The Pennsylvania State University
The Graduate School

DEBT MANAGEMENT PROBLEMS AND TOPICS IN STACKELBERG EQUILIBRIUM

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Abstract

The dissertation contains two parts. In the first part of the dissertation, we study optimal strategies for a borrower who needs to repay his debt, in an infinite time horizon. An instantaneous bankruptcy risk is present and the borrower refinances the debt by selling bonds to a pool of risk-neutral lenders. We consider both open-loop and feedback strategies. For open-loop strategies, we interpreted them as Stackelberg equilibria, where the borrower announces his repayment strategy at all future times, and lenders adjust the interest rate accordingly. Our analysis shows the existence of optimal open-loop controls, deriving necessary conditions for optimality and characterizing possible asymptotic limits as $t \to +\infty$. For feedback strategies, we study the solution of a Hamilton-Jacobi equations and construct it as the limit of viscous solutions. Under suitable assumptions, this (possibly discontinuous) limit can be interpreted as an equilibrium solution to a non-cooperative differential game with deterministic dynamics.

In the second part of the dissertation, we study the structure of the best reply map for the follower and the optimal strategy for the leader in a non-cooperative Stackelberg game. The two players choose their strategies within domains $X \subseteq \mathbb{R}^m$ and $Y \subseteq \mathbb{R}^n$. Two main cases are considered: either $X = Y = [0, 1]$, or $X = \mathbb{R}$, $Y = \mathbb{R}^n$ with $n \geq 1$. Using techniques from differential geometry, we prove that for an open dense set of cost functions the Stackelberg equilibrium is unique and is stable w.r.t. small perturbations of the two cost functions. Then we introduce a concept of "self consistent" Stackelberg equilibria for stochastic games in infinite time horizon, where the two players adopt feedback strategies and have exponentially discounted costs. We focus on games in continuous time, described by a controlled Markov process with finite state space. Under generic assumptions, we prove that a unique self-consistent Stackelberg equilibrium exists, provided that either (i) the leader is far-sighted, i.e. his exponential discount factor is sufficiently small, or (ii) the follower is narrow-sighted, i.e. his discount factor is large enough.

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Dedication

To my parents.

Chapter 1

Introduction

1 Optimal strategies in a Problem of Debt Management

Among all the financial instruments, the debt market is the largest in terms of the market size. According to Bank for International Settlements(BIS), as of 2009, the size of the worldwide bond market(total debt outstanding) is estimated at $82.2 trillion. Comparing to the GDP of the USA in 2009 which is $13.82 trillion, the global debt size is nearly six times as many as the GDP of the USA. Simply from the number, the whole world lives on debt, from a single family to a country. For example, companies may consult banks about issuing corporate bonds in order to maximize the company benefit and the governments also issue government bonds to support the country development. Small debts may cause insufficient funds while large debts may cause catastrophic bankruptcy. Thus the debt management problem has drawn great attention among both the academia and industry. The corresponding literature can be founded in [4, 7, 23, 28, 51]. In the debt management problem, the theory of optimal control has been widely used and achieves great success.

Given the dynamics of the state and the objective function to optimize, the optimal control theory gives equations that the optimal strategy should satisfy, thus people can construct the optimal strategies explicitly or numerically. However, the well-known dilemma is that the increase in the number of state variables in the model will generally make the model more realistic but harder to analyze both theoretically and numerically. Thus it is very important to get a model that can both capture the main features and give rich mathematical insights.

In this dissertation, we consider a model motivated by the paper [51] in both the deterministic and stochastic settings. The model we considered contains two state variables which capture the main features in the debt management problem and also yield fruitful mathematical results. We not only achieve quantitative properties of optimal strategies but also interpret different behaviors in the debt market. Specifically, we seek to minimize the cost to a borrower who needs to repay his debt in an infinite time horizon with presence of an instantaneous bankruptcy risk. The debt is financed by selling bonds. Instead of the fixed term bonds, we consider a zero-coupon bond which promises a stream of payments. By considering such type of zero-coupon bond, we can incorporate all the bonds issued at different times into a single bond price variable p. On the contrary, if the fixed term bonds are considered, one needs to add multiple variables to denote the change of the price for each bond and thus the model complexity becomes huge. In this setting, the price of the bond is determined by a pool of risk-neutral lenders and depends on the market belief of the borrower's bankruptcy risk at all future times. When bankruptcy happens, the borrower incurs a large cost B and the lenders only recover a fraction $\theta \in [0, 1]$ of the outstanding capital. So the bonds are sold at a fraction $p \in [0, 1]$ of their nominal value to compensate investors for the possible loss of their capital.

Two types of optimal strategies are considered, namely, the open-loop strategy and the feedback strategy. In the open-loop setting, we consider the borrower's repayment strategy $u = u(t)$ and the bond price $p = p(t)$ as functions of time. Then the optimal open-loop strategy is interpreted in the Stackelberg sense: the borrower announces a repayment strategy $u = u(t)$ for all future times and in turn, the discounted bond price $p(t)$ is determined by the competition among the risk-neutral lenders, based on the bankruptcy risk of the borrower at all future times. Since the bankruptcy risk grows with the size of the debt, we obtain a highly non-standard optimal control problem where the instantaneous dynamics depends on all the future trajectory.

In the feedback setting, we consider the borrower's repayment strategy $u = u(x)$

and the bond price $p = p(x)$ as functions of the debt size x. We then seek the pair of functions $(u^*(x), p^*(x))$ in the sense of a Nash equilibrium: 1. Given the bond price function $p^*(x)$, $u^*(x)$ is the optimal feedback strategy for the borrower. 2. $p^*(x)$ is an optimal strategy for the lender when the borrower adopts the feedback strategy $u^*(x)$. The main issue in the feedback setting is to solve the corresponding Hamilton-Jacobi system given by the optimal control theory. Such Hamilton-Jacobi system is highly nonlinear and coupled, and the traditional tools cannot be applied to the analysis. Instead, we use the technique called vanishing viscosity limit which adds a stochastic noise in the dynamics of the debt and achieves the limit as the noise go to zero. By this approach, since stochastic models yield smooth solutions, we can obtain various theoretical results for the vanishing viscosity limit. Most importantly, we show that the resulting vanishing viscosity limit can be interpreted as a Nash equilibrium in the deterministic case.

2 Stackelberg Equilibrium in stochastic games and its generic structure

The theory of non-cooperative games has been applied to many practical problems and has been widely studied, especially in economics and social science. In the theory of non-cooperative games, the main focus is the concept of equilibrium, which denotes a state such that no one has the motivation to change when other player's strategies stay in put.

In this dissertation, we focus on one typical equilibrium called Stackelberg equilibrium. Different from Nash equilibrium where all players are equal, Stackelberg games have a hierarchy. In a Stackelberg game there are two players, i.e. the leader and the follower. The game plays as follows. The leader has to first announce his strategy and then the follower responds to the leader's strategy. The hierarchy exists due to the order of taking strategies. Consider the bank regulation problem where the regulator (leader) announces regulation policies to the banks (followers). These regulation policies can be interest rate adjustments, changes to the bank's compliance and so on. In response to the regulation, banks change their investment strategies in order to maximize their profits. The Stackelberg equilibrium is a couple of strategies satisfying the follower conditions: 1. Given the leader's strategy, the follower always takes the strategy that minimizes his own cost and we call all such strategies best replies. 2. The leader's strategy minimizes his own cost with the follower's strategy being in the set of best replies. In the present and sequel, denote the control space of the leader and the follower to be $X \subset \mathbb{R}^m$ and $Y \subset \mathbb{R}^n$, respectively, and call $F, G : \mathbb{R}^m \times \mathbb{R}^n \mapsto \mathbb{R}$ the cost functions of the leader and the follower, respectively.

In the literature, the existence of a Stackelberg equilibrium is known under fairly general assumptions [21, 8, 44, 48, 56]. A major related issue is the uniqueness and stability of this equilibrium. Namely, if the cost functions F, G are slightly perturbed, does the new game still have a unique solution, close to the original one? This problem has been investigated in [44, 49, 48], within the general class of continuous cost functions. As pointed out in [44], it is not possible to obtain, under sufficiently general conditions, existence and stability results for the exact Stackelberg solutions. For this reason, in the above papers, a weaker concept of ϵ-solution was used.

In this dissertation, we study the stability for the best reply map and for exact Stackelberg solutions, within a class of smooth functions $F, G : \mathbb{R}^m \times \mathbb{R}^n \mapsto \mathbb{R}$. In this setting, examples of games with multiple equilibria are easy to construct. However, our main results show that, for "most" functions F, G (in a topological sense), the Stackelberg equilibrium is unique and is stable under small perturbations. While the results in [44, 49, 48] are based general topological principles, our stability results rely on completely different techniques, stemming from differential geometry; namely: Sard's theorem and a multi-jet version of Thom's transversality theorem [18, 38].

As a further direction in the theory of Stackelberg games, we introduce a new concept called Self-consistent Stackelberg equilibrium in the setting of stochastic games in infinite time horizon with discrete state variables, where the two players adopt feedback strategies and have exponentially discounted costs. We focus on the issue of the existence and uniqueness of the Self-consistent Stackelberg equilibrium. It is well known that the Stackelberg equilibrium in stochastic games depends on the initial distribution of the state. However, in the framework of stochastic optimal control, the optimal feedback control is independent of the initial distribution and is optimal for any initial distribution of the state. So the Self-consistent Stackelberg equilibrium is introduced in order to remove the reference to the initial distribution. Intuitively, a Self-consistent Stackelberg equilibrium is a Stackelberg equilibrium where the initial distribution equals to the stationary distribution of the stochastic system under the optimal feedback strategies. To the best of our knowledge, this is the first time that such kind of Stackelberg equilibrium is considered.

The rationale behind the Self-consistent Stackelberg equilibrium can be explained as follows. Assume that the feedbacks $(u_1^*(\cdot), u_2^*(\cdot))$ yield a Stackelberg equilibrium for a given probability distribution μ^0 on the initial state. Consider the stochastic dynamics determined by these feedbacks. As time progresses, the probability distribution μ^t on the state will keep changing. As a consequence, if the game were to be restarted anew at time $\tau > 0$, the leader would choose a different feedback control (optimal for μ^t and not for μ^0). This reflects the well known fact that Stackelberg equilibria in general are not time consistent. Even those in feedback form. However, in the setting of Self-consistent Stackelberg equilibrium, one has $\mu = \mu^t$ for all $t \geq 0$. The same feedback control $u_1^*(\cdot)$ thus remains the optimal choice of the leading player, at every positive time. In practice, Self-consistent Stackelberg equilibrium can model a situation where the leading player (a far-sighted legislator) either does not know or is not much concerned with the present state of the system. Rather, he wants to draft a regulation which, after a short transient period, will be recognized as "best possible" at all future times.

3 Organization of the dissertation

The dissertation is organized as follows.

In Chapter 2, we review some preliminary concepts in optimal control, non-cooperative games and differential geometry. In the optimal control part, we introduce the Pontryagin Maximum Principle and the Hamilton-Jacobi equation corresponding to open-loop and feedback controls, respectively. In the non-cooperative games part, we briefly go through the basics of the Nash equilibrium and the Stackelberg equilibrium. In the differential geometry part, we talk about Sard's theorem and the transversality lemma, which serve as working tools for Chapter

5.

Chapter 3 and 4 are dedicated to the debt management problem. Chapter 3 presents the results in [14] and Chapter 4 presents the results in [15]. Specifically, in Chapter 3 we first prove the existence and derive necessary conditions for the optimal open-loop control and then show some numerical results. In Chapter 4, we start with a stochastic model and then let the diffusion go to zero to achieve the viscosity limit. We show a bunch of analytical results for the vanishing viscosity limit and interpret the limit as a Nash equilibrium.

Chapter 5 contains the study in [16], concerning the generic structure and stability of Stackelberg equilibria. We first study a constrained optimization problem corresponding to the Stackelberg equilibrium and show the generic stability of the Stackelberg eqilibrium under small perturbations. Then we study the generic structure of the graph of the best reply map in cases where the control spaces are $X = Y = [0,1]$ and $X = Y = \mathbb{R}^n$ with $n > 1$, respectively. According to our analysis, in the case with $X = Y = [0,1]$, the follower's best reply, expressed as a function of the leader's control, is piecewise smooth.

Chapter 6 presents the results in [17], focusing on the Self-consistent Stackelberg equilibrium in continuous time stochastic games in infinite horizon with discrete state variables. We give necessary conditions for the existence of Self-consistent Stackelberg equilibrium and also show that under the generic conditions, a unique Self-consistent Stackelberg equilibrium exists when either the leader is far-sighted or the follower is narrow-sighted. That is, the exponential discount factor for the leader is sufficiently small, or the discount factor for the follower is sufficiently large.

Chapter 2

Preliminaries

1 Basics in optimal control theory

1.1 Open-loop control and Pontryagin Maximum Principle(PMP)

In the optimal control problem with a finite time horizon $T > 0$, we want to minimize the cost of the form

$$\min_{u \in \mathbb{U}} \int_0^T L(t, x(t), u(t))\, dt + \psi(x(T, u)). \tag{1.1}$$

subject to the dynamics

$$\dot{x}(t) = f(t, x(t), u(t)), \qquad x(0) = \bar{x},$$

where $x(t) \in R^n$ and $\mathbb{U}$ is the set of all the admissible control functions. $L(t, x, u)$ is called the running cost function, which is continuous in all variables and continuously differentiable w.r.t x. For a given set $\mathbf{U} \subset \mathbb{R}^m$, the family of admissible control functions is defined as

$$\mathbb{U} = \Big\{ u : [0, T] \to \mathbf{U} \text{ and is measurable} \Big\}.$$

In (1.1), $\psi(x)$ is called the terminal cost function which depends on the terminal state $x(T)$. The optimal control problem formulates a situation where people want to control the dynamics of x in order to achieve the lowest total cost defined in (1.1).

As what is described in the admissible set $\mathbb{U}$, when the admissible control u is a function only of time t, i.e. in the form of $u(t)$, we call it **open-loop**.

Theorem 1.1. Pontragin Maximum Principle *Under all the assumptions, let $u^*(\cdot)$ be a bounded admissable open control whose corresponding trajectory $x^*(\cdot)$ is optimal. There exits an adjoint variable $p : [0,T] \to \mathbb{R}^n$ satisfying the linear equation*

$$\dot{p}(t) = -p(t) \cdot D_x f(t, x^*(t), u^*(t)), \qquad p(T) = \nabla\psi(x^*(T)).$$

Also, the optimality condition

$$u^*(t) = arg\max_{\omega \in \mathbf{U}} \left\{ p(t) \cdot f(t, x^*(t), \omega) \right\}$$

holds for a.e $t \in [0,T]$.

1.2 Feedback control and Hamilton Jacobi Equation

Now, suppose that the controller can observe the state x at any time and the admissible control u is a function of state, i.e. $x \mapsto u(x)$. Then we say that u is a **feedback** control.

Consider the optimal control problem in infinite time horizon with the set of admissible feedback controls defined as

$$\mathbb{U}_1 \doteq \{u : x \mapsto \mathbf{U}\}.$$

Then the optimal control problem can be formulated as

$$\min_{u \in \mathbb{U}_1} J(x,u) \doteq \int_0^\infty e^{-rt} L(x(t), u(x(t)))\, dt, \tag{1.2}$$

where

$$\dot{x}(t) = f(x(t), u(x(t))), \qquad x(0) = x, \tag{1.3}$$

and r is the discounting factor. For any x, we define the value function as $V(x) \doteq$

$\min_{u \in \mathbb{U}_1} J(x,u)$.

Theorem 1.2. Hamilton-Jacobi equation *The value function V for the problem (1.2)-(1.3) is a viscosity solution to the Hmailton-Jacobi equation*

$$rV(x) - H(x, \nabla V(x)) = 0, \tag{1.4}$$

where the Hamiltonian function is defined as

$$H(x,p) \doteq \min_{\omega \in \mathbf{U}} \Big\{ L(x,\omega) + p \cdot f(x,\omega) \Big\}.$$

Furthermore, the optimal control $u^*(x)$ is given by the optimality condition

$$u^*(x) = \operatorname*{argmin}_{\omega \in \mathbf{U}} \Big\{ L(x,\omega) + \nabla V \cdot f(x,\omega) \Big\}.$$

2 Equilibria in non-cooperative games

In a basic setting, a game for two players can be formulated as follows.

- Player 1 chooses $x \in X$ and seeks to minimize his cost $F(x,y)$.
- Player 2 chooses $y \in Y$ and seeks to minimize his cost $G(x,y)$.

Here X, Y are topological spaces, while $F, G : X \times Y \mapsto \mathbb{R}$ are continuous functions. For a given $y \in Y$, the set of **best replies** for player 1 is defined as

$$R_1(y) \;\doteq\; \Big\{ x^* \in X \, ; \quad F(x^*, y) \le F(x,y) \ \text{ for all } x \in X \Big\}. \tag{2.5}$$

Similarly, for a given $x \in X$, the set of **best replies** for player 2 is defined as

$$R_2(x) \;\doteq\; \Big\{ y^* \in Y \, ; \quad G(x, y^*) \le G(x,y) \ \text{ for all } y \in Y \Big\}, \tag{2.6}$$

We say that a couple $(x^*, y^*) \in X \times Y$ is a **Nash equilibrium** if $x^* \in R_1(y^*)$ and $y^* \in R_2(x^*)$. This models a situation where both players announce their strategies simultaneously and want to minimize their own costs.

With a slight change of the notation, call player 1 the leader and player 2 the follower. Also, we rename the set of best replies for the follower in (2.6) by $R(x)$.

Then we say that a couple $(x^*, y^*) \in X \times Y$ is a **Stackelberg equilibrium** if $y^* \in R(x^*)$ and

$$F(x^*, y^*) \ \leq \ F(x, y) \qquad \text{for all } x \in X \text{ and } y \in R(x). \tag{2.7}$$

This models a situation where the leading player announces his strategy $x \in X$ in advance, and the follower chooses a reply $y \in Y$ which minimizes his own cost $G(x, y)$.

3 Working tools in differential geometry

3.1 Sard's lemma

Assume $f : \mathbb{R}^m \mapsto \mathbb{R}^n$

- $x \in \mathbb{R}^m$ is a **critical point** of f if the Jacobian matrix $Df(x)$ has rank $< n$. Equivalently: if the differential $Df(x)$ is not surjective.
- $y \in \mathbb{R}^n$ is a **critical value** of f if $y = f(x)$ for some critical point x.

Sard's Lemma [55].

For any $f \in \mathcal{C}^\infty(\mathbb{R}^m; \mathbb{R}^n)$, the set of critical values has n-dimensional measure zero.

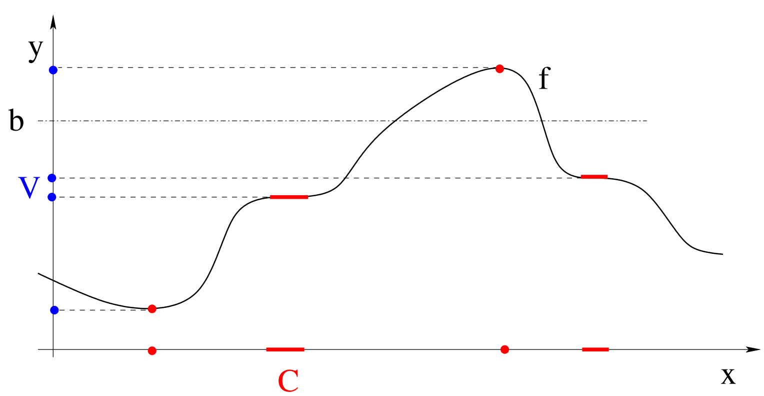

Figure 2.1. An illustration of Sard's theorem. Here the se of critical points of f is large, but its image has measure zero.

3.2 Transversality

Let $f : X \mapsto Y$ be a smooth map of manifolds and let W be a submanifold of Y.

We say that f is **transverse** to W at a point p, and write $f \pitchfork_p W$, if

- either $f(p) \notin W$,
- or else $f(p) \in W$ and $(df)_p(T_pX) + T_{f(p)}W = T_{f(p)}Y$.

We say that f is **transverse** to W, and write $f \pitchfork W$, if $f \pitchfork_p W$ for every $p \in X$.

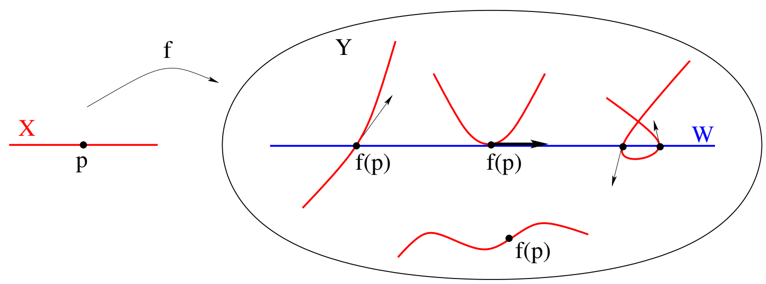

Figure 2.2. The map shown in the center, whose graph is tangent to W, is not transversal. All other functions are transversal.

Example. Take $X = \mathbb{R}^m$, $Y = \mathbb{R}^n$ with $m < n$. Assume $W = \{y_0\}$ for some $y_0 \in \mathbb{R}^n$. In this case, transversality implies

- either $f(x) \neq y_0$,
- or else $f(x) = y_0$ and rank $(Df(x)) = n$.

Since the second alternative is impossible,

$$f \pitchfork W \qquad \Longleftrightarrow \qquad f(x) \neq y_0 \qquad \text{for all } x \in \mathbb{R}^2$$

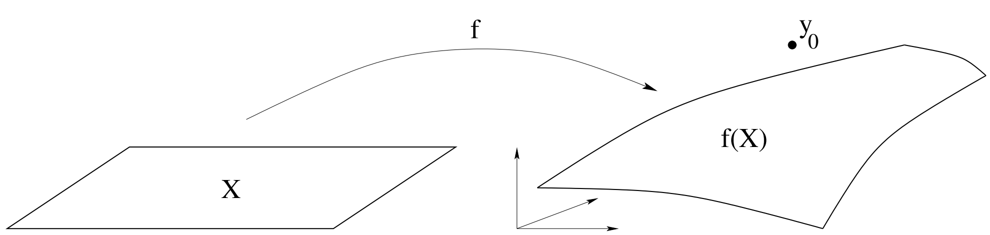

Figure 2.3. A map $f : \mathbb{R}^2 \mapsto \mathbb{R}^3$ is transversal to the manifold $W = \{y_0\}$ if and only if its image does not contain the point y_0.

Transversality lemma. *Let X, Θ, and Y be smooth manifolds, W a submanifold of Y. Let $\theta \mapsto \phi^\theta$ be a smooth map which to each $\theta \in \Theta$ associates a function $\phi^\theta \in \mathcal{C}^\infty(X, Y)$, and define $\Phi : X \times \Theta \mapsto Y$ by setting $\Phi(x, \theta) = \phi^\theta(x)$.*

If $\Phi \pitchfork W$ then the set $\{\theta \in \Theta, ; \quad \phi^\theta \pitchfork W\}$ is dense in Θ.

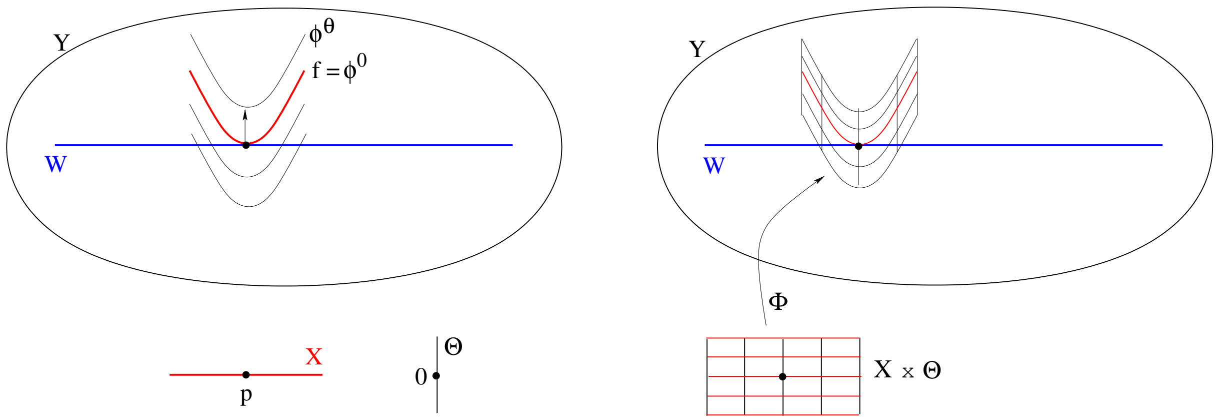

Figure 2.4. A family of maps $\theta \mapsto \phi^\theta$ depending on a parameter $\theta \in \Theta$ can also be seen as a single map Φ from the product space: $X \times \Theta \mapsto Y$. If Φ is transversal, then for a.e. θ the map ϕ^θ is transversal.

Chapter 3

Open-loop strategies in a debt magement problem

1 Introduction

We consider an optimization problem for a borrower, who needs to repay his debt, in an infinite time horizon. The main feature of our model is the presence of a bankruptcy risk. As a consequence, the interest rate payed on the loan is not a priori given, but must be determined as part of the solution.

The debt is financed by selling bonds, which promise a stream of payments to the investors. When bankruptcy happens, the borrower incurs in a very large cost B, while the lenders recover only a fraction $\theta \in [0,1]$ of their outstanding capital. For this reason, bonds are sold at a fraction $p \in [0,1]$ of their nominal value, to compensate investors for the possible loss of part of their capital.

A related model was recently considered in [20], formulated in terms of two state variables: the total amount of debt X and the average interest A payed on outstanding loans, at any given time t. In this chapter we introduce a simpler model, involving one single state variable x, describing the nominal value of the debt. This new model still captures the heart of the matter. While the analysis in [20] was mainly focused on existence of an optimal solution, here we can also derive necessary conditions for optimality, and determine the asymptotic limit of the solution as $t \to +\infty$.

Our solutions should be interpreted in a Stackelberg sense. The borrower announces a repayment strategy $u = u(t)$ for all future times. In turn, the discounted bond price $p(t)$ is determined by the competition among a pool of risk-neutral lenders, based on the bankruptcy risk at all future times. Since this risk grows with the total size of the debt, we obtain a highly non-standard optimal control problem for the borrower, where the instantaneous dynamics depends on the entire future trajectory.

The remainder of the chapter is organized as follows. In Section 2, we introduce the model and collect all the assumptions of the parameters and functions. Also, we formulate the optimal control problem for the borrower in open-loop form, given the initial value of the debt.

Section 3 contains a careful analysis of the dynamics of the system. Given an initial data $x(0) = x_0$ and a control function $u = u(t)$, we show that our evolution equations always admit at least one solution. Differently from usual control systems, here the solution may not be unique. Indeed, the competition among lenders may yield multiple Nash equilibria. In practical terms, this can be explained as follows.

If lenders regard their investment as safe, they will buy bonds at almost full price $p \approx 1$. This lowers the burden of servicing the debt, thus reducing the chance of bankruptcy.

On the other hand, given the same initial debt size, if lenders regard their investment as risky, they will buy bonds only at a deeply discounted price $p << 1$. In turn, this forces the borrower to sell a larger amount of bonds to raise the same amount of cash, pushing up the total debt and hence the risk of bankruptcy. In this respect, our model captures what is commonly called a “self-fulfilling prophecy”.

In Section 4 we prove that, for any given initial amount of debt, there exists at least one optimal open-loop control $u^*(\cdot)$ and a corresponding solution $x^*(\cdot)$, minimizing the expected total cost to the borrower.

Section 5 is concerned with necessary conditions for optimality. Two main cases are considered: either (i) bankruptcy occurs with probability one within a bounded interval of time, or (ii) as $t \to +\infty$, the debt approaches an asymptotic equilibrium state x_∞. In both cases, optimality conditions are obtained, relying on the Pontryagin Maximum Principle. Finally, Section 6 provides an example where the optimal strategies are analytically described and numerically computed.

For the basic theory of optimal control and the Pontryagin Maximum Principle we refer to [22, 30]. Necessary conditions for control problems with infinite time horizon were derived in [5, 6, 42, 59]. An introduction to non-cooperative differential games can be found in [21, 8]. In the economics literature, some related models of debt and bankruptcy can be found in [4, 7, 23, 28]. Very recently, in [51] Nuño and Thomas proposed a model where the yearly income of the debtor is a stochastic process. Bankruptcy occurs at the random time T_b when the debt-to-income ratio reaches a given threshold x^*. Optimal solutions in feedback form, and the dependence of the total expected cost on the choice of x^*, are studied in the paper [19].

2 The infinite horizon optimal control problem

The model includes the following variables:

- t = time, measured in years,
- $x(t)$ = total debt, measured as a fraction of the yearly income of the borrower,
- $u(t) \in [0, 1]$ = payment rate, as a fraction of the borrower's income,
- $p(t) \in [0, 1]$ = discounted bond price,
- T_b = random time when bankruptcy occurs.

In addition, we consider the functions

- $L(u)$ = cost to the borrower by implementing the control u,
- $\rho(x)$ = instantaneous risk of bankruptcy,
- $\theta(x(T_b)) \in [0, 1]$ = salvage rate, as a fraction of the outstanding capital which can be recovered by the lenders in case of bankruptcy,

and the constants

- r = discount rate,
- λ = rate at which the principal is payed back,
- B = bankruptcy cost to the borrower,
- M = maximum size of the debt, beyond which bankruptcy immediately occurs.

We regard $x(\cdot)$ as the one-dimensional state variable, while $u(\cdot)$ is the control variable for the borrower.

When an investor (i.e., a lender) purchases a coupon of unit nominal value at time $t = 0$, he receives the promise of a stream of payments for all future times. The primary capital is payed back at rate λ, so that the outstanding value of the loan at time $t > 0$ is $e^{-\lambda t}$. In addition, the borrower pays an interest r. The repayment rate is thus

$$\psi(t) \;=\; (\lambda + r)e^{-\lambda t}.$$

The total payoff for the lender, exponentially discounted in time, is computed by

$$\Psi \;=\; \int_0^{+\infty} e^{-rt}(\lambda + r)e^{-\lambda t}\,dt \;=\; 1\,. \tag{2.1}$$

However, if bankruptcy occurs at a random time T_b, payments will stop at time $t = T_b$, and the total payoff will be

$$\Psi \;=\; \int_0^{T_b} e^{-rt}(\lambda + r)e^{-\lambda t}\,dt + e^{-rT_b}\theta(x(T_b))e^{-\lambda T_b}\,. \tag{2.2}$$

Notice that the last term in (2.2) accounts for

[exponential discount] $\times$ [salvage rate]

$\times$ [outstanding capital at the time of bankruptcy]

If $\theta = 1$, so that the outstanding capital is recovered in full, then again $\Psi = 1$. In general, however, $\theta < 1$. Assuming that the market price results from the perfect competition of a pool of risk-neutral lenders, the coupon will be bought at the discounted price

$$p \;=\; E\left[\int_0^{T_b} e^{-rt}(\lambda+r)e^{-\lambda t}\,dt + e^{-rT_b}\theta(x(T_b))e^{-\lambda T_b}\right]. \tag{2.3}$$

Here E denotes the expected value of the given quantity, depending on the random variable T_b.

The distribution of random time T_b at which bankruptcy occurs is determined as follows. If at time τ the borrower is not yet bankrupt and the total debt is $x(\tau) = y$, then the probability that bankruptcy will occur shortly after time τ is measured by

$$\text{Prob.}\Big\{T_b \in [\tau,\, \tau+\varepsilon] \;\Big|\; T_b > \tau\,,\;\; x(\tau) = y\Big\} \;=\; \rho(y)\cdot\varepsilon + o(\varepsilon). \tag{2.4}$$

Here $\rho(y)$ measures an "instantaneous bankruptcy risk", while $o(\varepsilon)$ denotes a higher order infinitesimal as $\varepsilon \to 0$.

Assume that, as long as bankruptcy does not occur, the size of the debt $x(t)$ is determined at all future times. Define

$$T^M \;\doteq\; \inf\big\{t > 0\,;\;\; x(t) = M\big\} \;\in\; \mathbb{R}\cup\{+\infty\} \tag{2.5}$$

the first time when the debt reaches the maximum possible value M. The probability that the borrower is not yet bankrupt at a time $t > 0$ is then computed as

$$P(t) \;\doteq\; Prob.\{T_b > t\} \;=\; \begin{cases} \exp\left\{-\displaystyle\int_0^t \rho(x(\tau))\,d\tau\right\} & \text{if} \quad t < T^M, \\ \quad 0 & \text{if} \quad t \geq T^M. \end{cases} \tag{2.6}$$

Notice that this depends on the function $\tau \mapsto x(\tau)$. If debt is maintained at a higher level, then the probability of bankruptcy increases. Using (2.6) to compute the expectation in (2.3), for any $t \geq 0$ one obtains

$$\begin{aligned} p(t) &= 1 - \int_t^{T^M} \big[1-\theta(x(s))\big]\,\rho(x(s))\exp\left\{-\int_t^s \big[r+\lambda+\rho(x(\tau))\big]\,d\tau\right\}ds \\ &\qquad -\big[1-\theta(M)\big]\exp\left\{-\int_t^{T^M}[r+\lambda+\rho(x(\tau))]d\tau\right\} \\ &\doteq 1-\mathcal{L}_0(t)-\mathcal{L}_M(t)\,. \end{aligned} \tag{2.7}$$

The first integral $\mathcal{L}_0(t)$ on the right hand side of (2.7) accounts for the lost value if bankruptcy occurs at any time $s \in \,]t, T^M[$. The second integral $\mathcal{L}_M(t)$ accounts for the lost value if bankruptcy occurs exactly at time T^M. Clearly, if $T^M = +\infty$, then $\mathcal{L}_M(t) \equiv 0$.

Denoting by an upper dot the derivative w.r.t. time, from (2.7) we obtain

$$\begin{aligned} \dot p(t) &= \big[1-\theta(x(t))\big]\,\rho(x(t)) + \big[r+\lambda+\rho(x(t))\big](p(t)-1) \\ &= \big[p(t)-\theta(x(t))\big]\,\rho(x(t)) + (r+\lambda)(p(t)-1)\,. \end{aligned} \tag{2.8}$$

Next, we write an evolution equation for the nominal value $x(t)$ of the debt. Calling $u(t)$ the rate of payments made by the borrower, one has

$$\dot x(t) = \begin{cases} -\lambda x(t) + \dfrac{(\lambda+r)x(t)-u(t)}{p(t)} & \text{if } 0 < x(t) < M, \\ 0 & \text{if } x(t) \in \{0, M\}\,. \end{cases} \tag{2.9}$$

Indeed, (2.9) is motivated by the following observations.

- The nominal value of outstanding loans decreases, since they are payed back at the fixed rate λ.
- To service the current debt, the borrower should make a stream of payments with rate $(\lambda+r)x(t)$. If his instantaneous payment rate $u(t)$ is smaller than this value, new loans must be initiated. Accounting for the discounted price $p(t)$, the nominal value of these new loans is given by $\dfrac{(\lambda+r)x(t)-u(t)}{p(t)}$. It may also happen that $u(t) > (\lambda+r)x(t)$. In this case, the borrower is simply buying back some of his debt from the market.

- The second alternative in (2.9) guarantees that the state constraint $x(t) \in [0, M]$ is satisfied, for all $t \geq 0$. Clearly, if $x(\tau) = 0$, then at time τ the debt is completely extinguished and equals zero at all future times $t > \tau$. On the other hand, if $x(\tau) = M$, then again the evolution stops, because of immediate bankruptcy.

The equation (2.9) is supplemented by the initial datum

$$x(0) \;=\; x_0\,, \tag{2.10}$$

specifying the initial value of the debt.

Remark 1. In a standard control problem, the initial data (2.10) and the control function $u(\cdot)$ completely determine the evolution of the state $x(\cdot)$. This is not the case here. Indeed, the evolution equation (2.9) also involves the discounted price $p(t)$, which by (2.7) is determined by all future values of $x(\cdot)$. The existence of a solution is not an obvious fact, and will be proved in Section 3. Uniqueness does not hold, in general.

Given a control $u(\cdot)$ and a trajectory $x(\cdot)$, to compute the total expected cost to the borrower, exponentially discounted in time, we first introduce the function

$$\gamma(t) \;\doteq\; e^{-rt}\exp\left\{-\int_0^t \rho(x(s))\,ds\right\}. \tag{2.11}$$

Notice that this is the product of the exponential discount, times the probability of not being bankrupt at time t. We now compute

$$\begin{aligned} J(u,x) \;&=\; E\left[\int_0^{T_b} e^{-rt}L(u(t))\,dt + B\,e^{-rT_b}\right] \\ &=\; \int_0^{T^M} \gamma(t)\Big\{\rho(x(t))\,B + L(u(t))\Big\}\,dt \;+\; \gamma(T^M)\,B \\ &=\; B - \gamma(T^M)B - rB\int_0^{T^M}\gamma(t)\,dt + \int_0^{T^M}\gamma(t)L(u(t))\,dt + \gamma(T^M)B \\ &=\; B + \int_0^{T^M}\gamma(t)[L(u(t)) - rB]\,dt \end{aligned} \tag{2.12}$$

For a given initial value (2.10) of the debt we can now formulate the optimal control problem for the borrower, in open-loop form.

(DMP) Debt Management Problem with bankruptcy risk. *Given an initial size $x(0) = x_0$ of the debt, find a control $t \mapsto u(t) \in [0,1]$ and a corresponding map $t \mapsto (x(t), p(t))$, which minimize the expected cost*

$$J(u,x) \;\doteq\; B + \int_0^{T^M} \gamma(t)[L(u(t)) - rB]\,dt\,, \tag{2.13}$$

subject to the dynamics (2.9) and the constraint (2.7).

In the remainder of the chapter we will prove that this problem has at least one solution, and derive necessary conditions for optimality in the form of a maximum principle.

Concerning the functions ρ, L, θ, and the constants r, λ, B, M, we shall assume

(A1) *All constants r, λ, B, M are strictly positive. Moreover, $rM > 1$.*

(A2) *The function ρ is continuously differentiable. There exists $R_0 \geq 0$ such that*

$$\rho(x) = 0 \;\; for \;\; x \in [0, R_0], \quad \rho'(x) > 0 \;\; for \;\; x \in]R_0, M[, \quad \lim_{x \to M-} \rho(x) \;=\; +\infty. \tag{2.14}$$

(A3) *The map $\theta : [0, M] \mapsto [0,1]$ is Lipschitz continuous, nonincreasing, and strictly positive.*

(A4) *The cost function L is twice continuously differentiable for $u \in [0,1[$ and satisfies*

$$L(0) = 0, \qquad L' > 0, \quad L'' > 0, \qquad L(1) \;=\; \lim_{u \to 1-} L(u) \;=\; +\infty. \tag{2.15}$$

To motivate **(A3)**, assume that the borrower owns an amount R_0 of collateral (real estate, gold reserves, etc...) to back up his debt. In case of bankruptcy, this will be divided among lenders. In this case the function θ will have the form

$$\theta(x) \;=\; \min\left\{1, \frac{R_0}{x}\right\}. \tag{2.16}$$

3 Construction of solutions

Given a control $u(\cdot)$ and an initial datum x_0, the following analysis shows the existence of a solution to our system of evolution equations. Since here we are not solving a Cauchy problem, one should be aware that solutions may not be unique.

Theorem 1. *Let the assumptions* **(A1)-(A3)** *hold. Let a measurable function* $u : [0,\infty[\mapsto [0,1]$ *and an initial state* $x_0 \in [0,M]$ *be given. Then the equations (2.7), (2.9), (2.10) admit at least one solution* $t \mapsto (x(t), p(t))$*, defined for all* $t \geq 0$*.*

Proof. Let a measurable function u and an initial state x_0 be given.

1. Choose a constant $0 < \mu < r$ and let X be the Banach space of all continuous functions $f : [0,\infty[\mapsto \mathbb{R}$ such that

$$\|f\| \doteq \sup_{t\geq 0} e^{-\mu t}|f(t)| < +\infty. \tag{3.1}$$

Within the space X, consider the closed, convex subset

$$Y \doteq \Big\{p \in X\,;\quad p(t) \in \big[\theta(M),\, 1\big] \quad \text{for all } t \geq 0\Big\}. \tag{3.2}$$

We recall that, by **(A2)**, one has $\theta(M) > 0$.

2. For any $p \in Y$, let $x(\cdot) = \Lambda_1(p)$ be the solution to (2.9)-(2.10). We observe that this solution is unique. Indeed, for every $t \geq 0$, the map $x \mapsto -\lambda x + \frac{(\lambda+r)x-u(t)}{p(t)}$ is Lipschitz continuous. Hence the Cauchy problem has a unique solution defined up to the first time T where either $x(T) = 0$ or $x(T) = M$. By (2.9), the solution remains constant for all $t \geq T$.

Next, given a solution $x = \Lambda_1(p)$ of (2.9)-(2.10), let T^M be as in (2.5) and define the function $p^\sharp \doteq \Lambda_2(x)$ by setting

$$\begin{aligned} p^\sharp(t) &\doteq 1 - \int_t^{T^M} \big[1-\theta(x(s))\big]\,\rho(x(s))\exp\left\{-\int_t^s \big[r+\lambda+\rho(x(\tau))\big]\,d\tau\right\}ds \\ &\qquad -\big[1-\theta(M)\big]\exp\left\{-\int_t^{T^M}[r+\lambda+\rho(x(\tau))]d\tau\right\} \\ &= 1-\mathcal{L}_0(t)-\mathcal{L}_M(t), \end{aligned} \tag{3.3}$$

for $t \in [0, T^M[$, while

$$p^\sharp(t) = \theta(M) \qquad \text{for} \quad t \geq T^M. \tag{3.4}$$

In the next steps we shall prove that the composition

$$p \mapsto p^\sharp \doteq \Lambda_2(\Lambda_1(p)) \tag{3.5}$$

is a continuous, compact operator from Y into itself, with the distance induced by the norm (3.1).

3. We begin by proving that the map $p \mapsto x = \Lambda_1(p)$ is continuous. More precisely, for every $\varepsilon > 0$ there exists $\delta > 0$ such that, if $p, \tilde{p} \in Y$ and $\|p - \tilde{p}\| \leq \delta$, then the corresponding solutions of the Cauchy problem (2.9), (2.10) satisfy

$$e^{-\mu t}|x(t) - \tilde{x}(t)| \leq \varepsilon \qquad \text{for all } t \geq 0. \tag{3.6}$$

Observing that $x(t), \tilde{x}(t) \in [0, M]$, the inequality in (3.6) will certainly hold for all $t \geq \tau_\varepsilon \doteq \frac{1}{\mu}\log(\frac{M}{\varepsilon})$.

For $t \in [0, \tau_\varepsilon]$, recalling that $p, \tilde{p}$ take values inside the interval $[\theta(M)\,,\,1]$, from

(2.9) we deduce

$$\begin{aligned}
\frac{d}{dt}|x(t)-\tilde{x}(t)| &\leq \left|\frac{(\lambda+r)x(t)-u(t)}{p(t)} - \frac{(\lambda+r)\tilde{x}(t)-u(t)}{\tilde{p}(t)}\right| \\
&\leq \frac{r+\lambda}{\theta(M)}\,|x(t)-\tilde{x}(t)| + [(\lambda+r)M+1]\left|\frac{1}{p(t)} - \frac{1}{\tilde{p}(t)}\right| \\
&\leq \frac{r+\lambda}{\theta(M)}\,|x(t)-\tilde{x}(t)| + \frac{(\lambda+r)M+1}{\theta^2(M)}\,|p(t)-\tilde{p}(t)|.
\end{aligned}$$

Since $x(0)=\tilde{x}(0)=x_0$, if

$$\sup_{t\geq 0} e^{-\mu t}|p(t)-\tilde{p}(t)| \;\leq\; \delta$$

for some $\delta>0$ suitably small, an application of Gronwall's estimate on the interval $[0,\tau_\varepsilon]$ now yields (3.6).

4. In this step we prove that the set of functions $Z \doteq \{\Lambda_2(\Lambda_1(p))\,;\quad p\in Y\}$ is equicontinuous and $Z\subset Y$.

Let $\varepsilon>0$ be given, and let $p^\sharp=\Lambda_2(x)$, for some $x=\Lambda_1(p)$ with $p\in Y$. From (2.9) we deduce

$$-\frac{1}{\theta(M)} \;\leq\; \frac{rx(t)-1}{p(t)} \;\leq\; \dot{x}(t) \;\leq\; \frac{\lambda+r}{\theta(M)}\,M. \tag{3.7}$$

In particular, $x(\cdot)$ is uniformly Lipschitz continuous. In turn, by (2.16) the function $t\mapsto\theta(x(t))$ is Lipschitz continuous as well.

Since we assume $rM>1$, by the last inequality in (3.7) there exist constants $M_0<M$ and $c_0>0$ such that

$$x(t)\;\geq\;M_0 \qquad\Longrightarrow\qquad \dot{x}(t)\;\geq\;c_0\,. \tag{3.8}$$

Introducing the time

$$t_0 \;\doteq\; \inf\big\{t\geq 0\,;\quad x(t)\geq M_0\big\}, \tag{3.9}$$

by (3.8) we have

$$T^M \ \le\ t_0 + \frac{M - M_0}{c_0}\,. \tag{3.10}$$

To understand the properties of $p^\sharp(\cdot)$, we introduce the integral function

$$\varphi(x) \ \doteq\ \int_0^x \rho(y)\,dy\,. \tag{3.11}$$

Various cases will be considered.

Case 1: $T^M = +\infty$. In this case, by (3.10) we must have $x(t) < M_0$ for all $t \ge 0$. Differentiating (3.3), as in (2.8) we obtain

$$\dot p^\sharp(t) \ =\ \big[p^\sharp(t) - \theta(x(t))\big]\rho(x(t)) + (r+\lambda)(p^\sharp(t) - 1), \tag{3.12}$$

$$|\dot p^\sharp(t)| \ \le\ \rho(M_0) + (r+\lambda). \tag{3.13}$$

Hence $p^\sharp = \Lambda(p)$ is uniformly Lipschitz continuous.

Case 2: $T^M < +\infty$ and $\varphi(M) < +\infty$. In this case there is a positive probability that bankruptcy occurs exactly at time $t = T^M$. Moreover,

$$\lim_{t\to T^M-} \mathcal{L}_0(t) \ =\ 0, \qquad\qquad \lim_{t\to T^M-} \mathcal{L}_M(t) \ =\ 1 - \theta(M),$$

hence $p^\sharp$ is continuous at $t = T^M$.

Let t_0 be as in (3.9). By (3.13) the function $p^\sharp$ is uniformly Lipschitz continuous on the interval $[0, t_0]$. On the other hand, for $t_0 \le t < t' \le T^M$, by (3.3) we have

$$\begin{aligned} |p^\sharp(t') - p^\sharp(t)| &\le \int_t^{t'} \rho(x(s))\, ds + \int_t^{T^M} \rho(x(s)) \left(\int_t^{t'} r + \lambda + \rho(x(\tau))\, d\tau \right) ds \\ &\qquad + \int_t^{t'} r + \lambda + \rho(x(\tau))\, d\tau \\ &\le \left(\frac{\varphi(M)}{c_0} + 2 \right) \left[(r+\lambda)(t'-t) + \int_t^{t'} \rho(x(\tau))\, d\tau \right] \\ &\le \left(\frac{\varphi(M)}{c_0} + 2 \right) \left[(r+\lambda)(t'-t) + \int_{x(t)}^{x(t')} \left(\rho(x) \cdot \frac{d\tau}{dx} \right) dx \right] \\ &\le \left(\frac{\varphi(M)}{c_0} + 2 \right) \left[(r+\lambda)(t'-t) + \frac{\varphi(x(t')) - \varphi(x(t))}{c_0} \right]. \end{aligned} \tag{3.14}$$

By (3.7) it follows

$$|x(t') - x(t)| \le \frac{\lambda + r}{\theta(M)} M \cdot |t' - t|\,.$$

In turn, this yields a bound on the difference $\varphi(x(t')) - \varphi(x(t))$ in terms of $|t' - t|$ and the modulus of continuity of the function φ. By (3.14), every function $p^\sharp$ in (3.3) satisfies a uniform modulus of continuity.

Case 3: $T^M < +\infty$ and $\varphi(M) = +\infty$. Then, with probability one, bankruptcy occurs before time $t = T^M$. Moreover,

$$\lim_{t \to T^M -} \mathcal{L}_0(t) = 1 - \theta(M), \qquad\qquad \mathcal{L}_M(t) \equiv 0.$$

Therefore, we again conclude that $p^\sharp$ is continuous at $t = T^M$.

The expression (3.3) can now be rewritten as

$$p^\sharp(t) = 1 + \int_t^{T^M} [1 - \theta(x(s))] e^{-(r+\lambda)(s-t)} \cdot \frac{d}{ds} \exp\left\{ - \int_t^s \rho(x(\tau))\, d\tau \right\} ds\,. \tag{3.15}$$

Let t_0 be as in (3.9). By (3.13) the function $p^\sharp$ is Lipschitz continuous on $[0, t_0]$, and it is of course constant for $t \ge T^M$. We claim that $p^\sharp$ is uniformly Lipschitz continuous also on the time interval $[t_0, T^M]$. Namely,

$$|p^\sharp(\tau) - p^\sharp(t)| \le (C_1 + C_2)\, |\tau - t|, \tag{3.16}$$

where C_1, C_2 are constants such that

$$\begin{aligned} \left|\frac{d}{ds}\Big([1-\theta(x(s))]e^{-(r+\lambda)(s-t)}\Big)\right| &\leq C_1 \qquad \text{for all} \quad 0<t<s, \\ \frac{e^{(r+\lambda)s}-1}{s} &\leq C_2 \qquad \text{for all } 0<s<\tfrac{M-M_0}{c_0}. \end{aligned} \tag{3.17}$$

In view of (3.7) such constants exist, and are independent of $x(\cdot)$.

To prove our claim, observe that the function

$$s \mapsto \phi^t(s) \doteq \exp\left\{-\int_t^s \rho(x(\zeta))\,d\zeta\right\} \tag{3.18}$$

is monotone decreasing and satisfies

$$\phi^t(t) = 1\,, \qquad \phi^t(T^M) = 0$$

For each $0<y<1$, call $s^t(y)$ the time such that

$$\phi^t(s^t(y)) = y.$$

Notice that, since the function $t\mapsto \rho(x(t))$ is increasing, for $t_0<t<\tau<T^M$ we have

$$s^\tau(y) - s^t(y) \leq \tau - t\,.$$

Using (3.15) we thus obtain the estimate

$$\begin{aligned} &|p^\sharp(\tau)-p^\sharp(t)| \\ =\ & \left|\int_0^1 \Big\{\big[1-\theta(x(s^\tau(y)))\big]e^{-(r+\lambda)(s^\tau(y)-\tau)} - \big[1-\theta(x(s^t(y)))\big]e^{-(r+\lambda)(s^t(y)-t)}\Big\}\,dy\right| \\ \leq\ & C_1\int_0^1 |s^\tau(y)-s^t(y)|\,dy \\ & + \left|\int_0^1 \Big\{\big[1-\theta(x(s^t(y)))\big]e^{-(r+\lambda)(s^t(y)-t)}(e^{(r+\lambda)(\tau-t)}-1)\Big\}\,dy\right| \\ \leq\ & C_1\,|\tau-t| + |e^{(r+\lambda)(\tau-t)}-1| \\ \leq\ & (C_1+C_2)\,|\tau-t|, \end{aligned}$$

proving (3.16).

To check $Z \subset Y$, we need to show that

$$\theta(M) \ \leq \ p^\sharp(t) \ \leq \ 1 \qquad\qquad \text{for all} \quad t \geq 0. \tag{3.19}$$

Since $\theta(M) \leq \theta(x(t)) \leq 1$, for any $t \leq T^M$ we obtain

$$\begin{aligned} 1 \ \geq \quad p^\sharp(t) \ &\geq \ 1 - \int_t^{T^M} \big[1-\theta(M)\big]\, \rho(x(s)) \exp\left\{ - \int_t^s \big[r+\lambda+\rho(x(\tau))\big]\, d\tau \right\} ds \\ &\qquad -\big[1-\theta(M)\big] \exp\left\{ - \int_t^{T^M} [r+\lambda+\rho(x(\tau))] d\tau \right\} \\ &\geq \ 1-(1-\theta(M)) \ = \ \theta(M). \end{aligned}$$

Since by definition $p^\sharp(t) = \theta(M)$ for all $t \geq T^M$, we conclude that $Z \subset Y$.

5. In this step we prove that Λ_2 is continuous on the range of Λ_1. Let $x \in \Lambda_1(Y)$ and $\varepsilon > 0$ be given. We claim that there exists $\delta > 0$ such that, if $\tilde{x} \in \Lambda_1(Y)$ and $\|x - \tilde{x}\| \leq \delta$, then the corresponding functions $p = \Lambda_2(x)$ and $\tilde{p} = \Lambda_2(\tilde{x})$ satisfy

$$e^{-\mu t}|p(t) - \tilde{p}(t)| \ \leq \ \varepsilon \qquad \text{for all } t \geq 0. \tag{3.20}$$

To prove the claim, we first observe that, since $p(t), \tilde{p}(t) \in [0,1]$, the inequality (3.20) certainly holds for all $t \geq T_\varepsilon \doteq \frac{1}{\mu} \ln \frac{1}{\varepsilon}$.

Call T^M and $\widetilde{T}^M \in [0, +\infty]$ the first times when $x(t) = M$ and $\tilde{x}(t) = M$, respectively.

As long as $t < T^M$, the price $p(t)$ satisfies the linear ODE (2.8). Therefore, choosing any $T < T^M$, we have the representation

$$p(t) \ = \ p(T) e^{-\int_t^T [r+\lambda+\rho(x(\tau))]\, d\tau} + \int_t^T e^{-\int_t^s [r+\lambda+\rho(x(\tau))]\, d\tau} [r+\lambda+\theta(x(s))\rho(x(s))]\, ds.$$

Using a similar representation for $\tilde{p}$, for $t \leq T < \min\{T^M, \widetilde{T}^M\}$ the difference $p(t) - \tilde{p}(t)$ can thus be expressed as

$$p(t) - \tilde{p}(t) \ = \ I_1(t) + I_2(t)\,, \tag{3.21}$$

where

$$I_1(t) \doteq p(T)\exp\left\{-\int_t^T [r+\lambda+\rho(x(\tau))]\,d\tau\right\} \\ -\tilde p(T)\exp\left\{-\int_t^T [r+\lambda+\rho(\tilde x(\tau))]\,d\tau\right\} \tag{3.22}$$

and

$$\begin{aligned} I_2(t) \doteq & \int_t^T \exp\left\{-\int_t^s [r+\lambda+\rho(x(\tau))]\,d\tau\right\}[r+\lambda+\theta(x(s))\rho(x(s))]\,ds \\ & -\int_t^T \exp\left\{-\int_t^s [r+\lambda+\rho(\tilde x(\tau))]\,d\tau\right\}[r+\lambda+\theta(\tilde x(s))\rho(\tilde x(s))]\,ds. \end{aligned} \tag{3.23}$$

Two cases will be considered.

Case 1: $T^M = +\infty$. We then choose $T > T_\varepsilon \doteq \frac{1}{\mu}\ln\frac{1}{\varepsilon}$ large enough so that

$$\exp\left\{-\int_{T_\varepsilon}^T (r+\lambda)\,d\tau\right\} < \frac{\varepsilon}{3}.$$

For all $t \in [0,T]$, this choice implies $|I_1(t)| \le \frac{\varepsilon}{2}$. Then we choose $\delta > 0$ so small that $\|x-\tilde x\| \le \delta$ implies $|I_2(t)| \le \frac{\varepsilon}{2}$ for all $t \in [0,T]$. This achieves (3.20).

Case 2: $T^M < +\infty$.

By step **4**, all functions $p^\sharp \in \Lambda_2(\Lambda_1(Y))$ are equicontinuous. Hence there exists $\delta_1 > 0$ such that $M - \delta_1 \ge M_0$ and

$$|t-t'| \le \delta_1 \qquad \Longrightarrow \qquad |p^\sharp(t) - p^\sharp(t')| \le \frac{\varepsilon}{3}. \tag{3.24}$$

Recalling (3.8), define

$$M_\varepsilon \doteq M - \frac{c_0\delta_1}{2}$$

and call T_ε the unique time such that $x(T_\varepsilon) = M_\varepsilon$.

By choosing $\delta > 0$ sufficiently small, the inequality $\|x-\tilde x\| \le \delta$ implies

$$|T^M - \widetilde T^M| < \frac{\delta_1}{2}, \qquad\qquad |x(T_\varepsilon) - \tilde x(T_\varepsilon)| \le \frac{c_0\delta_1}{2}.$$

Since $p(T^M) = \tilde{p}(\widetilde{T}^M) = \theta(M)$, we have the estimate

$$|p(T_\varepsilon) - \tilde{p}(T_\varepsilon)| \;\leq\; |p(T^M) - p(T_\varepsilon)| + |\tilde{p}(\widetilde{T}^M) - \tilde{p}(T_\varepsilon)| \;\leq\; \frac{\varepsilon}{3} + \frac{\varepsilon}{3}. \tag{3.25}$$

Indeed, the last inequality follows from (3.24) together with

$$|T^M - T_\varepsilon| \;\leq\; \frac{1}{c_0}(M - x(T_\varepsilon)) \;=\; \frac{1}{c_0}\frac{c_0\delta_1}{2},$$

$$|\widetilde{T}^M - T_\varepsilon| \;\leq\; \frac{1}{c_0}(M - \tilde{x}(T)) \;\leq\; \frac{|M - x(T_\varepsilon)|}{c_0} + \frac{|x(T_\varepsilon) - \tilde{x}(T_\varepsilon)|}{c_0} \;\leq\; \frac{\delta_1}{2} + \frac{\delta_1}{2}.$$

For every $t \geq T$, the same argument used in (3.25) yields

$$|p(t) - \tilde{p}(t)| \;\leq\; \frac{2\varepsilon}{3}.$$

On the other hand, to estimate the difference (3.21) for $t < T$, we set

$$E(t) \;\doteq \exp\left\{-\int_t^T [r + \lambda + \rho(x(\tau))]\, d\tau\right\}$$

and

$$\widetilde{E}(t) \;\doteq \exp\left\{-\int_t^T [r + \lambda + \rho(\tilde{x}(\tau))]\, d\tau\right\}.$$

By possibly shrinking the value of δ, for every $t \in [0, T]$ we can achieve

$$|I_2(t)| \;\leq\; \frac{\varepsilon}{6}, \qquad\qquad |E(t) - \widetilde{E}(t)| \;\leq\; \frac{\varepsilon}{6}. \tag{3.26}$$

This implies

$$\begin{aligned} |I_1(t)| \;&=\; \left|p(T)E(T) - \tilde{p}(T)\widetilde{E}(T)\right| \\ &\leq\; |p(T) - \tilde{p}(T)| \cdot E(T) + \tilde{p}(T) \cdot |E(T) - \widetilde{E}(T)| \;\leq\; \frac{2\varepsilon}{3} + \frac{\varepsilon}{6}. \end{aligned} \tag{3.27}$$

Using (3.26)-(3.27), we conclude that the right hand side of (3.21) is $\leq \varepsilon$, for all $t \in [0, T]$.

