The Pennsylvania State University

The Graduate School

Eberly College of Science

ASYMPTOTICALLY CONTAINED REPRESENTATIONS AND THE SPHERICAL PLANCHEREL FORMULA

A Dissertation in

Mathematics

by

Qijun Tan

Submitted in Partial Fulfillment
of the Requirements
for the Degree of

Doctor of Philosophy

May 2019

The dissertation of Qijun Tan was reviewed and approved* by the following:

Nigel Higson
Evan Pugh Professor of Mathematics
Dissertation Advisor, Chair of Committee

Paul Baum
Evan Pugh Professor of Mathematics

Ping Xu
Distinguished Professor of Mathematics

Murat Günaydin
Professor of Physics

Mark Levi
Head of Mathematics Department

*Signatures are on file in the Graduate School.

Abstract

We introduce the notion of asymptotic containment of representations of C^*-algebras. The spectral measure of the ambient representation is closely related to the spectral measure of an asymptotically contained one. When G is a real reductive Lie group with Iwasawa decomposition KAN, and when M is the center of A in K, we show that the action of $C^*(G//K)$ on $L^2(K\backslash G/MN)$ is asymptotically contained in its action on $L^2(G//K)$. This fact can be used to prove Harish-Chandra's spherical Plancherel formula. Part of this thesis is a joint work with Nigel Higson.

Table of Contents

Acknowledgements

I am deeply grateful to my advisor Professor Nigel Higson for his guidance, unwavering support and generosity over the past six and a half years. I would also like to thank Professor Paul Baum for explaining in great detail various aspects of K-homology and many other things, and Professor Yi-Jun Yao for introducing the field of noncommutative geometry to me.

I am grateful to the mathematical department of the Penn State University for providing me all the resources that makes my graduate study a pleasure. I want to thank Shilin Yu, Yanli Song, Hsuan-Yi Liao, Eyal Subag, and many others I met at Penn State for their help and their discussions with me.

At last, I want to thank my parents and Yifan Li for their understanding and patience over many years.

The research described in this thesis was partially supported by the grant NSF 1101382 from the National Science Foundation.

Chapter 1

Introduction

The purpose of this thesis is to develop a new approach to Plancherel theorems, particularly in the context of real reductive groups. In this introductory chapter, we shall give a historical introduction to the topic, then give a first account for our approach and state our results.

1.1 Sturm-Liouville Theory and Weyl's Spectral Theorem

We shall start with an informal recount of the Sturm-Liouville theory. Let $[a, b]$ be a finite interval on $\mathbb{R}$. A *Sturm-Liouville* operator on $[a, b]$ is a second order linear differential operator of the form

$$D = -\frac{\mathrm{d}}{\mathrm{d}\,x} p \frac{\mathrm{d}}{\mathrm{d}\,x} + q. \tag{1.1.1}$$

For simplicity, we assume that p, q are real valued C^∞ functions with $p > 0$. The Sturm-Liouville theory is concerned with eigenvalues and eigenfunctions of D, that is, solutions of the equation

$$D\varphi_\lambda = \lambda\varphi_\lambda.$$

By imposing suitable boundary conditions on its domain, for instance

$$\varphi(a) = \varphi(b) = 0,$$

one makes D an unbounded self-adjoint operator. From a modern point of view, the Sturm-Liouville theorem is basically the spectral theorem for D:

Theorem 1.1.1 (Sturm-Liouville)**.** *The eigenvalues λ of D are real numbers, and each has multiplicity one. The eigenvalues are discrete, bounded below. If φ is a smooth function such that $\varphi(a) = \varphi(b) = 0$, then*

$$\varphi(x) = \sum_\lambda \frac{\langle \varphi_\lambda, \varphi \rangle}{\langle \varphi_\lambda, \varphi_\lambda \rangle} \varphi_\lambda(x)$$

for $\xi \in [a, b]$. Here φ_λ are eigenfunctions of D satisfying the above real boundary conditions.

Things are more complicated when one considers an infinite interval rather than a finite one. For one thing, the eigenfunctions of Sturm-Liouville operators in this case are not necessarily in L^2; for another, the spectrum might not be discrete.

In his habilitation, Weyl solved the problem of extending Sturm-Liouville theory to operators on a half-line, at least in an important special case. To obtain an eigenfunction expansion for D on $[0, \infty)$, Weyl made the assumption that $p(x)$ and $q(x)$ converge to constant values sufficiently fast as x tends to infinity [Wey10]. For simplicity, we shall assume in this introduction that

$$p(x) \equiv 1, \quad q(x) \equiv 0 \quad \text{if } x \gg 0. \tag{1.1.2}$$

Let φ_λ be the eigenfunction of D such that

$$\varphi_\lambda(0) = 0, \quad \frac{\mathrm{d}\,\varphi_\lambda}{\mathrm{d}\,x}(0) = 1. \tag{1.1.3}$$

With the boundary condition that $\varphi(0) = 0$, one can extend D into an unbounded self-adjoint operator on $L^2(0, \infty)$. In this case, D is positive modulo a compact perturbation, and therefore most of the spectrum consists of positive λ. For such a λ, from (1.1.2), together with the fact that φ_λ is real valued, we know that

$$\varphi_\lambda(x) = c(\lambda)\exp(i\sqrt{\lambda}x) + \overline{c(\lambda)}\exp(-i\sqrt{\lambda}x) \quad \text{for } x \gg 0, \tag{1.1.4}$$

where c is a complex valued function on $(0, \infty)$.

Theorem 1.1.2 (Weyl)**.** *Let D be a Sturm-Liouville operator on $[0, \infty)$ whose coefficients satisfy* (1.1.2)*. Let φ_λ be eigenfunctions of D that satisfy the boundary condition* (1.1.3) *and φ a C^∞ function compactly supported on $(0, \infty)$. We have*

$$\varphi(x) = \sum_{\lambda<0,\ \varphi_\lambda \in L^2} \frac{\langle \varphi_\lambda, \varphi\rangle}{\langle \varphi_\lambda, \varphi_\lambda\rangle}\varphi_\lambda(x) + \frac{1}{4\pi}\int_0^\infty \frac{\langle \varphi_\lambda, \varphi\rangle}{|c(\lambda)|^2}\varphi_\lambda(x)\frac{\mathrm{d}\,\lambda}{\sqrt{\lambda}}. \tag{1.1.5}$$

Weyl's spectral theorem, which is stronger than the version stated above (it applies to more general coefficients than (1.1.2)), has applications in the representation theory of reductive Lie groups. In fact, if one computes the radial part of the Casimir operator[1] on $SL(2, \mathbb{R})$, then (the stronger version of) the theorem is the spherical Plancherel formula of $SL(2, \mathbb{R})$. The theorem also suggested to Harish-Chandra the correct form of the Plancherel measure in general. As Borel [Bor01] wrote,

> "It was the reading of [Weyl's 1910 paper] which suggested to Harish-Chandra that the measure should be the inverse of the square modulus of a function in λ describing the asymptotic behaviour of the eigenfunctions . . . and I remember well from seminar lectures and conversations that he never lost sight of that principle, which is confirmed by his results in the general case."

[1] We consider the Casimir operator as an operator acting on, say, the upper half plane.

The approach Harish-Chandra took to prove the Plancherel formula for reductive Lie group was not a generalization of Weyl's theorem. The purpose of this thesis is to indicate that Weyl's spectral theorem can be approached from a perspective that can then be generalized to prove Harish-Chandra's spherical Plancherel formula.

1.2 Spherical Plancherel Formulae

We now give a brief introduction to Harish-Chandra's spherical Plancherel formula based on the examples of $SL(2,\mathbb{R})$ and $SL(n,\mathbb{R})$.

Let $G = SL(n,\mathbb{R})$, and let $\mathfrak{g}$ be its Lie algebra. The *Iwasawa decomposition* of G is

$$G = KAN,$$

where

$$K = SO(n),$$

A is the group of positive diagonal matrices in G, and N is the group of unipotent upper triangular matrices in G. Let $\mathfrak{a}$ be the Lie algebra of A. $\mathfrak{a}$ is the algebra of trace 0 diagonal matrices. The *restricted Weyl group* is

$$\mathcal{W} = N_K(\mathfrak{a})/Z_K(\mathfrak{a}),$$

where $N_K(\mathfrak{a})$ is the normalizer of $\mathfrak{a}$ in K, and $Z_K(\mathfrak{a})$ is the centralizer of $\mathfrak{a}$ in K. In our case $\mathcal{W} \cong S_n$. The Weyl group acts $\mathfrak{a}$, and the action is simply given by permutation of diagonal elements. We let

$$M = Z_K(A) = \left\{ \begin{pmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{pmatrix} : \det = 1 \right\}.$$

From now on we assume $G = SL(2,\mathbb{R})$. Important to us are certain subsets $\mathfrak{a}^+$ and $A^+ = \exp \mathfrak{a}^+$ of $\mathfrak{a}$ and A respectively. We let

$$\mathfrak{a}^+ = \left\{ \begin{pmatrix} \alpha & \\ & -\alpha \end{pmatrix} : \alpha > 0 \right\}.$$

One has polar decomposition of G:

$$G = KAK = K\overline{A^+}K. \tag{1.2.1}$$

Also recall that the *Casimir* of $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$ is the element

$$\Omega = X^2 + Y^2 - Z^2,$$

in the universal enveloping algebra of $\mathfrak{g}$, where

$$X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, Y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Z = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Let

$$C_c^\infty(G//K) = \{f \in C_c^\infty : f(k_1 g k_2) = f(g), \quad \forall k_1, k_2 \in K\}$$

be the vector space of K-bi-invariant, smooth, and compactly supported functions on G. There is a natural mapping

$$\psi_{KA^+K} \colon C_c^\infty(G//K) \longrightarrow C_c^\infty(A^+)$$

by restriction. And from (1.2.1), one can show that ψ_{KA^+K} extends (modulo a Radon-Nikodym factor) to an isomorphism from $L^2(G//K)$ to $L^2(A^+) \cong L^2(\mathfrak{a}^+)$. Therefore, to understand the spectral decomposition of Ω on $L^2(G//K)$, it suffices to understand the spectral decomposition of the *radial part* of Ω on $L^2(\mathfrak{a}^+)$.

To be more precise, let

$$D_{\text{radial}} = \frac{\mathrm{d}^2}{\mathrm{d}\,\alpha^2} + \frac{1}{\sinh^2 2\alpha},$$

and consider it as a differential operator on $C^\infty(\mathfrak{a}^+)$. Then the following diagram commutes:

$$\begin{array}{ccc} C^\infty(G//K) & \xrightarrow{\Omega} & C^\infty(G//K) \\ \downarrow{\scriptstyle \psi_{\text{radial}}} & & \downarrow{\scriptstyle \psi_{\text{radial}}} \\ C^\infty(\mathfrak{a}^+) & \xrightarrow[D_{\text{radial}}]{} & C^\infty(\mathfrak{a}^+). \end{array}$$

Here ψ_{radial} is the composition of restriction to A^+, the inverse of the exponential map and multiplication by $(\sinh 2\alpha)^{\frac{1}{2}}$. Thus, the radial part of Ω is a Sturm-Liouville operator. It is not difficult to show that for $\lambda > 0$ up to scalar, there is exactly one eigenfunction φ_λ of Ω with eigenvalue $-\lambda$. Hence, $\psi_{\text{radial}}(\varphi_\lambda)$ is a eigenfunction of D_{radial}. Similar to (1.1.4), it can be shown that

$$\varphi_\lambda(\alpha) = c_+(\lambda)\exp(i\sqrt{\lambda}\alpha) + c_-(\lambda)\exp(-i\sqrt{\lambda}\alpha) + o(\alpha). \tag{1.2.2}$$

Here c_+ and c_- are different parts of the *Harish-Chandra* **c***-function*, and

$$|c_+|^2 = |c_-|^2.$$

Given that D_{radial} has no negative spectrum, the spectral decomposition of D_{radial}, and therefore that of Ω on $L^2(G//K)$, is also almost the same as (1.1.5) (see [vdB08] for a more detailed account). We now arrive at the *spherical Plancherel formula for* $SL(2,\mathbb{R})$.

Theorem 1.2.1 (Harish-Chandra '57)**.** *If* $\varphi \in C_c(G//K)$*, then*

$$\|\varphi\|^2 = \int_{[0,\infty)} \frac{|\langle \varphi_\lambda, \varphi\rangle|^2}{|\mathbf{c}(\lambda)|^2}\,\mathrm{d}\,\lambda.$$

1.3 Asymptotic Containment

Let $\mathcal{A}$ be a C^*-algebra, (π, H) and (π_0, H_0) be two representations of $\mathcal{A}$.

Definition 1.3.1. (π_0, H_0) is *asymptotically contained* in (π, H) if

1. There is a one-parameter group $\{U_t\}$ of unitary operators on H_0 that commutes with $\pi_0(\mathcal{A})$.

2. There is a bounded operator $W\colon H_0 \longrightarrow H$ s.t. for any $a \in \mathcal{A}$ and $u, v \in H_0$,

$$\lim_{t\to\infty} \langle U_t u, \pi_0(a) U_t v\rangle - \langle W U_t u, \pi(a) W U_t v\rangle = 0. \tag{1.3.1}$$

The idea is that U_t corresponds to translation and that WU_t is asymptotically an intertwiner. One representation is asymptotically contained in another if in the limit of translation to infinity, it is included in the other. We shall illustrate with the example of Sturm-Liouville operator.

Example 1.3.2 (Sturm-Liouville operator)**.** Let D be a *Sturm-Liouville* operator as in (1.1.1) acting on $C^\infty(0,\infty)$. We assume asymptotic behavior of the coefficients as in (1.1.2), and we impose the boundary condition (1.1.3). We will denote the self-adjoint extension of D on $L^2([0,\infty))$ also by D. Since the spectrum of D is contained in $\mathbb{R}$, we can define a representation π of the C^*-algebra

$$\mathcal{A} = C_0(\mathbb{R})$$

on $H = L^2([0,\infty))$ via

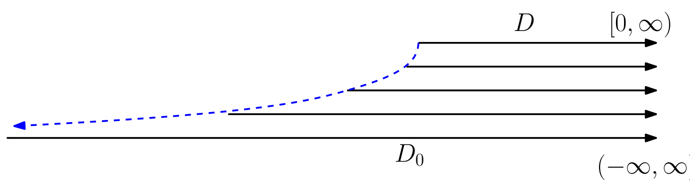
$$\pi(f) = f(D).$$

On the other hand, we let

$$D_0 = -\frac{\mathrm{d}^2}{\mathrm{d}\,x^2},$$

which we view as a self adjoint operator on $L^2(\mathbb{R})$. We define a second representation π_0 of $\mathcal{A}$ on $H_0 = L^2(\mathbb{R})$ via

$$\pi_0(f) = f(D_0).$$



Let U_t be translation towards the right by t. Since D_0 is translation invariant, $[U_t, f(D_0)] = 0$. We let W be the restriction of $L^2(\mathbb{R})$ to $L^2([0,\infty))$. For simplicity, suppose φ is compactly supported. For t large enough $U_t\varphi$ is supported on the region where (1.1.2) is true. In that case, $D\varphi = D_0\varphi$, and (1.3.1) is satisfied. Therefore, π_0 is asymptotically contained in π.

We give two reasons why asymptotic containments of representations are interesting.

First of all, recall that a representation π_0 of a C^*-algebra $\mathcal{A}$ is *weakly contained* in another representation π if

$$\pi(a) = 0 \Rightarrow \pi_0(a) = 0$$

for any $a \in \mathcal{A}$. The Plancherel measure is supported on representations weakly contained in the left regular representation. On the other hand, it is easy to check that asymptotic containment implies weak containment. Therefore, if we study a representation that is weakly contained in the left regular representation, then we restrict ourselves only to those representations with interesting Plancherel measure.

Secondly, suppose $\mathcal{A}$ is commutative and $W\colon H_0 \longrightarrow H$ is an isometric embedding of H_0 as a subrepresentation of H.

$$W\colon H_0 = \int^{\oplus} H_{0,\lambda}\,\mathrm{d}\,\mu_0(\lambda) \longrightarrow H = \int^{\oplus} H_\lambda\,\mathrm{d}\,\mu(\lambda)$$

descends to

$$W_\lambda \colon H_{0,\lambda} \longrightarrow H_\lambda,$$

and

$$W_\lambda^* W_\lambda = C \operatorname{Id}_{H_{0,\lambda}}. \tag{1.3.2}$$

$\langle W_\lambda u, W_\lambda v\rangle \, \mathrm{d}\,\mu(\lambda) = \langle u, v\rangle \, \mathrm{d}\,\mu_0(\lambda)$, that is $\frac{\mathrm{d}\,\mu_0}{\mathrm{d}\,\mu}(\lambda) = C$. If $H_{0,\lambda}$ are finite dimensional, we have

$$\frac{\mathrm{d}\,\mu_0}{\mathrm{d}\,\mu}(\lambda) = \frac{\operatorname{Tr}(W_\lambda^* W_\lambda)}{\dim H_{0,\lambda}}.$$

The important observation we made is that the above formula is still true even if π_0 is only asymptotically contained in π. Here is an informal version of our main result (see Theorem 3.1.6 for the precise statement):

Theorem 1.3.3. *If π_0 is asymptotically contained in π, and if a certain technical hypothesis holds, then*

$$\frac{\mathrm{d}\,\mu_0}{\mathrm{d}\,\mu}(\lambda) = \frac{\operatorname{Tr}(W_\lambda^* W_\lambda)}{\dim H_{0,\lambda}}. \tag{1.3.3}$$

In fact, we can *still* derive the formula (1.3.3) in essentially the same way as above, except that we must now *assume* the existence of operators W_λ that *asymptotically* decompose W in the sense that

$$\lim_{t\to+\infty} W_\lambda (U_t v)_{0,\lambda} - (W U_t v)_\lambda = 0. \tag{1.3.4}$$

This is the additional technical hypothesis. See Section 3.1 for details. As for (1.3.4), it is easiest to understand the meaning of this formula by examining the adjoint operators

$$C_\lambda = W_\lambda^* \colon H_\lambda \longrightarrow H_{0,\lambda}.$$

In the context of the Sturm-Liouville problem the spaces H_λ and $H_{0,\lambda}$ may be understood as the λ-eigenspaces of the operators D and D_0, respectively, and the asymptotic formula (1.3.4) simply asserts that each eigenfunction of D is mapped by C_λ to an eigenfunction of D_0 to which it is asymptotic in the sense of (1.1.4). This proves the existence of the operators C_λ in this context, and also computes the Radon-Nikodym derivative in (1.3.3) in terms of $|c(\lambda)|^2$. Weyl's formula (1.1.5) is a consequence of the Radon-Nikodym derivative formula (1.3.3) and the explicit formula for μ_0 that is provided by ordinary Fourier theory.[2]

We shall continue to use this strategy. In our applications, we assume spectral decomposition for π_0 is well understood. It will not be difficult to understand W_λ from a geometric

[2]To be accurate, the Radon-Nikodym derivative formula only determines μ on the positive part of the spectrum. A separate argument is required for the negative spectrum; see Section 3.3.

point of view. Therefore, (1.3.3) provides a convenient way to compute the Plancherel measure for the spectral decomposition of D_0, which is one of the core ingredients in proving a Plancherel formula.

Remark 1.3.4. In order to prove Theorem 1.3.3, we proved that a certain averaging of $W_\lambda^* W_\lambda$ has to be $C \cdot \mathrm{Id}$, just like in (1.3.2). This in the case of spherical Plancherel formula immediately implies the famous Maass-Selberg relation.

1.4 Asymptotic Containment and the Spherical Plancherel Formula

We now give a heuristic explanation how Theorem 1.3.3 can be used to prove the spherical Plancherel formula for $G = SL(2, \mathbb{R})$. We will give a complete argument later in the thesis. The idea is to compare the representation π we want to study, namely $C_r^*(G//K)$ on $H = L^2(G//K)$ to a simpler representation.

The simple representation π_0, as it turns out, is the left convolution action of $C_r^*(G//K)$ on $H_0 = L^2(K\backslash G/MN)$. According to Iwasawa's theorem, the multiplication map

$$K \times A \times N \longrightarrow G$$

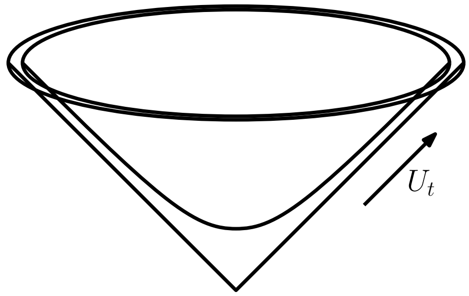
is a diffeomorphism. Also recall that $M = Z_K(A)$. Therefore,

$$L^2(K\backslash G/MN) \cong L^2(A).$$

In other words, we are simply studying a representation on L^2 functions over the real line. With a bit more work, one can show the spectral decomposition for the representation of $C^*(G//K)$ on $L^2(K\backslash G/MN)$ is nothing but the Fourier theory on a real line. Thus, we know the Plancherel measure for the spectral decomposition of H_0 with respect to π_0, which is $\mathrm{d}\,\mu_0$ in (1.3.3).

To use (1.3.3) to compute the Plancherel measure for the spherical Plancherel formula, we need to show that π_0 is asymptotically contained in π, or more intuitively π is asymptotic to π_0 after 'translation towards infinity'.

Note that G/K can be identified with one sheet in the two sheet hyperboloid coadjoint orbit while G/MN can be identified with a cone coadjoint orbit. The K action on the two spaces is just rotation along the z-axis in the figure below.



The actions of $C_r^*(G//K)$ on H and H_0 are induced from the actions of G on $L^2(G/K)$ and $L^2(G/MN)$, which can be visualized as the coadjoint action of G on the two coadjoint orbits in the figure above. We let W be first restricting the cone above a certain horizontal level, and then map it horizontally to the hyperboloid. We let U_t be the translation along the cone as shown above. Since the cone and the hyperboloid are geometrically close as we move along the cone towards infinity, and the coadjoint action is continuous on $\mathfrak{g}^*$, the action of G on $L^2(G/K)$ and $L^2(G/MN)$ are asymptotic towards infinity. Therefore, π_0 is asymptotically contained in π as required.

Chapter 2

Preliminaries

In this chapter we start with structure theory of real reductive Lie groups, and then review the part of representation theory of real reductive Lie groups that is relevant to the thesis. Most important to us is the connection between representations of G, those of $C^*(G)$, and common eigenfunctions of $U(\mathfrak{g})^K$. We also give a technical review of the Gelfand-Kostyuchenko method, which we will later use to compute isotypic components of spectral decompositions, and a review of Sturm-Liouville operators with real boundary value conditions.

2.1 Basic Lie Theory

We start with a brief review of basic Lie theory.

Definition 2.1.1. [War83] A *Lie group* G is a smooth manifold with a compatible group structure such that the map

$$\begin{aligned} G \times G &\longrightarrow G, \\ (g_1, g_2) &\longmapsto g_1 g_2^{-1} \end{aligned}$$

is smooth.

A *Lie algebra* $\mathfrak{g}$ is a real or complex non-associative algebra. Its multiplication is give by a *Lie bracket* $[\ ,\ ]$ which satisfies

1. $[X, Y] = -[Y, X]$ (anti-commutativity).
2. $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ (*Jacobi identity*).

A Lie algebra $\mathfrak{g}$ is *abelian* if for any $X, Y \in \mathfrak{g}$, $[X, Y] = 0$. The *universal enveloping algebra* of $\mathfrak{g}$ is $U(\mathfrak{g})$, which is the quotient of the free algebra generated by $\mathfrak{g}$ by the ideal generated by $\{[X, Y] - (XY - YX)\}$.

We shall only deal with finite dimensional Lie algebra in this document.

The *Lie algebra of a Lie group* G, as a vector space, is the collection of left-invariant vector fields on G. The Lie bracket is given by

$$[X, Y] = XY - YX.$$

From now on, we will use $\mathfrak{g}$ to denote the Lie algebra of the Lie group G.

The *exponential map* is a smooth map $\exp\colon \mathfrak{g} \longrightarrow G$ such that $tX \longmapsto \exp tX$ is a Lie group homomorphism from $\mathbb{R}$ to G, and such that the differential of $\exp$ at e is the identity map. If G is simply connected abelian, then $\exp$ is a bijection, and we denote the inverse by $\log$.

The Lie group G acts on itself by conjugation, which will be denoted by Ad. The differential of the action is the *adjoint action* Ad of G on $\mathfrak{g}$. The *coadjoint action* is the action of G on $\mathfrak{g}^*$ induced from the adjoint action.

The *adjoint action* ad of $\mathfrak{g}$ on itself is given by

$$\mathrm{ad}_X Y = [X, Y] \quad \text{for any } X, Y \in \mathfrak{g}.$$

It may be shown that ad is the differential of the group adjoint action. See [War83] for a more detailed account of basic Lie theory.

If ad_X is nilpotent, then $\exp \mathrm{ad}_X$ is algebraically defined, and is a Lie algebra automorphism of $\mathfrak{g}$. The subgroup of the automorphism group of $\mathfrak{g}$ generated by such $\exp \mathrm{ad}_X$is the *inner automorphism group* of $\mathfrak{g}$, and is denoted by $\mathrm{Int}\,\mathfrak{g}$.

The *Killing form of a Lie algebra* $\mathfrak{g}$ is the bilinear form

$$\kappa(X, Y) = \mathrm{Tr}(\mathrm{ad}_X \mathrm{ad}_Y). \tag{2.1.1}$$

Example 2.1.2. $\mathbb{R}$ is a Lie group with addition as the group structure. $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ is also a Lie group with the restriction of complex multiplication as the group structure. It may be shown that any abelian Lie group is a product copies of $\mathbb{R}$ and S^1.

