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ESSAYS ON TIME INCONSISTENCY AND INDUSTRIAL ORGANIZATION

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Abstract

This thesis consists of two chapters. In the first chapter, we provide an explanation for the common observation in the market for upgrades, which is that firms tend to offer small upgrades very frequently instead of significant ones less frequently. We explain this problem using the time-inconsistent behavior of consumers. We examine cases in the presence of naive hyperbolic preferences and sophisticated hyperbolic preferences separately. We show that it is optimal for the monopolist to offer the upgrades more frequently to more hyperbolic consumers under certain circumstances.

In the second chapter, we show that naive hyperbolic consumers might be unresponsive to interest rates and credit limits of credit card offers by companies because the offers have a grace period. Consequently, we demonstrate that there might be no competition on the interest rate and credit limit, even if more than one firm is in the market and even if the consumer accepts only one card. We determine whether the credit card companies can exploit time-inconsistent consumers and gain positive expected profits. We show that in fact there are circumstances in which both zero and positive expected profits could be possible.

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Dedication

For my dear parents and sister, and my beloved husband who filled my life with happiness.

Chapter 1

Time Inconsistency of Consumers and Excessive Upgrades by Durable Good Monopoly

1.1 Introduction

This paper explains the common observation in the market for upgrades by monopolists that they offer very frequent small upgrades instead of less frequent significant ones. Examples of this practice might be found in software, computer, and personal electronics. In the software industry, there are frequent upgrades that provide little extra value to the consumer. For example, consumers commonly complain about rushed and immature upgrades of Office by Microsoft. They believe that Microsoft is offering upgrades that are not tested enough and that do not have significant new features.[1]

We explain this problem by the time-inconsistent behavior of consumers. According to many laboratory and field studies, discount rates are much greater in the short run than in the long run, as reported by Harris and Laibson (2001). As illustrated by O'Donoghue and Rabin (1999), people update their discount rates as time passes and give more weight to the earlier days, as they get closer.[2]

[1] See the articles, "Office XP to ship just under the wire" by Mary Joe Foley and "Microsoft seeks revenue boost with rush Office release" by Joe Wilcox.

[2] O'Donoghue and Rabin (1999) write that "For example, when presented a choice between

Economists use different discounting functions to model this kind of behavior, one of which is hyperbolic discounting. In this paper, we use hyperbolic discounting to model the time-inconsistent behavior of consumers.

Strotz (1956) claims that people would not obey their optimal plan of the present moment if they would be allowed to reconsider their earlier plans at later dates. Because people are impatient, they give more weight to the earlier time as it gets closer, which causes time-inconsistent behavior. There is significant evidence that consumers' preferences are time inconsistent.[3] Loewenstein and Prelec (1992) stated that the discount function for the hyperbolic preferences is a generalized hyperbola, $\phi(t) = (1 + \alpha t)^{-\beta/\alpha},\ \alpha,\ \beta > 0$ and t is the time distance to the event. Therefore, as t decreases the discount factor increases. Laibson (1997) uses the so-called $\beta - \delta$ discount function which is given by $\{1, \beta\delta, \beta\delta^2, \beta\delta^3, ..\}$to model hyperbolic behavior. Since the $\beta - \delta$ discount function is a common way to model discrete time hyperbolic discounting, we use that method in our model.

There are papers about product upgrades for durable goods in the literature. Fudenberg and Tirole (1998) analyze the monopoly pricing of overlapping generations of a durable good. They state results describing when a monopolist would continue to sell the older version of a product along with the new one, and when they would offer discounts to the previous version owners to buy the new version of the product. Ellison and Fudenberg (2000) analyze two reasons for a monopoly supplier of software to offer upgrades more often than social optimal when the upgrades are backward compatible

Our model based on that of Fishman and Rob (2000) which analyzes "product innovation by a durable-good monopoly." Basically, they compared the frequency of product innovations with the social optimum under different cases: giving discounts to existing customers, planned obsolescence, and the case for which neither of these is possible. In their model, the utility the consumers get by the consumption of a good depends on the quality of that good. On the production side, a monopolist has to have an R&D stage to produce a new model; the quality of

doing seven hours of an unpleasant activity on April 1 versus eight hours on April 15, if asked on February 1 virtually everyone would prefer the seven hours on April 1. But come April 1, given the same choice, most of us apt to put off the work until April 15... When considering trade-offs between two future moments, present-biased preferences give stronger relative weight to the earlier moment as it gets closer"

[3]See Loewenstein and Thaler (1989), Ainslie (1991), and DellaVigna and Malmendier (2002)

the new model depends on the allocated time for R&D and the expenditures. So, the quality is given by a function that depends on R&D time and expenditures. Another important feature of their model is that the previous product forms a technological base for the development of the new product. In addition to R&D expenditures there is a fixed cost which they call "implementation cost", to be paid each time a new product is introduced.

In this paper, our aim is to explain a common observation, frequent and small upgrades. While others have explained this phenomenon as being due to network effects, we demonstrate, using the simplest possible model, that the time inconsistency of consumers also provides an explanation. Since we use the simplest model to explain the observed phenomenon, we do not attempt to accomplish a technical advancement. There are papers in the literature on the study of durable good upgrades or on the study of time inconsistency, but there has been no previous study about durable goods upgrades of a monopolist facing hyperbolic consumers.

1.2 The Model

Our model is based on that of Fishman and Rob (2000). There are two agents in the model: a time consistent profit maximizer monopolist and a time inconsistent consumer. The monopolist offers infinitely durable upgrades in a certain frequency for his product to the consumer who already had that product on hand.[4] The cost of creating upgrades is given by a linear function c(.).[5] The monopolist also incurs a fixed cost every time she offers an upgrade, F. We can think of this fixed cost as advertisement. Fishman & Rob (2000) interprets F as the cost of harnessing new knowledge into the present product. The monopolist discounts the future exponentially at the rate of δ and sells his upgrades to the consumer at price p by offering a payment plan of three equal payments to the consumer such that the consumer has to pay $p/3$ in the first period, the second period, and the third period.[6] If the consumer does not pay the designated amount in the first or second

[4]The model would be simpler if the upgrades were not durable. However, this assumption would not fit in software example.

[5]This is not a crucial assumption. This assumption is just to keep calculations easier and we suspect that our results will hold with convex cost functions as well. In fact, we have an example in which the results hold for a strictly convex cost.

[6]See Section 1.6 for the discussion of three period equal payments.

period, he delays it to the next periods with the interest rate r. For example, if the consumer fails to pay $p/3$ in the first period, he pays $(1+r)p/3+p/3$ in the second period such that $(1+r)p/3$ accrues from the first period with the interest rate r, and $p/3$ is the designated amount he needs to pay in the second period. If the consumer delays in the first and second periods, he pays $(1+r)^2p/3+(1+r)p/3+p/3$ in the third period such that $(1+r)^2p/3$ and $(1+r)p/3$ accrues from the first and second periods respectively, and $p/3$ is the designated amount he needs to pay in the third period. In our model, the consumer is not allowed to delay payments after the second period, which means he has to pay all he owes at the end of the third period.

There is an infinitely lived consumer who wants to maximize his lifetime utility when he decides to buy or not to buy an upgrade. The consumer's per period utility from an upgrade is given by a concave function $v(.)$. The consumer is a hyperbolic discounter with $\delta - \beta$ discount factor as in Laibson (1997). In other words, the sequence of discount factors for the hyperbolic consumer is $\{1, \beta\delta, \beta\delta^2, \beta\delta^3, ...\}$. Note that as β increases, the consumer gets closer to being an exponential discounter with discount factor δ. If the consumer decides to buy a certain amount of upgrade today, he will buy the same amount of upgrade every time it is offered. In other words, our problem is stationary. Since the upgrades are durable, the next time the consumer buys the same amount of upgrade, he adds that upgrade onto the previous one. This reflects the reality especially in the software market such that every time the software on hand is upgraded, new features/upgrades are added to the existing features/upgrades. If the consumer buys the upgrade, which payments he delays and which ones he pays on time depend on the interest rate and discount factors.

The monopolist offers the upgrades either in every period or in every two periods. Whenever an upgrade is offered, at the first period the consumer decides whether to buy the upgrade and then whether to delay his first payment. At the second period, he decides whether to delay his second payment together with his delayed first payment if there is any. At the third period, the consumer pays everything he owes.

We analyze this model first by assuming that the interest rate is endogenous and then under competitive financial markets implying a fixed interest rate. Under each of these cases, we study both naive and sophisticated hyperbolic consumers. We focus on pure strategy subgame perfect equilibrium of this game.

1.3 Analysis of the Model with an Endogenous Interest Rate

In this section, we assume that the monopolist is free to choose the interest rate for late payments.

1.3.1 Naive Hyperbolic Consumer

If the consumer is a naive hyperbolic discounter, at any time period his sequence of discount factors are $\{1, \beta\delta, \beta\delta^2, \beta\delta^3, ...\}$, but he thinks that tomorrow his sequence of discount factors will be $\{1, \delta, \delta^2, \delta^3, ...\}$, which causes the consumer to make time-inconsistent decisions. There are three different possible cases:

- Case 1: $1 > \delta(1 + r)$

  In this case, the consumer prefers to delay all the payments to the third period. In the first period, the consumer decides whether to pay the first payment in the designated period, which is the first period or to delay it until the second period. If he pays the first payment on time, he pays $p/3$, and if he delays it until the second period, the present value of his payment is $\beta\delta(1+r)p/3$. Since $1 > \beta\delta(1+r)$, he prefers to delay the first payment to the second period. Then he decides whether to delay the amount he owes at the second period to the third period. If he pays that amount in the second period, he pays $(1 + r)p/3 + p/3$ and the present value of that payment is $\beta\delta((1+r)p/3+p/3)$. If he decides to delay the amount he owes in the second period to the third period, he pays $(1+r)^2p/3+(1+r)p/3$ in the third period with the present value of $\beta\delta^2((1 + r)^2p/3 + (1 + r)p/3)$. Since $1 > \delta(1 + r)$, accordingly $\beta\delta((1 + r)p/3 + p/3) > \beta\delta^2((1 + r)^2p/3 + (1 + r)p/3)$, he prefers to delay all payments to the third period.

- Case 2: $1 < \beta\delta(1 + r)$

In this case, the consumer pays all payments in the designated periods. In the first period, the consumer decides whether to pay the first payment on time. If he pays the first payment on time he pays $p/3$, although if he delays it to the second period, the present value of his payment is $\beta\delta(1+r)p/3$. Since $1 < \beta\delta(1+r)$, he prefers to pay the first payment on time. After this, he decides whether to delay the second period payment to the third period. If he pays it in the second period, he pays $p/3$ with a present value of $\beta\delta p/3$. If he delays it, he pays $(1+r)p/3$ in the third period with the present value of $\beta\delta^2(1+r)p/3$. Since $1 < \delta(1+r)$, accordingly $\beta\delta p/3 < \beta\delta^2(1+r)p/3$, he prefers to pay the second payment on time, too. So, he pays all the payments in the designated periods.

- Case 3: $1 \geq \beta\delta(1+r)$ & $1 \leq \delta(1+r)$

  In this case, the consumer delays the first payment to the second period, but pays the second payment on time. In the first period, the consumer decides whether to delay the first payment to the second period. Due to the same reasoning we mentioned under Case 1, he delays it to the second period. After this, he decides whether to delay the second period payment to the third period. If he does not delay it, he pays $(1+r)p/3 + p/3$ with the present value of $\beta\delta((1+r)p/3 + p/3)$. If he delays it, he pays $(1+r)^2p/3 + (1+r)p/3$ in the third period with the present value of $\beta\delta^2((1+r)^2p/3 + (1+r)p/3)$. Since $1 \leq \delta(1+r)$, accordingly $\beta\delta((1+r)p/3 + p/3) \leq \beta\delta^2\left((1+r)^2p/3 + (1+r)p/3\right)$, he prefers not to delay the amount he owes in the second period to the third period. This payment strategy of the consumer is based on his beliefs in the first period. According to this plan, he believes that the present value of the total amount he will pay is $\left[\beta\delta(2+r+\delta)\right]p/3$. However, in reality his total payment will be more than $\beta\delta(2+r+\delta)$. (From now on, for convenience, when we say total payment, it means the present value of the total payment.) The consumer delays the first payment to the second period as we explained above, and there is no time inconsistency problem at this point. However, as soon as he reaches the second period, his discount factors becomes $\{1, \beta\delta, \beta\delta^2, ...\}$ again as opposed to his first period belief. Therefore, when he reaches the second period, he updates his plan and decides to delay the second payment to the third period too. In this case, his total payment is $[\beta\delta^2(r^2+3r+3)]\,p/3$ instead of $[\beta\delta(2+r+\delta)]\,p/3$. Note that $[\beta\delta^2(r^2+3r+3)]\,p/3$ is always bigger than $[\beta\delta(2+r+\delta)]\,p/3$ under the condition of $1 \leq \delta(1+r)$. Since he has to decide whether to buy the upgrade or not in the first period, if the present value of utility he will get from the upgrade is greater than or equal to his total payment, he decides to buy the upgrade. However, the problem is that the consumer underestimates his total payment. He thinks he will pay $[\beta\delta(2+r+\delta)]\,p/3$, but in reality he will pay $[\beta\delta^2(r^2+3r+3)]\,p/3$. Since the monopolist gets all the surplus, the monopolist offers such a quality level of upgrade that the utility the consumer gets from that upgrade is equal to $[\beta\delta(2+r+\delta)]\,p/3$, which is the total payment the consumer believes to make. Since the utility he gets from that upgrade is equal to the amount he believes to pay, he chooses to buy the upgrade. However, in reality since the consumer is a hyperbolic discounter, when he reaches the second period, he delays the second period payment to the third period as well, and ends up paying more than he expected and more than the total utility he gets from that upgrade.

We summarize these observations in the following lemma.

Lemma 1. *If the consumer is a naive hyperbolic discounter, the behavior of the consumer, depending on the discount factors, δ, β, and the interest rate, r, can be summarized as follows:*

if $1 > \delta(1+r) \implies$ the consumer delays all payments to the third period and the present value of his total payment is $[\beta\delta^2(r^2+3r+3)]\,p/3$,

if $1 < \beta\delta(1+r) \implies$ the consumer pays all payments in the designated periods and the present value of his total payments is $[1+\beta\delta+\beta\delta^2]\,p/3$, and

if $1 \geq \beta\delta(1+r)$ & $1 \leq \delta(1+r) \implies$ although the consumer believes in the first period that he delays the first payment but not the second one, he ends up delaying all payments to the third period. Moreover, although he believes that his total payment is $[\beta\delta(2+r+\delta)]\,p/3$, his actual total payment is $[\beta\delta^2(r^2+3r+3)]\,p/3$, which is more than what he expected.

Our aim is to see whether the frequency of the upgrades change if the consumer is hyperbolic discounter instead of exponential. In order to see this, we compare the total payoff if the upgrades are frequent and the total payoff if the upgrades are less frequent. For simplicity, we take every period upgrades as the frequent upgrades, and every two periods upgrades as the less frequent ones.

First, assume that the monopolist offers durable upgrades in every period and the amount of the upgrade is given by x. The per period value of that upgrade for the consumer is given by value function $v(.)$. Since the upgrade is durable and the consumer is infinitely lived, the total present value of x amount of upgrade for the hyperbolic consumer is $(1+\beta\delta\frac{1}{1-\delta})v(x)$.

Lemma 2. *If the consumer is a naive hyperbolic discounter and the monopolist offers the same amount of upgrade every period, we can write the payoff function of the monopolist as follows:*

if $1 > \delta(1+r)$

$\implies payoff_{1,case1} = \frac{1}{1-\delta}\left[v(x)(1+\beta\delta\frac{1}{1-\delta})\frac{1}{\beta} - c(x) - F\right]$,

if $1 < \beta\delta(1+r)$

$\implies payoff_{1,case2} = \frac{1}{1-\delta}\left[v(x)(1+\beta\delta\frac{1}{1-\delta})\frac{1+\delta+\delta^2}{1+\beta\delta+\beta\delta^2} - c(x) - F\right]$ *, and*

if $1 \geq \beta\delta(1+r)$ & $1 \leq \delta(1+r)$

$\implies payoff_{1,case3} = \frac{1}{1-\delta}\left[v(x)(1+\beta\delta\frac{1}{1-\delta})\frac{\delta}{\beta}\frac{r^2+3r+3}{2+r+\delta} - c(x) - F\right]$.