6. By the previous analysis, the map $p \mapsto p^\sharp \;\doteq \Lambda_2(\Lambda_1(p))$ is continuous from the closed convex set Y into itself. We now claim that this map is compact.

Indeed, let $(p_n)_{n\geq 1}$ be a sequence of functions in Y. By the equicontinuity of all functions $p^\sharp$, proved in step **4**, for every $\varepsilon > 0$ we can find a subsequence p_{n_j} such that $p^\sharp_{n_j} \doteq \Lambda_2(\Lambda_1(p_{n_j}))$ converges to some limit function $\bar{p}$ uniformly on bounded intervals. Since all p_n take values inside $[0,1]$, this implies the convergence in norm: $\|p^\sharp_{n_j} - \bar{p}\| \to 0$.

7. An application of Schauder's theorem yields the existence of a fixed point $\bar{p} = \Lambda_2(\Lambda_1(\bar{p}))$. We then consider the function $\bar{x} = \Lambda_1(\bar{p})$. By the definitions of Λ_1 and Λ_2 it follows that the map $t \mapsto (\bar{x}(t), \bar{p}(t))$ provides a solution to the equations (2.7), (2.9), (2.10). □

4 Existence of optimal solutions

In this section, given an initial size x_0 of the debt, we prove the existence of an optimal strategy that minimizes the expected cost to the borrower.

Theorem 2. *Under the assumptions* **(A1)–(A4)**, *the optimization problem* **(DMP)** *admits an optimal solution* (u, x, p), *which minimizes the expected cost (2.13).*

Proof. 1. For any initial data x_0, the trivial control $u(t) \equiv 0$ yields a total cost $J(u) \leq B$. Indeed, this cannot be worse than the cost of immediate bankruptcy. It thus suffices to prove the theorem assuming that

$$0 \;\leq\; m \;\doteq\; \inf_{(u,x,p)\in\mathcal{S}} \; J(u,x) \;<\; B\,. \tag{4.1}$$

Here the infimum is taken over the set $\mathcal{S}$ of all measurable controls $u : [0,\infty[\mapsto [0,1]$ and all solutions (x,p) of the system (2.7), (2.9), (2.10).

Consider a minimizing sequence $(u_n, x_n, p_n) \in \mathcal{S}$, so that

$$J(u_n, x_n) \;\to\; m \qquad \text{as} \qquad n \to \infty. \tag{4.2}$$

Define the corresponding functions γ_n as in (2.11), with x replaced by x_n. By the estimates derived in the proof of Theorem 1, the functions x_n, p_n are uniformly equicontinuous. By possibly taking a subsequence, we can thus assume the weak convergence $u_n \rightharpoonup u$ together with the convergence $x_n \to x$, $p_n \to p$ uniformly on bounded intervals $[0,T]$. Using the convexity of the cost function L and the fact that u enters linearly in the equations (2.9), we will prove that the triple of limit functions (u,x,p) is optimal.

2. We first consider the case where

$$\liminf_{n\to\infty} \; T_n^M \;=\; \infty. \tag{4.3}$$

By possibly taking a subsequence, we can assume that

$$(x_n,p_n,\gamma_n)(t) \;\to\; (x,p,\gamma)(t) \qquad\qquad \text{as} \;\; n\to\infty, \tag{4.4}$$

uniformly on every bounded interval $[0,\,T]$.

Since u enters linearly in the equations (2.9), by (4.4) and the weak convergence $u_n \rightharpoonup u$ it is clear that (2.7) and (2.9) are satisfied, together with the initial condition (2.10).

If $J(u,x) > m$, recalling (2.12) then there exists a bounded interval $[0,T]$ such that

$$B + \int_0^T \gamma(t)\Big\{L(u(t)) - rB\Big\}\,dt \;>\; m + e^{-rT}B\,. \tag{4.5}$$

Recalling that the cost function L is non-negative and convex, and observing that $\gamma(t) \le e^{-rt}$, we obtain

$$\begin{aligned} m \;&<\; B + \int_0^T \gamma(t)\Big\{L(u(t)) - rB\Big\}\,dt - e^{-rT}B \\ &\le\; B + \liminf_{n\to\infty} \int_0^T \gamma_n(t)\Big\{L(u_n(t)) - rB\Big\}\,dt - e^{-rT}B \\ &\le\; B + \liminf_{n\to\infty} \int_0^{T_n} \gamma_n(t)\Big\{L(u_n(t)) - rB\Big\}\,dt \;\le\; m\,. \end{aligned}$$

This contradiction shows that $J(u,x) \le m$, proving the optimality of the solution (u,x,p).

3. Next, consider the case where

$$T^M \doteq \liminf_{n\to\infty} T_n^M < \infty. \tag{4.6}$$

By possibly taking a subsequence, we can assume that $T_n^M \to T^M$, as well as the convergence (4.4) uniformly on every subinterval of the form $[0, T^M - \delta]$, with $\delta > 0$. As before, one checks that the limit functions (u, x, p) satisfy the conditions (2.7), (2.9), and (2.10).

If $J(u,x) > m$, then we can choose $\varepsilon, \delta > 0$ small enough so that

$$B + \int_0^{T^M-\delta} \gamma(t) \Big\{ L(u(t)) - rB \Big\}\, dt > m + \varepsilon\,, \qquad \delta r B < \varepsilon\,. \tag{4.7}$$

Using again the convexity of the cost function L and recalling that $\gamma_n(t) \leq 1$, by the two inequalities in (4.7) we obtain

$$\begin{aligned} m &= B + \liminf_{n\to\infty} \int_0^{T_n^M} \gamma_n(t) \Big(L(u_n(t)) - rB \Big)\, dt \\ &\geq B + \int_0^{T^M-\delta} \gamma(t) \Big\{ L(u(t)) - rB \Big\}\, dt - \delta r B > m\,. \end{aligned} \tag{4.8}$$

This contradiction shows that $J(u,x) \leq m$, completing the proof. □

5 Necessary conditions for optimality

Given an initial value of the debt $x_0 \in]0, M[$, let $t \mapsto u^*(t)$ be an optimal control, and let $t \mapsto (x^*(t), p^*(t))$ be a corresponding optimal solution of **(DMP)**. We seek necessary conditions for optimality. One easy case is covered by the next lemma. We recall that R_0 is the constant in (2.14).

Lemma 1. *Assume that $x^*(t) \leq R_0$ for all $t \geq \tau$. Then the optimal trajectory is constant:*

$$x^*(t) = x^*(\tau)\,, \qquad p^*(t) = 1, \qquad u^*(t) = rx^*(\tau) \qquad \textit{for all } t \geq \tau. \tag{5.1}$$

Proof. The assumption yields $\rho(x^*(t)) = 0$ and hence $p^*(t) = 1$ for all $t \geq \tau$. The control u^* thus provides the minimum cost for the optimal control problem

$$\text{minimize: } \int_\tau^\infty e^{-rt} L(u(t))\, dt,$$

with dynamics and state constraint

$$\dot{x} \;=\; rx - u, \qquad x(\tau) \;=\; \bar{x}, \qquad\qquad x(t) \in [0, R_0] \quad \text{for all} \quad t \geq \tau.$$

Here $\bar{x} = u^*(\tau)$. One readily checks that the value function for this problem is

$$V(\bar{x}) \;=\; \frac{1}{r} L(r\bar{x}),$$

and the unique optimal control is $u^*(t) \equiv r\bar{x}$, keeping the debt constant in time. This achieves the proof. □

Corollary 1. *If $x^*(t) = x_0 < R_0$ for some all $t \geq \tau$, then $x^*(t) = x_0$ for all $t \geq 0$.*

In the remainder of this section, we shall study optimal trajectories taking values in the interval $[R_0, M]$, where the bankruptcy risk is positive. Consider the function

$$\gamma^*(t) \;=\; e^{-rt} \cdot \exp\left\{ -\int_0^t \rho(x^*(s))\, ds \right\}, \tag{5.2}$$

and the time

$$T^M \;\doteq\; \inf\Big\{ t > 0\,; \quad x^*(t) = M \Big\} \;\in\; \mathbb{R} \cup \{+\infty\}\,. \tag{5.3}$$

For any $T < T^M$, a standard argument (see for example [42]) shows that the control u^* and the functions (x^*, p^*, γ^*) provide a solution to an optimization problem on the subinterval $[0, T]$, namely:

(OT) *Minimize the cost functional on the interval* $[0, T]$:

$$J(\gamma, u) \;\doteq\; \int_0^T f_0\big(u(t), \gamma(t)\big)\, dt\,, \qquad\qquad f_0(u, \gamma) \;\doteq\; [L(u) - rB]\gamma\,, \tag{5.4}$$

subject to

$$\begin{cases} \dot{x} &= f_1(u,x,p) &\doteq -\lambda x + \dfrac{(\lambda+r)x-u}{p}\,, \\ \dot{p} &= f_2(x,p) &\doteq \big[r+\lambda+\rho(x)\big]p - \big[r+\lambda+\theta(x)\rho(x)\big]\,, \\ \dot{\gamma} &\doteq f_3(x,\gamma) &\doteq -(r+\rho(x))\gamma\,, \end{cases} \tag{5.5}$$

with constraints and boundary data

$$u(t) \in [0,1]\,, \tag{5.6}$$

$$\begin{cases} x(0) &= x_0, \\ \gamma(0) &= 1, \end{cases} \tag{5.7}$$

$$\begin{cases} x(T) &= x^*(T), \\ p(T) &= p^*(T), \\ \gamma(T) &= \gamma^*(T). \end{cases} \tag{5.8}$$

Introducing the adjoint variables $(x^\dagger, p^\dagger, \gamma^\dagger)$ and a constant $\mu \geq 0$, we consider the Hamiltonian function

$$H(u,x,p,\gamma,x^\dagger,p^\dagger,\gamma^\dagger,\mu) \doteq \mu\, f_0(u,\gamma) + x^\dagger f_1(u,x,p) + p^\dagger f_2(x,p) + \gamma^\dagger f_3(x,\gamma). \tag{5.9}$$

An application of the Pontryagin Maximum Principle (PMP) to the above problem yields the existence of a constant $\mu \in \{0,1\}$ and functions $(x^\dagger(t), p^\dagger(t), \gamma^\dagger(t))$:

$[0,T] \to \mathbb{R}^3$ which satisfy the adjoint linear system

$$\begin{pmatrix} \dot{x}^\dagger \\ \dot{p}^\dagger \\ \dot{\gamma}^\dagger \end{pmatrix} = -\begin{pmatrix} \partial H/\partial x \\ \partial H/\partial p \\ \partial H/\partial \gamma \end{pmatrix}$$

$$= \begin{pmatrix} \lambda - \frac{\lambda+r}{p} & \theta'(x)\rho(x) - \rho'(x)(p-\theta(x)) & \rho'(x)\gamma \\ \frac{(\lambda+r)x-u}{(p)^2} & -(\lambda+r+\rho(x)) & 0 \\ 0 & 0 & r+\rho(x) \end{pmatrix} \begin{pmatrix} x^\dagger \\ p^\dagger \\ \gamma^\dagger \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 0 \\ L(u) - rB \end{pmatrix}, \tag{5.10}$$

together with the initial condition

$$p^\dagger(0) = 0, \tag{5.11}$$

and the non-degeneracy condition

$$(x^\dagger(0), p^\dagger(0), \gamma^\dagger(0), \mu) \neq (0,0,0,0). \tag{5.12}$$

Moreover, for a.e. $t \in [0,T]$, one has

$$u^*(t) = \arg\min_{\omega\in[0,1]} \left\{ -\frac{x^\dagger(t)}{p(t)}\omega + \mu\,\gamma(t) L(\omega) \right\}. \tag{5.13}$$

From the pointwise minimality condition (5.13) one can recover the optimal control u as a function of $x^\dagger$, p, and γ. More precisely, define the function $U^\sharp(x^\dagger, p, \gamma)$ implicitly by the relations

$$\begin{cases} L'\big(U^\sharp(x^\dagger,p,\gamma)\big) = \dfrac{x^\dagger}{p\,\gamma} & \text{if } \dfrac{x^\dagger}{p\,\gamma} > L'(0), \\ U^\sharp(x^\dagger,p,\gamma) = 0 & \text{if } \dfrac{x^\dagger}{p\,\gamma} \leq L'(0). \end{cases} \tag{5.14}$$

Inserting $u = U^\sharp(x^\dagger, p, \gamma)$ in (5.5) and (5.10), one obtains a system of 6 scalar ODEs for the variables $x, p, \gamma, x^\dagger, p^\dagger, \gamma^\dagger$, with the three boundary conditions in (5.7) and (5.11).

The goal of the following analysis is two-fold: (i) show that the optimization problem is "normal", hence in the (PMP) one can take $\mu = 1$, and (ii) determine three additional asymptotic conditions as $t \to +\infty$, in order to compute the optimal solution for all $t \geq 0$.

Lemma 2. *For every solution of (5.5)–(5.7) on an interval* $[0,T]$*, with* $x(t) \in [0,M]$ *and terminal data* $p(T) \in [\theta(M),\ 1]$*, one has*

$$p(t) \in [\theta(M), 1], \qquad \gamma(t) \in [0,1]\,. \tag{5.15}$$

Proof. The first inclusion follows from the implications

$$\begin{aligned} p = \theta(M) \quad &\Longrightarrow \qquad \dot{p} \ = \ (r+\lambda)(\theta(M)-1) + \rho(x)(\theta(M)-\theta(x)) \ \leq \ 0, \\ p = 1 \quad &\Longrightarrow \qquad \dot{p} \ = \ \rho(x)(1-\theta(x)) \ \geq \ 0\,. \end{aligned}$$

This shows that the interval $[\theta(M), 1]$ is backward invariant for the second ODE in (5.5).

Since $r + \rho(x) > 0$, the second inclusion in (5.15) is obvious. □

We will study the equations determined by the PMP in two main cases.

CASE 1: $T^M \doteq \inf\Big\{t > 0\,;\ \ x^*(t) = M\Big\} < +\infty\,.$ In this case $(u^*, x^*, p^*, \gamma^*)$ is optimal for the problem (5.4)–(5.7), with terminal constraints

$$\begin{cases} x(T) \ = \ M, \\ p(T) \ = \ \theta(M), \end{cases} \tag{5.16}$$

while T is regarded as a free terminal time. The PMP (see for example [22, 30]) now yields the existence of a constant $\mu \in \{0,1\}$ and an adjoint vector $(x^\dagger, p^\dagger, \gamma^\dagger)$ satisfying (5.10), so that the six boundary conditions in (5.7), (5.16), and

$$\begin{cases} p^\dagger(0) \ = \ 0, \\ \gamma^\dagger(T) \ = \ 0, \end{cases} \tag{5.17}$$

hold, together with the optimality condition (5.13). To show that $\mu = 1$, assume on the contrary that $\mu = 0$. Then we must have $x^\dagger(t) \equiv 0$. In turn, (5.10) and the boundary conditions (5.17) imply

$$p^\dagger(t) \;=\; \exp\left\{-\int_0^t \big(\lambda + r + \rho(x^*(t))\big)\,dt\right\} p^\dagger(0) \;=\; 0, \tag{5.18}$$

$$\gamma^\dagger(t) \;=\; \exp\left\{-\int_t^T \big(r + \rho(x^*(t))\big)\,dt\right\} \gamma^\dagger(T) \;=\; 0, \tag{5.19}$$

contradicting the non-degeneracy assumption.

Since the time T is free, we need an additional boundary condition. This can be obtained by the vanishing of the Hamiltonian function:

$$\begin{aligned} 0 \;&=\; H\big(u^*(t), x^*(t), p^*(t), \gamma^*(t), x^\dagger(t), p^\dagger(t), \gamma^\dagger(t)\big) \\ &=\; [L(u^*) - rB]\gamma^* + x^\dagger \left[-\lambda x^* + \frac{(\lambda + r)x^* - u^*}{p^*}\right] \\ &\quad + p^\dagger \Big[(r + \lambda + \rho(x^*))p^* - (r + \lambda + \theta(x^*)\rho(x^*))\Big] - \gamma^\dagger (r + \rho(x^*))\gamma^*. \end{aligned} \tag{5.20}$$

At time $t = 0$ we obtain

$$[L(u^*(0)) - rB] + x^\dagger(0)\left[-\lambda x_0 + \frac{(\lambda + r)x_0 - u^*(0)}{p^*(0)}\right] - r\gamma^\dagger(0) \;=\; 0. \tag{5.21}$$

By (5.13), this yields

$$\min_{\omega\in[0,1]} \left\{-\frac{x^\dagger(0)}{p^*(0)}\omega + L(\omega)\right\} \;=\; rB + r\gamma^\dagger(0) + x^\dagger(0)\lambda x_0 - \frac{(\lambda + r)x_0}{p^*(0)}\,. \tag{5.22}$$

Notice that in this case the quantity in (5.22) can be negative, without leading to any contradiction.

CASE 2: The optimal control u^* and a corresponding optimal trajectory (x^*, p^*) are defined for all $t \geq 0$. Moreover, we assume that, for some constants κ, τ,

$$0 \;<\; u^*(t) \;\leq\; \kappa \;<\; 1, \qquad\qquad \text{for all } t \geq \tau, \tag{5.23}$$

and that there exists the limit

$$x_\infty \;\doteq\; \lim_{t\to+\infty} x^*(t) \;>\; R_0. \tag{5.24}$$

According to the assumption **(A2)**, by (2.14) this implies $\rho(x_\infty) > 0$.

We claim that, in the necessary conditions (5.10)–(5.13), one can always take $\mu = 1$. Indeed, if $\mu = 0$, the minimum (5.13) can be attained only if $x^\dagger \equiv 0$. This identity, together with $p^\dagger(0) = 0$, implies $p^\dagger(t) \equiv 0$. Therefore, the non-degeneracy of the adjoint vector $(x^\dagger, p^\dagger, \gamma^\dagger)$ implies $\gamma^\dagger(t) \neq 0$ for all $t \geq 0$. However, since $\gamma(t) \neq 0$, the first equation in (5.10) yields $\rho'(x^*(t)) = 0$, and hence $x^*(t) \in [0, R_0]$ for all $t \geq 0$. This remaining case has already been covered in Lemma 1, showing that the optimal trajectory $x^*(\cdot)$ is constant. Our claim is thus proved.

Since the optimality condition determines the pointwise control value

$$u \;=\; U^\sharp(x^\dagger, p, \gamma) \;=\; (L')^{-1}(x^\dagger/p\gamma), \tag{5.25}$$

it is convenient to replace $x^\dagger, p^\dagger$ by the ratios $\tilde{x}^\dagger \doteq \frac{x^\dagger}{\gamma}$ and $\tilde{p}^\dagger \doteq \frac{p^\dagger}{\gamma}$. Using these rescaled variables, from the PMP we obtain a system of five ODEs:

$$\begin{cases} \dot{x} \;=\; -\lambda x + \dfrac{(\lambda + r)x - (L')^{-1}(\tilde{x}^\dagger/p)}{p}\,, \\[2mm] \dot{p} \;=\; \big[p - \theta(x)\big]\,\rho(x) + (r + \lambda)(p - 1)\,, \\[2mm] \dot{\tilde{x}}^\dagger \;=\; \left(\lambda + r + \rho(x) - \dfrac{\lambda + r}{p}\right)\tilde{x}^\dagger + \Big[\theta'(x)\rho(x) - \rho'(x)(p - \theta(x))\Big]\tilde{p}^\dagger + \rho'(x)\gamma^\dagger, \\[2mm] \dot{\tilde{p}}^\dagger \;=\; \dfrac{(\lambda + r)x - (L')^{-1}(\tilde{x}^\dagger/p)}{p^2}\tilde{x}^\dagger - \lambda\tilde{p}^\dagger, \\[2mm] \dot{\gamma}^\dagger \;=\; (r + \rho(x))\gamma^\dagger + L\Big((L')^{-1}(\tilde{x}^\dagger/p)\Big) - rB\,. \end{cases} \tag{5.26}$$

The initial data are

$$\begin{cases} x(0) \;=\; x_0\,, \\ \tilde{p}^\dagger(0) \;=\; 0. \end{cases} \tag{5.27}$$

Notice that in (5.26) the evolution equation for γ has been omitted, because the variable γ is not present in any of the equations for the remaining five variables.

By (5.24) and (2.8) it follows

$$\begin{aligned}\lim_{t\to+\infty} p(t) \;=\; p_\infty \;&\doteq\; 1-[1-\theta(x_\infty)]\rho(x_\infty)\int_0^\infty e^{-s[r+\lambda+\rho(x_\infty)]}\,ds\\ &=\; \frac{r+\lambda+\theta(x_\infty)\rho(x_\infty)}{r+\lambda+\rho(x_\infty)}\,.\end{aligned}\tag{5.28}$$

Moreover, by (5.23) and (5.25), the variable $\tilde x^\dagger$ remains uniformly bounded for $t\in[\tau,+\infty[\,$. In turn, the fifth equation in (5.26) implies that $\tilde p^\dagger$ remains bounded.

Finally, we claim that also the variable $\gamma^\dagger$ is uniformly bounded on $[\tau.+\infty[\,$. Otherwise, the last equation in (5.26) would imply that $\gamma^\dagger$ grows at an exponential rate, say

$$\big|\gamma^\dagger(t)\big| \;>\; \gamma_0\, e^{rt/2}. \tag{5.29}$$

Using (5.29) in the fourth equation, since $\rho'(x_\infty)>0$, one obtains $|\tilde x^\dagger(t)|\to+\infty$, reaching a contradiction.

The above arguments show that all variables $\tilde x^\dagger,\tilde p^\dagger,\gamma^\dagger$ remain uniformly bounded on $[\tau,+\infty[\,$. Since they satisfy a system of ODEs with Lipschitz continuous right hand side, we conclude that they are uniformly bounded on the entire domain $[0,+\infty[\,$.

In particular, the fourth equation in (5.26) implies that $\tilde x^\dagger$ is uniformly Lipschitz continuous. Hence the optimal control $u^*(t)=(L')^{-1}\big(\tilde x^\dagger(t)/p(t)\big)$ is Lipschitz continuous as well.

By the first equation in (5.26), the limits (5.24) and (5.28) and the Lipschitz continuity of the control function $u^*(\cdot)$ imply

$$\lim_{t\to+\infty} u^*(t) \;=\; u_\infty \;\doteq\; (\lambda+r-\lambda p_\infty)x_\infty\,. \tag{5.30}$$

From the optimality condition (5.13), if $u^*(t)>0$ it follows

$$\tilde x^\dagger(t) \;=\; p(t)L'(u(t)). \tag{5.31}$$

Letting $t\to+\infty$, this yields

$$\lim_{t\to+\infty}\tilde x^\dagger(t) \;=\; p_\infty L'(u_\infty). \tag{5.32}$$

The last two equations in (5.26) imply that the dual variables $\tilde{p}^\dagger(t)$ and $\gamma^\dagger$ also have limits as $t \to +\infty$. Indeed

$$\lim_{t\to+\infty} \tilde{p}^\dagger(t) \;=\; \tilde{p}^\dagger_\infty \;\doteq\; \frac{(\lambda+r)x_\infty - u_\infty}{\lambda p_\infty^2} L'(u_\infty)p_\infty \;=\; x_\infty L'(u_\infty). \tag{5.33}$$

$$\lim_{t\to+\infty} \gamma^\dagger(t) \;=\; \gamma^\dagger_\infty \;=\; \frac{rB - L(u_\infty)}{r+\rho(x_\infty)}. \tag{5.34}$$

By the previous analysis, from the asymptotic size of the debt x_∞ one can determine the limit values p_∞, u_∞ of the discounted bond price and of the control. We now show that the limit x_∞ cannot be arbitrary.

Observing that the right hand side of the fourth equation in (5.26) is uniformly Lipschitz continuous, and moreover $\tilde{x}(t) \to \tilde{x}_\infty$, we conclude that $\dot{\tilde{x}}^\dagger(t) \to 0$. This yields the identity

$$\left(\lambda + r + \rho(x_\infty) - \frac{\lambda+r}{p_\infty}\right)\tilde{x}^\dagger_\infty + \Big[\theta'(x_\infty)\rho(x_\infty) - \rho'(x_\infty)(p_\infty - \theta(x_\infty))\Big]\tilde{p}^\dagger_\infty + \rho'(x_\infty)\gamma^\dagger_\infty\,, \tag{5.35}$$

Using the identities (5.32)–(5.34), we eventually obtain the additional equation

$$\begin{aligned} L'(u_\infty)\Big[\rho(x_\infty)\theta(x_\infty) \;&+\; x_\infty\big[\theta'(x_\infty)\rho(x_\infty) - \rho'(x_\infty)(p_\infty - \theta(x_\infty))\big]\Big] \\ &+\; \rho'(x_\infty)\frac{rB - L(u_\infty)}{r+\rho(x_\infty)} \;=\; 0. \end{aligned}$$

Since $x_\infty\theta'(x_\infty) = -R_0/x_\infty = -\theta(x_\infty)$, the above equation can be written in the simpler form

$$-x_\infty L'(u_\infty)(p_\infty - \theta(x_\infty)) + \frac{rB - L(u_\infty)}{r+\rho(x_\infty)} \;=\; 0. \tag{5.36}$$

Summarizing the above analysis we obtain

Theorem 3. *Under the assumptions* **(A1)–(A4)**, *let* (u^*, x^*, p^*) *be an optimal solution to the debt management problem* **(DMP)**, *defined for all* $t \geq 0$. *Assume that, for some constants* κ, τ, *the conditions (5.23)-(5.24) hold. Then there exists adjoint variables* $\tilde{x}^\dagger, \tilde{p}^\dagger, \gamma^\dagger$ *satisfying the system (5.26), the initial conditions (5.27),* *and the asymptotic limits*

$$\Big(x^*, p^*, \tilde{x}^\dagger, \tilde{p}^\dagger, \gamma^\dagger\Big)(t) \ \to\ \left(x_\infty\,,\ p_\infty\,,\ L'(u_\infty)p_\infty\,,\ L'(u_\infty)x_\infty\,,\ \frac{rB - L(u_\infty)}{r+\rho(x_\infty)}\right), \tag{5.37}$$

as $t \to +\infty$, where

$$u_\infty = \lim_{t\to+\infty} u^*(t) = (\lambda + r - \lambda p_\infty)x_\infty\,, \qquad p_\infty = \lim_{t\to+\infty} p^*(t) = \frac{r+\lambda+\theta(x_\infty)\rho(x_\infty)}{r+\lambda+\rho(x_\infty)}\,. \tag{5.38}$$

Moreover, the limit value x_∞ satisfies the identity (5.36).

Proof. Consider a sequence of times $T_n \to +\infty$. For every $n \geq 1$, taking $T = T_n$, the functions u^*, x^*, p^* provide a solution to the optimization problem (5.4)–(5.8) on the time interval $[0, T_n]$. By the PMP, there exist an adjoint vector $(\tilde{x}_n^\dagger, \tilde{p}_n^\dagger, \gamma_n^\dagger)$ satisfying the system (5.26) with initial data (5.27). Thanks to the assumption (5.23), by the previous analysis this implies that on the domain $[\tau, T_n]$ all variables $\tilde{x}_n^\dagger, \tilde{p}_n^\dagger, \gamma_n^\dagger$ satisfy a uniform bound, independent of n. In turn, the equations (5.26) imply that all these functions are uniformly bounded and uniformly Lipschitz continuous on the entire half line $[0, +\infty[$. By Ascoli's theorem, we can thus extract a subsequence which converges uniformly on compact sets:

$$(\tilde{x}_n^\dagger, \tilde{p}_n^\dagger, \gamma_n^\dagger)(t) \ \to\ (\tilde{x}^\dagger, \tilde{p}^\dagger, \gamma^\dagger)(t).$$

It is now clear that the limit functions satisfy the last three equations in (5.26), and

$$u^*(t) \ =\ (L')^{-1}\left(\frac{\tilde{x}_n^\dagger(t)}{\tilde{p}_n(t)}\right) \ =\ (L')^{-1}\left(\frac{\tilde{x}^\dagger(t)}{\tilde{p}(t)}\right),$$

for all $n \geq 1$, $t \geq 0$.

By the previous analysis, as $t \to +\infty$ the limits (5.37)-(5.38) hold, together with the identity (5.36). □

Remark 2. The necessary conditions given in Theorem 2 consist of a system of ODEs for the five variables $(x, p, \tilde{x}^\dagger, \tilde{p}^\dagger, \gamma^\dagger)$, together with the two initial data for $x, \tilde{p}$ in (5.27) and the five equations in (5.36)–(5.38) for the asymptotic limits of these variables as $t \to +\infty$. At first sight, the problem may appear to be overdetermined (7 boundary values for 5 variables). However, this is the correct number of boundary conditions provided that the linearization of the system (5.26) around the asymptotic limit

$$\left(x_\infty\,,\ p_\infty\,,\ L'(u_\infty)p_\infty\,,\ L'(u_\infty)x_\infty\,,\ \frac{rB - L(u_\infty)}{r+\rho(x_\infty)}\right) \tag{5.39}$$

has a 3-dimensional unstable subspace.

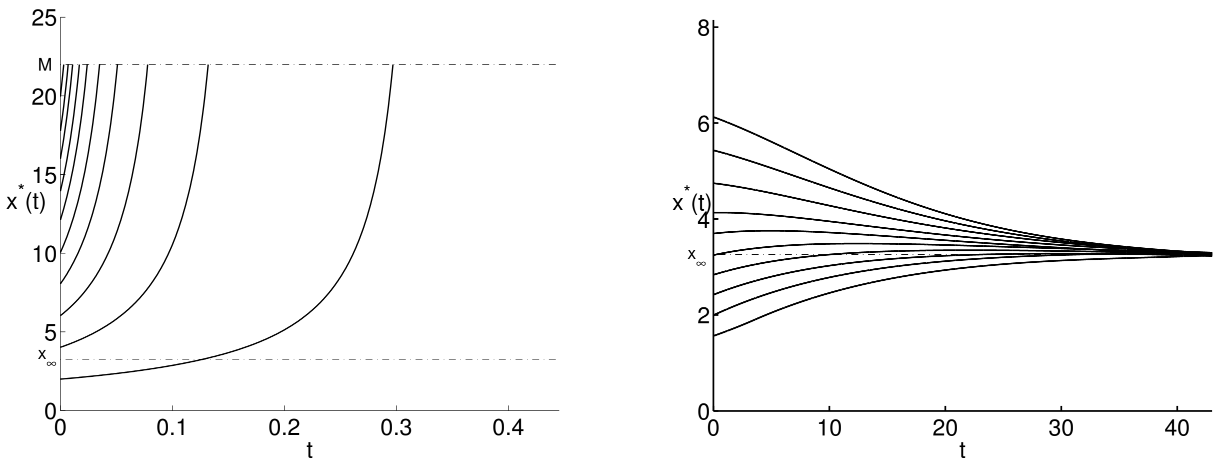

Figure 3.1. Left: solutions to (5.26) which reach the bankruptcy level M in finite time. Right: solutions which approach the steady state x_∞ as $t \to +\infty$.

6 An example

Consider the debt management problem **(DMP)**, taking

$$r = 0.05, \quad \lambda = 0.1, \quad R_0 = 0.1, \quad M = 22, \quad B = 10,$$

$$L(u) = \frac{u}{1-u}, \qquad \theta(x) = \min\left\{1, \frac{R_0}{x}\right\},$$

$$\rho(x) \ = \ \begin{cases} \ln\dfrac{M-R_0}{M-x} + \dfrac{R_0 - x}{M-R_0} & \text{if} \quad x \in [R_0, M[\,, \\ \qquad\qquad 0 & \text{if} \quad x \in [0, R_0]. \end{cases}$$

In this case, a numerical simulation yields the asymptotic values

$$x_\infty \approx 3.2598, \qquad p_\infty \approx 0.9308, \qquad u_\infty \approx 0.1855,$$

$$\tilde{x}^\dagger_\infty \approx 1.4032, \qquad \tilde{p}^\dagger_\infty \approx 4.9142, \qquad \gamma^\dagger_\infty \approx 4.4234.$$

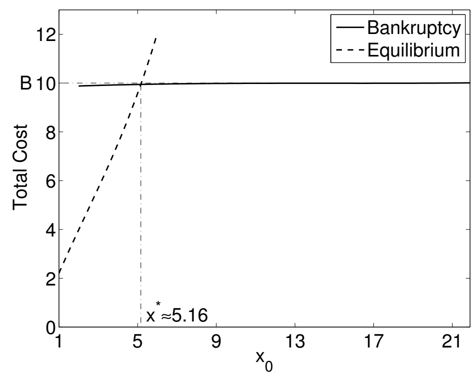

Figure 3.2. A comparison of the expected total cost of solutions to (5.26), depending on the initial value x_0 of the debt. The dashed line yields the cost of trajectories approaching a steady state x_∞ as $t \to +\infty$. The solid line yields the cost of trajectories reaching bankruptcy in finite time. Notice that this cost is just slightly smaller than the bankruptcy cost $B = 10$.

Linearizing the system (5.26) around these values, we obtain a 5×5 matrix whose eigenvalues are approximately computed as

$$\eta_1 \approx 0.1187, \qquad \eta_2 \approx 0.1150, \qquad \eta_3 \approx 0.1040, \qquad \eta_4, \eta_5 \approx -0.0765 \pm 0.0361\sqrt{-1}.$$

As expected, exactly three of these eigenvalues have positive real part.

Figure 3.1 shows some numerical results. The plots refer to the x-component of solutions to the system of equations (5.26), satisfied by optimal trajectories. Solutions where the debt $x(t)$ reaches the bankruptcy level M in finite time are shown on the left. Solutions where the debt $x(t)$ approaches asymptotically the steady state x_∞ are shown on the right. The total expected costs of these solutions are plotted in Fig. 3.2. When the initial size x_0 of the debt is small, i.e. $x_0 \leq x^* \approx 5.16$, strategies approaching the steady state x_∞ yield a lower cost. On the other hand, when the initial debt is large, i.e. $x_0 \geq x^*$, it is convenient to reach bankruptcy in finite time.

Chapter 4

Feedback strategies via vanishing viscosity limit

1 Introduction

This chapter is concerned with a Hamilton-Jacobi equation with discontinuous coefficients

$$(r + \rho(x))V \;=\; H(x, p(x), V'), \tag{1.1}$$

together with a family of smooth viscous approximations

$$(r + \rho(x))V_n \;=\; H(x, p_n(x), V_n') + \frac{\sigma_n^2 x^2}{2} V_n''. \tag{1.2}$$

The main motivation comes from a problem of optimal debt management, in infinite time horizon, with exponential discount and in the presence of a bankruptcy risk. As in [14, 19, 20, 51], this is modeled as a noncooperative game between a borrower and a pool of risk-neutral lenders. Here the independent variable x is the debt-to-income ratio, V is the value function for the borrower (i.e., his expected cost, under optimal play) while the function $\rho(x) \geq 0$ accounts for an instantaneous bankruptcy risk, which increases with the size of the debt. A distinguished feature of this model is that the discounted bond price $p(\cdot)$ is is not given a priori, but depends on the entire future evolution of the system, determined by the feedback control $u = u(x)$. Indeed, if the borrower chooses to keep the debt at a high level, then the lenders will buy the bonds at a deeply discounted price $p(x) < 1$, to compensate for the risk of losing their capital. The scalar equation (1.1) must thus be supplemented by an additional ODE for the function p. In particular, at points where the feedback control $u(\cdot)$ is discontinuous, also $p(\cdot)$ can have a jump.

In the first part of this chapter we construct a solution to the second order equation (1.2).This can be interpreted as a Nash equilibrium solution to a differential game with stochastic dynamics. In the final sections, we study the limits $V_n \to V$ and $p_n \to p$ as the diffusion coefficient $\sigma_n \to 0$. Under suitable assumptions, we show that this limit yields a solution to (1.1) and determines an equilibrium solution to a differential game with deterministic dynamics.

The main technical difficulties in the analysis stem from the fact that, while all functions p_n are smooth, the limit p may well be discontinuous. Hamilton-Jacobi equations with discontinuous hamiltonian function have attracted much interest in recent years [11, 9, 26, 13, 34, 39, 40, 54]. The key difference in our equations is that, since we are modeling a game, the hamiltonian function for the borrower is not given a priori, but is determined by the optimal reply adopted by the lenders. In turn, this reply depends on the feedback control chosen by the borrower.

The remainder of the chapter is organized as follows. In Section 2 we explain how the equations (1.1) arise from a problem of optimal debt management, modeled as a game between a borrower and a pool of risk-neutral lenders.

In Section 3 we prove the existence of solutions to the second order boundary value problem, for $\sigma > 0$. In the case where the bankruptcy risk is

$$\rho(x) \;=\; \begin{cases} 0 & \text{if } \; x < M, \\ +\infty & \text{if } \; x = M, \end{cases}$$

this result was proved in [19]. Here we consider more general functions $\rho(\cdot)$, using similar ideas. The main new contribution of the present chapter is the analysis of the limit $\sigma \to 0$, studied in Section 4. Our main result shows the existence of a limit, for a suitable subsequence $\sigma_n \to 0$. Under a "generic" assumption on the limit functions $V(\cdot)$ and $p(\cdot)$, we prove that this limit yields a Nash equilibrium solution to a noncooperative game, where the total debt follows a deterministic evolution equation. The analysis relies on classical techniques from the theory of viscosity solutions [12, 10, 22, 31], and a number of additional arguments to handle this specific problem.

2 Derivation of stochastic models

Assume that the yearly income $Y(t)$ of the borrower grows with rate μ and is subject to a stochastic perturbation:

$$dY(t) \;=\; \mu Y(t)\,dt + \sigma Y(t)\,dW. \tag{2.1}$$

Here W denotes standard Brownian motion. Calling $U(t)$ the repayment rate, the nominal value of the outstanding debt evolves according to the ODE

$$\dot{X}(t) \;=\; -\lambda X(t) + \frac{(\lambda+r)X(t) - U(t)}{p(t)}. \tag{2.2}$$

We define the debt-to-income ratio $x \doteq X/Y$, and set $u \doteq U/Y$. By Ito's formula, (2.1) and (2.2) yield the stochastic evolution equation

$$dx \;=\; \left[(\sigma^2-\lambda-\mu)x(t) + \frac{(\lambda+r)x(t)-u(t)}{p(t)}\right] dt + \sigma x(t)\,dW\,, \qquad x(0) \;=\; y\,. \tag{2.3}$$

We are mainly interested in controls in feedback form:

$$u \;=\; u^*(x) \quad \text{for} \quad x \in [0, M[\,.$$

Motivated by the optimality conditions, it is convenient to define

$$u^\sharp(p,\xi) \;\doteq\; \arg\min_{\omega\in[0,1]} \left\{L(\omega) - \frac{\xi}{p}\omega\right\} \;=\; \begin{cases} 0 & \text{if} \quad s \le L'(0), \\ (L')^{-1}\left(\dfrac{\xi}{p}\right) & \text{if} \quad \dfrac{\xi}{p} > L'(0). \end{cases} \tag{2.4}$$

$$\xi^\sharp(x,p) \;\doteq\; \operatorname*{argmax}_{\xi\ge 0} \; H(x,p,\xi), \tag{2.5}$$

Definition 1. *Consider a value function $V(\cdot)$, a feedback control $u^*(\cdot)$, and a price function $p(\cdot)$, defined for $x \in [0, M[$. We say that the triple (V, p, u^*) yields a Nash equilibrium solution to the debt management problem if the following holds.*

(i) Given the price $p = p(x)$*, one has that* $V(\cdot)$ *is the value function and* $u^*(x) = u^\sharp(p(x), V'(x))$ *is the optimal feedback control, in connection with the stochastic control problem*

$$\textit{minimize:}\ \ E\left[\int_0^{T_b} e^{-rt}L(u(t))\,dt + B\,e^{-rT_b}\right]$$

subject to

$$dx \;=\; \left[(\sigma^2-\lambda-\mu)x + \frac{(\lambda+r)x-u}{p(x)}\right]dt + \sigma x\,dW\,. \tag{2.6}$$

Here the distribution of the random time T_b *is determined by (2.4).*

(ii) Given the feedback control $u = u^*(x)$ *in (2.6), for every* $x_0 \in [0, M[$ *one has*

$$p(x_0) \;=\; E\left[\int_0^{T_b} e^{-rt}(\lambda+r)e^{-\lambda t}\,dt + e^{-rT_b}\theta(x(T_b))e^{-\lambda T_b}\right]_{x(0)=x_0}. \tag{2.7}$$

The theory of stochastic optimal control yields a second order ODE satisfied by the value function V. Using the Feynman-Kac formula [57] to derive an equation for $p(.)$, we eventually obtain the system

$$(r+\rho(x))V \;=\; \min_{\omega\in[0,1]}\left\{L(\omega) - \frac{V'}{p}\,\omega\right\} + \rho(x)B + V'\cdot\left(\frac{\lambda+r}{p} + \sigma^2-\lambda-\mu\right)x + \frac{\sigma^2x^2}{2}\cdot V''(x)\,, \tag{2.8}$$

$$\rho(x)(p-\theta(x)) \;=\; (r+\lambda)(1-p) + p'\cdot\left[(\sigma^2-\lambda-\mu)x + \frac{(\lambda+r)x - u^\sharp(p,V')}{p}\right] + \frac{\sigma^2x^2}{2}\cdot p''(x)\,. \tag{2.9}$$

This is supplemented by the boundary conditions

$$\begin{cases} V(0) \;=\; 0, \\ V(M) \;=\; B, \end{cases} \qquad \begin{cases} p(0) \;=\; 1, \\ p(M) \;=\; \theta(M)\,. \end{cases} \tag{2.10}$$

The optimal feedback control is then $u^*(x) = u^\sharp(p(x), V'(x))$, as in (2.4).

3 Solutions to the viscous equation

In this section we prove the existence of a solution to the boundary value problem (2.8)–(2.10), which in turn provides an equilibrium solution to the debt management problem.

Consider the functions

$$H(x,p,\xi) \doteq \min_{\omega\in[0,1]} \left\{ L(\omega) - \frac{\xi}{p}\omega \right\} + \xi \left(\frac{\lambda + r}{p} - (\lambda+\mu) \right) x + \rho(x) B \,, \tag{3.1}$$

$$\mathcal{H}(x,p,\xi,\sigma) \doteq H(x,p,\xi) + \sigma^2 x \xi \,. \tag{3.2}$$

Toward the analysis of H, observe that

$$\min_{\omega\in[0,1]} \{ L(\omega) - s\,\omega \} = L\big(\omega^\sharp(s)\big) - s\,\omega^\sharp(s), \tag{3.3}$$

where

$$\omega^\sharp(s) = \begin{cases} 0 & \text{if} \quad s \le L'(0), \\ (L')^{-1}(s) & \text{if} \quad s > L'(0). \end{cases} \tag{3.4}$$

An elementary computation yields

$$H_p(x,p,\xi) = \frac{\xi}{p^2} \left[\omega^\sharp\Big(\frac{\xi}{p}\Big) - (\lambda + r) x \right], \tag{3.5}$$

$$H_\xi(x,p,\xi) = -\frac{1}{p}\omega^\sharp\Big(\frac{\xi}{p}\Big) + \left(\frac{\lambda + r}{p} - (\lambda+\mu) \right) x \,, \tag{3.6}$$

$$H_{\xi\xi}(x,p,\xi) = \begin{cases} 0 & \text{if} \quad \dfrac{\xi}{p} \le L'(0), \\ \left[p^2 L'' \left(\omega^\sharp\Big(\dfrac{\xi}{p}\Big) \right) \right]^{-1} & \text{if} \quad \dfrac{\xi}{p} > L'(0). \end{cases} \tag{3.7}$$

Lemma 1. *Let the conditions (2.15) hold. Then, for all $\xi \ge 0$ and $0 < p \le 1$,* *the function $\mathcal{H}(x,p,\xi,\sigma)$ in (3.2) satisfies*

$$\begin{aligned}\rho(x)B + \left(\frac{(\lambda+r)x-1}{p} + (\sigma^2-\lambda-\mu)x\right)\xi \;&\leq\; \mathcal{H}(x,p,\xi,\sigma)\\ &\leq\; \rho(x)B + \left(\frac{\lambda+r}{p} + \sigma^2-\lambda-\mu\right)x\xi,\end{aligned} \tag{3.8}$$

$$\frac{(\lambda+r)x-1}{p} + (\sigma^2-\lambda-\mu)x \;\leq\; \mathcal{H}_\xi(x,p,\xi,\sigma) \;\leq\; \left(\frac{\lambda+r}{p} + \sigma^2-\lambda-\mu\right)x, \tag{3.9}$$

$$-\frac{\xi}{p^2}(\lambda+r)x \;\leq\; \mathcal{H}_p(x,p,\xi,\sigma) \;\leq\; \frac{\xi}{p^2}\left(1-(\lambda+r)x\right). \tag{3.10}$$

Moreover, for every $x,p>0,\sigma\geq 0$, the map $\xi\mapsto\mathcal{H}(x,\xi,p,\sigma)$ is concave down and satisfies

$$\mathcal{H}(x,p,0,\sigma) \;=\; \rho(x)B, \tag{3.11}$$

$$\mathcal{H}_\xi(x,p,0,\sigma) \;=\; \left(\frac{\lambda+r}{p} + \sigma^2-\lambda-\mu\right)x, \tag{3.12}$$

$$\lim_{\xi\to+\infty}\mathcal{H}(x,p,\xi,\sigma) \;=\; \begin{cases} -\infty, & \textit{if } \frac{1}{p} > \left(\frac{\lambda+r}{p}-\lambda+\sigma^2-\mu\right)x,\\ +\infty, & \textit{if } \frac{1}{p} \leq \left(\frac{\lambda+r}{p}-\lambda+\sigma^2-\mu\right)x.\end{cases} \tag{3.13}$$

Proof. The properties (3.8)-(3.9) and (3.11)-(3.13) follow directly from Lemma 2.1 in [19]. By (2.4), the estimate (3.10) follows from

$$\mathcal{H}_p(x,p,\xi,\sigma) \;=\; \frac{\xi}{p^2}\Big(u^\sharp(p,\xi)-(\lambda+r)x\Big).$$

□

Theorem 1. *Under the assumptions* **(A1)–(A3)**, *the boundary value problem (2.8)–(2.10) admits at least one solution $(V,p)\in\mathcal{C}^2$. Moreover $V'(x)\geq 0$ and $p'(x)\leq 0$, for all $0<x<M$.*

Proof. 1. In terms of the Hamiltonian function $\mathcal{H}$ at (3.2), the equations

(2.8)-(2.9) can be written as

$$\begin{cases} 0 &= -rV + \mathcal{H}(x, V_x, p, \sigma) + \dfrac{(\sigma x)^2}{2} V_{xx}, \\ 0 &= -\rho(x)\,(p - \theta(x)) + (r+\lambda)(1-p) + \mathcal{H}_\xi(x, V_x, p, \sigma) p_x + \dfrac{(\sigma x)^2}{2} p_{xx}\,. \end{cases} \tag{3.14}$$

Following an argument used in [19], a solution of (3.14) satisfying the boundary conditions (2.10) will be obtained by constructing stationary solutions of a related parabolic problem. For any $\varepsilon > 0$, let ρ_ε be a $\mathcal{C}^1$ function such that

$$\rho_\varepsilon(x) = \begin{cases} \rho(x) & \text{if } \rho(x) \le \varepsilon^{-1}, \\ 2\varepsilon^{-1} & \text{if } \rho(x) \ge 2\varepsilon^{-1}. \end{cases} \tag{3.15}$$

Consider the parabolic system

$$\begin{cases} V_t &= -(r + \rho_\varepsilon(x))V + \mathcal{H}(x, V_x, p, \sigma) + \left(\varepsilon + \dfrac{(\sigma x)^2}{2}\right) V_{xx}, \\ p_t &= -\rho_\varepsilon(x)\,(p - \theta(x)) + (r+\lambda)(1-p) + \mathcal{H}_\xi(x, V_x, p, \sigma) p_x + \left(\varepsilon + \frac{(\sigma x)^2}{2}\right) p_{xx}, \end{cases} \tag{3.16}$$

with boundary data

$$\begin{cases} V(t,0) &= 0, \\ V(t,M) &= B, \end{cases} \qquad \begin{cases} p(t,0) &= 1, \\ p(t,M) &= \theta(M)\,. \end{cases} \tag{3.17}$$

Notice that these ε-approximations remove the singularities at $x = 0$ (where the diffusion coefficient would vanish) and at $x = M$ (where $\rho = +\infty$). Indeed, the system (3.16) is uniformly parabolic, with $\mathcal{C}^1$ coefficients.

2. Adopting a semigroup notation, let $t \mapsto (V(t), p(t)) = S_t(V_0, p_0)$ be the solution of the parabolic system (3.16-3.17) with initial data

$$V(0,x) = V_0(x), \qquad p(0,x) = p_0(x).$$

Consider the closed, convex set of functions

$$\begin{aligned}\mathcal{D} \ &\doteq\ \Big\{(V,p) : [0,M] \mapsto [0,B] \times [\theta(M),1]; \quad V,p \in \mathcal{C}^2, \\ &\qquad V_x \geq 0 \ \text{ and the boundary conditions (2.10) hold}\Big\}.\end{aligned} \tag{3.18}$$

We claim that the domain $\mathcal{D}$ is positively invariant under the semigroup $\{S_t\}$. Indeed, defining the constant functions

$$V^+(t,x) \doteq B, \qquad V^-(t,x) \doteq 0, \qquad p^+(t,x) \doteq 1, \qquad p^-(t,x) \doteq \theta(M),$$

one easily checks that V^+ is a supersolution and V^- is a subsolution of the first equation in (3.16). Namely,

$$-(r+\rho(x))V^+ + \mathcal{H}(x, V_x^+, p, \sigma) + \left(\varepsilon + \frac{(\sigma x)^2}{2}\right) V_{xx}^+ \leq 0, \ V^+(t,0) \geq 0, \ V^+(t,M) \geq B,$$

$$-(r+\rho(x))V^- + \mathcal{H}(x, V_x^-, p, \sigma) + \left(\varepsilon + \frac{(\sigma x)^2}{2}\right) V_{xx}^- \geq 0, \ V^-(t,0) \leq 0, \ V^-(t,M) \leq B.$$

Similarly, p^+ is a supersolution and p^- is a subsolution of the second equation in (3.16). For any initial data $(V_0, p_0) \in \mathcal{D}$, the solution of (3.16)-(3.17) will satisfy

$$0 \leq V(t,x) \leq B, \qquad\qquad \theta(M) \leq p(t,x) \leq 1, \tag{3.19}$$

for all $x \in [0,M]$ and $t \geq 0$. As a consequence,

$$\begin{cases} V_x(t,0) \ \geq \ 0, \\ V_x(t,M) \ \geq \ 0, \end{cases} \qquad\qquad \begin{cases} p_x(t,0) \ \leq \ 0, \\ p_x(t,M) \ \leq \ 0. \end{cases} \tag{3.20}$$

3. Differentiating the first equation in (3.16) w.r.t. x one obtains

$$\begin{aligned} V_{xt} \ &= \ -(r+\rho(x))V_x + \rho'(x)(B-V) + (\mathcal{H}_x - \rho'(x)B) \\ &\qquad + \mathcal{H}_\xi V_{xx} + \mathcal{H}_p p_x + \sigma^2 x V_{xx} + \left(\varepsilon + \frac{(\sigma x)^2}{2}\right) V_{xxx}\,. \end{aligned} \tag{3.21}$$

By (3.5),(3.11), (3.12), and (3.19), we have

$$\mathcal{H}_x(x,p,0,\sigma) \;=\; \rho'(x)B, \qquad \mathcal{H}_\xi(x,p,0,\sigma) \;=\; \mathcal{H}_p(x,p,0,\sigma) \;=\; 0, \qquad V \leq B.$$

We thus conclude that $V_x = 0$ is a subsolution of (3.21), proving the positive invariance of the domain $\mathcal{D}$.

4. We observe that the functions $\mathcal{H}$ and $\mathcal{H}_\xi$ appearing in the equation (3.14) depend Lipschitz continuously on all their arguments. Moreover, according to the bounds (3.8)–(3.10), they have sublinear growth w.r.t. the variable $\xi = V_x$. One can thus apply Theorem 3 in [2] and obtain the existence of a steady state $(V^\varepsilon, p^\varepsilon) \in \mathcal{D}$ for the parabolic problem system (3.16)-(3.17).

We remark that the analysis in [2] applies to parabolic systems in any space dimension. In this general case, by Schauder's estimates, the solution has $\mathcal{C}^{2,\alpha}$ regularity. In our case, however, the domain $[0,M]$ is 1-dimensional. Solving (3.14) for V_{xx} and p_{xx} and using the fact that $\mathcal{H}, \mathcal{H}_\xi$ are Lipschitz continuous we conclude that these functions $V^\varepsilon, p^\varepsilon$ are twice differentiable with Lipschitz continuous second derivatives.

We claim that $p^\varepsilon_x(x) \leq 0$ for all $x \in [0,M]$. Indeed, assume by contradiction that $p^\varepsilon_x(x_0) > 0$ at some point x_0. Using the boundary condition (2.10) and the bounds (3.20), we can define the two points

$$x_1 \;\doteq\; \max\Big\{y;\;\; y < x_0,\;\; p^\varepsilon_x(y) = 0 \text{ and } p^\varepsilon_x(x) \geq 0 \text{ for all } \; x \in [y, x_0]\Big\} \;>\; 0,$$

$$x_2 \;\doteq\; \min\Big\{y;\;\; y > x_0,\;\; p^\varepsilon_x(y) = 0 \text{ and } p^\varepsilon_x(x) \geq 0 \text{ for all } \; x \in [x_0, y]\Big\} \;<\; M.$$

It is clear that $x_1 < x_2$, $p^\varepsilon_{xx}(x_2) \leq 0 \leq p^\varepsilon_{xx}(x_1)$ and $p^\varepsilon(x_1) < p^\varepsilon(x_2)$. On the other hand, by definition we have

$$\begin{aligned} & p^\varepsilon(x_1) - p^\varepsilon(x_2) \\ = \;& \frac{\rho(x_1)\theta(x_1) + r + \lambda + \frac{\varepsilon + (\sigma x_1)^2}{2} p^\varepsilon_{xx}(x_1)}{\rho(x_1) + r + \lambda} - \frac{\rho(x_2)\theta(x_2) + r + \lambda + \frac{\varepsilon + (\sigma x_2)^2}{2} p^\varepsilon_{xx}(x_2)}{\rho(x_2) + r + \lambda} \\ \geq \;& \frac{\rho(x_1)\theta(x_1) + r + \lambda}{\rho(x_1) + r + \lambda} - \frac{\rho(x_2)\theta(x_2) + r + \lambda}{\rho(x_2) + r + \lambda} \\ = \;& -\int_{x_1}^{x_2} \frac{\theta'(x)\rho(x)(\rho(x) + r + \lambda) + \rho'(x)(\theta(x) - 1)(r + \lambda)}{(\rho(x) + r + \lambda)^2}\, dx \;\geq\; 0, \end{aligned}$$

reaching a contradiction. We thus conclude that $p_x(x) \leq 0$ for all $x \in [0, M]$.

5. In the remainder of the proof, letting $\varepsilon \to 0$ we shall obtain a limit (V, p), which provides a $\mathcal{C}^2$ solution to the original boundary value problem (2.8)–(2.10). By setting

$$\zeta \;=\; \zeta(x, p, V') \;\doteq\; \left[\frac{(\lambda + r)x - u^\sharp(p, V')}{p} - (\lambda + \mu)x\right], \tag{3.22}$$

the system satisfied by $(V^\varepsilon, p^\varepsilon)$ can be written as

$$\left\{ \begin{aligned} rV \;&=\; (\zeta + \sigma^2 x)V' + \rho_\varepsilon(x)(B - V) + L(u^*) + \frac{\varepsilon + \sigma^2 x^2}{2} \cdot V''(x)\,, \\ \rho_\varepsilon(x)(p - \theta(x)) \;&=\; (\zeta + \sigma^2 x)\, p' \; + (r + \lambda)(1 - p) + \frac{\varepsilon + \sigma^2 x^2}{2} \cdot p''(x)\,. \end{aligned} \right. \tag{3.23}$$

By (3.12)-(3.13), we can find $\delta > 0$ sufficiently small and $\xi_0 > 0$ such that

$$x \in [0, \delta], \quad p \in [\theta(M), 1], \quad \xi \geq \xi_0 \qquad \Longrightarrow \qquad \mathcal{H}(x, p, \xi, \sigma) \leq 0. \tag{3.24}$$

This in turn implies that, if $V'(x) > \xi_0$ for some $x \in [0, \xi]$, then by (2.8) one has $V''(x) \geq 0$. We thus conclude that either $V'(x) \leq \xi_0$ for all $x \in [0, \delta]$, or else the global maximum of V' is attained on the subinterval $[\delta, M]$.

Notice that, since $u^\sharp < 1$ and $r > \mu$, by (3.22) one has

$$(r - \mu)x \;\geq\; 1 \quad \Longrightarrow \quad \zeta > 0.$$

Roughly speaking, this means that if the debt x is large enough it will keep increasing.

Consider the first equation in (3.23). On the region where $\zeta \geq 0$ the first three terms on the right hand side are all nonnegative. Recalling that $V' \geq 0$ and $0 \leq V \leq B$, we thus obtain

$$V''(x) \;\leq\; \frac{2rV}{\sigma^2 x^2} \;\leq\; \frac{2rB}{\sigma^2 x^2}. \tag{3.25}$$

This upper bound on the second derivative immediately yields a uniform bound on V' on $[(r-\mu)^{-1}+\delta, M]$.

Then we consider the interval $[\delta, (r-\mu)^{-1}+\delta]$. By the intermediate value theorem, there exists a point $\hat{x} \in [\delta, (r-\mu)^{-1}+\delta]$ such that

$$V'(\hat{x}) \;=\; \frac{V((r-\mu)^{-1}+\delta) - V(\delta)}{(r-\mu)^{-1}+\delta-\delta} \;\leq\; B(r-\mu). \tag{3.26}$$

The first equation in (3.23) yields an inequality of the form

$$|V''| \;\leq\; c_1|V'| + c_2, \qquad\qquad x \in [\delta, (r-\mu)^{-1}+\delta], \tag{3.27}$$

for suitable constants c_1, c_2 independent of ε. By Gronwall's lemma, (3.27) and (3.26) yield a uniform bound of V' on $[\delta, (r-\mu)^{-1}+\delta]$. We thus conclude that V' is uniformly bounded for all $x \in [\delta, M]$ and thus is uniformly bounded for all $x \in [0, M]$, independent of ε.

