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SYMPLECTIC REALIZATIONS OF NON–DEGENERATE POISSON–NIJENHUIS MANIFOLDS

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Abstract

Symplectic realization is a longstanding problem which can be traced back to Sophus Lie. In this dissertation, we present an explicit solution to this problem for an arbitrary holomorphic Poisson manifold. More precisely, for any holomorphic Poisson manifold $(\mathscr{X}, \pi)$, we prove there exists a holomorphic symplectic structure in a neighborhood Y of the zero section of the real cotangent bundle $T^\vee X$ such that the basepoint projection map is a symplectic realization of $(\mathscr{X}, \pi)$. We describe an explicit construction for such a new holomorphic symplectic structure on $Y \subset T^\vee X$.

Contents

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Chapter 1 Preliminaries

1.1 Conventions

In this dissertation, we will use the following conventions unless otherwise explicitly stated.

Let M be a real, smooth manifold. The $\mathbb{R}$-algebra of real valued smooth functions on M is denoted $\mathcal{C}^\infty(M)$. Let E be a vector bundle over M. The map $\mathrm{pr} : E \to M$ denotes the basepoint projection. The fibre of E at a point $m \in M$ is denoted E_m, so that $E_m = \mathrm{pr}^{-1}(m)$. The $\mathcal{C}^\infty(M)$-module of sections of E is denoted $\Gamma(E)$. The k-fold exterior tensor power of E is denoted $\wedge^k E$. By convention, $\wedge^0 E = M \times \mathbb{R}$ so that $\Gamma(\wedge^0 E) = \mathcal{C}^\infty(M)$. The $\mathcal{C}^\infty(M)$-module $\Gamma(\wedge^k E)$ is naturally isomorphic to $\wedge^k \Gamma(E)$, the $\mathcal{C}^\infty(M)$-module generated by elements of the form $s_1 \wedge \cdots \wedge s_k$ for $s_1, \ldots, s_k \in \Gamma(E)$.

Let n be the rank of the vector bundle E. The direct sum of vector bundles

$$\wedge^\bullet E := \bigoplus_{k=0,\ldots,n} \wedge^k E$$

is a bundle of graded anti-commutative $\mathbb{R}$-algebras over M with product given by the fibrewise exterior product. The $\mathcal{C}^\infty(M)$-module $\Gamma(\wedge^\bullet E)$ is canonically isomorphic to the direct sum

$$\Gamma^\bullet(E) := \bigoplus_{k=0,\ldots,n} \wedge^k \Gamma(E)$$

and is naturally a graded anti-commutative $\mathcal{C}^\infty(M)$-algebra, called the *exterior algebra of sections of E*. The degree of a homogeneous element X in $\Gamma^\bullet(E)$ is denoted $\mid X \mid$.

The $\mathbb{R}$-dual vector bundle of E is denoted $E^\vee$. There is a natural non-degenerate $\mathcal{C}^\infty(M)$-linear pairing

$$\langle \cdot, \cdot \rangle : \Gamma(E^\vee) \otimes \Gamma(E) \to \mathcal{C}^\infty(M)$$

induced by the fibrewise natural pairing $E_m^\vee \times E_m \to \mathbb{R}$ $(m \in M)$. More precisely for any $\lambda \in \Gamma(E^\vee)$ and any $X \in \Gamma(E)$:

$$\langle \lambda, X \rangle(x) := \lambda_x(X_x), \ \forall x \in M.$$

This pairing naturally extends to a non-degenerate $\mathcal{C}^\infty(M)$-linear pairing

$$\langle \cdot, \cdot \rangle : \Gamma(\wedge^\bullet E^\vee) \otimes \Gamma(\wedge^\bullet E) \to \mathcal{C}^\infty(M) \tag{1.1}$$

as follows. Let k be a non-negative natural integer with $1 \leq k \leq n$. Without loss of generality, let $\lambda \in \Gamma(\wedge^k E^\vee)$ and $X \in \Gamma(\wedge^k E)$ be such that

$$\lambda = \lambda_1 \wedge \cdots \wedge \lambda_k \ \text{ and } \ X = X_1 \wedge \cdots \wedge X_k,$$

where λ_i (resp. X_i) is in $\Gamma(E^\vee)$ (resp. $\Gamma(E)$) for any given i. Define

$$\langle \lambda, X \rangle := \frac{1}{k!} \sum_{\sigma \in P_k} (-1)^{\mathrm{sign}(\sigma)} \langle \lambda_i, X_{\sigma(i)} \rangle \tag{1.2}$$

where P_k is the group of permutations of $\{1, ..., k\}$ and

$$\mathrm{sign} : P_k \to \{-1, 1\}$$

is the signature map. For any $\lambda \in \Gamma(\wedge^k E^\vee)$ and any $X \in \Gamma(\wedge^{k'} E)$ with $k \neq k'$, we define $\langle \lambda, X \rangle = 0$. Finally, for f and g in $\mathcal{C}^\infty(M)$, we simply let $\langle f, g \rangle = fg$. It is classical that the pairing $\Gamma(\wedge^k E^\vee) \otimes \Gamma(\wedge^k E) \to \mathcal{C}^\infty(M)$ defined by (1.2) is non-degenerate. It follows that the $\mathcal{C}^\infty(M)$-module $\Gamma(\wedge^k E^\vee)$ is naturally isomorphic to the dual of $\Gamma(\wedge^k E)$ and that the pairing $\langle \cdot, \cdot \rangle : \Gamma(\wedge^\bullet E^\vee) \otimes \Gamma(\wedge^\bullet E) \to \mathcal{C}^\infty(M)$ is non-degenerate.

The tangent bundle of M is denoted TM. The fibre of TM over a fixed $m \in M$ is denoted $T_m M$ and called the *tangent space of M at m*. The $\mathcal{C}^\infty(M)$-module of sections of TM is denoted $\mathfrak{X}(M)$ and its elements are called *vector fields over M*. The $\mathcal{C}^\infty(M)$-module of derivations of $\mathcal{C}^\infty(M)$ is denoted $Der(M)$. There is a natural isomorphism $Der(M) \cong \mathfrak{X}(M)$. The value of a vector field X at a point m in M is denoted X_m or $X|_m$. The sections of $\wedge^k TM$ are called *k-multivector fields over M*. The collection of all k-multivector fields over M is denoted $\mathfrak{X}^k(M)$. Sections of $\wedge^2 TM$ are called *bivector fields over M*. The exterior algebra of sections of TM is denoted $\mathfrak{X}^\bullet(M)$ and is called the *algebra of multivector fields over M*.

The $\mathbb{R}$-dual vector bundle of TM is denoted $T^\vee M$ and is called the *cotangent bundle of M*. The fibre of $T^\vee M$ over a fixed $m \in M$ is denoted $T_m^\vee M$ and is called the *cotangent space of M at m*. The $\mathcal{C}^\infty(M)$-module $\Gamma(\wedge^k T^\vee M)$ is abbreviated $\Omega^k(M)$. An element $\omega \in \Omega^k(M)$ is called a *differential k-form on M* and its value at a particular point $m \in M$ is denoted ω_m. The exterior algebra of sections of $T^\vee M$ is denoted $\Omega^\bullet(M)$ and is called the *algebra of differential forms on M*.

Let p and q be natural integers. The sections of the vector bundle $\bigotimes^p TM \otimes \bigotimes^q T^\vee M$ are called *tensors of type* (p, q) *on* M or (p, q)*-tensors on* M. In particular, a $(1, 1)$-tensor on M is equivalent to a bundle map $TM \to TM$.

Let X and Y be vector fields over M. The *Lie bracket of* X *and* Y is denoted $[X, Y]$ and is defined by $[X, Y] = XY - YX$, the commutator of X and Y viewed as derivations of $\mathcal{C}^\infty(M)$. The *Lie derivative in the direction of* X, denoted $\mathcal{L}_X$, is the operator $\mathcal{L}_X : \mathfrak{X}(M) \to \mathfrak{X}(M)$ defined by

$$\mathcal{L}_X Y = [X, Y].$$

It extends naturally to an $\mathbb{R}$-linear operator $\mathcal{L}_X : \mathfrak{X}^\bullet(M) \to \mathfrak{X}^\bullet(M)$ satisfying the graded Leibniz identity

$$\mathcal{L}_X(Y_1 \wedge \cdots \wedge Y_k) = \sum_{i=1,\ldots,k} (-1)^{i-1} Y_1 \wedge \cdots \wedge \mathcal{L}_X(Y_i) \wedge \cdots \wedge Y_k \tag{1.3}$$

for $Y_1, \ldots, Y_k$ in $\mathfrak{X}(M)$. There is also a unique operator

$$\mathcal{L}_X : \Omega^\bullet(M) \to \Omega^\bullet(M)$$

satisfying the identity

$$\mathcal{L}_X(\omega_1 \wedge \cdots \wedge \omega_k) = \sum_{i=1,\ldots,k} (-1)^{i-1} \omega_1 \wedge \cdots \wedge \mathcal{L}_X(\omega_i) \wedge \cdots \wedge \omega_k \tag{1.4}$$

for $\omega_1, \ldots, \omega_k$ in $\Omega(M)$, and such that

$$\langle \mathcal{L}_X \alpha, Y \rangle = X\langle \alpha, Y \rangle - \langle \alpha, \mathcal{L}_X Y \rangle$$

for $X, Y \in \mathfrak{X}(M)$ and $\alpha \in \Omega(M)$. Finally, let N be a $(1, 1)$-tensor on M and fix $X \in \mathfrak{X}(M)$. We define a $(1, 1)$-tensor $\mathcal{L}_X N$ on M by the formula

$$(\mathcal{L}_X N)(Y) = \mathcal{L}_X N(Y) - N(\mathcal{L}_X Y).$$

All those versions of the operator $\mathcal{L}_X$ will be called Lie derivative (in the direction of X) and share the base same notation.

The Lie bracket of vector fields over M can be extended to a bracket on $\mathfrak{X}^\bullet(M)$. Indeed, there is a unique graded skew-symmetric bracket

$$[\![\cdot, \cdot]\!] : \mathfrak{X}^\bullet(M) \times \mathfrak{X}^\bullet(M) \to \mathfrak{X}^\bullet(M) \tag{1.5}$$

such that

$$\begin{aligned}
X \in \mathfrak{X}^k(M),\ Y \in \mathfrak{X}^l(M) &\implies [\![X, Y]\!] \in \mathfrak{X}^{k+l-1}(M) \\
X \in \mathfrak{X}(M),\ Y \in \mathfrak{X}(M) &\implies [\![X, Y]\!] = [X, Y] \\
X \in \mathfrak{X}(M),\ f \in \mathfrak{X}^0(M) &\implies [\![X, f]\!] = X(f) \\
f \in \mathfrak{X}^0(M),\ g \in \mathfrak{X}^0(M) &\implies [\![f, g]\!] = 0
\end{aligned}$$

and satisfying the identities

$$[\![X, Y \wedge Y']\!] = [\![X, Y]\!] \wedge Y' + (-1)^{(|X|-1)|Y|} Y \wedge [\![X, Y']\!] \tag{1.6}$$

and

$$[\![X, [\![Y, Z]\!]]\!] = [\![[\![X, Y]\!], Z]\!] + (-1)^{(|X|-1)(|Y|-1)} [\![Y, [\![X, Z]\!]]\!] \tag{1.7}$$

where X, Y, Z and Y' are homogeneous elements in $\mathfrak{X}^\bullet(M)$. The bracket $[\![\cdot, \cdot]\!]$ is called the *Schouten-Nijenhuis bracket of multivector fields over M*. For later reference, we note that (1.6) is called the graded Leibniz identity and (1.7) is called the graded Jacobi identity.

Let A be an algebra over a field k, with $k = \mathbb{R}$ or $k = \mathbb{C}$. A *Poisson bracket* on A is a k-bilinear skew-symmetric bracket $\{\cdot, \cdot\} : A \times A \to A$ such that

$$\{f, gh\} = \{f, g\}h + g\{f, h\}, \tag{1.8}$$

$$\{f, \{g, h\}\} = \{\{f, g\}, h\} + \{g, \{f, h\}\}, \tag{1.9}$$

for any f, g, h in A. For later reference we recall that (1.8) is called the Leibniz identity and (1.9) is called the Jacobi identity. An algebra A endowed with a Poisson bracket $\{\cdot, \cdot\}$ is called a *Poisson algebra*.

Let M be a smooth manifold. Its algebra $\mathcal{C}^\infty(M)$ of functions is an algebra over the field $\mathbb{R}$. By a Poisson bracket on M we mean a Poisson bracket on the $\mathbb{R}$-algebra $\mathcal{C}^\infty(M)$. Let $\{\cdot, \cdot\}$ be a Poisson bracket on M. There exists a unique bivector field π on M such that

$$\langle df \wedge dg, \pi \rangle = \{f, g\} \tag{1.10}$$

for all f, g in $\mathcal{C}^\infty(M)$. We call π the bivector field associated to $\{\cdot, \cdot\}$. The Jacobi identity (1.9) for the bracket $\{\cdot, \cdot\}$ is equivalent the condition $[\![\pi, \pi]\!] = 0$. Any bivector field π on M such that $[\![\pi, \pi]\!] = 0$ is called a *Poisson* bivector field. In particular, any Poisson bracket defines a Poisson bivector field. Conversely, any Poisson bivector field defines a unique Poisson bracket such that (1.10) holds.

Let M be a smooth manifold and $d_{DR} : \Omega^\bullet(M) \to \Omega^\bullet(M)$ be the De Rham differential. A differential 2-form ω on M is called *symplectic* if $d_{DR}\omega = 0$ and ω is non-degenerate.

Let π be a bivector field on M. Let m in M and fix $\lambda \in T^\vee_m M$. We define an element $\pi^\sharp(\lambda)$ in $T_m M$ by the equation

$$\langle \nu, \pi^\sharp(\lambda) \rangle = \langle \lambda \wedge \nu, \pi|_m \rangle$$

for any $\nu \in T^\vee_m M$. The correspondence $\lambda \mapsto \pi^\sharp(\lambda)$ defines a bundle map

$$\pi^\sharp : T^\vee M \to TM.$$

Dually, any differential 2-form ω on M defines a bundle map

$$\omega^\flat : TM \to T^\vee M.$$

If ω is a symplectic differential 2-form then $\omega^\flat$ is invertible (as a bundle map) and there exists a unique Poisson bivector field π such that $\pi^\sharp = (\omega^\flat)^{-1}$. Conversely, let π be a Poisson bivector field on M. We say π is *non-degenerate* if the bundle map $\pi^\sharp$ is invertible. In that case, there exists a unique symplectic differential 2-form ω on M such that $\omega^\flat = (\pi^\sharp)^{-1}$.

1.2 Poisson-Nijenhuis Structures

In this section, we review some standard facts regarding Poisson-Nijenhuis structures ('PN structures' for short) on smooth manifolds.

Poisson-Nijenhuis structures are pairs (π, N) consisting of a Poisson bivector field π and a compatible Nijenhuis tensor $N : TM \to TM$. They naturally arise in the theory of deformations of Lie brackets and constitute a natural framework for the theory of completely integrable systems [1]. Further, they yield entire hierarchies of deformed Poisson brackets obtained by twisting the original Poisson structure with N.

1.2.1 Lie Algebroids

In order to get a conceptual sense of PN structures, we need to introduce Lie algebroids.

Let M be a smooth manifold. A *Lie algebroid over* M is a triple $(E, \rho, [\cdot,\cdot])$ consisting of a vector bundle $E \to M$ together with a vector bundle map $\rho : E \to TM$, called the *anchor*, and a skew-symmetric bracket

$$[\cdot,\cdot] : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$$

such that the Jacobi identity

$$[\omega, [\nu, \lambda]] + [\lambda, [\omega, \nu]] + [\nu, [\lambda, \omega]] = 0 \tag{1.11}$$

the Leibniz identity

$$[\omega, f\nu] = (\rho(\omega)f)\nu + f[\omega, \nu] \tag{1.12}$$

hold for any $\omega, \nu, \lambda \in \Gamma(E)$ and any $f \in \mathcal{C}^\infty(M)$.

Example 1.2.1. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{R}$ with Lie bracket $[\cdot,\cdot]_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. Then the triple $(\mathrm{pt} \times \mathfrak{g}, \rho, [\cdot,\cdot]_{\mathfrak{g}})$, with $\rho : \mathfrak{g} \to T\mathrm{pt} = 0_{\mathrm{pt}}$ the zero map, is a Lie algebroid over $M = \mathrm{pt}$.

Example 1.2.2. Let M be a smooth manifold and $\mathfrak{g}$ be a Lie algebra with Lie bracket $[\cdot,\cdot]_{\mathfrak{g}}$. Define E as the trivial bundle over M with fibre $\mathfrak{g}$, i.e. $E := M \times \mathfrak{g}$. Then $\Gamma(E) = \mathcal{C}^\infty(M) \otimes \mathfrak{g}$ and we can define a Lie bracket

$$[\cdot,\cdot]_{\mathfrak{g}}^{\sim} : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$$

as the unique $\mathcal{C}^\infty(M)$-bilinear extension of $[\cdot,\cdot]_{\mathfrak{g}}$. Also, let $\rho : E \to TM$ be the zero map $\rho(e) = 0_{\mathrm{pr}(e)}$. Then the triple $(E, \rho, [\cdot,\cdot]_{\mathfrak{g}}^{\sim})$ is a Lie algebroid over M. From a more general point of view, the pair $(E, [\cdot,\cdot]_{\mathfrak{g}}^{\sim})$ is an example of a bundle of Lie algebras over M and it is easily seen that any such bundle is a Lie algebroid in a natural way.

Example 1.2.3. Let M be a smooth manifold. Let $E = TM$ and let $[\cdot,\cdot]$ be the Lie bracket of vector fields over M. Also, let $\rho = \mathrm{id}$ be the identity over TM. Then the triple $(E, \rho, [\cdot,\cdot])$ is a Lie algebroid called the *tangent Lie algebroid of M*.

The most relevant examples of Lie algebroids arise from a class of brackets outside of Example 1.2.2 and Example 1.2.3. To see this, let π be a Poisson bivector field on M. One can construct a Lie bracket on $\Omega^1(M)$, called the *Koszul bracket*, as follows. Let $\alpha, \beta \in \Omega^1(M)$ and define $\{\alpha,\beta\}_\pi^\vee$ by requiring that the identity

$$\langle \{\alpha,\beta\}_\pi^\vee, X\rangle = \langle \mathcal{L}_{\pi^\sharp\alpha}\beta, X\rangle - \langle \mathcal{L}_{\pi^\sharp\beta}\alpha, X\rangle + X\langle \alpha, \pi^\sharp\beta\rangle$$

holds for any X in $\mathfrak{X}(M)$. It is easily seen that the right hand side is $\mathcal{C}^\infty(M)$-linear in X and thus, *a fortiori*, that $\{\cdot,\cdot\}_\pi^\vee$ is a well defined bracket on $\Omega^1(M)$. It is also straightforward to check that the identities

$$\{\alpha, f\beta\}_\pi^\vee = f\{\alpha,\beta\}_\pi^\vee + (\pi^\sharp\alpha f)\beta, \tag{1.13}$$

and

$$\{df, dg\}_\pi^\vee = -d\{f,g\}_\pi \tag{1.14}$$

hold for any f and g in $\mathcal{C}^\infty(M)$ and any α and β in $\Omega^1(M)$. Here the bracket $\{\cdot,\cdot\}_\pi$ appearing in the right hand side of Identity 1.14 is the Poisson bracket defined by π.

An easy consequence of (1.14) is that the bracket $\{\cdot,\cdot\}_\pi^\vee$ satisfies the Jacobi identity on exact 1-forms. From (1.13) it then follows that the triple $(T^\vee M, \rho_\pi^\vee, \{\cdot,\cdot\}_\pi^\vee)$ is a Lie algebroid over M, where the anchor $\rho_\pi^\vee$ is the opposite of the Poisson contraction $\pi^\sharp : T^\vee M \to TM$. This Lie algebroid is called the *cotangent bundle Lie algebroid of* (M,π) and is denoted E_π.A lot of classical differential geometric notions can be extended to Lie algebroids, an example of which is given in the next definition.

Definition 1.2.1. Let E be a Lie algebroid over M with Lie bracket $[\cdot,\cdot]$ and anchor ρ. The *differential* of $(E,\rho,[\cdot,\cdot])$ is the operator $d : \Gamma(\wedge^\bullet E^\vee) \to \Gamma(\wedge^\bullet E^\vee)$ defined by the equation

$$\begin{aligned}\langle d\lambda, X\rangle = &\sum_{i=1}^{k+1}(-1)^{i+1}\rho(X_i)\langle \lambda, X_1\wedge\cdots\wedge\hat{X}_i\wedge\cdots\wedge X_{k+1}\rangle\\ &+\sum_{i<j}(-1)^{i+j}\langle\lambda,[X_i,X_j]\wedge X_1\wedge\cdots\wedge\hat{X}_i\wedge\cdots\wedge\hat{X}_j\wedge\cdots\wedge X_{k+1}\rangle,\end{aligned}$$

for any λ in $\Gamma(\wedge^k E^\vee)$ and any X in $\Gamma(\wedge^{k+1}E)$ such that $X = X_1\wedge\cdots\wedge X_{k+1}$ where $X_i \in \Gamma(E)$ $(i = 1,\ldots,k+1)$.

Example 1.2.4. Let M be a smooth manifold. The differential of the tangent Lie algebroid of M is the De Rham exterior derivative.

Example 1.2.5. Assume M is endowed with a Poisson bivector field π. The differential of the cotangent bundle Lie algebroid of (M,π) is the classical Poisson-Lichnerowiz differential $d = [\pi,\cdot] : \mathfrak{X}^\bullet(M) \to \mathfrak{X}^\bullet(M)$.

The construction of the Schouten-Nijenhuis bracket (1.5) generalizes trivially to Lie algebroids. Namely, let E be a Lie algebroid over M with bracket $[\cdot,\cdot]$ and anchor ρ. It is straightforward to check that there is a unique extension of $[\cdot,\cdot]$ to a graded skew-symmetric bracket

$$[\![\cdot,\cdot]\!] : \Gamma^\bullet(E)\times\Gamma^\bullet(E)\to\Gamma^\bullet(E)$$

satisfying the graded Jacobi identity (1.7) and the graded Leibniz identity (1.6) together with the following properties:

$$\begin{aligned}X\in\Gamma(\wedge^k E),\, Y\in\Gamma(\wedge^l E) &\implies [\![X,Y]\!]\in\Gamma(\wedge^{k+l-1}E)\\ X\in\Gamma(E),\, Y\in\Gamma(E) &\implies [\![X,Y]\!] = [X,Y]\\ X\in\Gamma(E),\, f\in\mathcal{C}^\infty(M) &\implies [\![X,f]\!] = \rho(X)(f)\\ f\in\mathcal{C}^\infty(M),\, g\in\mathcal{C}^\infty(M) &\implies [\![f,g]\!] = 0\end{aligned}$$

Before we close this section, let us mention the following definition.

Definition 1.2.2. Let $(E,[\cdot,\cdot],\rho)$ and $(E',[\cdot,\cdot]',\rho')$ be two Lie algebroids over the same base manifold M. A bundle map $f : E\to E'$ such that $\mathrm{pr}\circ f = \mathrm{id}$ is called a *morphism of Lie algebroids* if $\rho'\circ f = \rho$ and, for any X, Y in $\Gamma(E)$, we have

$$[f\circ X, f\circ Y]' = f\circ[X,Y].$$

1.2.2 Nijenhuis Tensors and Deformed Lie Brackets

Let us now turn to the 'N' in 'PN structures', namely the so-called Nijenhuis tensor. Let M be, as before, a smooth manifold and fix E a Lie algebroid over M with bracket $[\cdot,\cdot]$ and anchor ρ.

Definition 1.2.3. Let $N : E \to E$ be an endomorphism of vector bundles over M. The *Nijenhuis torsion* of N is the operator

$$T_N : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$$

defined by

$$T_N(X,Y) = [NX, NY] - N([NX, Y] + [X, NY]) + N^2[X,Y] \tag{1.15}$$

for X and Y in $\Gamma(E)$.

Remark 1.2.1. It is easily checked that (1.15) is $\mathcal{C}^\infty(M)$-bilinear and skew-symmetric in X and Y. It follows that T_N can be naturally viewed as a section of the vector bundle $\wedge^2 E^\vee \otimes E$.

Definition 1.2.4. Let E be a Lie algebroid over M. Let N be an endomorphism of E over M. We say that N is a *Nijenhuis operator* if its Nijenhuis torsion vanishes identically over M.