Proof. Since we know the present value of the possible total payment of the consumer for the upgrade in each possible case from proposition 1 and the present value of the total utility the consumer gets, it is possible to find the price for the upgrade. Since the consumer buys the upgrade only if the utility he gets from that is higher than or equal to his total payment and the monopolist get the entire surplus, the optimal price the monopolist charges under Case 1 is as follows:

$$\left[\beta\delta^2(r^2+3r+3)\right]p/3 = (1+\beta\delta\frac{1}{1-\delta})v(x) \implies p/3 = \frac{(1+\beta\delta\frac{1}{1-\delta})v(x)}{\left[\beta\delta^2(r^2+3r+3)\right]} \quad (1.1)$$

Since the monopolist receives all the payments at the third period with interests, she gains $[(1+r)^2p/3 + (1+r)p/3 + p/3]$ in the third period. Since the monopolist is an exponential discounter with δ factor, the present value of her revenue is $\delta^2\left[r^2+3r+3\right]p/3$. By **??**, the present value of the revenue is:

$$revenue_{case1} = \delta^2 \left[r^2 + 3r + 3\right] \frac{(1 + \beta\delta\frac{1}{1-\delta})v(x)}{[\beta\delta^2(r^2 + 3r + 3)]} = v(x)(1 + \beta\delta\frac{1}{1-\delta})\frac{1}{\beta}$$

Since the cost of an upgrade is $c(x)$ and the fixed cost is F, we can write the monopolist's payoff from one upgrade as follows:

$$payoff_{case1} = \left[v(x)(1 + \beta\delta\frac{1}{1-\delta})\frac{1}{\beta} - c(x) - F\right]$$

If the monopolist offers the same amount of upgrade every period, the present value of the total payoff is:

$$payoff_{1,case1} = \frac{1}{1-\delta}\left[v(x)(1 + \beta\delta\frac{1}{1-\delta})\frac{1}{\beta} - c(x) - F\right]$$

For Case 2, by following the same steps as above, we can find the present value of the total payoff as follows:

$$\left[1 + \beta\delta + \beta\delta^2\right] p/3 = (1 + \beta\delta\frac{1}{1-\delta})v(x) \implies p/3 = \frac{(1 + \beta\delta\frac{1}{1-\delta})v(x)}{[1 + \beta\delta + \beta\delta^2]}$$

$$revenue_{case2} = \left[1 + \delta + \delta^2\right] \frac{(1 + \beta\delta\frac{1}{1-\delta})v(x)}{[1 + \beta\delta + \beta\delta^2]} = v(x)(1 + \beta\delta\frac{1}{1-\delta})\frac{1 + \delta + \delta^2}{1 + \beta\delta + \beta\delta^2}$$

$$payoff_{1,case2} = \frac{1}{1-\delta}\left[v(x)(1 + \beta\delta\frac{1}{1-\delta})\frac{1 + \delta + \delta^2}{1 + \beta\delta + \beta\delta^2} - c(x) - F\right]$$

For Case 3, in the first period the consumer believes that his total payment will be $[\beta\delta(2 + r + \delta)]\, p/3$, so the optimal price to be charged can be found as follows:

$$[\beta\delta(2 + r + \delta)]\, p/3 = (1 + \beta\delta\frac{1}{1-\delta})v(x) \implies p/3 = \frac{(1 + \beta\delta\frac{1}{1-\delta})v(x)}{[\beta\delta(2 + r + \delta)]}$$

Then the revenue and payoff can be written as follows:

$$revenue_{case3} = \delta^2 \left[r^2 + 3r + 3\right] \frac{(1 + \beta\delta\frac{1}{1-\delta})v(x)}{[\beta\delta(2 + r + \delta)]} = v(x)(1 + \beta\delta\frac{1}{1-\delta})\frac{\delta}{\beta}\frac{r^2 + 3r + 3}{2 + r + \delta}$$

$$payoff_{1,case3} = \frac{1}{1-\delta}\left[v(x)(1 + \beta\delta\frac{1}{1-\delta})\frac{\delta}{\beta}\frac{r^2 + 3r + 3}{2 + r + \delta} - c(x) - F\right]$$

∎

Lemma 3. *If the consumer is a naive hyperbolic discounter and the monopolist offers the same amount of upgrade in every two periods, we can write the payoff function of the monopolist as follows:*

if $1 > \delta(1 + r)$

$\Longrightarrow payoff_{2,case1} = \frac{1}{1-\delta^2}\left[v(y)(1 + \beta\delta\frac{1}{1-\delta})\frac{1}{\beta} - \delta c(y) - F\right],$

if $1 < \beta\delta(1 + r)$

$\Longrightarrow payoff_{2,case2} = \frac{1}{1-\delta^2}\left[v(y)(1 + \beta\delta\frac{1}{1-\delta})\frac{1+\delta+\delta^2}{1+\beta\delta+\beta\delta^2} - \delta c(y) - F\right]$, *and*

if $1 \geq \beta\delta(1 + r)$ & $1 \leq \delta(1 + r)$

$\Longrightarrow payoff_{2,case3} = \frac{1}{1-\delta^2}\left[v(y)(1 + \beta\delta\frac{1}{1-\delta})\frac{\delta}{\beta}\frac{r^2+3r+3}{2+r+\delta} - \delta c(y) - F\right].$

Proof. For case 1, we know the present value of the revenue from the proof of the previous lemma:

$$revenue_{case1} = v(y)(1 + \beta\delta\frac{1}{1-\delta})\frac{1}{\beta}$$

The monopolist can spread the cost over two periods optimally if she offers the upgrades in every two periods, and since the cost function is linear, it will be optimal for her to produce the upgrade in the second period. The timing of this is as follows: at the beginning of the first period the monopolist knows that she will offer the upgrade at the end of the second period and according to that, she decides to create the upgrade in the second period. So the present value of the cost is $\delta c(y)$.Then we can write the payoff from one upgrade as follows:

$$payoff_{case1} = \left[v(y)(1 + \beta\delta\frac{1}{1-\delta})\frac{1}{\beta} - \delta c(y) - F\right]$$

If the monopolist offers the same amount of upgrade in every two periods, the present value of the total payoff is:

$$payoff_{2,case1} = \frac{1}{1-\delta^2}\left[v(y)(1+\beta\delta\frac{1}{1-\delta})\frac{1}{\beta} - \delta c(y) - F\right]$$

In the same way, the total payoff for Case 2 and Case 3 can be written as:

$$payoff_{2,case2} = \frac{1}{1-\delta^2}\left[v(y)(1+\beta\delta\frac{1}{1-\delta})\frac{1+\delta+\delta^2}{1+\beta\delta+\beta\delta^2} - \delta c(y) - F\right]$$

$$payoff_{2,case3} = \frac{1}{1-\delta^2}\left[v(y)(1+\beta\delta\frac{1}{1-\delta})\frac{\delta}{\beta}\frac{r^2+3r+3}{2+r+\delta} - \delta c(y) - F\right]$$

∎

Remark 1. *Note that the only difference between the payoff functions in every period (every two periods) upgrade is the first term in the parenthesis. So, we can write the payoff functions of case i as follows:*

$payoff_{1,casei} = \frac{1}{1-\delta}\left[v(x)A_i - c(x) - F\right]$

$payoff_{2,casei} = \frac{1}{1-\delta^2}\left[v(y)A_i - \delta c(y) - F\right]$

for $i = 1,\ 2$ *or* $3,$ *and* A_i *depends on* $\beta,\ \delta$ *and* r.

Lemma 4. *The optimal values of* x *and* y *are given by* $v'(x)A_i = c'(x)$ *and* $v'(y)A_i = \delta c'(y)$.

Proof. $\max_x payoff_{1,casei} = \frac{1}{1-\delta}\left[v(x)A_i - c(x) - F\right]$

$\max_y payoff_{2,casei} = \frac{1}{1-\delta^2}\left[v(y)A_i - \delta c(y) - F\right]$

From the first order conditions

$v'(x)A_i = c'(x)$

$v'(y)A_i = \delta c'(y)$

The second order condition is satisfied as $v(.)$ is concave and $c(.)$ is linear. ∎

Lemma 5. *The optimal value of* r *is equal to* $\frac{1-\beta\delta}{\beta\delta}$.

Proof. When the monopolist offers upgrades every period, the payoff function is:

$payoff_{1,casei} = \frac{1}{1-\delta}\left[v(x)A_i - c(x) - F\right]$, such that A_i is in terms of $\beta,\ \delta$ *and* r.

According to the above expression, the payoff is increasing with A. So, the monopolist chooses to create $Case\ i$ if $A_i \geq A_j, A_k\ i \neq j \neq k$ by choosing the appropriate value for r.

$\frac{1}{\beta} \geq \frac{1+\delta+\delta^2}{1+\beta\delta+\beta\delta^2} \implies A_1 \geq A_2$

$\frac{\delta}{\beta}\frac{r^2+3r+3}{2+r+\delta} \geq \frac{1}{\beta},\ for\ r\ values\ such\ that\ 1 \geq \beta\delta(1+r)\ \&\ 1 \leq \delta(1+r)\ is\ true. \implies A_3 > A_1$

So, the monopolist always chooses r such that $1 \geq \beta\delta(1+r)$ & $1 \leq \delta(1+r)$, is the condition for Case 3. Then we can write the payoff function with every period upgrades as follows:

$payoff_{1,case3} = \frac{1}{1-\delta}\left[v(x)(1+\beta\delta\frac{1}{1-\delta})\frac{\delta}{\beta}\frac{r^2+3r+3}{2+r+\delta} - c(x) - F\right]$

For the same reason as given above, we can write the payoff function with every two period upgrades as follows:

$payoff_{2,case3} = \frac{1}{1-\delta^2}\left[v(y)(1+\beta\delta\frac{1}{1-\delta})\frac{\delta}{\beta}\frac{r^2+3r+3}{2+r+\delta} - \delta c(y) - F\right]$

From now on, for convenience we will denote $payoff_{1,case3}$ as $payoff_1$ and $payoff_{2,case3}$ as $payoff_2$. Since the expression $\frac{r^2+3r+3}{2+r+\delta}$ is increasing with r, then $payoff_1$ and $payoff_2$ are increasing with r. Since the monopolist chooses the biggest r within the range given by $1 \geq \beta\delta(1+r)$ & $1 \leq \delta(1+r)$, the optimal value of r is $\frac{1-\beta\delta}{\beta\delta}$. ∎

Lemma 6. *There is a cutoff δ^* such that $payoff_2 - payoff_1$ is increasing with β for $\delta > \delta^*$.*

Proof. Let's first write the difference between the payoffs:

$$\begin{aligned} payoff_2 - payoff_1 &= \frac{1}{1-\delta^2}\left[v(y)(1+\beta\delta\frac{1}{1-\delta})\frac{\delta}{\beta}\frac{r^2+3r+3}{2+r+\delta} - \delta c(y) - F\right] \\ &- \frac{1}{1-\delta}\left[v(x)(1+\beta\delta\frac{1}{1-\delta})\frac{\delta}{\beta}\frac{r^2+3r+3}{2+r+\delta} - c(x) - F\right] \end{aligned}$$

$$\begin{aligned} payoff_2 - payoff_1 &= \frac{1}{1-\delta^2}\left[\underbrace{(1+\beta\delta\frac{1}{1-\delta})\frac{\delta}{\beta}\frac{r^2+3r+3}{2+r+\delta}}_{A}(v(y)-(1+\delta)v(x))\right] \\ &+ \frac{1}{1-\delta^2}\left[-\delta c(y) + (1+\delta)c(x) + \delta F\right] \end{aligned}$$

$$\begin{aligned}\frac{\partial\,(payoff_2 - payoff_1)}{\partial\beta} &= \frac{1}{1-\delta^2}A\left[v'(y)\frac{\partial y}{\partial\beta} - (1+\delta)v'(x)\frac{\partial x}{\partial\beta}\right] \\ &+\frac{1}{1-\delta^2}\left[v(y)-(1+\delta)v(x)\right]\frac{\partial A}{\partial\beta} \\ &+\frac{1}{1-\delta^2}\left[-\delta c'(y)\frac{\partial y}{\partial\beta} + (1+\delta)c'(x)\frac{\partial x}{\partial\beta}\right]\end{aligned}$$

If we write $v'(y)$ in terms of $c'(y)$ and $v'(x)$ in terms of $c'(x)$ by utilizing Lemma 4:

$$\frac{\partial\,(payoff_2 - payoff_1)}{\partial\beta} = \frac{1}{1-\delta^2}\left[v(y)-(1+\delta)v(x)\right]\frac{\partial A}{\partial\beta}$$

Since $A = (1+\beta\delta\frac{1}{1-\delta})\frac{\delta}{\beta}\frac{r^2+3r+3}{2+r+\delta}$ or $A = (1+\beta\delta\frac{1}{1-\delta})\frac{1+\beta\delta+(\beta\delta)^2}{\beta^2(1+\beta\delta+\beta\delta^2)}$

$$\frac{\partial A}{\partial\beta} < 0$$

$$\frac{\partial\,(payoff_2 - payoff_1)}{\partial\beta} = \frac{1}{1-\delta^2}\left[v(y)-(1+\delta)v(x)\right]\underbrace{\frac{\partial A}{\partial\beta}}_{-}$$

The limit of $y(\delta)$ as δ goes to 1 is equal to the limit of $x(\delta)$ as δ goes to 1. At this limit, $[v(y)-(1+\delta)v(x)]$ is strictly negative and equal to $-v(x)$ since $y(\delta)$ and $x(\delta)$ are continuous in δ. This means that there is a cutoff δ^* such that $[v(y)-(1+\delta)v(x)]$ is negative for $\delta > \delta^*$; consequently $payoff_2 - payoff_1$ is increasing with β for $\delta > \delta^*$. On the other hand, $payoff_2 - payoff_1$ is decreasing with β for $\delta < \delta^*$. ∎

Remark 2. *Since the value function, $v(.)$ is concave,*
$v'(x) = \frac{c'(x)}{A} \implies Av(x) > c(x)$
$v'(y) = \frac{\delta c'(y)}{A} \implies Av(y) > \delta c(y)$

Proposition 1. *If the exponential discount factor δ is big enough, there are some F and β^* which depends on F such that for $\beta < \beta^*$, the monopolist offers upgrades every period, and for $\beta > \beta^*$ she offers upgrades in every two periods. This means that the monopolist offers upgrades more frequently to more hyperbolic consumers for some fixed cost values if the exponential discount factor is high enough.*

Proof. Write the difference of the payoff functions as:

$payoff_2 - payoff_1 = \frac{1}{1-\delta^2}\left[A(v(y) - (1+\delta)v(x)) - \delta c(y) + (1+\delta)c(x) + \delta F\right]$

The limit of $y(\delta)$ is equal to the limit of $x(\delta)$ as δ goes to 1. At this limit, $[v(y) - (1+\delta)v(x)]$ is equal to $-v(x)$, and $[-\delta c(y) + (1+\delta)c(x)]$ is equal to $c(x)$, so $[A(v(y) - (1+\delta)v(x)) - \delta c(y) + (1+\delta)c(x)]$ is equal to $[-Av(x) + c(x)]$ since $y(\delta)$ and $x(\delta)$ are continuous in δ. We know that $[-Av(x) + c(x)]$ is less than zero from the previous remark. So, $[A(v(y) - (1+\delta)v(x)) - \delta c(y) + (1+\delta)c(x)]$ is less than zero for $\delta > \overline{\delta}$. If $\delta > \max\{\delta^*, \overline{\delta}\}$, $payoff_2 - payoff_1$ is increasing with β and for some F values, $payoff_2 - payoff_1 < 0$ for $\beta < \beta^*$ and $payoff_2 - payoff_1 > 0$ for $\beta > \beta^*$. This means that if δ is big enough, for some F values, the monopolist offers the upgrades more frequently to more hyperbolic consumers and less frequently to less hyperbolic consumers. ∎

Below we provide an example showing the cutoff points and the result.

Example 1. *Let $v(x) = \sqrt{x}$ and $c(x) = mx$*

$\frac{\partial payoff_2 - payoff_1}{\partial \beta} = \frac{1}{1-\delta^2}\left[v(y) - (1+\delta)v(x)\right]\frac{\partial A}{\partial \beta}$

$$\frac{\partial payoff_2 - payoff_1}{\partial \beta} = \underbrace{\frac{1}{1-\delta^2}}_{+}\left[\underbrace{\frac{A}{2m}}_{+}\underbrace{\left(\frac{1-\delta-\delta^2}{\delta}\right)}_{?}\right]\underbrace{\frac{\partial A}{\partial \beta}}_{-}$$

if $\delta > 0.61803 \implies payoff_2 - payoff_1$ is increasing with β

$payoff_2 - payoff_1 = \frac{1}{1-\delta^2}\left[\frac{1-\delta-\delta^2}{\delta}\frac{A^2}{4m} + \delta F\right]$

For $\delta > 0.61803$, there are some F values and a β^* cutoff value which depend on each F value such that the monopolist offers upgrades every period for $\beta < \beta^*$ and she offers upgrades in every two periods for $\beta > \beta^*$.

1.3.2 Sophisticated Hyperbolic Consumer

We now derive the results for the sophisticated hyperbolic consumer. These correspond to those derived already for the naive hyperbolic consumer. If the consumer is sophisticated hyperbolic, at any time period his sequence of discount factors are $\{1, \beta\delta, \beta\delta^2, \beta\delta^3, ...\}$ as in the naive hyperbolic consumer, but the sophisticated hyperbolic consumer is aware of his time inconsistency. In other words he knows that tomorrow his discount factors will be $\{1, \beta\delta, \beta\delta^2, \beta\delta^3, ...\}$ instead of $\{1, \delta, \delta^2, \delta^3, ...\}$.

For the sophisticated hyperbolic consumer, there are three possible cases with two subcases under Case 3.