Example 2.1.3. $SU(2)$ is a Lie group with group multiplication given by matrix multiplication. If we identify left-invariant vector fields with its restriction to the tangent space of $e \in G$, the Lie algebra of $SU(2)$ is $\mathfrak{su}(2) = \{M \in \mathfrak{gl}(2, \mathbb{C}) : \mathrm{Tr}\, M = 0 \text{ and } M^* = -M\}$. The Lie bracket is given by the commutator. It is one of the smallest simple Lie groups.

In this document, we only consider representations of Lie groups on Hilbert spaces.

Definition 2.1.4. [Kna86] A *representation* of a Lie group G is a pair (π, H) where H is a complex a Hilbert space, and π is a homomorphism of G into the group of bounded invertible linear operators on H such that the map

$$G \times H \longrightarrow H$$

is continuous.

A representation (π, H) is *irreducible* if there is no nontrivial invariant subspace, and is *unitary* if the range of π is a subset of unitary operators on H.

Two representations (π, H) and (π', H') are *equivalent* if there is an invertible operator $I\colon H \longrightarrow H'$ such that

$$I\pi(g) = \pi'(g)I$$

for any $g \in G$. The two representations are *unitarily equivalent* if I is a unitary operator.

The collection of unitary equivalence classes of irreducible unitary representations of a Lie group G is the *unitary dual* of G, and is denoted by $\widehat{G}$.

If (π, H) is a representation of G, its dual representation is (π, H^*). Here H^* is the same real vector space as H but with conjugate complex structure, i.e. $\lambda v^* \in H^*$ and $\overline{\lambda} v \in H$ are the same vector in the underlying real vector space if $v^* \in H^*$ and $v \in H$ are the same vector. The inner product is given by

$$\overline{\langle v^*, w^* \rangle} = \langle v, w \rangle,$$

where $v^*, w^* \in H^*$ are the same vectors as $v, w \in H$. The action π is the same on the underlying real vector space.

Given a representation (π, H) of Lie group G, the differential of π gives a representation of the Lie algebra $\mathfrak{g}$ of G on the space of smooth vectors $H^\infty \subset H$.

Example 2.1.5. Suppose G is unimodular. Fix a Haar measure on G, and let $H = L^2(G)$. The *left regular representation* is (λ_G, H), where

$$(\lambda_G(g)f)(\gamma) = f(g^{-1}\gamma)$$

for any $g, \gamma \in G$. Likewise the *right regular representation* is (ρ_G, H), where

$$(\rho_G(g)f)(\gamma) = f(\gamma g).$$

2.2 Semisimple and Reductive Lie Groups

In this section, we first review Cartan decomposition for semisimple Lie groups, and then then introduce the notion of real reductive Lie groups, which are motivated by Cartan decomposition. We then review Iwasawa decomposition and KAK decomposition.

Definition 2.2.1. A Lie algebra is *semisimple* if it is a direct sum of simple algebras. A Lie group is *semisimple* if it does not contain non-trivial abelian normal subgroups. A connected Lie group is semisimple iff its Lie algebra is.

In this thesis, unless otherwise noted, we assume semisimple Lie groups to be connected and have finite center.

2.2.1 Cartan Decomposition

Definition 2.2.2. A *Cartan involution* θ of a semisimple Lie algebra $\mathfrak{g}$ is an involution such that the bilinear form B

$$B_\theta(X,Y) = -\kappa(X,\theta Y), \quad X,Y \in \mathfrak{g} \tag{2.2.1}$$

is positive definite.

Theorem 2.2.3. *[Kna02, Corollary 6.18, 6.19] If $\mathfrak{g}$ is a real semisimple Lie algebra, $\mathfrak{g}$ has a Cartan involution. Any two Cartan involutions are conjugate via* $\operatorname{Int}\mathfrak{g}$.

Example 2.2.4. If $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{R})$, then a Cartan involution is given by $\theta(X) = -X^T$.

Definition 2.2.5. A *Cartan decomposition* of a semisimple Lie algebra $\mathfrak{g}$ is a vector space deposition of $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}_C$$

such that

- κ is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}_C$.
- $[\mathfrak{k},\mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k},\mathfrak{p}_C] \subset \mathfrak{p}_C, \quad [\mathfrak{p}_C,\mathfrak{p}_C] \subset \mathfrak{k}$.

Theorem 2.2.6. *[Kna02, p.359] There is a one-to-one correspondence between Cartan decompositions and Cartan involutions.*

If θ is a Cartan involution, and $\mathfrak{k}$ and $\mathfrak{p}_C$ are the $+1$ and -1 eigenspaces, then $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{p}_C$ is a Cartan decomposition.

Conversely, if $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}_C$ is a Cartan decomposition, then

$$\theta = \begin{cases} +1 & \text{on } \mathfrak{k} \\ -1 & \text{on } \mathfrak{p}_C \end{cases}$$

is a Cartan involution.

Example 2.2.7. $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{R})$, $\theta(X) = -X^T$. $\mathfrak{k} = \{$ skew symmetric matrices $\}$, $\mathfrak{p}_C = \{$ trace 0 symmetric matrices $\}$.

With the preparation of Cartan decomposition on Lie algebras, we now state the Cartan decomposition for Lie groups.

Theorem 2.2.8. *[Kna02, Theorem 6.31] Let G be a connected semisimple Lie group and θ be a Cartan decomposition of its Lie algebra $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}_C$ be the corresponding Cartan decomposition. Let K be the subgroup of G generated by* $\exp \mathfrak{k}$.

- *There is a* global Cartan involution Θ, *which is an automorphism of G such that* $\Theta^2 = \mathrm{Id}$ *and such that the differential of Θ is θ.*
- *K is the subgroup fixed by Θ.*
- *If G has finite center, K is a maximal compact subgroup of G.*
-

$$K \times \mathfrak{p}_C \longrightarrow G \tag{2.2.2}$$

$$(k, X) \longmapsto k \exp X \tag{2.2.3}$$

is a diffeomorphism. This is the global Cartan decomposition.

2.2.2 Reductive Lie Groups

Definition 2.2.9. A Lie algebra is *reductive* if it is a direct sum of a semisimple Lie algebra and an abelian Lie algebra.

A *reductive Lie group* is a 4-tuple (G, K, θ, κ) where G is a Lie group, K is a compact subgroup of G, θ is an involution of $\mathfrak{g}$, and κ is a non-degenerate bilinear form on $\mathfrak{g}$ such that

1. $\mathfrak{g}$ is reductive.
2. We have a decomposition $\mathfrak{g}$ as a vector space:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}_C. \tag{2.2.4}$$

   Here $\mathfrak{k}, \mathfrak{p}_C$ are the $+1$ and -1 eigenspaces of $\mathfrak{g}$ under θ, and $\mathfrak{k}$ is also the Lie algebra of K.

3. κ is invariant under $\mathrm{Ad}\, G$ and θ.
4. $\mathfrak{k}$ and $\mathfrak{p}_C$ are orthogonal under κ. κ is positive definite on $\mathfrak{p}_C$ and negative definite on $\mathfrak{k}$.

5. $$K \times \exp \mathfrak{p}_C \longrightarrow G \tag{2.2.5}$$

   is a diffeomorphism onto.

6. Every automorphism Ad_g of $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$ for $g \in G$ is in $Int\ \mathfrak{g}_{\mathbb{C}}$.

7. G_{ss}, the connected subgroup of G with Lie algebra $[\mathfrak{g}, \mathfrak{g}]$, has finite center.

For a reductive Lie group G, K is called the associated *maximal compact subgroup*, θ is called the *Cartan involution*, (2.2.4) is the *Cartan decomposition* at the algebra level, (2.2.5) is the *Cartan decomposition* at the Lie group level, and κ the *invariant bilinear form*.

Example 2.2.10. All closed subgroups of $GL(n, \mathbb{R})$ that are closed under transpose are reductive.

Therefore, connected semisimple Lie groups with finite centers are reductive. All reductive Lie groups are unimodular.

2.2.3 Iwasawa Decomposition

Suppose G is a connected real reductive Lie group with finite center. Let us fix a Cartan involution θ on $\mathfrak{g}$ and a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}_C$. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}_C$.

Definition 2.2.11. Given $\alpha \in \mathfrak{a}^*$, let

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : (\mathrm{ad}\, H)X = \alpha(H)X \text{ for all } H \in \mathfrak{g}\}.$$

If $\mathfrak{g}_\alpha \neq 0$, then α is a *restricted root* of $\mathfrak{g}$, and $\mathfrak{g}_\alpha$ is a *restricted root space*. The set of restricted roots is denoted by Σ. The *multiplicity* $n(\alpha)$ of a restricted root α is the dimension of $\mathfrak{g}_\alpha$. It is customary to denote the half sum (with multiplicity) of the positive restricted roots by ρ.

Let

$$M = Z_K(\mathfrak{a}). \tag{2.2.6}$$

The *restricted Weyl group* is

$$\mathcal{W} = N_K(\mathfrak{a})/Z_K(\mathfrak{a}). \tag{2.2.7}$$

Note that $\mathcal{W}$ acts on Σ.

Lemma 2.2.12. *[Kna02, Lemma 6.27]*

$$(\operatorname{ad} X)^* = -\operatorname{ad} \theta X$$

where the adjoint is taken over B_θ defined in (2.2.1).

$\mathfrak{a}$ acts as a commuting family of symmetric operators on $\mathfrak{g}$ due to the Lemma above. We therefore have:

Proposition 2.2.13 (Restricted Root Space Decomposition)**.** *[Kna02, Proposition 6.40]*

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha\in\Sigma} \mathfrak{g}_\alpha = \mathfrak{m} \oplus \mathfrak{a} \oplus \bigoplus_{\alpha\in\Sigma} \mathfrak{g}_\alpha \tag{2.2.8}$$

where $\mathfrak{m} = Z_{\mathfrak{k}}(\mathfrak{a})$.

Definition 2.2.14. A *simple system* Π of restricted roots Σ is a linearly independent set of restricted roots such that every restricted root $\alpha \in \Sigma$ is an integral linear combination of elements in the simple system, and that the coefficients are either non-positive or non-negative. A *simple root* is an element in a simple system.

Given a simple system, the *positive restricted roots* are those that are non-negative linear combinations of elements in the simple system, and the *negative restricted roots* are those that are non-positive linear combinations. We denote the collection of positive restricted roots Σ^+. We denote $\mathfrak{a}^+ = \{X \in \mathfrak{a} \colon \alpha(X) > 0 \;\text{ for any } \alpha \in \Sigma^+\} + (\mathfrak{p}_C \cap Z_{\mathfrak{g}})$, and $A^+ = \exp \mathfrak{a}^+$.

Theorem 2.2.15. *[Hum78, 10.1][Kna02, Corollary 6.53] A simple system always exists.*

Theorem 2.2.16. *[Hum78, 10.3] The restricted Weyl group $\mathcal{W}$ acts simply transitively on the set of all possible simple systems.*

Let us fix a set of positive roots Σ^+.

Theorem 2.2.17 (Iwasawa decomposition)**.** *[Kna02, Proposition 6.43, 6.46, 7.31] Let*

$$\mathfrak{n} = \bigoplus_{\alpha\in\Sigma^+} \mathfrak{g}_\alpha, \tag{2.2.9}$$

then $\mathfrak{n}$ *is nilpotent,* $[\mathfrak{a}\oplus\mathfrak{n}, \mathfrak{a}\oplus\mathfrak{n}] \subseteq \mathfrak{n}$*, and*

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}. \tag{2.2.10}$$

Furthermore, let $A,\ N$ *be subgroups of* G *generated by* $\exp\mathfrak{a},\ \exp\mathfrak{n}$*, then the identity components of* A *and* N *are simply connected, and*

$$K \times A \times N \longrightarrow G \tag{2.2.11}$$

$$(k, a, n) \longmapsto kan \tag{2.2.12}$$

is a diffeomorphism.

(2.2.10) *and* (2.2.11) *are Iwasawa decompositions of Lie algebras and Lie groups.*

Example 2.2.18. If $G = SL(n, \mathbb{R})$ and $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$, then $G = KAN$ where

- K may be taken to be $SO(n)$,
- A may be taken to be diagonal matrices with determinant 1,
- N may be taken to be upper triangular matrices with 1's on the diagonal,

and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ where

- $\mathfrak{k}$ is skew symmetric matrices,
- $\mathfrak{a}$ is trace 0 diagonal matrices,
- $\mathfrak{n}$ is upper triangular matrices with 0's on the diagonal.

The example above is actually representative for all Iwasawa decompositions.

Proposition 2.2.19. *[Kna02, Lemma 6.45] There exists a basis of* $\mathfrak{g}$ *such that the under the basis*

- *Elements of* $\operatorname{ad}\mathfrak{k}$ *are skew symmetric.*
- *Elements of* $\operatorname{ad}\mathfrak{a}$ *are diagonal.*
- *Elements of* $\operatorname{ad}\mathfrak{n}$ *are upper triangular with 0's on the diagonal.*

Finally, Iwasawa decomposition is unique up to conjugation by elements of K.

Theorem 2.2.20. *[Kna02, Theorem 6.51, 6.53, 7.29] Any two maximal abelian subspaces of* $\mathfrak{p}_C$ *are conjugate by some* $k \in K$*. If we fix an* $\mathfrak{a}$*, any two choices of* Σ^+ *(and hence* $\mathfrak{n}$*) are conjugate by some* $w \in \mathcal{W}$*.*

2.2.4 Polar Decomposition

Theorem 2.2.21. *[Kna02, Theorem 7.39]*

$$G = KAK. \tag{2.2.13}$$

If $g = k_1 a k_2$, then a is unique up to conjugation by an element in $\mathcal{W}$. k_i are unique up to conjugations by elements in M.

Since the restricted Weyl group $\mathcal{W}$ acts on $\mathfrak{a}$, and $\mathfrak{a}^+$ is a fundamental domain,

$$G = K\overline{A^+}K. \tag{2.2.14}$$

Example 2.2.22. For $G = SL(n, \mathbb{R})$, by polar decomposition, $g = kg_{sa}$ where g_{sa} is self-adjoint. $g_{sa} = k_1 a k_2$ by spectral decomposition.

Proposition 2.2.23. *[Hel08, Chapter II, Section 2][GV88, Section 2.4] The map*

$$\begin{aligned} \psi_{G/K} \colon K/M \times A^+ &\longrightarrow G/K \\ (kM, a) &\longmapsto kaK \end{aligned}$$

is an open imbedding with dense range. Here M is defined as in (2.2.6). *Fix a Haar measure on G, which descends to a G-invariant measure on G/K. With respect to the decomposition above, the measure on G/K is given by*

$$\int_{G/K} f(gK)\,\mathrm{d}\,gK = \int_{K\times A^+} f(kaK) J_K(a)\,\mathrm{d}\,k\,\mathrm{d}\,a. \tag{2.2.15}$$

Here $\mathrm{d}\,k$, $\mathrm{d}\,a$ are Haar measures on K, A, and

$$J_K(a) = \prod_{\alpha\in\Sigma^+} \big(\exp(\alpha(\log a)) - \exp(-\alpha(\log a))\big)^{n(\alpha)}$$

where $n(\alpha)$ is the multiplicity of the restricted root.

2.3 Levi-Langlands Decomposition

In this section, we start with the easier case of Levi-Langlands decomposition for the Lie algebra, and ends with the decomposition for Lie groups.

2.3.1 Decomposition for Complex Reductive Lie Algebra

Definition 2.3.1. If $\mathfrak{g}_{\mathbb{C}}$ is a complex Lie algebra, then a *Cartan subalgebra* (CSA) $\mathfrak{h}_{\mathbb{C}}$ is defined to be a nilpotent subalgebra of $\mathfrak{g}_{\mathbb{C}}$ whose normalizer is itself.

The existence of a CSA can be proved using generalized eigenspaces.

Theorem 2.3.2. *[Hum78, 16.4] The CSAs are conjugate via $Int\ \mathfrak{g}_{\mathbb{C}}$ for an arbitrary Lie algebra $\mathfrak{g}_{\mathbb{C}}$.*

Definition 2.3.3. A *Borel subalgebra* $\mathfrak{b}_{\mathbb{C}}$ in $\mathfrak{g}_{\mathbb{C}}$ is a maximal solvable Lie subalgebra.

Theorem 2.3.4. *[Hum78, 16.4] The Borel subalgebras are conjugate via $Int\ \mathfrak{g}_{\mathbb{C}}$ for an arbitrary Lie algebra $\mathfrak{g}_{\mathbb{C}}$.*

Example 2.3.5. If $\mathfrak{g}_{\mathbb{C}}$ is complex reductive, then

$$\mathfrak{b}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} + \sum_{\alpha \in \Sigma^+} \mathfrak{g}_{\mathbb{C},\alpha}$$

where Σ is the set of roots of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$[1].

Definition 2.3.6. A *parabolic subalgebra* $\mathfrak{p}_{\mathbb{C}}$ in $\mathfrak{g}_{\mathbb{C}}$ is a subalgebra that contains $\mathfrak{b}_{\mathbb{C}}$.

Example 2.3.7. Suppose $\mathfrak{g}_{\mathbb{C}}$ is complex reductive. Fix a simple system Π^+ and hence Σ^+. Suppose $\widehat{\Pi} \subset \Pi$, and $\Sigma^+_{\widehat{\Pi}}$ is the set of positive linear combinations of $\widehat{\Pi}$ in Σ^+.

$$\mathfrak{b}_{\mathbb{C}} + \sum_{\alpha \in -\Sigma^+_{\widehat{\Pi}}} \mathfrak{g}_{\mathbb{C},\alpha}$$

is a parabolic subalgebra.

Theorem 2.3.8. *[Kna02, Proposition 5.90] If $\mathfrak{g}_{\mathbb{C}}$ is complex reductive, then all parabolic subalgebra that contains $\mathfrak{b}_{\mathbb{C}}$ are given as above. Thus, the parabolic subalgebras are parameterized by subsets of* Π.

Example 2.3.9. $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}(n, \mathbb{C})$. $\mathfrak{h}_{\mathbb{C}} = \{$ trace 0 diagonal matrices $\}$. Let $e_i(\begin{pmatrix} h_1 & & \\ & \ddots & \\ & & h_n \end{pmatrix}) = h_i$, $\Sigma^+ = \{e_i - e_j : i < j\}$, then $\Pi = \{e_1 - e_2, \cdots, e_{n-1} - e_n\}$. $\mathfrak{g}_{e_i - e_j} = E_{ij}$, $\mathfrak{b}_{\mathbb{C}} = \{$ trace 0 upper triangular matrices$\}$. $\mathfrak{p}_{\mathbb{C}} = \{$ trace 0 block upper triangular matrices $\}$.

[1] See [Hum78, III] for a detailed account of root system.

2.3.2 Decomposition for Real Reductive Lie Algebra

From now on we assume $\mathfrak{g}$ to be real reductive, and that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}_C$ is the Cartan decomposition.

Definition 2.3.10. $\mathfrak{h}$ (resp. $\mathfrak{p}$) is a *CSA (resp. parabolic subalgebra)* of $\mathfrak{g}$ if $\mathfrak{h}_{\mathbb{C}}$ (resp. $\mathfrak{p}_{\mathbb{C}}$) is a CSA (resp. parabolic subalgebra) of $\mathfrak{g}_{\mathbb{C}}$.

From now on we shall only consider θ-stable CSA. The reason is that if $\mathfrak{h}$ is θ-stable, then $\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{k}) \oplus (\mathfrak{h} \cap \mathfrak{p}_C)$.

Theorem 2.3.11 ([Kna02] p.457)**.** *Any CSA of* $\mathfrak{g}$ *is conjugate to a* θ*-stable one via* $Int\ \mathfrak{g}$.

Definition 2.3.12. $\dim \mathfrak{h} \cap \mathfrak{p}$ and $\dim \mathfrak{h} \cap \mathfrak{k}$ are the *noncompact dimension* and the *compact* dimension.

Example 2.3.13 ([Kna02], p.458)**.**

1. If $\mathfrak{a}$ is maximal abelian in $\mathfrak{p}_C$, $\mathfrak{m} = Z_{\mathfrak{k}}(\mathfrak{a})$, and $\mathfrak{h}_{\mathfrak{m}}$ a maximal abelian subspace of $\mathfrak{m}$, then

$$\mathfrak{h}_{\mathfrak{m}} \oplus \mathfrak{a}$$

is a *maximally noncompact CSA*.

2. If $\mathfrak{t}$ is a maximal torus in $\mathfrak{k}$, then

$$\mathfrak{h} = Z_{\mathfrak{g}}(\mathfrak{t})$$

is a *maximally compact CSA*.

Theorem 2.3.14 ([Kna02], p.458)**.** *All maximally compact (resp. noncompact)* θ*-stable CSAs are conjugate via* K.

Note that compact/noncompact dimensions are obstructions to conjugacies via K. On the other hand, we have the following positive result.

Proposition 2.3.15 ([Kna02], Lemma 6.62, Prop 6.64)**.** *If* $\mathfrak{g}$ *is reductive,* $\mathfrak{h} \cap \mathfrak{p}_C = \mathfrak{h}' \cap \mathfrak{p}_C$*, then* $\mathfrak{h}$ *and* $\mathfrak{h}'$ *are conjugate via* K*. And there are finitely many conjugacy classes of CSA in* $\mathfrak{g}$.

Example 2.3.16. $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$. A complete list of θ-stable non-conjugate Cartan subalgebras is given by

$$\mathfrak{h}_j = \begin{pmatrix} t_1 & \theta_1 & & & & & & \\ -\theta_1 & t_1 & & & & & & \\ & & \ddots & & & & & \\ & & & t_j & \theta_j & & & \\ & & & -\theta_j & t_j & & & \\ & & & & & s_{2j+1} & & \\ & & & & & & \ddots & \\ & & & & & & & s_n \end{pmatrix},$$

where $\mathfrak{h}_j$ is also trace 0. The compact dimension of $\mathfrak{h}_j$ is j, and the noncompact dimension of $\mathfrak{h}_j$ is $n - j - 1$.

The set of parabolic subalgebras is partially ordered under inclusion. If $\mathfrak{a}$ is maximal abelian in $\mathfrak{p}_C$, $\mathfrak{n} = \oplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha$, and $\mathfrak{m} = Z_{\mathfrak{k}}(\mathfrak{a})$, then

$$\mathfrak{p}_0 = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n} \tag{2.3.1}$$

is a *minimal parabolic subalgebra*. Here Σ is the set of restricted roots. Any minimal parabolic subalgebra is conjugate to this one via Int $\mathfrak{g}$[Kna02, p.475].

Example 2.3.17. $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$,

$$\mathfrak{a} = \{ \text{ trace 0 diagonal matrices } \},$$

$$\mathfrak{n} = \{\text{upper triangular matrices with 0 on the diagonal}\},$$

$$\mathfrak{m} = 0,$$

$$\mathfrak{p}_0 = \{ \text{ trace 0 upper triangular matrices}\}.$$

Example 2.3.18. Let $\Pi \subset \Sigma^+$ be the set of simple roots, and $\widehat{\Pi} \subset \Pi$. Let $\Sigma^+_{\widehat{\Pi}}$ be the set of positive linear combinations of $\widehat{\Pi}$ in Σ^+.

$$\mathfrak{p} = \mathfrak{p}_0 + \sum_{\alpha \in -\Sigma^+_{\widehat{\Pi}}} \mathfrak{g}_\alpha$$

is a parabolic subalgebra of $\mathfrak{g}$. Every parabolic subalgebra that contains $\mathfrak{p}_0$ is of this form, and thus the parabolic subalgebras is parameterized by subsets of restricted roots.

Definition 2.3.19. Let $\mathfrak{n}_\mathfrak{p}$ be the nilradical (maximal nilpotent ideal) of $\mathfrak{p} \cap [\mathfrak{g}, \mathfrak{g}]$, and let $\mathfrak{m}_{1,\mathfrak{p}} = \mathfrak{p} \cap \theta(\mathfrak{p})$. Then,

$$\mathfrak{p} = \mathfrak{m}_{1,\mathfrak{p}} \oplus \mathfrak{n}_\mathfrak{p}$$

is the *Levi decomposition* of $\mathfrak{p}$.

Definition 2.3.20. Let $\mathfrak{a}_\mathfrak{p} = Z(\mathfrak{m}_{1,\mathfrak{p}}) \cap \mathfrak{p}_C$, and $\mathfrak{m}_\mathfrak{p}$ be the orthogonal complement of $\mathfrak{a}_\mathfrak{p}$ in $\mathfrak{m}_{1,\mathfrak{p}}$ under the inner product $\langle X, Y \rangle = -B(X, \theta Y)$. Then,

$$\mathfrak{p} = \mathfrak{m}_\mathfrak{p} \oplus \mathfrak{a}_\mathfrak{p} \oplus \mathfrak{n}_\mathfrak{p}$$

is the *Langlands decomposition* of $\mathfrak{p}$.

