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EQUIVARIANT ASYMPTOTIC MORPHISMS FOR THE SYMPLECTIC PLANE

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Alok Bakshi

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The dissertation of Alok Bakshi was reviewed and approved* by the following:

Nigel Higson
Evan Pugh Professor of Mathematics
Dissertation Advisor
Chair of Committee

Paul Baum
Evan Pugh Professor of Mathematics

Nathanial Brown
Professor of Mathematics

Martin Bojowald
Professor of Physics

Mark Levi
Professor of Mathematics
Head of the Mathematics Department

*Signatures are on file in the Graduate School

ABSTRACT

In this thesis we study equivariant asymptotic morphisms from the C_0–functions on the symplectic plane into the C^*–algebra of compact operators. We shall construct two asymptotic morphisms that are asymptotically equivariant for two different groups of symplectomorphisms, and prove (with additional hypotheses) that there does not exist any asymptotic morphism that is asymptotically equivariant for all symplectomorphisms. Furthermore, we shall prove that there exists a unique asymptotic morphism (with additional hypothesis on equivariance) that is equivariant for the group of affine symplectomorphisms of the plane.

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Chapter 1

Introduction

1.1 Philosophy of Quantization

By the end of nineteenth century, classical physics could explain almost all day-to-day phenomena. But as we probed deeper into the building blocks of nature, experiments revealed that nature is in fact quantum and we need entirely different mathematical framework to model hitherto unexplained quantum phenomena. Experimentally, thinking of the classical framework as a large–scale approximation to the quantum framework, one gets predictions of the latter by making quantum corrections (summarized by Planck's constant $\hbar$) to predictions of the former model.

According to Dirac, quantum corrections to predictions of classical theories should be reflected in the axioms of quantization as well. Here quantization refers to the recipe of deducing the "quantum framework model" from the "classical framework model" of the *same* system. Such a recipe, in general is incomplete as there might be many (or none) quantum models with the same limiting ($\hbar \to 0$) classical model.

1.1.1 Classical Framework

Mathematically, a system in the classical framework is modeled as a symplectic manifold (M, ω). Here M as a set can be thought of as phase space of the system. Moreover each $f \in C^\infty(M)$ can be thought of as an observable such that its value at state $m \in M$ is given by the evaluation $f(m)$. In particular, the distinguished energy observable named the Hamiltonian is also specified as a smooth function $H \in C^\infty(M)$.

Using the symplectic form ω one can associate to each observable $f \in C^\infty(M)$ its corresponding Hamiltonian vector field $X_f \in \Xi(M)$. Furthermore, one can define a non–degenerate Poisson-Lie bracket $\{\ ,\ \} : C^\infty(M) \times C^\infty(M) \longmapsto C^\infty(M)$ on the space of observables, and an observable f evolves in time as

$$\frac{df}{dt} = \{f, H\} = \omega(X_f, X_H).$$

1.1.2 Quantum Framework

Mathematically, a system in the quantum framework is modeled as a Hilbert space $\mathcal{H}$ whose set of one–dimensional subspaces $v \in \mathcal{PH}$ is interpreted as the phase space of the system.

Furthermore each symmetric (possibly unbounded) operator A can be thought of as a real observable. If the system is in state $v \in \mathcal{H},\ \|v\| = 1$, then the expected value of the observable is given by $\langle Av, v\rangle$.

In particular, the distinguished energy observable H, named the Hamiltonian, gives rise to time evolution for each observable $\mathcal{O}$ as

$$\frac{d\mathcal{O}}{dt} = i\hbar\,[\mathcal{O}, H].$$

1.1.3 Axioms of Quantization

Based upon the above discussion, a quantization map Q ought to associate to each observable $f \in C^\infty(M)$ (or a suitable Lie sub-algebra thereof) a symmetric operator $Q(f)$ with the following properties

Linearity: Q is $\mathbb{R}$ linear.

Non-triviality: $Q(1) = \mathbb{I}$.

Classical Apprxoimation: $Q(\{f,g\}) = \frac{i}{\hbar}\ [Q(f), Q(g)]$.

Self Adjointness: If the Hamiltonian vector field X_f of the function f is complete then operator $Q(f)$ is essentially self–adjoint on a suitable domain D.

Irreducibility: If $\{f_n\}$ is a complete system of classical observables then the quantum Hilbert space $\mathcal{H}$ must be irreducible under the action of $\{Q(f_n)\}$.

Below are the common approaches to quantize, each realizing a subset of the quantization axioms mentioned above.

1.2 Different Approaches to Quantization

For definiteness, we shall consider the system consisting of a one–dimensional particle interacting with the potential field $V \in C^\infty(\mathbb{R})$. For such a system the phase space is simply $\mathbb{R}^2$ and by choosing appropriate physical units one can without loss of generality choose $\omega = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ to be canonical symplectic form, leading to identification $\mathbb{R}^2 \cong T^*\mathbb{R}$.

1.2.1 Schrodinger Quantization

This classical quantization scheme interprets each quantum state ψ of the particle as the probability density function over its configuration space $q \in \mathbb{R}$.

A loose dictionary between the classical and quantum framework in this case would be

Particle in 1-Dim	Classical Framework	Quantum Framework
State Space	$(q,p) \in \mathbb{R}^2$	$\psi \in \mathcal{PH} \cong \mathcal{P}L^2(q)$
Position-type Observable	$F : (q,p) \longmapsto f(q)$	$\widehat{F} = f(q)$
Momentum-type Observable	$G : (q,p) \longmapsto g(p)$	$\widehat{G} = g\left(i\frac{\partial}{\partial q}\right)$
Hamiltonian	$H = \frac{p^2}{2m} + V(q)$	$H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q^2} + V(q)$
Observables Evolution	$\frac{df}{dt} = \frac{\partial f}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial H}{\partial q}\frac{\partial f}{\partial p}$	$\frac{dO}{dt} = i\hbar\,(OH - HO)$

Table 1.1: Mechanics of One Dimensional Particle in Classical and Quantum Framework

To quantize more general types of observable (say $h(p,q) = f(p)g(q)$) we run into the trouble, especially because $C^\infty(\mathbb{R}^2)$ is commutative under pointwise multiplication while symmetric operators do not commute under composition. One can choose to quantize the above product as

$$Q(f \cdot g) = \frac{1}{2}\left(Q(f) \circ Q(g) + Q(g) \circ Q(f)\right),$$

but the above axioms are not consistent, which is shown in [2] for the function $h(p,q) = p^2 q^2$.

In fact, one version of the "No–Go" theorem of Groenewold-van Hove [2] proves the impossibility of constructing such a map satisfying aforementioned quantization axioms. Nevertheless by restricting one or more axioms, one can get mathematically consistent quantization procedure.

1.2.2 Geometric Quantization

In order to get consistent quantization map, one can restrict the Poisson sub–algebra $\mathfrak{g} \subset C^\infty(\mathbb{R}^2)$ which is to be quantized. For instance, if one identifies $\mathbb{R}^2 \cong \mathbb{C}$ and takes the quantum Hilbert space to be the Fock space of holomorphic functions on $\mathbb{C}$, then one can successfully quantize the algebra of quadratic polynomials $\mathfrak{sp}(2,\mathbb{R}) \cong \mathrm{span}\{z^2, \bar{z}^2, z\bar{z}\}$ via the following map

$$Q\left(\frac{i}{2}z^2\right) = M_{\frac{1}{2}z^2}, \quad Q\left(\frac{i}{2}\bar{z}^2\right) = \frac{i}{2}\frac{\partial^2}{\partial z^2}, \quad Q\left(z\bar{z}\right) = M_z\frac{\partial}{\partial z} + \frac{1}{2}.$$

1.2.3 Deformation Quantization

Since quantum phenomena can be explained in terms of the spectrum of certain interesting operators one can relax the requirement of constructing Hilbert space from the classical phase space. Concretely if we interpret $\hbar$ as deformation parameter, then one can form new associative product $\star_\hbar$ on $A = C^\infty(\mathbb{R}^2)$ such that

$$f \star_\hbar g = f \cdot g + \frac{i\hbar}{2}\{f, g\} + \mathcal{O}(\hbar^2).$$

Note that in the above formulation $f \star_\hbar g$ is a formal power series in $\hbar$ whose coefficients are genuine functions on $\mathbb{R}^2$.

In particular, an associative product $\star_\hbar$ (called the Moyal product) can be written as

$$f \star_\hbar g = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{i\hbar}{2}\right)^n \Pi^n(f, g),$$

where Π^n is inductively defined as $\Pi^0(f, g) = f \cdot g,\ \ \Pi^1(f, g) = \{f, g\}$, and in general

$$\Pi^n(f, g) = \sum_{k=0}^{n}(-1)^k\, C(n, k)\ \left(\frac{\partial^k}{\partial p^k}\frac{\partial^{n-k}}{\partial q^{n-k}}\, f\right)\ \left(\frac{\partial^{n-k}}{\partial p^{n-k}}\frac{\partial^k}{\partial q^k}\, g\right).$$

1.3 No–Go Theorem

In the context of quantization, a No–Go theorem loosely states the impossibility of quantizing every smooth function on $\mathbb{R}^{2n}$. From now, we will restrict our attention to the case of $n = 1$, that is, to the case of $\mathbb{R}^2$.

We first recall that $C^\infty(\mathbb{R}^2)$ is a Poisson algebra under the following Poisson-Lie bracket operation

$$\{f, g\} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x} \qquad \forall f, g \in C^\infty(\mathbb{R}^2).$$

One of the fundamental Lie sub–algebras of $C^\infty(\mathbb{R}^2)$ is the one generated by the coordinate functions q and p which satisfy the famous Dirac commutation relation, namely

$$\{q, p\} = 1.$$

Without loss of generality one can choose these functions $q, p \in C^\infty(\mathbb{R}^2)$ to be the projection maps as follows

$$q : (x, y) \in \mathbb{R}^2 \longmapsto x,$$

$$p : (x, y) \in \mathbb{R}^2 \longmapsto y.$$

The functions q, p along with the identity function 1 generate the Heisenberg Lie algebra (with commutation relation $\{q, p\} = 1,\ \{q, 1\} = \{p, 1\} = 0$) i.e.

$$\mathfrak{heis}(3, \mathbb{R}) = \mathrm{span}_{\mathbb{R}} \{q, p, 1\}.$$

We will be interested in quantizing various Lie subalgebras $\mathfrak{L} \subset C^\infty(\mathbb{R}^2)$ which contains the Heisenberg Lie algebra $\mathfrak{heis}(3,\mathbb{R})$.

We first identify two such Lie algebras, namely the Jacobi Lie algebra $\mathfrak{hsp}(2,\mathbb{R})$ and the coordinate Lie algebra $\mathfrak{c}(2,\mathbb{R})$, which satisfy the above property. These are:

$$\mathfrak{hsp}(2,\mathbb{R}) = \operatorname{span}_{\mathbb{R}} \left\{1, q, p, q^2, p^2, qp\right\},$$

$$\mathfrak{c}(2,\mathbb{R}) = \{f(x)\ p + g(x) \ \mid\ f, g \in C^\infty(\mathbb{R})\}.$$

In fact for the case of $\mathbb{R}^2$ we don't have any other interesting choices. More specifically:

Theorem 1.3.1. *[2] Up to isomorphism, only non trivial Lie algebras of $C^\infty(\mathbb{R}^2)$ containing the Heisenberg Lie algebra are $\mathfrak{hsp}(2,\mathbb{R})$ and $\mathfrak{c}(2,\mathbb{R})$.*

1.3.1 Quantization of $\mathfrak{c}(2,\mathbb{R})$

We choose $\mathcal{H} = L^2(\mathbb{R})$ with domain $D = \mathcal{S}(\mathbb{R})$. Then for any $\eta \in \mathbb{R}$ one can define a quantization map

$$Q_\eta : \mathfrak{c}(2,\mathbb{R}) \longmapsto \operatorname{SymmOp}(L^2(\mathbb{R}))$$

as follows:

$$Q_\eta\left(f(x)\ p + g(x)\right) = -i\hbar\left(f(x)\frac{\partial}{\partial x} + \left(\frac{1}{2} + i\eta\right)\frac{\partial f}{\partial x}\right) + g(x).$$

Theorem 1.3.2. *[2] Every quantization of the $\mathfrak{c}(2,\mathbb{R})$ is unitarily equivalent to one of Q_η defined above. Moreover Q_η and $Q_{\eta'}$ are unitarily equivalent if and only if*

$\eta = \eta'$ and none of the quantization defined above can be extended to a larger Lie algebra containing $\mathfrak{c}(2, \mathbb{R})$.

1.3.2 Quantization of $\mathfrak{hsp}(2, \mathbb{R})$

As before, we choose $\mathcal{H} = L^2(\mathbb{R})$ with domain $D = \mathcal{S}(\mathbb{R})$. Jacobi Lie algebra can be quantized by the map

$$Q : \mathfrak{hsp}(2, \mathbb{R}) \longmapsto \mathrm{SymmOp}(L^2(\mathbb{R}))$$

defined as follows:

$$Q(q) = M_x, \quad Q(p) = -i\hbar\frac{\partial}{\partial x}, \quad Q(q^2) = M_{x^2}, \quad Q(p^2) = -\hbar^2\frac{\partial^2}{\partial x^2},$$
$$Q(1) = \mathbb{I}, \qquad Q(xp) = -i\hbar\left(M_x\frac{\partial}{\partial x} + \frac{1}{2}\right).$$

Theorem 1.3.3. *[2] The above representation of $\mathfrak{hsp}(2, \mathbb{R})$ exponentiates to the Lie group representation of double cover of Jacobi Lie group called metaplectic representation. Moreover by Stone–von Neumann theorem any unitary irreducible representation of $\mathfrak{hsp}(2, \mathbb{R})$ is unitarily equivalent to the above which cannot be extended to the larger Lie algebra containing $\mathfrak{hsp}(2, \mathbb{R})$.*

Keeping in view the results above, the No–Go theorem for $\mathbb{R}^2$ can be stated as

Theorem 1.3.4. *[2] It is impossible to quantize the full Lie algebra of functions on $\mathbb{R}^2$ namely $C^\infty(\mathbb{R}^{2n})$. Nevertheless it is possible to separately quantize $\mathfrak{hsp}(2, \mathbb{R})$ and $\mathfrak{c}(2, \mathbb{R})$, which are two maximal Lie subalgebras of it.*

1.4 Quantization via Asymptotic Morphism

In this thesis, we propose yet another way of quantizing symplectic phase space by demonstrating it for the case of $\mathbb{R}^2$.

Let M be a locally compact space and $\mathcal{H}$ be a Hilbert space. An asymptotic morphism (parameterized by $s \in (0, 1]$) is a collection of functions $T^s : C_0(M) \longrightarrow \mathcal{K}(\mathcal{H})$ such that

- $\lim\limits_{s \to 0} \|T^s(af + g) - aT^s(f) - T^s(g)\| = 0 \quad f, g \in C_0(M),\ a \in \mathbb{C}$
- $\lim\limits_{s \to 0} \|T^s(f \cdot g) - T^s(f) \circ T^s(g)\| = 0 \quad f, g \in C_0(M)$
- $\lim\limits_{s \to 0} \|T^s(\bar{f}) - T^s(f)^\star\| = 0 \quad f \in C_0(M)$

In case both the space M and Hilbert space $\mathcal{H}$ carry actions of a Lie group G, we require the above map to be either equivariant or asymptotically equivariant, that is, equivariant in the asymptotic sense. Any such asymptotic morphism $\{T^s\}_{s \in [0,1)}$ gives rise to an index map, namely

$$\text{ind} : K(C_0(M)) \longmapsto K(\mathcal{K}\mathcal{H}) \cong \mathbb{Z}.$$

For the next definitions, we will concern ourselves with the case of linear spaces i.e. as a symplectic space $M \cong T^*\mathbb{R}^n$. We further qualify T^s as being non–trivial and quantization map if

Non–trivial: An asymptotic morphism map T^s is non–trivial if and only if its associated index map sends the generator of $K(C_0(M)) \cong \mathbb{Z}$ to the generator of $K(\mathcal{KH}) \cong \mathbb{Z}$.

Quantization: An asymptotic morphism map T^s is a quantization of M if it is asymptotically equivariant with respect to all linear symplectomorphisms of $M \cong \mathbb{R}^{2n}$ i.e. $\mathrm{Sp}(2n, \mathbb{R})$ as well as all possible translations along position and momentum (cotangent) directions, that is Heisenberg group $\mathrm{Heis}(2n+1, \mathbb{R})$.

Ideally one would want to make T^s asymptotically equivariant with respect to all symplectic diffeomorphisms of M. But (as we shall see later) this is too strong a requirement. Thus we merely demand asymptotic equivariance with respect to both $\mathrm{Heis}(2n+1, \mathbb{R})$ and $\mathrm{Sp}(2n, \mathbb{R})$ groups. Taken together, we require the map to be asymptotically equivariant with respect to the double cover of $\mathrm{Heis}(2n+1, \mathbb{R}) \rtimes \mathrm{Sp}(2n, \mathbb{R}) \cong \mathrm{HSp}(2n, \mathbb{R})$ namely, the Jacobi Lie group.

1.4.1 Asymptotic Versus Other Types of Quantizations

In the case of geometric quantization, except for restricting the set of classical functions to be quantized, all other requirements of quantization are satisfied. Nevertheless, one requires phase spaces to satisfy additional topological requirement of integrality. Furthermore, one needs to incorporate additional information (a real or complex polarization) with classical data so as to uniquely identify the quantum Hilbert space.

