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MUTUAL NONLINEAR INTERACTION OF ULTRASONIC GUIDED WAVES IN PLATE WITH APPLICATIONS FOR NDE

A Dissertation in
Engineering Science and Mechanics

by

Mostafa Hasanian

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The dissertation of Mostafa Hasanian was reviewed and approved* by the following:

Clifford J. Lissenden  
Professor of Engineering Science and Mechanics  
Dissertation Advisor, Chair of Committee

Joseph L. Rose  
Paul Morrow Professor of Engineering Science and Mechanics

Bernhard R. Tittmann  
Shell Professor of Engineering Science and Mechanics

Parisa Shokouhi  
Associate Professor of Engineering Science and Mechanics

Mansour Solaimanian  
Professor of Civil and Environmental Engineering

Judith A. Todd  
Professor of Engineering Science and Mechanics  
P. B. Breneman Department Chair  
Head of the Department

*Signatures are on file in the Graduate School
ABSTRACT

Ultrasonic methods have an essential role in Nondestructive Evaluation (NDE) and Structural Health Monitoring (SHM) of infrastructure. Early damage in the structure can be diagnosed in microstructural scales whereas conventional NDE methods are less effective. The unique sensitivity of nonlinear ultrasonic waves to microstructural damage has intrigued scholars to search for how to fulfill its potential to prognose early damage in materials. The distorted ultrasonic wave packet, due to the material’s nonlinearity, can generate higher harmonics that are associated with incipient damage. The primary benefit of guided waves is that, relative to bulk waves, they can propagate long distances and interrogate inaccessible domains. However, the complex phenomenon of nonlinear guided waves makes the study more difficult. Ultrasonic higher order harmonics are generated due to the interaction of ultrasonic waves in the nonlinear material. Second order harmonics due to self-interactions are often corrupted by electronic nonlinearities. This has complicated the measurement of second harmonics associated with the material’s nonlinearity. The mutual interaction of ultrasonic waves generates sum and difference frequencies that are separated from higher harmonics generated by instrumentation. This significant characteristic of mutual interactions is incredibly helpful for the evaluation of microstructure since material nonlinearities are distinguishable from instrumentation nonlinearities. This dissertation is dedicated to the development of the mutual interactions of guided waves in a plate.

The general theory of nonlinear guided wave mixing in plate is developed in this dissertation. Two guided waves with different frequencies, wave numbers, and propagation directions interact in a plate. Herein, by vector-based calculations, the internal resonance criteria are formulated and evaluated for waves propagating in arbitrary directions in a plate. The non-zero power flux analysis reveals that non-collinear guided wave interactions transfer power to secondary guided wave modes that is impossible for collinear interactions, which is completely analogous to
bulk wave interactions. The phase matching condition, the resonance of nonlinear waves at sum/difference combinations of primary wavevectors, is incorporated to find the wave combinations having cumulative behavior. A list of the most feasible and interesting wave triplets (i.e., two primary waves and one secondary wave) is suggested based on prescribed conditions, and a few of them are studied. In contrast with the current theory that includes interaction of continuous wave packets, an analytical model is introduced for finite-sized interactions and used to demonstrate the effect of group velocity mismatch on the generation of secondary wave fields. Finite element results are compared to the analytical model observations, which gives insight into secondary wave formation. Furthermore, non-collinear mixing of two shear horizontal guided wave is studied by finite element simulation.

Shear Horizontal guided waves in the fundamental mode (SH0) are nondispersive and readily generated by magnetostrictive transducers. The interaction of two SH0 primary waves generates secondary Symmetric Lamb waves. Moreover, the different polarization of SH0 waves and secondary Lamb wave fields provides a unique opportunity to isolate material nonlinearity from most of the electronic distortions. The codirectional and counter-propagating interaction of SH0 waves are investigated through finite element simulations and experiments. SH0 waves are generated by magnetostrictive transducers, and secondary symmetric Lamb waves are measured by an angle beam and air-coupled sensors. The results indicate that the material nonlinearities can be isolated from system nonlinearities. Thus, the mutual interaction of guided waves can significantly enhance the quality of nonlinear measurements for material evaluation.
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To my dearest mother and father
Chapter 1
INTRODUCTION

1.1 Motivation

Ensuring the structural integrity of critical structures such as aircraft, power plants, and civil infrastructure is important for fleet management and operational decisions (Lissenden et al. 2015b). Halting power plants for maintenance is extremely expensive, particularly if it is not scheduled. Thus, material awareness and intimate knowledge of structural integrity are required for a reliable condition assessment. In current Non-Destructive Evaluation (NDE) and Structural Health Monitoring (SMH) methods, the presence of macro-scale damage is the main indication of imminent structural failure that demands immediate decision-making and action (García-Martín et al. 2011; Hanke et al. 2008; Rose 2014; Usamentiaga et al. 2014; Zoughi 2000). This leads to a costly procedure since the removal of the structure from duty is needed. As shown in Figure 1-1, the current detection limit of service life is based on macro-scale defects, while damage evaluation on the micro-scale could significantly shift the limits to lower damage progressions (Lissenden et al. 2014). Therefore, evaluating the material’s condition before reaching macro-scale damage can significantly help the prediction of the failure time (Golchinfar et al. 2017; Tashakori et al. 2018). This is called early damage detection and implies microstructural characterization of the material in the beginning stages of the damage formation.

Condition-based monitoring of structures, especially early damage detection, can improve scheduling for maintenance, resulting in budget and time savings. Eventually, condition-based maintenance will be practical for industries, where structural components will be repaired/replaced
based upon the material’s condition. The motivation of this dissertation is to devise/design new methods for microstructural material characterization where one of the applications is early damage detection.

Figure 1-1. Schematic diagram showing the microstructure evolution and damage progression (Lissenden et al. 2014).

Early damage detection was subject of several previous studies utilizing different methods and novel approaches (Donskoy and Ramezani 2018; Farhangdoust et al. 2017; Lissenden et al. 2015b; Moorthy et al. 2000; Palanichamy et al. 2000; Rai et al. 2004; Schlangen 2008; Yusa et al. 2006). Our focus is on nonlinear ultrasonics where the generation of higher harmonics is associated with damage levels in microstructures (Chillara and Lissenden 2015a; Ju et al. 2017; Matlack et al. 2014; Nagy 1998). The unique sensitivity of nonlinear ultrasonic waves to microstructural damage has been shown previously. We aim to push current barriers to establish new methods with premium functionality and enhanced accuracy and repeatability (Hasanian and Lissenden 2017, 2018a). This chapter introduces the prerequisite mathematics needed to understand the fundamentals of guided waves in a plate. In addition, the literature of nonlinear ultrasonics is reviewed with respect to the motivation for the current study. Finally, the structure and scope of this dissertation is given.
1.2 Mathematical preliminaries

Mathematical preliminaries for modeling guided wave in plates and some fundamental concepts from continuum mechanics are given in this section. The detailed formulations can be found elsewhere (Chillara and Lissenden 2012; Gurtin et al. 2009; Rose 2014). This section parallels that of my research partner Hwanjeong Cho (Cho 2017).

1.2.1 Guided waves in isotropic linear elastic plates

This section focuses on describing the fundamentals of guided waves in a plate in order to understand the propagation physics. The wavestructures and dispersion behavior of guided waves in plates are highlighted. Tensors and vectors are bold face capital and lower-case letters, respectively. Consider a plate with traction-free boundaries as shown in Figure 1-2.

![Figure 1-2. Plate in the Cartesian coordinate system with traction-free boundaries.](image)

The governing equation of dynamic motion for an isotropic linear elastic material is in the form of Navier’s equation (Chou and Pagano 1992):

$$\left(\lambda + \mu\right)u_{j,j} + \mu u_{i,j} = \rho \ddot{u}_i$$  \hspace{1cm} (1-1)

where \(\rho\) is the mass density, and \(u\) denotes the displacement field. \(\lambda\) and \(\mu\) are the Lame’s constants. Assuming the 2-dimensional (2D) plane strain in the \(X-Z\) plane, the Helmholtz decomposition (Rose 2014) can be used to solve the boundary value problem as:
\[ \mathbf{u} = \nabla \phi + \nabla \times \psi \] (1-2)

where \( \phi = \phi(X, Z) \) is the scalar field and \( \psi = (0, \psi(X, Z), 0) \) is a vector field. Assume a harmonic guided wave propagating in +X direction as,

\[
\phi(X, Z) = \phi(Z) e^{i(kX - \omega t)}
\]

\[
\psi(X, Z) = \psi(Z) e^{i(kX - \omega t)},
\] (1-3)

where \( \omega \) and \( k \) denote the angular frequency and the wavenumber of the guided wave mode respectively. Equation 1-3 is updated by \( \phi = A \cos(pZ) + B \sin(qZ) \) and \( \psi = C \cos(pZ) + D \sin(qZ) \) as the general solutions (Rose 2014). Then, Equations 1-1 and 1-2 can be solved based on the traction-free boundary conditions defined by the stress components,

\[ T_{XZ} = T_{ZZ} = 0 \text{ at } Z = \pm h \] (1-4)

Therefore, the homogeneous system of equations will be as (Rose 2014),

\[
\mu (-2ikp \sin(ph) + (k^2 - q^2) D \sin(qh)) = 0
\]

\[-\lambda (k^2 + p^2) A \cos(ph) - 2\mu (p^2 D \cos(ph) + ikD \cos(qh)) = 0
\]

\[
\mu (2ikp B \cos(ph) + (k^2 - q^2) C \cos(qh)) = 0
\]

\[-\lambda (k^2 + p^2) B \sin(ph) - 2\mu (p^2 C \sin(ph) + ikC \sin(qh)) = 0 ,
\] (1-5)

where

\[ p = \sqrt{\left(\frac{\omega^2}{c_L^2} - k^2\right)} \]

\[ q = \sqrt{\left(\frac{\omega^2}{c_T^2} - k^2\right)} \] (1-6)

and where \( c_L \) and \( c_T \) are longitudinal and shear wave velocities respectively. In order to have a meaningful solution for Equation 1-5, the determinant of the coefficient matrix (A-D) must vanish,
which leads to the following dispersion equations for symmetric and antisymmetric guided wave modes,

\[
\frac{\tan(qh)}{\tan(ph)} = \frac{-4k^2pq}{(q^2-k^2)^2} \quad \text{Symmetric modes}
\]

\[
\frac{\tan(qh)}{\tan(ph)} = \frac{-(q^2-k^2)^2}{4k^2pq} \quad \text{Antisymmetric modes}
\] (1-7)

Equation 1-7 can be solved numerically for each \(\omega\) and \(k\). Guided wave modes defined based on Equation 1-7 are denoted as Rayleigh-Lamb (RL) mode where displacement polarization is in \(X-Z\) plane. Similarly, Shear Horizontal (SH) guided waves, with in-plane displacement in the \(Y\) direction, can be derived if the plane strain condition is dismissed. Therefore, one can find the phase velocity of SH waves as (Rose 2014):

\[
c_p(f_d) = \pm 2c_T \left[ \frac{f_d}{\sqrt{4(f_d)^2 - n^2c_T^2}} \right]
\] (1-8)

When \(n=0\) (corresponding to the zeroth-order symmetric SH mode, SH0), \(c_p = c_T\) a the dispersionless wave propagating at the shear wave speed \(c_T\). Therefore, the SH0 wave is important since the fundamental mode is non-dispersive and easy to generate. Figure 1-3 shows the dispersion curves for a 1 mm thick aluminum plate \((E=70\text{GPa}, \nu=0.33, \rho=2700\text{kg/m}^3)\). Unlike bulk waves, guided waves have variable phase velocity based on frequency and wavenumber, which demands careful mode selection in order for nonlinear guided wave mixing to be successful.

Phase velocity refers to the velocity of the particles or the rate of the change in the phase of the wave motion. On the other hand, the group velocity is the velocity of the whole shape/envelope of the wave field (Hasanian and Lissenden 2018a; Müller et al. 2010). For bulk waves, since the velocity is independent of frequency, group velocities equal to phase velocity. However, for guided waves, the variability of phase velocity relative to frequency determines different group velocity modes. The group velocity is given by (Rose 2014) to be,
where \( c_p \) denotes the phase velocity, and \( fd \) is the frequency-thickness product. The group velocity of SH waves similarly is presented in an expression as:

\[
c_g(fd) = c_T \sqrt{1 - \frac{(n/2)^2}{(fd/c_T)^2}}
\]  

Numerical analysis of Equations 1-9 and 1-10 gives the group velocity dispersion curves as depicted in Figure 1-3.

**Figure 1-3.** Phase velocity dispersion curves and group velocity curves of an aluminum plate.

S: Symmetric Lamb wave, A: Antisymmetric Lamb wave, SH: Shear Horizontal.

While bulk waves demonstrate planar wave structure, guided waves in the plate can have more complex wave structures. Wavestructure is referred to as displacement distribution through the plate thickness. Out-of-plane, \( u_z \), and in-plane \( u_x \) and \( u_y \) are the two displacement forms that are
frequently addressed in this work. Wavestructures are determined by coefficients A-D which are eventually defining $u$ in Equation 1-2. Wavestructures of the A0 Lamb wave at 0.5 MHz are given in Figure 1-4, showing dominant out-of-plane displacement on the top surface. On the other hand, the S0 Lamb wave at 0.5 MHz has a dominant in-plane displacement field as shown in Figure 1-5. However, S Lamb waves at higher frequencies can have a large out-of-plane displacement component at the free surfaces (Figure 1-6 for instance) that would be quite helpful for reception by Air-Coupled (AC) sensors (Hasanian and Lissenden 2018b; Thiele et al. 2014). Finally, wavestructure of the SH0 guided wave is shown in Figure 1-7 indicating planar in-plane displacement field.

![Figure 1-4. Wavestructure of A0 Lamb wave at 0.5 MHz.](image-url)
Figure 1-5. Wavestructure of S0 Lamb wave at 0.5 MHz.

Figure 1-6. S Lamb waves with large out-of-plane displacements on top surface (a)-(c) Wavestructures of S0 Lamb waves with frequencies at 2, 2.28, and 3.3 MHz respectively, (d) Wavestructure of S1 Lamb wave at 2.84 MHz.
1.2.2 Continuum mechanics

Wave interaction happens due to the nonlinearity of stress/strain wave fields. For modeling and analyzing the nonlinear wave interactions, a continuum model is used in order to observe stress variations and strain perturbations. In this section, the fundamental continuum mechanics model is given, and in the next chapter we derive the vector-based equations for the general case of nonlinear guided wave mixing.

The position of a material particle in the reference frame is denoted by $\mathbf{X}$ and the current position after deformation is $\mathbf{x}(\mathbf{X}, t)$. Scalars are represented by normal Italic letters. In the case of waves in solid media, $\mathbf{u}(\mathbf{X}, t)$ presents the displacement field with respect to particle motion. Therefore, the displacement gradient is defined as,

$$\mathbf{H} = \frac{\partial \mathbf{u}}{\partial \mathbf{X}}$$

The nonlinear Lagrangian strain is related to the displacement gradient by,

$$\mathbf{E} = \frac{1}{2} \left( \mathbf{H} + \mathbf{H}^T + \mathbf{H} \mathbf{H}^T \right)$$
Consider a hyperelastic isotropic material with weakly nonlinear material properties, the energy density function (Landau and Lifshitz 1986) containing up to third order terms has the form,

\[ \tilde{W}(E) = \frac{1}{2} \lambda (\text{tr}(E))^2 + \mu \text{tr}(E^2) + \frac{1}{3} C (\text{tr}(E))^3 + B \text{tr}(E)\text{tr}(E^2) + \frac{1}{3} A \text{tr}(E^3) \]  

(1-13)

where \( \lambda \) and \( \mu \) are Lamé’s constants and \( A, B, \) and \( C \) are third order elastic constants (Norris 1998). \( \text{tr}(\cdot) \) denotes the trace operator. The second Piola-Kirchoff stress tensor is defined by,

\[ T_{RR}(E) = \frac{\partial \tilde{W}(E)}{\partial E} = \lambda \text{tr}(E)I + 2\mu E + C (\text{tr}(E))^2 I + B \text{tr}(E)E + AE^2 \]  

(1-14)

which can be written as a function of \( H \) up to second order,

\[ T_{RR}(H) = \frac{\lambda}{2} (H + H^T) + \mu (H + H^T) + \frac{C}{2} \text{tr}(H^TH)I + C \text{tr}(H^2)I + \mu H^TH + B \text{tr}(H)(H + H^T) + \frac{B}{2} \text{tr}(H^2 + H^TH)I + \frac{A}{4} \left( H^2 + H^TH + HH^T + (H^T)^2 \right). \]  

(1-15)

In order to extract nonlinear terms for characterization of the displacement field, we decompose this equation to linear (first order terms) and nonlinear (second order terms),

\[ T_{RR}(H) = T_{RR}^L(H) + T_{RR}^{NL}(H) \]  

(1-16)

\[ T_{RR}^L(H) = \frac{\lambda}{2} (H + H^T) + \mu (H + H^T), \]  

(1-17)

\[ T_{RR}^{NL}(H) = \frac{\lambda}{2} \text{tr}(H^TH)I + C \text{tr}(H^2)I + \mu H^TH + B \text{tr}(H)(H + H^T) + \frac{B}{2} \text{tr}(H^2 + H^TH)I + \frac{A}{4} \left( H^2 + H^TH + HH^T + (H^T)^2 \right). \]  

(1-18)

To define the stress wave field, the first Piola-Kirchoff stress \( S \) is related to \( T_{RR} \) through the deformation gradient \( F=I+H \) by \( S=FT_{RR} \) (Chillara and Lissenden 2012), so then,

\[ S(H) = (I + H) \left( (T_{RR}^L(H) + T_{RR}^{NL}(H)) \right) \]  

(1-19)

\[ S^L(H) = T_{RR}^L(H), \]  

(1-20)

\[ S^{NL}(H) = HT_{RR}^L(H) + T_{RR}^{NL}(H) \]  

(1-21)
Equation 1-21 contains nonlinear terms of the stress field that generate perturbations in the plate that eventually can accumulate and form secondary waveforms (Hasanian and Lissenden 2018a). In the next chapter, Equations 1-20 and 1-21 are substituted into the balance of the linear momentum, and the nonlinear wave propagation equations are solved by considering the boundary conditions.

1.3 Literature review

Ultrasonics is a well-known and effective method for NDE and SHM applications, especially for macro-scale damage detection. In general classification, bulk waves and guided waves are two different types of ultrasonic waves defined by the nature of the medium they travel within. Bulk waves propagate in a medium without any boundary, whereas guided waves are bounded between interfaces and free surfaces (Rose 2014). Bulk waves are well accepted and widely used in industry for inspection of plates, pipes, etc., due to the simplicity of wave structures that lead to the straightforward and direct generation and reception of bulk waves. However, bulk waves have limited utility for the inspection of thin plates and other waveguides, except for the pointwise scanning inspection method.

On the other hand, guided waves are constrained by the boundaries, and can be used for long distance inspections in bounded media. Moreover, they can interrogate otherwise inaccessible domains, and provide volumetric coverage from a single point (Rose 2004). Thus, guided waves can offer inspection of structures that is not practical with other methods, such as scanning of thin plates located in inaccessible areas (Rose 2002). Discussions and theories of guided waves in various waveguides for NDE and SHM applications can be found in many sources, for example (Auld 1990; Rose 2014).
Linear ultrasonics is a strong tool for the inspection of structures due to the sensitivity to defects on the order of a wavelength or larger. Despite great opportunities provided by linear ultrasonics, detection of small defects and material degradation that does not measurably change the linear elastic properties or have discontinuities of the order of the wavelength is hardly feasible. Fatigue damage, thermal degradation, embrittlement, and porosity are examples of damages that are known to be almost invisible to linear ultrasonic methods. Nevertheless, it has been shown that the aforementioned degradations are easily distinguishable from pristine materials by looking at microstructures using destructive techniques (Haque and A Saif 2002; Jhang 2009; Marino et al. 2016; Ruiz et al. 2013; Wang et al. 2011). However, the destructive methods are not applicable for SHM or any type of nondestructive inspection. The shape of ultrasonic waves is slightly distorted after propagating within the nonlinear elastic medium due to nonlinear properties of the material (Jhang 2009; Matlack et al. 2014; Zheng et al. 2000). The unique sensitivity of nonlinear ultrasonic waves to microstructural damage has intrigued scholars to search for how to fulfill its potential to detect incipient damage in materials (Chen et al. 2014; Matlack et al. 2014; Tang et al. 2014; Zhang et al. 2016a). There are many nonlinear ultrasonic methods to associate microstructures with different waveform distortions (Broda et al. 2014; Chillara and Lissenden 2015a; Jhang 2009; Matlack et al. 2014). Herein, we refer to nonlinear ultrasonics as the interrogation of signals at a frequency other than the excitation frequency, that includes integer multiples and sum/difference frequencies (Liu et al. 2014b). It has been proven that higher harmonics are significantly sensitive for the detection of micro-scale damage progression (Marino et al. 2016; Matlack et al. 2015; Nagy 1998; Solodov 1998). Consider generating two monochromatic waves in a nonlinear material at frequencies $\omega_a$ and $\omega_b$ as shown in Figure 1-8. Due to the material nonlinearity each wave will induce self-interactions that generate higher harmonics at integer multiples of the excitation frequencies. Furthermore, the mutual interaction of waves $a$ and $b$ will generate waves at the sum
and difference frequencies. While Figure 1-8 only shows second order interactions, third, fourth, and even higher order interactions also occur. The figure also shows amplitudes. The amplitudes of the secondary waves associated with material nonlinearity are typically orders of magnitude smaller than the primary waves, which requires precise measurement techniques.

Figure 1-8. Frequency domain recognition after wave interactions in a nonlinear material up to second order higher harmonics, frequencies ($\omega$) and corresponding amplitudes ($A$).

