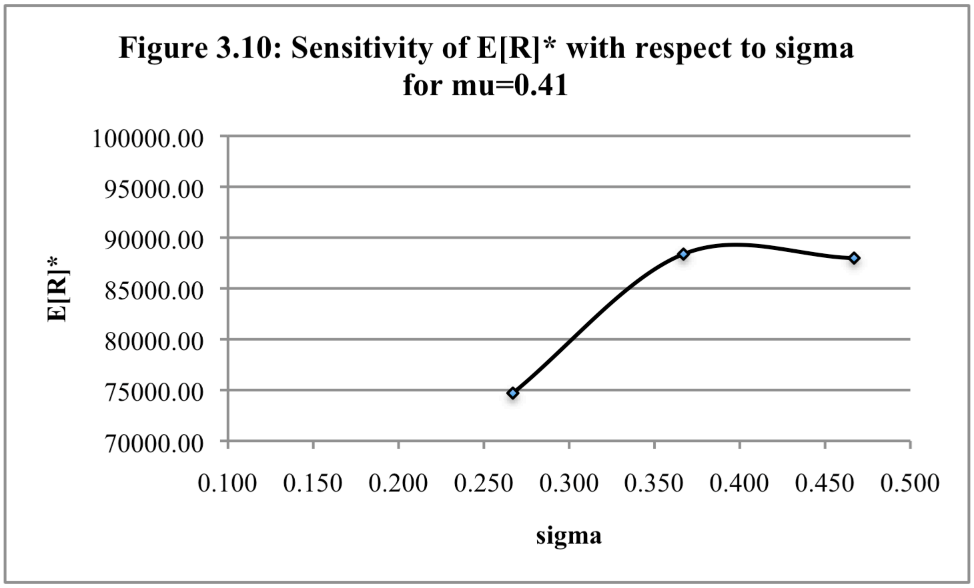
- If $\mu = 0.21$ (Figure 3.8)



- If $\mu = 0.31$ (Figure 3.9)



- If $\mu = 0.41$ (Figure 3.10)



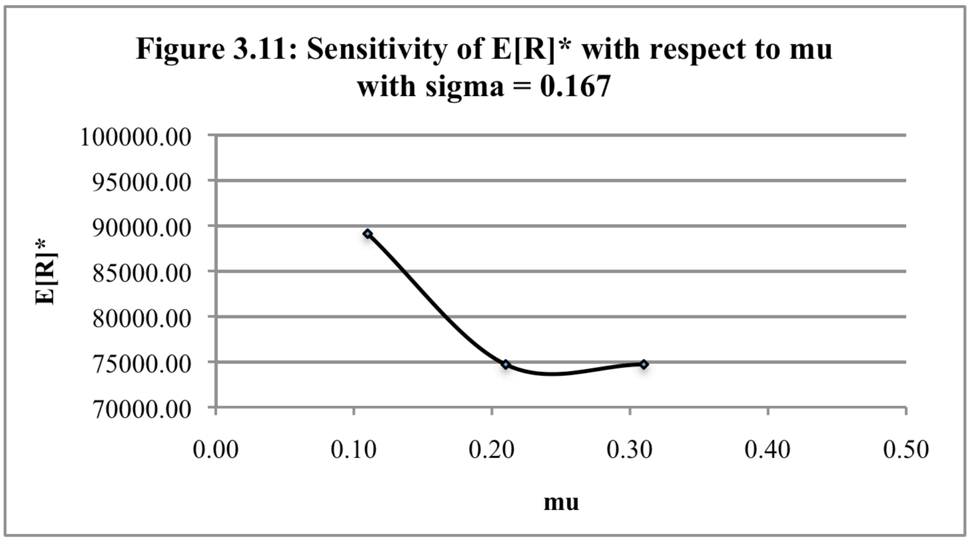
In this case, there are no data for $\sigma = 0.167$, because $\sigma = 0.167$ led to a negative probability p that the price will increase and thus was infeasible.

The sensitivity analysis conducted on σ for different values of μ demonstrates that the expected revenue behaves the same way for each value of μ, as shown in the charts. The expected revenue is maximized for the following range of σ : $0.267 \leq \sigma \leq 0.367$.