A similar analysis applies to p'. Since H_ξ is uniformly bounded, and $\rho_\varepsilon(x)$ is uniformly bounded on $x \in [\delta, M-\delta]$, for a given $\delta > 0$ independent of ε, we obtain an inequality of the form

$$|p''| \;\leq\; a_1|p'| + a_2, \qquad\qquad x \in [\delta,\; M-\delta], \tag{3.28}$$

for suitable constants a_1, a_2 independent of ε.

Summarizing the previous analysis, we thus have the bounds

$$\begin{cases} |V'(x)| \;\leq\; C & \text{for all } x \in [0, M], \\ |p'(x)| \;\leq\; C_\delta & \text{for all } x \in [\delta,\; M-\delta], \end{cases} \tag{3.29}$$

for some constants C and C_δ independent of ε.

6. By (3.29) we can extract a subsequence $\varepsilon_n \to 0$ and achieve the convergence

$$\begin{cases} V^{\varepsilon_n}(x) \;\to\; V(x) & \text{uniformly for } x \in [0, M], \\ p^{\varepsilon_n}(x) \;\to\; p(x) & \text{uniformly for } x \in [\delta, M-\delta], \text{ for any } \delta > 0. \end{cases} \tag{3.30}$$

Since all solutions $(V^\varepsilon, p^\varepsilon)$ satisfy the boundary condtions (2.10), by uniform convergence it is clear that the same boundary conditions are satisfied by V.

To prove that also $p(\cdot)$ satisfies the required boundary conditions (2.10), we construct a lower solution p^- and an upper solution p^+ of the second ODE equation in (3.23), independent of ε. Following the proof of Theorem 3.1 in [19], we have

$$1 \;\geq\; p^\varepsilon(x) \;\geq\; p^-(x) \;\doteq\; 1 - cx^\gamma \qquad \text{for all } x \in [0, x_1],$$

for some suitable constants $c > 0, \gamma > 0, x_1 > 0$.

Next, to construct an upper solution p^+, we choose $\tilde{x}_2$ such that $(r-\mu)^{-1} < x_2 < M$, and take $\delta > 0$ small enough such that $M - \delta > x_2$. Moreover, we introduce the constant

$$\kappa \;\doteq\; \max\left\{ \frac{R_0}{M^2}, \; \frac{(r+\lambda)[1-\theta(M)]}{x_2(r-\mu)-1}, \; \frac{1-\theta(M)}{M-x_2} \right\},$$

and define

$$p^+(x) \;\doteq\; \kappa(M-x) + \theta(M) \qquad \text{for } \; x \in [x_2, M].$$

By construction, $p^+(x_2) \geq 1 \geq p^\varepsilon(\tilde{x}_2)$ and

$$\begin{aligned} &\rho(x)[\theta(x) - \theta(M) - \kappa(M-x)] + (r+\lambda)(1-p^+) - \kappa\mathcal{H}_\xi(x, p^\varepsilon, V', \sigma) \\ &\quad \leq\; \rho(x)(M-x)\left(\tfrac{R_0}{M^2} - \kappa\right) + (r+\lambda)(1-\theta(M)) - \kappa\left[x_2(r+\mu) - 1\right] \;\leq\; 0, \end{aligned}$$

where we used the lower bound (3.9) on H_ξ. We thus conclude that $p^+(x)$ is a upper solution of the second equation in (3.23) on $x \in [x_2, M]$. Hence

$$\theta(M) \;\leq\; p^\varepsilon(x) \;\leq\; p^+(x) \;=\; \kappa(M-x) + \theta(M) \qquad \text{for } \; x \in [x_2, M],$$

for all $\varepsilon > 0$.

7. By the previous analysis, the limit functions V, p in (3.30) satisfy all boundary conditions in (2.10). Observing that the convergence is uniform and the coefficient $\frac{\sigma^2 x}{2}$ is uniformly positive, on every compact subinterval $[\delta,\, M-\delta]$, we now show that (V, p) satisfy the system of second order ODEs (2.8). This is accomplished in three steps.

(i) Using the bounds (3.29) in (3.23) we obtain that the functions V'', p'' are uniformly bounded on the interval $[\delta,\, M-\delta]$, by a constant independent of ε.

(ii) From the boundedness of V'', p'' it follows that the functions V', p' are uniformly Lipschitz continuous on $[\delta,\, M-\delta]$.

(iii) In turn, using the Lipschitz continuity of V', p' in (3.23) we conclude that the functions V'', p'' are uniformly Lipschitz continuous on $[\delta,\, M-\delta]$.

By possibly extracting a further subsequence, we thus obtain the convergence $(V^{\varepsilon_n}, p^{\varepsilon_n}) \to (V, p)$ in the space $\mathcal{C}^2([\delta,\, M-\delta])$. It is now clear that (V, p) provides a classical solution to the system of equations (3.14).

□

4 The vanishing viscosity limit

We now consider a sequence of solutions (V_n, p_n) to the system (2.8)–(2.10), with diffusion coefficient $\sigma = \sigma_n \to 0+$. Recalling (3.22), we write (2.8)-(2.9) in the equivalent form

$$\begin{cases} rV \;=\; (\zeta+\sigma^2 x)V' + \rho(x)(B-V) + L(u^\sharp(p,V')) + \dfrac{\sigma^2 x^2}{2}\cdot V''(x)\,, \\ \rho(x)(p-\theta(x)) \;=\; (\zeta+\sigma^2 x)\,p' \; + (r+\lambda)(1-p) + \dfrac{\sigma^2 x^2}{2}\cdot p''(x)\,. \end{cases} \tag{4.1}$$

Our first result shows the existence of a limit, as $\sigma \to 0$.

Theorem 2. *Let the assumptions* **(A1)–(A3)** *hold. Let* V_n, p_n *be a sequence of solutions to the boundary value problem (2.8)–(2.10), with diffusion coefficients* $\sigma_n \to 0$.

Then there exists one point $\hat{x} \in [0, M]$ *such that, by possibly extracting a subsequence, one has the uniform convergence* $V_n \to V$ *on every domain of the form* $[0, \hat{x} - \delta] \cup [\hat{x} + \delta, M]$*. In addition, one has the pointwise convergence* $p_n(x) \to p(x)$ *and* $V_n'(x) \to V'(x)$*, for a.e.* $x \in [0, M]$*. Also, the limit satisfies the H-J equation*

$$(r + \rho(x))V(x) \;=\; H(x, V'(x), p(x)) \tag{4.2}$$

for a.e. $x \in [0, M]$.

One may think of $\hat{x}$ as a threshold for the debt size. When $x < \hat{x}$, the optimal feedback strategy keeps the debt uniformly bounded in time. On the other hand, when $x > \hat{x}$, the debt keeps increasing and reaches the value M in finite time, where bankruptcy instantly follows.

Our second result shows that, under suitable assumptions, the above limit (V, p) yields a Nash equilibrium solution to a problem with deterministic dynamics. Toward this goal, we first introduce an auxiliary function $W : [0, M[\mapsto [0, +\infty]$, defined by

$$W(y) \;\doteq\; \sup_{\xi \geq 0} H(y, p(y), \xi) \;=\; \frac{1}{r + \rho(y)} \left\{ L\Big((\lambda + r)y - (\lambda + \mu)p(y)y \Big) + \rho(y)B \right\}. \tag{4.3}$$

Recalling (3.6), we can interpret $W(y)$ as the expected cost achieved by keeping the debt at the constant value $x(t) = y$. This is well defined as long as the control needed to keep the debt constant satisfies

$$\bar{u}(y) \;\doteq\; \Big(\lambda + r - (\lambda + \mu)p(y)\Big)y \;<\; 1. \tag{4.4}$$

On the other hand, if $\bar{u}(y) \geq 1$, then $W(y) = +\infty$. Since $p(\cdot)$ is monotone decreasing, the function

$$y \;\mapsto\; \left\{ L\Big((\lambda + r)y - (\lambda + \mu)p(y)y \Big) + \rho(y)B \right\} \;=\; (r + \rho(y))W(y)$$

is monotone increasing. Recalling that the instantaneous bankruptcy risk $\rho(\cdot)$ is a smooth function, we conclude that W has bounded variation on any subinterval $[0, \bar{y}]$ where $W < +\infty$. Downward jumps in p correspond to upward jumps in W.

We can now state the final result of this chapter:

Theorem 3. *In the same setting of Theorem 2, let $W(\cdot)$ be the function at (4.3). Then $V(x) \leq W(x-)$ for all $x \in [0, M]$.*

Moreover assume that the following two conditions hold:

(i) The set of points y where $V(y) = W(y-)$ is finite.

(ii) If p has a jump at a point $\bar{y}$ where $V(\bar{y}) < W(\bar{y}-)$, and if the time derivative

$$\dot{x} \;=\; \zeta(x) \;\doteq\; \frac{(\lambda + r)x - u^\sharp(p(x), V'(x))}{p(x)} - (\lambda + \mu)x$$

has the same sign to the right and to the left of $\bar{y}$, then

$$F^-(\bar{y}, p_1, V(\bar{y})) \;<\; \xi^\sharp(\bar{y}, p_2) \;<\; F^+(\bar{y}, p_3, V(\bar{y}))\,, \tag{4.5}$$

for all $p_1, p_2, p_3 \in [p(\bar{y}+),\, p(\bar{y}-)]$ with $\xi^\sharp$ defined in (2.5).

Then, setting $u^*(x) = u^\sharp(p(x), V'(x))$ as in (2.4), the functions (V, p, u^*) provide an equilibrium solution to the debt management problem, according to Definition 1.

A proof of Theorems 2 and 3 will be given in several steps.

1 - Existence of a pointwise limit.

Since each V_n is increasing and each p_n is decreasing, by Helly's compactness theorem we can choose a subsequence and achieve the pointwise convergence

$$V_n(x) \;\to\; V(x) \;\in\; [0, B], \qquad\qquad p_n(x) \;\to\; p(x) \;\in\; [\theta(M), 1] \tag{4.6}$$

for every $x \in [0, M]$. Clearly, V and p are monotone. The structure of this limit will be investigated in the following steps.

2 - Local Lipschitz continuity of V.

For every $n \geq 1$, since $p_n(\cdot)$ is non-increasing there exists a unique point $\hat{x}_n$ such that

$$\begin{cases} \Big(\lambda + r - (\lambda + \mu - \sigma_n^2)p_n(x)\Big)x < 1 & \text{if } x < \hat{x}_n\,, \\ \Big(\lambda + r - (\lambda + \mu - \sigma_n^2)p_n(x)\Big)x > 1 & \text{if } x > \hat{x}_n\,. \end{cases}$$

By possibly choosing a further subsequence, we can assume the convergence $\hat{x}_n \to \hat{x}$. We claim that the limit function V is Lipschitz continuous on every interval of the form $[0, \hat{x}-\varepsilon]$ or $[\hat{x}+\varepsilon, M]$. This will be proved by showing that all derivatives V_n' are uniformly bounded on these intervals.

To estimate V_n' on $[\hat{x}+\varepsilon, M]$, let ζ_n be as in (3.22), with V, p replaced by V_n, p_n. By construction, for any given $\varepsilon > 0$ there exists $\zeta^\dagger > 0$ such that

$$\zeta_n(x) + \sigma_n^2 x \;\geq\; \zeta^\dagger \qquad\qquad \text{for all } x \in [\hat{x}+\varepsilon,\, M], \tag{4.7}$$

for all $n \geq 1$ sufficiently large.

We observe that, if the maximum of V_n' on $[\hat{x}+\varepsilon,\, M]$ is attained at an interior point y, then

$$rV_n(y) \;\geq\; (\zeta_n(y) + \sigma_n^2 y)V_n'(y). \tag{4.8}$$

This yields the bound

$$V_n'(y) \;\leq\; \frac{rB}{\zeta_n(y) + \sigma_n^2 y} \;\leq\; \frac{rB}{\zeta^\dagger}\,, \tag{4.9}$$

and we are done.

If the maximum of V_n' is attained at $x = M$, then $V_n''(M^-) \geq 0$. Hence (4.8) holds at $y = M$. In turn, this implies (4.9).

It remains to study the case where V_n' attains its maximum at $\hat{x}+\varepsilon$. By the intermediate value theorem, there is a point $y^\dagger \in [\hat{x} + (\varepsilon/2),\ \hat{x}+\varepsilon]$ such that

$$V_n'(y^\dagger) \;=\; \frac{V_n(\hat{x}+\varepsilon) - V_n(\hat{x}+(\varepsilon/2))}{\varepsilon/2} \;\leq\; \frac{2B}{\varepsilon}\,. \tag{4.10}$$

Either

$$V_n'(x) \;\leq\; \frac{2B}{\varepsilon} \qquad\qquad \text{for all } x \in [y^\dagger, M], \tag{4.11}$$

and we are done. Or else V_n' attains a maximum at a point $y \in]y^\dagger, M]$. But in this case the bound (4.9) holds.

Since the bounds (4.9), (4.11) are independent of n, they imply the Lipschitz continuity of V on the interval $[\hat{x}+\varepsilon, M]$.

Next, we derive uniform bounds on V_n' on $[0, \hat{x}-\varepsilon]$. As before, there exists a point $y^\dagger \in [\hat{x}-\varepsilon,\ \hat{x}-(\varepsilon/2)]$ where (4.10) holds. By the argument at (3.24), there exists $\xi_0, \delta > 0$ such that either $V_n'(x) < \xi_0$ for all $x \in [0, \delta]$, or else the global maximum of V_n' is attained on the interval $[\delta, M]$.

It thus suffices to study local maxima of V_n' on the interval $[\delta, \hat{x}-(\varepsilon/2)]$. On this interval we have

$$\big(\lambda + r + (\sigma_n^2 - \lambda - \mu)p_n(x)\big)x \ < \ 1 - \epsilon_0 \,. \tag{4.12}$$

for some $\epsilon_0 > 0$ small enough and all n suitably large.

By (4.12), the definition of Hamiltonian function $\mathcal{H}$ at (3.2) yields

$$\lim_{\xi \to +\infty} \mathcal{H}(x, p, \xi, \sigma_n) \ = \ -\infty. \tag{4.13}$$

For $x \in [\delta,\ \hat{x}-(\varepsilon/2)]$ and $p \in [\theta(M), 1]$, call $\xi = \xi_n^0(x,p) > 0$ the value where the concave function $\xi \mapsto \mathcal{H}(x, p, \xi, \sigma_n)$ vanishes, i.e.

$$\mathcal{H}(x, p, \xi_n(x,p), \sigma_n) \ = \ 0\,.$$

Observe that these values are uniformly bounded. Namely, there exists some constant C_0 independent of n such that

$$\xi_n^0(x,p) \ \leq \ C_0\,. \tag{4.14}$$

If V_n' attains a local maximum at a point $y \in [\delta, \hat{x}-(\varepsilon/2)]$, then $V_n''(y) = 0$ and the first equation in (3.23) implies

$$\mathcal{H}(y, p_n(y), V_n'(y), \sigma_n) \ = \ (\zeta_n(y) + \sigma_n^2 y)V_n'(y) + L(u_n^*(y)) + \rho(y)B \ \geq \ 0. \tag{4.15}$$

Comparing (4.15) with (4.13) and using the uniform bound (4.14), we conclude that

$$V_n'(y) \ \leq \ C_0 \,. \tag{4.16}$$

Thanks to the uniform bounds on the derivatives V_n' in (4.16) and (3.24), letting $n \to \infty$ we obtain the Lipschitz continuity of the limit function V on the interval $[0, \, \hat{x} - \varepsilon]$.

For every $\varepsilon > 0$, we thus conclude that on $[0, \hat{x} - \varepsilon] \cup [\hat{x} + \varepsilon, M]$ the sequence V_n converges uniformly to a Lipschitz continuous limit function V. As a consequence, V is differentiable at a.e. point $x \in [0, M]$ and satisfies the boundary conditions in (2.10).

3 - An upper bound on V.

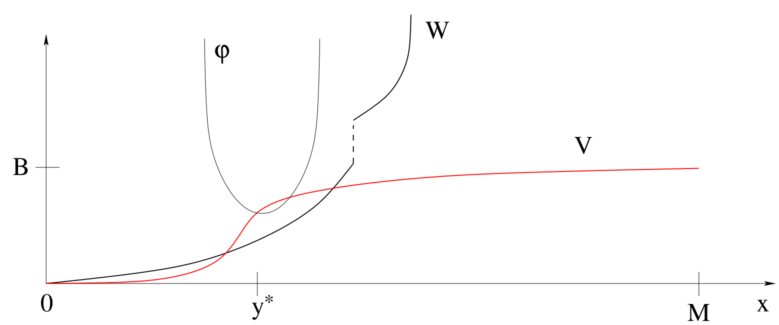

Figure 4.1. Proving that $V \leq W$.

We claim that, for every $y \in [0, M] \setminus \{\hat{x}\}$, one has

$$V(y) \ \leq \ W(y). \tag{4.17}$$

Intuitively, this should be obvious: V describes a minimum cost, while W is a cost achieved by one particular strategy that keeps the debt constant. For $y > \hat{x}$, recalling (4.7) we see that $W(y) = +\infty$ and there is nothing to prove.

To prove (4.17), assume on the contrary that (4.17) fails at some point $\bar{y} < \hat{x}$. Since V is continuous outside $\hat{x}$, we can find another point y^* such that $V(y^*) \ > \ W(y^*)$ and V, W, p are all continuous at y^*. We can choose $\varepsilon_0 > 0$ small enough so that

$$|x - y^*| \leq \varepsilon_0 \qquad \Longrightarrow \qquad V(x) \ > \ W(x) + \delta_0 \tag{4.18}$$

for some $\delta_0 > 0$. For the given $\delta_0 > 0$, there exists $\delta_1 > 0$ such that

$$\left|\sup_{\xi\geq 0} H(x,\xi,p) - (r+\rho(x))W(x)\right| \;\leq\; \frac{\delta_0}{2},$$

whenever $|p - p(x)| \leq 2\delta_1$ and $|x - y^*| \leq \varepsilon_0$. Given δ_1, by the continuity of $p(\cdot)$ at y^* we can further reduce ε_0 so that $|p(x) - p(y^*)| \leq \delta_1$ for all $|x - y^*| \leq \varepsilon_0$. Define

$$\varphi(x) \;=\; \frac{1}{\varepsilon_0^2 - (x-y^*)^2}\,,$$

for all $n \geq 1$ large enough, the function $V_n - \varphi$ will have a local maximum at some point x_n, with $|x_n - y^*| < \varepsilon_0$. Recalling (2.8) we now have

$$\begin{aligned}(r+\rho(x_n))V_n(x_n) \;&=\; H(x_n, V_n'(x_n), p_n(x_n)) + \sigma^2 V_n'(x_n)\,x_n + \frac{\sigma^2 x_n^2}{2} V_n''(x_n)\\ &\leq\; \sup_{\xi\geq 0} H(x_n,\xi,p_n(x_n)) + \sigma^2 V_n'(x_n)\,x_n + \frac{\sigma^2 x_n^2}{2}\varphi''(x_n).\end{aligned} \tag{4.19}$$

Letting $n \to \infty$ we can choose a subsequence such that $x_n \to x^*$ such that $p_n(x_n) \to p^*$, with $|x^* - y^*| \leq \varepsilon_0$ and $|p^* - p(x^*)| \leq |p^* - p(y^*)| + |p(y^*) - p(x^*)| \leq 2\delta_1$. Thanks to the boundedness of V_n' and φ'', we obtain

$$(r+\rho(x^*))V(x^*) \;\leq\; \sup_{\xi\geq 0} H(x^*,\xi,p^*) \;<\; (r+\rho(x^*))W(x^*) + \frac{\delta_0}{2}.$$

This contradicts the assumption (4.18), proving our claim.

4 - Behavior on the region where $V < W$.

To help the reader, we first give an overview of the forthcoming analysis. By (4.7), for $x > \hat{x}$ the debt is so large that it cannot be reduced. The dynamics is monotone increasing, and reaches the maximum value M in finite time. At that time, bankruptcy immediately occurs.

We thus focus on the region where $x < \hat{x}$. In general, we expect that $V(x) < W(x)$, except for a finite number of points where equality holds. These are the debt sizes at which the optimal strategy consists in keeping the debt constant in time.

In the following we consider any interval $[a, b]$ where the strict inequality $V(x) \leq W(x) - \delta$ holds for some $\delta > 0$. We will show that on this interval the limit dynamics is piecewise smooth. Namely, there exists an intermediate point $\bar{x} \in [a, b]$ such that V', p are continuous separately on $]a, \bar{x}[$ and on $]\bar{x}, b[$, and moreover

$$V'(x) \;=\; \begin{cases} F^+(x, V(x), p(x)), & \text{if} \quad x \in]a, \bar{x}[\,, \\ F^-(x, V(x), p(x)) & \text{if} \quad x \in]\bar{x}, b[\,, \end{cases} \tag{4.20}$$

$$\dot{x} \;=\; \zeta(x, p, V') \;\doteq\; \frac{(\lambda + r)x - u^\sharp(p(x), V'(x))}{p(x)} - (\lambda + \mu)x \begin{cases} < \;0 \;\; \text{if} \quad x \in]a, \bar{x}[\,, \\ > \;0 \;\; \text{if} \quad x \in]\bar{x}, b[\,. \end{cases} \tag{4.21}$$

In other words, each open interval $I =]a, b[$ where $V < W$ can be decomposed into a left portion $]a, \bar{x}[$, where the dynamics is strictly decreasing, and a right portion $]\bar{x}, b[$ where the dynamics is strictly increasing (see Fig. 4.2).

For technical reasons, we first prove the result assuming that all jumps in $p(\cdot)$ contained inside $]a, b[$ satisfy the inequality (4.5). At step 6 we will show that the assumption (4.5) is only needed in the case described by condition (ii) in Theorem 3.

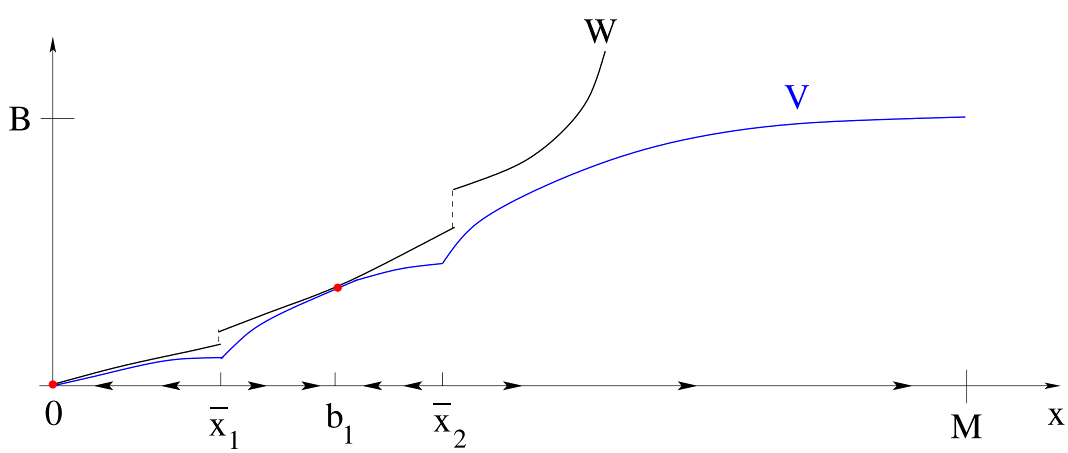

Figure 4.2. A limiting dynamics. Here the coincidence set, where $V = W$ and the dynamics is stationary, is $\Omega = \{0, b_1\}$. Restricted to the two open intervals $I_1 =]0, b_1[$ and $I_2 =]b_1, M[$ the dynamics is piecewise smooth. Namely, there exists intermediate points $\bar{x}_i \in I_i$ such that $\dot{x} < 0$ for $x < \bar{x}_i$ and $\dot{x} > 0$ for $x > \bar{x}_i$.

We now begin working out details. If $x < \hat{x}$ and $V(x) < W(x) < +\infty$, then as shown in [20] the equation

$$(r + \rho(x))V(x) \;=\; H(x, \xi, p(x)) \tag{4.22}$$

has two solutions, which we denote by

$$\xi^- \;=\; F^-(x, V(x), p(x)) \;<\; F^+(x, V(x), p(x)) \;=\; \xi^+. \tag{4.23}$$

Consider any open interval $I =]a, b[$ such that

$$V(x) \;<\; W(x) - \delta \tag{4.24}$$

for some $\delta > 0$ and all $a < x < b$. We claim that there exists $\bar{x} \in [a, b]$ such that (by possibly choosing a subsequence)

$$\begin{cases} V_n'(x) \;\to\; V'(x) \;=\; F^+(x, V(x), p(x)) \quad \text{for a.e. } \; x < \bar{x}, \\[2ex] V_n'(x) \;\to\; V'(x) \;=\; F^-(x, V(x), p(x)) \quad \text{for a.e. } \; x > \bar{x}\,. \end{cases} \tag{4.25}$$

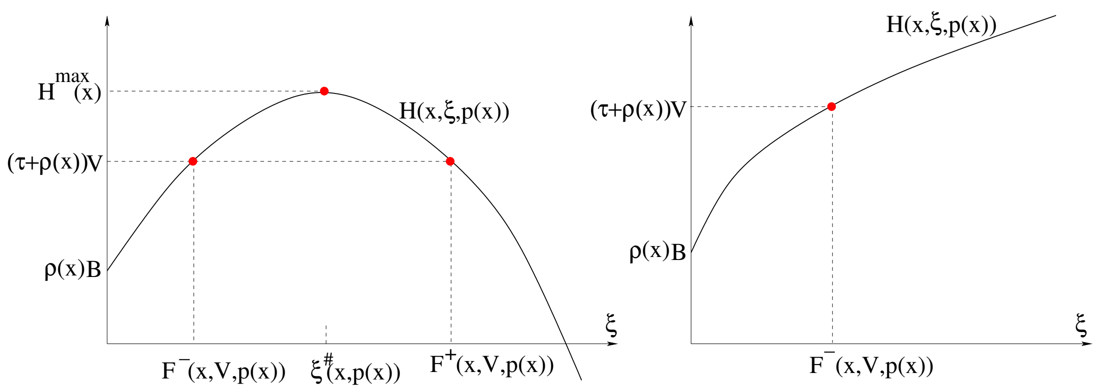

Figure 4.3. Left: the case where $x < \hat{x}$. For $(r + \rho(x))V > H^{max}(x, p(x))$ the equation (4.22) has no solution. At a point where $(r+\rho(x))V < H^{max}(x, p(x))$, it determines two distinct values F^-, F^+ for ξ. Right: the case where $x > \hat{x}$. For any $(r+\rho(x))V > \rho(x)B$, the equation (4.22) determines a unique solution $\xi = F^-$.

Toward this goal, consider a point $x \in I$ where V' exists and p is continuous, hence $p_n(x) \to p(x)$ as $n \to \infty$. We claim that (4.2) holds. Indeed, assume $V'(x) = \alpha$. Then we can choose test functions $\phi_1, \phi_2 \in \mathcal{C}^1$, such that

- $\phi_1'(x) = \alpha$ and $V - \phi_1$ has a strict local maximum at x,
- $\phi_2'(x) = \alpha$ and $V - \phi_2$ has a strict local minimum at x,

Given $\varepsilon > 0$, by approximating ϕ_1, ϕ_2 with $\mathcal{C}^2$ functions φ_1, φ_2, we can assume that

- $V - \varphi_1$ has a local maximum at a point x_1, with $|x_1 - x| < \varepsilon$ and $|\varphi_1'(x_1) - \alpha| < \varepsilon$,
- $V - \varphi_2$ has a local minimum at a point x_2, with $|x_2 - x| < \varepsilon$ and $|\varphi_2'(x_2) - \alpha| < \varepsilon$,

Letting $n \to \infty$ and using the convergence $V_n \to V$, we obtain

- $V_n - \varphi_1$ has a local maximum at a point $x_{1,n}$, with $|x_{1,n} - x| < \varepsilon$ and $|\varphi_1'(x_{1,n}) - \alpha| < \varepsilon$,
- $V_n - \varphi_2$ has a local minimum at a point $x_{2,n}$, with $|x_{2,n} - x| < \varepsilon$ and $|\varphi_2'(x_{2,n}) - \alpha| < \varepsilon$.

Then there exists a subsequence such that $x_{1,n} \to x_1^*$ with $|x_1^* - x| < \varepsilon$ and a similar estimate as (4.19) yields

$$(r + \rho(x_{1,n}))V_n(x_{1,n}) \geq \; H(x_n, \varphi_1'(x_{1,n}), p_n(x_{1,n})) + \sigma_n^2 \varphi_1'(x_{1,n})\, x_{1,n} + \frac{\sigma_n^2 x_{1,n}^2}{2} \varphi_1''(x_{1,n}).$$

By the convergence of $V_n \to V, p_n \to p$ and the boundedness of φ_1', φ_1'', we obtain

$$(r + \rho(x_1^*))V(x_1^*) \geq \; H(x_1^*, \varphi_1'(x_1^*), p(x_1^*)).$$

Since ε is arbitrary and $H(x, \xi, p)$ is continuous w.r.t. all variables, we have

$$(r + \rho(x))V(x) \geq \; H(x, \alpha, p(x)).$$

Using φ_2, a similar argument yields

$$(r + \rho(x))V(x) \leq \; H(x, \alpha, p(x)),$$

proving (4.2).

Next, assume that there exist two points $a < y_1 < y_2 < b$ such that

$$V'(y_1) \;=\; F^-(y_1, p(y_1), V(y_1)), \qquad\qquad V'(y_2) \;=\; F^+(y_2, p(y_2), V(y_2)).$$

Define the limit point

$$\bar{y} \;\doteq\; \sup\{y\;;\;V'(y) = F^-(y, p(y), V(y)),\;\; y < y_2\}.$$

Two cases will be considered:

CASE 1: p is continuous at $\bar{y}$. In this case, V' has an upward jump at $\bar{y}$. Since $\xi \mapsto H(x, \xi, p(x))$ is strictly concave, we can then find a $\mathcal{C}^1$ function ϕ with $F^-(\bar{y}, p(\bar{y}), V(\bar{y})) < \phi'(\bar{y}) < F^+(\bar{y}, p(\bar{y}), V(\bar{y}))$ such that

- $V - \phi$ has a strict local minimum at $\bar{y}$,
- $H(\bar{y}, \phi'(\bar{y}), p(\bar{y})) > (r + \rho(\bar{y}))V(\bar{y})$.

We then approximate ϕ by a $\mathcal{C}^2$ function φ. This yields the existence of a sequence of points y_n, all contained in a small neighborhood around $\bar{y}$, such that

- $V_n - \varphi$ has a local min at y_n,
- $H(y_n, \varphi'(y_n), p_n(y_n)) > (r + \rho(y_n))V_n(y_n) + \varepsilon_0$, for some $\varepsilon_0 > 0$ and all $n \geq 1$.

However, this implies

$$\begin{aligned} 0 \;&=\; -(r + \rho(y_n))V_n(y_n) + H(y_n, \varphi'(y_n), p_n(y_n)) + \sigma_n^2 \varphi'(y_n)\, y_n + \frac{\sigma^2 y_n^2}{2} V_n''(y_n) \\ &\geq\; \varepsilon_0 + \sigma_n^2 \varphi'(y_n)\, y_n + \frac{\sigma_n^2 y_n^2}{2} \varphi''(y_n). \end{aligned}$$

Letting $\sigma_n \to 0$, the right hand side converges to ε_0, providing a contradiction.

CASE 2: p has a downward jump at $\bar{y}$, with

$$F^-(\bar{y}, p_1, V(\bar{y})) \;<\; F^+(x, p_2, V(\bar{y})). \tag{4.26}$$

for any $p_1, p_2 \in [p(\bar{y}+), p(\bar{y}-)]$ such that $p_1 > p_2$. Notice that (4.26) is certainly true if (4.5) is satisfied.

As in CASE 1, using the strict concavity of $\xi \mapsto H(x,\xi,p(x))$, we can find a $\mathcal{C}^1$ function ϕ and some $\bar\varepsilon > 0$ such that

(i) $V - \phi$ has a strict local min at $\bar y$

(ii) $H(\bar y, \phi'(\bar y), p) > (r+\rho(\bar y))V(\bar y)$, for every $p \in [p(\bar y+) - \bar\varepsilon\,,\ p(\bar y-) + \bar\varepsilon]$.

Using the inequalities

$$F^-(x,p_1,V) \ < \ F^-(x,p_2,V) \ < \ F^+(x,p_2,V) \tag{4.27}$$

valid for any $p_1 < p_2$, the same arguments used for CASE 1 can be applied, obtaining again a contradiction. Notice that, due to the discontinuity of $p(y)$ at $\bar y$, the values $p_n(y_n)$ could be any points within the interval $[p(y-) - \bar\varepsilon\,,\ p(y+) + \bar\varepsilon]$. For this reason, in this case we need the stronger assumption (ii) above.

To prove the first inequality in (4.27) we observe that, by (4.22)-(4.23), the function $F^-(x,V,p)$ is implicitly defined by the equation

$$(r+\rho(x))V \ = \ H(x,F^-,p),$$

together with the inequality

$$\dot x \ = \ H_\xi(x,F^-,p) \ = \ -(\lambda+\mu)x + \frac{(\lambda+r)x - \omega^\sharp\left(\frac{F^-}{p}\right)}{p} \ > \ 0.$$

Hence, for fixed x, V, differentiating w.r.t. p one finds

$$H_\xi(x,F^-,p)\cdot F^-_p(x,V,p) + H_p(x,F^-,p) \ = \ 0.$$

By (3.10) it follows $H_p < 0$, hence $F^-_p > 0$. Integrating from p_1 to p_2, the first inequality in (4.27) is proved.

We now analyze the convergence $V'_n(x) \to V'(x)$ in (4.25). To fix the ideas, consider a point $y \in]a,\bar x[$ where V' exists and p is continuous. Assume, by contradiction, that there exists a subsequence such that

$$V'_n(y) \ \to \ \alpha \ \neq \ V'(y) \ = \ F^+(y,V(y),p(y)).$$

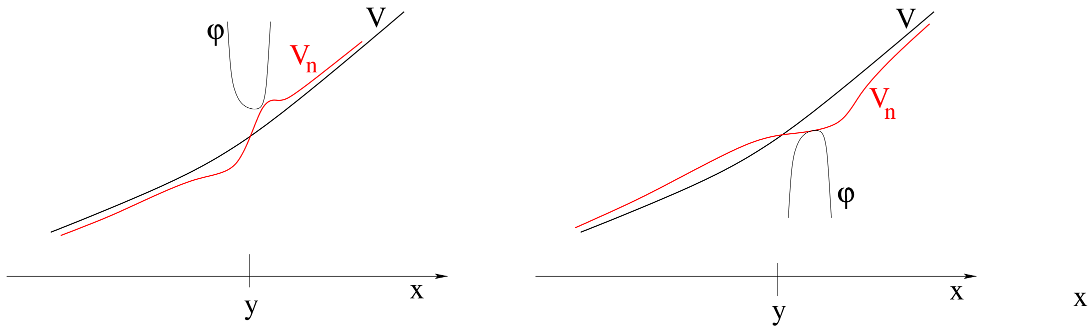

Figure 4.4. Proving the convergence $V_n' \to V'$.

CASE 1. $\alpha > V'(y)$ (see Fig. 4.4, left). Define

$$\hat{\delta} \doteq (r+\rho(y))V(y) - H\left(y, \frac{\alpha + V'(y)}{2}, p(y)\right).$$

Since $p(\cdot)$ is continuous at y, we can choose $\varepsilon, \delta > 0$ sufficiently small such that $|x-y| \leq \varepsilon$ implies $|p(x)-p(y)| < \delta$, and moreover

$$\Big|(r+\rho(y))V(y)-(r+\rho(x))V(x)\Big| < \frac{\hat{\delta}}{4}, \quad \left|H(x,\xi,p) - H\left(y, \frac{\alpha+V'(y)}{2}, p(y)\right)\right| < \frac{\hat{\delta}}{4}, \tag{4.28}$$

whenever $|x-y| < \varepsilon$, $|\xi - \frac{\alpha+V'(y)}{2}| < \varepsilon$, and $|p-p(y)| < \delta$. By the uniform convergence $V_n \to V$ and the intermediate value theorem, for the given $\varepsilon > 0$ and all $n \geq 1$ large enough there exists a point z_n such that $0 < z_n - y < \varepsilon$ such that $|V_n'(z_n) - V'(y)| < \varepsilon$.

Using the fact that $V_n'(y) \to \alpha$, we can find a $\mathcal{C}^2$ function φ and a sequence of points x_n such that

- $\left|\varphi'(x) - \frac{\alpha+V'(y)}{2}\right| < \varepsilon$ for all $|x-y| \leq \varepsilon$,
- $V_n - \varphi$ has a local maximum at a point x_n with $y < x_n < z_n < y+\varepsilon$.

Then (4.19) yields

$$(r+\rho(x_n))V_n(x_n) \;\leq\; H(x_n, \varphi'(x_n), p_n(x_n)) + \sigma_n^2\varphi'(x_n)\,x_n + \frac{\sigma^2 x_n^2}{2}\varphi''(x_n)\,.$$

Letting $n \to +\infty$, we can further choose a subsequence $x_n \to x^*$ with $|x^* - y| <$

ε such that $p_n(x_n) \to p^*$ with $|p^* - p(y)| < \delta$, and for the limit we have

$$(r + \rho(x^*))V(x^*) \ \leq \ H(x^*, \varphi'(x^*), p^*). \tag{4.29}$$

However, (4.28) implies that

$$\begin{aligned}(r + \rho(x^*))V(x^*) \ &\geq \ (r + \rho(y))V(y) - \frac{\hat{\delta}}{4} \\ &> H\left(y, \frac{\alpha + V'(y)}{2}, p(y)\right) + \frac{\hat{\delta}}{2} > H(x^*, \phi'(x^*), p^*) + \frac{\hat{\delta}}{4},\end{aligned}$$

providing a contradiction with (4.29).

CASE 2. $\alpha < V'(y)$ (see Fig. 4.4, right). In this case, we can find some value $\hat{\xi} \in (\alpha, V'(y))$ defined as

$$\hat{\xi} \doteq \beta_1 \alpha + (1 - \beta_1)V'(y),$$

for some $\beta_1 \in (0, 1)$ such that

$$H(y, \xi^\sharp, p(y)) > (r + \rho(y))V(y).$$

Then the case can be handled similarly. The main difference is that now we replace $\frac{\alpha+V'(y)}{2}$ by $\hat{\xi}$ and choose $z_n < y$, so the maximum becomes a minimum.

For the case where $V'(y) = F^-(y, V(y), p(y))$, The proof of the convergence $V_n'(y) \to V'(y)$ is entirely similar.

This step completes the proof of Theorem 2. In the remaining steps we work toward a proof of Theorem 3.

5 - Convergence of the dynamics, in the region where $V < W$.

By the previous analysis, for any interval $I =]a, b[$ where $V < W$, there exists a point $\bar{x}$ such that (4.25) holds. In particular, the feedback dynamics (4.21) induced by the limit value function V is strictly decreasing for $x < \bar{x}$ and strictly increasing for $x > \bar{x}$.

In this step we prove that this dynamics coincides with the limit of the stochastic dynamics

$$dx_n \;=\; (\zeta_n(x_n) + \sigma_n^2 x_n)\, dt + \sigma_n x_n\, dW\,, \tag{4.30}$$

where

$$\zeta_n(x) \;\doteq\; -(\lambda+\mu)x + \frac{(\lambda+r)x - u^\sharp(p_n(x), V_n'(x))}{p_n(x)}. \tag{4.31}$$

More precisely, consider an initial datum

$$x(0) \;=\; y_0 \;\in\;]a,b[\,\setminus\{\bar{x}\},$$

and let $x(\cdot)$, $x_n(\cdot)$ be the corresponding solutions to the ODE (4.21) and to the stochastic diffusion equation (4.30), respectively. We claim that, as long as $x(t) \in\,]a,b[$, we have

$$\lim_{n\to\infty} E\Big[|x_n(t) - x(t)|\Big] \;=\; 0. \tag{4.32}$$

Notice that, by the convergence $V_n' \to V'$ and $p_n \to p$, we have the pointwise a.e. convergence

$$\zeta_n(x) \;\to\; \zeta(x). \tag{4.33}$$

5.1 - lower bounds on the drift. To prove (4.32), we begin by showing that ζ_n remains bounded away from zero, uniformly w.r.t. n. To fix the ideas, assume $x_0 < \bar{x}$, so that the trajectory $t \mapsto x(t)$ is strictly decreasing. Notice that, if we had the uniform convergence $\zeta_n \to \zeta$, the result would be obvious.

To fix the ideas, assume that there exists $\delta > 0$ such that

$$V(x) \;\le\; W(x) - \delta \tag{4.34}$$

for all $x \in [a,b]$ and (4.20)-(4.21) hold for a.e. $x \in I$. At this stage, we still need the assumption (4.5) on all jumps in $p(.)$, namely at every point $y \in [a,b]$ where p has a jump, there holds

$$F^-(\bar{y}, p_1, V(\bar{y})) \;<\; \xi^\sharp(\bar{y}, p_2) \;<\; F^+(\bar{y}, p_3, V(\bar{y}))\,, \tag{4.35}$$

for all $p_1, p_2, p_3 \in [p(\bar{y}+),\, p(\bar{y}-)]$. Notice that this condition trivially holds at all points where p is continuous. In turn, by continuity and compactness, this implies that there exist $\varepsilon_0, \delta_0 > 0$ such that

$$F^-(x,p,V(x)) \ \leq \ \xi^\sharp(\tilde{x},\tilde{p}) - \delta_0, \tag{4.36}$$

$$F^+(x,p,V(x)) \ \geq \ \xi^\sharp(\tilde{x},\tilde{p}) + \delta_0\,, \tag{4.37}$$

for any $x, \tilde{x} \in \mathcal{I}(y,\varepsilon_0)$, and $p, \tilde{p} \in \mathcal{J}(y,\varepsilon_0)$. Here we define

$$\mathcal{I}(y,\varepsilon_0) \doteq \ \Big\{x\ ;\ x \in [a,b] \text{ with } |x-y| < \varepsilon_0\Big\},$$

$$\mathcal{J}(y,\varepsilon_0) \doteq \ \Big\{p\ ;\ p(x_r+) \leq p \leq p(x_l-)\ ,\ x_l = (y-\varepsilon_0)\vee a\ ,\ x_r = (y+\varepsilon_0)\wedge b\Big\}.$$

By (4.34), there exists $\delta_1 > 0$ such that

$$\zeta(x) \ \doteq \ \frac{(\lambda+r)x - u^\sharp(p(x),V'(x))}{p(x)} - (\lambda+\mu)x \ < \ \ -\delta_1 \tag{4.38}$$

at every point $x \in]a,\bar{x}[$ where V', p are well defined. We claim that a similar strict inequality also holds for the sequence of approximations. Namely, there exists a constant $\delta_2 > 0$ such that

$$\zeta_n(x) \ = \ \frac{(\lambda+r)x - u^\sharp(p_n(x),V_n'(x))}{p_n(x)} - (\lambda+\mu)x \ < \ \ -\delta_2 \tag{4.39}$$

for all $x \in]a,\bar{x}[$ and all $n \geq 1$ sufficiently large.

Arguing by contradiction, assume that there exists a sequence of points $z_n \in]a,\bar{x}[$ such that

$$\zeta_n(z_n) \to 0.$$

By possibly taking a subsequence, we can have $z_n \to z^* \in [a,\bar{x}]$ and $p_n(z_n) \to p^* \in \mathcal{J}(z^*,\varepsilon_0)$. Meanwhile, we also have

$$V_n'(z_n) \to \xi^\sharp(z^*,p^*).$$

Since V' is defined a.e, for all n large enough, there exists $z_n < x_n < (z^*+\varepsilon_0)\wedge b$ such that $|V_n'(x_n) - \xi_1| < \frac{\delta_0}{4}$ for some

$$\xi_1 \in \Big\{F^+(x,p,V(x))\ ;\ x \in \mathcal{I}(z^*,\varepsilon_0)\ ,\ p \in \mathcal{J}(z^*,\varepsilon_0)\Big\}\,. \tag{4.40}$$

Notice that the convex set defined in (4.40) contains the slopes of all the secant lines of $V(.)$ on the interval $\mathcal{I}(z^*, \varepsilon_0)$.

We define

$$\xi_0 = \left\{ \inf_{x\in\mathcal{I}(z^*,\varepsilon_0),\ p\in\mathcal{J}(z^*,\varepsilon_0)} F^+(x,p,V(x)) + \sup_{x\in\mathcal{I}(z^*,\varepsilon_0),\ p\in\mathcal{J}(z^*,\varepsilon_0)} \xi^\sharp(x,p) \right\} /2. \quad (4.41)$$

By the constructions of x_n and z_n with condition (4.37), for every $\bar{\varepsilon} > 0$ small, there exists a $\mathcal{C}^2$ function φ and a sequence of points y_n such that

- $\left|\varphi'(x) - \xi_0\right| < \bar{\varepsilon}$ for all $x \in \mathcal{I}(z^*, \varepsilon_0)$.
- $V_n - \varphi$ has a local minimum at y_n with $z_n < y_n < x_n$.

Then (4.19) yields that

$$(r + \rho(y_n))V_n(y_n) \geq H(y_n, \varphi'(y_n), p_n(y_n)) + \sigma_n^2 \varphi'(y_n)\, y_n + \frac{\sigma^2 y_n^2}{2}\varphi''(y_n)\,.$$

Again by possibly passing to the subsequence, $y_n \to y^* \in \mathcal{I}(z^*, \varepsilon_0)$ and $p_n(y_n) \to \hat{p} \in \mathcal{J}(z^*, \varepsilon_0)$, so the limit satisfies

$$(r + \rho(y^*))V(y^*) \ \geq\ H(y^*, \varphi'(y^*), \hat{p}). \quad (4.42)$$

Furthermore, (4.37) implies that

$$\xi^\sharp(y^*, \hat{p}) + \delta_0 \ \leq\ F^+(y^*, \hat{p}, V(y^*)).$$

Choosing $\delta_1 < \delta_0/4$, by the construction in (4.41) of ξ_0, we further obtain

$$\xi^\sharp(y^*, \hat{p}) + \frac{\delta_0}{4} \ \leq\ \varphi'(y^*) \ \leq\ F^+(y^*, \hat{p}, V(y^*)) - \frac{\delta_0}{4},$$

and thus there exists $\varepsilon_1 > 0$ such that

$$(r + \rho(y^*))V(y^*) \ \leq\ H(y^*, \varphi'(y^*), \hat{p}) - \varepsilon_1,$$

against (4.42). So we conclude that there exists $\delta_2 > 0$ depending on δ_0 such that (4.39) holds for any $x \in]a, \bar{x}[$ and all $n \geq 1$ sufficiently large.

By possibly reducing the value of $\delta_2 > 0$, an entirely similar argument yields

$$\zeta_n(x) \;=\; \frac{(\lambda+r)x - u^\sharp(p_n(x), V_n'(x))}{p_n(x)} - (\lambda+\mu)x \;>\; \delta_2\,, \tag{4.43}$$

for every $x \in]\bar{x}, b[$ and all n sufficiently large.

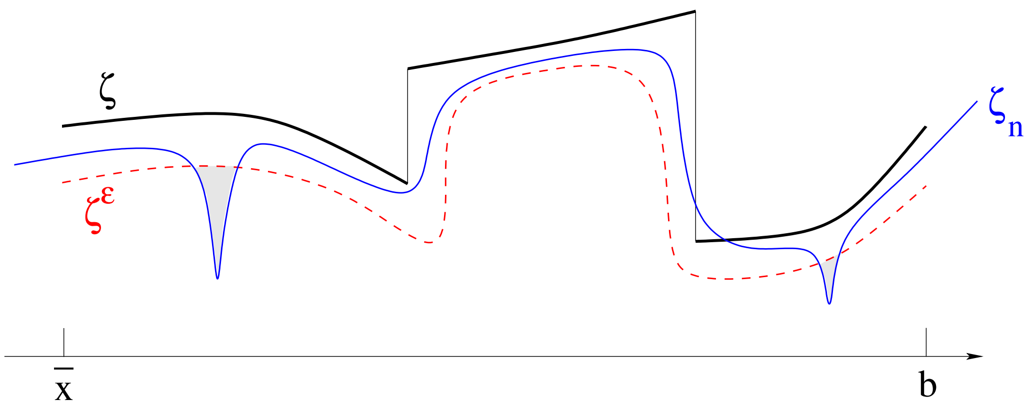

Figure 4.5. The functions ζ, ζ_n in (4.31)-(4.33) and the smooth approximation ζ^ε considered at (4.54). Setting $\zeta_n^\varepsilon = \min\{\zeta^\varepsilon, \zeta_n\}$, one has $\|\zeta^\varepsilon - \zeta_n^\varepsilon\|_{\mathbf{L}^1} \to 0$ as $n \to \infty$.

5.2 - BV bounds on the drift. Next, we claim that $\zeta(\cdot)$ has bounded variation on $]\bar{x}, b[$. Indeed, ζ is defined by (4.38), where $V' = F^-(x, p(x), V(x))$. Since $u^\sharp(p, \xi)$ defined in (2.4) is Lipschitz continuous w.r.t. p and ξ, while F^- is Lipschitz continuous w.r.t. x, p and V, by suitably choosing constants $\alpha_1, \alpha_2, \alpha_3$ we obtain the bound

$$\big|\zeta(x) - \zeta(y)\big| \;\leq\; \alpha_1|x-y| + \alpha_2\big|p(x) - p(y)\big| + \alpha_3\big|V(x) - V(y)\big|.$$

Since $V(\cdot)$ and $p(\cdot)$ are monotone, this implies that $\zeta(\cdot)$ has bounded variation.

5.3 - the distribution functions. Using the uniform positivity of the functions ζ_n and the pointwise a.e. convergence $\zeta_n \to \zeta$, where ζ has bounded variation, we can now prove the convergence (4.32).

To fix the ideas, assume that, for some $\delta > 0$,

$$\bar{x} + 2\delta \;<\; y_0 \;<\; b - 2\delta. \tag{4.44}$$

By the previous analysis, this implies

$$\zeta(x) \;>\; \delta_2\,, \qquad\qquad \zeta_n(x) \;>\; \delta_2\,, \tag{4.45}$$

for some $\delta_2 > 0$ and all $x \in [\bar{x} + 2\delta, \bar{x} - 2\delta]$ and all n sufficiently large.

Let $t \mapsto y(t)$ the solution to the Cauchy problem for the discontinuous ODE

$$\dot{y} \;=\; \zeta(y), \qquad\qquad y(0) = y_0. \tag{4.46}$$

Even if the function $\zeta(\cdot)$ is only measurable, the solution y is well defined. Indeed, since $\zeta(x) \geq \delta_2 > 0$, the value $y(t)$ is implicitly defined by

$$\int_{y_0}^{y(t)} \frac{dy}{\zeta(y)} \;=\; t\,. \tag{4.47}$$

Let $[0, T]$ be an interval of time such that

$$y(t) \;\leq\; b - 2\delta \qquad\qquad \text{for all } t \in [0, T]\,. \tag{4.48}$$

We will show that the convergence (4.32) holds, for all $t \in [0, T]$. This will be proved by establishing upper an lower bounds on the distribution functions of the random variables $x_n(t)$.

For each $n \geq 1$, consider the distribution function

$$Z_n(t, x) \;\doteq\; Prob.\Big\{x_n(t) < x \;\Big|\; x(0) = y_0\Big\}\,. \tag{4.49}$$

This function satisfies the parabolic equation

$$Z_{n,t} + \zeta_n(x) Z_{n,x} \;=\; \frac{\sigma_n^2 x^2}{2} Z_{n,xx} \tag{4.50}$$

with initial data

$$Z_n(0, x) \;=\; \begin{cases} 0 & \text{if} \quad x < y_0, \\ 1 & \text{if} \quad x > y_0\,, \end{cases} \tag{4.51}$$

We consider (4.50) restricted to the interval $[\bar{x}+\delta,\, b-\delta]$. The boundary conditions for this solution are not exactly known, but we can certainly say that

$$Z_x(t, \bar{x} + \delta) \;\geq\; 0, \qquad\qquad Z(t, b - \delta) \;\leq\; 1. \tag{4.52}$$

We claim that, for every $t > 0$

$$\lim_{n\to\infty} Z_n(t, x) \;=\; Z(t, x) \;\doteq\; \begin{cases} 0 & \text{if} \quad x < y(t), \\ 1 & \text{if} \quad x > y(t)\,, \end{cases} \tag{4.53}$$

To prove (4.53), since the map $x \mapsto \zeta(x)$ has bounded variation, for any given $\varepsilon > 0$ we can find a smooth function ζ^ε and a constant $\eta = \eta(\varepsilon) > 0$ such that (see Fig. 4.5)

$$\delta_2 \;\leq\; \zeta^\varepsilon(x) \;<\; \zeta(x), \qquad \text{for} \;\; x \in [\bar{x} + \delta, b - \delta], \tag{4.54}$$

and such that, calling $y^\varepsilon(\cdot)$ the unique solution to

$$\dot{y} \;=\; \zeta^\varepsilon(y), \qquad\qquad y(0) \;=\; y_0 - 2\eta\,. \tag{4.55}$$

one has

$$y^\varepsilon(t) \;\in\; [y(t) - \varepsilon,\; y(t)] \qquad\qquad \text{for all } t \in [0, T]. \tag{4.56}$$

We recall that, by the definition of T at (4.48), this holds as long as $y(t) \leq b - 2\delta$.

For every $n \geq 1$, define

$$\zeta_n^\varepsilon(x) \;\doteq\; \min\bigl\{\zeta^\varepsilon(x),\, \zeta_n(x)\bigr\}.$$

By (4.54) and the pointwise convergence $\zeta_n \to \zeta$, we have

$$\delta_2 \;\leq\; \zeta_n^\varepsilon(x) \;\leq\; \zeta^\varepsilon(x), \qquad\qquad \lim_{n\to\infty} \bigl\|\zeta^\varepsilon - \zeta_n^\varepsilon\bigr\|_{\mathbf{L}^1([\bar{x}+\delta, b-\delta])} \;=\; 0. \tag{4.57}$$

As shown in Fig. 4.6, consider the auxiliary functions Z_n^ε, $\widehat{Z}_n^\varepsilon$, defined as the solutions to the parabolic equations

$$Z^\varepsilon_{n,t} + \zeta^\varepsilon(x) Z^\varepsilon_{n,x} \;=\; \frac{\sigma_n^2 x^2}{2} Z_{n,xx}\,, \tag{4.58}$$

$$\widehat{Z}^\varepsilon_{n,t} + \zeta_n^\varepsilon(x) \widehat{Z}^\varepsilon_{n,x} \;=\; \frac{\sigma_n^2 x^2}{2} \widehat{Z}_{n,xx}\,, \tag{4.59}$$

on the interval $x \in [\bar{x}+\delta,\, b-\delta]$, with smooth initial data

$$Z_n^\varepsilon(0,x) \;=\; \widehat{Z}_n^\varepsilon(0,x) \;=\; \varphi_0(x), \tag{4.60}$$

where φ_0 is a $\mathcal{C}^\infty$ function such that

$$\varphi_0(x) \;=\; \begin{cases} 0 & \text{if} \quad x < y_0 - 2\eta, \\ 1 & \text{if} \quad x > y_0\,, \end{cases} \qquad\qquad 0 \le \varphi_0'(x) \le \eta\,. \tag{4.61}$$

Moreover, we impose the boundary conditions

$$Z_{n,x}^\varepsilon(t,\bar{x}+\delta) \;=\; \widehat{Z}_{n,x}^\varepsilon(t,\bar{x}+\delta) \;=\; 0, \tag{4.62}$$

$$Z_n^\varepsilon(t,b-\delta) \;=\; \widehat{Z}_n^\varepsilon(t,b-\delta) \;=\; 1. \tag{4.63}$$

Since $\zeta^\varepsilon(\cdot)$ is smooth, by the standard theory of parabolic equations it follows the existence of the limit

$$\lim_{n\to\infty} Z_n^\varepsilon(t,x) \;=\; Z^\varepsilon(t,x) \qquad \text{for all } t \in [0,T], \tag{4.64}$$

where Z^ε is the solution to the linear first order equation

$$Z_t^\varepsilon + \zeta^\varepsilon(x) Z_x^\varepsilon \;=\; 0, \tag{4.65}$$

with the same initial data (4.60). Moreover,

$$\lim_{n\to\infty} Z_{n,x}^\varepsilon(t,b-\delta) \;=\; 0, \qquad \lim_{n\to\infty} Z_n^\varepsilon(t,\bar{x}+\delta) \;=\; 0 \qquad \text{for all } t \in [0,T]. \tag{4.66}$$

By a comparison argument, we have the pointwise bound

$$Z_n^\varepsilon(t,x) \;\le\; \widehat{Z}_n^\varepsilon(t,x)\,. \tag{4.67}$$

In turn, this implies

$$0 \;\le\; \widehat{Z}_{n,x}^\varepsilon(t,b-\delta) \;\le\; Z_{n,x}^\varepsilon(t,b-\delta)\,. \tag{4.68}$$

Hence, by (4.66),

$$\lim_{n\to\infty} \widehat{Z}^{\varepsilon}_{n,x}(t, b-\delta) \;=\; 0 \qquad \text{for all } t\in[0,T]. \tag{4.69}$$

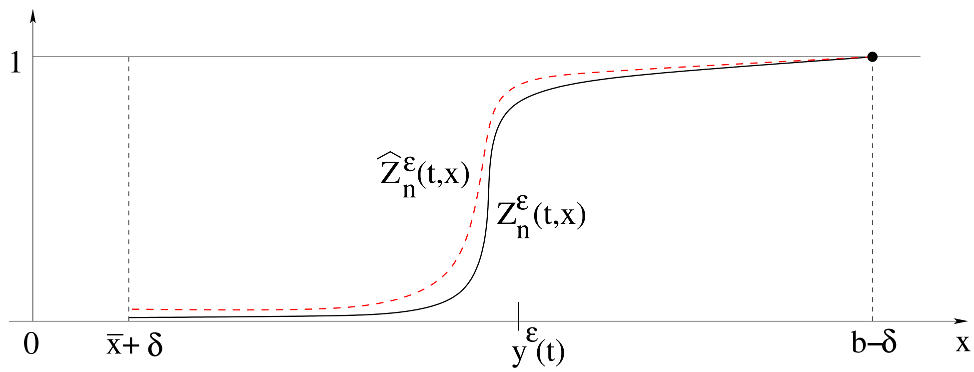

Figure 4.6. The functions Z^ε_n and $\widehat{Z}^\varepsilon_n$ in (4.67)-(4.68).

5.4 - upper bounds on the probability densities. Toward the main convergence proof, we still need to estimate the difference $\widehat{Z}^\varepsilon_n - Z^\varepsilon_n$. In this subsection, we show that the derivatives satisfy a uniform bound, independent of n:

$$\widehat{Z}^\varepsilon_{n,x} \;\le\; K_\varepsilon, \qquad\qquad Z^\varepsilon_{n,x} \;\le\; K_\varepsilon\,, \tag{4.70}$$

for some constant K_ε depending on ε but not on n. This is proved by a comparison argument.