- Case 1: $1 > \delta(1+r)$

  In this case, the consumer prefers to delay all the payments to the third period. The payoffs for the monopolist are:

  $$payoff_{1,case1} = \tfrac{1}{1-\delta}\left[v(x)(1+\beta\delta\tfrac{1}{1-\delta})\tfrac{1}{\beta} - c(x) - F\right]$$
  $$payoff_{2,case1} = \tfrac{1}{1-\delta^2}\left[v(y)(1+\beta\delta\tfrac{1}{1-\delta})\tfrac{1}{\beta} - \delta c(y) - F\right]$$

- Case 2: $1 < \beta\delta(1+r)$

  In this case, the consumer chooses to pay all payments in the designated periods. The payoffs for the monpolist are:

  $$payoff_{1,case2} = \tfrac{1}{1-\delta}\left[v(x)(1+\beta\delta\tfrac{1}{1-\delta})\tfrac{1+\delta+\delta^2}{1+\beta\delta+\beta\delta^2} - c(x) - F\right]$$
  $$payoff_{2,case2} = \tfrac{1}{1-\delta^2}\left[v(y)(1+\beta\delta\tfrac{1}{1-\delta})\tfrac{1+\delta+\delta^2}{1+\beta\delta+\beta\delta^2} - \delta c(y) - F\right]$$

- Case 3: $1 \geq \beta\delta(1+r)$ & $1 \leq \delta(1+r)$

  In this case, since the consumer is sophisticated, he knows that he will update his discount factors when he reaches the second period. Consequently, he knows that he will delay the second period payment to the third period. Now, he needs to determine whether to delay the first payment. Since he delays whatever he owes in the second period to the third period, delaying the first payment to the second period means delaying the first payment to the third period automatically. Since the sophisticated consumer is aware of this, he compares the amount he needs to pay if he does not delay the first payment and the amount he ends up paying if he delays the first payment until the third period. If he does not delay the first payment, he needs to pay $p/3$. If he delays the first payment, the present value of his payment is $\beta\delta^2(1+r)^2p/3$. When we compare $p/3$ and $\beta\delta^2(1+r)^2p/3$, under the conditions of $1 \geq \beta\delta(1+r)$ and $1 \leq \delta(1+r)$, we can see that $p/3 \leq \beta\delta^2(1+r)^2p/3$ for $\beta \geq \overline{\beta}$ such that $\overline{\beta}$ is between 0 and 1. This means that the consumer does not delay his first payment if $\beta \geq \overline{\beta}$, but delays it to the third period $\beta < \overline{\beta}$. As a result, we can write two subcases of Case 3:

Subcase 3a: $1 \geq \beta\delta(1+r)$ & $1 \leq \delta(1+r)$ & $\beta \geq \overline{\beta},\ 0 < \overline{\beta} < 1$

The sophisticated hyperbolic consumer does not delay the first payment, but delays the second payment until the third period. Therefore, the total payment of the consumer is $\left[1 + \beta\delta^2(1+r) + \beta\delta^2\right] p/3$. Since the total utility he gets from the upgrade is $(1+\beta\delta\frac{1}{1-\delta})v(x)$, the optimal price charged can be written as follows:

$\left[1 + \beta\delta^2(1+r) + \beta\delta^2\right] p/3 = (1+\beta\delta\frac{1}{1-\delta})v(x) \implies p/3 = \frac{(1+\beta\delta\frac{1}{1-\delta})v(x)}{[1+\beta\delta^2(1+r)+\beta\delta^2]}$

Revenue and payoffs are:

$$revenue_{subcase3a} = v(x)(1+\beta\delta\tfrac{1}{1-\delta})\tfrac{1+2\delta^2+\delta^2 r}{1+2\beta\delta^2+\beta\delta^2 r}$$
$$payoff_{1,subcase3a} = \tfrac{1}{1-\delta}\left[v(x)(1+\beta\delta\tfrac{1}{1-\delta})\tfrac{1+2\delta^2+\delta^2 r}{1+2\beta\delta^2+\beta\delta^2 r} - c(x) - F\right]$$
$$payoff_{2,subcase3a} = \tfrac{1}{1-\delta^2}\left[v(y)(1+\beta\delta\tfrac{1}{1-\delta})\tfrac{1+2\delta^2+\delta^2 r}{1+2\beta\delta^2+\beta\delta^2 r} - \delta c(y) - F\right]$$

Subcase 3b: $1 \geq \beta\delta(1+r)$ & $1 \leq \delta(1+r)$ & $\beta < \overline{\beta},\ 0 < \overline{\beta} < 1$

The sophisticated hyperbolic consumer delays all the payments until the third period. Therefore, the total payment of the consumer is $\beta\delta^2\left[r^2+3r+3\right] p/3$ and the total utility he gets is $(1+\beta\delta\frac{1}{1-\delta})v(x)$. As a result, we can find the optimal price charged as follows:

$\beta\delta^2\left[r^2+3r+3\right] p/3 = (1+\beta\delta\frac{1}{1-\delta})v(x) \implies p/3 = \frac{(1+\beta\delta\frac{1}{1-\delta})v(x)}{\beta\delta^2[r^2+3r+3]}$

Revenue and payoffs are:

$$revenue_{subcase3b} = v(x)(1+\beta\delta\tfrac{1}{1-\delta})\tfrac{1}{\beta}$$
$$payoff_{1,subcase3b} = \tfrac{1}{1-\delta}\left[v(x)(1+\beta\delta\tfrac{1}{1-\delta})\tfrac{1}{\beta} - c(x) - F\right]$$
$$payoff_{2,subcase3b} = \tfrac{1}{1-\delta^2}\left[v(y)(1+\beta\delta\tfrac{1}{1-\delta})\tfrac{1}{\beta} - \delta c(y) - F\right]$$

We can summarize these observations in the following lemma:

Lemma 7. *If the consumer is a sophisticated hyperbolic discounter, the behavior of the consumer depending on the discount factors, δ, β, and the interest rate, r, and the payoff function of the monopolist can be summarized as follows:*

if $1 > \delta(1+r) \implies$ *the consumer delays all payments until the third period and*

$$payoff_{1,case1} = \frac{1}{1-\delta}\left[v(x)(1+\beta\delta\frac{1}{1-\delta})\frac{1}{\beta} - c(x) - F\right]$$

$$payoff_{2,case1} = \frac{1}{1-\delta^2}\left[v(y)(1+\beta\delta\frac{1}{1-\delta})\frac{1}{\beta} - \delta c(y) - F\right]$$

if $1 < \beta\delta(1+r) \implies$ *the consumer pays all payments on the designated periods and*

$$payoff_{1,case2} = \frac{1}{1-\delta}\left[v(x)(1+\beta\delta\frac{1}{1-\delta})\frac{1+\delta+\delta^2}{1+\beta\delta+\beta\delta^2} - c(x) - F\right]$$

$$payoff_{2,case2} = \frac{1}{1-\delta^2}\left[v(y)(1+\beta\delta\frac{1}{1-\delta})\frac{1+\delta+\delta^2}{1+\beta\delta+\beta\delta^2} - \delta c(y) - F\right]$$

if $1 \geq \beta\delta(1+r)$ & $1 \leq \delta(1+r)$ & $\beta \geq \overline{\beta},\ 0 < \overline{\beta} < 1 \implies$ *the consumer pays the first payment on time but delays the second payment until the third period and*

$$payoff_{1,subcase3a} = \frac{1}{1-\delta}\left[v(x)(1+\beta\delta\frac{1}{1-\delta})\frac{1+2\delta^2+\delta^2 r}{1+2\beta\delta^2+\beta\delta^2 r} - c(x) - F\right]$$

$$payoff_{2,subcase3a} = \frac{1}{1-\delta^2}\left[v(y)(1+\beta\delta\frac{1}{1-\delta})\frac{1+2\delta^2+\delta^2 r}{1+2\beta\delta^2+\beta\delta^2 r} - \delta c(y) - F\right]$$

if $1 \geq \beta\delta(1+r)$ & $1 \leq \delta(1+r)$ & $\beta < \overline{\beta},\ 0 < \overline{\beta} < 1 \implies$ *the consumer delays all the payments until the third period and*

$$payoff_{1,subcase3b} = \frac{1}{1-\delta}\left[v(x)(1+\beta\delta\frac{1}{1-\delta})\frac{1}{\beta} - c(x) - F\right]$$

$$payoff_{2,subcase3b} = \frac{1}{1-\delta^2}\left[v(y)(1+\beta\delta\frac{1}{1-\delta})\frac{1}{\beta} - \delta c(y) - F\right]$$

Lemma 8. *The optimal value of* r *is either less than* $\frac{1-\delta}{\delta}$ *or equal to* $\frac{1-\delta}{\delta}$.

Proof. First, we can write the payoff functions as follows:

$$payoff_{1,casei} = \frac{1}{1-\delta}\left[v(x)A_i - c(x) - F\right]$$

$$payoff_{2,casei} = \frac{1}{1-\delta^2}\left[v(y)A_i - \delta c(y) - F\right]$$

such that A_i is in terms of $\beta,\ \delta$ and r

According to above expression, we can say that the monopolist's payoff is increasing with A. So, the monopolist will choose to create *case i* if $A_i \geq A_j, A_k$ $i \neq j \neq k$ by choosing appropriate value for r.

$\frac{1}{\beta} \geq \frac{1+2\delta^2+\delta^2 r}{1+2\beta\delta^2+\beta\delta^2 r}$ & $\frac{1}{\beta} \geq \frac{1+\delta+\delta^2}{1+\beta\delta+\beta\delta^2} \implies A_{3b} \geq A_{3a},\ A_1 \geq A_2\ and\ A_1 = A_{3b}$

So, the monopolist either chooses $r < \frac{1-\delta}{\delta}$ to create Case 1 or chooses $\frac{1-\delta}{\delta} \leq r \leq \frac{1-\beta\delta}{\beta\delta}$ with the condition $\beta < \overline{\beta}$ to create Case 3b. If the monopolist chooses r such that $\frac{1-\delta}{\delta} \leq r \leq \frac{1-\beta\delta}{\beta\delta}$ is true, she either creates Subcase 3a if $\beta \geq \overline{\beta}$ or creates Subcase 3b if $\beta < \overline{\beta}$. Since Subcase 3b gives a higher payoff than Subcase 3a, the monopolist wants to increase the possibility of Subcase 3b. In order to increase the possibility of Subcase 3b, she makes $\overline{\beta}$ as big as possible. Remember that in Case 3 the following condition holds:

$$\begin{aligned} 1 &\geq \beta\delta(1+r) \ \&\ 1 \leq \delta(1+r) \\ & and \\ 1 &> \underbrace{[\beta\delta(1+r)]}_{\leq 1}\underbrace{[\delta(1+r)]}_{\geq 1}\text{for } \beta < \overline{\beta} \end{aligned}$$

if $r = \frac{1-\delta}{\delta} \implies 1 > [\beta\delta(1+r)]\,[\delta(1+r)]$ for $\beta < \overline{\beta} = 1$

Therefore, in order to make $\overline{\beta}$ as big as possible, the monopolist should choose $r = \frac{1-\delta}{\delta}$. As a result, the optimal value of r is either less than $\frac{1-\delta}{\delta}$ or equal to $\frac{1-\delta}{\delta}$. ∎

Lemma 9. *There is a cutoff value δ^{**} such that $payoff_2 - payoff_1$ is increasing with β for $\delta > \delta^{**}$.*

Proof. From the proof of Lemma 6:

$$\frac{\partial payoff_2 - payoff_1}{\partial \beta} = \frac{1}{1-\delta^2}\left[v(y) - (1+\delta)v(x)\right]\frac{\partial A}{\partial \beta}$$

If the consumers are sophisticated, the monopolist would choose either $r < \frac{1-\delta}{\delta}$ or $r = \frac{1-\delta}{\delta}$. In both cases, A is equal to $\frac{1}{\beta}$. We know that $\frac{\partial A}{\partial \beta} = -\frac{1}{\beta^2} < 0$, consequently the sign of $\frac{\partial payoff_2 - payoff_1}{\partial \beta}$ depends on the sign of $[v(y) - (1+\delta)v(x)]$.

The limit of $y(\delta)$ is equal to the limit of $x(\delta)$ as δ goes to 1. At this limit, $[v(y) - (1+\delta)v(x)]$ is strictly negative and equal to $-v(x)$ since $y(\delta)$ and $x(\delta)$ are continuous in δ. This means that for a cutoff δ^{**}, $[v(y) - (1+\delta)v(x)]$ is negative for $\delta > \delta^{**}$; consequently $payoff_2 - payoff_1$ is increasing with β for $\delta > \delta^{**}$ and decreasing with β otherwise. ∎

Proposition 2. *If the exponential discount factor δ is big enough, there are some F and β^{**} which depends on F such that the monopolist offers upgrades every period for $\beta < \beta^{**}$, and she offers upgrades in every two periods for $\beta > \beta^{**}$. This means that the monopolist offers upgrades more frequently to more hyperbolic consumers for some fixed cost values if the exponential discount factor is high enough.*

Proof. Let's write the difference of the payoff functions:

$$payoff_2 - payoff_1 = \frac{1}{1-\delta^2}\left[A(v(y) - (1+\delta)v(x)) - \delta c(y) + (1+\delta)c(x) + \delta F\right]$$

The limit of $y(\delta)$ is equal to the limit of $x(\delta)$ as δ goes to 1. At this limit, $[v(y) - (1+\delta)v(x)]$ is equal to $-v(x)$, and $[-\delta c(y) + (1+\delta)c(x)]$ is equal to $c(x)$, so $[A(v(y) - (1+\delta)v(x)) - \delta c(y) + (1+\delta)c(x)]$ is equal to $[-Av(x) + c(x)]$ since $y(\delta)$ and $x(\delta)$ are continuous in δ. We know that $[-Av(x) + c(x)]$ is less than zero. So we can say that $[A(v(y) - (1+\delta)v(x)) - \delta c(y) + (1+\delta)c(x)]$ is less than zero for $\delta > \widetilde{\delta}$. If $\delta > \max\{\delta^{**}, \widetilde{\delta}\}$, $payoff_2 - payoff_1$ is increasing with β and for some F values, $payoff_2 - payoff_1 < 0$ for $\beta < \beta^{**}$ and $payoff_2 - payoff_1 > 0$ for $\beta > \beta^{**}$. This means that the monopolist will offer the upgrades more frequently to more hyperbolic consumers for some F values if δ is high enough. ∎

If we look at the same example we gave before, but with a sophisticated hyperbolic consumer instead of naive hyperbolic consumer, we can see that the cutoff values for δ are the same, which does not have to hold always.

Example 2. *Let $v(x) = \sqrt{x}$ and $c(x) = mx$ This is the same example as example 1, with the only difference being the expression for A.*

$\frac{\partial payoff_2 - payoff_1}{\partial \beta} = \frac{1}{1-\delta^2}\left[v(y) - (1+\delta)v(x)\right]\frac{\partial A}{\partial \beta}$

$$\frac{\partial payoff_2 - payoff_1}{\partial \beta} = \underbrace{\frac{1}{1-\delta^2}}_{+} \left[\underbrace{\frac{A}{2m}}_{+} \underbrace{\left(\frac{1-\delta-\delta^2}{\delta} \right)}_{?} \right] \underbrace{\frac{\partial A}{\partial \beta}}_{-}$$

if $\delta > 0.61803 \implies payoff_2 - payoff_1$ *is increasing with* β

$payoff_2 - payoff_1 = \frac{1}{1-\delta^2} \left[\frac{1-\delta-\delta^2}{\delta} \frac{A^2}{4m} + \delta F \right]$

For $\delta > 0.61803$, *there are some* F *values and a* β^{**} *cutoff value which depends on each* F *value such that for* $\beta < \beta^{**}$ *the monopolist offers upgrades every period and for* $\beta > \beta^{**}$ *she offers upgrades in every two periods.*

In this example, although the cutoff values for δ are the same for both naive and sophisticated consumers, we know that the cutoff values for β will be different, which is discussed in the following section.

1.3.3 Naive versus Sophisticated Consumers

For naive and sophisticated hyperbolic consumers, we found that if the δ is greater than some cutoff value the monopolist changes her strategy depending on the level of hyperbolicity of the consumer. This means that the monopolist offers the upgrades more frequently to more hyperbolic consumers, who have smaller β discount factors. An interesting question might be what would happen if the only difference is whether the consumer is naive or sophisticated. In other words, there are two possible kinds of consumers with the same discount factors and the only difference between them is whether they are naive of sophisticated.