The theory of geometric quantization closely parallels the orbit method in representation theory, which seeks to attach an unitary irreducible representation of Lie group G to its co–adjoint orbits in $\mathfrak{g}^*$.

With deformation quantization, one focuses on the algebra of commuting classical observables and continuously deforms those (with $\hbar$ as the formal parameter) to non–commuting quantum observables. Though one can quantize every classical observable in this way, it is not easy to re–interpret quantum observable (as a formal series in $\hbar$) as an operator on suitable Hilbert space.

In the case of quantization via an asymptotic morphism, one can quantize any function in the C_0 class. Furthermore, the quantum observables consists of sequence (parameterized by the parameter $\hbar \in (0, 1]$) of Hilbert space operators. Unfortunately here too, it is not always possible to reinterpret quantum observables as individual Hilbert space operators.

1.5 Thesis Results

Before considering general asymptotic morphisms, we first study two important asymptotic morphisms, denoted $T^{\hbar}_{\text{hsp}} : C_0(\mathbb{R}^2) \dashrightarrow \mathcal{K}\left(L^2(\mathbb{R})\right)$ and $T^{\hbar}_{\text{diff}} : C_0(\mathbb{R}^2) \dashrightarrow \mathcal{K}\left(L^2(\mathbb{R})\right)$ defined below.

1.5.1 Asymptotic Morphism Equivariant with Respect to Diffeomorphisms of the Line

Identifying $\mathbb{R}^2 \cong T^*\mathbb{R}$, for $a(x,\xi) \in C_0(T^*\mathbb{R}) \cong C_0(\mathbb{R}^2)$ and $g(x) \in L^2(\mathbb{R})$, the asymptotic morphism map

$$T^{\hbar}_{\text{diff}} : C_0(\mathbb{R}^2) \dashrightarrow \mathcal{K}\left(L^2(\mathbb{R})\right)$$

is defined[1] as follows

$$T^{\hbar}_{\text{diff}}(a)\, g(x) = \int_{\mathbb{R}} a(x, \hbar^{-1}\xi)\, e^{ix\xi}\, \hat{g}(\xi)\, d\xi = \int_{\mathbb{R}} k^{\hbar}_a(x,y)\, g(y)\, dy,$$

where the kernel $k^{\hbar}_a(x,y)$ is given by the following formula

$$k^{\hbar}_a(x,y) = \frac{\hbar}{2\pi} \int_{\mathbb{R}} a(x,\xi)\, e^{i\hbar(x-y)\xi}\, d\xi.$$

Via the identification, $\mathbb{R}^2 \cong T^*(\mathbb{R})$, the $\text{Diff}(\mathbb{R})$ action

$$\text{Diff}(\mathbb{R}) \times C_0(\mathbb{R}^2) \longrightarrow C_0(\mathbb{R}^2)$$

can be written (where $\phi \in \text{Diff}(\mathbb{R}),$ and $a(x,\xi) \in C_0(\mathbb{R}^2)$) as follows

$$\phi \cdot a(x,\xi) = a\left(\phi(x),\ \phi_*^{-1}\xi\right).$$

Similarly for $g \in L^2(\mathbb{R})$, the "change of coordinate" diffeomorphism group action $\text{Diff}(\mathbb{R}) \times L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$ (normalized to make it unitary) is

$$\phi \cdot g(x) = g\left(\phi(x)\right) |\phi'(x)|^{\frac{1}{2}}.$$

As verified in [1] the map defined above is an asymptotic morphism which is asymptotically equivariant with respect to the diffeomorphism group action defined above.

We note that one can place the symmetry group of the above–defined asymptotic morphism, namely $\text{Diff}(\mathbb{R})$, in the larger context of all symplectic diffeomorphisms of $T^*\mathbb{R} \cong \mathbb{R}^2$, that is, $\text{Diff}(\mathbb{R}) \subset \text{SympDiff}(T^*\mathbb{R})$. But we shall show that $T^\hbar_{\text{diff}}$ is not asymptotically equivariant with respect to this larger group.

1.5.2 Heisenberg Group Equivariant Asymptotic Morphism

First we note that using the natural diffeomorphism via the exponential map $\exp : \mathfrak{heis}(3,\mathbb{R}) \cong \mathbb{R}^3 \longrightarrow \text{Heis}(3,\mathbb{R})$, one can uniquely identify each $h \in \text{Heis}(3,\mathbb{R})$ with $(x_h, y_h, z_h) \in \mathfrak{heis}(3,\mathbb{R}) \cong \mathbb{R}^3$.

Choosing any unit norm function $\Theta \in L^2(\mathbb{R})$, for $g \in C_0(\mathbb{R}^2)$ and $\eta \in L^2(\mathbb{R})$ the asymptotic morphism map

$$T^{\hbar}_{\text{hsp}} : C_0(\mathbb{R}^2) \dashrightarrow \mathcal{K}\left(L^2(\mathbb{R})\right)$$

is defined[4] as follows

$$T^{\hbar}_{\text{hsp}}(g)\eta = \int_H \widehat{g}(x_h, y_h)\Theta(z_h)\left(\Pi_{\hbar}\left(h\right)\eta\right)\, dh.$$

We shall verify in chapter 4 that the above map is equivariant with respect to the following actions of $\text{HSp}(2,\mathbb{R})$ on $C_0(\mathbb{R}^2)$ and $L^2(\mathbb{R})$:

The groups $\text{Heis}(3,\mathbb{R})$, $\text{Sp}(2,\mathbb{R}) \subset \text{HSp}(2,\mathbb{R})$ act on $L^2(\mathbb{R})$ via Schrodinger and metaplectic representations respectively. On the other hand on $T^*\mathbb{R} \cong \mathbb{R}^2$ the standard symplectic and Heisenberg Lie group actions yields "change of coordinate" action of $\text{HSp}(2,\mathbb{R})$ on $C_0(\mathbb{R}^2)$.

Our main results are No–Go theorems in the context of quantization via asymptotic morphisms.

Theorem 1.5.1. *The two natural aforementioned asymptotic morphism maps*

$$T^{\hbar}_{hsp},\ T^{\hbar}_{diff} : C_0(\mathbb{R}^2) \dashrightarrow \mathcal{K}(L^2(\mathbb{R}^2)),$$

are asymptotically equivariant with respect to the groups $\text{Diff}(\mathbb{R})$ *and the double cover of Jacobi Lie group* $\text{HSp}(2,\mathbb{R}) \cong \text{Heis}(3,\mathbb{R}) \rtimes \text{Sp}(2,\mathbb{R})$*, respectively. Neither of* *the above maps is asymptotically equivariant with respect to the whole* $\mathrm{SympDiff}(\mathbb{R}^2)$ *group.*

In particular $T^{\hbar}_{diff}$ *is not asymptotically equivariant with respect to the group action of* $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{Sp}(2,\mathbb{R})$ *while* $T^{\hbar}_{hsp}$ *is not asymptotically equivariant with respect to the group action of* $\phi \in \mathrm{Diff}(\mathbb{R})$ *for* $\phi(x) = x + \frac{x^2}{2} + \frac{x^3}{3}$.

Theorem 1.5.2. *Any arbitrary linear topologically non–trivial asymptotic morphism*

$$T^{\hbar} : C_0(\mathbb{R}^2) \dashrightarrow \mathcal{K}(L^2(\mathbb{R}^2)),$$

with $\|T^{\hbar}\| \leq 1$ *that is rapidly asymptotically equivariant with respect to the* $\mathrm{HSp}(2,\mathbb{R})$ *group action, is equivariantly asymptotic equivalent to the asymptotic morphism* $T^{\hbar}_{hsp}$ *defined above.*

Thus above two theorems loosely speaking show the unique quantization of $\mathbb{R}^2$ via asymptotic morphisms, and hence that no quantization via asymptotic morphism of $\mathbb{R}^2$ is equivariant for the whole symplectic diffeomorphism group of $\mathbb{R}$.

Chapter 2

Background

In this chapter, some definitions and theorems are stated which will be relevant for the later chapters.

2.1 Symplectic Vector Space (V, ω)

A symplectic vector space (V, ω) is an even dimensional real vector space V equipped with the 2–form (called symplectic form) $\omega : V \times V \longrightarrow \mathbb{R}$, which is bilinear, alternating and non–degenerate.

One can globally coordinatize any V as $T^*\mathbb{R}^n \cong \mathbb{R}^{2n}$, identifying its symplectic form ω as

$$\omega = \begin{pmatrix} 0 & \mathbb{I}_n \\ -\mathbb{I}_n & 0 \end{pmatrix}.$$

2.2 Relevant Lie Sub–algebras of $C^\infty\left(\mathbb{R}^{2n}\right)$

Over the symplectic vector space $(\mathbb{R}^{2n}, \omega_0)$, one can choose linear coordinates $(q_1, q_2, \ldots, q_n)$ and $(p^1, p^2, \ldots, p^n)$ (position and momentum coordinates respectively), such that

$$\omega_0(q_i, p^j) = \delta_{i,j},$$
$$\omega_0(q_i, q_j) = \omega_0(p^i, p^j) = 0,$$

and in terms of these linear coordinates one can define a Lie–Poisson bracket on $C^\infty(\mathbb{R}^{2n})$ as follows:

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x} \frac{\partial g}{\partial y} - \frac{\partial f}{\partial y} \frac{\partial g}{\partial x} \right) \qquad \forall f, g \in C^\infty(\mathbb{R}^{2n}).$$

We will define and fix the notation for all Lie sub–algebras of $C^\infty(\mathbb{R}^{2n})$, which will make appearance in the later chapters.

2.2.1 Coordinate Lie Algebra $\mathfrak{c}(2n,\mathbb{R})$

The coordinate Lie–algebra $\mathfrak{c}(2n,\mathbb{R})$ is spanned by linear polynomials in momentum coordinate p^j and polynomials over position coordinates q_i i.e.

$$\mathfrak{c}(2n,\mathbb{R}) = \left\{ \sum_{i=1}^{n} f_i(q_1, q_2, \dots q_n)p^i + g(q_1, q_2, \dots, q_n) \;\mid\; f_i, g \in \mathbb{R}[q_1, q_2, \dots, q_n] \right\}.$$

2.2.2 Heisenberg Lie Algebra $\mathfrak{heis}(2n+1,\mathbb{R})$

The Heisenberg Lie algebra $\mathfrak{heis}(2n+1,\mathbb{R})$ is spanned by constant and linear polynomials over the position and momentum coordinates i.e.

$$\mathfrak{heis}(2n+1,\mathbb{R}) = \operatorname{span}\left\{1, q_i, p^j \;\mid 1 \leq i, j \leq n \right\}.$$

2.2.3 Symplectic Lie Algebra $\mathfrak{sp}(2n,\mathbb{R})$

The symplectic Lie algebra $\mathfrak{sp}(2n,\mathbb{R})$ is spanned by the second degree polynomials in position and momentum coordinates i.e.

$$\mathfrak{sp}(2n,\mathbb{R}) = \left\{ q_i q_j,\; q_i p^j,\; p^i p^j \;\mid 1 \leq i, j \leq n \right\}.$$

2.2.4 Jacobi Lie Algebra $\mathfrak{hsp}(2n, \mathbb{R})$

The Jacobi Lie algebra $\mathfrak{hsp}(2n, \mathbb{R})$ is spanned by all polynomials of degree at most two in the position and momentum coordinates i.e.

$$\mathfrak{hsp}(2n, \mathbb{R}) = \left\{1,\ q_i,\ p^j,\ q_i q_j,\ q_i p^j,\ p^i p^j\ \mid 1 \leq i, j \leq n\right\}.$$

In particular, we have

$$\mathfrak{hsp}(2n, \mathbb{R}) = \mathfrak{sp}(2n, \mathbb{R}) \bigoplus \mathfrak{heis}(2n+1, \mathbb{R}),$$

that is a semi–direct product decomposition of Lie algebras.

In this thesis, we will deal with the $n = 1$ case.

2.3 Asymptotic Morphism

Given two C^*–algebras A and B, an asymptotic morphism from A to B is a family of maps $\{T^s\}_{s\in(0,1]}$ such that the following conditions[7] hold:

- For all $a \in A$ the map $T_a : (0, 1] \longrightarrow B$ given by $s \mapsto T^s(a)$ is continuous.
- For all $a \in A$, $b \in B$ and scalar $\lambda \in \mathbb{C}$ the family $\{T^s\}_{s\in(0,1]}$ satisfies C^*–

morphism requirements asymptotically. That is:

$$\lim_{s\to 0} \|T^s(a+\lambda b) - T^s(a) - \lambda T^s(b)\| = 0,$$

$$\lim_{s\to 0} \|T^s(ab) - T^s(a)T^s(b)\| = 0,$$

$$\lim_{s\to 0} \|T^s(a^*) - T^s(a)^*\| = 0.$$

2.3.1 Equivariant Asymptotic Morphism

Suppose given two C^*–algebras A and B, both of which are acted upon by a Lie group G. We define the following two closely related notions of equivariance and asymptotic equivariance.

Definition 2.3.1. An asymptototic morphism $T^s : A \dashrightarrow B$ is **equivariant** with respect to the group actions G if for every $a \in A$ and every $g \in G$ one has

$$T^s(g \cdot a) = g \cdot T^s(a).$$

Definition 2.3.2. An asymptototic morphism $T^s : A \dashrightarrow B$ is **asymptotically equivariant** with respect to the group actions G if for every $a \in A$ and every $g \in G$ one has

$$\lim_{s\to 0} \|T^s(g \cdot a) - g \cdot T^s(a)\| = 0.$$

Definition 2.3.3. For the special case of C^*–algebras $A = C_0(\mathbb{R}^{2n})$ and $B = \mathcal{K}\left(L^2(\mathbb{R}^n)\right)$, let $\mathcal{A} = \mathcal{S}(\mathbb{R}^n)$ be the dense Schwartz subspace of A. Moreover we let $\mathcal{K}^\infty$ be the Frechet subspace of compact operators that has integral kernels $k(x,y)$

lying in the Schwartz space of $\mathbb{R}^n \times \mathbb{R}^n$. If we choose the Hermite functions basis of $L^2(\mathbb{R})$ then $\mathcal{K}^\infty$ can be shown to consist of operators $K = (a_{i,j})_{i,j=1}^\infty$ on $L^2(\mathbb{R}^n)$ topologized by the following semi–norms:

$$\|K\|_{p,q} = \sum_{i,j=1}^{\infty} i^p j^q \, |a_{i,j}| < \infty \qquad \forall p, q \in \mathbb{N}.$$

Asymptotic morphism $T^s : A \dashrightarrow B$ is defined to be **rapidly asymptotically equivariant** with respect to the group actions G if for all $a \in \mathcal{A}$ and $g \in G$ one has $T^s(g \cdot a) - g \cdot T^s(a) \in \mathcal{K}^\infty$ such that

$$\lim_{s \to 0} \|T^s(g \cdot a) - g \cdot T^s(a)\|_{p,q} = 0 \quad \forall a \in \mathcal{A}, \quad \forall p, q \in \mathbb{N}.$$

We note that the equivariant asymptotic morphisms are by definition (rapidly) asymptotically equivariant.

2.3.2 Equivalence of Asymptotic Morphisms

Definition 2.3.4. Asymptotic morphisms $T^s, U^s : A \dashrightarrow B$ are **equivalent**[7] (denoted as $T^s \cong U^s$) if for all $a \in A$ we have

$$\lim_{s \to 0} \|T^s(a) - U^s(a)\| = 0.$$

Definition 2.3.5. Asymptotic morphisms $T^s, U^s : A \dashrightarrow B$ are **homotopy equivalent**[7] (denoted as $T^s \sim U^s$) if there exists an asymptotic morphism

$\Phi^s : A \dashrightarrow C([0,1], B)$ such that for all $a \in A$

$$\Phi_s(a)(0) = T^s(a),$$

$$\Phi^s(a)(1) = U^s(a).$$

We note that equivalent asymptotic morphisms are automatically homotopy equivalent. For equivariant (respectively asymptotically equivariant) asymptotic morphisms one requires homotopy equivalence map Φ^s to be equivariant (respectively asymptotically equivariant).

2.4 Heisenberg, Symplecic and Jacobi Lie Groups

2.4.1 Symplectic Lie Group

Given a symplectic vector space (V, ω), the symplectic group $\mathrm{Sp}(V, \omega)$ is the subgroup of all invertible linear transformations of V which preserve the form ω.

For the two–dimensional case ($n = 1$) using global coordinates one can identify the corresponding symplectic Lie group as special linear group i.e.

$$\mathrm{Sp}(2, \mathbb{R}) = \mathrm{Sp}(T^*\mathbb{R}, \omega) \cong \mathrm{SL}(2, \mathbb{R}).$$

2.4.2 Heisenberg Lie Group

given a symplectic vector space (V, ω), the corresponding Heisenberg Lie algebra $\mathfrak{heis} = V \oplus \mathbb{R}Z$ is defined with the following bracket relations:

- $[X, Z] = 0, \quad \forall X \in \mathfrak{heis}$,
- $[X, Y] = \omega(X, Y)\, Z, \quad \forall X, Y \in V$.