The probability densities $\phi^\varepsilon_n = Z^\varepsilon_{n,x}$ and $\widehat{\phi}^\varepsilon_n = \widehat{Z}^\varepsilon_{n,x}$ satisfy the parabolic equations

$$\phi^\varepsilon_{n,t} + (\zeta^\varepsilon(x)\phi^\varepsilon_n)_x \;=\; \left(\frac{\sigma_n^2 x^2}{2}\phi^\varepsilon_{n,x}\right)_x, \qquad\qquad \widehat{\phi}^\varepsilon_{n,t} + (\zeta^\varepsilon_n(x)\widehat{\phi}^\varepsilon_n)_x \;=\; \left(\frac{\sigma_n^2 x^2}{2}\widehat{\phi}^\varepsilon_{n,x}\right)_x. \tag{4.71}$$

By (4.60), all these functions have the same initial data

$$\phi^\varepsilon_n(0,x) \;=\; \widehat{\phi}^\varepsilon_n(0,x) \;=\; \begin{cases} \eta^{-1} & \text{if} \quad x\in[y_0-\eta,\, y_0]\,, \\ 0 & \text{if} \quad x\notin[y_0-\eta,\, y_0]\,, \end{cases} \tag{4.72}$$

while from (4.62), and (4.66), (4.69) we deduce the boundary conditions

$$\phi^\varepsilon_n(t,\bar x+\delta) \;=\; \widehat{\phi}^\varepsilon_n(t,\bar x+\delta) \;=\; 0, \tag{4.73}$$

$$\lim_{n\to\infty} \phi_n^\varepsilon(t, b-\delta) \;=\; \lim_{n\to\infty} \widehat{\phi}_n^\varepsilon(t, b-\delta) \;=\; 0, \tag{4.74}$$

uniformly for $t \in [0,T]$. A uniform upper bound for $\phi_n^\varepsilon, \widehat{\phi}_n^\varepsilon$ will be proved by constructing suitable upper solutions, independent of time.

The equations in (4.71) admit infinitely many stationary solutions $\psi(x)$. These are found by solving the equations

$$\frac{\sigma_n^2 x^2}{2}\psi_x \;=\; \zeta^\varepsilon(x)\psi + C_0 \qquad \text{or} \qquad \frac{\sigma_n^2 x^2}{2}\psi_x \;=\; \zeta_n^\varepsilon(x)\psi + C_0\,.$$

where C_0 is an arbitrary constant. Define

$$\zeta^{max} \;\doteq\; \sup_{\bar{x}+\delta\le x\le b-\delta} \zeta^\varepsilon(x), \qquad\qquad C_0 \;=\; -\,\eta^{-1}\zeta^{max}\,.$$

To construct a suitable upper bound, we consider the backward Cauchy problem on $[\bar{x}, b]$

$$\psi'(x) \;=\; \frac{2}{\sigma_n^2 x^2}\big(\zeta^\varepsilon(x)\psi - \eta^{-1}\zeta^{max}\big)\,, \qquad\qquad \psi(b) \;=\; \eta^{-1}. \tag{4.75}$$

Observe that the solution of (4.75) satisfies

$$\frac{1}{\eta} \;\le\; \psi(x) \;\le\; \frac{\zeta^{max}}{\eta\delta_2} \qquad\qquad \text{for all } x \in\,]\bar{x}, b]. \tag{4.76}$$

Indeed, this follows immediately from the implications

$$\psi \le \eta^{-1} \qquad \Longrightarrow \qquad \psi' \le 0,$$

$$\psi \ge (\eta\delta_2)^{-1}\zeta^{max} \quad \Longrightarrow \qquad \psi' \ge 0\,.$$

Thanks to the boundary conditions (4.73)-(4.74), using a comparison argument together with (4.76) we deduce

$$Z_{n,x}^\varepsilon(t,x) \;=\; \phi_n^\varepsilon(t,x) \;\le\; \frac{\zeta^{max}}{\eta\delta_2}\,. \tag{4.77}$$

The same argument applies to $\widehat{Z}^\varepsilon_{n,x}$, yielding

$$\widehat{Z}^\varepsilon_{n,x}(t,x) \;=\; \widehat{\phi}^\varepsilon_n(t,x) \;\leq\; \frac{\zeta^{max}}{\eta\delta_2}\,. \tag{4.78}$$

5.5 - convergence estimates. Comparing (4.58) with (4.59), integrating by parts, and using (4.68), we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_{\bar x+\delta}^{b-\delta}\big(\widehat{Z}^\varepsilon_n(t,x)-Z^\varepsilon_n(t,x)\big)\,dx\\
&= \int_{\bar x+\delta}^{b-\delta}\Big(\zeta^\varepsilon(x)Z^\varepsilon_{n,x}(t,x)-\zeta^\varepsilon_n(x)\widehat{Z}^\varepsilon_{n,x}(t,x)+\frac{\sigma_n^2x^2}{2}(\widehat{Z}^\varepsilon_{n,xx}-Z^\varepsilon_{n,xx})\Big)\,dx\\
&= \int_{\bar x+\delta}^{b-\delta}\big(\zeta^\varepsilon(x)-\zeta^\varepsilon_n(x)\big)\widehat{Z}^\varepsilon_{n,x}(t,x)\,dx+\sigma_n^2x\left(\widehat{Z}^\varepsilon_{n,x}(t,x)-Z^\varepsilon_{n,x}(t,x)\right)\Big|_{x=b-\delta}\\
&\qquad+\int_{\bar x+\delta}^{b-\delta}(\zeta^\varepsilon(x)+\sigma_n^2x)\Big(Z^\varepsilon_{n,x}(t,x)-\widehat{Z}^\varepsilon_{n,x}(t,x)\Big)\,dx\\
&\leq\ \|\zeta^\varepsilon-\zeta^\varepsilon_n\|_{\mathbf{L}^1}\|\widehat{Z}_{n,x}\|_{\mathbf{L}^\infty}+(\|\zeta^\varepsilon_x\|_{\mathbf{L}^\infty}+1)\int_{\bar x+\delta}^{b-\delta}\big(\widehat{Z}^\varepsilon_n(t,x)-Z^\varepsilon_n(t,x)\big)\,dx\\
&\qquad+(\zeta^\varepsilon(x)+\sigma_n^2x)\big(\widehat{Z}^\varepsilon_n(t,x)-Z^\varepsilon_n(t,x)\big)\Big|_{x=\bar x+\delta}\,.
\end{aligned}\tag{4.79}$$

Here and in the following formulas all the norms $\mathbf{L}^1,\mathbf{L}^\infty$ refer to the interval $[\bar x+\delta,\,b-\delta]$. Using the bound

$$\widehat{Z}^\varepsilon_n(t,\bar x+\delta)\ \leq\ Z^\varepsilon_n(t,\bar x+2\delta)+\delta^{-1}\left\|\widehat{Z}^\varepsilon_n(t,\cdot)-Z^\varepsilon_n(t,\cdot)\right\|_{\mathbf{L}^1}$$

to estimate the last term in (4.79), and recalling (4.70), for all n sufficiently large we obtain

$$\begin{aligned}
&\frac{d}{dt}\left\|\widehat{Z}^\varepsilon_n(t,\cdot)-Z^\varepsilon_n(t,\cdot)\right\|_{\mathbf{L}^1}\\
&\quad\leq\ \|\zeta^\varepsilon-\zeta^\varepsilon_n\|_{\mathbf{L}^1}K_\varepsilon+(\|\zeta^\varepsilon_x\|_{\mathbf{L}^\infty}+1)\left\|\widehat{Z}^\varepsilon_n(t,\cdot)-Z^\varepsilon_n(t,\cdot)\right\|_{\mathbf{L}^1}\\
&\qquad\quad+(\zeta^{max}+1)\Big(Z^\varepsilon_n(t,\bar x+2\delta)+\delta^{-1}\left\|\widehat{Z}^\varepsilon_n(t,\cdot)-Z^\varepsilon_n(t,\cdot)\right\|_{\mathbf{L}^1}\Big)\\
&\quad=\ A\cdot\left\|\widehat{Z}^\varepsilon_n(t,\cdot)-Z^\varepsilon_n(t,\cdot)\right\|_{\mathbf{L}^1}+B_n\,,
\end{aligned}\tag{4.80}$$

with

$$A \;=\; (\|\zeta^\varepsilon_x\|_{\mathbf{L}^\infty}+1)+\delta^{-1}(\zeta^{max}+1),$$

$$B_n \;=\; \|\zeta^\varepsilon-\zeta^\varepsilon_n\|_{\mathbf{L}^1}K_\varepsilon+(\zeta^{max}+1)\,Z^\varepsilon_n(t,\bar x+2\delta).$$

By Gronwall's lemma, for all $t\le T$ it follows

$$\big\|\widehat Z^\varepsilon_n(t,\cdot)-Z^\varepsilon_n(t,\cdot)\big\|_{\mathbf{L}^1}\;\le\;\big(e^{At}-1\big)A^{-1}B_n\,. \tag{4.81}$$

letting $n\to\infty$, one has $B_n\to 0$, hence the right hand side of (4.81) approaches zero as well.

Summarizing the previous analysis, for any $\varepsilon>0$ we have shown that

$$\begin{aligned}\lim_{n\to\infty}\int_{\bar x+\delta}^{b-\delta}\Big[Z_n(t,x)-Z(t,x)\Big]_+dx\;&\le\;\int_{\bar x+\delta}^{b-\delta}\Big(Z^\varepsilon(t,x)-Z(t,x)\Big)\,dx\\ &+\lim_{n\to\infty}\int_{\bar x+\delta}^{b-\delta}\Big|Z^\varepsilon_n(t,x)-Z^\varepsilon(t,x)\Big|\,dx+\lim_{n\to\infty}\int_{\bar x+\delta}^{b-\delta}\Big(\widehat Z^\varepsilon_n(t,x)-Z^\varepsilon_n(t,x)\Big)\,dx\\ &\le\;\varepsilon+0+0,\end{aligned} \tag{4.82}$$

where $Z(t,x)$ is defined in (4.53) and we used the notation $[s]_+\doteq\max\{s,0\}$.

A similar estimate on the distribution of Z_n from below yields the complementary upper bound on the distribution of $x_n(t)$. We only sketch the argument, because it is almost identical to the previous one. Given $\varepsilon>0$, we can find a smooth function ζ^ε and a constant $\eta=\eta(\varepsilon)$ such that

$$\delta_2\;\le\;\zeta(x)\;<\;\zeta^\varepsilon(x)\;\le\;1+\sup_{\bar x+\delta\le x\le b-\delta}\zeta(x)$$

and such that the solution $y^\varepsilon(\cdot)$ of the Cauchy problem

$$\dot y\;=\;\zeta(y),\qquad\qquad y(0)\;=\;y_0+2\eta$$

satisfies

$$y^\varepsilon(t)\;\in\;[y(t),\,y(t)+\varepsilon]\qquad\qquad\text{for all } t\in[0,T].$$

We define the corresponding drifts as $\zeta^\varepsilon_n\;=\;\max\{\zeta^\varepsilon(x),\zeta_n(x)\}$. Then we consider the same parabolic equations (4.58)-(4.59) replacing the boundary conditions

(4.62)-(4.63) with

$$Z_n^\varepsilon(t,\bar x+\delta) \;=\; \widehat Z_n^\varepsilon(t,\bar x+\delta) \;=\; 0, \tag{4.83}$$

$$Z_{n,x}^\varepsilon(t,b-\delta) \;=\; \widehat Z_{n,x}^\varepsilon(t,b-\delta) \;=\; 0. \tag{4.84}$$

As initial conditions we take (4.60), replacing (4.61) with

$$\varphi_0(x) \;=\; \begin{cases} 0 & \text{if} \quad x<y_0, \\ 1 & \text{if} \quad x>y_0+2\eta\,, \end{cases} \qquad\qquad 0\le \varphi_0'(x)\le \eta\,. \tag{4.85}$$

In place of (4.82) we now obtain

$$\begin{aligned}\lim_{n\to\infty}\int_{\bar x+\delta}^{b-\delta}\Big[Z(t,x)-Z_n(t,x)\Big]_+\,dx \;\le\; &\int_{\bar x+\delta}^{b-\delta}\Big(Z(t,x)-Z^\varepsilon(t,x)\Big)\,dx\\ &+\lim_{n\to\infty}\int_{\bar x+\delta}^{b-\delta}\Big|Z_n^\varepsilon(t,x)-Z^\varepsilon(t,x)\Big|\,dx+\lim_{n\to\infty}\int_{\bar x+\delta}^{b-\delta}\Big(Z_n^\varepsilon(t,x)-\widehat Z_n^\varepsilon(t,x)\Big)\,dx\\ \le\;\; &\varepsilon+0+0.\end{aligned} \tag{4.86}$$

Combining (4.82) with (4.86) we obtain (4.32), in the case where $y_0\in\,]\bar x,b[\,$.

The case $y_0\in\,]a,\bar x[\,$ can be handled in an entirely similar way.

6 - Analysis of the jumps in p, in the region where $V<W$.

The characterization of the limit dynamics at (4.25) was derived under the assumption that all jumps in $p(\cdot)$ contained inside the open interval $]a,b[$ where $V<W$ satisfy (4.5).

Indeed, assume that p has large jump at a point $y\in\,]a,b[\,$. Being non-increasing, the function p has bounded variation. Hence there exists $\varepsilon>0$ such that all the jumps of $p(\cdot)$ contained inside $]y-\varepsilon,y[\,\cup\,]y,y+\varepsilon[$ are suitably small, so that (4.26) holds. By the previous steps, the function $\zeta(\cdot)$ at (4.38) has a constant sign on $]y-\varepsilon,y[$ and on $]y,y+\varepsilon[$.

Here we relax the smallness assumption (4.5) on all the jumps of p to just the jumps under condition (ii) in Theorem 3, which assumes that large jumps in p can only happen at point $y\in]a,b[$ where the dynamics separates or converges in its neighbourhood, namely, $\zeta(x)<0$ on $]y-\varepsilon,y[$ and $\zeta(x)>0$ on $]y,y+\varepsilon[$ or $\zeta(x)>0$ on $]y-\varepsilon,y[$ and $\zeta(x)<0$ on $]y,y+\varepsilon[$. All the other jumps should satisfy the condition (4.5).

Various cases must be considered.

CASE 1. The speed $\zeta(x)$ has the same sign on the intervals $]y-\varepsilon, y[$ and $]y, y+\varepsilon[$. Due to the smallness assumption in p, in this case, for initial points x_0 close y, the solution $t \mapsto x(t, x_0)$ of the Cauchy problem

$$\dot{x} \;=\; \zeta(x), \qquad\qquad x(0) = x_0 \tag{4.87}$$

depends continuously on x_0. By the convergence (4.32), the same is true of the limit of the random variables x_n in (4.30).

Recalling that $p(x_0)$ is defined at (2.3) as the expected return for the investors, when the initial debt size is x_0, the continuous dependence on x_0 implies that p is continuous near y. Therefore, no jump occurs.

CASE 2. $\zeta(x) > 0$ on $]y-\varepsilon, y[$ and $\zeta(x) < 0$ on $]y, y+\varepsilon[$. Though p might have large jumps at y, this still implies that the solutions of (4.87) depend continuously on the initial point x_0. As in the previous case, by (4.32) and (2.3) we conclude that p is continuous at the point y.

CASE 3. The remaining case is when $\zeta(x) < 0$ on $]y-\varepsilon[$ and $\zeta(x) > 0$ on $]y+\varepsilon[$. This is the only possible case where p can have a jump. Indeed, $\bar{x} = y$ is precisely the point inside $[a, b]$ where the dynamics changes sign. This achieves a proof of (4.20), removing the smallness assumption on the jumps of $p(\cdot)$.

7 - Behavior on the region where $V = W$.

We now consider the coincidence set

$$\Omega \;\doteq\; \{x \in [0, M]\,; \quad V(x) = W(x-)\}, \tag{4.88}$$

and make the assumption that Ω contains finitely many points. We claim that $p(y) = p^\sharp(y)$ for all $y \in \Omega$.

From the previous analysis, on the finitely many open intervals where $V < W$

we either have

$$V'(x) \;=\; F^+(x, V(x), p(x)), \qquad \frac{(\lambda + r)x - u^\sharp(p(x), V'(x))}{p(x)} - (\lambda + \mu)x \;<\; 0, \tag{4.89}$$

or

$$V'(x) \;=\; F^-(x, V(x), p(x)), \qquad \frac{(\lambda + r)x - u^\sharp(p(x), V'(x))}{p(x)} - (\lambda + \mu)x \;>\; 0. \tag{4.90}$$

We again consider various cases.

CASE 1. There exists $\delta > 0$ such that (4.90) holds on $]y - \delta, y[$ and (4.89) holds on $]y, y + \delta[$.

Notice that, for any open interval I where (4.89) holds, p satisfies the ODE

$$p'(x) \;=\; \frac{(\rho(x) + r + \lambda)p - (r + \lambda + \theta(x)\rho(x))}{\zeta(x)} \;\leq\; 0. \tag{4.91}$$

Since $\zeta(x) < 0$ on I, (4.91) implies

$$p(x) \;\geq\; p^\sharp(x) \doteq \frac{r + \lambda + \theta(x)\rho(x)}{r + \lambda + \rho(x)}. \tag{4.92}$$

Similarly, for any I where (4.90) holds, $p(x) \leq p^\sharp(x)$. Since $p(\cdot)$ is nonincreasing, we conclude that p is continuous at y and $p(y) = p^\sharp(y)$.

CASE 2. There exists $\delta > 0$ such that (4.89) holds on $]y - \delta, y[\cup]y, y + \delta[$. If $p(\cdot)$ is continuous at y, then

$$\zeta(x) \;\to\; 0$$

as $x \to y$. If $p(y) \neq p^\sharp(y)$, since (4.91) is satisfied on $]y, y + \delta[$, we must have $p(y) > p^\sharp(y)$. This implies that $p'(x) \to +\infty$ as $x \to y$, and thus $W'(x) \to +\infty$ as $x \to y$. However, $V(x)$ is Lipschitz continuous and $V(\cdot) \leq W(\cdot)$, reaching a contradiction.

If $p(\cdot)$ is discontinuous at y, by construction we have $\zeta(x) \to 0$ as $x \to y-$. Then the similar analysis shows that $p(y-) = p^\sharp(y)$ and $p^\sharp(y+) \geq p^\sharp(y)$, providing again a contradiction.

We conclude that in this case $p(\cdot)$ must be continuous at y and $p(y) = p^\sharp(y)$.

CASE 3. There exists $\delta > 0$ such that (4.90) holds on $]y - \delta, y[\cup]y, y + \delta[$.

If $p(\cdot)$ is continuous at y, we obtain that

$$\zeta(x) \to 0$$

as $x \to y$. If $p(y) \neq p^\sharp(y)$, since (4.91) is satisfied on $]y, y+\delta[$, we must have $p(y) < p^\sharp(y)$. A similar contradiction is obtained as in CASE 2.
If $p(\cdot)$ is discontinuous at y, by construction we have $\zeta(x) \to 0$ as $x \to y-$. A similar analysis now shows that $p(y-) = p^\sharp(y)$.
We conclude that in this case $p(y-) = p^\sharp(y)$ and p can have a downward jump at y.

8 - Behavior near $\hat{x}$.

By the previous analysis, V is continuous everywhere on $[0, M]$ except for a possible jump at $\hat{x}$. By the definition of $\hat{x}$, for any $x > \hat{x}$ one has $\big(\lambda + r - (\lambda + \mu)p(x)\big)x > 1$. In other words, if the debt becomes larger than $\hat{x}$, then it must keep increasing, reaching the bankruptcy level M in finite time.

On the other hand, for x close to $\hat{x}$ but smaller that this threshold, various cases can arise.

CASE 1. $\zeta(x) < 0$ for $x \in]\hat{x} - \varepsilon, \hat{x}[$. In this case, the evolution generated by the optimal feedback is discontinuous at $\hat{x}$. As a result, the discounted price p will have a jump at $\hat{x}$ so that $\big(\lambda + r - (\lambda + \mu)p(\hat{x}-)\big)\hat{x} < 1$. This guarantees that it is possible to decrease the debt, starting from $\hat{x}$. Notice that in this case, V may also have a jump at $\hat{x}$.

CASE 2. $\zeta(x) > 0$ on $]\hat{x} - \varepsilon, \hat{x}[$ and $V(\hat{x}-) = W(\hat{x}-)$. In this case, we claim that V is Lipschitz continuous on $]\hat{x} - \varepsilon, \hat{x}[$ and has a jump at $\hat{x}$. Moreover, one has

$$p(\hat{x}-) \;=\; p^\sharp(\hat{x}), \qquad\qquad \big(\lambda + r - (\lambda + \mu)p^\sharp(\hat{x})\big)\hat{x} \;<\; 1.$$

Notice that the assumption $V(\hat{x}-) = W(\hat{x}-)$ implies $W(\hat{x}-) < +\infty$. In turn this implies

$$\big(\lambda + r - (\lambda + \mu)p(\hat{x}-)\big)\hat{x} < 1, \quad V'(\hat{x}^-) = F^-(\hat{x}, V(\hat{x}-), p(\hat{x}-)) < \xi^\sharp(\hat{x}, p(\hat{x}-)) < +\infty,$$

so V is Lipschitz continuous on $]\hat{x} - \varepsilon, \hat{x}[$. Then we show that $p(\hat{x}-) = p^\sharp(\hat{x})$. Using the similar argument in the previous analysis, if $p(\hat{x}-) \neq p^\sharp(\hat{x})$, then we have $p'(x) \to +\infty$ as $x \to \hat{x}-$, and thus $W'(x) \to +\infty$ as $x \to \hat{x}-$. However, $V(x)$ is Lipschitz continuous on $]\hat{x}-\varepsilon, \hat{x}[$ and $V(\hat{x}-) = W(\hat{x}-)$, reaching a contradiction.

CASE 3. $\zeta(x) > 0$ on $]\hat{x} - \varepsilon, \hat{x}[$ and $V(\hat{x}-) < W(\hat{x}-)$. By the previous analysis, this implies that there exists $\delta > 0$ such that $V(x) < W(x) - \delta$ on $]\hat{x} - \varepsilon, \hat{x}[$. By the assumption as in (4.36) that the jump of p at $\hat{x}$ is sufficiently small, we obtain that the solutions of (4.87) depend continuously on the initial point x_0 for any $x_0 \in]\hat{x} - \varepsilon, \hat{x} + \varepsilon[$. By (2.3) and (2.12) we conclude that both p and V are continuous at the point $\hat{x}$. In this case, the evolution generated by the optimal feedback control is continuous at $\hat{x}$ and there exists some $\hat{x}^* < \hat{x}$ such that for any $x > \hat{x}^*$, the debt must keep increasing, reaching the bankruptcy level M in finite time.

9 - A limit semigroup.

By the previous analysis, if $]a, b[$ is an interval where $V < W$, then it can be partitioned into two subintervals, where either (4.89) or (4.90) hold.

Given the limit functions V, p, and $u^*(x) = u^\sharp(p(x), V'(x))$, we consider the ODE

$$\dot{x} \;=\; \frac{(\lambda + r)x - u^*(x)}{p(x)} - (\lambda + \mu)x\,. \tag{4.93}$$

Since the control u^* is not continuous in general, for any initial datum x_0 several Carathéodory solutions may exist. We can define a semigroup of solutions by first setting

$$S_t x_0 \;=\; x_0 \qquad\qquad \text{for all } x_0 \in \Omega \;\doteq\; \{x\,;\;\; V(x) = W(x-)\}\,. \tag{4.94}$$

On the other hand, if $x_0 \notin \Omega$, let $]a, b[$ be the connected component of $[0, M[\,\backslash\Omega$ which contains x_0. Let $\bar{x} \in [a, b]$ and $\{V'(x) | x \neq \bar{x}\}$ be as in (4.25). We then define

$$t \;\mapsto\; x(t) \;=\; S_t x_0$$

to be the unique solution of

$$\dot{x} \;=\; \left[\frac{(\lambda + r)x - u^\sharp(p, V'(x-))}{p(x-)} - (\lambda + \mu)x\right], \tag{4.95}$$

as long as $x(t) \notin \Omega$. We then extend the definition of S by setting

$$\tau(x_0) \doteq \sup\Big\{t \geq 0\,; \quad S_t x_0 \notin \Omega\Big\},$$

$$S_t x_0 \doteq S_{\tau(x_0)} x_0 \qquad\qquad \text{for all } t \geq \tau(x_0).$$

In other words, the evolution stops as soon as the trajectory hits Ω.

10 - A Nash equilibrium solution.

Finally, we claim that the limit values V, p, u^* provide a Nash equilibrium solution to the debt management problem. More precisely:

(i) The discounted price $p(\cdot)$ is consistent with the dynamics defined at (4.94)-(4.95). Indeed, for every initial point x_0 (with the possible exception of points $\bar{x}$ in (4.21) where the dynamics is discontinuous), one has

$$\begin{aligned} p(x_0) = \;& 1 - \big[1 - \theta(M)\big] \exp\left\{ - \int_0^{T^M} [r + \lambda + \rho(S_t(x_0))] dt \right\} \\ & - \int_0^{T^M} \big[1 - \theta(S_t(x_0))\big]\, \rho(S_t(x_0)) \exp\left\{ - \int_0^t \big[r + \lambda + \rho(S_s(x_0))\big]\, ds \right\} dt. \end{aligned} \tag{4.96}$$

(ii) Given $p(\cdot)$, the function V is the value function for the corresponding deterministic optimal control problem, namely

$$\text{minimize:} \;\; E\left[\int_0^{T_b} e^{-rt} L(u(t))\, dt + B\, e^{-rT_b} \right] \tag{4.97}$$

subject to

$$dx = \left[-(\lambda + \mu)x + \frac{(\lambda + r)x - u(t)}{p(x)} \right] dt\,, \quad x(0) = x_0, \tag{4.98}$$

and $u^*(x) = u^\sharp(p(x), V'(x))$ is an optimal feedback control.

To prove (i), we observe that by construction the result is true for all x_0 in the finite set Ω where V and W coincide. The boundary conditions imply it is also true for $x_0 = M$.

To show that it remains true on any connected component $]a,b[\ \subset\ [0,M]\setminus\Omega$, with $a,b\in\Omega$, we observe that $p(x)$ satisfies the ODE

$$p'(x) \ = \ \frac{(\rho(x)+r+\lambda)p-(r+\lambda+\theta(x)\rho(x))}{\zeta(x)} \tag{4.99}$$

separately on the subintervals $]a,\bar{x}[$ and $]\bar{x},b[$, where $\bar{x}$ is as in (4.25). For any point $x_0\in]a,\bar{x}[$, we have

$$\frac{d}{dt}p(S_t(x_0)) \ = \ p'(S_t(x_0))\cdot\frac{d}{dt}S_t(x_0) \ = \ (\rho(x)+r+\lambda)p(S_t(x_0))-(r+\lambda+\theta(x)\rho(S_t(x_0))). \tag{4.100}$$

Since $\lim_{t\to+\infty} S_t(x_0)=a\in\Omega$, we have

$$\lim_{t\to+\infty} p(S_t(x_0)) \ = \ p(a) \ = \ \frac{r+\lambda+\theta(a)\rho(a)}{r+\lambda+\rho(a)}. \tag{4.101}$$

Solving (4.99) with boundary condition (4.101) we achieve (4.96). Notice that in (4.101), the continuity of $p(\cdot)$ at a is shown under CASE 1 and CASE 2 in step 7. The analysis of $p(\cdot)$ on $]\bar{x},b[$ is entirely similar. The only difference is that according to the analysis under CASE 1 and CASE 3 in step 7, $p(\cdot)$ is either continuous at b or $p(b-)=p^\sharp(b)$.

Finally, we prove (ii). For the given function $p(\cdot)$, call V^* the value function for the optimal control problem (4.97)-(4.98). For any initial value $x_0\in[0,M]$, the feedback control $u^*(x)\doteq u^\sharp(p(x-),V'(x-))$ yields the cost $V(x_0)$. Thus we have

$$V^*(x_0)\leq V(x_0).$$

To prove optimality we need to show that $V^*(x_0)\leq V(x_0)$. Consider any measurable control function $u(t):[0,+\infty[\to[0,1]$. Calling $t\to x(t)$ the solution to

$$\dot{x} \ = \ \left(-\lambda-\mu+\frac{\lambda+r}{p(x)}\right)x-\frac{u(t)}{p(x)}, \qquad x(0)=x_0,$$

a standard computation yields that

$$E\left[\int_0^{T_b} e^{-rt}L(u(t))\,dt + B\,e^{-rT_b}\right] = \int_0^{T^M} \gamma(t)\Big\{\rho(x(t))\,B+L(u(t))\Big\}\,dt + \gamma(T^M)\,B,$$

where $T^M = \inf\big\{t \geq 0; x(t) = M\big\} \in]0,+\infty]$ is the bankruptcy time and

$$\gamma(t) \doteq e^{-rt}\exp\left\{-\int_0^t \rho(x(s))\,ds\right\}.$$

Then we need to show that

$$\int_0^{T^M} \gamma(t)\Big\{\rho(x(t))\,B + L(u(t))\Big\}\,dt \;+\; \gamma(T^M)\,B \geq V(x_0). \tag{4.102}$$

For $t \in [0, T^M]$, consider the function

$$\phi^u(t) \doteq \int_0^t \gamma(s)\Big\{\rho(x(s))\,B + L(u(s))\Big\}\,ds \;+\; \gamma(t)V^*(x(t)).$$

Notice that ϕ^u is absolutely continuous except for a possible upward jump at $\hat{t} \doteq \sup_{t<T^M}\{x(t) < \hat{x}\}$. By the definition of $\hat{x}$, for any control $u(\cdot)$ we have $\dot{x}(t) > 0$ for all $t \geq \hat{t}$. Indeed, $\hat{x}$ is a "point of no return": when the debt crosses $\hat{x}$, there is no way to reduce it.

At any Lebesgue point t of $u(\cdot)$, with $t \neq \hat{t}$, we compute

$$\begin{aligned}
\tfrac{d}{dt}\phi^u(t) \;&=\; \gamma(t)\left[\rho(x(t))B + L(u(t)) - (r+\rho(x(t)))V(x(t)) + V'(x(t))\dot{x}(t)\right] \\
&=\; \gamma(t)\Bigg[\rho(x(t))B + L(u(t)) - (r+\rho(x(t)))V(x(t)) \\
&\qquad +V'(x(t))\left(\left(-\lambda-\mu+\frac{\lambda+r}{p(x)}\right)x - \frac{u(t)}{p(x)}\right)\Bigg] \\
&\geq\; \gamma(t)\Bigg[\min_{\omega\in[0,1]}\{L(\omega) - \frac{V'(x(t))}{p(x)}\omega\} - (r+\rho(x(t)))V(x(t)) + \rho(x(t))B \\
&\qquad +V'(x(t))\left(\left(-\lambda-\mu+\frac{\lambda+r}{p(x(t))}\right)x\right)\Bigg] \\
&=\; \gamma(t)\left[H(x(t),V'(x(t)),p(x(t))) - (r+\rho(x(t)))V(x(t)) + \rho(x(t))B\right] \;=\; 0.
\end{aligned}$$

Since $V(\hat{x}+) \geq V(\hat{x}-)$, we also have $\phi^u(\hat{t}-) \leq \phi^u(\hat{t}+)$. Therefore

$$V(x_0) \;=\; \phi^u(0) \;\leq\; \lim_{t\to T^M-}\phi^u(t) \;=\; \int_0^{T^M}\gamma(t)\Big\{\rho(x(t))\,B + L(u(t))\Big\}\,dt + \gamma(T^M)\,B.$$

This completes the proof of Theorem 3. □

Chapter 5

Generic structure and stability of Stackelberg equilibrium

1 Introduction

In the theory of non-cooperative games, the concept of Stackelberg equilibrium [58] has been widely investigated, due to its several applications to economic models [21]. In a basic setting, a game for two players can be formulated as follows.

- Player 1 (the leader) chooses $x \in X$ and seeks to minimize his cost $F(x, y)$.
- Player 2 (the follower) chooses $y \in Y$ and seeks to minimize his cost $G(x, y)$.

Here X, Y are topological spaces, while $F, G : X \times Y \mapsto \mathbb{R}$ are continuous functions.

For a given $x \in X$, the set of **best replies** for the follower is defined as

$$R(x) \doteq \Big\{ y^* \in Y\,; \quad G(x, y^*) \leq G(x, y) \ \text{ for all } y \in Y \Big\}. \tag{1.1}$$

We say that a couple $(x^*, y^*) \in X \times Y$ is a **Stackelberg equilibrium** if $y^* \in R(x)$ and

$$F(x^*, y^*) \ \leq \ F(x, y) \qquad \text{for all } x \in X \text{ and } y \in R(x).$$

This models a situation where the leading player announces his strategy $x \in X$ in advance, and the follower chooses a reply $y \in Y$ which minimizes his own cost

$G(x, y)$.

In the literature, the existence of a Stackelberg equilibrium is known under fairly general assumptions [21, 8, 44, 48, 56]. A major related issue is the uniqueness and stability of this equilibrium. Namely, if the cost functions F, G are slightly perturbed, does the new game still have a unique solution, close to the original one? This problem has been investigated in [44, 49, 48], within the general class of continuous cost functions. As pointed out in [44], it is not possible to obtain, under sufficiently general conditions, existence and stability results for the exact Stackelberg solutions. For this reason, in the above papers, a weaker concept of ϵ-solution was used.

Aim of the present chapter is to study stability for the best reply map and for exact Stackelberg solutions, within a class of smooth functions $F, G : \mathbb{R}^m \times \mathbb{R}^n \mapsto \mathbb{R}$. In this setting, examples of games with multiple equilibria are easy to construct. However, our main results show that, for "most" functions F, G (in a topological sense), the Stackelberg equilibrium is unique and is stable under small perturbations. While the results in [44, 49, 48] are based general topological principles, our stability results rely on completely different techniques, stemming from differential geometry; namely: Sard's theorem and a multi-jet version of Thom's transversality theorem [18, 38].

The analysis of Stackelberg equilibria can be accomplished in two steps.

(i) Study the graph of the best reply map $R(\cdot)$, namely

$$\text{Graph}(R) \;=\; \{(x, y)\,;\; y \in R(x)\} \;\subset\; \mathbb{R}^m \times \mathbb{R}^n. \tag{1.2}$$

Show that, for a generic function $G \in \mathcal{C}^3(R^{m+n})$, this graph can be expressed in terms of finitely many equalities or inequalities, in generic position.

(ii) Study the constrained minimization problem for the function F, restricted to Graph(R). Show that, for a generic function $F \in \mathcal{C}^2(R^{m+n})$, a unique global minimum exists, which is stable under small perturbations.

We recall that a property is said to be **generic** if it holds on the intersection of countably many open dense sets. The main goal of the present chapter is to study the stability of the equilibrium (x^*, y^*) under perturbations of the cost functions

F, G, in the following sense.

Definition. *Given the cost functions F, G, we say that the Stackelberg equilibrium (x^*, y^*) is* **strongly stable** *if, for any $\varepsilon > 0$ there exists $\delta > 0$ such that the condition*

$$\|\widetilde{F} - F\|_{\mathcal{C}^2} \leq \delta, \qquad \|\widetilde{G} - G\|_{\mathcal{C}^3} \leq \delta, \tag{1.3}$$

implies that the perturbed game, with F, G replaced by $\widetilde{F}, \widetilde{G}$, has a unique Stackelberg equilibrium $(\tilde{x}, \tilde{y}) \in X \times Y$. Moreover,

$$|\tilde{x} - x^*| \leq \varepsilon, \qquad |\tilde{y} - y^*| \leq \varepsilon.$$

We first consider the basic case where $X = Y = [0, 1]$. Using Thom's transversality theorem we prove that the set of couples $(F, G) \in \mathcal{C}^2 \times \mathcal{C}^3$ that yield a unique, strongly stable Stackelberg equilibrium is open and dense. In a later section we prove similar results in the case where the strategy of the follower takes values in a multi-dimensional set. Namely: $X = \mathbb{R}$ while $Y = \mathbb{R}^n$.

The remainder of the chapter is organized as follows. Section 2 collects some basic stability results on the stability of the minimizer of a $\mathcal{C}^2$ function F restricted to a set $\mathcal{A} \subset \mathbb{R}^N$ which is defined in terms of finitely many equalities or inequalities. In Section 3 we begin the analysis of the best reply map. When $X = \mathbb{R}^m$, $Y = \mathbb{R}^n$, the necessary conditions for optimality imply that

$$\text{Graph}(R) \subseteq \mathcal{M} \doteq \{(x, y) \in \mathbb{R}^{m+n}; \quad \nabla_y G(x, y) = 0\}. \tag{1.4}$$

As a first step, we observe that, by Sard's theorem, for a generic function $G \in \mathcal{C}^3$ the set $\mathcal{M}$ in (1.4) is a $\mathcal{C}^2$ manifold. Theorem 4.1 in Section 4 describes in detail the generic structure of the best reply map, in the one-dimensional case where $X = Y = [0, 1]$. In turn, this yields the generic stability of the Stackelberg equilibrium, proved in Section 5.

The analysis in Section 6 shows that similar results on the structure of the best reply map and on the stability of the Stackelberg equilibrium still hold, in the case where the follower chooses his strategy within a multi-dimensional space. Namely, $X = \mathbb{R}$ while $Y = \mathbb{R}^n$. Section 7 contains some concluding remarks. In particular, we discuss the possible extension of our results to the case where the strategy of the leader also lies in a multi-dimensional set $X \subseteq \mathbb{R}^m$. Finally, an Appendix collects some basic results from differential geometry. A multi-jet version of Thom's transversality theorem is proved, which provides a key ingredient for our analysis.

In connection with Nash equilibria for special classes of non-cooperative games, generic properties of solutions have been studied in [41, 50].

2 Minima of generic functions on generic sets

In a Stackelberg game, the leader seeks to minimize his own cost $F(x, y)$, restricted to the graph of the best reply map $R(\cdot)$. As it will be shown in a later section, for a generic function G in (1.1), this graph can be expressed in terms of a finite set of equalities and inequalities.

We thus consider here a constrained minimization problem of the general form

$$\min_{x \in \mathcal{A}} f(x), \tag{2.1}$$

under the following assumptions.

(B1) *$f : \mathbb{R}^N \mapsto \mathbb{R}$ is twice continuously differentiable. Moreover,* $\lim_{|x|\to\infty} f(x) = +\infty$.

(B2) *$\mathcal{A} \subset \mathbb{R}^N$ is a nonempty closed set, described in terms of finitely many equalities and inequalities:*

$$\mathcal{A} \;=\; \Big\{ x \in \mathbb{R}^N\,; \quad \phi_i(x) = 0, \quad \psi_j(x) \geq 0, \qquad i \in \mathcal{I}, \quad j \in \mathcal{J} \Big\}. \tag{2.2}$$

(B3) *All functions ϕ_i, ψ_j are in $\mathcal{C}^2(\mathbb{R}^N)$. Moreover, for any subset $\mathcal{J}' \subseteq \mathcal{J}$ and any point x such that*

$$\phi_i(x) = \psi_j(x) = 0 \qquad \text{for all} \quad i \in \mathcal{I}, \quad j \in \mathcal{J}', \tag{2.3}$$

the gradients

$$\nabla\phi_i(x), \quad \nabla\psi_j(x), \qquad\quad i \in \mathcal{I}, \quad j \in \mathcal{J}', \tag{2.4}$$

are linearly independent.

Roughly speaking, we would like to prove that, for all functions f, ϕ_i, ψ_j in an open dense set of $\mathcal{C}^2$, the problem (2.1) admits a unique minimizer, which is stable under perturbations. With this goal in mind, we denote by $\mathcal{F}^\infty$ the family of all functions f satisfying **(B1)**. Since $\mathcal{F}^\infty$ is not a vector space, a suitable topology must first be defined.

We recall that $\mathcal{C}^k(\mathbb{R}^N)$ is a Banach space with the norm

$$\|f\|_{\mathcal{C}^k} \doteq \sup_{|\alpha|\leq k} \sup_{x\in\mathbb{R}^N} \Big| D^\alpha f(x) \Big| . \tag{2.5}$$

Using a standard notation (see for example [35]), here

$$D^\alpha f(x) \doteq \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1} \cdots \partial_{x_N}^{\alpha_N}} f(x)$$

denotes a partial derivative of order $|\alpha| = \alpha_1 + \cdots + \alpha_N$. On the set $\mathcal{F}^\infty$ we now consider the distance

$$d_2(f,g) \doteq \min\Big\{1,\ \|f-g\|_{\mathcal{C}^2}\Big\}. \tag{2.6}$$

With this distance, $(\mathcal{F}^\infty, d_2)$ is a complete metric space. Notice that the convergence $f_n \to f$ holds iff $\|f_n - f\|_{\mathcal{C}^2} \to 0$.

Next, we introduce

Definition 2.1. *Under the assumptions* **(B2)-(B3)**, *we say that the global minimum (2.1) is attained at a point* $\bar{x}$ **in generic position** *if the following conditions hold.*

(i) $\bar{x} \in \mathcal{A}$ *is the unique point where the global minimum is attained.*

(ii) Setting $\mathcal{J}' \doteq \{j \in \mathcal{J}\,;\quad \psi_j(\bar{x}) = 0\}$, *there exists constants* α_i, β_j, $i \in \mathcal{I}$, $j \in \mathcal{J}'$ *such that*

$$\nabla f(\bar{x}) = \sum_{i\in\mathcal{I}} \alpha_i \nabla\phi_i(\bar{x}) + \sum_{j\in\mathcal{J}'} \beta_j \nabla\psi_j(\bar{x}), \tag{2.7}$$

with $\beta_j > 0$ *for all* $j \in \mathcal{J}'$.

(iii) There exists $\rho, \varepsilon > 0$ such that

$$f(x) - f(\bar{x}) \ \geq \ \varepsilon\,|x-\bar{x}|^2 \qquad \textit{for all } x \in \mathcal{A} \textit{ with } |x-\bar{x}| \leq \rho\,. \tag{2.8}$$

We remark that, by the assumption **(B3)**, the equations (2.3) define a manifold $\mathcal{M}_{\mathcal{I}\cup\mathcal{J}'} \subset \mathbb{R}^N$ of dimension $N - |\mathcal{I}| - |\mathcal{J}'|$. By assumption, the restriction of f to this manifold has a local minimum at $\bar{x}$. According to (2.8), the Hessian matrix $D^2 f(\bar{x})$ has full rank (hence it is strictly positive definite) at $\bar{x}$. We remark that this rank condition is invariant under coordinate changes on the submanifold $\mathcal{M}_{\mathcal{I}\cup\mathcal{J}'}$.

As shown by the following theorem, minima attained in generic position (according to Definition 2.1) are stable w.r.t. small $\mathcal{C}^2$ perturbations in the cost function f or in the constraints φ_i, ψ_j.

Theorem 2.1. *Let $f, \phi_i, \psi_j \in \mathcal{C}^2(\mathbb{R}^N)$, $i \in \mathcal{I}$, $j \in \mathcal{J}$, be such that the assumptions* **(B1)-(B3)** *hold. Moreover, assume that the global minimum (2.1) is attained at a point $\bar{x}$ in generic position. Then there exists $\delta > 0$ such that, if*

$$\|\tilde{f} - f\|_{\mathcal{C}^2} \ \leq \ \delta, \qquad \|\tilde{\phi}_i - \phi_i\|_{\mathcal{C}^2} \ \leq \ \delta, \qquad \|\tilde{\psi}_j - \psi_j\|_{\mathcal{C}^2} \ \leq \ \delta \tag{2.9}$$

for all $i \in \mathcal{I}$, $j \in \mathcal{J}$, then the corresponding optimization problem

$$\min_{x \in \widetilde{\mathcal{A}}} \tilde{f}(x) \tag{2.10}$$

has a unique minimizer $\tilde{x}$, also in generic position. Here $\widetilde{\mathcal{A}}$ is the set defined as in (2.2), with ϕ_i, ψ_j replaced by $\tilde{\phi}_i, \tilde{\psi}_j$, respectively. Moreover, for some constant C one has

$$|\tilde{x}-\bar{x}| \ \leq \ C\cdot\max\Big\{\|\tilde{f}-f\|_{\mathcal{C}^2}, \ \ \|\tilde{\phi}_i-\phi_i\|_{\mathcal{C}^2}, \ \ \|\tilde{\psi}_j-\psi_j\|_{\mathcal{C}^2}\,; \quad i \in \mathcal{I}, \ j \in \mathcal{J}\Big\}. \tag{2.11}$$

Proof. 1. As a first step we claim that, for every $r > 0$, there exists $\delta > 0$ such that the inequalities in (2.9) imply that (2.10) has a minimizer $\tilde{x}$ with

$$|\tilde{x} - \bar{x}| \ \leq \ r. \tag{2.12}$$

Indeed, by **(B1)-(B3)** and since $\bar{x}$ is the unique global minimizer of f, by choosing $\delta_1 > 0$ sufficiently small we have the implication

$$\|\tilde{f} - f\|_{\mathcal{C}^2} \;\leq\; \delta_1 \qquad \Longrightarrow \qquad \sup_{|x-\bar{x}|\leq\delta_1} \tilde{f}(x) \;<\; \inf_{|x-\bar{x}|\geq r} \tilde{f}(x). \tag{2.13}$$

Next, by choosing $\delta_2 > 0$ small enough, we have the implication

$$\begin{aligned} &\|\tilde{\phi}_i - \phi_i\|_{\mathcal{C}^2} \;\leq\; \delta_2, \qquad \|\tilde{\psi}_j - \psi_j\|_{\mathcal{C}^2} \;\leq\; \delta_2 \qquad \text{for all } i \in \mathcal{I}, \;\; j \in \mathcal{J} \\ &\Longrightarrow \qquad \widetilde{\mathcal{A}} \cap B(\bar{x}, \delta_1) \neq \emptyset \quad \text{and} \quad \tilde{\psi}_j(x) > 0 \quad \text{for all } x \in B(\bar{x}, \delta_1), \;\; j \notin \mathcal{J}'. \end{aligned} \tag{2.14}$$

Moreover, $B(\bar{x}, \delta_1)$ denotes the open ball centered at $\bar{x}$ with radius δ_1.

Now choose any $x_1 \in B(\bar{x}, \delta_1) \cap \widetilde{\mathcal{A}}$. For any x such that $|x - \bar{x}| \geq r$ by (2.13) it follows $\tilde{f}(x) > \tilde{f}(x_1)$. Hence the global minimum of $\tilde{f}$ cannot be attained outside $B(\bar{x}, r)$.

2. We now prove (2.11). Let $\tilde{x} \in B(\bar{x}, \delta_1)$ be a global minimizer of $\tilde{f}$, restricted to $\widetilde{\mathcal{A}}$. The first order necessary condition for optimality imply

$$\nabla\tilde{f}(\tilde{x}) \;=\; \sum_{i\in\mathcal{I}} \tilde{a}_i \nabla\widetilde{\phi}_i(\tilde{x}) + \sum_{j\in\mathcal{J}^\sharp} \tilde{b}_j \nabla\widetilde{\psi}_j(\tilde{x}), \tag{2.15}$$

for some coefficients $\tilde{a}_i, \tilde{b}_j$. Here $\mathcal{J}^\sharp \subseteq \mathcal{J}'$ is the set of indices $j \in \mathcal{J}$ such that $\widetilde{\psi}_j(\tilde{x}) = 0$.

By continuity, choosing $0 < \delta < \min\{\delta_1, \delta_2\}$ small enough, we obtain

$$\mathcal{J}^\sharp \;=\; \mathcal{J}', \qquad\qquad \tilde{b}_j \;>\; 0 \quad \text{for all } j \in \mathcal{J}'.$$

3. We now apply the implicit function theorem to the map

$$\Lambda: \;(\phi, \psi, x) \;\mapsto\; (\phi_i(x),\, \psi_j(x))_{i\in\mathcal{I}, j\in\mathcal{J}'}\,, \tag{2.16}$$

from a space $\mathcal{C}^2 \times \mathcal{C}^2 \times \mathbb{R}^N$ into $\mathbb{R}^{|\mathcal{I}|+|\mathcal{J}'|}$. We choose coordinates

$$x \;=\; (x', x'') \;=\; (x_1, \ldots, x_\nu\,,\; x_{\nu+1}, \ldots, x_N), \tag{2.17}$$

with $\nu = N - |\mathcal{I}| - |\mathcal{J}'|$ such that the equation $\Lambda(\phi, \psi, x', x'') = 0$ defines a $\mathcal{C}^2$ function

$$x'' \;=\; \varphi(x'),$$

for x' in a neighborhood of $\bar{x}'$.

By the implicit function theorem, one has

$$D_{x'}\varphi \;=\; -\left[\frac{\partial\Lambda(\phi,\psi)}{\partial x''}\right]^{-1}\frac{\partial\Lambda(\phi,\psi)}{\partial x'}.$$

By continuity, for all $\delta > 0$ small enough the matrix of partial derivatives $\frac{\partial\Lambda(\tilde{\phi},\tilde{\psi})}{\partial x''}$ still has full rank for all $x = (x', x'')$ in a neighborhood of $\bar{x} = (\bar{x}', \bar{x}'')$. Hence, the vector equation

$$\Lambda(\widetilde{\phi}, \widetilde{\psi}, x', x'') \;=\; 0,$$

defined as in (2.16), determines a $\mathcal{C}^2$ function

$$x'' \;=\; \tilde{\varphi}(x'), \qquad \text{with} \qquad D_{x'}\tilde{\varphi} = -\left[\frac{\partial\Lambda(\tilde{\phi},\tilde{\psi})}{\partial x''}\right]^{-1}\frac{\partial\Lambda(\tilde{\phi},\tilde{\psi})}{\partial x'},$$

for x' in a neighborhood of $\bar{x}'$.

Computing the second order derivatives of the implicit functions φ and $\tilde{\varphi}$, by (2.9) we obtain an estimate of the form

$$\|\tilde{\varphi} - \varphi\|_{\mathcal{C}^2(B(\bar{x}',\rho))} \;\leq\; C_0\delta, \tag{2.18}$$

for some constant C_0.

4. Now consider the map $x' \mapsto F(x') \doteq f(x', \varphi(x'))$. By the assumption **(B1)**, this has a strict minimum at $x' = \bar{x}'$. Moreover, by (2.8) the Hessian matrix $D^2_{x'}F(\bar{x}')$ is strictly positive definite.

By continuity, for $\delta > 0$ small enough, if (2.9) holds then the corresponding function $\widetilde{F}(x') = \tilde{f}(x', \tilde{\varphi}(x'))$ has strictly positive definite Hessian matrix $D^2_{x'}\widetilde{F}(x')$ at every point x' in a neighborhood of $\bar{x}'$. By possibly shrinking the value of δ, we conclude that $D^2_{x'}\widetilde{F}(\tilde{x}')$ is strictly positive definite. Observing that (2.15) holds with $\tilde{b}_j > 0$ for all $j \in \mathcal{J}'$, we obtain

$$\tilde{f}(x) - \tilde{f}(\tilde{x}) \;\geq\; \tilde{\varepsilon}\,|x - \tilde{x}|^2 \qquad \text{for all } x \in \tilde{\mathcal{A}} \text{ with } |x - \tilde{x}| \leq \tilde{\rho}\,, \tag{2.19}$$

for some $\tilde{\varepsilon}, \tilde{\rho} > 0$. Hence the global minimum of $\tilde{f}$ on $\widetilde{\mathcal{A}}$ is attained at a point $\tilde{x}$ in generic position.

5. It remains to prove the inequality (2.11), showing that the minimizer depends in a Lipschitz continuous way on the functions f, ϕ_i, ψ_j, w.r.t. the $\mathcal{C}^2$ norm. This follows from the first order necessary conditions for optimality, together with the implicit function theorem. Indeed, the point $\bar{x} = (\bar{x}', \bar{x}'')$ where the constrained minimum is attained is uniquely determined by the N equations

$$\phi_i(\bar{x}) = 0, \qquad \psi_j(\bar{x}) \;=\; 0, \qquad \nabla_{\bar{x}'} F(\bar{x}', \varphi(\bar{x}')) \;=\; 0. \tag{2.20}$$

In other words, $\bar{x}$ can be represented as a zero of the map

$$\Gamma : (f, \phi, \psi, x) \;\mapsto\; \Big(\phi_i(x),\, \psi_j(x),\, \partial_{x_k} F(x', \varphi(x'))\Big) \;\in\; \mathbb{R}^N. \tag{2.21}$$

Here $i \in \mathcal{I}$, $j \in \mathcal{J}'$, $k \in \{1, \ldots, \nu\}$ with $\nu = N - |\mathcal{I}| - |\mathcal{J}'|$, as in (2.17). We regard (2.21) as a map from $\mathcal{C}^2 \times \mathcal{C}^2 \times \mathcal{C}^2 \times \mathbb{R}^N$ into $\mathbb{R}^N$. In order to apply the implicit function theorem on a Banach space [32] and achieve the estimate (2.11), it suffices to check that the $N \times N$ matrix of partial derivatives $\frac{\partial \Gamma}{\partial x}$ has full rank in a neighborhood of $(f, \phi, \psi, \bar{x})$. Indeed, by the condition (2.8) one has

$$F(x', \varphi(x')) \;\geq\; F(\bar{x}', \varphi(\bar{x}')) + \varepsilon |x' - \bar{x}'|^2$$

for all x' in a neighborhood of $\bar{x}'$. This implies that the $\nu \times \nu$ Hessian matrix $D^2_{x'} F(x', \varphi(x'))$ has full rank at $\bar{x}'$. Furthermore, since the $\nu \times (N - \nu)$ matrix of partial derivatives

$$\left(\frac{\partial [\phi_i, \psi_j]}{\partial x''} \right)_{i \in \mathcal{I}, j \in \mathcal{J}'}$$

also has full rank at $\bar{x}''$, the $N \times N$ matrix of the partial derivatives $\frac{\partial \Gamma}{\partial x}$ has full rank at $(f, \phi, \psi, \bar{x})$. By continuity, $\frac{\partial \Gamma}{\partial x}$ has full rank in a neighborhood of $(f, \phi, \psi, \bar{x})$. Observing that Γ is Lipschitz continuous w.r.t. f, ϕ_i, ψ_j (in the $\mathcal{C}^2$ distance), this achieves the proof of (2.11). □

We now show that the conditions (i)–(iii) in Definition 2.1 are "generic". Indeed, they hold for an open, dense set of $\mathcal{C}^2$ functions f, φ, ψ.

Theorem 2.2. *Let $\phi_i, \psi_j \in \mathcal{C}^2(\mathbb{R}^N)$, $i \in \mathcal{I}$, $j \in \mathcal{J}$ be such that the assumptions* **(B2)-(B3)** *hold. Then, for an open dense set of functions $f \in \mathcal{F}^\infty$, the global minimum in (2.1) is attained in generic position.*

Proof. 1. Consider any $f \in \mathcal{F}^\infty$. Let $\bar{x} \in \mathcal{A}$ be a point where f attains its global minimum. Consider the set of indices

$$\mathcal{J}' \doteq \{j \in \mathcal{J}\,; \quad \psi_j(\bar{x}) = 0\}.$$

Introduce two smooth functions $\rho, \eta : \mathbb{R}_+ \mapsto [0,1]$, satisfying

$$\rho(s) = \begin{cases} 1 & \text{if } s \in [0, r_0], \\ 0 & \text{if } s \geq 2r_0\,, \end{cases} \qquad \rho'(s) \leq 0 \quad \text{for all } s > 0, \tag{2.22}$$

$$\eta(s) = \begin{cases} s^2 & \text{if } s \in [0, 1/2], \\ 1 & \text{if } s \geq 1\,, \end{cases} \qquad \eta'(s) \geq 0 \quad \text{for all } s > 0, \tag{2.23}$$

We then define a family of perturbed functions

$$f_\varepsilon(x) = f(x) + \varepsilon\rho(|x - \bar{x}|) \cdot \sum_{j \in \mathcal{J}'} \psi_j(x) + \varepsilon\eta(|x - \bar{x}|) \tag{2.24}$$

Choosing $r_0 > 0$ suitably small, it follows that $f_\varepsilon(x) > f(x)$ for all $x \neq \bar{x}$ and $\varepsilon > 0$.

Thanks to the properties of the cut-off functions ρ, η we have $\|f_\varepsilon - f\|_{\mathcal{C}^2} \to 0$ as $\varepsilon \to 0$.

2. It remains to check that, for every $\varepsilon > 0$, the global minimum of f_ε, which is attained at the single point $\bar{x}$, is in generic position. If $\bar{x}$ is a global minimizer of $f(\cdot)$ on $\mathcal{A}$, the first order necessary condition (2.7) implies

$$\nabla f_\varepsilon(\bar{x}) \;=\; \nabla f(\bar{x}) + \varepsilon \sum_{j\in\mathcal{J}'} \nabla\psi_j(\bar{x}) \;=\; \sum_{i\in\mathcal{I}} \alpha_i \nabla\phi_i(\bar{x}) + \sum_{j\in\mathcal{J}'} (\beta_j+\varepsilon)\nabla\psi_j(\bar{x}).$$

Therefore, for every $\varepsilon > 0$, the condition (2.7) is satisfied.

On the other hand, the inequality (2.8) follows from

$$f_\varepsilon(x) \;\geq\; f(x) + \varepsilon|x-\bar{x}|^2 \;\geq\; f_\varepsilon(\bar{x}) + \varepsilon|x-\bar{x}|^2 \qquad \text{for all } x\in\mathcal{A} \text{ with } |x-\bar{x}|\leq 1/2.$$

3. The two previous steps show that the set of functions f, for which the minimun is attained in generic position, is dense on $\mathcal{F}^\infty$. The fact that it is open follows from Theorem 2.1. □

Corollary 2.1. *Consider the set $\mathcal{F}^\sharp$ of functions $(f, \phi_i, \psi_j)_{i\in\mathcal{I}, j\in\mathcal{J}}$ such that either (i) the domain $\mathcal{A}$ in (2.2) is empty, or else (ii) $\mathcal{A}\neq\emptyset$ and the minimization problem (2.1) has a unique minimizer in generic position. Then $\mathcal{F}^\sharp$ is open and dense in $\mathcal{F}^\infty \times \mathcal{C}^2(\mathbb{R}^N)\times\ldots\times\mathcal{C}^2(\mathbb{R}^N)$.*

Proof. **1.** The openness of the set of functions (f, ϕ_i, ψ_j) that satisfy the conditions in Definition 2.1 is again a consequence of Theorem 2.1.