Example 2.3.21 ([GV88] p.67)**.** Let

$$\mathfrak{p}_{\widehat{\Pi}} = \mathfrak{p}_0 + \sum_{\alpha \in -\Sigma^+_{\widehat{\Pi}}} \mathfrak{g}_\alpha.$$

Then,

$$\mathfrak{a}_{\widehat{\Pi}} = \{H \in \mathfrak{a} : \alpha(H) = 0 \text{ for all } \alpha \in \widehat{\Pi}.\}$$

$$\mathfrak{m}_{1,\widehat{\Pi}} = \mathfrak{a} + \sum_{\alpha \in \Sigma_{\widehat{\Pi}}} \mathfrak{g}_\alpha, \quad \Sigma_{\widehat{\Pi}} = \Sigma^+_{\widehat{\Pi}} \cup -\Sigma^+_{\widehat{\Pi}}.$$

$$\mathfrak{n}_{\widehat{\Pi}} = \sum_{\alpha \in \Sigma^+ \setminus \Sigma^+_{\widehat{\Pi}}} \mathfrak{g}_\alpha.$$

Example 2.3.22. $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$.

$$\mathfrak{p} = \begin{pmatrix} {\color{red}\bullet} & {\color{red}\bullet} & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ {\color{red}\bullet} & {\color{red}\bullet} & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & {\color{green}\bullet} & {\color{green}\bullet} & {\color{green}\bullet} & \bullet & \bullet & \bullet \\ 0 & 0 & {\color{green}\bullet} & {\color{green}\bullet} & {\color{green}\bullet} & \bullet & \bullet & \bullet \\ 0 & 0 & {\color{green}\bullet} & {\color{green}\bullet} & {\color{green}\bullet} & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & 0 & 0 & {\color{blue}\bullet} & {\color{blue}\bullet} & {\color{blue}\bullet} \\ 0 & 0 & 0 & 0 & 0 & {\color{blue}\bullet} & {\color{blue}\bullet} & {\color{blue}\bullet} \\ 0 & 0 & 0 & 0 & 0 & {\color{blue}\bullet} & {\color{blue}\bullet} & {\color{blue}\bullet} \end{pmatrix} \quad \text{trace } 0,$$

$$\mathfrak{a}_\mathfrak{p} = \begin{pmatrix} a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_3 \end{pmatrix} \qquad a_1 + a_2 + a_3 = 0,$$

$$\mathfrak{n}_\mathfrak{p} = \begin{pmatrix} 0 & 0 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & 0 & 0 & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & 0 & 0 & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & 0 & 0 & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathfrak{m}_\mathfrak{p} = \begin{pmatrix} \bullet & \bullet & 0 & 0 & 0 & 0 & 0 & 0 \\ \bullet & \bullet & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \bullet & \bullet & \bullet & 0 & 0 & 0 \\ 0 & 0 & \bullet & \bullet & \bullet & 0 & 0 & 0 \\ 0 & 0 & \bullet & \bullet & \bullet & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & 0 & 0 & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & 0 & 0 & \bullet & \bullet & \bullet \end{pmatrix} \qquad \mathrm{Tr}\begin{pmatrix} \bullet & \bullet \\ \bullet & \bullet \end{pmatrix} = \mathrm{Tr}\begin{pmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{pmatrix} = \mathrm{Tr}\begin{pmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{pmatrix} = 0.$$

Proposition 2.3.23 ([Kna86] p.133)**.**

$$N_\mathfrak{g}(\mathfrak{a}) = Z_\mathfrak{g}(\mathfrak{a}_\mathfrak{p}) = \mathfrak{m}_{1,\mathfrak{p}},$$

$$\mathfrak{g} = \mathfrak{m}_\mathfrak{p} \oplus \mathfrak{a}_\mathfrak{p} \oplus \mathfrak{n}_\mathfrak{p} \oplus \theta(\mathfrak{n}_\mathfrak{p})$$

is an orthogonal decomposition.

2.3.3 Decomposition for Real Reductive Lie Group

Definition 2.3.24. A *parabolic subgroup* P is the subgroup of normalizers of a parabolic subalgebra $\mathfrak{p}$, meaning $P = \{g\colon \operatorname{Ad}_g[\mathfrak{p}] = \mathfrak{p}\}$.

Note that the Lie algebra of P is $\mathfrak{p}$.

Theorem 2.3.25 ([GV88] p.68)**.** *Let $M_{1,\mathfrak{p}} = P \cap \theta(P),\ N_{\mathfrak{p}} = \exp(\mathfrak{n}_{\mathfrak{p}})$, then we have a group version of Levi decomposition*

$$P = M_{1,\mathfrak{p}} N_{\mathfrak{p}} \quad M_{1,\mathfrak{p}} \cap N_{\mathfrak{p}} = e.$$

Furthermore, $M_{1,\mathfrak{p}}$ is reductive.

Definition 2.3.26. Given any reductive Lie group G, define ${}^0G = K \exp(\mathfrak{p}_C \cap [\mathfrak{g}, \mathfrak{g}])$.

Theorem 2.3.27 ([GV88] p.68)**.** *Let $A_{\mathfrak{p}} = \exp(\mathfrak{a}_{\mathfrak{p}})$, and $M_{\mathfrak{p}} = {}^0(M_{1,\mathfrak{p}}) = Z_K(\mathfrak{a}_{\mathfrak{p}})[\exp \mathfrak{m}_{\mathfrak{p}}]$. We have*

$$P = M_{\mathfrak{p}} A_{\mathfrak{p}} N_{\mathfrak{p}},$$

$$M_{1,\mathfrak{p}} = M_{\mathfrak{p}} A_{\mathfrak{p}},$$

and they are diffeomorphisms. The Lie algebra of $M_{\mathfrak{p}}$ is $\mathfrak{m}_{\mathfrak{p}}$, and $M_{1,\mathfrak{p}}$ is the normalizer of $\mathfrak{m}_{1,\mathfrak{p}}$. The map

$$K \times M_{\mathfrak{p}} \times A_{\mathfrak{p}} \times N_{\mathfrak{p}} \longrightarrow G$$

is surjective, proper, and everywhere submersive. Besides, $k_1 m_1 a_1 n_1 = k_2 m_2 a_2 n_2$ iff

$$k_2 = k_1 k \quad m_2 = k^{-1} m_1 \quad a_2 = a_1 \quad n_2 = n_1.$$

$M_{1,\mathfrak{p}}$ normalizes $N_{\mathfrak{p}}$.

Proposition 2.3.28. *$M_{1,\mathfrak{p}}$ is the normalizer of $\mathfrak{a}_{\mathfrak{p}}$ and $A_{\mathfrak{p}}$.*

Example 2.3.29.

$$P = \begin{pmatrix} \color{red}\bullet & \color{red}\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \color{red}\bullet & \color{red}\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & \color{green}\bullet & \color{green}\bullet & \color{green}\bullet & \bullet & \bullet & \bullet \\ 0 & 0 & \color{green}\bullet & \color{green}\bullet & \color{green}\bullet & \bullet & \bullet & \bullet \\ 0 & 0 & \color{green}\bullet & \color{green}\bullet & \color{green}\bullet & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & 0 & 0 & \color{blue}\bullet & \color{blue}\bullet & \color{blue}\bullet \\ 0 & 0 & 0 & 0 & 0 & \color{blue}\bullet & \color{blue}\bullet & \color{blue}\bullet \\ 0 & 0 & 0 & 0 & 0 & \color{blue}\bullet & \color{blue}\bullet & \color{blue}\bullet \end{pmatrix} \qquad \det P = 1$$

$$M_{\mathfrak{p}} = \begin{pmatrix} \bullet & \bullet & 0 & 0 & 0 & 0 & 0 & 0 \\ \bullet & \bullet & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \bullet & \bullet & \bullet & 0 & 0 & 0 \\ 0 & 0 & \bullet & \bullet & \bullet & 0 & 0 & 0 \\ 0 & 0 & \bullet & \bullet & \bullet & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & 0 & 0 & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & 0 & 0 & \bullet & \bullet & \bullet \end{pmatrix} \quad \det \begin{pmatrix} \bullet & \bullet \\ \bullet & \bullet \end{pmatrix} = \det \begin{pmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{pmatrix} = \det \begin{pmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{pmatrix} = \pm 1$$

$$A_{\mathfrak{p}} = \begin{pmatrix} a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_3 \end{pmatrix} \quad a_1, a_2, a_3 > 0, a_1 a_2 a_3 = 1,$$

$$N_{\mathfrak{p}} = \begin{pmatrix} 1 & 0 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ 0 & 1 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & 1 & 0 & 0 & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & 1 & 0 & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & 0 & 1 & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

There is another way to construct the Langlands decomposition. We can start with a θ-stable CSA $\mathfrak{h}$. Let $\mathfrak{a}_{\mathfrak{h}} = \mathfrak{h} \cap \mathfrak{p}_C$, $\mathfrak{m}_{\mathfrak{h}} =$ orthogonal complement of $\mathfrak{a}$ in $Z_{\mathfrak{g}}(\mathfrak{a})$. Form a "root space decomposition" w.r.t. $\mathrm{ad}_{\mathfrak{a}}$, and let $\mathfrak{n}$ to be the span of positive roots.

$$\mathfrak{p}_{\mathfrak{h}} = \mathfrak{m}_{\mathfrak{h}} \oplus \mathfrak{a}_{\mathfrak{h}} \oplus \mathfrak{n}_{\mathfrak{h}}$$

is the Langlands decomposition of the parabolic subalgebra $\mathfrak{p}_{\mathfrak{h}}$. And if $A_{\mathfrak{h}} = \exp \mathfrak{a}_{\mathfrak{h}}$, $N_{\mathfrak{h}} = \exp \mathfrak{n}_{\mathfrak{h}}$, $M_{\mathfrak{h}} = Z_K(\mathfrak{a}_{\mathfrak{h}})[\exp \mathfrak{m}_{\mathfrak{h}}]$, then

$$P_{\mathfrak{h}} = M_{\mathfrak{h}} A_{\mathfrak{h}} N_{\mathfrak{h}}$$

is also the Langlands decomposition of the parabolic subgroup $P_{\mathfrak{h}}$.

2.3.4 Integral Formula

Let M be as in (2.2.6), A and N as in the Iwasawa decomposition. In view of (2.3.1) and the discussion above,

$$P_0 = MAN$$

is a minimal parabolic subgroup, and the formula above is its Levi-Langlands decomposition.

Proposition 2.3.30. *[Hel08, Chapter II, Section 2][GV88, Section 2.4] The map*

$$\begin{aligned} \psi_{G/MN} \colon K/M \times A &\longrightarrow G/MN \\ (kM, a) &\longmapsto kaMN \end{aligned}$$

is a homeomorphism. The G-invariant measure on G/MN is given by

$$\int_{G/MN} f(xMN) \,\mathrm{d}\, xMN = \int_{K\times A} f(kaMN) J_{MN}(a) \,\mathrm{d}\, k \,\mathrm{d}\, a. \tag{2.3.2}$$

Here $\mathrm{d}\, k$, $\mathrm{d}\, a$ *are Haar measures on* K, A*, and*

$$J_{MN}(a) = \exp(2\rho(\log(a))).$$

2.4 Some Representation Theory of Reductive Lie Groups

In this section, we suppose G is real reductive. We fix a Haar measure on G.

Definition 2.4.1. [Dix77, 1.2.1] An *involutive Banach algebra* is a Banach algebra with an involution $*$ such that $\|x^*\| = \|x\|$.

The convolution algebra $L^1(G)$ is an involutive Banach algebra with involution given by $f^*(g) := \overline{f(g^{-1})}$.

Definition 2.4.2. Let (π, H) be a unitary representation of G. The *associated representation* π of $L^1(G)$ is defined by

$$\pi(f) = \int_G f(g)\pi(g) \,\mathrm{d}\, g$$

where $f \in L^1(G)$ and $g \in G$.

Proposition 2.4.3. *[Dix77, 13.3.1, 13.3.4] Associated representations give a one-to-one correspondence between unitary representations of G and non-degenerate *-representations of $L^1(G)$ on Hilbert spaces.*

Definition 2.4.4. Let $\mathcal{A}$ be an involutive Banach algebra with an approximate identity, and let $\mathrm{Repn}(\mathcal{A})$ be the set of representations of $\mathcal{A}$. Let $I = \{x \in \mathcal{A} : \pi(x) = 0 \text{ for any } \pi \in \mathrm{Repn}(\mathcal{A})\}$. The completion of $\mathcal{A}/I$ w.r.t. the norm

$$\|x\|' = \sup_{\pi \in \mathrm{Repn}(\mathcal{A})} \|\pi(x)\|$$

is the *enveloping C^*-algebra* of $\mathcal{A}$.

The enveloping C^*-algebra of $L^1(G)$ is the *group C^*-algebra of* G, denoted $C^*(G)$.

Proposition 2.4.5. *[Dix77, 2.7.4] Let $\mathcal{A}$ be an involutive Banach algebra, $\widetilde{\mathcal{A}}$ be its enveloping C^*-algebra, and $\iota_{C^*} : \mathcal{A} \longrightarrow \widetilde{\mathcal{A}}$ the canonical embedding. There is a bijection between representations π of $\mathcal{A}$ and representations $\widetilde{\pi}$ of $\mathcal{A}$ given by*

$$\widetilde{\pi} \longmapsto \widetilde{\pi} \circ \iota_{C^*}.$$

The map preserves non-degeneration and being topological irreducible.

Therefore, the study of unitary representations of G is equivalent to the study of non-degenerate representations of its group C^*-algebra $C^*(G)$. We shall use the same symbol to denote either a representation of G, or the corresponding representation of $L^1(G)$ or $C^*(G)$.

We denote the subalgebra of $C^*(G)$ generated by K-bi-invariant elements by $C^*(G//K)$. Let λ_G be the left regular representation of G (and the corresponding algebras) on $L^2(G)$. Let $C^*_r(G//K) = C^*(G//K)/(C^*(G//K) \cap \ker \lambda_G)$. The representations of $C^*_r(G//K)$ are the focus of this thesis.

In view of the polar decomposition (2.2.13), the following result is not a big surprise.

Proposition 2.4.6. *[GV88, Corollary 1.5.4] $C^*(G//K)$ is commutative.*

We now compute the spectrum of $C^*_r(G//K)$.

$L^2(G//K)$, the subspace of $L^2(G)$ generated by K-bi-invariant elements, is invariant under the action of $C^*(G//K)$. We denote the restriction of λ_G to $C^*(G//K)$ and $L^2(G//K)$ by $\lambda_{G//K}$. The action of $C_c(G//K) \subset C^*(G//K)$ on $C_c(G//K) \subset L^2(G//K)$ is given by right convolution. Since $C^*_r(G//K)$ is abelian, the irreducible representations of it are one dimensional, therefore intuitively elements in $\lambda \in Spec(C_r^*(G//K))$ are in bijection with convolution eigenfucntions F_λ such that

$$\varphi * F_\lambda = \lambda(\varphi)F_\lambda \text{ for any } \varphi \in C_c(G//K)$$

and normalized in a certain way. On the other hand, let D be a G-invariant differential operator that descends to G/K. If $F_\lambda \in C^\infty(G)$, We have

$$D(\varphi * F_\lambda) = \varphi * D(F_\lambda) = \lambda(\varphi)D(F_\lambda),$$

and therefore F_λ is an eigenfunction of D. Hence, intuitively, there is a correspondence between $Spec(C_r^*(G//K))$ and certain eigenfunctions of G-invariant differential operators on G/K, $Diff_G(G/K)$. We now make this heuristic argument precise.

Let

$$\mathfrak{Q} = U(\mathfrak{g})^K, \tag{2.4.1}$$

the K invariant elements in $U(\mathfrak{g})$ where K acts by adjoint action. Elements in $\mathfrak{Q}$ descend to elements in $Diff_G(G/K)$.

We first make an identification of $Diff_G(G/K)$ with quotients of $\mathfrak{Q}$.

Proposition 2.4.7. *[GV88, Proposition 1.7.5] $q \in \mathfrak{Q}$ acts as 0 on $C^\infty(G//K)$ iff $q \in \mathfrak{k}U(\mathfrak{g}) \cap \mathfrak{Q} = U(\mathfrak{g})\mathfrak{k} \cap \mathfrak{Q}$. Therefore*

$$Diff_G(G/K) = \mathfrak{Q}/(\mathfrak{Q} \cap \mathfrak{k}U(\mathfrak{g})) = \mathfrak{Q}/(\mathfrak{Q} \cap U(\mathfrak{g})\mathfrak{k}). \tag{2.4.2}$$

Making the heuristic argument above more precise, we have the following.

Theorem 2.4.8. *[GV88, Lemma 1.3.3, Lemma 1.3.4, Theorem 1.3.5, Theorem 1.4.5] The following statements are equivalent*

- *λ is an irreducible continuous representation of $C_c(G//K)$.*
- *There is an $F_\lambda \in C(G//K)$ such that $\varphi * F_\lambda = \lambda(\varphi)F_\lambda$ for any $\varphi \in C_c(G//K)$, and that $F_\lambda(e) = 1$.*
- *$F_\lambda \in C^\infty(G//K)$ is a common eigenfunction of $\mathfrak{Q}$ with $F_\lambda(e) = 1$.*

Therefore $Spec(C_r^*(G//K))$ is in bijection with a subset of common eigenfunctions of $\mathfrak{Q}$. We continue by the following characterization of such functions.

Theorem 2.4.9. *[GV88, Lemma 1.4.2, Theorem 1.4.5] For any $\lambda \in \mathrm{Hom}(\mathfrak{Q}/(\mathfrak{Q}\cap\mathfrak{k}U(\mathfrak{g})), \mathbb{C})$, up to scalars there is exactly one F_λ s.t. $qF_\lambda = \lambda(q)F_\lambda$. Furthermore, $F_\lambda \neq 0$ iff $F_\lambda(e) \neq 0$.*

The celebrated Harish-Chandra homomorphism among other things moves our computation further.

Theorem 2.4.10. *Let $\beta_{\mathfrak{n}}$ be the projection of*

$$U(\mathfrak{g}) \cong U(\mathfrak{a}) \oplus (\mathfrak{k}U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n})$$

onto $U(\mathfrak{a})$ and $\gamma_{\Delta^+}\colon U(\mathfrak{a}) \longrightarrow U(\mathfrak{a})$ is the algebra homomorphism generated by

$$\gamma_{\Delta^+}\colon X \longmapsto X - \rho(X).$$

The map

$$\gamma = \gamma_{\Delta^+} \circ \beta_{\mathfrak{n}}\colon \mathfrak{Q} \longrightarrow S(\mathfrak{a})^{\mathcal{W}}$$

is an algebra homomorphism that is onto and whose kernel is $\mathfrak{Q} \cap \mathfrak{k}U(\mathfrak{g})$. γ is independent of the choice of $\mathfrak{n}$, and for any $D \in S(\mathfrak{a})^{\mathcal{W}}$, we can always find $q_D \in \mathfrak{Q}$ s.t. $\deg q_D = \deg D$ and $\gamma(q_D) = D$.

Since $\mathcal{W}$ acts as a finite reflection group on $\mathfrak{a}$, by Chevalley's theorem

$$\mathrm{Hom}(S(\mathfrak{a})^{\mathcal{W}}, \mathbb{C}) = \mathfrak{a}^*_{\mathbb{C}}/\mathcal{W}.$$

Thus we know that $Spec(C^*_r(G//K))$ is in bijection with a subset of

$$\mathrm{Hom}(\mathfrak{Q}/(\mathfrak{k}U(\mathfrak{g}) \cap \mathfrak{Q}), \mathbb{C}) \cong \mathfrak{a}^*_{\mathbb{C}}/\mathcal{W}.$$

On the other hand, it is known that the K-finite matrix coefficients of unitary representations weakly contained in the left regular representation belong to $L^{2+\varepsilon}(G)$ for any $\varepsilon \in \mathbb{R}^+$ (compare [CHH88]). We will show that if $\lambda \in \mathfrak{a}^*_{\mathbb{C}} \setminus i\mathfrak{a}^*$ is a non-degenerate representation of $C^*_r(G//K)$, it extends to a non-degenerate irreducible representation of $C^*_r(G)$ which has K-finite matrix coefficients not in $L^{2+\varepsilon}(G)$. Furthermore, we will show the following.

Theorem 2.4.11.

$$Spec(C^*_r(G//K)) \cong i\mathfrak{a}^*/\mathcal{W}. \tag{2.4.3}$$

2.5 Plancherel Formula and Spherical Plancherel Formula

2.5.1 Plancherel Formula

Definition 2.5.1. [Dix77] Let $\mathcal{M}$ be a measurable space and μ a positive measure on it. A *measurable field of Hilbert spaces* over $\mathcal{M}$ is a family of Hilbert spaces $\{H_m\}_{m\in\mathcal{M}}$ index by $\mathcal{M}$ and subset $\mathcal{X}$ of sections of $\{H_m\}_{m\in\mathcal{M}}$ such that

- $\mathcal{X}$ is a vector subspace of all sections of $\{H_m\}_{m\in\mathcal{M}}$.
- There is a countable subset $\{x_i\}$ of $\mathcal{X}$ such that $\{x_i|_m\}$ spans a dense subset of H_m.
- For any $x, x' \in \mathcal{X}$, $\langle x, x'\rangle$ is a measurable function on $\mathcal{M}$.

Definition 2.5.2. Let $(\{H_m\}_{m\in\mathcal{M}}, \mathcal{X})$ be a measurable field of Hilbert spaces. A *measurable field of operators* is a family of operators $\{T_m \in B(H)\}_{m\in\mathcal{M}}$ that maps $\mathcal{X}$ into $\mathcal{X}$.

A *measurable field of representations* of a C^*-algebra $\mathcal{A}$ is a family of representations $\{\pi_m\}$ such that $\{\pi_m(a)\}$ is a measurable field of operators for any $a \in \mathcal{A}$.

Given a measurable field of representations, the *direct integral* of it is the representation $\int^{\oplus} \pi_m \,\mathrm{d}\,\mu(m)$ acting on $\int^{\oplus} H_m \,\mathrm{d}\,\mu(m)$.

Definition 2.5.3. A von Neumann algebra is *type I* if every non-zero projection it dominates a non-zero abelian projection.

A C^*-algebra $\mathcal{A}$ is *type I* if for any representation π of $\mathcal{A}$, the von Neumann algebra generated by $\pi(\mathcal{A})$ is type I.

With the vocabularies, we can now state the spectral decomposition for type I C^*-algebras.

Theorem 2.5.4. *[Dix77, 8.6.6, 9.1] Let $\mathcal{A}$ be a separable type I C^*-algebra and π_0 a non-degenerate representation of $\mathcal{A}$ in a separable Hilbert space. Then there exist mutually singular positive measures $\mu_1, \mu_2, \cdots, \mu_\infty$ on $\widehat{\mathcal{A}}$ such that*

$$\pi_0 \cong \int^{\oplus} \pi \,\mathrm{d}\,\mu_1(\pi) \bigoplus 2\int^{\oplus} \pi \,\mathrm{d}\,\mu_1(\pi) \bigoplus \cdots \bigoplus \aleph_0 \int^{\oplus} \pi \,\mathrm{d}\,\mu_\infty(\pi) \tag{2.5.1}$$

Since unitary representations of G are in bijection with representations of $C^*(G)$, the abstract Plancherel formula is essentially an application of the theorem above.

Theorem 2.5.5. *[Dix77, 18.8] Let G be a unimodular separable locally compact type I group, λ_G and ρ_G be the left and right regular representations of G, then*

$$L^2(G) \cong \int_{\widehat{G}}^{\oplus} H_\pi \otimes H_\pi^* \,\mathrm{d}\,\mu(\pi) \tag{2.5.2}$$

The isomorphism intertwines λ_G with $\pi \otimes \mathrm{Id}$ and ρ_G with $\mathrm{Id} \otimes \pi^*$.

Example 2.5.6. $\widehat{S^1} = \mathbb{Z}$ with $\pi_n(g)v = \exp(2\pi i n g)\ v$ for every $v \in H_{\pi_n} = \mathbb{C}$, and we have

$$L^2(S^1) \cong \bigoplus_{\pi \in \widehat{S^1}} H_\pi \otimes H_\pi^*. \tag{2.5.3}$$

The formula above is the same as the Plancherel formula in its simplest form

$$L^2(S^1) = \bigoplus_{n \in \mathbb{Z}} \operatorname{span}\{\exp(2\pi i n g)\}. \tag{2.5.4}$$

To make the identification, note that $\operatorname{span}\{\exp(-2\pi i n g)\}$ can be identified with $H_{\pi_n} \otimes H_{\pi_n}^*$ with λ_G identified with $\pi_n \otimes \operatorname{Id}$ and ρ_G identified with $\operatorname{Id} \otimes \pi_n^*$.

Example 2.5.7. The dual of a compact Lie group is discrete, and (2.5.2) specializes to Peter-Weyl theorem.

Theorem 2.5.8. *If G is a compact Lie group, then*

$$L^2(G) \cong \bigoplus_{\pi \in \widehat{G}} H_\pi \otimes H_\pi^*.$$

2.5.2 Spherical Plancherel Formula

If G is a real reductive Lie group, Harish-Chandra and Knapp-Zuckerman showed that it is possible to give a concrete description of the subset of $\widehat{G}$ on which μ is supported, and μ itself.

We now describe Harish-Chandra's spherical Plancherel formula. Abstractly, the spherical Plancherel formula is the spectral decomposition of the representation of $C_r^*(G//K)$ on $L^2(G//K)$. We have already computed the spectrum of $C_r^*(G//K)$ in (2.4.3), therefore we now only need to describe the Plancherel measure. In order to do that, we need to introduce Harish-Chandra's $\mathbf{c}$-function. Like before, we shall introduce this function from a differential operator point of view.