The Heisenberg Lie algebra $\mathfrak{heis}$ is a nilpotent Lie algebra and we can identify it as a manifold with the Heisenberg Lie group Heis via the diffeomorphism provided by the exponential map

$$\exp : \mathfrak{heis} \xrightarrow{\cong} \text{Heis}.$$

Moreover, the above defined exponential diffeomorphism induces the group structure on the Heisenberg Lie algebra elements

$$(x_1, x_2) \oplus s \in \mathbb{R}^{2n} \oplus \mathbb{R} \cong \mathfrak{heis}(2n+1, \mathbb{R}),$$

as follows:

$$\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, s \right) \circ \left(\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, t \right) = \left(\begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix}, s + t + \frac{1}{2} \left(\langle x_1, y_2 \rangle - \langle x_2, y_1 \rangle \right) \right).$$

The three dimensional Heisenberg Lie group $\text{Heis}(3, \mathbb{R})$ can also be identified with the group of 3×3 upper triangular matrices, that is

$$\mathrm{Heis}(3, \mathbb{R}) = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; a, b, c \in \mathbb{R} \right\}.$$

2.4.3 Jacobi Lie Group

The Jacobi Lie group $\mathrm{HSp}(V, \omega)$ is a semi–direct product of the symplectic Lie group with the Heisenberg Lie group, as follows:

The symplectic Lie group $\mathrm{Sp}(V, \omega)$ acts on the Heisenberg Lie group as follows:

$$T \cdot \exp(v \oplus z) = \exp(Tv) \oplus z \qquad \forall T \in \mathrm{Sp}(V, \omega),\; v \in V,\; z \in \mathbb{R}Z.$$

Using the above action one defines Jacobi group as the semidirect product $\mathrm{HSp}(V, \omega) = \mathrm{Heis}(V, \omega) \rtimes \mathrm{Sp}(V, \omega)$.

Explicitly if $\Phi : \mathrm{Sp}(V, \omega) \longrightarrow \mathrm{Aut}\,(\mathrm{Heis}(V, \omega))$ is the map constructed above, then for all $(h_1, g_1),\; (h_2, g_2) \in \mathrm{Heis}(V, \omega) \rtimes \mathrm{Sp}(V, \omega)$ we have

$$(h_1, g_1) \cdot (h_2, g_2) = (h_1\, \Phi(g_1) h_2,\; g_1 g_2)\,.$$

For sake of notational convenience, in case of $\mathbb{R}^{2n}$ equipped with the canonical symplectic form ω we use the following notation

$$\mathrm{HSp}(2n, \mathbb{R}) = \mathrm{HSp}(\mathbb{R}^{2n}, \omega).$$

For future calculation, we note the following lemma.

Lemma 2.4.1. *The Jacobi group* $\mathrm{HSp}(2,\mathbb{R})$ *is generated by the subgroup* $\mathrm{Sp}(2,\mathbb{R})$ *and one parameter subgroup* $\{M^a\}_{a\in\mathbb{R}}$

$$M^a = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{Heis}(3,\mathbb{R}).$$

Proof. We first note that

$$\left(e, g^{-1}\right) \cdot (h, e) \cdot (e, g) = \left(\Phi(g^{-1})h,\, e\right).$$

In particular for $h = M^a$ and $g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{Sp}(2,\mathbb{R})$ we have

$$\left(e, g^{-1}\right) \cdot (M^a, e) \cdot (e, g) = (N^a, e), \text{ where } N^a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{Heis}(3,\mathbb{R}). \quad (2.4.1)$$

Since $\{M^a, N^a\}_{a\in\mathbb{R}}$ generate the Heisenberg group $\mathrm{Heis}(3,\mathbb{R})$ the result follows.

□

2.5 Hermite Functions and Ladder Operators

Consider the Hilbert space $L^2(\mathbb{R})$, inside of it we fix Schwartz space $\mathcal{S}(\mathbb{R})$ as the dense subspace over which the multiplication and derivative differential operators are defined.

The Hermite operator $L^\hbar$ (parameterized by $\hbar > 0$) with domain in Schwartz space is defined as:

$$L^\hbar = \frac{1}{2\hbar}\left(x^2 - \hbar^2 \frac{d^2}{dx^2}\right),$$

and it follows that Hermite operator is symmetric and positive definite. The Hermite functions are the eigenfunctions of the Hermite operator with the boundary condition that solution be polynomially bounded at the infinity. We give the details below.

2.5.1 Hermite Functions

The normalized Hermite functions $\{\psi_n\}_{n=0}^{\infty}$ for the case of $\hbar = 1$ can be explicitly defined as follows

$$\psi_0(x) = \pi^{-\frac{1}{4}} \exp\left(-\frac{x^2}{2}\right),$$
$$\psi_1(x) = \sqrt{2}\,\pi^{-\frac{1}{4}}\, x \exp\left(-\frac{x^2}{2}\right),$$
$$\psi_2(x) = \left(\sqrt{2}\,\pi^{\frac{1}{4}}\right)^{-1} (2x^2 - 1) \exp\left(-\frac{x^2}{2}\right),$$
$$\vdots$$
$$\psi_n(x) = (-1)^n \left(2^n n! \sqrt{\pi}\right)^{-\frac{1}{2}} \exp\left(\frac{x^2}{2}\right) \frac{d^n}{dx^n} \exp\left(-x^2\right).$$

Starting from the above, we define generalized Hermite functions $\psi_n^{\hbar}$ by the following:

$$\psi_n^{\hbar}(x) = \hbar^{-\frac{1}{4}}\, \psi_n(x\hbar^{-\frac{1}{2}}).$$

Below we list the important identities[11] satisfied by the Hermite functions:

- Normalized Hermite functions satisfy the following second order differential equation:

$$\psi_n''(x) + (2n + 1 - x^2)\psi_n(x) = 0 \qquad \forall n \geq 0.$$

- Normalized Hermite functions diagonalize the unitary Fourier transform operator $\mathcal{F}$ as follows

$$\mathcal{F}(\psi_n)(x) = (-i)^n\, \psi(x) \qquad \forall n \geq 0.$$

- Hermite functions are the eigenfunctions of the Hermite operator

$$L^{\hbar}\,\psi_n^{\hbar} = \left(n + \frac{1}{2}\right)\psi_n^{\hbar} \qquad \forall n \geq 0.$$

- Hermite functions are rapidly decreasing i.e. $\psi_n^{\hbar} \in \mathcal{S}(\mathbb{R}) \quad \forall n \geq 0$.

- Hermite functions form an orthonormal basis of the Hilbert space $L^2(\mathbb{R})$.

2.5.2 A Useful Identity Involving Hermite Functions

Hermite polynomials $\{H_n(x)\}_{n=0}^{\infty}$ are defined as follows

$$H_n(x) = (-1)^n\, e^{x^2}\, \frac{d^n}{dx^n} e^{-x^2}.$$

One has following identity relating Hermite polynomials with monomials

$$x^n = \frac{n!}{2^n} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{k!(n-2k)!}\, H_{n-2k}(x),$$

and using it we get the following identity (emphasized here for the future calculation)

$$x^n e^{-x^2} = \frac{n!}{2^n} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^{n-2k}}{k!(n-2k)!}\, \frac{d^{n-2k}}{dx^{n-2k}} e^{-x^2}. \tag{2.5.1}$$

2.5.3 Ladder Operators

The ladder operators, also called the creation operator $a^\dagger$ and annihilation operator a, are defined as unbounded operators on $\mathcal{S}(\mathbb{R})$ as follows:

$$a^\dagger = \frac{1}{\sqrt{2\hbar}}\left(x - \hbar\frac{d}{dx}\right),$$
$$a = \frac{1}{\sqrt{2\hbar}}\left(x + \hbar\frac{d}{dx}\right).$$

We note that

$$L_\hbar = aa^\dagger - \frac{1}{2}\mathbb{I}.$$

Furthermore, the creation and annihilation operators are subdiagonal and superdiagonal operators in the Hermite function basis. Indeed

$$a^\dagger \cdot \psi_n^\hbar(x) = \sqrt{n+1}\,\psi_{n+1}^\hbar(x),$$
$$a \cdot \psi_n^\hbar(x) = \sqrt{n}\,\psi_{n-1}^\hbar(x) \qquad \forall n \geq 0.$$

Finally these operators satisfy the following two useful identities

$$[a, a^\dagger] = \mathbb{I},$$
$$\langle af, g\rangle_{L^2(\mathbb{R})} = \left\langle f, a^\dagger g\right\rangle_{L^2(\mathbb{R})}$$

2.6 Commutation Relation

We fix an orthonormal basis in the Hilbert space $L^2(\mathbb{R})$.

Let D be an infinite dimensional diagonal operator i.e. $D = (d_{n,n})_{n=1}^{\infty}$. Similarly let $L = (l_{n+1,n})_{n=1}^{\infty}$ be a subdiagonal operator and $U = (u_{n,n+1})_{n=1}^{\infty}$ be a superdiagonal operator.

We have the following commutation relations between the operators introduced above.

- Commutator of subdiagonal and diagonal operator is again a subdiagonal operator given by the following equation

$$[L, D] = (l_{n+1,n} \cdot (d_{n,n} - d_{n+1,n+1}))_{n=1}^{\infty} .$$

- Commutator of superdiagonal and diagonal operator is again a superdiagonal operator given by the following equation

$$[U, D] = (u_{n,n+1} \cdot (d_{n+1,n+1} - d_{n,n}))_{n=1}^{\infty} .$$

- Commutator of subdiagonal and superdiagonal operators is a diagonal operator

$$[L, U] = (H_{n,n})_{n=1}^{\infty} ,$$

where one has

$$H_{n,n} = \begin{cases} -u_{1,2} \cdot l_{2,1} & n = 1 \\ -u_{n,n+1} \cdot l_{n+1,n} + u_{n-1,n} \cdot l_{n,n-1} & n > 1 \end{cases} .$$

- Commutator of two subdiagonal operators is a sub-subdiagonal (with another shift below the diagonal) operator

$$[L^1, L^2] = (H_{n+2,n})_{n=1}^{\infty},$$

where one has

$$H_{n+2,n} = l^1_{n+2,n+1} \cdot l^2_{n+1,n} - l^1_{n+1,n} \cdot l^2_{n+2,n+1}.$$

- Commutator of superdiagonal operators is a super–superdiagonal (with another shift above the diagonal) operator

$$[U^1, U^2] = (H_{n,n+2})_{n=1}^{\infty},$$

where one has

$$H_{n,n+2} = u^1_{n,n+1} \cdot u^2_{n+1,n+2} - u^1_{n+1,n+2} \cdot u^2_{n,n+1}.$$

- Commutator of sub–subdiagonal and diagonal operator is again a sub–subdiagonal operator given by the following equation

$$[L, D] = (l_{n+2,n} \cdot (d_{n,n} - d_{n+2,n+2}))_{n=1}^{\infty}.$$

- Commutator of super–superdiagonal and diagonal operator is again a super–superdiagonal operator given by the following equation

$$[U, D] = (u_{n,n+2} \cdot (d_{n+2,n+2} - d_{n,n}))_{n=1}^{\infty}.$$

Let us consider the three special operators

$$a = (u_{n,n+1})_{n=1}^{\infty} \text{ where } u_{n,n+1} = \sqrt{n},$$

$$a^{\dagger} = (l_{n+1,n})_{n=1}^{\infty} \text{ where } l_{n+1,n} = \sqrt{n},$$

$$T = (t_n)_{n=1}^{\infty} \text{ where we have used the notation } t_{n,n} = t_n.$$

Commutator relations for these special operators are

- Commutator of a and T is superdiagonal with

$$[a, T] = (u_{n,n+1})_{n=1}^{\infty},$$

  where

$$u_{n,n+1} = \sqrt{n}\,(t_{n+1} - t_n).$$

- Commutator of $a^{\dagger}$ and T is subdiagonal with

$$[a^{\dagger}, T] = (l_{n+1,n})_{n=1}^{\infty},$$

  where

$$l_{n+1,n} = \sqrt{n}\,(t_n - t_{n+1}).$$

- Commutator of a^2 and T is super–superdiagonal with

$$[a^2, T] = (u_{n,n+2})_{n=1}^{\infty},$$

  where

$$u_{n,n+2} = \sqrt{n(n+1)}\,(t_{n+2} - t_n).$$

- Commutator of $a^{\dagger^2}$ and T is sub–subdiagonal with

$$[a^{\dagger^2}, T] = (l_{n+2,n})_{n=1}^{\infty},$$

where

$$l_{n+2,n} = \sqrt{n(n+1)}\,(t_n - t_{n+2}).$$

- Commutator of a and $a^\dagger$ is identity by definition

$$[a, a^\dagger] = \mathbb{I}.$$

Next we calculate few more commutators, which would come up in the later calculation

- Commutator of a and $[a, T]$ is a super–superdiagonal operator

$$[a,\ [a, T]\,] = (H_{n,n+2})_{n=1}^{\infty},$$

where

$$\begin{aligned} H_{n,n+2} &= \sqrt{n}\cdot\sqrt{n+1}\,(t_{n+2} - t_{n+1}) - \sqrt{n+1}\cdot\sqrt{n}\,(t_{n+1} - t_n) \\ &= \sqrt{n^2+n}\,(t_{n+2} - 2t_{n+1} + t_n)\,. \end{aligned}$$

- Commutator of $a^\dagger$ and $[a^\dagger, T]$ is a sub–subdiagonal operator

$$[a^\dagger,\ [a^\dagger, T]\,] = (H_{n+2,n})_{n=1}^{\infty},$$

where

$$\begin{aligned} H_{n+2,n} &= \sqrt{n+1}\cdot\sqrt{n}\,(t_n - t_{n+1}) - \sqrt{n}\cdot\sqrt{n+1}\,(t_{n+1} - t_{n+2}) \\ &= \sqrt{n^2+n}\,(t_{n+2} - 2t_{n+1} + t_n)\,. \end{aligned}$$

- Commutator of a and $[a^\dagger, T]$ is a diagonal operator with

$$[a,\, [a^\dagger, T]\,] = (H_{n,n})_{n=1}^{\infty}\,,$$

where

$$H_{n,n} = \begin{cases} (t_1 - t_2) & n = 1 \\ \\ (2n-1)t_n - nt_{n+1} - (n-1)t_{n-1} & n > 1. \end{cases}$$

- Commutator of $a^\dagger$ and $[a, T]$ is a diagonal operator with

$$[a^\dagger,\, [a, T]\,] = (H_{n,n})_{n=1}^{\infty}\,,$$

where

$$H_{n,n} = \begin{cases} (t_1 - t_2) & n = 1 \\ \\ (2n-1)t_n - nt_{n+1} - (n-1)t_{n-1} & n > 1. \end{cases}$$

2.7 Schrodinger Representation of the Heisenberg Group Heis$(3, \mathbb{R})$

On the Hilbert space $L^2(\mathbb{R})$ one has position Q and momentum P as canonical unbounded operators with domain in the Schwartz subspace $\mathcal{S}(\mathbb{R})$. The are defined as follows:

$$Q\, f(x) = x\ f(x),$$

$$P\, f(x) = \frac{\hbar}{i} f'(x).$$

One can verify that Q and P are essentially self–adjoint operators which satisfy the following "canonical commutation relation (CCR)"

$$[Q,P] = i\hbar\,\mathbb{I}.$$

For the sake of convenience, we define the skew–adjoint operators $\hat{Q}, \hat{P}$ and $\hat{C}$ as follows

$$\hat{Q} = -iQ,\ \ \hat{P} = -iP,\ \ \hat{C} = -i\hbar\,\mathbb{I}.$$

The Lie algebra spanned by the operators $\hat{Q}, \hat{P}$ and $\hat{C}$ is isomorphic to the Heisenberg Lie algebra. Indeed the choice of the above basis gives following faithful Lie algebra representation $\pi_\hbar : \mathfrak{heis}(3,\mathbb{R}) \cong \mathrm{span}_{\mathbb{R}}\left(\hat{Q}, \hat{P}, \hat{C}\right) \longmapsto \mathrm{DiffOp}(\mathbb{R})$

$$\begin{aligned}
\pi_\hbar(\hat{Q}) f(x) &= -ix\, f(x),\\
\pi_\hbar(\hat{P}) f(x) &= -\hbar\, f'(x),\\
\pi_\hbar(\hat{C}) f(x) &= -i\hbar\, f(x).
\end{aligned}$$

The above Lie algebra representation can be exponentiated to give the so–called Schordinger representation $\Pi_\hbar : \mathrm{Heis}(3,\mathbb{R}) \longmapsto \mathcal{U}(L^2(\mathbb{R}))$. That is,

$$\begin{aligned}
\Pi_\hbar\left(\exp(u\hat{Q})\right)\, f(x) &= \exp(-iux)\, f(x),\\
\Pi_\hbar\left(\exp(v\hat{P})\right)\, f(x) &= f(x - v\hbar),\\
\Pi_\hbar\left(\exp(w\hat{C})\right)\, f(x) &= \exp(-i\hbar w)\, f(x).
\end{aligned}$$