2. To prove that this set is dense in the $\mathcal{C}^2$ topology, in view of Theorem 2.2 it suffices to show that the set of $\mathcal{C}^2$ functions $(\phi_i, \psi_j)_{i\in\mathcal{I}, j\in\mathcal{J}}$ that satisfy the condition **(B3)** is dense. Toward this goal, let functions $\phi_i, \psi_j\in\mathcal{C}^2(\mathbb{R}^N)$ be given. For each subset $\mathcal{J}'\subset\mathcal{J}$ consider the map

$$x \;\mapsto\; (\phi_i(x),\, \psi_j(x))_{i\in\mathcal{I}, j\in\mathcal{J}'}$$

from $\mathbb{R}^N$ into $\mathbb{R}^{|\mathcal{I}|+|\mathcal{J}'|}$. By Sard's theorem [18, 38], a.e. $(y,z) = (y_i, z_j)\in\mathbb{R}^{|\mathcal{I}|}\times\mathbb{R}^{|\mathcal{J}'|}$ is a regular value of this map. Hence, taking the perturbations

$$\tilde{\phi}_i(x) \;=\; \phi_i(x) + y_i\,, \qquad\qquad \tilde{\psi}_j(x) \;=\; \psi_j(x) + z_j\,,$$

one has the alternative:

- either $(\tilde{\phi}_i(x), \tilde{\psi}_j(x))_{i\in\mathcal{I}, j\in\mathcal{J}'} \;\neq\; (0,\ldots,0)\in\mathbb{R}^{|\mathcal{I}|+|\mathcal{J}'|}\,,$

- or else $(\tilde{\phi}_i(x), \tilde{\psi}_j(x))_{i\in\mathcal{I}, j\in\mathcal{J}'} = (0,\ldots,0)$ and the matrix of partial derivatives
$$\left(\frac{\partial[\phi_i,\psi_j]}{\partial x_k}\right)_{i\in\mathcal{I}, j\in\mathcal{J}', 1\le k\le N}$$
has rank $|\mathcal{I}| + |\mathcal{J}'|$.

3. We now observe that the second alternative cannot hold if $|\mathcal{I}|+|\mathcal{J}'| > N$. Since the vector (y_i, z_j) can be taken arbitrarily small, we conclude that there exists an open dense set of functions $(\phi_i, \psi_j)_{i\in\mathcal{I}, j\in\mathcal{J}}$ such that, for any $\mathcal{J}' \subseteq \mathcal{J}$, the following conditions hold.

- If $|\mathcal{I}| + |\mathcal{J}'| > N$, then $(\phi_i(x), \psi_j(x))_{i\in\mathcal{I}, j\in\mathcal{J}'} \neq (0,\ldots,0) \in \mathbb{R}^{|\mathcal{I}|+|\mathcal{J}'|}$ for every $x \in \mathbb{R}^N$,
- If $(\phi_i(x), \psi_j(x))_{i\in\mathcal{I}, j\in\mathcal{J}'} = (0,\ldots,0) \in \mathbb{R}^{|\mathcal{I}|+|\mathcal{J}'|}$, then the $|\mathcal{I}|+|\mathcal{J}'|$ gradients $\nabla\phi_i(x)$, $\nabla\psi_j(x)$ are linearly independent.

This achieves the proof. □

Remark 2.1. Although the family of sets $\mathcal{A}$ that can be represented as in **(B2)-(B3)** is quite general, there are simple examples where the graph of best reply map does not fit within this framework. For example, let $X = Y = [0,1]$ and take $G(x,y) = y^2 - 4xy$. Then the graph of the best reply map is

$$\text{graph}(R) \;=\; \{x \in [0,1/2],\quad y = 2x\} \cup \{x \in [1/2,1],\quad y = 1\}. \tag{2.25}$$

For this reason, we need to extend the previous results to this slightly more general setting. In this direction, we observe that the set in (2.25) can be equivalently written as

$$\text{graph}(R) \;=\; \{1 - y \geq 0,\quad 2x - y = 0\} \cup \{1 - y = 0,\quad 2x - y \geq 0\}. \tag{2.26}$$

In the following, we shall thus consider more general domains $\mathcal{A}$ which admit the following characterization.

(C) *There exists a finite open covering $\mathbb{R}^N = V_1 \cup \cdots \cup V_m$ such that, for each $k \in \{1,\ldots,m\}$, the intersection $\mathcal{A} \cap V_k$ admits a representation in terms of* *finitely many functions* $\phi_i \in \mathcal{C}^2(\mathbb{R}^N)$, $i \in I$. *Namely, the following properties hold:*

(i) *For any* $x \in V_k$ *and any subset* $I' \subseteq I$, *if* $\phi_i(x) = 0$ *for all* $i \in I'$ *then the gradients* $\nabla\phi_i(x)$, $i \in I'$, *are linearly independent.*

(ii) *There exists finitely many subsets of indices* $I_1, \ldots, I_\nu \subseteq I$, *such that*

$$\mathcal{A} \cap V_k \;\doteq\; \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_\nu \tag{2.27}$$

where, for each $\ell = 1, \ldots, \nu$,

$$\mathcal{A}_\ell \;=\; \{x \in V_k\,; \quad \phi_i(x) = 0, \quad \phi_j(x) \geq 0 \quad \text{for all} \;\; i \in I_\ell, \quad j \notin I_\ell\}. \tag{2.28}$$

Notice that the example (2.26) fits in this framework, with $\phi_1 = 1 - y$, $\phi_2 = 2x - y$.

We now extend Definition 2.1 to this more general setting.

Definition 2.2. *Consider a domain* $\mathcal{A}$ *which admits the characterization in* **(C)**. *We say that the global minimum (2.1) is attained at a point* $\bar{x}$ **in generic position** *if the following conditions hold.*

(i) $\bar{x} \in \mathcal{A}$ *is the unique point where the global minimum is attained.*

(ii) *Let* $\bar{x} \in V_k$, *so that (2.27)-(2.28) holds. Then, defining* $J \doteq \{j \in I\,; \;\; \phi_j(\bar{x}) = 0\}$, *for any* ℓ *such that* $\bar{x} \in \mathcal{A}_\ell$, *there exists constants* α_i, β_j, $i \in I_\ell$, $j \in J \setminus I_\ell$ *such that*

$$\nabla f(\bar{x}) \;=\; \sum_{i \in I_l} \alpha_i \, \nabla\phi_i(\bar{x}) + \sum_{j \in J \setminus I_l} \beta_j \, \nabla\phi_j(\bar{x}), \tag{2.29}$$

with $\beta_j > 0$ *for all* $j \in J \setminus I_\ell$.

(iii) *There exists* $\rho, \varepsilon > 0$ *such that*

$$f(x) - f(\bar{x}) \;\geq\; \varepsilon\, |x - \bar{x}|^2 \qquad \text{for all } x \in \mathcal{A} \text{ with } |x - \bar{x}| \leq \rho\,. \tag{2.30}$$

The results proved in Theorem 2.1 and in Theorem 2.2 can be easily extended to this more general setting.

Corollary 2.2. *Let $f \in \mathcal{C}^2(\mathbb{R}^N)$ and let $\mathcal{A} \subset \mathbb{R}^N$ be a compact domain which admits the characterization* **(C)**. *Moreover, assume that the global minimum (2.1) is attained at a point $\bar{x}$ in generic position. Then the conclusions in Theorem 2.1 remain valid.*

Proof. Assume that the global minimum is attained at a point $\bar{x} \in \mathcal{A} \cap V_k$, so that (2.27)-(2.28) holds. If $\bar{x} \in \mathcal{A}_\ell$, then the proof of Theorem 2.1 shows that, for any sufficiently small perturbations $\tilde{f}, \tilde{\phi}_i$, the corresponding minimum is still achieved at a point in $\mathcal{A}_\ell$. The conclusion is thus obtained by applying Theorem 2.1 to each domain $\mathcal{A}_\ell$ which contains $\bar{x}$. □

Corollary 2.3. *Let $\mathcal{A} \subset \mathbb{R}^N$ be a compact domain which admits the characterization* **(C)**. *Then, for an open dense set of functions $f \in \mathcal{F}^\infty$, the global minimum in (2.1) is attained in generic position.*

Proof. Given the open covering $\mathbb{R}^N = V_1 \cup \cdots \cup V_m$, we can find a new covering $\mathbb{R}^N = V_1' \cup \cdots \cup V_m'$, where each $V_k' \subset V_k$ is a closed set. For any given $k \in \{1, \ldots, m\}$, consider the representation (2.27)-(2.28). Using Theorem 2.2 we obtain an open dense set of functions $\mathcal{F}_{\ell,k}$ such that, if $f \in \mathcal{F}_{\ell,k}$ and the global minimum of f on $\mathcal{A}$ is achieved at a point $\bar{x} \in \mathcal{A}_\ell \cap V_k'$, then it is attained in generic position. Taking the intersection of these finitely many open dense sets of functions, the result is proved. □

3 Generic properties of the best reply map

Starting with this section, we consider a cost function for the follower $G : \mathbb{R}^m \times \mathbb{R}^n \mapsto \mathbb{R}$, and study the structure of the **best reply map:** $x \mapsto R(x) \subset \mathbb{R}^n$. We seek a description of the graph

$$\mathcal{A} \doteq \text{Graph}(R) = \{(x, y);\ y \in R(x)\} \tag{3.1}$$

for a generic function $G \in \mathcal{C}^3$. The eventual goal is to show that, for a generic function G, the above graph can be expressed in terms of finitely many equalities or inequalities, as in (2.2)–(2.4). In the following, our basic assumptions will be

(A1) *The function $G = G(x, y)$ lies in $\mathcal{C}^3(\mathbb{R}^m \times \mathbb{R}^n)$.*

(A2) *There exists* $\rho > 0$ *such that*

$$G(x,y) \; > \; G(x,0) \qquad\qquad \textit{for all} \;\; x \in \mathbb{R}^m, \;\; |y| \geq \rho, \tag{3.2}$$

We shall denote by $\mathcal{G}$ the family of all functions $G = G(x,y)$ which satisfy **(A1)-(A2)**. Notice that, if $G \in \mathcal{G}$, then by (3.2), for each $x \in \mathbb{R}^m$ the set of best replies $R(x)$ is a nonempty compact set contained in the open ball centered at the origin with radius ρ, namely

$$R(x) \;\subseteq\; B_\rho \;\subset\; \mathbb{R}^n. \tag{3.3}$$

As a consequence, the admissible set $\mathcal{A}$ in (3.1) is closed. We seek to understand the structure of this set $\mathcal{A}$, for a generic function $G \in \mathcal{C}^3(\mathbb{R}^{m+n})$. As a preliminary, we observe that the necessary conditions for optimality imply

$$\mathcal{A} \;\subseteq\; \mathcal{M} \;\doteq\; \Big\{(x,y) \in \mathbb{R}^{m+n}\,; \;\;\nabla_y G(x,y) = 0\Big\}. \tag{3.4}$$

Here and in the sequel we write $\nabla_y G = (G_{y_1}, \ldots, G_{y_n})$. The next lemma establishes the generic regularity of $\mathcal{M}$.

Lemma 3.1. *Let* $\kappa > 0$ *be given. There exists an open, dense subset* $\mathcal{G}^\sharp \subset \mathcal{G}$ *such that, for every* $G \in \mathcal{G}^\sharp$*, the set*

$$\mathcal{M}_\kappa \;\doteq\; \Big\{(x,y)\,;\; \nabla_y G(x,y) = 0, \quad |x| < \kappa\Big\} \tag{3.5}$$

is an m*-dimensional* $\mathcal{C}^2$ *manifold, embedded in* $\mathbb{R}^{m+n}$*.*

Proof. 1. Given $G \in \mathcal{G}$ and $\varepsilon > 0$, by a mollification procedure we can construct $g \in \mathcal{G} \cap \mathcal{C}^\infty$ with $\|g - G\|_{\mathcal{C}^3} < \varepsilon$.

2. Consider the map $(x,y) \mapsto \nabla_y g(x,y)$ from $\mathbb{R}^{m+n}$ into $\mathbb{R}^n$. By Sard's theorem [18, 38], the set of critical values of this map has measure zero. As a consequence, we can find $\theta = (\theta_1, \ldots, \theta_n)$ with $|\theta| < \varepsilon$ such that, at every point where

$\nabla_y g(x,y) \;=\; \theta$, the $n \times (m+n)$ Jacobian matrix $D(\nabla_y g)$ has full rank, namely

$$\text{rank}\begin{pmatrix} \partial_{x_1}\partial_{y_1} g & \cdots & \partial_{x_m}\partial_{y_1} g & \partial_{y_1}\partial_{y_1} g & \ldots & \partial_{y_n}\partial_{y_1} g \\ \vdots & & \cdots & & & \vdots \\ \partial_{x_1}\partial_{y_n} g & \cdots & \partial_{x_m}\partial_{y_n} g & \partial_{y_1}\partial_{y_n} g & \ldots & \partial_{y_n}\partial_{y_n} g \end{pmatrix} \;=\; n. \qquad (3.6)$$

3. We can now consider a smooth function $\tilde g : \mathbb{R}^{m+n} \mapsto \mathbb{R}$ such that

$$\tilde g(x,y) \;=\; g(x,y) - \sum_{i=1}^{n} \theta_i y_i \qquad\qquad |x| \le \kappa, \quad |y| \le \rho.$$

If $\varepsilon > 0$ was chosen sufficiently small, this can be extended to the entire space, still remaining in $\mathcal{G}$.

The above construction yields a function $\tilde g$, arbitrarily close to G in the $\mathcal{C}^3$ norm, for which the following implication holds:

$$|x| \le \kappa, \quad |y| \le \rho, \quad \nabla_y \tilde g(x,y) \;=\; 0 \qquad \Longrightarrow \qquad \text{rank}\big(D(\nabla_y \tilde g)(x,y)\big) \;=\; n. \qquad (3.7)$$

By the implicit function theorem, this implies that the set

$$\widetilde{\mathcal{M}}_\kappa \;\doteq\; \Big\{(x,y)\,;\; \nabla \tilde g_y(x,y) = 0, \quad |x| < \kappa\Big\}$$

is still a smooth manifold.

4. Let now $\mathcal{G}^\sharp$ be the set of all functions $G \in \mathcal{G}$ for which the implication (3.7) holds. By the previous steps, this set is dense in $\mathcal{G}$. It remains to show that $\mathcal{G}^\sharp$ is open.

Assume, on the contrary, that there exists a sequence of functions $G_k \notin \mathcal{G}^\sharp$, with $G_k \to G$ in $\mathcal{C}^3$ and $G \in \mathcal{G}^\sharp$. This implies that, for each $k \ge 1$, there exists (x_k, y_k) with

$$|x_k| \le \kappa, \quad |y_k| \le \rho, \quad \nabla G_k(x_k, y_k) \;=\; 0, \qquad \text{rank}\big(D(\nabla_y G_k)(x_k, y_k)\big) \;<\; n.$$

Taking a subsequence we can assume the convergence $x_k \to x$, $y_k \to y$. By continuity, it follows

$$|x| \le \kappa, \quad |y| \le \rho, \quad \nabla G(x,y) \,=\, 0, \qquad \mathrm{rank}\big(D(\nabla_y G)(x,y)\big) \,<\, n.$$

against the assumptions. This contradiction completes the proof. □

4 The best reply map in one-dimensional strategies

In this section we study the generic structure of the best reply map, and the stability of the Stackelberg equilibrium, starting with a simple one-dimensional framework. Namely, we assume that the strategies x, y for both the leader and the follower range over a closed interval, say

$$x \,\in\, X \,=\, [0,1], \qquad\qquad y \,\in\, Y \,=\, [0,1]. \tag{4.1}$$

Given a function $G \in \mathcal{C}^3(\mathbb{R}^2)$, consider the best reply map

$$R(x) \;\doteq\; \Big\{ y^* \in [0,1]\,;\;\; G(x,y^*) \le G(x,y) \quad \text{for all} \;\; y \in [0,1] \Big\}. \tag{4.2}$$

By Lemma 3.1 there exists an open dense subset $\mathcal{G}^\sharp \subset \mathcal{C}^3(\mathbb{R}^2)$ such that, for every $G \in \mathcal{G}^\sharp$, the set

$$\mathcal{M} \;\doteq\; \Big\{ (x,y)\,;\; G_y(x,y) = 0, \;\; |x| < 2,\; |y| < 2 \Big\} \tag{4.3}$$

is a $\mathcal{C}^2$ manifold. Indeed, in analogy with (3.7), for a generic function $G \in \mathcal{C}^3$ we have

$$|x| \le 2, \quad |y| \le 2, \quad G_y(x,y) \,=\, 0 \qquad \Longrightarrow \qquad \nabla G_y(x,y) \,\ne\, 0. \tag{4.4}$$

We now prove a structure theorem for the best reply map, valid for a generic cost function G of the follower.

Theorem 4.1. *There exists an open dense subset of cost functions $G \in \mathcal{C}^3(\mathbb{R}^2)$ such that the best reply map (4.2) has the following structure.*

There exists finitely many points $0 = x_0 < x_1 < \cdots < x_\nu = 1$, *and functions* $\varphi_k \in \mathcal{C}^2(\mathbb{R})$ *such that*

$$\Big\{(x,y)\,;\quad y\in R(x)\,,\quad x\in[0,1]\Big\} \;=\; \bigcup_{k=1}^{\nu}\Big\{(x,\varphi_k(x))\,;\quad x\in[x_{k-1},x_k]\Big\}. \tag{4.5}$$

Moreover, either $\varphi_k(x_k)\neq\varphi_{k+1}(x_k)$, *or else*

$$\varphi_k(x_k) \;=\; \varphi_{k+1}(x_k) \;\in\; \{0,1\}, \qquad \varphi_k'(x_k) \;\neq\; \varphi_{k+1}'(x_k). \tag{4.6}$$

Finally, for all $k = 1,\ldots,\nu-1$, *one has*

$$G\big(x_k,\varphi_k(x_k)\big) \;=\; G\big(x_k,\varphi_{k+1}(x_k)\big), \qquad \frac{d}{dx}G\big(x,\varphi_k(x)\big)\Big|_{x=x_k} > \frac{d}{dx}G\big(x,\varphi_{k+1}(x)\big)\Big|_{x=x_k}. \tag{4.7}$$

Proof. 1. We introduce a set of conditions such that, if none of them is satisfied, (for any choice of x, y, y_1, y_2, y_3 in $[0,1]$), then the representation (4.5) holds (Fig. 5.1). By showing that each of these conditions is NOT satisfied by all functions G in an open dense subset of $\mathcal{C}^3$, the theorem will be proved.

(i) $G_y(x,y) = 0$, $G_{yy}(x,y) = 0$, and $x\in\{0,1\}$ or $y\in\{0,1\}$.

(ii) $G_y(x,y) = 0$, $G_{yx}(x,y) = 0$, and $x\in\{0,1\}$ or $y\in\{0,1\}$.

(iii) $G_y(x,y) = 0$, $x\in\{0,1\}$ and $y\in\{0,1\}$.

(iv) $G_y(x,y) = G_{yy}(x,y) = G_{yyy}(x,y) = 0$.

(v) $G_y(x,y) = G_{yy}(x,y) = G_{xy}(x,y) = 0$.

(vi) $G_y(x,y_1) = G_y(x,y_2) = 0$, $G(x,y_1) = G(x,y_2)$, $G_{yy}(x,y_1) = 0$, for some $y_1\neq y_2$.

(vii) $G_y(x,y_1) = G_y(x,y_2) = 0$, $G(x,y_1) = G(x,y_2)$, $G_x(x,y_1) = G_x(x,y_2)$, for some $y_1\neq y_2$.

(viii) $G_y(x,y_1) = 0$, $G_{yy}(x,y_1) = 0$, $G(x,y_1) = G(x,y_2)$, $y_2\in\{0,1\}$, for some $y_1\neq y_2$.

(ix) $G_y(x,y_1) = 0$, $G(x,y_1) = G(x,y_2)$, $G_x(x,y_1) = G_x(x,y_2)$, $y_2 \in \{0,1\}$, for some $y_1 \neq y_2$.

(x) $G_y(x,y_1) = G_y(x,y_2) = 0$, $G(x,y_1) = G(x,y_2)$, $y_2 \in \{0,1\}$, for some $y_1 \neq y_2$.

(xi) $G_y(x,y_1) = 0$, $G(x,y_1) = G(x,y_2)$, $y_1, y_2 \in \{0,1\}$, for some $y_1 \neq y_2$.

(xii) $G(x,y_1) = G(x,y_2)$, $G_x(x,y_1) = G_x(x,y_2)$, $y_1, y_2 \in \{0,1\}$, for some $y_1 \neq y_2$.

(xiii) There are three distinct points (x,y_1), (x,y_2), (x,y_3) such that $G(x,y_1) = G(x,y_2) = G(x,y_3)$ and for each $i = 1,2,3$ one has either $G_y(x,y_i) = 0$ or $y_i \in \{0,1\}$.

As the reader will easily check, each of these conditions involves a number of identities which is strictly larger than the corresponding number of variables. Hence, for "most" functions G, this set of equations will have no solution. As shown in the following steps, a rigorous proof of this fact can be achieved thanks to a multi-jet version of Thom's transversality theorem.

2. The conditions (i)–(v) are all handled in a similar way. Given a function $G \in \mathcal{C}^\infty(\mathbb{R}^2)$, its third order jet prolongation is the vector function whose components are all its derivatives up to order three:

$$j^3G(x,y) = (G,\ G_x, G_y,\ G_{xx}, G_{xy}, G_{yy},\ G_{xxx}, G_{xxy}, G_{xyy}, G_{yyy})(x,y). \tag{4.8}$$

The map j^3G is thus a section of the vector bundle $J^3(\mathbb{R}^2,\mathbb{R})$ of all third order jets of maps from $\mathbb{R}^2$ into $\mathbb{R}$.

For each of the conditions in (i)–(v) we shall consider a smooth submanifold $W \subset J^3(\mathbb{R}^2,\mathbb{R})$. This will be defined in terms of three independent equalities, hence it will have codimension 3. By Thom's transversality theorem, there is a dense set of $\mathcal{C}^\infty$ functions G whose prolongation j^3G is transversal to W. Since j^3G is a section of $J^3(\mathbb{R}^2,\mathbb{R})$, it is a two-dimensional manifold. In this case, transversality implies that the intersection is empty. In other words, for a dense set of $\mathcal{C}^\infty$ functions G, the three identities that define W are never simultaneously satisfied.

For example, for condition (i) we consider four distinct sub-manifolds. Each of them is defined by the two identities

$$G_y = 0, \qquad G_{yy} = 0,$$

together with one of the four equalities $x = 0$, $x = 1$, $y = 0$, or $y = 1$. Condition (ii) is entirely similar.

For condition (iii) we need again to consider four distinct sub-manifolds. Each of them is defined by the identity $G_y = 0$, plus a choice of $x \in \{0, 1\}$ and $y \in \{0, 1\}$.

To handle condition (iv), it suffices to consider the linear sub-manifold $W \subset J^3(\mathbb{R}^2; \mathbb{R})$ consisting of all jets such that $G_y = 0$, $G_{yy} = 0$, $G_{yyy} = 0$. Finally, condition (v) is handled by defining $W \subset J^2(\mathbb{R}^2; \mathbb{R})$ to be the linear manifold of all jets such that $G_y = 0$, $G_{yy} = 0$, $G_{xy} = 0$.

3. Conditions (vi)–(xi) refer to the values of G and its first two derivatives at two different points. For this reason, we shall need a multi-jet transversality theorem, proved in the Appendix. We start by introducing the manifold

$$Z^{(2)} \doteq \big\{(x, y_1, y_2)\,; \quad y_1 \neq y_2\big\}.$$

On $Z^{(2)}$ we consider the multi-jet bundle $\widehat{J}^2_2(\mathbb{R}^2, \mathbb{R})$, consisting of couples of 2-jets of maps from $\mathbb{R}^2$ to $\mathbb{R}$ with sources (x, y_1), (x, y_2), $y_1 \neq y_2$. Notice that every function $G \in \mathcal{C}^\infty(\mathbb{R}^2, \mathbb{R})$ determines a map

$$\widehat{j}^2_2 G : (x, y_1, y_2) \;\mapsto\; \big(j^2 G(x, y_1)\,,\; j^2 G(x, y_2)\big). \tag{4.9}$$

Each of the conditions (vi) and (vii) yields a manifold $W \subset \widehat{J}^2_2(\mathbb{R}^2, \mathbb{R})$, consisting of multijets which satisfy the four given identities. By Theorem 0.1, there is a dense set of functions $G \in \mathcal{C}^\infty(\mathbb{R}^2, \mathbb{R})$ whose second order jet prolongation $\widehat{j}^2_2 G$ is transversal to W.

Since W is defined in terms of four identities, it has codimension 4. On the other hand, the graph of $\widehat{j}^2_2 G$ has dimension 3. In this case, transversality implies that the intersection is empty. In other words, for a dense set of functions G, the four conditions in (v) or in (vi) are not simultaneously satisfied at any couple of distinct points (x, y_1), (x, y_2).

For each of the conditions (viii), (ix), and (x) we obtain two distinct sub-manifolds $W_0, W_1 \subset \widehat{J}_2^2(\mathbb{R}^2, \mathbb{R})$, imposing the equality $y_2 = 0$ or $y_2 = 1$, respectively. Again, by Theorem 0.1, there is a dense set of functions $G \in \mathcal{C}^\infty(\mathbb{R}^2, \mathbb{R})$ whose second order jet prolongation $\widehat{j}_2^2 G$ is transversal to W_0 or W_1, respectively. By dimensionality this implies that, for such functions G, for every couple of points $(x, y_1) \neq (x, y_2)$ at least one of the four conditions in (viii) fails. Similarly, at least one of the four conditions in (ix) and at least one in (x) must fail.

Condition (xi) leads to four affine sub-manifolds W, each of codimension 4, depending on the choices of $y_1, y_2 \in \{0, 1\}$. The analysis is entirely similar to the previous cases. Condition (xii) is entirely straightforward.

4. Condition (xiii) refers to the values of G and its first derivatives at three different points. For this reason, we introduce the manifold

$$Z^{(3)} \doteq \{(x, y_1, y_2, y_3)\,; \quad y_i \neq y_j \ \text{ for } \ i < j\}.$$

On $Z^{(3)}$ we consider the multi-jet bundle $\widehat{J}_3^1(\mathbb{R}^2, \mathbb{R})$, consisting of couples of 1-jets of maps from $\mathbb{R}^2$ to $\mathbb{R}$ with three distinct sources (x, y_1), (x, y_2), and (x, y_3). Notice that every function $G \in \mathcal{C}^\infty(\mathbb{R}^2, \mathbb{R})$ determines a map

$$\widehat{j}_3^1 G : (x, y_1, y_2, y_3) \ \mapsto \ \Big(j^1 G(x, y_1)\,,\ j^1 G(x, y_2)\,,\ j^1 G(x, y_3)\Big).$$

On $\widehat{J}_3^1(\mathbb{R}^2, \mathbb{R})$ we consider a finite number of sub-manifolds W, each defined by 5 identities. The first two identities are

$$G(x, y_1) \ = \ G(x, y_2), \qquad\qquad G(x, y_1) \ = \ G(x, y_3).$$

The remaining three identities are obtained by choosing, for each $i = 1, 2, 3$, either $G_y(x, y_i) = 0$, or $y_i = 0$, or else $y_i = 1$.

By Theorem 0.1, there is a dense set of functions $G \in \mathcal{C}^\infty(\mathbb{R}^2, \mathbb{R})$ whose first order jet prolongation $\widehat{j}_3^1 G$ is transversal to any of the above manifolds W.

We now observe that each W is defined in terms of five identities, and thus has codimension 5. On the other hand, the graph of $\widehat{j}_3^1 G$ has dimension 4. Once again, transversality implies that the intersection is empty. In other words, for a dense set of functions G, the five conditions in (xiii) are not simultaneously satisfied at any triple of distinct points (x, y_1), (x, y_2), (x, y_3).

5. In this step we prove that the set of functions $G \in \mathcal{C}^3(\mathbb{R}^2, \mathbb{R})$ for which none of the conditions (i)–(xiii) is satisfied on the domain $Q = \{(x, y) \in [0,1] \times [0,1]\}$ is open in $\mathcal{C}^3$.

Let $(G^{(n)})_{n\geq 1}$ be a sequence of functions converging to G in $\mathcal{C}^3$. Assume that, for every n, at least one of the conditions (i)–(xiii) is satisfied, within the domain Q. We need to show that the same holds for G.

We start with the easy case where, for some sequence $(x^{(n)}, y^{(n)}) \in Q$, each $G^{(n)}$ satisfies one of the conditions (i)–(v). By taking a subsequence, we can assume $(x^{(n)}, y^{(n)}) \to (\bar{x}, \bar{y}) \in Q$. By continuity, the limit function G satisfies the same condition at $(\bar{x}, \bar{y})$, proving our claim.

Concerning the remaining conditions (vi)–(xiii), a more careful analysis is needed, because these conditions involve two or three distinct points.

Assume that, for each $n \geq 1$, the function $G^{(n)}$ satisfies one of the conditions (vi)–(xiii), for distinct points $(x^{(n)}, y_i^{(n)})$, $i = 1, 2, 3$. By taking a subsequence, we can assume the convergence

$$(x^{(n)}, y_i^{(n)}) \;\to\; (\bar{x}, \bar{y}_i), \qquad\qquad i = 1, 2, 3.$$

If the limit points $(\bar{x}, \bar{y}_i)$ are distinct, then by continuity G still satisfies the same condition, and we are done. Notice that this is certainly the case for (xi) and (xii), because here we require $y_1^{(n)}, y_2^{(n)} \in \{0, 1\}$ with $y_1^{(n)} \neq y_2^{(n)}$.

To complete the proof, we need to consider the cases where two of the points $(\bar{x}, \bar{y}_i)$ coincide, say,

$$\lim_{n\to\infty} y_1^{(n)} \;=\; \lim_{n\to\infty} y_2^{(n)} \;=\; \bar{y} \;\in [0,1]. \tag{4.10}$$

- If $G^{(n)}$ satisfies all identities in (vi), for every $n \geq 1$, then by taking limits we conclude

  $$G_y(\bar{x}, \bar{y}) \;=\; G_{yy}(\bar{x}, \bar{y}) \;=\; G_{yyy}(\bar{x}, \bar{y}) \;=\; 0. \tag{4.11}$$

  Hence the limit function G satisfies (iv).

- If $G^{(n)}$ satisfies all identities in (vii), for every $n \geq 1$, then

$$G_y(\bar{x}, \bar{y}) \;=\; G_{yy}(\bar{x}, \bar{y}) \;=\; G_{xy}(\bar{x}, \bar{y}) \;=\; 0.$$

  Hence the limit function G satisfies (v).

- Next, assume that $G^{(n)}$ satisfies all identities in(viii), or in (ix), or in (x), for every $n \geq 1$. Taking the limit, in all cases we conclude

$$G_y(\bar{x}, \bar{y}) \;=\; G_{yy}(\bar{x}, \bar{y}) \;=\; 0, \qquad \bar{y} \in \{0, 1\}.$$

  Hence (i) holds.

- Finally, assuming that all functions $G^{(n)}$ satisfy (xiii), different cases need to be considered.

  If all three sequences of points $y_1^{(n)}$, $y_2^{(n)}$, $y_3^{(n)}$, converge to the same limit $\bar{y}$, then the convergence $G^{(n)} \to G$ in $\mathcal{C}^3$ implies (4.11). Hence the limit function G satisfies (iv).

  The remaining possibility is that

$$y_1^{(n)}, y_3^{(n)} \;\to\; \bar{y}_1\,, \qquad\qquad y_2^{(n)} \;\to\; \bar{y}_2 \;\neq\; \bar{y}_1\,. \tag{4.12}$$

  Four sub-cases must be considered.

  - If $\bar{y}_1 \in \{0, 1\}$ and $\bar{y}_2 \in \{0, 1\}$, then

$$G_y(\bar{x}, \bar{y}_1) \;=\; 0, \qquad G(\bar{x}, \bar{y}_1) \;=\; G(\bar{x}, \bar{y}_2), \qquad \bar{y}_1, \bar{y}_2 \in \{0, 1\}.$$

  Hence all identities in (xi) hold.

  - If $\bar{y}_1 \notin \{0, 1\}$ and $\bar{y}_2 \in \{0, 1\}$, then

$$G_y(\bar{x}, \bar{y}_1) = 0, \qquad G_{yy}(\bar{x}, \bar{y}_1) = 0, \qquad G(\bar{x}, \bar{y}_1) = G(\bar{x}, \bar{y}_2),$$

  for some $y_1 \neq y_2 \in \{0, 1\}$. Hence (viii) holds.

- If $\bar{y}_1 \in \{0,1\}$ and $\bar{y}_2 \notin \{0,1\}$, then

$$G_y(\bar{x},\bar{y}_1) = G_y(\bar{x},\bar{y}_2) = 0, \qquad\qquad G(\bar{x},\bar{y}_1) = G(\bar{x},\bar{y}_2),$$

for some $y_1 \neq y_2 \in \{0,1\}$. Hence (x) holds.

- If $\bar{y}_1 \notin \{0,1\}$ and $\bar{y}_2 \notin \{0,1\}$, then

$$G_y(\bar{x},\bar{y}_1) = G_{yy}(\bar{x},\bar{y}_1) = G_y(\bar{x},\bar{y}_2) = 0, \qquad\qquad G(\bar{x},\bar{y}_1) = G(\bar{x},\bar{y}_2),$$

Hence (vi) holds.

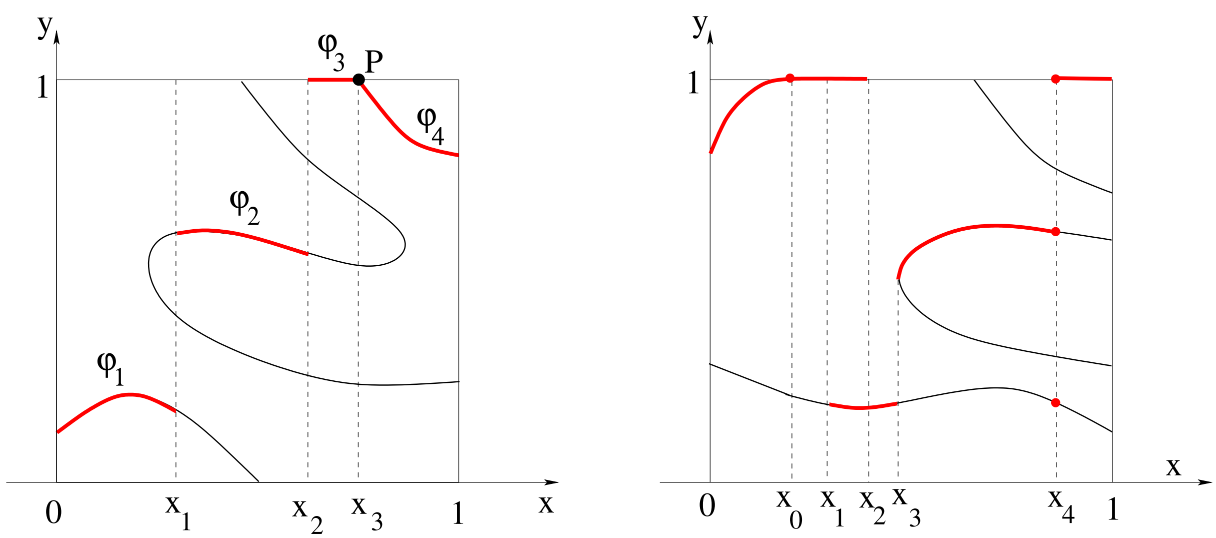

Figure 5.1. Left: the graph of the "best reply map" (in red), for a generic cost function G. Right: a sample of non-generic cases. At x_0 the curve where $G_y = 0$ touches the line $y = 1$ tangentially, so that (ii) holds. On the whole interval $[x_1, x_2]$ the function $G(x,\cdot)$ has two equal minimizers, and (ix) holds. At the point x_3 two global minima are attained, where one of these is along a curve where $G_y = 0$, with vertical tangent. This happens when $G_{yy} = 0$, so that (iv) holds. At x_4 the function $G(x_4,\cdot)$ achieves the minimum at three distinct points, hence (xiii) holds.

6. To conclude the proof of the theorem, consider a function $G \in \mathcal{C}^3(\mathbb{R}^2,\mathbb{R})$ in the open, dense set where none of the conditions (i)–(xiii) holds. We claim that, in this case, the best reply map satisfies the conclusions of the theorem.

By the necessary conditions for optimality, the graph of the best reply map is a closed set, contained in the union of the three sets

$$\{(x,y)\,;\ G_y(x,y)=0,\quad x,y\in[0,1]\}\ \cup\ \{(x,0)\,;\ x\in[0,1]\}\ \cup\ \{(x,1)\,;\ x\in[0,1]\}.$$

Consider any point $\bar x \in [0,1]$. By (xii), the minimum of the function $y \mapsto G(\bar x, y)$ over $[0,1]$ can be attained at most at two distinct points. Various cases will be considered in the remaining steps.

7. We first assume that the global minimum is attained at a single point $\bar y$. Two main cases can occur.

CASE 1: $\bar y \in]0,1[$. We claim that in this case $G_{yy}(\bar x,\bar y)\neq 0$, hence for x in a neighborhood of $\bar x$ the best reply map is single valued:

$$R(x) \;=\; \{\phi(x)\},$$

where $y=\phi(x)$ is the function implicitly defined by

$$G_y(x,y) \;=\; 0. \tag{4.13}$$

Indeed, assume on the contrary that $G_{yy}(\bar x,\bar y)=0$. Since we also have $G_y(\bar x,\bar y)=0$, by (i) it follows that $\bar x\notin\{0,1\}$. Moreover, from (iv) and (v) it follows that

$$G_{xy}(\bar x,\bar y)\;\neq\;0, \qquad\qquad G_{yyy}(\bar x,\bar y)\;\neq\;0. \tag{4.14}$$

Hence, by (4.14), the equation (4.13) can be solved in a neighborhood of $(\bar x,\bar y)$ in terms of a function $x=\psi(y)$, with

$$\psi(\bar y)\;=\;\bar x, \qquad \psi'(\bar y)\;=\;-\frac{G_{yy}(\bar x,\bar y)}{G_{xy}(\bar x,\bar y)}\;=\;0, \qquad \psi''(\bar y)\;=\;-\frac{G_{yyy}(\bar x,\bar y)}{G_{xy}(\bar x,\bar y)}\;\neq\;0.$$

To fix the ideas, assume $\psi''(\bar y)>0$, the other case being entirely similar. Since $\bar x>0$, we can find a strictly increasing sequence $x_n\uparrow\bar x$, with $x_n>0$ for every n. Call $y_n\in[0,1]$ a point where $G(x_n,\cdot)$ attains its global maximum. This implies that either $y_n\in\{0,1\}$ or else $G_y(x_n,y_n)=0$. Choosing a subsequence, we achieve the convergence $(x_n,y_n)\to(\bar x,y^*)$ for some $y^*\in[0,1]$. By continuity, $G(x^*,\cdot)$ attains its global minimum at y^*. Hence, by uniqueness, $y^*=\bar y$. Since

$\bar y \notin \{0,1\}$, we conclude that $G_y(x_n, y_n) = 0$ for all sufficiently large n. This yields a contradiction, because the equation (4.13) does not admit any solution with $x < \bar x$, in a suitably small neighborhood of $(\bar x, \bar y)$.

CASE 2: $\bar y = 0$. (The case where where $\bar y = 1$ is entirely similar.)

In this case, a necessary condition is $G_y(\bar x, 0) \geq 0$. If $G_y(\bar x, 0) > 0$, then we immediately conclude that $R(x) = \{0\}$ on an entire neighborhood of $\bar x$.

The remaining case is where $G_y(\bar x, 0) = 0$. By (i) and (ii) we then have $G_{yy}(\bar x, 0) \neq 0$ and $G_{xy}(\bar x, 0) \neq 0$. Hence, in a neighborhood of $(\bar x, 0)$, the equation (4.13) uniquely defines a function $y = \phi(x)$, with

$$\phi(\bar x) \;=\; 0, \qquad\qquad \phi'(\bar x) \;=\; -\frac{G_{xy}(\bar x, 0)}{G_{yy}(\bar x, 0)} \neq 0.$$

To fix the ideas, let $\phi'(\bar x) > 0$. Then, in a neighborhood of $\bar x$, the best reply map is single-valued and has the form

$$R(x) \;=\; \begin{cases} \{0\} & \text{if} \quad x \leq \bar x, \\[2mm] \{\phi(x)\} & \text{if} \quad x > \bar x. \end{cases} \tag{4.15}$$

8. Finally, we assume that the global minimum of $G(\bar x, \cdot)$ is attained at the two distinct points $\bar y_1 \neq \bar y_2$.

CASE 1: $\bar y_1 = 0$, $\bar y_2 = 1$.

Using (xi) and (xii), together with the necessary conditions for optimality, we obtain

$$G_y(\bar x, 0) \;>\; 0, \qquad G_y(\bar x, 1) \;<\; 0, \qquad G_x(\bar x, 0) \neq G_x(\bar x, 1).$$

To fix the ideas, assume $G_x(\bar x, 0) < G_x(\bar x, 1)$, the other case being similar. Then, for all x in a neighborhood of $\bar x$, the best reply map has the form

$$R(x) \;=\; \begin{cases} \{1\} & \text{if} \quad x < \bar x, \\[2mm] \{0,1\} & \text{if} \quad x = \bar x, \\[2mm] \{0\} & \text{if} \quad x > \bar x. \end{cases} \tag{4.16}$$

CASE 2: $0 < \bar{y}_1 < \bar{y}_2 = 1$. (The case $0 = \bar{y}_1 < \bar{y}_2 < 1$ is entirely similar.)

Since $G_y(\bar{x}, \bar{y}_1) = 0$ and (viii) fails, we must have $G_{yy}(\bar{x}, \bar{y}_1) \neq 0$. Hence, in a neighborhood of $(\bar{x}, \bar{y}_1)$ the equation (4.13) is solved by a function $y = \phi(x)$.

At the point $x = \bar{x}$ we now compute

$$\frac{d}{dx}G(x, \phi(x)) \;=\; G_x + G_y\phi'(x) \;=\; G_x(\bar{x}, \bar{y}) \;\neq\; G_x(\bar{x}, 1)$$

Note that the last inequality stems from the fact that (ix) fails. To fix the ideas, assume $G_x(\bar{x}, \bar{y}_1) < G_x(\bar{x}, 1)$, the other case being similar. Then, for all x in a neighborhood of $\bar{x}$, the best reply map has the form

$$R(x) \;=\; \begin{cases} \{1\} & \text{if} \quad x < \bar{x}, \\ \{\bar{y}_1, 1\} & \text{if} \quad x = \bar{x}, \\ \{\phi(x)\} & \text{if} \quad x > \bar{x}. \end{cases}$$

CASE 3: $0 < \bar{y}_1 < \bar{y}_2 < 1$.

By the necessary conditions for a minimum, this implies $G_y(\bar{x}, \bar{y}_1) = G_y(\bar{x}, \bar{y}_2) = 0$. Since both (vi) and (vii) fail, this implies

$$G_{yy}(\bar{x}, \bar{y}_1) \;\neq\; 0, \qquad G_{yy}(\bar{x}, \bar{y}_2) \;\neq\; 0, \tag{4.17}$$

$$G_x(\bar{x}, \bar{y}_1) \;\neq G_x(\bar{x}, \bar{y}_2). \tag{4.18}$$

By (4.17), near the points $(\bar{x}, \bar{y}_1)$ and $(\bar{x}, \bar{y}_1)$ the equation (4.13) implicitly defines two functions $y = \phi_1(x)$ and $y = \phi_2(x)$. By (4.18) at $x = \bar{x}$ we have

$$\begin{aligned} \tfrac{d}{dx}G(x, \phi_1(x))\Big|_{x=\bar{x}} \;&=\; G_x(\bar{x}, \bar{y}_1) + G_y(\bar{x}, \bar{y}_1)\phi_1'(\bar{x}) \;=\; G_x(\bar{x}, \bar{y}_1) \\ &\neq\; G_x(\bar{x}, \bar{y}_2) \;=\; \frac{d}{dx}G(x, \phi_2(x))\Big|_{x=\bar{x}}. \end{aligned}$$

To fix the ideas, assume $G_x(\bar{x}, \bar{y}_1) < G_x(\bar{x}, \bar{y}_2)$, the other case being similar. Then, for all x in a neighborhood of $\bar{x}$, the best reply map has the form

$$R(x) = \begin{cases} \{\phi_2(x)\} & \text{if} \quad x < \bar{x}, \\ \{\bar{y}_1, \bar{y}_2\} & \text{if} \quad x = \bar{x}, \\ \{\phi_1(x)\} & \text{if} \quad x > \bar{x}. \end{cases} \tag{4.19}$$

9. By the previous analysis, if $G \in \mathcal{C}^3(\mathbb{R}^2, \mathbb{R})$ is a function that does not satisfy any of the conditions (i)–(xiii), then for x in a neighborhood of any point $\bar{x} \in [0,1]$ the best reply map has the structure described in (4.5)-(4.6). Covering the compact domain $[0,1]$ with a finite number of open interval, the theorem is proved. □

5 Generic stability of one-dimensional Stackelberg equilibria

We again consider a noncooperative game, where the players choose strategies in $X = Y = [0,1]$. In order to apply the results in Section 2, we shall need

Lemma 5.1. *Assume that the graph of the best reply map has the structure described at (4.5)-(4.6) in Theorem 4.1. Then this graph can be also written in the form (2.27)-(2.28), where the functions φ_i satisfy property (i) in the characterization* **(C)**.

Proof. Let (4.5)-(4.6) hold. By suitably covering the square $[0,1] \times [0,1]$ with finitely many open sets $V_1, \ldots, V_m$, for each $q \in \{1, \ldots, m\}$ the intersection $V_q \cap \text{graph}(R)$ takes one of the forms described below.

CASE 1 (see Fig. 5.2, left). Assume that

$$\mathcal{A} = \{y = \varphi_k(x), \quad a < x \leq x_k\}.$$

To treat this case, it suffices to consider the function

$$\phi_1(x, y) \doteq y - \varphi_k(x)$$

and an additional function ϕ_2, with the following properties:

$$\begin{cases} \phi_2(x,\varphi_k(x)) > 0 \quad \text{for} \quad x \in \,]a, x_k[\,, \\ \phi_2(x,\varphi_k(x_k)) = 0 \,, \end{cases} \tag{5.1}$$

$$\frac{d}{dx}\phi_2(x,\varphi_k(x))\bigg|_{x=x_k} \; < \; 0. \tag{5.2}$$

We then have the representation

$$\mathcal{A} \;=\; \{(x,y)\,; \quad \phi_1(x,y) = 0, \qquad \phi_2(x,y) \geq 0\}. \tag{5.3}$$

CASE 2 (see Fig. 5.2, center). Assume that

$$\mathcal{A} \;=\; \{y = \varphi_k(x), \quad a < x_k \leq x_k\} \;\cup\; \{y = 1, \quad x_k \leq x < b\},$$

with $\varphi_k(x_k) = 1$, $\varphi_k'(x_k) > 1$. Without loss of generality, we can assume $\varphi_k(x) > 1$ for $x > x_k$.

To handle this case we consider the two functions

$$\phi_1(x,y) \;\doteq\; \varphi_k(x) - y, \qquad \phi_2(x,y) \;=\; 1 - y.$$

We then have the representation

$$\begin{aligned} \mathcal{A} \;&=\; \mathcal{A}_1 \cup \mathcal{A}_2 \\ &= \Big\{(x,y)\,;\; \phi_1(x,y) = 0,\; \phi_2(x,y) \geq 0\Big\} \;\cup\; \Big\{(x,y)\,;\; \phi_1(x,y) \geq 0,\; \phi_2(x,y) = 0\Big\}. \end{aligned}$$

All other cases (see for example Fig. 5.2, right) are entirely similar. □

As an immediate consequence of Corollaries 2.2 and 2.3 we now obtain the generic stability of the Stackelberg equilibrium, in this particular setting (see Fig. 5.3).

Theorem 5.1. *Let $G \in \mathcal{C}^3(\mathbb{R}^2)$ be a function in the open dense set considered in Theorem 4.1, for which the best reply map R has the structure described at (4.5)-(4.6). Then there exists an open dense set of functions $\mathcal{F} \subset \mathcal{C}^2(\mathbb{R}^2)$ such that, for every $F \in \mathcal{F}$, the following holds.*

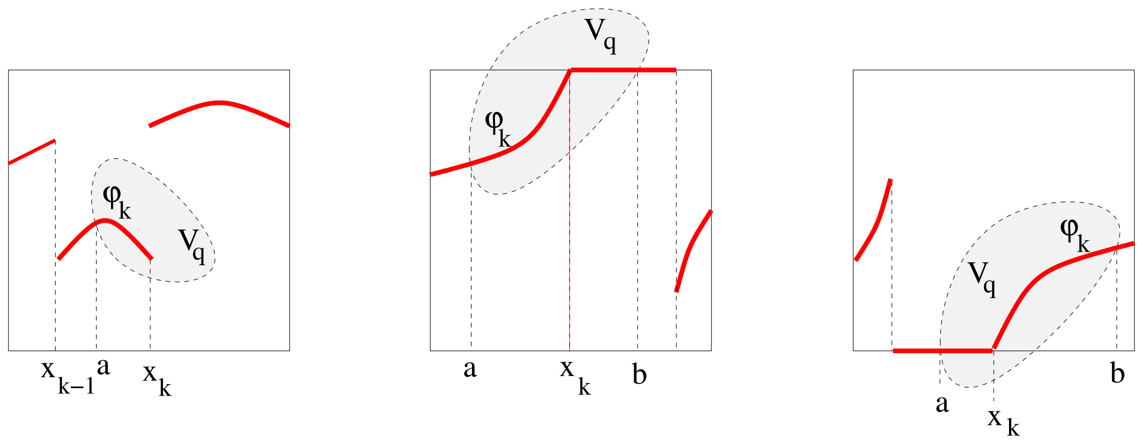

Figure 5.2. Left and center: the two main cases considered in the proof of Lemma 5.1. Right: another configuration, entirely similar to the one considered in CASE 2.

The global minimum of F on $\mathcal{A} = \mathrm{graph}(R)$ is attained at a point $(\bar{x}, \bar{y})$ in generic position, as defined in (2.2). Moreover there exists constants $C, \delta > 0$ such that, if

$$\|\widetilde{F} - F\|_{\mathcal{C}^2} \leq \delta, \qquad \|\widetilde{G} - G\|_{\mathcal{C}^3} \leq \delta, \tag{5.4}$$

then the corresponding perturbed optimization problem

$$\min_{(x,y)\in\widetilde{\mathcal{A}}} \widetilde{F}(x,y) \tag{5.5}$$

has a unique minimizer $(\tilde{x}, \tilde{y})$, also in generic position. Here $\widetilde{\mathcal{A}} = graph(\widetilde{R})$ is the best reply map corresponding to the cost function $\widetilde{G}$. In addition

$$|\tilde{x} - \bar{x}| + |\tilde{y} - \bar{y}| \leq C \cdot \Big(\|\widetilde{F} - F\|_{\mathcal{C}^2} + \|\widetilde{G} - G\|_{\mathcal{C}^3} \Big). \tag{5.6}$$

Proof. Consider a cost function $G \in \mathcal{C}^3(\mathbb{R}^2)$ in the open dense set where none of the conditions (i)–(xiii) in the proof of Theorem 4.1 are satisfied. Then the corresponding best reply map $R(\cdot)$ has the generic structure described at in (4.5)-(4.6). In particular (see Fig. 5.2), there is a finite covering of the square $[0,1] \times [0,1]$ with open sets V_q, $q = 1, \ldots, \nu$, such that the restriction of the graph of $R(\cdot)$ to each set V_q can be described by a finite set of equalities and inequalities involving

- functions of the form $\phi(x,y) = y - \varphi_k(x)$, where φ_k is implicitly defined by the identity $\nabla_y G(x, , \varphi_k(x)) = 0$.

- the two functions $\phi(x, y) = y$ and $\phi(x, y) = y - 1$,
- functions of the form $\phi(x, y) = x - x_k$, where x_k is a point where the best reply map has a jump.

If now $\widetilde{G}$ is a another cost function with $\|\widetilde{G} - G\|_{\mathcal{C}^3}$ sufficiently small, then by repeated applications of the implicit function theorem we check that the best reply map $\widetilde{\mathcal{A}}$ determined by $\widetilde{G}$ also admits the structure (4.5)-(4.6). Furthermore, all the corresponding functions $\tilde{\phi}_i$ in the characterization **(C)** satisfy

$$\|\phi_i - \tilde{\phi}_i\|_{\mathcal{C}^2} \;\leq\; C\,\|G - \widetilde{G}\|_{\mathcal{C}^3} \qquad \text{for all} \quad i \in I,$$

for some constant $C > 0$. The estimate (5.6) now follows from Corollaries 2.2 and 2.3. □

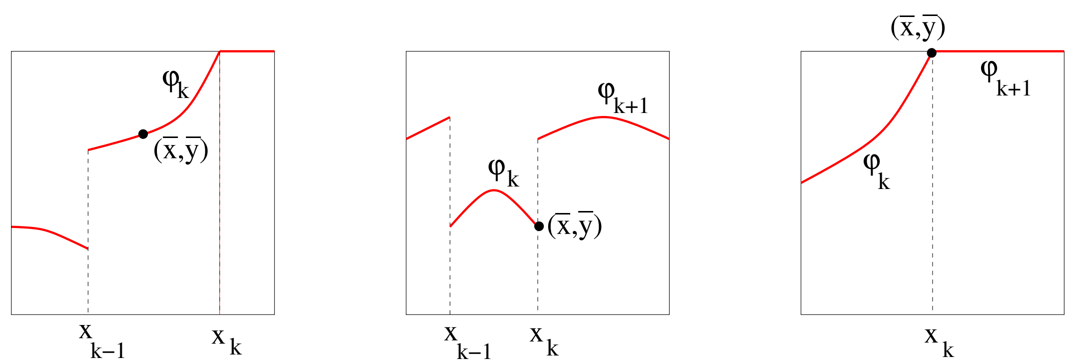

Figure 5.3. Examples of a best reply map $R(\cdot)$ with generic structure, according to Theorem 4.1. Three different cases where the function F can attain a global minimum at a point $(\bar{x}, \bar{y})$ on the graph of $R(\cdot)$, in generic position.

6 Multi-dimensional strategies for the follower

We now extend the analysis to the case where the follower chooses his strategy within a multi-dimensional space. To avoid technicalities associated with the boundary of the sets X, Y, we will simply assume that $x \in X = \mathbb{R}$, $y \in Y = \mathbb{R}^n$. The follower will thus solve a minimization problem on $\mathbb{R}^n$, depending on a scalar parameter. The generic structure of the best reply map, shown in Fig. 5.4, can be readily described. As in Section 3, we denote by $\mathcal{G}$ the set of all functions $G : \mathbb{R} \times \mathbb{R}^m \mapsto \mathbb{R}$ which satisfy **(A1)-(A2)**.

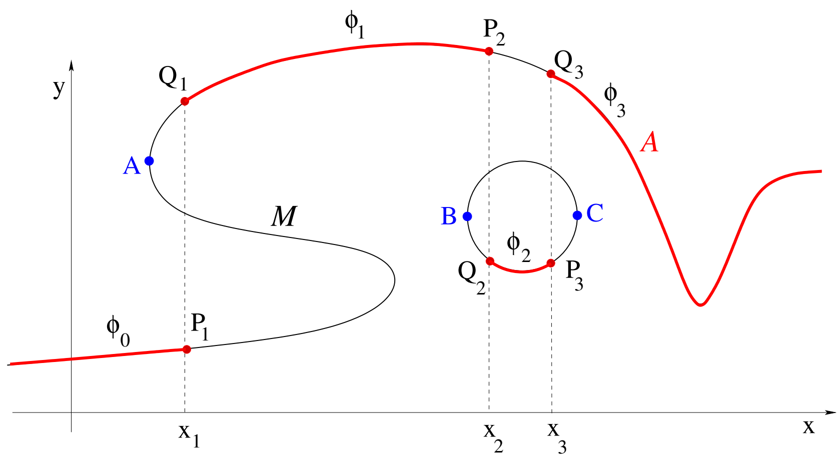

Figure 5.4. The generic structure of the best reply map $x \mapsto R(x) \subset \mathbb{R}^n$, depending on a scalar parameter $x \in \mathbb{R}$. As x varies, at the points A, B, C, D, where a new pair of local minima of $G(x, \cdot)$ is created, the value of G must be strictly larger than the global minimum.

Theorem 6.1. *Given $\alpha > 0$, there exists an open dense subset $\mathcal{G}^\sharp \subset \mathcal{G}$ of cost functions such that, for any $G \in \mathcal{G}^\sharp$, restricted to the interval $[-\alpha, \alpha]$, the best reply map has the following structure.*

There exists finitely many points $-\alpha = x_0 < x_1 < \cdots < x_\nu = \alpha$, and functions $\varphi_k \in \mathcal{C}^2(\mathbb{R};\ \mathbb{R}^n)$ such that

$$\{(x,y)\,;\quad y \in R(x)\,,\quad |x| \le \alpha\} \;=\; \bigcup_{k=1}^{\nu} \Big\{(x, \varphi_k(x))\,;\quad x \in [x_k, x_{k+1}]\Big\}. \tag{6.1}$$

Moreover, $\varphi_k(x_k) \neq \varphi_{k-1}(x_k)$ for all $k = 1, \ldots, \nu - 1$.

Proof. 1. Let a function $G_0 \in \mathcal{G}$ be given, together with constants $\kappa, \rho > 0$. Using Lemma 3.1, for any $\varepsilon > 0$ we can find a function $g \in \mathcal{G} \cap \mathcal{C}^\infty$ with $\|g - G_0\|_{\mathcal{C}^3} < \varepsilon$ for which the following implication holds:

$$|x| \le \kappa, \quad |y| \le \rho, \quad \nabla_y g(x,y) \,=\, 0, \qquad \Longrightarrow \qquad \mathrm{rank}\big(D(\nabla_y g)(x,y)\big) \,=\, n. \tag{6.2}$$

As a consequence, for every sufficiently small $\mathcal{C}^3$ perturbation of g, the implication (6.2) still holds.

For a given function $G \in \mathcal{C}^3$, we consider the $n \times (n+1)$ matrix of partial derivatives

$$A = \Big(A_0\Big|A_1\Big|\cdots\Big|A_n\Big) \doteq \begin{pmatrix} \partial_x\partial_{y_1}G & \partial_{y_1}\partial_{y_1}G & \ldots & \partial_{y_n}\partial_{y_1}G \\ \vdots & \vdots & & \vdots \\ \partial_x\partial_{y_n}G & \partial_{y_1}\partial_{y_n}G & \ldots & \partial_{y_n}\partial_{y_n}G \end{pmatrix}. \tag{6.3}$$

The condition $\text{rank}(A) = n$ at every point in the set

$$\mathcal{M} \doteq \Big\{(x,y) \in \mathbb{R}^{1+n}; \quad \nabla_y G(x,y) = 0\Big\}, \tag{6.4}$$

guarantees that $\mathcal{M}$ is a 1-dimensional embedded manifold in $\mathbb{R}^{1+n}$. By the implicit function theorem, near points where

$$\text{rank}(A_1|\cdots|A_n) = n, \tag{6.5}$$

this manifold $\mathcal{M}$ can be represented as the graph of a function $x \mapsto (y_1(x), \ldots, y_n(x))$.