Remark 1.2.2. In the particular case where E is the tangent Lie algebroid of M, a Nijenhuis operator $N : TM \to TM$ is also called a Nijenhuis tensor on M.

Let E be a Lie algebroid over M with bracket $[\cdot,\cdot]$ and anchor ρ. Assume $N : E \to E$ is an endomorphism of vector bundles over M (not necessarily a Nijenhuis operator). We define [1] an N-deformed bracket $[\cdot,\cdot]_N$ as follows. For any $\epsilon \geq 0$, we first define a bracket $[\cdot,\cdot]_\epsilon$ by the expression

$$[X,Y]_\epsilon = \exp(-\epsilon N)\left([\exp(\epsilon N)(X), \exp(\epsilon N)(Y)]\right). \tag{1.16}$$

where $\exp(\epsilon N) : E \to E$ is the isomorphism of vector bundles exponentiating ϵN fibrewise. Trivially, $[\cdot,\cdot]_\epsilon$ is isomorphic to $[\cdot,\cdot]$. By Taylor expanding (1.16) at $\epsilon = 0$ we get the identity

$$[X,Y]_\epsilon = [X,Y] + [X,Y]_N + (\text{"higher order terms"})$$

where

$$[X,Y]_N = [NX, Y] + [X, NY] - N[X,Y]. \tag{1.17}$$

The bracket in (1.17) is an "N-twisted" version of the original bracket $[\cdot,\cdot]$ on $\Gamma(E)$.

Let $\rho_N := \rho \circ N$. It is easy to check that the Leibniz identity (1.12) is satisfied for $[\cdot,\cdot]_N$ and ρ_N. However, the Jacobi identity (1.11) is not automatic for $[\cdot,\cdot]_N$ and thus the triple $(E, \rho_N, [\cdot,\cdot]_N)$ is not necessarily a Lie algebroid. The following theorem gives a sufficient condition for (1.11) to hold. The proof is classical and will be omitted here for brevity [1].

Theorem 1.2.1. *Let M be a smooth manifold and E a Lie algebroid over M with Lie bracket $[\cdot,\cdot]$ and anchor ρ. Let $N : E \to E$ be a Nijenhuis operator on E. Then the bracket $[\cdot,\cdot]_N$ defined in (1.17) satisfies the Jacobi identity. In particular, the triple $(E, \rho_N, [\cdot,\cdot]_N)$ is a Lie algebroid over M.*

1.2.3 The Compatibility Condition and Lie Bialgebroids

In order to concisely state the compatibility condition between the Poisson bivector and the Nijenhuis tensor at the centre of a Poisson-Nijenhuis structure, we need to introduce the notion of *Lie bialgebroids*.

By a *pair of Lie algebroids in duality* we mean a pair $(E, E^\vee)$ where E and $E^\vee$ are both Lie algebroids over M. Lie bialgebroids are particular pairs of Lie algebroids in duality that satisfy a certain compatibility condition with respect to their mutual Lie brackets. They were introduced by Mackenzie-Xu [2] in their seminal work on extending the correspondence between Lie bialgebras and Poisson Lie groups to Poisson Lie *groupoids* (see Section **??**).

Definition 1.2.5. Let $(E, E^\vee)$ be a pair of Lie algebroids in duality. Let $[\cdot,\cdot]$ (resp. ρ) be the bracket (resp. anchor) of E and let $[\cdot,\cdot]^\vee$ (resp. $\rho^\vee$) be the bracket (resp. anchor) of $E^\vee$. Let

$$d : \Gamma^\bullet(E^\vee) \to \Gamma^\bullet(E^\vee) \quad (\text{resp. } d^\vee : \Gamma^\bullet(E) \to \Gamma^\bullet(E))$$

denote the differential of E (resp. $E^\vee$) and let $[\![\cdot,\cdot]\!]$ (resp. $[\![\cdot,\cdot]\!]^\vee$) be the natural extension of $[\cdot,\cdot]$ (resp. $[\cdot,\cdot]^\vee$) to $\Gamma^\bullet(E)$ (resp. $\Gamma^\bullet(E^\vee)$). We say the pair $(E, E^\vee)$ is a *Lie bialgebroid* if the equation

$$d^\vee[\![X, Y]\!] = [\![d^\vee X, Y]\!] + [\![X, d^\vee Y]\!] \tag{1.18}$$

holds for any X and Y in $\Gamma(E)$.

Remark 1.2.3. An operator $d^\vee$ which satisfies (1.18) is called a degree $+1$ (graded) derivation of the (graded) Lie bracket $[\![\cdot,\cdot]\!]$. Because $d^\vee$ is automatically a (graded) derivation of the (graded) algebra structure on $\Gamma^\bullet(E)$ (see Proposition **??**), it follows that Definition 1.2.5 is equivalent to asking that $d^\vee$ be a (graded) derivation of the Gerstenhaber algebra $(E, [\![\cdot,\cdot]\!], \wedge)$. This is a common way of defining Lie bialgebroids in the literature [3].

Remark 1.2.4. It is a non-trivial result of Mackenzie-Xu (Theorem 3.10 in [2]) that Definition 1.2.5 is symmetric in the roles of E and $E^\vee$. More precisely, the pair $(E, E^\vee)$ is a Lie bialgebroid if and only if the pair $(E^\vee, E)$ is as well.

Example 1.2.6. Let M be a smooth manifold and π be a Poisson bivector field on M. Let E_π be the cotangent bundle Lie algebroid of (M, π), as defined in the previous section. Recall that, as vector bundles, $E_\pi = T^\vee M$. Consider the tangent Lie algebroid $(TM, [\cdot,\cdot], \mathrm{id})$. Obviously, the pair (E_π, TM) is a pair of Lie algebroids in duality. Let δ be the Lie algebroid differential of TM, which acts on $\Gamma(\wedge^\bullet E_\pi)$. It is easy to see that δ identifies with the De Rham differential and from the latter fact it is easy to check that the pair (E_π, TM) is indeed a Lie bialgebroid.

Definition 1.2.6. Let $(E, E^\vee)$ and $(E', E'^\vee)$ be two Lie bialgebroids such that E and E' are Lie algebroids over the same base manifold M. A bundle map $f : E \to E'$ such that $\mathrm{pr} \circ f = \mathrm{id}$ is called a *morphism of Lie bialgebroids* if

– the map $f : E \to E'$ is a morphism of Lie algebroids and,

– the dual map $f^\top : E'^\vee \to E^\vee$ is a morphism of Lie algebroids.

We are now ready to define Poisson-Nijenhuis structures. For this purpose, let M be a smooth manifold and $N : TM \to TM$ a Nijenhuis operator on TM (viewed as the tangent Lie algebroid). Define the bracket $[\cdot,\cdot]_N$ as in (1.17) from twisting the Lie bracket $[\cdot,\cdot]$ of vector fields with N. From Theorem 1.2.1, it follows that the triple $(TM, N, [\cdot,\cdot]_N)$ is a Lie algebroid over M. Let $(TM)_N$ denote that Lie algebroid and $d_N : \Omega^\bullet(M) \to \Omega^\bullet(M)$ the associated differential.

Now let π be a Poisson bivector field on M. Let $\{\cdot,\cdot\}^\vee_\pi$ be the Koszul bracket on $\Omega^1(M)$ and define $\rho^\vee_\pi := \pi^\sharp : T^\vee M \to TM$. From the discussion in the previous section it follows that the triple $(T^\vee M, \rho^\vee_\pi, \{\cdot,\cdot\}^\vee_\pi)$ is a Lie algebroid, called the cotangent bundle Lie algebroid of (M, π). Let E_π denote the latter. Note that the pair $(E_\pi, (TM)_N)$ is a pair of Lie algebroids in duality.

Definition 1.2.7. Let M be a smooth manifold endowed with a Poisson bivector field π and a Nijenhuis tensor $N : TM \to TM$. The pair (π, N) is called a *Poisson-Nijenhuis (PN) structure* on M if the pair $(E_\pi, (TM)_N)$ is a Lie bialgebroid.

Remark 1.2.5. We should add a brief note on the history of Definition 1.2.7. The original definition of PN structures goes back to the work of Magri-Morosi [4] where it was defined by the vanishing of a rather complicated tensor quantity. In 1994, Mackenzie-Xu [2] introduced Lie bialgebroids and in 1996, Kosmann-Schwarzbach [5] proved that the vanishing of this quantity was equivalent to a certain form of the Lie bialgebroid condition (1.18).

1.2.4 Deforming the Poisson Bracket on a PN Manifold

An important feature of PN structures is that they allow us to deform the original Poisson bracket as well. To see this, we first recall the following classical fact (see [2]).

Proposition 1.2.1. *Let M be a smooth manifold and let $(E, E^\vee)$ be a Lie bialgebroid over M. Let ρ (resp. $\rho^\vee$) denote the anchor of E (resp. $E^\vee$). There is a unique Poisson bracket $\{\cdot,\cdot\}_{E,E^\vee}$ on M satisfying the equation*

$$\{f, g\}_{E,E^\vee} = \langle df, (\rho \circ (\rho^\vee)^\top)(dg)\rangle$$

for any f and g in $\mathcal{C}^\infty(M)$, where $(\rho^\vee)^\top : T^\vee M \to E$ is the transpose of $\rho^\vee$.

Applying Proposition 1.2.1 to the Lie bialgebroid associated to a PN structure gives the following corollary.

Corollary 1.2.1. *Let M be a smooth manifold endowed with a Poisson bivector field π and a Nijenhuis tensor $N : TM \to TM$ such that the pair (π, N) is a Poisson-Nijenhuis structure on M. There is a unique Poisson bivector field π_N on M such that*

$$\langle \pi_N, \alpha \otimes \beta \rangle = -\langle \beta, (N \circ \pi^\sharp)(\alpha) \rangle$$

for any α and β in $\Omega(M)$.

Let M be a smooth manifold, π be a Poisson bivector field on M and N a Nijenhuis tensor on M. From Corollary 1.2.1 it follows that, when the pair (π, N) is a Poisson-Nijenhuis structure on M, there exists a unique Poisson bivector field π_N such that $\pi_N^\sharp = N \circ \pi^\sharp$. The next classical theorem [6] can be viewed as a partial converse.

Theorem 1.2.2. *Let M be a smooth manifold, π be a Poisson bivector field on M and N a $(1,1)$-tensor on M. Let π_N be the $(0,2)$-tensor on M defined by the equation*

$$\langle \lambda \otimes \mu,\, \pi_N \rangle = \langle \lambda, N(\pi^\sharp(\mu)) \rangle,$$

for all λ, μ in $\Gamma(T^\vee M)$. Then π_N is skew-symmetric if and only if $N \circ \pi^\sharp = \pi^\sharp \circ N^\top$. In that case, we also have the following assertions:

- *The bivector field π_N is Poisson if and only if N is a Nijenhuis tensor.*
- *Assume N is a Nijenhuis tensor. Then the Schouten-Nijenhuis bracket $[\![\pi, \pi_N]\!]$ vanishes if the pair (π, N) is a Poisson-Nijenhuis structure on M, and the converse holds if π is non-degenerate.*

The following theorem is a result (Corollary 1.5) of Vaisman [6].

Theorem 1.2.3. *Let M be a smooth manifold and let π and π' be Poisson bivector fields on M. Assume π is non-degenerate and let N be the $(1,1)$-tensor defined by the equation $N = \pi'^\sharp \circ (\pi^\sharp)^{-1}$. If the Schouten-Nijenhuis bracket $[\![\pi, \pi']\!]$ vanishes, then the pair (π, N) is a PN structure on M.*

The next theorem is a result of Kosmann-Schwarzbach [5].

Theorem 1.2.4. *Let M be a smooth manifold and let (π, N) be a PN structure on M. Let $(TM)_N$ denote the N-twisted tangent Lie algebroid. Let π_N be the Poisson bivector field defined by the equation $\pi_N^\sharp = N \circ \pi^\sharp$ and let E_π (resp. E_{π_N}) denote the cotangent bundle Lie algebroid of (M, π) (resp. (M, π_N)). The maps*

$$N : (TM)_N \to TM, \quad N^\top : E_{\pi_N} \to E_\pi$$

are morphisms of Lie algebroids. Here, TM is viewed as the tangent Lie algebroid.

In particular, the dual pair of maps $(N^\top, N)$ is a morphism of Lie bialgebroids from the pair (E_{π_N}, TM) to the pair $(E_\pi, (TM)_N)$.

1.3 Symplectic Realizations and Symplectic Groupoids

1.3.1 Symplectic Realizations of Poisson Manifolds

Sophus Lie [7] defines a "function group" as a subset $\mathcal{F} \subset \mathcal{C}^\infty(\mathbb{R}^{2n})$ such that

– the collection $\mathcal{F}$ is a subalgebra under the canonical Poisson bracket on $\mathbb{R}^{2n}$, i.e. $\{\mathcal{F}, \mathcal{F}\}_{\text{can}} \subset \mathcal{F}$; and

– it is generated by a finite number of independent functions that belong to $\mathcal{F}$, i.e. there exist functions $\phi_1, \ldots, \phi_r$ in $\mathcal{F}$ such that for any $f \in \mathcal{F}$, there is a smooth function $F : \mathbb{R}^r \to \mathbb{R}$ satisfying $f = F(\phi_1, \ldots, \phi_r)$.

These two conditions together guarantee that there exist functions $\pi_{ij} : \mathbb{R}^r \to \mathbb{R}$ such that $\{\phi_i, \phi_j\} = \pi_{ij}(\phi_1, \ldots, \phi_r)$, for any $i, j = 1, \ldots, r$. In modern terms, this endows $\mathbb{R}^r$ with the structure of a Poisson manifold, with Poisson bivector field $\pi = (\pi_{ij})_{ij}$, such that the map

$$\Phi : (\phi_1, \ldots, \phi_r) : \mathbb{R}^{2n} \to \mathbb{R}^r$$

is a Poisson map. This construction yields an instance of a so-called *symplectic realization* in the following sense.

Definition 1.3.1. Let M be a smooth manifold endowed with a Poisson bivector field π. A *symplectic realization* of M is a triple (W, Ω, q) consisting of a smooth manifold W endowed with a symplectic 2-form Ω and a surjective submersion $q : W \twoheadrightarrow M$ such that

$$q_* \Pi = \pi,$$

where Π is the Poisson bivector field defined by the equation $\Pi^\sharp = (\Omega^\flat)^{-1}$.

Let M be Poisson manifold with Poisson bracket $\{\cdot, \cdot\}$. Conceptually a symplectic realization of M amounts to an embedding of the Poisson algebra $(\mathcal{C}^\infty(M), \{\cdot, \cdot\})$ into the "symplectic algebra" $\mathcal{C}^\infty(W)$ of a symplectic manifold (W, ω). This process is akin to a *desingularization* of the Poisson structure [8] and plays an important role in a larger program to quantize Poisson manifolds.

Example 1.3.1. Let $\mathfrak{g}$ be a finite dimensional real Lie algebra with Lie bracket $[\cdot,\cdot]$. For any $X \in \mathfrak{g}$, let $\ell_X : \mathfrak{g}^\vee \to \mathbb{R}$ be the function defined by

$$\ell_X(\alpha) = \langle \alpha, X \rangle$$

for any α in $\mathfrak{g}^\vee$. Let $\{\cdot,\cdot\}^\vee$ be the unique Poisson bracket on $\mathfrak{g}^\vee$ satisfying the equation

$$\{\ell_X, \ell_Y\}^\vee = \ell_{[X,Y]}$$

for all X and Y in $\mathfrak{g}$. The Poisson bracket $\{\cdot,\cdot\}^\vee$ is called the *Lie-Poisson bracket* defined by $[\cdot,\cdot]$. Let π_{LP} denote the Poisson bivector field on $\mathfrak{g}^\vee$ defined by the Lie-Poisson bracket. Now let G be a real Lie group with Lie algebra $\mathfrak{g}$. The cotangent bundle $T^\vee G$ has a canonical symplectic 2-form ω_{can} defined as the exterior derivative of the Liouville form. Let $L_g : G \to G$ denote left-translation by $g \in G$. Let $q_L : T^\vee G \to \mathfrak{g}^\vee$ be the map defined by

$$q_L(\lambda) = L^*_{g^{-1}}\lambda$$

for any $\lambda \in T^\vee G$ ($g \in G$). From the fact that L_g is a diffeomorphism, it follows that q_L restricts to fibrewise isomorphisms $T^\vee_g G \to \mathfrak{g}$ for all $g \in G$. In particular, q_L is a surjective submersion. It is also simple to see that $(q_L)_*\,\pi_{\mathrm{can}} = \pi_{\mathrm{LP}}$, where π_{can} is the Poisson bivector field defined by the equation $\pi^\sharp_{\mathrm{can}} = (\omega^\flat_{\mathrm{can}})^{-1}$. In conclusion, the Lie group G yields a canonical symplectic realization $(T^\vee G, \omega_{\mathrm{can}}, q_L)$ of $(\mathfrak{g}^\vee, \pi_{\mathrm{LP}})$.

The construction of the Poisson bivector field π_{LP} on $\mathfrak{g}^\vee$ stated in Example 1.3.1 above extends trivially when we replace $\mathfrak{g}$ by a general Lie algebroid. We record this fact in the next remark for later reference.

Remark 1.3.1. Let M be a smooth manifold and E a vector bundle over M. Assume the dual vector bundle $E^\vee$ has a Lie algebroid structure with anchor $\rho^\vee : E^\vee \to TM$ and bracket $[\cdot,\cdot]^\vee : \Gamma(E^\vee) \times \Gamma(E^\vee) \to \Gamma(E^\vee)$. There exists a unique Poisson bivector field π_{LP} on E (as a manifold) such that

$$\{\ell_\lambda, \ell_\nu\}_{\mathrm{LP}} = \ell_{[\lambda,\nu]^\vee}$$

for any $\lambda, \nu \in \Gamma(E^\vee)$. Here $\{\cdot,\cdot\}_{\mathrm{LP}}$ is the Poisson bracket associated to π_{LP} and, for any $\mu \in \Gamma(E^\vee)$ we let ℓ_μ denote the fibrewise linear function on E defined by

$$\ell_\mu(X) = \langle \mu(m), X \rangle$$

for all $X \in E_m$ ($m \in M$). We will call π_{LP} the *Lie-Poisson bivector field on E associated to the Lie algebroid structure on $E^\vee$*.

Lie [7] proved the existence of local symplectic realizations for regular Poisson structures. Much later, in 1983, Weinstein [9] generalized that result to arbitrary Poisson manifolds using the splitting theorem. The existence of *global* symplectic realizations was proved in 1987 independently by Karasev [10] and Weinstein [11] using gluing arguments. Strikingly they discovered that any symplectic realization carries a local groupoid structure compatible with the symplectic 2-form, a device called a *symplectic local groupoid* (see Definition 1.3.8 below). Further, this groupoid structure is canonical up to the choice of a Lagrangian section. This general perspective sheds light on Example 1.3.1 and confirms that the problem of finding symplectic realizations is tied to the Lie theory of Lie groupoids in a subtle way.

1.3.2 Lie Groupoids

A groupoid is a set G_1 endowed with a partially defined multiplication satisfying all the axioms of a group wherever the product is defined. More precisely:

Definition 1.3.2. A *groupoid*, denoted $G_1 \rightrightarrows G_0$, is a pair of sets (G_1, G_0) endowed with the following maps:

- a source (resp. target) map $\alpha : G_1 \to G_0$ (resp. $\beta : G_1 \to G_0$),
- a units injection $\epsilon : G_0 \hookrightarrow G_1$,
- an inverse map $\iota : G_1 \to G_1 : g \mapsto \iota(g) = g^{-1}$ and,
- a multiplication map μ defined on all pairs $(g, g') \in G_1 \times G_1$ such that $\beta(g) = \alpha(g')$;

altogether satisfying the following conditions:

- $\alpha(\mu(g, g')) = \alpha(g)$, $\beta(\mu(g, g')) = \beta(g')$,
- $\alpha(\epsilon(m)) = m = \beta(\epsilon(m))$,
- $\mu(\mu(g, g'), g'') = \mu(g, \mu(g', g''))$ (associativity)
- $\iota(\iota(g)) = g$, $\mu(g, \iota(g)) = \epsilon(\alpha(g))$, $\mu(\iota(g), g) = \epsilon(\beta(g))$ (inverse)
- $\mu(g, \epsilon(\beta(g))) = g = \mu(\epsilon(\alpha(g)), g)$ (unit)

for any m in G_0, and any g, g', g'' in G_1 for which the relevant operations are defined.

Terminology. In a groupoid $G_1 \rightrightarrows G_0$ we call G_1 the *arrow space* and G_0 the *object space*.

A *Lie groupoid* is a groupoid $\Sigma \rightrightarrows M$ where Σ and M are smooth manifolds and all structures maps are smooth maps. It is further assumed that the source and target maps are surjective submersions and that the multiplication $\mu : \Sigma \times_{\beta,M,\alpha} \Sigma \to \Sigma$ is a smooth map. Note that assuming that both α and β are surjective submersions is necessary in order to guarantee that the fibre product $\Sigma \times_{\beta,M,\alpha} \Sigma$ exists as a smooth manifold.

Definition 1.3.3. Let $\Sigma \rightrightarrows M$ and $\Sigma' \rightrightarrows M'$ be Lie groupoids. Let α and β (resp. α' and β') denote the source and target maps of $\Sigma \rightrightarrows M$ (resp. $\Sigma' \rightrightarrows M'$). A smooth map $\varphi : \Sigma \to \Sigma'$ is called a *morphism of Lie groupoids* if for any g and g' in Σ such that $\beta(g) = \alpha(g')$, we have $\beta(\varphi(g)) = \alpha(\varphi(g'))$ and $\varphi(g)\varphi(g') = \varphi(gg')$.

Remark 1.3.2. Let $\Sigma \rightrightarrows M$ and $\Sigma' \rightrightarrows M'$ be two Lie groupoids and let $\varphi : \Sigma \to \Sigma'$ be a morphism of Lie groupoids. Let α, β and ϵ (resp. α', β' and ϵ') denote the source, target and unit map of Σ (resp. Σ'). For any $g \in \Sigma$, from the equations

$$\begin{aligned} \varphi(g) &= \varphi(g)\varphi(\epsilon(\beta(g))) \\ &= \varphi(\epsilon(\alpha(g)))\varphi(g), \end{aligned}$$

it follows that there exists a unique (smooth) map $\varphi_0 : M \to M'$ such that the diagrams

$$\begin{array}{ccc} \Sigma & \xrightarrow{\varphi} & \Sigma' \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha} \\ M & \xrightarrow{\varphi_0} & M' \end{array} \qquad \begin{array}{ccc} \Sigma & \xrightarrow{\varphi} & \Sigma' \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\beta} \\ M & \xrightarrow{\varphi_0} & M' \end{array}$$

commute. We say that φ *induces* φ_0 *on the object space.*

It is possible to "localize" the axioms of a Lie groupoid in the following way.

Definition 1.3.4. A *local Lie groupoid* $\Sigma \rightrightarrows M$ is a pair of smooth manifolds (Σ, M) endowed with the following:

- a surjective submersion $\alpha : \Sigma \to M$ (resp. $\beta : \Sigma \to M$) called the source (resp. target),
- a smooth embedding $\epsilon : M \hookrightarrow \Sigma$ called the units,
- an open neighbourhood U of $\epsilon(M)$ in Σ and a diffeomorphism $\iota : U \to U : g \mapsto \iota(g) = g^{-1}$ called the inverse, and
- an open neighbourhood W of $\mathrm{Diag}(\epsilon(M))$ in $\Sigma \times_{\beta,M,\alpha} \Sigma$ and a smooth map $\mu : W \to \Sigma$ called the multiplication.

It is further assumed that all the above maps satisfy the same conditions as the corresponding maps of Definition 1.3.2 wherever the respective operations are defined.

Notation. The open subset W of Definition 1.3.4 will be denoted $\Sigma^{(2)}$. Note that, in general, $\Sigma^{(2)} \neq \Sigma \times_{\beta,M,\alpha} \Sigma$.

We can also localize Definition 1.3.3.

Definition 1.3.5. Let $\Sigma \rightrightarrows M$ and $\Sigma' \rightrightarrows M'$ be local Lie groupoids. A *morphism of local Lie groupoids* is a pair (V, φ) consisting of an open neighbourhood V of $\epsilon(M)$ in Σ and a smooth map $\varphi : V \to \Sigma'$ such that for any (g, g') in $(V \times_{\beta,M,\alpha} V) \cap \Sigma^{(2)}$, the pair $(\varphi(g), \varphi(g'))$ belongs to $\Sigma'^{(2)}$ and the equation

$$\varphi(g)\varphi(g') = \varphi(gg')$$

holds.