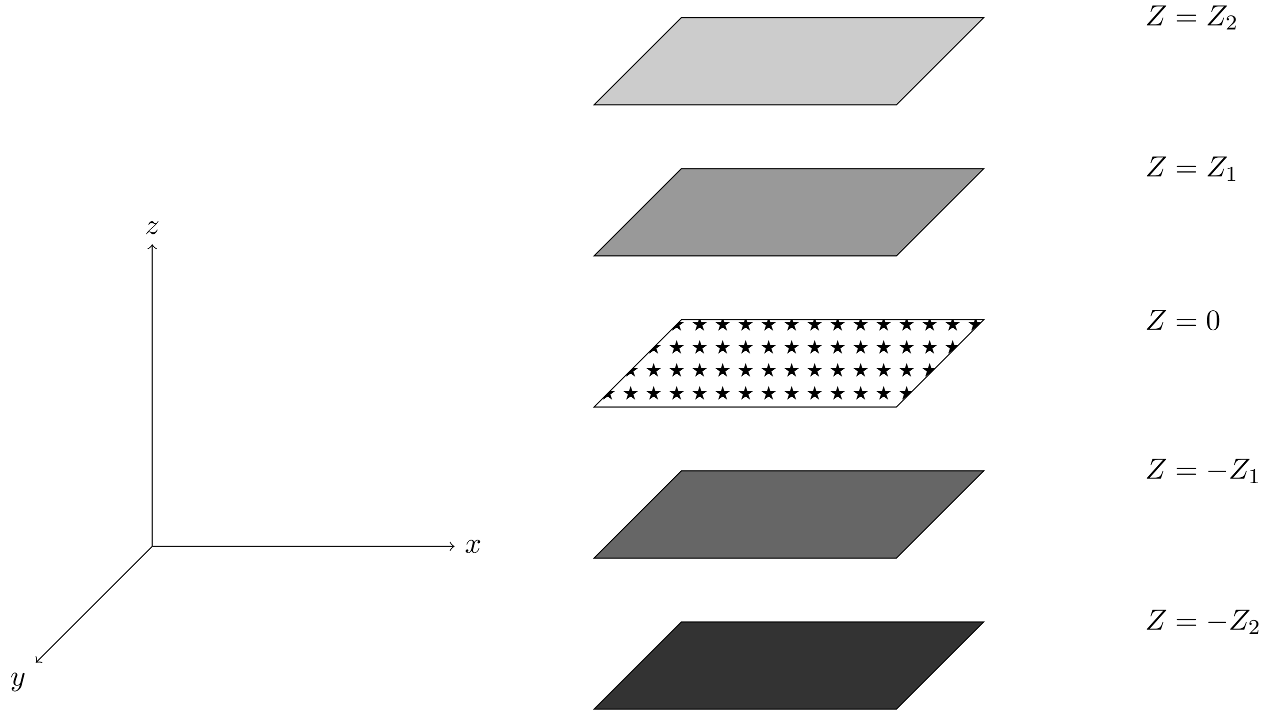

Figure 2.1: Irreducible unitary representations of $\mathrm{Heis}(3, \mathbb{R})$ depicted in terms of its coadjoint orbits. To each symplectic plane $\left(z = z_0,\, \omega = \dfrac{1}{z_0} dx \wedge dy\right)$, $z_0 \neq 0$ one can canonically attach the infinite–dimensional unitary irreducible representation of central character $\exp(iz_0)$. The one–dimensional unitary irreducible representations of $\mathrm{Heis}(3, \mathbb{R})$ can be associated with the points in XY plane, namely the plane $Z = 0$.

It satisfies the following Weyl relation

$$\Pi_\hbar\left(\exp(v\hat{P})\right)\Pi_\hbar\left(\exp(u\hat{Q})\right) = \exp\left(iuv\hbar\right)\Pi_\hbar\left(\exp(u\hat{Q})\right)\Pi_\hbar\left(\exp(v\hat{P})\right).$$

2.8 Classification of Representations of the Heisenberg Group $\mathrm{Heis}(3,\mathbb{R})$

The irreducible unitary representations of $\mathrm{Heis}(3,\mathbb{R})$ can be sorted into the following two categories [3]:

Finite–dimensional representations Being a nilpotent Lie group, all finite–dimensional representations of $\mathrm{Heis}(3,\mathbb{R})$ are one–dimensional. Given any $(m,n) \in \mathbb{R}^2$, there is an associated one–dimensional representation of $\mathrm{Heis}(3,\mathbb{R})$ that can be written as

$$\xi_{m,n} \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} v = \exp(-imx - iny)\, v \quad \forall v \in \mathbb{C}.$$

Infinite–dimensional representation Given any $\hbar \in \mathbb{R}^*$ we say that the center of Heisenberg group acts by $\hbar$ if

$$\Pi_\hbar \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \eta(t) = \exp(-i\hbar z)\, \eta(t).$$

According to the Stone von–Neumann theorem there exists (up to unitary equivalence) only one–infinite dimensional unitary irreducible representation of $\mathrm{Heis}(3,\mathbb{R})$ for which the center acts by $\hbar$. For definiteness, in our thesis we shall choose our representation $\Pi_\hbar : \mathrm{Heis}(3,\mathbb{R}) \longmapsto \mathcal{U}(L^2(\mathbb{R}))$ to be the following

$$\Pi_\hbar \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \eta(t) = \exp(-i\hbar z)\exp(-iyt)\,\eta(t-\hbar x), \tag{2.8.1}$$

where $\eta(t) \in \text{Ł}^2(\mathbb{R})$. Note that the above representation is equivalent to the Schrodinger representation via the following identification

$$\widehat{P} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \widehat{Q} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \text{ and } \widehat{C} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

2.9 Metaplectic Representation of the Symplectic Group $\mathrm{Sp}(2,\mathbb{R})$

For a given central character, we recall the unique (up to unitary equivalence) irreducible unitary representation of the Heisenberg group

$$\Pi_\hbar \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \eta(t) = \eta(t-\hbar a)\ \exp(ibt)\ \exp(-i\hbar c).$$

We denote by $\pi_\hbar = (\Pi_\hbar)_*$ the corresponding Lie algebra representation, that is

$$\pi_\hbar \begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} = -a\hbar \frac{d}{dt} + ib\,t - i\hbar c.$$

We also recall the action of $\mathrm{Sp}(2,\mathbb{R})$ on $\mathfrak{heis}(3,\mathbb{R})$ and hence (by exponential diffeomorphism) on $\mathrm{Heis}(3,\mathbb{R})$. The group action is given by

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix} \cdot \exp \begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} = \exp \begin{pmatrix} 0 & ax+by & c \\ 0 & 0 & az+bw \\ 0 & 0 & 0 \end{pmatrix}.$$

In other words, for $\begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \mathrm{Sp}(2,\mathbb{R})$ and $\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{Heis}(3,\mathbb{R})$ the group action $\mathrm{Sp}(2,\mathbb{R}) \times \mathrm{Heis}(3,\mathbb{R}) \longrightarrow \mathrm{Heis}(3,\mathbb{R})$ is given by

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix} \cdot \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & ax+by & c + \frac{1}{2}\left((ax+by)(az+bw) - ab\right) \\ 0 & 1 & az+bw \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus given any representation $\Pi_\hbar$ of $\mathrm{Heis}(3,\mathbb{R})$ and $g \in \mathrm{Sp}(2,\mathbb{R})$, we can form another representation of $\mathrm{Heis}(3,\mathbb{R})$ namely

$$\Pi_\hbar^g(h) = \Pi_\hbar(g \cdot h) \quad \forall h \in \mathrm{Heis}(3,\mathbb{R}).$$

One can see that $\Pi_\hbar^g$ has same central character given by $\phi_\hbar$ hence by Stone von–Neumann theorem there exists an unitary equivalence between $\Pi_\hbar$ and $\Pi_\hbar^g$.

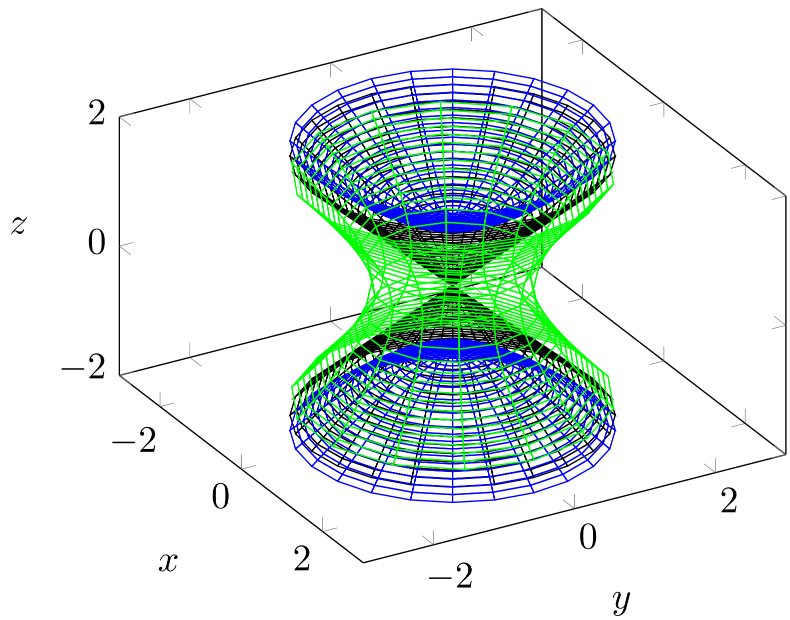

Figure 2.2: Representations of $\mathrm{Sp}(2,\mathbb{R})$ depicted in terms of its coadjoint orbits. Elliptic orbits correspond to the discrete series while hyperbolic orbits correspond to the principal series. Finally **nilpotent** orbit corresponds to the metaplectic representation. We note that some representations (complementary series and limit of discrete series) are not attached to any coadjoint orbit at all.

The metaplectic representation $\Psi_\hbar : Sp(2,\mathbb{R}) \longrightarrow \mathcal{U}(L^2(\mathbb{R}))$ is (up to a phase) defined as

$$\Psi_\hbar(g) = \text{Intertwiner between } \Pi_\hbar(h) \text{ and } \Pi_\hbar(g\cdot h)$$

Concretely for $\eta(t) \in L^2(\mathbb{R})$ we must have

$$\Pi_\hbar(g\cdot h)\Psi_\hbar(g)\,\eta = \Psi_\hbar(g)\Pi_\hbar(h)\,\eta.$$

One can verify that the projective unitary representation (called metaplectic representation) $\Psi_\hbar : \mathrm{Sp}(2,\mathbb{R}) \longmapsto \mathcal{PU}(L^2(\mathbb{R}))$ is given as[3]

$$\begin{aligned}
\Psi_\hbar \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \eta(t) &= \left(\frac{i}{2\pi\hbar}\right)^{\frac{1}{2}} \int_{\mathbb{R}} \exp\left(-\frac{i\,xt}{\hbar}\right)\,\eta(x)\;dx,\\
\Psi_\hbar \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \eta(t) &= \exp\left(\frac{ix}{2\hbar}t^2\right)\eta(t),\\
\Psi_\hbar \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \eta(t) &= \frac{1}{\sqrt{x}}\;\eta(x^{-1}t),\\
\Psi_\hbar \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \eta(t) &= \Psi_\hbar \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Psi_\hbar \begin{pmatrix} 1 & 0 \\ -x & 1 \end{pmatrix} \Psi_\hbar \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \eta(t).
\end{aligned} \tag{2.9.1}$$

The Lie algebra action for the metaplectic representation is given by

$$\psi_\hbar(h) = \psi_\hbar \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -\left(t\,\frac{d}{dt} + \frac{1}{2}\,\mathbb{I} \right),$$

$$\psi_\hbar(f) = \psi_\hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \frac{it^2}{2\hbar}\,\mathbb{I},$$

$$\psi_\hbar(e) = \psi_\hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \frac{i\hbar}{2}\frac{d^2}{dt^2}.$$

Using the alternate basis for $\mathfrak{sp}(2,\mathbb{R}) \otimes \mathbb{C}$, namely

$$\widetilde{h} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \widetilde{e} = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \quad \widetilde{f} = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix},$$

the Lie algebra representation $\psi_\hbar : \mathfrak{sp}(2,\mathbb{R}) \otimes \mathbb{C} \longrightarrow \mathrm{DiffOp}(\mathbb{R})$ for the above is

$$\psi_\hbar(\widetilde{h}) = -\frac{1}{2\hbar}\left(t^2 - \hbar^2 \frac{d^2}{dt^2} \right),$$

$$\psi_\hbar(\widetilde{e}) = -\frac{1}{4}\left(\mathbb{I} + 2t\frac{d}{dt} + \frac{1}{\hbar}t^2 + \hbar\frac{d^2}{dt^2} \right),$$

$$\psi_\hbar(\widetilde{f}) = -\frac{1}{4}\left(\mathbb{I} + 2t\frac{d}{dt} - \frac{1}{\hbar}t^2 - \hbar\frac{d^2}{dt^2} \right).$$

2.10 Schrodinger and Metaplectic Representation of the Jacobi Group in Terms of the Ladder Operators

Using ladder operators, one can rewrite the Lie algebra representation of the Jacobi Lie algebra $\mathfrak{hsp}(2,\mathbb{R}) \cong \mathfrak{sp}(2,\mathbb{R}) \oplus \mathfrak{heis}(3,\mathbb{R})$, that is induced from the Schrodinger and the metaplectic representations as follows:

$$\pi_\hbar \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \sqrt{\frac{\hbar}{2}} \left(a^\dagger - a\right),$$

$$\pi_\hbar \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = i\sqrt{\frac{\hbar}{2}} \left(a^\dagger + a\right),$$

$$\psi_\hbar \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{1}{2}\left(a^{\dagger^2} - a^2\right),$$

$$\psi_\hbar \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = \left(a^\dagger a + \frac{1}{2}\right),$$

$$\psi_\hbar \begin{pmatrix} -1 & -i \\ -i & 1 \end{pmatrix} = a^2,$$

$$\psi_\hbar \begin{pmatrix} -1 & i \\ i & 1 \end{pmatrix} = a^{\dagger^2}. \tag{2.10.1}$$

2.11 Segal–Bargmann Space

Segal–Bargmann space $\mathcal{H}L^2(\mathbb{C}, \mu_\hbar)$ is the Hilbert space of holomorphic functions[6] on $\mathbb{C}$ such that

$$\mathcal{H}L^2\left(\mathbb{C}, \mu_\hbar\right) = \left\{ F \in \mathrm{Hol}(\mathbb{C}) \;\middle|\; \int_{\mathbb{C}} |F(z)|^2 \, d\mu_\hbar(z)\, dz < \infty \right\},$$

where the measure $\mu_\hbar$ is defined as

$$d\mu_\hbar = \frac{1}{\pi\hbar} \exp\left(-\frac{|z|^2}{\hbar}\right) \, dz,$$

along with the inner product

$$\langle F, G\rangle = \frac{1}{\pi\hbar}\int_{\mathbb{C}} \overline{F(z)}\, G(z) \, d\mu_\hbar(z)\, dz \qquad F, G \in \mathrm{Hol}(\mathbb{C}).$$

Orthonormal Basis

Apart from the convenient fact that Segal–Bargmann Hilbert space $\mathcal{H}L^2\left(\mathbb{C}, \mu_\hbar\right)$ consists of holomorphic functions (rather than their equivalence classes), one has monomials (with suitable scaling) as an orthonormal basis. In particular, the family

$$\left\{\sqrt{\frac{1}{\hbar^n n!}}\; z^n\right\}_{n=0}^{\infty}$$

is an orthonormal basis.

There exists an explicit unitary isomorphism[6]

$$A_\hbar : L^2(\mathbb{R}, dx) \longmapsto \mathcal{H}L^2(\mathbb{C}, \mu_\hbar),$$

given by the formula

$$A_\hbar f(x) = \left(\frac{1}{\pi\hbar}\right)^{\frac{1}{4}} \int_{\mathbb{R}} \exp\left(-\frac{z^2 - 2\sqrt{2}zx + x^2}{2\hbar}\right) f(x)\; dx, \qquad f(x) \in L^2(\mathbb{R}, dx).$$

It is called the Bargmann transform.

2.11.1 Schrodinger and the Metaplectic Representation of the Jacobi Group in the Segal–Bargmann Space

Under the Bargmann transform $A_\hbar$ the creation a and annihilation $a^\dagger$ operators on $L^2(\mathbb{R})$ are transformed into the following operators on the Bargmann space

$\mathcal{H}L^2(\mathbb{C}, \mu_\hbar)$:

$$A_\hbar \cdot a \cdot A_\hbar^{-1} = \sqrt{\hbar}\,\frac{d}{dz},$$
$$A_\hbar \cdot a^\dagger \cdot A_\hbar^{-1} = \frac{1}{\sqrt{\hbar}}\,z.$$

Moreover under the Segal–Bargmann space, the Lie algebra version of the Schrodinger metaplectic representation of the Jacobi group $\Pi_\hbar : \mathrm{HSp}(2,\mathbb{R}) \longrightarrow \mathcal{PU}\left(L^2(\mathbb{R})\right)$ is transformed as follows:

$$A_\hbar \cdot \pi_\hbar \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot A_\hbar^{-1} = \frac{1}{\sqrt{2}}\left(z - \hbar\frac{d}{dz}\right),$$

$$A_\hbar \cdot \pi_\hbar \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \cdot A_\hbar^{-1} = \frac{i}{\sqrt{2}}\left(z + \hbar\frac{d}{dz}\right),$$

$$A_\hbar \cdot \psi_\hbar\left(\widetilde{h}\right) \cdot A_\hbar^{-1} = z\frac{d}{dz} + \frac{1}{2}\,\mathbb{I},$$
$$A_\hbar \cdot \psi_\hbar\left(\widetilde{e}\right) \cdot A_\hbar^{-1} = -\frac{\hbar}{2}\,\frac{d^2}{dz^2},$$
$$A_\hbar \cdot \psi_\hbar\left(\widetilde{f}\right) \cdot A_\hbar^{-1} = -\frac{1}{2\hbar}z^2.$$

Chapter 3

Asymptotic Morphism over $\mathbb{R}^2$

Here we shall fix the notation in preparation for in depth discussion about the quantization of $\mathbb{R}^2$ via asymptotic morphisms in the next two chapters.