Theorem 2.5.9. *[GV88, Theorem 4.4.11] Suppose G is real reductive, and F_λ is as in Theorem 2.4.8. There is a unique function,* Harish-Chandra's $\mathbf{c}$-function, *defined on an open dense subset ${}^*\mathcal{F}'$[2] of $\mathfrak{a}_{\mathbb{C}}^*$ such that for any fixed $\lambda \in {}^*\mathcal{F}'$ and $\xi \in \mathfrak{a}^+$*

$$F_\lambda(\exp t\xi) = \sum_{\sigma \in \mathcal{W}} \mathbf{c}(\sigma\lambda)\left(\exp((\sigma\lambda - \rho)(t\xi)) + o(\exp((\sigma\lambda - \rho)(t\xi)))\right) \tag{2.5.5}$$

as $t \to \infty$.

[2]See [GV88, p. 150] for a detailed description of the set

Theorem 2.5.10. *[HC58a][HC58b] Suppose G is real reductive, and $h \in C_c(G//K)$.*

$$\|h\|^2 = \frac{1}{|\mathcal{W}|} \int_{i\mathfrak{a}^*} \frac{|Hh(-\lambda)|^2}{|\mathbf{c}(\lambda)|^2} \, \mathrm{d}\, \mu_{i\mathfrak{a}^*} \lambda.$$

Here $\mathbf{c}$ is the Harish-Chandra $\mathbf{c}$-function,

$$H \colon h \longmapsto \int F_\lambda h$$

is the Harish-Chandra transform, and $\mu_{i\mathfrak{a}^*}$ is a Haar measure on $i\mathfrak{a}^*$.

2.6 Gelfand-Kostyuchenko Method

One application of Gelfand-Kostyuchenko method is making the connection between spectral decomposition and eigenfunction decomposition for operators with continuous spectrum. For instance, if T is a self-adjoint operator on $L^2(G)$ with continuous spectrum, the Hilbert spaces in its spectral H_λ can no longer be described as the eigenspace of T since there will be no eigenfunctions in $L^2(G)$. To solve the problem, one first observes that in some case, the eigenfunctions are tempered distributions, and it is possible to understand the spectral decomposition as a decomposition of tempered distribution.

Now we make things more precise. For more details see [Ber88].

Definition 2.6.1. Let $H = \int H_\lambda \,\mathrm{d}\,\mu(\lambda)$ and $\mathcal{S}$ a separable topological vector space. A morphism $\iota \colon \mathcal{S} \longrightarrow H$ is *pointwise defined* if there exists a family of morphisms $\iota_\lambda \colon \mathcal{S} \longrightarrow H_\lambda$ that integrates to ι.

Lemma 2.6.2. *Suppose $\mathcal{S}$, H, and H_λ are modules of a separable algebra $\mathcal{A}$, and the action of $\mathcal{A}$ on H_λ integrates to the action of $\mathcal{A}$ on H. Suppose ι is a morphism of $\mathcal{A}$-modules, then ι descends to morphism of $\mathcal{A}$-modules. The same holds for actions of separable groups.*

Definition 2.6.3. A morphism $\zeta \colon \mathcal{S} \longrightarrow \mathcal{S}'$ of topological vector spaces is *fine* if $\mathcal{S}$ is separable and if for each morphism $\iota \colon \mathcal{S}' \longrightarrow H = \int H_\lambda$, the composition $\iota \circ \zeta$ is pointwise defined.

Theorem 2.6.4. *Let $\mathcal{S}$ be a separable Hilbert space and $\zeta \colon \mathcal{S} \longrightarrow H$ a Hilbert-Schmidt morphism. Then ζ is fine.*

Now suppose $\iota \colon \mathcal{S} \longrightarrow H = \int H_\lambda$ is fine. We have a family of morphisms $\iota_\lambda \colon \mathcal{S} \longrightarrow H_\lambda$. Let $\mathcal{S}^*$ be the dual of $\mathcal{S}$. Correspondingly, we have adjoint maps $\iota_\lambda^* \colon H_\lambda \longrightarrow \mathcal{S}^*$.

Furthermore, the integration of Hilbert spaces becomes integration of elements in $\mathcal{S}^*$:

$$\varphi = \int \iota^* \iota(\varphi). \tag{2.6.1}$$

Chapter 3

Asymptotic Containment and Weyl's Spectral theorem

In this chapter, we first develop some results of asymptotically contained representations. This will be the abstract foundation of our approach to Weyl's spectral theorem and later Harish-Chandra's spherical Plancherel formula. The key result here is Theorem 3.1.9. We then use asymptotic containment described in Section 3.1 to compute the Plancherel measure for most part of the spectrum. In the last section, we deal with the part of the spectrum that is not covered by asymptotic containment.

This chapter is joint work with Nigel Higson. The chapter is almost a verbatim copy of part of our manuscript [HT16], with only minor changes.

3.1 Asymptotic Containment of Representations

In this section we shall describe our alternative approach to the computation of the continuous part of the spectral measure in Weyl's theorem. We shall formulate our method in fairly general terms, applicable to examples beyond Weyl's theorem, although we shall not strive for the upmost generality in the assumptions that we make. We shall check those assumptions in the case of Sturm-Liouville operators in Section 3.2.

Let $\mathcal{A}$ be a separable, commutative C^*-algebra with Gelfand spectrum Λ, so that of course

$$\mathcal{A} \cong C_0(\Lambda).$$

We shall view elements of $\mathcal{A}$ as continuous functions on Λ without further comment.

Let us suppose that we are given two non-degenerate representations of $\mathcal{A}$ on separable Hilbert spaces:

$$\pi\colon \mathcal{A} \longrightarrow B(H) \quad \text{and} \quad \pi_0\colon \mathcal{A} \longrightarrow B(H_0).$$

We shall assume that π_0 is asymptotically contained in π, as in Definition 1.3.1, with unitary group $\{U_t\}$ on H_0 and operator $W\colon H_0 \to H$ as described in the definition. Our analysis of the relation between π and π_0 will be based on following formula, which is an immediate consequence of (1.3.1).

Lemma 3.1.1. *If $a \in \mathcal{A}$ and if $g, h \in H_0$, then*

$$\langle g, \pi_0(a)h\rangle_{H_0} = \lim_{T\to+\infty} \frac{1}{T}\int_0^T \langle WU_t g, \pi(a)WU_t h\rangle_H \, dt. \qquad \square$$

We shall make the following assumptions concerning the representation π_0; in practice they will be easy to verify using Fourier analysis.

Assumption 3.1.2. *We shall suppose that we are given:*

(i) *An open subset* $\Lambda_0 \subseteq \Lambda$ *and a locally trivial continuous field of finite-dimensional Hilbert spaces* $\{H_{0,\lambda}\}_{\lambda\in\Lambda_0}$ *over* Λ_0 *(or in other words a Hermitian vector bundle).*

(ii) *A dense subspace* $\mathfrak{H}_0 \subseteq H_0$ *and a linear map* $h \mapsto \{h_{0,\lambda}\}$ *from* $\mathfrak{H}_0$ *into the continuous sections of* $\{H_{0,\lambda}\}$ *such that*

$$H_{0,\lambda} = \{\, h_{0,\lambda} \,:\, h \in \mathfrak{H}_0 \,\}$$

for every $\lambda \in \Lambda_0$.

(iii) *A Borel measure* μ_0 *on* Λ_0 *such that* $\langle h_{0,\lambda}, g_{0,\lambda}\rangle$ *is a* μ_0*-integrable function of* λ*, for every* $h, g \in \mathfrak{H}_0$*, and such that*

$$\big\langle h, \pi_0(a)g\big\rangle_{H_0} = \int_{\Lambda_0} \big\langle h_{0,\lambda}, g_{0,\lambda}\big\rangle_{H_{0,\lambda}} a(\lambda)\, d\mu_0(\lambda)$$

for every $h, g \in \mathfrak{H}_0$ *and every* $a \in \mathcal{A}$.

Assumption 3.1.3. *We shall assume that the action of the one-parameter unitary group* $\{U_t\}$ *on the Hilbert space* H_0 *maps the subspace* $\mathfrak{H}_0$ *into itself, and that the continuous field* $\{H_{0,\lambda}\}_{\lambda\in\Lambda_0}$ *carries a continuous, unitary action of* $\mathbb{R}$ *such that*

$$(U_t h)_{0,\lambda} = U_{t,\lambda} h_{0,\lambda}$$

for every $h \in \mathfrak{H}_0$ *and every* $\lambda \in \Lambda_0$.

We shall make assumptions on the representation π that similar to those in Assumption 3.1.2, except that we shall in addition assume that π has *multiplicity one*: the fibers in the field of Hilbert spaces that decomposes π have dimension one. This assumption isn't altogether necessary (finite-dimensionality would suffice), but it simplifies the statements of the results that follow, along with their proofs, and it is satisfied in the situations of interest to us. Here are the details.

Assumption 3.1.4. *We shall suppose that there is given:*

(i) *A locally trivial continuous field of* one*-dimensional Hilbert spaces* $\{H_\lambda\}_{\lambda\in\Lambda}$ *over* Λ *(that is, a Hermitian line bundle over* Λ*).*

(ii) *A dense subspace* $\mathfrak{H} \subseteq H$ *such that if* $h \in \mathfrak{H}_0$ *then* $WU_t h \in \mathfrak{H}$ *for all* $t \gg 0$.

(iii) *A linear map* $h \mapsto \{h_\lambda\}$ *from* $\mathfrak{H}$ *into the continuous sections of* $\{H_\lambda\}$ *and a* σ*-finite measure* μ *on the Borel subsets of* Λ *such that* $\langle h_\lambda, g_\lambda\rangle$ *is a* μ*-integrable function of* λ*,* *for every $h, g \in \mathfrak{H}$, and such that*

$$\big\langle h, \pi(a)g\big\rangle_H = \int_\Lambda \big\langle h_\lambda, g_\lambda\big\rangle_{H_\lambda} a(\lambda)\, d\mu(\lambda)$$

for every $h, g \in \mathfrak{H}$ and every $a \in \mathcal{A}$.

Finally, we shall make the following assumption concerning the asymptotic relation between the fields $\{H_\lambda\}$ and $\{H_{0,\lambda}\}$. As we noted in the introduction, and as we shall see clearly in the next section, in the Sturm-Liouville context this means that C_λ maps each λ-eigenfunction of D to a λ-eigenfunction for D_0 that is asymptotic to it.

Assumption 3.1.5. *We shall assume that there is given a continuous family of* injective *linear maps*

$$C_\lambda \colon H_\lambda \longrightarrow H_{0,\lambda} \qquad (\lambda \in \Lambda_0)$$

with the property that if h belongs to $\mathfrak{H}_0$, and if $\{v_\lambda\}$ is a continuous section of $\{H_\lambda\}$, and K is a compact subset of Λ_0, then

$$\lim_{t\to+\infty} \sup_{\lambda\in K} \Big| \big\langle C_\lambda v_\lambda, (U_t h)_{0,\lambda}\big\rangle_{H_{0,\lambda}} - \big\langle v_\lambda, (WU_t h)_\lambda\big\rangle_{H_\lambda} \Big| = 0.$$

Using the four assumptions listed above we shall prove:

Theorem 3.1.6. *The measure μ_0 is absolutely continuous with respect to μ on the open set Λ_0, with Radon-Nikodym derivative*

$$\frac{d\mu_0}{d\mu}(\lambda) = \frac{\operatorname{Trace}(C_\lambda^* C_\lambda)}{\dim(H_{0,\lambda})}.$$

Here is the proof. We are assuming that

$$\langle h, \pi_0(\varphi)h\rangle_{H_0} = \int_{\Lambda_0} \big\langle h_{0,\lambda}, h_{0,\lambda}\big\rangle_{H_{0,\lambda}} \varphi(\lambda)\, d\mu_0(\lambda). \tag{3.1.1}$$

We shall obtain from our remaining assumptions a new integral formula, namely

$$\langle h, \pi_0(\varphi)h\rangle_{H_0} = \int_{\Lambda_0} \big\langle h_{0,\lambda}, \operatorname{Av}\big[C_\lambda C_\lambda^*\big] h_{0,\lambda}\big\rangle_{H_{0,\lambda}} \varphi(\lambda) d\mu(\lambda), \tag{3.1.2}$$

where the operator $\operatorname{Av}[C_\lambda C_\lambda^*] \colon H_{0,\lambda} \to H_{0,\lambda}$ is defined by the averaging formula

$$\operatorname{Av}\big[C_\lambda C_\lambda^*\big] = \lim_{T\to\infty} \frac{1}{T} \int_0^T U_{-t,\lambda} C_\lambda C_\lambda^* U_{t,\lambda}\, dt$$

(since we are dealing here with operators on the finite-dimensional space $H_{0,\lambda}$ the limit certainly exists). At this point, we can appeal to the following uniqueness result for spectral decompositions:

Lemma 3.1.7. *Let $\{T_\lambda\}$ be a measurable field of positive operators on $\{H_{0,\lambda}\}_{\lambda\in\Lambda_0}$. Suppose that*

$$\int_{\Lambda_0} \left\langle h_{0,\lambda}, T_\lambda h_{0,\lambda}\right\rangle_{H_{0,\lambda}} \varphi(\lambda)\, d\mu(\lambda) = \int_{\Lambda_0} \left\langle h_{0,\lambda}, h_{0,\lambda}\right\rangle_{H_{0,\lambda}} \varphi(\lambda)\, d\mu_0(\lambda)$$

for every $h \in \mathfrak{H}_0$ and every continuous and compactly supported function φ. Then T_λ is a scalar multiple of the identity for μ-almost all $\lambda \in \Lambda_0$. In addition μ_0 is absolutely continuous with respect to μ on Λ_0 and

$$T_\lambda = \frac{d\mu_0}{d\mu}(\lambda) \cdot I_{H_{0,\lambda}}$$

μ-almost everywhere on Λ_0.

Proof. For each point $\lambda_0 \in \Lambda_0$ there exists $h \in \mathfrak{H}_0$ for which the section $h_{0,\lambda}$ is nonzero at λ_0. It follows immediately from the uniqueness part of the Riesz representation theorem that μ_0 is absolutely continuous with respect to μ near λ_0 with Radon-Nikodym derivative

$$\frac{d\mu_0}{d\mu}(\lambda) = \frac{\left\langle h_{0,\lambda}, T_\lambda h_{0,\lambda}\right\rangle_{H_{0,\lambda}}}{\left\langle h_{0,\lambda}, h_{0,\lambda}\right\rangle_{H_{0,\lambda}}}.$$

Since the derivative is independent of $\{h_{0,\lambda}\}$ this implies that

$$T_\lambda = \frac{d\mu_0}{d\mu}(\lambda) \cdot I_{H_{0,\lambda}},$$

almost everywhere, as required. □

Returning to the proof of Theorem 3.1.6, Lemma 3.1.7 tells us that the operator $\mathrm{Av}[C_\lambda C_\lambda^*]$ is a scalar multiple of the identity for μ-almost-all $\lambda \in \Lambda_0$. The computation

$$\mathrm{Trace}\left(\mathrm{Av}\left[C_\lambda C_\lambda^*\right]\right) = \mathrm{Trace}(C_\lambda C_\lambda^*) = \mathrm{Trace}(C_\lambda^* C_\lambda),$$

determines the multiple, and the theorem follows. So it remains to establish the integral formula (3.1.2):

Lemma 3.1.8. *If $h \in \mathfrak{H}_0$ and if φ is a continuous and compactly supported function on Λ_0,* *then*

$$\langle h, \pi_0(\varphi)h\rangle_{H_0} = \int_{\Lambda_0} \big\langle h_{0,\lambda}, \mathrm{Av}\big[C_\lambda C_\lambda^*\big] h_{0,\lambda}\big\rangle_{H_{0,\lambda}} \varphi(\lambda) d\mu(\lambda).$$

Proof. It suffices to prove this formula for all functions φ that are supported on compact sets $K \subseteq \Lambda_0$ over which the field $\{H_\lambda\}$ is trivializable, and so we shall assume that here. In addition we shall use the notation

$$W_t = WU_t \colon H_0 \longrightarrow H.$$

According to Lemma 3.1.1,

$$\langle h, \pi_0(\varphi)h\rangle_{H_0} = \lim_{T\to+\infty} \frac{1}{T}\int_0^T \big\langle W_t h, \pi(\varphi) W_t h\big\rangle_H dt. \tag{3.1.3}$$

Now use Assumption 3.1.4 to write the integrand in the right hand side of (3.1.3) as

$$\big\langle W_t h, \pi(\varphi) W_t h\big\rangle_H = \int_{\Lambda_0} \big\langle (W_t h)_\lambda, (W_t h)_\lambda\big\rangle_{H_\lambda} \varphi(\lambda) d\mu(\lambda).$$

Since the space H_λ are one-dimensional, we can write

$$\big\langle (W_t h)_\lambda, (W_t h)_\lambda\big\rangle_{H_\lambda} = \big\langle (W_t h)_\lambda, v_\lambda\big\rangle_{H_\lambda} \cdot \big\langle v_\lambda, (W_t h)_\lambda\big\rangle_{H_\lambda},$$

where $\{v_\lambda\}$ is a continuous section of $\{H_\lambda\}$ with $\|v_\lambda\|_{H_\lambda} = 1$ for all $\lambda \in K$. So

$$\begin{aligned}&\langle h, \pi_0(\varphi)h\rangle_{H_0}\\ &\quad = \lim_{T\to+\infty} \frac{1}{T}\int_0^T \int_{\Lambda_0} \big\langle (W_t h)_\lambda, v_\lambda\big\rangle_{H_\lambda} \cdot \big\langle v_\lambda, (W_t h)_\lambda\big\rangle_{H_\lambda} \varphi(\lambda) d\mu(\lambda) dt.\end{aligned} \tag{3.1.4}$$

Consider now the difference

$$\begin{aligned}&\big\langle (U_t h)_{0,\lambda}, C_\lambda v_\lambda\big\rangle_{H_{0,\lambda}} \cdot \big\langle C_\lambda v_\lambda, (U_t h)_{0,\lambda}\big\rangle_{H_{0,\lambda}}\\ &\qquad - \big\langle (W_t h)_\lambda, v_\lambda\big\rangle_{H_\lambda} \cdot \big\langle v_\lambda, (W_t h)_\lambda\big\rangle_{H_\lambda},\end{aligned} \tag{3.1.5}$$

which we can write as

$$\begin{aligned}&\big\langle (U_t h)_{0,\lambda}, C_\lambda v_\lambda\big\rangle_{H_{0,\lambda}} \Big[\big\langle C_\lambda v_\lambda, (U_t h)_{0,\lambda}\big\rangle_{H_{0,\lambda}} - \big\langle v_\lambda, (W_t h)_\lambda\big\rangle_{H_\lambda}\Big]\\ &\qquad + \Big[\big\langle (U_t h)_{0,\lambda}, C_\lambda v_\lambda\big\rangle_{H_{0,\lambda}} - \big\langle (W_t h)_\lambda, v_\lambda\big\rangle_{H_\lambda}\Big] \big\langle v_\lambda, (W_t h)_\lambda\big\rangle_{H_\lambda}.\end{aligned}$$

The terms in the square brackets converge to 0, as $t \to +\infty$, uniformly in $\lambda \in K$. In addition, since

$$\left|\left\langle (U_t h)_{0,\lambda}, C_\lambda v_\lambda \right\rangle_{H_{0,\lambda}}\right| = \left|\left\langle U_{t,\lambda} h_{0,\lambda}, C_\lambda v_\lambda \right\rangle_{H_{0,\lambda}}\right| \le \|h_\lambda\| \cdot \|C_\lambda v_\lambda\|$$

we see that $\langle (U_t h)_{0,\lambda}, C_\lambda v_\lambda \rangle_{H_{0,\lambda}}$ is uniformly bounded in t and $\lambda \in K$. It follows from this and Assumption 3.1.5 that $\langle (W_t h)_\lambda, v_\lambda \rangle_{H_\lambda}$ is uniformly bounded too. So the expression (3.1.5) converges to zero as $t \to \infty$, uniformly in $\lambda \in K$.

Observe next that

$$\begin{aligned}\left\langle (U_t h)_{0,\lambda}, C_\lambda v_\lambda \right\rangle_{H_{0,\lambda}} &\left\langle C_\lambda v_\lambda, (U_t h)_{0,\lambda} \right\rangle_{H_{0,\lambda}} \\ &= \left\langle U_{t,\lambda} h_{0,\lambda}, C_\lambda v_\lambda \right\rangle_{H_{0,\lambda}} \cdot \left\langle C_\lambda v_\lambda, U_{t,\lambda} h_{0,\lambda} \right\rangle_{H_{0,\lambda}} \\ &= \left\langle h_{0,\lambda}, U_{-t,\lambda} C_\lambda C_\lambda^* U_{t,\lambda} h_{0,\lambda} \right\rangle_{H_{0,\lambda}}.\end{aligned}$$

So it follows from our analysis of (3.1.5) that the inner integral in (3.1.4) is asymptotic to the integral

$$\int_{\Lambda_0} \left\langle h_{0,\lambda}, U_{-t,\lambda} C_\lambda C_\lambda^* U_{t,\lambda} h_{0,\lambda} \right\rangle_{H_{0,\lambda}} \varphi(\lambda)\, d\mu(\lambda)$$

(that is, the difference converges to zero as $t \to +\infty$). As a result,

$$\langle h, \pi_0(\varphi) h \rangle_{H_0} = \lim_{T \to +\infty} \frac{1}{T} \int_0^T \Big(\int_{\Lambda_0} \left\langle h_{0,\lambda}, U_{-t,\lambda} C_\lambda C_\lambda^* U_{t,\lambda} h_{0,\lambda} \right\rangle_{H_{0,\lambda}} \varphi(\lambda)\, d\mu(\lambda) \Big)\, dt,$$

It now follows from Fubini's theorem, that

$$\langle h, \pi_0(\varphi) h \rangle_{H_0} = \lim_{T \to +\infty} \int_{\Lambda_0} \Big(\frac{1}{T} \int_0^T \left\langle h_{0,\lambda}, U_{-t,\lambda} C_\lambda C_\lambda^* U_{t,\lambda} h_{0,\lambda} \right\rangle_{H_{0,\lambda}} dt \Big) \varphi(\lambda) d\mu(\lambda).$$

The integral in the parentheses is uniformly bounded in T. Therefore we can interchange the limit as $T \to +\infty$ and the integral over Λ_0 to obtain (3.1.2), as required. □

Theorem 3.1.6 gives a formula for the measure μ_0 in terms of the measure μ. But since our goal is to obtain information about the measure μ, we should invert this formula:

Theorem 3.1.9. *The measue μ is absolutely continuous with respect to the measure μ_0 on*

Λ_0, *and the Radon-Nikodym derivative of* μ *with respect to* μ_0 *on* Λ_0 *is*

$$\frac{d\mu}{d\mu_0}(\lambda) = \frac{\dim(H_{0,\lambda})}{\operatorname{Trace}(C_\lambda^* C_\lambda)}. \qquad \square$$

3.2 Sturm-Liouville Operators

In this section we shall apply the approach of Section 3.1 to Sturm-Liouville operators on the half-line. So let

$$D = -\frac{d}{dx} \cdot p(x) \cdot \frac{d}{dx} + q(x),$$

where the coefficient functions $p(x)$ and $q(x)$ are smooth and real-valued on $[0, \infty)$, and where $p(x)$ is everywhere positive. We shall begin by assuming in addition that

$$\lim_{x\to\infty} p(x) = 1, \quad \lim_{x\to\infty} p'(x) = 0 \quad \text{and} \quad \lim_{x\to\infty} q(x) = 0. \tag{3.2.1}$$

Later on in the section we shall make stronger assumptions about the rates of convergence in the limits above.

We shall take for granted the following result, which from a modern perspective is straightforward:

Theorem 3.2.1. *The operator* D *is essentially self-adjoint on the domain of smooth, compactly supported functions on* $[0, \infty)$ *that vanish at* 0. $\square$

Remark 3.2.2. We have made a simple and explicit choice of boundary conditions in the theorem, but nothing in what follows depends on the boundary conditions, as long as they determine an essentially self-adjoint operator.

Associated to the unbounded self-adjoint operator D on the Hilbert space $H = L^2[0, \infty)$ is the functional calculus representation

$$\pi\colon C_0(\mathbb{R}) \longrightarrow B(H)$$
$$\pi\colon \varphi \longmapsto \varphi(D).$$

We shall compare π to the functional calculus representation

$$\pi_0\colon C_0(\mathbb{R}) \longrightarrow B(H_0)$$
$$\pi_0\colon \varphi \longmapsto \varphi(D_0),$$

where $D_0 = -d^2/dx^2$, which we shall treat as an essentially self-adjoint operator on the Hilbert space $H_0 = L^2(-\infty,\infty)$ with domain the smooth compactly supported functions.

Define $U_t\colon H_0 \to H_0$ to be the translation operator

$$(U_t h)(x) = h(x-t).$$

Obviously each $\varphi(D_0)$ commutes with each U_t. Denote by

$$W\colon H_0 \longrightarrow H$$

the orthogonal projection (which restricts functions on $(-\infty,\infty)$ to functions on $(0,\infty)$, of course). The following computation checks that π_0 is asymptotically contained in π.

Lemma 3.2.3. *If $\varphi \in C_0(\mathbb{R})$, and if $g, h \in L^2(-\infty,\infty)$, then*

$$\lim_{t\to+\infty} \Big[\big\langle WU_t g, \varphi(D) WU_t h \big\rangle_{L^2(0,\infty)} - \big\langle g, \varphi(D_0) h \big\rangle_{L^2(-\infty,\infty)} \Big] = 0.$$

Proof. We shall prove that

$$\lim_{t\to+\infty} \big\| \varphi(D) WU_t h - WU_t \varphi(D_0) h \big\|_{L^2(0,\infty)} = 0 \tag{3.2.2}$$

for every $h \in L^2(-\infty,\infty)$, which will suffice. The set of all $\varphi \in C_0(\mathbb{R})$ satisfying (3.2.2) is a norm-closed subalgebra of $C_0(\mathbb{R})$, and it therefore suffices to show that the resolvent functions $\varphi(\lambda) = (\lambda \pm i)^{-1}$ belong to it. Moreover it suffices to check (3.2.2) for each of these two functions φ and for a dense set of functions h in $L^2(-\infty,\infty)$.