Terminology. Let $\Sigma \rightrightarrows M$ be a local Lie groupoid. We call the manifold Σ the *arrow space* and the manifold M the *object space*. Also, we will often write the product of two elements g and g' of Σ (for which the product is defined) as $g \cdot g'$ or gg'.

Terminology. Let $\Sigma \rightrightarrows M$ and $\Sigma' \rightrightarrows M'$ be local Lie groupoids. We say a smooth map $\varphi : \Sigma \to \Sigma'$ is a morphism of local Lie groupoids if there exists an open subset V of Σ such that $(V, \varphi|_V)$ is a morphism of local Lie groupoids.

Remark 1.3.3. The content of Remark 1.3.2 carries over *verbatim* when one replaces "Lie groupoid" with "local Lie groupoid" and "$g \in \Sigma$" with "for any $g \in \Sigma$ sufficiently close to $\epsilon(M)$".

Let $\Sigma \rightrightarrows M$ and $\Sigma' \rightrightarrows M'$ be local Lie groupoids and let $(V, \varphi : V \to \Sigma')$ be a morphism of local Lie groupoids. We say φ is an *isomorphism of local Lie groupoids* if there exists an open neighbourhood V' of $\epsilon(M)$ in Σ such that $V' \subset V$ and such that $\varphi|_{V'}$ is a diffeomorphism. We then say Σ and Σ' are *isomorphic local Lie groupoids.*

Definition 1.3.6. Let $\Sigma \rightrightarrows M$ be a (local) Lie groupoid. We say Σ is *source-connected source-simply connected* if for any $m \in M$, the source fibre $\alpha^{-1}(m)$ is a connected simply connected topological space.

Let $\Sigma \rightrightarrows M$ be a local Lie groupoid. Let α, β and ϵ be, respectively, the source, target and units map of $\Sigma \rightrightarrows M$. By *the Lie algebroid of* $\Sigma \rightrightarrows M$ we mean the triple $(E, \rho, [\cdot,\cdot]_E)$ defined as follows. Let $T^\alpha\Sigma$ be the vector bundle over Σ defined as the kernel of the differential of α and let E be the vector bundle defined as the pullback by $\epsilon : M \to \Sigma$ of $T^\alpha \Sigma$. Let $\rho : E \to TM$ be the restriction to E of the differential of β.

For any g in Σ, let $L_g : \Sigma \to \Sigma$ be the left translation map, defined by $L_g(g') = gg'$ for any $g' \in \Sigma$ close to $\epsilon(M)$. To any element $\gamma \in \Gamma(E)$, we associate a vector field $\overleftarrow{\gamma}$ defined over a neighbourhood of $\epsilon(M)$ by the expression

$$\overleftarrow{\gamma}_g = L_{g,*}(\gamma(\beta(g))), \tag{1.19}$$

for any $g \in \Sigma$ sufficiently close to $\epsilon(M)$. Let $[\cdot, \cdot]_E$ be the unique bracket on $\Gamma(E)$ satisfying the equation:

$$\left.\overleftarrow{[\gamma, \gamma']_E}\right|_g = [\overleftarrow{\gamma}, \overleftarrow{\gamma}']|_g$$

for any γ, γ' in $\Gamma(E)$ and any $g \in \Sigma$ sufficiently close to $\epsilon(M)$. Here the bracket in the right-hand side is the Lie bracket of vector fields. We refer the reader to [12] for a detailed proof that the triple $(E, \rho, [\cdot, \cdot]_E)$ is indeed a well-defined Lie algebroid.

Lie algebroids are to local Lie groupoids what Lie algebras are to Lie group germs. More precisely, let $\Sigma \rightrightarrows M$ and $\Sigma' \rightrightarrows M$ be two local Lie groupoids with Lie algebroid (respectively) E and E'. There is a one-one correspondence between morphisms $\Sigma \to \Sigma'$ of local Lie groupoids and morphisms $E \to E'$ of Lie algebroids. Indeed, for any morphism $\varphi : \Sigma \to \Sigma'$ of local Lie groupoids, let $A\varphi : E \to E'$ be the only map making the following diagram:

$$\begin{array}{ccc} E & \xrightarrow{A\varphi} & E' \\ \downarrow{\scriptstyle i_E} & & \downarrow{\scriptstyle i_{E'}} \\ T\Sigma & \xrightarrow{\varphi_*} & T\Sigma' \end{array}$$

commute. Here i_E (resp. $i_{E'}$) is the natural embedding of E (resp. E') into $T\Sigma$ (resp. $T\Sigma'$). Then $A\varphi$ is a morphism of Lie algebroids. Conversely, given a Lie algebroid morphism $f : E \to E'$, there exists a morphism $\varphi : \Sigma \to \Sigma'$ of local Lie groupoids such that $A\varphi = f$. Furthermore, the map φ is uniquely defined in a neighbourhood of the units in Σ. The correspondence of E to Σ (resp. $A\varphi$ to φ) defines a functor called the Lie functor. In particular, we will call the map $A\varphi$ the *map obtained by applying the Lie functor to* φ. The facts stated above are classical and we again refer the reader to [12] for more details.

Let $\Sigma \rightrightarrows M$ be a local Lie groupoid with Lie algebroid E. Before we close this section, we record two important examples of local Lie groupoids constructed out of $\Sigma \rightrightarrows M$.

Example 1.3.2. The tangent bundle $T\Sigma$ is the arrow space of a natural local Lie groupoid $T\Sigma \rightrightarrows TM$. The source (resp. target, unit, multiplication and inverse) map of $T\Sigma \rightrightarrows TM$ is defined as the tangent map of the source (resp. target, unit, multiplication and inverse) of $\Sigma \rightrightarrows M$. The local Lie groupoid $T\Sigma \rightrightarrows M$ is called the *tangent local Lie groupoid of* $\Sigma \rightrightarrows M$.

Example 1.3.3. The cotangent bundle $T^\vee\Sigma$ is the arrow space of a natural local Lie groupoid $T^\vee\Sigma \rightrightarrows E^\vee$, the structure maps of which are defined as follows. Let $\alpha : \Sigma \to M$ (resp. $\beta : \Sigma \to M$) be the source (resp. target) map of $\Sigma \rightrightarrows M$. For any $g \in \Sigma$, there is an embedding $L_g : E_{\beta(g)} \to T_g\Sigma$ (resp. $R_g : E_{\alpha(g)} \to T_g\Sigma$) defined as the tangent map of left (resp. right) multiplication by g. We define the source (resp. target) map $\alpha^\vee : T^\vee\Sigma \to E^\vee$ (resp. $\beta^\vee : T^\vee\Sigma \to E^\vee$) of $T^\vee\Sigma \rightrightarrows E^\vee$ by

$$\alpha^\vee|_{T^\vee_g\Sigma} = (R_g)^\top \quad (\text{resp. } \beta^\vee|_{T^\vee_g\Sigma} = (L_g)^\top).$$

The unit map $\epsilon^\vee : E^\vee \to T^\vee\Sigma$ is the vector bundle morphism over $\epsilon : M \to \Sigma$ defined by the property that

$$\epsilon^\vee(\lambda)|_{E_m} = \lambda, \quad \text{and } \epsilon^\vee(\lambda)|_{\epsilon_*(M)} = 0,$$

for all $m \in M$ and all $\lambda \in E^\vee_m$. Finally, the multiplication is defined as follows: for any $\lambda \in T^\vee_g\Sigma$ and any $\nu \in T^\vee_{g'}\Sigma$ such that $\beta^\vee(\lambda) = \alpha^\vee(\nu)$, we let their product $\lambda \cdot \nu$ be the unique element of $T^\vee_{gg'}\Sigma$ defined by the property that

$$\langle \lambda \cdot \nu, \mu_*(X, Y)\rangle = \langle \lambda, X\rangle + \langle \nu, Y\rangle$$

for any $X \in T_g\Sigma$ and any $Y \in T_g\Sigma'$ such that $\beta_*(X) = \alpha_*(Y)$. It is standard [12] that $T^\vee\Sigma \rightrightarrows E^\vee$ endowed with the structure maps defined above is a local Lie groupoid, called the *cotangent bundle local Lie groupoid of* $\Sigma \rightrightarrows M$.

1.3.3 Symplectic Groupoids and a Third Lie Theorem

We now turn to the notion of *symplectic groupoids*. The latter are Lie groupoids endowed with a symplectic 2-form compatible with the groupoid structure. They were originally introduced [11], [10] in an effort to unify many constructions in symplectic and Poisson geometry. Also, they are inherently tied to symplectic realizations as we will see below. Interestingly, the precise extent to which symplectic groupoids relate to the larger program of quantization of Poisson manifolds was clarified only relatively recently by the work of Cattaneo-Felder [13] on the Poisson sigma model (see the next chapter).

Definition 1.3.7. Let $\Sigma \rightrightarrows M$ be a Lie groupoid and let Ω be a symplectic 2-form on Σ. Let $\mu : \Sigma \times_M \Sigma \to \Sigma$ denote the multiplication map of the groupoid $\Sigma \rightrightarrows M$ and let $\bar{\Sigma}$ denote the manifold Σ endowed with the opposite symplectic form $-\Omega$. We say that Ω is *multiplicative* if the graph of μ is a Lagrangian submanifold of $(\Sigma \times \Sigma \times \bar{\Sigma}, \Omega \times \Omega \times -\Omega)$.

Definition 1.3.8. A *symplectic groupoid* is a pair $(\Sigma \rightrightarrows M, \Omega)$ consisting of a Lie groupoid $\Sigma \rightrightarrows M$ and a multiplicative symplectic 2-form Ω on Σ.

Remark 1.3.4. Both Definition 1.3.7 and Definition 1.3.8 can be trivially adapted to *local* Lie groupoids, in which case we will talk about symplectic *local* groupoids.

Let $(\Sigma \rightrightarrows M, \Omega)$ be a symplectic groupoid. The following standard result of Weinstein [11] shows that the unit submanifold of Σ naturally inherits a (suitably compatible) Poisson structure.

Theorem 1.3.1. *Let $\Sigma \rightrightarrows M$ be a Lie groupoid endowed with a symplectic 2-form ω such that $(\Sigma \rightrightarrows M, \Omega)$ is a symplectic groupoid. Let $\alpha : \Sigma \twoheadrightarrow M$ (resp. $\beta : \Sigma \twoheadrightarrow M$) denote the source (resp. target) map of the groupoid $\Sigma \rightrightarrows M$. There exists a unique Poisson bivector field π on M such that*

$$\alpha_* \Pi = \pi \quad \textit{and} \quad \beta_* \Pi = -\pi.$$

where Π is the Poisson bivector field defined by the equation $\Pi^\sharp = (\Omega^\flat)^{-1}$.

Since the source and target maps of a Lie groupoid are surjective submersions (by definition), it follows from Theorem 1.3.1 that the source (resp. target) map of a symplectic groupoid with object space M automatically yields symplectic realizations of (M, π) (resp. $(M, -\pi)$) where π is the Poisson bivector field ensured by Theorem 1.3.1.

Remark 1.3.5. The proof of Theorem 1.3.1 does not rely on the global structure of the groupoid. Hence it is easily seen that the result also holds when one makes the replacement "symplectic Lie groupoid" $\longleftrightarrow$ "symplectic *local* groupoid" everywhere.

Let $(\Sigma \rightrightarrows M, \Omega)$ and $(\Sigma' \rightrightarrows M, \Omega')$ be two symplectic local groupoids over the same object space M. We say that Σ and Σ' are *isomorphic symplectic local groupoids* if there exists an isomorphism $\varphi : \Sigma \to \Sigma'$ of local Lie groupoids such that $\varphi^* \Omega' = \Omega$ and such that φ induces the identity on M.

The next theorem was proved idependently by Weinstein [11] and Karasev [10] and is a local converse to Theorem 1.3.1 that can be viewed as a third local Lie theorem for Poisson manifolds.

Theorem 1.3.2. *Let M be a smooth manifold and π be a Poisson bivector field on M. There exists a unique (up to isomorphism) symplectic local groupoid $(\Sigma \rightrightarrows M, \Omega)$ such that the source and target maps $\alpha, \beta : \Sigma \to M$ of Σ give symplectic realizations*

$$(\Sigma, \Omega, \alpha) \quad \textit{and} \quad (\Sigma, -\Omega, \beta)$$

of (M, π).

In the situation of Theorem 1.3.2, we will say that the symplectic local groupoid $(\Sigma \rightrightarrows M, \Omega)$ realizes (M, π). Note that, although the groupoid $\Sigma \rightrightarrows M$ is local, it covers the entirety of M and thus defines global symplectic realizations.

1.4 Holomorphic Poisson Manifolds

In this section, we briefly recall, for the sake of completeness, some standard notions on holomorphic Poisson structures. As most of those elementary facts closely parallel the real, smooth, Poisson case, we will simply point the reader to the appropriate references for further details.

In what follows, we let $\mathscr{X}$ (resp. $\mathscr{Y}$) be a complex manifold and X (resp. Y) its underlying real manifold. We will denote the structure sheaf of $\mathscr{X}$ by $\mathcal{O}_{\mathscr{X}}$. Recall that a *complex structure* on $\mathscr{X}$ is equivalent to an integrable almost complex structure J on X, i.e. an endomorphism $J : TX \to TX$ of the underlying real tangent bundle TX with $J^2 = -1$ and vanishing Nijenhuis torsion with respect to the Lie bracket defined by the commutator of vector fields. Furthermore the *holomorphic* tangent bundle $T\mathscr{X}$ is isomorphic, as a complex vector bundle, to the complex eigenbundle $T^{1,0}X$ of the complexified operator $J \otimes \mathbb{C}$ with eigenvalue $+i$.

Definition 1.4.1. A *holomorphic Poisson bracket* on a complex manifold $\mathscr{X}$, is a bracket $\{\cdot,\cdot\}$ on the structure sheaf $\mathcal{O}_{\mathscr{X}}$ of $\mathscr{X}$ such that the pair $(\mathcal{O}_{\mathscr{X}}, \{\cdot,\cdot\})$ is a sheaf of Poisson algebras.

A *holomorphic Poisson manifold* is a complex manifold $\mathscr{X}$ endowed with a holomorphic Poisson bracket. The latter is equivalent to a certain holomorphic bivector field on $\mathscr{X}$, as the next proposition shows. We refer the reader to [14] or [15] for a proof.

Proposition 1.4.1. *Let $\mathscr{X}$ be a complex manifold and let $\{\cdot,\cdot\}$ be a holomorphic Poisson bracket on $\mathscr{X}$. There exists a unique bivector field $\pi \in \Gamma(\wedge^2 T^{1,0}X)$ satisfying*

$$\bar{\partial}\pi = 0 \text{ and } [\pi,\pi] = 0, \tag{1.20}$$

and such that

$$\{f,g\}_U = \langle \pi, \partial f \wedge \partial g \rangle \tag{1.21}$$

for any open subset U of X and any two holomorphic functions f and g defined on U. Here $\{\cdot,\cdot\}_U$ denotes the restriction of the bracket $\{\cdot,\cdot\}$ to the sections of $\mathcal{O}_{\mathscr{X}}$ over U.

Conversely, let $\pi \in \Gamma(\wedge^2 T^{1,0}X)$ be a bivector field satisfying (1.20). There exists a unique holomorphic Poisson bracket on $\mathscr{X}$ such that (1.21) holds.

A bivector field π on $\mathscr{X}$ satisfying (1.20) is called a *holomorphic Poisson bivector field* and the pair $(\mathscr{X}, \pi)$ is called a holomorphic Poisson manifold. The relationship between holomorphic Poisson bivector fields and Poisson-Nijenhuis structures was exhibited in the work of Laurent-Gengoux–Stienon–Xu [14]. The gist of it is summarized in the following proposition.

Proposition 1.4.2. *Let $\mathscr{X}$ be a complex manifold with almost complex structure J and let π be any smooth section of $\wedge^2 T^{1,0}X$. Denote by π_R (resp. π_I) the real (resp. pure imaginary) part of π. Then π is a holomorphic Poisson bivector field if and only if*

– *the pair (π_I, J) defines a Poisson-Nijenhuis structure on X, and*

– *the equation $\pi_R^\sharp = \pi_I^\sharp \circ J^\top$ holds,*

where $J^\top : T^\vee X \to T^\vee X$ denotes the $\mathbb{R}$-dual of J.

Let π be a holomorphic bivector field on $\mathscr{X}$. We have an associated morphism of holomorphic vector bundles $\pi^\sharp : T^\vee \mathscr{X} \to T\mathscr{X}$. Indeed, just as in the real case, for any ω in $T_x^\vee \mathscr{X}$ $(x \in \mathscr{X})$ we let $\pi^\sharp(\omega)$ be defined by the equation

$$\langle \nu, \pi^\sharp(\omega) \rangle = \langle \nu \wedge \omega, \pi \rangle$$

for any $\nu \in T_x^\vee \mathscr{X}$. When $\pi^\sharp$ is invertible we say that the bivector field π is *non-degenerate.*

Definition 1.4.2. Let $\mathscr{X}$ be a complex manifold. A *holomorphic symplectic* 2*-form* on $\mathscr{X}$ is a Dolbault-closed holomorphic differential 2-form ω such that the pairing

$$T\mathscr{X} \times T\mathscr{X} \to \mathbb{C} : (X, Y) \mapsto \langle \omega, X \wedge Y \rangle$$

is non-degenerate.

Naturally (and analoguously to the real case) a holomorphic symplectic 2-form is equivalent to a non-degenerate holomorphic Poisson bivector field, a fact we record in the following easy proposition.

Proposition 1.4.3. *Let $\mathscr{X}$ be a complex manifold and let ω be a holomorphic symplectic* 2*-form on $\mathscr{X}$. Let $\omega^\flat : T\mathscr{X} \to T^\vee \mathscr{X}$ be the map defined by the equation*

$$\langle \omega^\flat(X), Y \rangle = \langle \omega, X \wedge Y \rangle,$$

for all $X, Y \in T_x \mathscr{X}$ $(x \in \mathscr{X})$. There exists a unique non-degenerate holomorphic Poisson bivector field π on $\mathscr{X}$ such that $\pi^\sharp = (\omega^\flat)^{-1}$.

Conversely, let π be a non-degenerate holomorphic Poisson bivector field on $\mathscr{X}$. There exists a unique holomorphic symplectic 2*-form ω on $\mathscr{X}$ such that $\omega^\flat = (\pi^\sharp)^{-1}$.*

The proof of Proposition 1.4.3 parallels directly the corresponding statement in the real case and thus will be omitted here.

1.4.1 Symplectic Realizations in the Holomorphic Setting

Let $\mathscr{X}$ and $\mathscr{Y}$ be two complex manifolds. Let $\pi_{\mathscr{X}}$ (resp. $\pi_{\mathscr{Y}}$) be a holomorphic Poisson bivector on $\mathscr{X}$ (resp. $\mathscr{Y}$).

Definition 1.4.3. A holomorphic map $f : \mathscr{X} \to \mathscr{Y}$ is said to be *Poisson* if $f_* \pi_{\mathscr{X}} = \pi_{\mathscr{Y}}$.

It is straightforward to extend the definition of symplectic realizations to the holomorphic Poisson case.

Definition 1.4.4. Let $\mathscr{X}$ be a holomorphic Poisson manifold. A *holomorphic symplectic realization* of $\mathscr{X}$ is a triple $(\mathscr{Y}, \Omega, q)$ consisting of a complex manifold $\mathscr{Y}$, a holomorphic symplectic 2-form Ω on $\mathscr{Y}$ and a holomorphic map $q : \mathscr{Y} \to \mathscr{X}$ such that

– the corresponding real map $q_{\mathbb{R}} : Y \to X$ is a surjective submersion, and

– we have $q_* \Pi = \pi$, where Π is the holomorphic Poisson bivector field on $\mathscr{Y}$ defined by the equation $\Pi^\sharp = (\Omega^\flat)^{-1}$.

Example 1.4.1. Let $\mathscr{X}$ be a complex manifold. Let $\pi = 0$ be the zero bivector field. Then π is a Poisson bivector field. On the other hand, the holomorphic cotangent bundle $T^\vee \mathscr{X}$ has a canonical symplectic structure ω_{can} defined as $\partial \theta$ where θ is the holomorphic Liouville 1-form on $T^\vee \mathscr{X}$ and ∂ is the Dolbault operator. The triple $(T^\vee \mathscr{X}, \omega_{\text{can}}, \text{pr})$ where $\text{pr} : T^\vee \mathscr{X} \to \mathscr{X}$ is the basepoint projection, gives a holomorphic symplectic realization of $(\mathscr{X}, \pi)$.

Example 1.4.2. Let $\mathfrak{g}$ be a complex Lie algebra with Lie bracket $[\cdot, \cdot]$ and denote by $\mathfrak{g}^\vee$ its dual as a vector space over $\mathbb{C}$. For X in $\mathfrak{g}$, let ℓ_X be the linear functional on $\mathfrak{g}^\vee$ defined by

$$\ell_X(\lambda) = \langle \lambda, X \rangle,$$

for any $\lambda \in \mathfrak{g}$. There is a unique holomorphic Poisson bivector field π on $\mathfrak{g}^\vee$ such that

$$\langle \partial \ell_X \wedge \partial \ell_Y, \pi \rangle = \ell_{[X,Y]}$$

for any X and Y in $\mathfrak{g}$, where ∂ denotes the Dolbault operator. Now let G be a complex Lie group with Lie algebra $\mathfrak{g}$. Then G is a complex manifold. Let $T^\vee G$ be the holomorphic cotangent bundle of G and let ω_{can} be the holomorphic symplectic 2-form on $T^\vee G$ defined in Example 1.4.1. Let

$$q : T^\vee G \to T_1^\vee G \cong \mathfrak{g}^\vee$$

be the map defined by $q(\lambda) = g^{-1} \lambda$ for any $\lambda \in T_g^\vee G$ $(g \in G)$. Then the triple $(T^\vee G, \omega_{\text{can}}, q)$ is holomorphic symplectic realization of $(\mathfrak{g}^\vee, \pi)$.

Chapter 2 The Cattaneo-Felder Construction

Let M be a smooth manifold and let π be a Poisson bivector field on M. The Poisson sigma model (PσM for short) is a topological field theory associated to (M, π). In their seminal work on the PσM, Cattaneo-Felder [13] strikingly proved that the natural phase space of the theory, provided it is smooth, has automatically the structure of a symplectic groupoid with object space M. A short while later, Crainic-Fernandes [16] found a criterion for the integrability of an arbitrary Lie algebroid E. That criterion can be viewed as a natural obstruction to a certain carefully constructed groupoid, called the *Weinstein groupoid of* E, being smooth. When one applies the construction of the Weinstein groupoid from [16] to the special case where E is the cotangent bundle Lie algebroid E_π defined by (M, π), one recovers (mostly *verbatim*) the groupoid constructed by Cattaneo-Felder, albeit without its symplectic form.

Let M be a smooth manifold and let E be a Lie algebroid over M. In this section, we define the Weinstein local groupoid $\mathscr{W}_{\mathrm{loc}}(E) \rightrightarrows M$ of E and we show that any local Lie groupoid $\Sigma \rightrightarrows M$ with Lie algebroid E is naturally isomorphic to $\mathscr{W}_{\mathrm{loc}}(E)$. Let π be a Poisson bivector field on M and let E_π denote the cotangent bundle Lie algebroid of (M, π). We show how the construction of Cattaneo-Felder for the phase space of the PσM of (M, π) can be localized and viewed as a way to induce a multiplicative symplectic 2-form on the Weinstein local groupoid of E_π.

2.1 The Weinstein Local Groupoid

Let E be a vector bundle over M. Let n be the dimension of M and d be the fibre dimension of E. Let $\mathscr{P}(E)$ denote the collection of all paths, of class $\mathcal{C}^1$, valued in E. The collection $\mathscr{P}(E)$ is naturally a Banach manifold modeled on the space of continuously differentiable paths valued in $\mathbb{R}^n \times \mathbb{R}^d$ with the topology of uniform convergence.