3.1 Jacobi Group Action on $C_0\left(\mathbb{R}^2\right)$

The symplectic group $\mathrm{Sp}(2, \mathbb{R})$ acts on $\mathbb{R}^2$ via standard matrix multiplication. That is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}.$$

Moreover the one parameter subgroup $\{M^a\}_{a\in\mathbb{R}} = \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ of the Heisenberg Lie group $\mathrm{Heis}(3,\mathbb{R})$ acts on $\mathbb{R}^2$ via translation along the configuration space. That is

$$\begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+a \\ y \end{pmatrix}.$$

Inspired by (2.4.1), we define

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y-a \end{pmatrix}.$$

Moreover let us also define

$$\begin{pmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Lemma 3.1.1. *The above group actions taken together define an action of* $\mathrm{HSp}(2,\mathbb{R})$ *on* $\mathbb{R}^2$*, inducing in turn the action* $\Phi : \mathrm{HSp}(2,\mathbb{R}) \times C_0(\mathbb{R}^2) \longrightarrow C_0(\mathbb{R}^2)$ *given by:*

$$g \cdot f(x) = f(g^{-1} \cdot x) \qquad \forall g \in \mathrm{HSp}(2,\mathbb{R}),\ f \in C_0(\mathbb{R}^2),\ x \in \mathbb{R}^2.$$

□

3.2 Jacobi Group Representation on $\mathcal{K}\left(L^2(\mathbb{R})\right)$

We recall that the Schrodinger metaplectic representation of the Jacobi group $\mathrm{HSp}(2,\mathbb{R})$, which is a projective unitary irreducible representation, is generated as follows:

$$\Pi_\hbar \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \eta(t) = \exp(-i\hbar z)\exp(i\,yt)\,\eta(t-\hbar\,x),$$

$$\Pi_\hbar \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \eta(t) = \left(\frac{i}{2\pi\hbar}\right)^{\frac{1}{2}} \int_{\mathbb{R}} \exp\left(-\frac{i\,xt}{\hbar}\right)\,\eta(x)\;dx,$$

$$\Pi_\hbar \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \eta(t) = \exp\left(\frac{ix}{2\hbar}t^2\right)\eta(t),$$

$$\Pi_\hbar \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \eta(t) = \frac{1}{\sqrt{x}}\;\eta(x^{-1}t).$$

Since these formulas define a projective unitary representation on $L^2(\mathbb{R})$, the Jacobi group $\mathrm{HSp}(2,\mathbb{R})$ acts as automorphisms of the

C^*–algebra of compact operators on $L^2(\mathbb{R})$. Concretely, the group action $\mathrm{HSp}(2,\mathbb{R})\times \mathcal{K}\left(L^2(\mathbb{R})\right) \longrightarrow \mathcal{K}\left(L^2(\mathbb{R})\right)$ is given as

$$g\cdot T = \Pi_\hbar(g)\,T\,\Pi_\hbar(g)^* \qquad \forall g\in \mathrm{HSp}(2,\mathbb{R}),\; T\in\mathcal{K}\left(L^2(\mathbb{R})\right).$$

3.3 Diffeomorphism Group Action

Identifying $T^*(\mathbb{R}) \cong \mathbb{R}^2$, the diffeomorphism group Diff $(\mathbb{R})$ acts upon $\mathbb{R}^2$ as follows

$$\phi \cdot (x, y) \longrightarrow \left(\phi(x),\ (\phi'(x))^{-1} y\right) \quad \forall x, y \in \mathbb{R}.$$

Thus the induced action $\mathrm{Diff}(\mathbb{R}) \times C_0^\infty(\mathbb{R}^2) \longrightarrow C_0^\infty(\mathbb{R}^2)$ becomes

$$\phi \cdot g \longrightarrow \left((x, y) \mapsto g\left(\phi(x),\ (\phi'(x))^{-1} y\right)\right) \quad \forall \phi \in \mathrm{Diff}(\mathbb{R}),\ g \in C_0^\infty(\mathbb{R}^2).$$

This is an action by symplectic diffeomorphisms of $\mathbb{R}^2$ under the above identification of $T^*\mathbb{R}^2 \cong \mathbb{R}^2$.

Similarly the diffeomorphism group acts on $L^2(\mathbb{R})$ via unitary transform as follows:

$$\Pi(\phi)\,\eta(t) = |\phi'(t)|^{\frac{1}{2}}\ f(\phi(x)) \qquad \phi \in \mathrm{Diff}(\mathbb{R}),\ f \in L^2(\mathbb{R}),$$

inducing the group action $\mathrm{Diff}(\mathbb{R}) \times \mathcal{K}(L^2(\mathbb{R})) \longrightarrow \mathcal{K}(L^2(\mathbb{R}))$ as follows:

$$\phi \cdot T = \Pi(\phi)\,T\,\Pi(\phi)^* \qquad \forall \phi \in \mathrm{Diff}(\mathbb{R}),\ T \in \mathcal{K}\left(L^2(\mathbb{R})\right).$$

3.4 Asymptotic Morphisms over $C^*(\mathbb{R}^2)$

For any asymptotic morphism $T^\hbar : C_0(\mathbb{R}^2) \dashrightarrow \mathcal{K}(L^2(\mathbb{R}))$ one can compose it with the symplectic Fourier transformation

$$\mathcal{F} : C^*(\mathbb{R}^2) \xrightarrow{\cong} C_0(\mathbb{R}^2)$$

to obtain an asymptotic morphism over $C^*(\mathbb{R}^2)$ given by

$$T^\hbar \circ \mathcal{F} : C^*(\mathbb{R}^2) \dashrightarrow \mathcal{K}(L^2(\mathbb{R})),$$

and vice versa.

The symplectic Fourier transformation is defined as follows:

$$\mathcal{F}(f)(v) = \frac{1}{4\pi^2}\int_{\mathbb{R}^2} f(w)\, e^{-i\omega(v,w)} d\, w \qquad \forall f \in C_0(\mathbb{R}^2).$$

Since we have already fixed $\omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ as the standard symplectic form on $\mathbb{R}^2$, we have

$$\mathcal{F}(f)(x,y) = \int_{\mathbb{R}^2} f(\alpha,\beta)\, e^{i(\alpha y - \beta x)}\; d\alpha\, d\beta.$$

This map is equivariant under the $\mathrm{Sp}(2,\mathbb{R})$ group actions on $C_0(\mathbb{R}^2)$ and $C^*(\mathbb{R}^2)$ associated to the standard action on $\mathbb{R}^2$.

Chapter 4

Heisenberg–Equivariant Asymptotic Morphism

4.1 Introduction

We recall that for $g \in L^1(\mathbb{R}^2)$ its symplectic Fourier transform

$\mathcal{F} : L^1(\mathbb{R}^2) \longrightarrow C_0(\mathbb{R}^2)$ is defined to be the following:

$$\mathcal{F}(g)(x,y) = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} g(\alpha,\beta) \exp\left(i(\alpha y - \beta x)\right) \; d\alpha \; d\beta.$$

This extends to a C^*–algebra isomorphism

$$\mathcal{F} : C^*(\mathbb{R}^2) \xrightarrow{\cong} C_0(\mathbb{R}^2).$$

One can similarly define a C^*–algebra isomorphism for the Heisenberg group,

$$\mathcal{F} : \mathrm{Heis}(3,\mathbb{R}) \xrightarrow{\cong} \mathcal{A},$$

where we recall, the C^*–algebra of any Lie group G is the completion of the convolution algebra $C_c^\infty(G)$ in the norm

$$\|f\|_{C^*(G)} = \sup_{\Pi \in \widehat{G}} \left\| \int_G f(h)\Pi(h)\, dh \right\|_{\mathcal{B}(\mathcal{H}_\Pi)} \qquad \forall f \in C_c^\infty(G).$$

The target C^*–algebra $\mathcal{A}$ for the Fourier transform is the algebra of continuous sections (vanishing at infinity) of a continuous field of C^*–algebras $\mathcal{A}_t$ over $\mathbb{R}$. The full details will not concern us, but we shall use several features of the Fourier transform. First, the fibers of the continuous field are as follows:

$$\mathcal{A}_0 = C_0(\mathbb{R}^2)$$
$$\mathcal{A}_t = \mathcal{K}(L^2(\mathbb{R})) \qquad \text{for } t \neq 0$$

Second the values of the Fourier transform in these fibers are

$$\mathcal{F}(f)_t = \begin{cases} (u,v) \mapsto \widehat{f}(u,v,0) \in C_0(\mathbb{R}^2) & t = 0 \\ \Pi_t(f) \in \mathcal{K}(L^2(\mathbb{R})) & t \in \mathbb{R}^*. \end{cases}$$

Moreover the map $\hbar \longmapsto \mathcal{F}(\hbar)$ is norm continuous. This is a consequence of the definition of continuous field.

Fix $\Theta \in \mathcal{S}(\mathcal{Z}(\mathfrak{heis}(3,\mathbb{R}))) \cong \mathcal{S}(\mathbb{R})$ such that $\|\Theta\|_1 = 1$. The exponential map leads to an isomorphism $\exp : \mathcal{S}(\mathfrak{heis}(3,\mathbb{R})) \cong \mathcal{S}(\text{Heis}(3,\mathbb{R}))$ which gives rise to an inclusion $i : C^*(\mathbb{R}^2) \longmapsto C^*(\text{Heis}(3,\mathbb{R}))$ defined as follows:

$$i(g) = (g \cdot \Theta) \circ \exp^{-1} \qquad g \in C^*(\mathbb{R}^2).$$

Under the diffeomorphism $\exp : \mathfrak{heis}(3,\mathbb{R}) \cong \mathrm{Heis}(3,\mathbb{R})$, for any $h \in \mathrm{Heis}(3,\mathbb{R})$, let $(x_h, y_h, z_h) \in \mathfrak{heis}(3,\mathbb{R})$ be such that

$$\exp\left(x_h, y_h, z_h\right) = h.$$

We define $T^{\hbar}_{\mathrm{hsp}} : C_0(\mathbb{R}^2) \dashrightarrow \mathcal{K}(L^2(\mathbb{R}))$ as follows:

$$\begin{aligned} T^{\hbar}_{\mathrm{hsp}}(g) &= \int_{\mathrm{Heis}(3,\mathbb{R})} (\widehat{g} \cdot \Theta) \circ \exp^{-1}(h)\, \Pi_{\hbar}(h)\, dh \\ &= \int_{\mathfrak{heis}(3,\mathbb{R})} (\widehat{g} \cdot \Theta)\, (x_h, y_h, z_h)\, \Pi_{\hbar}(\exp(x_h, y_h, z_h))\; dx_h\, dy_h\, dz_h \\ &= \int_{\mathbb{R}^3} \widehat{g}(x,y)\Theta(z)\, \Pi_{\hbar} \begin{pmatrix} 1 & x & z + \frac{xy}{2} \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} dx\, dy\, dz. \end{aligned}$$

By construction, $T^{\hbar}_{\mathrm{hsp}}$ is an asymptotically equivariant asymptotic morphism under the Heisenberg group actions described earlier. Equivariance under the symplectic group action follows from the following lemma:

Lemma 4.1.1. *The asymptotic morphism $T^{\hbar}_{hsp}$ constructed above is* ***equivariant*** *with respect to the symplectic group action.*

Proof. The symplectic Fourier transformation $\mathcal{F}$ is equivariant with respect to symplectic group $\mathrm{Sp}(2,\mathbb{R})$:

$$\widehat{\Omega \cdot g} = \Omega \cdot \widehat{g} \qquad \forall\ \Omega \in \mathrm{Sp}(2,\mathbb{R}),\ g \in C_0(\mathbb{R}^2).$$

Now given any $\Omega \in \mathrm{Sp}(2,\mathbb{R})$, we have

$$T^{\hbar}_{\mathrm{hsp}}(\Omega \cdot g) = \int_{\mathfrak{heis}(3,\mathbb{R})} \widehat{g}(\Omega^{-1} \cdot (x_h, y_h))\ \Theta(z_h)\ \Pi_\hbar \circ \exp(x_h, y_h, z_h)\ dx_h\, dy_h\, dz_h.$$

We recall that the symplectic group acts trivially upon $\mathcal{Z}\left(\mathfrak{heis}(3,\mathbb{R})\right)$. Therefore by change of coordinates we have

$$\begin{aligned} T^{\hbar}_{\mathrm{hsp}}(\Omega \cdot g) &= \int_{\mathfrak{heis}(3,\mathbb{R})} \widehat{g}(x_h, y_h)\ \Theta(z_h)\ \Pi_\hbar \circ \exp\left(\Omega \cdot (x_h, y_h, z_h)\right)\ dx_h\, dy_h\, dz_h \\ &= \int_{\mathrm{Heis}(3,\mathbb{R})} \widehat{g} \cdot \Theta\left(\exp^{-1} h\right)\ \Pi_\hbar\left(\Omega \cdot h\right)\ dh. \end{aligned}$$

We can further write

$$\Pi_\hbar(\Omega \cdot h) = \Phi_\hbar(\Omega)\, \Pi_\hbar(h)\, \Phi_\hbar(\Omega)^*,$$

where $\Phi_\hbar : \mathrm{Sp}(2,\mathbb{R}) \longrightarrow \mathcal{PU}(L^2(\mathbb{R}))$ is the metaplectic representation of the symplectic group $\mathrm{Sp}(2,\mathbb{R})$. We note that $\Phi_\hbar(\Omega)$ has ambiguity only up to a sign and so conjugation through it is well defined. Using the above fact we get

$$\begin{aligned} T^{\hbar}_{\mathrm{hsp}}(\Omega \cdot g) &= \int_{\mathrm{Heis}(3,\mathbb{R})} \widehat{g} \cdot \Theta\left(\exp^{-1} h\right)\ \Phi_\hbar(\Omega)\, \Pi_\hbar(h)\, \Phi_\hbar(\Omega)^*\ dh \\ &= \Phi_\hbar(\Omega)\, T^{\hbar}_{\mathrm{hsp}}(g)\, \Phi_\hbar(\Omega)^*, \end{aligned}$$

which proves the symplectic group equivariance. □

4.2 Non–Equivariance Under Diff($\mathbb{R}$)

We will show that the asymptotic morphism $T^{\hbar}_{\mathrm{hsp}}$ is not equivariant with respect to the full diffeomorphism group of $\mathbb{R}$.

Let us revisit the asymptotic morphism

$$T_{\text{hsp}}^{\hbar}(g)(\eta) = \int_{\text{Heis}(3,\mathbb{R})} \mathcal{F}(g)(x_h, y_h)\Theta(z_h)\,\Pi_{\hbar}(h)\eta\; dh.$$

From (2.8.1), we recall the Heisenberg group representation $\Pi_{\hbar} : \text{Heis}(3,\mathbb{R}) \longmapsto \mathcal{U}(L^2(\mathbb{R}))$ fixed earlier:

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \cdot \eta(t) = \exp(-i\hbar c)\exp(ibt)\; \eta(t-\hbar a).$$

Then we have

$$\Pi_{\hbar}(h)\eta(t) = \Pi_{\hbar}(\exp(x_h, y_h, z_h))\,\eta(t) = \begin{pmatrix} 1 & x_h & z_h + \frac{1}{2}x_h y_h \\ 0 & 1 & y_h \\ 0 & 0 & 1 \end{pmatrix} \cdot \eta(t).$$

Simplifying the above we get

$$\Pi_{\hbar}(h)\eta(t) = \exp\left(-i\hbar(z_h + \frac{1}{2}x_h y_h)\right)\exp\left(iy_h t\right)\; \eta(t-\hbar x_h).$$

Let us fix $\Theta(z) = \exp(-\pi z^2) \in \mathcal{S}(\mathbb{R})$ (note that $\|\Theta\|_1 = 1$).

With the above choices the asymptotic morphism $T_{\text{hsp}}^{\hbar} : C_0(\mathbb{R}^2) \dashrightarrow \mathcal{K}(L^2(\mathbb{R}))$ can be explicitly written down as follows:

$$\begin{aligned} T_{\text{hsp}}^{\hbar}(g)\eta(t) = \int_{\mathbb{R}^3} \mathcal{F}\left(g\right)\left(x,y\right)\exp(-\pi z^2)\exp(-i\hbar z) \\ \exp\left(-\frac{i\hbar xy}{2}\right)\exp(iyt)\;\eta(t-\hbar x)\; dx\; dy\; dz. \quad (4.2.1) \end{aligned}$$

We now prove our result on non-equivariance of asymptotic morphism $T^{\hbar}_{\text{hsp}}$ with respect to the $\text{Diff}(\mathbb{R})$ group action.

Theorem 4.2.1. *The asymptotic morphism* $T^{\hbar}_{hsp} : C_0(\mathbb{R}^2) \dashrightarrow \mathcal{K}\left(L^2(\mathbb{R})\right)$ *is* ***not equivariant*** *with respect to the* $\text{Diff}(\mathbb{R})$ *group action. In particular* $T^{\hbar}_{hsp}$ *is not equivariant for* $g(x,y) = \exp\left(-\frac{x^2+y^2}{2}\right) \in C_0(\mathbb{R}^2)$ *and* $\phi(x) = x + \frac{x^2}{2} + \frac{x^3}{3} \in$ $\text{Diff}(\mathbb{R})$.