Let $\varphi(x) = (x \pm i)^{-1}$. We shall calculate the limit (3.2.2) when

$$h = (D_0 \pm iI) f \quad \text{and} \quad f \in C_c^\infty(-\infty,\infty).$$

If $f \in C_c^\infty(-\infty,\infty)$, and if $t \gg 0$, then $WU_t f$ is a smooth and compactly supported function on $(0,\infty)$, and we compute that

$$\begin{aligned} \varphi(D) WU_t h - \varphi(D_0) WU_t h &= (D \pm iI)^{-1} WU_t (D_0 \pm iI) f - WU_t f \\ &= (D \pm iI)^{-1} (D - D_0) WU_t f, \end{aligned}$$

where, in the last line, we are regarding D_0 as a differential operator acting on the smooth and compactly supported functions on $(0,\infty)$. The conditions (3.2.1) imply that

$$\lim_{t\to+\infty} \|(D - D_0) WU_t f\| = 0,$$

and so (3.2.2) is proved for $\varphi(x) = (x \pm i)^{-1}$, as required. $\square$

Assumptions 3.1.2 and 3.1.3 about the representation π_0 from the previous section are easily obtained from the Fourier transform

$$\widehat{h}(\xi) = \int_{-\infty}^{\infty} h(x) e^{-i\xi x}\, dx,$$

as follows. To begin, let $\Lambda_0 = (0,\infty)$, and for $\lambda \in \Lambda_0$ define $H_{0,\lambda}$ to be the two-dimensional vector space of functions on the line spanned by $e^{i\sqrt{\lambda}x}$ and $e^{-i\sqrt{\lambda}x}$. Equip $H_{0,\lambda}$ with the inner product that makes these two functions an orthonormal basis. The family $\{H_{0,\lambda}\}_{\lambda>0}$ obviously forms a continuous field of Hilbert spaces over Λ_0 with constant and finite fiber dimension.

Now let $\mathfrak{H}_0$ be space of smooth and compactly supported functions in H_0. The Fourier transform associates to each $h \in \mathfrak{H}_0$ a continuous section $\{h_{0,\lambda}\}$ of the continuous field, namely

$$h_{0,\lambda} = \widehat{h}(\sqrt{\lambda}) e^{i\sqrt{\lambda}x} + \widehat{h}(-\sqrt{\lambda}) e^{-i\sqrt{\lambda}x}.$$

Moreover it follows from Plancherel's formula that

$$\langle h, \varphi(D_0) g\rangle_{L^2(-\infty,\infty)} = \int_{\Lambda_0} \langle h_{0,\lambda}, g_{0,\lambda}\rangle_{H_{0,\lambda}}\, \varphi(\lambda)\, d\mu_0(\lambda),$$

where

$$d\mu_0(\lambda) = \frac{1}{4\pi}\frac{d\lambda}{\sqrt{\lambda}}. \tag{3.2.3}$$

So Assumption 3.1.2 is satisfied. The unitary actions

$$U_{t,\lambda}\colon ae^{i\sqrt{\lambda}x} + be^{-i\sqrt{\lambda}x} \longmapsto e^{-i\sqrt{\lambda}t} a e^{i\sqrt{\lambda}x} + e^{i\sqrt{\lambda}t} b e^{-i\sqrt{\lambda}x}$$

on the fibers $H_{0,\lambda}$ decompose the translation action on $L^2(-\infty,\infty)$, as in Assumption 3.1.3.

Let us turn now to the representation π of $C_0(\mathbb{R})$. General theory guarantees that the π a measurable direct integral decomposition

$$L^2(0,\infty) \cong \int_{\mathbb{R}}^{\oplus} H_\lambda\, d\nu(\lambda). \tag{3.2.4}$$

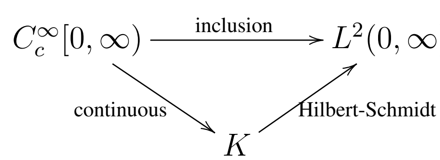
See Section 2.5.1 and [Dix81, Part II, Chapter 6, Theorem 2] for details. The direct integral decomposition is not unique, and for instance if ρ is any positive, measurable function on

Λ, then we can rescale inner products and the measure by

$$\langle\ ,\ \rangle_{L_\lambda} \to \rho(\lambda)^{-1}\langle\ ,\ \rangle_{L_\lambda} \quad \text{and} \quad d\nu(\lambda) \to \rho(\lambda)d\nu(\lambda)$$

so as to obtain a different decomposition (the map $h \mapsto \{s_\lambda(h)\}$ is not changed). We shall make such a *conformal change* below.

Our first task is to upgrade the measurable decompositon given above to a continuous decomposition, as required in Assumption 3.1.4, and we shall use the method outlined in Section 2.6. The inclusion of the topological vector space $C_c^\infty[0,\infty)$ into $L^2(0,\infty)$ factors through a Hilbert-Schmidt operator. That is, there is a commuting diagram



where K is a Hilbert space.

Lemma 3.2.4 (See for example [Mau67, Chapter VII, Section 1])**.** *For all $\lambda \in \mathbb{R}$ there exist continuous linear operators*

$$\varepsilon_\lambda \colon C_c^\infty[0,\infty) \longrightarrow H_\lambda \tag{3.2.5}$$

such that if $h \in C_c^\infty[0,\infty)$, and if $\{s_\lambda(h)\}_{\lambda\in\mathbb{R}}$ is the associated square-integrable section of $\{H_\lambda\}_{\lambda\in\mathbb{R}}$, then $s_\lambda(h) = \varepsilon_\lambda(h)$ for almost every $\lambda \in \mathbb{R}$. □

Let us for a moment restrict the operators ε_λ in the lemma to the space $C_c^\infty(0,\infty)$ of test functions on $(0,\infty)$, so as to obtain operators

$$\varepsilon_\lambda \colon C_c^\infty(0,\infty) \longrightarrow H_\lambda. \tag{3.2.6}$$

Since $C_c^\infty(0,\infty)$ is dense in the Hilbert space $L^2(0,\infty)$, these have dense range for every λ. The adjoint operators

$$\varepsilon_\lambda^* \colon H_\lambda^* \longrightarrow C_c^\infty(0,\infty)^* \tag{3.2.7}$$

are therefore injective for every λ. That is, for every $\lambda \in \mathbb{R}$ the map ε_λ^* is defined and embeds H_λ^* into the space of distributions on $\mathbb{R}$.

Keeping in mind the Hilbert space isomorphism $H_\lambda^* \cong \overline{H_\lambda}$ we find that if $h \in C_c^\infty(0,\infty)$, and if $\{v_\lambda\}_{\lambda\in\mathbb{R}}$ is a measurable section of $\{H_\lambda\}_{\lambda\in\mathbb{R}}$ then

$$\langle v_\lambda, s_\lambda(h)\rangle_{H_\lambda} = \langle v_\lambda, \varepsilon_\lambda(h)\rangle_{H_\lambda} = \int_0^\infty \overline{\varepsilon_\lambda^*(v_\lambda)} \cdot h,$$

for every $\lambda \in \mathbb{R}$, where the right-hand integral is the pairing between distributions and test functions. Using this and the propery (iii) above, we find that if $V_\lambda = \varepsilon_\lambda^*(v_\lambda)$, then

$$\int_0^\infty DV_\lambda \cdot h = \int_0^\infty \lambda V_\lambda \cdot h$$

for all $\lambda \in \mathbb{R}$ (the operator D is applied to V_λ in the sense of distributions) and since $C_c(0,\infty)$ is separable it follows that

$$DV_\lambda = \lambda V_\lambda$$

for all λ.

By linear ODE theory the space of λ-eigendistributions for the Sturm-Liouville operator D is 2-dimensional for every λ, and consists of smooth functions on $[0,\infty)$. So H_λ is at most 2-dimensional for every λ.

Lemma 3.2.5. *For ν-almost every $\lambda \in \mathbb{R}$ the operator ε_λ^* is defined and embeds L_λ^* into the one-dimensional space of (smooth) solutions of the differential equation $DG_\lambda = \lambda G_\lambda$ that satisfy the boundary condition $G_\lambda(0) = 0$.*

Proof. If $g, h \in C_c^\infty[0,\infty)$. Assume that $g(0) = h(0) = 0$ so that both g and h are in the domain of D. Then for every λ we can write

$$\begin{aligned}\langle \varepsilon_\lambda(Dh), s_\lambda(g)\rangle_{H_\lambda} &- \langle \varepsilon_\lambda(h), s_\lambda(Dg)\rangle_{H_\lambda} \\ &= \langle \varepsilon_\lambda(Dh), s_\lambda(g)\rangle_{H_\lambda} - \langle \varepsilon_\lambda(h), \lambda s_\lambda(g)\rangle_{H_\lambda} \\ &= \int_0^\infty \overline{(Dh)(x)} G_\lambda(x)\,dx - \int_0^\infty \overline{h(x)} \lambda G_\lambda(x)\,dx \\ &= \int_0^\infty \overline{(Dh)(x)} G_\lambda(x)\,dx - \int_0^\infty \overline{h(x)} (DG_\lambda)(x)\,dx,\end{aligned} \tag{3.2.8}$$

where $G_\lambda = \varepsilon_\lambda^*(s_\lambda(g))$. Assume now that in addition $h'(0) = 1$. Calculating the difference of integrals using the fundamental theorem of calculus we find that

$$\int_0^\infty \overline{(Dh)(x)} G_\lambda(x)\,dx - \int_0^\infty \overline{h(x)} (DG_\lambda)(x)\,dx = p(0)G_\lambda(0).$$

The top expression in (3.2.8) is an integrable function of λ, and therefore so is $G_\lambda(0)$. If φ

is any continuous and compactly supported function on $\mathbb{R}$, then by (iii) above the integral of the left-hand side of (3.2.8), times $\varphi(\lambda)$, is equal to zero, and so

$$\int_0^\infty G_\lambda(0)\,\varphi(\lambda)\,d\mu(\lambda) = 0,$$

It follows that $G_\lambda(0) = 0$ for almost every λ. The lemma follows from this because the elements $s_\lambda(g)$ span H_λ, for almost all λ. □

Definition 3.2.6. Denote by H_λ the one-dimensional space of (smooth) solutions of the differential equation $DG_\lambda = \lambda G_\lambda$ that satisfy the boundary condition $G_\lambda(0) = 0$. Equip H_λ with the inner product for which the eigenfunction F_λ with $F'_\lambda(0) = 1$ has norm 1, and view the spaces H_λ as the fibers of the unique continuous field of one-dimensional Hilbert spaces over $\mathbb{R}$ for which $\lambda \mapsto F_\lambda$ is a continuous section.

We are ready to verify Assumption 3.1.4. We shall use the continuous field $\{H_\lambda\}$ defined above, and we shall take

$$\mathfrak{H} = C_c^\infty(0,\infty).$$

If $h \in \mathfrak{H}$, then define $h_\lambda \in H_\lambda$ by

$$h_\lambda = \int_0^\infty \overline{F_\lambda(x)} h(x)\,dx \cdot F_\lambda.$$

Certainly $\{h_\lambda\}$ is a continuous section of the field $\{H_\lambda\}$ since the function F_λ depends continuously (in fact analytically) on λ.

If $v_\lambda \in L_\lambda$, and if $\varepsilon^*_\lambda(v_\lambda) = F_\lambda$, then for $g, h \in \mathfrak{H}$ we can write

$$\langle h_\lambda, g_\lambda\rangle_{H_\lambda} = \langle s_\lambda(h), v_\lambda\rangle_{L_\lambda} \cdot \langle v_\lambda, s_\lambda(g)\rangle_{L_\lambda}$$

So if we define a positive, measurable function $\rho(\lambda)$ (on the complement of a ν-nullset) by

$$\rho(\lambda) = \|v_\lambda\|^2_{L_\lambda},$$

then, keeping in mind that L_λ is one-dimensional for almost all λ, we find that

$$\langle h_\lambda, g_\lambda\rangle_{H_\lambda} = \langle s_\lambda(h), v_\lambda\rangle_{L_\lambda} \cdot \langle v_\lambda, s_\lambda(g)\rangle_{L_\lambda} = \rho(\lambda) \cdot \langle s_\lambda(h), s_\lambda(g)\rangle_{L_\lambda}.$$

As a result of this, if we define a σ-finite measure μ, absolutely continuous with respect to

ν, by

$$d\mu(\lambda) = \rho(\lambda)^{-1} d\nu(\lambda),$$

then

$$\begin{aligned}\langle h, \varphi(D) g\rangle^2_{L^2(0,\infty)} &= \int_{-\infty}^{\infty} \langle s_\lambda(h), s_\lambda(g)\rangle_{L_\lambda} \varphi(\lambda)\, d\nu(\lambda) \\ &= \int_{-\infty}^{\infty} \langle s_\lambda(h), s_\lambda(g)\rangle_{L_\lambda} \varphi(\lambda)\, \rho(\lambda)\, d\mu(\lambda) = \int_{-\infty}^{\infty} \langle h_\lambda, g_\lambda\rangle_{H_\lambda} \varphi(\lambda)\, d\mu(\lambda).\end{aligned}$$

This verifies Assumption 3.1.4.

Finally, we need to analyze the asymptotics of the λ-eigenfunctions of D and verity Assumption 3.1.5. For this purpose we shall need to assume a bit more about the coefficients of D, namely that

$$\int_{x_0}^{\infty} |1 - p(x)^{-1}|\, dx < \infty \quad \text{and} \quad \int_{x_0}^{\infty} |q(x)|\, dx < \infty. \tag{3.2.9}$$

Proposition 3.2.7. *Let $\lambda > 0$ and let F_λ be the λ-eigenfunction of D with $F'(0) = 1$. There is a unique nonzero λ-eigenfunction $F_{0,\lambda}$ of D_0 such that*

$$\lim_{x\to\infty} \big| F_\lambda(x) - F_{0,\lambda}(x) \big| = 0. \tag{3.2.10}$$

The convergence is uniform over compact sets of eigenvalues λ in $(0, \infty)$.

For completness, we shall prove this result at the end of the section, but our method is essentially the same as Weyl's [Wey10], and it is in any case standard in differential equations. Of course, in the simple case that we described in the introduction, where the coefficients of D are eventually constant, Proposition 3.2.7 is obvious.

In any case, using Proposition 3.2.7 we define injective operators

$$C_\lambda \colon H_\lambda \longrightarrow H_{0,\lambda}$$

by $C_\lambda \colon F_\lambda \mapsto F_{0,\lambda}$ where F_λ and $F_{0,\lambda}$ are as in Proposition 3.2.7. If h is a smooth, compactly supported function on $\mathbb{R}$, and if $v_\lambda = F_\lambda$, then

$$\big\langle C_\lambda v_\lambda, (U_t h)_{0,\lambda} \big\rangle_{H_{0,\lambda}} - \big\langle v_\lambda, (W_t h)_\lambda \big\rangle_{H_\lambda} = \int_0^{\infty} (\overline{F_{0,\lambda}}(x) - \overline{F_\lambda}(x)) h(x-t) dx \tag{3.2.11}$$

(this formula holds as long as t is large that $h(x-t)$ is supported on the positive x-axis). Proposition 3.2.7 implies that if λ is confined to a compact set in $(0, \infty)$, then the integral converges to zero, uniformly in λ, as required by Assumption 3.1.5.

Theorem 3.2.8. *Assume that the coefficient functions of the Sturm-Liouville operator D satisfy* (3.2.1) *and* (3.2.9) *above. Let g and h be smooth and compactly supported functions on $[0,\infty)$. If $\beta > \alpha > 0$, and if $P_{[\alpha,\beta]}$ is the spectral projection for D associated to the interval $[\alpha,\beta]$, then*

$$\langle g, P_{[\alpha,\beta]} h\rangle = \frac{1}{4\pi}\int_\alpha^\beta \langle g, F_\lambda\rangle\langle F_\lambda, h\rangle \frac{1}{|c(\lambda)|^2}\frac{d\lambda}{\sqrt{\lambda}}$$

where F_λ is the unique λ-eigenfunction with $F_\lambda(0) = 0$ and $F'_\lambda(0) = 1$, and $c(\lambda)$ is characterized by

$$\lim_{x\to+\infty}\left(F_\lambda(x) - c(\lambda)e^{i\sqrt{\lambda}x} - \overline{c(\lambda)}e^{-i\sqrt{\lambda}x}\right) = 0.$$

Proof. We shall compute $\|P_{[\alpha,\beta]}h\|^2$ (the formula in the statement of the theorem will follow by polarization). First, according to the definition of a direct integral decomposition,

$$\|P_{[\alpha,\beta]}h\|^2 = \int_\alpha^\beta \|h_\lambda\|^2_{H_\lambda}\, d\mu(\lambda).$$

Now let $\{v_\lambda\}$ be the section of $\{H_\lambda\}$ for which $\varepsilon^*_\lambda(\overline{v_\lambda}) = \overline{F_\lambda}$, with F_λ as in the statement of the theorem. Then

$$\int_\alpha^\beta \|h_\lambda\|^2_{H_\lambda}\, d\mu(\lambda) = \int_\alpha^\beta \frac{\left|\langle v_\lambda, h_\lambda\rangle_{H_\lambda}\right|^2}{\langle v_\lambda, v_\lambda\rangle_{H_\lambda}}\, d\mu(\lambda) = \int_\alpha^\beta \frac{|\langle F_\lambda, h_\lambda\rangle|^2}{\langle v_\lambda, v_\lambda\rangle_{H_\lambda}}\, d\mu(\lambda),$$

and applying Theorem 3.1.9 we get

$$\begin{aligned}\int_\alpha^\beta \|h_\lambda\|^2_{H_\lambda}\, d\mu(\lambda) &= \int_\alpha^\beta \frac{|\langle F_\lambda, h_\lambda\rangle|^2}{\langle v_\lambda, v_\lambda\rangle_{H_\lambda}} \frac{2d\mu_0(\lambda)}{\mathrm{Trace}(C^*_\lambda C_\lambda)}\\ &= 2\int_\alpha^\beta \frac{|\langle F_\lambda, h_\lambda\rangle|^2}{\langle C_\lambda v_\lambda, C_\lambda v_\lambda\rangle_{H_{0,\lambda}}}\, d\mu_0(\lambda).\end{aligned}$$

It follows from our definition of C_λ that this is

$$\int_\alpha^\beta \frac{|\langle F_\lambda, h_\lambda\rangle|^2|}{|c(\lambda)|^2}\, d\mu_0(\lambda),$$

and the theorem follows from the explicit formula for μ_0 in (3.2.3). □

It remains to prove Proposition 3.2.7. We begin with the following preliminary computation.

Lemma 3.2.9. *Suppose that a smooth function* $u\colon [0,\infty) \to \mathbb{C}^n$ *is a solution of the differential equation*

$$u'(x) = Mu(x) + Q(x)u(x)$$

where M *is a constant* $n \times n$ *matrix and* Q *is a smooth* $n \times n$ *matrix-valued function. Let*

$$k(x) = \int_x^\infty \|\exp(-xM)Q(x)\exp(xM)\|\,dx \tag{3.2.12}$$

and assume that $k(0) < \infty$. *If*

$$s(x) = \exp(-xM)u(x)$$

then the limit

$$s(\infty) = \lim_{x \to +\infty} s(x)$$

exists. Moreover,

$$\|s(\infty) - s(x)\| \leq \text{constant} \cdot k(x) \cdot \|s(\infty)\|, \tag{3.2.13}$$

where the constant can be chosen to be a continuous function of $k(0)$. *In particular, if the limit* $s(\infty)$ *is zero, then* $s(x)$ *is identically zero.*

Proof. Since all norms on finite-dimensional spaces are equivalent, we can choose any norm in the statement of the lemma. We shall choose the Hilbert-Schmidt norm, defined by

$$\|T\|^2 = \operatorname{Trace}(T^*T).$$

This is a submultiplicative norm, and it is a smooth function on the space of nonzero matrices.

Consider the linear differential equation

$$U'(x) = MU(x) + Q(x)U(x)$$

in which $U(x)$ is a smooth, $n \times n$ matrix-valued function. There exists a unique solution for the initial condition

$$U(0) = I,$$

and it is defined for all x. Moreover each $U(x)$ is invertible: the inverse matrices can be obtained by solving the linear differential equation

$$V'(x) = -V(x)M - U(x)Q(x)$$

with initial condition $V(0) = I$. If we write

$$S(x) = \exp(-xM)U(x),$$

then

$$S'(x) = \exp(-xM)Q(x)\exp(xC)S(x),$$

and so of course

$$\|S'(x)\| \leq \|\exp(-xM)Q(x)\exp(xC)\| \cdot \|S(x)\|. \tag{3.2.14}$$

This, together with the simple inequality

$$\left|\frac{d}{dx}\|S(x)\|\right| \leq \|S'(x)\|$$

gives us the estimate

$$\left|\frac{d}{dx}\log\|S(x)\|\right| \leq \|\exp(-xM)Q(x)\exp(xM)\|. \tag{3.2.15}$$

The integrability hypothesis of the lemma now implies that

$$\sup_{x\in[0,\infty)} \log\|S(x)\| < \infty,$$

and therefore

$$\sup_{x\in[0,\infty)} \|S(x)\| < \infty. \tag{3.2.16}$$

Both suprema are bounded by a continuous function of $k(0)$. Returning to (3.2.14), it follows from (3.2.16) that

$$\|S'(x)\| \leq \text{constant} \cdot \|\exp(-xM)Q(x)\exp(xM)\|, \tag{3.2.17}$$

where the constant can be chosen to be a continuous function of $k(0)$. So by applying the integrability hypothesis a second time we find that $S(x)$ converges to a limit $S(\infty)$ as x tends to infinity.

The limit $S(\infty)$ is an invertible matrix. Indeed we can apply the argument we've just given to the matrix-valued function

$$T(x) = S(x)^{-1} = U(x)^{-1}\exp(xM),$$

in place of $S(x)$. The function $T(x)$ is a solution of the differential equation

$$T'(x) = -T(x)\exp(-xM)Q(x)\exp(xM),$$

and the argument above shows that $T(x)$ converges to a limit as x tends to infinity, and is bounded by a continuous function of $k(0)$. And of course

$$\lim_{x\to\infty} T(x) \cdot \lim_{x\to\infty} S(x) = \lim_{x\to\infty} T(x)S(x) = I.$$

To complete the proof, it follows from the uniqueness of solutions property for ODE's that

$$u(x) = U(x)u(0),$$

so that

$$s(x) = \exp(-xM)u(x) = \exp(-xM)U(x)u(0) = S(x)u(0).$$

So the limit $s(\infty)$ exists. As for (3.2.13), we can write

$$\|s(\infty) - s(x)\| \leq \|S(\infty)\| \cdot \|I - T(\infty)S(x)\| \cdot \|u(0)\|,$$

and then estimate the middle norm on the right-hand side by

$$\int_x^\infty \|T(\infty)\|\|S'(x)\|\,dx.$$

From (3.2.17) we obtain

$$\|s(\infty) - s(x)\| \leq \text{constant} \cdot k(x) \cdot \|u(0)\|.$$

Since $u(0) = T(\infty)s(\infty)$, we obtain

$$\|s(\infty) - s(x)\| \leq \text{constant} \cdot k(x) \cdot \|s(\infty)\|,$$

for a constant that is a continuous function of $k(0)$, as required. □

Let us now apply the lemma to the Sturm-Liouville eigenvalue equation. If $\lambda \in \mathbb{C}$ and if F_λ is any λ-eigenfunction for D, then the vector-valued function

$$u_\lambda(x) = \begin{bmatrix} p(x)F'_\lambda(x) \\ F_\lambda(x) \end{bmatrix} \tag{3.2.18}$$

is a solution of the differential equation

$$u'(x) = M_\lambda u(x) + Q(x)u(x), \tag{3.2.19}$$

where

$$M_\lambda = \begin{bmatrix} 0 & 1 \\ -\lambda & 0 \end{bmatrix} \quad \text{and} \quad Q(x) = \begin{bmatrix} 0 & q(x) \\ 1 - p(x)^{-1} & 0 \end{bmatrix}. \tag{3.2.20}$$

Proof of Proposition 3.2.7. If $\lambda > 0$, then the matrices M_λ in (3.2.20) are skew-adjoint for inner products on $\mathbb{C}^2$ that depend continuously on λ. So if K is a compact subset of $(0, \infty)$, then (using the fixed norm on matrices that we chose earlier)

$$\sup_{\lambda \in K, x \in \mathbb{R}} \| \exp(x M_\lambda) \| < \infty. \tag{3.2.21}$$

As a result, the hypotheses on $p(x)$ and $q(x)$ in the proposition imply that if $M = M_\lambda$ and Q are as in (3.2.20), and if $k(x) = k_\lambda(x)$ is as in Lemma 3.2.9, then

$$\sup_{\lambda \in K} k_\lambda(0) < \infty.$$

Let us now apply Lemma 3.2.9 to the vector-valued function $u_\lambda(x)$ in (3.2.18). Define the function $F_{0,\lambda}(x)$ to be the bottom entry of the vector-valued function

$$F_{0,\lambda}(x) = \exp(x C_\lambda) s_\lambda(\infty)$$

with $s_\lambda(\infty)$ as in the lemma. This is a linear combination of $\exp(\pm i\sqrt{\lambda} x)$, and hence an element of $H_{0,\lambda}$. In fact if $s_\lambda(\infty) = \left[\begin{smallmatrix} a_\lambda \\ b_\lambda \end{smallmatrix}\right]$, and if we write

$$\begin{bmatrix} a_\lambda \\ b_\lambda \end{bmatrix} = \left(a_\lambda/2 + b_\lambda/2i\sqrt{\lambda}\right) \begin{bmatrix} 1 \\ i\sqrt{\lambda} \end{bmatrix} + \left(a_\lambda/2 - b_\lambda/2i\sqrt{\lambda}\right) \begin{bmatrix} 1 \\ -i\sqrt{\lambda} \end{bmatrix},$$

and note that the vectors on the right are eigenvectors for a_λ with eigenvalues $\pm i\sqrt{\lambda}$, then we find that

$$F_{0,\lambda}(x) = \frac{1}{2}(b_\lambda + a_\lambda i\sqrt{\lambda}) \exp(i\sqrt{\lambda} x) + \frac{1}{2}(b_\lambda - a_\lambda i\sqrt{\lambda}) \exp(-i\sqrt{\lambda} x). \tag{3.2.22}$$

This formula also shows that the norm of $s_\lambda(\infty)$ is uniformly bounded by a multiple of the norm of $F_{0,\lambda}$ as λ varies over K. The required estimate follows from this fact together with the conclusion of Lemma 3.2.9 and another application of (3.2.21). $\square$

3.3 Non-Positive Spectrum

In this section we shall briefly examine the non-positive part of the spectrum of a Sturm-Liouville operator D of the type considered in the previous section, with coefficient functions satisfying (3.2.1) and (3.2.9).