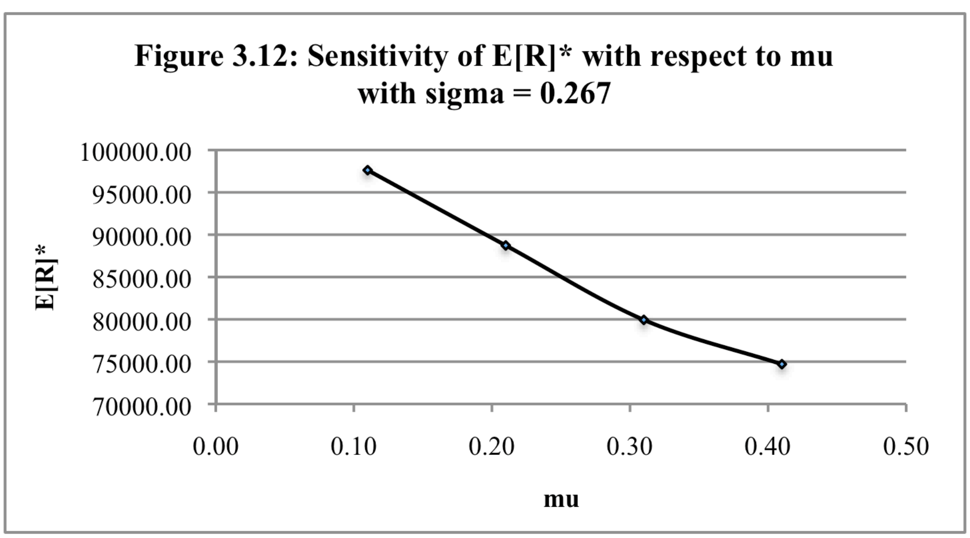
Sensitivity analysis with respect to μ

The sensitivity analysis with respect to μ is conducted for different values of σ, as follows:

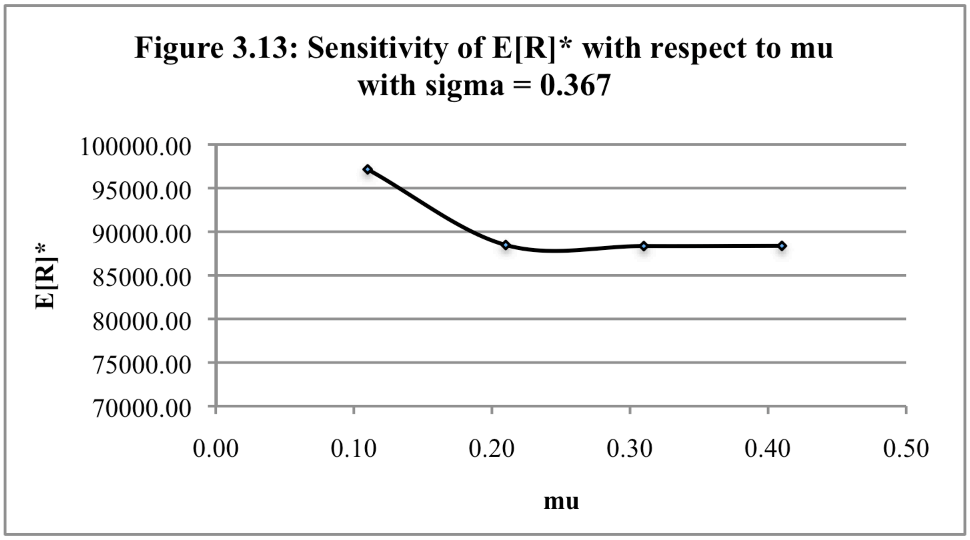
- If $\sigma = 0.167$ (Figure 3.11)



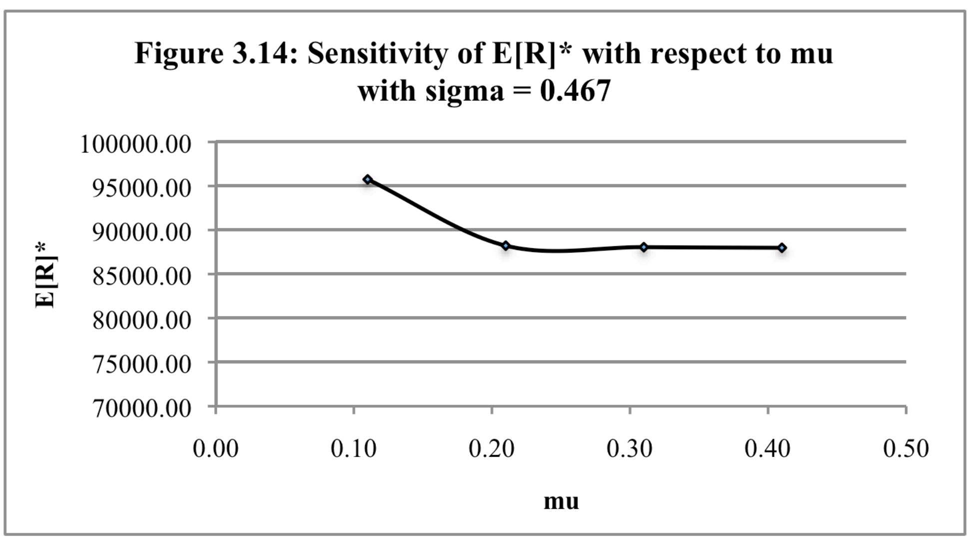
- If $\sigma = 0.267$ (Figure 3.12)