Remark 2.1.1. The Banach manifold $\mathscr{P}(TE)$ is naturally a Banach bundle over $\mathscr{P}(E)$ with basepoint projection

$$\mathrm{pr} : \mathscr{P}(TE) \to \mathscr{P}(E) : (X : I \to TE) \mapsto (\mathrm{pr} \circ X : I \to E).$$

Further, the tangent bundle of $\mathscr{P}(E)$ naturally identifies with $\mathscr{P}(TE)$ as a Banach bundle. To see this, for any $e : I \to E$ in $\mathscr{P}(E)$, let $T_e\mathscr{P}(E)$ denote the tangent space of $\mathscr{P}(E)$ at e. By definition, we have

$$T_e\mathscr{P}(E) = \{\dot{s}(0) | s : I \to \mathscr{P}(E),\ \text{s differentiable at } u = 0\}.$$

For any path $s : I \to \mathscr{P}(E)$ let $\tilde{s} : I \times I \to E$ be the function defined by

$$\tilde{s}(t, u) = (s(t))(u).$$

Let $\mathscr{P}_e(TE)$ denote the collection of all paths $X : I \to TE$ of class $\mathcal{C}^1$ such that $\mathrm{pr} \circ X = e$. The map

$$\mathfrak{I}_e : T_e\mathscr{P}(E) \to \mathscr{P}_e(TE) : \dot{s}(0) \mapsto \left.\frac{d\tilde{s}}{dt}\right|_{t=0}$$

defines an isomorphism $\mathfrak{I} : T\mathscr{P}(E) \to \mathscr{P}(TE)$ of Banach bundles over $\mathscr{P}(E)$.

From now on and until the end of this chapter (unless otherwise explicitly stated), we let E be a Lie algebroid over M.

Definition 2.1.1. Let E be a Lie algebroid with anchor ρ. A path $e : I \to E$ in $\mathscr{P}(E)$ is said to be an *A-path* if

$$\rho\Big(e(t)\Big) = \frac{d\gamma_e(t)}{dt}, \tag{2.1}$$

where $\gamma_e = \mathrm{pr} \circ e$.

The collection of all A-paths in $\mathscr{P}(E)$ is denoted $\mathscr{A}(E)$. It is simple to see that $\mathscr{A}(E)$ is an (infinite dimensional) closed Banach submanifold of $\mathscr{P}(E)$. The tangent bundle of $\mathscr{A}(E)$ is described explicitly in the next proposition.

Proposition 2.1.1. *Let E be a Lie algebroid and let $e : I \to E$ be an A-path. Let $X : I \to TE$ be a path such that $pr(X(t)) = e(t)$ for all t in I. Then the tangent vector $\mathfrak{I}_e^{-1}(X)$ belongs to $T_e\mathscr{A}(E)$ if and only if X satisfies the equation*

$$\rho_*\Big(X(t)\Big) = \frac{d}{dt}(pr_* \circ X)(t) \tag{2.2}$$

for all t in I.

Proof. Let $\bar{e} : I \times I \to E$ such that $\bar{e}(0,t) = e(t)$ for all $t \in I$ and such that each slice $e(s,-) : I \to E$ is an A-path. The claim follows from differentiating the equation

$$(\rho \circ e)(s,t) = \frac{d}{dt}(\mathrm{pr} \circ e)(s,t)$$

in s. □

Remark 2.1.2. Let $\mathcal{C}(TE)$ be the Banach bundle over $\mathscr{A}(E)$ defined as the image by the isomorphism $\mathfrak{I}$ of $T\mathscr{A}(E)$. From Proposition 2.1.1 it follows that a path $X : I \to TE$ belongs to $\mathcal{C}(TE)$ if and only if X satisfies (2.2). Since $\mathcal{C}(TE)$ is isomorphic to $T\mathscr{A}(E)$, we can transport the Lie bracket of vector fields on $\mathscr{A}(E)$ to a Lie bracket on the collection of all sections of $\mathcal{C}(TE)$ over $\mathscr{A}(E)$. We will denote the latter bracket by $[\cdot,\cdot]_{\mathcal{C}}$. In particular, $\mathfrak{I}$ interchanges the Lie bracket on $\mathfrak{X}(\mathscr{A}(E))$ with the bracket $[\cdot,\cdot]_{\mathcal{C}}$.

Let $\Sigma \rightrightarrows M$ be a local Lie groupoid and let E be the Lie algebroid of $\Sigma \rightrightarrows M$. Let α (resp. β, ϵ, μ) be the source (resp. target, unit, multiplication) map of Σ. Let n denote the dimension of Σ as a manifold. By a *source*-path in Σ we mean a path $g : I \to \Sigma$ of class $\mathcal{C}^2$ such that $g(0) = \epsilon(\alpha(g(0)))$ and such that $\alpha(g(t)) = \alpha(g(0))$ for any t in I. Let $\mathscr{S}(\Sigma)$ denote the collection of all source-paths in Σ. It is endowed with a natural Banach manifold structure modeled on the space of paths valued in $\mathbb{R}^n$ with the topology of uniform convergence.

The relationship between source-paths valued in Σ and A-paths valued in E is the same as that of function to derivative. To see this, let $\mathscr{C}(M)$ denote the collection of all constant paths valued in M. There is a natural embedding of $\mathscr{C}(M)$ into $\mathscr{S}(\Sigma)$ (resp. $\mathscr{A}(E)$) as constant paths valued in $\epsilon(M)$ (resp. 0_M). The following proposition shows that, locally around $\mathscr{C}(M)$, source-paths in Σ and A-paths in E are in smooth one-one correspondence.

Proposition 2.1.2. *Let $\Sigma \rightrightarrows M$ be a local Lie groupoid with Lie algebroid E. There is an open neighbourhood U of $\mathscr{C}(M)$ in $\mathscr{S}(\Sigma)$ such that the map*

$$\delta : U \to \mathscr{A}(E) : (g : I \to \Sigma) \mapsto (\delta g)(t) = (g(t))^{-1}\dot{g}(t)$$

is well-defined and a diffeomorphism onto its image.

Proof. Let g be a source-path in Σ. The path

$$(\Delta_t g)(\epsilon) = (g(t))^{-1} g(t+\epsilon)$$

is well-defined for any ϵ and any t both sufficiently close to 0. Equivalently, there exists an open subset U of $\mathscr{C}(M)$ in $\mathscr{S}(\Sigma)$ such that for any $g \in U$, the path $(\Delta_t g)(\epsilon)$ is well-defined for all ϵ close to 0 and any t in I. In particular, for any $g \in U$, the path

$$(\delta g)(t) = \frac{d}{d\epsilon}(\Delta_t g)(\epsilon)\Big|_{\epsilon=0}$$

is well-defined for all t in I.

Let e be an A-path in $\delta(U)$ and let $m_0 = \mathrm{pr}(e(0))$. Let $\gamma = \mathrm{pr} \circ e$ and let $\Sigma^\gamma = \beta^{-1}(\gamma(I))$. Note that Σ^γ is an immersed closed submanifold of Σ. From the Tubular Neighbourhood Theorem, it follows that there exists an open subset W of Σ such that $\Sigma^\gamma \cap W$ is an embedded closed submanifold of Σ. Consider the initial value problem:

$$\text{find } g : I \to \Sigma^\gamma \cap W \text{ such that } \begin{cases} \dot{g}(t) = g(t)e(t), \\ g(0) = \epsilon(m_0). \end{cases} \tag{2.3}$$

Note that the vector $g(t)e(t)$ is well-defined provided $g(t)$ is sufficiently close to $\epsilon(M)$ and is tangent to $\Sigma^\gamma \cap W$ since

$$\begin{aligned} \beta_*(g(t)e(t)) &= \beta_*(e(t)) \\ &= \rho(e(t)) \\ &= \dot{\gamma}(t). \end{aligned}$$

In particular, there exists an open neighbourhood W of $\epsilon(M)$ in Σ such that the initial value problem (2.3) is well-defined. From classical results on ODEs it easily follows that there exists an open neighbourhood U' of $\mathscr{C}(M)$ in $\mathscr{A}(E)$ such that for any $e \in U'$, there exists a unique solution g_e to (2.3) that belongs to U. Further the correspondence $e \mapsto g_e$ is smooth. It is clear from the first equation in (2.3) that $\delta(g_e) = e$ so that $\delta^{-1}(e) = g_e$. Thus, after redefining U to be $\delta^{-1}(U')$, the map $\delta : U \to \delta(U)$ is a diffeomorphism, as required. □

Let $\mathscr{G}(E)$ denote the vector space of all continuously differentiable functions $\sigma : M \times I \to E$ such that 1) for any $t \in I$, the slice $\sigma_t := \sigma(-, t) : M \to E$ is a section of E, and 2) for any $m \in M$, we have $\sigma(m, 0) = \sigma(m, 1) = 0_m$. We define a Lie bracket $[\cdot, \cdot]_{\mathrm{T}}$ on $\mathscr{G}(E)$ by the formula

$$[\sigma, \nu]_{\mathrm{T}}(m, t) = [\sigma_t, \nu_t]_E(m), \tag{2.4}$$

for any σ, ν in $\mathscr{G}(E)$, where $[\cdot, \cdot]_E$ is the Lie bracket on sections of E. In particular, the space $\mathscr{G}(E)$ is an infinite dimensional Lie algebra.

Let $X \in \Gamma(E)$. Define a vector field ad_X on E as follows. Fix $e_0 \in E$ and let $m_0 = \mathrm{pr}(e_0)$. Choose a smooth section $\bar{e} \in \Gamma(E)$ such that $\bar{e}(m_0) = e_0$ and let

$$\mathrm{ad}_X|_{e_0} := \mathrm{vert}_{e_0}\Big([X, \bar{e}]_E(m_0)\Big) - \bar{e}_*(\rho(X(m_0))) \tag{2.5}$$

where $\mathrm{vert}_{e_0} : E_{m_0} \to T_{e_0}E$ is the natural embedding of the fibre E_{m_0} as the vertical tangent space of E at e_0. It is easily seen that the dependence of the right-hand side in $\bar{e}$ is $\mathcal{C}^\infty(M)$-linear and thus that (2.5) only depends on the value of $\bar{e}$ at m_0. We call ad_X the *adjoint vector field* to X.

There is a canonical (infinitesimal) action of $\mathscr{G}(E)$ on $\mathscr{A}(E)$. Indeed, such an action is encoded in a Lie algebra morphism

$$\alpha : \mathscr{G}(E) \to \mathfrak{X}(\mathscr{A}(E))$$

which we shall define now. Fix σ in $\mathscr{G}(E)$. In order to define the vector field $\alpha(\sigma)$ it is enough to define a section

$$\hat{\sigma} : \mathscr{A}(E) \to \mathcal{C}(TE).$$

In particular, for any e in $\mathscr{A}(E)$, we have to define a corresponding path

$$\hat{\sigma}(e) : I \to TE$$

satisfying (2.2). Let $\gamma_e : I \to M$ be the base path $\gamma_e = \mathrm{pr} \circ e$. We define $\hat{\sigma}(e)$ by the expression

$$\mathrm{ev}_t(\hat{\sigma}(e)) := \mathrm{ad}_{\sigma_t}\big|_{e(t)} + \partial_t \sigma(\gamma_e(t), t) \tag{2.6}$$

for any $t \in I$, where $\partial_t \sigma(\gamma_e(t), t)$, an element of the vertical tangent bundle of E at $\sigma(\gamma_e(t), t)$, is viewed as an element of the vertical tangent bundle of E at $e(t)$ by translating along the difference $e(t) - \sigma(\gamma_e(t), t)$. It is straightforward to check that $\hat{\sigma}(e)$ satisfies (2.2).

The following lemma is originally due to Cattaneo-Felder [16], who proved it in the special case $E = E_\pi$ (where π is a Poisson bivector field on M) by a computation in local coordinates. Its extension to general Lie algebroids first appeared in the work of Crainic-Fernandes [16] where it was proved via to use of exponential charts. The proof below is intrinsic.

Lemma 2.1.1. *The map $\alpha : \mathscr{G}(E) \to \mathfrak{X}(\mathscr{A}(E))$ defined by $\alpha(\sigma) = \mathfrak{I}^{-1} \circ \hat{\sigma}$ is a morphism of Lie algebras. In particular, $\mathscr{G}(E)$ acts on $\mathscr{A}(E)$.*

Proof. For any σ in $\mathscr{G}(E)$, let $\mathrm{ad}_\sigma : \mathscr{A}(E) \to \mathscr{P}(TE)$ (resp. $\dot{\sigma} : \mathscr{A}(E) \to \mathscr{P}(TE)$) be the map defined by

$$\mathrm{ev}_t\left(\mathrm{ad}_\sigma(e)\right) = \mathrm{ad}_{\sigma_t}\big|_{e(t)} \qquad (\text{resp. } \mathrm{ev}_t\left(\dot{\sigma}(e)\right) = \partial_t \sigma(\gamma_e(t), t))$$

for all e in $\mathscr{A}(E)$ and any t in I, where ev_t denotes the evaluation of a path at time t. It is easily seen that, for any $e \in \mathscr{A}(E)$, the paths $\mathrm{ad}_\sigma(e)$ and $\dot{\sigma}(e)$ belong to $\mathcal{C}(TE)$.

Now let σ, ν be elements of $\mathscr{G}(E)$ and let $\psi = [\sigma, \nu]_{\mathrm{T}}$. It is clear that we have

$$\widehat{\sigma} = \mathrm{ad}_\sigma + \dot{\sigma} \text{ and } \widehat{\nu} = \mathrm{ad}_\nu + \dot{\nu}.$$

In particular,

$$\begin{aligned} \mathfrak{I} \circ [\alpha(\sigma), \alpha(\nu)] &= [\hat{\sigma}, \hat{\nu}]_{\mathcal{C}} \\ &= [\mathrm{ad}_\sigma + \dot{\sigma},\, \mathrm{ad}_\nu + \dot{\nu}]_{\mathcal{C}} \\ &= [\mathrm{ad}_\sigma, \mathrm{ad}_\nu]_{\mathcal{C}} + [\mathrm{ad}_\sigma, \dot{\nu}]_{\mathcal{C}} + [\dot{\sigma}, \mathrm{ad}_\nu]_{\mathcal{C}} + [\dot{\sigma}, \dot{\nu}]_{\mathcal{C}}. \end{aligned}$$

A direct computation shows that $[\mathrm{ad}_\sigma, \mathrm{ad}_\nu]_{\mathcal{C}} = \mathrm{ad}_\psi$. Further, one can check that $[\dot\sigma, \dot\nu] = 0$. Finally, we have the equation

$$[\mathrm{ad}_\sigma, \dot\nu]_{\mathcal{C}} + [\dot\sigma, \mathrm{ad}_\nu]_{\mathcal{C}} = \dot\psi.$$

It thus follows that

$$\begin{aligned} \mathfrak{I} \circ [\alpha(\sigma), \alpha(\nu)] &= \mathrm{ad}_\psi \, + \, \dot\psi \\ &= \widehat{\psi} \\ &= \mathfrak{I} \circ \alpha(\psi) \\ &= \mathfrak{I} \circ \alpha([\sigma, \nu]_T) \end{aligned}$$

which concludes the proof. □

Let $\mathcal{D}(E)$ be the involutive distribution on $\mathscr{A}(E)$ defined by the action of $\mathscr{G}(E)$. More precisely, for any $e \in \mathscr{A}(E)$, let $\mathcal{D}(E)_e \subset T_e\mathscr{A}(E)$ be the collection of all $\alpha(\sigma)|_e$ as σ travels in $\mathscr{G}(E)$. From the infinite dimensional Frobenius theorem [17], it follows that there exists a foliation $\mathcal{F}(E)$ integrating $\mathcal{D}(E)$. Let ${}^{\mathscr{A}(E)}/_{\mathcal{F}(E)}$ denote the (topological) space of leaves of $\mathcal{F}(E)$. For brevity, we denote the latter by $\mathscr{W}(E)$ and, for any $e \in \mathscr{A}(E)$, we let $[e] \in \mathscr{W}(E)$ denote the $\mathcal{F}(E)$-equivalence class of e. In general, $\mathscr{W}(E)$ is not a smooth manifold. The next theorem shows that locally around $\mathscr{C}(M)$, it indeed is.

Theorem 2.1.1. *Let E be a Lie algebroid. Let $\mathcal{D}(E)$ be the involutive distribution on $\mathscr{A}(E)$ defined by action of $\mathscr{G}(E)$ and let $\mathcal{F}(E)$ denote the foliation integrating $\mathcal{D}(E)$. There exists an open neighbourhood V of $\mathscr{C}(M)$ in $\mathscr{A}(E)$ such that the topological space ${}^{V}/_{V \cap \mathcal{F}(E)}$ has a finite dimensional smooth structure, where $V \cap \mathcal{F}(E)$ denotes the foliation defined as the intersection of $\mathcal{F}(E)$ with V.*

Proof. For any σ in $\mathscr{G}(E)$, a computation in local charts shows that the vector field $\alpha(\sigma)$ coincides with the vector field "X_σ" of Lemma 4.16 in [18]. In particular, the result follows easily from the proof of Theorem 4.23 in [18]. □

Notation. Let V be as in Theorem 2.1.1. We let $\mathscr{W}_{\mathrm{loc}}(E)$ denote the quotient ${}^{V}/_{V \cap \mathcal{F}(E)}$, which is a finite dimensional smooth manifold. Let q denote the canonical surjection

$$q : \mathscr{A}(E) \to \mathscr{W}_{\mathrm{loc}}(E) : e \mapsto [e],$$

which is defined in a neighbourhood of $\mathscr{C}(0_M)$ in $\mathscr{A}(E)$. Then q is a surjective submersion. For any e sufficiently close to $\mathscr{C}(M)$ and any $X \in T_e\mathscr{A}(E)$, we let $[X]$ denote the tangent vector $q_*(X)$.

It is standard that $\mathscr{W}_{\mathrm{loc}}(E)$ is the arrow space of a local Lie groupoid with object space M. To recall this, define the maps

$$\begin{aligned}\tilde{\alpha} &: V \to M : \tilde{\alpha}(e) = \mathrm{pr}(e(0)),\\ \tilde{\beta} &: V \to M : \tilde{\beta}(e) = \mathrm{pr}(e(1)),\\ \tilde{\iota} &: V \to V : \tilde{\iota}(e) = \bar{e},\\ \tilde{\epsilon} &: M \to V : \tilde{\epsilon}(m) = c_m,\end{aligned}$$

where $\bar{e}$ is the path $\bar{e}(t) = -e(1-t)$ and where c_m is the constant path $c_m(t) = \epsilon(m)$. Let τ be any smooth real-valued function such that $\tau(t) = 1$ for $t \geq 1$, $\tau(t) = 0$ for $t \leq 0$ and $\dot{\tau}(t) > 0$ for $0 < t < 1$. For any e in $\mathscr{A}(E)$, let e^τ denote the path defined by $e^\tau(t) = \dot{\tau}(t)e(\tau(t))$. For any e, e' in $\mathscr{A}(E)$ such that $\tilde{\beta}(e) = \tilde{\alpha}(e')$, let $e \odot e'$ be the path defined by the concatenation

$$e \odot e' = \begin{cases} 2e(2t), & 0 \leq t \leq \frac{1}{2},\\ 2e'(2t-1), & \frac{1}{2} < t \leq 1.\end{cases}$$

and let $e \cdot e'$ be the path defined by

$$e \cdot e' = e^\tau \odot e'^\tau.$$

It is easy to see that $e \cdot e'$ is an A-path. Further, it is clear that there exists an open neighbourhood W of $\mathrm{Diag}(M)$ in $V \times_{\tilde{\beta},M,\tilde{\alpha}} V$ such that, for any (e, e') in W, the A-path $e \cdot e'$ belongs to V. We let $\tilde{\mu}$ be the map

$$\tilde{\mu} : W \to V : \tilde{\mu}(e, e') = e \cdot e'.$$

We are now ready to state the main theorem of this section, which is a straightforward adaptation to local groupoids of a classical result due to Crainic-Fernandes [19].

Theorem 2.1.2. *Let E be a Lie algebroid and let V be the open subset ensured by Theorem 2.1.1. Define $\mathscr{W}_{loc}(E) := {}^{V}\!/_{V \cap \mathcal{F}(E)}$, which is a smooth manifold, and let $\tilde{\alpha}, \tilde{\beta}, \tilde{\epsilon}$ and $\tilde{\mu}$ be the maps defined above. The maps*

$$\begin{aligned}\bar{\alpha} &: \mathscr{W}_{loc}(E) \to M : \bar{\alpha}([e]) = [\tilde{\alpha}(e)],\\ \bar{\beta} &: \mathscr{W}_{loc}(E) \to M : \bar{\beta}([e]) = [\tilde{\beta}(e)],\\ \bar{\iota} &: \mathscr{W}_{loc}(E) \to \mathscr{W}_{loc}(E) : \bar{\iota}([e]) = [\tilde{\iota}(e)],\\ \bar{\epsilon} &: M \to \mathscr{W}_{loc}(E) : \bar{\epsilon}(m) = [\tilde{\epsilon}(m)],\\ \bar{\mu} &: (q \times q)(W) \to \mathscr{W}_{loc}(E) : \bar{\mu}([e], [e']) = [\tilde{\mu}(e, e')].\end{aligned}$$

are well-defined and smooth. Further, with $\bar{\alpha}$ (resp. $\bar{\beta}$, $\bar{\iota}$, $\bar{\epsilon}$, $\bar{\mu}$) as source (resp. target, inverse, unit and multiplication), the smooth manifold $\mathscr{W}_{loc}(E)$ is the arrow space of a local Lie groupoid $\mathscr{W}_{loc}(E) \rightrightarrows M$.

Remark 2.1.3. Crainic-Fernandes [19] proved that the multiplication map $\bar{\mu}$ of Theorem 2.1.2 does not depend on the particular choice of the test function τ.

The local Lie groupoid $\mathscr{W}_{\text{loc}}(E) \rightrightarrows M$ of Theorem 2.1.2 is called the *Weinstein local Lie groupoid of E*. Note that, insofar as we're only interested in the isomorphism class of $\mathscr{W}_{\text{loc}}(E) \rightrightarrows M$, it is justified for us to hide the dependency on the open subset V ensured by Theorem 2.1.1. Indeed it is trivial that different choices of V lead to isomorphic Weinstein local Lie groupoids.

Let $\Sigma \rightrightarrows M$ be a local Lie groupoid with Lie algebroid E. The next theorem shows that the canonical relation between source-paths and A-paths (see Proposition 2.1.2) induces an ismorphism between $\Sigma \rightrightarrows M$ and the Weinstein local groupoid of E.

Theorem 2.1.3. *Let $\Sigma \rightrightarrows M$ be a local Lie groupoid with Lie algebroid E. Let δ be the map defined in Proposition 2.1.2. The map*

$$w_\Sigma : \Sigma \to \mathscr{W}_{loc}(E) : z \mapsto [\delta(g_z)]$$

where $g_z : I \to \Sigma$ is any source-path such that $g_z(1) = z$, is a well-defined isomorphism of local Lie groupoids.

Proof. For any g and g' in $\mathscr{S}(\Sigma)$, we say *g and g' are homotopic* if there exists a twice continuously differentiable function $H : I \times I \to \Sigma$ such that

$$\begin{aligned} H(0,t) &= g(t) \\ H(1,t) &= g'(t) \\ H(s,0) &= g(0) = g'(0) \\ H(s,1) &= g(1) = g'(1) \end{aligned}$$

for all s and t in I. Let z be an element of Σ and let g_z and g'_z be two source-paths such that $g_z(1) = z = g'_z(1)$. In order to prove w_Σ is well-defined, we need to check that $[\delta(g_z)] = [\delta(g'_z)]$. Without loss of generality, assume that Σ is source-connected source-simply connected. It follows that g_z is homotopic to g'_z. In particular, it suffices to check that for any two homotopic source-paths g and g', we have $[\delta(g)] = [\delta(g')]$.

Let g, g' be two homotopic source-paths of Σ and let $H : I \times I \to \Sigma$ be an homotopy from g to g'. In particular, $H(0,t) = g(t)$ and $H(1,t) = g'(t)$. Provided that g and g' as well as the surface spanned by H are sufficiently close to the units in Σ, we get an associated collection $(f_s)_{s\in I}$ of paths $f_s : I \to \Sigma$ defined by

$$f_s(t) = H(0,t)^{-1} H(s,t).$$

For any s in I, note that $f_s(0) = \epsilon(m_0)$ and $f_s(1) = \epsilon(m_1)$ where $m_0 = \alpha(g(0))$ and $m_1 = \beta(g(1))$. Let $\Gamma(s,t) : M \to \Sigma$ be a twice differentiable family of local bisections [12] of $\Sigma \rightrightarrows M$ such that, at each fixed s and t in I: $\Gamma(s,t)$ is a local bisection through $f_s(t)$. For each fixed $s \in I$, the adjoint action of the time-dependent local bisection $\Gamma_s = \Gamma(s,-)$ on g gives a path $\mathrm{Ad}_{\Gamma_s} g : I \to \Sigma$. More precisely, modulo obvious limitations due to the fact Σ is a local Lie groupoid, we have

$$\begin{aligned}(\mathrm{Ad}_{\Gamma_s} g)(t) &= \Gamma(s,0)^{-1} g(t) \Gamma(s,t)\\ &= \epsilon(m_0) g(t) H(0,t)^{-1} H(s,t)\\ &= g(t) g(t)^{-1} H(s,t)\\ &= H(s,t).\end{aligned}$$

Here the second line follows from the fact that $\Gamma(s,0)$ is a local bisection through $f_s(0) = \epsilon(m_0)$. In particular, it follows that

$$\mathrm{Ad}_{\Gamma_0}(g) = g \quad \text{and} \quad \mathrm{Ad}_{\Gamma_1}(g) = g'. \tag{2.7}$$

Let $\mathcal{G}$ denote the semi-group of time-dependent local bisections of $\Sigma \rightrightarrows M$. We conclude from the above that if g and g' are homotopic then there exists a $\mathcal{C}^2$-path $\Gamma : I \to \mathcal{G}$ such that (2.7) is satisfied. The above discussion can easily be reversed to obtain the converse.