Study of the nonlinear interaction of elastic waves in solids traces back to (Gol’dberg 1961) and (Breazeale and Ford 1965; Hikata and Elbaum 1966). For generation of combinational harmonics through mutual-interaction Jones and Rollins derived a resonance criterion for scattered bulk waves based on the interaction volume, elastic properties, intersection angle, and frequency ratio (Jones and Kobett 1963; Rollins 1963). Jones also followed the same path for generation of combinational harmonics via mutual interaction of two waves (Jones and Kobett 1963). Important reasons for studying these interactions are their extraordinary potential to nondestructively characterize a variety of modes of material degradations at early stages as indicated in a number of articles (McGovern and Reis 2015; Zheng et al. 2000). The mutual interaction of bulk waves has also been studied extensively (Demčenko et al. 2012; Ju et al. 2017; McGovern and Reis 2015; Tang et al. 2014; Zhang et al. 2016b). Bulk wave interaction studies can be categorized into non-collinear and collinear methods based on the direction of the primary waves. Thus, the concepts
and insights in the current study have been inherited from bulk wave self- and mutual interactions (Korneev and Demčenko 2014).

Unlike bulk waves, the complex nature of guided waves, including wave structures and group velocities, requires explicit theoretical studies to find the best combination of frequencies and wave numbers for generating a strong nonlinear field. Deng, as a pioneer, analyzed cumulative second harmonic generation from guided waves in plate (Deng 1998). Later de Lima and Hamilton (de Lima and Hamilton 2005; De Lima and Hamilton 2003) employed a perturbation approach to obtain solutions of nonlinear equations of motion. The Normal Mode Expansion (NME) method enabled de Lima to define two important resonance conditions to have cumulative second harmonics. Later, Muller (Müller et al. 2010) suggested the group velocity matching condition is additionally required for generation of cumulative second harmonics through the interaction of a finite wave packet. Chillara (Chillara and Lissenden 2012) developed the nonlinear elastic wavefield equations in a new formulation that makes it extremely straightforward to follow/develop the equations for further studies. Based on these studies, one of the important conditions to have strong and cumulative second harmonics is the phase matching condition. The phase matching condition ensures the resonances of wave numbers between primary and secondary wavefields in order to excite the secondary wave field perfectly. Liu has investigated the selection of primary modes for the generation of strong internally resonant second harmonics in plate (Liu et al. 2013a). The resonance conditions were demonstrated along with mode selection methodology, where it is shown that finding a perfect mode pair is challenging because there are many considerations in addition to internal resonance. Chillara and Lissenden further investigated the role of group velocity for the generation of second harmonics suggesting the necessity of group velocity matching condition (Chillara and Lissenden 2014, 2015a). Prior to this dissertation, the guided wave interactions studies were on collinear guided wave mixing with associated model developments
(Chillara and Lissenden 2012; De Lima and Hamilton 2003; Müller et al. 2010) and simulations (Lissenden et al. 2015a; Liu et al. 2014a), but no experimental studies of mutual interaction of guided waves are found in the literature. Note that some experimental studies of guided wave self-interaction (second harmonic generation) have been conducted, e.g. (Matlack et al. 2012).

Using high power amplifiers to generate strong primary waves can cause the generation of initial second harmonics related to electronics that are mixed with a material’s nonlinearity. Not only high-power amplifiers, but also transducers, sensors, and filters generate and take in higher harmonics that confound measurement of material nonlinearity (Demčenko et al. 2012; Hasanian and Lissenden 2018b; McGovern and Reis 2014; Mostavi et al. 2017). In this regard, mutual wave interactions are attractive because they allow the material interrogation to be at frequencies far removed from known nonlinearities in measurement systems (Demčenko et al. 2012; McGovern and Reis 2015). Previously, arbitrary angle mixing of bulk waves and Rayleigh waves have been studied (Korneev and Demčenko 2014; Morlock et al. 2015; Thiele et al. 2014), whereas the general theory of guided wave mixing with arbitrary mixing angle has not been investigated with vector-based calculations. While researchers have addressed the effect of group velocity combinations on generation of second harmonics (Matsuda and Biwa 2011; Müller et al. 2010; Xiang et al. 2016; Zhu et al. 2018), a clear insight on the mechanism of guided wave mixing with respect to group velocities of wave packets is still needed.

This dissertation is an important step forward to the goal of applying mutual interaction of guided waves to nondestructively characterize the changes in material microstructure that precede macro-scale damage. By introducing the vector-based analysis of nonlinear guided wave interactions, we first analyze the internal resonance criteria in general terms, which requires the formulation of equations of waves propagating in arbitrary directions. This has not been addressed previously, and our study provides new insight into guided wave mixing. Normalizing the power
flux from the primary modes to the secondary mode provides a useful tool for comparing the different wave mixing possibilities. Furthermore, the results can be specialized to the more well-known cases of bulk wave interaction, which provides confidence that the approach is viable. The resulting equations enable mode, frequency, and direction selection and we find that non-collinear interactions can generate internally resonant second order waves that are not generated by collinear interactions. Next, the important aspect for mutual mixing of the finite size wave packets is investigated through a simple model of group velocity by representation of wave fields based on the superposition principle. Some mode triplets, mainly Shear Horizontal primary waves, are investigated by finite element simulations and experimental studies. We conclude that mutual guided wave mixing in plate introduces very effective methods for interrogating material nonlinearity separate from system nonlinearities. The proposed methods can almost completely isolate the nonlinearity associated with mutual interactions.

1.4 Scope and outline of the dissertation

The goal of this dissertation is to develop the NDE methods for reliable and accurate measurement of material nonlinearity using the mutual interaction of guided waves in plate structures. To achieve this goal, this study contains the following chapters with the following topics:

Chapter 2: Formulation of the mutual interaction between guided waves propagating in arbitrary directions within a plate is given. Guided wave mode selection methodology is illustrated to find the best mode triplets for efficient measurement of material nonlinearity.
Chapter 3: Interaction of guided wave packets having finite size is considered instead of continuous waves. The wave packets with defined size and group velocity are constructed based on the superposition principle. The results reveal that the size of the mixing zone is crucial to accumulate sufficient nonlinear waves that are detectible.

Chapter 4: Codirectional nonlinear mixing of two SH0 waves is considered. First of all, finite element results display the phenomenon and how the secondary wave field develops. Next, the experimental method is explained and the results are shown.

Chapter 5: Counter-propagating SH0 waves are studied. Finite element studies confirm the effectiveness of the mode selection. The experimental procedure is explained and is followed by the results. Also, the role of the phase matching condition for effective generation of secondary wave at the sum frequency is evaluated experimentally. Through the experiments, it is demonstrated that nonlinear waves cannot be generated if matching conditions are not achieved. The observations using the same experimental setup reveals the majority of nonlinearity can come from material if the correct mixing combination and testing method are selected.

Chapter 6: Non-collinear mixing of SH0 guided waves at a 90º mixing angle is studied by finite element simulation. Special boundary conditions are applied to the plate to ensure single mode excitation of SH0 waves. Employing a subtraction method, the out of plane displacement field caused by nonlinear interactions is visualized. The characteristics of the extracted secondary wave field confirms the generation of the designed nonlinear mixing pattern.
Chapter 2

THEORY OF DIRECTIONAL NONLINEAR INTERACTION OF GUIDED WAVES IN PLATE

2.1 Introduction

In this chapter, the nonlinear interaction of guided waves in plate is theoretically modeled. The general case of guided wave mixing, with an arbitrary mixing angle, is introduced. Finally, by deriving the internal resonance conditions, a list of possible guided wave mixing candidates is provided for further studies.

The nonlinear interaction of planar ultrasonic guided waves propagating in lossless isotropic elastic materials has been formulated previously for co-directional continuous waves (Chillara and Lissenden 2012; De Lima and Hamilton 2003; Müller et al. 2010) and for non-collinear bulk waves (Korneev and Demčenko 2014). However, the general nonlinear interaction of guided waves, including collinear and non-collinear, has not been introduced prior to this study. The interaction between two primary waves \(a\) and \(b\) is considered, which leads to the generation of a secondary wave due to geometric and material nonlinearities. While the main focus is on guided wave interaction in traction-free plates, the introduced formulation is also applicable to bulk waves as will be shown. To generate a strong nonlinear ultrasonic wave associated with material nonlinearity, the combination of wave characteristics has to meet certain conditions that are recognized as perfect resonance conditions. Perfect resonance condition refers to ideal phase matched mixing waves that results in linear increase in nonlinear harmonics. Thus, perfect resonance conditions are discussed in this chapter that concluded to a list of possible mixing modes.
This chapter is based primarily on two articles published in the *Journal of Applied Physics* (Hasanian and Lissenden 2017, 2018a).

### 2.2 Theoretical formulation

Consider planar waves having angular frequency $\omega$, wave structure $\mathbf{U}$, and wavevector $\mathbf{K}$ in the $XY$-plane. The displacement fields representing continuous waves $a$ and $b$ are given by,

\[
\mathbf{u}_a(X,Y,Z,t) = \frac{1}{2}[\mathbf{U}_a(Z)e^{i[\mathbf{K}_a\cdot\mathbf{p}(X,Y) - \omega a t]} + \text{c.c.}]
\]  
(2-1)

\[
\mathbf{u}_b(X,Y,Z,t) = \frac{1}{2}[\mathbf{U}_b(Z)e^{i[\mathbf{K}_b\cdot\mathbf{p}(X,Y) - \omega b t]} + \text{c.c.}]
\]  
(2-2)

respectively, where $\mathbf{p}$ is the position vector for a point in space with respect to the coordinate system shown in Figure 2-1. The term c.c. represents the complex conjugate, and thus $\mathbf{u}_a$ and $\mathbf{u}_b$ are the real parts of their respective displacement fields. The wave structures $\mathbf{U}_a$ and $\mathbf{U}_b$ are the displacement profiles through the plate thickness that have arbitrary amplitudes and are not normalized. Wavevectors $\mathbf{K}_a$ and $\mathbf{K}_b$ are defined by their scalar wave number $k$ and unit vector $r$ associated with the direction of propagation,

\[
\mathbf{K}_a = k_a \mathbf{r}_a
\]  
(2-3)

\[
\mathbf{K}_b = k_b \mathbf{r}_b
\]  
(2-4)

The phase velocity, $C_p$, is computed from

\[
|\mathbf{K}| = \frac{\omega}{C_p}
\]  
(2-5)

The total displacement field is decomposed into primary and secondary wave fields,

\[
\mathbf{u} = \mathbf{u}_a + \mathbf{u}_b + \mathbf{u}_{aa} + \mathbf{u}_{bb} + \mathbf{u}_{ab}
\]  
(2-6)
where the secondary fields are assumed to be much smaller than the primary fields. To second order, the self-interactions are denoted with subscripts $aa$ and $bb$, while the mutual interactions are given by subscript $ab$. The field equations and boundary conditions for nonlinear wave propagation in a plate are provided below. The equations provide the foundation for the internal resonance criteria. Decompose the displacement gradient in an analogous way to the displacement field in Equation 2-6,

$$\nabla \mathbf{u} = \mathbf{H} = \mathbf{H}_a + \mathbf{H}_b + \mathbf{H}_{aa} + \mathbf{H}_{bb} + \mathbf{H}_{ab} \quad (2-7)$$

The Lagrangian strain is related to the displacement gradient,

$$\mathbf{E} = \frac{1}{2} [\mathbf{H} + \mathbf{H}^T + \mathbf{H}^T \mathbf{H}] \quad (2-8)$$

For hyperelastic materials the strain energy function can be written:

$$\tilde{W}(\mathbf{E}) = \frac{1}{2} \lambda [\text{tr}(\mathbf{E})]^2 + \mu \text{tr}(\mathbf{E}^2) + \frac{1}{3} C [\text{tr}(\mathbf{E})]^3 + B \text{tr}(\mathbf{E}) \text{tr}(\mathbf{E}^2) + \frac{1}{3} A \text{tr}(\mathbf{E}^3) \quad (2-9)$$

where $\lambda$ and $\mu$ are Lame’s constants, and $A$, $B$, and $C$ are the Landau-Lifshitz elastic constants. The strain energy function gives the second Piola-Kirchhoff stress:

$$\mathbf{T}_{RR} = \frac{\partial \tilde{W}(\mathbf{E})}{\partial \mathbf{E}} \quad (2-10)$$

which is related to the first Piola-Kirchhoff stress, $\mathbf{S}$, through the deformation gradient, $(\mathbf{I} + \mathbf{H})$:

$$\mathbf{S} = [\mathbf{I} + \mathbf{H}] \mathbf{T}_{RR} \quad (2-11)$$

Next, the stress is decomposed into linear and nonlinear terms,

$$\mathbf{S}(\mathbf{H}) = \mathbf{S}^L(\mathbf{H}_a) + \mathbf{S}^L(\mathbf{H}_b) + \mathbf{S}^L(\mathbf{H}_{aa}) + \mathbf{S}^L(\mathbf{H}_{bb}) + \mathbf{S}^L(\mathbf{H}_{ab}) + \mathbf{S}^{NL}(\mathbf{H}_a, \mathbf{H}_a, 2) + \mathbf{S}^{NL}(\mathbf{H}_b, \mathbf{H}_b, 2) + \mathbf{S}^{NL}(\mathbf{H}_a, \mathbf{H}_b, 2) \quad (2-12)$$

The superscript $L$ denotes stress terms that are linear with respect to the displacement gradient. The terms $\mathbf{S}^{NL}(\mathbf{H}_a, \mathbf{H}_a, 2)$ and $\mathbf{S}^{NL}(\mathbf{H}_b, \mathbf{H}_b, 2)$ represent nonlinear stress terms with respect to the displacement gradient due to second order self-interactions, but will not be written here because
our interest is in mutual interactions. Likewise, $S^{NL}(\mathbf{H}_a, \mathbf{H}_b, 2)$ denotes second order mutual interaction of displacement fields $\mathbf{u}_a$ and $\mathbf{u}_b$, which can be shown to be:

$$S^{NL}(\mathbf{H}_a, \mathbf{H}_b, 2) = \frac{\lambda}{2} \text{tr}(\mathbf{H}_b + \mathbf{H}_b^T)\mathbf{H}_a + \mu \mathbf{H}_a [\mathbf{H}_b + \mathbf{H}_b^T] + \frac{\lambda}{2} \text{tr}(\mathbf{H}_a + \mathbf{H}_a^T)\mathbf{H}_b +$$

$$\mu \mathbf{H}_b [\mathbf{H}_a + \mathbf{H}_a^T] + \frac{\lambda}{2} \text{tr}(\mathbf{H}_a^T \mathbf{H}_b + \mathbf{H}_b^T \mathbf{H}_a) \mathbf{I} + 2C \text{tr}(\mathbf{H}_a) \text{tr}(\mathbf{H}_b) \mathbf{I} + \mu \left[ \mathbf{H}_a^T \mathbf{H}_b + \mathbf{H}_b^T \mathbf{H}_a \right] +$$

$$B \text{tr}(\mathbf{H}_a)[\mathbf{H}_b + \mathbf{H}_b^T] + B \text{tr}(\mathbf{H}_b)[\mathbf{H}_a + \mathbf{H}_a^T] + \frac{B}{2} \text{tr}(\mathbf{H}_a \mathbf{H}_b + \mathbf{H}_b \mathbf{H}_a + \mathbf{H}_a^T \mathbf{H}_b + \mathbf{H}_b^T \mathbf{H}_a) \mathbf{I} +$$

$$\frac{\lambda}{4} \left[\mathbf{H}_a \mathbf{H}_b + \mathbf{H}_b \mathbf{H}_a + \mathbf{H}_a^T \mathbf{H}_b^T + \mathbf{H}_b^T \mathbf{H}_a^T + \mathbf{H}_a^T \mathbf{H}_b + \mathbf{H}_b^T \mathbf{H}_a + \mathbf{H}_a \mathbf{H}_b^T + \mathbf{H}_b \mathbf{H}_a^T\right]$$

(2-13)

It is worth noting that Equation 2-13 applies to both guided waves and bulk waves.

Now consider the balance of linear momentum for a material with mass density $\rho_\kappa$ in the reference configuration,

$$\text{Div}(\mathbf{S}(\mathbf{H})) = \rho_\kappa \ddot{\mathbf{u}}$$

(2-14)

which also applies to both guided waves and bulk waves. Apply the traction-free boundary conditions on the top and bottom surfaces of a plate,

$$\mathbf{S}(\mathbf{H}) \mathbf{n}_Z = 0 \quad \text{for} \quad Z = \pm h$$

(2-15)

where $\mathbf{n}_Z$ is the unit normal to the free surfaces, and $h$ is half of the plate thickness.

Assuming that the nonlinearity is weak, a perturbation solution can be applied. Here, we are only solving the mutual interaction problem, thus the secondary portion of the boundary value problem has the nonlinear stress term from Equation 2-13 as the driving force. The driving force appears in both the wave equation and the boundary conditions:

$$\text{Div}(\mathbf{S}^L(\mathbf{H}_{ab})) - \rho_\kappa \ddot{\mathbf{u}}_{ab} = -\text{Div}(\mathbf{S}^{NL}(\mathbf{H}_a, \mathbf{H}_b, 2))$$

(2-16)

$$\mathbf{S}^L(\mathbf{H}_{ab}) \mathbf{n}_Z = -\mathbf{S}^{NL}(\mathbf{H}_a, \mathbf{H}_b, 2) \mathbf{n}_Z \quad \text{for} \quad Z = \pm h$$

(2-17)
Of course, Equations 2-16 and 2-17 represent the portion of the boundary value problem related to mutual interaction. The other portions are for self-interactions of waves a and b as well as the linear portion of the stress in Equation 2-14 and 2-15.

Figure 2-1. (a) Mixing two guided waves in the XY plane generates a secondary wave based on vector summation, (b) definition of three types of bulk waves (L = Longitudinal, SV = Shear Vertical, SH = Shear Horizontal) and their directions obtained by rotation about a vertical axis.

Substituting the \( u_a \) and \( u_b \) displacement fields, Equation 2-1 and 2-2, into the nonlinear stress term associated with mutual interactions (i.e., Equation 2-13) results in terms with sum and difference exponential components:

\[
e^{i\left[\text{\boldsymbol{K}}_a \pm \text{\boldsymbol{K}}_b\right]\cdot \text{\boldsymbol{p}}(X,Y) - [\omega_a \pm \omega_b]t}\quad e^{-i\left[\text{\boldsymbol{K}}_a \pm \text{\boldsymbol{K}}_b\right]\cdot \text{\boldsymbol{p}}(X,Y) - [\omega_a \pm \omega_b]t}
\]

for \( \omega_a \geq \omega_b \quad (2-18) \)
These components represent potential second order harmonic waves at the sum and difference frequencies \((\omega_m = \omega_a \pm \omega_b)\), and form a secondary harmonic displacement field related to wavevectors \((\mathbf{K}_a \pm \mathbf{K}_b)\). This time varying pattern changes during each instant of wave interaction, and consequently can create different nonlinear scattering fields.

Each wavevector is directed at an arbitrary angle within the XY-plane as illustrated in Figure 2-1a. The interaction angle, \(\theta\), gives the rotation angle about the Z-axis between the two primary waves and \(\gamma\) is the angle between wave \(a\) and the generated secondary waves. To provide a reference, wave \(a\) propagates in the \(X\)-direction. Similarly, planar L, SV, and SH types of bulk waves in an infinite medium are defined in Figure 2-1b, and can interact at different angles by rotating the primary wavevectors around a vertical axis.

### 2.2.1 Problem solution

The objective is to describe the secondary waves, \(u_{ab}\), that are generated at the sum and difference frequencies of waves \(a\) and \(b\). Therefore, likewise previous studies (Auld 1990; Chillara and Lissenden 2012; De Lima and Hamilton 2003), the normal mode expansion is used to represent the secondary linear stress and velocity fields in terms of all possible propagating modes,

\[
\mathbf{S}^l(\mathbf{H}_{ab}) = \frac{1}{2} \left[ \sum_{m=1}^{\infty} A_m(X,Y) \mathbf{S}_m(Z) e^{-i[(\omega_a \pm \omega_b)t] + c.c.} \right] 
\]

\[
\mathbf{u}_{ab} = \frac{1}{2} \left[ \sum_{m=1}^{\infty} A_m(X,Y) \mathbf{V}_m(Z) e^{-i[(\omega_a \pm \omega_b)t] + c.c.} \right] 
\]

where \(\mathbf{S}_m\) and \(\mathbf{V}_m\) are the modal stress and velocity fields, and \(A_m(X,Y)\) is a scalar giving the modal amplitude of wave \(m\) with displacement field \(u_m\). The modal velocity field \(\mathbf{V}_m\) is related to the modal stress field \(\mathbf{S}_m\) through the displacement, displacement gradient, linearized strain, and linearized material behavior:
\[
\begin{align*}
\mathbf{e}_m &= \frac{1}{2} [\mathbf{H}_m + \mathbf{H}_m^T] \\
\mathbf{S}_m &= [\lambda + \mu] \mathbf{e}_m \mathbf{I} + \mu \mathbf{e}_m
\end{align*}
\] (2-21)

The normal mode expansion and complex reciprocity theorem of (Auld 1990) can be used to derive a partial differential equation in \(X\) and \(Y\),

\[
4 \mathbf{P}_{mn}^t \cdot \mathbf{n}_X \left[ \frac{\partial}{\partial X} - i \mathbf{K}_n^* \cdot \mathbf{n}_X \right] A_m(X, Y) + 4 \mathbf{P}_{mn}^t \cdot \mathbf{n}_Y \left[ \frac{\partial}{\partial Y} - i \mathbf{K}_n^* \cdot \mathbf{n}_Y \right] A_m(X, Y) = \\
\left[ f_n^{surf} + f_n^{vol} \right]
\] (2-23)

from which the modal amplitudes can be determined. The full derivation is provided in Appendix A. In Equation 2-23 \(\mathbf{n}_X\) and \(\mathbf{n}_Y\) are unit vectors in the \(X\) and \(Y\) directions respectively, an asterisk (*) denotes the complex conjugate, and:

\[
\begin{align*}
\mathbf{P}_{mn}^t &= \frac{-1}{4} \int_{-h}^h S_m \mathbf{V}_n^* + S_n^* \mathbf{V}_m \, dZ \\
\mathbf{P}_{mn} &= \mathbf{P}_{mn}^t \cdot \mathbf{r}_m \\
f_n^{surf} &= \frac{-1}{2} S^{NL}(\mathbf{H}_a, \mathbf{H}_b, 2) \mathbf{V}_n^* \cdot \mathbf{n}_Z \bigg|_{-h}^h \\
f_n^{vol} &= \frac{1}{2} \int_{-h}^h \text{Div}(S^{NL}(\mathbf{H}_a, \mathbf{H}_b, 2)) \cdot \mathbf{V}_n \, dZ
\end{align*}
\] (2-24-27)

where \(\mathbf{P}_{mn}^t\) is the power flux density vector integrated through the plate thickness, and \(f_n^{surf}\) and \(f_n^{vol}\) represent the nonlinear driving forces associated with power flux from the primary wave fields to the secondary wave field through the lateral boundaries and the volume respectively. For the secondary wave field to exist \(P_{mn}\) in Equation 2-25 must be nonzero. Considering the special case of waves \(a\) and \(b\) being co-directional, where \(\theta = 0\), the following simplifications occur:

\[
\mathbf{P}_{mn}^t \cdot \mathbf{n}_X = P_{mn}, \mathbf{P}_{mn}^t \cdot \mathbf{n}_Y = 0, \mathbf{K}_n^* \cdot \mathbf{n}_X = k_n^*, \text{and} \begin{bmatrix} k_a \pm k_b \end{bmatrix} \cdot \mathbf{p}(X,Y) = [k_a \pm k_b].
\]

and Equation 2-23 reduces to:

\[
4 P_{mn} \left[ \frac{d}{dX} - i k_n^* \right] A_m(X) = \left[ f_n^{surf} + f_n^{vol} \right] e^{i[k_a \pm k_b]X} 
\] (2-28)
which has been given in the literature (De Lima and Hamilton 2003).