We observe that, by removing any column from the matrix A in (6.3), we obtain an $n \times n$ matrix whose rank can be either n or $n-1$.

2. Let g be a smooth function for which (6.1) holds. Given any $\delta > 0$, we claim that there exists $G \in \mathcal{C}^\infty$ with $\|G - g\|_{\mathcal{C}^3} < \delta$ and such that, for $|x| \leq \alpha$ and $|y| \leq \rho$, none of the following statements holds true.

(i) *There exist three distinct points* (x, y^1), (x, y^2), (x, y^3) *such that*

$$\begin{aligned} G(x,y^1) &= G(x,y^2) = G(x,y^3), \\ \nabla_y G(x,y^1) &= \nabla_y G(x,y^2) = \nabla_y G(x,y^3) = 0. \end{aligned} \tag{6.6}$$

Notice that, by requiring that (i) fails, we rule out the possibility that, for some x_1, the global minimum of $G(x_1,\cdot)$ is attained at three or more distinct points (see Fig. 5.5, left).

(ii) *There exist two distinct points* (x, y^1), (x, y^2) *such that*

$$G(x, y^1) \;=\; G(x, y^2), \qquad \nabla_y G(x, y^1) \;=\; \nabla_y G(x, y^2) \;=\; 0, \tag{6.7}$$

$$\text{rank}\big(A_1|\cdots|A_n\big)(x, y_1) \;<\; n. \tag{6.8}$$

By requiring that (ii) fails, we preclude the existence of x_2 such that the minimum of $G(x_2, \cdot)$ is attained at two distinct points, and at one of these points the tangent vector to $\mathcal{M}$ is vertical (see again Fig. 5.5).

(iii) *There exist two distinct points* (x, y^1), (x, y^2) *such that*

$$rank\big(A_1|\cdots|A_n\big)(x, y^1) \;=\; rank\big(A_1|\cdots|A_n\big)(x, y^2) \;=\; n, \tag{6.9}$$

and moreover

$$\begin{aligned} G(x, y^1) \;&=\; G(x, y^2), \qquad \nabla_y G(x, y^1) \;=\; \nabla_y G(x, y^2) \;=\; 0, \\ \frac{d}{dx} G(x, \phi_1(x)) \;&=\; \frac{d}{dx} G(x, \phi_2(x)). \end{aligned} \tag{6.10}$$

Here ϕ_1, ϕ_2 *are the functions implicitly defined by the system of* n *equations*

$$\nabla_y G(x, \phi(x)) \;=\; 0, \tag{6.11}$$

with $\phi_1(x) = y^1$ *and* $\phi_2(x) = y^2$.

Notice that, by (6.9), both functions ϕ_1, ϕ_2 are well defined in a neighborhood of x, and have $\mathcal{C}^2$ regularity. To understand the meaning of (iii), assume that the function $G(x_3, \cdot)$ attains its global minimum at two distinct points (x_3, y^1) and (x_3, y^2). By (6.9), on a neighborhood of x_3 the equation (6.11) implicitly defines two functions $y = \phi_1(x)$, $y = \phi_2(x)$. If (iii) fails, at every point $x \neq x_3$ with $|x - x_3|$ small enough, we have

$$G(x, \phi_1(x)) \;\neq\; G(x, \phi_2(x)).$$

In particular, the minimum of $G(x, \cdot)$ cannot be simultaneously attained at more then one point, for all x in a neighborhood of x_3 (see Fig. 5.5).

Our last condition implies that there cannot exist $x_4 \in \mathbb{R}$ such that the 1-dimensional manifold $\mathcal{M}$ has a second order tangency with the vertical hyperplane $\{(x,y)\,;\; x = x_4\}$, as shown in Fig. 5.5, right.

(iv) *There exists a point* $(\bar{x}, \bar{y}) \in \mathbb{R}^{1+n}$ *such that*

$$\nabla_y G(\bar{x}, \bar{y}) \;=\; 0, \quad \text{rank}(A_1|A_2|\cdots|A_n)(\bar{x}, \bar{y}) \;=\; n-1. \tag{6.12}$$

Moreover, if $s \mapsto (x(s), y(s))$ *is a local arc-length parameterization of* $\mathcal{M}$ *with* $(x(0), y(0)) = (\bar{x}, \bar{y})$*, then*

$$\frac{d^2}{ds^2} x(s)\bigg|_{s=0} \;=\; 0. \tag{6.13}$$

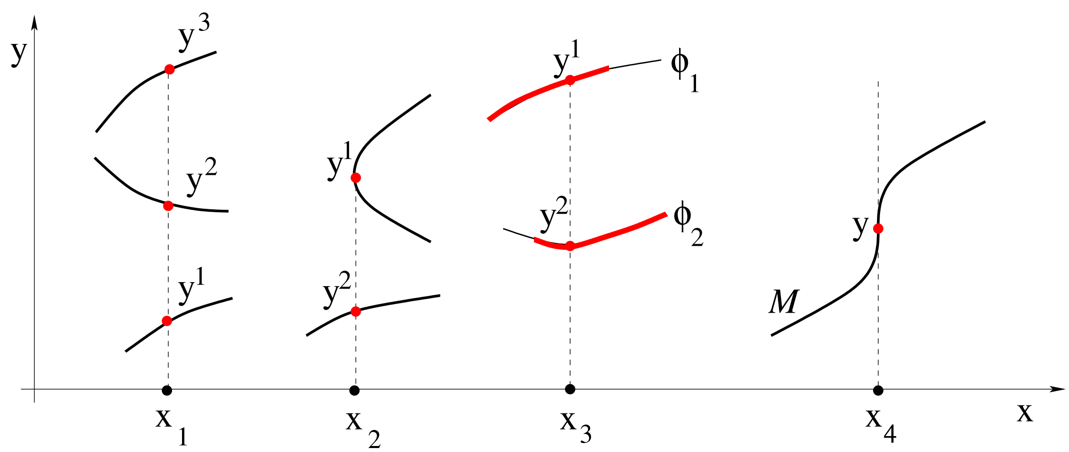

Figure 5.5. Four non-generic configurations, described at (i)–(iv) of the proof of Theorem 6.1.

Our claims will be proved in the forthcoming steps. Notice that in all cases (i)–(iv), we are considering sets of points (x, y^i) defined by a number of equations which is strictly larger than the dimension of the spaces they live in. Hence, for a generic function G, these sets are empty.

3. In this step we prove that the set of functions G which do not satisfy (i) at any point $x \in \mathbb{R}$ is dense. Indeed, consider the open domain

$$X^{(3)} \;\doteq\; \Big\{(x, y^1, y^2, y^3) \in \mathbb{R}^{1+3n}\,;\;\; y^i \neq y^j \text{ for } 1 \leq i < j \leq 3\Big\}. \tag{6.14}$$

Call $J^1(X^{(3)}, \mathbb{R})$ the bundle of all 1-jets of functions $f : X^{(3)} \mapsto \mathbb{R}$. Given a smooth map $G : \mathbb{R} \times \mathbb{R}^n \mapsto \mathbb{R}$, this yields a map

$$j^1G(x, y^1, y^2, y^3) \doteq \Big(G(x,y^1), G(x,y^2), G(x,y^3), \nabla_y G(x,y^1), \nabla_y G(x,y^2), \nabla_y G(x,y^3)\Big).$$

Consider the set

$$\mathcal{M}_1 \doteq \Big\{(z_1, z_2, z_3, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \in \mathbb{R}^3 \times \mathbb{R}^{3n}\,; \quad z_1 - z_2 = z_2 - z_3 = 0, \quad \mathbf{v}_1 = \mathbf{v}_2 = \mathbf{v}_3 = 0\Big\}. \tag{6.15}$$

It is immediate to check that $\mathcal{M}_1$ is a smooth submanifold of $J^1(X^{(3)}, \mathbb{R})$. By the multi-jet version of Thom's transversality theorem (proved in the Appendix), the set of all $G \in \mathcal{C}^\infty(\mathbb{R}^{1+n}, \mathbb{R})$ such that j^1G is transversal to $\mathcal{M}_1$ is residual in the $\mathcal{C}^\infty$ topology. Observing that $\dim(X^{(3)}) = 1 + 3n$ while $\text{codim}(\mathcal{M}_1) = 2 + 3n$, transversality implies that the intersection is empty:

$$\big\{j^1G(x, y^1, y^2, y^3)\,; \quad (x, y^1, y^2, y^3) \in X^{(3)}\big\} \cap \mathcal{M}_1 = \emptyset. \tag{6.16}$$

4. Next, we show that the set of functions G which do not satisfy (ii) at any point $(x, y) \in \mathbb{R} \times \mathbb{R}^n$ is dense. For this purpose consider the open set

$$X^{(2)} \doteq \Big\{(x, y^1, y^2) \in \mathbb{R}^{1+2n}\,; \quad y^1 \neq y^2\Big\}. \tag{6.17}$$

Call $J^2(X^{(2)}, \mathbb{R})$ the bundle of all 2-jets of functions $f : X^{(2)} \mapsto \mathbb{R}$. Given a smooth map $G : \mathbb{R} \times \mathbb{R}^n \mapsto \mathbb{R}$, this yields a map

$$j^2G(x, y^1, y^2) \doteq \Big(G(x,y^1), G(x,y^2), \nabla_y G(x,y^1), \nabla_y G(x,y^2), D_y^2 G(x,y^1), D_y^2 G(x,y^2)\Big).$$

where $D_y^2 G \doteq (\partial^2_{y_i y_j} G)$ denotes the $n \times n$ Hessian matrix of second derivatives of

G w.r.t. $y_1, \ldots, y_n$. For each $k = 1, 2, \ldots, n-1$, consider the set

$$\mathcal{M}_{2,k} \doteq \Big\{(z_1, z_2, \mathbf{v}_1, \mathbf{v}_2, A_1, A_2)\,; \quad z_1 - z_2 = 0, \quad \mathbf{v}_1 = \mathbf{v}_2 = 0, \ \operatorname{rank}(A_1) = k\Big\}. \tag{6.18}$$

Using Proposition 3.2.6 in [18], p. 33, we check that $\mathcal{M}_2$ is a smooth submanifold of $J^2(X^{(2)}, \mathbb{R})$. By a multi-jet version of Thom's transversality theorem, the set of all $G \in \mathcal{C}^\infty(\mathbb{R}^{1+n}, \mathbb{R})$ such that j^2G is transversal to $\mathcal{M}_{2,k}$ is residual in the $\mathcal{C}^\infty$ topology. We now observe that $\dim(X^{(2)}) = 1 + 2n$ while $\operatorname{codim}(\mathcal{M}_{2,k}) = 1 + 2n + (n-k)^2$. Hence, for every $k = 0, 1, \ldots, n-1$, transversality implies that the intersection is empty:

$$\big\{j^2G(x, y^1, y^2)\,; \quad (x, y^1, y^2) \in X^{(2)}\big\} \cap \mathcal{M}_{2,k} = \emptyset. \tag{6.19}$$

5. We now prove that the set of all functions G which do not satisfy (iii) at any point $(x, y^1, y^2) \in X^{(2)}$ is dense.

Call $J^2(X^{(2)}, \mathbb{R})$ the bundle of all 1-jets of functions $f : X^{(2)} \mapsto \mathbb{R}$. Given a smooth map $G : \mathbb{R} \times \mathbb{R}^n \mapsto \mathbb{R}$, consider the map

$$j^1G(x, y^1, y^2) \doteq \Big(G(x, y^1),\ G(x, y^2),\ G_x(x, y^1),\ G_x(x, y^2),\ \nabla_y G(x, y^1),\ \nabla_y G(x, y^2)\Big).$$

Define the set

$$\mathcal{M}_3 \doteq \Big\{(z_1, z_2, w_1, w_2, \mathbf{v}_1, \mathbf{v}_2) \in \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^{2n}\,; \quad z_1 - z_2 = 0, \quad w_1 - w_2 = 0, \quad \mathbf{v}_1 = \mathbf{v}_2 = 0\Big\}.$$

It is clear that $\mathcal{M}_3$ a smooth submanifold of $J^1(X^{(2)}, \mathbb{R})$. By a multi-jet version of Thom's transversality theorem (proved in the Appendix), the set of all $G \in \mathcal{C}^\infty(\mathbb{R}^{1+n}, \mathbb{R})$ such that j^1G is transversal to $\mathcal{M}_3$ is residual in the $\mathcal{C}^\infty$ topology. Since $\dim(X^{(2)}) = 1 + 2n$ while $\operatorname{codim}(\mathcal{M}_3) = 2 + 2n$, transversality implies that the intersection is empty:

$$\big\{j^1G(x, y^1, y^2);\ (x, y^1, y^2) \in X^{(2)}\big\} \cap \mathcal{M}_3 = \emptyset. \tag{6.20}$$

Notice that the assumption (6.9) guarantees that ϕ_1, ϕ_2 are well defined. Since $\nabla_y G(x, \phi_1(x)) = \nabla_y G(x, \phi_2(x)) = 0$, one has the equivalence

$$\frac{d}{dx}G(x, \phi_1(x)) = \frac{d}{dx}G(x, \phi_2(x)) \qquad \Longleftrightarrow \qquad G_x(x, \phi_1(x)) = G_x(x, \phi_2(x)).$$

Therefore, the set of all functions $G \in \mathcal{C}^\infty(\mathbb{R}^{1+n}, \mathbb{R})$ which do not satisfy (iii) at any point $(x, y^1, y^2) \in X^{(2)}$ is the intersection of all the set of functions G such that j^2G is transversal to $\mathcal{M}_{2,k}$ for $k = 0, \ldots, n-1$, and moreover j^1G is transversal to $\mathcal{M}_3$. We thus conclude that the set of functions $G \in \mathcal{C}^\infty(\mathbb{R}^{1+n}, \mathbb{R})$ which do not satisfy (iii) at any point $(x, y^1, y^2) \in X^{(2)}$ is dense in $\mathcal{C}^3(\mathbb{R}^{1+n}, \mathbb{R})$.

6. Finally, we show that the set of functions G which do not satisfy (iv) at any point $(\bar{x}, \bar{y}) \in \mathbb{R}^{1+n}$ is dense. For this purpose, we need to show that the family of third order jets of functions $G : \mathbb{R}^{1+n} \mapsto \mathbb{R}$, which satisfy all the conditions in (iv), is a smooth manifold with codimension $n - 2$.

We recall that the assumption $\text{rank}(A) = n$ at every point $(x, y) \in \mathcal{M}$ implies that the set $\mathcal{M}$ in (4.3) is a 1-dimensional manifold, embedded in $\mathbb{R}^{1+n}$. The condition $\text{rank}(A_1|\cdots|A_n)(\bar{x}, \bar{y}) = n - 1$ implies that the tangent vector to $\mathcal{M}$ at $(\bar{x}, \bar{y})$ is vertical, i.e.:

$$\frac{d}{ds}x(s)\bigg|_{s=0} = 0. \tag{6.21}$$

To obtain an expression for the second derivative in (6.13), we first differentiate the identity $\nabla_y G(x, y) = 0$ and obtain the linear system

$$\Big(A_0\Big|A_1\Big|\cdots\Big|A_n\Big)\begin{pmatrix} dx/ds \\ dy_1/ds \\ \vdots \\ dy_n/ds \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{6.22}$$

Calling A_{-i} the $n \times n$ matrix obtained from $A = (A_0|A_1|\cdots|A_n)$ by deleting the i-th column, the solution to (6.22) can be written as

$$\frac{dx}{ds} = \kappa \cdot \det(A_1|A_2|\cdots|A_n), \qquad \frac{dy_i}{ds} = \kappa \cdot \det(A_{-i}), \qquad i = 1, \ldots, n, \tag{6.23}$$

where $\kappa = \kappa(A)$ is a normalizing factor, chosen in order to achieve

$$\left(\frac{dx}{ds}\right)^2 + \sum_{i=1}^{n} \left(\frac{dy_i}{ds}\right)^2 \; = \; 1.$$

At a point $(\bar{x}, \bar{y})$ where $\det(A_1|A_2|\cdots|A_n) = 0$, using (6.23) the second derivative (6.13) is computed by

$$\begin{aligned} \frac{d^2}{ds^2} x(s) \; &= \; \kappa\, \nabla_y \Big[\det(A_1|A_2|\cdots|A_n) \Big] \cdot \frac{d}{ds} y(s) \\ &= \; \kappa \sum_{i=1}^{n} \partial_{y_i} \Big[\det(A_1|A_2|\cdots|A_n) \Big] \cdot \frac{dy_i}{ds} \\ &= \; \kappa^2 \sum_{i=1}^{n} \sum_{j=1}^{n} \det(H_{i,j}) \, \det(A_{-i}). \end{aligned} \tag{6.24}$$

Here

$$H_{i,j} \doteq \begin{pmatrix} \partial_{y_1}\partial_{y_1} G & \cdots & \partial_{y_{j-1}}\partial_{y_1} G & \partial_{y_i}\partial_{y_j}\partial_{y_1} G & \partial_{y_{j+1}}\partial_{y_1} G & \ldots & \partial_{y_n}\partial_{y_1} G \\ \vdots & & & \cdots & & & \vdots \\ \partial_{y_1}\partial_{y_n} G & \cdots & \partial_{y_{j-1}}\partial_{y_n} G & \partial_{y_i}\partial_{y_j}\partial_{y_n} G & \partial_{y_{j+1}}\partial_{y_n} G & \ldots & \partial_{y_n}\partial_{y_n} G \end{pmatrix} \tag{6.25}$$

is the matrix obtained by differentiating the j-th column of $(A_1|\cdots|A_n)$ by y_i.

Call $J^3(\mathbb{R}^{1+n};\mathbb{R})$ the bundle of all 1-jets of functions $f : \mathbb{R}\times\mathbb{R}^n \mapsto \mathbb{R}$. Given a smooth map $G : \mathbb{R}\times\mathbb{R}^n \mapsto \mathbb{R}$, its third-order jet at a point (x, y) is

$$j^3 G(x,y) \; = \; \Big(G(x,y),\, \nabla_y G(x,y),\, D_y^2 G(x,y),\, D_y^3 G(x,y) \Big).$$

For any $(z, \mathbf{v}, A, T) \in J^3(\mathbb{R}^{1+n};\mathbb{R})$, denote by $\widetilde{H}_{i,j}(A,T)$ the $n \times n$ matrix obtained from (6.25) by replacing the second and third derivatives of G with the corresponding elements in A and T.

With this notation, the family of third order jets of functions satisfying (iv) can thus be expressed by

$$\mathcal{M}_4 \doteq \Bigg\{(z, \mathbf{v}, A, T) \in J^3(\mathbb{R}^{1+n}; \mathbb{R}) \; ; \quad \mathbf{v} = 0, \qquad \text{rank}(A_1|A_2|\cdots|A_n) = n-1, \\ \sum_{i=1}^{n}\sum_{j=1}^{n} \det(\widetilde{H}_{i,j}(A,T)) \, \det(A_{-i}) \; = \; 0\Bigg\}. \tag{6.26}$$

In order to apply Thom's transversality theorem, we need to show that $\mathcal{M}_4$ is a smooth manifold with $\text{codim}(\mathcal{M}_4) = n+2$. Indeed, the vector equation $\mathbf{v} = 0$ yields n scalar equations, while the condition

$$\text{rank}(A_1|A_2|\cdots|A_n) \; = \; n-1, \tag{6.27}$$

provides one more, independent scalar equation. The last condition in (6.26) can be written as

$$\Psi(A,T) \; \doteq \; \sum_{i=1}^{n}\sum_{j=1}^{n} \det\big(\widetilde{H}_{i,j}(A,T)\big) \, \det(A_{-i})\,. \tag{6.28}$$

We claim that the above equation determines a smooth manifold of codimension 1, transversal to the manifolds determined by the previous equations in (6.26). This will be true if at least one of the partial derivatives of Ψ does not vanish. Namely

$$\frac{\partial}{\partial G_{y_1 y_i y_j}} \; \Psi(A,T) \; \neq \; 0 \tag{6.29}$$

for some indices i, j.

To simplify the computations, we observe that the property (iv) which defines $\mathcal{M}_4$ is invariant w.r.t. rotations of the coordinates $y = (y_1, \ldots, y_n)$. Without loss of generality, we can thus assume that the tangent vector to the 1-dimensional manifold $\mathcal{M}$ at the point $(\bar{x}, \bar{y})$ is

$$\left(\frac{dx}{ds}, \; \frac{dy_1}{ds}, \; \frac{dy_2}{ds}, \; \ldots \; , \frac{dy_n}{ds}\right) \; = \; (0, 1, 0, \ldots, 0). \tag{6.30}$$

In this case, one has

$$\det(A_{-2}) \;=\; \cdots \;= \det(A_{-n}) \;=\; 0. \tag{6.31}$$

For notational convenience, we shall here denote by

$$\widetilde{A} \;=\; (a_{ij})_{1\leq i,j\leq n} \;=\; (A_1|\cdots|A_n)$$

is the $n\times n$ symmetric matrix of second derivatives $G_{y_iy_j}$. Using (6.31), from (6.28) one obtains

$$\frac{\partial}{\partial G_{y_1y_ky_\ell}}\,\Psi(A,T) \;=\; \sum_{\ell=1}^{n}\frac{\partial}{\partial G_{y_1y_ky_\ell}}\det\big(\widetilde{H}_{1,\ell}(A,T)\big) \;=\; \begin{cases} \frac{\partial}{\partial a_{k\ell}}\det(\widetilde{A}) & \text{if } i=j,\\ 2\frac{\partial}{\partial a_{k\ell}}\det(\widetilde{A}) & \text{if } i\neq j.\end{cases}$$

By assumption, $\widetilde{A}$ has rank $n-1$. Hence there exists a minor $M^{k\ell}$ obtained from $\widetilde{A}$ by removing the k-th row and the ℓ-th column, which has rank $n-1$.

If $k=\ell$, then

$$\frac{\partial}{\partial G_{y_1y_ky_k}}\,\Psi(A,T) \;=\; \frac{\partial}{\partial a_{kk}}\det(\widetilde{A}) \;=\; \det(M^{kk}) \;\neq\; 0.$$

If $k\neq\ell$, recalling that $\widetilde{A}$ is a symmetric matrix, then by symmetry

$$\frac{\partial}{\partial G_{y_1y_ky_\ell}}\Psi(A,T) \;=\; \left(\frac{\partial}{\partial a_{k\ell}}+\frac{\partial}{\partial a_{\ell k}}\right)\det(\widetilde{A}) \;=\; 2\cdot(-1)^{k+\ell}\det(M^{k\ell}) \;\neq\; 0.$$

In both cases, we find a partial derivative which does not vanish. As a result, $\mathcal{M}_4$ is a smooth manifold with $\mathrm{codim}(\mathcal{M}_4)=n+2$.

We can now use Thom's transversality theorem and conclude that the set of all $G\in\mathcal{C}^\infty(\mathbb{R}^{1+n},\mathbb{R})$ such that j^3G is transversal to $\mathcal{M}_4$ is residual in the $\mathcal{C}^\infty$ topology. Since $\mathrm{codim}(\mathcal{M}_4)=n+2$, transversality implies that the intersection is empty:

$$\{j^3G(x,y);\quad (x,y)\in\mathbb{R}^{1+n}\}\cap\mathcal{M}_4 \;=\; \emptyset.$$

7. In this step we show that the family $\mathcal{G}^\sharp\subset\mathcal{C}^3(\mathbb{R}^{1+n},\mathbb{R})$ of all cost functions G that do not satisfy any of the conditions (i)–(iv), at any point (x,y) where $G(x,\cdot)$

attains its global minimum, is open in the topology induced by $\mathcal{C}^3$.

Assume, on the contrary, that there exists a convergent sequence of functions $G_k \to G$ in $\mathcal{C}^3$, with $G_k \notin \mathcal{G}^\sharp$ for every $k \geq 1$. We claim that $G \notin \mathcal{G}^\sharp$ as well. Since there are four conditions that each G_k can satisfy, we consider them separately.

CASE 1: Suppose that all functions G_k satisfy condition (i). Hence there exists a sequence of points $(x_k, y_k^1, y_k^2, y_k^3)$ satisfying (6.6) for G_k, with

$$|x_k| \leq \kappa, \quad |y_k^i| \leq \rho \text{ for } i = 1, 2, 3, \quad y_k^i \neq y_k^j \text{ for all } \ 1 \leq i < j \leq 3.$$

By possibly taking a subsequence, one has

$$x_k \to x_*, \ y_k^1 \to y_*^1, \ y_k^2 \to y_*^2, \ y_k^3 \to y_*^3 \quad \text{with} \quad |x_*| \leq \kappa, |y_*^1| \leq \rho, |y_*^2| \leq \rho, |y_*^3| \leq \rho.$$

Three sub-cases can arise.

Case 1a: $y_*^i \neq y_*^j$ for all $1 \leq i < j \leq 3$. In this case, by continuity, G satisfies (i) at $(x_*, y_*^1, y_*^2, y_*^3)$.

Case 1b: Two of the points y_*^1, y_*^2, y_*^3 coincide with each other while the third point is different. Without loss of generality, suppose $y_*^1 = y_*^2 \neq y_*^3$. In this case, since $\nabla_y G_k(x_k, y_k^1) = \nabla_y G_k(x_k, y_k^2) = 0$, in a neighborhood of (x_*, y_*) for every k large enough there must be a point $(x_k^\sharp, y_k^\sharp) \in \mathcal{M}_k$ where the tangent vector to $\mathcal{M}_k$ is vertical. This means

$$\nabla_y G(x_k^\sharp, y_k^\sharp) \ = \ 0, \qquad\qquad \det\bigl(\partial_{y_i y_j} G\bigr)(x_k^\sharp, y_k^\sharp) \ = \ 0.$$

Moreover, $(x_k^\sharp, y_k^\sharp) \to (x_*, y_*^1)$ as $k \to \infty$. By continuity, we conclude that G satisfies (ii) at (x_*, y_*^1, y_*^3).

Case 1c: $y_*^1 = y_*^2 = y_*^3$. As in the previous case, in a neighborhood of (x_*, y_*) for every k large enough there must be a point $(x_k^\sharp, y_k^\sharp) \in \mathcal{M}_k$ where the tangent vector to $\mathcal{M}_k$ is vertical. Letting $k \to \infty$ we again conclude

$$\det\bigl(\partial_{y_i y_j} G\bigr)(x_*, y_*) \ = \ 0. \tag{6.32}$$

By (6.32), the limit manifold $\mathcal{M}$ has a vertical tangent at (x_*, y_*). Let $s \mapsto (x(s), y(s))$ be a local arc-length parameterization of $\mathcal{M}$ with $(x(0), y(0)) = (x_*, y_*)$.

If (6.13) holds, then the condition (iv) is verified and we are done (Fig,5.6, left). Otherwise, to fix the idea assume

$$\frac{d^2}{ds^2}x(s)\bigg|_{s=0} \;>\; 0.$$

We claim that (x_*, y_*) cannot be a point of global minimum for $G(x_*, \cdot)$. Indeed, consider a strictly increasing sequence $x_\nu \to x_*$. Let y_ν be a value where the function $G(x_\nu, \cdot)$ attains its global minimum. By taking a subsequence we can assume the convergence $(x_\nu, y_\nu) \;\to\; (x_*, y_\sharp)$ for some $y_\sharp$ (see Fig. 5.6, right). Notice that we must have $y_\sharp \neq y_*$, because the manifold $\mathcal{M}$ does not contain any point (x, y) with $x < x_*$, in a neighborhood of (x_*, y_*). By continuity, $(x_*, y_\sharp)$ must provide a global minimum to $G(x_*, \cdot)$. We conclude that all conditions in (ii) are now satisfied, with

$$(x, y^1) \;=\; (x_*, y_*), \qquad\qquad (x, y^2) \;=\; (x_*, y_\sharp).$$

CASE 2: Suppose that all G_k satisfy condition (ii). This implies that there exists a sequence of points $(x_k, y_{1,k}, y_{2,k})$ satisfying (ii) for G_k with

$$|x_k| \leq \kappa, \quad |y_k^i| \leq \rho \text{ for } i = 1, 2, \quad y_k^1 \neq y_k^2\,.$$

By possibly taking a subsequence, one has

$$x_k \to x_*,\; y_k^1 \to y_*^1,\; y_k^2 \to y_*^2, \quad \text{with} \quad |x_*| \leq \kappa,\; |y_*^1| \leq \rho,\; |y_*^2| \leq \rho.$$

Two sub-cases must be considered.

Case 2a: $y_*^1 \neq y_*^2$. In this case, by continuity the limit function G satisfies (ii) at (x_*, y_*^1, y_*^2).

Case 2b: $y_*^1 = y_*^2$. Since G_k satisfies (ii) at (x_k, y_k^1, y_k^2), this case can be handled by the same arguments as case 1c.

CASE 3: Suppose that all G_k satisfy condition (iii), say at points (x_k, y_k^1, y_k^2), with

$$|x_k| \leq \kappa, \quad |y_k^i| \leq \rho \text{ for } i = 1, 2, \quad y_k^1 \neq y_k^2.$$

By possibly taking a subsequence, one has

$$x_k \to x_*, \quad y_k^1 \to y_*^1, \quad y_k^2 \to y_*^2, \quad \text{with} \quad |x_*| \le \kappa,\ |y_*^1| \le \rho,\ |y_*^2| \le \rho.$$

Two sub-cases must be considered.

Case 3a: $y_*^1 \neq y_*^2$. In this case, if both ϕ_1 and ϕ_2 are still well defined at (x_*, y_*^1) and (x_*, y_*^2), then by continuity G satisfies (iii) at (x_*, y_*^1, y_*^2). The remaining possibility is that at least one of the functions ϕ_1 or ϕ_2 that is not well defined. Without loss of generality, assume that ϕ_1 is not well defined. Namely, $\det(\partial^2_{y_i,y_j} G(x, y_*^1)) = 0$ and hence G satisfies (ii). In all cases we conclude that $G \notin \mathcal{G}^\sharp$.

Case 3b: $y_*^1 = y_*^2 = y_*$. In this case, for all k large enough $(x_k, \phi_1(x_k))$ and $(x_k, \phi_2(x_k))$ should lie on the same component of a smooth one-dimensional manifold $\mathcal{M}_k$ defined by the equation $\nabla_y G_k(x, y) = 0$. In a neighborhood of (x_*, y_*), for every k large enough there must be a point $(x_k^\sharp, y_k^\sharp) \in \mathcal{M}_k$ where the tangent vector to $\mathcal{M}_k$ is vertical. Letting $k \to \infty$ we again conclude that (6.32) holds. Assuming that $G(x_*, \cdot)$ attains its global minimum at (x_*, y_*), the same arguments used in case 1c again imply that either (ii) or (iv) hold.

CASE 4: Suppose that every function G_k satisfies condition (iv) at some point (x_k, y_k) with

$$|x_k| \le \kappa, \quad |y_k| \le \rho.$$

By possibly taking a subsequence, we can assume $(x_k, y_k) \to (x_*, y_*)$. By the convergence $G_k \to G$ in $\mathcal{C}^3$, we conclude that G satisfies the condition (iv) at the point (x_*, y_*).

8. In this step we claim that, if none of the conditions (i)–(iv) holds true at any point where $G(x, \cdot)$ attains its global minimum, then the graph of the best reply map satisfies the conclusion of the theorem.

Indeed, consider any point $\bar{x}$. By (i) the minimum of the function $y \mapsto G(\bar{x}, y)$ is attained at most at two distinct points.

CASE 1: The global minimum is attained at the two points $y_1 \neq y_2$. Then, since (ii) fails, near (x, y_1) and (x, y_2) the manifold $\mathcal{M}$ coincides with the graph

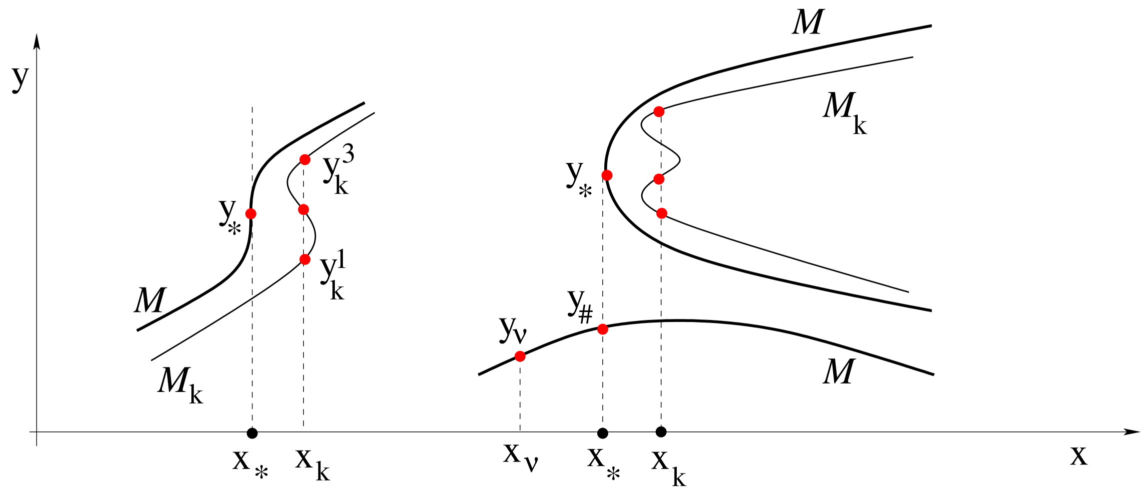

Figure 5.6. The two main cases considered in step **7** of the proof of Theorem 6.1.

of two functions $y = \phi_1(x)$ and $y = \phi_2(x)$. By the previous steps, we can assume that (iii) fails. To fix the ideas let

$$\frac{d}{dx}G(x, \phi_1(x)) \;>\; \frac{d}{dx}G(x, \phi_2(x)).$$

This implies that, for some $\delta > 0$ small enough, the global minimum is attained at $(x, \phi_1(x))$ for $x \in [\bar{x} - \delta, \bar{x}]$ and at $(x, \phi_2(x))$ for $x \in [\bar{x}, \bar{x} + \delta]$.

CASE 2: The global minimum is attained at a single point $\bar{y}$. We claim that

$$\det\left(\partial^2_{y_i y_j} G(\bar{x}, \bar{y})\right) \;\neq\; 0, \tag{6.33}$$

hence in a neighborhood of $\bar{x}$ the best reply map is single-valued: $R(x) = \{\phi(x)\}$, for some $\mathcal{C}^2$ function $\phi(\cdot)$.

Indeed, if (6.33) fails, since G does not satisfy (iv) at $(\bar{x}, \bar{y})$, we can assume that $\frac{d^2x}{ds^2}|_{s=0} > 0$, the other case is entirely similar. By the same arguments as case 1c, $G(\bar{x}, \cdot)$ achieves the global minimum at two points $\bar{y} \neq \hat{y}$, reaching a contradiction.

Combining the above two cases, the proof of the theorem is completed. $\square$

An application of Theorems 2.1 and 2.2 now yields the generic stability of Stackelberg equilibria. As in Section 2, we denote by $\mathcal{F}^\infty$ the family of functions satisfying **(B1)**, with the distance (2.6).

Theorem 6.2. *Consider a generic function* $G \in \mathcal{G}^\sharp \subset \mathcal{C}^3(\mathbb{R} \times \mathbb{R}^N)$*, satisfying* *the conclusion of Theorem 6.1. Then there exists an open dense set of functions* $\mathcal{F}^\sharp \subset \mathcal{F}^\infty$ *such that, for every* $F \in \mathcal{F}^\sharp$*, the following holds.*

The global minimum of F *on* $\mathcal{A} = \text{graph}(R)$ *is attained in generic position, as defined in (2.2). Moreover there exists constants* $C, \delta > 0$ *such that, if*

$$\|\widetilde{F} - F\|_{\mathcal{C}^2} \leq \delta, \qquad \|\widetilde{G} - G\|_{\mathcal{C}^3} \leq \delta, \tag{6.34}$$

then the corresponding perturbed optimization problem

$$\min_{(x,y)\in\widetilde{\mathcal{A}}} \widetilde{F}(x,y) \tag{6.35}$$

has a unique minimizer $(\tilde{x}, \tilde{y})$*, also in generic position. Here* $\widetilde{\mathcal{A}} = graph(\widetilde{R})$ *is the best reply map corresponding to the cost function* $\widetilde{G}$*. In addition*

$$|\tilde{x} - \bar{x}| + |\tilde{y} - \bar{y}| \leq C \cdot \Big(\|\widetilde{F} - F\|_{\mathcal{C}^2} + \|\widetilde{G} - G\|_{\mathcal{C}^3} \Big). \tag{6.36}$$

Proof. The same arguments used in the proof of Theorem 5.1 apply here as well. We remark that, in the present case, the strategy of the follower is not constrained to a closed set, hence the best reply map has the simpler structure described in Theorem 6.1 (compare Fig. 5.4 with Fig. 5.2). The conclusion of Theorem 6.2 thus follows already from Theorems 2.1 and 2.2, without using Corollaries 2.2 and 2.3. □

7 Concluding remarks

In this chapter we used techniques from differential geometry to analyze the generic structure of solutions to noncooperative games. In particular, we described the structure of the best reply map and proved the uniqueness and stability of the Stackelberg equilibrium, for an open dense set of cost functions to the leader and to the follower.

While our results only cover some specific settings, it is clear that these techniques can have broader applications. To keep the discussion simple, we considered Stackelberg games where the leader's strategy lies in a one-dimensional space. We expect that a similar analysis can be performed in the case where the leader chooses his strategy within a two-dimensional manifold. In this case, the follower solves a minimization problem depending on two parameters. The best reply map will then contain not only "fold" but also "cusp" singularities [1, 38]. On the other hand, when the leader's strategy has dimension $m \geq 3$, the generic structure of this map may be more difficult to describe.

The main motivation for the present work came from the analysis of infinite-horizon stochastic games with discrete state space [17], which will be introduced int he next chapter. In the next chapter, a key role is played by a stability assumption on the Stackelberg equilibrium solution, when the follower adopts a myopic strategy. The arguments used in the proof of Theorem 4.1 show that this assumption is not very restrictive. Indeed, it is satisfied by "almost all" cost functionals.

Chapter 6

Self-consistent Feedback Stackelberg Equibria for Infinite Horizon Stochastic Games

1 Introduction

Because of their usefulness in economic modeling, Stackelberg games have been the topic of several investigations [21, 24, 25, 27, 43, 46, 47, 53]. In the present chapter we consider a stochastic game for two players in infinite time horizon, with exponentially discounted costs. Our main focus will be on Stackelberg equilibrium solutions in feedback form. The state of the system is denoted by $x \in X$, and can take either discrete or continuum values. We assume that the two players (the leader and the follower, respectively) choose time-independent feedback strategies

$$u_1(x) \in U_1\,, \qquad u_2(x) \in U_2,$$

which affect the random evolution of the system. Two main settings can be considered:

1 - A discrete state space. Here $X = \{1, 2, 3, \ldots, N\}$. Given the feedback controls $u_k : X \mapsto U_k$, $k = 1, 2$, we assume that the evolution is described by a

Markov process in continuous time, with transition probabilities

$$\text{Prob.}\Big\{x(t+\varepsilon)=j \;\Big|\; x(t)=i\Big\} \;=\; \phi_{ij}(u_1(i),u_2(i))\,\varepsilon + o(\varepsilon), \qquad j\neq i, \quad (1.1)$$

for some given functions $\phi_{ij}\geq 0$.

2 - A continuum state space. In this case $X=\mathbb{R}^d$. Given the controls u_1,u_2, we assume that the evolution is described by a diffusion process of the form

$$dx \;=\; f\big(x,u_1(x),u_2(x)\big)\,dt+\sigma(x)\,dW, \qquad (1.2)$$

where f is a smooth function and W denotes standard Brownian motion [36, 37].

In both cases, the expected costs to the two players take the form

$$J_k \;\doteq\; E^{\mu}\left[\int_0^{+\infty} r_k\,e^{-r_k t}L_k\Big(x(t),u_1(x(t)),u_2(x(t))\Big)\,dt\right], \qquad k=1,2. \quad (1.3)$$

Here E^{μ} denotes the expectation for a given probability measure μ on the initial state $x(0)$. Moreover, r_1,r_2 are the exponential discount factors.

We now review the concept of **Stackelberg equilibrium solution**, based on the notion of **best reply**. This covers both cases of discrete and continuum state space.

Definition 1.1. *We say that $u_2^*: X\mapsto U_2$ is a* **best reply** *for Player 2 to a strategy $u_1^*: X\mapsto U_1$ implemented by Player 1 if, for every initial datum x_0, the control u_2^* is an optimal feedback for the following stochastic optimization problem.*

$$\textit{Minimize:}\quad E^{x_0}\left[\int_0^{+\infty} r_2 e^{-r_2 t}L_2\big(x(t),u_1^*(x(t)),u_2(t)\big)\,dt\right], \qquad (1.4)$$

subject to the dynamics (1.1) or (1.2), with $u_1=u_1^*(x)$ and with initial condition

$$x(0) \;=\; x_0. \qquad (1.5)$$

The set of best replies for Player 2 will be denoted by $\mathcal{R}_2(u_1^*)$.

Note that Player 2 solves a stochastic optimal control problem. The same feedback control u_2^* will thus be optimal simultaneously for every initial datum $x_0 \in X$, and hence also for every probability distribution μ on the initial data.

Definition 1.2. *We say that a pair of functions $u_1^* : X \mapsto U_1$, $u_2^* : X \mapsto U_2$ is a* **feedback Stackelberg equilibrium** *for the game with dynamics (1.1) or (1.2), cost functions (1.3), and probability distribution μ on the initial data, if the following holds.*

(i) $u_2^* \in \mathcal{R}_2(u_1^*)$.

(ii) For every other feedback control $u_1^\sharp : X \mapsto U_1$ of the leader, and every optimal reply $u_2^\sharp \in \mathcal{R}_2(u_1^\sharp)$ of the follower, one has

$$E^\mu \left[\int_0^{+\infty} r_1 e^{-r_1 t} L_1\big(x(t), u_1^*(x(t)), u_2^*(t)\big)\, dt \right] \leq E^\mu \left[\int_0^{+\infty} r_1 e^{-r_1 t} L_1\big(x(t), u_1^\sharp(x(t)), u_2^\sharp(t)\big)\, dt \right]. \tag{1.6}$$

Notice that here we assume that the control of the leader $u_1 = u_1(x)$ is assigned only as a function of the state x. This is very different from a feedback $u_1 = u_1(x, u_2(x))$ which also depends on the instantaneous control chosen by the follower. For alternative concepts of Stackelberg equilibria, see for example [24, 25].

The main issue motivating the present chapter is the dependence of the Stackelberg equilibrium on the initial probability distribution μ. In the case of an optimal control problem, or a Nash equilibrium to a differential game, the same feedback solution remains valid for every initial datum. However, this is no longer true for Stackelberg equilibria, which are not "time consistent", in general. In other words, if the leader were allowed to restart the game at time $\tau > 0$ from the state $x(\tau) \neq x_0$, he would likely choose a feedback control $u = \widehat{u}_1(x) \neq u_1^*(x)$, different from the original one.

A typical example occurs in a bank regulation problem. At a time when the overall banking system is healthy, the government (the leading player) announces a policy of "no intervention", ruling out any bailout. This should discourage bank managers (the followers) from taking excessive risks to boost their profits. However, if at a later time some large banks are at risk of bankruptcy, the government may wish to reverse its own policy and provide a bailout.

Aim of the present chapter is to propose a concept of feedback equilibrium which does not make reference to any initial probability distribution, but depends only on the dynamics of the stochastic system and the cost functions to the players.

Definition 1.3. *Consider the stochastic game with dynamics (1.1) or (1.2), and cost functions (1.3). We say that a pair of functions* (u_1^*, u_2^*)*, with* $u_k^* : X \mapsto U_k$*,* $k = 1, 2$*, is a* **self-consistent Stackelberg equilibrium in feedback form** *if the conditions (i)-(ii) in Definition 1.2 hold, for some probability* μ *on the set of initial data which is* invariant *for the Markov process (1.1) with transition intensities*

$$\phi_{ij}\big(u_1^*(i), u_2^*(i)\big), \tag{1.7}$$

or, respectively, for the diffusion process

$$dx \;=\; f\big(x, u_1^*(x), u_2^*(x)\big)\, dt + \sigma(x)\, dW. \tag{1.8}$$

Remarks. 1. The above definition can model a situation where the leading player (a far-sighted legislator) either does not know or is not much concerned with the present state of the system. Rather, he wants to draft a regulation which, after a short transient period, will be recognized as "best possible" at future times.

2. Definition 1.3 could be strengthened by requiring that the measure μ is the unique, asymptotically stable invariant probability distribution for the Markov process (1.1) with transition intensities (1.7), or for the diffusion process (1.8), respectively.

3. An alternative interpretation goes as follows. Consider a government (the leading player) announcing a legislation u_1^* which affects a large number of agents (industrial companies, financial institutions, etc.). If each of these agents acts independently, implementing a best reply u_2^*, the random states of these agents will evolve in time according to (1.7) or (1.8). Call μ^t the random distribution of these agents at time t, and assume that, as $t \to +\infty$, one has the asymptotic convergence $\mu^t \to \mu$. If the leading player is a far-sighted legislator, he can choose a feedback u_1^* which is optimal w.r.t. the asymptotic probability distribution μ. This feedback may not be optimal at the initial time $t = 0$, but it will get closer and closer to an optimal one as $t \to +\infty$.

4. Up to this stage, we tacitly assumed that the feedback strategies could be arbitrary measurable functions $u_k : X \mapsto U_k$, as long as the stochastic evolution equations (1.1) are well defined.

This assumption is entirely appropriate for $u_2^*(\cdot)$, which is the optimal feedback reply of the follower and can thus be determined by solving an equation of Hamilton-Jacobi type. On the other hand, if $u_1(\cdot)$ models a strategy publicly announced by a legislator, one may wish to restrict the choice of u_1 within a finite-dimensional set $\mathcal{U}_1$ of functions which can be more easily implemented in practice. For example, think of a central bank who announces the future values $u_1 = u_1(x)$ of the prime rate as a function of leading economic indicators $x = (x_1, \ldots, x_d)$. In a realistic setting, this function $x \mapsto u_1(x)$ must have a very simple structure. Say, piecewise constant or piecewise affine, with a small number of jumps.

If the feedback control $u_1(\cdot)$ is restricted to a small subset $\mathcal{U}_1$ of all measurable maps $X \mapsto U_1$, the concept of feedback Stackelberg equilibrium (u_1^*, u_2^*) remains valid with one obvious modification. Namely, in Definition 1.2 one should now require that $u_1^* \in \mathcal{U}_1$, and moreover (1.6) holds for every feedback control $u_1^\sharp \in \mathcal{U}_1$. Our Definition 1.3 of self-consistent equilibrium can be modified accordingly.

In the remainder of this chapter we focus on the model (1.1) with discrete state space. Self-consistent equilibria for the diffusion process (1.2) will be studied in the future.

Section 2 reviews some basic properties of the best reply map and of Stackelberg equilibria. Toward the existence of a self-consistent equilibrium, we consider the composite map

$$\mu \;\mapsto\; (u_1^*, u_2^*) \;\mapsto\; \mu^\infty\,, \tag{1.9}$$

where

- μ is a probability distribution on the initial state,
- (u_1^*, u_2^*) is a Stackelberg equilibrium solution in feedback form, relative to the initial distribution μ,
- μ^∞ is the asymptotic probability distribution as $t \to +\infty$ for the dynamical system (1.1), with feedback controls u_1^*, u_2^*.

According to Definition 1.3, (u_1^*, u_2^*) is a self-consistent Stackelberg equilibrium if $\mu^\infty = \mu$. To prove the existence of self-consistent equilibrium, under suitable assumptions we will show that the composite map in (1.9) is single-valued and has a fixed point. Three results in this direction will be provided.

In Section 3 we derive conditions that ensure that the best reply map $u_1 \mapsto \mathcal{R}_2(u_1)$ as well as the map $\mu \mapsto (u_1^*, u_2^*)$ in (1.9) are always single-valued. The existence of a fixed point for (1.9), stated in Theorem 3.1, is then a consequence of Brouwer's theorem. Unfortunately, this approach requires quite restrictive convexity assumptions, which are not always easy to check.

The analysis in the remaining sections follows an entirely different direction. Namely, we seek results on the existence of a self-consistent Stackelberg equilibrium which are valid for a *generic game*. That means: our results hold true for all triples (L_1, L_2, ϕ) of cost functions and transition functions, in an open dense set of a suitable space of continuously differentiable maps.

Our analysis shows that composite map (1.9) is a strict contraction, and hence has unique fixed point, in two main cases:

(i) The follower is narrow-sighted. Namely, the exponential discount factor r_2 is sufficiently large.

(ii) The best reply for the follower is single-valued, and the leader is far-sighted Namely, the discount factor r_1 is sufficiently small.

As an intermediate step, in Section 4 we first consider a one-shot game where the leader and the follower choose controls $x, y \in [0,1]^N$, while the cost functions have the form

$$F(x,y), \qquad G_\varepsilon(x,y) = \sum_{i=1}^{N} G_i(x_i, y_i) + \varepsilon \widetilde{G}(x,y), \tag{1.10}$$

respectively. Here we assume $F \in \mathcal{C}^2(\mathbb{R}^{2N})$, $G_i \in \mathcal{C}^3(\mathbb{R}^2)$, $\widetilde{G} \in \mathcal{C}^3(\mathbb{R}^{2N})$. Building upon the analysis in Chapter 5, we provide a detailed description of the best reply map for the follower, first for $\varepsilon = 0$, then for $\varepsilon > 0$ small, valid in a generic setting. Namely, our results apply to cost functions $G_i, \widetilde{G}$ in an open dense subset of $\mathcal{C}^3$. In all these cases, the Stackelberg equilibrium is unique and is stable w.r.t. small perturbations of the cost functions. These results, and the techniques developed to achieve them, can have independent interest.

In Section 5 the previous analysis is applied to the case of feedback equilibria stochastic game with dynamics (1.1) and cost functions (1.3). The case of a myopic follower corresponds to the one-shot game (1.10) with $\varepsilon = 0$, while $\varepsilon = r_2^{-1} > 0$ small corresponds to the case of a narrow-sighted follower. Under generic assumptions on the cost functions L_1, L_2 in (1.3), and on the transition functions ϕ_{ij} in (1.1), we prove that the map (1.9) is single-valued. Indeed, for $r_2 >> 1$ large enough, this map is a strict contraction. Its fixed point yields the unique self-consistent Stackelberg equilibrium.

Finally, in Section 6 we consider a far-sighted leader. Namely, we take discount factors $r_2 > 0$ arbitrary but $r_1 > 0$ suitably small. Assuming that the best reply map of the follower is single-valued, we show that, under generic conditions on the cost L_1 to the leader, a unique self-consistent Stackelberg equilibrium exists. Again, the proof is achieved by showing that the composed map in (1.9) is a strict contraction.

2 Feedback equilibria with discrete state space

In this section we consider a continuous-time Markov process with finitely many states $i \in \{1, ..., N\}$ and transition functions ϕ_{ij} as in (1.1). To simplify our notation, from now on we denote by u, v the controls of the leader and of the follower, respectively. We assume that these controls take values in the sets $U_1 = U_2 = [0, 1]$. A pair of feedbacks controls $u, v : \{1, \ldots, N\} \mapsto [0, 1]$ can thus be identified with two vectors

$$u = (u_1, \ldots, u_N), \qquad v = (v_1, \ldots, v_N). \tag{2.1}$$

For any given pair of feedback controls $(u, v) \in [0, 1]^N \times [0, 1]^N$ and any initial state $x(0) = i \in \{1, \ldots, N\}$, the expected cost $J_{2,i}$ to Player 2 is

$$J_{2,i} \doteq E^{\{x(0)=i\}} \left[\int_0^{+\infty} r_2 e^{-r_2 t} L_2\Big(x(t), u_{x(t)}, v_{x(t)}\Big)\, dt \right]. \tag{2.2}$$

To shorten the notation, from now on we shall write

$$L_{2,i} \;=\; L_2(i, u_i, v_i), \qquad \phi_{ij} \;=\; \phi(u_i, v_i), \qquad \phi_i \;=\; \sum_{j\neq i} \phi_{ij}\,.$$

Calling τ the first time when the system jumps from state i to some other state, we have the identities

$$\begin{aligned} J_{2,i} \;&=\; \int_0^\infty \left(\int_0^\tau r_2 e^{-r_2 t} L_{2,i}\, dt + r_2 e^{-r_2\tau} \sum_{j\neq i} \frac{\phi_{i,j}}{\phi_i} J_{2,j} \right) \phi_i e^{-\phi_i \tau}\, d\tau \\ &=\; \int_0^\infty \left((1 - e^{-r_2\tau}) L_{2,i} + e^{-r_2\tau} \sum_{j\neq i} \frac{\phi_{i,j}}{\phi_i} J_{2,j} \right) \phi_i e^{-\phi_i \tau}\, d\tau \qquad (2.3) \\ &=\; L_{2,i} + \frac{1}{\phi_i + r_2} \left(\sum_{j\neq i} \phi_{i,j} J_{2,j} - \phi_i L_{2,i} \right), \end{aligned}$$

$$J_{2,i} - \sum_{j\neq i} \frac{\phi_{ij}}{\phi_i + r_2} J_{2,j} \;=\; \frac{r_2}{\phi_i + r_2} L_{2,i}\,, \qquad (2.4)$$

or equivalently:

$$J_{2,i} \;=\; L_{2,i} + \frac{1}{r_2} \sum_{j\neq i} \phi_{ij} (J_{2,j} - J_{2,i}). \qquad (2.5)$$

From (2.5) it follows the limit

$$\lim_{r_2\to+\infty} J_{2,i} \;=\; L_{2,i}\,. \qquad (2.6)$$

Here and in the sequel, it will be convenient to adopt a vector notation and write

$$\mathbf{J_k} \;\doteq\; (J_{k,1}, \ldots, J_{k,N})^T, \qquad \mathbf{L_k} \doteq (L_{k,1}, \ldots, L_{k,N})^T, \qquad k = 1, 2. \qquad (2.7)$$

Notice that (2.5) is a system of linear equations for the components of the column vector $\mathbf{J_2}$, namely

$$\Big(\mathbf{I} - \frac{1}{r_2} \boldsymbol{\Phi} \Big) \mathbf{J_2} \;=\; \mathbf{L_2}, \qquad (2.8)$$

where $\mathbf{\Phi}$ is the matrix with components

$$\Phi_{ij} \;=\; \begin{cases} -\phi_i & \text{if} \quad i = j, \\ \phi_{ij} & \text{if} \quad i \neq j. \end{cases} \tag{2.9}$$

Notice that $\mathbf{\Phi}$ is the generator matrix of the continuous-time finite-state Markov process at (1.1).

Remark 2.1. For any given feedback control vectors u, v, if the continuous-time Markov chain is irreducible and positive recurrent, there exists a unique probability distribution

$$\mu^\infty \;\in\; \Delta_N \;\doteq\; \Big\{(p_1, p_2, ..., p_N)\,;\;\; p_i \geq 0,\;\; \sum p_i = 1\Big\}, \tag{2.10}$$

which is stationary, i.e. it satisfies $\mu^\infty \cdot \mathbf{\Phi} = \mathbf{0}$. For the basic theory of continuous-time Markov chains we refer to [60].

For any feedback control $u \in [0,1]^N$ chosen by the leader, the value function of the follower is defined as

$$V_2(i) \;=\; \min_v \; E^{\{x(0)=i\}} \left[\int_0^{+\infty} r_2 \, e^{-r_2 t} L_2\big(x(t), u(x(t)), v(t)\big)\, dt \right]. \tag{2.11}$$

By a standard argument, V_2 satisfies the dynamic programming equation

$$V_2(i) \;=\; \min_{\omega \in [0,1]} \left\{ L_2(i, u_i, \omega) + \frac{1}{r_2} \sum_{j \neq i} \phi_{ij}(u_i, \omega)\Big(V_2(j) - V_2(i)\Big) \right\} \tag{2.12}$$

for every $i \in \{1, ..., N\}$. Without loss of generality, the minimization in (2.11) can be restricted to optimal controls of feedback type: $v \in [0,1]^N$. A feedback control v^* is optimal if it satisfies

$$v_i^* \;=\; \arg\min_{\omega \in [0,1]} \left\{ L_2(i, u_i, \omega) + \frac{1}{r_2} \sum_{j \neq i} \phi_{ij}(u_i, \omega)\Big(V_2(j) - V_2(i)\Big) \right\} \tag{2.13}$$

for every $i = 1, \ldots, N$. Notice that in this case the same feedback control v^* is optimal for every probability distribution $\mu \in \Delta_N$ on the initial data. We denote by $\mathcal{R}_2(u) \subseteq [0,1]^N$ the set of feedback controls v^* for the follower which satisfy (2.13). These are the *best replies* for the follower to the strategy u chosen by the leader.

Remark 2.2. Assume that the Markov chain is reducible. For example, starting from state 1 one can reach states 2,3, but never state 4. Then, if the initial state is $x(0) \in \{1,2,3\}$, we can arbitrarily choose the control v_4 and still obtain a best reply for the follower. These kind of controls are here ruled out, if they do not satisfy (2.13).

Lemma 2.1. *Consider a game with dynamics (1.1) and cost functions (1.3), where the functions ϕ_{ij} and L_2 are continuous. Then the best reply map $u \mapsto \mathcal{R}_2(u)$ is a multivalued function with closed graph.*

Proof. We first observe that the set $\mathcal{R}_2(u)$ of best replies is independent of the probability distribution μ on the initial state. Indeed, for any given feedback control u of the leader, the best reply solves a standard optimal control problem for the follower.

Now consider a sequence of feedback controls $(u^n)_{n\geq 1}$ for the leading player, with $u^n \to u \in [0,1]^N$, as $n \to \infty$. Calling V_2^n, V_2 the corresponding value functions, we then have the convergence $V_2^n(i) \to V_2(i)$ for all $i \in \{1,\ldots,N\}$.

Assume that $v^n \in \mathcal{R}_2(u^n)$ is a best reply, for every $n \geq 1$. Moreover, assume that $v_i^n \to v_i$ for every component $i \in \{1,\ldots,N\}$ as $n \to \infty$. Then the assumption

$$v_i^n = \arg\min_{\omega\in[0,1]} \left\{ L_2(i,u_i^n,\omega) + \frac{1}{r_2}\sum_{j\neq i}\phi_{ij}(u_i^n,\omega)\Big(V_2^n(j) - V_2^n(i)\Big)\right\}$$

by continuity implies

$$v_i = \arg\min_{\omega\in[0,1]} \left\{ L_2(i,u_i,\omega) + \frac{1}{r_2}\sum_{j\neq i}\phi_{ij}(u_i,\omega)\Big(V_2(j) - V_2(i)\Big)\right\},$$

showing that the graph of $\mathcal{R}_2$ is closed. □

Lemma 2.2. *Consider a game with dynamics (1.1) and cost functions (1.3). Assume that:*

(i) Every transition function ϕ_{ij} is affine w.r.t. v_i. Namely, it has the form

$$\phi_{ij}(u_i, v_i) \;=\; a_i(u_i) + b_i(u_i)v_i\,, \qquad \text{with} \;\; a_i, b_i \in \mathcal{C}^3(\mathbb{R}).$$

(ii) For every fixed $i \in \{1, \ldots, N\}$, and $u_i \in [0,1]$, the function $L_2(i, u_i, \cdot)$ is strictly convex in the variable v_i.