For any $s \in I$, let $g_s : I \to \Sigma$ be the intermediate source-path

$$g_s(t) = H(s,t).$$

Also let γ_s be the path traced by the targets of g_s, i.e. $\gamma_s(t) = \beta(g_s(t))$ and let e_s be the A-path associated to g_s, i.e. $e_s(t) = (\delta g_s)(t)$. It is easily seen that

$$e_s(t) = \left(\mathrm{Ad}_{\Gamma_s} e_0\right)(t) + \delta_t \Gamma(s,t)|_{\gamma_s(t)} \tag{2.8}$$

for all s in I, where $\delta_t \Gamma(s,t)$ is the local section of E defined by

$$\delta_t \Gamma(s,t) = \Gamma(s,t)^{-1}\, \partial_t \Gamma(s,t)$$

and $\mathrm{Ad}_{\Gamma_s} e_0 : I \to E$ is the A-path defined by

$$(\mathrm{Ad}_{\Gamma_s} e_0)(t) = \Gamma(s,t)^{-1}\, e_0(t)\, \Gamma(s,t).$$

Let $\delta_s \Gamma(s,t)$ be the local section of E defined by

$$\delta_s \Gamma(s,t) = \Gamma(s,t)^{-1} \partial_s \Gamma(s,t).$$

By differentiating (2.8) with respect to s, we get the equation

$$(\mathrm{ev}_t \circ \mathfrak{I}) \left(\frac{d}{ds} \delta g_s\right) = \mathrm{ad}_{\delta_s \Gamma(s,t)}\Big|_{e_s(t)} + \partial_s(\delta_t \Gamma)(s,t)|_{\gamma_s(t)}\,. \tag{2.9}$$

It is easily checked that

$$\partial_t(\delta_s\Gamma) = \partial_s(\delta_t\Gamma)$$

and thus we can rewrite (2.9) as

$$(\mathrm{ev}_t \circ \mathfrak{I})\left(\frac{d}{ds}\delta g_s\right) = \mathrm{ad}_{\sigma_s(-,t)}\Big|_{e_s(t)} + \partial_t\sigma_s(\gamma_s(t), t)$$

where $\sigma_s(m,t) = \delta_s\Gamma(s,t)|_m$. Comparing with (2.6) it follows that the path

$$s \in I \mapsto \delta g_s = e_s \in \mathscr{A}(E)$$

lies entirely in a single leaf of $\mathcal{F}(E)$. Thus δg is in the same leaf of $\mathcal{F}(E)$ as $\delta g'$, as desired. Conversely, let V be the open subset of Theorem 2.1.2 and let e and e' be two elements of V connected by a path in a single leaf of $V \cap \mathcal{F}(E)$. It is easy to proceed through the argument above in reverse to obtain an homotopy from $\delta^{-1}e$ to $\delta^{-1}e'$. In particular, it follows that w_Σ is invertible locally around M.

That w_Σ is compatible with the local groupoid structures is standard and can be shown as in the proof Theorem 3.20 of [16]. This concludes the proof. □

Let E be a Lie algebroid over M and let $\mathscr{W}_{\mathrm{loc}}(E) \rightrightarrows M$ be the Weinstein local groupoid of E. Let $\mathscr{E}$ be the Lie algebroid of $\mathscr{W}_{\mathrm{loc}}(E)$, which (as a vector bundle) is a bundle over M. Let $m \in M$. By definition, the fibre of $\mathscr{E}$ at m is the collection of all elements of $T\mathscr{W}_{\mathrm{loc}}(E)$ of the form

$$\frac{d}{ds}[e_s]\Big|_{s=0}$$

where, for all $s \geq 0$, the path $e_s : I \to E$ is an A-path such that

$$\mathrm{pr}(e_s(0)) = m$$

and where $e_0 : I \to E$ is the constant zero A-path $e_0(t) = 0_m$.

Remark 2.1.4. Let $\Sigma \rightrightarrows M$ be a local Lie groupoid with Lie algebroid E and let $w_\Sigma : \Sigma \to \mathscr{W}_{\mathrm{loc}}(E)$ be the map of Theorem 2.1.3. Let $Aw_\Sigma : E \to \mathscr{E}$ be the map obtained by applying the Lie functor [12] to w_Σ. Let e_0 be an element of E and let e be an A-path such that $e(0) = e_0$. Further, for any $s \geq 0$, let $e_s : I \to E$ be the A-path defined by

$$e_s(t) = se(st),$$

for all $t \in I$. It is easily seen that $Aw_\Sigma(e_0)$ is the derivative at $s = 0$ of the path $I \to \mathscr{W}_{\mathrm{loc}}(\Sigma) : s \mapsto [e_s]$. Note that

$$\frac{d}{ds}e_s(t)\Big|_{s=0} = e(0) + t\dot{e}(0) \tag{2.10}$$

for all t in I, where $e(0)$ is viewed as a vertical tangent vector to E at 0_m and where $\dot{e}(0)$ is viewed as a horizontal tangent vector to E at 0_m. Since w_Σ is an isomorphism of local Lie groupoids, Aw_Σ is an isomorphism of Lie algebroids. Thus from (2.10) it follows that any element X of the fibre of $\mathscr{E}$ over m is of the form

$$X = \left[\mathfrak{I}^{-1}\ell_{e_0}\right]$$

for a unique e_0 in the fibre of E over m, where $\ell_{e_0} : I \to T_{0_m}E$ is the path

$$\ell_{e_0}(t) = e_0 \oplus t\rho(e_0)$$

for all $t \in I$, where $\rho : E \to TM$ is the anchor of E and where we have, as in (2.10), used the canonical splitting $T_{0_m}E = E_m \oplus T_mM$. In particular, the map

$$\mathfrak{w}_E : E \to \mathscr{E} : e_0 \mapsto [\mathfrak{I}^{-1}\ell_{e_0}]$$

is an isomorphism of Lie algebroids over M. It is clear that $\mathfrak{w}_E$ only depends on E.

Proposition 2.1.3. *Let $\Sigma \rightrightarrows M$ (resp. $\Sigma' \rightrightarrows M$) be a local Lie groupoid with Lie algebroid E (resp. E') and let $\mathscr{W}_{loc}(E) \rightrightarrows M$ (resp. $\mathscr{W}_{loc}(E') \rightrightarrows M$) be the Weinstein local Lie groupoid of E (resp. E'). Let $\varphi : \Sigma \to \Sigma'$ be a morphism of local Lie groupoids. The map*

$$\mathscr{W}_{loc}(\varphi_*) : \mathscr{W}_{loc}(E) \to \mathscr{W}_{loc}(E') : [e] \mapsto [\varphi_* \circ e]$$

is a well-defined morphism of local Lie groupoids. Further, the following diagram:

$$\begin{array}{ccc} \Sigma & \xrightarrow{\varphi} & \Sigma' \\ \downarrow{\scriptstyle w_\Sigma} & & \downarrow{\scriptstyle w_{\Sigma'}} \\ \mathscr{W}_{loc}(E) & \xrightarrow{\mathscr{W}_{loc}(\varphi_*)} & \mathscr{W}_{loc}(E') \end{array} \tag{2.11}$$

commutes.

Proof. For any g in $\mathscr{S}(\Sigma)$ sufficiently close to $\mathscr{C}(M)$, let $g' = \varphi \circ g$. We then have the easy chain of equalities

$$\begin{aligned} \delta g'(t) &= g'^{-1}(t) \cdot \dot{g}'(t), \\ &= (\varphi(g^{-1}(t)) \cdot (\varphi_*(\dot{g}(t))), \\ &= \varphi_*(\delta g(t)), \end{aligned}$$

from which it follows easily that the diagram (2.11) commutes. Finally, from the fact that both w_Σ and $w_{\Sigma'}$ are isomorphisms of local Lie groupoids, it follows that $\mathscr{W}_{\mathrm{loc}}(\varphi_*)$ is a morphism of local Lie groupoids. ◻

2.2 The Cattaneo-Felder Groupoid

Let M be a smooth manifold and let θ be the Liouville 1-form on $T^\vee M$. Let ω_{can} be the symplectic 2-form on $T^\vee M$ defined as the exterior derivative of θ. There is an associated canonical symplectic 2-form $\tilde{\omega}_{\mathrm{can}}$ on the path-space $\mathscr{P}(T^\vee M)$. To see this, for any e in $\mathscr{P}(T^\vee M)$, we first define a skew-symmetric pairing

$$\tilde{\omega}'_{\mathrm{can},e} : \mathscr{P}_e(TT^\vee M) \times \mathscr{P}_e(TT^\vee M) \to \mathbb{R}$$

by the formula

$$\tilde{\omega}'_{\mathrm{can},e}(X, Y) = \int_0^1 \omega_{\mathrm{can}}(X(t), Y(t))dt \tag{2.12}$$

for any X and Y in $\mathscr{P}_e(TT^\vee M)$.

The next lemma is due to Cattaneo-Felder [13].

Lemma 2.2.1. *Let $\tilde{\omega}_{can}$ be the differential 2-form on $\mathscr{P}(T^\vee M)$ defined by*

$$\tilde{\omega}_{can,e} = (\wedge^2 \mathfrak{I}_\psi)^\top \, \tilde{\omega}'_{can,e}$$

for any ψ in $\mathscr{P}(T^\vee M)$. The 2-form $\tilde{\omega}_{can}$ is closed and non-degenerate.

In particular, the path space $\mathscr{P}(T^\vee M)$ is a (infinite dimensional) symplectic manifold. Recall that a vector field X on $\mathscr{P}(T^\vee M)$ is called *Hamiltonian* if there exists a smooth function $H \in \mathcal{C}^\infty(\mathscr{P}(T^\vee M))$ such that $\tilde{\omega}_{\mathrm{can}}(X, -) = dH$. In that case, we will say X is the Hamiltonian vector field *generated by* H, and will denote X by X_H.

Let π be a Poisson bivector field on M and let E_π be the cotangent bundle Lie algebroid of (M, π). Let $\mathscr{A}(E_\pi)$ be the Banach manifold of all A-paths in E_π and let $(\mathscr{G}(E_\pi), [\cdot, \cdot]_{\mathrm{T}})$ be the Lie algebra defined in the previous section. Recall that, as vector bundles, $E_\pi = T^\vee M$ so that $\mathscr{A}(E_\pi)$ is a Banach submanifold of $\mathscr{P}(T^\vee M)$.

For any ξ in $\mathscr{G}(E_\pi)$, define a functional H_ξ on $\mathscr{P}(T^\vee M)$ by the formula

$$H_\xi[\psi] = \int_0^1 \langle \xi(\psi(u), u), \frac{d}{du}\mathrm{pr}(\psi(u)) - \pi^\sharp(\psi(u)) \rangle \, du \tag{2.13}$$

where ψ is an element of $\mathscr{P}(T^\vee M)$. The following theorem is due to Cattaneo-Felder [13].

Theorem 2.2.1. *For any ξ in $\mathscr{G}(E_\pi)$, there exists a unique Hamiltonian vector field X_ξ generated by the functional H_ξ. Further, the map $\xi \mapsto X_\xi$ defines an action of $\mathscr{G}(E_\pi)$ on $\mathscr{P}(T^\vee M)$.*

The above theorem ensures that there is a Hamiltonian action of $\mathscr{G}(E_\pi)$ on $\mathscr{P}(T^\vee M)$. The next proposition shows that this action restricts to the action α of Lemma 2.1.1 on the submanifold $\mathscr{A}(E_\pi)$.

Proposition 2.2.1. *Let α be the action of $\mathscr{G}(E_\pi)$ on $\mathscr{A}(E_\pi)$ ensured by Lemma 2.1.1 and, for any ξ in $\mathscr{G}(E_\pi)$, let X_ξ be the Hamiltonian vector field ensured by Theorem 2.2.1. For all $\xi \in \mathscr{G}(E_\pi)$, the restriction $X_\xi|_{\mathscr{A}(E_\pi)}$ is tangent to $\mathscr{A}(E_\pi)$ and coincides with the vector field $\alpha(\xi)$.*

Proof. By a simple computation in local charts, it is seen that $\alpha(\xi)$ coincides with the formula (3.6) from [13] for X_ξ. □

Let $\tilde{\omega}_{\text{can}}$ be the symplectic 2-form ensured by Lemma 2.2.1 and let ω denote the (closed) differential 2-form obtained by restricting $\tilde{\omega}_{\text{can}}$ to the submanifold $\mathscr{A}(E_\pi)$. A simple consequence of Theorem 2.2.1 is that ω is invariant under the action of $\mathscr{G}(E_\pi)$. Further, Cattaneo-Felder [13] proved that the null spaces of ω are exactly the tangent spaces to the orbits of that action. In particular, one has the following corollary.

Corollary 2.2.1. *Let M be a smooth manifold and π be a Poisson bivector field on M. Let E_π be the cotangent bundle Lie algebroid of (M, π) and let $\mathscr{W}_{loc}(E_\pi) \rightrightarrows M$ be the Weinstein local groupoid of E_π. There exists a multiplicative symplectic 2-form Ω on the arrow space $\mathscr{W}_{loc}(E_\pi)$ such that*

$$\Omega_{[\psi]}\left([X],\ [Y] \right) = \tilde{\omega}_{can,\psi}\left(X, Y\right) \tag{2.14}$$

for any ψ in $\mathscr{A}(E_\pi)$ sufficiently close to $\mathscr{C}(M)$, and any X, Y in $T_\psi\mathscr{A}(E_\pi)$.

Proof. The existence of Ω follows from the discussion above. The multiplicativity of Ω is an easy consequence of the definition of the product operation in $\mathscr{W}_{\text{loc}}(E_\pi)$ and the fact the formula defining ω over the concatenation of two A-paths is the sum of the formula defining ω over each segment. □

Definition 2.2.1. Let M be a smooth manifold endowed with a Poisson bivector field π. The *Cattaneo-Felder symplectic local groupoid* ("CF local groupoid" for short) of (M, π) is the local groupoid

$$\mathscr{CF}_{\text{loc}}(M) \rightrightarrows M,$$

defined as the Weinstein local groupoid of E_π, endowed with the multiplicative symplectic 2-form Ω ensured by Corollary 2.2.1.

Before we close this section, we briefly spell out how the Cattaneo-Felder symplectic local groupoid relates to the phase space of the Poisson sigma model. In order to do this, let M be any smooth manifold. Note that a path $[-T, T] \to \mathscr{P}(T^\vee M)$ is the same thing as a map $[-T, T] \times I \to T^\vee M$. Let R be the rectangle $[-T, T] \times I$. We will call the $[-T, T]$-direction the "time direction", and the I-direction the "spatial direction". The associated coordinate functions will be denoted

t and u respectively. By a *field* in the theory we mean a continuously differentiable map $\psi : R \to T^\vee M$ thought of as a path $\psi : [-T, T] \to \mathscr{P}(T^\vee M)$.

Let π be a Poisson bivector field on M and let E_π denote the cotangent bundle Lie algebroid of (M, π). Let $\xi : R \to T^\vee M$ be a field satisfying the boundary conditions

$$\xi(t, 0) = 0 = \xi(t, 1)$$

for any t in $[-T, T]$. We define a functional $\mathrm{Con}[\psi, \xi]$ by the formula

$$\mathrm{Con}[\psi, \xi] = \int_R \langle \xi(t, u), \frac{d}{du}\mathrm{pr}(\psi(t, u)) - \pi^\sharp(\psi(t, u)) \rangle \; dudt.$$

and a functional $\mathrm{S}[\psi, \xi]$ by

$$\mathrm{S}[\psi, \xi] = \mathrm{S}_0[\psi] + \mathrm{Con}[\psi, \xi] \tag{2.15}$$

where

$$\mathrm{S}_0[\psi] = \int_R -\langle \psi(t, u), \frac{d}{dt}\mathrm{pr}(\psi(t, u)) \rangle \; dudt, \tag{2.16}$$

We can view ξ as Lagrange multipliers: the equations of motion for (2.15) will enforce the constraint

$$\frac{d}{du}\mathrm{pr}(\psi(t, u)) - \pi^\sharp(\psi(t, u)) = 0$$

which guarantees that, for a solution ψ, the slice $\psi(t, -)$ is an A-path for all $t \in I$.

The proof of the next proposition is a straightforward computation for which we refer the reader to [13].

Proposition 2.2.2. *Let M be a smooth manifold and π be a Poisson bivector field on M. The functional (2.15) is a special case of the action functional of the Poisson sigma model associated to (M, π).*

Chapter 3
Non-Degenerate Poisson-Nijenhuis Manifolds

3.1 Non-Degeneracy and Symplectic Realizations

In this section we define non-degenerate Poisson-Nijenhuis structures and their symplectic realizations.

Definition 3.1.1. Let M be a smooth manifold and let (π, N) be a PN structure on M. We say the Poisson-Nijenhuis structure (π, N) is *non-degenerate* if the Nijenhuis tensor N is invertible as a bundle morphism $N : TM \to TM$.

Example 3.1.1. Let $\mathscr{M}$ be a complex manifold and let $\pi = \pi_R + i\pi_J$ be a holomorphic Poisson bivector field on $\mathscr{M}$. Let M denote the underlying real manifold of $\mathscr{M}$ and let J be the integrable almost complex structure of $\mathscr{M}$. From Proposition 1.4.2 it follows that the pair (π_I, J) is a PN structure on M. It is clearly non-degenerate since $J^{-1} = -J$.

Definition 3.1.2. Let M (resp. M') be a smooth manifold and let (π, N) (resp. (π', N')) be a PN structure on M (resp. M'). A smooth map $f : M \to M'$ is called a *Poisson-Nijenhuis map* if

$$f_* \pi = \pi' \quad \text{and} \quad f_* \circ N = N' \circ f_*.$$

In other words, f is a Poisson-Nijenhuis map if and only if f is a Poisson map whose differential commutes with the Nijenhuis tensors.

Remark 3.1.1. In the notations of Definition 3.1.2, if $f : M \to M'$ is a Poisson-Nijenhuis map, then $f_* \pi_N = \pi'_N$ where π_N (resp. $\pi'_{N'}$) is the Poisson bivector field on M (resp. M') defined by the equation $\pi_N^\sharp = N \circ \pi^\sharp$ (resp. $\pi'_{N'} = N' \circ \pi'^\sharp$).

Definition 3.1.3. Let M be a smooth manifold and let (π, N) be a non-degenerate PN structure on M. Let π_N be the Poisson bivector field on M defined by the equation $\pi_N^\sharp = N \circ \pi^\sharp$. A *symplectic realization of the non-degenerate PN structure* (π, N) *on* M is a quadruple $(W, \Omega, \Omega', \varphi)$ consisting of a smooth manifold W, two symplectic 2-forms Ω and Ω' on W, and a surjective submersion $\varphi : W \to M$ such that

– the triple (W, Ω, φ) (resp. (W, Ω', φ)) is a symplectic realization of (M, π) (resp. (M, π_N));

– let Π (resp. Π') be the Poisson bivector field on W defined by the equation $\Pi^\sharp = (\Omega^\flat)^{-1}$ (resp. $\Pi'^\sharp = (\Omega'^\flat)^{-1}$), then the Schouten-Nijenhuis bracket $[\![\Pi, \Pi']\!]$ vanishes.

Definition 3.1.4. Let M be a smooth manifold. A *symplectic-Nijenhuis structure* on M is a pair (ω, N) consisting of a symplectic 2-form ω on M and a $(1, 1)$-tensor N on M such that the pair (π, N) is a PN structure on M, where π is the Poisson bivector field defined by the equation $\pi^\sharp = (\omega^\flat)^{-1}$.

We say the symplectic-Nijenhuis structure (ω, N) is *non-degenerate* if N is invertible as a bundle morphism $N : TM \to TM$.

Remark 3.1.2. Let (π, N) be a PN structure on M. Let $(W, \Omega, \Omega', \varphi)$ be a symplectic realization of (π, N) and let $\mathcal{N}$ be the $(1, 1)$-tensor on W defined by the "difference" $(\Omega'^\flat)^{-1}\Omega^\flat$. From Proposition 1.2.2 it follows that $(W, \mathcal{N})$ is a non-degenerate symplectic-Nijenhuis structure and it is easy to see that $\varphi : W \to M$ is a Poisson-Nijenhuis map with respect to the underlying PN structures. The reverse statement is true, in an obvious sense.

3.2 The Main Theorem

Let M be a smooth manifold and (π, N) a Poisson-Nijenhuis structure on M. Let ω_{can} denote the canonical symplectic 2-form on $T^\vee M$. Let E_π be the cotangent bundle Lie algebroid of (M, π) and let ∇ be an E_π-connection on E_π. Let φ_t^∇ denote the flow of the geodesic vector field of ∇. As we will see below (in Proposition 3.4.1), there is an open neighbourhood U of $0_M \subset T^\vee M$ such that $\varphi_t^\nabla : U \to T^\vee M$ is defined for all t in I.

The following theorem is the main result of this dissertation.

Theorem 3.2.1. *Assume the pair* (π, N) *is a non-degenerate Poisson-Nijenhuis structure on* M*. Let* ω*,* ω_N *be the differential* 2*-forms on* U *defined by the formulae*

$$\omega = \int_0^1 (\varphi_t^\nabla)^* \omega_{can} \, dt, \tag{3.1}$$

$$\omega_N = \int_0^1 ((N^{-1})^\top \circ \varphi_t^\nabla)^* \omega_{can} \, dt, \tag{3.2}$$

Then ω and ω_N are symplectic. Further the tensor $\mathbb{N}$ on U defined by

$$\mathbb{N} = (\omega_N^\flat)^{-1} \circ \omega^\flat, \tag{3.3}$$

is Nijenhuis and the pair $(\omega, \mathbb{N})$ is a symplectic-Nijenhuis structure on U. Finally, the quadruple $(U, \omega, \omega_N, pr|_U : U \to M)$ is a symplectic realization of the Poisson-Nijenhuis structure (π, N) on M.

It is highly surprising that such simple formulae as (3.1) and (3.2) lead to as much structure as that is needed for a symplectic realization. Our approach relies on proving that the symplectic forms ω and ω_N are both part of an implicit *symplectic-Nijenhuis local groupoid* (see Definition 3.5.2 below) subtly related to (π, N).

The goal of this chapter is to prove Theorem 3.2.1.

3.3 Poisson Groupoids and Lie Bialgebroids

Poisson groupoids were introduced by Weinstein [20] as a generalization of both Poisson Lie groups and the symplectic groupoids which arise in the integration theory of Poisson manifolds.

Definition 3.3.1. Let $\Sigma \rightrightarrows M$ be a local Lie groupoid and let π be a Poisson bivector field on Σ. We say π is *multiplicative* if the the bundle map

$$\pi^\sharp : T^\vee\Sigma \to T\Sigma$$

is a morphism of local Lie groupoids.

A (local) Lie groupoid endowed with a multiplicative Poisson bivector field is called a *Poisson (local) groupoid*. By a *(iso-)morphism of Poisson local groupoids* we mean a (iso-)morphism of local Lie groupoids that is, in addition, a Poisson map.

The next proposition is elementary.

Proposition 3.3.1. *Let $\Sigma \rightrightarrows M$ be a local Lie groupoid and let Π be a multiplicative Poisson bivector field on Σ such that $\Pi^\sharp : T^\vee\Sigma \to T\Sigma$ is invertible. Then the symplectic 2-form Ω on Σ defined by the equation $\Omega^\flat = (\Pi^\sharp)^{-1}$ is multiplicative.*

Conversely, let $\Sigma \rightrightarrows M$ be a local Lie groupoid and let Ω be a multiplicative symplectic 2-form on Σ. Then the Poisson bivector field Π, defined by the equation $\Pi^\sharp = (\Omega^\flat)^{-1}$, is multiplicative.