Proof. For $g(\alpha,\beta) = \exp\left(-\frac{\alpha^2+\beta^2}{2}\right)$, its Fourier transform is given by

$$\mathcal{F}(g)(x,y) = \frac{1}{2\pi}\exp\left(-\frac{x^2+y^2}{2}\right).$$

Thus the above explicit expression for the asymptotic morphism $T^{\hbar}_{\text{hsp}}$ becomes

$$T^{\hbar}_{\text{hsp}}(g)\eta(t) = \frac{1}{2\pi}\int_{\mathbb{R}^3} \exp\left(-\frac{x^2+y^2}{2}\right)\exp(-\pi z^2)\exp(-i\hbar z) \exp\left(-\frac{i\hbar xy}{2}\right)\exp(iyt)\ \eta(t-\hbar x)\ dx\ dy\ dz.$$

We recall that

$$\int_{\mathbb{R}} \exp(-ax^2+bx)\ dx = \sqrt{\frac{\pi}{a}}\ \exp\left(\frac{b^2}{4a}\right),$$

so that

$$\int_{\mathbb{R}} \exp(-\pi z^2)\ \exp(i\hbar z)\ dz = \exp\left(-\frac{\hbar^2}{4\pi}\right),$$

$$\int_{\mathbb{R}} \exp\left(-\frac{y^2}{2}\right)\ \exp\left(iy\left(-\frac{\hbar x}{2}+t\right)\right)\ dy = \sqrt{2\pi}\ \exp\left(-\frac{(t-\frac{\hbar x}{2})^2}{2}\right).$$

Substituting the above expression into above integrals the explicit expression for the asymptotic morphism $T^{\hbar}_{\mathrm{hsp}}$ becomes

$$T^{\hbar}_{\mathrm{hsp}}(g)\eta(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp\left(-\frac{x^2}{2}\right) \exp\left(-\frac{(t-\frac{\hbar x}{2})^2}{2}\right) \eta(t-\hbar x)\, dx,$$

that is

$$T^{\hbar}_{\mathrm{hsp}}(g)\eta(t) = \frac{1}{\sqrt{2\pi}\hbar} \int_{\mathbb{R}} \exp\left(-\frac{x^2}{2\hbar^2}\right) \exp\left(-\frac{(2t-x)^2}{8}\right) \eta(t-x)\, dx.$$

We further make a choice for $\eta^h_c \in L^2(\mathbb{R})$ with $\|\eta^h_c\|_2 = 1$ as follows:

$$\eta^h_c(x) = \frac{\pi^{-\frac{1}{4}}}{\sqrt{h}} \exp\left(-\frac{(x-c)^2}{2h^2}\right),$$

where the parameters c and h can vary with $\hbar$ and will be specified later in the proof. With the above choice the expression for asymptotic morphism can be further simplified to the following:

$$\begin{aligned}
T^{\hbar}_{\mathrm{hsp}}(g)\eta^h_c(t) &= \frac{\pi^{-\frac{3}{4}}}{\hbar\sqrt{2h}} \int_{\mathbb{R}} \exp\left(-\frac{x^2}{2\hbar^2}\right) \exp\left(-\frac{(2t-x)^2}{8}\right) \exp\left(-\frac{(t-x-c)^2}{2h^2}\right) dx \\
&= \frac{\pi^{-\frac{3}{4}}}{\hbar\sqrt{2h}} \exp\left(-\frac{t^2}{2}\right) \exp\left(-\frac{(t-c)^2}{2h^2}\right) \int_{\mathbb{R}} \exp\left(-x^2\left(\frac{4h^2+\hbar^2h^2+4\hbar^2}{8\hbar^2h^2}\right)\right. \\
&\quad \left. + x\left(\frac{h^2t+2t-2c}{2h^2}\right)\right) dx \\
&= \frac{\pi^{-\frac{3}{4}}}{\hbar\sqrt{2h}} \exp\left(-\frac{t^2}{2}\right) \exp\left(-\frac{(t-c)^2}{2h^2}\right) \sqrt{\frac{8\hbar^2h^2\pi}{4h^2+\hbar^2h^2+4\hbar^2}} \\
&\quad \exp\left(\frac{\hbar^2(h^2t+2t-2c)^2}{2h^2(4h^2+\hbar^2h^2+4\hbar^2)}\right) \\
&= 2\pi^{-\frac{1}{4}} \exp\left(-\frac{t^2}{2}\right) \exp\left(-\frac{(t-c)^2}{2h^2}\right) \sqrt{\frac{h}{4h^2+\hbar^2h^2+4\hbar^2}} \\
&\quad \exp\left(\frac{\hbar^2h^4t^2+4\hbar^2(t-c)^2+4\hbar^2h^2t(t-c)}{8h^4+2\hbar^2h^4+8h^2\hbar^2}\right).
\end{aligned}$$

Next we let h vary with the parameter $\hbar$ via $h(\hbar) = \hbar$, while keeping the parameter

c fixed. In that case we have

$$\limsup_{\hbar\to 0}\left\|T^{\hbar}_{\text{hsp}}(g)\eta^{s}_{c}(t)-\frac{1}{\sqrt{2}\pi^{\frac{1}{4}}}\exp\left(-\frac{t^2}{2}\right)\frac{1}{\sqrt{\hbar}}\exp\left(-\frac{(t-2c)^2+\epsilon_1(\hbar)}{4\hbar^2+\epsilon_2(\hbar)}\right)\right\|=0,$$

where $\epsilon_1(\hbar)$ and $\epsilon_2(\hbar)$ are the polynomial functions of $\hbar$ such that $\epsilon_1(0)=\epsilon_2(0)=0$. Furthermore, we recall that

$$\frac{\pi^{-\frac{1}{4}}}{\sqrt{2\hbar}}\exp\left(-\frac{(t-2c)^2}{4\hbar^2}\right)=\eta^{\sqrt{2}\hbar}_{2c},$$

and that for small enough $\hbar$, the function $\eta^{\sqrt{2}\hbar}_{2c}$ is concentrated near $t=2c$, so that $\exp\left(-\frac{t^2}{2}\right)=\exp\left(-2c^2\right)+\mathcal{O}(\hbar)$. Thus we have the following norm equalities

$$\limsup_{\hbar\to 0}\left\|T^{\hbar}_{\text{hsp}}(g)\eta^{\hbar}_{c}(t)-\frac{1}{\sqrt{2}\pi^{\frac{1}{4}}}\exp\left(-\frac{t^2}{2}\right)\frac{1}{\sqrt{\hbar}}\exp\left(-\frac{(t-2c)^2}{4\hbar^2}\right)\right\|_2=0.$$

$$\limsup_{\hbar\to 0}\left\|T^{\hbar}_{\text{hsp}}(g)\eta^{\hbar}_{c}(t)-\exp\left(-2c^2\right)\eta^{\sqrt{2}\hbar}_{2c}\right\|_2=0.$$

We finally will fix the parameter c such that $\exp\left(2c^2\right)=2$, so that we have

$$\limsup_{\hbar\to 0}\left\|T^{\hbar}_{\text{hsp}}(g)\eta^{\hbar}_{c}(t)-\frac{1}{2}\eta^{\sqrt{2}\hbar}_{2c}\right\|_2=0. \tag{4.2.2}$$

Consider the diffeomorphism $\phi\in\text{Diff}(\mathbb{R})\subset\text{SympDiff}(\mathbb{R}^2)$ defined as follows

$$\phi(x)=\frac{x^3}{3}+\frac{x^2}{2}+x\quad\forall x\in\mathbb{R}.$$

The diffeomorphism ϕ acts on $g\in C_0(\mathbb{R}^2)$ as follows

$$\phi\cdot g(x,y)=\exp\left(-\frac{1}{2}\left(\frac{x^3}{3}+\frac{x^2}{2}+x\right)^2\right)\exp\left(-\frac{1}{2}\left(\frac{y}{x^2+x+1}\right)^2\right),$$

which in particular yields the following norm inequality

$$\|\phi\cdot g-g\|_{C_0(\mathbb{R}^2)}<\frac{1}{2}. \tag{4.2.3}$$

We note that

$$\begin{aligned}\limsup_{\hbar\to 0}\left\|U_\phi\, T^{\hbar}_{\mathrm{hsp}}(g) - T^{\hbar}_{\mathrm{hsp}}(g)\, U_\phi\right\| &= \limsup_{\hbar\to 0}\left\|T^{\hbar}_{\mathrm{hsp}}(g) - U^*_\phi\, T^{\hbar}_{\mathrm{hsp}}(g)\, U_\phi\right\| \\ &\leq \limsup_{\hbar\to 0}\left\|T^{\hbar}_{\mathrm{hsp}}(g) - T^{\hbar}_{\mathrm{hsp}}(\phi\cdot g)\right\| + \limsup_{\hbar\to 0}\left\|T^{\hbar}_{\mathrm{hsp}}(\phi\cdot g) - U^*_\phi\, T^{\hbar}_{\mathrm{hsp}}(g)\, U_\phi\right\| \\ &\leq \|\phi\cdot g - g\|_{C_0(\mathbb{R}^2)} + \limsup_{\hbar\to 0}\left\|T^{\hbar}_{\mathrm{hsp}}(\phi\cdot g) - U^*_\phi\, T^{\hbar}_{\mathrm{hsp}}(g)\, U_\phi\right\| \\ &< \frac{1}{2} + \limsup_{\hbar\to 0}\left\|T^{\hbar}_{\mathrm{hsp}}(\phi\cdot g) - U^*_\phi\, T^{\hbar}_{\mathrm{hsp}}(g)\, U_\phi\right\|.\end{aligned}$$

This gives the following norm estimate

$$\limsup_{\hbar\to 0}\left\|T^{\hbar}_{\mathrm{hsp}}(\phi\cdot g) - U^*_\phi\, T^{\hbar}_{\mathrm{hsp}}(g)\, U_\phi\right\| > \limsup_{\hbar\to 0}\left\|U_\phi\, T^{\hbar}_{\mathrm{hsp}}(g) - T^{\hbar}_{\mathrm{hsp}}(g)\, U_\phi\right\| - \frac{1}{2} \tag{4.2.4}$$

The diffeomorphism $\phi\in\mathrm{Diff}(\mathbb{R})$ acts on η^h_c as follows:

$$U_\phi\,\eta^h_c(x) = \phi'(x)^{\frac{1}{2}}\,\frac{\pi^{-\frac{1}{4}}}{\sqrt{h}}\,\exp\left(-\frac{(\phi(x)-c)^2}{2h^2}\right),$$

giving rise to the following norm limit:

$$\limsup_{h\to 0}\left\|U_\phi\,\eta^h_c(x) - \phi'(\phi^{-1}(c))^{\frac{1}{2}}\,\frac{\pi^{-\frac{1}{4}}}{\sqrt{h}}\,\exp\left(-\frac{(x-\phi^{-1}(c))^2\phi'(\phi^{-1}(c))^2}{2h^2}\right)\right\|_2 = 0,$$

which, in turn can be succinctly expressed as:

$$\limsup_{h\to 0}\left\|U_\phi\,\eta^h_c - \eta^{\Xi(h)}_{\phi^{-1}(c)}\right\|_2 = 0 \text{ where } \Xi(h) = \frac{h}{(\phi'(\phi^{-1}(c)))}.$$

We note that

$$\begin{aligned}
\limsup_{\hbar\to 0} \left\| U_\phi \, T^{\hbar}_{\text{hsp}}(g) - T^{\hbar}_{\text{hsp}}(g) \, U_\phi \right\| &\geq \limsup_{\hbar\to 0} \left\| U_\phi \, T^{\hbar}_{\text{hsp}}(g) \, \eta^{\hbar}_c - T^{\hbar}_{\text{hsp}}(g) \, U_\phi \, \eta^{h}_c \right\|_2 \\
&= \limsup_{\hbar\to 0} \left\| U_\phi \left(\frac{1}{2} \, \eta^{\sqrt{2}\hbar}_{2c} \right) - T^{\hbar}_{\text{hsp}}(g) \eta^{\Theta(\hbar)}_{\phi^{-1}(c)} \right\|_2 \\
&= \frac{1}{2} \limsup_{\hbar\to 0} \left\| \eta^{\Theta(\sqrt{2}\hbar)}_{\phi^{-1}(2c)} - \eta^{\sqrt{2}\Theta(\hbar)}_{2\phi^{-1}(c)} \right\|_2 \\
&= 1.
\end{aligned}$$

The last equality follows from the fact that $\phi^{-1}(2c) \neq 2\phi^{-1}(c)$ and $\left|\eta^{h}_c\right|^2(t)$ is a normal density with mean c and variance $2h^2$.

From (4.2.4), we have

$$\limsup_{\hbar\to 0} \left\| T^{\hbar}_{\text{hsp}}(\phi \cdot g) - U^{*}_\phi \, T^{\hbar}_{\text{hsp}}(g) \, U_\phi \right\| > 1 - \frac{1}{2} > 0.$$

Thus the asymptotic morphism $T^{\hbar}_{\text{hsp}}$ is not equivariant with respect to the diffeomorphism group $\text{Diff}(\mathbb{R})$. $\square$

Chapter 5

Diffeomorphism–Equivariant Asymptotic Morphism

5.1 Definition

Consider the $\mathrm{Diff}(\mathbb{R})$ asymptotically equivariant asymptotic morphism $T^{\hbar}_{\mathrm{diff}} : C_0(\mathbb{R}^2) \dashrightarrow \mathcal{K}\left(L^2(\mathbb{R})\right)$ that is defined as follows:

$$T^{\hbar}_{\mathrm{diff}}(a)\, g(x) = \int_{\mathbb{R}} a(x, \hbar\xi)\, e^{ix\xi}\, \widehat{g}(\xi)\, d\xi \qquad \forall a \in C_0(\mathbb{R}^2),\ g \in L^2(\mathbb{R}). \tag{5.1.1}$$

Intuitively, the asymptotic morphism $T^{\hbar}_{\mathrm{diff}}$ can be thought of as the extension of the map $C_c^{\infty}(\mathbb{R}^2) \longrightarrow \mathrm{DiffOp}(L^2(\mathbb{R}))$ defined as follows:

$$a_1(x) \otimes a_2(y) \longmapsto a_1(x) \otimes a_2\left(\hbar \frac{d}{dx}\right).$$

As shown in [1], this asymptotic morphism is asymptotically equivariant with respect to the full diffeomorphism group of $\mathbb{R}$. In particular $T^{\hbar}_{\text{diff}}$ is asymptotically equivariant with respect to the scaling subgroup $\cong \mathbb{R}^*$ of $\text{Sp}(2, \mathbb{R})$.

Nevertheless it is not equivariant with respect to the linear symplectic subgroup $\text{Sp}(2, \mathbb{R}) \subset \text{SympDiff}(\mathbb{R}^2)$. This is proved in the theorem below.

Theorem 5.1.1. *The asymptotic morphism $T^{\hbar}_{\text{diff}} : C_0(\mathbb{R}^2) \dashrightarrow \mathcal{K}(L^2(\mathbb{R}))$ is **not** asymptotically equivariant with respect to the action of the linear symplectomorphism $\omega = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \text{Sp}(2, \mathbb{R})$. In particular, it is not equivariant with respect to the symplectic diffeomorphism group of $\mathbb{R}$.*

Proof. Let us consider the function $a(x, \xi) = \exp(-x^2 - \xi^2) \in C_0(\mathbb{R}^2)$ and the symplectomorphism $\omega = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \text{Sp}(2, \mathbb{R})$. The symplectomorphism ω acts on the asymptotic morphism $T^{\hbar}_{\text{diff}} : C_0(\mathbb{R}^2) \dashrightarrow \mathcal{K}(L^2(\mathbb{R}))$ as follows:

$$(\omega \cdot a)(x, \xi) = a(-\xi, x)$$

$$\left(\omega \cdot T^{\hbar}_{\text{diff}}\right)(a) = \mathcal{F}\, T^{\hbar}_{\text{diff}}(a)\, \mathcal{F}^*.$$

As $a(x, \xi) = a(-\xi, x)$ we have

$$\begin{aligned} \limsup_{\hbar \to 0} \left\| T^{\hbar}_{\text{diff}}(\omega \cdot a) - \omega \cdot T^{\hbar}_{\text{diff}}(a) \right\| &= \limsup_{\hbar \to 0} \left\| T^{\hbar}_{\text{diff}}(a) - \mathcal{F}\, T^{\hbar}_{\text{diff}}(a)\, \mathcal{F}^* \right\| \\ &= \limsup_{\hbar \to 0} \left\| T^{\hbar}_{\text{diff}}(a)\, \mathcal{F} - \mathcal{F}\, T^{\hbar}_{\text{diff}}(a) \right\|. \end{aligned}$$

Let us choose $g(x) = \pi^{-\frac{1}{4}} \exp\left(-\frac{x^2}{2}\right) \in L^2(\mathbb{R}),\ \|g\|_2 = 1$ so that

$$\mathcal{F}\, g = \mathcal{F}^*\, g = g.$$

Moreover from (5.1.1) we can explicitly calculate that

$$T^{\hbar}_{\text{diff}}(a)\, \mathcal{F}\, g = T^{\hbar}_{\text{diff}}(a)\, g = \pi^{-\frac{1}{4}} \sqrt{\frac{2\pi}{1+2\hbar^2}}\, \exp\left(-\left(\frac{3+4\hbar^2}{1+2\hbar^2}\right)\frac{x^2}{2}\right).$$

From the above, we also get that

$$\mathcal{F}\, T^{\hbar}_{\text{diff}}(a)\, g = \pi^{-\frac{1}{4}} \sqrt{\frac{2\pi}{3+4\hbar^2}}\, \exp\left(-\left(\frac{1+2\hbar^2}{3+4\hbar^2}\right)\frac{x^2}{2}\right).$$

Using the Lebesgue dominated convergence theorem, we have that

$$\limsup_{\hbar\to 0} \left\|T^{\hbar}_{\text{diff}}(\omega\cdot a) - \omega\cdot T^{\hbar}_{\text{diff}}(a)\right\| = \limsup_{\hbar\to 0} \left\|T^{\hbar}_{\text{diff}}(a)\, \mathcal{F} - \mathcal{F}\, T^{\hbar}_{\text{diff}}(a)\right\|$$

$$\geq \limsup_{\hbar\to 0} \left\|T^{\hbar}_{\text{diff}}(a)\, \mathcal{F}\, g - \mathcal{F}\, T^{\hbar}_{\text{diff}}(a)\, g\right\|_2$$

$$= \limsup_{\hbar\to 0} \left\|\pi^{-\frac{1}{4}} \sqrt{\frac{2\pi}{1+2\hbar^2}}\, \exp\left(-\left(\frac{3+4\hbar^2}{1+2\hbar^2}\right)\frac{x^2}{2}\right) - \pi^{-\frac{1}{4}} \sqrt{\frac{2\pi}{3+4\hbar^2}}\, \exp\left(-\left(\frac{1+2\hbar^2}{3+4\hbar^2}\right)\frac{x^2}{2}\right)\right\|_2$$

$$= \sqrt{2\pi^{\frac{1}{2}}}\, \left\|\exp\left(-\frac{3x^2}{2}\right) - \frac{1}{\sqrt{3}}\, \exp\left(-\frac{x^2}{6}\right)\right\|_2 = (64\pi^3)^{\frac{1}{4}} \left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{5}}\right).$$

Thus we have

$$\limsup_{\hbar\to 0} \left\|T^{\hbar}_{\text{diff}}(\omega\cdot a) - \omega\cdot T^{\hbar}_{\text{diff}}(a)\right\| > 0,$$

and this completes the proof. □

Chapter 6

Main Result

In this chapter, we shall state and prove the main result of our thesis, which, loosely speaking, is a No–Go theorem in the context of asymptotic quantization of $\mathbb{R}^2$.