Let $n$ be the mode that is not orthogonal to mode $m$, i.e., $P_{mn} \neq 0$, which is guaranteed for propagating modes when $m = n$. Then the solution of Equation 2-23 for $A_m(X,Y)$ can be written:

$$A_m(X,Y) = -\frac{[f_n^{surf} + f_n^{vol}]}{4P_{mn}|K_n - (K_a \pm K_b)|} |e^{i|K_n|p(X,Y) - e^{i|K_a \pm K_b|p(X,Y)}|} \text{ if } K_n \neq K_a \pm K_b \quad (2-29)$$

$$A_m(X,Y) = \frac{[f_n^{surf} + f_n^{vol}]}{4P_{mn}} |K_a \pm K_b| |e^{i|K_a \pm K_b|p(X,Y)} \text{ if } K_n = K_a \pm K_b \quad (2-30)$$

Interpreting Equations 2-29 and 2-30 provides the internal resonance criteria for the generation of cumulative second order harmonics in plates. Equation 2-29 represents bounded oscillation, while Equation 2-30 is a wave with an amplitude that increases linearly as it propagates.

1. The *phase matching* condition requires that the wavevectors do in fact add as vectors, $K_n = K_a \pm K_b$.

2. The *nonzero power flux* condition, $f_n^{surf} + f_n^{vol} \neq 0$, requires that the nonlinear driving forces transfer power from the primary modes to the secondary mode.

The nonzero power flux condition assures the excitation of the secondary wave field by the interaction of the two primary waves. In the literature, parity analysis has been used to identify the mode pairs with zero power flux in order to know which types of secondary waves cannot be generated (Chillara and Lissenden 2012; Liu et al. 2013a; Müller et al. 2010). In the present study, a parametric analysis leveraging the symmetric and antisymmetric nature of the displacement profiles is used to assess the nonzero power flux condition, in lieu of a formal parity analysis. Likewise, the phase matching condition provides a basis for selecting the type of primary waves, their directions, and their frequencies in order to generate secondary fields.

Now the internally resonant secondary wave field can be written:
\[ \mathbf{u}_{ir} = \frac{1}{2} \left[ A_{ir}(X, Y) \mathbf{u}_{ir}(Z)e^{-i[\omega_a \pm \omega_b]t} + \text{c.c.} \right] \]  

(2-31)

when mode \( m \) satisfies both criteria, thus the subscript \( m \) is replaced with \( ir \), and all the other propagating modes are orthogonal and do not affect the modal expansion. Substituting Equation 2-30 into Equation 2-31 gives,

\[ \mathbf{u}_{ir} = \frac{1}{2} \left[ f_{s}^{ir} + f_{v}^{ir} \right] \left( \frac{(K_{a} \pm K_{b})}{|K_{a} \pm K_{b}|} \right) \mathbf{U}_{ir}(Z)e^{i[|K_{a} \pm K_{b}|p(X,Y) - [\omega_a \pm \omega_b]t]} + \text{c.c.} \]  

(2-32)

which can be re-organized by defining three intermediate terms: normalized wave structure, position vector, and ‘mixing power’, which are, respectively:

\[ \mathbf{U}_{ir}(Z) = \frac{\mathbf{u}_{ir}(Z)}{\text{Amp}(\mathbf{u}_{ir}(Z))} \]  

(2-33)

\[ r_{ir} = \frac{(K_{a} \pm K_{b})}{|K_{a} \pm K_{b}|} \]  

(2-34)

\[ M_{ir} = \frac{[f_{s}^{ir} + f_{v}^{ir}]}{4P_{ir,ir}} \frac{\text{Amp}(\mathbf{u}_{ir})}{\text{Amp}(\mathbf{u}_{a})\text{Amp}(\mathbf{u}_{b})} \]  

(2-35)

resulting in

\[ \mathbf{u}_{ir}(X, Y, Z, t) = \frac{1}{2} \left[ M_{ir}\text{Amp}(\mathbf{u}_{a})\text{Amp}(\mathbf{u}_{b})\mathbf{U}_{ir}(Z)[r_{ir} \cdot \mathbf{p}(X,Y)]e^{i[|K_{ir}|p(X,Y) - \omega_{ir}t]} + \text{c.c.} \right] \]  

(2-36)

where \( \text{Amp}(\cdot) \) means the maximum amplitude, whether it is in-plane or out-of-plane.

The new factor defined by Equation 2-35, called the *mixing power*, quantifies the power transferred from the interacting primary waves to the internally resonant secondary waves. The numerator of the first fraction contains the power transferred to the secondary wave field and the denominator of the first fraction is the power associated with the secondary waves, while the second ratio provides normalization.

With the exception of the boundary conditions and the nonlinear surface force, all of the equations above can be applied to bulk waves. Thus, this special case is analyzed in the next section.
2.2.2 Special case of bulk wave interactions

Bulk waves propagate in infinite media, thus there are no boundary conditions and no surface forces. The wave structures are uniform. Due to the differences in wave structures, dispersion, and the presence of multiple modes, guided waves are more complicated to analyze. However, the equations formulated in the previous section apply to both guided and bulk waves, except for Equations 2-16, 2-17, and 2-26. This unification provides some confidence that the guided wave analysis is correct. Considering the first fraction in Equation 2-30,

\[
\frac{f_n^{surf} + f_n^{vol}}{4\rho_{mn}} \Rightarrow \frac{f_n^{vol}}{4\rho_{mn}} = -2 \frac{\text{Div}(S_{\text{NL}}(H_n, H_n^2) \cdot V_n^2)}{[S_m V_n^2 + S_n V_m] \tau_m}
\] (2-37)

which enables assessment of the nonzero power flux condition based on the type of bulk waves; i.e., L, SV, or SH (Longitudinal, Shear-Vertical, and Shear-Horizontal). One merit of this approach is that the matrix-based calculations are straightforward and flexible. Korneev (Korneev and Demčenko 2014) described the second order nonlinear interactions of planar bulk waves based on wavevectors and third order elastic constants. The equations developed here simplify to their results. Nonzero power flux analysis and phase matching conditions were considered parametrically, and the features of the candidate wave triplets are displayed in Table 2-1 for L-L and L-SH primary wave interactions. The term “Wave triplet” refers to the combination of two primary waves and the secondary wave that their interaction generates. Table 2-1 is a simplified version of the table provided by (Korneev and Demčenko 2014). First, wave triplets (i.e., the set of primary and secondary waves) with zero power flux are marked as “0”. Second, the phase matching condition was investigated parametrically, which is simplified by the nondispersive nature of bulk waves. This leads to categorizing wave triplets based on phase matching:

“N” denotes nonzero power flux, but not phase-matching;

“✓” denotes that for some frequencies and angles phase matching is satisfied;
“=” denotes where only collinear mixing gives phase matching.

One interesting attribute of Table 2-1 is that the interaction of L and SH bulk waves can generate L or SH waves at the difference frequency. This implies it is possible to generate two different types of waves out of a certain mixing pair. It is shown, for the first time in this study, that different guided wave types can be excited simultaneously by interaction of two primary guided waves (Table 2-2).

Table 2-1. Possible wave interactions for planar waves (Figure 2-1b) in an isotropic bulk material: zero power flux (0), power flux but not phase-matched (N), certain range of frequencies and mixing angles provide phase matching (√), and phase matching only occurs for collinear mixing (=).

<table>
<thead>
<tr>
<th>Primary waves</th>
<th>Secondary wave Resonance frequency (ω₁+ω₂)</th>
<th>Secondary wave Resonance frequency (ω₁-ω₂)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>L</td>
<td>SH</td>
</tr>
<tr>
<td>L-L</td>
<td>=</td>
<td>N</td>
</tr>
<tr>
<td>L-SH</td>
<td>✓</td>
<td>N</td>
</tr>
</tbody>
</table>

2.2.3 Parallel study on non-collinear guided wave mixing

Concurrent to this thesis work, Ishii et al were independently approaching the same problem in a different way (Ishii et al. 2018a). The fundamental physics of the problem is undeniably important to study. Investigation of guided wave interactions with arbitrary mixing angles can be inspirational from a theoretical and fundamental point of view.

Both this dissertation and Ishii et al.’s approach replace the scalar wave number term with its wave vector form in order to consider arbitrary wave propagation directions. Ishii et al.’s approach is to numerically determine at what interaction angle two primary waves generate combinational harmonics. Thus, they present maps that indicate the necessary interaction angle for various types of guided wave modes. Furthermore, within the context of full wave interaction (i.e., continuous plane waves) they investigate the amplitudes of the combinational harmonics generated. In the
present dissertation, the internal resonance criteria (i.e., phase matching and nonzero power flux) are formulated and then assessed analytically. While we conducted the parity analysis parametrically, Ishii et al. employ the internal resonance criteria numerically. Furthermore, since guided waves for nondestructive characterization are typically tone burst pulses there is a finite wave interaction (mixing) zone, which is discussed in chapter 3.

2.3 Guided wave selection

Wave mixing involves two primary waves that interact to generate a secondary wave, thus we refer to the set as a wave triplet. In order for the secondary waves to be cumulative, they need to be internally resonant. There are three independent variables that define each wave in a wave triplet:

(i) guided wave type and mode;
(ii) frequency;
(iii) wavevector direction.

The process of guided wave selection of a wave triplet for a plate application starts by determining the primary wave types that transfer power to the secondary mode using the nonzero power flux condition. Then the phase matching condition is employed to search the dispersion relations for the potential wave triplets based on frequency and direction. Finally, the potential wave triplets are compared in order to identify which ones have the strongest power transfer to the secondary mode. Nondestructive material characterization applications based on wave mixing call for maximizing the weak nonlinearity. Wave structures and available commercial transducers are important in the selection process. As an example, waves with dominant out-of-plane displacements are more visible to air-coupled and angle beam transducers. Therefore, a secondary wave field with dominant out-of-plane displacement can be selected since it is easily detectible with angle beam transducers.
2.3.1 Parity analysis

A parametric analysis employing the symmetry and antisymmetry features of Lamb-type and SH-type (shear-horizontal) guided waves in plates was performed. The MATLAB code is given in Appendix B. For convenience, we define shorthand notation that uses S and A for symmetry and antisymmetry respectively. Lamb waves are the default type; thus S and A alone refer to symmetric and antisymmetric Lamb waves. Likewise, SSH and ASH denote symmetric and antisymmetric SH waves respectively. Table 2-2 provides the results obtained by analyzing the nonzero power flux condition, $f_{n}^{\text{surf}} + f_{n}^{\text{vol}} \neq 0$, for all types of guided wave mixing. The primary wave pair types are given in the first column. The possible secondary wave types that meet the nonzero power flux condition are listed in the second column for any interaction angle $\theta$ and in the third column for non-collinear mixing. The results in Table 2-2 are significant and show that power flux is more limited for collinear wave mixing (either co-directional or counter-propagating) than it is for non-collinear wave mixing. That is, if we remove the possibility of collinear mixing then there are additional types of secondary waves (shown in the third column of Table 2-2) that have nonzero power flux. For example, the non-collinear interaction between waves of the same type has nonzero power flux to second order SSH waves, which is not the case for collinear interaction. Another example is that non-collinear interaction between S and SSH waves or A and ASH waves transfer nonzero power to secondary S wave types, which does not occur for collinear interaction. The same thing is true for bulk waves, as indicated in Table 2-1; L-L interaction can generate SH waves, while L-SH interaction can generate both L and SH waves at the difference frequency. There are also examples of guided wave mixing that closely resemble bulk wave mixing; i.e. when the wave structures are uniform, such as for the SH0 mode and for the S0 mode at low frequency (Figures 1-5 and 1-7). Thus, SH0-SH0, SH0-S0, and S0-S0 mixing results could approach those for bulk waves if the S0 mode is at low frequency.
Table 2. Secondary wave types of second order that satisfy the nonzero power flux condition (second and third columns) for all possible types of mutual wave interactions (first column).

<table>
<thead>
<tr>
<th>Mixing pairs</th>
<th>Arbitrary mixing angle</th>
<th>Non-collinear angle $(\theta \neq 0^\circ \text{ and } \theta \neq 180^\circ)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>S-S</td>
<td>S</td>
<td>S, SSH</td>
</tr>
<tr>
<td>A-A</td>
<td></td>
<td></td>
</tr>
<tr>
<td>SSH-SSH</td>
<td></td>
<td></td>
</tr>
<tr>
<td>ASH-ASH</td>
<td></td>
<td></td>
</tr>
<tr>
<td>S-A</td>
<td>A</td>
<td>A, ASH</td>
</tr>
<tr>
<td>SSH-ASH</td>
<td></td>
<td></td>
</tr>
<tr>
<td>S-SSH</td>
<td>SSH</td>
<td>SSH, S</td>
</tr>
<tr>
<td>A-ASH</td>
<td></td>
<td></td>
</tr>
<tr>
<td>S-ASH</td>
<td>ASH</td>
<td>ASH, A</td>
</tr>
<tr>
<td>A-SSH</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2.3.2 Phase matching condition

The analysis of phase matching requires knowing the dispersion relations for the guided waves of interest, as the dispersion curves for an aluminum plate are plotted in Figure 1-3 based on the material properties given in Table 2-3. Phase velocity dispersion curves are replicated in Figure 2-2 for illustrating the phase matching mode selection.

Table 2-3. Properties of the elastic aluminum plate.

<table>
<thead>
<tr>
<th>Material</th>
<th>Plate thickness (mm)</th>
<th>Density $(\text{kg/m}^3)$</th>
<th>Lame’s constants (GPa)</th>
<th>Landau-Lifshitz constants (GPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aluminum</td>
<td>1</td>
<td>2700</td>
<td>55.27</td>
<td>25.95</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\lambda$</td>
<td>$\mu$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$A$</td>
<td>$B$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$C$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>-351.2</td>
<td>-149.4</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>-102.8</td>
<td></td>
</tr>
</tbody>
</table>

The wave numbers are used to compute the phase velocities for the dispersion curves, while the introduced equations are based on wavevectors. Also, the phase matching condition for
directional guided wave mixing is a vector based on $\mathbf{K}_n = \mathbf{K}_a \pm \mathbf{K}_b$. Therefore, the absolute value of wavevectors must be used for assessing phase matching condition on dispersion curves,

$$
\mathbf{K}_n = k_n \mathbf{r}_n = |\mathbf{K}_a \pm \mathbf{K}_b| \mathbf{r}_n
$$

(2-38)

$$
k_n = \sqrt{k_a^2 + k_b^2 \pm 2k_a k_b \cos(\theta)}
$$

(2-39)

The phase matching condition is imposed on the $(\omega_n, k_n)$ dispersion relations for each propagating mode given a frequency and direction. The phase velocity of the corresponding wavenumber and frequency is

$$
c_n = \frac{\omega_n}{k_n} = \frac{\omega_a \pm \omega_b}{\sqrt{k_a^2 + k_b^2 \pm 2k_a k_b \cos(\theta)}}
$$

(2-40)

Note that the secondary wave field exists as a propagating wave only when its wavenumber (or phase velocity) corresponds to a propagating mode at the frequency defined as $\omega_a \pm \omega_b$.

The mode selection process for a representative wave mixing case (case 17 in Table 2-4) is described as:

a) For non-collinear interaction of two SH0 waves, the possible secondary wave fields are symmetric Lamb waves and symmetric SH waves (see the parity analysis in Table 2-2). We wish to generate a symmetric Lamb wave.

b) Choose frequencies of 1.5 MHz and 0.78 MHz for the primary SH0 waves having wave numbers of 3040 m$^{-1}$ and 1581 m$^{-1}$ in a 1 mm aluminum plate. The associated sum and different frequencies of 2.28 and 0.72 MHz are distinct from the second harmonics of the primary waves.

c) Now calculate wavenumbers at the sum and difference frequencies based on $\mathbf{K}_n = \mathbf{K}_a \pm \mathbf{K}_b$. For the mixing angle of $90^\circ$, the wave numbers are 3227 m$^{-1}$ for both sum and difference
frequencies. The corresponding phase velocities, based on Equation 2-40, are 4181 m/s (call it point 1) and 1320 m/s (call it point 2) for the sum and difference frequencies respectively.

d) Test whether the obtained combination of frequencies and wave numbers are propagating guided waves or not (does it lie on the dispersion curve?). The calculated frequencies and phase velocities are identified in Figure 2-2 Point 1, corresponding to the sum frequency, lies on the S0 Lamb wave mode. Therefore, it is a valid propagating wave. On the other hand, Point 2 in Figure 2-2 is not on a propagating mode. Hence, Point 2 results in bounded oscillations described by Equation 2-29. Thus, the mixing of SH0 waves with 0.78 and 1.5 MHz frequencies at 90º angle will generate a secondary S0 wave at the sum frequency.

Figure 2-2. Phase velocity dispersion curves. Two points associated with two primary waves (red circles), and points corresponding to sum (point 1) and difference (point 2) frequencies with related wave numbers and phase velocities (cross signs) are shown on the dispersion curves.

Due to the source influence (Rose 2014) associated with finite size transducers, the phase matching condition $K_n = K_a \pm K_p$ only needs to be satisfied approximately (Matsuda and Biwa 2011). Therefore, if the combination of $(\omega_n, k_n)$ lands in the vicinity of a dispersive curve, mode selection is acceptable as a legitimate wave triplet.
2.3.3 Selecting wave triplets

There are numerous wave triplets that satisfy both conditions for generation of internally resonant secondary waves given the different possible wave types, frequencies, and directions. As pointed out for Table 2-2, there are possible wave triplets for non-collinear wave interaction that were unknown prior to this work. A partial list of the potential wave triplets for a 1 mm thick aluminum plate is offered in Table 2-4.

Table 2-4. Selected wave triplets for second order nonlinear interaction in 1 mm thick aluminum plate. The angle $\theta$ is between the two primary waves and the angle $\gamma$ is between the secondary waves and Wave $a$.

<table>
<thead>
<tr>
<th>Set</th>
<th>$\theta$</th>
<th>$\gamma$</th>
<th>Wave $a$ Type &amp; Freq. (MHz)</th>
<th>Wave $b$ Type &amp; Freq. (MHz)</th>
<th>Wave $m$ Type &amp; Freq. (MHz)</th>
<th>$M_r \times 10^6$ (m$^2$)</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>L 1.0</td>
<td>L 1.0</td>
<td>L 2.0</td>
<td>1.89</td>
<td>Longitudinal bulk waves to provide a reference</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>S0 0.5</td>
<td>S0 0.5</td>
<td>S0 1.00</td>
<td>1.13</td>
<td>Low frequency self-interaction, but not perfectly phase matched</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>S1 3.6</td>
<td>S1 3.6</td>
<td>S2 7.2</td>
<td>15.4</td>
<td>High frequency self-interaction</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>SH0 1.65</td>
<td>SH0 1.65</td>
<td>S0 3.3</td>
<td>4.48</td>
<td>SH0 self-interaction, but group velocities are not matched</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>0</td>
<td>SH0 2.6</td>
<td>SH0 0.7</td>
<td>S0 3.3</td>
<td>2.81</td>
<td>Co-directional SH0 (many other frequency combinations are available)</td>
</tr>
<tr>
<td>6</td>
<td>180</td>
<td>0</td>
<td>SH0 1.72</td>
<td>SH0 0.34</td>
<td>S0 2.06</td>
<td>2.12</td>
<td>SH0 counter-propagating (many other frequency combinations are available)</td>
</tr>
<tr>
<td>7</td>
<td>180</td>
<td>0</td>
<td>S0 1.16</td>
<td>A0 0.26</td>
<td>A0 0.9</td>
<td>2.56</td>
<td>Three Lamb waves at low frequencies</td>
</tr>
<tr>
<td>8</td>
<td>0</td>
<td>0</td>
<td>A1 2.78</td>
<td>S0 1.10</td>
<td>A0 1.70</td>
<td>5.11</td>
<td>Lamb wave mixing – Difference harmonics</td>
</tr>
<tr>
<td>9</td>
<td>0</td>
<td>0</td>
<td>A0 1.46</td>
<td>S0 1.06</td>
<td>A1 2.52</td>
<td>4.63</td>
<td>Lamb wave mixing – Sum frequencies</td>
</tr>
</tbody>
</table>
### 2.3.4 Comparing mixing power

Table 2-4 shows that there are many possible wave triplets that generate internally resonant secondary waves. In order to aid in the selection of a wave triplet for a specific application, the mixing power from Equation 24, can be considered along with the type of transducer and the excitability of the wave structure for that type of transducer. The mixing power provides a quantitative comparison of the power flux to the secondary mode based on the wave interactions and is given in the seventh column of Table 2-4. It’s instructive to compare mixing power for internally resonant wave triplets; take Sets 1 and 3 for example. It seems reasonable to expect
longitudinal waves to generate the largest mixing power. However, that is not the case in Table 2-4 because mixing power depends strongly on both wave structure and frequency. If we consider L and S1 self-interactions at the same frequency, 3.6 MHz, then the mixing power for the L waves is 24.59×10^6 m^2, which is significantly larger than the mixing power of S1 waves (15.4×10^6 m^2).