The value $\lambda{=}0$ belongs to the spectrum of D, of course, because the spectrum is closed. But for the purposes of fully determining the measure μ we need to determine whether or not 0 is an eigenvalue.

The answer is that $\lambda{=}0$ is not an eigenvalue, at least if we assume a bit more about the rate of convergence of the coefficients $p(x)$ and $q(x)$ to their asymptotic values.

Lemma 3.3.1. *Suppose that*

$$\int_1^\infty x^2|1-p(x)^{-1}|\,dx<\infty \quad \textit{and} \quad \int_1^\infty x^2|q(x)|\,dx<\infty.$$

Then D has no non-zero square-integrable 0-eigenfunctions.

Proof. This is a consequence of Lemma 3.2.9. When $\lambda=0$ the integral in the statement of Lemma 3.2.9 is finite for the matrices M and Q in (3.2.20). So any 0-eigenfunction for D is asymptotic to a 0-eigenfunction for D_0. But the latter are the functions $ax+b$, and we find that no 0-eigenfunction for D can be square-integrable. □

Let us consider now the negative part of the spectrum of D. The argument below is not optimal,[1] but it uses the same ideas we have already developed to handle the continuous spectrum. Moreover it is adequate to handle the operators that arise in harmonic analysis.

Lemma 3.3.2. *If we assume that*

$$\int_1^\infty e^{\alpha x}|1-p(x)^{-1}|\,dx<\infty \quad \textit{and} \quad \int_1^\infty e^{\alpha x}|q(x)|\,dx<\infty.$$

for some $\alpha>0$, then $\lambda{=}0$ is not a limit point of the set of eigenvalues of the self-adjoint Hilbert space operator D.

Proof. For every $\lambda\in\mathbb{C}$ the matrix M_λ in (3.2.20) satisfies $M_\lambda^2=-\lambda I$, and therefore

$$\exp(xM_\lambda)=\cosh(x\sqrt{\lambda})I+\frac{\sinh(x\sqrt{\lambda})}{\sqrt{\lambda}}M_\lambda \tag{3.3.1}$$

[1] See Weyl's paper [Wey10] or [DS88, Chapter XII, Section 7] for sharper results.

for any square root of λ. It follows from this that if $\varepsilon > \delta > 0$, then

$$|\lambda| \le \delta \quad \Rightarrow \quad \|\exp(xM_\lambda)\| \le \text{constant} \cdot e^{\varepsilon x} \tag{3.3.2}$$

for all x and some constant independent of λ and x. Now choose $\varepsilon = \alpha/4$. The estimate (3.3.2) and Lemma 3.2.9 imply that for $u_\lambda(x)$ as in (3.2.18) the limit

$$w_\lambda := \lim_{x\to\infty} \exp(-xM_\lambda)u_\lambda(x)$$

exists whenever $|\lambda| \le \delta$, and moreover

$$|\lambda| \le \delta \quad \Rightarrow \quad \big\|\exp(-xM_\lambda)u_\lambda(x) - w_\lambda\big\| \le \text{constant} \cdot e^{-2\varepsilon x} \tag{3.3.3}$$

for some constant that is again independent of λ and x. If we write

$$u_\lambda(x) = \exp(xM_\lambda)w_\lambda + \exp(xM_\lambda)\big[\exp(-xM_\lambda)u_\lambda(x) - w_\lambda\big]$$

then we find from (3.3.2) and (3.3.3) that

$$u_\lambda \in L^2 \quad \Leftrightarrow \quad \exp(xM_\lambda)w_\lambda \in L^2,$$

and so f_λ is square-integrable if and only if the second entry of the vector-valued function $\exp(xM_\lambda)w_\lambda$ is a square-integrable function.

Now if $z > 0$ and if $\lambda = -z^2$, and if we write $w_\lambda = \left[\begin{smallmatrix} a_\lambda \\ b_\lambda \end{smallmatrix}\right]$, then

$$\exp(xM_\lambda)w_\lambda = (a_\lambda/2 + b_\lambda/2z)\, e^{xz} \begin{bmatrix} 1 \\ z \end{bmatrix} + (a_\lambda/2 - b_\lambda/2z)\, e^{-zx} \begin{bmatrix} 1 \\ -z \end{bmatrix},$$

as we noted in (3.2.22). The second term on the right is always square-integrable. So we find that the second entry of $\exp(xM_\lambda)w_\lambda$ is a square-integrable function if and only if the first term on the right-hand side is zero, or in other words

$$f_\lambda \in L^2 \quad \Leftrightarrow \quad a_\lambda z + b_\lambda = 0$$

(as long as $z > 0$). But now w_λ, and therefore the quantity $a_\lambda z + b_\lambda$, is holomorphic in a sufficiently small neighborhood of $0 \in \mathbb{C}$. It is not identically zero because for example if z is nonzero and purely imaginary (so that $\lambda = -z^2$ is positive), then a_λ and b_λ are real and at least one is nonzero. So there are at most finitely many L^2-eigenvalues in a sufficiently small neighborhood of $0 \in \mathbb{C}$. $\square$

We can say more using perturbation theory. The operator D is a semi-bounded and relatively compact perturbation of the positive operator

$$-d/dx \cdot p(x) \cdot d/dx.$$

So the negative part of its spectrum consists of an at most countably infinite set of eigenvalues accumulating only at 0. Compare [Kat76, Chapter IV, Theorem 5.35]. But Lemma 3.3.2 rules out the possibility of accumulation at 0. Hence:

Theorem 3.3.3. *If we assume that*

$$\int_1^\infty e^{\alpha x}|1 - p(x)^{-1}|\, dx < \infty \quad \textit{and} \quad \int_1^\infty e^{\alpha x}|q(x)|\, dx < \infty.$$

for some $\alpha > 0$, then the operator D has at most finitely many L^2-eigenfunctions satisfying the boundary condition $f_\lambda(0) = 0$, all associated to negative eigenvalues. □

Chapter 4

Asymptotic Containment and the Spherical Plancherel Formula

In this chapter, we apply the method of asymptotic containment to prove Harish-Chandra's spherical Plancherel formula. Let G be a real reductive group with maximal compact subgroup K. We fix

$$\mathcal{A} = C_r^*(G//K).$$

Let $G = KAN$ be the Iwasawa decomposition, and let $M = Z_K(\mathfrak{a})$. We define π and π_0 be the convolution action of $\mathcal{A}$ on ${}^K L^2(G/K)$ and on ${}^K L^2(G/MN)$, respectively. We shall verify that π_0 is asymptotically contained in π in Section 4.1. In Section 4.2, we shall show that the spectral decomposition of π_0 is essentially the Fourier theory on $\mathbb{R}^n$, and in Section 4.3, we verify other technical assumptions proposed in Section 3.1, and prove Harish-Chandra's spherical Plancherel formula on a dense open subset of $Spec(C_r^*(G//K))$. Finally, in Section 4.4, we show that the rest of the spectrum has measure 0.

4.1 Representations of $C_r^*(G//K)$

Let us start with a connected real reductive Lie group G. Let $\mathfrak{g}$ be its Lie algebra with $\mathfrak{g}_{\mathbb{C}}$ its complexification. By choosing a set of positive restricted roots Σ^+, we also fix an the Iwasawa decomposition $G = KAN$. See Section 2.2 and Section 2.3 for details of the structure theory. We use $\mathcal{W}$ to denote the restricted Weyl group, and let $\mathfrak{Q} = U(\mathfrak{g})^K$, as in Chapter 2. We also use the notation $\mathfrak{a}_\varepsilon^+ = \{\xi \in \mathfrak{a} : \alpha(\xi) > \varepsilon > 0, \quad \forall \alpha \in \Sigma^+\} + (\mathfrak{p}_C \cap Z_{\mathfrak{g}})$, $A_\varepsilon^+ = \exp(\mathfrak{a}_\varepsilon^+)$, and $A_\varepsilon = \overline{N_K(A) A_\varepsilon^+ N_K(A)}$. It is standard to write $\mathfrak{a}_0^+$ as $\mathfrak{a}^+$, A_0^+ as A^+, and $M = Z_K(\mathfrak{a})$.

Fix a Haar measure on G. We will let

$$\mathcal{A} = C_r^*(G//K), \ H = {}^K L^2(G/K),$$

and we define π to be the restriction of the representation of $\mathcal{A}$ on $L^2(G)$ to H. Likewise, we define

$$H_0 = {}^K L^2(G/MN),$$

and let π_0 be the representation of $\mathcal{A}$ on H_0 by convolution.

Consider the K-equivariant submanifold of G/MN

$$\iota \colon (KA_\varepsilon^+ K)/K \subset G/K \longrightarrow (KA_\varepsilon MN)/MN \subset G/MN.$$

induced by inclusion $KA_\varepsilon^+ \longrightarrow KA_\varepsilon$. Since the complement of $(KA^+K)/K$ in G/K has measure zero, we can define a linear map W to be the composition of the following three maps:

$$\begin{array}{ccc} & {}^K L^2(G/MN) \xrightarrow{\text{restriction}} {}^K L^2((KA_\varepsilon MN)/MN)\,. \\ & \swarrow \iota^* \\ {}^K L^2((KA_\varepsilon^+ MN)/MN) \xrightarrow{\text{extension}} {}^K L^2(G/K) & \end{array}$$

From now on, we fix once and for all a $\xi \in \mathfrak{a}^+$, and denote

$$a_t = \exp(t\xi). \tag{4.1.1}$$

The group A normalizes MN, and therefore acts on G/MN from the right. Let $\widetilde{U}_t$ denote right translation of G/MN by a_t, that is

$$\widetilde{U}_t\colon kaMN \longmapsto kaa_t^{-1}MN.$$

With the integral formulae Proposition 2.2.23 and Proposition 2.3.30 in mind, we let U_t be a one-parameter unitary group on $L^2(G/MN)$ defined by

$$U_t f = \widetilde{U}_t^*(f)\sqrt{J_{MN}(a_t^{-1})}.$$

Here J_{MN} is as in Proposition 2.3.30. We shall also use J_K as in Proposition 2.2.23 later on. U_t commutes with K actions, and therefore restricts to an action on

$$H_0 = {}^K L^2(G/MN)$$

which we shall also denote by U_t. Right actions commute with left ones. Therefore if $\varphi \in C^*(G//K)$, then

$$[\pi_0(\varphi), U_t] = 0.$$

The following proposition shows π_0 is asymptotically contained in π.

Proposition 4.1.1.

$$\lim_{t\to+\infty}(W_t^*\pi(\varphi)W_t - U_t^*\pi_0(\varphi)U_t)h = 0$$

where $W_t = WU_t$.

The key idea is that

$$\text{“}\lim_{t\to\infty} \operatorname{Ad}_{a_t} K = N\text{”}.$$

For a matrix group, one can choose a matrix in K, compute the left hand side and see with his own eyes that it converges to a matrix in N. For an arbitrary real reductive Lie group, consider the action of the element $kana_t$ on $eK \in G/K$. The isotropy group of $a_t.eK$ is asymptotic to N by the above formula, and therefore $(kana_t).eK$ is asymptotic to $(kaa_t).eK$. More precisely, we have:

Lemma 4.1.2. *Let $a_{KA^+K}\colon KA^+K \subset G \longrightarrow A^+$ be the function that maps k_1ak_2 to a, where $a \in A^+$.*

$$\lim_{t\to\infty} a_{KA^+K}(kana_t)a^{-1}a_t^{-1} = e. \tag{4.1.2}$$

The convergence is uniform on compact subsets of $K \times A \times N$ in the sense that if $\mathcal{K} \subset K \times A \times N$ is compact and $\mathcal{A}_e \subset A$ is a neighborhood of $e \in A$, then for all $t > T_{\mathcal{A}_e}$ and $(k,a,n) \in \mathcal{K}$

$$a_{KA^+K}(kana_t)a^{-1}a_t^{-1} \in \mathcal{A}_e. \tag{4.1.3}$$

To prove the lemma above, we need the following preparations.

Lemma 4.1.3.

$$\lim_{t\to\infty} \operatorname{Ad}_{a_t^{-1}} n = 0 \tag{4.1.4}$$

The convergence is uniform on compact subsets of N.

Proof. Immediate from Proposition 2.2.19. □

Lemma 4.1.4. *There is a smooth map*

$$\zeta\colon [0,1] \times \mathbb{R}^{\dim \mathfrak{k}} \longrightarrow \mathfrak{g} \tag{4.1.5}$$

such that

$$\zeta(\{\tau\} \times \mathbb{R}^{\dim \mathfrak{k}}) = \mathfrak{k}_{-\log\tau} \quad \text{for any } \tau > 0.$$

where $\mathfrak{k}_t = \operatorname{Ad}_{a_t^{-1}} \mathfrak{k}$.

Proof. One can define a topology on the space of all $\dim \mathfrak{k}$ subspaces of $\mathfrak{g}$. In fact, choose any inner product on $\mathfrak{g}$, the set of all ordered $\dim \mathfrak{k}$-tuple of orthonormal vectors in $\mathfrak{g}$ has a natural topology. The quotient of that set by the equivalence relation that two tuple are equivalent if they span the same subspace gives a topology on the space of all $\dim \mathfrak{k}$ subspaces of $\mathfrak{g}$. By Proposition 2.2.19, the family of space $\mathfrak{k}_{-\log\tau}$ has a limit in the topology described above. □

Proof of Lemma 4.1.2. Denote the projection of $\mathcal{K}$ to A by $\mathcal{K}_A$. Consider the map

$$\begin{aligned}\widetilde{\zeta}\colon \mathcal{K}_A \times [0,1] \times \mathbb{R}^{\dim \mathfrak{k}} \times \mathcal{A}_e \times K &\longrightarrow G,\\ \widetilde{\zeta}(a,\tau,v,a',k) &= \operatorname{Ad}_{a^{-1}}(\exp(\zeta(\tau,v)))a'k.\end{aligned}$$

By (4.1.5), $\widetilde{\zeta}$ is smooth, and thus there is a smooth family of tangent maps

$$\mathrm{d}\,\widetilde{\zeta}_{a,\tau}\colon T_{(0,e,e)}\mathbb{R}^{\dim\mathfrak{k}}\times\mathcal{A}_e\times K\longrightarrow T_eG.,$$

If $aa_t\in A^+$, then $\mathrm{Ad}_{(aa_t)^{-1}}\,\mathfrak{k}+\mathfrak{a}+\mathfrak{k}=\mathfrak{g}$. Since $\mathcal{K}_A$ is compact, $\mathrm{d}\,\widetilde{\zeta}_{a,\tau}$ are isomorphisms for $(a,\tau)\in\mathcal{K}_A\times[0,\varepsilon_{\mathcal{K}_A}]$ for some $\varepsilon_{\mathcal{K}_A}>0$. Hence, we proved for some $T_{\mathcal{K}_A}>0$,

$$\bigcap_{t>T_{\mathcal{K}_A},a\in\mathcal{K}_A}\mathrm{Ad}_{(aa_t)^{-1}}(K)\mathcal{A}_eK \tag{4.1.6}$$

contains a neighborhood of $e\in G$.

Now we go back to (4.1.3). Since $a_{KA^+K}(kana_t)=a_{KA^+K}(ana_t)$, we need to show that $ana_t\in K\mathcal{A}_eaa_tK$, or equivalently

$$Ad_{a_t^{-1}}n\in Ad_{(aa_t)^{-1}}(K)\mathcal{A}_eK$$

if $(k,a,n)\in\mathcal{K}$ and $t>T_{\mathcal{A}_e}$. This follows from (4.1.4) and (4.1.6). □

To prove Proposition 4.1.1, we need one more preparation. Note that W is injective on $L^2(A_1^+MN/MN)$, and thus we can define the left inverse of the restriction to $L^2(A_1^+MN/MN)$,

$$W^{-1}\colon L^2(KA_1^+K/K)\longrightarrow L^2(G/MN).$$

W^{-1} is a bounded linear operator. If $\operatorname{supp}h\subset KA_1^+MN$, then $W^{-1}Wh=h$.

Lemma 4.1.5. *If h is a compactly supported function on G/MN and M_t a family of operators $M_t\colon L^2(G/MN)\longrightarrow L^2(G/K)$ such that*

- $\|M_t\|\le C$ *for any* $t>0$,
- *for any* $\tau>0$, $\operatorname{supp}M_th\subset KA_1^+a_\tau K$ *when* $t>T_\tau$,

then

$$\lim_{t\to\infty}\|(W^{-1}-W^*)M_th\|=0.$$

Proof. $W^*M_th=\frac{J_K}{J_{MN}}W^{-1}M_th$, and $\lim_{\tau\to\infty}\frac{J_K}{J_{MN}}\Big|_{A_1^+a_\tau}=1$. □

Proof of Proposition 4.1.1. We can assume that $h,\varphi\in C_c(G//K)$. By (4.1.2), the operator $\pi(\varphi)W_t$ satisfies the conditions of M_t in Lemma 4.1.5, and therefore we need to prove

$\lim_{t\to\infty} \|U_t^* W^{-1}\pi(\varphi)W_t h - U_t^*\pi_0(\varphi)U_t h\| = 0.$

$$\begin{aligned} &\|U_t^* W^{-1}\pi(\varphi)W_t h - U_t^*\pi_0(\varphi)U_t h\| \\ \leq &\int_{\operatorname{supp}\varphi} |\varphi(g)| \|U_t^* W^{-1}\lambda_g W U_t h - U_t^*\lambda_g U_t h\| \,\mathrm{d}\, g. \end{aligned}$$

Therefore, we further reduce the problem to proving

$$\lim_{t\to\infty} \sup_{g\in\operatorname{supp}\varphi} \|U_t^* W^{-1}\lambda_g W U_t h - U_t^*\lambda_g U_t h\| = 0. \tag{4.1.7}$$

Note that

$$\begin{aligned} (U_t^* W^{-1}\lambda_g W U_t h - U_t^*\lambda_g U_t h)(ka) =& h(a_{KA^+K}(g^{-1}kaa_t)a_t^{-1}) - h(g^{-1}ka) \\ =& h(a_{KA^+K}(k'a'n'a_t)a_t^{-1}) - h(a') \end{aligned}$$

if $g^{-1}ka = k'a'n'$. Since the convergence in (4.1.2) is uniform in $g \in \operatorname{supp}\varphi$, (4.1.7) follows from dominated convergence theorem.

□

4.2 Spectral Theory of π_0

In this section we calculate the spectral decomposition of π_0. The calculation is divided into three steps. First we show that common eigenfunctions to a collection of differential operators correspond to elements in the spectrum in Proposition 4.2.2. Then in Proposition 4.2.3 we show the differential operators in question have constant coefficients after we restrict them to A^+. Finally, the Fourier theory on Euclidean spaces is invoked to calculate the spectral decomposition of π_0.

Recall that in Section 2.4 we computed that

$$Spec(C_r^*(G//K)) \cong i\mathfrak{a}^*/\mathcal{W}.$$

By an abuse of notation, from now on we will use λ to denote an element in $Spec(C_r^*(G//K))$ or $\operatorname{Hom}(\mathfrak{Q}/\mathfrak{k}\mathfrak{Q}, i\mathbb{R})$, or $i\mathfrak{a}^*$. In particular,

$$\lambda(\gamma(q)) = \lambda(q)$$

where γ is the Harish-Chandra homomorphism (see Theorem 2.4.10), and $q \in \mathfrak{Q}$.

Now we examine the spectral theory of π_0.

We first address the issue of right invariant differential operator since they are going to be the operators that acts on ${}^K C^\infty(G/MN)$. Denote the Lie algebra of right invariant vector fields on G by $\mathfrak{g}_r$. There is a natural algebra isomorphism

$$R\colon U(\mathfrak{g}) \longrightarrow U(\mathfrak{g}_r)$$

generated by

$$R\colon X \longmapsto -X_r.$$

Everything on $U(\mathfrak{g})$ can be transferred to $U(\mathfrak{g}_r)$ via the isomorphism R. For instance, R restricts to an isomorphism from $\mathfrak{Q}$ to $\mathfrak{Q}_r = U(\mathfrak{g}_r)^K$. If $q \in \mathfrak{Q}$, we shall denote

$$q_r = R(q).$$

Lemma 4.2.1. *If $F_\lambda \in C^\infty(G//K)$, then*

$$q_r F_\lambda = q^* F_\lambda.$$

Proof. It is straightforward to verify that

$$f * (q_r h) = (q^* f) * h \tag{4.2.1}$$

where q^* is the formal adjoint of q, or in other words the image of q under the anti-automorphism of $U(\mathfrak{g})$ generated by $X \longmapsto -X$.

Let $\varphi \in C^\infty(G//K)$ such that $F_\lambda * \varphi = F_\lambda$. Since $C^\infty(G//K)$ is commutative under convolution,

$$q_r F_\lambda = q_r(F_\lambda * \varphi) = (q_r F_\lambda) * \varphi = \varphi * q_r F_\lambda = (q^* \varphi) * F_\lambda = F_\lambda * q^* \varphi = q^*(F_\lambda * \varphi) = q^* F_\lambda.$$

□

Proposition 4.2.2. *If $F_{0,\lambda} \in {}^K C^\infty(G/MN)$ is a common eigenfunction of $\mathfrak{Q}_r$ with*

$$q_r F_{0,\lambda} = \lambda(q^*) F_{0,\lambda},$$

then

$$\varphi * F_{0,\lambda} = \lambda(\varphi) F_{0,\lambda} \tag{4.2.2}$$

for any $\varphi \in C_C(G//K)$.

Proof. Consider the function

$$\Phi_y(x) = \int_K F_{0,\lambda}(xky)\,\mathrm{d}\,k. \tag{4.2.3}$$

We have $\Phi_y \in C^\infty(G//K)$ and

$$q_r\Phi_y = \lambda(q^*)\Phi_y.$$

By Lemma 4.2.1, Φ_y is a common eigenfunction of $\mathfrak{Q}$. By Theorem 2.4.9, the eigenspace in $C^\infty(G//K)$ are one dimensional, and therefore Φ_y is a multiple of F_λ. According to (4.2.3),

$$\Phi_y(x) = F_\lambda(x)F_{0,\lambda}(y).$$

Since φ is right K-invariant,

$$\begin{aligned}
\varphi * F_{0,\lambda}(y) =& \int_{G\times K} \varphi(x^{-1}k)F_{0,\lambda}(xy)\,\mathrm{d}\,k\,\mathrm{d}\,x \\
=& \int_G \varphi(x^{-1}) \int_K F_{0,\lambda}(xky)\,\mathrm{d}\,k\,\mathrm{d}\,x \\
=& \int_G \varphi(x^{-1})F_\lambda(x)\,\mathrm{d}\,xF_{0,\lambda}(y) \\
=& \lambda(\varphi)F_{0,\lambda}(y).
\end{aligned}$$

□

We now push forward $\mathfrak{Q}_r$ onto A in order to use the Fourier theory on Euclidean spaces to calculate the spectral measure μ_0. Let γ_r be the right version of Harish-Chandra isomorphism:

$$\gamma_r = \gamma \circ R^{-1} \colon \mathfrak{Q}_r \longrightarrow S(\mathfrak{a})^{\mathcal{W}}. \tag{4.2.4}$$

By definition,

$$\gamma(q) = \gamma_r(q_r). \tag{4.2.5}$$

By the integral formula (2.3.2), the map

$$\iota_{MN} \colon f \longmapsto f|_{AMN}\sqrt{J_{MN}} \tag{4.2.6}$$

is an isomorphism from $H_0 = {}^K L^2(G/MN)$ to $L^2(A)$.

Proposition 4.2.3.

$$\iota_{MN}(q_r f) = \gamma(q^*)\iota_{MN} f. \tag{4.2.7}$$

Proof. Of course, $U(\mathfrak{g}_r)\mathfrak{k}_r f = 0$. But $q_r f \in {}^K C(G/MN)$, and thus is determined by its value on AMN. Since $U(\mathfrak{g}_r)f \in C(G/MN)$, and AM normalizes N, $\mathfrak{n}_r U(\mathfrak{g}_r)f|_{AMN} = 0$.