- If $\sigma = 0.367$ (Figure 3.13)



- If $\sigma = 0.467$ (Figure 3.14)



The expected revenue is a decreasing function of μ, as shown in Figures 3.11, 3.12, 3.13 and 3.14. This property is all the more emphasized when $\sigma = 0.267$ (see Figure 3.12).

The sensitivity analysis emphasizes the importance of accuracy in the parameters μ and σ when the airline must decide upon a strategy. For instance, according to Table 3.9, holding everything constant and when $\mu = 0.21$, but increasing σ from 0.167 to 0.267 changes the airline's strategy as follows:

- if $\sigma = 0.167$, put options are not recommended,
- if $\sigma = 0.267$, the model recommends the use of 45 put options with a strike price of $L^* = \$ 200$, which results in an increase in the expected revenue of 18.77%, as compared to the previous scenario.

Likewise, holding everything constant and when $\sigma = 0.167$, but increasing μ from 0.21 to 0.31, changes the airline's strategy to the following:

- if $\mu = 0.21$, it is recommended to use 45 put options with a strike price $L^* = \$ 250$,
- if $\mu = 0.31$, the model recommends 35 put options with a strike price $L^* = \$ 200$, which results in a decrease in the expected revenue of 9.91%, as compared to the previous scenario.

This sensitivity analysis shows that the airline must be careful in the input data it uses, in order to make the model as accurate as possible. Estimating μ and σ must be considered a very important issue, in addition to forecasting the demand distribution.

CHAPTER 4

CONCLUSIONS AND FUTURE WORK

4.1 Summary and Conclusions

The review of current research in airline revenue management revealed the need for a method that could more accurately account for the uncertainty of demand and ticket price and offer more flexibility to the airline. The work done by others in use of the options in other areas of revenue management has also been examined.

The model we present incorporates call and put options in order to provide the airline with flexibility in the management of an aircraft capacity. The two types of options are by definition mutually exclusive. When, at the beginning of the booking process, the total expected demand is greater than the capacity of the aircraft, the airline hedges with call options. These tickets can be recalled and resold in the market at a higher fare, one period before the flight departs. When, at the beginning of the booking process, the total expected demand is smaller than the capacity of the aircraft, the airline hedges with put options. These tickets can then be sold to the travel agent, one period before the flight departs. The initial number of options purchased to hedge some tickets, the option premiums, and the exercise prices (prices at which the airline will either recall the "call-optioned" tickets or sell the "put-optioned" tickets to a travel agent) are important decision variables. The assumption of the ticket price of a fare class following a random walk allows the application of the binomial option pricing model (BOPM), resulting in reducing the number of decision variables to four: the two numbers of options purchased and the two exercise prices. The numerical search method proposed aims at finding the optimal values of the decision variables that maximize the expected revenue of the airline for a specific flight.

We give a numerical example to illustrate the application of the model and method proposed and we show the efficiency of using call and put options. Based on the data used in the example, a sensitivity analysis is performed to determine how responsive the results obtained from the method to changes in some variables. We find that changing the skewness of the distribution has a significant effect on the expected revenue. Moreover, the numerical example shows that the strike price (for the put option model) does not have a major influence in the optimization process, as the expected revenue is an increasing function of it, and thus will most often be set at its upper bound. It has also been shown that the parameters μ (drift of the random walk) and σ (standard deviation of the logarithmic change in the ticket price) directly affect the utilization of options, because they have a direct effect on the probability that the ticket price will increase. It is therefore essential for the airline to have accurate historical data in order to estimate μ and σ.

4.2 Future research

A number of areas exists in which future work could extend the effectiveness and utilization of our model. Incorporation of uncertainty in the binomial parameters would be a valuable addition. Moreover, we opted for European options, which can only be exercised at the maturity date; yet American options, which offer more flexibility in the exercise date (anytime between the purchasing date and the expiration date), would be an interesting extended research.

Moreover, the model could be modified so that margins (i.e., collaterals) could be introduced, allowing the airline to use both types of options when the expected demand is slightly different from the capacity of the aircraft. The probability of no-show could be considered as a random variable, in order to be more realistic. Different fare classes could be easily considered and more complex overbooking techniques could be introduced.