The caveat for the mixing power comparison is that it does not account for the duration over which the wave interactions occur. In this chapter, guided waves are considered as continuous plane waves, whereas interaction of wave packets is of interest for most practical cases. For continuous plane waves, the primary waves can interact everywhere, enabling the “nonlinear forces” to act as distributed sources for the secondary mode. However, if the primary waves are actuated by tone bursts, then the wave interactions are confined to finite zones as depicted in Figure 2-3, with the size of the interaction zone being dictated by the duration of the tone burst, mixing angle, and travel distances as well as the group velocities and dispersion of the three interacting waves. Chapter 3 considers the effect of finite-size wave mixing zones, but first some additional remarks on the wave interaction triplets shown in Table 2-4 are appropriate.
2.3.5 Remarks on wave interaction triplets

**SH0 wave interactions (Sets 4-6, 15, 17, and 18)** – As indicated in Table 2-2, the interaction of two SH0 waves in any direction will transfer power to symmetric Lamb waves. Phase matching dictates which symmetric mode is generated, dependent upon the interaction angle and frequencies. From a measurement perspective, it is advantageous to have primary and secondary waves with different polarizations. In most cases it is the S0 mode that is phase-matched, which has the additional advantage that for many (relatively high) frequencies there is a large $u_z$ component at the plate surface, enhancing reception options. Furthermore, since the SH0 mode is nondispersive
there is an infinite number of mutual interaction combinations analogous to Sets 5 and 6. Assessing the mixing power helps determine which combination is optimal, and maximum mixing power is obtained for Set 5 when waves $a$ and $b$ have the same frequency (i.e., self-interaction).

**Co-directional guided wave interactions** (Sets 2-5, 8-14) – Both self and mutual interactions can generate secondary waves at the sum and difference frequencies. For self-interaction, the difference frequency is zero, hence it is known as the quasi-static pulse (Wan et al. 2018), and the sum frequency is the second harmonic. The advantage of co-directional interaction is that the wave mixing zone can be significantly larger than for counter-propagating and non-collinear waves. The disadvantage is that it is less suitable for sensing localized nonlinearity.

**One-way wave mixing** (Sets 11-14) – When co-directional waves generate a secondary wave that propagates back to the source it is called one-way mixing (Chen et al. 2014). This potentially enables transduction for actuation and reception to be at the same point, which is useful for interrogation of inaccessible regions.

**Counter-propagating waves** (Sets 6, 7, 15, 16) – Two transducers actuate waves that propagate toward the other transducer. The mixing zone size is determined by the duration of the tonebursts (and pulse spreading for dispersive modes), and its location can be moved by adjusting the time delays used for each of the primary waves. Thus, the material between the transducers can be scanned for localized nonlinearities. Leaving the transducers at fixed locations can eliminate inconsistencies associated with variable coupling conditions.

**Low frequency guided wave mixing** (Sets 2, 7) – Low frequency $S_0$, $A_0$, and $SH_0$ waves can be readily actuated and received, and it is easier to avoid actuation of unwanted modes. Furthermore, the wave structures are simple.

**Non-collinear wave mixing** (Sets 17-22) – While Table 2-4 only includes wave triplets that have an interaction angle of 90º, any angle is possible, and can be analyzed as described in this
dissertation. Ishii (Ishii et al. 2018a) analyzed the continuous range of angles. The mixing power and wave mixing zone size are important considerations.

**Zero group velocity excitation** (Sets 15, 16) – There are some Lamb wave modes that have a Zero Group Velocity (ZGV) for a limited range of frequencies. If a secondary wave is generated at a frequency where the group velocity is zero, then the wave energy is trapped at the mixing location, presumably making it easier to measure. There have been a few attempts to leverage Lamb waves at zero group velocity points for nondestructive testing (Grünsteidl et al. 2015). In addition to ZGV waves, standing waves are also considered as stationary waves exhibiting the feature that wave energy does not disperse into the plate (Gallot et al. 2015; Zheng et al. 2000). However, standing waves are typically not studied as a part of classical guided waves in plates, but in this dissertation the nonlinear generation of standing waves in plates is discussed in Appendix C.

### 2.4 Conclusions

The general theory of nonlinear elastic wave interactions was given by including wavevectors to account for the directivity of different waves. While this approach was followed for bulk waves previously, it was the first time for guided waves. A list of major contributions is given here:

- Formulation of nonlinear guided wave interactions based on wavevectors;
- Introduced perfect resonance conditions, nonzero power flux and phased matching conditions, with respect to wavevectors;
- Proposed a list of possible wave triplets for mutual interaction of guided waves in plates;
- Remarked on the utility of the listed wave triplets.

Mixing of guided waves in plates with arbitrary propagation directions shows the possibility of generating more wave types rather than what is possible for second harmonic generation from
self-interactions. This can be explored in detail in future studies. A list of potential wave triplets was introduced based on internal resonance conditions for cumulative wave mixing, whereas more mixing combinations are possible. The present study focuses on plates, but similar studies can be initiated on pipes.
Chapter 3

MUTUAL GUIDED WAVE INTERACTION OF FINITE SIZE WAVE PACKETS

3.1 Introduction

The theoretical efforts in the previous chapter to formulate equations for the elastic wave mixing are based on the interaction of two continuous wave fields. In other words, it is assumed that each planar wave field is a uniform harmonic oscillation of an elastic wave over an infinite domain. While in few cases large wave packets can be considered as the continuous wave fields (Jhang 2009; Jingpin et al. 2015), in reality, toneburst excitation is the most prevalent way to generate an ultrasonic wave. This means that the current form of the presented theory in Chapter 2 is incomplete because it does not comprise the interaction of finite-sized elastic wave packets. Particularity in case of mutual interaction between ultrasonic waves, the localized wave mixing is important for NDE applications (Lissenden et al. 2014; Tang et al. 2014). In addition, the dispersive nature of guided waves, which determines the group velocity of the wave packet, is not observable within continuous wave fields (Chillara and Lissenden 2015a; Müller et al. 2010; Xiang et al. 2016). Therefore, an analytical model is extended in this chapter to study and observe the role of limited size mixing packets and the mixing zone in which secondary wave fields are generated.

In this chapter the interaction of finite size wave packets is analyzed using a simplified analytical model. The role of group velocities and a finite size mixing zone are discussed, and it is described how selection of wave triplets can be influenced by them. The results of the analytical study are validated by finite element simulations. Since finite element simulations are widely used in this dissertation, the general approaches for finite element modeling are given in this chapter as
The proposed analytical model and the related results have been published in the Journal of Applied Physics (Hasanian and Lissenden 2018a).

3.2 Effect of finite wave mixing zone size: Group velocity

If we embrace the fundamental physics that power cannot be transferred from the primary waves to the secondary waves through the nonlinear surface tractions or body forces unless all three waves are physically interacting, it behooves us to develop models to represent the size of the wave interaction domain. In the case of continuous plane waves the wave mixing zone is large and not a concern, but if tonebursts are used, especially for nonzero interaction angles, the size of the wave mixing zone depends on the interaction angle, the number of cycles in the toneburst, the group velocities of the waves, and the dispersion effect. In this section a simple model to help understand the effect of group velocity is presented. While the model is presented in 1D for simplicity, it also applies to non-collinear wave interactions.

3.2.1 Simplified analytical model

Group velocity is defined to be $\frac{d\omega}{dk}$, which in turn has been shown to be the speed at which the energy of the wave packet travels. In other words, the group velocity is the speed of the wave packet’s envelope. For a given phase velocity, the group velocity can have any value; positive, zero, or even negative. Assume that the wave packet can be represented by a continuous wave, shaped by an envelope that travels independently at its own speed, i.e., the group velocity. Since the nonlinear effects that we are trying to promote work best for waves with low dispersion, the pulse spreading associated with dispersion is ignored. Therefore, as illustrated in Figure 3-1, the wave packet is the linear combination of a continuous harmonic wave and a time-dependent
envelope that shapes the wave packet and propagates at the group velocity. For 1D wave propagation in the $X$-direction, a Gaussian envelope is used to form the wave packet $\mathbf{u}^p$ associated with the continuous wave $\mathbf{u}$,

$$\mathbf{u}^p(X,t) = \mathbf{u}(c_p, \omega, X, t)e^{-W_0[X-X_0(t)]^2}$$  \hspace{1cm} (3-1)

where $W_0$ defines the width of the wave packet, and $X_0$ gives the location of its center. $X_0$ travels at the group velocity. The width of the wave packet is defined to be the distance between the wave packet head and tail, which are taken to be the points where the displacement is 1% of its maximum value (Figure 3-1).

Figure 3-1. Illustration of the wave packet formed by superimposing an envelope onto a continuous harmonic function.

Now consider the secondary wave field $\mathbf{u}_{ir}^{inc}$ generated by the interaction of primary waves $a$ and $b$, which can be computed by Equation 2-36 specialized for wave propagation in the $X$-direction,

$$\mathbf{u}_{ir}^{inc} = \frac{1}{2}[M_{ir} \text{Amp}(U_a)\text{Amp}(U_b)\mathcal{U}_{ir}(Z)d_{int}e^{i(k_{ir}X-\omega_{ir}t)} + c. c.]$$  \hspace{1cm} (3-2)

The mixing power, wave structure, and amplitudes of waves $a$ and $b$ at the current time are the inputs, and $d_{int}$ is the spatial distance over which the waves interact during the time step $\Delta t$. Let the
secondary wave packet be formed by a Gaussian envelope that travels at the group velocity of the
secondary waves analogous to Equation 3-1. The width of the envelope is determined by the
interacting waves $a$ and $b$. The second assumption is that the superposition principle is applicable
to both the primary and secondary wave fields. Therefore, the total secondary wave field becomes
the summation of all secondary waves, $\mathbf{u}_{ir}$, generated by the wave interactions for each time step
$\Delta t$,

$$
\mathbf{u}_{ir}^p = \sum_{inc=1}^{n} \mathbf{u}_{ir}^{inc}
$$

(3-3)

where $n = t_{total}/\Delta t$ and $t_{total}$ is the total duration of the wave mixing. As a result, the secondary wave
packet is the summation of all infinitesimal secondary wave packets that are generated during the
sequence of time steps. This simple analytical model can provide insight into the effect of the wave
mixing zone size.

Consider wave triplet Set 4 in Table 2-4, which is self-interaction of the SH0 mode at 1.65
MHz that generates the internally resonant S0 mode. But the S0 mode at 3.3 MHz has a lower group
velocity (2700 m/s) than the SH0 mode (3100 m/s). SH0 wave packets with $W = 27$ mm are
generated using Equation 3-1 as primary waves in a 1 mm aluminum plate. Equations 3-2 and 3-3
simulate the generation and propagation of the secondary wave field. Figure 3-2 shows the in-plane
amplitudes of the SH0 waves, in addition to the out-of-plane displacements of the secondary S0 in
snapshots of the wave field at the plate top surface taken at 5, 20, and 50 $\mu$s. The results indicate
that while cumulative generation occurs based on the interaction of the primary waves, the SH0
wave packet separates from the S0 wave packet due to its higher group velocity. As a result, the
wave interactions at the peak of the secondary wave field cease and the amplitude of the secondary
S0 wave stops increasing. However, the generation of S0 waves continue in a new place, which is
right in front of the S0 wave packet. This leads to an extension of S0 wave packet rather than a
further increase in peak amplitude. After this saturation point, the secondary wave amplitude
remains constant. Most importantly, the maximum amplitude correlates directly with the width (W) of the primary wave packet. In other words, the longer the wave packet, the higher the amplitude of the secondary wave packet. This phenomenon has been observed in an experimental investigation conducted by (Xiang et al. 2016) for Lamb waves in an aluminum plate.

![Figure 3-2](image_url)

Figure 3-2. Propagation of primary SH0 waves [a and b] at 1.65 MHz (W = 27 mm) and secondary S0 waves at 3.3 MHz in the spatial domain at: (a) 5 µs, (b) 20 µs and (c) 50 µs. SH0 wave self-interaction begins at X = 50 mm.

3.2.2 Comparing codirectional mixing methods

The preceding example in the previous section (Figure 3-2) shows that group velocity mismatch resulting in a finite-sized wave mixing zone limits the amplitude of the secondary wave. The cumulative behavior ends when wave interactions cease. To put this result into context, the four self-interaction wave triplets shown in Table 2-4 (i.e., Sets 1-4) are compared. The simplified
analytical model and a finite element model are compared in order to show that the simplified model results are reasonable. Details of the finite element modeling and methodology are given in section 3.4. The self-interaction cases are selected for clarity of presentation of the results, analogous results are expected for mutual wave interactions. We found the second ratio of the mixing power, which is \( \frac{\text{Amp}(U_{ir})}{\text{Amp}(U_a)\text{Amp}(U_b)} \) in Equation 2-35, to be a useful metric when plotted as a function of propagation distance. For self-interaction, it reduces to \( \frac{\text{Amp}(U_{ir})}{[\text{Amp}(U_a)]^2} \), which is more commonly written as \( A_2 / A_1^2 \) and is commonly known as the relative nonlinearity parameter. The results from both analytical and finite element models are shown in the combination of Table 3-1 and Figure 3-3, which clearly indicates that when the group velocities match (Cases D and E) the secondary waves are cumulative, and when they do not match (Cases A and B) the relative nonlinearity parameter approaches an asymptote. Note that in Cases D and E of Table 3-1 that the pulse width is labeled N.A., which means not applicable. A pulse width is used in the analysis, but because the three waves all have the same group velocity the pulse width is irrelevant. Case C is not phase matched but is attractive because the S0 mode at 0.5 MHz is relatively easy to actuate preferentially (Wan et al. 2016).

The finite element results shown in Figure 3-3 validate the analytical model. The results reflected in Figure 3-3 are conclusive and support the interpretation of results given earlier in the chapter on the effect of mixing zone size and group velocity combinations. As stated before, the total amplitude of the generated secondary wave field has a strong and direct relationship with the size of the mixing zone size. According to Table 3-1, both Cases A and B in Figure 3-3 are the same except they have different pulse widths. However, the total displacement fields of Cases A and B are different, which relates to the length of the wave packets, or mixing zone size in other words. Thus, the larger the mixing zone, the larger the amplitude of the secondary displacement wave field, provided the phase matching condition is satisfied. In the case of a nearly phase-
matched condition, Case C, the size of the mixing zone must be optimized. Otherwise, a larger mixing zone may have a negative effect (Chillara and Lissenden 2015a).

Table 3-1. Description of the self-interaction wave triplets as results are shown in Figure 3-3.

<table>
<thead>
<tr>
<th>Case</th>
<th>Wave Triplet</th>
<th>Primary Frequency (MHz)</th>
<th>Mixing Power $\times 10^6$ (m$^2$)</th>
<th>Pulse Width (mm)</th>
<th>Phase Matched</th>
<th>Group Velocity Matched</th>
<th>Model Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>SH0-SH0→S0</td>
<td>1.65</td>
<td>4.48</td>
<td>27</td>
<td>Yes</td>
<td>No</td>
<td>Analytical and Finite Element</td>
</tr>
<tr>
<td>B</td>
<td>SH0-SH0→S0</td>
<td>1.65</td>
<td>4.48</td>
<td>83</td>
<td>Yes</td>
<td>No</td>
<td>Analytical</td>
</tr>
<tr>
<td>C</td>
<td>S0-S0→S0</td>
<td>0.5</td>
<td>1.13</td>
<td>73</td>
<td>No</td>
<td>No</td>
<td>Finite Element</td>
</tr>
<tr>
<td>D</td>
<td>L-L→L</td>
<td>1.0</td>
<td>1.89</td>
<td>N.A.</td>
<td>Yes</td>
<td>Yes</td>
<td>Analytical and Finite Element</td>
</tr>
<tr>
<td>E</td>
<td>S1-S1→S2</td>
<td>3.6</td>
<td>15.4</td>
<td>N.A.</td>
<td>Yes</td>
<td>Yes</td>
<td>Analytical and Finite Element</td>
</tr>
</tbody>
</table>

Figure 3-3. The relative nonlinearity parameter plotted as a function of wave propagation distance for five different cases (A-E) of self-interaction shown in Table 3-1. Lines show analytical results, where symbols demonstrate finite element results.
3.3 Subtraction method

Mutual interaction of guided waves provides a large number of possible mixing modes and opportunities for innovative experimental methods. Relative to the self-interaction of guided waves, mutual interactions reduce the effect of instrumentation nonlinearities by separating the material and system nonlinearities. Since second and third order harmonic nonlinearities associated with self-interactions are in most cases convoluted with primary signals, extraction of material-related nonlinearities and the interpretations are complicated. The approach in this dissertation is to facilitate experiments to obtain material nonlinearity through direct measurements and less complicated post-processing techniques. The mutual interaction of ultrasonic waves offers particular methods which significantly simplifies the extraction and interpretation of nonlinear features (Demčenko et al. 2012; Li et al. 2018; McGovern and Reis 2014).

3.3.1 Defining the difference signal

In the subsequent numerical simulations and experiments to assess material nonlinearity via mutual wave interaction we collect data from three tests:

- Test A+B: both waves \(a\) and \(b\) are excited in the plate;
- Test A: only wave \(a\) is excited;
- Test B: only wave \(b\) is excited.

Test A, Test B, and Test A+B correspond to generating waves \(a\) and waves \(b\) separately and then generating both waves \(a\) and \(b\) simultaneously. Therefore, we can observe the wave fields and frequency content in the presence and absence of mutual interactions. Also, we define the *Difference signal* which comes from the subtraction of Test A and Test B from Test A+B:

\[
D_{\text{diff}} = D_{\text{Test A+B}} - D_{\text{Test A}} - D_{\text{Test B}} \tag{3-4}
\]
where $D_{\text{diff}}$ is the residue of the displacement fields, which ought to enable visualization of the displacements from the mutual interaction of Waves $a$ and $b$. Note that $D$ is the displacement field in the direction of a specific axis $X$, $Y$, or $Z$. In finite element analysis, self-interaction is easily modeled by making Test A and Test B identical. This approach was used to obtain the results in Figure 3-3. Therefore, Figure 3-3 presents the actual maximum displacements of each wave field.

### 3.3.2 Shear Horizontal and Lamb wave isolation

The subtraction method isolates the nonlinearity associated with mutual wave interaction. As such, the *difference* signal should contain only the material nonlinearity associated with the mutual interaction of waves $a$ and $b$, which is only in Test A+B. However, energy transform from primary waves to secondary waves decreases the amplitude of waves $a$ and $b$. These slight changes in the primary amplitudes are evident in the *difference* signal. Therefore, in addition to mutual interaction nonlinearities in *difference* signal, the residue of the primary wave fields can be observed in the *difference* signals. When the primary waves are SH0, then out-of-plane *difference* signal does not involve with displacements of primary wave fields. As a result, the *difference* signal does not contain any residue of the primary wave packets.

Consider interaction of SH0 waves which leads to the generation of Symmetric Lambs waves (S). Referring to Figure 3-4a, the displacement fields of the SH0 primary waves and S secondary waves have different polarizations, which enables the isolation of S waves from the SH waves in reception. Thus, evaluating the difference signal of the out-of-plane displacement fields enables us to isolate nonlinearities associated with mutual interactions to a very high degree, since displacements related to the primary waves are excluded by the nature of the problem. Using angle beam transducers coupled by gel, or air-coupled transducers, the SH waves propagate past the
receiver without leakage, while the S secondary mode with dominant out-of-plane displacement is collected by the transducer (Figure 3-4b). This is an important characteristic of SH-SH wave interactions.

![Diagram](image)

Figure 3-4. (a) Different polarization of SH0 waves and S Lamb waves in a plate, (b) isolated S wave from SH0 wave in reception.

### 3.4 Finite element methodology

The finite element models are established in COMSOL Multiphysics V. 5.2 where nonlinear material behavior is included through a hyperelastic model. The Landau-Lifshitz third order elastic constants are given in Table 2-3 along with Lame’s constants. Geometric and material nonlinearities are both included in the simulation. Transient dynamics nonlinear ultrasonic wave propagation simulations require both element sizes and time steps to be small, thus an efficient model is highly desirable. So, mesh sizes and time steps must be as fine as practical to provide precise generation of secondary waves. In this regard, mesh sizes are at least 10 times smaller than the shortest wavelength in the simulation. In addition, time steps are defined to have at least 40 calculation points per the shortest time period.

The applied displacement loading is designed to create wave packets. Thus, boundary conditions, edge of models, are imposed by Gaussian shaped pulses of different cycles and
frequencies. Since we mainly worked on SH0-SH0 wave interactions, a brief description of the modeling method is given here. The details of each model are given in subsections of that specific subject. Since particle displacements of SH and Rayleigh Lamb waves are in different polarizations a 3D model is required. In cases of codirectional and counter-propagating wave mixing, to keep the model compact, periodic boundary conditions are applied along the lateral boundaries.

In-plane and out-of-plane displacements, corresponding to SH0 and S waves respectively, are collected in each case. The subtraction method is used to visualize the secondary wave field that facilities the amplitude measurements and group velocity calculations. Moreover, the Fast-Fourier Transform (FFT) is used to evaluate the frequency content of wave packets. In some cases, wave structures are visualized and group velocities are calculated in order to verify the guided wave mode.

3.5 Conclusions

The nonlinear interaction of guided wave packets is studied in this chapter, while ultrasonic waves are mainly considered as continuous waves in previous nonlinear interaction works. A list of the significant contributions in this chapter is given below:

- The superposition principle is defined for nonlinear wave field;
- A simplified analytical model is presented to observe the formation of secondary wave field as a result of mixing two finite size wave packet;
- The effect of wave packet dimensions and group velocity mismatching condition is discussed conceptually.