On the other hand, if

$$q^* = (q^*)_{U(\mathfrak{a})} + (q^*)_{U(\mathfrak{g})\mathfrak{n}+\mathfrak{k}U(\mathfrak{g})},$$
$$q_r = (q_r)_{U(\mathfrak{a}_r)} + (q_r)_{\mathfrak{n}_r U(\mathfrak{g}_r)+U(\mathfrak{g}_r)\mathfrak{k}_r}$$

where $q'_{U'}$ means $q' \in U'$, then

$$(q_r)_{\mathfrak{n}_r U(\mathfrak{g}_r)+U(\mathfrak{g}_r)\mathfrak{k}_r} = R\left(\left((q^*)_{U(\mathfrak{g})\mathfrak{n}+\mathfrak{k}U(\mathfrak{g})}\right)^*\right).$$

Since $\mathfrak{a}$ is commutative, $U(\mathfrak{a}) = U(\mathfrak{a}_r)$ when considered as differential operators on A, and

$$(q_r)_{U(\mathfrak{a}_r)} = R\left(\left((q^*)_{U(\mathfrak{a})}\right)^*\right) = (q^*)_{U(\mathfrak{a})}.$$

□

By now we have converted the problem of computing the spectral theory of π_0 to computing the spectral theory of $S(\mathfrak{a})^{\mathcal{W}}$, which is the image of the Harish-Chandra isomorphism. The later is essentially the Fourier theory.

Proposition 4.2.4. *Let $\mathfrak{a}_0^*$ be a fundamental domain of $\mathfrak{a}^*$ under the Weyl group action such that*

$$\eta(\mathfrak{a}_0^+) > 0$$

for all $\eta \in \mathfrak{a}_0^*$. Take

$$\Lambda_0 = i\mathfrak{a}_0^*, \tag{4.2.8}$$

and

$$\mathfrak{H}_0 = {}^K C_c(G/MN).$$

For any $\lambda \in \Lambda_0$, let

$$H_{0,\lambda} = span\{F_{0,\sigma.\lambda} : \iota_{MN} F_{0,\sigma.\lambda} = \exp(\sigma.\lambda),\ \sigma \in \mathcal{W}\},$$

and equip $H_{0,\lambda}$ with the inner product that makes $\{F_{0,\sigma.\lambda}\}$ an orthonormal basis.

If $h \in \mathfrak{H}_0$, define

$$\varepsilon_{0,\lambda}(h) = \sum_{\sigma\in\mathcal{W}} \left(\int_{G/MN} h\overline{F_{0,\sigma.\lambda}}\right) F_{0,\sigma.\lambda}.$$

Obviously $\varepsilon_0(h)$ is a continuous section over the trivial bundle $\mathbb{C}^{|\mathcal{W}|} \times \Lambda_0$. Furthermore, *we have for any $h, h' \in \mathfrak{H}_0$ and any $\varphi \in C_c(G//K)$*

$$\langle h, \pi_0(\varphi)h'\rangle = \int_{\Lambda_0} \langle \varepsilon_{A,\lambda}(h), \varepsilon_{A,\lambda}(h')\rangle \lambda(\varphi)\, \mathrm{d}\,\mu_0(\lambda). \tag{4.2.9}$$

The proposition above verifies Assumption 3.1.2.

Proof. For any $\lambda \in i\mathfrak{a}_0^*$ (where $i = \sqrt{-1}$), there are exactly $|\mathcal{W}|$ linearly independent eigenfunctions of $S(\mathfrak{a})^{\mathcal{W}}$ with eigenvalue λ, namely

$$\{F_{A,\sigma.\lambda} = \exp(\sigma.\lambda) : \sigma \in \mathcal{W}\}.$$

Let $H_{A,\lambda} = span\{F_{A,\sigma.\lambda}\}$ equipped with the inner product that makes $\{F_{A,\sigma.\lambda}\}$ an orthonormal basis. By Fourier theory, if $h_A, h'_A \in C_c(A)$, define

$$\varepsilon_{A,\lambda}\colon h_A \longmapsto \sum_{\sigma\in\mathcal{W}} \left(\int h_A \overline{F_{A,\sigma.\lambda}}\right) F_{A,\sigma.\lambda},$$

then

$$\langle h_A, h'_A\rangle = \int_{i\mathfrak{a}_0^*} \langle \varepsilon_{A,\lambda}(h_A), \varepsilon_{A,\lambda}(h'_A)\rangle\, \mathrm{d}\,\mu_0(\lambda) \tag{4.2.10}$$

where μ_0 is the restriction of the admissible measure $\mu_{i\mathfrak{a}^*}$ on $i\mathfrak{a}^*$ to $i\mathfrak{a}_0^*$. Here admissible means exactly that (4.2.10) holds (compare [GV88, Section 6.4]).

Combining Proposition 4.2.2 and Proposition 4.2.3, we know that if

$$\gamma(q)\iota_{MN}(F_{0,\lambda}) = \lambda(q)\iota_{MN}(F_{0,\lambda})$$

for any $q \in \mathfrak{Q}$, then

$$\varphi * F_{0,\lambda} = \lambda(\varphi)F_{0,\lambda}$$

for any $\varphi \in C_c(G//K)$. Therefore,

$$H_{0,\lambda} = span\{F_{0,\lambda} : \gamma(q)\iota_{MN}(F_{0,\lambda}) = \lambda(q)\iota_{MN}(F_{0,\lambda}), \quad \forall q \in \mathfrak{Q}\},$$

and the proposition follows from (4.2.2) and (4.2.10). □

As for assumption (3.1.3), by definition of $F_{0,\sigma.\lambda}$, we have

$$U_t F_{0,\sigma.\lambda} = \exp(-\sigma.\lambda(t\xi))F_{0,\sigma.\lambda}.$$

Therefore, U_t descends to a unitary action on $H_{0,\lambda}$.

4.3 Spectral Theory of π

Next, we examine the spectral theory of π. We shall again invoke the Gelfand-Kostyuchenko method described in Section 2.6. Suppose $\mathcal{S} \longrightarrow H$ is fine. For almost all λ (with respect to the measure μ) there are continuous operators

$$\varepsilon_\lambda \colon \mathcal{S} \longrightarrow H_\lambda \tag{4.3.1}$$

with dense range such that if $h \in \mathcal{S}$, then $h_\lambda = \varepsilon_\lambda(h)$ for almost every $\lambda \in \Lambda$. In our case we can take $\mathcal{S} = {}^K C_c^\infty(G/K)$ (compare [Ber88, Section 1, Lemma 2.3]). Based on the fact that

$$H_\lambda^* \cong \overline{H_\lambda},$$

the adjoint map

$$\varepsilon_\lambda^* \colon H_\lambda^* \longrightarrow \mathcal{S}^* \tag{4.3.2}$$

identifies H_λ as the space of eigendistributions of left convolution with $C^*(G//K)$(see [Ber88, Section 1, Lemma 1.3 b)]). Since $C_c^\infty(G//K)$ is dense in $C^*(G//K)$, and convolutions of distributions with compactly supported smooth functions are smooth function, H_λ consists of ordinary smooth functions. According to Theorem 2.4.8, H_λ are common eigenfunctions of $\mathfrak{Q}$, and by Theorem 2.4.9

$$\dim H_\lambda = 1. \tag{4.3.3}$$

We therefore verified Assumption 3.1.4.

We shall now restrict the eigenfunctions of π and π_0 to A^+ and A respectively and compare the differential equations they satisfy. Such a comparison will lead us to the asymptotic maps A_λ.

As we have already seen in Section 4.2,

$$H_{0,\lambda} = \{F_{0,\lambda} \in {}^K C^\infty(G/MN) : \gamma(q)\iota_{MN}F_{0,\lambda} = \lambda(q)\iota_{MN}F_{0,\lambda}, \quad \forall q \in \mathfrak{Q}\}. \tag{4.3.4}$$

By the integral formula (2.2.15), the map

$$\iota_K \colon f \longmapsto f|_{A^+K}\sqrt{J_K} \tag{4.3.5}$$

is an isomorphism from $H = {}^K L^2(G/K)$ to $L^2(A^+)$.

The radial part of $\mathfrak{Q}$ is well known. Let

$$f_\alpha = \frac{\exp(-\alpha \circ \log)}{1 - \exp(-2\alpha \circ \log)}, \quad g_\alpha = \frac{\exp(-2\alpha \circ \log)}{1 - \exp(-2\alpha \circ \log)}$$

with $\alpha \in \Sigma^+$, $\mathcal{R}_d$ be the vector space of degree d homogeneous polynomials in f_α and g_α, and

$$\mathcal{R}^+ = \sum_{d>0} \mathcal{R}_d.$$

Theorem 4.3.1. *Suppose $f \in {}^K C^\infty(G/K)$ that is supported on KA^+K and $q \in \mathfrak{Q}$, then*

$$(qf)|_{A^+} = \left(\exp(-\rho)\gamma(q)\exp(\rho) + \sum_{1 \le i \le n} \varphi'_{q,i} u'_{q,i} \right) (f|_{A^+}).$$

Here $\varphi'_i \in \mathcal{R}^+$, and u'_i are constant coefficient differential operators. Furthermore, $\deg u'_i < \deg q$ *and* $\deg \varphi'_i + \deg u'_i \le \deg q$.

A straightforward calculation gives the following corollary.

Corollary 4.3.2. *Suppose $f \in {}^K C_c^\infty(G/K)$ that is supported on KA^+K, then*

$$\iota_K(qf) = \left(\gamma(q) + \sum_{1 \le i \le n} \varphi_{q,i} u_{q,i} \right) (\iota_K f).$$

Here $\varphi_i \in \mathcal{R}^+$, and u_i are constant coefficient differential operators. Furthermore, $\deg u_i < \deg q$ *and* $\deg \varphi_i + \deg u_i \le \deg q$.

We are now able to conclude that

$$H_\lambda = \{F_\lambda \in {}^K C^\infty(G/K) : \left(\gamma(q) + \sum_i \varphi_{q,i} u_{q,i} \right) (\iota_K F_\lambda) = \lambda(q) \iota_K F_\lambda, \quad \forall q \in \mathfrak{Q}\}. \tag{4.3.6}$$

To proceed, we will use the standard technique of converting the differential equations satisfied by

$$f_{0,\lambda} = \iota_{MN} F_{0,\lambda}, \; f_\lambda = \iota_K F_\lambda$$

into first order differential equations on vectors. By Chevalley's theorem [GB85a, Theorem 7.3.5], $S(\mathfrak{a})$ is a free $S(\mathfrak{a})^{\mathcal{W}}$-module. Let

$$\{b_1 = 1, b_2, \cdots, b_{|\mathcal{W}|}\}$$

be a set of free generators, and

$$\overrightarrow{f} = \begin{pmatrix} f \\ b_2 f \\ \vdots \\ b_{|\mathcal{W}|} f \end{pmatrix}.$$

Let

$$\theta\colon S(\mathfrak{a}) \longrightarrow M_{|\mathcal{W}|}(S(\mathfrak{a})^{\mathcal{W}})$$

be the homomorphism s.t.

$$\theta(b)\overrightarrow{f} = b\overrightarrow{f}.$$

If $\{\xi_i\}$ is a basis of $\mathfrak{a}$, then by (4.3.4)

$$\xi_i \overrightarrow{f_{0,\lambda}} = \lambda(\theta(\xi_i))\overrightarrow{f_{0,\lambda}}. \tag{4.3.7}$$

On the other hand, $\overrightarrow{f_\lambda}$ satisfies the following differential equation.

Proposition 4.3.3.

$$\xi_i \overrightarrow{f_\lambda} = \big(\lambda(\theta(\xi_i)) + \Phi_i\big)\overrightarrow{f_\lambda} \tag{4.3.8}$$

where Φ_i is a matrix with entries in $\mathcal{R}^+$.

Proof. It suffices to prove that if $b \in S(\mathfrak{a})$, then

$$\gamma(q)bf_\lambda = \lambda(q)bf_\lambda + \sum_k \varphi_{q,b,k} b_k f_\lambda \tag{4.3.9}$$

where $\varphi_{q,b,i} \in \mathcal{R}^+$. To achieve this, we will do an induction on $\deg \gamma(q)b$. When the degree is 0, (4.3.9) trivially holds. Without loss of generality, we assume

$$\deg q = \deg \gamma(q).$$

By (4.3.6),

$$\gamma(q)bf_\lambda = b\gamma(q)f_\lambda = \lambda(q)bf_\lambda + b(\sum_i \varphi_{q,i}u_{q,i})f_\lambda = \lambda(q)bf_\lambda + (\sum_j \widetilde{\varphi_{q,j}}\widetilde{u_{q,j}})f_\lambda$$

where $\widetilde{\varphi_{q,j}} \in C^\infty(A^+)$ and $\widetilde{u_{q,j}}$ are constant coefficient differential operators. By a straightforward computation $S(\mathfrak{a})\mathcal{R}^+ \subset \mathcal{R}^+$, and thus

$$\widetilde{\varphi_{q,j}} \in \mathcal{R}^+.$$

On the other hand, $\deg \widetilde{u_{q,j}} \leq \deg b + \deg u_{q,i} < \deg b + \deg q$. By induction and by writing $\widetilde{u_{q,j}}$ as a $S(\mathfrak{a})^{\mathcal{W}}$-linear combination of b_k, we conclude that

$$\gamma(q) b f_\lambda = \lambda(q) b f_\lambda + (\sum_j \widetilde{\varphi_{q,j}} \widetilde{u_{q,j}}) f_\lambda = \lambda(q) b f_\lambda + \sum_k \varphi_{q,b,k} b_k f_\lambda$$

□

Corollary 4.3.4. *If $\lambda \in \Lambda_0$, then f_λ is asymptotic to some $f_{*,\lambda} \in \iota_{MN} H_{0,\lambda}$ in the sense that for any compact set $\mathcal{K} \subset A^+$*

$$|f_\lambda(a_t x) - f_{*,\lambda}(a_t x)| \leq \varphi_{\mathcal{K}}(t) \|f_{*,\lambda}\|_{H_{0,\lambda}}, \quad \forall x \in \mathcal{K} \tag{4.3.10}$$

where $\varphi_{\mathcal{K}}$ is independent of λ and has exponential decay and a_t is as in (4.1.1). *Furthermore, if $f_\lambda \neq 0$, then $f_{*,\lambda} \neq 0$.*

Proof. By (4.3.8),

$$\xi \overrightarrow{f_\lambda} = \big(\lambda(\theta(\xi)) + \Phi_\xi\big) \overrightarrow{f_\lambda}.$$

Since $\lambda \in i\mathfrak{a}^*$, and $\iota_{MN} \overrightarrow{F_{0,\sigma.\lambda}}$ are $|\mathcal{W}|$ linearly independent solutions to

$$\xi \overrightarrow{f} = \lambda(\theta(\xi)) \overrightarrow{f},$$

$\lambda(\theta(\xi))$ is diagonalizable and its eigenvalues are purely imaginary. Therefore,

$$\| \exp(x \lambda(\theta(\xi)))\| = |\mathcal{W}|.$$

Since the entries of Φ_ξ are in $\mathcal{R}^+$, the following function is well defined:

$$\widetilde{\varphi_{\mathcal{K}}}(t) = \max_{x \in \mathcal{K}} \left| \int_t^\infty \|\Phi_\xi(a_t x)\| \, \mathrm{d}\, t \right|.$$

Recall that $\|ST\| \leq \|S\| \|T\|$. Therefore, by the previous lemma, for each $x \in \mathcal{K}$ there is $f_{x,\lambda} \in \iota_{MN} H_{0,\lambda}$ s.t.

$$\|\overrightarrow{f_{x,\lambda}}(a_t x) - \overrightarrow{f_\lambda}(a_t x)\| \leq \varphi_{\mathcal{K}}(t) \|f_{x,\lambda}\|_{H_{0,\lambda}}$$

where $\varphi_{\mathcal{K}} = \text{constant} \cdot \widetilde{\varphi_{\mathcal{K}}}$ is independent of x or λ.

Fix $x_1, x_2 \in \mathcal{K}$, and suppose $x_2 = x_1 \exp(\xi_{1,2})$. $\overrightarrow{f_\lambda}$ restricted to

$$L_t = \{a_t x_1 \exp(\tau \xi_{1,2}) : \tau \in [0,1]\}$$

is a vector-valued function that satisfies

$$\xi_{1,2}\overrightarrow{f_\lambda} = (\gamma(\theta(\xi_{1,2})) + \Phi_{\xi_{1,2}})\overrightarrow{f_\lambda}.$$

Since $\|\overrightarrow{f_\lambda}(a_t x_1)\|$ is bounded as $t \to \infty$,

$$\lim_{t\to\infty} \|\overrightarrow{f_\lambda}(a_t x_1) - \exp(-\gamma(\theta(\xi_{1,2})))\overrightarrow{f_\lambda}(a_t x_2)\| = 0.$$

Therefore

$$\lim_{t\to\infty} \|\overrightarrow{f_{x_1,\lambda}}(a_t x_2) - \overrightarrow{f_{x_2,\lambda}}(a_t x_2)\| = 0,$$

and

$$f_{x_1,\lambda} = f_{x_2,\lambda} = f_{*,\lambda}.$$

□

Now define

$$A_\lambda \colon H_\lambda \longrightarrow H_{0,\lambda}$$

to be the map such that

$$\iota_{MN} A_\lambda \iota_K^{-1} \colon f_\lambda \longmapsto f_{*,\lambda}$$

where f_λ and $f_{*,\lambda}$ are as in (4.3.10). A_λ are injective by Corollary 4.3.4 as required in Assumption 3.1.5. Furthermore, if $\widetilde{h} \in \mathfrak{H}_0$, denote

$$h = \iota_{MN}\widetilde{h},$$

and we have

$$\iota_{MN} U_t \widetilde{h} = \rho_t(h), \quad \iota_K W_t \widetilde{h} = \rho_t(h)\frac{\sqrt{J_K}}{\sqrt{J_{MN}}}$$

where $\rho_t(h)(x) = h(a_t^{-1}x)$. By (4.3.10)

$$\left|\left\langle A_\lambda v_\lambda, \varepsilon_{0,\lambda}(U_t h)\right\rangle_{H_{0,\lambda}} - \left\langle v_\lambda, (W_t h)_\lambda\right\rangle_{H_\lambda}\right| = \left|\int \overline{f_{*,\lambda}}\rho_t(h) - \int \overline{f_\lambda}\rho_t(h)\frac{\sqrt{J_K}}{\sqrt{J_{MN}}}\right|$$

is bounded by $\|\rho_0(h)\|_2 \varphi_{\mathrm{supp}(h)}(t)\|A_\lambda v_\lambda\|$. Thus, we have verified Assumption 3.1.5.

Let us summarize.

Theorem 4.3.5. *Suppose G is a connected real reductive Lie group, $\varphi \in C^*(G//K)$ is supported on Λ_0, and h, h' are compactly supported K-invariant functions on G/K, then*

$$\langle h, \pi(\varphi)h'\rangle_{{}^K L^2(G/K)} = \frac{1}{|\mathcal{W}|}\int_{i\mathfrak{a}^*} \overline{\mathcal{H}h(-\lambda)}\mathcal{H}h'(-\lambda)\varphi(\lambda)\frac{1}{|\mathbf{c}(\lambda)|^2}\,\mathrm{d}\,\mu_{i\mathfrak{a}^*}\lambda.$$

Here **c** *is the Harish-Chandra* **c***-function,*

$$\mathcal{H}\colon h \longmapsto \int F_\lambda h$$

(F_λ as in Theorem 2.4.8) is the Harish-Chandra transform, and on the right hand side of the formula, we view φ as a $\mathcal{W}$-invariant function on $i\mathfrak{a}^*$.

Proof. By Theorem 3.1.6, we have

$$\langle h, \pi(\varphi)h' \rangle_{{}^K L^2(G/K)} = |\mathcal{W}| \int_{\Lambda_0} \overline{\varepsilon_\lambda(h)} \varepsilon_\lambda(h') \varphi(\lambda) \frac{1}{\| \overrightarrow{c} \|^2} \,\mathrm{d}\,\mu_0 \lambda$$

where

$$\varepsilon_\lambda(h) = \int_G \overline{F_\lambda(x)} h(x) \,\mathrm{d}\, x,$$

$$\iota_K F_\lambda \to \sum_{\sigma \in \mathcal{W}} c_\sigma \exp(\sigma.\lambda)$$

as $a \to \infty$, and

$$\overrightarrow{c} = \begin{pmatrix} c_{\sigma_1} \\ \vdots \\ c_{\sigma_{|\mathcal{W}|}} \end{pmatrix}.$$

Note that $F_\lambda = F_{\sigma.\lambda}$ by Theorem 2.4.9, and $\overline{F_\lambda} = F_{-\lambda}$ if $\lambda \in i\mathfrak{a}^*$. Therefore

$$\langle h, \pi(\varphi)h' \rangle_{{}^K L^2(G/K)} = \int_{i\mathfrak{a}^*} \overline{\mathcal{H}h(-\lambda)} \mathcal{H}h'(-\lambda) \varphi(\lambda) \frac{1}{\| \overrightarrow{c} \|^2} \,\mathrm{d}\,\mu_0 \lambda.$$

Now the theorem follows from the well known facts that $\mathbf{c}(\sigma.\lambda) = c_\sigma(\lambda)$ and the Mass-Selberg relation $|\mathbf{c}(\sigma.\lambda)|^2 = |\mathbf{c}(\lambda)|^2$, which is an immediate consequence of Lemma 3.1.7. □

4.4 The Plancherel Measure of the Boundary

We want to prove the boundary of the Weyl chamber has Plancherel measure 0. We are only going to show that for any λ on the boundary, there is a neighborhood of it where the measure is 0. If this is true, then any compact set will have measure 0, and hence the entire boundary.

The boundary of the Weyl chamber can be divided into several walls. To be more precise, let $\Sigma \subset \Sigma^+$ be the set of simple roots. Choose any subset S of Σ, the set

$$\partial_S = \{\lambda \in \mathfrak{a}^* : \lambda = \sum_{\alpha \in \Sigma \setminus S} r_\alpha \alpha,\ r_\alpha > 0\}$$

is the Weyl wall that is invariant under $\{\sigma_\alpha : \alpha \in S\}$ where σ_α is the simple reflection by α.

First we will show that $f_\lambda - f_{0,\lambda}$ restricted to a cone is in L^2. We fix a norm on $\mathfrak{a}^*$ to make it into a normed space (any norm will do). Let

$$\mathcal{C}_\varepsilon = \{\exp(t\xi_\varepsilon) : t \geq 1, \|\xi_\varepsilon\| = 1, \langle \alpha, \xi_\varepsilon \rangle \geq \varepsilon \ \forall \alpha \in \Sigma\}.$$

Of course, $\mathcal{C}_\varepsilon$ is non empty if $\varepsilon > 0$ is small enough.

Proposition 4.4.1. *Let $\widetilde{K} \subset \partial_S$ by any compact space. For any $\lambda \in \widetilde{K}$, there is an eigenfunction $f_{*,\lambda}$ of $\mathrm{Stab}_{\mathcal{W}} S$-invariant constant coefficient differential operators such that*

$$\|(f_\lambda - f_{*,\lambda})|_{\mathcal{C}_\varepsilon}\|_2 \leq \mathrm{constant} \cdot \|f_{*,\lambda}\|. \tag{4.4.1}$$

Here constant *depends only on $\widetilde{K}$. Choose any continuous family of basis for eigenfunctions of $\mathrm{Stab}_{\mathcal{W}} S$-invariant constant coefficient differential operators. $\|f_{*,\lambda}\|$ is the norm of the coordinate vector.*

Proof. Let ξ_ε be any element in $\mathfrak{a}$ such that $\|\xi_\varepsilon\| = 1$ and that $\langle \alpha, \xi_\varepsilon \rangle \geq \varepsilon \ \forall \alpha \in \Sigma$. An argument similar to the one in the proof of Corollary 4.3.4 shows that for any $\lambda \in \Lambda$ and $t \geq 1$

$$\|\overrightarrow{f_\lambda}(\exp t\xi_\varepsilon) - \overrightarrow{f_{*,\lambda}}(\exp t\xi_\varepsilon)\| \leq \mathrm{contant} \cdot \exp(-\frac{t\varepsilon}{2})\|\overrightarrow{f_{*,\lambda}}\|.$$

Here constant is independent of ξ_ε. Now the proposition follows from an integration using the polar coordinate. ☐

Using the integration formula of f_λ, we also know that $f_{*,\lambda}$ is a continuous family of functions for $\lambda \in \partial_S$.

From now on we fix

$$\mathcal{C} = \mathcal{C}_\varepsilon.$$

Suppose a compact set $\widetilde{K} \subset \partial_S$ has positive μ measure. By simple measure theory, we are guaranteed a subset K of it such that

$$\mu(K) > 0$$

and

$$\|f_\lambda\|_{H_\lambda} < \text{constant}$$

on K for some fixed constant. By the direct integral decomposition of $\mathcal{H}$, for any φ supported on K,

$$\|\int_K \varphi(\lambda) f_\lambda \,\mathrm{d}\,\mu(\lambda)\|_2 \leq \sqrt{\int_K |\varphi(\lambda)|^2 \text{constant}^2 \,\mathrm{d}\,\mu(\lambda)} = \text{constant} \cdot \|\varphi\|_2.$$

Combing this estimate with (4.4.1), we have

$$\begin{aligned} \|\int_K \varphi(\lambda) f_{*,\lambda} \,\mathrm{d}\,\mu(\lambda)\|_2 \leq & \|\int_K \varphi(\lambda) f_\lambda \,\mathrm{d}\,\mu(\lambda)\|_2 \\ & + \|\int_K \varphi(\lambda)(f_\lambda - f_{*,\lambda}) \,\mathrm{d}\,\mu(\lambda)\|_2 \leq \text{constant} \cdot \|\varphi\|_2. \end{aligned} \tag{4.4.2}$$

The constant here is independent of φ.