Proposition 3. *The monopolist's strategy does not only depend on the level of the time inconsistency of the consumers, but also depends on whether the consumer is aware of his time-inconsistent behavior. For some* β *values, the monopolist offers upgrades more frequently to a naive hyperbolic consumer than to a sophisticated hyperbolic consumer, although the level of time inconsistency is the same for both.*

Proof. Remember the expression for the difference of the payoffs:

$$payoff_2 - payoff_1 = \frac{1}{1-\delta^2} \left[\underbrace{A(v(y) - (1+\delta)v(x)) - \delta c(y) + (1+\delta)c(x)}_{R} + \delta F \right]$$

Since the monopolist chooses the r value, we know A_N and A_S from the previous findings,[7]

$$\begin{aligned}
&A_N = (1+\beta\delta\frac{1}{1-\delta})\frac{1+\beta\delta+(\beta\delta)^2}{\beta^2\left(1+\beta\delta+\beta\delta^2\right)}, \\
&A_S = (1+\beta\delta\frac{1}{1-\delta})\frac{1}{\beta}, \\
&\text{therefore } A_N > A_S.
\end{aligned}$$

We can find how the R part of the expression for difference of payoffs changes with A.

$$\begin{aligned}
\frac{\partial R}{\partial A} &= (v(y)-(1+\delta)v(x)) + A\left(v'(y)\frac{\partial y}{\partial A} - (1+\delta)v'(x)\frac{\partial x}{\partial A}\right) \\
&\quad -\delta c'(y)\frac{\partial y}{\partial A} + (1+\delta)c'(x)\frac{\partial x}{\partial A}
\end{aligned}$$

From the first order conditions given in Lemma 4, we can write $\partial R/\partial A$ as follows:

$$\frac{\partial R}{\partial A} = (v(y)-(1+\delta)v(x))$$

If the consumer is a naive hyperbolic consumer,

$$\begin{aligned}
&(v(y)-(1+\delta)v(x)) < 0, \text{ and} \\
&payoff_2 - payoff_1 \text{ is increasing with } \beta \text{ for } \delta > \max\{\delta^*, \overline{\delta}\}
\end{aligned}$$

If the consumer is a sophisticated hyperbolic consumer,

$$\begin{aligned}
&(v(y)-(1+\delta)v(x)) < 0, \text{ and} \\
&payoff_2 - payoff_1 \text{ is increasing with } \beta \text{ for } \delta > \max\{\delta^{**}, \widetilde{\delta}\}
\end{aligned}$$

[7]Subscript N and S denote naive and sophisticated consumers respectively.

R part of the expression in the difference of the payoffs is decreasing with A for $\delta > \max\{\delta^*, \delta^{**}\}$:

$$\begin{aligned} A_N > A_S &\implies R_N < R_S \\ &\implies [payoff_2 - payoff_1]_N < [payoff_2 - payoff_1]_S \end{aligned}$$

Let's assume that there is an F such that $[payoff_2 - payoff_1]_N < 0$ and $[payoff_2 - payoff_1]_S < 0$ for $\delta > \max\{\delta^*, \overline{\delta}, \delta^{**}, \widetilde{\delta}\}$ and β is very small. Since $\delta > \max\{\delta^*, \delta^{**}\}$, both $[payoff_2 - payoff_1]_N$ and $[payoff_2 - payoff_1]_S$ increase with β and $[payoff_2 - payoff_1]_N$ is always less than $[payoff_2 - payoff_1]_S$. This means that $[payoff_2 - payoff_1]_S$ reaches zero (at $\beta = \beta^{**}$) before $[payoff_2 - payoff_1]_N$ reaches zero (at $\beta = \beta^*$), as β increases.

For $\delta > \max\{\delta^*, \overline{\delta}, \delta^{**}, \widetilde{\delta}\}$, there are F, β^*, and β^{**} for naive and sophisticated hyperbolic consumers respectively such that:

1. If $\beta^{**} < \beta < \beta^* \implies$ the monopolist offers upgrades every period for a naive consumer although she offers upgrades in every two periods for a sophisticated consumer.

2. If $\beta < \beta^{**} \implies$ she offers upgrades every period for both kinds of consumers.

3. If $\beta^* < \beta \implies$ she offers upgrades in every two periods for both kinds of consumers.

Different cutoff values for naive and sophisticated consumers suggest that the monopolist's strategy does not only depend on the hyperbolic behavior of the consumer but also on whether the consumer is aware of his time inconsistency. ∎

1.4 Exogenous Interest Rate

In this section we analyze the case in which there is a competitive financial market, implying a fixed interest rate. In this case the monopolist cannot choose the interest rate. Since our concern is to see whether there can be a change in the strategy of the monopolist if there are time-inconsistent consumers, we only analyze the cases in which we observe time inconsistency in the consumer's behavior. Therefore, the case we analyze is Case 3 such that $1 \geq \beta\delta(1+r)$ and $1 \leq \delta(1+r)$ is true. Since we only get results for Case 3 under naive hyperbolic consumer and Subcase 3b under sophisticated hyperbolic consumer, we have a partial result if the interest rate is exogenous.

First, we solve the problem if the consumer is naive hyperbolic. From the previous findings we know that $A = (1 + \beta\delta\frac{1}{1-\delta})\frac{\delta}{\beta}\frac{r^2+3r+3}{2+r+\delta}$ if the consumer is naive hyperbolic and if $1 \geq \beta\delta(1+r)$ and $1 \leq \delta(1+r)$ is true. Since $\frac{\partial A}{\partial \beta} < 0$, Lemma 6 and Proposition 1 holds for this case as well. As a result, we get the same result we found in Section 1.3.1.

Now, we analyze the problem if the consumer is sophisticated hyperbolic. Again we consider Case 3 in which $1 \geq \beta\delta(1+r)$ and $1 \leq \delta(1+r)$ holds. We know that there are two subcases if the consumer is sophisticated. If Subcase 3b occurs, A is equal to $(1 + \beta\delta\frac{1}{1-\delta})\frac{1}{\beta}$ and $\frac{\partial A}{\partial \beta} < 0$. Consequently Lemma 9 and Proposition 2 hold for this case as well and we get the same result we found in Section 1.3.2.

1.5 Welfare Analysis

In this section, we determine what frequency of upgrades, either every period or in every two periods, is socially better. Moreover, we show that sometimes monopolist offers upgrades every period although to offer the upgrades in every two periods is socially better. We evaluate the consumer's welfare with long term preferences in the sense that $\beta = 1$. This welfare measure is used by O'Donoghue & Rabin (2001) and Della Vigna & Malmendier (2004).

Proposition 4. *When the consumer is naive hyperbolic, if the exponential discount factor δ is big enough, there are some F and $\overline{\beta}$, which depends on F such that for $\beta < \overline{\beta}$, the monopolist offers the upgrades every period although to offer upgrades in every two periods is socially better.*

Proof. Define the consumer welfare of a consumer with $\beta - \delta$ discount factor every time he buys an upgrade as:

$$\frac{1}{1-\delta}v(x) - p$$

such that p is the total price he pays. This is the welfare evaluated with long-term preferences. We can write the welfare of the monopolist every time he offers the upgrades when he offers it every period and in every two periods as follows:

$$p - (c(x) + F)$$

$$p - (\delta c(y) + F)$$

If the monopolist offers upgrades every period, the social welfare is:

$$w_1 = \frac{1}{1-\delta}\left[\frac{1}{1-\delta}v(x) - (c(x) + F)\right]$$

If the monopolist offers upgrades in every two periods, the social welfare is:

$$w_2 = \frac{1}{1-\delta^2}\left[\frac{1}{1-\delta}v(y) - (\delta c(y) + F)\right]$$

Lets write down the difference of the social welfare in each case:

$$w_2 - w_1 = \frac{1}{1-\delta^2}\left[\frac{1}{1-\delta}\left[v(y) - (1+\delta)v(x)\right] - \delta c(y) + (1+\delta)c(x) + \delta F\right]$$

Note that $w_2 - w_1$ does not depend on β since $\frac{\partial(w_2-w_1)}{\partial\beta} = 0$. Remember that we can write the difference of the payoffs when the monopolist offers upgrades every period and in every two period as:

$$p_2 - p_1 = \frac{1}{1-\delta^2}\left[\begin{array}{c} \underbrace{(1+\beta\delta\frac{1}{1-\delta})\frac{\delta}{\beta}\frac{r^2+3r+3}{2+r+\delta}}_{A}(v(y) - (1+\delta)v(x)) \\ -\delta c(y) + (1+\delta)c(x) + \delta F \end{array}\right]$$

We already showed that $\frac{\partial(p_2-p_1)}{\partial\beta} > 0$ for $\delta > \delta^*$. Note that the only difference between $(w_2 - w_1)$ and $(p_2 - p_1)$ is the factor of $(v(y) - (1+\delta)v(x))$. Let $\beta = 1$, then we can show that there is a $\widehat{\delta}$ such that for $\delta > \widehat{\delta}$, $A > \frac{1}{1-\delta}$. Since $\frac{\partial A}{\partial\beta} < 0$, for $0 < \beta < 1$, $A > \frac{1}{1-\delta}$ is always true. Therefore, there is $\widetilde{\delta} = \max\{\delta^*, \widehat{\delta}\}$ such that for $\delta > \widetilde{\delta}$, the following holds:

$$p_2 - p_1 < w_2 - w_1$$

From proposition (1), we know that for some F values, $payoff_2 - payoff_1 < 0$ for $\beta < \beta^*$ and $payoff_2 - payoff_1 > 0$ for $\beta > \beta^*$.Therefore, there are some F and $\overline{\beta}$, which depends on F such that for $\beta < \overline{\beta}$, the monopolist will offer the upgrades in every period although it is socially better to offer upgrades in every two periods. ∎

Proposition 5. *When the consumer is sophisticated hyperbolic, if the exponential discount factor δ is big enough, there are some F and $\widehat{\beta}$, which depends on F such that for $\beta < \widehat{\beta}$, the monopolist offers the upgrades every period although to offer upgrades in every two periods is socially better.*

Proof. The proof is similar to the proof of proposition (4) with the difference on expression A. We can write $p_2 - p_1$ as follows:

$$p_2 - p_1 = \frac{1}{1-\delta^2}\begin{bmatrix} \underbrace{(1+\beta\delta\frac{1}{1-\delta})\frac{1}{\beta}}_{A}(v(y) - (1+\delta)v(x)) \\ -\delta c(y) + (1+\delta)c(x) + \delta F \end{bmatrix}$$

For all δ and β, $A \geq \frac{1}{1-\delta}$. For high enough values of δ, the following holds

$$p_2 - p_1 < w_2 - w_1$$

From proposition (2), we know that for some F values, $payoff_2 - payoff_1 < 0$ for $\beta < \beta^{**}$ and $payoff_2 - payoff_1 > 0$ for $\beta > \beta^{**}$.Therefore, there are some F and $\widehat{\beta}$, which depends on F such that for $\beta < \widehat{\beta}$, the monopolist will offer the upgrades in every period although it is socially better to offer upgrades in every two periods. ∎

1.6 Conclusion and Discussion

In this paper, using a simple model, we explain frequent and small upgrades observed in the market for certain goods such as software. Our model assumes that the upgrades are build on each other. This assumption is very close to the one in Fishman and Rob (2000).

In our model, we assume the monopolist divides the total price of the upgrade into three equal payments because three periods are minimal to see the time-inconsistent behavior of hyperbolic agents. For more than three periods, we would get the same results but with more complicated calculations. Therefore, for the sake of simplicity we assume the monopolist divides the payments into three equal ones.

There might be a question about the optimal number of payments that the monopolist offers. In fact, it is not optimal for the monopolist to divide the payments into three if she determines the value of r and if the consumer is naive hyperbolic. Because in that case, the monopolist will choose r such that the naive hyperbolic consumer believes that he will delay the first payment but not the second, although he will end up delaying both the first and the second payments until the third period. In such a situation, the consumer ends up paying more than what he was planning. The more division of the payment the monopolist offers, the more the consumer pays. Therefore, the monopolist would prefer to offer as large a number of payments (maybe infinite) as possible to the consumer and use him as a money pump if the consumer does not have a budget constraint. In our model, the monopolist and the consumer have δ exponential discount factor and the consumer has additional β hyperbolic discount factor. If the discount factor for the monopolist is less than δ, then we cannot expect the monopolist to offer an infinite number of payments. As the consumer delays the payments, the amount the monopolist gets will be discounted more severely. As a result, we can say that as the discount factor of the monopolist decreases, the optimal number of payments the monopolist offers also decreases. The other case is if the monopolist determines the value of r and if the consumer is a sophisticated hyperbolic. In this case, the number of periods of payment does not change the monopolist's profit, as long as it is more than one. This is quite intuitive. Since the consumer is a sophisticated, the monopolist cannot use him as a money pump and get more profit by making him delay more. Even if the monopolist makes him delay more by giving more periods to delay, the sophisticated consumer will be able to calculate correctly how much he will end up paying as opposed to the naive hyperbolic consumer.

Another question might be about the optimal division of payments. If the monopolist determines the value of r and if the consumer is naive hyperbolic, it is not optimal to divide the payments equally. In that scenario the consumer will end up delaying all payments until the last period. Since the more the consumer delays, the more the monopolist earns, the monopolist maximizes her profit by offering the consumer to pay the entire amount in the first period but giving him the flexibility to delay it with a determined number of periods. Now, consider the case in which the monopolist determines the value of r and if the consumer is sophisticated hyperbolic. As explained in the previous paragraph, the monopolist does not get more or less profit by making the consumer delay more or less, as long as he delays at least one period. Division of payments will give the same profit to the monopolist.

In our model, although we solved the case for durable upgrades, we would get the same result if the upgrades were not durable but with different cutoff values for δ and β.

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Chapter 2

Credit Card Competition and Naive Hyperbolic Consumers

2.1 Introduction

The credit card industry has evoked much interest in the last few years because of the conjunction of persistent high interest rates and what appears to be vigorous competition among credit card providers (Ausubel, 1991). The two questions that have been most puzzling are, first, why interest rates stay high despite the competition and, second, why consumers continue to borrow at these high rates. Moreover, there is no agreement in the literature as to whether the banks/companies earn competitive profits in this market (Evans & Schmalensee, 2000). Motivated by these questions, we present a theoretical model of credit card competition to determine the possibility of positive expected profits in the market equilibrium. The main features of the model are time-inconsistent consumers and a grace period offer in credit card contracts.[1] We, therefore, also contribute to the debate as to whether time-inconsistent consumers are "money pumps" or whether competition

[1]Strotz (1956) claims that people would not obey their optimal plan of the present moment if they were allowed to reconsider their plans in future periods. Because people are impatient; they give more weight to the earlier time as it gets closer, and this causes time inconsistent behavior. There is a significant amount of evidence for the existence of time-inconsistent preferences. See Hausman (1979), Loewenstein and Thaler (1989), Ainslie (1991), and DellaVigna and Malmendier (2002).

effectively eliminates the disadvantage from this inconsistency.[2]

We show that there are multiple equilibria with zero and positive expected profits for some parameter values. We also find that there is a unique positive profit equilibrium for some other parameter values, and a unique zero profit equilibrium for still some other parameter values.

In our model, there is an initial period for contracting, followed by three consumption periods. Two credit card companies simultaneously offer contracts that are defined by the interest rate and the credit limit and these contracts have a grace period. We allow the consumer to accept more than one contract in the contracting period and to declare bankruptcy. The consumer is time inconsistent and has a constant income at each period. We model time inconsistency using quasi-hyperbolic discount structure of Laibson (1997). A time-inconsistent consumer underestimates his future debt at any given period. If the contracting between the consumer and a company occurs before the consumption takes place (e.g. the consumer is allowed to borrow on his new card after the contracting period only) and if the contract includes a grace period offer, then these circumstances may cause the consumer to believe that he will not pay interest on his credit card debt; that is, he believes that he is just a convenience user.[3] If the consumer believes that he is just a convenience user, then he is indifferent among different interest rates on different credit card contracts. This indifference eliminates the competition on the interest rate. In addition, the consumer would be indifferent to credit limit variations if all the credit limits offered are higher than his believed amount of future debt in any period.

In the literature, there are experimental and empirical studies that analyze consumer behavior in the credit card market.[4] There are also a few theoretical papers that provide alternative explanations for the phenomenon we consider. Parlour and Rajan (2001) construct a model in which the competing firms cannot sustain zero-profit equilibria under certain conditions. In their model, there are three stages. In the first stage, companies offer contracts with a credit line and interest

[2] See Laibson and Yariv (2004)

[3] A grace period is a period in which the consumer may pay his debt without interest. The consumer who always pays his debt within the grace period is called a convenience user.

[4] See Calem and Mester (1995), Ausubel (1999), Laibson, Repetto and Tobacman (2001) and Ausubel and Shui (2004)

rate, and in the second stage the consumer decides which credit card contracts to accept and whether to default. If he is going to default, he accepts all contracts offered in the second stage. In Parlour and Rajan's (2001) model, the consumer has an incentive to default given by $\alpha \geq 0$ such that αd is the amount of the consumer's shielded assets from bankruptcy when he defaults on total loans of d. If the incentive to default is high enough and if there is multiple contracting, they show that there are positive profits. Moreover, the interest rates are sticky and above the risk-adjusted cost of funds. Parlour and Rajan's (2001) results depend on a moral hazard problem because the consumer decides whether to default or not at the second stage when he decides which offers to accept.

In our model, as opposed to that of Parlour and Rajan (2001), the consumer may decide to default in later periods and the result does not depend on the moral hazard problem. We also include the cost of bankruptcy for the consumer because of the costly bankruptcy procedure, as well as future costs of declaring bankruptcy—e.g. a bad credit score.

Dellavigna and Malmendier (2004) analyze the firms' profit-maximizing contract design when the consumers are partially naive hyperbolic discounters. They show that in the optimal two-part tariff, firms price "investment goods" less than the marginal cost and "leisure goods"—e.g. credit card financed consumptions—higher than the marginal cost. They also show that this result is robust to competition. However, the company earns more than a competitive profit only under a monopoly case.

We show the possibility of more than marginal cost pricing in a different model than that of Dellavigna and Malmendier (2004), although we suggest the same phenomenon —consumer's time inconsistency—as an explanation. We do not restrict the contracts with two-part tariffs so that the consumer has a preference on the credit limit as well. We also include the risk of bankruptcy and show that a positive expected profit equilibrium might be possible even if there is competition.