Just as local Lie groupoids are in one-one correspondence with Lie algebroids (up to isomorphism), Poisson local groupoids are in one-one correspondence with Lie bialgebroids (up to isomorphism). In what follows, we only need one direction of this correspondence namely, going from a Lie bialgebroid to a Poisson local groupoid. This result, which is due to Mackenzie-Xu [21], is summarized in Theorem 3.3.1. We will also need to study the behavior of this correspondence under morphisms of Lie bialgebroids, which is the content of Theorem 3.3.2.

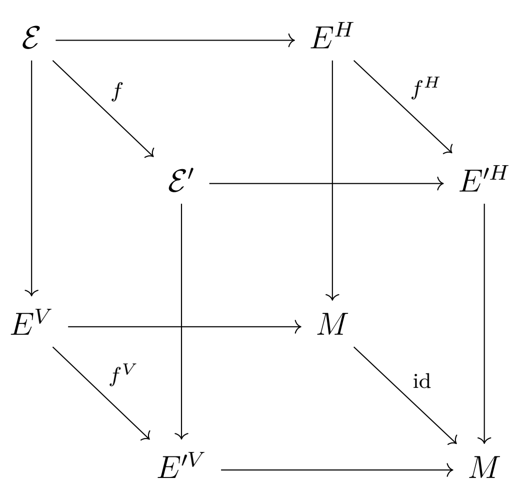
3.3.1 A Few Double Vector Bundles

Recall that a double vector bundle over M is a commuting diagram

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\quad h \quad} & E^H \\ {\scriptstyle v}\big\downarrow & & \big\downarrow{\scriptstyle v_0} \\ E^V & \xrightarrow{\quad h_0 \quad} & M \end{array} \tag{3.4}$$

where $\mathcal{E}$ is a vector bundle over E^H (resp. E^V) with basepoint projection h (resp. v), and E^H (resp. E^V) is a vector bundle over M with basepoint projection v_0 (resp. h_0). We further require that h (resp. v) is a vector bundle morphism over h_0 (resp. v_0). We call $\mathcal{E}$ the total space of the double vector bundle and we call the vector bundle $\mathcal{E} \to E^H$ (resp. $\mathcal{E} \to E^V$) the *horizontal bundle* (resp. the *vertical bundle*).

A morphism of double vector bundles (over the identity on M) is a triple (f, f^H, f^V) of maps fitting in a commuting diagram



(3.5)

such that f is a vector bundle morphism over both f^V and f^H and such that f^H (resp. f^V) is a vector bundle morphism over id. Note that f^H and f^V are uniquely determined by f since $f^H = f|_{0_H}$ and $f^V = f|_{0_V}$ where 0_H (resp. 0_V) denotes the zero section of the horizontal (resp. vertical) vector bundle structure. Thus we will call a map $f : \mathcal{E} \to \mathcal{E}'$ between the total spaces of two double vector bundles a *morphism of double vector bundles* if the triple $(f, f|_{0_H}, f|_{0_V})$ is a morphism of double vector bundles.

Let $\bar{\mathcal{E}}$ denote the total space of the double vector bundle obtained by mirroring the diagram (3.4) across the diagonal. We say a map $f : \mathcal{E} \to \mathcal{E}'$ is a *mirroring morphism of double vector bundles* if f is a morphism of double vector bundles from $\bar{\mathcal{E}}$ to $\mathcal{E}'$.

Let E be a vector bundle over a smooth manifold M. The tangent bundle TE is the total space of a natural double vector bundle

$$\begin{array}{ccc} TE & \xrightarrow{\mathrm{pr}_*} & TM \\ {\scriptstyle \mathrm{pr}_{TE}}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_{TM}} \\ E & \xrightarrow{\mathrm{pr}_E} & M \end{array} \tag{3.6}$$

where pr_* is the differential of the basepoint projection $\mathrm{pr} : E \to M$. Similarly, the cotangent bundle $T^\vee E$ is the total space of the double vector bundle

$$\begin{array}{ccc} T^\vee E & \xrightarrow{\mathrm{pr}_{T^\vee E}} & E \\ {\scriptstyle \mathrm{pr}^!}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_E} \\ E^\vee & \xrightarrow{\mathrm{pr}_{E^\vee}} & M \end{array} \tag{3.7}$$

where $\mathrm{pr}^! : T^\vee E \to E^\vee$ is the map defined by

$$\mathrm{pr}^!\big|_{T_e^\vee E} = \mathrm{vert}_e^\top$$

for all e in E. Recall $vert_e : E_{\mathrm{pr}(e)} \to T_e E$ is the embedding of the fibre as the vertical tangent subspace of E at e.

Remark 3.3.1. Let Σ be a smooth manifold (not necessarily a local Lie groupoid). Applying the double vector bundle (3.6) to $E = T\Sigma$ we get the diagram

$$\begin{array}{ccc} TT\Sigma & \xrightarrow{\mathrm{pr}_*} & T\Sigma \\ {\scriptstyle \mathrm{pr}_{TT\Sigma}}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_{T\Sigma}} \\ T\Sigma & \xrightarrow{\mathrm{pr}_{T\Sigma}} & \Sigma \end{array} \tag{3.8}$$

Thus the horizontal bundle of $TT\Sigma$ is, by convention, the vector bundle with projection $\mathrm{pr}_* : TT\Sigma \to T\Sigma$.

3.3.2 The Lie Algebroids of $T^\vee\Sigma$ and $T\Sigma$

Let $\Sigma \rightrightarrows M$ be a local Lie groupoid with Lie algebroid E. Let $T^\vee\Sigma \rightrightarrows E^\vee$ (resp. $T\Sigma \rightrightarrows TM$) denote the cotangent (resp. tangent) local Lie groupoid of $\Sigma \rightrightarrows M$. Let

$$AT^\vee\Sigma \quad (\text{resp. } AT\Sigma)$$

denote the Lie algebroid of $T^\vee\Sigma \rightrightarrows E^\vee$ (resp. $T\Sigma \rightrightarrows TM$). Note that $AT^\vee\Sigma$ (resp. $AT\Sigma$) is a bundle over $E^\vee$ (resp. TM).

The Lie algebroid $AT\Sigma$ is the total space of the double vector bundle

$$\begin{array}{ccc} AT\Sigma & \xrightarrow{\mathrm{pr}_{AT\Sigma}} & TM \\ {\scriptstyle A\mathrm{pr}}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_{TM}} \\ E & \xrightarrow{\mathrm{pr}_E} & M \end{array} \tag{3.9}$$

where $A\mathrm{pr}$ is the map obtained by applying the Lie functor to the basepoint projection $\mathrm{pr}_{T\Sigma} : T\Sigma \to \Sigma$. The Lie algebroid $AT^\vee\Sigma$ is the total space of the double vector bundle

$$\begin{array}{ccc} AT^\vee\Sigma & \xrightarrow{A\mathrm{pr}} & E \\ {\scriptstyle \mathrm{pr}_A}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_E} \\ E^\vee & \xrightarrow{\mathrm{pr}_{E^\vee}} & M \end{array}$$

where $A\mathrm{pr}$ is the map obtained by applying the Lie functor to the basepoint projection $\mathrm{pr}_{T^\vee\Sigma} : T^\vee\Sigma \to \Sigma$ and pr_A is the basepoint projection of the Lie algebroid $AT^\vee\Sigma$.

Let $j : TT\Sigma \to TT\Sigma$ denote the canonical flip of the double tangent bundles. Recall that the map j interchanges the horizontal and vertical bundles of $TT\Sigma$, i.e. j is a mirroring isomorphism. Let

$$i : AT\Sigma \to TT\Sigma$$

be the natural embedding of $AT\Sigma$ into $TT\Sigma$. Note that i is a mirroring morphism of double vector bundles where $AT\Sigma$ is considered as the total space of (3.9). Also, let

$$i' : TE \to TT\Sigma$$

be the differential of the natural embedding of E into $T\Sigma$. Note that i' is a morphism of double vector bundles where TE is considered as the total space of (3.6).

Recall from [2] that there is a (unique) natural isomorphism $j_\Sigma : AT\Sigma \to TE$ of double vector bundles such that the diagram of double vector bundles

$$\begin{array}{ccc} AT\Sigma & \xrightarrow{j_\Sigma} & TE \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i'} \\ \overline{TT\Sigma} & \xrightarrow{j} & TT\Sigma \end{array}$$

commutes.

The canonical pairing of a vector bundle with its dual induces a morphism of Lie groupoids

$$P : T^\vee\Sigma \times_\Sigma T\Sigma \to \mathbb{R}$$

By applying the Lie functor to P we obtain a pairing

$$AP : AT^\vee\Sigma \times_E AT\Sigma \to \mathbb{R}$$

which is non-degenerate [2]. In particular AP induces an isomorphism of vector bundles $i_\Sigma : AT^\vee\Sigma \to (AT\Sigma)^\dagger$ where $(AT\Sigma)^\dagger$ is the dual of the vertical vector bundle structure $AT\Sigma \to E$. Composing with the dual of j_Σ over E we get a map

$$j'_\Sigma = j_\Sigma^\top \circ i_\Sigma : AT^\vee\Sigma \to T^\vee E$$

which can be shown [2] to be a natural isomorphism of double vector bundles, where $T^\vee E$ is considered as the total space of the double vector bundle (3.7).

3.3.3 From Lie Bialgebroids to Poisson Local Groupoids

The following theorem is a straightforward adaptation to local Lie groupoids of a classical result due to Mackenzie-Xu [21]. Its proof is identical to that of Theorem 4.1 in [21], be it for obvious adjustments.

Theorem 3.3.1. *Let $(E, E^\vee)$ be a Lie bialgebroid and let $\Sigma \rightrightarrows M$ be a local Lie groupoid with Lie algebroid E. Let π_{LP} be the Lie-Poisson bivector field on E defined by the Lie algebroid structure on $E^\vee$. There is a unique multiplicative Poisson bivector field Π on Σ such that the following diagram:*

$$\begin{array}{ccc} AT^\vee\Sigma & \xrightarrow{A\Pi^\sharp} & AT\Sigma \\ {\scriptstyle j'_\Sigma}\downarrow & & \downarrow{\scriptstyle j_\Sigma} \\ T^\vee E & \xrightarrow{\pi_{LP}^\sharp} & TE \end{array} \qquad (3.10)$$

commutes. Here $A\Pi^\sharp : AT^\vee\Sigma \to AT\Sigma$ is the map obtained by applying the Lie functor to $\Pi^\sharp : T^\vee\Sigma \to T\Sigma$.

Let M be a smooth manifold and let π be a Poisson bivector field on M. Let E_π be the cotangent bundle Lie algebroid of (M, π). Recall that, as vector bundles, $E_\pi = T^\vee M$. In particular, we endow $E_\pi^\vee$ with a Lie algebroid structure isomorphic to the tangent Lie algebroid structure on TM. It is easy to see that the dual pair of Lie algebroids $(E_\pi, E_\pi^\vee)$ is a Lie bialgebroid. Let $\Sigma \rightrightarrows M$ be any local Lie groupoid with Lie algebroid E_π and let Π be the multiplicative Poisson bivector field on Σ ensured by Theorem 3.3.1 applied to the pair $(E_\pi, E_\pi^\vee)$. From the commutativity of the diagram (3.10) it is trivial that Π is non-degenerate. In particular, let Ω be the symplectic 2-form defined by the equation $\Omega^\flat = (\Pi^\sharp)^{-1}$. Then Ω is multiplicative and the pair $(\Sigma \rightrightarrows M, \Omega)$ is a symplectic local groupoid realizing (M, π). The following corollary summarizes this paragraph for later reference.

Corollary 3.3.1. *Let M be a smooth manifold and let π be a Poisson bivector field on M. Let E_π be the cotangent bundle Lie algebroid of (M, π) and let $\Sigma \rightrightarrows M$ be a local Lie groupoid with Lie algebroid E_π. There exists a unique multiplicative symplectic 2-form Ω on Σ such that the Poisson bivector field Π defined by the equation $\Pi^\sharp = (\Omega^\flat)^{-1}$ coincides with the Poisson bivector field ensured by Theorem 3.3.1 applied to the natural Lie bialgebroid $(E_\pi, E_\pi^\vee)$.*

Let $(\mathscr{CF}_{\mathrm{loc}}(M) \rightrightarrows M, \Omega)$ be the Cattaneo-Felder symplectic local groupoid of (M, π) and let $\mathscr{E}_\pi$ be the Lie algebroid of $\mathscr{CF}_{\mathrm{loc}}(M) \rightrightarrows M$. Let Π be the Poisson bivector field defined by the equation $\Pi^\sharp = (\Omega^\flat)^{-1}$. Let $A\Pi^\sharp$ (resp. $A\Omega^\flat$) be the map obtained by applying the Lie functor to the map $\Pi^\sharp : T^\vee\Sigma \to T\Sigma$ (resp. $\Omega^\flat : T\Sigma \to T^\vee\Sigma$). It is clear that $A\Pi^\sharp = (A\Omega^\flat)^{-1}$. Our goal in the next few paragraphs is to prove that the dual vector bundle $\mathscr{E}_\pi^\vee$ has a natural Lie algebroid structure such that the pair $(\mathscr{E}_\pi, \mathscr{E}_\pi^\vee)$ is a Lie bialgebroid and with respect to which the multiplicative Poisson bivector field on $\mathscr{CF}_{\mathrm{loc}}(M)$ ensured by Theorem 3.3.1 is Π.

Recall from Remark 2.1.4 that there is an isomorphism $E_\pi \to \mathscr{E}_\pi$ of Lie algebroids, which we shall denote $\mathfrak{w}_\pi$. Dualizing we get an isomorphism $\mathfrak{w}_\pi^\top : \mathscr{E}_\pi^\vee \to E_\pi^\vee$ of vector bundles. Since $E_\pi^\vee = TM$, this allows us to transport the Lie algebroid structure $(TM, [\cdot,\cdot], \mathrm{id})$ to $\mathscr{E}_\pi^\vee$. Since $\mathfrak{w}_\pi$ is an isomorphism of Lie algebroids it is clear that the dual pair $(\mathscr{E}_\pi, \mathscr{E}_\pi^\vee)$ of Lie algebroids is a Lie bialgebroids and that the dual pair of maps $(\mathfrak{w}_\pi, \mathfrak{w}_\pi^\top)$ is an isomorphism of Lie bialgebroids from the pair $(E_\pi, E_\pi^\vee)$ to the pair $(\mathscr{E}_\pi, \mathscr{E}_\pi^\vee)$.

Let π_{LP} be the Lie-Poisson bivector field on $\mathscr{E}_\pi$ defined by the Lie algebroid structure on $\mathscr{E}_\pi^\vee$. It is clear that π_{LP} is non-degenerate. Let ω_{LP} be the symplectic 2-form on $\mathscr{E}_\pi$ defined by the equation $\omega_{\mathrm{LP}}^\flat = (\pi_{\mathrm{LP}}^\sharp)^{-1}$.

Lemma 3.3.1. *The following diagram:*

$$
\begin{array}{ccc}
AT^{\vee}\Sigma & \xleftarrow{A\Omega^{\flat}} & AT\Sigma \\
\downarrow{\scriptstyle j'_{\Sigma}} & & \downarrow{\scriptstyle j_{\Sigma}} \\
T^{\vee}\mathscr{E}_{\pi} & \xleftarrow{\omega^{\flat}_{LP}} & T\mathscr{E}_{\pi}
\end{array}
\tag{3.11}
$$

commutes, where $\Sigma = \mathscr{CF}_{loc}(M)$.

Proof. Choose a path $\gamma : I \to M$ and, for all s, r in I, let $e_{s,r} : I \to E_{\pi}$ (resp. $f_{s,r} : I \to E_{\pi}$) be an A-path such that

– 1) the element $e_{s,r}(0)$ (resp. $f_{s,r}(0)$) belongs to the fibre of E_{π} over $\gamma(r)$,

– 2) we have $e_{0,r}(t) = 0_{\gamma(r)}$ (resp. $f_{0,r}(t) = 0_{\gamma(r)}$) for all $r, t \in I$, and

– 3) we have $e_{s,0}(t) = f_{s,0}(t)$ for all $s, t \in I$.

We further assume that the map

$$I \times I \times I \to E_{\pi} : (s, r, t) \mapsto e_{s,r}(t) \quad (\text{resp. } I \times I \times I \to E_{\pi} : (s, r, t) \mapsto f_{s,r}(t))$$

is twice continuously differentiable. For any $s \in I$ (resp. any $r \in I$), we define

$$
\begin{aligned}
e_s(t) &:= \frac{d}{dr} e_{s,r}(t)\bigg|_{r=0} \quad \left(\text{resp. } e^r(t) := \frac{d}{ds} e_{s,r}(t)\bigg|_{s=0}\right) \\
f_s(t) &:= \frac{d}{dr} f_{s,r}(t)\bigg|_{r=0} \quad \left(\text{resp. } f^r(t) := \frac{d}{ds} f_{s,r}(t)\bigg|_{s=0}\right).
\end{aligned}
$$

For any $s \in I$ (resp. any $r \in I$), let X_s, Y_s (resp. X^r, Y^r) be the tangent vectors defined by

$$X_s = \left[\mathfrak{I}^{-1} e_s\right], Y_s = \left[\mathfrak{I}^{-1} f_s\right], \quad \left(\text{resp. } X^r = \left[\mathfrak{I}^{-1} e^r\right], Y^r = \left[\mathfrak{I}^{-1} f^r\right]\right).$$

Let

$$
\begin{aligned}
X &:= \frac{d}{ds} X_s\bigg|_{s=0} \in AT\Sigma, \quad x := \frac{d}{dr} X^r\bigg|_{r=0} \in T\mathscr{E}_{\pi}, \\
Y &:= \frac{d}{ds} Y_s\bigg|_{s=0} \in AT\Sigma \quad \text{and} \quad y := \frac{d}{dr} Y^r\bigg|_{r=0} \in T\mathscr{E}_{\pi}.
\end{aligned}
$$

By definition, we have

$$j_{\Sigma}(X) = x \quad \text{and} \quad j_{\Sigma}(Y) = y.$$

The commutativity of the diagram (3.11) will be ensured by proving that

$$\frac{d}{ds}\,\Omega(X_s,Y_s)\bigg|_{s=0} = \omega_{\mathrm{LP}}(x,y). \tag{3.12}$$

Note that, from the first assumption on $e_{s,r}(t)$ (resp. $f_{s,r}(t)$), it follows that $e^r(0)$ (resp. $f^r(0)$) belong to the vertical tangent space to E_π at $0_{\gamma(r)}$. Let $\hat{e}(r)$ (resp. $\hat{f}(r)$) denote the element of the fibre of E_π over $\gamma(r)$ corresponding to $e^r(0)$ (resp. $f^r(0)$) under the canonical isomorphism of the vertical tangent space of E_π at $0_{\gamma(r)}$ with the fibre of E_π at $\gamma(r)$. Further, let $j_E : TTE \to TTE$ be the canonical flip of the double tangent bundle of E. We have

$$\begin{aligned}\frac{d}{ds}\,\Omega(X_s,Y_s)\bigg|_{s=0} &= \Omega_*\left(j_\Sigma\left(\frac{d}{dr}X^r\bigg|_{r=0}\right), j_\Sigma\left(\frac{d}{dr}Y^r\bigg|_{r=0}\right)\right)\\ &= \Omega_*\left(j_\Sigma\left(\frac{d}{dr}[\mathfrak{I}^{-1}\ell_{\hat{e}(r)}]\bigg|_{r=0}\right), j_\Sigma\left(\frac{d}{dr}[\mathfrak{I}^{-1}\ell_{\hat{f}(r)}]\bigg|_{r=0}\right)\right)\\ &= \int_0^1 \omega_{\mathrm{can},*}\left(j_E\left(\frac{d}{dr}\ell_{\hat{e}(r)}(t)\bigg|_{r=0}\right), j_E\left(\frac{d}{dr}\ell_{\hat{f}(r)}(t)\bigg|_{r=0}\right)\right)\,dt\end{aligned} \tag{3.13}$$

where the second to last equation follows from Remark 2.1.4 and where

$$\begin{aligned}\ell_{\hat{e}(r)}(t) &= e^r(0) + t\rho_*(e^r(0)),\\ \ell_{\hat{f}(r)}(t) &= f^r(0) + t\rho_*(f^r(0)),\end{aligned}$$

as in the remark. It is easy to compute, using local charts, that the integrand of (3.13) is

$$\omega_{\mathrm{can}}\left(\frac{d}{dr}\hat{e}(r)\bigg|_{r=0}, \frac{d}{dr}\hat{f}(r)\bigg|_{r=0}\right),$$

so that, in particular, it is constant in t. On the other side of the desired equality (3.12), we have, by definition:

$$\omega_{\mathrm{LP}}(x,y) = \omega_{\mathrm{can}}\left(\frac{d}{dr}\hat{e}(r)\bigg|_{r=0}, \frac{d}{dr}\hat{f}(r)\bigg|_{r=0}\right).$$

This concludes the proof. □

We summarize the discussion above in the next proposition.

Proposition 3.3.2. *Let M be a smooth manifold and π be a Poisson bivector field on M. Let E_π be the cotangent bundle Lie algebroid of (M,π). Let $(\mathscr{CF}_{loc}(M) \rightrightarrows M, \Omega)$ be the Cattaneo-Felder symplectic local groupoid of (M,π) and let $\mathscr{E}_\pi$ be the Lie algebroid of $\mathscr{CF}_{loc}(M)$. The following assertions hold.*

- *There is a natural Lie algebroid structure on the vector bundle $\mathscr{E}_\pi^\vee \to M$ such that the pair $(\mathscr{E}_\pi, \mathscr{E}_\pi^\vee)$ is a Lie bialgebroid.*

- *Let $\mathfrak{w}_\pi : E_\pi \to \mathscr{E}_\pi$ denote the isomorphism of Lie algebroids of Remark 2.1.4 and $\mathfrak{w}_\pi^\top : \mathscr{E}_\pi^\vee \to E_\pi^\vee$ the dual of $\mathfrak{w}_\pi$. Then the dual pair of maps $(\mathfrak{w}_\pi, \mathfrak{w}_\pi^\top)$ is an isomorphism of Lie bialgebroids from the pair $(E_\pi, E_\pi^\vee)$ to the pair $(\mathscr{E}_\pi, \mathscr{E}_\pi^\vee)$.*

- *Let Π be the Poisson bivector field on $\mathscr{CF}_{loc}(M)$ defined by the equation $\Pi^\sharp = (\Omega^\flat)^{-1}$. Then Π coincides with the Poisson bivector field ensured by Theorem 3.3.1 applied to the Lie bialgebroid $(\mathscr{E}_\pi, \mathscr{E}_\pi^\vee)$.*

Let $\Sigma \rightrightarrows M$ and $\Sigma' \rightrightarrows M$ be local Lie groupoids. Let E (resp. E') be the Lie algebroid of Σ (resp. Σ'). Assume that $E^\vee$ and $E'^\vee$ have Lie algebroids structure such that the pairs $(E, E^\vee)$ and $(E', E'^\vee)$ are Lie bialgebroids. Let Π (resp. Π') be the multiplicative Poisson bivector field on Σ (resp. Σ') ensured by Theorem 3.3.1.

Theorem 3.3.2. *Let $\varphi : \Sigma \to \Sigma'$ be a morphism of local Lie groupoids and let $A\varphi : E \to E'$ be the map obtained by applying the Lie functor to φ. Then*

$$\varphi_* \Pi = \Pi'$$

if and only if the dual map

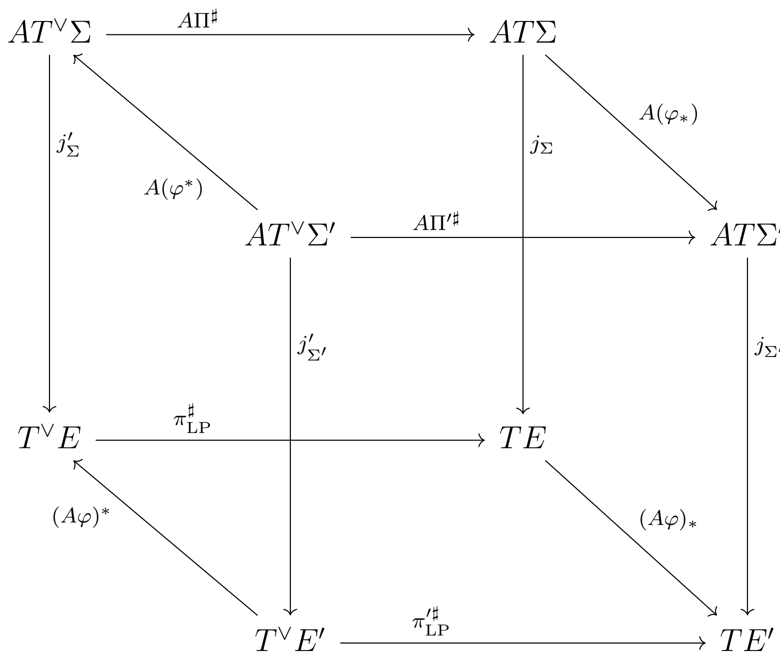
$$(A\varphi)^\top : E'^\vee \to E^\vee$$

is a morphism of Lie algebroids. In other words, φ is a Poisson map if and only if the dual pair of maps $(A\varphi, (A\varphi)^\top)$ is a morphism of Lie bialgebroids from $(E, E^\vee)$ to $(E', E'^\vee)$.