6.1 Compact Group Equivariance

We first observe that every asymptotic morphism that is asymptotically equivariant under the action of a Lie group G, can be made exactly equivariant with respect to the maximal compact subgroup K of G.

Lemma 6.1.1. *Consider any* $\mathrm{HSp}(2,\mathbb{R})$*–asymptotically equivariant asymptotic morphism* $T^\hbar : C_0(\mathbb{R}^2) \dashrightarrow \mathcal{K}(L^2(\mathbb{R}))$*. Then* $T^\hbar$ *is asymptotic to an* $\mathrm{HSp}(2,\mathbb{R})$*–asymptotically equivariant asymptotic morphism that is exactly equivariant with respect to the*

SO(2) *group action.*

Proof. Since SO(2) is a compact group with an invariant Haar measure dk we can construct a new asymptotic morphism $\widetilde{T^\hbar} : C_0(\mathbb{R}^2) \dashrightarrow \mathcal{K}(L^2(\mathbb{R}))$ as follows:

$$\widetilde{T^\hbar}(f) = \int_{\mathrm{SO}(2)} T^\hbar(k \cdot f)\, dk.$$

The new asymptotic morphism $\widetilde{T^\hbar}$ can be easily seen to be exactly SO(2)–equivariant, and is still HSp(2, $\mathbb{R}$)–asymptotically equivariant. □

6.2 Matrix Entries

The asymptotic morphism $T^\hbar_{\mathrm{hsp}} : C_0(\mathbb{R}^2) \dashrightarrow \mathcal{K}(L^2(\mathbb{R}))$ is equivariant with respect to the $\mathrm{SO}(2, \mathbb{R}) \subset \mathrm{Sp}(2, \mathbb{R})$ group action. Using this, we can infer some facts about the matrix elements of the operators $T^\hbar_{\mathrm{hsp}}(g)$ for $g \in C_0(\mathbb{R}^2)$.

We first fix $\{e_n\}_{n\geq 0}$ to be the Hermite function basis of $L^2(\mathbb{R})$. Because of the SO(2) equivariance, if $g_{0,0}(x, y) = \exp(-x^2 - y^2)$ then $T^\hbar(g_{0,0})$ is a diagonal operator. From the explicit expression of $T^\hbar_{\mathrm{hsp}}$ in (4.2.1), we can, in principle, explicitly calculate the diagonal entries of $T^\hbar(g_{0,0})$.

Furthermore, for $m, n \in \mathbb{N}$, if $g_{m,n}(x, y) = x^n y^m \exp(-x^2 - y^2)$ then by using the identity

$$x^n y^m \exp(-x^2-y^2)$$
$$= \frac{m!n!}{2^{m+n}} \left(\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^{n-2k}}{k!(n-2k)!} \frac{d^{n-2k}}{dx^{n-2k}} e^{-x^2} \right) \left(\sum_{l=0}^{\lfloor \frac{m}{2} \rfloor} \frac{(-1)^{m-2l}}{l!(m-2l)!} \frac{d^{m-2l}}{dx^{m-2l}} e^{-y^2} \right),$$

we see that in the Hermite function basis the operator $T^{\hbar}_{\text{hsp}}(g_{m,n})$ is a finite band matrix, whose entries again can be calculated explicitly.

Finally as the family $\{g_{m,n}\}_{m,n=0}^{\infty}$ is dense in the C^*–algebra $C_0(\mathbb{R}^2)$, we can get the explicit formula for matrix entries of $T^{\hbar}_{\text{hsp}}(g)$ for any $g \in C_0(\mathbb{R}^2)$.

In particular we note the following:

$$\limsup_{\hbar \to 0} \left\langle T^{\hbar}_{\text{hsp}} \left(\exp(-x^2-y^2) \right) \, e_m, \, e_n \right\rangle = \delta_{m,n} \qquad \forall m, n \geq 0.$$

6.3 Topological Non–triviality

Since we are dealing with the concrete case of $\mathbb{R}^2$, and since we can assume rotation group equivariance, we shall use the following result as a definition of topological non–triviality

Definition 6.3.1. A SO(2)–equivariant asymptotic morphism $T^{\hbar} : C_0(\mathbb{R}^2) \dashrightarrow \mathcal{K}(L^2(\mathbb{R}))$ is said to be topologically non–trivial if for the function $f(x,y) = \exp(-x^2-$

y^2) one has

$$\limsup_{\hbar \to 0} \left\langle T^{\hbar}(f)\, e_n,\ e_n \right\rangle = 1 \qquad \forall n \geq 0,$$

where $\{e_n\}_{n=0}^{\infty}$ are the Hermite functions basis of $L^2(\mathbb{R})$. Note that since f is SO(2) invariant, operator $T^{\hbar}(f)$ is diagonal with respect to the Hermite functions basis.

6.4 Lie Algebra Representation of Vector Fields on $\mathbb{R}^2$

We recall the action of Heis$(3, \mathbb{R})$ on $\mathbb{R}^2$ defined in (3.1.1):

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \cdot (x, y) \longmapsto (x + a, y - b).$$

From the above, generators of the Heisenberg Lie algebra act as vector fields on $\mathbb{R}^2$ via the map $\mathfrak{heis}(3, \mathbb{R}) \longmapsto \mathfrak{X}(\mathbb{R}^2)$ as follows:

$$M \cdot f(x, y) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot f = \left.\frac{d}{dt}\right|_{t=0} f\left(\exp(-tM) \cdot (x, y)\right)$$

$$= \left.\frac{d}{dt}\right|_{t=0} f\left(x - t, y\right) = -\frac{\partial}{\partial x}(f),$$

$$
\begin{aligned}
N \cdot f(x,y) &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \cdot f = \left.\frac{d}{dt}\right|_{t=0} f\left(\exp(-tN)\cdot(x,y)\right) \\
&= \left.\frac{d}{dt}\right|_{t=0} f\left(x, y+t\right) = \frac{\partial}{\partial y}(f).
\end{aligned}
$$

Similarly from the natural action of $\mathrm{Sp}(2,\mathbb{R})$ on $\mathbb{R}^2$, the action of $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{sp}(2,\mathbb{R})$ as a vector field on $\mathbb{R}^2$ is:

$$
\begin{aligned}
h \cdot f(x,y) &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot f = \left.\frac{d}{dt}\right|_{t=0} f\left(\exp(-th)\cdot(x,y)\right) \\
&= \left.\frac{d}{dt}\right|_{t=0} f\left(\exp(-t)\cdot x, \exp(t)\cdot y\right) = \left(-x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) f.
\end{aligned}
$$

Equipping $\mathbb{R}^2 \cong T^*\mathbb{R}$ with the complex structure given by $ix = y$, we get two special complex vector fields, namely

$$
\begin{aligned}
\frac{\partial}{\partial z} &= \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \\
\frac{\partial}{\partial \bar{z}} &= \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).
\end{aligned}
$$

From (2.10.1), the Lie algebra representation of these special vector fields on $\mathbb{R}^2$

interpreted as elements of $\mathfrak{hsp}(2,\mathbb{R})$ is summarized below:

$$\begin{aligned}
\pi_\hbar\left(\frac{\partial}{\partial x}\right) &= -\pi_\hbar(M) = -\sqrt{\frac{\hbar}{2}}\ \left(a^\dagger - a\right), \\
\pi_\hbar\left(\frac{\partial}{\partial y}\right) &= \pi_\hbar(N) = i\sqrt{\frac{\hbar}{2}}\ \left(a^\dagger + a\right), \\
\pi_\hbar\left(\frac{\partial}{\partial z}\right) &= -\pi_\hbar(M) - i\pi_\hbar(N) = \sqrt{\frac{\hbar}{2}}\ a, \\
\pi_\hbar\left(\frac{\partial}{\partial \bar{z}}\right) &= -\pi_\hbar(M) + i\pi_\hbar(N) = -\sqrt{\frac{\hbar}{2}}\ a^\dagger, \\
\pi_\hbar(h) = \pi_\hbar\left(-x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) &= \frac{1}{2}\left(a^{\dagger 2} - a^2\right). \qquad (6.4.1)
\end{aligned}$$

With the help of these ancillary results, we finally state and prove the main result.

6.5 Statement and Proof of the Theorem

Theorem 6.5.1. *Consider a topologically non-trivial, linear asymptotic morphism* $T^\hbar : C_0(\mathbb{R}^2) \dashrightarrow \mathcal{K}(\mathcal{H})$ *with* $\|T^\hbar\| \leq 1$ *that is rapidly asymptotically equivariant with respect to the* $\mathrm{HSp}(2,\mathbb{R})$*. The asymptotic morphism* $T^\hbar$ *is asymptotically equivalent to the asymptotic morphism* $T^\hbar_{hsp}$ *defined before.*

Proof. Using the lemma (6.1.1) we can – without loss of generality – assume that the asymptotic morphism $T^\hbar$ is equivariant with respect to the action of $\mathrm{SO}(2) \subset \mathrm{Sp}(2,\mathbb{R})$ subgroup.

For sake of notation, let us denote by $U^\hbar = T^\hbar - T^\hbar_{\text{hsp}}$, the difference of two asymptotic morphisms.

We note that the C^*–algebra $C_0(\mathbb{R}^2)$ is generated as a C^*–algebra by the functions f, g, and $\bar{g}$, where

$$f(z) = \exp\left(-|z|^2\right),$$
$$g(z) = z\exp\left(-|z|^2\right).$$

As $\left\|T^\hbar\right\| \leq 1$, an asymptotic equivalence between $T^\hbar$ and $T^\hbar_{\text{hsp}}$ is implied by the following:

$$\limsup_{\hbar\to 0} \|U^\hbar(F)\| = 0 \qquad \forall F \in \{f, g, \bar{g}\}. \tag{6.5.1}$$

We fix the Hermite functions as an orthonormal basis of $L^2(\mathbb{R})$. Both of the asymptotic morphisms $T^\hbar$ and $T^\hbar_{\text{hsp}}$ are equivariant with respect to the SO(2) group. Therefore with respect to the Hermite function basis the operators $U^\hbar(f), U^\hbar(g)$ and $U^\hbar(\bar{g})$ are diagonal, subdiagonal and superdiagonal operators, respectively.

Rapidly Asymptotic Equivariance For $f(z) = \exp(-|z|^2)$

We look for a differential operator X in the universal enveloping algebra of $\mathfrak{hsp}(2, \mathbb{R})$ such that the following conditions hold:

- X is not rotationally invariant. In particular $\left[X, T^{\hbar}(f)\right] \neq 0$.

- $X(f) = 0$.

One particular choice for X is:

$$X = \frac{1}{2}\left(h + N^2\right) - \frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z}.$$

X is not rotationally invariant. Moreover

$$\begin{aligned} Xf &= \frac{1}{2}\left(h + N^2\right)\exp\left(-x^2 - y^2\right) - \left(\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z}\right)\exp(-|z|^2) \\ &= \frac{1}{2}\left(-x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{\partial^2}{\partial y^2}\right)\exp\left(-x^2 - y^2\right) - \left(\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z}\right)\exp(-z\bar{z}) \\ &= 0 \end{aligned}$$

X acts on $U^{\hbar}(f)$ via representation of the universal enveloping algebra of $\mathfrak{hsp}(2, \mathbb{R})$.

In particular, we have

$$\begin{aligned} X \cdot U^{\hbar}(f) &= \frac{1}{2}\left[N,\, [N, U^{\hbar}(f)]\,\right] + \frac{1}{2}[h,\, U^{\hbar}(f)] - \left[\frac{\partial}{\partial \bar{z}},\, \left[\frac{\partial}{\partial z}, U^{\hbar}(f)\right]\right] \\ &= -\frac{\hbar}{4}\left[a^{\dagger} + a,\, [a^{\dagger} + a, U^{\hbar}(f)]\,\right] + \frac{1}{4}[a^{\dagger 2} - a^2,\, U^{\hbar}(f)] + \frac{\hbar}{2}\left[a^{\dagger},\, [a, U^{\hbar}(f)]\,\right] \\ &= \frac{1}{4}[a^{\dagger 2},\, U^{\hbar}(f)] - \frac{1}{4}[a^2,\, U^{\hbar}(f)] \\ &- \frac{\hbar}{4}[a^{\dagger},\, [a^{\dagger},\, U^{\hbar}(f)]] - \frac{\hbar}{4}[a,\, [a,\, U^{\hbar}(f)]]. \end{aligned}$$

Let $U^{\hbar}(f) = (u_n(\hbar))_{n=1}^{\infty}$ be the diagonal entries of $U^{\hbar}(f)$. We note that

- Both of the asymptotic morphisms $T^{\hbar}$ and $T^{\hbar}_{\text{hsp}}$ are topologically non–trivial.

As $U^\hbar(f) = T^\hbar(f) - T^\hbar_{\text{hsp}}(f)$ we have

$$\limsup_{\hbar\to 0} u_n(\hbar) = 0 \qquad \forall n \in \mathbb{N}$$

- Both $[a^\dagger,\, [a^\dagger,\, U^\hbar(f)]]$ and $[a^{\dagger^2},\, U^\hbar(f)]$ are sub-sub diagonal operators with $(n+2, n)^{\text{th}}$ entries given by $\sqrt{n^2+n}\,(u_{n+2}(\hbar) - 2u_{n+1}(\hbar) + u_n(\hbar))$ and $\sqrt{n^2+n}\,(u_n(\hbar) - u_{n+2}(\hbar))$ respectively.

- Similarly both $[a,\, [a,\, U^\hbar(f)]]$ and $[a^2,\, U^\hbar(f)]$ are super-super diagonal operators.