Brito and Hartley (1995) show that rational individuals may choose to pay interest on a credit card rather than pay the transaction costs on regular bank loans. They show that the demand for credit card debts is likely to be less sensitive to a change in credit card interest rates than to a change in interest rates of alternative loans. Since a decrease in cost of funds will decrease the interest rates in alternative loans, existing customers' demand for the credit card debts will decrease, consequently the banks will seek additional customers with higher risk of default and the credit card interest rates will reflect this higher risk.

In contrast to the approach of Brito and Hartley (1995), we look at the competition among credit card companies rather than the competition between credit card financing and other forms of financing. Moreover, the explanation we provide for the lack of competition on the interest rate does not depend on the consumers with high risk of default.

Eliaz and Spiegler (2006) characterize the menu of contracts when a principal faces dynamically inconsistent heterogenous consumers. They show, under a monopoly case, that a principal can exploit more naive consumers more intensely. They also provide some examples, including the credit card "teaser rates", for the application of their results.

In our model, we include the grace period feature of the credit card market. To the best of our knowledge, this feature was not included in the previous credit card competition models. The interest rate would have affected the consumer's decision had we not include the grace period in our model. In reality, however, the interest rate does not affect the decision of a convenience user.

Summarizing our contribution, we show that the naive hyperbolic consumer might be unresponsive to the interest rate and the credit limit of the credit card offers by companies because the offers have a grace period. We determine whether the credit card companies can exploit time-inconsistent consumers and gain positive expected profits. We show that in fact there are circumstances in which both zero and positive expected profits could be possible.

2.2 The Model

There are three periods of consumption preceded by an initial period/period zero which is for contracting only (We need at least three periods in order to see an interest bearing debt). There is one good at each consumption period. There are three agents in the model: one consumer and two companies who compete against each other for the credit card business of the consumer. Companies know the consumer's type so that this is a model of competition for each different type of consumer.

2.2.1 The Consumer

Following Phelps and Pollak (1968) and Laibson (1997), we use hyperbolic discounting to model the time-inconsistent consumer. In the hyperbolic discounting literature, starting with Strotz (1956), two kinds of consumers are discussed. The first kind is called "naive" as he is not aware of his time inconsistency. Specifically, he knows that his future discounting today is $\{1, \beta\delta, \beta\delta^2, \beta\delta^3, ..\}$, and believes that from tomorrow on it will be $\{1, \delta, \delta^2, \delta^3, ..\}$, although in reality it will be $\{1, \beta\delta, \beta\delta^2, \beta\delta^3, ..\}$ again. The second, called "sophisticated," is aware of his time inconsistency. He knows that his future discounting today is $\{1, \beta\delta, \beta\delta^2, \beta\delta^3, ..\}$, and he correctly anticipates that it will be $\{1, \beta\delta, \beta\delta^2, \beta\delta^3, ..\}$ from tomorrow on. O'Donoghue and Rabin (2001) introduce a model to represent a partially naive hyperbolic consumer, who is aware of his time inconsistency but underestimates its severity. According to O'Donoghue and Rabin (2001), the partially naive hyperbolic consumer knows that his future discounting today is $\{1, \beta\delta, \beta\delta^2, \beta\delta^3, ..\}$, and he incorrectly believes that it will be $\{1, \beta'\delta, \beta'\delta^2, \beta'\delta^3, ..\}$ from tomorrow on such that $\beta < \beta'$.

In our model, we analyze the naive hyperbolic consumer only with $\beta \in [0, 1)$. This consumer chooses contracts at the initial period and consumes the consumption good in the following three consumption periods. There is an arbitrarily small cost for accepting a contract at period zero. The consumer's total utility is also affected by whether he defaults in the last period.

At each period $t = 0, 1, 2$, the consumer aims to maximize U_t, where

$$
\begin{aligned}
&U_0 = \beta\delta\left[u(c_1^0) + \delta u(c_2^0) + \delta^2 u(c_3^0) + \delta^2 v(d_1^0, d_2^0)\right],\\
&U_1 = u(c_1^1) + \beta\delta u(c_2^1) + \beta\delta^2 u(c_3^1) + \beta\delta^2 v(d_1^1, d_2^1),\\
&\textit{and}\\
&U_2 = u(c_2^2) + \beta\delta u(c_3^2) + \beta\delta v(d_1^2, d_2^2).
\end{aligned}
$$

Here $c_\tau^t \in \mathbb{R}_+$ is the time t plan for consumption at time τ, and $d_j^t \in \{-1, 0\}$

is the default decision: $d_j^t = -1$ denotes default and $d_j^t = 0$ denotes no default. The function u is strictly increasing and concave, and satisfies the standard Inada conditions. The function v satisfies $v(-1,0) = v(0,-1) = v(-1,-1) = -C$ and $v(0,0) = 0$, where $C > 0$ is the exogenous cost of default. We interpret this cost as the cost of bankruptcy proceedings and of having unfavorable terms in any contract in the future after declaring bankruptcy. As the consumer becomes more of a risky prospect, the cost of bankruptcy will be lower for the consumer.

We do not need to write U_3 because period three is the terminal period which implies that the period two plan is implemented.

The consumer chooses a trade $y_j^t = (s_{j0}, n_{j1}^t, p_{j2}^t, n_{j2}^t, p_{j3}^t, d_j^t)$ for each company's card $j = 1, 2$ at each period. Here $s_{j0} \in \{0, 1\}$, where $s_{j0} = 1$ denotes the contract is signed with company j at period zero and $s_{j0} = 0$ denotes it is rejected; $n_{j\tau}^t \in \mathbb{R}_+$ is the time t plan for the amount of new debt from company j at time τ; $p_{j\tau}^t \in \mathbb{R}_+$ is the time t plan for the repayment to company j at time τ. At each consumption period, the consumer receives an income of m.

These trades of the consumer determine the consumption:

$$c_1^t = m + \sum_{j=1}^{2} s_{j0} n_{j1}^t; \; t = 0, 1.$$

$$c_2^t = m + \sum_{j=1}^{2} s_{j0} \left(n_{j2}^t - p_{j2}^t\right); \; t = 0, 1, 2.$$

$$c_3^t = m - \left[\left(1 + d_1^t\right)\left(1 + d_2^t\right)\right] \sum_{j=1}^{2} s_{j0} p_{j3}^t; \; t = 0, 1, 2, 3.$$

The consumer chooses the trades sequentially. The time t plan does not affect the subsequent period's plan except for the trades completed at time t —those for which $t = \tau$.

2.2.2 The Companies and the Class of Contracts

For simplicity, we assume that the only source of revenue is from the interest payments. Company $j's$ profit is $\Pi_j = \left(1 + d_j^3\right)(n_{j1}^1 - p_{j2}^2) r_j + d_j^3 \left(n_{j1}^1 + n_{j2}^2 - p_{j2}^2\right).$ Informally, each credit card company j charges an interest rate of r_j for loans for more than one period, although it is not permitted to charge interest for only one-period loans. A credit card company loses everything lent if the consumer defaults.

The company's strategy set is not the set of all contracts but consists only of contracts specified by a credit limit l and interest rate $r \in [0,1]$,[5] which have the following form:

$$\begin{aligned} n_{j1} &\leq l_j \\ n_{j1} + n_{j2} - p_{j2} &\leq l_j \end{aligned}$$

which means that the consumer's total debt cannot be greater than his credit limit l_j at any period. Also,

$$\begin{aligned} p_{j2} &\leq \min\{m, n_{j1}\} \\ p_{j3} &\leq \min\{m, (n_{j1} - p_{j2})(1 + r_j) + n_{j2}\} \end{aligned}$$

which means that the consumer's payment cannot be higher than his income or his total debt at any period.

2.2.3 Strategic Interaction

The two companies make simultaneous contract offers and the consumer decides which to choose at the initial period; subsequently the consumer makes two sequential decisions as described in Section 2.2.1. Therefore, the only strategic game between the companies and the consumer takes place at period zero.

We focus on the pure strategy subgame perfect equilibria of this game.[6] From this point on we will examine only the subgame perfect equilibria with three tie breaking conventions. One is on consumer's decision and two on the companies' credit limit offers;

- if the consumer wants to choose one contract and is indifferent between the

[5] Our results would not change as long as the upper bound is finite.

[6] Players act optimally at each decision node. However, as a difference from the standard SPNE, a naive hyperbolic consumer has wrong beliefs about his future decisions.

contracts, then each choice is equally likely;

- if a company is indifferent among two credit limits which are higher than m, the company chooses the lower one;
- if a company is indifferent among two credit limits which are lower than m, the company chooses the higher one.

2.3 Analysis

We first determine how the consumer chooses the contract(s) given that he does not default. Our approach is as if there is only one interest rate affecting the consumer's problem. We later confirm that in equilibrium only one interest rate is relevant for the consumer even if he has two different contracts. Second, we analyze the consumer's decision on default. Third, we explain the companies' objective functions and determine the best responses under two different cases; under the first case, one contract is chosen and under the second case, both contracts are chosen. Fourth, we show the existence of the equilibrium by construction. We assume that a particular case holds and calculate the best responses, and then check whether the strategies are consistent with that assumed case. If not, we do the same exercise for the other case.

2.3.1 The Consumer's Behavior with no Default Option

In this model, we analyze only the first-period debt because interest revenue comes only from the first-period debt. Since it is a dominant strategy for the consumer to pay as much of his debt as possible within the grace period, we can write the consumer's repayment in each period in terms of his income and total debt of the previous period. In particular, if $n_1 \leq m$, then $p_2 = n_1$ and $p_3 = n_2$; if $n_1 > m$, then $p_2 = m$ and $p_3 = (n_1 - m)(1 + r)$.

At each period, the consumer underestimates the future period debts. Therefore, the initial-period self's believed amount of debt for the first period, n_1^0, is always less than the actual amount of debt, n_1^1.[7]

[7]In the hyperbolic discounting literature, the consumer in different periods is considered as different selves of the consumer.

Proposition 6. *There is a cutoff exponential discount factor δ' such that*

$$\text{for all } \delta \geq \delta', \ n_1^0 \leq m.$$

There is another cutoff exponential discount factor $\delta'' \geq \delta'$ and a cutoff hyperbolic discount factor β' such that

$$\text{for all } (\delta, \beta) \text{ where } \delta > \delta'' \text{ and } \beta < \beta', \ n_1^0 \leq m < n_1^1.$$

Proof. See the appendix for the proof of the proposition ∎

Accordingly, the initial-period self believes that he can pay the first-period debt within the grace period without interest, therefore he is not responsive to interest rates. However, the consumer ends up paying interest as opposed to the initial-period self's belief.

We consider the consumer with discount factors $\delta > \delta''$ and $\beta < \beta'$ only. Since there is an arbitrarily small cost for accepting a contract, the consumer will choose only one contract if there is at least one company offering enough credit limit. Suppose the equilibrium credit limits are l_1 and l_2,

- if $\max\{l_1, l_2\} \geq \max\{n_1^0, n_2^0\} \equiv n^0 > \min\{l_1, l_2\}$, then the consumer accepts only the contract with higher credit limit;
- if $\min\{l_1, l_2\} \geq n^0$, then the consumer accepts one contract randomly;
- if $\max\{l_1, l_2\} < n^0$, then the consumer accepts both contracts.

2.3.2 The Consumer's Decision on Default

At each period, the consumer compares his total utility from defaulting and not defaulting, which gives three different cutoff cost of default values, namely C_0, C_1, and C_2. If the consumer's exogenously given cost of default (C) is higher than these cutoffs, he chooses not to default.

The relevant cutoff (the highest one of the three) depends on the number of contracts chosen. Suppose l_1 and l_2 are in appropriate ranges that the consumer chooses only one contract. For convenience, we take the first company as the chosen company with interest rate r if there is only one contract chosen. The cutoffs and the relation among them are then given in the following two lemmas.

Lemma 10. *When only one contract is accepted, the cutoff cost of default for each period is as follows*[8]*:*

$$
\begin{aligned}
C_0 &= \frac{1}{\delta^2}\left[\begin{array}{c} \max\limits_{n_1 \leq l_1 + l_2} [u(m+n_1) + \delta u(m + l_1 + l_2 - n_1) + \delta^2 u(m)] \\ - \max\limits_{n_1 \leq m, n_2} [u(m+n_1) + \delta u(m - n_1 + n_2) + \delta^2 u(m - n_2)] \end{array}\right] \\
C_1 &= \frac{1}{\beta\delta^2}\left[\begin{array}{c} \max\limits_{n_1 \leq l_1} [u(m+n_1) + \beta\delta u(m + l_1 - n_1) + \beta\delta^2 u(m)] \\ - \max\limits_{n_1 \leq l_1, n_2} [u(m+n_1) + \beta\delta u(n_2) + \beta\delta^2 u(m - (n_1 - m)(1+r) - n_2)] \end{array}\right] \\
C_2 &= \frac{1}{\beta\delta}\left[\begin{array}{c} [u(m + l_1 - n_1) + \beta\delta u(m)] \\ - \max\limits_{n_2} [u(n_2) + \beta\delta u(m - (n_1 - m)(1+r) - n_2)] \end{array}\right]
\end{aligned}
$$

Proof. See the appendix for the proof of the lemma. ∎

There are two entities which make the consumer closer to default: higher credit limit to default on, and spending more than anticipated amount because of time inconsistency. If the consumer had only one card to default on at each period, then we could easily say that C_2 is the highest among the three because the second-period self spends more than what the previous-period selves expected. However, the total credit limit to default on is not the same at each period; it is $l_1 + l_2$ at the initial period and only l_1 at later periods. Therefore C_0 might be higher than C_2 depending on the value of l_2.

Lemma 11. *If the consumer accepts only one contract, then $C_1 < C_2$, therefore relevant cutoff $C' = \max\{C_0(l_1, l_2), C_2(l_1, r)\}$.*

Proof. See the appendix for the proof of the lemma. ∎

Suppose l_1 and l_2 are in appropriate ranges that the consumer chooses both of the contracts. When the consumer chooses both contracts and if the total credit limit offered is more than m, it is a dominant strategy for the consumer to pay his

[8]When the credit limit of the accepted contract is less than m,then C_1 and C_2 expressions will be different. However, the Lemma 11 would still hold.

debt on the higher interest rate card first. Note that the consumer is able to pay all his debt on the higher interest rate card within the grace period. This is because the credit limit is less than the income. Therefore, there will be no interest revenue for the higher interest rate card. For convenience, we take the first company as the company with lower interest rate (r) if both contracts are chosen. The cutoffs and the relation among them are then given in the following two lemmas.

Lemma 12. *When both contracts are accepted, the cutoff cost of default for each period is as follows*[9]*:*

$$
\begin{aligned}
C_0 &= \frac{1}{\delta^2}\left[\begin{array}{c} \max\limits_{n_1 \le l_1+l_2} [u(m+n_1) + \delta u(m+l_1+l_2-n_1) + \delta^2 u(m)] \\ -\max\limits_{n_1 \le m, n_2} [u(m+n_1) + \delta u(m-n_1+n_2) + \delta^2 u(m-n_2)] \end{array}\right] \\
C_1 &= \frac{1}{\beta\delta^2}\left[\begin{array}{c} \max\limits_{n_1 \le l_1+l_2} [u(m+n_1) + \beta\delta u(m+l_1+l_2-n_1) + \beta\delta^2 u(m)] \\ -\max\limits_{n_1 \le l_1+l_2, n_2} [u(m+n_1) + \beta\delta u(n_2) + \beta\delta^2 u(m-(n_1-m)(1+r)-n_2)] \end{array}\right] \\
C_2 &= \frac{1}{\beta\delta}\left[\begin{array}{c} [u(m+l_1+l_2-n_1) + \beta\delta u(m)] \\ -\max\limits_{n_2} [u(n_2) + \beta\delta u(m-(n_1-m)(1+r)-n_2)] \end{array}\right]
\end{aligned}
$$

Proof. The proof of this lemma is the same as the proof of Lemma 10 except that the credit limit at periods one and two will be $l_1 + l_2$ instead of only l_1. ∎

The credit limit to default on is the same at each period, therefore we can easily show that the highest and the relevant cutoff is C_2.

Lemma 13. *If the consumer accepts both contracts, then $C_0 < C_1 < C_2$, therefore $C' = C_2(l_1, l_2, r)$.*

Proof. See the appendix for the proof of the lemma. ∎

---

[9] When the total credit limit is less than m,then C_1 and C_2 expressions will be different. However, the Lemma 13 would still hold.

2.3.3 The Credit Card Companies' Behavior and Best Responses

Under the first case (when only one contract is chosen), the consumer's interest bearing debt is l_1 on the first contract if $l_1 > m$, and zero otherwise. We can write the first company's objective function as follows:

$$\max_{l_1,r} (l_1 - m)r \tag{2.1}$$

$$s.t.$$

$$m \leq l \leq n_1^1 \tag{2.2}$$

$$\max\{C_0(l_1,l_2), C_2(l_1,r)\} \leq C \tag{2.3}$$

Note that l_2 affects the first company's decision only when $C_0(l_1,l_2)$ is the relevant cutoff.