We next construct a family of φ_t so that (4.4.2) is impossible to hold, which allows us to conclude $\mu(\partial_S) = 0$.

By Theorem A.9,

$$f_{*,\lambda}(\exp \xi) = \sum_{\sigma,j} c_{\sigma,j}(\lambda) \exp\langle \sigma.\lambda, \xi \rangle p_{\sigma,j}(\exp \xi) \tag{4.4.3}$$

where the polynomials $p_{\sigma,j}$ are homogeneous. Let $c = c_{\sigma_0,j_0}, p = p_{\sigma_0,j_0}$ be such that

- $\{\lambda \in K : c_{\sigma_0,j_0}(\lambda) \neq 0\} > 0$.
- $\deg p_{\sigma_0,j_0} \geq \deg p_{\sigma_1 j_1}$ if (σ_1, j_1) is another pair that satisfies the above requirement.

Without loss of generality, We assume $\sigma = e$ and $c(\lambda) > \varepsilon > 0$ on K. Let $\lambda \in K$. The span of a Weyl wall has the same stabilizer group as that of the Weyl wall. Therefore, if $\sigma.\lambda \in \text{span } \partial_S$, then

$$\text{Stab}_{\mathcal{W}} \sigma.\lambda = \text{Stab}_{\mathcal{W}} \partial_S = \text{Stab}_{\mathcal{W}} \lambda.$$

By Theorem A.9, we can regroup the terms in (4.4.3) as

$$\begin{aligned} f_{*,\lambda}(\exp\xi) =& c(\lambda)\exp\langle\lambda,\xi\rangle p(\exp\xi) \\ &+ \sum_{\{\sigma:\sigma.K\subset \mathrm{span}\,\partial S,\sigma\neq e\}} c_{\sigma,j_0}(\lambda)\exp\langle\sigma.\lambda,\xi\rangle p(\exp\xi) \\ &+ \sum_{\{\sigma:\sigma.K\subset \mathrm{span}\,\partial S\},j\neq j_0} c_{\sigma,j}(\lambda)\exp\langle\sigma.\lambda,\xi\rangle p_j(\exp\xi) \\ &+ \sum_{\{\widetilde{\sigma}:\widetilde{\sigma}.K\not\subset \mathrm{span}\,\partial S\},\widetilde{j}} c_{\widetilde{\sigma},\widetilde{j}}(\lambda)\exp\langle\widetilde{\sigma}.\lambda,\xi\rangle p_{\widetilde{\sigma},\widetilde{j}}(\exp\xi). \end{aligned} \tag{4.4.4}$$

Let

$$\mathfrak{a}^{\perp} = \{\xi^{\perp}\in\mathfrak{a} : \langle\partial_S,\xi^{\perp}\rangle = 0\}.$$

We have

$$\mathfrak{a}^{\perp}\subset\ker\sigma.\lambda \Leftrightarrow \sigma.\lambda\in\mathrm{span}\,\partial_S.$$

Let

$$\mathfrak{a} = \mathfrak{a}^{\perp}\oplus\mathfrak{a}_{//},$$

$$B_{\varepsilon} = \{\xi_{//}\in\mathfrak{a}_{//} : \|\xi_{//}\|\leq\varepsilon\}.$$

We fix

$$\exp\xi_1\in\mathcal{C},\ \lambda_1\subset\mathfrak{a}^{\perp}$$

such that l_1 is a compact subset whose interior is not empty and that

$$\exp(\xi_1+l_1+B_{\varepsilon})\subset\mathcal{C}.$$

By perturbing x_1, we may assume that $\|p|_{\exp(x_1+l_1)}\|_2 > 0$ and p_j are orthogonal to p on x_1+l_1 if $\deg p_j = \deg p$.

Let

$$\varphi_t(\lambda) = \frac{\exp-\langle\lambda,t\xi_1\rangle}{c(\lambda)}.$$

Note that

$$\|\varphi_t\|_2\leq\text{constant}.$$

The rest of the section is to show that

$$\limsup_{t\to\infty}\|\int_K\varphi(\lambda)f_{*,\lambda}\,\mathrm{d}\,\mu(\lambda)\|_2 = \infty \tag{4.4.5}$$

which contradicts (4.4.2).

$$
\begin{aligned}
\int_K \varphi_t(\lambda) f_{*,\lambda}(\exp\xi)\,\mathrm{d}\,\mu(\lambda) = &\int_K \varphi_t(\lambda)c(\lambda)\exp\langle\lambda,\xi\rangle p(\exp\xi)\,\mathrm{d}\,\mu(\lambda) \\
&+\sum\int_K \varphi_t(\lambda)c_\sigma(\lambda)\exp\langle\sigma.\lambda,\xi\rangle p(\exp\xi)\,\mathrm{d}\,\mu(\lambda) \\
&+\sum\int_K \varphi_t(\lambda)c_{\sigma,j}(\lambda)\exp\langle\sigma.\lambda,\xi\rangle p_j(\exp\xi)\,\mathrm{d}\,\mu(\lambda) \\
&+\sum\int_K \varphi_t(\lambda)c_{\widetilde{\sigma},\widetilde{j}}(\lambda)\exp\langle\widetilde{\sigma}.\lambda,\xi\rangle p_{\widetilde{\sigma},\widetilde{j}}(\exp\xi)\,\mathrm{d}\,\mu(\lambda) \\
=&\beta_t+\sum\beta_{t,\sigma}+\sum\beta_{t,\sigma,j}+\sum\beta_{t,\widetilde{\sigma},\widetilde{j}}.
\end{aligned}
$$

Denote

$$L_t = \exp(t\xi_1 + tl_1 + B_\varepsilon)$$

and

$$\xi = \xi^\perp + \xi_{//}$$

where $\xi^\perp \in \mathfrak{a}^\perp, \xi_{//} \in \mathfrak{a}_{//}$.

$$
\begin{aligned}
\|\beta_t\|^2_{L_t} =&\int_{K\times K}\overline{\varphi_t(\lambda_1)c(\lambda_1)}\varphi_t(\lambda_2)c(\lambda_2) \\
&\int_{t\xi_1+tl_1+B_\varepsilon}\exp\langle\lambda_2-\lambda_1,\xi\rangle|p(\exp\xi)|^2\,\mathrm{d}\,\xi\,\mathrm{d}\,\mu\times\mu(\lambda_1,\lambda_2) \\
=&\int_{K\times K}\int_{tl_1+B_\varepsilon}\exp\langle\lambda_2-\lambda_1,\xi\rangle|p(\exp(t\xi_1+\xi))|^2\,\mathrm{d}\,\xi\,\mathrm{d}\,\mu\times\mu(\lambda_1,\lambda_2) \\
=&t^{2\deg p+\dim l_1}\int_{K\times K}\int_{B_\varepsilon}\exp\langle\lambda_2-\lambda_1,\xi_{//}\rangle \\
&\int_{l_1}|p(\exp(\xi_1+\xi^\perp+\frac{\xi_{//}}{t}))|^2\,\mathrm{d}\,\xi^\perp\,\mathrm{d}\,\xi_{//}\,\mathrm{d}\,\mu\times\mu(\lambda_1,\lambda_2).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lim_{t\to\infty}\frac{\|\beta_t\|^2_{L_t}}{t^{2\deg p+\dim l_1}} =&\|p|_{\exp(\xi_1+\lambda_1)}\|^2\int_{K\times K}\int_{B_\varepsilon}\exp\langle\lambda_2-\lambda_1,\xi_{//}\rangle\,\mathrm{d}\,\xi_{//}\,\mathrm{d}\,\mu\times\mu(\lambda_1,\lambda_2) \\
=&\text{positive constant.}
\end{aligned}
$$

Similarly,

$$\langle \beta_t, \beta_{t,\sigma}\rangle_{L_t} = t^{2\deg p + \dim l_1} \int_{K\times K} \frac{c_\sigma(\lambda_2)}{c(\lambda_2)} \exp\langle \sigma.\lambda_2 - \lambda_2, t\xi_1\rangle \int_{B_\varepsilon} \exp\langle \sigma.\lambda_2 - \lambda_1, \xi_{//}\rangle$$
$$\int_{l_1} |p(\exp(\xi_1 + \xi^\perp + \frac{\xi_{//}}{t}))|^2 \, \mathrm{d}\,\xi^\perp \, \mathrm{d}\,\xi_{//} \, \mathrm{d}\,\mu \times \mu(\lambda_1, \lambda_2).$$

$$\int_{t_1}^{t_2} \frac{\langle \beta_t, \beta_{t,\sigma}\rangle_{L_t}}{t^{2\deg p + \dim l_1}} \, \mathrm{d}\, t = \| p|_{\exp(\xi_1 + \lambda_1)} \|^2$$
$$\int_{K\times K} \frac{c_\sigma(\lambda_2)(\exp\langle \sigma.\lambda_2 - \lambda_2, t_2\xi_1\rangle - \exp\langle \sigma.\lambda_2 - \lambda_2, t_1\xi_1\rangle)}{c(\lambda_2)\langle \sigma.\lambda_2 - \lambda_2, \xi_1\rangle}$$
$$\int_{B_\varepsilon} \exp\langle \sigma.\lambda_2 - \lambda_1, \xi_{//}\rangle \, \mathrm{d}\,\xi_{//} \, \mathrm{d}\,\mu \times \mu(\lambda_1, \lambda_2) + o(\frac{1}{t_1})$$

The integral is uniformly bounded, and therefore

$$\limsup_{t\to\infty} \mathrm{Re} \, \frac{\langle \beta_t, \beta_{t,\sigma}\rangle_{L_t}}{t^{2\deg p + \dim l_1}} \geq 0.$$

$$\langle \beta_t, \beta_{t,\sigma,j}\rangle_{L_t} = t^{\deg p + \deg p_j + \dim l_1}$$
$$\int_{K\times K} \frac{c_\sigma(\lambda_2)}{c(\lambda_2)} \exp\langle \sigma.\lambda_2 - \lambda_2, t\xi_1\rangle \int_{B_\varepsilon} \exp\langle \sigma.\lambda_2 - \lambda_1, \xi_{//}\rangle$$
$$\int_{l_1} \overline{p(\exp(\xi_1 + \xi^\perp + \frac{\xi_{//}}{t}))} p_j(\exp(\xi_1 + \xi^\perp + \frac{\xi_{//}}{t}))$$
$$\mathrm{d}\,\xi^\perp \, \mathrm{d}\,\xi_{//} \, \mathrm{d}\,\mu \times \mu(\lambda_1, \lambda_2).$$

Since $\langle p, p_j\rangle_{\exp(\xi_1 + l_1)} = 0$,

$$\lim_{t\to\infty} \frac{\langle \beta_t, \beta_{t,\sigma,j}\rangle_{L_t}}{t^{\deg p + \deg p_j + \dim l_1}} = 0.$$

$$\langle \beta_t, \beta_{t,\widetilde{\sigma},\widetilde{j}}\rangle_{L_t} = t^{\deg p + \deg p_{\widetilde{\sigma},\widetilde{j}} + \dim l_1}$$
$$\int_{K\times K} \frac{c_{\widetilde{\sigma},\widetilde{j}}(\lambda_2)}{c(\lambda_2)} \exp\langle \widetilde{\sigma}.\lambda_2 - \lambda_2, t\xi_1\rangle \int_{B_\varepsilon} \exp\langle \widetilde{\sigma}.\lambda_2 - \lambda_1, \xi_{//}\rangle$$
$$\int_{l_1} \overline{p(\exp(\xi_1 + \xi^\perp + \frac{\xi_{//}}{t}))} p_{\widetilde{\sigma},\widetilde{j}}(\exp(\xi_1 + \xi^\perp + \frac{\xi_{//}}{t}))$$
$$\exp\langle \widetilde{\sigma}.\lambda_2, t\xi^\perp\rangle \, \mathrm{d}\,\xi^\perp \, \mathrm{d}\,\xi_{//} \, \mathrm{d}\,\mu \times \mu(\lambda_1, \lambda_2).$$

By Stokes theorem, $\lim_{t\to\infty}\int_{l_1}\overline{p}p_{\widetilde{\sigma},\widetilde{j}}\exp\langle\widetilde{\sigma}.\lambda_2, t\xi^{\perp}\rangle\,\mathrm{d}\,\xi^{\perp}=0$, and hence

$$\lim_{t\to\infty}\frac{\langle\beta_t,\beta_{t,\widetilde{\sigma},\widetilde{j}}\rangle_{L_t}}{t^{\deg p+\deg p_{\widetilde{\sigma},\widetilde{j}}+\dim l_1}}=0.$$

To summarize,

$$\begin{aligned}&\limsup_{t\to\infty}\frac{\|\int_K\varphi(\lambda)f_{*,\lambda}\,\mathrm{d}\,\mu(\lambda)\|^2}{t^{2\deg p+\dim l_1}}\\ \geq&\limsup_{t\to\infty}\frac{\|\beta_t\|_{L_t}^2+\sum\mathrm{Re}\,\langle\beta_t,\beta_{t,\sigma}\rangle_{L_t}+\sum\mathrm{Re}\,\langle\beta_t,\beta_{t,\sigma,j}\rangle_{L_t}+\sum\mathrm{Re}\,\langle\beta_t,\beta_{t,\widetilde{\sigma},\widetilde{j}}\rangle_{L_t}}{t^{2\deg p+\dim l_1}}\\ \geq&\text{positive constant,}\end{aligned}$$

and therefore we proved (4.4.5).

Appendix

Common Eigenfunctions of Invariant Differential Operators

In this appendix we are going to use the following notations. Let S denote the algebra of constant coefficient differential operators on $\mathbb{R}^n$ and P the algebra of polynomials on $\mathbb{R}^n$. We have the natural grading $S = \oplus S_n$, $P = \oplus P_n$. Let W be a finite reflection group on $\mathbb{R}^n$. W also acts on S and P, and we denote $I = S^W \cap \oplus_{n\geq 1} S_n$.

General Theory

Finite reflection groups and constant coefficient differential operators invariant under these groups were extensively studied. One can find references for the results in this subsection in [GV88, Section 5.5] and [GB85b, Chapter 7].

Theorem A.1. I is a polynomial algebra with n homogeneous generators.

Theorem A.2. All algebraic homomorphisms from I to $\mathbb{C}$ are restrictions of homomorphisms from S to $\mathbb{C}$.

Theorem A.3. S is a free I-module with $w = |W|$ homogeneous generators.

Let $D_1 = 1,\ D_2, \cdots, D_w$ be a set of homogeneous generators.

Proposition A.4. Fix $\lambda \in \hom(I, \mathbb{C})$, then there are at most w linear independent common eigenfunctions of I with eigenvalue λ.

Proof. Suppose for any $D^W \in I$,

$$D^W f = \lambda(q) f.$$

Let $x \in \mathbb{R}^n$. Consider $\overrightarrow{f}(x) = \begin{pmatrix} D_1 f(x) \\ \vdots \\ D_w f(x) \end{pmatrix}$. If $\overrightarrow{f}(x) = 0$, then $Sf(x) = 0$. Note $\Sigma \in I$. Since f is an eigenfunction of Σ, f is analytic. Thus, f is constant 0 on an open set. Since $\mathbb{R}^n$ is connected, $f \equiv 0$. ☐

Theorem A.5. For each $\lambda \in \hom(I, \mathbb{C})$, there are exactly w linearly independent eigenfunctions.

Proof. Let $\rho \colon S \longrightarrow M_w(I)$ be such that $\rho(D) \begin{pmatrix} D_1 \\ \vdots \\ D_w \end{pmatrix} = \begin{pmatrix} D \circ D_1 \\ \vdots \\ D \circ D_w \end{pmatrix}$. It is easy to check that

$$D^W f = \lambda(D^W) f \quad \forall D^W \in I \quad \iff \quad \partial_i(\overrightarrow{f}) = \lambda(\rho(\partial_i))(\overrightarrow{f}) \quad \forall 0 \leq i \leq n.$$

Now that $\rho(\partial_i)$ commutes with each other, the partial differential equations on the right always has a solution. ☐

Actually, we can write down all the eigenfunctions very explicitly if λ is regular. $D^W \in I$ means $\forall \sigma \in W$, $D^W(f) \circ \sigma = D^W(f \circ \sigma)$. Now if $D^W(f) = \lambda(D^W) f$, then $D^W(f \circ \sigma) = D^W(f) \circ \sigma = \lambda(D^W) f \circ \sigma$. Given $\lambda \in hom(S, \mathbb{C})$, one common eigenfunction of I is $\exp(\lambda(x))$, and thus the w eigenfunctions are $\exp(\sigma.\lambda(x))$.

Eigenfunctions of Singular λ

Now we consider common eigenfunctions of invariant differential operators I associated to a singular eigenvalue λ.

We first address the case when $\lambda = 0$. Define an inner product on P_n:

$$\langle \prod_i x_i^{m_i}, \prod_i x_i^{m_i'} \rangle = \prod_i \delta_{m_i, m_i'} \prod_i m_i!.$$

Here we take the convention that $0! = 1$. It is obvious that if $D \in S_n$ and $p \in P_n$, then

$$Dp = \langle \overline{\text{symbol}(D)}, p \rangle.$$

Lemma A.6. Suppose $D \in S_m$ and $p \in P_n$ with $m \leq n$. If for any $D' \in S_{n-m}$

$$D' D p = 0,$$

then

$$Dp = 0.$$

Proof. Let $\overline{\text{symbol}(D')} = Dp$. □

Proposition A.7. There are w homogeneous polynomials p_i such that

$$Ip_i = 0$$

and that $\{p_i\}$ forms a basis of P over $P^W \cap \oplus_{n\geq 1} P_n$.

Proof. Let $D_1 = 1,\ D_2, \cdots, D_w$ be a set of homogeneous generators of S over I. For any n, let F_n be the restriction to S_n the sub-I-module that is generated by D_j with $\deg D_j < n$. Let p_i be the polynomial determined by symbol$(D_i) \in P_n$ that is orthogonal to symbol(F_n). By Lemma A.6, $Ip_i = 0$. □

Now we deal with the general case. Let $W' = Stab_\lambda$ and $I' = S^{W'} \cap \oplus_{n\geq 1} S_n$.

Proposition A.8. If p is a polynomial such that

$$I'p = 0,$$

then for any $D^W \in I$,

$$D^W\Big(\exp(\lambda(x))p\Big) = \lambda(D)\Big(\exp(\lambda(x))p\Big).$$

Proof. Let $\rho_\lambda \colon S \longrightarrow S$ be the algebra homomorphism generated by

$$\rho_\lambda(X) = X + \lambda(X)$$

for any $X \in S_1$. By a straight forward calculation,

$$D\Big(\exp(\lambda(x))p\Big) = \exp(\lambda(x))\Big(\rho_\lambda(D)p\Big)$$

for any $D \in S$ and $p \in P$.

If $\sigma' \in W'$, then $[\sigma', \rho_\lambda] = 0$. $\sigma'.\rho(D^W) = \rho(\sigma'.D^W) = \rho(D^W)$, and

$$\rho_\lambda(D^W) - \lambda(D^W) \in I'.$$

Now we conclude that

$$(D^W - \lambda(D))\Big(\exp(\lambda(x))p\Big) = \exp(\lambda(x))\Big((\rho_\lambda(D^W) - \lambda(D))p\Big) = 0.$$

□

Since W' by itself is a finite reflection group, we conclude from Proposition A.7 and Proposition A.8 that there are $|W'|$ common eigenfunction of I of the form $\exp(\lambda(x))p_j$ with eigenvalue λ.

To summarize,

Theorem A.9. The common eigenfunctions of I associated to the eigenvalue λ with the stabilizer group $W' \subset W$ are

$$\exp\langle \sigma.\lambda, \cdot \rangle p_{\mathrm{Ad}_\sigma W', j}$$

where $\{p_{\widetilde{W},j}\}$ is a homogeneous basis of P over $P^{\widetilde{W}} \cap \oplus_{n\geq 1} P_n$.

Bibliography

[Ber88] J. N. Bernstein. On the support of Plancherel measure. *J. Geom. Phys.*, 5(4):663–710 (1989), 1988.

[Bor01] A. Borel. *Essays in the history of Lie groups and algebraic groups*, volume 21 of *History of Mathematics*. American Mathematical Society, Providence, RI; London Mathematical Society, Cambridge, 2001.

[CHH88] M. Cowling, U. Haagerup, and R. Howe. Almost L^2 matrix coefficients. *J. Reine Angew. Math.*, 387:97–110, 1988.

[Dix77] J. Dixmier. C^**-algebras*. North-Holland Publishing Co., Amsterdam-New York-Oxford, 1977. Translated from the French by Francis Jellett, North-Holland Mathematical Library, Vol. 15.

[Dix81] J. Dixmier. *von Neumann algebras*, volume 27 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam-New York, 1981. With a preface by E. C. Lance, Translated from the second French edition by F. Jellett.

[DS88] N. Dunford and J. T. Schwartz. *Linear operators. Part II*. Wiley Classics Library. John Wiley & Sons, Inc., New York, 1988. Spectral theory. Selfadjoint operators in Hilbert space, With the assistance of William G. Bade and Robert G. Bartle, Reprint of the 1963 original, A Wiley-Interscience Publication.

[GB85a] L. C. Grove and C. T. Benson. *Finite reflection groups*, volume 99 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1985.

[GB85b] L. C. Grove and C. T. Benson. *Finite reflection groups*, volume 99 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1985.

[GV88] Ramesh Gangolli and V. S. Varadarajan. *Harmonic analysis of spherical functions on real reductive groups*, volume 101 of *Ergebnisse der Mathematik und* *ihrer Grenzgebiete [Results in Mathematics and Related Areas]*. Springer-Verlag, Berlin, 1988.

[HC58a] Harish-Chandra. Spherical functions on a semisimple Lie group. I. *Amer. J. Math.*, 80:241–310, 1958.

[HC58b] Harish-Chandra. Spherical functions on a semisimple Lie group. II. *Amer. J. Math.*, 80:553–613, 1958.

[Hel08] S. Helgason. *Geometric analysis on symmetric spaces*, volume 39 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, second edition, 2008.

[HT16] N. Higson and Q. Tan. On a spectral theorem of Weyl. Preprint, 2016. arXiv: 1611.03396.

[Hum78] James E. Humphreys. *Introduction to Lie algebras and representation theory*, volume 9 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1978. Second printing, revised.

[Kat76] T. Kato. *Perturbation theory for linear operators*. Springer-Verlag, Berlin-New York, second edition, 1976. Grundlehren der Mathematischen Wissenschaften, Band 132.

[Kna86] Anthony W. Knapp. *Representation theory of semisimple groups*, volume 36 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1986. An overview based on examples.

[Kna02] Anthony W. Knapp. *Lie groups beyond an introduction*, volume 140 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, second edition, 2002.

[Mau67] K. Maurin. *Methods of Hilbert spaces*. Translated from the Polish by Andrzej Alexiewicz and Waclaw Zawadowski. Monografie Matematyczne, Tom 45. Państwowe Wydawnictwo Naukowe, Warsaw, 1967.

[vdB08] Erik van den Ban. Weyl, eigenfunction expansions and harmonic analysis on non-compact symmetric spaces. In *Groups and analysis*, volume 354 of *London Math. Soc. Lecture Note Ser.*, pages 24–62. Cambridge Univ. Press, Cambridge, 2008.

[War83] Frank W. Warner. *Foundations of differentiable manifolds and Lie groups*, volume 94 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1983. Corrected reprint of the 1971 edition.

[Wey10] H. Weyl. Über gewöhnliche Differentialgleichungen mit Singularitäten und die zugehörigen Entwicklungen willkürlicher Funktionen. *Math. Ann.*, 68(2):220–269, 1910.

Vita
Qijun Tan

Education:

- **Penn State Univesity** 2012 - date
  Ph.D. candidate in Mathematics
  Advisor: Nigel Higson
- **Fudan University** 2008 - 2012
  B.A. in Mathematics and Applied Mathematics

Publications:

- Asymptotically Contained Representations and Spherical Plancherel Formula, with N. Higson, In preparation
- On a Spectral Theorem of Weyl, with N. Higson, arXiv:1611.03396
- Mackey Analogy via $\mathcal{D}$-modules in the Example of $SL(2, \mathbb{R})$, with Y. Yao and S. Yu, *International Journal of Mathematics* 28: 1750055 (2017)
- Potential Function in a Continuous Dissipative Chaotic System: Decomposition Scheme and Role of Strange Attractor, with Y. Ma, R. Yuan, B. Yuan, and P. Ao, *International Journal of Bifurcation and Chaos* 24: 1450015 (2014)

Honors and Awards:

- Jack and Eleanor Pettit Scholarship in Science, Penn State University, 2017 - 2018
- Nomination, Harold F.Martin Graduate Assistant Outstanding Teaching Award (university level teaching award), Penn State University, 2016 - 2017
- August and Ruth Homeyer Graduate Fellowship, Penn State University, 2015 - 2017
- Jack and Eleanor Pettit Scholarship in Science, Penn State University, 2015 - 2016
- Vollmer-Kleckner Scholarship, Penn State University, 2014 - 2015
- Merit Award, Mathematics Department, Penn State University, June 2013
- University Graduate Fellowship, Penn State University, 2012 - 2013