The analytical model shows the importance of the mixing zone size, or wave packet size in other words. The results of finite element simulations validate the presented analytical model.
Introducing the analytical model, for the first time, enabled the prediction of wave packet interactions. As a result, the effect of group velocities and finite mixing zone are observed. This gave a new insight about the role of different group velocities and the effect of mixing zone size. The observations conclude that limited size of wave packets and group velocity mismatched condition can limit the maximum reachable amplitude of secondary wave field, but it cannot have a destructive effect. In other words, limited mixing zone and group velocity conditions set a saturation limit on secondary wave field amplitude. Therefore, to get to higher amplitudes, larger wave packets (or mixing zone) are necessary. This concept is considered in the mode selection process to collect the most appropriate wave triplets for more feasible experiments. Some basic techniques and instructions, that are used in the finite element simulations, are introduced in this chapter. For the rest of this dissertation, the main focus is on SH0-SH0 wave mixing because of the special merits of the method. Besides the straightforward generation of SH0 waves, the different polarization of the primary and secondary wave fields is a very useful feature.
Chapter 4

MIXING OF CODIRECTIONAL SHEAR HORIZONTAL WAVES

4.1 Introduction

The present chapter investigates mixing codirectional SH0 waves, whose interaction generates S0 Lamb waves at the sum frequency. Localized wave mixing methods, such as counter-propagating, provide advantages for the detection of localized material degradation by scanning, but the wave mixing zone is necessarily small, potentially limiting sensitivity. On the other hand, codirectional mixing of SH0 waves has the advantage of a large wave mixing zone, resulting in more power flux to the secondary S0 waves that should provide better sensitivity to situations where material degradation occurs globally. Codirectional wave mixing has been proved to be a truly strong tool for nonlinear material characterization (Kim et al. 2014; Liu et al. 2013a; b; Matlack et al. 2012; Morlock et al. 2015). The main common problem with current techniques is the reception of primary and secondary wave fields at the same time. In other words, primary and multiple harmonics are merged into one signal which needs post processing to separate the frequencies and corresponding values. Usually this process is complicated, especially considering the convoluted signals with system nonlinearities such as second harmonics generated by amplifiers. SH0-SH0 wave mixing generates a secondary wave field with a different polarization. Therefore, the right choice of measurement tools facilitates the separate reception of the primary and secondary wave fields. This alone can significantly enhance the quality of measurement and lead to a straightforward method to evaluate material nonlinearity and integrity.
This chapter begins with codirectional mode triplet selection and a description of the methodology to be implemented. Then, the finite element analysis and related discussions are provided, mainly to observe the phenomenon of secondary wave field generation in the absence of group velocity matching. The corresponding experimental technique is demonstrated with discussion and explanations. In addition, nonlinear measurements are conducted on aluminum specimens with various fatigue levels. The experimental program described in this chapter has been submitted to the journal Ultrasonics (Shan et al. 2019) for possible publication.

4.2 Mode selection

Based on the parity analysis, the codirectional mixing of SH0 waves, either by self or mutual interactions, generates Symmetric Lamb waves (S). The phase matching condition is described thoroughly in Chapter 2, and here it is explained how multiple mode triplets are available for codirectional SH0 wave mixing. The phase matching condition is considered based on wavevector calculations as follows,

\[ \mathbf{K}_m = \mathbf{K}_a \pm \mathbf{K}_b \] (4-1)

in order that the wavelengths are phased. So, wavevectors are replaced by wavenumbers which can be positive or negative,

\[ k_m = k_a \pm k_b \] (4-2)

where wavenumbers can be expressed as the terms of angular frequency and phase velocities as,

\[ \frac{\omega_m}{c_m} = \frac{\omega_a}{c_a} \pm \frac{\omega_b}{c_b}. \] (4-3)
For codirectional interaction of SH0 waves, there is just one way to satisfy the phase matching condition, which is when the phase velocity of the S wave is the same as the SH0 waves. Thus, the Equation 4-3 is simplified to following term:

\[ \omega_m = \omega_a \pm \omega_b \]  

(4-4)
Based on Equation 4-4, there are multiple candidates for codirectional SH0 mixing, as far as the frequency combination of the primary waves sums to the frequency of the S0 Lamb wave with the phase velocity of the SH0 wave (3.1 mm/μs for aluminum). Thus, the \( f_m d \) product would be around 3.3 to 3.4 MHz-mm based on the elastic properties of aluminum. Figure 4-1 illustrates selected mode combinations for the current study, where the S0 secondary wave has a 3.3 MHz-mm center frequency for the aluminum plate. While any sum and difference combination of frequencies that can set 3.3 MHz-mm is possible, we chose 1.65 MHz-mm self-interaction. In addition, combination of SH0 waves with 1.0 MHz-mm and 2.3 MHz-mm is selected for mutual interaction. Table 4-1 summarizes the wave triplets used for finite element simulations and experiments. Experimental study on mutual interaction of selected SH0 waves is done on a 3.125 mm aluminum plate, so final selected frequencies are divided by 3.125 to accommodate with thickness of the plate (as shown in Table 4-1 line two).

Table 4-1. Self- and mutual interaction SH0 mixing modes investigated by finite element simulation and experiments respectively.

<table>
<thead>
<tr>
<th>Test mode</th>
<th>Study</th>
<th>Plate thickness</th>
<th>Wave (a)</th>
<th>Wave (b)</th>
<th>Wave (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Self-interaction</td>
<td>Finite element</td>
<td>1 mm</td>
<td>SH0 1.65 MHz</td>
<td>-</td>
<td>S0 3.3 MHz</td>
</tr>
<tr>
<td>Mutual-interaction</td>
<td>Experiments</td>
<td>3.125 mm</td>
<td>SH0 0.75 MHz</td>
<td>SH0 0.32 MHz</td>
<td>S0 1.07 MHz</td>
</tr>
</tbody>
</table>

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4.3 Finite element simulation

Finite element modeling of self-interaction of SH0 waves, based on Table 4-1, is conducted to observe the mixing phenomenon considering the different group velocities of SH0 and S0 waves (3.1 mm/µs for SH0 waves and 2.39 mm/µs for S0 3.3 MHz-mm). In addition, results validate the developed analytical model for Chapter 3, when compared with Figure 3-2.

4.3.1 Model description

The principal details of finite element simulations are given in Chapter 3. Figure 4-2 illustrates the 3D model for self-interaction of the SH0 mode to generate the S0 mode. Periodic boundary conditions on the front and back faces help to minimize the model size. The maximum element size is 0.1 mm and the maximum time step is 0.01 µs. The displacement signal with 50 nm amplitude is applied to the boundary of the plate to generate the SH0 waves. Displacement wave fields are
acquired at the top surface of the model where the S0 mode at 3.3 MHz has significant out-of-plane displacement as shown in Figure 1-6c.

![Finite element model](image)

Figure 4-2. Finite element model which is SH0 self-interaction to generate the S0 mode. Wave fields are acquired from the top surface of the aluminum plate.

### 4.3.2 Results

The $Y$-displacement components of the primary SH0 wave at $X = 25$ mm are shown in Figure 4-3. Also, $Z$-displacement components, out-of-plane displacements, are shown in Figure 4-3 which represent the S0 wave at 3.3 MHz. The corresponding frequency spectra of the wave packets in Figure 4-3 are provided in Figure 4-4, using an FFT calculated over the time window of 7-17 µs.

![Frequency spectra](image)

Figure 4-3. In-plane and out-of-plane displacements on the top surface of the aluminum plate at $X = 0.025$ mm.
Figure 4-4. Frequency content of wave packets shown in Figure 4-3.

The Y- displacement components of the primary SH0 wavefield are shown in Figure 4-5 at 10, 20, and 50 µs. The Z- displacement components of the S0 secondary wave field are also shown in Figure 4-5. The combination of the calculated frequency spectra and group velocities confirm that the secondary wave generated by SH0 nonlinear interactions is the S0 Lamb mode as in Liu et al (Liu et al. 2013a).

The Figure 4-5 shows that the primary and secondary wave fields separate due to their group velocity mismatch. As they separate, power is transferred to the S0 mode over a smaller domain, which prevents the secondary wave amplitude from increasing further. The excellent agreement between Figure 3-2 (simplified analytical model) and Figure 4-5 (finite element simulation) indicates that the simplified analytical model is realistic, which was discussed previously in Chapter 3. The exclusion of dispersive pulse spreading in the analytical model is evident in the wave packets at 50 µs, where the finite element simulation indicates some pulse spreading and a very minor reduction in amplitude.
Figure 4-5. SH0 in-plane primary displacement field and S0 out-of-plane displacement field at (a) 10 µs, (b) 20 µs, and (c) 50 µs.

The maximum measured out-of-plane displacement for each location is shown in Figure 4-6, which indicates the initial cumulative increase in the S0 mode amplitude. However, this cumulative increase reaches a saturation point. Since the S0 wave accumulates along its propagation in the X direction to a certain point, this limited cumulative behavior can be defined by the slope of the fitted line in Figure 4-6. In the next sections, it is shown that this slope is sensitive to material’s nonlinearity and therefore it is useful for early damage detection.
4.4 Experiments

Experiments are conducted to assess the cumulative behavior of the secondary S0 Lamb wave. Wave triplets presented in the second line of Table 4-1 are used for experiments based upon mutual-interactions, so that the subtraction method can be applied to visualize the secondary wave field. In this chapter, the details of the experimental setup and the corresponding results are discussed.

4.4.1 Methodology

All experiments conducted in this dissertation follow some common rules to perform an efficient and successful nonlinear ultrasonic test. Moreover, SH0 waves are actuated by Magnetostrictive Transducers (MST) in both Chapters 4 and 5. Therefore, before illustrating the codirectional SH0 wave mixing setup, the general approach to the experiments is given.
4.4.1.1 General experimental approach

The general technique for conducting the experiments is pivotal for measuring material nonlinearities. While the details of each experiment are given in subsequent sections, the general considerations are discussed here. The underlying goal associated with investigating secondary waves generated by mutual wave interactions is to assess their potential to characterize material nonlinearity at frequencies away from integer multiples of the excitation frequencies. One of the main concerns is whether the Signal to Noise Ratio (SNR) of the secondary wave is large enough to make high quality measurements. A list of measurement practices intended to provide an acceptable SNR is given below:

- Using amplifiers and oscilloscopes with the minimum internal noise level. The RITEC RAM 5000 Snap (Warwick, RI, USA) system is used to generate two signals. The RITEC output amplifier is used to gain the received signals.

- The number of signals averaged together can significantly improve the SNR, so 512 signals are averaged in the current study. Also use the maximum possible sampling rate, since it defines the resolution of the frequency content. In the current study, a minimum sampling rate of 250 MHz is used.

- Isolate of the system from environmental electrical noise. The ground connection significantly reduces the noise level, especially on metallic samples.

- Sensors can be a significant source of noise, whereas some specific piezoelectric sensors introduce minimum noise levels. Select sensors that are appropriate for nonlinear material evaluation. Assess the noise level of the transducers prior to conducting the experiments.

To generate SH waves, Magnetostrictive Transducers (MSTs) are used that are similar to those used in previous work (Lissenden et al. 2014; Liu et al. 2013a). A meander electric coil placed on top of the iron-cobalt magnetostrictive foil adhesively bonded to the top of the plate forms the MST. Figure 4-7a illustrates schematically the MST setup for SH wave generation. SH0 wavelength is defined by the coil spacing.
Figure 4-7. Experimental setup for SH0 wave mixing: (a) Schematic view of wave actuation by MST, (b) View of actuation of two waves on one foil for codirectional SH0 mixing (c) Photo showing AC sensor, dual MSTs, gel, and putty for codirectional mixing.

Material and electronic nonlinearities from the input amplifier, foil, plate’s material under the foil, and the epoxy glue could shadow the material nonlinearity from interested part of the plate. These aforementioned nonlinearities are addressed as source nonlinearities. Assuming that we use a sensor that solely collects out-of-plane displacement, e.g. air-coupled and angle beam transducers, the source nonlinearities would pass through the sensor and interrupt the true measurements (if major out-of-plane displacement exists). On the other hand, SH waves do not express any out-of-
plane displacements but just in-plane. Therefore, a gel layer covering the near field of MST actuator can absorb the unwanted out-of-plane components, while in-plane SH wave packet remains untouched (As illustrated in Figure 4-7a). This simple but effective method enabled us to diminish a lot of out-of-plane nonlinearities that are generated at the MST. In the this and the next chapters, the role of gel to recognize the secondary wave field is demonstrated.

4.4.1.2 Experimental setup for codirectional SH0 mixing

The experimental setup is illustrated schematically in Figure 4-7b. Two meander coils were placed on an iron-cobalt foil in order to generate $f_a=0.75$ MHz and $f_b=0.32$ MHz SH0 waves in a 3.125 mm thick 2024-T3 aluminum plate. The meander spacings of electrical coils are 3.6 mm and 10 mm respectively, corresponding to the SH0 wavelengths. A time delay of 15 µs was applied to the signal sent to MST A so that SH0 wave packets travel as one. A photograph of the setup is given in Figure 4-7c. A Lorentz force Electromagnetic Acoustic Transducer (EMAT) was used to measure the in-plane displacements associated with the SH0 waves. Air-coupled (AC) sensor (1MHz center frequency, Ultran, USA) received the out-of-plane displacements corresponding to the 1.07 MHz S0 Lamb in a 3.125 mm aluminum plate, which has significant out-of-plane displacement (similar to Figure 1-6c for 1mm aluminum plate). The EMAT is a noncontact transducer that is used with zero liftoff and manually positioned for these tests. The AC sensor is mounted on a fixed stage. The AC orientation angle is based on Snell’s law to find the best angle receiving associated S0 wave. By sliding the plate sample relative to the AC sensor, signals at different propagating distances are measured. The sampling rate of the system was set to 250 MHz, and 512 signals were averaged together. Damping materials were applied to the edges of the plate to reduce the boundary reflections.
As discussed in the previous section, the SH0 wave mixing is almost immune to instrument nonlinearities, when AC sensors or angle beam transducers are used for reception. However, in the current setup, some serious nonlinearities appear to be injected into the plate due to electric and magnetic inductions and material nonlinearities of MSTs. The close proximity of the two electrical coils could result in undesirable inductions. In addition, the generated waves interact in the foil, adhesive, and aluminum plate just under the transducer. Due to the suspected generation of nonlinearities at the MST, a layer of gel is applied close to the MSTs to damp the initial out-of-plane displacements (Figure 4-7b). As a result, the AC sensor would not receive the initial out-of-plane nonlinearities, since they are damped by the gel. This ensures the acquisition of the S0 Lamb wave associated with the material nonlinearity in the plate.

4.4.2 Results

4.4.2.1 Linear and nonlinear responses

First, the linear responses of the two individual SH waves $a$ and $b$ at different sensing points are captured with the EMAT as shown in Figure 4-8, exemplified by the signals acquired with the edge of the EMAT housing located at 20, 50 and 80 mm from the end of the gel. The amplitudes of waves $a$ and $b$, denoted as $A_a$ and $A_b$, can be considered independent of propagation distance between 20-80 mm, since the amplitude variations for waves $a$ and $b$ are just 5% and 2% respectively. Thus, the values of $A_a$ and $A_b$ are assumed to be constant over the range of measurements.
Figure 4-8. The linear SH wave signals at different sensing positions: (a) SH0 0.75 MHz wave \( a \); (b) SH0 0.32 MHz wave \( b \).

After measuring the primary wave amplitudes, the AC transducer is used to receive out-of-plane components of the nonlinear Lamb waves. Using the subtraction method (Equation 3-4), the nonlinear Lamb wave signal is obtained, shown in Figure 4-9a, which was received 90 mm from the end of the gel. The FFT of the windowed signal in Figure 4-9a gives the frequency spectrum plotted in Figure 4-9b, which indicates that the signal propagates with a center frequency of 1.07 MHz – the sum frequency of the primary SH0 waves, \( f_a + f_b \).
Figure 4-9. Difference signal computed from signals received by the AC sensor 90 mm away from the end of the gel: (a) time-domain signal; (b) frequency spectrum of the windowed wave packet in (a). Peak value of signal in 1.075 MHz is $A_{ab}$, the amplitude of the S0 Lamb wave.

4.4.2.2 Cumulative effect of the S0 Lamb wave

Experiments are conducted to assess whether the nonlinear S0 Lamb wave is cumulative for this internally resonant wave triplet, as expected for the SH0 primary waves and S0 secondary waves employed. First, tests were conducted with the gel layer in place. According to the analytical and finite element results, the linearly cumulative effect stops after a certain propagation distance due to the group velocity mismatch. Therefore, eight sensing points were selected from 20-90 mm away from the gel in 10 mm intervals, as shown by the red dots in Figure 4-7c. After extracting the difference signals with the subtraction method at each point, FFT analysis was used to determine the amplitude $A_{ab}$ at the sum frequency. The result of Figure 4-10a indicates accumulation of the
difference signal at 1.075 MHz that correlates to the S0 Lamb wave. Next, the gel layer was simply
removed without changing anything else. Following the same process, the amplitudes $A_{ab}$ at
different locations are plotted in Figure 4-10b. In this case, the amplitudes are larger, but no
cumulative effect is observed, suggesting that the plate material nonlinearity is mixed with other
nonlinear sources. This demonstrates the importance of the gel in the system design. A line fit to
the measured points in Figure 4-10a characterizes the nonlinearity. Next, the slope of the fitted line
is used as the material nonlinearity index.

![Graph](image.png)

Figure 4-10. Amplitude of out-of-plane displacement obtained from the difference signal
computed from the AC sensor as a function of propagation distance: (a) with the gel; (b) without
the gel.
4.4.2.3 Sensitivity to tensile fatigue damage

According to the theoretical analysis, the slope of the line fitted to the accumulated secondary wave amplitude is directly related to the material nonlinearity. This is identified in the current study as the mixing power in Equation 2-35. Thus, the slope of the fitted line must be normalized by the amplitude of the primary waves $A_a$ and $A_b$ in order to have a universal value to compare different plates. In the current study, we observed that variations of $A_a$ and $A_b$ are considerably smaller than changes in amplitude of the secondary wave field $A_{ab}$. Therefore, the slope of the line is directly used for material state assessment.

A set of aluminum plate samples were subjected to increasing numbers of tensile loading cycles (maximum stress = yield strength, stress ratio = 0.1), creating fatigue damage but no visible cracks. First, cyclic loading was applied to a 3.125 mm thick 2024-T3 aluminum plate sample with a servohydraulic test frame until it fractured. The gage section of the samples have straight sides and no notches, resulting in a uniform stress field until the time when a macroscale crack initiates. Cycling to 25%, 50% and 75% of the fatigue life was applied to three identical plate samples. The ultrasound measurements described in this Chapter were taken on a pristine sample and the three fatigued samples. The gel layer was implemented in all tests and measurements were made at the same eight sensing points described above (i.e., 20-90 mm from the end of the gel). Using the same procedure, the amplitude $A_{ab}$ of the sum frequency is plotted as a function of the propagation distance in Figure 4-11a and linear regression is used to determine the slope for each of the four samples. Then, the slopes are normalized with respect to the values obtained from the pristine sample and plotted as a function of percentage of fatigue life. Four independent sets of measurements were made, then the mean and standard deviation are plotted in Figure 4-11b.

The results show a dramatic increase in the slopes; around 20% increase after the plate is fatigued to just 25% of its life. The slope increases 45% for the sample fatigued to 75% of its life.
No cracks were observed by visual inspection of the samples. The changes in slopes demonstrate the effectiveness of the proposed nonlinear method for early damage detection and evaluation.

Figure 4-11. (a) Amplitude $A_{ab}$ obtained from the difference signal computed from the AC sensor as a function of propagation distance for aluminum plates cycled to 0, 25, 50, and 75% of the fatigue life; (b) normalized slopes for the four samples.

4.5 Conclusions

Codirectional SH0-SH0 wave mixing mode selection offers various combination of frequencies for a certain resonance condition. This chapter is dedicated to comprehensive study of
SH0-SH0 codirectional guided wave mixing throughout finite element analysis and experiments. The primary milestones accomplished are:

- Codirectional SH0-SH0 wave mixing modes are identified;
- The generation of the secondary wave field in the absence of group velocity matching is observed and discussed;
- A unique experimental methodology is proposed, which enables the separate reception of SH0 waves and Lamb waves;
- The sensitivity of the proposed method to early fatigue damage is demonstrated.

The finite element analysis was done to demonstrate that the self-interaction of SH waves generates an S0 secondary wave that is cumulative up to a certain distance. The results validate the simplified analytical model and show the effect of mismatched group velocities. Experiments are performed to validate the proposed method as well as demonstrate its use for nondestructive material characterization. The experimental observations indicate cumulative generation of S0 Lamb waves as the result of codirectional interaction of two SH0 waves. The experiments require gel to eliminate the undesired sources of nonlinearity. With the gel in place, the cumulative effect of the secondary S0 Lamb wave is verified and the corresponding slope is extracted and further used to characterize the material status of the fatigue specimens. Due to the different polarizations between the primary SH0 waves and the secondary S0 wave, the nonlinear waves can be separated from the primary waves to simplify experimental post processing. Results demonstrate the proposed method provides high sensitivity to early fatigue damage, which makes it promising for future early damage detection applications.
Chapter 5

MIXING OF COUNTER-PROPAGATING SHEAR HORIZONTAL WAVES

5.1 Introduction

Counter-propagating wave mixing can be used to scan a plate by moving the mixing zone, whose position is controlled by adjusting the triggering time delay between the two transducers (Lissenden et al. 2014). As a result, scanning and imaging are possible with the transducers and sensors at a fixed position, which means more consistent and reliable tests. Once counterpropagating ultrasonic waves interact to each other, the mixing zone is defined by the size of the wave packets. Thus, the mixing zone dimension can be variable based on the size of the inspection region (Cho et al. 2019; Tang et al. 2014). Since primary waves propagate in different directions, the nonlinear generated wave travels independently from the primary wave packets. Consequently, it would be possible to detect the nonlinear wave packet in the received time domain signals (McGovern and Reis 2015). Prior to this study, only counter-propagating bulk waves were studied; and guided waves were not considered due to the complexity of selecting the right set of guided waves and frequencies. Here, the mixing possibilities are addressed and demonstrated through finite element studies and experimental investigation.