Lemma 14. *Let us denote the first company's proft maximizing credit limit with l_1^* for $l_2 = 0$. Given that only one contract is chosen, the best response of the first company is:*

- *a straight line at l_1^* for $l_2 < l_2''$,*
- *a decreasing line with -1 slope for $l_2'' \leq l_2 < l_2'''$,*
- *zero for $l_2''' \leq l_2$.*

Proof. Given that only one contract is chosen, if $l_2 = 0$, then $C_0 < C_2$. We already know that $C_1 < C_2$ and we can show that $C_0 < C_1$ by following the steps in the proof of Lemma 13, consequently $C_0 < C_2$.

For $l_2 = 0$, the relevant cutoff is C_2. Note that l_2 does not affect C_2, but C_0 :

$$\frac{\partial C_0}{\partial l_2} = \frac{u'(m + l_1 + l_2 - n_1)}{\delta} > 0$$

Therefore, there is a cutoff l_2' such that:

$$\text{if } l_2 \ < \ l_2', \text{ then } C_0 < C_2 \text{ and the relevant cutoff is } C_2$$

$$\text{if } l_2' \;\leq\; l_2, \text{ then } C_2 \leq C_0 \text{ and the relevant cutoff is } C_0$$

For $l_2 \leq l_2'$, the best response is a straight line at l_1^* since the relevant cutoff C_2 is not affected by the second company's credit limit. Once the relevant cutoff becomes C_0, this cutoff increases with l_2 and starts to bind the consumer's cost of default at some l_2''. The best response stays as a straight line for $l_2' \leq l_2 < l_2''$ since default constraint (2.3) is not binding in this region.

Note that the relevant cutoff C_0 is not affected by the interest rate but by the credit limits, specifically $0 < \frac{\partial C_0}{\partial l_1} = \frac{\partial C_0}{\partial l_2}$. Therefore, the best response decreases with -1 slope for $l_2'' \leq l_2$—to satisfy the binding default constraint.

Once the best response becomes zero at $l_2 = l_2'''$, it stays as zero for $l_2''' \leq l_2$. ∎

Under the second case (when both contracts are chosen), the consumer's interest bearing debt is $(\min\{l_1 + l_2, n_1^1\} - m)$ on the lower interest rate card if $l_1 + l_2 \geq m$, and zero otherwise. We can write the first company's objective function as follows:

$$\begin{aligned}
&\max_{l_1,r} \; \left(\min\{l_1 + l_2, n_1^1\} - m\right) r && (2.4)\\
&s.t\\
&m \leq l_1 + l_2\\
&C_2(l_1, l_2, r) \leq C && (2.5)
\end{aligned}$$

Both companies offer an interest rate of zero. This is because of one of the following two reasons:

- if $l_1 + l_2 > m$, then there will not be an interest revenue for the higher interest rate card, which will create competition and drive the interest rates down to zero,
- if $l_1 + l_2 \leq m$, then none of the companies will get interest revenue. Since we consider only the higher credit limit when a company is indifferent between any two credit limits less than m, the interest rate of the contract with

the highest credit limit would be zero among the ones which give the same revenue (zero revenue).[10]

Now, we determine the best responses in the following lemma.

Lemma 15. *Let us denote the first company's profit maximizing credit limit with* $l_1^* < n^0$ *if* $l_2 = 0$. *Given that both contracts are chosen, the best response of the first company is:*

- *a decreasing line with* -1 *slope for* $l_2 < l_2'''$,
- *zero for* $l_2''' \leq l_2$.

Proof. Given that both contracts are chosen, the relevant cutoff is C_2. The reason for not offering credit limit more than n^0 is the binding default constraint. Therefore, as the second company's credit limit increases, the first company should decrease the credit limit since the interest rate is already zero. From the expression for C_2 given in Lemma 12, we know that $0 < \frac{\partial C_0}{\partial l_1} = \frac{\partial C_0}{\partial l_2}$. Therefore, the best response is a decreasing straight line with -1 slope until it hits zero at l_2''', and stays at zero for $l_2''' \leq l_2$. ∎

2.3.4 Description of the Equilibria

In this section, we show the existence of the positive profit equilibria and zero profit equilibria. Before showing that, we provide, first, a numerical example which gives a unique positive profit equilibrium if the consumer's cost of default is infinitely high and, second, another example which gives a unique zero profit equilibrium if the consumer's cost of default is zero.

Example 3. *If the consumer's cost of default is infinitely high, the default constraint will never be binding* ($l_2'' = \infty$)*, and the best response of each company will be a straight line at the profit maximizing credit limit.*[11] *Let us take the utility*

[10] If an interest rate were a positive value, then the corresponding credit limit could be increased by decreasing the interest rate.

[11] The cost of default does not have to be infinitely high in order to have a unique positive profit equilibrium.

function as $u(x) = x^{1/2}$ *and the discount factors as* $\delta = 0.7$ *and* $\beta = 0.65$*. Then,*

$$\begin{aligned}
\textit{believed amount of first period debt, } n_1^0 &= 0.73m, \\
\textit{believed amount of second period debt, } n_2^0 &= 0.58m, \\
\textit{minimum credit limit required for only one contract to be chosen, } n^0 &= 0.73m, \\
\textit{actual amount of first period debt, } n_1^1 &= 1.14m, \\
\textit{profit maximizing interest rate, } r &= 0.13, \\
\textit{profit maximizing minimum credit limit, } l &= 1.14m, \\
\textit{the profit for the chosen company, } \Pi &= 0.0182m.
\end{aligned}$$

We show the best responses as follows:

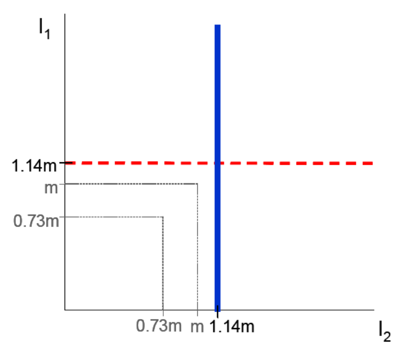

Figure 2.1. Example of best responses when C is infinitely high

Example 4. *If the consumer's cost of default is zero, then any contract with a positive credit limit would trigger default. Therefore, there is unique zero profit equilibrium with zero credit limit offers.*

We now demonstrate three basic equilibria and show more complicated ones in two examples later in this section.

Proposition 7. *If the best response* l_i *of the company* i *is greater than* m *for* $l_j \leq m$*,* $(\{i, j\} = \{1, 2\})$*, then there is positive expected profit equilibria without competition on interest rate.*

Proof. In this case, the consumer chooses only one contract. As we explained above, the best response of each company is a linear line first, then a decreasing line with -1 slope and then zero. The best response curves are depicted in Figure 2.2.

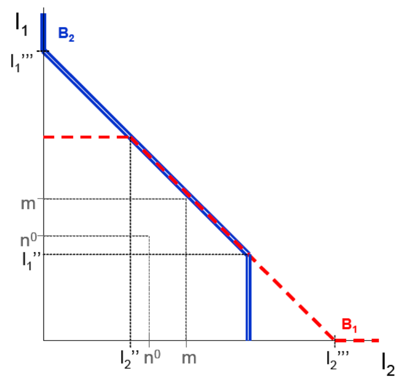

Figure 2.2. Positive profit equilibria without competition

Since there is no competition on the interest rate and the consumer chooses the contract with a credit limit higher than the income with a positive probability, there are positive expected profit equilibria. ∎

Proposition 8. *If the best response* $l_i \in [n^0, m]$ *for* $l_j \leq n^0$*, then there is zero profit equilibria without competition on interest rate.*

Proof. The consumer chooses only one contract and the best response of each company is as we explained above.

If the best response $l_i \in [n^0, m]$ for $l_j = 0$, this means that the default constraint is binding (i.e. $C = C_2 > C_0$ for $l_j < l_j'$), otherwise the company i would offer a credit limit more than m. We know that $C = C_2 = C_0$ at $l_j = l_j'$. Note that l_j'' denotes the cutoff at which C starts to bind the consumer's cost of default C_0. Therefore $l_j' = l_j''$. As a result, the best response curves are as follows:

Since the consumer chooses only one contract and the offered credit limit is less than m, there is no interest revenue generated for the chosen company. As a result, there is zero profit equilibria. ∎

Proposition 9. *If the best response* l_i *of the company* i *is less than* n^0 *for* $l_j = 0$*, then there is zero profit equilibria with competition on the interest rate*

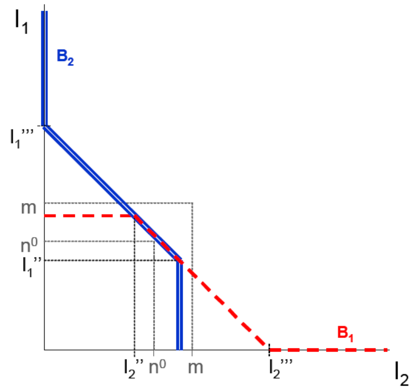

Figure 2.3. Zero profit equilibria without competition

Proof. Since none of the companies offer more than n^0, the consumer accepts two contracts. Therefore, each company offers a zero interest rate as we explained at the beginning of this section. If the profit maximizing credit limit of the company i is less than n^0 even for $l_j = 0$, that would be because of the binding default constraint, specifically $C = C_2$ (C_2 is always greater than C_0 from Lemma 13). Therefore, as l_j increases, the company i's best response should decrease starting from $l_j = 0$. In other words, $l_j'' = 0$. The best response curves are as follows:

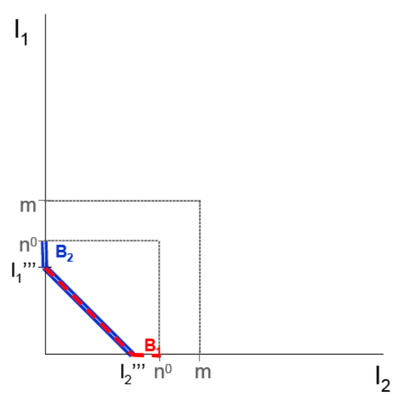

Figure 2.4. Zero profit equilibria with competition

Since $l_1 + l_2 < m$ and $r = 0$, the equilibria give zero profit. ∎

There may be cases in which more than one of these three equilibria are possible under the same parameter values.

Example 5. *Let l_i^* denote the profit maximizing credit limit for company i for $l_j = 0$. Let us denote the best response curves with B_1 and B_2 for company 1 and 2 respectively under the first case:*

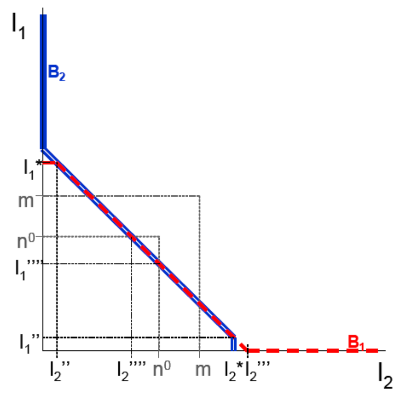

Figure 2.5. Multiple equilibria 1

Note that $l_i < n^0$ for $l_j'''' < l_j < n^0$. This means that the consumer should have accepted both contracts when $l_j'''' < l_j < n^0$, and consequently B_i cannot be the best response in $l_j \in (l_j''', n^0)$. We need to draw another best response, B_i', in $l_j \in (l_j''', n^0)$ under the second case.

For $l_j'''' \leq l_j \leq n^0$, B_i' is always under B_i. This is because of the following relation among the cutoff cost of defaults:

$$C_0^r = C_0^0 < C_2^0$$

B_i is a decreasing line in (l_j'''', n^0), which means C binds the consumer's cost of default, $C = C_0^r$. Since $C_0^r < C_2^0$, B_i' should give a lower value than B_i in (l_j'''', n^0) to satisfy the default constraint.

As a result, we can draw the best responses together with B_i' in (l_j'''', n^0) as follows:

In this case, both zero and positive profit equilibria are possible.

Note that B_i' values may not be in (l_i'''', n^0) for $l_j \in (l_j'''', n^0)$ In that case, zero profit equilibrium with competition does not exist, but there are other equilibria.

Example 6. *Let the profit maximizing credit limit for company i is $l_i^* = n^0$ for $l_j = 0$. Let us denote the best response curves with B_1 and B_2 under the first case:*

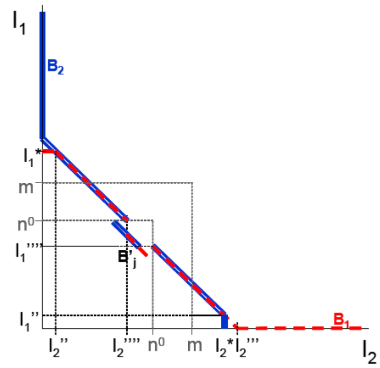

Figure 2.6. Multiple equilibria 2

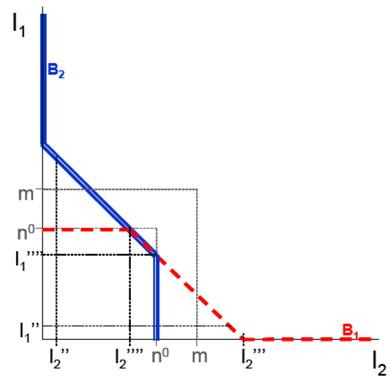

Figure 2.7. Multiple equilibria 3

Because of the same reasoning we explained in the previous example, B_1 *and* B_2 *cannot be the best responses in the area of* $(l_2'''', n^0) \times (l_1'''', n^0)$*. We need to draw best responses* B_1' *and* B_2' *in that area under the second case. Therefore, the best responses are as in either of the following graphs.*

In the first graph, there are zero profit equilibria at (n^0, l_1'''') *and* (l_2'''', n^0)*. In the second graph, there are additional zero profit equilibria in the region* $(l_2'''', n^0) \times (l_1'''', n^0)$*.*

2.4 Extensions

We first determine whether the consumer would have been better off had we restricted the credit limits to the income. Second, we check whether the equilibria in

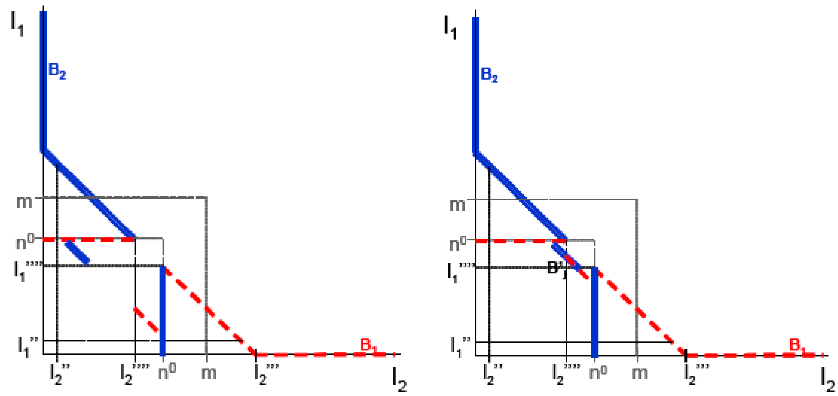

Figure 2.8. Multiple equilibria 4

the original model would survive if the grace period interest rate were endogenous. Third, we examine the equilibria by allowing for uncertainty on the consumer's cost of default.

2.4.1 Consumer's Welfare

We calculate the consumer's welfare according to the initial-period self.[12] Whether he is better off with the restriction on the credit limits depends on the utility function and discount factors δ and β. We provide an example for both sides.

Example 7. *Let us take the utility function as* $u(x) = x^{1/2}$. *For* $\delta = 0.61$ *and* $\beta = 0.8$, *if the consumer's cost of default is high enough and if we do not restrict credit limits, we find the total utility according to the initial-period self as follows:*

$$\begin{aligned}
\textit{profit maximizing interest rate} &\Rightarrow r = 0.11 \\
\textit{believed amount of debt} &\Rightarrow n_1^0 = 0.986021m \\
\textit{actual amount of debt} &\Rightarrow n_1^1 = 1.12491m \\
\textit{total utility} &\Rightarrow U_0' = 2.11778m^{1/2}
\end{aligned}$$

If we restrict credit limits with the income, then the total utility according to

[12] Welfare calculations are different for each different self of the consumer.

the initial-period self is as follows:

$$\text{actual amount of debt with max credit limit of } m \;\Rightarrow\; k_1^1 = m$$
$$\text{total utility with restriction on credit limit} \;\Rightarrow\; U_0'' = 2.12737m^{1/2}$$

Since $U_0' = 2.11778m^{1/2} < U_0'' = 2.12737m^{1/2}$, *the consumer would have been better off had we restricted the credit limits to the income.*

The company's profit by offering this contract is:

$$\Pi = 0.0137401m$$

Example 8. *Let us take the same utility function* $u(x) = x^{1/2}$ *and the same exponential discount factor* $\delta = 0.61$, *but a different hyperbolic discount factor* $\beta = 0.9$. *If the consumer's cost of default is high enough and if we do not restrict credit limits, we find the total utility according to the initial-period self as follows:*

$$\text{profit maximizing interest rate} \;\Rightarrow\; r = 0.05$$
$$\text{believed amount of debt} \;\Rightarrow\; n_1^0 = 0.986021m$$
$$\text{actual amount of debt} \;\Rightarrow\; n_1^1 = 1.05996m$$
$$\text{total utility} \;\Rightarrow\; U_0' = 2.12624m^{1/2}$$

If we restrict credit limits with the income, then the total utility according to the initial-period self is as follows:

$$\text{actual amount of debt with max credit limit of } m \;\Rightarrow\; k_1^1 = m$$
$$\text{total utility with restriction on credit limit} \;\Rightarrow\; U_0'' = 2.12586m^{1/2}$$

Since $U_0' = 2.12624m^{1/2} > U_0'' = 2.12586m^{1/2}$, *the consumer would not have been better off had we restricted credit limits to the income.*

The company's profit by offering this contract is:

$$\Pi = 0.002998m$$

2.4.2 Endogenous Grace Period

Our base model treats the introductory interest rate as an exogenous value set to be zero, which corresponds to the grace period. However, incorporating the competition on the introductory interest rate is something left to future research. In this section, we determine which of the equilibria would survive if the introductory interest rate were endogenous as well. We specify each contract j with a credit limit l_j, interest rate r_{j1}—introductory— for one-period loans and interest rate r_{j2}—regular— for two-period loans, and we write the company $j's$ profit as $\Pi_j = \min\{n_{j1}^1 r_{j1}, p_{j2}^2\}/\delta + \left(1+d_j^3\right)(n_{j1}^2(1+r_{j1}) - p_{j2}^2)r_{j2} + d_j^3\left(n_{j1}^1 + n_{j2}^2 - p_{j2}^2\right)$ as opposed to $\Pi_j = \left(1+d_j^3\right)(n_{j1}^1 - p_{j2}^2)r_j + d_j^3\left(n_{j1}^1 + n_{j2}^2 - p_{j2}^2\right)$. Note that interest revenue from one-period loans is possible in this endogenous introductory interest rate case. Moreover, the introductory interest rate may also affect the number of cards to be accepted in addition to credit limits. We divide the discussion by the following three cases.