Proof. Let π_{LP} (resp. π'_{LP}) be the Lie-Poisson bivector field on E (resp. E') defined by the Lie algebroid structure on $E^\vee$ (resp. $E'^\vee$). Assume that $(A\varphi)^\top : E'^\vee \to E$ is a morphism of Lie algebroids. It is simple to see that

$$(A\varphi)_* \, \pi_{\mathrm{LP}} = \pi'_{\mathrm{LP}}, \tag{3.14}$$

i.e. that $A\varphi : E \to E'$ is a Poisson map. Consider the following cube:



where $A(\varphi^*)$ (resp. $A(\varphi_*)$) is the map obtained by applying the Lie functor to $\varphi^* : T^\vee\Sigma' \to T^\vee\Sigma$ (resp. $\varphi_* : T\Sigma \to T\Sigma'$). From (3.14) it follows that the bottom face commutes. From the naturality of j_Σ and j'_Σ, it is easily seen that the side-faces commute. Since the back and front faces commute as well (by assumption), it follows that the top face commutes. In particular, we have that

$$\begin{aligned} A(\varphi_* \circ \Pi^\sharp \circ \varphi^*) &= A(\varphi_*) \circ A\Pi^\sharp \circ A(\varphi^*) \\ &= A\Pi'^\sharp \end{aligned}$$

and thus

$$\varphi_* \circ \Pi^\sharp \circ \varphi^* = \Pi'^\sharp$$

as desired. The converse is obtained by going through the argument backward. This concludes the proof. □

3.4 Connections and Exponential Maps

In Lie theory, the classical exponential map establishes a local diffeomorphism from an open neighbourhood of zero in a Lie algebra to an open neighbourhood of the unit in the corresponding local Lie group. This construction extends to Lie algebroids and local Lie groupoids. Unlike the Lie algebra case, however, one needs to choose some additional geometrical structure, namely a connection. In this section, we recall some basic facts about the exponential map for Lie groupoids and describe the latter explicitly using the Weinstein local Lie groupoid.

Let M be a smooth manifold and let E be a Lie algebroid over M with anchor ρ and bracket $[\cdot, \cdot]$. By an *E-connection on E* we mean a $\mathbb{R}$-bilinear map

$$\nabla : \Gamma(E) \times \Gamma(E) \to \Gamma(E) : (X, Y) \mapsto \nabla_X Y$$

satisfying the conditions

$$\nabla_{fX} Y = f \nabla_X Y,$$
$$\nabla_X (fY) = (\rho(X) f) Y + f \nabla_X Y,$$

for any X, Y in $\Gamma(E)$ and any f in $\mathcal{C}^\infty(M)$.

Example 3.4.1. Let $\tilde{\nabla}$ be a linear connection on E as a vector bundle and let $\nabla_X Y := \tilde{\nabla}_{\rho(X)} Y$ for any X, Y in $\Gamma(E)$. Then ∇ is an E-connection on E. However, not every E-connection on E is of this form.

Definition 3.4.1. Let E be a Lie algebroid and ∇ be an E-connection on E. Let $e : I \to E$ be an A-path. We say e is *A-geodesic* if it satisfies the geodesic equation

$$\nabla_{e(t)} e(t) = 0 \tag{3.15}$$

for any t in I.

Remark 3.4.1. Let $e, e' : I \to E$ be two A-geodesic A-paths such that $e(0) = e'(0)$. From the definition of the underlying initial value problem to Equation 3.15 it easily follows that $e(t) = e'(t)$ for all $t \in I$.

An E-connection on E defines an bundle map

$$h : E \times_M E \to TE$$

called the *horizontal lifting map*. Explicitly: fix $m \in M$ and fix X, Y in E_m. Choose a section $\bar{Y} \in \Gamma(E)$ such that $\bar{Y}(m) = Y$ and let

$$h(X, Y) = \bar{Y}_*(\rho(X)) - \mathrm{vert}_Y(\nabla_X \bar{Y}(m)). \tag{3.16}$$

It is not hard to check that the right-hand side of (3.16) is independent of the choice of $\bar{Y}$.

Definition 3.4.2. Let E be a Lie algebroid and ∇ be an E-connection on E. The *geodesic vector field of* ∇ is the vector field ξ^∇ on E defined by

$$\xi^\nabla_X = h(X, X),$$

for any X in E.

Notation. Let E be a Lie algebroid and ∇ be an E-connection on E. The flow of the geodesic vector field of ∇ is denoted φ_t^∇. In other words, φ^∇ is the function

$$\varphi^\nabla : I \times E \to E : (t, e) \mapsto \varphi_t^\nabla(e)$$

that solves the initial value problem

$$\frac{d}{dt}\varphi_t^\nabla(e)\bigg|_{t=t_0} = \xi^\nabla\Big|_{\varphi_{t_0}^\nabla(e)}$$
$$\varphi_0^\nabla(e) = e,$$

for any fixed $e \in E$. Naturally, φ^∇ is only defined *a priori* on a neighbourhood of $\{0\} \times E$ in $I \times E$.

Proposition 3.4.1. *Let E be a Lie algebroid and ∇ be an E-connection on E. There is a neighbourhood U of 0_M in E such that $\varphi_t^\nabla : E \to E$ is defined for all $t \in I$. Further, for all $e \in U$, the path*

$$\varphi^\nabla(e) : I \to E : t \mapsto \varphi_t^\nabla(e)$$

is A-geodesic.

Proof. Let s be a real number and let $m_s : E \to E$ be the map defined by $m_s(e) = se$. It is easily checked that $s\xi_e^\nabla = (m_s)_*^{-1}\xi_{se}^\nabla$ for all $s > 0$ and all $e \in E$. It then follows that $s\varphi_{ts}^\nabla(e) = \varphi_t^\nabla(se)$ where one side is defined exactly when the other is. The existence of U as in the claim easily follows.

For the second part, let $e \in U$ and let $e(t) = \varphi_t^\nabla(e)$ for any $t \in I$. Let γ_e denote the base path of $e(t)$, i.e. $\gamma_e(t) = \mathrm{pr}(e(t))$. We have

$$\begin{aligned}\dot\gamma_e(t) &= \mathrm{pr}_*(\xi_{e(t)}^\nabla),\\ &= \mathrm{pr}_*(h(e(t), e(t))),\\ &= \rho(e(t)).\end{aligned}$$

Hence $e(t)$ is indeed an A-path. Choose a time-dependent section

$$\bar e : I \times M \to E$$

such that $\bar e(t, \gamma_e(t)) = e(t)$. For any $t \in I$, let $\bar e_t : M \to E$ be the slice of $\bar e$ at time t, i.e. $\bar e_t(m) = \bar e(t, m)$. Then

$$\begin{aligned}\nabla_{e(t)}e(t) &= \frac{d}{dt}\bar e(t, \gamma_e(t)) + (\nabla_{e(t)}\bar e_t)(\gamma_e(t)),\\ &= [\dot e(t) - \bar e_{t,*}(\dot\gamma_e(t))] + (\nabla_{e(t)}\bar e_t)(\gamma_e(t)),\\ &= \dot e(t) - \xi_{e(t)}^\nabla,\\ &= 0,\end{aligned}$$

as claimed. ∎

Example 3.4.2. Let M be a smooth manifold and π a Poisson bivector field on M. Let E_π denote the cotangent bundle Lie algebroid of (M, π) and let ρ be the anchor of E_π. Choose a connection $\tilde{\nabla}$ on TM as a vector bundle. Let $\tilde{\nabla}^\vee$ be the linear connection on $T^\vee M$ induced by $\tilde{\nabla}$. Let ∇ be the E_π-connection on E_π defined by

$$\nabla_X Y = \tilde{\nabla}^\vee_{\rho(X)} Y,$$

for any $X, Y \in \Gamma(E_\pi)$. The latter makes sense since, as vector bundles, $E_\pi = T^\vee M$.

Let $\{q^i\}$ be a set of local coordinates on M and let $\{\Gamma^k_{ij}\}$ be the Christoffel symbols of $\tilde{\nabla}$. Further, assume π has the following local chart expression

$$\pi = \sum_{ij} \pi^{ij} \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial q^j}.$$

where π^{ij} are local functions of the coordinates $\{q^i\}$. Then the geodesic vector field of ∇ admits the following local expression:

$$\xi^\nabla = \sum_{ij} p_i \pi^{ij} \frac{\partial}{\partial q^j} + \sum_{ijkl} p_k p_l \pi^{ki} \Gamma^l_{ij} \frac{\partial}{\partial p_j},$$

where $\{q^i, p_i\}$ are the induced local coordinates on $T^\vee M$. We call ξ^∇ the *Poisson geodesic vector field* of $\tilde{\nabla}$. It was called Poisson spray in [22].

Let $\Sigma \rightrightarrows M$ be a local Lie groupoid with Lie algebroid E. Let α (resp. β) be the source (resp. target) map of Σ. Assume, without loss of generality, that, for any $m \in M$ the source fibre $\Sigma_m = \alpha^{-1}(m)$ is a closed submanifold of Σ. Finally, let ∇ be an E-connection on E. There is a unique affine connection ∇^m on Σ_m such that

$$(\nabla^m)_{\overleftarrow{X}} \overleftarrow{Y}(g) = \overleftarrow{\nabla_X Y}(g) \tag{3.17}$$

for any X, Y in $\Gamma(E)$ and any $g \in \Sigma_m$ sufficiently close to $\epsilon(M)$.

Definition 3.4.3. In the notations of the previous paragraph, the *groupoid exponential map of* ∇ is the unique map $\exp^\nabla : E \to \Sigma$ such that for any $m \in M$, the restriction

$$\exp^\nabla\Big|_{E_m} : E_m \to \Sigma_m \subset \Sigma$$

is the ordinary exponential of the connection ∇^m on Σ_m.

Remark 3.4.2. Weinstein-Xu [23] proved that $\exp^\nabla$ is a well-defined diffeomorphism in a neighbourhood of 0_M in E. Also, by definition, $\alpha \circ \exp^\nabla = \mathrm{pr}$ and $\exp^\nabla(0_m) = \epsilon(m)$ for any $m \in M$. In particular, for any e in E, the exponential path $t \mapsto \exp^\nabla(te)$ is a source-path.

Let U be the open subset of E ensured by Proposition 3.4.1. We define a map $\Phi : U \to \mathscr{A}(E)$ as follows. For any e_0 in U, let $\Phi(e_0)$ be the A-path defined by

$$(\Phi(e_0))(t) = \varphi_t^{\nabla}(e_0)$$

for all $t \in I$. One should think of Φ as a kind of exponential map at the level of A-paths. Formally, the relation between Φ and the groupoid exponential from Definition 3.4.3 is summarized in Theorem 3.4.1. Its proof is a consequence of the following simple lemma, which relates, for a given element $e_0 \in E$, the groupoid exponential path $t \mapsto \exp^{\nabla}(te_0)$ to the A-path $\Phi(e_0)$.

Lemma 3.4.1. *Let $\Sigma \rightrightarrows M$ be a local Lie groupoid with Lie algebroid E and let ∇ be an E-connection on E. Let $U \subset E$ be as in Proposition 3.4.1 and let e_0 be an element of U. Let $g : I \to \Sigma$ be the source-path defined by $g(t) = \exp^{\nabla}(te_0)$ and let $e : I \to E$ be the A-path defined by $e = \Phi(e_0)$. Then g is a solution to the initial value problem*

$$\begin{aligned} \dot{g}(t) &= g(t)e(t) \\ g(0) &= \epsilon(pr(e_0)). \end{aligned}$$

for all $t \in I$ sufficiently close to 0.

Proof. Let $m = \mathrm{pr}(e_0)$. From Equation 3.17 it follows that

$$\begin{aligned} 0 &= {\nabla^m}_{\dot{g}(t)}\, \dot{g}(t) \\ &= g(t) \cdot \nabla_{g^{-1}(t)\dot{g}(t)}\, g^{-1}(t)\dot{g}(t). \end{aligned}$$

Let $e' : I \to E$ be the A-path $e'(t) = g^{-1}(t)\dot{g}(t)$. Then from the above equation it follows that e' is A-geodesic with respect to ∇. From Proposition 3.4.1 it follows that e is A-geodesic as well. Since

$$e(0) = e_0 = e'(0)$$

the claim follows from Remark 3.4.1. □

Theorem 3.4.1. *Let $\Sigma \rightrightarrows M$ be a local Lie groupoid with Lie algebroid E and let ∇ be an E-connection on E. Let $w_\Sigma : \Sigma \to \mathscr{W}_{loc}(E)$ be the isomorphism defined in Theorem 2.1.3. There exists an open neighbourhood U of $0_M \subset E$ such that the maps*

$$\begin{aligned} \exp^{\nabla} &: U \to \Sigma, \\ [\Phi] &: U \to \mathscr{W}_{loc}(E) : e_0 \mapsto [\Phi(e_0)], \end{aligned}$$

are well-defined diffeomorphisms onto their respective images. Further, we have

$$w_\Sigma \circ \exp^{\nabla} = [\Phi]. \tag{3.18}$$

Proof. Let U be a sufficiently small open neighbourhood of $0_M \subset E$ such that both $\exp^\nabla$ and Φ are well-defined. For any $u \in U$, let $\exp_u$ denote the path $\exp_u : I \to \Sigma$ defined by $\exp_u(t) = \exp^\nabla(tu)$. From Lemma 3.4.1 it follows that $\delta \exp_u = \Phi(u)$. Thus we have that

$$\begin{aligned} w_\Sigma(\exp^\nabla(u)) &= w_\Sigma(\exp_u(1)) \\ &= [\delta \exp_u] \\ &= [\Phi(u)] \end{aligned}$$

which concludes the proof. □

The above theorem, when applied to a symplectic local groupoid realizing a given Poisson manifold, can be used to give a (deceptively simple) explicit construction for a symplectic realization. This is the content of Corollary 3.4.1 below, which follows immediately from the next theorem.

Theorem 3.4.2. *Let M be a smooth manifold and let π be a Poisson bivector field on M. Let E_π be the cotangent bundle Lie algebroid of (M, π) and let $\Sigma \rightrightarrows M$ be a local Lie groupoid with Lie algebroid E_π. Let Ω be the multiplicative symplectic 2-form ensured by Corollary 3.3.1. For any E_π-connection ∇ on E_π there exists an open neighbourhood U of $0_M \subset T^\vee M$ such that the exponential $\exp^\nabla : U \to \Sigma$ is well-defined and the equation*

$$(\exp^\nabla)^* \, \Omega = \int_0^1 (\varphi_t^\nabla)^* \omega_{can} \, dt \tag{3.19}$$

holds.

Proof. Let U be as in Theorem 3.4.1 and let $(\mathscr{CF}_{\mathrm{loc}}(M) \rightrightarrows M, \Omega_{\mathscr{CF}})$ be the Cattaneo-Felder symplectic local groupoid of (M, π). Recall that, as a local Lie groupoid, $\mathscr{CF}_{\mathrm{loc}}(M)$ is the Weinstein local Lie groupoid of E_π. It follows from Proposition 3.3.2 and Theorem 3.3.2 that the isomorphism $w_\Sigma : \Sigma \to \mathscr{CF}_{\mathrm{loc}}(M)$ of local Lie groupoids satisfies

$$(w_\Sigma)^* \, \Omega_{\mathscr{CF}} = \Omega.$$

From (3.18) it follows that (3.19) is equivalent to the integral formula

$$[\Phi]^* \, \Omega_{\mathscr{CF}} = \int_0^1 (\varphi_t^\nabla)^* \omega_{\mathrm{can}} \, dt. \tag{3.20}$$

In order to prove (3.20), let X be an element of TU and let $x : I \to TU$ be the path defined by

$$x(t) = (\varphi_t^\nabla)_* \, X$$

for all $t \in I$. It is easy to see that

$$[\Phi]_* \, X = [\mathfrak{I}^{-1} x]. \tag{3.21}$$

The formula (3.20) then follows immediately from the definition of $\Omega_{\mathscr{CF}}$ (see Corollary 2.2.1) and (3.21). □

As an immediate consequence of Theorem 3.4.2, we recover the following result, all parts of which but the last sentence were proved in 2010 by Crainic-Mărcut by a rather complicated direct computation [22]. Here, it follows at once from the more conceptual approach used in Theorem 3.4.2 which is grounded in the construction of the Cattaneo-Felder local groupoid.

Corollary 3.4.1. *Let M be a smooth manifold and π be a Poisson bivector field on M. Let E_π denote the cotangent bundle Lie algebroid of (M, π) and let ∇ be a E_π-connection on E_π. There exists an open neighbourhood U of $0_M \subset E_\pi$ such that the differential 2-form defined by the formula*

$$\omega := \int_0^1 (\varphi_t^\nabla)^* \omega_{can}\, dt$$

is symplectic. Further, the triple $(T^\vee M, \omega, pr|_U : U \to M)$ is a symplectic realization of (M, π). Finally, the zero section $0_M \subset T^\vee M$ is a Lagrangian submanifold of (U, ω).

3.5 Symplectic-Nijenhuis Local Groupoids

There is a one-to-one correspondence between Poisson manifolds and symplectic local groupoids (see Theorem 1.3.1 and Theorem 1.3.2). This correspondence can be extended to a one-one correspondence between Poisson-Nijenhuis manifolds and symplectic-Nijenhuis local groupoids (to be defined below). This is a result due to Stienon-Xu [24] which we review in this section.

Let $\Sigma \rightrightarrows M$ be a local Lie groupoid with source, target and unit maps respectively denoted α, β and ϵ. Let $T\Sigma \rightrightarrows TM$ be the tangent local groupoid of $\Sigma \rightrightarrows M$.

Definition 3.5.1. Let $\Sigma \rightrightarrows M$ be a local Lie groupoid. A $(1,1)$-tensor $\mathcal{N}$ on Σ is said to be *multiplicative* if the bundle map $\mathcal{N} : T\Sigma \to T\Sigma$ is a morphism of local Lie groupoids.

Definition 3.5.2. Let $\Sigma \rightrightarrows M$ be a local Lie groupoid. A *multiplicative symplectic-Nijenhuis structure* on $\Sigma \rightrightarrows M$ is a symplectic-Nijenhuis structure $(\Omega, \mathcal{N})$ on Σ such that both Ω and $\mathcal{N}$ are multiplicative.

A local Lie groupoid endowed with a multiplicative symplectic-Nijenhuis structure will be called a *symplectic-Nijenhuis local groupoid*.

Remark 3.5.1. Let $\Sigma \rightrightarrows M$ be a local Lie groupoid. Let Ω be a symplectic 2-form on Σ and let $\mathcal{N}$ be a $(1,1)$-tensor on Σ such that the pair $(\Omega, \mathcal{N})$ is a multiplicative symplectic-Nijenhuis structure on $\Sigma \rightrightarrows M$. The Poisson bivector field Π on Σ defined by the equation $\Pi^\sharp = (\Omega^\flat)^{-1}$, is multiplicative. Moreover, let $\Pi_{\mathcal{N}}$ be the Poisson bivector field on Σ defined by the equation $\Pi^\sharp_{\mathcal{N}} = \mathcal{N} \circ \Pi^\sharp$. Clearly, $\Pi_{\mathcal{N}}$ is multiplicative. Note that if $\mathcal{N}$ has non-trivial kernel as a bundle map $T\Sigma \to T\Sigma$, then the Poisson bivector field $\Pi_{\mathcal{N}}$ is degenerate. Thus in general a symplectic-Nijenhuis local groupoid gives rise to one symplectic local groupoid $(\Sigma \rightrightarrows M, \Omega)$ and one (generally degenerate) Poisson local groupoid $(\Sigma \rightrightarrows M, \Pi_{\mathcal{N}})$.

The next two theorems imply a result due to Stienon-Xu [24] which states that there is a one-one correspondence between Poisson-Nijenhuis manifolds and symplectic-Nijenhuis groupoids. The proofs of Theorem 3.5.1 and Theorem 3.5.2 are elaborations of the proof of Theorem 5.2 in [24].

Theorem 3.5.1. *Let $\Sigma \rightrightarrows M$ be a local Lie groupoid and let α be the source map of $\Sigma \rightrightarrows M$. Let Ω be a symplectic 2-form on Σ and let $\mathcal{N}$ be a $(1,1)$-tensor on Σ such that the pair $(\Omega, \mathcal{N})$ is a multiplicative symplectic-Nijenhuis structure on $\Sigma \rightrightarrows M$. There exists a unique Poisson-Nijenhuis structure (π, N) on M such that*

$$\alpha_* \Pi = \pi \;\; \textit{and} \;\; N \circ \alpha_* = \alpha_* \circ \mathcal{N}. \tag{3.22}$$

where Π is the Poisson bivector field defined by the equation $\Pi^\sharp = (\Omega^\flat)^{-1}$.

In other words, there exists a unique PN structure (π, N) on M such that the source map α is a Poisson-Nijenhuis map $(\Sigma, \Pi, \mathcal{N}) \to (M, \pi, N)$.

In the situation of Theorem 3.5.1 above, we will say that *the symplectic-Nijenhuis local groupoid $(\Sigma \rightrightarrows M, \Omega, \mathcal{N})$ induces the Poisson-Nijenhuis structure (π, N) on M.*

Proof. Note that $(\Sigma \rightrightarrows M, \Omega)$ is a symplectic local groupoid. Let Π be the Poisson bivector field on Σ defined by the equation $\Pi^\sharp = (\Omega^\flat)^{-1}$. From Theorem 1.3.1 it follows that there exists a unique Poisson bivector field π on M such that $\pi := \alpha_* \Pi$. On the other hand, because $\mathcal{N} : T\Sigma \to T\Sigma$ is multiplicative, we have

$$\mathcal{N}(\epsilon_*(TM)) \subset \epsilon_*(TM).$$

Since ϵ is an embedding, the latter implies there exists a unique $(1,1)$-tensor $N : TM \to TM$ such that $\epsilon_* \circ N = \mathcal{N} \circ \epsilon_*$. It is clear that $\alpha_* \circ \mathcal{N} = N \circ \alpha_*$ and that $N \circ \pi^\sharp = \pi^\sharp \circ N^\top$.

We claim that the pair (π, N) is a Poisson-Nijenhuis structure on M. The vanishing of the Nijenhuis torsion of N follows trivially from that of $\mathcal{N}$. From Theorem 1.2.2 it follows that there exists a unique Poisson bivector field π_N such that $\pi_N^\sharp = N \circ \pi^\sharp$. It is clear that $\pi_N = \alpha_* \Pi_{\mathcal{N}}$ where $\Pi_{\mathcal{N}}$ is the Poisson bivector field on Σ defined by the equation $\Pi_{\mathcal{N}}^\sharp = \mathcal{N} \circ \Pi^\sharp$. Furthermore, note that

$$\begin{aligned} [\![\pi, \pi_N]\!] &= \alpha_* \left([\![\Pi, \Pi_{\mathcal{N}}]\!] \right), \\ &= \alpha_* (0) \text{ from Theorem 1.2.2 applied to } (\Pi, \mathcal{N}), \\ &= 0. \end{aligned}$$

Again, from Theorem 1.2.2, it follows that the pair (π, N) is a Poisson-Nijenhuis structure, as desired. □

Theorem 3.5.2. *Let M be a smooth manifold and let (π, N) be a Poisson-Nijenhuis structure on M. There exists a symplectic-Nijenhuis local groupoid $(\Sigma \rightrightarrows M, \Omega, \mathcal{N})$ inducing the PN structure (π, N) on M.*

Proof. Let $(\Sigma \rightrightarrows M, \Omega)$ be the symplectic local groupoid realizing (M, π) ensured by Corollary 3.3.1. Let Π be the Poisson bivector field defined by the equation $\Pi^\sharp = (\Omega^\flat)^{-1}$.