In this chapter, the two counter-propagating guided wave mixing case are introduced and mode selections are explained. Next, the finite element tool is utilized to investigate the wave propagation and frequency content. The nonlinear wave generations are validated by assessing the frequency content, wave structures, and group velocities. An experimental procedure is carefully addressed including applying a gel layer to examine the source of nonlinearities. The experimental results are meticulously investigated to ensure the origin of generated nonlinearities are from the material
rather than instrumentation. The main body of this chapter was published in the Journal of Applied Physics, as it is acknowledged to be the first experimental study on mutual interaction of guided waves in a plate (Hasanian and Lissenden 2017).

5.2 Mode selection

The mode selection methodology is elaborated on in Chapter 2, so here some mode triplets given in Table 2-4 are investigated. Wave triplets 6 and 15 are selected for finite element simulation, and wave triplet 6 is considered for experiments. SH0 guided waves are again selected as the primary wave fields because they are non-dispersive and can be actuated as a single mode excitation. Nonlinear S0 generation, wave triplet 6, uses two relatively low frequency SH0 waves (1.72 MHz and 0.34 MHz) to generate S0 wave at 2.06 MHz frequency on a 1 mm aluminum plate. S0 wave at 2 MHz has a significant out-of-plane displacement that makes it detectible by a variety of different sensors. Nonlinear generation of Zero-Group-Velocity (ZGV) Lamb wave at S1 2.84 MHz-mm in an aluminum plate is considered. Since ZGV Lamb wave fluctuates locally and do not propagate their signal is expected to have a long duration. Therefore, measuring the displacement signals of nonlinear ZGV waves is straight-forward, at least in concept (Grünsteidl et al. 2015).

5.2.1 SH0-SH0-S0 wave triplet

Wavevector calculations play an important role when the mixing waves have different propagating directions. Counter-propagating guided wave mixing means that one of the waves has a negative wavenumber to account for its propagation in the opposite direction. By following this simple rule, and imposing the phase matching condition, the SH0-SH0-S0 wave triplet was discovered (mode 6 in Table 2-4). Figure 5-1 shows the three waves that comprise the triplet on the
phased velocity dispersion curves for an aluminum plate. The wave structure of S0 wave at 2 MHz-mm is given in the inset the Fig 5-1 and also in Figure 1-6a. The dominant out-of-plane displacement of the wave in top surface is one of the top reasons it is selected. However, S0 at 2 MHz-mm is dispersive which a negative aspect of this wave triplet.

Figure 5-1. Wave triplet 6 in Table 2-4 shown on the dispersion curves for a 1 mm aluminum plate. Inset shows the structure of the S0 mode at 2MHz (Figure 1-6a).

5.2.2 ZGV wave triplet

Zero-group-velocity modes appear when the slope of the dispersion curve changes from positive to negative values. The S1 Lamb mode at 2.84 MHz is a point known to have zero group velocity and is shown in Figure 5-2. If the S1 mode at 2.84 MHz were to be generated as a secondary wave from wave mixing it might be quite easy to measure since it does not propagate. So what primary modes can be mixed together to generate it? As indicated in Table 2-4 case 15, two counter-propagating SH0 waves whose frequencies add up to 2.84 MHz will satisfy both internal resonance criteria.
5.3 Finite element simulations

The finite element modeling approach is explained in Chapter 3. The COMSOL Multiphysics V 5.2 models are built with specific mesh resolutions for each case. Wave propagation in the simulation is in the X direction, while particle displacements of SH and Lamb waves are in the Y direction and X/Z directions respectively, thus a 3D model is required; but to keep the model compact, periodic boundary conditions are applied along the lateral boundaries as shown in Figure 5-3 to ensure a planar wavefront.

![Figure 5-3. Schematic of the 3D strip finite element model with periodic boundary conditions. Displacements in X and Y directions are in-plane displacements, while in the Z direction they are out of plane displacement.]()
5.3.1 Nonlinear generation of the S0 secondary wave

5.3.1.1 Model description

To ensure the accuracy of results and efficient generation of the secondary wave field, maximum element size of 0.1 mm and time steps of 0.02 µs are used. Applying displacement boundary conditions at opposite ends of the model, SH0 waves are actuated as shown in Figure 5-4. The applied displacement loading is designed to create pulses of equal time duration. Thus Gaussian-shaped pulses of 8 and 40 cycles at 0.34 and 1.72 MHz respectively are employed. The four positions where wave signals are received are shown in Figure 5-4.

5.3.1.2 Results

The primary SH0 wave packets for Test A+B are shown in Figure 5-5 from point 4 where primary waves arrive separately. Then Figure 5-6 shows the \( u_z \) displacements at point 2 in the mixing zone as out-of-plane displacements that contain S0 2.06 MHz. The form of the \( u_z \) wave packet received at point 2 calls for in-depth analysis, thus in addition to Tests A+B, A, and B, a fourth test called ‘Linear A+B’ with waves \( a \) and \( b \) propagating in a linear elastic material is simulated. The Fast Fourier Transform (FFT) of the windowed region shown in Figure 5-6 is used.
to analyze results from each of the four tests in the frequency domain. But first, consider the spectral results from the primary SH0 waves shown in Figure 5-7 (instead of the $u_z$ displacement) obtained from the same time window. Only peaks at the excitation frequencies of the SH0 waves at 0.34 and 1.72 MHz exist since shear waves do not generate second harmonic shear waves.

Figure 5-5. Test A+B SH0 displacements, $(u_y)_b$ and $(u_y)_a$, at point 4 corresponding to SH0 0.34 MHz and 1.72 MHz respectively.

Figure 5-6. Test A+B out of plane displacement, $u_x$, at point 2 inside the mixing zone. The dashed window represents the time window for FFT.
Figure 5-7. Frequency content of SH displacements, $u_y$, for Tests A+B, A, B, and Linear A+B at point 2.

Turning our attention back to the out-of-plane displacements shown in Figure 5-6, consider the spectral results plotted in Figure 5-8. Test A+B has four second order harmonic peaks at $2f_b$, $f_a - f_b$, $f_a + f_b$, and $2f_a$ in addition to the quasistatic pulse at zero frequency. Test A shares the $2f_a$ peak and Test B shares the $2f_b$ peak, and these peaks come from the SH0 waves generating S0 waves at the second harmonic frequency due to shear-normal coupling as shown by (Liu et al. 2013a) and described by (Chillara and Lissenden 2015b). The peak at $2f_a$ is significantly higher than at $2f_b$ because the phase matching condition is nearly satisfied at that frequency, but not at $2f_b$. These peaks are eliminated by the subtraction method, and therefore are of no further interest. Neither Test A nor Test B share the peaks at $f_a \pm f_b$ because these peaks come from mutual wave interaction. The peak for Test A+B is approximately four orders of magnitude higher than the Test A value at $f_a + f_b$. Test Linear A+B has no quasistatic pulse and no peaks at the second harmonic frequencies; the very small peaks that occur at $f_a$ and $f_b$ are apparently associated with shear-coupling induced by the nonlinear strain-displacement relation as discussed in (Chillara and Lissenden 2015b). It is noteworthy that the Test A+B peak at $f_a + f_b$ is roughly two orders of
magnitude higher than the peak at \( f_a - f_b \). Calculated vectors in Table 2-4 indicate that the secondary S0 2.06 MHz wave at frequency \( f_a + f_b \) will propagate in the 0\(^\circ\) direction, while there is no propagating mode at \( f_a - f_b \). The Test A+B results plotted at points 1, 2, and 4 in Figure 5-9 show a \( f_a + f_b \) peak at point 4 outside the wave mixing zone, but not at point 1. In fact, the peak at point 4 is larger than the peak at point 2 in accordance with the S0 mode at \( f_a + f_b \) being cumulative in the mixing zone. It seems that the \( f_a - f_b \) peak at point 2 shown in Figure 5-8 is from the modulation that is confined to the mixing zone.

![Figure 5-8](image.png)

Figure 5-8. Frequency content of out of plane displacements, \( u_z \), for Tests A+B, A, B, and Linear A+B at point 2.
The role of phase matching can be demonstrated by exciting wave $b$ at 0.5 MHz instead of 0.34 MHz, which results in the combinational frequencies $f_a \pm f_b$ not satisfying Equation 2-30. Figure 5-9 shows the frequency content at points 2 and 4 for Test A+B for the primary frequencies 1.72 and 0.5 MHz. Clearly, there is nonlinear modulation in the mixing zone (Point 2), but this nonlinearity is unable to propagate outside the mixing region (to point 4) because there are not propagating waves that satisfy the wave vector matching required. Note that the frequency spectrum received at point 4 has peaks at $2f_a$ and $2f_b$ as described above, plus one at 2 MHz that appears to be due to the bandwidth of the tone burst excitation (with 0.5 MHz center frequency) including some energy at 0.34 MHz, which then leads to generation at the sum frequency of 0.34 + 1.72 MHz.

A subtraction method can be used to compute the group velocity of the secondary wave and confirm that it is the S0 mode. The subtraction method isolates the nonlinearity associated with mutual wave interaction, which now can only come from the nonlinear material constitutive relation and the nonlinear strain-displacement relation. The Test A+B, A, B, and Difference signals

![Frequency content of out of plane displacements, $u_z$, for Test A+B at points 1, 2 and 4.](image)
are plotted in Figure 5-11. Now the time of flight can be determined from the *Difference* signals computed at two different points; and knowing the distance between the points, the group velocity can be computed. Figure 5-11 shows the *Difference* signal, $D_{diff}$, at point 4; and group velocity is calculated based on difference in time-of-arrivals at points 3 and 4, where the distance between points 3 and 4 is 40 mm. Thus, the group velocity is $(40 \text{ mm})/(13.94 \mu\text{s}) = 2869 \text{ m/s}$, which agrees well with the group velocity of 2860 m/s predicted by dispersion analysis for the S0 mode at 2.06 MHz.

![Graph showing frequency content](image)

Figure 5-10. Frequency content of out of plane displacements, $u_z$, for Test A+B at points 2 and 4, where $f_b = 0.5 \text{ MHz}$.
5.3.2 Nonlinear generation of the S1 ZGV secondary wave

5.3.2.1 Model description

The same periodic boundary conditions in the previous section are used to minimize the model size for this simulation. The schematic of the model, which is 200 mm long and 1 mm thick, is shown in Figure 5-12. SH0 waves are similarly excited at opposite ends of the plate, and displacements from different tests are collected from the center of the mixing zone.
5.3.2.2 Results

In-plane and out-of-plane displacements from the center of mixing zone are shown in Figure 5-13. The S1 ZGV Lamb wave has considerable out of plane displacement component (Figure 1-6d). Therefore, this can be used to distinguish primary waves from the secondary wave field. To inspect if the ZGV wave is generated and stays in the mixing zone, the out-of-plane displacements are depicted for a point inside and another one outside of the mixing zone. Figure 5-13 shows difference signals at the top surface inside and outside of the mixing zone. The difference signal outside the mixing zone is essentially zero because the ZGV S1 mode at 2.84 MHz does not propagate, but rather it remains in the mixing zone.
Figure 5-13. In-plane displacements in the center of the mixing zone for Test A+B and difference signals inside and out of the mixing zone.

While nonlinear displacements inside the mixing zone show stationary oscillations, outside of it they are nearly zero. Once SH0 waves leave the mixing zone, the ZGV mode is trapped inside the zone and continues to oscillate. Therefore, in practice, it is possible to select a particular window when only the ZGV mode is oscillating in the mixing zone. This could simplify the measurement of material nonlinearity without complication from instrumentation nonlinearities.

5.4 Experimental studies of nonlinear S0 generation

In this section wave triplet 6 of Table 2-4 will be investigated using laboratory experiments to confirm that the second harmonic S0 mode is measurable. The underlying goal associated with investigating secondary waves generated by mutual wave interactions is to assess their potential to characterize material nonlinearity at frequencies away from integer multiples of the excitation...
frequencies. One of the main concerns is whether the signal to noise ratio of the secondary wave is large enough to make high quality measurements. In the case of mode triplet 6 the primary waves (SH0 mode) and secondary waves (S0 mode) have different polarities, thus the S0 mode will have to be measurable on its own as opposed to the measurement of the distortion of a primary wave (as is the case for self-interaction measurements). Therefore, by using angle beam or air-coupled transducers we assure to explicitly receive displacements associated to S0 Lamb wave and not SH0 displacements.

5.4.1 Methodology

Some introductory aspects of our experimental procedure are given in Chapter 4. A polished 1 mm thick 7075-0 aluminum plate 900 mm long and 50 mm wide is used for the waveguide. Mounted 500 mm apart are two magnetostrictive transducers (MSTs). To generate the SH0 wave \( a \) at 1.7 MHz a 1.8 mm coil spacing is used. Likewise, to generate the SH0 wave \( b \) at 0.31 MHz a 9 mm coil spacing is used. Unlike the planar wave finite element simulations, the 50 mm wide MSTs actuate SH waves having a curved wavefront. Frequency tuning optimized the SH0 wave packets and resulted in a slight deviation from the frequency values given in Table 2-4 for mode triplet 6. Because the transducers excite finite size domains in the dispersion curve space rather than points, the slight deviation in frequencies is believed to have no appreciable effect on the generation of the secondary S0 wave mode.
5.4.2 Angle beam sensor

5.4.2.1 Setup and procedure

The contact transducer and wedge are coupled to the waveguide with sonographic gel, and a dead load is used to maintain a consistent couplant thickness as the transducer is moved to different points for each of Test A+B, Test A, and Test B. The gel does not attenuate the SH0 waves because they have shear displacements. The experimental setup is shown in Figure 5-14a. The Ritec RAM 5000 Snap (Warwick, RI, USA) system is used to generate two signals that are sent through matching networks and then to the transducers. Collected signals from the angle beam transducer are amplified by 40 dB and then 700 signals are averaged together and recorded from the oscilloscope. A 2.25 MHz center-frequency contact transducer mounted on a 36-degree acrylic wedge is placed between the transmitters to measure the secondary S0 wave signals at the sum frequency of 2.01 MHz, where the S0 mode has a large out-of-plane displacement component at the surface as shown in the inset of Figure 5-2. A photo of the experimental setup is provided in Figure 5-14b.
5.4.2.2 Experimental results

Tests A+B, A, and B are conducted with the angle beam transducer positioned at different points relative to MST $a$ as shown in Figure 5-14. The SH wave signals received at the opposite MST location are shown in Figure 5-15, where the signal before 100 µs is just electromagnetic interference. The signals received when the angle beam transducer is located at 380 mm, which is outside the mixing zone (214-286 mm), are shown in Figure 5-16 for Tests A+B, A, B, as well as the computed *Difference* signal based on the subtraction method. Wave reception at $f_a$ in Test A and $f_b$ in Test B is unexpected because the gel-coupled angle beam transducer receives the out of plane displacement component, while the MST generates an in-plane displacement component. In previous MST measurements (Liu et al. 2013a), it is found that the SH wavefront is curved and that the finite size of the receiver led to measurement of small $X$-direction displacement components of the ($Y$-direction dominated) SH wave. The same could be happening here. While the signals from
Tests A+B, A, and B can contain this component, geometric nonlinearities, and instrument nonlinearities, the *Difference* signal should contain only the nonlinearity associated with mutual wave interaction. The important feature of Figure 5-16d is that the *Difference* signal is quite clean except between 174 and 214 µs, which corresponds to the arrival of the S0 secondary wave at the sum frequency \((f_a + f_b)\).

![Graph](image)

Figure 5-15. SH0 \(u_t\) signals received at 0.5 m by MST. The signals at 50 µs are electromagnetic interference.
Figure 5-16. Time domain signals for Tests A+B, A, B, and the Difference from the angle beam transducer at 380mm location.

The fast Fourier transform is applied to each signal shown in Figure 5-16 using the 40 µs window size and location indicated there, with the results shown in Figure 5-17a for Test A+B, Figure 5-17b for Test A and Test B, and Figure 5-17c for the Difference signal. Notice in Figure 5-17b that the frequency for Test A is $f_a$ and that the frequency for Test B is $f_b$. This is consistent with the discussion about a curved SH wavefront having a small $X$-direction displacement component, but not understood why they are received by the angle beam transducer. Most importantly, the frequency spectrum for the Difference signal is completely dominated by the frequency peak at $f_a + f_b$ associated with mutual wave interaction. Conspicuous in its absence is a peak at the difference frequency $f_a - f_b$; but a peak is not expected here because the data were acquired at a point outside the mixing zone and because no propagating mode exists at that frequency.
A simple test was devised to demonstrate that the signal measured at $f_a + f_b$ is indeed the S0 mode. The top surface of the plate was covered with ultrasonic gel (Ultragel II) because the S0 mode at 2 MHz is known to have a large $u_z$ component, out-of-plane displacement, at the surface. Thus, the S0 mode will be quickly attenuated as it leaks into the gel. Three zones for gel placement are identified in Figure 5-14: (1) from MST $a$ to the mixing zone, (2) from the mixing zone to the angle beam transducer, and (3) from the angle beam transducer to MST $b$. The results shown in Figure 5-17 are based on the plate top surface being bare in Figure 5-17a-c, gel being on the top surface in Zones 1 and 3 in Figure 5-17d-f, and gel being on the top surface in Zone 2 in Figure 5-17g-i. The difference signal is received when there is no gel in Zone 2 (Figures 5-17c and 5-17f), but it is not received when there is gel in Zone 2 (Figure 5-17i). Thus, the interpretation that the difference signal is the S0 secondary mode generated in the mixing zone and propagating to the right is correct. The source of the nonlinearity will be discussed further in next section.
Figure 5-17. Frequency content obtained from angle beam transducer for Tests (a) A+B, (b) A and B, (c) and Difference signals. Tests were repeated with the plate top surface in Zones 1 and 3 covered with gel for Tests (d) A+B, (e) A and B, and (f) Difference signals and again for gel only in Zone 2 for Tests (g) A+B, (h) A and B, and (i) Difference signals.

Now the Short Time Fourier Transform (STFT) is used to estimate the group velocity of the $f_a + f_b$ harmonic wave. A Tukey time window size of 40 µs is used for the STFT. The amplitude of the 2.01 MHz frequency is plotted as a function of time for Tests A+B, A, and B in Figure 5-18. The small peak prior to 100 µs in all three tests is associated with the electromagnetic interference. The dominant peak at approximately 150 µs only occurs for Test A+B and is therefore attributed to mutual wave interaction. The time of flight (marked in Figure 5-18) of the energy at frequency $f_a + f_b$ from Test A+B enables the group velocity to be estimated provided the travel distance is known. The time of flight results obtained from STFT of Test A+B signals acquired at five different
locations are plotted in Figure 5-19. Linear regression indicates that the slope of Figure 5-19 is 
\[ c_g = 2758 \text{ m/s}, \] which is within 4% of the value obtained from dispersion analysis for the S0 wave mode at 2.01 MHz (2860 m/s).

Finally, the STFT is applied to signals obtained from Test A+B at 15 different points and the frequency peak at \( f_a + f_b \) is plotted in Figure 5-20 as a function of spatial position. We interpret Figure 5-20 based on the second order harmonic \( f_a + f_b \) being the S0 mode that is generated in the mixing zone and propagates to the right. The four points at locations before 240 mm represent the noise floor, the six points between 240 and 300 mm are in the mixing zone and the high amplitudes are associated with modulation, while the five points at locations greater than 300 mm represent propagation of the S0 mode to the right. The decreasing S0 wave amplitude after 300 mm is most likely due to diffraction, attenuation, and lack of a nonlinear driving force outside of the mixing region.

![Figure 5-18. STFT-obtained spectrum for frequency \( f_a + f_b \). The peaks before 100 µs are related to electromagnetic interference.](image)

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Finally, the STFT is applied to signals obtained from Test A+B at 15 different points and the frequency peak at \( f_a + f_b \) is plotted in Figure 5-20 as a function of spatial position. We interpret Figure 5-20 based on the second order harmonic \( f_a + f_b \) being the S0 mode that is generated in the mixing zone and propagates to the right. The four points at locations before 240 mm represent the noise floor, the six points between 240 and 300 mm are in the mixing zone and the high amplitudes are associated with modulation, while the five points at locations greater than 300 mm represent propagation of the S0 mode to the right. The decreasing S0 wave amplitude after 300 mm is most likely due to diffraction, attenuation, and lack of a nonlinear driving force outside of the mixing region.

![Figure 5-18. STFT-obtained spectrum for frequency \( f_a + f_b \). The peaks before 100 µs are related to electromagnetic interference.](image)
Figure 5-19. Time of flight of the second order harmonics $f_a + f_b$ from five different measurement locations from Test A+B. Group velocity is estimated by the slope of the line.

Figure 5-20. Amplitude of second order harmonic versus location of the receiver

5.4.2.3 Discussion on the source of nonlinearity

The evidence that material nonlinearity is being measured and not artifacts of instrument nonlinearities is overwhelming. The most robust support for this claim is due to the primary and secondary waves having different polarities, and that the angle beam transducer does not receive
the primary SH0 waves. Therefore, the instrument nonlinearity associated with actuating the primary waves is not received by the angle beam transducer. Granted, there are small signals received in Test A and Test B at the primary frequencies, but those are of the same order as the \( f_a + f_b \) peak and therefore could not generate it. In some Tests A+B the \( f_a + f_b \) peak is even larger than the peaks at \( f_a \) and \( f_b \). Although obvious from Figures 5-17b, 5-17e, and 5-17h, it is worth stating that the signals from Tests A and B contain no peaks at \( f_a + f_b \), thus the peaks at \( f_a + f_b \) in Tests A+B must be from mutual wave interaction.

The energy in the Difference signal only occurs at the sum frequency (Figures 5-17c and 5-17f) and this energy is traveling at the group velocity of the S0 wave mode at 2.01 MHz (Figure 5-19). In accordance with the S0 wave mode being generated in the mixing zone and traveling to the right, the energy is largest in the mixing zone due to the local modulation and then as it travels to the right its amplitude decreases consistent with diffraction and attenuation (Figure 5-20). Furthermore, Figure 5-21 compares the Difference signal in the time domain at three locations: 280, 350, and 430 mm and shows that the Difference signal is independent from the signals of Tests A and B since they travel with different velocities. For instance, Figure 5-21c shows that the Difference signal arrives significantly later than the signal in Test A. The same thing was shown in Figure 5-16 where the S0 wave (Figure 5-16d) arrives later than the wave package in Test A (Figure 5-16b).