Claim 1. *The positive expected profit equilibrium without competition on the interest rate survives when we endogenize the introductory interest rate.*

Solution 1. *Let $((l_1, 0, r_{12}), (l_2, 0, r_{22}))$ denote a positive profit equilibrium with $l_1, l_2 > m$ and check whether there is any profitable deviation from this equilibrium. If a company offers an introductory interest rate higher than zero, then the probability of that contract being chosen will be zero. Therefore, the expected profit for that company will be zero. As a result, there is no profitable deviation from the current state and the positive expected profit equilibrium survives.*

Claim 2. *The zero profit equilibrium without competition on the interest rate survives when we endogenize the introductory interest rate if both of the credit limits are higher than n_1^0. On the other hand, the zero profit equilibrium does not survive if one of the credit limits is less than n_1^0 and the other is more than n_1^0.*

Solution 2. *Let $((l_1, 0, r_{12}), (l_2, 0, r_{22}))$ denote a zero profit equilibrium without competition on the interest rate such that $m > l_1, l_2 \geq n_1^0$. We want to check whether there is a profitable deviation from this equilibrium. If a company offers an introductory interest rate of more than zero, then the consumer does not choose* *this company with a positive probability anymore. Therefore, a profitable deviation is not possible and the current equilibrium survives.*

Let $((l_1, 0, r_{12}), (l_2, 0, r_{22}))$ *denote a zero profit equilibrium without competition on the interest rate such that* $m > l_1 \geq n_1^0 > l_2$*. Let us check the possibility of a profitable deviation from this equilibrium. If the first company increases its introductory interest rate more than zero, then it increases its interest revenue. This is because the other company's credit limit is not enough for the consumer, and therefore he has to accept both contracts and pay interest on the first credit card. Therefore, a profitable deviation is possible and this equilibrium does not survive.*

Claim 3. *The zero profit equilibrium with competition on the interest rate does not survive when we endogenize the introductory interest rate.*

Solution 3. *Let* $((l_1, 0, 0), (l_2, 0, 0))$ *denote a zero profit equilibrium with competition on the interest rate such that* $l_1, l_2 < n_1^0$*. We want to check whether there is a profitable deviation from this equilibrium. If a company offers an introductory interest rate higher than zero, then this company can increase its interest revenue from zero to a positive number. This is because the other company cannot satisfy the consumer's borrowing need alone, so accordingly the consumer borrows the rest of the amount he needs from the company with a positive introductory interest rate. Therefore, zero profit equilibrium with competition on the interest rate does not survive.*

2.4.3 The Model with Uncertainty

In this section, we analyze the model with uncertainty such that the companies know only the distribution of the consumer's cost of default rather than its actual value. In this case, we can write the first company's optimization problem under the first case as follows:

$$\begin{aligned} &\max_{l_1, r} M = (l_1 - m) r (1 - F(C)) - l_1 F(C) \\ &s.t. \\ &m \leq l_1 \leq n_1^1 \end{aligned} \tag{2.6}$$

$$C = \max\{C_0, C_2\}$$

The first company's problem under the second case is:

$$\max_{l_1, r} M = (\min\{l_1 + l_2, n_1^1\} - m)r(1 - F(C)) - \delta^3 l_1 F(C) \tag{2.7}$$
$$s.t.$$
$$m \leq l_1 + l_2$$
$$C = C_2$$

We can make two comments on this model of uncertainty. Firstly, there is no contract offer with a credit limit in the range of $[n^0, m]$ in equilibrium. If a company offers such a contract, the consumer chooses that contract alone with a positive probability, but the chosen company does not gain any interest revenue. Moreover, the positive probability of the default creates a negative expected profit. Note that the term $-l_1 F(C)$ captures the effect of the default on the profit.

Secondly, if both companies offer a contract with a credit limit in the range of $(0, n^0)$ and if the equilibrium exists, then it is zero profit equilibrium. The consumer accepts both contracts if both companies offer a credit limit less than n^0. Once the consumer has two credit cards on hand, the competition will drive the interest rates down. To determine how far the interest rates decrease because of this competition, we need to analyze the objective function (2.7) for each company $j = 1, 2$. Note that the company with a lower credit limit can decrease his interest rate more. Therefore, competition for the consumer also decreases credit limits until expected profits become zero. Note that a decrease in the interest rate diminishes the probability of default as well as the interest revenue. Therefore, a decrease in the credit limit offers is not because of the more risky consumer but because of the less profitable consumer.

There is no intuitive evidence against the possibility of the contracts with a credit limit that is more than the consumer's income in equilibrium. However, we have not been able to show the existence of equilibrium in this model of uncertainty since the problem is analytically not tractable.

2.5 Discussion & Conclusion

In our model, we focus on two aspects of a credit card contract, the interest rate and the credit limit. Cash back and reward points are other aspects of credit card contracts that were not common 10 years ago. If we would include these aspects as well, then positive expected profits might not be possible, even if there were no competition on the interest rate.

Consumer's time inconsistency and naivete, and credit card companies' grace period offer are essential to have a positive expected profit equilibrium. If the introductory interest rate is not zero (no grace period), then there would be no convenience users in the credit card market and everybody would be responsive to the interest rate. The small cost of applying for a credit card makes the consumer choose only one card when the offered credit limits are higher than the consumer's believed amount of debt, and eliminates competition for the chosen card.

The consumer chooses one of the contracts randomly when he wants to choose only one contract and when he is indifferent among the offered contracts. If the consumer believes that he might make a mistake and borrows more than his income, then he would choose the contract with lower interest rate and our result of noncompetitive interest rates and positive profit equilibrium would not hold. One might describe this as the "trembling-hand perfectness" problem with our equilibria, though there is no universally accepted definition of such a concept in the infinite strategy space setting of this paper. An alternative concept in the extensive form, also defined for finite games, has been discussed by van Damme (1987) and Perea (2001). This concept, known as "quasi-perfectness" appears to have some nice features in terms of eliminating some of the anomalies between normal form and extensive form notions. The definition of quasi-perfect equilibrium assumes that a player, when moving at a particular node, assumes that he will always follow the given strategy with probability 1 in the future, though the other players are believed to be constrained to put positive probability on all their actions at all their succeeding information sets. The heuristic transfer of this concept to our framework appears to be consistent with the behavior of the consumer we have modeled in this paper.

In our model, the consumer's cost of bankruptcy is exogenously given. With the recent changes in the bankruptcy law, we expect this cost to increase for the consumer.[13] If this cost (C) increases, the consumer is less likely to default and credit card companies are more likely to get positive expected profits.

Since our results depend on the consumer's wrong belief about his future consumption, we can easily get the same results for more than three periods of consumption with appropriate δ and β. Moreover, because of exactly the same reasoning, we can produce similar results with partially naive hyperbolic consumers as well.

In our model, we do not allow contracting after the initial period. If we did, our results would hold in this three periods of consumption model, because applying for a lower interest rate card does not decrease the interest payment on the first-period debt since the consumer can only start using the new card in the following period. As a result, the only consumer who applies for a new card in the first period is the one planning to default. Therefore, there would be no credit card offers to this consumer after the initial period.

Now let us consider the case of more than three periods of consumption. When we allow contracting after the initial period, we suspect that positive expected profits may still be possible, depending on the severity of the consumer's time inconsistency. Although applying for a lower interest rate card does not help to decrease the interest payment on the current period's debt, it helps to decrease the interest payment on future periods' debt. If the consumer cannot foresee the interest payments for future periods because of the severe time inconsistency, then he does not apply for a lower interest rate card and our results still hold. On the other hand, if he can foresee his future period interest payments, then there are two different possible outcomes. Firstly, if the initially chosen company can prevent other companies from offering contracts after the initial period by giving a high credit limit to the consumer and accordingly increasing the consumer's risk of default, then positive profits may still be possible. Secondly, if we allow the initially chosen company to change his interest rate at the first period, then the competition drives the interest rates down to zero. Note that there is first-mover advantage and the companies do not need to compete on the interest rate in order

[13]More strict bankruptcy proceedings such as more strict conditions to file under chapter 7 and mandatory financial management education program.

to be the first-mover in any of these cases. As a result, positive profit equilibria may still exist.

2.6 Appendix

Proof of Proposition 1. 1. if $n_1 = \sum_{j=1}^{2} n_{j1} \leq m \Rightarrow \sum_{j=1}^{2} p_{j2} = n_1$ and $\sum_{j=1}^{2} p_{j3} = \sum_{j=1}^{2} n_{j2} = n_2$. Therefore, we can write the objective function of the consumer at $t = 0$ as follows:

$$\max_{n_1,\ n_2} \beta\delta\left[u(m+n_1) + \delta u(m-n_1+n_2) + \delta^2 u(m-n_2)\right]$$

$$s.t. \tag{2.8}$$

$$n_1 \leq m \tag{2.9}$$

$$-n_1 \leq 0 \tag{2.10}$$

$$-n_2 \leq 0 \tag{2.11}$$

FOCs if the constraints are not binding:

$$u'(m - n_1^0 + n_2^0) = \delta u'(m - n_2^0) \tag{2.12}$$

$$u'(m + n_1^0) = \delta u'(m - n_1^0 + n_2^0) \tag{2.13}$$

We know that $n_1^0 = n_2^0 = 0$, when $\delta = 1$. In order to see how n_1^0 and n_2^0 change with δ, we can take the derivative of (2.12) and (2.13) with respect to δ :

$$u''(m - n_1^0 + n_2^0)(-\frac{\partial n_1^0}{\partial \delta} + \frac{\partial n_2^0}{\partial \delta}) = u'(m - n_2^0) + \delta u''(m - n_2^0)(-\frac{\partial n_2^0}{\partial \delta}) \tag{2.14}$$

$$u''(m+n_1^0)\frac{\partial n_1^0}{\partial \delta} = u'(m-n_1^0+n_2^0)+\delta u''(m-n_1^0+n_2^0)(-\frac{\partial n_1^0}{\partial \delta}+\frac{\partial n_2^0}{\partial \delta}) \quad (2.15)$$

- If $\frac{\partial n_1^0}{\partial \delta} > 0 \Rightarrow (-\frac{\partial n_1^0}{\partial \delta}+\frac{\partial n_2^0}{\partial \delta}) > 0 \; and \; \frac{\partial n_2^0}{\partial \delta} > 0$ by (2.15). However, the signs of these derivatives give a contradiction in (2.14). Therefore, $\frac{\partial n_1^0}{\partial \delta} < 0$ must be correct.
- Given that $\frac{\partial n_1^0}{\partial \delta} < 0$, if $\frac{\partial n_2^0}{\partial \delta} > 0 \Rightarrow (-\frac{\partial n_1^0}{\partial \delta}+\frac{\partial n_2^0}{\partial \delta}) > 0$. However, the sign of these derivatives give a contradiction in (2.14). Therefore, $\frac{\partial n_2^0}{\partial \delta} < 0$ must be also correct.

From (2.12):

$$n_2^0 > \frac{n_1^0}{2} \quad (2.16)$$

As δ decreases, n_1^0 *and* n_2^0 increase. We know that n_2^0 will never be greater than m from the first order condition. Moreover, if we do not allow n_2^0 to be greater than $m/2$, by (2.16) we can make sure that n_1^0 will not be greater than m. So, there is a lower bound for δ, namely δ^*, such that for $\delta > \delta^*$ the constraint (2.9) will not be binding and the consumer believes that he will not keep a positive balance on his credit card. Moreover, from the definition of the maximum, we can write the following inequality for $\delta > \delta^*$:

$$\begin{aligned} &\max_{n_1<m,\; n_2} \beta\delta\left[u(m+n_1)+\delta u(m-n_1+n_2)+\delta^2 u(m-n_2)\right] \quad (2.17)\\ > &\max_{n_1=m,\; n_2} \beta\delta\left[u(m+n_1)+\delta u(m-n_1+n_2)+\delta^2 u(m-n_2)\right] \end{aligned}$$

2. if $n_1 = \sum_{j=1}^{2} n_{j1} \geq m \implies \sum_{j=1}^{2} p_{j2} = m$ and $\sum_{j=1}^{2} p_{j3} = \left(\sum_{j=1}^{2} n_{j1} - m\right)(1+r)$. Therefore the objective function is:

$$\max_{n_1,\; n_2} \beta\delta\left[\begin{array}{c} u(m+n_1)+\delta u(n_2) \\ +\delta^2 u(m-(n_1-m)(1+r)-n_2) \end{array}\right] \quad (2.18)$$

$s.t.$

$$-n_1 \leq -m \quad (2.19)$$

$$-n_1 \leq 0 \tag{2.20}$$

$$-n_2 \leq 0 \tag{2.21}$$

FOCs if the constraints are not binding:

$$u'(n_2^0) = \delta u'(m(2+r) - n_1^0(1+r) - n_2^0) \tag{2.22}$$

$$u'(m + n_1^0) = \delta(1+r)u'(n_2^0) \tag{2.23}$$

From (2.23), we can say that there will be a cutoff δ^{**} such that for $\delta > \delta^{**}$, the following inequality will be true:[14]

$$n_1^0 < n_2^0$$

Since we know that n_2^0 cannot be higher than m from (2.22), the following inequality will also be true for $\delta > \delta^{**}$:

$$n_1^0 < m$$

This means that the constraint (2.19) will be binding for $\delta > \delta^{**}$ and we can write the objective function of the consumer as follows:

$$\max_{n_2} \beta\delta \left[u(2m) + \delta u(n_2) + \delta^2 u(m - n_2)\right] \tag{2.24}$$

As a result, for $\delta > \delta' = \max\{\delta^*, \delta^{**}\}$, from (2.17) and (2.24):

$$\begin{aligned}
& \max_{n_1<m,\ n_2} \beta\delta \left[u(m+n_1) + \delta u(m - n_1 + n_2) + \delta^2 u(m - n_2)\right] \\
> & \max_{n_1=m,\ n_2} \beta\delta \left[u(m+n_1) + \delta u(m - n_1 + n_2) + \delta^2 u(m - n_2)\right] \\
= & \max_{n_2} \beta\delta \left[u(2m) + \delta u(n_2) + \delta^2 u(m - n_2)\right]
\end{aligned}$$

[14] For example, take $\delta \geq 1/(1+r)$. For those δ values it is easy to see that $m + n_1^0 < n_2^0$, consequently $n_1^0 < n_2^0$.

This means that the consumer will believe that he will not pay interest on his credit card debt when he is at the initial period.

3. We showed that a consumer with an exponential discount factor $\delta > \delta' = \max\{\delta^*,\ \delta^{**}\}$ believes that he will not keep a positive balance. In order to show that this consumer will keep a positive balance and pay interest, we will analyze the consumer with $\delta > \delta'$ only.