Finally, this thesis has shown, through sensitivity analysis, the importance of an accurate estimation of the random walk parameters μ and σ. Forecasting techniques could be implemented in order to further refine the accuracy of the model.

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APPENDIX A

VISUAL BASIC FOR APPLICATIONS CODE

```
'/* Declare global variables */
Option Explicit
Option Base 1
Dim thesisup As Excel.Workbook
Private initial_put As Integer
Private initial_call As Integer
Private exercised_put(6 ^ 4) As Integer
Private exercised_call(6 ^ 4) As Integer
Private recalled_tickets As Integer
Private sell_on_market As Integer
Private sell_to_travel_agent As Integer
Private lastsoldc(6 ^ 5) As Integer
Private lastsoldp(6 ^ 5) As Integer
Private lastsold(2, 6 ^ 5) As Single
Private capacity As Integer
Private noperiods As Integer
Private mu As Single
Private sigma As Single
Private d(6) As Integer
Private p(6)
Private a(5) As Integer
Private N(6, 5) As Integer
Private h As Single
Private strike_L As Integer
Private strike_K As Integer
Private rate As Single
Private R As Single
```

```
Private jj As Integer
Private lastdemand(6) As Integer
Private lastdemand_p(6) As Single
Private s As Single
Private m As Integer
Private dd As Single
Private uu As Single
Private arr_stdrev(2, 155521) As Single
'Private one_revenue(2, 6 * 30 ^ 4)
'Private exp_revenue(6 * 30)
Private expected_revenue As Single
Private Br As Single
Private p_price As Single
Private ticketprice(2, 5, 6) As Single
Private rev_per(2, 4, 30)
Private denied_boarding_cost As Integer
Private call_op_price As Single
Private Put_op_price As Single
Private index_mu As Integer
Private index_dd As Integer
Private numberm(2, 6 ^ 4) As Single
Private nb_denied_boarding As Single
Private one_Revenue As Single
Private one_Revenue_prob As Single
Private expected_lastdemand As Single
Private expected_R0 As Single
Private column As Long
Private line As Long
Private expected_exercised_call As Single
Private expected_exercised_put As Single
```

```
Private mm As Integer
Private last_revenue As Single
'Private kk As Long

Private Sub musensitivity()
sigma = Worksheets("Code").Range("sigma")
mu = 0.11
index_mu = 2

  '/* Read input parameters */
  jj = Worksheets("Code").Range("numberofperiods").Value
  capacity = Worksheets("Code").Range("capacity").Value
  h = Worksheets("Code").Range("time_h").Value
  rate = Worksheets("Code").Range("rate").Value
  s = Worksheets("code").Range("initial_price").Value
  Br = Exp(rate * h)
  uu = Exp(mu * h + sigma * Sqr(h))
  dd = Exp(mu * h - sigma * Sqr(h))
  p_price = (Br ^ (-1) - dd) / (uu - dd)
  initial_call = Worksheets("code").Range("initial_call").Value
  initial_put = Worksheets("code").Range("initial_put").Value
  strike_K = Worksheets("code").Range("strike_K").Value
  strike_L = Worksheets("code").Range("strike_L").Value
  denied_boarding_cost = Worksheets("code").Range("denied_boarding_cost").Value
Do
  Call optima
  Call reportmu
  Call sigmasensitivity
  mu = mu + 0.1
  uu = Exp(mu * h + sigma * Sqr(h))
```

```
    dd = Exp(mu * h - sigma * Sqr(h))
    p_price = (Br ^ (-1) - dd) / (uu - dd)
Loop While mu < 0.5
End Sub

Private Sub sigmasensitivity()
sigma = 0.067
'mu = Worksheets("Code").Range("mu")
uu = Exp(mu * h + sigma * Sqr(h))
dd = Exp(mu * h - sigma * Sqr(h))
p_price = (Br ^ (-1) - dd) / (uu - dd)
Do
    Call optima
    Call reportsigma
    sigma = sigma + 0.1
    uu = Exp(mu * h + sigma * Sqr(h))
    dd = Exp(mu * h - sigma * Sqr(h))
    p_price = (Br ^ (-1) - dd) / (uu - dd)
Loop While sigma < 0.5
End Sub

Private Sub optima()
    '/* Read input parameters */
    'jj = Worksheets("Code").Range("numberofperiods").Value
    'capacity = Worksheets("Code").Range("capacity").Value
    'mu = Worksheets("Code").Range("mu").Value
    'sigma = Worksheets("Code").Range("sigma").Value
    'h = Worksheets("Code").Range("time_h").Value
    'rate = Worksheets("Code").Range("rate").Value