Then for every $u \in [0,1]^N$, the set of best replies $\mathcal{R}_2(u)$ is a singleton.

Proof. Under the above assumptions (i)-(ii), for every i the map

$$\omega \;\mapsto\; L_2(i, u_i, \omega) + \frac{1}{r_2}\sum_{j\neq i} \phi_{ij}(u_i, \omega)\Big(V_2(j) - V_2(i)\Big) \tag{2.14}$$

is strictly convex, being the sum of a strictly convex and an affine function. Its minimum is thus attained at a single point $v_i^* \in [0,1]$. □

Lemma 2.3. *In addition to the assumptions of Lemma 2.2, assume that the cost functions $L_2(i, \cdot, \cdot)$ are $\mathcal{C}^3$ and satisfy*

$$\lim_{\omega\to 0+} L_2(i, u_i, \omega) \;=\; \lim_{\omega\to 1-} L_2(i, u_i, \omega) \;=\; +\infty. \tag{2.15}$$

Then the best reply map $u \mapsto \mathcal{R}_2(u)$ is single-valued and has $\mathcal{C}^2$ regularity.

Proof. 1. Given $u \in [0,1]^N$, let $v^*(u)$ be the unique best reply of the follower. Under the assumptions (i)-(ii) in Lemma 2.2, by (2.15), the minimizer of (2.14) cannot be achieved at the boundary of $[0,1]$, i.e $v_i^*(u) \notin \{0,1\}$. As a result, $v^*(u)$ remains strictly in the interior of $[0,1]^N$.

2. It remains to show that the best reply map $u \mapsto v^*(u)$ has $\mathcal{C}^2$ regularity. To fix the ideas, consider the initial distribution $\mu = (\frac{1}{N}, \ldots, \frac{1}{N}) \in \Delta_N$. Then the cost function for the follower can be expressed as

$$G(u,v) \doteq \mu^T \cdot \mathbf{C}^{-1}(u,v) \cdot \mathbf{L}_2(u,v),$$

where

$$\mathbf{C}(u,v) \;\doteq\; \mathbf{I} - \frac{1}{r_2}\mathbf{\Phi}(u,v), \tag{2.16}$$

and $\mathbf{\Phi}$ is the matrix at (2.9). For any fixed u, the feedback control of the follower is independent of the probability distribution μ on the initial state. By the necessary conditions for optimality, the unique minimizer $v^*(u)$ for the map $G(u, \cdot)$ is characterized by the equation

$$\nabla_v G(u, v^*(u)) \;=\; 0. \tag{2.17}$$

The regularity assumptions on L_2 and ϕ imply that $G \in \mathcal{C}^3$. By the implicit function theorem, to prove that the map $u \to v^*(u)$ has $\mathcal{C}^2$ regularity it thus suffices to check that the Hessian matrix $D_v^2 G$ of second order derivatives of G w.r.t. $v = (v_1, \ldots, v_N)$ is everywhere strictly positive definite. Toward this goal, observe that for every $i \in \{1, \ldots, N\}$ we have the implication

$$[\mathbf{C}^{-1}\mathbf{L}_2]_{v_i} \;=\; \mathbf{0} \qquad \Longrightarrow \qquad [\mathbf{C}^{-1}]_{v_i}\mathbf{L}_2 \;=\; -\mathbf{C}^{-1}\mathbf{L}_{2,v_i}\ .$$

We now compute

$$[\mathbf{C}^{-1}]_{v_i} \;=\; \mathbf{C}^{-1}\mathbf{C}_{v_i}\mathbf{C}^{-1}$$

where, by (2.16) and (2.9), the matrix $\mathbf{C}_{v_i}$ has the form

$$[\mathbf{C}_{v_i}]_{j,k} = \begin{cases} 0 & \text{if} \quad j \neq i, \\ -\partial_{v_i}\phi_{ik} & \text{if} \quad j = i, k \neq i, \\ \sum_{p\neq i} \partial_{v_i}\phi_{ip} & \text{if} \quad j = k = i. \end{cases}$$

Since ϕ_{ij} is affine w.r.t. v_i, all second derivatives vanish: $\mathbf{C}_{v_i v_j} = \mathbf{0}$. At the point

$(u, v^*(u))$, the second order derivatives can thus be computed as

$$\begin{aligned}
[\mathbf{C}^{-1}\mathbf{L}_2]_{v_i,v_j} &= [\mathbf{C}^{-1}]_{v_i,v_j}\mathbf{L}_2 + [\mathbf{C}^{-1}]_{v_i}\mathbf{L}_{2,v_j} + [\mathbf{C}^{-1}]_{v_j}\mathbf{L}_{2,v_i} + \mathbf{C}^{-1}\mathbf{L}_{2,v_iv_j} \\
&= [\mathbf{C}^{-1}]_{v_j}\mathbf{C}_{v_i}\mathbf{C}^{-1}\mathbf{L}_2 + \mathbf{C}^{-1}\mathbf{C}_{v_i}[\mathbf{C}^{-1}]_{v_j}\mathbf{L}_2 + [\mathbf{C}^{-1}]_{v_i}\mathbf{L}_{2,v_j} + [\mathbf{C}^{-1}]_{v_j}\mathbf{L}_{2,v_i} + \mathbf{C}^{-1}\mathbf{L}_{2,v_iv_j} \\
&= -\mathbf{C}^{-1}\mathbf{C}_{v_j}\mathbf{C}^{-1}\mathbf{L}_{2,v_i} - \mathbf{C}^{-1}\mathbf{C}_{v_i}\mathbf{C}^{-1}\mathbf{L}_{2,v_j} + \mathbf{C}^{-1}\mathbf{C}_{v_i}\mathbf{C}^{-1}\mathbf{L}_{2,v_j} \\
&\quad + \mathbf{C}^{-1}\mathbf{C}_{v_j}\mathbf{C}^{-1}\mathbf{L}_{2,v_i} + \mathbf{C}^{-1}\mathbf{L}_{2,v_iv_j} \\
&= \mathbf{C}^{-1}\mathbf{L}_{2,v_iv_j}\,.
\end{aligned} \tag{2.18}$$

By observing that

$$\mathbf{L}_{2,v_iv_j} = \begin{cases} (0, \;\ldots\;, 0) & \text{if} \quad i \neq j, \\ \Big(0, \;\ldots, \; 0, \; \frac{\partial^2}{\partial v_i \partial v_i} L_2(i, u_i, v_i), \; 0, \;\ldots, \; 0\Big)^T & \text{if} \quad i = j, \end{cases}$$

we conclude that the Hessian matrix of the cost function G with respect to v at the point $(u, v^*(u))$ is diagonal. Namely,

$$D_v^2 G(u, v^*(u)) \;=\; \mathrm{diag}(d_1(u), d_2(u)\ldots, d_n(u)),$$

where the diagonal element $d_i(u)$ is defined as

$$d_i(u) \;=\; \frac{\partial^2}{\partial v_i^2} L_2(i, u_i, v_i^*(u)) \cdot \frac{1}{N}\sum_{j=1}^{N}[\mathbf{C}^{-1}]_{j,i}\,.$$

Due to the fact that the continuous Markov chain is irreducible and positive recurrent, for any $r_2 > 0$ the matrix $\mathbf{C}$ is strictly diagonally dominant with positive diagonal elements and nonpositive off-diagonal elements, thus $\mathbf{C}$ is a nonsingular M-matrix. In turn, this implies that $\mathbf{C}^{-1}$ has nonnegative elements and the thus $\sum_{j=1}^{N}[\mathbf{C}^{-1}]_{j,i} > 0$ for all $i \in \{1, \ldots, N\}$.

On the other hand, since $L_2(i, u_i, v_i)$ is strictly convex with respect to v_i, $D_v^2 G(u, v^*(u))$ is strictly positive definite. This completes the proof. □

Remark 2.3 In the above lemmas, for simplicity the controls u_i, v_i were assumed to take values in $[0, 1]$. The same results clearly remain valid more generally if they take values in compact convex sets $U_1, U_2 \subset \mathbb{R}^d$, and $L_2(i, u_i, \omega) \to +\infty$ as ω approaches the boundary of U_2.

We conclude this section by briefly reviewing the existence of Stackelberg equilibrium solutions, for a game with finite state space $X = \{1, \ldots, N\}$. For the basic theory of multifunctions we refer to [3] or the Appendix in [22].

Theorem 2.1. *Consider the game with dynamics (1.1) and exponentially discounted payoffs (1.3). For every probability distribution μ on the initial state, there exists a Stackelberg equilibrium in feedback form. Calling $\mathcal{S}(\mu)$ the set of all these equilibria, the multifunction $\mu \mapsto \mathcal{S}(\mu)$ is upper semicontinuous with compact values.*

Proof. **1.** Let an initial probability distribution $\mu \in \Delta_N$ as in (2.10) be given. To construct a Stackelberg equilibrium, consider a sequence (u^n, v^n) with $v^n \in \mathcal{R}_2(u^n)$ for all $n \geq 1$, and assume

$$\begin{aligned}
&\lim_{n\to\infty} E^\mu \Big[\int_0^{+\infty} r_1 e^{-r_1 t} L_1\big(x(t), u^n(x(t)), v^n(x(t))\big)\, dt \Big] \\
&\quad = \inf_u \inf_{v \in \mathcal{R}_2(u)} E^\mu \Big[\int_0^{+\infty} r_1 e^{-r_1 t} L_1\big(x(t), u(x(t)), v(x(t))\big)\, dt \Big].
\end{aligned}$$

By possibly taking a subsequence we can assume $(u^n, v^n) \to (u^*, v^*) \in [0,1]^{2N}$. The upper semicontinuity of the multifunction $\mathcal{R}_2$, proved in Lemma 2.1, now implies $v^* \in \mathcal{R}_2(u^*)$. Moreover the continuity of the expected value yields

$$\begin{aligned}
&E^\mu \Big[\int_0^{+\infty} r_1 e^{-r_1 t} L_1\big(x(t), u^*(x(t)), v^*(x(t))\big)\, dt \Big] \\
&\quad = \inf_u \inf_{v \in \mathcal{R}_2(u)} E^\mu \Big[\int_0^{+\infty} r_1 e^{-r_1 t} L_1\big(x(t), u(x(t)), v(x(t))\big)\, dt \Big].
\end{aligned}$$

Hence (u^*, v^*) provides a Stackelberg equilibrium.

2. We now prove the upper semicontinuity of the map $\mu \mapsto \mathcal{S}(\mu)$, and show that it takes compact values. Consider any convergent sequence $(\mu^n, u^n, v^n) \to (\mu^*, u^*, v^*)$ where $\mu^n \in \Delta_N, (u^n, v^n) \in \mathcal{S}(\mu^n)$. Notice that here we again use the upper semicontinuity of $\mathcal{R}_2$ to guarantee $v^* \in \mathcal{R}_2(u^*)$. By the definition of $\mathcal{S}(\mu)$, we have

$$E^{\mu^n}\Big[\int_0^{+\infty} r_1 e^{-r_1 t} L_1\big(x(t), u^n(x(t)), v^n(x(t))\big)\, dt\Big]$$
$$= \inf_u \inf_{v\in\mathcal{R}_2(u)} E^{\mu^n}\Big[\int_0^{+\infty} r_1 e^{-r_1 t} L_1\big(x(t), u(x(t)), v(x(t))\big)\, dt\Big].$$

Since the expected cost J_1 defined in (1.3) is continuous with respect to μ, u, v, we achieve

$$E^{\mu^*}\Big[\int_0^{+\infty} r_1 e^{-r_1 t} L_1\big(x(t), u^*(x(t)), v^*(x(t))\big)\, dt\Big]$$
$$= \inf_u \inf_{v\in\mathcal{R}_2(u)} E^{\mu^*}\Big[\int_0^{+\infty} r_1 e^{-r_1 t} L_1\big(x(t), u(x(t)), v(x(t))\big)\, dt\Big].$$

Hence the map $\mu \mapsto \mathcal{S}(\mu)$ is upper semicontinuous. For any fixed $\mu \in \Delta_N$ and any sequence $(u^n, v^n) \in \mathcal{S}(\mu)$, by possibly taking a subsequence, we can assume $(u^n, v^n) \to (u^*, v^*)$ with $v^* \in \mathcal{R}_2(u^*)$. By the same argument used in step **1**, $(u^*, v^*) \in \mathcal{S}(\mu)$. Since $(u, v) \in [0,1]^N \times [0,1]^N$ which is a bounded set, the upper semicontinuous map $\mu \mapsto \mathcal{S}(\mu)$ has compact values. □

3 Existence of Self-Consistent Stackelberg equilibria

As before, we consider a finite state space $X = \{1, \ldots, N\}$, so that a feedback control can be identified with a vector $u \in [0,1]^N$. Moreover, a probability distribution on X is identified with a point on the unit simplex Δ_N at (2.10).

For convenience, we shall denote by $\mathcal{S}(\mu)$ the set of all the Stackelberg equilibria (u, v) for the given initial distribution μ. Moreover, for any subset $\mathcal{S} \subseteq [0,1]^N \times [0,1]^N$ we define the family of invariant probability distributions

$$\mu^\infty(\mathcal{S}) \doteq \Big\{\mu \in \Delta_N\,;\quad \mu \text{ is an invariant distribution for the Markov chain (1.1)}$$
$$\text{generated by some } (u,v) \in \mathcal{S}\Big\}.$$

According to Definition 1.3, to construct a self-consistent Stackelberg equilibrium one needs to find a fixed point of the (possibly multivalued) transformation

$$\mu \;\mapsto\; \mathcal{S}(\mu) \;\mapsto \mu^\infty(\mathcal{S}(\mu)) \;\doteq\; \Psi(\mu). \tag{3.1}$$

In order to prove the existence of a fixed point for the above multifunction $\Psi : \Delta_N \mapsto \Delta_N$, a natural approach is to show that every $\Psi(\mu)$ is single-valued, and then use Brouwer's fix point theorem. In this section we examine some cases where this approach is successful.

Lemma 3.1. *Suppose that, for any given controls $(u, v) \in [0, 1]^N \times [0, 1]^N$, the continuous time Markov chain at (1.1) is irreducible and positive recurrent. Then the map Ψ defined at (3.1) is an upper semicontinuous multifunction with nonempty, compact values.*

Proof. By Lemma 2.2 the multifunction $\mu \to \mathcal{S}(\mu)$ is upper semicontinuous with nonempty, compact values. It thus remains to check the continuity of the single-valued map $(u, v) \mapsto \mu^\infty(u, v)$. For any fixed pair of controls $(u, v) \in [0, 1]^N \times [0, 1]^N$, due to the assumption that the Markov chain is irreducible and positive recurrent, the kernel of $\mathbf{\Phi}$ in (2.9) has dimension 1. Therefore the invariant distribution $\mu^\infty(u, v)$ is the unique solution to the linear system

$$\mathbf{p} \cdot \mathbf{\Phi}(u, v) \;=\; \mathbf{0} \qquad \text{with constraint} \quad \mathbf{p} \cdot \mathbf{e}^T = 1,$$

where $\mathbf{e}$ is the row vector with all elements equal to 1. An equivalent way to express the constraint is

$$\mathbf{p} \cdot \mathbf{E} \;=\; \mathbf{e},$$

where $\mathbf{E}$ is an $n \times n$ matrix with all elements equal to 1. This yields

$$\mathbf{p} \;\doteq\; \mathbf{e} \cdot (\mathbf{G} + \mathbf{E})^{\mathbf{-1}}.$$

Since the jump rates $\phi_{ij}(u_i, v_i)$ depend continuously on the controls u_i, v_i, the invertible matrix $(\mathbf{G} + \mathbf{E})(u, v)$ is a continuous function of u and v. In turn, the asymptotic probability distribution $\mu^\infty(u, v)$ depends continuously on u, v. This completes the proof of the upper semicontinuity of Ψ. □

Toward the existence of a self-consistent Stackelberg equilibrium, we shall assume that the best reply map satisfies the conclusions of Lemma 2.3, namely

(A1) For every $u \in [0,1]^N$, the best reply $R_2(u) = \{v^*(u)\}$ is a singleton. Furthermore, the map $u \mapsto v^*(u)$ has $\mathcal{C}^2$ regularity.

For every $u \in [0,1]^N$ and any initial state $x(0) = i \in \{1, \ldots, N\}$, the expected cost to the leader is

$$J_1(i,u) \doteq E^{x(0)=i}\left[\int_0^{+\infty} r_1 e^{-r_1 t} L_1\Big(x(t), u_{x(t)}, v^*_{x(t)}(u)\Big)\, dt\right].$$

In analogy with (2.5), this expected cost is computed by

$$J_1(i,u) = L_1(i, u_i, v_i^*(u)) + \frac{1}{r_1}\sum_{j\neq i} \phi_{ij}(u_i, v_i^*(u)\Big)\big(J_1(j,u) - J_1(i,u)\big).$$

The above identities can be written in vector form

$$\left(\mathbf{I} - \frac{1}{r_1}\mathbf{\Phi}\right)\cdot \mathbf{J_1} = \mathbf{L_1}.$$

Therefore

$$\mathbf{J_1} = \left(\mathbf{I} - \frac{1}{r_1}\mathbf{\Phi}\right)^{-1} \mathbf{L_1}\,. \tag{3.2}$$

For any initial distribution $\mu \in \Delta_N$, the expected cost to the leader is now computed by the inner product $\mu \cdot \mathbf{J_1}(u, v^*(u))$. Notice that this is a linear combination of the components $J_1(i,u)$ with coefficients given by the components of the vector $\mu \in \Delta_N$. To prove the uniqueness of the Stackelberg equilibrium $S(\mu)$, for every initial distribution $\mu \in \Delta_N$, it now suffices to require the strict convexity of the maps $u \mapsto J_1(i,u)$. for every $i \in \{1, ..., N\}$.

To better understand this convexity requirement, consider the Jacobian matrix of the best reply map $u = (u_1, u_2, ..., u_N) \mapsto v^*(u) = (v_1^*, v_2^*, ..., v_N^*)$, namely

$$\widetilde{\mathbf{J}}_{\mathbf{2}}(i,j) \doteq \frac{\partial v_i^*(u)}{\partial u_j}.$$

In addition, consider the matrix

$$\mathbf{A} \;\doteq\; \left(\mathbf{I} - \frac{1}{r_1}\mathbf{\Phi}\right)^{-1}.$$

For every $k \in \{1, ..., N\}$, the Jacobian matrix of partial derivatives of the expected cost $J_1(k, u)$ starting at state k is computed by

$$\widetilde{J}_k(i,j) \;=\; \frac{\partial}{\partial u_j}\left(\frac{\partial(\mathbf{A}\cdot\mathbf{L_1})_k}{\partial u_i} + \sum_{k=1}^{N} \frac{\partial(\mathbf{A}\cdot\mathbf{L_1})_k}{\partial v_k}\widetilde{\mathbf{J}}_{\mathbf{2}}(k,i)\right). \tag{3.3}$$

Assuming the positive definiteness of the above matrices $\widetilde{J}_k$, we can prove a first result on the existence of self-consistent equilibria.

Theorem 3.1. *Consider the game with dynamics (1.1) and exponentially discounted payoffs (1.3). Assume that* **(A1)** *holds and moreover, for every* $k \in \{1, ..., N\}$ *and* $u \in [0,1]^N$*, the matrix* $\widetilde{J}_k$ *in (3.3) is strictly positive definite, namely*

$$w^T \widetilde{J}_k w \;>\; 0 \qquad \textit{for all} \;\; w \in \mathbb{R}^N \setminus \{0\}. \tag{3.4}$$

Then a self-consistent Stackelberg equilibrium exists.

Proof. By (3.4), $\widetilde{J}_1(k, u, v^*(u))$ is strictly convex with respect to u for all $k \in \{1, 2, ..., N\}$ and all $\mu \in \Delta_N$. Hence the set $S(\mu)$ of Stackelberg equilibria reduces to a single point. Lemma 3.1 further implies that the map Ψ in (3.1) is a continuous, single-valued map from Δ_N to itself. By Brouwer's theorem, Ψ has a fixed point μ^*. Therefore, the corresponding feedback controls $(u^*, v^*) \in S(\mu^*)$ determine a self-consistent Stackelberg equilibrium. □

Unfortunately, the computation of the matrix $\widetilde{J}_k$ in (3.3) can be far from easy. In particular, it requires finding the inverse of the matrix $\mathbf{I} - \frac{1}{r_1}\mathbf{\Phi}$. Moreover, in several cases the best reply map can be multi-valued. For these reasons, a different approach will be developed in the remaining sections of the chapter.

4 Generic structure of the best reply map

Our eventual goal is to understand the detailed structure of the best reply map $u \mapsto \mathcal{R}_2(u)$, with

$$u = (u_1, \ldots, u_N) \in [0,1]^N, \qquad \mathcal{R}_2(u) \subseteq [0,1]^N.$$

In the myopic case this map is decoupled. Indeed, for each $i \in \{1, \ldots, N\}$,

$$v_i \in \mathcal{R}_{2,i}(u_i) \doteq \arg\min_{\omega \in [0,1]} L_2(i, u_i, \omega)$$

depends only on the component u_i. Under generic assumptions on the cost function L_2, the structure of the multifunctions $\mathcal{R}_{2,i}$ has been analyzed in Chapter 5. These results are here used as a starting point, to describe the best reply map in the more general case of a short-sighted follower with discount factor $r_2 >> 1$.

We begin by reviewing the results in Chapter 5. Consider a one-shot Stackelberg game where the leader and the follower have cost functions $F(x,y)$ and $G(x,y)$. Here $x \in [0,1]$ and $y \in [0,1]$ are the strategies chosen by the leader and by the follower, respectively. The best reply map is the multifunction

$$x \mapsto R(x) \doteq \left\{ y^* \in [0,1]\,;\; G(x,y^*) = \min_{y \in [0,1]} G(x,y) \right\}. \tag{4.1}$$

In turn, the goal of the leader is to minimize the cost function F, restricted to the graph of R, namely

$$\min \Big\{ F(x,y)\,; \quad x \in [0,1], \quad y \in R(x) \Big\}. \tag{4.2}$$

For a generic cost function $G \in \mathcal{C}^3(\mathbb{R}^2)$, the structure of the map $R(\cdot)$ was described Theorem 4.1 in Chapter 5.

Next, we consider a game where the strategies of the leader and the follower both range in $[0,1]^N$. We start with the simple case where the cost function has the form

$$G(x,y) = \sum_{i=1}^{N} G_i(x_i, y_i). \tag{4.3}$$

Let $x_i \mapsto R_i(x_i) \subseteq [0,1]$ be the best reply map corresponding to the cost function G_i. Then the vector-valued reply map $x \mapsto R(x) \subseteq [0,1]^N$ has the product structure:

$$R(x_1, \ldots, x_N) \;=\; R_1(x_1) \times \cdots \times R_N(x_N). \tag{4.4}$$

By Theorem 4.1, if $G_i \in \mathcal{G}$ for every $i = 1, \ldots, N$, then the structure of the multifunction R can be immediately described in terms of the maps R_i.

In the remainder of this section we analyze the best reply maps for a family of perturbed cost functions of the form

$$G_\varepsilon(x,y) \;=\; \sum_{i=1}^{N} G_i(x_i, y_i) + \varepsilon \widetilde{G}(x,y) + o(\varepsilon), \tag{4.5}$$

where $G_i \in \mathcal{G}$ for every $i = 1, \ldots, N$, while $\widetilde{G} \in \mathcal{C}^3(\mathbb{R}^{2N})$ and $o(\varepsilon)$ denotes an additional term whose $\mathcal{C}^3$ norm vanishes faster than ε. Namely, $\varepsilon^{-1}\|o(\varepsilon)\|_{\mathcal{C}^3} \to 0$ as $\varepsilon \to 0$.

4.1 Regular Stratified Domains.

Definition 4.1. *Let $V \subset \mathbb{R}^d$ be a compact set. Given $k \geq 1$, by a $\mathcal{C}^k$-**stratification** of V we mean a finite collection $\mathcal{S} = \{\mathcal{M}_{ij}\,;\;\; i = 0, \ldots, d,\; j = 1, \ldots, N_i\}$ of manifolds $\mathcal{M}_{ij} \subset \mathbb{R}^d$ with the following properties.*

(i) The manifolds $\mathcal{M}_{ij}$ provide a disjoint covering of V. Namely

$$V \;=\; \bigcup_{i,j} \mathcal{M}_{ij}\,, \qquad\qquad (i,j) \neq (i',j') \quad \Longrightarrow \quad \mathcal{M}_{ij} \cap \mathcal{M}_{i'j'} \;=\; \emptyset. \tag{4.6}$$

(ii) Each $\mathcal{M}_{ij}$, $1 \leq j \leq N_i$ is an embedded manifold of class $\mathcal{C}^k$, with dimension i. More precisely, every point $\bar{x} \in \mathcal{M}_{ij}$ has a neighborhood $\mathcal{N}_{\bar{x}}$ with the following property. There exist a $\mathcal{C}^k$ map $\varphi : \mathbb{R}^d \mapsto \mathbb{R}^{d-i}$ such that

$$\text{rank}(D\varphi(x)) \;=\; d - i \quad \text{for every } x, \qquad\qquad \mathcal{M}_{ij} \cap \mathcal{N}_{\bar{x}} \;=\; \{x\,;\;\; \varphi(x) = 0\} \cap \mathcal{U}_{\bar{x}}.$$

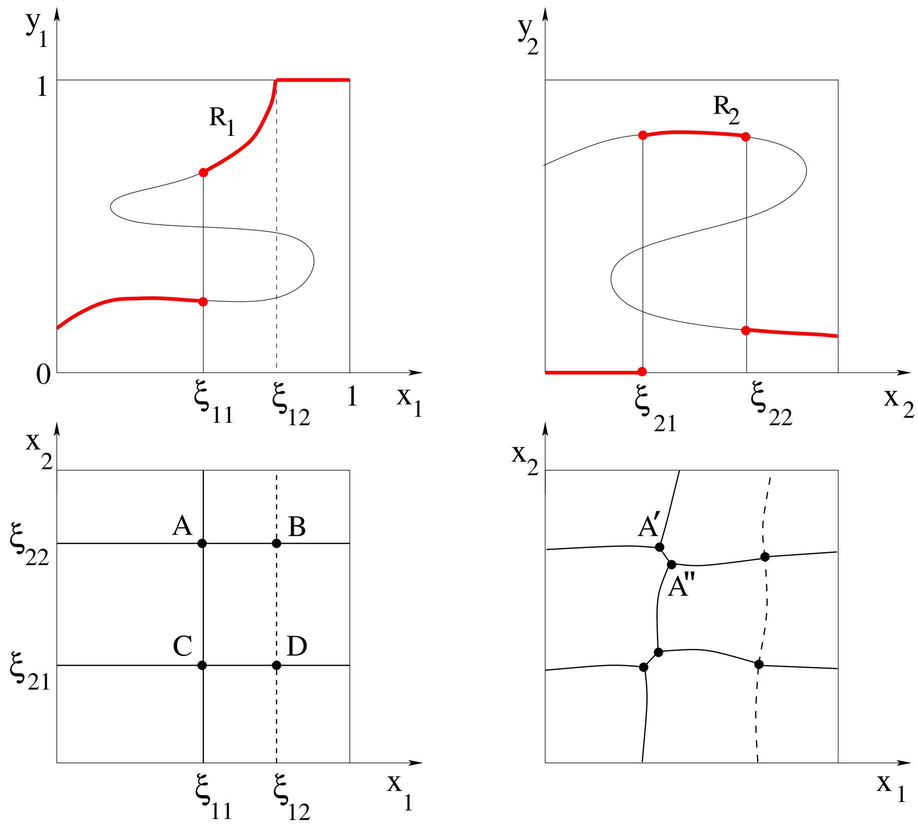

Figure 6.1. Above: for a myopic follower, the best reply maps are decoupled. We can thus consider separately the graphs of the best reply maps $x_1 \mapsto R_1(x_1)$ and $x_2 \mapsto R_2(x_2)$. The thin lines denote the solutions to $\nabla_y G_i = 0$. The thick lines denote the global minima of the functions $G_i(x_i, \cdot)$. For x_1 in a neighborhood of ξ_{11}, the function $G(x_1, \cdot)$ has two local minima, which coincide at $x_1 = \xi_{11}$. The best reply map R_1 is two-valued at $x_1 = \xi_{11}$ and has a kink at $x_1 = \xi_{12}$. The best reply map R_2 is two-valued at $x_2 = \xi_{21}$ and at $x_2 = \xi_{22}$. Still for a myopic follower, we can look at the best reply map R as a multivalued map from $[0,1]^2$ into itself. The figure al the bottom left shows the stratification of $[0,1]^2$ corresponding to this map: $R(x_1, x_2) = R_1(x_1) \times R_2(x_2)$. Notice that R is 4-valued at A and at C, and it is 2-valued along on the lines $x_1 = \xi_{11}$, $x_2 = \xi_{12}$ and $x_2 = \xi_{22}$. Moreover, it has a kink along the line $x_1 = \xi_{12}$. Bottom right: the stratification of $[0,1]^2$ induced by the best reply map R_ε corresponding to a small, smooth perturbation of the cost function to the follower. Notice that, for a generic perturbation $\widetilde{G}$ in (4.5), the point A is replaced by the two points A' and A''.

Example 4.1. Consider the best reply map in (4.4), obtained as a product of the best replies R_i. This map can be associated with a stratification of the hypercube $[0,1]^N$ as follows.

For each $i = 1, \ldots, N$, as in (4.5) assume that the best reply map R_i is described by

$$\Big\{(x,y)\,;\quad y\in R_i(x)\,,\quad x\in[0,1]\Big\} \;=\; \bigcup_{k=1}^{\nu(i)}\Big\{(x,\varphi_{i,k}(x))\,;\quad x\in[\xi_{i,k-1},\xi_{i,k}]\Big\}, \quad (4.7)$$

for suitable points

$$0 \;=\; \xi_{i,0} \;<\; \xi_{i,1}<\ldots<\xi_{i,\nu(i)} \;=\; 1.$$

For each i, the interval $[0,1]$ can be written as a disjoint union of single points and open intervals:

$$\begin{aligned}[0,1] \;&=\; \{0\}\,\cup\,]\xi_{i,0},\xi_{i,1}[\,\cup\{\xi_{i,1}\}\,\cup\,\cdots\,\cup\,\{\xi_{i,\nu(i)-1}\}\,\cup\,]\xi_{i,\nu(i)-1},\xi_{i,\nu(i)}[\,\cup\,\{1\}\\ &=\; \bigcup_{\ell=1}^{1+2\nu(i)} V_{i,\ell}\,.\end{aligned} \quad (4.8)$$

Notice that $\nu(i)+1$ of the sets on the right hand side of (4.8) are singletons, and $\nu(i)$ of these are open intervals.

In turn, this yields a stratification of the cube $[0,1]^N$ in terms of manifolds of the form

$$\mathcal{M}_\ell \;=\; V_{1,\ell(1)}\times V_{2,\ell(2)}\times\cdots\times V_{N,\ell(N)}\,. \quad (4.9)$$

A covering of $[0,1]^N$ is obtained by taking all possible choices of multi-indices

$$\ell \;=\; (\ell(1),\ldots,\ell(N)), \qquad \text{with} \quad \ell(i)\in\{1,2,\ldots,1+2\nu(i)\} \quad \text{for every } i.$$

The total number of these submanifolds is $\prod_{i=1}^N(1+2\nu(i))$. The dimension of the manifold $\mathcal{M}_\ell$ is

$$\dim(\mathcal{M}_\ell) \;=\; \#\Big\{i\,;\quad V_{i,\ell(i)} \text{ is an open interval}\Big\} \;\in\; \{0,1,2,\ldots,N\}.$$

The best reply map R in (4.4) can now be described in terms of this stratification. Consider any manifold $\mathcal{M}_\ell$ of dimension N. By construction, this is the cartesian product of N open intervals, say

$$\mathcal{M}_\ell \;=\;]\xi_{1,\ell(1)-1},\, \xi_{1,\ell(1)}[\;\times\;\cdots\;\times\;]\xi_{N,\ell(N)-1},\, \xi_{N,\ell(N)}[\,. \tag{4.10}$$

Restricted to $\mathcal{M}_\ell$, the map R is single-valued, and coincides with a smooth function. Indeed, by (4.7) we have

$$R(x_1,\ldots,x_N) \;=\; \Big\{ \big(\varphi_{1,\ell((1)}(x_1)\,,\; \varphi_{2,\ell(2)}(x_2)\,,\;\ldots\;,\, \varphi_{N,\ell(N)}(x_N)\big) \Big\}. \tag{4.11}$$

The graph of R is now obtained by taking the closure of its restriction on the N-dimensional open boxes $\mathcal{M}_\ell$ in (4.10).

We observe that, along manifolds of dimension $< N$, the best reply can be multi-valued. The cardinality of $R(x)$ is computed as follows. For each $i = 1,\ldots,N$, split

$$\{0,1,2,\ldots,\nu(i)\} \;=\; J_i \cup K_i\,,$$

where

- J_i is the set of indices $j \in \{1,2,\ldots,\nu(i)-1\}$ such that R_i has a jump at ξ_{ij}. Namely, the set $R_i(\xi_{ij})$ contains two distinct values.
- K_i is the set of indices j such that either $\xi_{ij} \in \{0,1\}$ or else $R_i(\xi_{ij}) \in \{0,1\}$. Notice that, if the second alternative holds, then in a neighborhood of ξ_{ij} the best reply map R_i is single-valued, with a kink at ξ_{ij}.

For example, for the best reply maps R_1, R_2 shown at the top of Fig. 6.1, one has

$$J_1 = \{\xi_{11}\}, \qquad K_1 = \{0, \xi_{12}, 1\}, \qquad J_2 = \{\xi_{21}, \xi_{22}\}, \qquad K_2 = \{0,1\}.$$

Given any manifold $\mathcal{M}_\ell$ of the form (4.9), for all $x = (x_1,\ldots,x_N) \in \mathcal{M}_\ell$ the cardinality of $R(x)$ is

$$\#R(x) \;=\; 2^q, \qquad \text{where} \qquad q \;=\; \#\Big\{ i\,;\;\; V_{i,\ell(i)} = \{\xi_{ij}\} \;\text{ for some } j \text{ with } \xi_{ij} \in J_i \Big\}.$$

4.2 Structure of the best reply map, for a generic perturbation.

Next, consider a one-shot Stackelberg game where the leader and the follower choose their strategies $x, y \in [0,1]^N$. Let $F, G_\varepsilon : [0,1]^{2N} \mapsto \mathbb{R}$ be respectively the cost functions for the leader and for the follower, with G_ε as in (4.5).

Our goal is to prove that, under generic assumptions on the functions $F, G_i, \widetilde{G}$, for every $\varepsilon > 0$ sufficiently small the Stackelberg equilibrium is unique and can be characterized by a set of $2N$ independent scalar equations in the variables $x_1, \ldots, x_N, y_1, \ldots, y_N$.

To fix the ideas, assume that, in the unperturbed case (4.3), the game has a unique Stackelberg equilibrium, attained at (x^*, y^*) with $x^* \in \mathcal{M}_\ell = V_{1,\ell(1)} \times \cdots \times V_{N,\ell(N)}$. In general, the manifold $\mathcal{M}_\ell$ will have dimension

$$\dim(\mathcal{M}_\ell) \;=\; N-p-q, \qquad \begin{cases} p \;=\; \#\Big\{i\,;\;\; V_{i,\ell(i)} = \{\xi_{ij}\} \quad \text{for some } j = \ell(i) \in K_i\Big\}, \\ q \;=\; \#\Big\{i\,;\;\; V_{i,\ell(i)} = \{\xi_{ij}\} \quad \text{for some } j = \ell(i) \in J_i\Big\}. \end{cases} \tag{4.12}$$

To help the reader, we first explain the main ideas with the aid of some figures. For simplicity, let $N = 2$ and assume that, when the cost function G has the decoupled form (4.3), the best reply map induces the stratification shown in Fig. 6.1, bottom left. In particular: for every (x_1, x_2) in a neighborhood of the point $A = (x_1^*, x_2^*) \doteq (\xi_{11}, \xi_{22})$, the equation

$$\nabla_y G(x,y) \;=\; 0 \tag{4.13}$$

has four distinct solutions corresponding to local minima of $G(x,\cdot)$ (together with other solutions, which do not yield local minima of G). For notational convenience, these four solutions of (4.13) will be denoted by $x \mapsto y_{\mathcal{I}}(x)$, where $\mathcal{I} \subseteq \{1,2\}$.

We wish to understand what happens to the best reply map when (4.3) is replaced by (4.5), for a generic perturbation $\widetilde{G}$ and for $\varepsilon > 0$ small enough. In this case, in a neighborhood of the point $A = (x_1^*, x_2^*) \doteq (\xi_{11}, \xi_{22})$, the equation (4.13) still has four distinct solutions $x \mapsto y^\varepsilon_{\mathcal{I}}(x)$ corresponding to local minima of the function G_ε in (4.5). The four domains where each of the functions $y^\varepsilon_{\mathcal{I}}$ yields a global minimum of G_ε will be denoted by

$$\Omega^\varepsilon_{\mathcal{I}} \doteq \Big\{x\,;\quad G_\varepsilon(x, y^\varepsilon_{\mathcal{I}}(x)) = \min_{\mathcal{I}'\subseteq\{1,2\}} G_\varepsilon(x, y^\varepsilon_{\mathcal{I}'}(x))\Big\}. \tag{4.14}$$

When $\varepsilon = 0$, these four domains have a rectangular shape, as shown in Fig. 6.1, bottom left. At the point A, all four functions $G(x^*, y_{\mathcal{I}}(x^*))$ coincide, and the best reply map is thus quadruple-valued. On the other hand, for a generic perturbation $\widetilde{G}$, when $\varepsilon > 0$ the domains $\Omega^\varepsilon_{\mathcal{I}}$ will have a more complicated shape, shown in Fig. 6.1, bottom right. In particular, the point A splits into 2 distinct points $A'_\varepsilon, A''_\varepsilon$, where the best reply map R_ε is triple-valued.

We now analyze the general case $N \geq 1$. For clarity of exposition, we first consider the case where $p = 0$, so that all singularities of the best reply map through $x^* = (x^*_1, \ldots, x^*_N)$ are jumps. Recalling (4.12), let $\mathcal{J} \subseteq \{1, \ldots, N\}$ be the subset of indices i such that $\ell(i) \in J_i$. These are the components i such that the best reply map for G_i has a jump at x^*_i. According to (4.12), its cardinality is $\#\mathcal{J} = q$. For notational convenience, if $i \in \mathcal{J}$ we define the functions y^-_i, y^+_i in a neighborhood of the jump point x^*_i so that

$$R_i(x_i) = \begin{cases} \{y^-_i(x_i)\} & \text{if} \quad x_i < x^*_i, \\ \{y^+_i(x_i)\} & \text{if} \quad x_i > x^*_i, \\ \{y^-_i(x_i), y^+(x_i)\} & \text{if} \quad x_i = x^*_i. \end{cases} \tag{4.15}$$

On the other hand, if $i \notin \mathcal{J}$, hence x^*_i is not a jump point, we simply write $R_i(x_i) = \{y_i(x_i)\}$ for x_i close to x^*_i. For every subset $\mathcal{I} \subseteq \mathcal{J}$, we consider the function

$$y_{\mathcal{I}}(x) \doteq (y_1, \ldots, y_N), \qquad \text{where} \qquad y_i = \begin{cases} y^+_i(x) & \text{if} \;\; i \in \mathcal{I}, \\ y^-_i(x) & \text{if} \;\; i \in \mathcal{J}\setminus\mathcal{I}, \\ y_i(x) & \text{if} \;\; i \notin \mathcal{J}. \end{cases} \tag{4.16}$$

Consider the 2^q linear functions of the variable $z = (z_1, \ldots, z_N)$ defined as follows. For every subset $\mathcal{I} \subseteq \mathcal{J}$, define

$$\begin{aligned}\Lambda_{\mathcal{I}}(z) &= \sum_{i\in\mathcal{I}} \left[\frac{d}{dx_i} G_i(x_i, y_i^+(x_i))\right]_{x_i=x_i^*} z_i + \sum_{i\in\mathcal{J}\setminus\mathcal{I}} \left[\frac{d}{dx_i} G_i(x_i, y_i^-(x_i))\right]_{x_i=x_i^*} z_i \\ &\quad + \sum_{i\notin\mathcal{J}} \left[\frac{d}{dx_i} G_i(x_i, y_i(x_i))\right]_{x_i=x_i^*} z_i\,.\end{aligned} \tag{4.17}$$

By construction, for $x \approx x^*$ and every $\mathcal{I} \subseteq \mathcal{J}$ we have the linear approximations

$$G(x, y_{\mathcal{I}}(x)) \;=\; G(x^*, y_{\mathcal{I}}(x^*)) + \Lambda_{\mathcal{I}}(x - x^*) + \mathcal{O}(1) \cdot |x - x^*|^2. \tag{4.18}$$

On the other hand, for $\varepsilon > 0$, the corresponding approximation for the perturbed function G_ε in (4.5) takes the form

$$G_\varepsilon(x, y_{\mathcal{I}}^\varepsilon(x)) \;=\; G(x^*, y_{\mathcal{I}}(x^*)) + \Lambda_{\mathcal{I}}(x - x^*) + \varepsilon\widetilde{G}(x^*, y_{\mathcal{I}}(x^*)) + \mathcal{O}(1) \cdot (|x - x^*| + \varepsilon)^2. \tag{4.19}$$

By assumption, at the point $x = x^*$ all the 2^q functions $G(x, y_{\mathcal{I}}(x))$, $\mathcal{I} \subseteq \mathcal{J}$, coincide.

Given a general perturbation $\widetilde{G}$ as in (4.5), consider the 2^q-tuple of numbers

$$\lambda_{\mathcal{I}} \;=\; \widetilde{G}\big(x^*, y_{\mathcal{I}}(x^*)\big). \tag{4.20}$$

Under generic conditions we expect that, in a neighborhood of x^*, the domains

$$\Omega_{\mathcal{I}}^\varepsilon \;\doteq\; \Big\{x\,;\;\; G_\varepsilon(x, y_{\mathcal{I}}^\varepsilon(x)) \;=\; \min_{\mathcal{I}'\subseteq\mathcal{J}} G_\varepsilon(x, y_{\mathcal{I}'}^\varepsilon(x))\Big\} \tag{4.21}$$

will produce the same stratification as the polytopes

$$\overline{\Omega}_{\mathcal{I}} \;\doteq\; \Big\{x \in \mathbb{R}^N\,;\;\; \Lambda_{\mathcal{I}}(z) + \lambda_{\mathcal{I}} \;=\; \min_{\mathcal{I}'\subseteq\mathcal{J}} \Lambda_{\mathcal{I}'}(z) + \lambda_{\mathcal{I}'}\Big\}. \tag{4.22}$$

This motivates the following

Definition 4.2. *Given a finite family of linear functions $\{\Lambda_{\mathcal{I}}\}_{\mathcal{I}\subseteq\mathcal{J}}$, we say that a 2^q-tuple of real numbers $(\lambda_{\mathcal{I}})_{\mathcal{I}\subseteq\mathcal{J}}$ is* generic *if, for any collection of $q+2$ distinct sets $\mathcal{I}_1, \mathcal{I}_2, \ldots \mathcal{I}_{q+2} \subseteq \mathcal{J}$, the linear system of $q+1$ equations for the q variables*

$z = (z_i)_{i\in\mathcal{J}}$

$$\Lambda_{\mathcal{I}_1}(z) + \lambda_{\mathcal{I}_1} \;=\; \Lambda_{\mathcal{I}_2}(z) + \lambda_{\mathcal{I}_2} \;=\; \cdots \;=\; \Lambda_{\mathcal{I}_{q+2}}(z) + \lambda_{\mathcal{I}_{q+2}} \tag{4.23}$$

has no solutions.

Remark 4.1. According to (4.17), the linear functions $\Lambda_{\mathcal{I}}$ are defined on the entire space $\mathbb{R}^N$. However, for any $\mathcal{I}, \mathcal{I}' \subseteq \mathcal{J}$, the difference $\Lambda_{\mathcal{I}}(z) - \Lambda_{\mathcal{I}'}(z)$ does not depend on the components z_i with $i \notin \mathcal{J}$. With a slight abuse of notation, one can thus regard (4.23) as a set of equations for the q variables $z = (z_i)_{i\in\mathcal{J}}$ only.

Lemma 4.1. *Given the 2^q linear functions $\Lambda_{\mathcal{I}}$ in (4.17), the subset $\mathcal{S}$ of generic 2^q-tuples $(\lambda_{\mathcal{I}})_{\mathcal{I}\subseteq\mathcal{J}}$ is open and dense.*

As a consequence, for all functions $\widetilde{G}$ in an open dense subset of $\mathcal{C}^3(\mathbb{R}^{2N})$, the values in (4.20) satisfy $(\lambda_{\mathcal{I}})_{\mathcal{I}\subseteq\mathcal{J}} \in \mathcal{S}$.

Proof. For any collection of distinct subsets $\mathcal{I}_1, \mathcal{I}_2, \ldots \mathcal{I}_{q+2} \subseteq \mathcal{J}$, the system (4.23) is equivalent to

$$\Lambda_{\mathcal{I}_1}(z) - \Lambda_{\mathcal{I}_i}(z) \;=\; \lambda_{\mathcal{I}_1} - \lambda_{\mathcal{I}_i}, \qquad i \in \{2, \ldots, q+2\}, \tag{4.24}$$

which is a linear system of $q+1$ equations and q variables. Hence the set $\widetilde{\mathcal{S}}$ of 2^q-tuples $(\lambda_{\mathcal{I}})_{\mathcal{I}\subseteq\mathcal{J}}$ for which (4.24) has no solutions is dense in $\mathbb{R}^{2^q}$. Furthermore, by continuity, $\widetilde{\mathcal{S}}$ is also open. By definition, $\mathcal{S}$ is the intersection of all such open dense sets $\widetilde{\mathcal{S}}$, corresponding to all finite collections of distinct subsets $\mathcal{I}_1, \mathcal{I}_2, \ldots \mathcal{I}_{q+2} \subseteq \mathcal{J}$. Therefore, $\mathcal{S}$ itself is open and dense. The second statement of the lemma follows immediately. □

4.3 Generic structure of the Stackelberg equilibrium

In a Stackelberg game, the goal of the leading player is to minimize the cost function $F = F(x, y)$ restricted to the graph of the best reply map for the follower. More specifically, when the cost function G_ε for the follower has the form (4.5), the leader seeks

$$\min\left\{F(x, y)\,; \quad x \in [0,1]^N, \quad y \in R_\varepsilon(x)\right\}. \tag{4.25}$$

Here

$$R_\varepsilon(x) \doteq \left\{y^* \in [0,1]^N\,; \quad G_\varepsilon(x,y^*) \le G_\varepsilon(x,y) \text{ for all } \; y\in[0,1]^N\right\} \tag{4.26}$$

denotes the best reply map. Under generic assumptions on $F, G_1, \ldots, G_N$, and $\widetilde{G}$, we want to show that, for all $\varepsilon \in [0, \varepsilon_0]$ sufficiently small, the constrained optimization problem (4.25) has a unique global minimizer $(x^*_\varepsilon, y^*_\varepsilon)$. Moreover, this minimizer depends Lipschitz continuously on ε and is stable w.r.t. small perturbations of the cost function F.

Theorem 4.1. *Consider a cost function $G(x,y)$ of the form (4.3) where for every $i = 1,\ldots,N$, the function $G_i \in \mathcal{G}$ satisfies all generic properties listed in Theorem 4.1. Then there exists $\varepsilon_0 > 0$ and a open dense subset of functions $(F(x,y), \widetilde{G}(x,y)) \in \mathcal{C}^2 \times \mathcal{C}^3$ such that the problem (4.25) has a unique minimizer $(x^*_\varepsilon, y^*_\varepsilon) \in [0,1]^N$ for all $\varepsilon \in [0, \varepsilon_0]$ sufficiently small.*

Proof. 1. Call

$$\mathcal{A} \doteq \{(x,y)\,; \quad y \in R(x)\}, \qquad\qquad \mathcal{A}_\varepsilon \doteq \{(x,y)\,; \quad y\in R_\varepsilon(x)\} \tag{4.27}$$

the graphs of the best reply maps corresponding to the cost functions G and G_ε, respectively. Let $F \in \mathcal{C}^2(\mathbb{R}^{2N})$ be given. Let $(x^*, y^*) \in [0,1]^{2N}$ be a Stackelberg equilibrium for the game with costs F, G, so that

$$F(x^*,y^*) \;=\; \min_{(x,y)\in\mathcal{A}} F(x,y).$$

For clarity of exposition, we first assume that $(x^*, y^*) \in \,]0,1[^{2N}$ is an interior point. The modifications required to treat the general case where some of the components x^*_i, y^*_i take values 0 or 1 will be discussed at the end. As before, call $\mathcal{J} \subset \{1,\ldots,N\}$ the set of indices such that the best reply map $x_i \mapsto y_i(x_i)$ has a jump at x^*_i. Recalling the notation introduced at (4.16), assume

$$y^* \;=\; y_{\mathcal{I}^*}(x^*),$$

for some $\mathcal{I}^* \subseteq \mathcal{J}$. For every $\mathcal{I} \subseteq \mathcal{J}$, define the composite map $F_\mathcal{I}(x) \doteq F(x, y_\mathcal{I}(x))$. Then the first order necessary conditions for a constrained local minimum imply

$$\frac{\partial}{\partial x_i} F_{\mathcal{I}^*}(x^*) \begin{cases} \geq 0 & \text{for } \ i \in \mathcal{I}, \\ \leq 0 & \text{for } \ i \in \mathcal{J} \setminus \mathcal{I}, \\ = 0 & \text{for } \ i \notin \mathcal{J}. \end{cases} \tag{4.28}$$

As shown Theorem 2.2 in Chapter 5, by performing an arbitrarily small $\mathcal{C}^2$ modification of F we can assume that

(a1) The point (x^*, y^*) where the constrained global minimum is attained is unique.

(a2) The necessary conditions in (4.28) are satisfied, with all inequalities being strict.

(a3) The $(N-q) \times (N-q)$ Hessian matrix of second partial derivatives

$$\left(\frac{\partial^2}{\partial x_j \partial x_k} F_{\mathcal{I}^*}(x^*) \right)_{j,k \notin \mathcal{J}}$$

is strictly positive definite.

Indeed, all this can be achieved by replacing F with

$$F(x,y) + \left[a_0 \big(|x - x^*|^2 + |y - y^*|^2 \big) + \sum_{k=1}^{N} a_k (x_k - x_k^*) \right] \varphi(x,y). \tag{4.29}$$

Here $a_0 > 0$ and $a_k \in \mathbb{R}$ can be chosen arbitrarily small, while $\varphi \in \mathcal{C}_c^\infty$ is a cutoff function such that $\varphi(x,y) = 1$ for $(x,y) \in [0,1]^{2N}$.

2. Assume that the perturbation $\widetilde{G}$ satisfies the conclusion of Lemma 4.1. Consider the polytope (see Fig. 6.2, left)

$$\overline{\Omega}_{\mathcal{I}^*} \ \doteq \ \left\{ z = (z_i)_{i \in \mathcal{J}} \, ; \ \ \Lambda_{\mathcal{I}^*}(z) + \lambda_{\mathcal{I}^*} \ = \ \min_{\mathcal{I} \subseteq \mathcal{J}} \Lambda_\mathcal{I}(z) + \lambda_\mathcal{I} \right\} \ \subset \ \mathbb{R}^q . \tag{4.30}$$

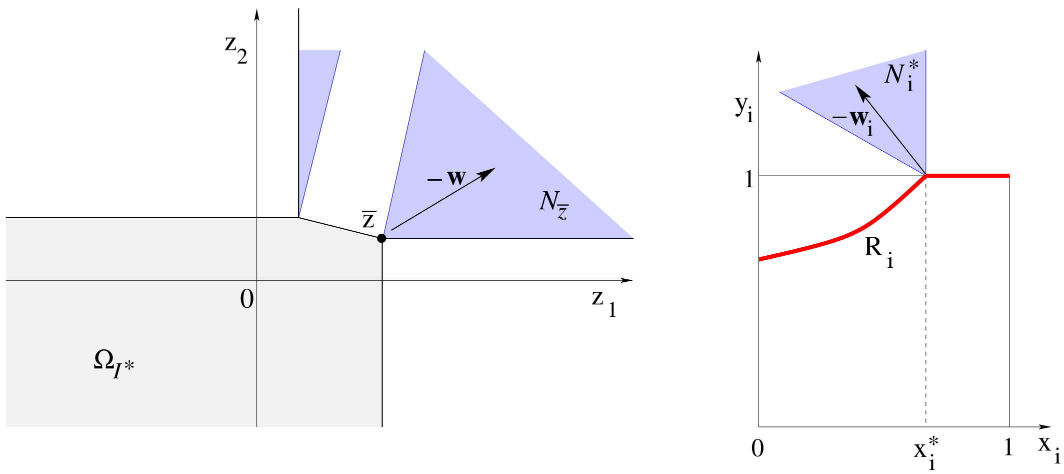

Figure 6.2. Left: the polytope $\overline{\Omega}_{\mathcal{I}^*}$ defined at (4.30). Under generic assumptions, there exists a unique point $\overline{z} \in \overline{\Omega}_{\mathcal{I}^*}$ which minimizes the inner product with $\mathbf{w}$. Notice that the vector $-\mathbf{w}$ lies in the interior of the normal cone $\mathcal{N}_{\overline{z}}$ at $\overline{z}$. Right: a point x_i^* where the best reply map $x_i \mapsto R_i(x_i)$ has a kink. By performing an arbitrarily small perturbation of the cost function F, the negative gradient $-\mathbf{w}_i \doteq -(F_{x_i}, F_{y_i})$ is contained in the interior of the normal cone $\mathcal{N}_i^*$.

Consider the vector of partial derivatives at the point x^*

$$\mathbf{w} \;\doteq\; \left(\frac{\partial}{\partial x_i} F_{\mathcal{I}^*}(x^*, y_{\mathcal{I}^*}(x^*))\right)_{i\in\mathcal{J}} \;\in\; \mathbb{R}^q. \tag{4.31}$$

According to the generic assumption (a2), all of these derivatives are nonzero. By a further, small $\mathcal{C}^2$ perturbation of the function F, still of the form (4.29), we achieve the existence of a unique point $\overline{z} \in \overline{\Omega}_{\mathcal{I}^*}$ such that

$$\langle \overline{z}, \mathbf{w}\rangle \;=\; \min_{z\in\overline{\Omega}_{\mathcal{I}^*}} \langle z, \mathbf{w}\rangle, \qquad\qquad -\mathbf{w} \in \text{int}\, \mathcal{N}_{\overline{z}}\,. \tag{4.32}$$

Here $\mathcal{N}_{\overline{z}}$ denotes the outer normal cone to $\overline{\Omega}_{\mathcal{I}}$ at the point $\overline{z}$. Since we are assuming that the values $\lambda_{\mathcal{I}}$ in (4.20) satisfy the conclusion of Lemma 4.1, we can identify a family of q subsets $\mathcal{I}_1, \ldots, \mathcal{I}_q \subseteq \mathcal{J}$ such that the point z^*, where the constrained minimum is achieved, is characterized by the q linearly independent equations

$$\Lambda_{\mathcal{I}^*}(z) + \lambda_{\mathcal{I}^*} \;=\; \Lambda_{\mathcal{I}_k}(z) + \lambda_{\mathcal{I}_k}\,, \qquad\qquad k = 1, \ldots, q\,. \tag{4.33}$$

3. We will show that, for all $\varepsilon > 0$ sufficiently small, the Stackelberg equilibrium for the cost functions F, G_ε is achieved at a unique point $(x^*_\varepsilon, y^\varepsilon_{\mathcal{I}}(x^*_\varepsilon))$, determined as follows.

Recall that, for every $\mathcal{I} \subseteq \mathcal{J}$ the equations (4.15)-(4.16) determine a function $y_{\mathcal{I}}$ on a neighborhood of x^*. All these 2^q functions satisfy the implicit equation $\nabla_y G(x, y_{\mathcal{I}}(x)) = 0$. Similarly, for $\varepsilon > 0$, we obtain 2^q distinct functions $y^\varepsilon_{\mathcal{I}}$, all defined on a neighborhood of x^*. Each of them satisfies the implicit equation

$$\nabla_y G_\varepsilon(x, y^\varepsilon_{\mathcal{I}}(x)) \;=\; 0 \;\in\; \mathbb{R}^N. \tag{4.34}$$

Next, define the $(N-q)$-dimensional manifold

$$\mathcal{M}_\varepsilon \;=\; \Big\{(x,y)\,;\quad y = y^\varepsilon_{\mathcal{I}^*}(x)\,,\quad G_\varepsilon(x, y^\varepsilon_{\mathcal{I}^*}(x)) \;=\; G_\varepsilon(x, y^\varepsilon_{\mathcal{I}_k}(x)),\quad k = 1,\ldots,q\Big\}. \tag{4.35}$$

It will be convenient to use the notation $x = (z, \zeta) \in \mathbb{R}^q \times \mathbb{R}^{N-q}$, where

$$z \;=\; (x_i)_{i\in\mathcal{J}}\,, \qquad \zeta \;=\; (x_i)_{i\notin\mathcal{J}}\,. \tag{4.36}$$

Accordingly, we write $x^* = (z^*, \zeta^*)$. We can now solve the $N+q$ equations in (4.35), expressing the variables z, y as functions of ζ. This yields a parameterization of $\mathcal{M}_\varepsilon$ of the form

$$\zeta \;\mapsto\; \Big(x^\varepsilon(\zeta),\, y^\varepsilon_{\mathcal{I}^*}(x^\varepsilon(\zeta))\Big). \tag{4.37}$$

4. We claim that, for $\varepsilon > 0$ sufficiently small, the point $(x^*_\varepsilon, y^*_\varepsilon)$ where the Stackelberg equilibrium is achieved can be determined as the unique minimizer of the function F restricted to the manifold $\mathcal{M}_\varepsilon$.

Indeed, when $\varepsilon = 0$ the point $(x^*, y^*) = (x^*, y_{\mathcal{I}^*}(x^*))$ is the unique minimizer of F constrained to the set $\mathcal{A}$ in (4.27). Define the set corresponding to (4.30) as

$$\Omega^\varepsilon_{\mathcal{I}^*} \;\doteq\; \Big\{(x,y);\quad y = y^\varepsilon_{\mathcal{I}^*}(x)\,,\quad G_\varepsilon(x, y^\varepsilon_{\mathcal{I}^*}(x)) = \min_{\mathcal{I}\subseteq\mathcal{J}} G_\varepsilon(x, y^\varepsilon_{\mathcal{I}}(x))\Big\}. \tag{4.38}$$

By continuity, for $\varepsilon > 0$ small enough any minimizer of F on $\mathcal{A}_\varepsilon$ must lie in a small neighborhood of (x^*, y^*). This already implies that $(x^*_\varepsilon, y^*_\varepsilon) \in \Omega^\varepsilon_{\mathcal{I}^*}$.