   When the consumer comes to the first period, his objective function will be as follows:

$$\max\ u(m+n_1) + \beta\delta u(m-p_2+n_2) + \beta\delta^2 u(m-p_3)$$

   if $n_1 \leq m \implies p_2 = n_1,\ p_3 = n_2$. Therefore, the problem is as follows:

$$\max_{n_1,\ n_2}\ u(m+n_1) + \beta\delta u(m-n_1+n_2) + \beta\delta^2 u(m-n_2) \tag{2.25}$$

$$s.t.$$

$$n_1 \leq m \tag{2.26}$$

$$-n_1 \leq 0 \tag{2.27}$$

$$-n_2 \leq 0 \tag{2.28}$$

   FOCs if the constraints are not binding:

$$u'(m-n_1^1+n_2^1) = \delta u'(m-n_2^1) \tag{2.29}$$

$$u'(m+n_1^1) = \beta\delta u'(m-n_1^1+n_2^1) \tag{2.30}$$

   For $\delta > \delta'$, in order to determine how n_1^1 and n_2^1 change with β, we take the derivative of (2.29) and (2.30) with respect to β:

$$u''(m-n_1^1+n_2^1)(-\frac{\partial n_1^1}{\partial\beta} + \frac{\partial n_2^1}{\partial\beta}) = \delta u''(m-n_2)(-\frac{\partial n_2^1}{\partial\beta}) \tag{2.31}$$

$$u''(m+n_1^1)\frac{\partial n_1^1}{\partial \beta} = \delta u'(m-n_1^1+n_2^1)+\beta\delta u''(m-n_1^1+n_2^1)(-\frac{\partial n_1^1}{\partial \beta}+\frac{\partial n_2^1}{\partial \beta}) \quad (2.32)$$

- If $\frac{\partial n_1^1}{\partial \beta} > 0 \Rightarrow (-\frac{\partial n_1^1}{\partial \beta} + \frac{\partial n_2^1}{\partial \beta}) > 0$ *and* $\frac{\partial n_2^1}{\partial \beta} > 0$ by (2.32). However, these inequalities do not satisfy (2.31). Therefore, $\frac{\partial n_1^1}{\partial \beta} < 0$ must be true.
- Given that $\frac{\partial n_1^1}{\partial \beta} < 0$, if $\frac{\partial n_2^1}{\partial \beta} > 0$, these inequalities do not satisfy (2.31). Therefore, $\frac{\partial n_2^1}{\partial \beta} < 0$ must be also true.

As a result, as β decreases, n_1^1 and n_2^2 increases.

Write down the difference between the left- and right-hand side of the inside of the utility function in (2.29) as ε_1:

$-n_1^1 + n_2^1 + n_2^1 = \varepsilon_1 \implies n_1^1 = 2n_2^2 - \varepsilon_1$

For any β as $\delta \to 1$, then $\varepsilon_1 \to 0$, $n_1^1 \to 2n_2^1$.

So, we can write (2.30) as:

$u'(m + 2n_2^1) = \beta u'(m - n_2^1)$

As $\beta \to 0$, $\frac{u'(m+2n_2^1)}{u'(m-n_2^1)} \to 0$, then $n_2^1 \to m$, since the denominator will be infinity and the numerator will be a finite number. Consequently, $n_1^1 \to 2m$ and we can say that there will be a $\widetilde{\delta}$ and β^* such that for $\delta > \widetilde{\delta}$ and $\beta < \beta^*$ the constraint (2.26) will be binding. Then, for those δ and β values, we can write the objective function as follows:

$$\max_{n_1=m,\ n_2} u(m+n_1) + \beta\delta u(m-n_1+n_2) + \beta\delta^2 u(m-n_2) \quad (2.33)$$

4. if $n_1 \geq m \implies p_2 = m$ & $p_3 = (n_1 - m)(1+r) + n_2$

   problem is

$$\max_{n_1,\ n_2} \begin{bmatrix} u(m+n_1) + \beta\delta u(n_2) \\ +\beta\delta^2 u(m-(n_1-m)(1+r)-n_2) \end{bmatrix} \quad (2.34)$$

$s.t.$

$$-n_1^1 \leq -m \tag{2.35}$$
$$-n_1^1 \leq 0 \tag{2.36}$$
$$-n_2^1 \leq 0 \tag{2.37}$$

FOCs if the constraints are not binding:

$$u'(n_2^1) = \delta u'(m(2+r) - n_1^1(1+r) - n_2^1) \tag{2.38}$$

$$u'(m + n_1^1) = \beta\delta(1+r)u'(n_2^1) \tag{2.39}$$

For $\delta > \delta'$, in order to determine how n_1^1 and n_2^1 change with β, we take the derivative of (2.38) and (2.39) with respect to β:

$$u''(n_2^1)(\frac{\partial n_2^1}{\partial \beta}) = \delta u''(m(2+r) - n_1^1(1+r) - n_2^1)(-\frac{\partial n_1^1}{\partial \beta}(1+r) - \frac{\partial n_2^1}{\partial \beta}) \tag{2.40}$$

$$u''(m + n_1^1)(\frac{\partial n_1^1}{\partial \beta}) = \delta(1+r)u'(n_2^1) + \beta\delta(1+r)u''(n_2^1)\frac{\partial n_2^1}{\partial \beta} \tag{2.41}$$

- If $\frac{\partial n_1^1}{\partial \beta} > 0 \Rightarrow \frac{\partial n_2^1}{\partial \beta} > 0$ by (2.41). However, these inequalities do not satisfy (2.40). Therefore, $\frac{\partial n_1^1}{\partial \beta} < 0$ must be true.
- Given that $\frac{\partial n_1^1}{\partial \beta} < 0$, if $\frac{\partial n_2^1}{\partial \beta} < 0$, the equation (2.40) gives a contradiction. Therefore, $\frac{\partial n_2^1}{\partial \beta} > 0$ must be also true.

As a result, n_1^1 decreases and n_2^1 increases with β.

Write down the difference between the left- and right-hand side of the inside of the utility function in (2.38) as γ_1:

$n_2^1 - m(2+r) + n_1^1(1+r) + n_2^1 = \gamma_1 \implies n_1^1 = \frac{\gamma_1 - 2n_2^1 + m(2+r)}{1+r}$

For any β as $\delta \to 1,$ then $\gamma_1 \to 0,$ $n_1^1 \to \frac{-2n_2^1 + m(2+r)}{1+r}$

So, we can write (2.39) as:

$u'(m+\frac{-2n_2^1+m(2+r)}{1+r})=\beta(1+r)u'(n_2^1)$

As $\beta\rightarrow 0$, $\frac{u'\left(\frac{-2n_2^1+m(2+r)}{1+r}\right)}{u'(n_2^1)}\rightarrow 0$, then $n_2^1\rightarrow 0$, since the denominator will be infinity and the numerator will be a finite number. Consequently, $n_1^1\rightarrow m\frac{2+r}{1+r}>m$. Therefore, we can say that there will be a $\widehat{\delta}$ and β^{**} such that for $\delta>\widehat{\delta}$ and $\beta<\beta^{**}$, the constraint (2.35) will not be binding. From the definition of the maximum, we can write the following inequality for $\delta>\widehat{\delta}$ and $\beta<\beta^{**}$:

$$\begin{aligned} &\max_{n_1>m,\ n_2}\left[\begin{array}{c} u(m+n_1)+\beta\delta u(n_2) \\ +\beta\delta^2u(m-(n_1-m)(1+r)-n_2) \end{array}\right] \\ > &\max_{n_1=m,\ n_2}\left[\begin{array}{c} u(m+n_1)+\beta\delta u(n_2) \\ +\beta\delta^2u(m-(n_1-m)(1+r)-n_2) \end{array}\right] \end{aligned} \tag{2.42}$$

As a result, for $\delta>\delta''=\max\{\delta',\ \widetilde{\delta},\ \widehat{\delta}\}$ and $\beta<\beta'=\min\{\beta^*,\ \beta^{**}\}$, from (2.33) and (2.42):

$$\begin{aligned} &\max_{n_1>m,\ n_2} u(m+n_1)+\beta\delta u(n_2)+\beta\delta^2u(m-(n_1-m)(1+r)-n_2) \\ > &\max_{n_1=m,\ n_2} u(m+n_1)+\beta\delta u(n_2)+\beta\delta^2u(m-(n_1-m)(1+r)-n_2) \\ = &\max_{n_1=m,\ n_2} u(m+n_1)+\beta\delta u(m-n_1+n_2)+\beta\delta^2u(m-n_2) \end{aligned}$$

This means that the consumer will end up paying interest on his credit card debt as opposed to his belief at the initial period that he would not.

∎

Proof of Lemma 1. 1. At the initial period

The consumer's total utility if he plans not to default and if he plans to default is as follows:

$$\max_{n_1 \le m,\ n_2} \beta\delta \left[u(m+n_1) + \delta u(m-n_1+n_2) + \delta^2 u(m-n_2)\right]$$
$$\max_{n_1} \beta\delta \left[u(m+n_1) + \delta u(m+l_1+l_2-n_1) + \delta^2 u(m)\right]$$

Therefore, the cutoff cost of default in order not to plan to default at the first period can be found as follows:

$$C_0 = \frac{1}{\delta^2}\begin{bmatrix} \max_{n_1} [u(m+n_1) + \delta u(m+l_1+l_2-n_1) + \delta^2 u(m)] \\ \max_{n_1 \le m,\ n_2} [u(m+n_1) + \delta u(m-n_1+n_2) + \delta^2 u(m-n_2)] \end{bmatrix}$$

2. At the first period

When the consumer reaches the first period, he realizes that his actual debt is more than his income and consequently he will pay interest. In this case, his first period debt will be more than his income if he plans to default as well. Therefore, the consumer's total utility if he plans not to default and if he plans to default respectively is as follows:

$$\max_{n_1 \le l_1,\ n_2} \left[u(m+n_1) + \beta\delta u(n_2) + \beta\delta^2 u(m-(n_1-m)(1+r)-n_2)\right]$$
$$\max_{n_1 \le l_1} \left[u(m+n_1) + \beta\delta u(m+l_1-n_1) + \beta\delta^2 u(m)\right]$$

Therefore, the cutoff cost of default in order not to plan to default at the first period can be found as follows:

$$C_1 = \frac{1}{\beta\delta^2}\begin{bmatrix} \max_{n_1 \le l_1} [u(m+n_1) + \beta\delta u(m+l_1-n_1) + \beta\delta^2 u(m)] \\ - \max_{n_1 \le l_1,\ n_2} \begin{bmatrix} u(m+n_1) + \beta\delta u(n_2) \\ +\beta\delta^2 u(m-(n_1-m)(1+r)-n_2) \end{bmatrix} \end{bmatrix}$$

3. At the second period

When the consumer comes to the second period, the consumer's total utility if he plans not to default and if he plans to default respectively as follows:

$$\begin{array}{l} \max\limits_{n_2} [u(n_2) + \beta\delta u(m - (n_1 - m)(1 + r) - n_2)] \\ [u(m + l_1 - n_1) + \beta\delta u(m)] \end{array}$$

Therefore, we can find the cutoff cost of default in order not to plan to default at the second period as follows:

$$C_2 = \frac{1}{\beta\delta} \left[\begin{array}{c} [u(m + l_1 - n_1) + \beta\delta u(m)] \\ -\max\limits_{n_2} [u(n_2) + \beta\delta u(m - (n_1 - m)(1 + r) - n_2)] \end{array} \right]$$

∎

Proof of Lemma 2. If β would be equal to 1 from the second period on as the first-period self believes, then the consumer would not be time inconsistent and C_2 would be equal to C_1. Therefore, it will be enough to look at how C_2 changes with β to find out the relation between C_1 and C_2 since we know that C_2 is equal to C_1 when $\beta = 1$.

$$\frac{\partial C_2}{\partial \beta} = \frac{\left[\begin{array}{c} \left(u'\left(m + l_1 - n_1\right)\left(-\frac{\partial n_1}{\partial \beta}\right) + \delta u(m)\right) \\ -\left(\begin{array}{c} \delta u(m - (n_1 - m)(1 + r) - n_2^*) \\ +\beta\delta u'\left(m - (n_1 - m)\left(1 + r\right) - n_2^*\right)\left(1 + r\right)\left(-\frac{\partial n_1}{\partial \beta}\right) \end{array} \right) \end{array} \right]\beta\delta}{\left(\beta\delta\right)^2} - \frac{\left(\begin{array}{c} [u(m + l_1 - n_1) + \beta\delta u(m)] \\ -[u(n_2^*) + \beta\delta u(m - (n_1 - m)(1 + r) - n_2^*)] \end{array} \right)\delta}{\left(\beta\delta\right)^2}$$

such that n_2^* represents the profit maximizing n_2 in case of planning not to default. If we simplify the previous equation:

$$\frac{\partial C_2}{\partial \beta} = \left(-\frac{\partial n_1}{\partial \beta}\right) \frac{\left[u'\left(m+l_1-n_1\right)-\beta\delta u'\left(m-\left(n_1-m\right)\left(1+r\right)-n_2\right)\left(1+r\right)\right]\beta\delta}{\left(\beta\delta\right)^2} - \frac{\left[u(m+l_1-n_1)-u(n_2^*)\right]\delta}{\left(\beta\delta\right)^2}$$

We know that $\left(-\frac{\partial n_1}{\partial \beta}\right) > 0$ from the proof of Proposition 2. From the definition of n_2^* :

$$u'(n_2^*) = \beta\delta u'\left(m-\left(n_1-m\right)\left(1+r\right)-n_2^*\right) \tag{2.43}$$

In period two, if the consumer plans to default, he borrows more than the optimal amount if he plans not to default:

$$m+l_1-n_1 \geq n_2^* \tag{2.44}$$

From (2.43) and (2.44):

$$u'\left(m+l_1-n_1\right) < \beta\delta u'\left(m-\left(n_1-m\right)\left(1+r\right)-n_2^*\right)$$

$$\Rightarrow \left[u'\left(m+l_1-n_1\right)-\beta\delta u'\left(m-\left(n_1-m\right)\left(1+r\right)-n_2\right)\left(1+r\right)\right] < 0$$

$$\Rightarrow \frac{\left(-\frac{\partial n_1}{\partial \beta}\right)\left[\begin{array}{c} u'\left(m+l_1-n_1\right) \\ -\beta\delta u'\left(m-\left(n_1-m\right)\left(1+r\right)-n_2\right)\left(1+r\right)\end{array}\right]\beta\delta}{\left(\beta\delta\right)^2} < 0 \tag{2.45}$$

From (2.44):

$$u(m+l_1-n_1) \geq u(n_2^*)$$

$$\Rightarrow -\frac{\left[u(m+l_1-n_1)-u(n_2^*)\right]\delta}{\left(\beta\delta\right)^2} < 0 \tag{2.46}$$

From (2.45) and (2.46):

$$\frac{\partial C_2}{\partial \beta} < 0$$

Accordingly,

$$C_2 > C_1 \text{ for } \beta < 1$$

Finally, we can define the cutoff C' as follows:

$$C' = \max\{C_0,\ C_2\}$$

∎

Proof of Lemma 4. If $\beta = 1$, then C_1 would be written as follows:

$$C_1 = \frac{1}{\delta^2}\left[\begin{array}{c} \max\limits_{n_1 \le l_1 + l_2} [u(m+n_1) + \delta u(m + l_1 + l_2 - n_1) + \delta^2 u(m)] - \\ \max\limits_{m \le n_1 \le l_1 + l_2, n_2} \left[\begin{array}{c} u(m+n_1) + \delta u(n_2) \\ +\delta^2 u(m - (n_1 - m)(1+r) - n_2) \end{array}\right] \end{array}\right] \tag{2.47}$$

In order to compare C_0 and C_1, let's rewrite the expression for C_0 below:

$$C_0 = \frac{1}{\delta^2}\left[\begin{array}{c} \max\limits_{n_1 \le l_1 + l_2} [u(m+n_1) + \delta u(m + l_1 + l_2 - n_1) + \delta^2 u(m)] \\ - \max\limits_{n_1 \le m, n_2} [u(m+n_1) + \delta u(m - n_1 + n_2) + \delta^2 u(m - n_2)] \end{array}\right] \tag{2.48}$$

When we carefully analyze (2.47) and (2.48), we can see that the first expressions in brackets are the same for both. Therefore, any possible difference between C_0 and C_1 is because of the second expressions in the brackets. From Proposition 1, we know that the profit maximizing n_1 is less than m for $\delta > \delta'$. Therefore, the second expression in the bracket of (2.48) is greater than the second expression in the bracket of (2.47) for $\delta > \delta'$. Consequently $C_0 < C_1$.

Now, if we show how C_1 changes with β, then we might be able to compare C_0 and C_1.

$$\frac{\partial C_1}{\partial \beta} = \frac{\left[\begin{array}{c} [\delta u(m + l_1 + l_2 - n_1^*) + \delta^2 u(m)] \\ - [\delta u(n_2^{**}) + \delta^2 u(m - (n_1^{**} - m)(1+r) - n_2^{**})] \end{array}\right] \beta\delta^2}{(\beta\delta^2)^2}$$

$$-\frac{\begin{pmatrix} [u(m+n_1^*)+\beta\delta u(m+l_1+l_2-n_1^*)+\beta\delta^2 u(m)] \\ -[u(m+n_1^{**})+\beta\delta u(n_2^{**})+\beta\delta^2 u(m-(n_1^{**}-m)(1+r)-n_2^{**})] \end{pmatrix}\delta^2}{\left(\beta\delta^2\right)^2}$$

such that n_1^*, n_1^{**} and n_2^{**} represent the profit maximizing n_1 and n_2 in case of planning default and not to default respectively. If we simplify the previous equation as:

$$\frac{\partial C_1}{\partial \beta} = -\frac{[u(m+n_1^*)-u(m+n_1^{**})]\,\delta^2}{\left(\beta\delta^2\right)^2} < 0$$

Accordingly;

$$C_1 > C_0 \text{ for } \beta \leq 1$$

Finally, we can write $C' = C_2$ since we already know that $C_2 > C_1$ from the previous proof. ∎

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