    's = Worksheets("code").Range("initial_price").Value
```

```
'Br = Exp(-rate * h)
'uu = Exp(mu * h + sigma * Sqr(h))
'dd = Exp(mu * h - sigma * Sqr(h))
'dd = Worksheets("Code").Range("price_down")
'uu = Worksheets("Code").Range("price_up")
'p_price = (Br ^ (-1) - dd) / (uu - dd)
initial_call = Worksheets("code").Range("initial_call").Value
initial_put = Worksheets("code").Range("initial_put").Value
strike_K = Worksheets("code").Range("strike_K").Value
strike_L = Worksheets("code").Range("strike_L").Value
'denied_boarding_cost = Worksheets("code").Range("denied_boarding_cost").Value
Dim revenue1 As Double
Dim revenue2 As Double
Dim revenue3 As Double
Dim revenue4 As Double
Dim rev1 As Double
Dim rev2 As Double
Dim rev3 As Double
Dim rev4 As Double
Call last_period_demand
Call Numbersold
Call ticketsold
Call priceofticket
Call stdrevenue(s, jj - 1)
Call Revenue
Do
  Do
    Do
      Do
        revenue1 = expected_revenue
```

```
        strike_L = strike_L + 5
        If (strike_L > 250) Then
            strike_L = strike_L - 5
            Exit Do
        End If
        Call Revenue
        rev1 = expected_revenue
        If rev1 < 1.001 * revenue1 Then
            strike_L = strike_L - 5
        End If
    Loop While (rev1 > 1.001 * revenue1)
    revenue2 = revenue1
    strike_K = strike_K + 5
    If (strike_K > 150) Then
        strike_K = strike_K - 5
        Exit Do
    End If
    Call Revenue
    rev2 = expected_revenue
    If rev2 < 1.001 * revenue2 Then
        strike_K = strike_K - 5
    End If
Loop While (rev2 > 1.001 * revenue2)
revenue3 = revenue2
initial_put = initial_put + 1
If (initial_put > 150) Then
    initial_put = initial_put - 1
    Exit Do
End If
Call Revenue
```

```
        rev3 = expected_revenue
        If rev3 < 1.001 * revenue3 Then
            initial_put = initial_put - 1
        End If
    Loop While (rev3 > 1.001 * revenue3)
    revenue4 = revenue3
    initial_call = initial_call + 1
    If (initial_call > 150) Then
        initial_call = initial_call - 1
        Exit Do
    End If
    Call Revenue
    rev4 = expected_revenue
    If rev4 < 1.001 * revenue4 Then
        initial_call = initial_call - 1
    End If
  Loop While (rev4 > 1.001 * revenue4)
  expected_revenue = revenue4
  Call report
End Sub

Private Sub Revenue()
  '/* Read input parameters */
  'jj = Worksheets("Code").Range("numberofperiods").Value
  'capacity = Worksheets("Code").Range("capacity").Value
  'mu = Worksheets("Code").Range("mu").Value
  'sigma = Worksheets("Code").Range("sigma").Value
  'h = Worksheets("Code").Range("time_h").Value
  'rate = Worksheets("Code").Range("rate").Value
  's = Worksheets("code").Range("initial_price").Value
```

```
'initial_call = Worksheets("code").Range("initial_call").Value

'initial_put = Worksheets("code").Range("initial_put").Value

'Br = Exp(-rate * h)

'uu = Exp(mu * h + sigma * Sqr(h))

'dd = Exp(mu * h - sigma * Sqr(h))

'dd = Worksheets("Code").Range("price_down")

'uu = Worksheets("Code").Range("price_up")

'p_price = (Br ^ (-1) - dd) / (uu - dd)

'strike_K = Worksheets("code").Range("strike_K").Value

'strike_L = Worksheets("code").Range("strike_L").Value

'denied_boarding_cost = Worksheets("code").Range("denied_boarding_cost").Value

Dim l As Integer

Dim k As Integer

expected_exercised_call = 0

expected_exercised_put = 0

'Call Numbersold

'Call ticketsold

'Call priceofticket

'Call stdrevenue(s, jj - 1)

'Call last_period_demand

Call call_op

Call Put_op


Call last_cashflow

    expected_revenue = initial_call * (s - call_op_price) - initial_put * Put_op_price + expected_R0 + Exp(-5 * h * rate) * last_revenue - strike_K * expected_exercised_call + strike_L * expected_exercised_put - nb_denied_boarding * denied_boarding_cost

    'Call report

End Sub


'/* minimum  function*/
```

```
Function Min(x As Variant, y As Variant)
  '/* Dim x As Single
  '/* Dim y As Single
   If x < y Then Min = x Else Min = y
End Function

'/* maximum  function*/
Function Max(x As Variant, y As Variant)