Table 5-1 summarizes the results of the experiments with gel placed on the plate surface (Figure 5-17) by showing the decrease in peak amplitude values at the primary and secondary frequencies relative to the comparable test on bare aluminum plate. The results shown in Table 5-1 for the peak at \( f_a \) are consistent with MST A generating a small amplitude wave having a \( u_z \) component as well as the primary SH0 wave, since the presence of gel in Zone 1 and Zone 2 (separately) both partially reduce the peak. However, the complete loss of a peak at \( f_a + f_b \) for gel in Zone 2 alone, along with
only a nominal reduction for gel in Zone 1, can only be explained by the $f_a+f_b$ peak being the S0 mode generated in the mixing zone.

Table 5-1. Percentage decrease in frequency peak relative to bare surface for Test A+B.

<table>
<thead>
<tr>
<th>Test cases</th>
<th>$f_b = 0.31$ MHz</th>
<th>$f_a = 1.7$ MHz</th>
<th>$f_a+f_b = 2.01$ MHz</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gel in Zones 1&amp;3</td>
<td>0%</td>
<td>39%</td>
<td>16%</td>
</tr>
<tr>
<td>Gel in Zone 2</td>
<td>0%</td>
<td>23%</td>
<td>98%</td>
</tr>
</tbody>
</table>

Figure 5-21. Time domain signals from different receiver locations: (a) 280 mm, (b) 350 mm, (c) 430 mm. Signal amplitudes are offset for clarity. Note that the signal from Test A+B is not shown.
The same experimental methodology was applied to the sample without using the Ritec receiver gain, i.e., the angle beam transducer signal was sent directly to the oscilloscope without amplification. Due to high sensitivity of the sensor and excellent SNR, reception of secondary wave at the sum frequency is possible in the absence of an amplifier. The spectral results were unchanged, confirming that the signal at 2.01 MHz is not caused by the Ritec receiver gain. Figure 5-22a represents an sample of measured signals without output gain, that perfectly shows the nonlinear wave packet in Difference signal.

If the measured nonlinearity comes from the instrumentation, it should exist at some other frequencies rather than just at selected frequencies for guided wave mixing based on the phase matching condition (Liu et al. 2012). To evaluate this, the amplifier of the received signal is bypassed. The angle beam transducer is located at 280 mm (Figure 5-20). A wide range of frequency combinations were tested, where appropriate meander coils are selected for each frequency assuring the effective generation of SH0 waves. It is confirmed that the distinct Difference signal shows up only at the frequencies selected based on the phase matching condition. Three selected cases along with the actual phase matched condition experiment are shown in Figure 5-22. Only the Difference signal in Figure 5-22a has a distinct wave packet because it is phase matched, the other non-phase matched cases do not have a wave packet. If the source of the A-scan signal were instrument related, there would be no bias to 2 MHz as is the case for the S0 mode.
5.4.2.4 Sum frequency sensitivity to thermal damage

The most certain way to confirm that the measured nonlinearity is related to the material nonlinearity is to change the material nonlinearity by changing the microstructure. There are many methods to alter the microstructure (Kim et al. 2014; Li et al. 2012; Lissenden et al. 2014; Marino et al. 2016), where we used localized thermal aging. The grain size and precipitation in aluminum alloys changes due to heat treatment (Marino et al. 2016). One more experiment demonstrates the sensitivity of the secondary S0 wave to localized microstructure changes. An aluminum plate was heated locally for 3 hours at 327°C as illustrated in Figure 5-23 to cause thermal aging, and then
after it cooled down ultrasonic tests were conducted with the transducers at the locations shown. Within one test setup, two different time delays were applied to the excitation signals such that mixing occurred in zone 1 (inside where local heating occurred) and then in zone 2 (outside where heating occurred). Then, all transducers were removed and replaced, and replicate data acquired.

Figure 5-23. Test setup for localized heating; (a) size and location (in mm) of heated region and wave mixing zones and (b) difference signal from zone 1 normalized with respect to difference signal from zone 2, and (c) frequency spectra for two difference signals from zone 1.

Figure 5-23b shows difference signals obtained from zone 1 normalized with respect to difference signals from zone 2 for the same setup for both before and after localized heating. The normalized difference signal is larger after heating, indicating that microstructure has changed.
Also shown in Figure 5-23c are frequency spectra for two difference signals from zone 1, which clearly show the dominance of the 2.01 MHz S0 mode and increase of that after heating.

5.4.3 Air-coupled sensor

Air-coupled transducers can sense airborne acoustic waves that leak out of the plate at specific angles and frequency ranges. Since it is a perfect noncontact method and does not pick up in-plane displacements, it is a decent receptor to avoid electronic nonlinearities when SH0 waves are mixing (Morlock et al. 2015). Thus, the angle beam transducer was replaced with an air-coupled transducer and the results were essentially the same. The only difference being that no signal was received in Test B at $f_b$, because the ACT was oriented to receive right-propagating waves rather than left-propagating waves. This means that the source of the measured secondary wave at the sum frequency is not a result of the interaction the signals at $f_a$ and $f_b$ inside the electronics.

The results of the 2 MHz center frequency air-coupled transducer shown in Figure 5-24 confirm the validity of the results acquired by the angle-beam transducer. In Figure 5-24, the Difference signal shows a strong signal at 2 MHz, which is the S0 wave. The Difference signals in both experiments, angle beam and air-coupled, are large compared to the out-of-plane signals in Tests A and B, which indicates that material, rather than instrument, nonlinearity is the source of the signal. It is worth mentioning that the first signal packages in Figure 5-24 is just electromagnetic interference.
Figure 5-24. A-scan signals and corresponding frequency content from an air-coupled transducer. Frequency content is calculated by Fast Fourier Transform obtained from the rectangular window.

5.5 Conclusions

A wavevector analysis of guided waves was performed that identified the counter-propagating Shear-Horizontal waves (SH0 at 1.72 MHz and SH0 at 0.34 MHz) whose interaction generates a symmetric Lamb wave at the sum frequency (S0 at 2.06 MHz). Also, the nonlinear generation of ZGV Lamb waves are considered through interaction of two SH0 waves. The significant contributions of this chapter are:

- Introduced the wave triplets suitable for counter-propagating wave interactions;
- Finite element study and validation of selected wave triplets with a unique modeling approach;
- Designed the experimental methodology, and proposed nonlinear assessment techniques;
- Successfully made nonlinear measurements using different types of sensors.

The finite element simulations in both cases confirm the nonlinear generation of the theoretically expected Lamb waves, which propagate or fluctuate independently from the primary wave packets. Experiments are conducted to assess the feasibility of wave triplet 6 from Table 2-4, so then generation of S0 2.07 MHz is observed. Using the *difference* signal it is shown for the first time that the secondary guided wave generated by the mutual interaction of two guided waves is measurable on its own. The different polarization the S0 Lamb wave at the sum frequency from the primary SH0 waves ensures the reception of material nonlinearity isolated from the instrumentation system nonlinearly. Various experimental tests were performed to make sure the secondary waves are material-related rather than from instrumentation nonlinearities. The thin gel layer applied over the aluminum absorbs the out-of-plane displacements while in-plane displacements are untouched. Therefore, the gel layer is capable of damping S0 2.007 MHz Lamb wave, which is used to isolate the source of nonlinearities. The demonstrated methods can be used/developed in future works to identify the source of sum/difference harmonics.

Our exploration was extended to employing new sensors such that Polyvinylidene Difluorine (PVDF) sensors. PVDF sensors are capable to receive both SH0 and S0 wave at the same time, so to measure the normalized material nonlinearity. Furthermore, they are more affordable and easier to work with. The proposed scanning methodology was applied to image a plate with localized damaged nonlinearity. The results are published in the recent paper (Cho et al. 2019), and details can be found in Cho’s dissertation (Cho 2017).
Chapter 6

MIXING OF NON-COLLINEAR SHEAR HORIZONTAL WAVES

6.1 Introduction

The interaction of bulk waves with arbitrary mixing angle has been studied through theory, numerical analysis, and experiments (Demčenko et al. 2015; McGovern and Reis 2015). The most recent theory, which is reviewed in this work, was developed by (Korneev and Demčenko 2014). More importantly, non-collinear bulk wave mixing was investigated by both finite element studies and experiments (Blanloeuil et al. 2015; Croxford et al. 2009; Demčenko et al. 2015; McGovern and Reis 2015; Zhang et al. 2016b). While the method promises straightforward detection and isolation of material nonlinearity, the complex nature of the test setups makes it difficult to development the method to more practical extensions.

Non-collinear guided wave mixing is investigated theoretically in this work (and related publications), and by Ishii et al in a parallel study (Hasanian and Lissenden 2018a; Ishii et al. 2018a), although experiments are not yet reported. This chapter presents a finite element simulation of non-collinear SH wave mixing in a plate. It is primarily based on the work published recently by (Hasanian and Lissenden 2018a). In another parallel study, Ishii has addressed the FEA of non-collinear Lamb wave mixing (Ishii et al. 2018b), whereas focus of this study is SH guided wave mixing. In this chapter, we demonstrate a modeling approach to efficiently and clearly simulate SH guided wave mixing with 90º mixing angle. The main challenge is to define the correct boundary conditions to ensure the single mode SH0 wave excitation. After defining the boundary conditions, the simulations are performed and the nonlinear wave field is visualized.
6.2 Mode selection

We demonstrated earlier in this thesis that by non-collinear guided wave mixing the excitation of more types of secondary guided waves is possible, compared with collinear guided wave mixing (Table 2-2). The analogous non-collinear mixing types have been observed in bulk waves, which were explored in Table 2-1. Our focus in this chapter is on the well-known mixing triplet SH-SH-S that benefits from different polarizations of the primary and secondary wave fields. Therefore, wave triplet 17 from Table 2-4 is selected, which is interesting for both fundamental studies and NDE applications. Details of mode selection and phase matching conditions were explained in Section 2.2.2 and Figure 2-2. Finally, SH0 waves having frequencies of 1.5 MHz and 0.78 MHz are selected for mixing in a 1 mm aluminum plate with a 90 degree mixing angle. We expect to generate and observe a secondary S0 Lamb wave at 2.28 MHz with 27.5° scattering angle with respect to SH0 1.5 MHz line of propagation.

6.3 Finite element model

The general methodology of FE modeling is given in Section 3.3. A small 12 mm by 12 mm area of the plate is discretized for SH0-SH0 non-collinear wave mixing simulation. Key aspects of the 3D finite element model of a 1 mm thick aluminum plate are shown in Figure 6-1. Boundary conditions for SH0 wave generation were created, and are shown in Figure 6-1a to ensure single mode excitation and uniform secondary wave generation at the sum frequency. In-plane displacements are applied along the left and bottom edges of the model using the tonebursts shown in Figure 6-1b. The maximum element edge length of 0.2 mm and the time step of 0.02 μs provide sufficient spatial and temporal resolution of the secondary waves. There are ten elements through the plate thickness.
Figure 6-1. 3D finite element model of aluminum plate for non-collinear SH0 wave mixing; (a) wave directions and boundary conditions, (b) applied displacement excitation on left edge, $u_Y$ with 1.5 MHz center frequency, and applied displacement excitation on bottom edge, $u_X$ with 0.78 MHz center frequency.
6.4 Results of finite element simulation

The displacement fields are recorded at different heights (Z axis). The secondary wave field is visualized by plotting the $u_Z$ displacement component of the difference computed by Equation 3-4. Figure 6-2 shows contour maps, or snapshots, of the displacement field on the top surface of the plate after 6 µs. The primary wave fields are planar as shown in Figures 6-2a and 6-2b for the $u_X$ (Test B) and $u_Y$ (Test A) displacement components respectively.

Figure 6-2. Wave triplet Set 17 finite element simulation results for the displacement field on the plate top surface at 7 µs: (a) primary $u_X$, (b) primary $u_Y$, (c) secondary $u_Z$, (d) sectional view A through the secondary waves.

The contour plot of the secondary wave (difference in the $u_Z$ components) depicts a wave propagating at an angle of $28.4^\circ$ from wave $a$. This is within $1^\circ$ of the angle predicted by the wave vector analysis (Table 2-4). The sectional view through the secondary waves shown in Figure 6-2d
indicates that the wavelength is 1.82 mm. Based on the phase velocity of the S0 mode at 2.28 MHz has a wavelength of 1.84 mm. Furthermore, the in-plane (resultant of $u_x$ and $u_y$) and out-of-plane ($u_z$) displacement profiles shown in Figure 6-3 for the secondary wave field are in excellent agreement with the wave structure of the S0 mode at 2.28 MHz (Figure 1-6b). The wavelength and wave structure indicate that the secondary wave is indeed the S0 mode.

The simulation was repeated on an aluminum plate that is linearly elastic, except in one subdomain, which is nonlinearly elastic. Thus, we can check if generated perturbations can travel individually in linear part of the plate to assure existence of the guided wave. The nonlinearly elastic subdomain is a 6 mm square, while the modeled domain is a 15 mm square as shown in Figure 6-4. Secondary waves should only be generated within the nonlinear elastic material, but they should then propagate on their own through the linearly elastic material. The simulation results for the secondary $u_z$ displacement field at 8 $\mu$s (computed by Equation 3-4) are shown in Figure 6-4, where we see that the secondary wave does propagate into the linearly elastic material and that the wavefront has some curvature due to its oblique incidence on the interface between the nonlinear and linear domains.
Figure 6-4. Wave triplet 17 finite element simulation results for plate having only localized nonlinearity: (a) secondary $u_Z$ contours on top surface at 8 $\mu$s, (b) sectional view A of the secondary wave field.

### 6.5 Conclusions

Non-collinear mixing of two SH0 guided waves with 90° mixing angle is demonstrated in this chapter as the main contribution. SH0 waves with 1.5 and 0.78 MHz frequencies are mixed and the nonlinear generation of S0 2.28 MHz Lamb wave is observed. The model simulates single mode excitation of SH0 waves in a 3D plate, thanks to the specific boundary conditions. Visualization of the secondary wave field, the S0 wave, is achieved by incorporating the subtraction method. This was straightforward because of the separate polarizations of SH0 and S0 waves. Consequently, difference out-of-plane displacement fields presented in this chapter are purely the products of
mutual interactions between SH0 waves. The non-collinear guided wave mixing has rarely been studied in the literature (Ishii et al. 2018b), which makes the current simulations unique. Moreover, non-collinear guided wave mixing experiments are yet to be reported in the literature. Experiments can be done by employing MSTs for SH wave generation and using angle beam transducers for reception.
Chapter 7

CONCLUSIONS

7.1 Summary

This dissertation introduces the fundamentals and practical aspects of using the nonlinear mutual interaction of guided waves in plates for applications in NDE (Nondestructive Evaluation) and SHM (Structural Health Monitoring). Self-interaction of ultrasonic waves has been studied in many works, however the separation of material nonlinearity and instrument distortion remains a challenge. One of the key advantages of mutual interaction of ultrasonic waves is the ability to isolate material nonlinearity. This, in addition to the diversity of mutual mixing mode triplets, was the primary motivation behind this research.

Chapter 2 discussed the theoretical aspects of the vector-based nonlinear mutual interaction of guided waves in a plate. The internal resonance conditions for ultrasonic guided waves propagating in arbitrary directions were formulated based on wavevector analysis. Based on the nonzero power flux condition and use of the guided wave dispersion relations to satisfy the phase matching condition, wave triplets consisting of two primary waves and secondary waves were identified and explained briefly. In Chapter 3, the effect of a finite-sized wave interaction region was studied with respect to the group velocities of the guided waves triplets. The proposed analytical model enabled the observation of secondary wave fields from the interaction of two primary wave packets of any length and group velocity. The analytical model compared favorably to finite element simulations and quantitatively confirmed that the amplitude of the secondary waves is dependent upon the size of the wave interaction region. As a result, the wave triplets with potential practical applications
for NDE were identified, where they would not have been noticed due to their mismatching group
velocities previously. As an example, co-directional SH guided wave mixing is a strong candidate
for further investigations, even though the group velocity of the S0 Lamb wave at the sum
frequency is not equal to the group velocities of the primary SH waves.

Chapter 4 and 5 extensively studied codirectional and counter-propagating mutual interaction
of SH (Shear Horizontal) waves. First, finite element simulations confirmed the accuracy of
selected mode triplets, as well as to design the experiments. Next, laboratory investigations were
conducted to measure the materials nonlinearity. Most of the experimental investigations were
aimed at the identification of secondary waves at the sum frequency related to the material. After
establishing the methods, the applications for NDE were evaluated in each case.

In chapter 6, the capability of the SH0-SH0 non-collinear wave triplet (with secondary S0 wave
generation) was validated with finite element simulations. A unique boundary condition was
applied which enabled the single mode excitation of SH0 waves. Finally, single mode primary
waves were mixed, and the secondary wave field is visualized by incorporating the subtraction
method. Results indicate the generation of S0 waves due to the interaction of SH0 waves, where
S0 wave can independently propagate in the plate separately from SH0 waves.

7.2 Key contributions

There are a number of contributions made in this dissertation that advance the knowledge of
nonlinear guided wave mixing in plates. The effort has resulted in new guided wave mixing
methods that are less susceptible to system nonlinearity. This research addresses the use of the
proposed methods for early damage detection in plate-like structures. The most significant
contributions of this work are:
1. Prior to this research nonlinear guided wave propagation was analyzed using a scalar approach, which limited it to self-interaction of waves or mutual interaction of co-directional waves. A vector approach was implemented to extend the analysis of wave interactions to an arbitrary mixing angle. The culmination of this analysis is the derivation of a partial differential equation (Equation 2-23) whose solution dictates the vector-based internal resonance conditions (Equation 2-30) for cumulative secondary waves.

2. Non-zero power flux for non-collinear interactions was explored and results in new types of mixing modes that have not been reported for guided waves. An analogy was provided to compare non-collinear guided wave mixing in plates with bulk wave mixing.

3. The vector-based phase-matching condition of the internal resonance criteria was used in conjunction with the nonzero power flux condition to select wave triplets comprised of two primary waves that generate a cumulative secondary wave. Table 2-4 provides 22 mode triplets, each having unique features that make it worthwhile to test. This is one of the advantages of mutual wave interactions over self-interactions; that there are many mixing combinations having a wide range of features for different applications.

4. An analytical model was introduced to predict the interaction of guided waves with prescribed characteristics. The model is capable of demonstrating the interaction of different wave packets with prescribed characteristics and group velocities. It was shown that the group velocity mismatched condition would not affect the power flux or the cumulative nature of secondary wave field generation as long as the waves are interacting, but once the interaction decreases, then a maximum amplitude of the secondary waves is obtained. This new insight was influential to choosing which wave triplets to list in Chapter 2, and should impact the future use of guided wave mixing for early detection of material degradation.
5. New methods were introduced for nonlinear material evaluation, and two of them were implemented in the laboratory. The most challenging part was to verify the source of the nonlinearities, which can be either from material or instrumentation. The selected mode triplets and experimental methodology were meant to ensure the measurement of material nonlinearity. The introduced experimental techniques isolate material nonlinearities from system nonlinearity. This represents significant progress since distinguishing different nonlinearities has always been a challenge.

6. SH0 guided wave mixing was selected as it devotes several benefits for nonlinear material characterization. Most importantly, the different polarizations of the SH0 waves and secondary Lamb waves make it possible to acquire nonlinearities associated with the material by using air-coupled or angle beam transducers. Moreover, single mode excitation of the SH0 wave is straightforward and feasible.

7.3 Future work suggestions

1. There are a lot of mode triplets that have not yet been explored, and many of these are quite interesting. Experiments for many of them should be designed. One interesting case would be counter-propagating Lamb waves, identified as case number 7 in Table 2-4.

2. Conducting finite element studies and experiments to prove the new reported mixing combinations based on the non-zero power flux analysis introduced in Table 2-2 and detailed in Table 2-4. One example is S-S mixing at 90º, which can generate secondary SH guided waves. Wave triplet 20 in Table 2-4 can be investigated in this regard.

3. Effect of energy transfer, from primary to secondary wave field, on the amplitude of the primary waves. It is interesting to investigate the change in the amplitude of primary waves due to wave mixing with non-phase matched conditions. Results will be helpful for phased-
array nonlinear scanning in plates. This can be done theoretically and verified by finite
element modeling.

4. It is interesting to modify the current theory and reformulate it based on the superposition
principle of real and complex infinitesimal perturbations. In other words, the real and
complex stress fields would generate very small perturbations in each time step, and the
nonlinear wave field is the total superposition of them. Therefore, the secondary wave field
can be formed in a way that is disconnected from the internal resonance conditions.

5. Noncontact methods for nonlinear guided wave material evaluation still remain
challenging. Capacitance based air-coupled transducer is a promising technology that
would be more dominant in future with enhanced SNR. In addition, immersion methods
have not explored for nonlinear mutual guided wave mixing, but introduce nonlinearity in
the fluid.

6. Investigation of the acoustic scattering of guided waves in the plate remains of great
interest. SH0 and S0 waves at low frequencies, with nondispersive characteristics, are two
feasible candidates to study.

7. Extend the wavevector based theory to investigate composite plates. The codirectional
guided wave mixing would be helpful for evaluating the material integrity, and counter-
propagating mixing mode can be used for plate scanning.
BIBLIOGRAPHY


nondestructive evaluation of alkali–silica reaction damage in concrete prism samples.”

*Materials and Structures*, 50, 60.


*NDT & E International*, 67, 64–70.


Appendix A: DERIVATION OF EQUATION 2-23

As discussed by normal mode expansion; we represent the secondary linear stress and velocity fields by superposition of all possible propagating modes,

\[ S^L(H_{ab}) = \frac{1}{2} \sum_{m=1}^{\infty} A_m(X, Y) S_m(Z) \]  \hspace{1cm} (A-1)

\[ \dot{u}_{ab} = \frac{1}{2} \sum_{m=1}^{\infty} A_m(X, Y) V_m(Z) \]  \hspace{1cm} (A-2)

where \( S_m \) and \( V_m \) are the modal stress and velocity fields, and \( A_m(X, Y) \) is a scalar giving the modal amplitude of wave \( m \) with displacement field \( u_m \). The time dependent exponential terms are omitted for brevity since they cancel out in equations.