By definition, the pair $(E_\pi, (TM)_N)$ is a Lie bialgebroid. Let Ω' denote the multiplicative Poisson bivector field on Σ ensured by Theorem 3.3.1 applied to the Lie bialgebroid $(E_\pi, (TM)_N)$. Let $\mathcal{N} : T\Sigma \to T\Sigma$ be the $(1,1)$-tensor defined by

$$\mathcal{N} = (\Pi')^\sharp \circ \Omega^\flat.$$

We claim that the pair $(\Omega, \mathcal{N})$ is a multiplicative symplectic-Nijenhuis structure on $\Sigma \rightrightarrows M$. It is clear that both Ω and $\mathcal{N}$ are multiplicative.From Theorem 1.2.3 it follows that it is enough to check that

$$[\![\Pi, \Pi']\!] = 0.$$

Let $\delta_N : \Gamma(\wedge^\bullet T^\vee M) \to \Gamma(\wedge^\bullet T^\vee M)$ be the differential of the Lie algebroid $(TM)_N$ and let $\delta : \Gamma(\wedge^\bullet T^\vee M) \to \Gamma(\wedge^\bullet T^\vee M)$ be the differential of tangent Lie algebroid TM. A computation in [24] shows that the graded commutator $[\delta_N, \delta] = \delta_N \circ \delta + \delta \circ \delta_N$ vanishes. From the Universal Lifting Theorem [25] it then follows that $[\![\Pi, \Pi']\!] = 0$, as needed.

We claim the symplectic-Nijenhuis local groupoid $(\Sigma \rightrightarrows M, \Omega, \mathcal{N})$ induces the Poisson-Nijenhuis structure (π, N) on M. Indeed, from Theorem 1.3.2 it follows that $\alpha_* \Pi = \pi$. On the other hand, it easily seen that $\alpha_* \Pi' = \pi_N$ as well, from which the second part of (3.22) follows. This concludes the proof. □

Remark 3.5.2. Let $\Sigma \rightrightarrows M$ be a local Lie groupoid endowed with a multiplicative symplectic-Nijenhuis structure $(\Omega, \mathcal{N})$ and let (π, N) be the corresponding PN structure on M. Let α be the source map of $\Sigma \rightrightarrows M$. Let $\Pi_{\mathcal{N}}$ be the Poisson bivector field defined by the equation $\Pi_{\mathcal{N}}^\sharp = \mathcal{N} \circ (\Omega^\flat)^{-1}$. It is trivial that $\mathcal{N}$ is invertible (as a local Lie groupoid morphism $\mathcal{N} : T\Sigma \to T\Sigma$) if and only $\Pi_{\mathcal{N}}$ is non-degenerate as a Poisson bivector field. In that case, let $\Omega_{\mathcal{N}}$ be the symplectic 2-form defined by the equation $\Omega_{\mathcal{N}}^\flat = (\Pi_{\mathcal{N}}^\sharp)^{-1}$. It is clear that $\Omega_{\mathcal{N}}$ is multiplicative and, as in Remark 3.1.2, it follows that the quadruple $(\Sigma, \Omega, \Omega_{\mathcal{N}}, \alpha)$ is a symplectic realization of the PN structure (π, N).

3.6 The Proof

In this section, we assume fixed the following structure.

$$\begin{aligned}
M &: \text{ a smooth manifold,}\\
(\pi, N) &: \text{ a non-degenerate Poisson-Nijenhuis structure on } M,\\
\pi_N &: \text{ the Poisson bivector field defined by the equation } \pi_N^\sharp = N \circ \pi^\sharp,\\
E_\pi &: \text{ the cotangent bundle Lie algebroid of } (M, \pi),\\
E_{\pi_N} &: \text{ ——————————————— } (M, \pi_N),\\
(TM)_N &: \text{ the N-twisted tangent Lie algebroid of } M,\\
\nabla &: \text{ a } E_\pi - \text{connection on } E_\pi.
\end{aligned}$$

We now turn to the proof of Theorem 3.2.1. Let $\Sigma \rightrightarrows M$ be a local Lie groupoid with Lie algebroid E_π, endowed with a multiplicative symplectic-Nijenhuis structure $(\Omega, \mathcal{N})$ inducing the PN structure (π, N) on the manifold M, as is ensured by Theorem 3.5.2. Let Π (resp. $\Pi_{\mathcal{N}}$) be the multiplicative Poisson bivector field on Σ defined by the equation $\Pi^\sharp = (\Omega^\flat)^{-1}$ (resp. $\Pi_{\mathcal{N}}^\sharp = \mathcal{N} \circ \Pi^\sharp$). We can asume that Π (resp. $\Pi_{\mathcal{N}}$) is the multiplicative Poisson bivector field on Σ ensured by Theorem 3.3.1 applied to the Lie bialgebroid (E_π, TM) (resp. $(E_\pi, (TM)_N)$). From Theorem 3.4.2 it follows that there exists a neighbourhood U of the zero section in $T^\vee M$ such that the exponential map $\exp^\nabla : U \to \Sigma$ is a diffeomorphism onto its image and such that the symplectic 2-form $(\exp^\nabla)^* \Omega$ satisfies

$$(\exp^\nabla)^* \Omega = \int_0^1 (\varphi_t^\nabla)^* \omega_{\text{can}} \, dt. \tag{3.23}$$

We claim that $\Pi_{\mathcal{N}}$ is non-degenerate and that, letting $\Omega_{\mathcal{N}}$ be the multiplicative symplectic 2-form defined by the equation $\Omega_{\mathcal{N}}^\flat = (\Pi_{\mathcal{N}}^\sharp)^{-1}$, we have the formula

$$(\exp^\nabla)^* \Omega_{\mathcal{N}} = \int_0^1 ((N^\top)^{-1} \circ \varphi_t^\nabla)^* \omega_{\text{can}} \, dt. \tag{3.24}$$

Let $(\Sigma_N \rightrightarrows M, \Omega')$ be a symplectic local groupoid realizing (M, π_N) and let Π' be the multiplicative Poisson bivector field on Σ_N defined by the equation $\Pi'^\sharp = (\Omega'^\flat)^{-1}$. We can assume that Π' is the bivector field ensured by Theorem 3.3.1 applied to the Lie bialgebroid (E_{π_N}, TM). From Lemma 1.2.4, it follows that the dual pair of maps $(N^\top, N)$ defines an isomorphism of Lie bialgebroids from the pair (E_{π_N}, TM) to the pair $(E_\pi, (TM)_N)$. Let $\bar{N} : \Sigma_N \to \Sigma$ be the isomorphism of local Lie groupoids such that the morphism of Lie algebroids $E_{\pi_N} \to E_\pi$ obtained by applying the Lie functor to $\bar{N}$ coincides with $N^\top$. From Theorem 3.3.2 it follows that

$$\bar{N}_* \Pi' = \Pi_{\mathcal{N}}.$$

Since $\bar{N}$ is invertible locally around the units submanifold of Σ and Π' is non-degenerate, it follows that the Poisson bivector field $\Pi_{\mathcal{N}}$ is non-degenerate in a neighbourhood $\epsilon(\Sigma) \subset \Sigma$. Further, in that neighbourhood, we have

$$\left(\bar{N}^{-1}\right)^* \Omega' = \Omega_{\mathcal{N}}.$$

In particular, we have the equation

$$\begin{aligned} (\exp^\nabla)^* \Omega_{\mathcal{N}} &= (\exp^\nabla)^* (\bar{N}^{-1})^* \Omega' && (3.25)\\ &= (\bar{N}^{-1} \circ \exp^\nabla)^* \Omega'. && (3.26) \end{aligned}$$

Note that, from Proposition 2.1.3 it follows that

$$w_{\Sigma_N} \circ \bar{N}^{-1} = \mathscr{W}_{\text{loc}}\left((N^\top)^{-1}\right) \circ w_\Sigma.$$

and thus, from Theorem 3.4.1 it follows that

$$\begin{aligned} w_{\Sigma_N} \circ \bar{N}^{-1} \circ \exp^\nabla &= \mathscr{W}_{\text{loc}}\left((N^\top)^{-1}\right) \circ [\Phi] \\ &= [(N^\top)^{-1}\Phi] \end{aligned}$$

where the map $[(N^\top)^{-1}\Phi] : U \to \mathscr{W}_{\text{loc}}(E_{\pi_N})$ is defined by

$$[(N^\top)^{-1}\Phi](u) = [t \in I \mapsto \left((N^\top)^{-1} \circ \varphi_t^\nabla\right)(u)]$$

for all u in U.

Let $(\mathscr{CF}_{\text{loc}}(M) \rightrightarrows M, \Omega'_{\mathscr{CF}})$ be the Cattaneo-Felder symplectic local groupoid of (M, π_N). Exactly like in the proof of Theorem 3.4.2, we have

$$(w_{\Sigma_N})^* \Omega'_{\mathscr{CF}} = \Omega'.$$

In particular, it follows immediately that

$$\begin{aligned} [(N^\top)^{-1}\Phi]^* \Omega'_{\mathscr{CF}} &= (w_{\Sigma_N} \circ \bar{N}^{-1} \circ \exp^\nabla)^* \Omega'_{\mathscr{CF}}, \\ &= (\bar{N}^{-1} \circ \exp^\nabla)^* (w_{\Sigma_N})^* \Omega'_{\mathscr{CF}}, \\ &= (\bar{N}^{-1} \circ \exp^\nabla)^* \Omega', \end{aligned}$$

and thus we conclude that

$$[(N^\top)^{-1}\Phi]^* \Omega'_{\mathscr{CF}} = (\exp^\nabla)^* \Omega_{\mathcal{N}}.$$

Formula (3.24) then follows directly from the definition of $\Omega'_{\mathscr{CF}}$ and the equation above.

Note that, from (3.23) and (3.24) it follows that

$$\omega = (\exp^{\nabla})^* \,\Omega \quad \text{and} \quad \omega_N = (\exp^{\nabla})^* \,\Omega_{\mathcal{N}},$$

where ω and ω_N are as in the statement of the theorem. Since $\exp^{\nabla}$ is a diffeomorphism onto its image it is trivial that the pair $(\omega, \mathbb{N})$ is a symplectic-Nijenhuis structure on U, where $\mathbb{N}$ is defined as in the statement of the theorem. From Remark 3.4.2 it follows that $\alpha \circ \exp^{\nabla} = \mathrm{pr}$ and thus the quadruple $(U, \omega, \omega_N, \mathrm{pr})$ is a symplectic realization of the PN-structure (π, N) on M. This concludes the proof.

3.7 Symplectic Realizations of Holomorphic Poisson Manifolds

Let $\mathscr{M}$ be a complex manifold and let π be a holomorphic Poisson bivector field on $\mathscr{M}$. Let M denote the underlying real manifold of $\mathscr{M}$ and let $J : TM \to TM$ be the (integrable) almost complex structure of $\mathscr{M}$. Let π_R (resp. π_I) denote the real (resp. pure imaginary) part of π. Recall, from Proposition 1.4.2, that the pair (π_I, J) is a Poisson-Nijenhuis structure on M. Let E_{π_I} denote the cotangent bundle Lie algebroid of (M, π_I) and let ω_{can} denote the canonical (real) symplectic 2-form on $T^\vee M$. The main theorem of this section is the following general result.

Theorem 3.7.1. *For any E_{π_I}-connection ∇ on E_{π_I} there exists an open neighbourhood U of the (real) cotangent bundle $T^\vee M$ such that the differential 2-forms ω_R, ω_I defined by the formulae*

$$\omega_R(X,Y) = -\int_0^1 \left(J^\top \circ \varphi_t^{\nabla}\right)^* \omega_{can}\,(X,Y)\; dt$$
$$\omega_I(X,Y) = \int_0^1 \left(\varphi_t^{\nabla}\right)^* \omega_{can}\,(X,Y)\; dt$$

for any X, Y in T_uU $(u \in U)$, are symplectic. Let $\mathbb{J}$ be the $(1,1)$-tensor on U defined by

$$\mathbb{J} = (\omega_R^\flat)^{-1} \circ \omega_I^\flat.$$

Then $\mathbb{J}$ is an integrable almost complex structure on U. In particular, the pair $(U, \mathbb{J})$ defines a complex manifold, which we will denote $\mathscr{U}$. Let ω be the complex differential 2-form on U defined by

$$\omega = \frac{1}{4}\left(\omega_R - i\,\omega_I\right).$$

Then ω is a holomorphic symplectic 2-form on $\mathscr{U}$ and the triple $(\mathscr{U}, \omega, pr)$ is a holomorphic symplectic realization of $(\mathscr{M}, \pi)$.

Note that Theorem 3.7.1 is a straightforward consequence of Theorem 3.2.1 and of Proposition 1.4.2 provided it is shown that $\mathbb{J}$ is an almost complex structure, i.e. that $\mathbb{J}^2 = -\text{id}$. In light of the proof of Theorem 3.2.1, the latter will follow immediately from the next theorem.

Theorem 3.7.2. *Let $\mathscr{M}$ be a complex manifold endowed with a holomorphic Poisson bivector field $\pi = \pi_R + i\pi_I$. Let M be the underlying real manifold of $\mathscr{M}$ and let (π_I, J) be the PN structure on M ensured by Proposition 1.4.2. Let $\Sigma \rightrightarrows M$ be a local Lie groupoid endowed with a symplectic-Nijenhuis structure $(\Omega, \mathcal{J})$ inducing the PN structure (π_I, J) on M. Then, in a neighbourhood of the units in Σ, we have*

$$\mathcal{J}^2 = -id.$$

In particular, locally around the units in Σ, the tensor $\mathcal{J}$ is an integrable almost complex structure on Σ.

The goal of this section is to prove Theorem 3.7.2, from which Theorem 3.7.1 follows. Our strategy relies on showing that, up to natural isomorphisms, applying the Lie functor to the map $\mathcal{J} : T\Sigma \to T\Sigma$, yields the complete lift to $T^\vee M$ of the $(1,1)$-tensor J.

Let M be a smooth manifold. We start by recalling the definition of the complete lift of $(1,1)$-tensors on M to the cotangent bundle. For details, we refer the reader to [26] and the references therein.

Let θ denote the Liouville 1-form. Let N be a $(1,1)$-tensor on M. Let θ_N be the differential 1-form on $T^\vee M$ defined by the equation

$$\langle \theta_N,\, X \rangle = \langle \lambda,\, N(\text{pr}_*(X)) \rangle$$

for all X in $T_\lambda T^\vee M$ $(\lambda \in T^\vee M)$.

Definition 3.7.1. The *complete lift to the cotangent bundle* of N is the $(1,1)$-tensor $N^c : TT^\vee M \to TT^\vee M$ defined by the equation

$$\omega_{\text{can}}(N^c X, Y) = (d\theta_N)(X, Y), \tag{3.27}$$

for all X, Y in $T_\lambda T^\vee M$ $(\lambda \in T^\vee M)$.

The next lemma is central to our proof of Theorem 3.7.2.

Lemma 3.7.1. *Let M be a smooth manifold and let (π, N) be a PN structure on M. Let E_π be the cotangent bundle Lie algebroid of (M, π) and let π_{LP} be the Lie-Poisson bivector field on $T^\vee M$ defined by the Lie algebroid structure $(TM)_N$. The following equation holds:*

$$N^c = \pi_{LP}^\sharp \circ \omega_{can}^\flat.$$

Remark 3.7.1. Let π_{can} be the Poisson bivector field defined by the equation $\pi^\sharp_{\text{can}} = (\omega^\flat_{\text{can}})^{-1}$. According to a theorem of Vaisman [6], the Schouten-Nijenhuis bracket $[\![\pi_{\text{LP}}, \pi_{\text{can}}]\!]$ vanishes. From Theorem 1.2.3 it follows that the pair (π_{can}, N^c) is a Poisson-Nijenhuis structure on $T^\vee M$.

Proof. Recall that, by definition, for any $X, Y \in \mathfrak{X}(M)$ and any $f, g \in \mathcal{C}^\infty(M)$, we have:

$$\{\ell_X, \ell_Y\}_{\pi_{\text{LP}}} = \ell_{[N(X),Y]+[X,N(Y)]-N([X,Y])}, \tag{3.28}$$
$$\{\ell_X, \text{pr}^* f\}_{\pi_{\text{LP}}} = \text{pr}^* df(N(X)), \tag{3.29}$$
$$\{\text{pr}^* f, \text{pr}^* g\}_{\pi_{\text{LP}}} = 0, \tag{3.30}$$

where $\{\cdot,\cdot\}_{\pi_{\text{LP}}}$ is the Poisson bracket defined by the Poisson bivector field π_{LP}. For any smooth function $\psi : T^\vee M \to \mathbb{R}$, we let $\mathcal{H}_\psi$ be the vector field on $T^\vee M$ defined by

$$\mathcal{H}_\psi := (\omega^\flat_{\text{can}})^{-1}(d\psi).$$

Note that

$$\begin{aligned} N^c = \pi^\sharp_{\text{LP}} \circ \omega^\flat_{\text{can}} &\iff N^c \circ (\omega^\flat_{\text{can}})^{-1} = \pi^\sharp_{\text{LP}} \\ &\iff N^c(\mathcal{H}_\psi) \cdot \phi = (\pi^\sharp_{\text{LP}}(d\psi)) \cdot \phi \end{aligned}$$

for all $\psi, \phi \in \mathcal{C}^\infty(T^\vee M)$. As an algebra over $\mathbb{R}$, the space of functions $\mathcal{C}^\infty(T^\vee M)$ is generated by functions of the form

$$\ell_X \in \mathcal{C}^\infty(T^\vee M) \quad \text{and} \quad \text{pr}^* f \in \mathcal{C}^\infty(T^\vee M)$$

for any $X \in \mathfrak{X}(M)$ and $f \in \mathcal{C}^\infty(M)$. It thus suffices to prove the following identities

$$N^c(\mathcal{H}_F) \cdot G = (\pi_{\text{LP}})^\sharp(dF) \cdot G, \tag{3.31}$$
$$N^c(\mathcal{H}_{\ell_X}) \cdot \ell_Y = (\pi_{\text{LP}})^\sharp(d\ell_X) \cdot \ell_Y, \tag{3.32}$$
$$N^c(\mathcal{H}_{\ell_X}) \cdot F = (\pi_{\text{LP}})^\sharp(d\ell_X) \cdot F, \tag{3.33}$$
$$N^c(\mathcal{H}_F) \cdot \ell_X = (\pi_{\text{LP}})^\sharp(F) \cdot \ell_X, \tag{3.34}$$

for all $F, G \in \mathcal{C}^\infty(T^\vee M)$ such that $F = \text{pr}^* f$ and $G = \text{pr}^* g$ for some f, g in $\mathcal{C}^\infty(M)$, and all $X, Y \in \mathfrak{X}(M)$. Note that (3.33) is equivalent to (3.34). Indeed, it suffices to note that the tensor $\omega^\flat_{\text{can}} \circ N^c$ is skew-symmetric. The latter is clear from the defining equation (3.27) of N^c.

- To prove (3.31). Note that $\text{pr}_*(\mathcal{H}_{\text{pr}^* f}) = 0$, for any $f \in \mathcal{C}^\infty(M)$. From the defining equation of N^c it easily follows that $\text{pr}_*(N^c(\mathcal{H}_{\text{pr}^* f})) = 0$. In particular, both sides of (3.31) vanish.

– To prove (3.32). We use the following standard relation (see Proposition 5.4.3 in [26]): $N^c(\mathcal{H}_{\ell_X}) - \mathcal{H}_{\ell_{N(X)}} = (\omega^\flat_{\text{can}})^{-1}(\theta_{\mathcal{L}_X N})$, for any $X \in \mathfrak{X}(M)$. Also, a direct computation shows that $(\omega^\flat_{\text{can}})^{-1}(\theta_{\mathcal{L}_X N}) \cdot \ell_Y = \ell_{(\mathcal{L}_X N)(Y)}$, for all $X, Y \in \mathfrak{X}(M)$. It follows directly that

$$\begin{aligned} N^c(\mathcal{H}_{\ell_X}) \cdot \ell_Y &= \mathcal{H}_{\ell_{N(X)}} \cdot \ell_Y + (\omega^\flat_{\text{can}})^{-1}(\theta_{\mathcal{L}_X N}) \cdot \ell_Y \\ &= \ell_{[N(X),Y]} + \ell_{(L_X N)(Y)} \\ &= \ell_{[N(X),Y]+[X,N(Y)]-N([X,Y])}, \end{aligned}$$

as desired.

– To prove (3.33). We have $(\omega^\flat_{\text{can}})^{-1}(\theta_{\mathcal{L}_X N}) \cdot F = 0$, for any $X \in \mathfrak{X}(M)$ and any F such that $F = \text{pr}^* f$ for some $f \in \mathcal{C}^\infty(M)$. Thus:

$$\begin{aligned} N^c(\mathcal{H}_{\ell_X}) \cdot F &= \mathcal{H}_{\ell_{N(X)}} \cdot F + (\omega^\flat_{\text{can}})^{-1}(\theta_{\mathcal{L}_X N}) \cdot F \\ &= \text{pr}^* \left((NX) \cdot f\right) \\ &= \pi^\sharp_{\text{LP}}(d\ell_X) \cdot F, \end{aligned}$$

as required.

This concludes the proof. □

Lemma 3.7.2. *Let M be a smooth manifold and let (π, N) be a PN structure on M. Let E_π denote the cotangent bundle Lie algebroid of (M, π). Let $\Sigma \rightrightarrows M$ be a local Lie groupoid endowed with a multiplicative symplectic-Nijenhuis structure $(\Omega, \mathcal{N})$ inducing (π, N) on M. Let $A\mathcal{N} : AT\Sigma \to AT\Sigma$ denote the map obtained by applying the Lie functor to $\mathcal{N}$. Then, up to replacing $\Sigma \rightrightarrows M$ by an isomorphic symplectic-Nijenhuis local groupoid, the following diagram:*

$$\begin{array}{ccc} AT\Sigma & \xrightarrow{A\mathcal{N}} & AT\Sigma \\ \downarrow{\scriptstyle j_\Sigma} & & \downarrow{\scriptstyle j_\Sigma} \\ TE_\pi & \xrightarrow{N^c} & TE_\pi \end{array} \tag{3.35}$$

commutes.

Proof. Let Π (resp. $\Pi_{\mathcal{N}}$) be the Poisson bivector field defined by the equation $\Pi^\sharp = (\Omega^\flat)^{-1}$ (resp. $\Pi^\sharp_{\mathcal{N}} = \mathcal{N} \circ (\Omega^\flat)^{-1}$). Up to replacing $\Sigma \rightrightarrows M$ by an isomorphic symplectic-Nijenhuis local groupoid, we can assume Π (resp. $\Pi_{\mathcal{N}}$) is the multiplicative Poisson bivector field ensured by Theorem 3.3.1 applied to the Lie bialgebroid (E_π, TM) (resp. $(E_\pi, (TM)_N)$). In particular, the following diagrams

$$\begin{array}{ccc} AT^\vee\Sigma & \xrightarrow{A\Pi^\sharp} & AT\Sigma \\ \Big\downarrow{\scriptstyle j'_\Sigma} & & \Big\downarrow{\scriptstyle j_\Sigma} \\ T^\vee T^\vee M & \xrightarrow{\pi^\sharp_{\mathrm{can}}} & TT^\vee M \end{array} \qquad \begin{array}{ccc} AT^\vee\Sigma & \xrightarrow{A\Pi^\sharp_{\mathcal{N}}} & AT\Sigma \\ \Big\downarrow{\scriptstyle j'_\Sigma} & & \Big\downarrow{\scriptstyle j_\Sigma} \\ T^\vee T^\vee M & \xrightarrow{\pi^\sharp_{\mathrm{LP}}} & TT^\vee M \end{array} \tag{3.36}$$

commute. Here π_{can} (resp. π_{LP}) is the Poisson bivector field on $T^\vee M$ corresponding to the Lie–Poisson structure defined by the Lie algebroid TM (resp. $(TM)_N$). Also, we have used implicitly that $E_\pi = T^\vee M$ as vector bundles. Since $\pi^\sharp_{\mathrm{can}} = (\omega^\flat_{\mathrm{can}})^{-1}$, the commutativity of the diagram (3.35) follows directly from Lemma 3.7.1 and the commutativity of the diagrams (3.36). □

We are now ready to prove Theorem 3.7.2.

Proof. Up to an isomorphism of symplectic-Nijenhuis local groupoids, the diagram (3.35) commutes. It follows directly that

$$(J^c)^2 = -\mathrm{id} \implies (A\mathcal{J})^2 = -\mathrm{id},$$

where $A\mathcal{J} : AT\Sigma \to AT\Sigma$ is the map obtained by applying the Lie functor to $\mathcal{J}$. Recall [27] that, for any integrable almost complex structure J on M, the complete lift to the cotangent bundle is an almost complex structure on $T^\vee M$. Thus $(J^c) = -\mathrm{id}$ and indeed $(A\mathcal{J})^2 = -\mathrm{id}$. It follows directly that $\mathcal{J}^2 = -\mathrm{id}$ in a neighbhourhood of the units in Σ. This concludes the proof. □

THE END.

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