   If x < y Then Max = y Else Max = x
End Function

'/* compute the binomial coefficient */
Function binoCoeff(J, k)
        Dim i As Integer
        Dim b As Double
          b = 1
        For i = 0 To k - 1
                b = b * (J - i) / (k - i)
        Next i
        binoCoeff = b
End Function

'/* compute the price of call option */
Private Sub call_op()
   Dim k As Integer
   Dim bicomp As Single
   Dim sumbi As Single
   Dim Jk As Double
   Dim firstBicomp As Single
```

```
call_op_price = 0

    For k = 0 To jj
        Jk = binoCoeff(jj, k)
        bicomp = Jk * (p_price ^ k) * ((1 - p_price) ^ (jj - k)) * (s * (uu ^ k) * (dd ^ (jj - k)) - strike_K)
        If bicomp < 0 Then
            bicomp = 0
        End If
        sumbi = sumbi + bicomp
    Next k
   call_op_price = sumbi / (Br ^ jj)
End Sub

'/* compute the price of put option */
Private Sub Put_op()
    Dim k As Integer
    Dim bicomp1 As Single
    Dim sumbi1 As Single
    Dim Jk As Double
    Dim firstBicomp1 As Single
Put_op_price = 0
    For k = 0 To jj
        Jk = binoCoeff(jj, k)
        bicomp1 = Jk * (p_price ^ k) * ((1 - p_price) ^ (jj - k)) * (strike_L - (s * (uu ^ k) * (dd ^ (jj - k))))
        If bicomp1 < 0 Then
            bicomp1 = 0
        End If
        sumbi1 = sumbi1 + bicomp1
    Next k
   Put_op_price = sumbi1 / (Br ^ jj)
```

```
End Sub

'/* compute the price Sj associated dwith the probability pj*/
Private Sub priceofticket()
   '/*Dim J As Integer
   Dim k As Integer
   Dim J As Integer

   '/*Dim p As Single
   Dim Jk As Double
   For J = 1 To jj
      For k = 1 To J + 1
         Jk = binoCoeff(J, (k - 1))
         ticketprice(1, J, k) = (uu) ^ (k - 1) * (dd) ^ (J - (k - 1)) * s
         ticketprice(2, J, k) = Jk * p_price ^ (k - 1) * (1 - p_price) ^ (J - k + 1)
      Next k
   Next J

End Sub

'/* compute the standard revenue from period 1 to period jj-1 */
Private Sub stdrevenue(s, J)
   '/*Dim s As Single
   '/*Dim J As Integer
   Dim number As Long
   Dim i As Integer
   Dim q As Integer
   Dim m As Integer
   Dim k As Integer
   Dim sum As Double
```

```
   Dim l As Integer
   Dim ll As Integer
   Dim prob As Double
   Dim sum1 As Double
   Dim prob1 As Double

expected_R0 = 0
number = 1
l = 1
sum = 0
prob = 1
For k = 1 To J
   number = 1
   For l = 1 To k + 1
      For i = 1 To 6
         rev_per(1, k, number) = ticketprice(1, k, l) * N(i, k)
         rev_per(2, k, number) = ticketprice(2, k, l) * p(i)
         number = number + 1
      Next i
   Next l
Next k
number = 1
For i = 1 To 12
   For k = 1 To 18
      For l = 1 To 24
         For ll = 1 To 30
            arr_stdrev(1, number) = Exp(-rate * h) * rev_per(1, 1, i) + Exp(-2 * rate * h) * rev_per(1, 2, k) + Exp(-3 * rate * h) * rev_per(1, 3, l) + Exp(-4 * rate * h) * rev_per(1, 4, ll)
            arr_stdrev(2, number) = rev_per(2, 1, i) * rev_per(2, 2, k) * rev_per(2, 3, l) * rev_per(2, 4, ll)
            expected_R0 = expected_R0 + arr_stdrev(1, number) * arr_stdrev(2, number)
            number = number + 1
```

```
            Next ll
        Next l
    Next k
Next i
End Sub

Private Sub last_cashflow()
    'Dim m As Integer
    last_revenue = 0
    mm = 0
    Do While mm < 6
        mm = mm + 1
            Call exercised_options
            Call updatelastsold
         For m = 1 To 6 ^ 5
            last_revenue = last_revenue + lastsold(1, m) * lastsold(2, m) * ticketprice(1, 5, mm) * ticketprice(2,
5, mm)
        Next m
    Loop
End Sub

Private Sub alevel()
Dim k As Integer
For k = 1 To jj
a(k) = Worksheets("ticketsold").Cells(k + 1, 2).Value
Next k
End Sub

Private Sub Numbersold()
Call demandarray
Call alevel
```

```
Dim k As Integer
Dim i As Integer
For i = 1 To jj
   For k = 1 To 6
   N(k, i) = Min(d(k), a(i))
   Next k
Next i
End Sub

Private Sub demandarray()
Dim k As Integer
'Dim Arr_new(2, 11)
For k = 1 To 6
d(k) = Worksheets("Demand").Cells(k + 1, 1).Value
p(k) = Worksheets("Demand").Cells(k + 1, 2).Value
Next k
'Arr = Arr_new
End Sub

Private Sub ticketsold()
Dim number As Long
Dim i As Integer
Dim k As Integer
Dim l As Integer
Dim ll As Integer
number = 1
For i = 1 To 6
   For k = 1 To 6
      For l = 1 To 6
         For ll = 1 To 6
```

```
            numberm(1, number) = N(i, 1) + N(k, 2) + N(l, 3) + N(ll, 4)
            numberm(2, number) = p(i) * p(l) * p(k) * p(ll)
            number = number + 1
        Next ll
      Next l
   Next k
Next i
End Sub

Private Sub last_period_demand()
Dim k As Integer
Dim number1 As Integer
Dim kk As Integer
expected_lastdemand = 0

For k = 1 To 6
   lastdemand(k) = Worksheets("LastDemand").Cells(k + 1, 1).Value
   lastdemand_p(k) = Worksheets("LastDemand").Cells(k + 1, 2).Value
   expected_lastdemand = expected_lastdemand + lastdemand(k) * lastdemand_p(k)
Next k
End Sub

Private Sub exercised_options()
Dim number1 As Integer
For number1 = 1 To 6 ^ 4
   'no call option should be exercised if out-of-the-money
   If strike_K > ticketprice(1, 5, mm) Then
      exercised_call(number1) = 0
   Else
      exercised_call(number1) = Max(Min(numberm(1, number1) + expected_lastdemand - capacity, initial_call), 0)
```

```
    End If
     'no put option should be exercised if out-of-the-money
    If strike_L < ticketprice(1, 5, mm) Then
        exercised_put(number1) = 0
    Else
        exercised_put(number1) = Max(Min(capacity - (numberm(1, number1) + expected_lastdemand), initial_put), 0)
    End If
    expected_exercised_put = expected_exercised_put + numberm(2, number1) * exercised_put(number1)
    expected_exercised_call = expected_exercised_call + numberm(2, number1) * exercised_call(number1)
Next number1
    'Call updatelastsold
End Sub


Private Sub updatelastsold()
     Dim k As Integer
     Dim number1 As Integer
     Dim kk As Integer
     nb_denied_boarding = 0
     kk = 1
     For number1 = 1 To 6 ^ 4
        For k = 1 To 6
            lastsoldc(kk) = Max(0, Min(lastdemand(k), capacity - initial_call - numberm(1, number1) + exercised_call(number1)))
            lastsoldp(kk) = Max(0, Min(capacity - numberm(1, number1) - exercised_put(number1), lastdemand(k)))
            lastsold(1, kk) = lastsoldc(kk) + lastsoldp(kk)
            lastsold(2, kk) = numberm(2, number1) * lastdemand_p(k)
            nb_denied_boarding = nb_denied_boarding + lastsold(2, kk) * numberm(2, number1) * Max(0, initial_call + numberm(1, number1) + exercised_put(number1) + lastsold(1, kk) - capacity)
            kk = kk + 1
        Next k
```

```
    Next number1
End Sub

'/*report the result */
Private Sub report()
Dim k As Long
   With Worksheets("Code")
      .Cells(20, 2) = expected_revenue
      .Cells(21, 2) = initial_call
      .Cells(22, 2) = initial_put
      .Cells(23, 2) = strike_K
      .Cells(24, 2) = strike_L
      .Cells(25, 2) = call_op_price
      .Cells(26, 2) = Put_op_price
   End With
End Sub

Private Sub reportmu()
   With Worksheets("sensitivity")
   index_mu = index_mu + 1
      .Cells(index_mu, 1) = mu
      .Cells(index_mu, 2) = sigma
      .Cells(index_mu, 3) = dd
      .Cells(index_mu, 4) = uu
      .Cells(index_mu, 5) = expected_revenue
      .Cells(index_mu, 6) = initial_call
      .Cells(index_mu, 7) = strike_K
      .Cells(index_mu, 8) = initial_put
      .Cells(index_mu, 9) = strike_L
      .Cells(index_mu, 10) = call_op_price
```

```
        .Cells(index_mu, 11) = Put_op_price
        .Cells(index_mu, 12) = p_price
    End With
End Sub

Private Sub reportsigma()
    With Worksheets("sensitivity")
    index_mu = index_mu + 1
        .Cells(index_mu, 1) = mu
        .Cells(index_mu, 2) = sigma
        .Cells(index_mu, 3) = dd
        .Cells(index_mu, 4) = uu
        .Cells(index_mu, 5) = expected_revenue
        .Cells(index_mu, 6) = initial_call
        .Cells(index_mu, 7) = strike_K
        .Cells(index_mu, 8) = initial_put
        .Cells(index_mu, 9) = strike_L
        .Cells(index_mu, 10) = call_op_price
        .Cells(index_mu, 11) = Put_op_price
        .Cells(index_mu, 12) = p_price
    End With
End Sub
```