Recalling (4.35), we now show that $\mathcal{M}_\varepsilon \subset \Omega^\varepsilon_{\mathcal{I}^*}$. Using the notation introduced at (4.37), for $\mathcal{I} \notin \{\mathcal{I}^*, \mathcal{I}_1, \ldots, \mathcal{I}_q\}$ we have

$$\lim_{\varepsilon\to 0+} \frac{x^\varepsilon(\zeta^*) - x^*}{\varepsilon} \;=\; \overline{z}\,, \tag{4.39}$$

and

$$\lim_{\varepsilon\to 0+} \frac{G_\varepsilon\big(x^\varepsilon(\zeta^*), y^\varepsilon_{\mathcal{I}^*}(x^\varepsilon(\zeta^*))\big) - G_\varepsilon\big(x^\varepsilon(\zeta^*), y^\varepsilon_{\mathcal{I}}(x^\varepsilon(\zeta^*))\big)}{\varepsilon} \;<\; 0.$$

If we replace ζ^* with a nearby value ζ, the limit in (4.39) will take some value z close to $\overline{z}$, while the second limit remains strictly negative. This shows that, along $\mathcal{M}_\varepsilon$, we have $G_\varepsilon(x, y^\varepsilon_{\mathcal{I}^*}(x)) \leq G_\varepsilon(x, y^\varepsilon_{\mathcal{I}}(x))$ for all $\mathcal{I} \subseteq \mathcal{J}$, proving our claim.

5. Calling $F^\varepsilon_{\mathcal{I}^*}(x) \doteq F(x, y^\varepsilon_{\mathcal{I}^*}(x))$, the necessary conditions for optimality imply

$$-\nabla F^\varepsilon_{\mathcal{I}^*}(x^*_\varepsilon) \;\in\; \mathcal{N}_{x^*_\varepsilon}, \tag{4.40}$$

where $\mathcal{N}_{x^*_\varepsilon}$ denotes the outer normal cone to $\Omega^\varepsilon_{\mathcal{I}^*}$ at the point x^*_ε.

To determine this normal cone, for $\mathcal{I} \subset \mathcal{J}$ call $\mathbf{v}^\varepsilon_{\mathcal{I}}(x)$ the gradient of the composed map

$$x \;\mapsto\; G_\varepsilon(x, y^\varepsilon_{\mathcal{I}}(x)).$$

As $x \to x^*$ and $\varepsilon \to 0+$ we have the convergence

$$\mathbf{v}^\varepsilon_{\mathcal{I}}(x) \;\to\; \mathbf{v}_{\mathcal{I}},$$

where, as in (4.17),

$$\mathbf{v}_{\mathcal{I}} \;=\; (v_1, \ldots, v_N), \qquad\qquad v_i \;=\; \begin{cases} \Big[\frac{d}{dx_i} G_i(x_i, y_i^+(x_i))\Big]_{x_i = x_i^*} & \text{if} \quad i \in \mathcal{I}\,, \\ \Big[\frac{d}{dx_i} G_i(x_i, y_i^-(x_i))\Big]_{x_i = x_i^*} & \text{if} \quad i \in \mathcal{J} \setminus \mathcal{I}\,, \\ \Big[\frac{d}{dx_i} G_i(x_i, y_i(x_i))\Big]_{x_i = x_i^*} & \text{if} \quad i \notin \mathcal{J}\,. \end{cases}$$

Because of (4.32) there exists $c_0 > 0$ such that

$$\langle \mathbf{w}, z - \overline{z}\rangle \;\geq\; c_0 |z - \overline{z}| \qquad \text{for all } z \in \overline{\Omega}_{\mathcal{I}^*}\,,. \tag{4.41}$$

By the continuity of the gradients of the functions $x \mapsto G_\varepsilon(x, y^\varepsilon_{\mathcal{I}}(x))$ and $x \mapsto F(x, y^\varepsilon_{\mathcal{I}}(x))$, from (4.41) we deduce that, for every $x = (z, \zeta)$ sufficiently close to x^* such that $(x, y^\varepsilon_{\mathcal{I}^*}(x)) \in \Omega^\varepsilon_{\mathcal{I}^*}$ and $\varepsilon > 0$ small, one has

$$F\big(x, y^\varepsilon_{\mathcal{I}^*}(x)\big) - F\big(x^\varepsilon(\zeta), y^\varepsilon_{\mathcal{I}^*}(x^\varepsilon(\zeta))\big) \;\geq\; \frac{c_0}{2}|x - x^\varepsilon(\zeta)|.$$

Hence the minimum of F can only be attained along $\mathcal{M}_\varepsilon$.

Finally, (a3) implies that the map $\zeta \mapsto F\big(x^\varepsilon(\zeta), y^\varepsilon_{\mathcal{I}^*}(x^\varepsilon(\zeta))\big)$ is strictly convex in a neighborhood of ζ^*, hence it has a unique local minimum, which must provide the global minimum of F on the entire domain $\mathcal{A}_\varepsilon$.

6. We now discuss the modifications needed to handle the case where the myopic equilibrium is attained at a point $(x^*, y^*) = (x_1^*, \ldots, x_N^*, y_1^*, \ldots, y_N^*)$ on the boundary of the domain $[0,1]^{2N}$. To fix the ideas, consider the sets of indices

$$\begin{array}{llll} \mathcal{J}_0 \doteq \{i\,;\;\; x_i^* = 0\}, & \qquad \mathcal{J}'_0 \doteq \{i\,;\;\; y_i^* = 0\}, \\ \mathcal{J}_1 \doteq \{i\,;\;\; x_i^* = 1\}, & \qquad \mathcal{J}'_1 \doteq \{i\,;\;\; y_i^* = 1\}. \end{array}$$

As shown by the analysis in Theorem 4.1 in Chapter 5, for $G_1, \ldots, G_N$ in an open dense subset of $\mathcal{C}^3([0,1]^{2N}$, all these sets are disjoint.

For every $i \in \mathcal{J}_0 \cap \mathcal{J}_1$, the necessary conditions for optimality imply that at $x = x^*$ the gradient of F satisfies

$$\frac{\partial}{\partial x_i} F(x, y_{\mathcal{I}^*}(x)) \;=\; \left\{ \begin{array}{ll} \dfrac{\partial F}{\partial x_i} + \dfrac{\partial F}{\partial y_i} \cdot \dfrac{\partial}{\partial x_i} y_{\mathcal{I}^*} \;\geq\; 0 & \text{if} \quad x_i^* = 0, \\ \dfrac{\partial F}{\partial x_i} + \dfrac{\partial F}{\partial y_i} \cdot \dfrac{\partial}{\partial x_i} y_{\mathcal{I}^*} \;\leq\; 0 & \text{if} \quad x_i^* = 1. \end{array} \right. \tag{4.42}$$

By possibly performing a further small perturbation of F, always of the form (4.29), we can assume that all the above inequalities are strict.

Similarly, for each $i \in \mathcal{J}'_0 \cup \mathcal{J}'_1$, call $\mathcal{A}_i \subset [0,1]^2$ the graph of the best reply map $u_i \mapsto R_i(u_i)$. Let $\mathcal{N}_i^* \subset \mathbb{R}^2$ be the normal cone to $\mathcal{A}_i$ at the point (x_i^*, y_i^*). Since F attains its minimum at (x^*, y^*) restricted to $\mathcal{A}_1 \times \cdots \times \mathcal{A}_N$, the necessary conditions for optimality imply

$$-\mathbf{w}_i \;\in\; \mathcal{N}_i^*\,, \qquad\qquad \mathbf{w}_i \;\doteq\; (F_{x_i}, F_{y_i})(x^*, y^*) \;\in\; \mathbb{R}^2. \tag{4.43}$$

By possibly performing a further small perturbation of F, as in (4.29), we can assume that $-\mathbf{w}_i$ lies strictly in the interior, namely (see Fig. 6.2, right)

$$-\mathbf{w}_i \ \in \ \mathrm{int}\, \mathcal{N}_i^*. \tag{4.44}$$

As before, in a neighborhood of x^* the implicit equation (4.34) determines 2^q distinct functions $x \mapsto y^\varepsilon_{\mathcal{I}}(x)$, with $\mathcal{I} \subseteq \mathcal{J}$. In the present case, however, these maps may not take values inside $[0,1]^N$. Because of (4.42), for $\varepsilon > 0$ small enough, the minimum of F on the graph $\mathcal{A}_\varepsilon$ of the best reply map will be attained at a point $(x^*_\varepsilon, y^*_\varepsilon)$ with

$$\begin{array}{llll} x^*_{\varepsilon,i} = 0 & \text{for } i \in \mathcal{J}_0, & \qquad y^*_{\varepsilon,i} = 0 & \text{for } i \in \mathcal{J}_0', \\ x^*_{\varepsilon,i} = 1 & \text{for } i \in \mathcal{J}_1\,, & \qquad y^*_{\varepsilon,i} = 1 & \text{for } i \in \mathcal{J}_1'\,. \end{array} \tag{4.45}$$

Let $\mathcal{M}_\varepsilon$ be the manifold in (4.35). The same arguments used in the previous steps now show that the global minimum of F restricted to $\mathcal{A}_\varepsilon$ is attained on the sub-manifold $\mathcal{M}'_\varepsilon \subset \mathcal{M}_\varepsilon$ consisting of all points $(x,y) \in \mathcal{M}_\varepsilon$ which satisfy the additional p identities in (4.45). Notice that $\mathcal{M}'_\varepsilon$ has dimension

$$\dim(\mathcal{M}'_\varepsilon) \;=\; \dim(\mathcal{M}_\varepsilon) - p \;=\; N - p - q, \qquad p \;=\; |\mathcal{J}_0| + |\mathcal{J}_1| + |\mathcal{J}_0'| + |\mathcal{J}_1'|, \quad q \;=\; |\mathcal{J}|.$$

Indeed, it consists of all points (x,y) which satisfy the $N+q$ equations

$$y = y^\varepsilon_{\mathcal{I}^*}(x)\,, \qquad\qquad G_\varepsilon(x, y^\varepsilon_{\mathcal{I}^*}(x)) \;=\; G_\varepsilon(x, y^\varepsilon_{\mathcal{I}_k}(x)), \quad k = 1, \ldots, q, \tag{4.46}$$

together with the p equations in (4.45). As in (4.36), it is convenient to use the notation $x = (z, \zeta) \in \mathbb{R}^{p+q} \times \mathbb{R}^{N-p-q}$, where

$$z \;=\; (x_i)_{i \in \widehat{\mathcal{J}}}\,, \quad \zeta \;=\; (x_i)_{i \in \mathcal{J}^\dagger}\,, \quad \widehat{\mathcal{J}} \;=\; \mathcal{J} \cup \mathcal{J}_0 \cup \mathcal{J}_1 \cup \mathcal{J}_0' \cup \mathcal{J}_1'\,, \quad \mathcal{J}^\dagger \;=\; \{1, \ldots, N\} \setminus \widehat{\mathcal{J}}.$$

Accordingly, we write $x^* = (z^*, \zeta^*)$. We can now solve the $N+p+q$ equations in (4.45)-(4.46), expressing the variables z, y as functions of ζ. Namely

$$x \;=\; x^\varepsilon(\zeta) \;=\; (z^\varepsilon(\zeta), \zeta), \qquad\qquad y \;=\; y^\varepsilon_{\mathcal{I}^*}(x^\varepsilon(\zeta)). \tag{4.47}$$

This yields a parameterization of $\mathcal{M}'_\varepsilon$ in terms of the $N-p-q$ variables ζ_i, $i \in \mathcal{J}^\dagger$. The unique point $(x^*_\varepsilon, y^*_\varepsilon) \in \mathcal{M}'_\varepsilon \subset [0,1]^N$, where the global constrained minimum is attained, is determined by the additional $N-p-q$ equations

$$\nabla_\zeta F\Big(x^\varepsilon(\zeta), y^\varepsilon_{\mathcal{I}^*}(x^\varepsilon(\zeta))\Big) \;=\; 0. \tag{4.48}$$

Assuming that the Hessian matrix of second partial derivatives

$$\left(\frac{\partial^2}{\partial x_j \partial x_k} F_{\mathcal{I}^*}(x^*)\right)_{j,k\in\mathcal{J}^\dagger} \tag{4.49}$$

is strictly positive definite in a neighborhood of (x^*, y^*), we obtain a unique point of global minimum $(x^*_\varepsilon, y^*_\varepsilon)$, for all $\varepsilon > 0$ sufficiently small.

7. The previous analysis has shown that, given arbitrary cost functions $F \in \mathcal{C}^2(\mathbb{R}^{2N}$, $G = \sum_i G_i$ with $G_i \in \mathcal{C}^3(\mathbb{R}^2)$, and $\widetilde{G} \in \mathcal{C}^3(\mathbb{R}^{2N})$, we can slightly modify each of these functions so that the following holds. The game with costs F and G_ε as in (4.5) admits a unique Stackelberg equilibrium $(x^*_\varepsilon, y^*_\varepsilon)$ for all $\varepsilon > 0$ small enough. In other words, the set of triples $(F, G, \widetilde{G})$, for which the Stackelberg equilibrium is unique, is dense in the product space $\mathcal{C}^2 \times \mathcal{C}^3 \times \mathcal{C}^3$. To complete the proof, it now suffices to observe that all our previous conditions have been formulated in terms of strict inequalities. Therefore the set of triples $(F, G, \widetilde{G})$ that satisfy all these conditions is open. □

5 Stackelberg equilibria with a narrow-sighted follower

Now we are ready to study the existence of self-consistent Stackelberg equilibria for a narrow-sighted follower, i.e. with discount factor $r_2 >> 1$. We first formulate the optimization problem for a myopic follower, which yields a cost function in diagonal form. Then, using the results proved in Section 4, we show the generic stability of the Stackelberg equilibrium w.r.t. perturbations of the cost functions L_1, L_2 for the leader and for the follower, respectively. In turn, this will yield the existence of a unique self-consistent Stackelberg equilibrium, as a fixed point of the transformation (1.9). We still denote by $u, v \in [0,1]^N$ the feedbacks adopted respectively by the leader and by the follower, as in (2.1).

Let $\mu = (\mu_1, \ldots, \mu_N) \in \Delta_N$ be a probability distribution on the initial state, and assume that $\mu_i > 0$ for every $i = 1, \ldots, N$. We define the cost function for a myopic follower as

$$G(u,v) \;=\; \sum_{i=1}^{N} \mu_i L_{2,i}(u_i, v_i). \tag{5.1}$$

In this case, the best reply $v^*(u) = (v_1^*, \ldots, v_N^*)(u)$ is the one that minimizes the instantaneous running cost, namely

$$v_i \;\in\; R_i(u_i) \;\doteq\; \Big\{\omega^* \in [0,1]\,;\;\; L_{2,i}(u_i,\omega^*) \;=\; \min_{\omega\in[0,1]} L_{2,i}(u_i,\omega)\Big\}. \tag{5.2}$$

Notice that this does not depend on the initial distribution μ.

On the other hand, if the system is initially in state i, the expected cost to the leading player is then

$$J_{1,i}(u,v) \;\doteq\; E^{x(0)=i}\left[\int_0^{+\infty} r_1 e^{-r_1 t} L_{1,x(t)}\big(u_{x(t)}, v_{x(t)}\big)\,dt\right]. \tag{5.3}$$

A similar computation as in (2.3)–(2.5) now yields

$$J_{1,i} \;=\; L_{1,i} + \frac{1}{r_1}\sum_{j\neq i} \phi_{ij}(J_{1,j} - J_{1,i}), \qquad i \in \{1, \ldots, N\}. \tag{5.4}$$

Recalling (2.7) we can write (5.4) in vector notation:

$$\mathbf{J_1}(u,v) \;=\; \Big(\mathbf{I} - \frac{1}{r_1}\mathbf{\Phi}(u,v)\Big)^{-1}\mathbf{L_1}(u,v). \tag{5.5}$$

Given the probability distribution μ on the initial state, the expected cost to the leader is

$$F(u,v) \;\doteq\; \sum_{i=1}^{N} \mu_i\, J_{1,i}(u,v). \tag{5.6}$$

Recalling (5.2), we say that a pair of feedback strategies, denoted by $(u^*, v^*) = (u_1^*, \ldots, u_N^*, v_1^*, \ldots, v_N^*)$, is a **Stackelberg equilibrium** if (u^*, v^*) is an optimizer for the constrained minimization problem for the leader

$$\min_{(u,v)\in\mathcal{A}_{\text{myopic}}} F(u,v) \tag{5.7}$$

where

$$\mathcal{A}_{\text{myopic}} \doteq \Big\{(u,v)\in[0,1]^{2N}\ ;\ v_i\in R_i(u_i),\ i=1,\ldots,N\Big\}.$$

Next, we consider the case of a short-sighted follower, whose discount factor is $r_2 >> 1$. If the initial state is $x(0)=i$, the expected cost $J_{2,i}$ was computed at (2.2)–(2.8). More generally, given a probability distribution μ on the initial state, the expected cost is computed as

$$\sum_{i=1}^{N}\mu_i J_{2,i}(u,v) \;=\; \mu^T\Big(\mathbf{I}-\frac{1}{r_2}\mathbf{\Phi}(u,v)\Big)^{-1}\mathbf{L_2}(u,v).$$

Setting $\varepsilon = r_2^{-1}$, this can be written in the form

$$G_\varepsilon(u,v) \;=\; G(u,v)+\varepsilon\widetilde{G}(u,v)+o(\varepsilon), \tag{5.8}$$

where G is the myopic cost in (5.1), while

$$\widetilde{G}(u,v) \;=\; \sum_{i,j}\mu_i\Phi_{ij}(u,v)L_{2,j}(u,v). \tag{5.9}$$

In view of (5.8), we are thus in the same framework studied in Section 4. We shall use the results of the previous section to analyze the uniqueness and stability of the Stackelberg equilibrium, for generic cost functions L_1, L_2 and transition intensity functions ϕ_{ij}, in the case of a narrow-sighted follower with $r_2 >> 1$.

Remark 5.1. It is important to observe that, while the cost functions G, G_ε for the follower depend on the initial probability distribution μ, the best reply does not. Indeed, for every $\varepsilon = r_2^{-1} > 0$ the follower solves a stochastic optimization problem, and an optimal feedback $v_\varepsilon^* = (v_{\varepsilon,1}^*,\ldots,v_{\varepsilon,N}^*)$ is simultaneously optimal for every $\mu\in\Delta_N$. When studying the generic structure of the best reply map for the follower, it is thus not restrictive to assume that the probability distribution on the initial state is

$$\overline{\mu} \;=\; \left(\frac{1}{N},\;\ldots\;,\frac{1}{N}\right). \tag{5.10}$$

On the other hand, one should be aware that, for $\varepsilon > 0$, the optimal strategy of the leader does depend on the initial distribution $\mu \in \Delta_N$, in general.

5.1 Generic stability of the Stackelberg equilibrium.

Our next goal in this section is to show that, for an open dense set of functions $L_{1,i} \in \mathcal{C}^2(\mathbb{R}^2)$ and $L_{2,i} \in \mathcal{C}^3(\mathbb{R}^2)$, and transition functions $\phi_{ij} \in \mathcal{C}^3(\mathbb{R}^2)$, the results in Theorem 4.1 can be applied to the cost functions G, F, G_ε in (5.1), (5.6), and (5.8).

Theorem 5.1. *There exists an open dense set $\mathcal{F}$ of cost functions $L_{1,i} \in \mathcal{C}^2(\mathbb{R}^2)$, $L_{2,i} \in \mathcal{C}^3(\mathbb{R}^2)$ and transition functions $\phi_{ij} \in \mathcal{C}^2(\mathbb{R}^2)$ with $\phi_{ij}(u_i, v_i) \geq 0$, such that, for $(L_1, L_2, \phi) \in \mathcal{F}$, the game with cost functions F, G in (5.6), (5.1), modeling a myopic follower, has a unique Stackelberg equilibrium.*

In addition, for every $(L_1, L_2, \phi) \in \mathcal{F}$ and every probability distribution $\mu \in \Delta_N$ on the initial state, there exists $\varepsilon_0 > 0$ small enough so that, for any $0 < \varepsilon \leq \varepsilon_0$, the game with cost functions F, G_ε in (5.6), (5.8), modeling a narrow-sighted follower, has a unique Stackelberg equilibrium.

All these solutions are stable w.r.t. small perturbations of the cost functions $L_{1,i} \in \mathcal{C}^2$, $L_{2,i} \in \mathcal{C}^3$, and of the transition functions $\phi_{ij} \in \mathcal{C}^3$, respectively.

Proof. 1. Let any $(\mathbf{L}_1, \mathbf{L}_2, \phi) \in \mathcal{C}^2 \times \mathcal{C}^3 \times \mathcal{C}^2$ be given, with $\phi_{ij} \geq 0$ for all $i, j \in \{1, \ldots, N\}$.

By Theorem 4.1 there exists an open dense set $\mathcal{G} \subset \mathcal{C}^3(\mathbb{R}^2)$ such that, for $G_i \in \mathcal{G}$, the best reply maps $u_i \mapsto R_i(u_i)$ have the regularity properties listed in (4.5)–(4.7). We now observe that, in the myopic case, the cost function (5.1) for the follower has the diagonal form (4.3), with $G_i(u_i, v_i) = \mu_i L_{2,i}(u_i, v_i)$. Hence, by an arbitrarily small $\mathcal{C}^3$ perturbation of each of the functions $L_{2,i}$ we can achieve $G_i \in \mathcal{G}$ for $i = 1, \ldots, N$.

2. Let now (u^*, v^*) be a (possibly non-unique) Stackelberg equilibrium. According to the analysis in Section 4, there exists an open dense set of functions

$(F, \widetilde{G}) \in \mathcal{F} \subset \mathcal{C}^2 \times \mathcal{C}^3$ such that the conclusion of Theorem 4.1 holds. In the steps **3 − 5** of the proof we will show that, by performing arbitrarily small perturbations of the running costs $L_{1,i}, L_{2,i}$ and of the transition functions ϕ_{ij}, the corresponding cost functions $(F, \widetilde{G}) \in \mathcal{F}$ satisfy the conclusion of Theorem 4.1, with F given by (5.6) and $\widetilde{G}$ as in (5.9).

3. As observed in Remark 5.1, to study the best reply map it is not restrictive to assume that the initial probability distribution is $\overline{\mu} = (\frac{1}{N}, \ldots, \frac{1}{N})$. As in Section 4, call $\mathcal{J} \subset \{1, \ldots, N\}$ the set of indices i such that the best reply map R_i has a jump at $u_i = u_i^*$, say from v_i^- to v_i^+. Setting $q = |\mathcal{J}|$, for each of the 2^q subsets $\mathcal{I} \subseteq \mathcal{J}$, define the function $u \mapsto v_{\mathcal{I}}(u)$ as in (4.16), with (x, y) replaced by (u, v). Moreover, recalling (2.9), consider the 2^q linear functions $\Lambda_{\mathcal{I}}(z)$ as in (4.17), and the scalar numbers

$$\lambda_{\mathcal{I}} \doteq \sum_{i,j} \frac{1}{N} \Phi_{ij}(u_i^*, v_i^{\pm}) L_{2,j}(u_j^*, v_j^{\pm}), \tag{5.11}$$

with the understanding that we are taking the value v_k^+ for $k \in \mathcal{I}$ and the value v_k^- for $k \in \mathcal{J} \setminus \mathcal{I}$, while $v_k^- = v_k^+$ for $k \notin \mathcal{J}$.

By slightly changing the transition functions ϕ_{ij} at the points $(u_i^*, v_i^{\pm})$, and by possibly performing a further small perturbation of the cost functions to the follower, we can ensure that the numbers $\lambda_{\mathcal{I}}$ in (5.11) satisfy the generic conditions in Definition 4.2.

4. According to (5.5), for every initial probability distribution $\mu \in \Delta_N$, the cost function for the leader can be written as

$$F^{\mu}(u, v) = \sum_{i=1}^{N} c_i(u, v) L_{1,i}(u_i, v_i), \tag{5.12}$$

where

$$c_i(u, v) = \sum_{k=1}^{N} \mu_k \left[\left(\mathbf{I} - \frac{1}{r_1} \mathbf{\Phi}(u, v) \right)^{-1} \right]_{ki} > 0.$$

Notice that $c_i(u, v)$ is independent of the cost function $\mathbf{L}_1$ and only depends on μ and on the transition functions $\phi_{ij}(u, v)$. The strict positivity of c_i follows from the fact that the Markov chain is irreducible and positive recurrent.

5. We now analyze how a small $\mathcal{C}^\infty$ perturbation of the functions $L_{1,i}$ affects the cost function $F = F^\mu$ in (5.12).

Fix a cutoff function $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^2)$ such that

$$\varphi(x,y) \in [0,1], \qquad \varphi(x,y) = 1 \quad \text{for all } (x,y) \in [0,1]^2. \tag{5.13}$$

Assume that the cost functions $L_{1,i}$ for the leader are replaced by

$$L^\sharp_{1,i}(u_i,v_i) = L_{1,i}(u_i,v_i) + \Big(\varepsilon'\big(|u_i-u_i^*|^2+|v_i-v_i^*|^2\big)+a_i(u_i-u_i^*)+b_i(v_i-v_i^*)\Big)\,\varphi(u_i,v_i), \tag{5.14}$$

for some small constants $\varepsilon' > 0$ and $a_i, b_i \in \mathbb{R}$, $i = 1,\ldots,N$. Calling

$$F^\sharp(u,v) = \sum_{i=1}^N c_i(u,v) L^\sharp_{1,i}(u_i,v_i),$$

we find

$$\begin{cases} \partial_{u_i} F^\sharp(u^*,v^*) - \partial_{u_i} F(u^*,v^*) = c_i(u^*,v^*) a_i\,, \\ \partial_{v_i} F^\sharp(u^*,v^*) - \partial_{v_i} F(u^*,v^*) = c_i(u^*,v^*) b_i\,. \end{cases} \tag{5.15}$$

Since $c_i > 0$, from (5.15) it follows that, by taking any $\varepsilon' > 0$, the conditions (a1) and (a3) in the proof of Theorem 4.1 are satisfied. Moreover, by suitably choosing the constants a_i, b_i in (5.14), we can perform arbitrary modifications of the gradient of the function F at the point (u^*,v^*), and achieve condition (a2) as well. Finally, by a further small modification of the gradient of F, we obtain that the vector $\mathbf{w}$ in (4.31) satisfies (4.32). This establishes our claim made in step **2.**

6. We now recall that the set $\mathcal{F}$ of functions $(F, G, \widetilde{G})$ which satisfy the conclusion of Theorem 4.1 is open in the spaces $\mathcal{C}^2 \times \mathcal{C}^3 \times \mathcal{C}^3$. In turn, the family of triples

$$(L_{1,i},\ L_{2,i},\ \phi_{ij})_{i,j=1,\ldots,N} \in \mathcal{C}^2 \times \mathcal{C}^3 \times \mathcal{C}^3, \tag{5.16}$$

for which the corresponding functions defined at (5.6), (5.1), and (5.9) satisfy $(F, G, \widetilde{G}) \in \mathcal{F}$, is open. This completes the proof. $\square$

5.2 Self-Consistent Stackelberg equilibria with a narrow-sighted follower.

Relying on the previous analysis we now prove

Theorem 5.2. *There exists an open dense set $\mathcal{F} \subset \mathcal{C}^2 \times \mathcal{C}^3 \times \mathcal{C}^3$ such that, if the cost functions and the transition intensities satisfy $(L_1, L_2, \phi_{ij}) \in \mathcal{F}$, then for every discount factor $r_2 > 0$ sufficiently large the stochastic game with dynamics (1.1) and cost functions (1.3) admits a unique self-consistent Stackelberg equilibrium.*

Proof. **1.** Let $(L_{1,i}, L_{2,i}, \phi_{ij}) \in \mathcal{F}$ be given, where $\mathcal{F}$ is the open dense set considered in Theorem 5.1. Let $(u^*, v^*) \in [0,1]^{2N}$ be the unique Stackelberg equilibrium corresponding to a myopic follower, i.e. to $\varepsilon = 0$. We observe that in this case both the leader and the follower solve an optimal control problem, hence the feedbacks u^*, v^* are simultaneously optimal for every probability distribution on the initial data.

Let μ^∞ the asymptotic stationary probability distribution, for the dynamics (1.1) corresponding to the feedbacks u^*, v^*. When $\varepsilon = 0$, μ^∞ trivially achieves the unique self-consistent equilibrium.

2. Next, for $\varepsilon \geq 0$ small, consider the maps

$$\mu \;\mapsto\; (u_\varepsilon^*(\mu), v_\varepsilon^*(\mu)), \qquad\qquad (u,v) \;\mapsto\; \mu^\infty(u,v).$$

Here $(u_\varepsilon^*(\mu), v_\varepsilon^*(\mu))$ are the feedback controls corresponding to the unique Stackelberg equilibrium, with discount factors r_1 and $r_2 = \varepsilon^{-1}$, and with probability distribution μ assigned on the initial state. Moreover, $\mu^\infty(u,v) \in \Delta_N$ is the asymptotic probability distribution as $t \to +\infty$, for the Markov process (1.1) with transition functions $\phi_{ij}(u_i, v_i)$. Under generic assumptions on the functions $\phi_{ij} \geq 0$, this process has has a unique stationary distribution, for every $(u,v) \in [0,1]^{2N}$.

For $\varepsilon > 0$ small, we claim that

$$\frac{\partial}{\partial\mu}\big(u_\varepsilon^*(\mu), v_\varepsilon^*(\mu)\big) \;=\; \mathcal{O}(1)\cdot\varepsilon, \tag{5.17}$$

where the Landau symbol $\mathcal{O}(1)$ denotes a uniformly bounded function. Indeed, recalling the notation used at (4.48), we can write

$$u \;=\; u^\varepsilon(\zeta) \;=\; (z^\varepsilon(\zeta),\zeta), \qquad\qquad v \;=\; v^\varepsilon_{\mathcal{I}^*}(u^\varepsilon(\zeta)), \tag{5.18}$$

where

$$\zeta \;=\; (\zeta_i)_{i\in\mathcal{J}^\dagger}\,, \qquad\qquad \mathcal{J}^\dagger \doteq \{1,\ldots,N\}\setminus(\mathcal{J}\cup\mathcal{J}_0\cup\mathcal{J}_1\cup\mathcal{J}_0'\cup\mathcal{J}_1')$$

is the variable along the $(N-p-q)$-dimensional manifold $\mathcal{M}'_\varepsilon$.

This yields a parameterization of $\mathcal{M}'_\varepsilon$ in terms of the $N-p-q$ variables ζ_i, $i\in\mathcal{J}^\dagger$. The unique point $(u^*_\varepsilon(\mu), v^*_\varepsilon(\mu))\in\mathcal{M}'_\varepsilon$, where the global constrained minimum of F^μ is attained, is determined by the additional $N-p-q$ equations

$$\nabla_\zeta F^\mu\big(u^\varepsilon(\zeta), v^\varepsilon_{\mathcal{I}^*}(u^\varepsilon(\zeta))\big) \;=\; 0. \tag{5.19}$$

In order to prove (5.17) it suffices to show that, at a point where (5.19) holds, one has

$$\frac{\partial}{\partial\mu_j}\frac{\partial}{\partial\zeta_i}F^\mu\big(u^\varepsilon(\zeta), v^\varepsilon_{\mathcal{I}^*}(u^\varepsilon(\zeta))\big) \;=\; \mathcal{O}(1)\cdot\varepsilon, \tag{5.20}$$

for every $i\in\mathcal{J}^\dagger$, $j\in\{1,\ldots,N\}$, and $\varepsilon\geq 0$ sufficiently small.

To prove (5.20), we first observe that in (5.11) the initial probability distribution μ enters linearly in the function F^μ. Moreover, the function $c_i(u,v)$ is $\mathcal{C}^3$ w.r.t. all variables, and all functions $L_{1,i}(u_i,v_i)$, $u^\varepsilon(\zeta)$ and $v^\varepsilon_{\mathcal{I}^*}(u)$ have $\mathcal{C}^2$ regularity. Thus we conclude that the left hand side of (5.20) is a $\mathcal{C}^1$ function of ζ.

When $\varepsilon=0$, i.e. in the myopic case, the best reply is decoupled: $v^*(u) = (v_1^*(u_1),\ldots,v_N^*(u_N))$. As a result, under the best reply, at any state i the running cost and the dynamics depend only on u_i. Thus the leader's optimization problem becomes a standard stochastic optimal control problem and this shows that the Stackelberg equilibrium (u^*,v^*) is independent of the initial distribution μ. Under our generic assumptions, for $\varepsilon>0$ small enough the best reply can now be written as

$$v^*_{\varepsilon,i}(u) \;=\; v^*_i(u_i)+\varepsilon\widetilde{v}_i(u)+o(\varepsilon), \tag{5.21}$$

where the second term depends simultaneously on all components of u. In turn, the optimal control for the leader has the expansion

$$u^*_\varepsilon(\mu) = u^* + \varepsilon\tilde{u}(\mu) + o(\varepsilon).$$

Combining above estimates, one achieves (5.20).

4. Next, we observe that the map $(u, v) \mapsto \mu^\infty(u, v)$ is Lipschitz continuous. By (5.17), for all $\varepsilon > 0$ sufficiently small the composed map

$$\mu \;\mapsto\; \mu^\infty\big(u^*_\varepsilon(\mu), v^*_\varepsilon(\mu)\big)$$

is a strict contraction and thus has a unique fixed point. By definition, this provides the self-consistent Stackelberg equilibrium. □

6 A far-sighted leader

In this section we consider the case where the discount factor $r_1 \geq 0$ of the leader is very small. Meanwhile, the discount factor $r_2 > 0$ is any positive number.

Throughout this section, we assume all the conditions in Lemma 2.3, which means that the best reply map $u \mapsto v^*(u)$ is single-valued and satisfies **(A1)**.

For $u, v \in [0,1]^N$, set

$$F^\infty(u,v) \;=\; \sum_{i=1}^{N} \mu_i^\infty(u,v) \cdot L_i(u_i, v_i). \tag{6.22}$$

where $\mu^\infty(u, v)$ is the asymptotic probability distribution for the dynamics (1.1), by the feedbacks (u, v).

Lemma 6.1. *For any fixed ϕ and $\mathbf{L}_2$ satisfying assumption* **(A2)**-**(A3)***, there exists an open dense set of functions $\mathbf{L}_{1,i} \in \mathcal{C}^2(\mathbb{R}^2)$ such that the optimization problem*

$$\min_{u\in[0,1]^N} F^\infty(u, v^*(u)) \tag{6.23}$$

has a unique minimizer $u^* = (u^*_1, \ldots, u^*_N) \in [0,1]^N$ in generic position. More *precisely, the following implications hold:*

$$\begin{cases} u_i^* = 0 \implies & \dfrac{\partial}{\partial u_i} F(u, v^*(u))\Big|_{u=u^*} > 0, \\ u_i^* = 1 \implies & \dfrac{\partial}{\partial u_i} F(u, v^*(u))\Big|_{u=u^*} < 0, \end{cases} \tag{6.24}$$

In addition, calling $I = \{i \in \{1, \ldots, N\}\,; \quad 0 < u_i < 1\}$*, the Hessian matrix of second derivatives*

$$\left(\frac{\partial^2}{\partial u_i \partial u_j} F^\infty(u, v^*(u)) \right)_{i,j\in I}$$

computed at $(u^*, v^*(u^*))$ *is strictly positive definite.*

Proof. Indeed, all the above conditions can be achieved by an arbitrarily small, smooth perturbation of the functions $L_{1,i}$, as in (5.14). In this case, one could even use a simpler kind of perturbations, having the form

$$L^\flat_{1,i}(u_i, v_i) \;=\; L_{1,i}(u_i, v_i) + \Big(\varepsilon' |u_i - u_i^*|^2 + a_i (u_i - u_i^*) \Big)\, \varphi(u_i, v_i). \tag{6.25}$$

□

Theorem 6.1. *For any fixed* ϕ *and* $\mathbf{L}_2$ *satisfying assumption* **(A2)-(A3)***, there exists an open dense set of functions* $\mathbf{L}_1 \in \mathcal{C}^2(\mathbb{R}^2) \times \ldots \times \mathcal{C}^2(\mathbb{R}^2)$ *such that for all discount exponents* $r_1 > 0$ *sufficiently small a self-consistent Stackelberg equilibrium exists.*

Proof. For any initial probability distribution $\mu \in \Delta_N$ and $r_1 > 0$, define the function

$$J^\mu_{1,r_1}(u, v) \doteq E^\mu \Big[\int_0^\infty r_1 e^{-r_1 t} L_{1,x(t)}(u_{x(t)}, v_{x(t)})\, dt \Big].$$

Recalling (3.2), we have

$$J^\mu_{1,r_1} \;=\; \mu \cdot r_1 \Big(r_1 \mathbf{I} - \mathbf{\Phi} \Big)^{-1} \cdot \mathbf{L}_1. \tag{6.26}$$

Call $\mathbf{A}$ the adjunct matrix of $\mathbf{\Phi}$, defined as the transpose of the cofactor matrix:

$$[\mathbf{A}]_{ij} = (-1)^{i+j} M_{ji}\,,$$

where M_{ij} is the (i,j)-minor of order $n-1$ of $\mathbf{\Phi}$. Since the continuous-time Markov chain is irreducible and positive recurrent, $\mathbf{\Phi}$ has rank $N-1$. As a result, both the left and right null space of $\mathbf{\Phi}$ have dimension 1, being spanned by the vectors $\mu^\infty = (\mu_1^\infty, \ldots, \mu_N^\infty)$ and $(1/N, \ldots, 1/N)^T$, respectively. Since

$$\mathbf{\Phi} \cdot \mathbf{A} = \mathbf{A} \cdot \mathbf{\Phi} = \mathbf{0},$$

for all $i, j = 1, \ldots, N$ one has

$$[\mathbf{A}]_{ij} = \beta\, \mu_j^\infty,$$

for some constant β. To determine β, let

$$-\lambda_N \leq \cdots \leq -\lambda_2 < -\lambda_1 = 0$$

be the eigenvalues of $\mathbf{\Phi}$. For $r_1 > 0$ small we then have the expansion

$$r_1\Big(r_1\mathbf{I} - \mathbf{\Phi}(u,v)\Big)^{-1} = \frac{1}{\prod_{i=2}^N \lambda_i}\mathbf{A}(u,v) + r_1\widetilde{\mathbf{\Phi}}(u,v) + o(r_1),$$

for a suitable function $\widetilde{\mathbf{\Phi}}$, whose expression can be derived from $\mathbf{\Phi}$. Since

$$\mu^\infty \cdot (r_1\mathbf{I} - \mathbf{\Phi}) = r_1\mu^\infty \qquad \Longrightarrow \qquad \mu^\infty = \mu^\infty \cdot r_1(r_1\mathbf{I} - \mathbf{\Phi})^{-1},$$

in the limit $r_1 \to 0+$ one has the implication

$$\mu^\infty = \frac{1}{\prod_{i=2}^N \lambda_i}\mu^\infty \cdot \mathbf{A} \qquad \Longrightarrow \qquad \beta = \prod_{i=1}^N \lambda_i.$$

Thus by setting $\varepsilon = r_1$, the cost function has the expansion

$$J_{1,\varepsilon}^\mu(u,v) = \frac{1}{\prod_{i=2}^N \lambda_i}\mu \cdot \mathbf{A} \cdot \mathbf{L}_1 + \varepsilon\mu \cdot \tilde{\mathbf{\Phi}} \cdot \mathbf{L}_1 + o(\varepsilon) = F^\infty(u,v) + \varepsilon\widetilde{F}(u,v) + o(\varepsilon),$$

where $\widetilde{F} = \mu \cdot \widetilde{\mathbf{\Phi}} \cdot \mathbf{L}_1$.

Call u^* the minimizer in (6.23) and v^* the unique best reply $v^*(u^*)$. Due to the form of F^∞ defined in (6.22) and the fact that $\mu_i^\infty > 0$, we can perturb the cost functions $L_{1,i}$ as in (6.25), and achieve arbitrary modifications of the gradient of the function F^∞ at point (u^*, v^*). As a result, all conditions (a1)-(a3) in the proof of Theorem (4.1) will be satisfied.

By the stability results in Chapter 5, for all $r_1 = \varepsilon$ small enough, the minimization problem

$$\min_{u\in[0,1]^N} J^\mu_{1,\varepsilon}(u, v^*(u)) \tag{6.27}$$

has a unique global minimizer $u^*_\varepsilon(\mu)$.

When $\varepsilon = 0$, by construction $\mu^\infty(u^*, v^*)$ is the asymptotic probability distribution, hence this achieves the self-consistent equilibrium. Also when $\varepsilon = 0$, the Stackelberg equilibrium does not depend on the initial distribution μ, i.e. $u^*_0(\mu) = u^*$.

For any fixed $\mu \in \Delta_N$, the global minimizer $u^*_\varepsilon(\mu)$ of (6.27) is determined by the equations

$$\nabla_u J^\mu_{1,\varepsilon}\Big(u, v^*_\varepsilon(u)\Big) \;=\; 0.$$

Similar arguments as in the proof of (5.20), now yield

$$\frac{\partial}{\partial \mu_j}\frac{\partial}{\partial u_i} J^\mu_{1,\varepsilon}\Big(u^*_\varepsilon(\mu), v^*_\varepsilon(u^*_\varepsilon(\mu))\Big) \;=\; \mathcal{O}(1)\cdot\varepsilon, \tag{6.28}$$

for any pair $(i,j) \in \{1,\ldots,N\}^2$. Since in the limit $\varepsilon = 0$ the left hand side of (6.28) is zero, by continuity we achieve

$$\frac{\partial}{\partial \mu} u^*_\varepsilon(\mu) \;=\; \mathcal{O}(1)\cdot\varepsilon.$$

In turn, the composed map

$$\mu \;\mapsto\; u^*_\varepsilon(\mu) \;\mapsto\; \mu^\infty\Big(u^*_\varepsilon(\mu), v^*(u^*_\varepsilon(\mu))\Big)$$

is a strict contraction, hence it has a unique fixed point. □

7 An example

Consider a stochastic process where the state can take only two possible values: $x(t) \in \{0, 1\}$. The two players choose controls $u_1, u_2 \in [0, 1]$. Given an initial state

$x_0 \in \{0,1\}$, the cost functions are

$$\begin{cases} J_1 \doteq E^{x_0}\left[\int_0^{+\infty} e^{-rt} L_1\big(x(t), u_1(t)\big)\, dt\right], \\ J_2 \doteq E^{x_0}\left[\int_0^{+\infty} e^{-rt} L_2\big(x(t), u_1(t), u_2(t)\big)\, dt\right], \end{cases} \tag{7.1}$$

where

$$\begin{cases} L_1(1, u_1) = u_1\,, \\ L_1(0, u_1) = 1 + u_1\,, \end{cases} \qquad \begin{cases} L_2(1, u_1, u_2) = -\, u_2\,, \\ L_2(0, u_1, u_2) = K_1 u_1 + u_2\,. \end{cases}$$

The system can randomly jump from one state to the other. The transition probabilities are:

$$\begin{cases} \text{Prob.}\Big\{x(t+\varepsilon) = 1 \;\Big|\; x(t) = 0\Big\} = 0, \\ \text{Prob.}\Big\{x(t+\varepsilon) = 0 \;\Big|\; x(t) = 1\Big\} = K_2 u_2\, \varepsilon + o(\varepsilon). \end{cases} \tag{7.2}$$

Here K_1, K_2 are suitably large constants. We think of u_1 as the control of the leader and u_2 as the control of the follower.

Interpretation of the model: Consider a game between a government (the leading player) and a bank manager (the follower). Two states are possible: $x \in \{0,1\}$. When $x = 1$ the bank manager (player 2) is active, and can achieve a profit which increases with the risk level u_2 that he is choosing. However, by increasing u_2, the manager also increases the risk of bankruptcy, i.e., of falling back into state $x = 0$.

We think of $x = 0$ as a bankruptcy state, where the bank manager can make no profit, and the government pays an additional social cost (unemployment, lost wages, etc.). The control u_1 allows the government to punish the manager in case of bankruptcy. Of course, one expects $u_1 = 0$ when $x = 1$. On the other hand, choosing a large value of u_1 when $x = 0$ will deter the bank manager from taking risks.

We observe that this is a typical case where the Stackelberg equilibrium is not "time consistent". Different equilibria in feedback form exists, depending on the initial state. Indeed, announcing the punishment $u_1(0) = 1$ is an effective preemptive strategy only when $x(t) = 1$. If the manager is already bankrupt, wasting money to punish him is useless.

Claim: Depending on the initial condition, we have two distinct Stackelberg solutions, in feedback form:

CASE 1: $x_0 = 0$. Then the state is $x(t) \equiv 0$, and an optimal Stackelberg solution is

$$u_1(0) = u_1(1) = u_2(0) = 0, \qquad u_2(1) = 1.$$

CASE 2: $x_0 = 1$. Then, if $K_1K_2 \geq r$, the Stackelberg solution is

$$u_1(0) = 1, \qquad u_1(1) = u_2(1) = u_2(0) = 0.$$

Proof. Denote by $V_i : \{0, 1\} \mapsto \mathbb{R}$ the value function of player i.

CASE 1. In this case, since $x(t) \equiv 0$, for any leader's control u_1, the response of player 2 is $u_2(0) = 0$ and the cost for player 2 is $V_2(0) = K_1u_1(0)/r$. However, since the leader's strategy is simply $u_1(0) = 0$ and $V_1(0) = 1/r$, we have $V_2(0) = 0$. The standard HJB equation implies that

$$rV_2(1) = \min_{\omega\in[0,1]} \big\{-\omega + K_2\omega(V_2(0) - V_2(1))\big\} = \min_{\omega\in[0,1]} \{\omega\{-1 - K_2V_2(1)\}\}. \quad (7.3)$$

Thus the corresponding optimal control and value function are $u_2(1) = 1$ and $V_2(1) = -1/(r + K_2)$. Actually, due to the fact that $x = 0$ is a absorption state, the leader does not care about what happens at $x = 1$ so the control $u_1(1)$ can be any value in $[0, 1]$. So there are infinitely many Stackelberg equilibrium solutions, namely

$$u_1(0) = u_2(0) = 0, \qquad u_1(1) \in [0, 1], \qquad u_2(1) = 1.$$

CASE 2. In this case, no matter what the leader's control is, the follower's control at state 0 is always $u_2(0) = 0$. Thus $V_2(0) = K_1u_1(0)/r$ and $u_2(1)$ is determine by the HJB equation (7.3). Notice that $u_1(0)$ is not necessarily 0 since the leader wants to minimize the cost starting at state 1. According to (7.3), the follower reacts differently to the control $u_1(0)$:

$$u_2(1) = \begin{cases} 0, \text{ if } u_1(0) > \frac{r}{K_1 K_2}, \\ 1, \text{ if } u_2(0) < \frac{r}{K_1 K_2}. \end{cases} \tag{7.4}$$

Back to the leader, it is obvious that $u_1(1) = 0$ and $V_1(0) = \frac{1+u_1(0)}{r}$. Thus we have

$$rV_1(1) = K_2 u_2(1)(V_1(0) - V_1(1)) \Longrightarrow V_1(1) = \frac{K_2 u_2(1)(1 + u_1(0))}{r(r + K_2 u_2(1))}.$$

It is obvious that $V_1(1) \geq 0$ and according to (7.4), the equality is achieved only if $u_2(1) = 0$. As a result, the leader should impose any control $u_1(0) \geq \frac{r}{K_1 K_2}$ so that $u_2(1) = 0$. Then there are still infinitely many optimal Stackelberg solutions, namely

$$u_1(1) = u_2(0) = u_2(1) = 0, \;\; u_1(0) = \text{any value in } \left[\frac{r}{K_1 K_2}, 1\right].$$

□

Remark. All of the above Stackelberg solutions are self-consistent, simply because the system stays at the initial state forever.

In CASE 2, if the leader announces a punishment strategy $u_1(0) \geq \frac{r}{K_1 K_2}$, then the follower will not take any risks, because bankruptcy becomes too costly.

A transitive Markov process. In the previous example, when $u_2(1) = 0$ the evolution is entirely deterministic. The existence of distinct Stackelberg solutions, both in feedback form, depending on the initial data, is clear. We now show that similar conclusions remain valid also in cases where the dynamics is transitive. For example, let the transition probabilities (7.2) be replaced by

$$\begin{cases} \text{Prob.}\Big\{x(t+\varepsilon) = 1 \;\Big|\; x(t) = 0\Big\} &= \; \delta\,\varepsilon + o(\varepsilon), \\ \text{Prob.}\Big\{x(t+\varepsilon) = 0 \;\Big|\; x(t) = 1\Big\} &= \; (K_2 u_2 + \delta)\varepsilon + o(\varepsilon), \end{cases} \tag{7.5}$$

for some $\delta > 0$ sufficiently small.

For any control u_1 of the leader, the best reply u_2 and the value function V_2 for the follower should satisfy the following dynamic programming equation:

$$\begin{cases} rV_2(0) &= K_1u_1(0) + \delta(V_2(1) - V_2(0)), \\ rV_2(1) &= \delta(V_2(0) - V_2(1)) + \min_{\omega\in[0,1]} \Big\{\omega\big(-1 + K_2(V_2(0) - V_2(1))\big)\Big\}. \end{cases} \tag{7.6}$$

A simple calculation shows that the best replies for player 2 are

$$u_2(1) = \begin{cases} 0 & \text{if } \; u_1(0) > \frac{r+2\delta}{K_1K_2}, \\ \in [0,1] & \text{if } \; u_1(0) = \frac{r+2\delta}{K_1K_2}, \\ 1 & \text{if } \; u_1(0) < \frac{r+2\delta}{K_1K_2}. \end{cases} \tag{7.7}$$

Denoting $W_1(u_1, \mu)$ the expected cost to the leader with initial distribution $x_0 \sim \mu = (\mu_0, \mu_1)$ and under optimal reply u_2, we have

$$\begin{cases} W_1(u_1,(0,1)) &= \dfrac{(K_2u_2(1) + \delta)(1 + u_1(0)) + (r + \delta)u_1(1)}{r(r + K_2u_2(1) + 2\delta)}, \\ W_1(u_1,(1,0)) &= \dfrac{(r + K_2u_2(1) + \delta)(1 + u_1(0)) + \delta u_1(1)}{r(r + K_2u_2(1) + 2\delta)}. \end{cases} \tag{7.8}$$

and thus by linearity for general $\mu \in \Delta_2$, we have

$$\begin{aligned} W_1(u_1, \mu) &= \mu_0 W_1(u_1,(1,0)) + \mu_1 W_1(u_1,(0,1)) \\ &= \frac{(\mu_0 r + K_2u_2(1) + \delta)(1 + u_1(0)) + (\mu_1 r + \delta)u_1(1)}{r(r + K_2u_2(1) + 2\delta)}. \end{aligned}$$

From (7.7) we have that $u_2(1)$ only depends on $u_1(0)$, so in any type of strategy $u_1(1)$ should be 0. By simple computation, the value function of the leader for any initial distribution μ, i.e. $V_1(\mu) \doteq \min_{u_1\in\Delta_2} W_1(u_1, \mu)$, has the form

$$V_1(\mu) = r^{-1} \min \left\{ \frac{(\mu_0 r + \delta)(1 + \frac{r+2\delta}{K_1K_2})}{r + 2\delta}, \frac{(\mu_0 r + K_2 + \delta)}{r + K_2 + 2\delta} \right\}.$$

Specifically, by defining

$$\tilde{\mu}_0 \doteq \frac{1}{r}\left(\frac{K_2(r+2\delta)}{\frac{r+2\delta}{K_1}+\frac{(r+2\delta)^2}{K_1K_2}+K_2}-\delta\right),$$

the Stackelberg solutions are

$$\begin{aligned} &\text{CASE 1 :}\mu_0 \le \tilde{\mu}_0 \Longrightarrow u_1(1)=u_2(1)=u_2(0)=0, u_1(0)=\frac{r+2\delta}{K_1K_2},\\ &\text{CASE 2 :}\mu_0 \ge \tilde{\mu}_0 \Longrightarrow u_1(1)=u_1(0)=u_2(0)=0, u_2(1)=1. \end{aligned} \tag{7.9}$$

The invariant distribution for CASE 1 in (7.9) is $(1/2, 1/2)$ and that for CASE 2 is $(\frac{K_2+\delta}{K_2+2\delta}, \frac{\delta}{K_2+2\delta})$. So we obtain the following self-consistent Stackelberg solutions:

- SC Stackelberg Equilibrium 1: if $\tilde{\mu}_0 \ge \frac{1}{2}$,

$$\mu = (1/2, 1/2), u_1(1)=u_2(1)=u_2(0)=0, u_1(0)=\frac{r+2\delta}{K_1K_2}.$$

- SC Stackelberg Equilibrium 2: if $\tilde{\mu}_0 \le \frac{K_2+\delta}{K_2+2\delta}$,

$$\mu = \left(\frac{K_2+\delta}{K_2+2\delta}, \frac{\delta}{K_2+2\delta}\right), u_1(1)=u_1(0)=u_2(0)=0, u_2(1)=1.$$

Appendix A

A multi-jet transversality theorem.

We state here a version of the multi-jet transversality theorem which is used several times in the disstertation. In the following, for given integers $m, n \geq 1$ we consider maps $f : \mathbb{R}^{m+n} \mapsto \mathbb{R}$. Let $\mathcal{P}^k$ the space of polynomials of degree $\leq k$ in the $m+n$ variables $(x, y) \in \mathbb{R}^m \times \mathbb{R}^n$. Identifying a polynomial with its coefficients, it is clear that $\mathcal{P}^k$ is a finite dimensional vector space. The product space

$$J^k(\mathbb{R}^{m+n}; \mathbb{R}) \doteq \mathbb{R}^{m+n} \times \mathcal{P}^k$$

is a jet bundle over the space R^{m+n}. Any function $f : \mathbb{R}^{m+n} \mapsto \mathbb{R}$ determines a section of this bundle, defined as $j^k f(x, y) = P^{(x,y)}$, where the polynomial $P^{(x,y)}(\cdot)$ is the k-th order Taylor approximation of the function f at the point (x, y).

Next, for $s \geq 1$ we call

$$Z^{(s)} \doteq \{(x, y_1, \ldots, y_s) \in \mathbb{R}^m \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n ; \quad y_i \neq y_j \text{ for all } 1 \leq i < j \leq s\}. \tag{0.1}$$

Notice that $Z^{(s)}$ is an open subset of a vector space of dimension $m + sn$; hence it is a manifold. The set

$$\begin{aligned} J_s^k(\mathbb{R}^{m+n}; \mathbb{R}) = \Big\{ &(x, y_1, \ldots, y_s, P_1, \ldots, P_s) \in \mathbb{R}^m \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n \\ &\times \mathcal{P}^k \times \cdots \times \mathcal{P}^k, y_i \neq y_j \text{ for all } 1 \leq i < j \leq s \Big\} \end{aligned} \tag{0.2}$$

is a k-th order jet bundle over $Z^{(s)}$. Any smooth function $f : \mathbb{R}^{m+n} \mapsto \mathbb{R}$ determines a section of this bundle defined as

$$j_s^k f(x, y_1, \ldots, y_s) \;=\; (Q^{(x,y_1)}, \;\ldots\;, Q^{(x,y_s)}), \tag{0.3}$$

where $Q^{(x,y_i)}$ is the polynomial of degree $\leq k$ determined by the k-th order Taylor approximation to f at the point (x, y_i). We can now state a version of the multi-jet transversality theorem which is used in our paper. The proof is similar to the one on p. 57–59 of [38], with some simplifications due to the fact that our maps are defined on Euclidean spaces, rather than on general manifolds.

Theorem 0.1. *Let W be a smooth submanifold of $J_s^k(\mathbb{R}^{m+n}; \mathbb{R})$. Then the set of functions $f \in \mathcal{C}^\infty(\mathbb{R}^{m+n}; \mathbb{R})$ which are transversal to W is dense, in the $\mathcal{C}^\infty$ topology.*

Proof. 1. Cover the open set $Z^{(s)}$ with countably many open sets V_ν, $\nu \geq 1$, such that

- If $(x, y_1, \ldots, y_s) \in V_\nu$ and $(\tilde{x}, \tilde{y}_1, \ldots, \tilde{y}_s) \in V_\nu$, then $y_i \neq \tilde{y}_j$ for all $1 \leq i < j \leq s$.

Construct $\mathcal{C}^\infty$ functions $\phi_\nu : \mathbb{R}^{m+n} \mapsto [0, 1]$, $\nu \geq 1$, with the following properties.

- $\mathrm{Supp}(\phi_\nu) \subset V_\nu$.
- For each $\nu \geq 1$, call $V'_\nu \subset V_\nu$ the interior of the set where $\phi_\nu = 1$. Then $\bigcup_{\nu \geq 1} V'_\nu = Z^{(s)}$.

2. For each $\nu \geq 1$, and any polynomials $P_1, \ldots, P_s$ of degree $\leq k$, define the function

$$f^{(P)}(x, y) \;=\; f(x, y) + \sum_{\ell=1}^{s} \phi_k(x, y) P_\ell(x, y_\ell). \tag{0.4}$$

We now consider the map

$$(x, y_1, \ldots, y_s, P_1, \ldots, P_s) \;\mapsto\; \Big(j^k f^{(P)}(x, y_1), \;\ldots\; j^k f^{(P)}(x, y_s) \Big). \tag{0.5}$$

The right hand sides are the coefficients of the k-th order Taylor approximations to the maps $(x, y) \mapsto f^{(P)}(x, y)$ at the points (x, y_ℓ). Since $\phi_\nu \equiv 1$ on V'_ν, it is clear that the differential of the map (0.5) has full rank. Hence this map is transversal to any manifold W, restricted to V'_ν. By the transversality theorem, there is a residual set $\mathcal{S}_\nu \subset \mathcal{C}^\infty(\mathbb{R}^{m+n}; \mathbb{R})$ such that, for every $f \in \mathcal{S}_\nu$, the map $j^k_s f$ in (0.3) is transversal to W at every point $(x, y_1, \ldots, y_s, P_1, \ldots, P_s) \in W$ such that $(x, y_1, \ldots, y_s) \in V'_\nu$.

3. Repeating the same argument for every $\nu \geq 1$, we obtain a sequence of residual subsets $\mathcal{S}_\nu$. The intersection $\mathcal{S} \doteq \bigcap_{\nu \geq 1} \mathcal{S}_\nu$ is still residual in $\mathcal{C}^\infty(\mathbb{R}^{m+n}; \mathbb{R})$. By construction, for every $f \in \mathcal{S}$ the map $j^k_s f$ is transversal to W. □

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