Since $X \cdot f = 0$ therefore

$$X \cdot U^\hbar(f) - U^\hbar(X \cdot f) = X \cdot U^\hbar(f) \in \mathcal{K}^\infty.$$

Comparing the sub-sub diagonal entries of $X \cdot U^\hbar(f)$ we get (for the sequence $(\hat{c}_n)_{n=1}^\infty \in \mathcal{K}^\infty$)

$$\frac{1}{4}\sqrt{n^2+n}\,(u_n(\hbar) - u_{n+2}(\hbar)) - \frac{\hbar}{4}\sqrt{n^2+n}\,(u_{n+2}(\hbar) - 2u_{n+1}(\hbar) + u_n(\hbar)) = \hat{c}_{n+2}(\hbar),$$

which leads to the following recurrence relation

$$u_{n+2}(\hbar) = \frac{2\hbar\, u_{n+1}(\hbar) + (1-\hbar)\, u_n(\hbar)}{1+\hbar} - \frac{4\hat{c}_{n+2}(\hbar)}{(1+\hbar)\sqrt{n^2+n}}. \tag{6.5.2}$$

For the sake of convenience, let us denote

$$c_{n+2}(\hbar) = -\frac{4\hat{c}_{n+2}(\hbar)}{(1+\hbar)\sqrt{n^2+n}}, \tag{6.5.3}$$

$$u_m(\hbar) = \max\left(|u_1(\hbar)|, |u_2(\hbar)|\right), \tag{6.5.4}$$

$$C(\hbar) = \sum_{n=3}^{\infty} |c_n(\hbar)|, \tag{6.5.5}$$

where rapid equivariance property implies

$$\limsup_{\hbar\to 0} u_m(\hbar) = \limsup_{\hbar\to 0} C(\hbar) = 0. \tag{6.5.6}$$

Moreover the recurrence relation (6.5.2) can be written as

$$u_{n+2}(\hbar) = \frac{2\hbar u_{n+1}(\hbar) + (1-\hbar)u_n(\hbar)}{1+\hbar} + c_{n+2}(\hbar). \tag{6.5.7}$$

Lemma 6.5.2. *The recurrence relation above implies the following system of inequalities for* $0 < \hbar < 1$

$$|u_n(\hbar)| \leq u_m(\hbar) + \sum_{k=3}^{n} |c_k(\hbar)| \qquad \forall n \geq 3.$$

Proof. Above inequality clearly holds for $n \leq 2$. We notice that for $\hbar \in (0,1)$ and $x, y \in (0, \infty)$ one has

$$\frac{2\hbar x + (1-\hbar)y}{1+\hbar} \leq \max\left(x, y\right).$$

Now assuming the inequality holds till $k \leq n$, we inductively get the following chain of inequalities

$$\begin{aligned}
|u_{n+1}(\hbar)| &= \left|\frac{2\hbar u_n(\hbar) + (1-\hbar)u_{n-1}(\hbar)}{1+\hbar} + c_{n+1}(\hbar)\right| \\
&\leq \frac{2\hbar\,|u_n(\hbar)| + (1-\hbar)\,|u_{n-1}(\hbar)|}{1+\hbar} + |c_{n+1}(\hbar)| \\
&\leq \frac{2\hbar\left(u_m(\hbar) + \sum_{k=3}^{n}|c_k(\hbar)|\right) + (1-\hbar)\left(u_m(\hbar) + \sum_{k=3}^{n-1}|c_k(\hbar)|\right)}{1+\hbar} \\
&\quad + |c_{n+1}(\hbar)| \\
&\leq u_m(\hbar) + \sum_{k=3}^{n}|c_k(\hbar)| + |c_{n+1}(\hbar)| \\
&\leq u_m(\hbar) + \sum_{k=3}^{n+1}|c_k(\hbar)|,
\end{aligned}$$

thus proving the above lemma. □

Applying the above lemma repeatedly, we get the following uniform bound on $u_n(\hbar)$:

$$|u_n(\hbar)| \leq u_m(\hbar) + C(\hbar) \qquad n \in \mathbb{N}.$$

Hence we have

$$\begin{aligned}
\limsup_{\hbar\to 0}\left\|U^\hbar(f)\right\| &= \limsup_{\hbar\to 0}\sup_{n\in\mathbb{N}}|u_n(\hbar)| \leq \limsup_{\hbar\to 0}\sup_{n\in\mathbb{N}}|u_m(\hbar) + C(\hbar)| \\
&\leq \limsup_{\hbar\to 0}|u_m(\hbar) + C(\hbar)| = 0. \qquad (6.5.8)
\end{aligned}$$

Rapidly Asymptotic Equivariance For $g(z) = z\exp(-|z|^2)$

We note that

$$g(z) = -\frac{\partial}{\partial \bar{z}} \exp(-|z|^2) = -\frac{\partial f}{\partial \bar{z}},$$

from which rapid asymptotic equivariance with respect to the vector field $-\frac{\partial}{\partial \bar{z}}$ we have

$$U^{\hbar}(g) + \sqrt{\frac{\hbar}{2}} \left[a^{\dagger}, U^{\hbar}(f)\right] \in \mathcal{K}^{\infty}. \tag{6.5.9}$$

Furthermore $\left[a^{\dagger}, U^{\hbar}(f)\right]$ is a subdiagonal operator whose $(n+1,n)$ entry is given by $\sqrt{n}\,(u_{n+1}(\hbar) - u_n(\hbar))$ where $u_n(\hbar)$'s are the diagonal entries of $U^{\hbar}(f)$.

The recurrence relations (6.5.7) on $u_n(\hbar)$ gives

$$u_{n+1}(\hbar) - u_n(\hbar) = -\frac{1-\hbar}{1+\hbar}\left(u_n(\hbar) - u_{n-1}(\hbar)\right) + c_{n+1}(\hbar).$$

Denoting $\triangle u_{n+1}(\hbar) = u_{n+1}(\hbar) - u_n(\hbar)$ the above recurrence relation can be rewritten as

$$\triangle u_{n+1}(\hbar) = -\frac{1-\hbar}{1+\hbar}\triangle u_n(\hbar) + c_{n+1}(\hbar).$$

Solving the above system we get

$$\triangle u_{n+2}(\hbar) = (-1)^{n-1}\left(\frac{1-\hbar}{1+\hbar}\right)^{n-1}\triangle u_2(\hbar) + \sum_{k=0}^{n-2}\left(\frac{1-\hbar}{1+\hbar}\right)^{k}(-1)^k c_{n+1-k}(\hbar),$$

giving us the following estimate

$$|\triangle u_{n+2}(\hbar)| \leq \left(\frac{1-\hbar}{1+\hbar}\right)^{n-1}|\triangle u_2(\hbar)| + \sum_{k=0}^{n-2}\left(\frac{1-\hbar}{1+\hbar}\right)^{k}|c_{n+1-k}(\hbar)|.$$

Therefore from (6.5.9) we have

$$\limsup_{\hbar\to 0} \|U^{\hbar}(g)\| = \limsup_{\hbar\to 0} \sqrt{\frac{\hbar}{2}} \sup_{n\geq 1} \left(\sqrt{n}\ |\triangle u_{n+1}(\hbar)|\right)$$

$$\leq \limsup_{\hbar\to 0} \sqrt{\frac{\hbar}{2}} \sup_{n\geq 1} \left(\sqrt{n} \left(\frac{1-\hbar}{1+\hbar}\right)^{n-1} |\triangle u_2(\hbar)| + \sqrt{n} \sum_{k=0}^{n-2} \left(\frac{1-\hbar}{1+\hbar}\right)^{k} |c_{n+1-k}(\hbar)|\right)$$

$$\leq \limsup_{\hbar\to 0} \sqrt{\frac{\hbar}{2}} \sup_{n\geq 1} \left(\sqrt{n} \left(\frac{1-\hbar}{1+\hbar}\right)^{n-1} |\triangle u_2(\hbar)| + \sqrt{n} \sum_{k=3}^{n+1} \left(\frac{1-\hbar}{1+\hbar}\right)^{n+1-k} |c_{k}(\hbar)|\right)$$

$$\leq \limsup_{\hbar\to 0} \sqrt{\frac{\hbar}{2}} \sup_{n\geq 1} \left(\sqrt{n} \left(\frac{1-\hbar}{1+\hbar}\right)^{n-1} |\triangle u_2(\hbar)| + \sqrt{n} \sum_{k=3}^{\lfloor n/2\rfloor} \left(\frac{1-\hbar}{1+\hbar}\right)^{n/2} |c_{k}(\hbar)|\right)$$

$$+ \limsup_{\hbar\to 0} \sqrt{\frac{\hbar}{2}} \sup_{n\geq 1} \left(\sqrt{n} \sum_{k=\lfloor n/2\rfloor+1}^{n-2} |c_k(\hbar)|\right). \tag{6.5.10}$$

As $\{c_n(\hbar)\}_{n\in\mathbb{N}}$ is a Schwartz sequence therefore there exist bounds $M_1(\hbar), M_2(\hbar)$ such that $\limsup_{\hbar\to 0} M_1(\hbar) = \limsup_{\hbar\to 0} M_2(\hbar) = 0$ with following inequalities

$$\sum_{k=3}^{\lfloor n/2\rfloor} |c_k(\hbar)| \leq M_1(\hbar),$$

$$\sum_{k=\lfloor n/2\rfloor+1}^{n-2} |c_k(\hbar)| \leq \frac{1}{n^2} M_2(\hbar). \tag{6.5.11}$$

Using the above two inequalities last term of the inequality (6.5.10) can be seen to be vanish:

$$\limsup_{\hbar\to 0} \sqrt{\frac{\hbar}{2}} \sup_{n\geq 1} \left(\sqrt{n} \sum_{k=\lfloor n/2\rfloor+1}^{n-2} |c_k(\hbar)|\right) \leq \limsup_{\hbar\to 0} \sqrt{\frac{\hbar}{2}} \left(\sup_{n\geq 1} \left(\frac{1}{n^{\frac{3}{2}}} M_2(\hbar)\right)\right) = 0.$$

Therefore we have

$$\limsup_{\hbar\to 0}\|U^{\hbar}(g)\| = \limsup_{\hbar\to 0}\sqrt{\frac{\hbar}{2}}\ \sup_{n\geq 1}\sqrt{n}\ |\triangle u_{n+1}(\hbar)|$$
$$\leq \limsup_{\hbar\to 0}\left(\sqrt{\frac{\hbar}{2}}\ \sup_{n\geq 1}\left(\sqrt{n}\left(\frac{1-\hbar}{1+\hbar}\right)^{n-1}|\triangle u_2(\hbar)| + \sqrt{n}\sum_{k=0}^{n-2}\left(\frac{1-\hbar}{1+\hbar}\right)^{k}|c_{n+1-k}(\hbar)|\right)\right)$$
$$\leq \limsup_{\hbar\to 0}\sqrt{\frac{\hbar}{2}}\ \sup_{n\geq 1}\left(\sqrt{n}\left(\frac{1-\hbar}{1+\hbar}\right)^{n-1}|\triangle u_2(\hbar)| + \sqrt{n}\sum_{k=3}^{\lfloor n/2\rfloor}\left(\frac{1-\hbar}{1+\hbar}\right)^{n/2}|c_k(\hbar)|\right)$$
$$\leq \limsup_{\hbar\to 0}\sqrt{\frac{\hbar}{2}}\ \sup_{n\geq 1}\left(\sqrt{n}\left(\frac{1-\hbar}{1+\hbar}\right)^{n-1}|\triangle u_2(\hbar)| + \sqrt{n}\left(\frac{1-\hbar}{1+\hbar}\right)^{n/2}M_1(\hbar)\right).$$

The functions $k_1^{\hbar}(n) = \sqrt{n}\left(\frac{1-\hbar}{1+\hbar}\right)^{n-1}$ and $k_2^{\hbar}(n) = \sqrt{n}\left(\frac{1-\hbar}{1+\hbar}\right)^{\frac{n}{2}}$ are maximized when $n_1 = \frac{1}{2\log\left(\frac{1+\hbar}{1-\hbar}\right)}$, and $n_2 = \frac{1}{\log\left(\frac{1+\hbar}{1-\hbar}\right)}$ respectively. Their maximum values are the following

$$k_1^{\hbar}(n) \leq k_1^{\hbar}(n_1) = \left(\frac{1}{2\log(\frac{1+\hbar}{1-\hbar})}\right)^{\frac{1}{2}}\left(\frac{1-\hbar}{1+\hbar}\right)^{\frac{1}{2\log\left(\frac{1+\hbar}{1-\hbar}\right)}-1},$$
$$k_2^{\hbar}(n) \leq k_2^{\hbar}(n_2) = \left(\frac{1}{\log(\frac{1+\hbar}{1-\hbar})}\right)^{\frac{1}{2}}\left(\frac{1-\hbar}{1+\hbar}\right)^{\frac{1}{2\log\left(\frac{1+\hbar}{1-\hbar}\right)}}.$$

Thus the norm estimate becomes

$$\limsup_{\hbar\to 0}\|U^{\hbar}(g)\| \leq \limsup_{\hbar\to 0}\left(\sqrt{\frac{\hbar}{2}}\ k_1^{\hbar}(n_1)\,|\triangle u_2(\hbar)| + \sqrt{\frac{\hbar}{2}}\ k_2^{\hbar}(n_2)M_1(\hbar)\right).$$

Moreover we have following limits

$$\limsup_{\hbar\to 0}\sqrt{\frac{\hbar}{2}}\,k_1^{\hbar}(n_1) = \limsup_{\hbar\to 0}\sqrt{\frac{\hbar}{2}}\left(\frac{1}{2\log(\frac{1+\hbar}{1-\hbar})}\right)^{\frac{1}{2}}\left(\frac{1-\hbar}{1+\hbar}\right)^{\frac{1}{2\log\left(\frac{1+\hbar}{1-\hbar}\right)}-1} = \frac{1}{2\sqrt{2e}},$$
$$\limsup_{\hbar\to 0}\sqrt{\frac{\hbar}{2}}\,k_2^{\hbar}(n_2) = \limsup_{\hbar\to 0}\sqrt{\frac{\hbar}{2}}\left(\frac{1}{\log(\frac{1+\hbar}{1-\hbar})}\right)^{\frac{1}{2}}\left(\frac{1-\hbar}{1+\hbar}\right)^{\frac{1}{2\log\left(\frac{1+\hbar}{1-\hbar}\right)}} = \frac{1}{2\sqrt{e}}.$$

Substituting these into the norm estimates, we get

$$\limsup_{\hbar\to 0}\|U^{\hbar}(g)\| \leq \limsup_{\hbar\to 0}\frac{1}{2\sqrt{2e}}\left(|\triangle u_2(\hbar)| + \sqrt{2}M_1(\hbar)\right) = 0. \qquad (6.5.12)$$

A similar calculation as above shows that

$$\limsup_{\hbar\to 0} \|U^{\hbar}(\bar{g})\| = 0. \tag{6.5.13}$$

From (6.5.1), (6.5.8), (6.5.12), and (6.5.13) the asymptotic morphisms $T^{\hbar}$ and $T^{\hbar}_{\text{hsp}}$ are asymptotically equivalent. $\square$

Chapter 7

Future Work

7.1 Current Work

We recall the statement of the main theorem proved earlier.

Theorem 7.1.1. *Consider a topologically non-trivial, linear asymptotic morphism $T^\hbar : C_0(\mathbb{R}^2) \dashrightarrow \mathcal{K}(\mathcal{H})$ with $\|T^\hbar\| \leq 1$, that is rapidly asymptotically equivariant with respect to the standard action of* $\mathrm{HSp}(2,\mathbb{R})$ *Lie group. Then the asymptotic morphism $T^\hbar$ is asymptotically equivalent to the asymptotic morphism $T^\hbar_{hsp}$, that is*

$$T^\hbar(f) \cong \int_{\mathrm{Heis}(3,\mathbb{R})} \left(\widehat{f} \cdot \Theta\right) \circ \exp^{-1}(g)\, \Pi_\hbar(g)\, dg.$$

An immediate goal is to strengthen the result of this thesis by getting rid of the *seemingly unnecessary* assumption of *rapidly asymptotic equivariance* on $T^\hbar$. This assumption was needed in the proof to alleviate the analytical difficulties while deriving the norm estimates. More specifically, we would like to experiment with various differential operators $D : C^\infty(\mathbb{R}^2) \longrightarrow C^\infty(\mathbb{R}^2)$ that belong to the universal enveloping algebra of $\mathfrak{hsp}(2, \mathbb{R})$ such that the following conditions hold:

- D is not equivariant with respect to the SO(2) rotation group.
- $D \circ \exp\left(-x^2 - y^2\right) = 0$

Following the same recipe as in proof, for each operator D, one would get different norm estimates and perhaps with a bit more work, one won't need to assume the *rapidly asymptotic equivariance* of asymptotic morphisms anymore.

In other words, a possible improved version of same theorem would be the following

Conjecture 7.1.2. *Any topologically non–trivial asymptotic morphism $T^\hbar : C_0(\mathbb{R}^2) \dashrightarrow \mathcal{K}(L^2(\mathbb{R}))$ which is asymptotically equivariant with respect to the Jacobi group is asymptotically equivalent to the asymptotic morphism $T^\hbar_{hsp}$ constructed before.*

7.2 Future Goals

A more distant goal is to generalize the above result over to $\mathbb{R}^{2n}$ and more generally over the arbitrary cotangent bundle T^*M. An even more distant goal is to formulate and derive an analogous result in the context of coadjoint orbits of reductive Lie groups.

While generalizing over $\mathbb{R}^{2n}$, we expect to face the technical difficulties of a similar nature.

On the other hand, to generalize the result over to cotangent bundles or coadjoint orbits, one needs to rephrase non-triviality of asymptotic morphism in topological terms. Furthermore, one needs to make a choice on the symmetries to be preserved by the asymptotic morphism.

A possible extension of the above result in the context of $\mathbb{R}^{2n}$ would be the following

Conjecture 7.2.1. *There exists a unique topologically non–trivial asymptotic morphism $T^\hbar : C_0(\mathbb{R}^{2n}) \dashrightarrow \mathcal{K}(L^2(\mathbb{R}^n))$ which is asymptotically equivariant with respect to the Jacobi Lie group* $\mathrm{HSp}(2n, \mathbb{R})$ *action.*

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Vita — Alok Bakshi

Contact Information

Mobile	(+1) 814–321–2026
E-mail	alok.kumar.bakshi@gmail.com
Website	https://alokbakshi.github.io

Education

PhD in Mathematics from Penn State University	2019
Master of Mathematics from ISI Kolkata	2012
B.Tech. in Industrial and Production Engineering from IIT Delhi	2005

Work Experience

Software Engineer in Intel MKL–DNN	April 2019 onwards
Mathematics Instructor at Penn State University	2013—2018

Technical Skills

Programming Languages	C, C++, Python
Programming Idioms	Parallel Computing (OpenMP, MPI, CUDA)
Big Data	Machine Learning (tensorflow library), Statistics
OR and Probability	Markov Chains, Linear Programming