APPENDIX B

GLOSSARY OF THE FINANCIAL OPTIONS TERMINOLOGY USED

American option: option that can be exercised any time before or at the maturity date.

Binomial parameter: parameter that multiplies the price, which follows a random walk, at each binomial step.

Call option: right for the owner of the option to buy the underlying asset at the strike price before or at the maturity date.

Drift: mean of the logarithmic change of the price over the lifetime of the option.

European option: option that can be exercised only at maturity date.

Exercise price: price paid buy the owner of the option to exercise it.

Expiration date: date at which the option expires.

Maturity date: see expiration date.

Margin: collateral that the holder of a position in options has to deposit to cover the credit risk of his counterparty.

Option: financial instrument that convey the right, but not the obligation, to engage in a future transaction on some underlying asset.

Premium: price paid to buy an option.

Put option: right for the owner of the option to sell the underlying asset at the strike price before or at the maturity date.

Random walk parameter: drift and volatility.

Strike price: see exercise price.

Volatility: standard deviation of the logarithmic change of the price over the lifetime of the option.

APPENDIX C

ALPHABETICAL LIST OF VARIABLES

a_j **:** authorization level to limit the number of tickets sold per period j, j=1,… , $J-1$.

$B_r(j)$ **:** discount factor between time the initiation of the booking process and period j.

C **:** capacity of the plane.

c_{db} **:** cost of denying boarding to a passenger.

c_S **:** premium of one call option.

d **:** binomial parameter when the ticket price decreases.

d_j **:** demand during period j, j=1,…, J.

K : strike price associated with a call option.

L **:** strike price associated with a put option.

m **:** number of tickets from the first period until the end of the penultimate period ($j = 1,..., J-1$).

μ **:** drift of the random walk.

n_{0C} **:** number of call options purchased from the customers by the airline at the beginning of the airline booking process (t=0).

n_{0P} **:** number of put options purchased from the agent by the airline at the beginning of the airline booking process (t=0).

n_C^e **:** number of tickets recalled from the customers.

n_{db} **:** number of passengers who are denied boarding.

n_j **:** number of tickets sold by the airline during period j, j=1,… , $J-1$, at price S_j.

n_J **:** number of tickets sold during the last period: $n_J = n_J^C + n_J^P$.

n_J^C **:** number of tickets sold by the airline during the last period at price S_J in the call option model.

n_J^P **:** number of tickets sold by the airline during the last period at price S_J in the put option model.

$\boldsymbol{n_P^e}$ **:** number of tickets under put options sold to the travel agent.

$\boldsymbol{p_S}$ **:** premium of one put option.

$\boldsymbol{r}$ **:** nominal interest rate (continuous compounding).

$\boldsymbol{s}$ **:** probability of a no-show.

$\boldsymbol{S_0}$ **:** price paid for each ticket per customer at the beginning of the airline booking process.

$\boldsymbol{\sigma}$ **:** standard deviation of the price logarithmic change.

$\boldsymbol{S_j}$ **:** price paid for each ticket per customer during period j of the booking process, $j = 1, \ldots, J$.

$\boldsymbol{T}$ **:** expiration or maturity date of the option.

$\boldsymbol{u}$ **:** binomial parameter when the ticket price increases.

$\boldsymbol{\Delta K}$ **:** change in the strike price K in the optimization process.

$\boldsymbol{\Delta L}$ **:** change in the strike price L in the optimization process.

$\boldsymbol{\Delta n_{0C}}$ **:** change in the number of call options purchased n_{0C}.

$\boldsymbol{\Delta n_{0P}}$: change in the number of call options purchased n_{0P}.

$\Delta\boldsymbol{\mu}$ **:** change in the parameter μ.

$\Delta\boldsymbol{\sigma}$ **:** change in the parameter σ.