The forced guided wave problem has been subject of different studies, where forced excitation of the waveguide generates a set of guided waves (Auld 1990; Rose 2014). The theory is used for nonlinear guided waves for mode extraction with respect to nonlinear interactions due to mutual guided wave mixing. The complex reciprocity relations (Auld 1990) between different solutions of the elastic wave propagations is given as,

\[ \nabla \cdot (V_2^* S_1 + V_1^* S_2^*) = -(V_2^* f_1 + V_1^* f_2^*) \]  \hspace{1cm} (A-3)

where \( f \) is the body force exposed to secondary wave field by nonlinear stresses due to wave interactions, which is a divergence of the nonlinear stresses.

\[ f_1 = \text{Div}(S^{NL}(H_a, H_b, 2)) \]  \hspace{1cm} (A-4)

Assume solution “1” is the secondary wave field that is

\[ V_1 = \dot{u}_{ab} \]  \hspace{1cm} (A-5)

\[ S_1 = S^L(H_{ab}) \]  \hspace{1cm} (A-6)

and assume solution “2” is an arbitrary secondary plate mode with \( f_2^* = 0 \) (zero body load), then
\[ V_z = \frac{1}{2} V_n(Z) e^{iK_n p(X,Y)} \]  
\[ S_z = \frac{1}{2} S_n(Z) e^{iK_n p(X,Y)} \]  

Note that time dependent exponential terms are dropped from all equations for brevity. Expanding Equation A-3,

\[ \frac{\partial}{\partial x} (V_z^* \cdot S_z + V_1 \cdot S_z^*) \cdot n_x + \frac{\partial}{\partial y} (V_z^* \cdot S_z + V_1 \cdot S_z^*) \cdot n_y + \]

\[ \frac{\partial}{\partial z} (V_z^* \cdot S_z + V_1 \cdot S_z^*) \cdot n_z = -(V_z^* \cdot f_1) \]  

Equation A-9 can be reformed by integrating over thickness \(Z\),

\[ \int_{-h}^{h} \left( \frac{\partial}{\partial x} (V_z^* \cdot S_z + V_1 \cdot S_z^*) \cdot n_x + \frac{\partial}{\partial y} (V_z^* \cdot S_z + V_1 \cdot S_z^*) \cdot n_y \right) dZ = \]

\[ -(V_z^* \cdot S_z + V_1 \cdot S_z^*) \cdot n_z \bigg|_{-h}^{h} = \int_{-h}^{h} ((V_z^* \cdot f_1)) \cdot dZ \]  

Substituting Equations A-7 and A-8 in Equation A-10, we get,

\[ \int_{-h}^{h} \left( \frac{1}{2} e^{-iK_n p(X,Y)} (V_n^* (Z) \cdot S_z + V_1 \cdot S_n^* (Z)) \right) \cdot n_x + \frac{\partial}{\partial y} \left( \frac{1}{2} e^{-iK_n p(X,Y)} (V_n^* (Z) \cdot S_z + V_1 \cdot S_n^* (Z)) \right) \cdot n_y \bigg|_{-h}^{h} = \]

\[ \begin{aligned} \frac{1}{2} e^{-iK_n p(X,Y)} \int_{-h}^{h} ((V_n^* (Z) \cdot f_1)) \cdot dZ & \end{aligned} \] 

Now substitute solution “1” into the integral parts of Equation A-11 sets,

\[ \int_{-h}^{h} \left( \frac{\partial}{\partial x} \left( \frac{1}{2} e^{-iK_n p(X,Y)} (V_n^* (Z) \cdot [\sum_{m=1}^{\infty} A_m (X,Y) S_m (Z)] + \right. \]

\[ \left. [\sum_{m=1}^{\infty} A_m (X,Y) V_m (Z) \cdot S_n^* (Z)) \right) \cdot n_x + \frac{\partial}{\partial y} \left( \frac{1}{2} e^{-iK_n p(X,Y)} (V_n^* (Z) \cdot [\sum_{m=1}^{\infty} A_m (X,Y) S_m (Z)] + \right. \]

\[ \left. [\sum_{m=1}^{\infty} A_m (X,Y) V_m (Z) \cdot S_n^* (Z)) \right) \cdot n_y \right) dZ = \]

\[ \begin{aligned} \frac{1}{2} e^{-iK_n p(X,Y)} \int_{-h}^{h} ((V_n^* (Z) \cdot f_1)) \cdot dZ & \end{aligned} \]  

By rearranging the above equation, we get the equation:
The de of Equation
\[
\sum_{m=1}^{\infty} \left( \frac{\partial}{\partial X} A_m(X,Y) e^{-iK_n p(X,Y)} \int_{-h}^{h} \left( -\frac{1}{4} (V_n^*(Z) \cdot S_m(Z) + V_m(Z) \cdot S_n^*(Z)) \right) dZ \right) \cdot n_X +
\]
\[
\sum_{m=1}^{\infty} \left( \frac{\partial}{\partial Y} A_m(X,Y) e^{-iK_n p(X,Y)} \int_{-h}^{h} \left( -\frac{1}{4} (V_n^*(Z) \cdot S_m(Z) + V_m(Z) \cdot S_n^*(Z)) \right) dZ \right) \cdot n_Y =
\]
\[
\frac{1}{2} e^{-iK_n p(X,Y)} \left\{ (V_n^*(Z) \cdot S_1) \cdot n_Z \right|_{-h}^{h} + \int_{-h}^{h} ((V_n^*(Z) \cdot f_1)) \cdot dZ \right\}
\]
where we assumed \( S_n^*(Z) \) is negligible in compared with other forces in the left side of the equation.

Notice that the summation on the left side of Equation A-13 has only one solution due to the orthogonality relations (Auld 1990; Rose 2014), and that is when \( m=n \). Thus, it can be rewritten in the form,
\[
\frac{\partial}{\partial X} \left( A_m(X,Y) e^{-iK_n p(X,Y)} 4P'_{mn} \right) \cdot n_X + \frac{\partial}{\partial Y} \left( A_m(X,Y) e^{-iK_n p(X,Y)} 4P'_{mn} \right) \cdot n_Y =
\]
\[
\frac{1}{2} e^{-iK_n p(X,Y)} \left\{ (V_n^*(Z) \cdot S_1) \cdot n_Z \right|_{-h}^{h} + \int_{-h}^{h} ((V_n^*(Z) \cdot f_1)) \cdot dZ \right\}
\]
where,
\[
P'_{mn} = \frac{1}{4} \int_{-h}^{h} \left( -\frac{1}{4} (V_n^*(Z) \cdot S_m(Z) + V_m(Z) \cdot S_n^*(Z)) \right) dZ
\]
Rearranging the equations,
\[
\frac{\partial}{\partial X} \left( A_m(X,Y) e^{-iK_n p(X,Y)} 4P'_{mn} \right) \cdot n_X + \frac{\partial}{\partial Y} \left( A_m(X,Y) e^{-iK_n p(X,Y)} 4P'_{mn} \right) \cdot n_Y =
\]
\[
\frac{1}{2} e^{-iK_n p(X,Y)} \left\{ (V_n^*(Z) \cdot S_1) \cdot n_Z \right|_{-h}^{h} + \int_{-h}^{h} ((V_n^*(Z) \cdot f_1)) \cdot dZ \right\}
\]
Applying the derivation, and using Equation 2-17 to simplify the Equation A-16, we have the final form of the equation as:
\[
4P'_{mn} \cdot n_X \left( \frac{\partial}{\partial X} - iK_n^* \cdot n_X \right) A_m(X,Y) + 4P'_{mn} \cdot n_Y \left( \frac{\partial}{\partial Y} - iK_n^* \cdot n_Y \right) A_m(X,Y) = f_n^{\text{surf}} + f_n^{\text{vol}} (A-17)
\]
where,
\[
f_n^{\text{surf}} = -\frac{1}{2} S^{NL}(H_a, H_b, 2) V_n^* \cdot n_Z \right|_{-h}^{h}
\]
\[
f_n^{\text{vol}} = \frac{1}{2} \int_{-h}^{h} \text{Div}(S^{NL}(H_a, H_b, 2)) \cdot V_n^* dZ
\]

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Appendix B: PARITY ANALYSIS, MATLAB CODE

Sample code for symbolic parity analysis is given below. The final internal and surface power flux is calculated based on symbolic parameters comprising material properties, frequencies, wavevectors, and mixing angle.

The wavestructures are presented in the form of sinusoidal and cosinusoidal functions with respect to symmetric or antisymmetric characteristics. Thus, with modifying the wavestructures, we can examine different wave forms with different types of symmetry. Once all of the parameters are defined in the form of symbols, the nonlinear stress field is calculated, and consequently the power fluxes. Finally, we substitute a real number for the mixing angle $\theta$ (teta in code). If the mixing mode triplet is valid, the power flux substituted equation by that particular teta is not supposed to be nil. If symbolic power flux with respect to that specific mixing angle is zero, then the selected combination of guided mode types is not obtainable. The copy of the MATLAB code is as,

```matlab
syms L u A B C ro
assume([L u A B C ro], 'real');
assume([L u ro], 'positive');
syms x y z t
assume(x, 'real'); assume(y, 'real'); assume(z, 'real');

%%
syms wa wb h teta ka kb;
assume([wa wb h teta ka kb], 'real');
assume([wa wb h teta ka kb], 'positive');
assume(wa>wb);
T=1; % 1:summ -1:difference
Ka=ka*[1 0]; Kb=kb*[cos(teta) sin(teta)];
wm=wa+T*wb; Km=Ka+T*Kb;
Landa=angle(Km(1,1)+i*Km(1,2));
rot_t=[cos(teta) -sin(teta) 0;
      sin(teta) cos(teta) 0;
      0       0       1];
rot_L=[cos(Landa) -sin(Landa) 0;
      sin(Landa) cos(Landa) 0]
```
{
0 0 1];  

%%%  

TYPEa='SSH'; TYPEb='SSH'; TYPEm='SSH';

if TYPEa=='S'
    Ua=[cos(z); 0; i*sin(z)]*exp(i*(dot(Ka,[x y])));
end

if TYPEa=='A'
    Ua=[sin(z); 0; i*cos(z)]*exp(i*(dot(Ka,[x y])));
end

if TYPEa=='SSH'
    Ua=[0; cos(z); 0]*exp(i*(dot(Ka,[x y])));
end

if TYPEa=='ASH'
    Ua=[0; sin(z); 0]*exp(i*(dot(Ka,[x y])));
end

if TYPEb=='S'
    Ub=rot_t*[cos(z); 0; i*sin(z)]*exp(i*(dot(T*Kb,[x y])));
end

if TYPEb=='A'
    Ub=rot_t*[sin(z); 0; i*cos(z)]*exp(i*(dot(T*Kb,[x y])));
end

if TYPEb=='SSH'
    Ub=rot_t*[0; cos(z); 0]*exp(i*(dot(T*Kb,[x y])));
end

if TYPEb=='ASH'
    Ub=rot_t*[0; sin(z); 0]*exp(i*(dot(T*Kb,[x y])));
end

if TYPEm=='S'
    Vm=rot_L*[cos(z); 0; i*sin(z)]*(-i)*wm*exp(i*dot(Km,[x y])); %S
end

if TYPEm=='A'
    Vm=rot_L*[sin(z); 0; i*cos(z)]*(-i)*wm*exp(i*dot(Km,[x y])); %A
end

if TYPEm=='SSH'
    Vm=rot_L*[0; cos(z); 0]*(-i)*wm*exp(i*dot(Km,[x y])); %SSH
end

if TYPEm=='ASH'
    Vm=rot_L*[0; sin(z); 0]*(-i)*wm*exp(i*dot(Km,[x y])); %ASH
end

%%%  

Ha=((diff(Ua,x) diff(Ua,y) diff(Ua,z)));
Hb=((diff(Ub,x) diff(Ub,y) diff(Ub,z)));

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SNL=L/2*trace(Hb+Hb.')*Ha+u*Ha*(Hb+Hb.')... 
+L/2*trace(Ha+Ha.)*Hb+u*Hb*(Ha+Ha.')... 
+L/2*trace(Ha.'*Hb+Hb.'*Ha)*eye(3)... 
+2*C*trace(Ha)*trace(Hb)*eye(3)+u*(Ha.'*Hb+Hb.'*Ha)... 
+B*trace(Ha)*(Hb+Hb.).'... 
+B*trace(Hb)*(Ha+Ha.)... 
+B/2*trace(Ha*Hb+Hb*Ha+Ha.'*Hb+Hb.'*Ha)*eye(3)... 
+A/4*(Ha*Hb+Hb*Ha+Ha.'*Hb.+Hb.'*Ha.'... 
+Ha.'*Hb+Hb.'*Ha+Ha*Hb.'+Hb*Ha.'); 

DIV=[diff(SNL(1,1),x)+diff(SNL(1,2),y)+diff(SNL(1,3),z); 
    diff(SNL(2,1),x)+diff(SNL(2,2),y)+diff(SNL(2,3),z); 
    diff(SNL(3,1),x)+diff(SNL(3,2),y)+diff(SNL(3,3),z)]; 

DIV=simplify(DIV); 
N2=[0;0;1]; 
Power_Surface=-1*N2.'*(SNL*conj(Vm)) 
Fs=(subs(Power_Surface,z,h)-subs(Power_Surface,z,-h)); 
Fs=real(eval(simplify(Fs))) 

Power_body=(1*conj(Vm).*DIV); 
Pb=eval(simplify(Power_body)); 
INT0=int(Pb,z); 
INT0=real(subs(INT0,z,h)-subs(INT0,z,-h)); 
tt=90; disp(["teta='",num2str(tt)]); 
INT=subs(INT0,teta(tt/180*pi)) 
SURF=subs(Fs,teta(tt/180*pi)
Appendix C: NONLINEAR STANDING WAVES

Standing waves in the plate, as described by their name, do not propagate but demonstrate a stationary fluctuation. Once a shear or longitudinal bulk wave is trapped between two perfectly reflecting boundaries, walls, the standing waves are formed. This means the stationary fluctuation forms a stress gradient across the plate thickness, but zero in-plane stress gradient. The possible standing waves are formed when the superposition of reflected waves generates a sinusoidal displacement field with stationary zero displacement nodes. This can be express in the form of the following equation:

\[ 2h = \frac{n}{2} \lambda \quad n=1, 2, \ldots \]  

(C-1)

where \( 2h \) is plate thickness, and \( \lambda \) is the wavelength of the planar fluctuating wave along the plate thickness. Thus, the frequency of corresponding standing waves is defined by:

\[ f = \frac{nc}{4h} \quad n=1, 2, \ldots \]  

(C-2)

where the \( c \) is shear or longitudinal wave velocity of bulk waves. Assuming that shear and longitudinal bulk wave velocities are 3100 and 6300 m/s respectively, the corresponding standing wave frequencies for a 1 mm aluminum plate are given in Table C-1.

Table C-1. Frequencies of shear and longitudinal standing waves.

<table>
<thead>
<tr>
<th>Wave types</th>
<th>Frequencies (MHz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shear standing wave</td>
<td>1.55, 3.1, 4.65, 6.2</td>
</tr>
<tr>
<td>Longitudinal standing wave</td>
<td>3.15, 6.3, 9.45, 12.6</td>
</tr>
</tbody>
</table>

The other approach to investigate the standing waves is looking into the guided wave dispersion curves of a plate. Since standing waves are formed by superposition of plane waves, then the wave field is uniform at a fixed height \( (u_z) \). Therefore, the wave number of the corresponding guided
wave is theoretically zero, because there is no change in wavefield along the plate. Wave number is defined as \( k = \frac{2\pi f}{c} \). Therefore, in order to have a zero-wave-number guided wave, either the frequency must be zero or guided wave phase velocity goes to infinity. The unbounded phase velocities are also defined as cutoff frequencies of guided waves, where the group velocity goes to zero. Therefore, it is concluded that standing waves are actually guided waves located at cutoff frequencies. Figure C-1 shows where zero wave number guided wave modes are located. Guided waves with zero frequency resonance conditions represent the rigid motion of the plate along X, Y, and Z axes, that are called bulk resonances. Actual fluctuating standing waves are located at very low wave numbers and high phase velocities (Cutoff frequencies). Figure C-1 shows the standing waves located at cutoff frequencies.

Figure C-1. Aluminum dispersion curves, with the illustration of zero wave number resonances. The standing waves are located at guided waves with cutoff frequencies.
The wave structures are derived based on the described method in Chapter 1. The wave structure of some guided waves at cutoff frequencies are extracted and presented in Figure C-2. The coordinate system is inherited from Figure 1-2 in chapter 1, that is also shown in Figure C-3. Thus, the $u_Z$ demonstrates out-of-plane displacement. Standing waves can be categorized based on displacement polarization, in-plane and out-of-plane standing waves. Interestingly, the in-plane standing waves can have either $X$ or $Y$ axis displacement component (Due to symmetry of pure in-plane displacement). Therefore, Figure C-2 contains in-plane standing waves in $X$ and $Y$ direction with same frequency. For example, Figures C-2d and C-2e both are in-plane standing wave with same excitation frequency but different polarization.

Comparing the numbers of Table C-1 and Figure C-2 validates our assumptions and standing wave extractions from dispersion curves. The standing wave introduced in Figure C-2f, the frequency of 3.0989 MHz with major out of plan displacements, represents symmetric in-plane displacement which means it carries the symmetric Lamb wave structure (S1 Lamb wave in this case). Therefore, it can be generated by nonlinear mixing of two SH0 guided waves based on the parity analysis. To include phase matching conditions for nonlinear generation of the corresponding standing wave, the only combination will be the counter-propagation of two SH0 waves at the half frequency of the corresponding standing wave as illustrated in Figure C-3. The 3D finite element model is built based on the explained method in section 3-4, and the three tests were performed. First two SH0 waves were generated separately, and then both of them together. The maximum in-plane displacement of SH0 waves is 10 nm, with 10 µs wave packet length.
Figure C-2. Wave structures of selected guided waves with close to zero wave number ($k=0.0001 \text{ 1/m}$). Bulk resonances at zero frequency (a, b, and c). Guided waves with very high phase velocities (d-h). The overlapped displacement components are labeled.
Figure C-3. Schematic illustration of mixing two SH0 waves at 1.545 MHz in a 1 mm nonlinear aluminum plate.

Figure C-4 shows the displacements at $X=50$ mm which is located in the center of mixing zone. Figure C-4a shows two SH0 waves mixed (superimposed) on each other, thus the total amplitude is 20 nm, where it was 10 nm for each SH0 wave. Figure C-4b presents out-of-plane displacements, both linear and nonlinear, generated by a single SH0 wave excitation and propagation. So, it does not belong to standing wave. Standing wave is generated by mixing of two SH0 waves, which is also combined by other perturbations including linear and nonlinear second harmonics (Shown in Figure C-4c). Difference out-of-plane displacements after subtraction are shown in Figure C-4d. The subtraction method is approached based on the demonstrated method in section 3-3 (Test(A+B) - Test(A) - Test(B)). The fluctuating pattern observed in Figure C-4d corresponds to the standing wave that is generated, and oscillates locally.
Figure C-4. (a) In-plane displacement of SH0 waves mixing, (b) linear and second harmonic nonlinear out-of-plane displacements due to single SH0 generation, (c) out-of-plane displacements during SH0 wave mixing, (d) difference out-of-plane displacement derived by subtraction method.
In case of localized excitation and fluctuation of standing waves, the in-plane stress gradient is not zero. Also, the wave number is not zero, because the fluctuation is not uniform in plate. As a result, the stress field tends to spread in the plate to comply with the zero in-plane stress gradient or in other words to form a planar wave field. This means localized standing waves are not stable and the energy of the wave dissipates gradually. This can be marked as the main disadvantageous of the current demonstrated nonlinear standing wave generation. Nevertheless, including a larger mixing zone can significantly help to have a higher and more stable wave field.
Appendix D: NONTECHNICAL ABSTRACT

Nondestructive Evaluation (NDE) and Structural Health Monitoring (SHM) of materials are investigated through nonlinear ultrasonics. The material nonlinearity associated with microstructural features distorts the ultrasonic wave packet, altering the frequency spectrum. Since microstructural changes precede macroscale damage, nonlinear ultrasonics provides the unique opportunity to diagnose material degradation early in its progression. Thus, the unique sensitivity of nonlinear ultrasonic waves to microstructural damage is the motivation for the current study to develop more reliable methods to detect early damage in materials. Ultrasonic guided waves, which travel in a bounded medium, are of great interest due to particular advantages such as rapid scanning of plates. The focus of this dissertation is on the mutual interaction of guided waves in a plate, which occurs when multiple waves propagating in the plate mix together.

The wave vectors are considered to form the nonlinear stress fields associated with the interaction of two different guided waves with arbitrary mixing angle. By solving the differential equations, the conditions that generate the strong and cumulative nonlinear perturbations are found. Based on internal resonance conditions, the wave triplets (two primary waves and the secondary wave) determined form the basis for experimental studies and numerical simulations. An analytical model is given which simulates the interaction of guided waves with finite wave packets.

The rest of the dissertation is on demonstrating the proposed experimental techniques and then exploring the results. Codirectional shear horizontal guided wave mixing is considered, where limited cumulative behavior of out-of-plane displacements is used for early damage characterization. In addition, interaction counter-propagating shear horizontal guided waves is studied. The results indicate the strong generation of a secondary wave field that travels independently from primary wave packets.
VITA

Mostafa Hasanian

Education:

Ph.D., Engineering Science and Mechanics, The Pennsylvania State University, University Park, PA, US May 2019

M.S., Mechanical Engineering, Sharif University of Technology, Tehran, Iran Dec 2010

B.S., Mechanical Engineering, Sharif University of Technology, Tehran, Iran Sep 2008

Awards:

- Scholarship Dr. Richard E. Llorens award, 2018
- Graduate Robert A. Sebrosky fellowship 2016 students, 2016
- ASNT Fellowship Award to support “Robotic Laser Ultrasonics for Remote Detection of Stress Corrosion Cracking in Harsh Environments”, 2016
- Fund for Excellence in Graduate Recruitment (FEGR) Scholarship, ESM, 2013
- International Fee Remission Scholarship and Melbourne research Scholarship, The University of Melbourne, Australia 2013

Selected Publications: