UNIVERSITY STUDENTS’ SCHOOL MATHEMATICS UNDERSTANDING AND ITS GROWTH IN THEIR LEARNING OF COLLEGIATE MATHEMATICS: THROUGH THE LENS OF A TRANSFORMATIVE TRANSITION FRAMEWORK

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ABSTRACT

In the context of addressing Klein’s (1908/1924) double discontinuity, the goal of this study was to address the question of “how might university students come to see school mathematics from an advanced viewpoint in their learning of collegiate mathematics?” by observing, documenting, and explaining important growth in university students’ understandings that builds on connections between school and collegiate mathematics. To this end, a construct called transformative transition and its accompanying categorical framework were developed based on the literature on the nature and development of an individual’s mathematical knowledge. A transformative transition involves a qualitative leap in existing understandings as an individual encounters a new (to the cognizing subject) construct and integrates it into his/her cognitive system. Its four categories—extending, unifying, strengthening, and deepening—delineate different ways in which that qualitative leap might take place; and a qualitative leap in each category is described as advancement in levels of the categories, derived from two existing frameworks—APOS theory (Arnon et al., 2014) and Piaget and Garcia’s triad (1983/1989).

The study was designed to allow close observation of participants’ understandings and possible transformative transitions in their learning of collegiate mathematics. The mathematical tasks used in the teaching interviews [TIs] were designed using the Abstraction-in-Context [AiC] framework (Dreyfus, Hershkowitz, & Schwarz, 2015). The design of tasks was intended to provide a setting for participants building on what they had previously known about factorization and polynomial equations in order to construct the unique factorization theorem for polynomials. Pre-TI and post-TI interviews were also conducted to identify any changes in participants’ relevant understandings before and after the TIs. Data collection included conducting a total of 40 interviews with six mathematics-intensive majors. The categorical framework of transformative transitions was used as a lens to interpret and analyze participants’ progress, if any, under each of the four categories of transformative transitions.
Five of the six participants were observed to enrich their prior understandings of factorization and polynomial equations through extending, unifying, strengthening, and deepening in the course of interviews. In the beginning of interviews, some participants had demonstrated assumptions and norms that seemingly had been exercised and established in their school mathematics (e.g., separating polynomial factorization from number factorization, thinking factorization always results in a product of lesser degree polynomials). These assumptions and norms appeared to be reexamined by the participants from a different angle during the TIs, and as a result, their prior understandings seemed to be reconstructed and interrelated to form an enhanced understanding.

Some features of the AiC-grounded interview context appeared to have supported and underlain participants’ transformative transitions. The contextual features are (i) a sequence of tasks in the order of defining-applying-describing, (ii) an unusual or unexpected encounter with a problem beyond their routines or assumptions, (iii) a task with a variety of instantiations of a single entity on which to reflect, and (iv) a task for determining the truth of a general statement regarding a construct. When participants did not seem to make transformative transitions, individual tendencies that appeared to act as obstacles to making transformative transitions were identified as (i) tendency to overlook the mathematical entities under consideration, (ii) conflated use of different concepts or procedures, (iii) compartmentalization accompanied by reliance on formulas, and (iv) utilitarian/instrumentalist view.

A major contribution of this study is an empirical elaboration of the categorical framework, which I hope could provide mathematics educators with ways to think about how to interpret and organize their observations of university students’ learning and how to support university students to see school mathematics from an advanced viewpoint.
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Chapter 1 Rationale

Mathematics is a field of the study that is characterized by its logical coherence and the interconnectedness of ideas. Such a characterization of mathematics is associated with the perspective that mathematics is a shared body of knowledge that has been created and developed through the centuries via successive intellectual activities by individuals and groups (Ernest, 1998). In this succession of intellectual activities, development of new ideas builds on, extends, and is connected to the previous ones in both individual and social dimensions. Also, the integration of different branches within mathematics, such as algebra and geometry, results in a system such as a Cartesian coordinate system. Mathematicians’ new theorems are tested for their logical coherence and develop into a logically coherent form through a professional review and reformulation process. As a result, mathematics continuously grows as an interconnected and coherent body.

The mathematics education community reflects the importance of logical coherence and interconnectedness in mathematics through its claims about the ways in which the teaching and learning of mathematics should take place. For example, in *Connecting Mathematics across the Curriculum*, House and Coxford (1995) noted that mathematics should be presented as “a unified discipline, a woven fabric rather than a patchwork of discrete topics” (p. vii). This commitment to logical coherence and interconnectedness underlies the recommendations about the teaching and learning of mathematics for students at both the school and college levels from professional mathematics education and mathematics organizations. The National Council of Teachers
of Mathematics (NCTM, 2000) through *Principles and Standards for School Mathematics* recommended that students learn mathematics in such a way that they come to view mathematics as an integrated and coherent entity. The Committee on the Undergraduate Program in Mathematics [CUPM] of the Mathematical Association of America (2015) recommended majors in the mathematical sciences be taught to see connections among mathematical areas.

The aforementioned sources collectively suggest that it is desirable and important for learners of mathematics at any level K–16 to construct mathematical knowledge in such a way that ideas are connected and logically coherent. Despite the importance of building connections in students mathematical understandings, it is questionable whether university students come naturally to construct their mathematical knowledge in such a way that newly encountered ideas and ideas they have previously learned in their school mathematics are connected and that the resulting system is logically coherent in their minds. This doubt is rooted in a long-standing problem in the field of undergraduate mathematics education, which is introduced in the following section. The problem, namely the double discontinuity, motivated the conceptualization of the current study.

**The problem**

Klein (1908/1924) remarked on the problem of the *double discontinuity* that prospective teachers experience in their transitions from high school to university and in returning to school in order to teach mathematics. He stated that
[a] young university student found himself, at the outset, confronted with problems which did not suggest, in any particular, the things with which he had been concerned at school.... When, after finishing his course of study, he became a teacher, he suddenly found himself expected to teach the traditional elementary mathematics in the old pedantic way. (p. 1)

Klein’s statement suggests that university students may not come to see how what they have previously learned in high school can be possibly connected to new ideas they encounter in university mathematics (i.e., the first one of the double discontinuity). It also suggests that, even after completing their coursework, what they learned in collegiate mathematics may seem, from their perspectives, far removed from what they come to teach in school mathematics (i.e., the second discontinuity). Although this problem, namely a “double discontinuity,” was posed more than a century ago by Klein, individuals and organizations in the field (e.g., the Conference Board of the Mathematical Sciences [CBMS], 2012; Cuoco, 2001; CUPM, 2015; Hodgson, 2001) acknowledge the problem still exists in the current mathematics education of university students.

Scholars in the field of mathematics education have responded to and approached the long-standing issue of the double discontinuity from various angles. The problem, as initially proposed by Klein, seems to refer to a disconnection that resides in a learner’s perception of mathematics content. However, some researchers have approached the problem by interpreting the discontinuity as a disconnection in a learner’s mathematical understandings from a knowledgeable outsider’s perspective. In the following section, I provide a synthesis of existing approaches to the double discontinuity problem in the literature (including the dichotomy between perception and understandings) and illustrate
the field’s efforts in addressing the issue so far. Based on the examination of a body of research on the double discontinuity problem, I argue for the importance of investigating the following question: *How does a university student come to see school mathematics from an advanced viewpoint in his/her learning of collegiate mathematics?*

### Existing approaches to the problem of the double discontinuity

The current section consists of two parts: first, a synthesis of empirical evidence related to the double discontinuity problem; second, an overview of what has been done in the field of mathematics and mathematics education in order to deal with the issue of the double discontinuity. First, empirical studies suggest the problem of the double discontinuity still persists in both dimensions of *perception* and *understanding*.

Researchers who focused on a *perceptual* dimension (e.g., Goulding, Hatch, & Rodd, 2003; Latterell, 2008; Shin, Lee, Han, & Lyou, 2005) documented the discontinuity in students’ perceptions in both transitional directions: (1) from learning of school mathematics to learning of collegiate mathematics, (2) from learning of collegiate mathematics to teaching of school mathematics. To investigate the existence and the nature of the issue of double discontinuity in a perceptual dimension, researchers collected and analyzed survey data or written responses that consist of participants’ (preservice or inservice teachers) self-reported perception of collegiate mathematics contents or coursework experiences. For example, Goulding and colleagues quoted students expressing their disappointment in being less successful in university mathematics courses than in school years and saying “it was such a jump [in content] …
we’d done nothing like it ['it refers to analysis] before” (p. 376, brackets added). The student’s voice suggests that the transition from school mathematics to analysis (a representative area of collegiate mathematics) was not very smooth for him and, from his perspective, school mathematics and analysis seemed to be quite different in their nature.

Latterell (2008) attended to some prospective teachers’ severely negative perception of undergraduate mathematics and their difficulties in studying it. From her survey asking what her participants liked best and least about mathematics, Latterell found that prospective teachers tended to associate their secondary mathematics experiences with what they liked best about mathematics while associating their undergraduate mathematics experiences with what they liked least about mathematics. For example, her participants reported they liked best “being given an algorithm to follow,” “computations in which you just plug in and get an answer out,” or “being able to go through the manipulation” (p. 5), which Latterell interpreted as reflecting elements of mathematics present in secondary mathematics. In contrast, when asked what they liked least about mathematics, participants indicated an intense dislike for collegiate mathematics, especially with respect to proofs and abstract nature of contents, as follows: “I don’t like abstract math. Those crazy proofs that we had to do. I didn’t understand it at all” “I like proofs least. I have a tendency, as long as I got the right answer, I didn’t care why it was true.” (p. 5). It is conceivable that the prospective teachers whose view of proofs or abstract mathematics is consistent with the preceding quotes would not recognize the connection between what they learn in collegiate mathematics courses and their future teaching.
In a relatively larger-scale study (Shin, Lee, Han, & Lyoo, 2005), researchers also found inservice teachers’ negative perception of the effects of the collegiate mathematics coursework on their preparation for teaching. Shin and colleagues noted that 56.4% of 110 secondary mathematics teachers in their Likert-scale survey answered that the courses in their teacher education programs, which included a number of mathematics content courses as well as mathematics methods courses and general pedagogy courses, did not help them much in teaching their students. This survey result supports the existence of the issue of double discontinuity, at least in a perceptual dimension, because more than half of the teachers in the survey seemed to think that what they learned in collegiate mathematics was not connected to their job of teaching.

Researchers who use both dimensions of perception and understanding to better understand the issue of the double discontinuity offer additional explanations (Wasserman, Villanueva, Mejia-Ramos, & Weber, 2015; Zazkis & Leikin, 2010). In an effort to understand how the teachers might connect the knowledge acquired in collegiate mathematics to their teaching of school mathematics, these researchers not only gathered information through interviews and surveys but also asked teachers to complete several mathematical or pedagogical tasks. Wasserman and colleagues interpreted their results to mean that inservice secondary teachers who completed a real analysis course in their teacher preparation programs might not have been given the opportunity to develop real analysis knowledge that is relevant to school mathematics or that teachers might not perceive as important some of the connections that experts view valuable. In the study of Zazkis and Leikin (2010), practicing secondary teachers were asked to describe the ways in which their advanced mathematical knowledge was implemented in their teaching and
to provide specific content-related examples that illustrate their usage of advanced mathematical knowledge in their teaching. Many practicing secondary teachers who participated in their study (i.e., 36 out of 42) were able to explain some general potential benefits of their advanced mathematical knowledge, such as a confidence-boost in teaching and problem-solving ability, but a majority of teachers were not successful in providing specific examples of using their advanced mathematical knowledge. From this, one might hypothesize that teachers, regardless of whether or not they perceive the value in what was taught in their collegiate mathematics coursework, might rarely use their knowledge acquired in collegiate mathematics courses in their teaching practice.

Such rare use of collegiate mathematics knowledge in teaching of school mathematics might be a matter of choice, but the lack of use may more likely be a function of a lack of understanding how school mathematics can be viewed differently from an advanced standpoint. This explanation can be supported by a set of literature (Cofer, 2015; Lee, 2010; Ticknor, 2012) in which researchers focused on prospective teachers’ mathematical understandings, in particular, how they related (or did not relate) their college-level mathematical understandings to secondary-level problem contexts (e.g., division by zero, simplifying an expression). These studies document meager effects of university students’ learning of collegiate mathematics on advancing their existing understandings of school mathematics.

For example, participants in Cofer’s study, when asked to explain school-mathematics-related concepts such as division by zero, were able to provide some abstract mathematical explanations as well as the explanations based on the mathematics they had previously known (i.e., analogy-based and rule-based explanations). However,
they gave no evidence of the ability to integrate that abstract explanation with the rule and analogy and, sometimes, the explanations that individuals gave conflicted with each other. Cofer concluded that students’ coursework experience in abstract algebra did not transform their existing understandings of school mathematics in the way that existing and new understandings cohere in entirety. Ticknor also found that learning of essential properties of algebraic structures—commutativity, associativity, and existence of inverses—in an abstract algebra course did not affect the students’ ways of solving secondary mathematical problems and their justification of each step. The participants in Ticknor’s study solved the problems, focusing on symbolic manipulation steps, in the same way that they would have done without learning these properties of algebraic structures. For example, when students were asked to simplify the following expression

\[2(x – 3) – 3(5 – 3x)\]

and to provide properties they used in their simplification, students universally identified the distributive property but none of them provided justification for why the minus three in front of the parenthesis becomes a negative three when distributed or why distributing the negative three to the term negative 3x results in a positive 9x using an inverse concept. It seemed that the students’ understandings of algebraic structures did not play a significant role in sophisticating their school mathematics understandings related to symbolic manipulation in algebra. These studies suggest that university students’ school mathematics understandings may remain largely unchanged and unaffected by new knowledge constructed in collegiate mathematics.

If mathematics is an area of study that is characterized by its logical coherence and interconnectedness of ideas, and if mathematics educators want students of collegiate
mathematics to construct their understanding that reflects such coherence and interconnectedness, it seems natural that recognition of the double discontinuity be followed by a call for action. In a body of literature (which is discussed in the subsequent paragraphs), individuals and organizations proposed their perspectives on how collegiate mathematics could be taught to address this issue of the double discontinuity. One subset of this body consists of policy documents and textbook materials in which the authors explain explicitly how collegiate mathematics is connected to school mathematics. The other subset of this literature reports empirical studies in which researchers implement their own approaches for addressing the issue of the double discontinuity and discuss student learning outcomes.

In policy-related documents such as *The Mathematical Education of Teachers* and *The Mathematical Education of Teachers II* (CBMS, 2001 and 2012, respectively), it is acknowledged that the problem of the double discontinuity is a long-standing issue that needs to be addressed. Accordingly, the authors of these documents recommended that collegiate mathematics courses for Prospective Secondary Mathematics Teachers [PSMTs] should help them to “make insightful connections between the advanced mathematics they are learning and the high school mathematics they will be teaching” (CBMS, 2001, p. 39). Specifically, CBMS (2001, 2012) made the following recommendations as ways to alleviate the issue of the double discontinuity: First, reform mathematics content courses for teachers in a way that allows PSMTs to connect ideas in school mathematics to the ideas in undergraduate mathematics; second, design and offer a capstone course sequence for teachers in which PSMTs examine ideas in school
mathematics from an advanced perspective; and third, offer three courses with a primary focus on high school mathematics from an advanced viewpoint.

Despite such a demand for change proclaimed by the authors of *The Mathematical Education of Teachers* and *The Mathematical Education of Teachers II* (CBMS, 2001, 2012), it appears that only a small percentage of teacher preparation programs are aligned with such recommendations. In a national survey in which more than 2,000 institutions participated (Blair, Kirkman, & Maxwell, 2013), 22% of the participating mathematics departments (that offered courses for PSMTs) offered a special geometry course for PSMTs; 7% offered a special modern algebra course for PSMTs; 4% offered a special advanced calculus or analysis course for PSMTs. In the survey conducted by Newton, Maeda, Alexander, and Senk (2014), only 10% of 78 programs preparing high school teachers met the recommendation of three courses with a primary focus on high school mathematics from an advanced viewpoint.

Although the fact that such a small percentage of programs meet the recommendations made in CBMS (2001, 2012) might seem discouraging, there have been growing efforts in supporting PSMTs to broaden and deepen their mathematical knowledge in mathematics content courses in a way that could be useful for their future teaching practices (Murray & Star, 2013). Some groups of scholars (e.g., Heid & Wilson, 2015; Nicodemi, Sutherland, & Towsley, 2007; Sultan & Artzt, 2011; Usiskin, Peressini, Marchisotto, & Stanley, 2003) developed instructional materials and resources that could be used in the types of courses recommended in the CBMS documents (i.e., capstone courses or mathematics content courses designed for teachers). These practical materials are valuable in that they provide explicit and specific examples of how mathematical
ideas in collegiate mathematics can be considered as deeply connected to the ideas in school mathematics.

Others report on their laboratory approaches to teaching mathematics content courses or capstone courses for PSMTs in response to concerns about issues related to the double discontinuity (Deiser & Reiss, 2014; Huntley & Flores, 2010; Kaiser & Bunchholtz, 2014; Vale, McAndrew, & Krishnan, 2011; Winsløw & Grønbæk, 2014). Although capstone courses for PSMTs are intended to help them to connect relevant parts of their university-level mathematics courses to high school mathematics, recent findings (Deiser & Reiss, 2014; Winsløw & Grønbæk, 2014) suggest that doing so can be significantly challenged by lack of students’ foundational knowledge. Winsløw and Grønbæk noted that “the knowledge that they [students] are supposed to use [in their capstone course] is not always what is most familiar to them” (p. 23, brackets added). When Winsløw and Grønbæk tried to implement the activity of defining what it means to have a nonrational, real exponent (e.g., $a^x$ for $a > 0$, with $x$ a nonrational, real number), they could not fully implement what they intended because of students’ lack of background knowledge needed. According to Winsløw and Grønbæk, PSMTs in their study were not successful in some basic algebraic reasoning at the high school level as well as in constructing theorem-based arguments even after completing a substantial number of collegiate mathematics courses. This lack of students’ foundational knowledge had led students to work only on local parts of the big picture. Deiser and Reiss also concluded that “the mathematical knowledge acquired in secondary schools does not necessarily constitute a reliable foundation for university mathematics” (p. 60) and this
might hinder university students from making a successful transition from high school mathematics to collegiate mathematics.

Winsløw and Grønbæk suggested that having students take remedial courses could be one way to address this issue. Such a suggestion seems to imply a perspective that developing school mathematics understandings should be a separate business from taking collegiate mathematics courses or capstone courses. The current study chooses an alternative view that their school mathematics understandings can be consolidated, reconstructed, and developed further concurrently with their learning of collegiate mathematics and reports how university students’ school mathematics and collegiate mathematics understandings may codevelop.

Meanwhile, some studies (Huntley & Flores, 2010; Kaiser & Bunchholtz, 2014; Vale, McAndrew, & Krishnan, 2011) report positive outcomes of their efforts in taking laboratory approaches to teaching mathematics content courses or capstone courses for PSMTs. For example, Huntley and Flores outlined the projects an instructor used in a history of mathematics course for PSMTs and reported how it was helpful for PSMTs’ knowledge development for teaching. One of the students voiced:

Something else that I discovered when doing this research is the connection between algebra and geometry, and how the two intertwine. Until doing this, I had never realized the strong connection between the two, and that this is something that could really help students in seeing mathematics as a big picture, instead of in parts, as I did. (p. 607)

The student’s comment “until doing this, I had never realized the strong connection between [algebra and geometry]” indicates that what she learned in the
collegiate mathematics class could have changed her conception about school mathematics. It seems that the students were deliberately and explicitly guided to investigate and trace historical perspectives on the mathematical ideas they already knew, and so their existing understandings were enhanced in the collegiate mathematics course. In their professional learning program for junior secondary mathematics (presumably, grade 7 to 10) teachers in Australia, Vale, McAndrew, and Krishnan (2011) designed their program so that teachers worked on mathematical tasks for exploring concepts, developing generalizations, and deriving/proving formulas at the senior secondary school level (Grades 11 and 12). Teachers’ self-reported reflections and observation of their classroom teaching suggested that the professional learning program helped teachers deepen and broaden their understanding of junior secondary mathematics content and see more mathematical connections and general structure within the mathematics they teach. For example, one teacher expressed that she was “beginning to see how concepts link up and ‘fit in’. I also feel as though I am generating a better understanding of ‘how’ and ‘why’” (p. 203). Although neither study (Huntley & Flores, 2010; Vale et al., 2011) provided a discussion regarding growth in participants’ understandings of connections between school and collegiate mathematics, participants in those studies articulated they developed further the mathematics they had previously known in the context of dealing with higher level mathematics. The results from the studies in the preceding paragraph (Huntley & Flores, 2010; Vale et al., 2011) are in contrast with the results from Cofer (2015) and Ticknor (2012). As described previously, school mathematics understandings of participating prospective teachers in both Cofer’s and Ticknor’s studies seemed to remain unchanged and
unaltered by their learning of collegiate mathematics. I note that the instruction of collegiate mathematics courses in Cofer’s and Ticknor’s studies did not involve any experimental approaches in response to the issue of the double discontinuity. This marked contrast between the two groups of studies suggests that learning of collegiate mathematics might take place with or without advancing the learner’s existing understandings of school mathematics. However, the contrast also suggests that, given deliberate and explicit guidance to reflect on existing understandings in a new or higher mathematical context, students may be able to develop their mathematical knowledge into a coherent whole that is tightly interconnected.

In summary, the problem of the double discontinuity has been widely studied in the field of mathematics education, in part, with efforts of helping students to make some connection between school and collegiate mathematics. Notwithstanding some promising findings, the existing discussion seems to be centered on a deficit in prospective teachers’ understandings and their perceptions that school and college mathematics are largely disconnected. In the following section, I discuss a limitation in the existing body of literature on the double discontinuity problem and propose an open question that guided conceptualization of the current study.

**An open question related to the problem of the double discontinuity**

A question that needs to be raised in looking back a whole body of literature discussed in the previous section is: What empirical basis has the field established for studying and addressing the issue of the double discontinuity? When researchers reported
the double discontinuity residing in participants’ perceptions or claimed a formation of a connected view in participants’ minds, empirical bases on which they formed their arguments were largely from self-reported data such as participants’ responses to a survey or an interview. When specific mathematical tasks were implemented to closely examine participants’ understandings related to connections between school and collegiate mathematics, the studies mainly focused on reporting a deficit in participants’ understandings. In spite of a body of research on the issue of double discontinuity, there is little empirical basis in the literature for describing and explaining a process through which university students’ school mathematics understandings and collegiate mathematics understandings cohere in the context of learning collegiate mathematics. Existing studies in which participants voiced that they saw connections in their collegiate mathematics learning (e.g., Huntley & Flores, 2010) do not account for how such connections were constructed in the students’ minds.

Another question that seems worth considering is: Is the disconnect between school mathematics understandings and collegiate mathematics understandings a problem restricted to the population of prospective mathematics teachers? To put it differently, do mathematics majors (whose intended career trajectory does not include teaching school mathematics) experience a similar phenomenon of disconnection when taking collegiate mathematics courses such as abstract algebra and analysis? One might wonder why such disconnection might matter for a mathematics major who is not a prospective teacher. I argue that examination of this underexplored population in the discussion of the double discontinuity problem might be crucial for the following three reasons.
First, by developing coherent and connected mathematical knowledge, mathematics majors could grow to be professional users of mathematics who are empowered to utilize flexibly and broadly their mathematical knowledge in their future career. This claim can be supported by Skemp’s (1976/2006) notion of relational understanding. He noted understanding what underlies seemingly disparate mathematical ideas and seeing them as “parts of a connected whole” enables one to develop mathematical knowledge that is adaptable to new tasks and more lasting (p. 92). A curricular recommendation for the undergraduate program in mathematical sciences (CUPM, 2004) also explicitly states “communicate the breadth and interconnections of the mathematical sciences” and also “conceptual understanding of mathematical ideas … [is] essential for both applications and further study of mathematics” (p. 6). Such recommendations reflect the importance of developing coherent and connected mathematical knowledge in mathematics majors that could prevent them from being “lost in a long list of required topics and computational techniques” (p. 6).

Second, the majority of lecturers of upper level collegiate mathematics courses such as abstract algebra and analysis, serve both populations (i.e., mathematics majors and mathematics education majors) in their classes (Blair, Kirkman, & Maxwell 2013). Hence, development of an empirical basis for improving the teaching of those classes should entail explanations for how students in both populations come to develop connected understandings. Third, from a researcher’s perspective, this underexplored population of mathematics majors might serve as an information-rich data source. The inclusion of this new population can potentially provide different insights into the phenomenon of disconnection and also into possibly fruitful ways of developing
connected mathematical knowledge. Hence, the population under consideration in this study is university students who are mathematics-intensive majors, which include both mathematics majors and secondary mathematics education majors.

As previously discussed, a principle underlying some curricular recommendations for mathematics-intensive majors (e.g., CBMS, 2012; CUPM, 2004, 2015) is that they should, in their undergraduate program, build connections in their evolving mathematical knowledge and, furthermore, develop an advanced perspective on their preexisting mathematical knowledge. Given this principle, it is crucial for the field of mathematics education to establish empirically-grounded explanations for how phenomena such as building connections and advancing existing knowledge take place. The current study adds to a growing body of literature on the double discontinuity problem by investigating and documenting growth in university students’ understandings—in particular, their formation of advanced perspectives on school mathematics in their learning of collegiate mathematics. To this end, the following question is pursued in the current study:

*How do university students come to see school mathematics from an advanced viewpoint in their learning of collegiate mathematics?*

Here I use the phrase “come to see” to refer mainly to changes in their understandings (rather than changes in their perceptions) because, as previously mentioned, changes in understanding is a significantly less explored research area on the double discontinuity problem. As the field begins to establish more solid empirical bases for explaining important growth in university students’ understandings that builds on connections between school and collegiate mathematics, mathematics educators could potentially gain insight into how to interpret and organize their experiences of university students’
learning and how to support university students to see school mathematics from an advanced viewpoint. In Chapter 2, I provide theoretical and conceptual underpinnings for investigating the question proposed in this paragraph.
Chapter 2 Conceptual and theoretical underpinnings

The general question that guided the current study is as follows: *How do university students come to see school mathematics from an advanced viewpoint in their learning of collegiate mathematics?* In this chapter, I discuss the theories, conceptual frameworks, and findings that guided and informed my study and explain how I draw on those conceptual and theoretical underpinnings to frame the current study. I first discuss perspectives on mathematics and on mathematical knowledge construction underlying the current study. Second, I introduce a categorical framework that I derived from the existing body of literature on the nature and development of one’s mathematical understanding, which delineates ways in which university students’ school mathematics understandings may be advanced and reorganized in the context of their learning of collegiate mathematics. Third, I describe how I used the Abstraction-in-Context (Dreyfus, Hershkowitz, & Schwarz, 2015) framework, in order to explain the context in which the observation of targeted phenomenon of the current study takes place. Fourth, I explain factorization and, in large, a decomposition perspective as an important and appropriate mathematical context for studying university students’ growth in mathematical understandings. The chapter concludes with specific research questions involving key constructs that are introduced in Chapter 2.

The interconnected nature of mathematics

In this study, mathematics is viewed as a shared body of knowledge that has been created and developed through the centuries by means of individuals’ intellectual
activities and collective reflection on products of the individuals’ intellectual activities. This view reflects Ernest’s (1998) social constructivism, which emphasizes the essential role of successive interactions and conversations among individuals and public domains in construction of both subjective and objective knowledge of mathematics. It is in the succession of intellectual activities that mathematics continuously grows as an interconnected and coherent body. The current section (a) builds on two mathematical examples (one with a concept of cyclic group, the other with a concept of group) to highlight interconnected nature of mathematics in both individual’s knowledge and in the field of mathematics; (b) exemplifies how one might be able to reconsider elementary concepts connected to a higher order concept (such as a cyclic group) from an advanced viewpoint; and (c) discusses Piaget’s reflective abstraction as a fundamental process that explains construction of a higher order concept in one’s mind with immanent connection to the relevant elementary concepts.

In the following discussion of successive intellectual activities, an intellectual activity refers to a set of goal-directed actions that collectively lead to a meaningful mathematical conclusion or construction of a concept for the cognizing subject. One example of a succession of intellectual activities that highlights connection between elementary concepts and higher order concepts can be found in Dienes (1967). A succinct presentation of the example is provided in the following excerpts and detailed in subsequent discussion.

The abstraction resulting from one cycle may provide a basis for the next, higher order, cycles. The experience cycle leading to natural number could, as we have said, lead to the ideas of even and odd. These, in turn,
could lead to the connection between even and odd that might, then be recognized as isomorphic to multiplication with equivalence classes of positive and negative numbers. The mathematic entity (two-group) invariant under this isomorphism could lead to cyclic groups of order 3, 4, 5, .... (Dienes, 1967, p. 24)

In Dienes’ example, the first intellectual activity has to do with understanding natural numbers. Dienes explained that a child might become aware of a concept of natural number by engaging in actions such as manipulating sets of objects, comparing sets to observe one-to-one correspondence between them (i.e., having the same number of objects), and ordering sets by which one has more objects than the other. A set of such experiences may lead to the realization that the quantity (or the natural number) that corresponds to a set of objects is an invariant property that does not depend on the kinds of objects s/he counts. Building on what is obtained from the first intellectual activity, a child may then engage in another activity that consists of arranging a set of objects into pairs and observing that doing so results in only two possibilities—either there is one left over or none. The natural numbers are then viewed as a collection of two types of numbers, which are called odd and even. The third intellectual activity in his example is for understanding the relationship between odd and even numbers. By counting on the objects in two sets consecutively, a child might come to realize a pattern that “odd plus odd equals even,” “odd plus even equals odd,” “even plus odd equals odd,” and “even plus even equals even.” Dienes then continued with activities involving higher order concepts (such as equivalence class, isomorphism, cyclic groups) that may be more appropriate for older students. For example, when one partitions the set of natural
numbers into two equivalence classes, a set of even numbers and a set of odd numbers, it is possible to consider the set as a group under addition, \{even numbers, odd numbers\}—often denoted by \{\bar{0}, \bar{1}\} or \(\mathbb{Z}/2\mathbb{Z}\). Then the previous observation of the relationship between odd and even numbers under the operation of addition may lead to a finding that the whole set, \{\bar{0}, \bar{1}\}, can be generated by a single element, \bar{1}, because \bar{1} generates itself and \bar{1} + \bar{1} generates the other element, \bar{0}. The fact that a single element generates an entire set also applies to \{positive numbers, negative numbers\} under the operation of multiplication. Dienes pointed out such structural similarity between \{even numbers, odd numbers\} and \{positive numbers, negative numbers\} may provide a basis for constructing a higher order concept, a cyclic group, which means a group that is generated by a single element. I depicted the succession of mathematical ideas used in Dienes’s example in detail in Figure 2.1.
In Dienes’s example the final product of a series of intellectual activities seems to be the notion of cyclic group. Although reasoning with cyclic group does not need to rely on thinking about natural numbers, these two concepts, cyclic group and natural numbers, can be seen as connected through a series of successive intellectual activities. Intermediate products such as \{even numbers, odd numbers\} with addition and \{positive numbers, negative numbers\} with multiplication are also closely connected to the notion of cyclic group because they are specific examples in which the structure of cyclic group is embodied.
Dienes’s example shows that understandings of elementary concepts may provide a basis for constructing higher order concepts, but would it also be possible that construction of, for example, a cyclic group allows one to reconsider and understand elementary concepts such as integers from an advanced viewpoint? One possible way to utilize a cyclic group to have a deeper understanding of integers is discussed in the next few paragraphs.

A cyclic group, as is for many other abstract algebra concepts, is a concept that reflects a fundamental structural property of a set of elements. For a group under an operation denoted by addition, say \((G, +)\), if one knows \(G\) is generatable by a single element, say \(g\), then it can be said that \(g\) is a fundamental building block of the whole set with \(G = \{\ldots, -4g, -3g, -2g, -g, 0, g, 2g, 3g, 4g, \ldots\}\).

In particular, examining the nature of a finite cyclic group can help one to reexamine base ten system that uses only finitely many symbols to represent infinitely many integers. First, if a group \(G\) is finite, say \(|G| = n\), then it must mean that \(n\)-many elements in \(\{\ldots, -4g, -3g, -2g, -g, 0, g, 2g, 3g, 4g, \ldots\}\) repeat themselves in a cycle of \(n\). To be specific, if a group \(G\) of order 5 is cyclic with a generator \(g\), the following holds true and one element from each of the equivalence classes can be chosen to represent the corresponding equivalence class.

\[-5g = 0 = 5g = 10g = \ldots\]
\[-4g = g = 6g = 11g = \ldots\]
\[-3g = 2g = 7g = 12g = \ldots\]
\[-2g = 3g = 8g = 13g = \ldots\]
\[-g = 4g = 9g = 14g = \ldots\]
Figure 2.2 shows a cyclic group of order 5 represented in two different ways: first line with all elements generated by $g$ allowing repetition, and the second line with five representative elements.

\[
G = \{..., -6g, -5g, -4g, -3g, -2g, -g, 0, g, 2g, 3g, 4g, 5g, 6g, 7g, 8g, 9g, 10g, 11g, ...\} \\
= \{0, g, 2g, 3g, 4g\}
\]

Figure 2.2. A cyclic group of order 5

Understanding of a cyclic group can be a powerful tool for developing an advanced viewpoint on school mathematics, in particular, the fundamental cyclic nature in the base ten number system. First of all, the notion of a generator of a cyclic group makes it salient that 1 is the element that can generate the whole set of integers under addition (note the negative numbers are generated by the additive inverse of 1; 1 and -1 are the only generators of the cyclic group, $\mathbb{Z}$). 1 is the fundamental building block that generates infinitely many elements in the set of integers. On top of seeing the structural importance of 1, the notion of having a repeated cycle in a cyclic group can be connected to deeply understanding how only ten symbols (0, 1, 2, 3, ..., 8, 9) afford representing infinitely many numbers in the base ten system. By repeatedly adding 1, one at some point reaches 9 and, after adding another 1, there is no longer a free symbol available among 0, 1, 2, 3, ..., 8, 9 for representing that subsequent number (say X, for now). In reusing and combining the existing symbols to represent X, the cyclic nature comes into play. What comes after 9 in the cycle of \{0, 1, 2, 3, ..., 8, 9\} would be 0; hence it makes sense to use 0 in the representation of X. Also, the number X is exactly one (1) cycle
away from 0; hence using 1 in the representation of X would allow one to uniquely determine which quantity X refers to. As a result, a combination of 0 and 1 (0 for the representative in the equivalence class of X; 1 for the length of a cycle from 0 to X) is used to represent the number X, namely 10. As such, numbers in base ten system can be understood in light of a concept of cyclic group; and doing so reveals a key structural property of the base ten system, which is the cyclic nature in the written representation of numbers.

A succession of intellectual activities that forms interconnectedness among mathematical ideas may take centuries of efforts and involve multiple resources. For example, in his survey about the evolution of group theory, Kleiner (1986) noted, “The abstract point of view in group theory emerged slowly. It took over one hundred years from the time of Lagrange’s implicit group-theoretic work of 1770 for the abstract group concept to evolve” (p. 207). Kleiner discussed four different sources for the emergence of group theory: (a) Classic algebra with respect to studies of polynomials gave rise to groups of permutations; (b) number theory gave rise to groups of numbers, $n$th roots of unity, integers mod $n$, and equivalence classes; (c) geometry gave rise to groups of transformations; and (d) analysis gave rise to groups of transformations. Kleiner remarked that as common features among these four sources had begun to be noted, the abstract concept of a group had emerged in the last decades of the 19th century by Arthur Cayley and Heinrich Weber.

In both Dienes’s and Kleiner’s examples, the development of new knowledge seems to have involved a significant reflection process in the succession of intellectual activities. In Dienes’s example, the way that odd and even numbers behave under
addition operation and the way that negative and positive numbers behave under the
operation of multiplication served as a basis for the subsequent activity, and they were
reflected on to result in an emergence of new, higher order idea. In Kleiner’s example,
reflection on the four sources gave rise to the abstract concept of group. A theory that
explains such a reflective process in the knowledge development was proposed by Piaget.
It is called the theory of reflective abstraction (Piaget, 1980, 2001). That theory laid the
groundwork for theories that constitute important parts of the current study’s
framework—that is, APOS theory (Arnon et al., 2014; Dubinsky, Dautermann, Leron, &
Zazkis, 1994), Piaget and Garcia’s triad (1983/1989), and Abstraction-in-Context
(Dreyfus, Hershkowitz, & Schwarz, 2015). Piaget’s work as it applies to this study is
discussed in the following section.

**Perspectives on mathematical knowledge construction**

The work of Piaget has had a significant impact on mathematics education
research for more than the past half-century by providing the field with theory explaining
how knowledge develops in an individual’s mind and what constitutes the genesis of
knowledge. According to Piagetian theory, knowledge consists of cognitive structures
and knowledge develops when the knowing subject applies a cognitive structure to the
environment. Then, a mechanism for knowledge development can be explained by the
knowing subject’s adaptation process to the environment (via assimilation and
accommodation process) toward equilibrium status. This entire process is called
equilibration and, as a result of equilibration, a new cognitive structure is constructed and so knowledge develops.

However, Campbell, in the book *Studies in Reflecting Abstraction* (Piaget, 2001) that he translated, noted that the equilibration process does not fully account for all kinds of knowledge development. To be specific, Campbell suggested that equilibration process did not provide explanation for “How do we build new cognitive structures that are actually about old structures?” (p. 4). A new cognitive structure may develop not only when adapting to the environment but also when reflecting on one’s own existing structures. Thus, the sources from which one may draw the elements of new knowledge can be either external environment or internal cognitive system.

Piaget (1972, 1980) took into account these two possible sources (i.e., external and internal) in his explanation for knowledge development. He proposed, “All new knowledge presupposes an abstraction, .... Two kinds of abstraction are distinguishable, then according to their exogenous or endogenous sources” (1980, p. 89). When the sources are exogenous, that is, when information is drawn directly from external objects, the abstraction process is called *empirical abstraction*. In contrast, when the sources are endogenous, that is, when information is drawn from the very actions of the learner and the coordination of the actions, the abstraction is called *reflective abstraction*. In detail, reflective abstraction involves two phases as follows:

On the one hand there is *projection* (as though by a reflective surface) onto a higher plane of what is drawn from the lower plane (for instance, from the plane of action to the plane of representation). On the other hand, there is *reflection*, a mental act of reconstructing and reorganizing on the
higher plane what has been transferred by projection from the lower one.

(Piaget, 2001, p. 303, emphases in original)

Piaget (1980) noted reflective abstraction is a fundamental process in the development of mathematical knowledge and it is the process that “explains the formation of new structures starting with existing ones” (Piaget, 1985, p. 149).

Hence, Piaget’s theory provides a general theoretical perspective that underlies the targeted phenomenon of the current study. Since reflective abstraction is the process through which one constructs new cognitive structures as connected to existing ones (Piaget, 1985), one’s learning of collegiate mathematics can utilize this process to construct an inherent connection between school mathematics and collegiate mathematics. Furthermore, reconstruction and reorganization of familiar school mathematics ideas in the reflection phase might involve a student in examining the familiar ideas from a notably different, advanced perspective.

What is important in this process is that the source of reflective abstraction does not exist in external objects themselves but lies in the actions of a learner and the coordination of the actions. These actions and coordination of the actions embody mathematical structures to be abstracted by the individuals. Thus it becomes evident that learners need to have many opportunities to engage in activities and reflect on their actions and coordination of actions within the activities. Lack of connections in university students’ minds documented in several existing studies (Cofer, 2015; Lee, 2010; Ticknor, 2012) might be explained by lack of such opportunities given to university students.
Then, what could be examples of actions and coordination of actions? I use a context of factorization to exemplify actions and coordination of actions that might give rise to an abstract algebra concept, an irreducible polynomial. The word “irreducible” in a factorization situation may evoke a relevant construct that could be intuitively approachable by collegiate students, by the time they are taking an abstract algebra class; but defining it requires a sophisticated attention to the set of polynomials in which the factorization is considered and a possibility of having trivial factorization. A polynomial \( f(x) \) is called irreducible in \( D[x] \) if the following two conditions are met: First, \( f(x) \) is not a unit\(^1\) and, second, the only ways to factor \( f(x) \) in \( D[x] \) are involving a unit. Hence, \( x - 1 \) is considered irreducible in \( \mathbb{Q}[x] \) and the factorization \( 2 \cdot \left( \frac{x-1}{2} \right) \) does not prevent \( x - 1 \) from being an irreducible polynomial in \( \mathbb{Q}[x] \).

For constructing a conception of an irreducible polynomial, a university student can utilize his/her existing school mathematics understandings of factorization and build on their actions and coordination of actions on some polynomial expressions in the factorization context. Consider a student engaging in the following set of questions:

- What is the “completely factored” form of the polynomial, \( p_1(x) = x^4 - 12x^2 + 36 \) over \( \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \) and \( \mathbb{C} \)?
- What about \( x^5 - x^2 \)?
- What about \( 4x + 16x^3 \)?
- What about \( 4x^2 + 2x - 20 \)?

---

\(^1\) A *unit* in \( D \) is an element that is a factor of every element in \( D \). A unit may be also considered a trivial factor such as 1 or -1 in \( \mathbb{Z} \).
• What do you consider to be irreducible polynomials in each solution?

• How would you define an irreducible polynomial?

In this series of questions, the possible actions of the student could be (a) factoring multiple polynomials until concluding that it cannot be factored further and (b) determining what the irreducible polynomials are in each of the solutions specifically (this would include determining whether polynomials such as $4$, $4x$, and $2x - 4$ would be considered irreducible or not). The possible coordination of the actions could be looking across the factorizations and comparing the determinations of irreducible polynomials that happened in different sets of polynomials depending on the types of coefficients. These actions of the student and the coordination of the actions might lead to the abstraction of what s/he might consider as irreducible or reducible in general.

During the process, the student’s school mathematics understandings may be reorganized and reconstructed (this is what happens in the reflection phase in the reflective abstraction process). For example, the student may realize that complete factorization is relative to the coefficient set and reconstruct what s/he used to think of as complete factorization to include this relativeness in his/her schema of factorization. Also, the student may realize that zeros of a polynomial and factors of a polynomial are closely related by engaging in the activities and reorganize his/her knowledge structure by linking these two (i.e., given that $\alpha$ is a zero of a polynomial, $(x - \alpha)$ is a factor of the polynomial). The current study is interested in this kind of change in university students’ understandings of school mathematics in their learning of collegiate mathematics and is intended to provide an empirical basis for explaining how such change might take place.
Whereas Piaget’s reflective abstraction provides a general, macro-level perspective for the current study, APOS theory (Arnon et al., 2014; Dubinsky, Dautermann, Leron, & Zazkis, 1994) and Piaget and Garcia’s triad (1983/1989) serve as micro-level tools and are used in the framework for the current study (which is introduced in the section entitled \textit{A categorical framework for transformative transitions} on page 50). In what follows, I discuss APOS theory and the triad, both of which are rooted in Piaget’s reflective abstraction.

The application of the theory of reflective abstraction in the context of learning collegiate mathematics was intensively discussed by Dubinsky (1991). Noting that Piaget’s research mainly focused on children’s logical thinking, Dubinsky proposed a theory of the development of concepts in advanced mathematical thinking based on Piaget’s reflective abstraction. The constructs he used in the explanation of his theory are action, process, object, and schema.

Dubinsky viewed one’s cognitive system as composed of a number of schemas that are interrelated, wherein he defined a schema as a collection of actions, processes, objects, and relevant subschemas. Objects—for example, numbers, variables, functions, vectors, groups, limits—are mathematical entities to which a specified action of the student is directed. A process is an interiorized action, which means that the student can recognize general steps to take in that action without needing to actually perform it.

For example, in the context of evaluating a definite integral of a function, calculating the area under a curve using the limit of the sum of the areas of approximating rectangles is an action that is directed to the function that determined the curve. With an action conception, a student tends to rely on external cues to perform each step in the
calculation—in this case, expressions such as the function rule or the formula for a Riemann sum. If the action becomes an internal part of the student, it can now be called a process. Dubinsky viewed this “interiorization” of an action into a process as a form of construction in reflective abstraction (p. 105). Another important form of construction in reflective abstraction is called “encapsulation” of a process into an object (p. 105).

Encapsulation of a process into an object means the student is coming to think of the process as a totality as if the process is a thing. Dubinsky pointed out that encapsulating the entire area calculation process involved in a definite integral into an object is difficult for some university students. Once the encapsulation is achieved, the object can be viewed as varying as one of its parameters varies (e.g., the upper limit of the integral).

This object then can be used in a higher level process that involves the function given by the integral as follows:

\[ \ln x = \int_1^x \frac{dt}{t}, \quad x > 0 \]

It is also important to note that a schema is not to be understood as a list of ideas but rather “a circular feedback system” (Dubinsky, 1991, p. 105). It is so because a schema is composed of relevant processes and objects that are continuously reconstructed through interiorization of existing actions (on the objects) into processes and through encapsulation of existing processes into objects (see Figure 2.3).
In the later literature of Dubinsky and colleagues (e.g., Dubinsky, Dautermann, Leron, & Zazkis, 1994), these four constructs—action, process, object, and schema—were used to explain different levels at which a mathematical concept could be understood. The theory of Dubinsky and colleagues, namely APOS theory, was adapted in the current study because it offered an analytical lens for understanding how university students’ existing school mathematics understandings might transform and codevelop in the construction of new knowledge at the collegiate level. Details about how it was used in the current study are explained in the section entitled A categorical framework for transformative transitions on page 50.

Whereas Dubinsky and colleagues dealt with levels of understanding of a single mathematical entity (such as a function), Piaget and Garcia (1983/1989) described levels of understanding signified by the relationships among various objects and coherence of schema to which those objects belong. In a mathematical problem situation, an individual may need to invoke a number of mathematical conceptions (which could be actions,
processes, and objects) at the same time and use them collectively to deal with the problem situation. Invoking multiple conceptions may occur even more so in learning of collegiate mathematics than at the secondary level because the level of the complexity of a problem situation is generally higher at the tertiary level than at a secondary level. At the collegiate level, one may need to invoke and relate existing conceptions together in a novel (to the student) way. For example, to solve a quadratic equation with imaginary coefficients (e.g., $x^2 - (1 + i)x + i = 0$), a conception of complex numbers and a conception of quadratic formula need to be invoked together. Of course, it is possible that one’s activated conception is, depending on its level, not sufficiently developed to deal with the situation, such as not being able to simplify the expression, $\sqrt{(1 + i)^2 - 4i}$, in the application of the quadratic formula to solve this equation. However, when successfully activated and coordinated, a schema for solving a quadratic equation may evolve into a coherent whole in which the coefficients are now allowed to be any complex numbers when applying the quadratic formula. In this example, the existing conceptions of complex numbers and of quadratic formula were invoked and reconstructed on the higher level than the one on which they were previously constructed. Freudenthal (1978/2004) noted that “Mathematics exercised on a lower level becomes mathematics observed on the higher level” (p. 61) and emphasized the importance of students’ making conscious efforts to “raise[e] the level” (p. 71) in mathematical learning process. Thus, it seems important to study how students’ existing schemas constructed in their school mathematics learning can be invoked, observed, reflected on, and interrelated at a higher level to form an overarching schema in the learning of collegiate mathematics.
Piaget and Garcia (1983/1989) proposed that a triad of levels— intra-object, inter-object, and trans-object levels—that can be used to describe such developing schema.

According to Piaget and Garcia, knowledge grows through a “transitional mechanism… that leads from intra-object (object\(^2\) analysis) to inter-object (analyzing relations or transformations) to trans-object (building of structures) levels of analysis” (p. 28, parentheses in original). In Piaget and Garcia’s triad, the *intra-object level* is characterized by a learner’s focusing on the object in isolation from other objects and providing “only local and particular explanations” for discoveries related to the object (p. 273). At this level, because the learner focuses on individual objects, his/her analysis of the objects is limited to some internal properties *within* the object. As a schema develops, a new element is incorporated in the existing schema followed by an assimilation or accommodation process. The *inter-object level* is characterized by coordinating the existing object with other similar, new ones and finding the relations *between* the objects. An organized grouping begins at this level through the coordination process, but the learner may not be able to explain yet how the composition of new and existing components can be seen as a whole. Finally, in the *trans-object level*, relations *across* the objects are established in the learner’s mind and a collection of the object and others cohere and form a developed schema. This level can be characterized by synthesis across the objects that leads to “the building of structures” (p. 178).

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\(^2\) It should be noted that the word, “object,” here does not necessarily refer to the same entity as the object in the APOS theory. Piaget used the word, “object,” in a much more general sense to include a physical object such as a stick and also a cognitive object such as a number.
Because of the triad’s explanatory power with respect to transitional mechanism of schema development, several studies employed Piaget and Garcia’s triad together with APOS theory as an analytical framework for explaining university students’ schema development in calculus (Baker, Cooley, & Trigueros, 2000; Clark et al., 1997; Cooley, Trigueros, & Baker, 2007). Their empirical findings attested that the triad served as an effective framework for analyzing levels of university students’ understandings of calculus concepts. In the current study, Piaget and Garcia’s triad was adapted in addition to the APOS theory to develop a categorical framework that could serve as an analytical lens through which one observes different kinds of schema development processes. Details about how the triad was used in the current study are developed further in the section entitled *A categorical framework for transformative transitions* on page 50.

Literature mentioned in the current section collectively provides some theoretical insights into the open question proposed in the end of Chapter 1, which was:

*How do university students come to see school mathematics from an advanced viewpoint in their learning of collegiate mathematics?*

From a Piagetian perspective, constructs that preexisted in a learner’s mind are *reorganized* and *reconstructed* to give rise to a new construct through reflective abstraction as the learner perseveres in a new problem context. A change through intra-inter-trans levels as in the Piaget and Garcia’s triad seems to be a way to characterize the *reorganization*; a change through action-process-object levels as in APOS theory seems to be a way to characterize the *reconstruction*. It might be through this reorganization and reconstruction that a university student come to see school mathematics from an advanced viewpoint in their learning of collegiate mathematics. Because such
reorganization and reconstruction processes are internal and not directly accessible, I introduce a construct called transformative transition as a way of approximating such internal processes for the purpose of the current study. In the next section, the new construct, transformative transition, with four categories of it is introduced as an analytical tool for organizing, documenting, and describing empirically university students’ understandings of school mathematics in their learning of collegiate mathematics.

**Transformative transitions**

The main goal of this section is to introduce a new construct called *transformative transition* and a literature-based categorical framework for the transformative transition. In bridging the review of existing literature in the previous sections and an introduction to the new construct and framework, explanation seems to be needed with respect to how the two terms, *knowledge* and *understanding*, are used in this study. The reason why these two terms are explained at this point is as follows: First, in my review of literature in Chapter 1 and Chapter 2, the two terms have been used to reflect the original authors’ choice of the term in their articles. Because these terms are often used without being defined in those original resources, it seemed inappropriate to impose my definitions of these terms on the literature reviewed in the previous sections. Second, although knowledge and understanding are closely related, I define transformative transition using “understanding” rather than “knowledge.” This choice needs to be justified based on explanation of how *knowledge* and *understanding* have been used in the field and how
they are used in the current study. Hence this section first discusses meanings of knowledge and understanding, defines a construct of transformative transition, and introduces a literature-based framework for transformative transition.

**Knowledge and understanding**

The following dictionary definitions of *knowledge* and *understanding* suggest the two concepts are closely related to each other in their everyday meanings.

Knowledge: “Facts, information, and skills acquired through experience or education; the theoretical or practical *understanding* of a subject” (Oxford Dictionary, n.d., emphasis added).

Understanding: “Comprehension = an ability to understand the meaning or importance of something (or the *knowledge* acquired as a result)” (Oxford Dictionary, n.d., emphasis added).

As such, knowledge seems to be possibly defined using the term understanding, and understanding may be defined using the term knowledge. The goal of this section is to make clear a subtle distinction between the two for the purpose of this study, building on their usage in some important literature in the field of mathematics education. (I note providing a comprehensive review of how the two terms are used in the field of mathematics education, needless to say in other fields such as philosophy, is beyond the scope of the current study.)
In this study, knowledge is viewed as a dynamic entity that consistently evolves as a learner interacts with social environment. This view is consistent with a constructivist perspective that knowledge is not fully residing in an individual’s mind nor the environment but is instead actively constructed by the individual as a result of his/her interaction with the environment (Millroy, 1992). Hence, describing knowledge as a collection of facts, information, and skills acquired by an individual (as previously defined in the dictionary) is not as important, in the current study, as explaining how knowledge might develop and examining the kinds of interactions through which knowledge might be evolving. However, viewing knowledge as a dynamic entity that is constantly evolving might cause a challenge for a researcher in two ways: (a) an observer cannot have a direct access to the entity that is being called up by the cognizing subject (von Glasersfeld, 1995) and (b) even with an indirect access, an observer cannot obtain a comprehensive picture of one’s knowledge because the cognizing subject would activate and evoke a portion of the knowledge (or a schema) in a way that s/he sees relevant to the given situation (Dubinsky & Lewin, 1986). Hence, a claim such as “a student’s knowledge of factorization has developed” is considered very difficult, if not impossible, to substantiate directly.

The kind of language that is used in the current study is, instead, “a student’s understanding of factorization has grown from level X to level Y,” where understanding is considered as a way of knowing, or a way in which the relevant knowledge is operationalized. Understanding then can be inferred from observations of what an individual does with what s/he knows. The direct access to “what an individual can articulate and demonstrate at the moment of insight itself” allows an observer to infer the
cognizing subject’s understanding that supplies reasons for the observed behaviors (Dubinsky & Lewin, 1986, p. 57). As more and more contacts are made with a cognizing subject while engaging in various mathematical situations, an observer may use the inferred understandings from a sequence of multiple observations as a reasonable approximation of the cognizing subject’s knowledge.

Defining an understanding as a way of knowing is discussed in Sierpinska (1991) in her review of how prominent scholars such as Skemp (1976/2006) and Dewey (1988) have used the word “understanding” in mathematics education. APOS theorists’ use of the term, understanding (or conception as interchangeably used by them) is also in line with defining an understanding as a way of knowing. However, some literature such as Principles and Standards for School Mathematics (NCTM, 2000) uses the term, understanding, in a particular way that may conflict with use of the term in the current study. In what follows, I first discuss different views on understanding and then explain why defining understanding as a way of knowing makes sense in the current study.

Sierpinska (1990) noted that understanding is often regarded as a two-value function: either one understands a concept or does not. Such a view is implicit in a statement like “Unfortunately, learning mathematics without understanding has long been a common outcome of school mathematics instruction” (NCTM, 2000, p. 20, emphasis in original). In this view, understanding seems to be interpreted as a type of knowledge such as “knowing why something is true or appropriate” (Simon, 2006, p. 360), “knowledge of causes” (Pritchard, 2014, p. 315), or a conceptual knowledge that is “rich in relationships… a network in which the linking relationships are as prominent as the discrete pieces of information” (Hiebert & Lefevre, 1986, pp. 3–4). This view conflicts
with the current study’s view on understanding. For example, a student who demonstrates knowing *how* to do factorization but not *why* doing so makes sense may be considered as not understanding factorization from this dichotomous perspective; however, in my study, the same student would be considered demonstrating his/her own understanding in the way that s/he operationalized his/her knowledge regardless of whether the knowledge is about how or why.

If one interprets understanding as a way of knowing, it is possible to think about types or styles of understandings. Skemp (1976/2006) proposed two types of understandings: “relational understanding” and “instrumental understanding” and defined them in terms of knowing. The former refers to “knowing both what to do and why” (p. 89) and the latter refers to “knowing what methods work” (p. 92). Such different qualifications of understandings (as in “relational” and “instrumental”) reflects different styles in which one might operationalize his/her own knowledge.

APOS theorists also distinguish “deep understanding” from understanding in general; that is, understanding is treated as something that could be demonstrated at different levels. For APOS theorists, “a person who demonstrates a deep understanding of a concept is capable of dealing with unfamiliar and even new situations using the concept or concepts in question” (Arnon et al., 2014, p. 181) and the depth level of one’s understanding increases from action, to process, and to object. An individual and intrapersonal understanding is also phrased as *conception* in order to distinguish it from a *concept*, which is “the collective understanding of that content by the community of mathematicians” (p. 18). Hence, in APOS literature, investigating demonstrated
conceptions (or understandings) of a student can be considered a tool for approximating the student’s knowledge in evolving status.

Since the current study is also in need of an analytical tool for investigating a kind of knowledge development—in particular, how university students might come to see school mathematics from an advanced viewpoint in their learning of collegiate mathematics—I choose to investigate university students’ understandings as demonstrated in what they do with their own knowledge and document possible changes in their existing school mathematics understandings in their learning of collegiate mathematics. Understanding as a way of knowing can be considered inferable through the observations of the cognizing subject’s intellectual activities, and such observations provide a way to approximate an internal process of knowledge development. The definition of a new construct, transformative transition, in the next section builds on the meaning of understanding as discussed in the current section.

**Meaning of a transformative transition**

In this study, a transformative transition refers to a particular kind of change in one’s mathematical understandings that occurs as an individual encounters and integrates a new (to the learner) construct to his/her cognitive system. The change is of a particular kind in the sense that in order for the change to qualify as a transformative transition, (a) his/her existing understandings are transformed as a new construct is integrated into a learner’s cognitive system and (b) the transformation involves a qualitative leap in the existing understandings. The use of the word transition in the transformative transition
has to do with my goal of conceptualizing a notion that contrasts with a transition characterized by disconnection in one’s understandings as in the double discontinuity problem. In the discussion of kinds of the “qualitative leap” in a transformative transition (which is provided in the next section), I exemplify four different ways in which a university student may come to form an advanced viewpoint on his/her school mathematics understandings in his/her learning of collegiate mathematics.

One assumption underlying my definition of transformative transition is that construction of new knowledge may or may not result in transformation of one’s existing understandings. Figure 2.4 depicts my point that there could be two different ways transitions may occur when new knowledge is constructed: simple transition and transformative transition.
Figure 2.4. Two possible types of transitions in one’s cognitive system when a new construct is added

In this study, a simple transition means a change in one’s cognitive system in such a way that existing understandings do not go through any change while new knowledge is constructed and added to the system. In the upper row of Figure 2.4 the fact that there is no change in existing understandings is illustrated as the preservation of the circular shape. Meanwhile, a transformative transition means a change in one’s cognitive system that accompanies a transformation of existing understandings, together with the construction of new knowledge. In the lower row of Figure 2.4 the transformation of existing understandings is illustrated as the change from a circular shape into a polygonal shape and the new knowledge cohering with existing ones is represented as the fit between a quadrilateral and a star-shaped polygon.
Four categories—deepening, extending, strengthening, and unifying—listed in Figure 2.4 are considered possible ways in which a qualitative leap might occur in existing understandings. Each of them can be briefly described as follows: (1) a qualitative leap in one’s existing understandings could involve increasing the depth level of understanding of the concept that one has been already introduced to; (2) it could take the form of increasing the boundary of contexts for considering and situating a set of existing understandings; (3) it could mean making new or stronger connections between what one has previously known; (4) it could mean that existing understandings that had no apparent relations before are unified under a certain overarching notion. Details about these categories follow in the next section.

Prior to the conclusion of this section is a discussion of similarities and differences between simple/transformative transitions and assimilation/accommodation in Piagetian theory and why it makes sense to use the construct of simple/transformative transitions for the purpose of the current study. Piaget explains that, when a cognizing subject experiences new objects in the environment, two possible mechanisms may take place to obtain equilibrium status: either “assimilation of objects to a scheme” or “accommodation of this scheme to the objects” (Piaget, 1980, p. 102).

The first mechanism, assimilation, takes place in the following situation: when the new object or an experience around the new object is perceived by the cognizing subject as absorbable to and fitting the existing schema. In this situation, the cognizing subject would “assimilate only what it can fit into the structures it already has” (von Glasersfeld, 1995, p. 63). von Glasersfeld noted that such a description should be understood as a statement from an observer’s point of view rather than as a conscious
choice made by the cognizing subject. He further noted “when an organism assimilates, [the organism] remains unaware of, or disregards, whatever does not fit into the conceptual structures [the organism] possesses” (p. 63). Hence, the “fit” used by the cognizing subject during assimilation may be missing a potentially more powerful (from an observer’s point of view) way of linking his/her existing schema and the new experience. The notion of simple transition is crafted in this study to emphasize a missed chance of linking an existing schema and a new experience during the assimilation. Hence, although a simple transition is similar to assimilation in the sense that the existing understanding/schema does not change as a result of dealing with a new situation, a simple transition has a narrower focus than what assimilation describes.

A hypothetical situation of encountering a modulus concept can be characterized by assimilation in a global sense and simple transition in a local sense. For a complex number $z = a + bi$, where $a$ and $b$ are real, the modulus of $z$ (denoted by $|z|$) is defined as $\sqrt{a^2 + b^2}$; in other words, the distance from the origin to the point represented by $z$ on the complex plan. What might happen if a university student, who had previously understood an absolute value of a number, $x$, (denoted by $|x|$) as simply flipping the sign when $x$ is negative, encountered this definition of modulus? Would he recognize any “fit” between the modulus and the absolute value possibly triggered by the same vertical-bar notation? It seems imaginable that the student may assimilate the new definition and still remain unaware of a way to link the modulus concept and the absolute value concept. Perhaps, based on the similarity between the formula used in the definition of the modulus ($|z| = \sqrt{a^2 + b^2}$) and the Pythagorean theorem ($c = \sqrt{a^2 + b^2}$), the student might perceive the new definition familiar and assimilate it to his prior cognitive
structure. The kind of learning that occurred in this hypothetical situation is what I call a *simple transition* (in contrast with *transformative transition*) because of the missed chance of linking the modulus concept and the absolute value concept and of further refining his existing understandings of the absolute value. In sum, a simple transition should be understood as describing a particular kind of assimilation that is characterized by a potentially relevant fit remaining unnoticed between existing understandings and a new object.

A similar relation holds between the transformative transition and accommodation. Although a transformative transition is similar to accommodation in the sense that the existing understanding/schema is modified in the experience of new objects, a transformative transition has a narrower focus than accommodation and refers to a specific kind of accommodation. The mechanism, *accommodation*, takes place when assimilation is impossible. In other words, if a cognizing subject finds the experience of a new object unexpected, s/he would reconstruct the existing schema in order to deal with the unexpected experience and reach equilibrium status. von Glasersfeld noted “if the unexpected outcome of the activity was disappointing, one or more of the newly noticed characteristics may effect a change in the recognition pattern” in his explanation of an accommodation mechanism (p. 65). Ways in which “change in the recognition pattern” takes place might vary; and a transformative transition refers to a particular kind of such change (or a particular kind of accommodation) in the sense that the change involves a qualitative leap in the existing understandings. I present a hypothetical situation that is characterized by accommodation but not by transformative transition, in order to explain the difference between them.
In the previous hypothetical situation related to the modulus and absolute value concepts, a student might find the modulus notation, \(|z|\), reminiscent of the absolute value of a real number \(|x|\). If the student had previously understood \(|x|\) as a procedure of negating \(x\) when \(x\) is negative and leaving \(x\) as is when \(x\) is positive, then she might perceive the definition of modulus, \(|z| = \sqrt{a^2 + b^2}\), as unexpected. Perhaps, the new definition might be disappointing for her, since her prior procedure of negating the argument or leaving it as is no longer works for the modulus of a complex number. She might feel the need of changing the recognition pattern when she sees the vertical bar notation. At this point, questions of interest in this study are what kinds of change take place and whether the change involves a qualitative leap in the existing understandings.

Accommodation without a qualitative leap in the existing understanding could take place as follows. The student may change her recognition pattern by coming up with a new procedure that could replace the original procedure of deciding whether a given number is negative or not and negating its sign accordingly. A new recognition pattern might be first, deciding whether a given number is a nonreal number, a real number that is positive, or a real number that is negative and then, applying a relevant subprocedure including the new formula, \(\sqrt{a^2 + b^2}\). This sort of accommodation does not appear to have advanced her existing understanding of the absolute value, because it is still understood as simply flipping the sign when the argument is negative. On the other hand, accommodation could have involved a qualitative leap in her existing understanding in the following ways (and I call them transformative transitions). The student could have come to be able to see \(|x|\) as equivalent to \(\sqrt{x^2}\) in light of the new definition \(|z| = \)
\[ \sqrt{a^2 + b^2} \] (\( z = a + bi \)), since \( x = x + 0 \cdot i \). Also, it is possible that she came to understand both \(|x|\) and \(|z|\) as representing the distance from the origin to the point represented by the number. Absolute value of \( x \) is now visually conceptualized using a number line and perceived as the distance from 0 to \( x \) (rather than a procedure of flipping the sign of a number when it is negative).

In sum, assimilation and accommodation are global mechanisms in which a simple transition and a transformative transition are observed, respectively. However, assimilation and accommodation did not seem to capture the local phenomena of interest in the current study that underlie the problem of the double discontinuity. A transformative transition can be considered as a tool for explaining the growth in university students’ understandings in their learning of collegiate mathematics. Further characterization of a transformative transition was informed by my review of literature on the nature and development of one’s mathematical understanding. In the next section, I discuss how four categories of transformative transition were derived from the literature and exemplify how each category applies to understanding school mathematics from an advanced viewpoint.

**A categorical framework for transformative transitions**

To carry out this study, it was necessary to complete a critical review of the literature that informs how to characterize and typify the growth of university students’ existing understandings of mathematics in their learning of collegiate mathematics. In what follows, I discuss how the literature areas were selected for the review and what
steps were involved in conducting the review and in constructing the categorical framework. It should be noted that no assumption was made that the categories developed from the review comprise all of the areas of growth.

First of all, I started with a set of literature on “mathematical connections” (e.g., Businskas, 2008; Evitts, 2004; Singletary, 2012; Suominen, 2015) and “mathematical knowledge for teaching” (e.g., Ball, 1993; Ball & Bass, 2009; Fernandez & Figueiras, 2014; Jakobsen, Thames, & Ribeiro, 2013; Wasserman & Mamolo, 2015; Zazkis & Mamolo, 2011, 2012). This initial choice seemed to make sense because the motivation of the current study—the double discontinuity problem—concerns the disconnect in preservice and inservice teachers’ mathematical knowledge between school mathematics and collegiate mathematics. Reviewing the literature informed some important attributes of mathematical knowledge that would be considered potentially desirable for prospective mathematics teachers to have.

In the search for generally desirable attributes of mathematical knowledge (i.e., not necessarily restricted to the knowledge of mathematics teachers) and how those attributes might develop, I considered the literature on “mathematical knowledge construction” (as discussed in section Perspectives on mathematical knowledge construction starting on page 27) and “abstraction process” (e.g., Brown, Collins, & Duguid, 1989; Davydov, 1972/1990; Hershkowitz, Schwarz, & Dreyfus, 2001; Lobato, 2005; Noss & Hoyles, 1996; Noss, Hoyles, & Pozzi, 2002; Ohlsson & Lehtinen, 1997; Roth & Hwang, 2006a, 2006b; van Oers, 1998, 2001). I also conducted an internet-based search in Google Scholar with the following keywords and located an additional body of literature:
• Mathematical understanding or knowledge
• Levels of mathematical understanding or knowledge
• Types of mathematical understanding or knowledge
• Qualities of mathematical understanding or knowledge
• Development of mathematical understanding or knowledge

In this search, the literature focusing on the secondary or collegiate level (that is, the grade levels relevant to the problem of the double discontinuity) was included in my review of the literature while the literature focusing on children’s or elementary mathematics was excluded from it.

Throughout the review, I analyzed how each of the articles or book chapters treated mathematical understanding/knowledge and its development. The questions I asked of each of the sources are: What are the qualifications of mathematical understanding/knowledge—such as “relational” and “instrumental” in Skemp (1976/2006)—or relevant constructs used in this source? How are the qualifications or the relevant constructs defined or described in this source? If any desirable attributes of mathematical knowledge are present, how is the development of such attributes characterized in this source? In the end, the constructs used in each of the sources were listed, and a brief description of those constructs or other relevant information was recorded.

As a final step, comparisons among the sources were conducted to see whether there exist commonly applicable constructs for typifying the quality of mathematical understanding/knowledge. The process of identifying commonly applicable constructs was iterative in that an emerging construct from a group of sources was consistently
applied to another source to see if an additional construct is needed to explain the incoming source. As a result, four common constructs (depth, breadth, connectedness, coherence) were derived to generate the four categories—deepening, extending, strengthening, and unifying. In the review, I also looked for ways to characterize and describe qualitative leaps under each category so that a student’s progress can be identified in their understandings under each category. Literature rooted in the Piagetian constructivism informed characterization of the qualitative leaps in the development of mathematical understanding/knowledge\(^3\) (e.g., encapsulating process into object, developing schematic understandings). In sum, my literature review resulted in the four categories and levels for identifying a student’s progress under each category. The four categories were the only ones suggested by my review, and I note the strategy in my review for generating those categories was not designed to be comprehensive.

When revisiting the literature with the emerging four categories, a number of studies in the review suggested one of the following four ways to characterize the development of one’s understandings: deepening, extending, strengthening, and unifying. The following sections discuss how each of these categories builds on existing literature and explain how APOS theory or Piaget and Garcia’s triad can be used to identify student’s progress with respect to each category.

\(^3\) Some of the theories for characterizing the progress in mathematical knowledge development (e.g., Dubinsky’s APOS theory and Piaget and Garcia’s triad) were discussed in the previous section entitled Perspectives on mathematical knowledge construction. In the current section (entitled A categorical framework for transformative transitions), I focus on how each of the categories draws on the theories.
**Deepening category**

Depth of mathematical understanding or knowledge is a commonly used construct for characterizing mathematical understanding or knowledge that can exist in different levels. For example, Star (2005) contrasted deep mathematical knowledge with a superficial one, noting that a learner’s initial knowledge of a concept or a procedure can be superficial and fragile but over time the knowledge can deepen. Building on the work of De Jong and Ferguson-Hesseeler (1996), Star characterized deep-level knowledge as the knowledge with “comprehension, flexibility, and critical judgment” (p. 408). From this perspective, deep understanding or knowledge is certainly the desired state of a learner’s mind in educational settings.

Ma (1999) also developed a notion that she called *profound understanding of fundamental mathematics* from the comparison between two groups of elementary teachers’ mathematical knowledge. Her findings suggest that numerical operations can be understood at qualitatively different levels. On one hand, four basic numerical operations can be understood *superficially* in terms of how to complete an algorithmic procedure given a basic operation task. On the other hand, one may “reach the essence” of the numerical operations by understanding the fundamental ideas underlying the four operations (p. 112). According to Ma, understanding a topic with depth means to have the capacity to connect the topic to the ideas of “greater conceptual power” (p. 124). For example, understanding the four basic operations with *depth* means to be able to understand how the ideas such as inverse operations, place value, associativity, commutativity, and distributivity underlie the four operations.
Gray, Pinto, Pitta, and Tall (1999) also used the same phrase *greater conceptual power* to characterize mathematical knowledge of high achievers. They coined a notion called *procept*—a symbol that evokes either process or concept—and argued that flexible proceptual understandings allow one to concentrate on “concepts that give greater conceptual power” (p. 128). Examples of procept are symbols such as +, D (determinant of a quadratic equation), and sin (sine function), which is used to represent either process (adding, calculating \( b^2 - 4ac \), and dividing the length of a side that is opposite a given angle by the length of the hypotenuse in a right triangle, respectively) or concept (addition operation, an entity the sign of which determines the number of real roots for the equation, and a ratio assigned to a set of similar right triangles, respectively). Gray and colleagues argued that proceptual understanding develops through three phases: at the least sophisticated level, a learner does routine mathematical procedures associated with a symbol; at the next level, the learner performs a mathematical process associated with the same symbol flexibly; and, at the most sophisticated level, the learner treats the symbol as representing a concept (i.e., proceptual understanding). The notion of proceptual understanding highlights the fact that the same mathematical ideas may be understood with different levels of depth, which is consistent with Ma’s observation.

Although the contrast between superficial and deep understanding (De Jong & Ferguson-Hesseeler, 1996; Star, 2005) and connection between deep understanding and having greater conceptual power (Gray et al., 1999; Ma, 1999) offer some perspectives on how to characterize deep mathematical understandings, they do not elaborate theories of how the depth of understanding might be increased, roughly speaking, from a lesser
degree of depth to a greater degree.\(^4\) APOS theory offers a framework for elaborating on what it means to deepen the level of existing understandings. As previously mentioned, APOS theory posits that a mathematical concept is learned through the “developmental progression” from Action to Process to Object to Schema, hence abbreviated as APOS (Arnon et al., p. 10).

In the most basic level, \emph{action level}, a mathematical idea can be understood in terms of an action, that is, in terms of “a repeatable physical or mental manipulation that transforms objects in some way” (Dubinsky, Dautermann, Leron, & Zazkis, 1994, p. 270). The action level can be characterized by the learner’s need for specific formulas, recipes or external cues for performing steps in thinking about the mathematical idea. By repeating and reflecting upon an action, the learner interiorizes the action into a process. At the \emph{process level}, the action takes place in the mind of the learner and s/he does not need to actually perform the action for thinking about the idea. At the \emph{object level}, the student reflects on the process and thinks of it as a totality so that s/he can encapsulate the process of applying a transformation to an object. Then, the student can draw on the object and apply it to take another action on the object. Finally, a mathematical idea can be understood at the \emph{schema level}, if a collection of processes, objects, and the other schemas works in the learner’s mind coherently and structurally to handle a mathematical situation.

\(^4\) I note that the procept theory (Gray et al., 1999) offers some explanation of how the depth of understanding might be increased. However, the procept theory is not generally applicable because a mathematical concept may not be associated with a symbol that represents the concept.
From the perspective of APOS theory, the progression from action level to process level or from process level to object level suggests increased quality and power of the learner’s understanding of the same concept. To borrow the wording used by Ma (1999), the progression means to get closer to or “reach the essence” of the concept (p. 112). It also suggests increased capability for dealing with unfamiliar problem situations using the same concept and hence, development of a deep understanding of the concept (Arnon et al., p. 181). In this study, I regard the progression from action to process to object level as increase in the depth level of existing understandings of a certain, single mathematical notion.

If deep understanding is what mathematics educators hope that university students develop in their learning of mathematics as suggested by researchers (De Jong & Ferguson-Hesseeler, 1996; Ma, 1999; Star, 2005), it seems reasonable to investigate one’s transformative transitions in terms of how the depth of one’s understanding can be increased. Thus, the first category of transformative transitions is as follows:

**Deepening category:** A learner makes a transformative transition by increasing the depth level of existing understandings of a certain, single mathematical notion in the learning of a new construct.

As mentioned earlier, in this study, increase in the depth level of existing understandings of a certain mathematical notion means to make a progression from action level to process level or from process level to object level. Thus, I adopted action, process, and object levels from the work of Dubinsky and colleagues to interpret the data
and describe how university students make transformative transitions in the deepening category.

I adopted only the first three of the four APOS levels—action, process, and object, and not schema—because the advancement from object level to schema level is more about relational understandings of multiple mathematical notions changing its status than about understanding of a single notion taking a higher order status. In this study, the strengthening, extending, and unifying categories capture how existing understandings of multiple notions may form a different relationship in one’s mind.

Entries in Table 2.1 exemplify in the context of the mathematical notion of factorization how action, process, and object levels in APOS theory can be adapted to identify a student’s making transformative transitions in the deepening category.

Table 2.1. Levels in deepening category and descriptions of each level with respect to understandings of factorization

<table>
<thead>
<tr>
<th>Levels in the deepening category</th>
<th>Descriptions (with respect to understandings of factorization)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Action level</strong></td>
<td>A student at this level likely demonstrates high reliance on formulas and mechanical aspects when factoring. Once s/he does not remember the formula or does not recognize any applicable recipes, it might be difficult for him/her to move the discussion forward.</td>
</tr>
<tr>
<td><strong>Process level</strong></td>
<td>A student at this level can express the process in a generic way and explain what it means to factor without actually performing the factorization. It might be that the student uses a specific example but the explanation is generic.</td>
</tr>
<tr>
<td><strong>Object level</strong></td>
<td>A student at this level can compare factorization in a certain situation with factorization in a different situation–factorizations themselves can become objectified so that they can be compared to each other. At this level, factorization may be an object to be</td>
</tr>
</tbody>
</table>
characterized, for example, by its primeness, completeness, and uniqueness.

Sfard’s (1991) reification theory also conforms to the APOS theory view that emergence of an object level understanding is likely preceded by a process conception of that mathematical entity. Sfard regarded this ontological genesis in one’s mind (i.e., being able to recognize the process as “a thing”) as an important leap in the development of one’s understandings. I consider this kind of ontological leap with respect to a certain, single mathematical notion as a key feature of my category of deepening.

Some researchers (Dubinsky et al., 1994; Sfard, 1991; Gray & Tall, 1994) pointed out that many mathematical concepts have dual aspects (as an object and as a process) and that flexible shifting between object level and process level is essential for one’s knowledge development. Thus, a student with object-level understanding may reveal only process-level understanding depending on how the problem situation is perceived by the student. In this study, it is acknowledged that certain mathematical situations require the flexible use of the two conceptions (i.e., as an object and as a process) and that there is a possibility that a student with object-level understandings may be operating at the process level. Thus, evidence of object suggests object level but evidence of action or process does not preclude the existence of object level. In terms of making transformative transitions, when an instance clearly indicates one’s formation of a notably different view on the same entity (e.g., from process to object), it can be interpreted as making transformative transitions in terms of the deepening category.
*Extending category*

In my review of the literature, another important category for describing growth of mathematical understandings arose: the *extent* of the context in which the understandings are situated. Pratt and Noss (2002) argued the process of “broadening of contextual neighborhood” is essential to the trajectories of mathematical learning (p. 48). According to Pratt and Noss, mathematical knowledge develops through the abstraction process and the abstraction process is to be understood as extending the context in which mathematical ideas are situated rather than deleting contexts surrounding the ideas and drawing away some mathematical features—so-called *decontextualization.*

Some researchers (Kaminski, Sloutsky, & Heckler, 2009; Lobato, 2008) also contend that extending the boundary of situations in which students recognize and apply their mathematical knowledge should be an important part of their knowledge development. In their studies on the *transfer* of mathematical knowledge, Kaminski and colleagues studied the effects of generic versus concrete instantiations on undergraduate students’ ability to transfer the learned concept to novel situations. They found learning with generic instantiations helped students to attend to relational structures and recognize the learned structure in novel contexts. This might mean, when a school mathematics idea is learned in a generic way, the idea may be transferred effectively to a collegiate mathematics context embedding the idea.

In fact, if a teacher can see how a school mathematics idea is embedded in a larger context, it may help him or her to teach the idea in a generic way, which may later make his or her students’ transfer easier. The importance of a teacher being able to see
the larger context for a school mathematics idea is reflected in the idea of horizon knowledge of mathematics (Ball & Bass, 2009). Ball and Bass proposed horizon knowledge of mathematics as a component of mathematical knowledge for teaching and noted the horizon knowledge contributes to the teacher’s instructional choices. Horizon knowledge of mathematics means “an awareness of the large mathematical landscape in which the present experience and instruction is situated” (p. 6). Then developing horizon knowledge means to broaden one’s horizon farther away by extending to a setting in which the concept was not originally proposed. It also means to gain in-depth appreciation for how the ideas along the learning trajectories are related and fit together (Zazkis & Mamolo, 2011). Hence, increasing the boundary of the contexts in which school mathematics ideas are considered seems to be what needs to be encouraged in prospective mathematics teachers’ learning of collegiate mathematics.

This notion of extending the context in which one’s mathematical understandings are situated led me to think of a case that is unlikely to be explained by the deepening category and yet viewed as a transformative transition. There could be a case in which university students come to be aware of a larger context in which what they have previously known can be situated. For example, suppose that a university student in his complex analysis course encounters a definition of logarithmic function in the domain of complex numbers (i.e., \( \ln z = \ln |z| + i \arg z \) for a complex number \( z \)) and tries to make sense of the definition. He previously thought that a logarithmic function can be situated only in the context of having nonnegative input values, because \( \log_b a \) means the power to which \( b \) would have to be raised to equal to \( a \), and that means \( a \) must be positive. Now,
with Euler’s formula, he can see that the way that log of a complex number \( z \) is defined is consistent with the way that log was defined previously, since

\[
e^{\ln|z| + i\arg z}
\]

\[
= e^{\ln|z|} \cdot (\cos(\arg z) + i \sin(\arg z))
\]

\[
= |z| \cdot (\cos(\arg z) + i \sin(\arg z))
\]

\[
= z
\]

Then it can be said he is now able to understand logarithmic function in a broader context (i.e., logarithmic function with the domain of complex numbers) and see the consistency between the way logarithm was defined previously and the new way. In other words, in the construction of this new knowledge, his existing understandings of logarithmic function go through a transformative change. Such a change in existing understandings may not be explained by the deepening category with action, process, and object levels because his understandings of logarithmic functions may be at object level both before and after being aware of a larger context in which logarithms can be situated. Thus, the second category of transformative transitions is formed as follows:

**Extending category:** A learner makes a transformative transition by increasing the boundary of contexts in which a set of existing understandings are situated in the individual’s mind in the learning of a new construct.

To analyze how one might come to extend the context in which existing understandings can be situated in his/her mind, I adopted Piaget and Garcia’s triad to describe how university students make transformative transitions in terms of the
extending category (and also strengthening and unifying categories in subsequent sections). In an *intra-object level of extending*, a learner may recognize and work with the object in only a familiar context that may be quite local and particular. In a way, the context may be bounded even without the boundary being noticed by the learner. In an *inter-object level of extending*, the stability of an initial boundary is lessened by the discovery or observation of ones similar to the object that do not fit in with familiar ones in the boundary. A learner tries to coordinate these objects and find the relationship between them. In a *trans-object level of extending*, the initial boundary is rebuilt to include and cohere to all these objects in a same group and so, the learner’s existing understandings of the object now can be situated in a broader context.

In Table 2.2, I exemplify how Piaget and Garcia’s triad was adapted to identify a student’s making transformative transitions in terms of the *extending* category using the mathematical notion of factorization.

Table 2.2. Levels in the extending category and descriptions of each level with respect to understandings of factorization

<table>
<thead>
<tr>
<th>Levels in extending category</th>
<th>Descriptions (with respect to understandings of factorization)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intra-object level</td>
<td>At the intra-object level, the student can perform factorization in a particular and localized context such as over integers. For example, s/he may assume a particular context, ( \mathbb{R}[x] ), in his/her consideration of factorization and think ( x^2 + x + 1 ) cannot be factored because it cannot be expressed as a product of linear expressions with real coefficients.</td>
</tr>
<tr>
<td>Inter-object level</td>
<td>At the inter-object level, the student begins to coordinate an unfamiliar (to him/her) context for factorization with existing factorization schema. The student begins to see how the existing</td>
</tr>
</tbody>
</table>
understandings related to the familiar context of factorization can be extended and applied to the unfamiliar context.

A student at this level, however, cannot coordinate all of his/her constructs related to factorization in order to produce the desired results in the problems given. For example, the student may not think it is possible to obtain a factorization such as \((x + 1)(x - i)\) because of its conflict with the usual conjugate pairs appearing in factorization such as \((x + i)(x - i)\) and \((x - \frac{1+i\sqrt{3}}{2})(x - \frac{-1-i\sqrt{3}}{2})\).

| Trans-object level | At the trans-object level, the student constructs a coherent structure/schema of his/her constructs related to factorization. The student demonstrates awareness of the relevant (to the given problem) contexts for factorization and ability to examine how the choice of different contexts might affect the way factorization operates. |

**Strengthening category**

A third important construct suggested in the literature for characterizing the quality of one’s mathematical understanding is connectedness, or the state of knowledge network being connected. The capacity to “weave together” all parts in one’s knowledge system is valued as well as depth and breadth of the knowledge (Ma, 1999, p. 121). The existence of integrated and well-connected knowledge has been regarded as an indicator of high quality knowledge by mathematics education researchers (Baroody, Feil, & Johnson, 2007). Baroody and colleagues used five levels—not connected, sparsely connected, somewhat connected, well connected, richly connected—to describe Star’s (2005) reconceptualization of the quality of conceptual or procedural knowledge of mathematics.
Describing the quality of knowledge in terms of the level of connectedness seems to suggest an underlying view that knowledge construction is somewhat like building a structural network that consists of nodes representing concepts, procedures, or generalizations and conceptual links between the nodes. Hiebert and Carpenter (1992) remarked that “understanding in mathematics is making connections between ideas, facts, or procedures” and that the development of one’s understanding is identified with constructing more connections and making the connections stronger (p. 67). In this study, the strengthening category focuses on how the strength of the connections between existing constructs in a learner’s mind may codevelop with the construction of new knowledge. Then, the question arises concerning how we can describe the strength of a connection and identify when a connection is getting stronger.

Because interconnectedness is an essential characteristic of mathematics as a field of study, it is possible that students come to construct some links between mathematical ideas, explicitly or implicitly, in their learning of mathematics. For example, when solving a quadratic equation, a student may apply factorization and find solutions. The series of the student’s actions may involve multiple constructs such as equation, factorization, coefficients, and roots, and so he or she may have formed certain cognitive links between these constructs in the student’s mind. I propose that these cognitive links have different strength in at least three different ways (see Figure 2.5).
The weakest type of link between two constructs A and B, I think, is characterized by A and B connecting through a medium, without a direct link connecting A and B. I call this kind of link *secondary*. For example, when solving a polynomial equation, students might think that the only way to connect from an equation to its root(s) is through a medium of factorization. In such a case, students have no direct cognitive link that connects from an equation to its root(s). When there is no direct link, a student who is solving an equation such as

\[ x^3 - 2x^2 + 2x - 1 = 0 \]

may seem clueless unless he or she knows how to factor it. In contrast, if a student has formed a direct cognitive link that connects from an equation to its root(s) and knows what a root of an equation means, then she or he will look for a value of x for which the equation holds true and see \( x = 1 \) as one of its roots. A direct link may be viewed as stronger than a secondary link.
It is also important to note that these cognitive links may have *directionality* (Evitts, 2005)—that is, it may flow one way along the link but not the other way. Evitts used examples from Knuth’s (2000a; 2000b) study to exemplify the directionality of connections. Students in Knuth’s study who were able to connect a symbolic linear equation with the graph of the equation mainly did so by connecting *from equation to* graph and had difficulties in connecting *from graph to* equation.

Directionality can be illustrated in the context of solving a quadratic equation. Given a specific quadratic equation, a student may easily link the equation to the corresponding roots, if they exist. However, the link the student has formed may be *unidirectional*—from an equation to its roots. To be specific, if he or she were to start with a root $2 + \sqrt{3}$ and to find a corresponding quadratic equation with integer coefficients, he or she may find the link that she had previously formed is not sufficient to solve this problem. Being able to move from a root to its equation will make the connection between two constructs stronger. A bidirectional link may be considered as stronger than a unidirectional link.

To summarize, I identify having no direct link but a secondary link as the least strong type of connection (see the far left image in Figure 2.5); having a direct link that is only unidirectional as stronger but not the strongest (see the middle image in Figure 2.5); and having a bidirectional link as the strongest (see the far right image in Figure 2.5).

In the learning of collegiate mathematics, university students can have chances to revisit and strengthen the links between existing constructs in their mind because the existing constructs are illuminated from a different angle. It is similar to one being able to see, from a higher place, a road between two places that one could not figure out how to
link on the ground level. Sfard’s (1991) reification theory supports this view—her theory suggests that existing understandings can be solidified and glued together when higher level concepts begin to be interiorized. Thus, the third category of transformative transitions is formed as follows:

**Strengthening category:** A learner makes a transformative transition by increasing the strength of the link between existing understandings of more than one mathematical construct in the learning of a new construct.

Because the strengthening category also relates to how existing understandings of multiple notions may form a different, new schema in one’s mind, I adopted Piaget and Garcia’s triad to describe how university students make transformative transitions in terms of the strengthening category.

In an *intra-object level of strengthening*, a learner focuses on a construct in isolation from other constructs, and so, constructs in the learner’s mind may be barely and weakly linked at this level (e.g., secondary link). Here the learner’s understanding can be characterized as “instrumental” (Skemp, 1976/2006) in the sense that the learner may know how to apply certain rules to attain the desired answer but may not know why the rules work in relation to other constructs. In an *inter-object level of strengthening*, a learner begins to coordinate and find the relations between two or more constructs but may demonstrate only a unidirectional link between them or cannot fully coordinate a bidirectional relationship. In a *trans-object level of strengthening*, a learner establishes a coherent structure about how existing constructs are related to each other in his/her mind (that is, fully coordinates bidirectional relationships)—which can be regarded as
“relational understanding” from Skemp’s perspective. Thus, the strength of the links between existing constructs is greater than the strength of the links at the inter- and intra-object level.

In Table 2.3, I exemplify how Piaget and Garcia’s triad was adapted to identify student’s making transformative transitions in terms of strengthening category using the mathematical notions related to polynomial equation.

Table 2.3. Levels in the strengthening category and descriptions of each level with respect to understandings of polynomial equation

<table>
<thead>
<tr>
<th>Levels in strengthening category</th>
<th>Descriptions (with respect to understandings of polynomial equation)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Intra-object level</strong></td>
<td>At the intra-object level in terms of polynomial equation, the student can solve a given polynomial equation in a particular and localized context and can utilize a construct involved in solving an equation such as quadratic formula, discriminant, and factorization individually and in isolation from other constructs. A student at this level cannot relate different constructs involved in solving an equation such as equations, roots of equation, quadratic formula, discriminant, and factorization in order to produce the desired results in the problems given.</td>
</tr>
<tr>
<td><strong>Inter-object level</strong></td>
<td>At the inter-object level in terms of polynomial equation, the student begins to relate and coordinate two or more constructs simultaneously. A student at this level, however, cannot coordinate all of his/her constructs related to solving a polynomial equation in order to produce the desired results in the problems given. For example, a student cannot generate a polynomial equation starting with the corresponding roots, even though s/he can begin with a polynomial equation and generate roots (i.e., unidirectional).</td>
</tr>
</tbody>
</table>
**Trans-object level**

At the trans-object level in terms of polynomial equation, the student constructs a coherent structure/schema of his/her constructs related to polynomial equation.

A student at this level can demonstrate how one construct is related to the other construct bidirectionally with full coordination. A student can readily recognize and utilize what relationship is to be used to produce the desired result.

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**Unifying category**

The fourth category in the transformational transitions framework was derived from the literature that treats successful knowledge development as associated with developing a hierarchical, organizational knowledge structure in which upper level, overarching constructs organize lower level constructs. McGowen and Tall (1999), in their studies observing students’ concept map change, found that having “basic anchoring structures” and the ability to restructure knowledge in a way that integrates and organizes new constructs effectively was a common feature of the more successful group of collegiate students in these studies (p. 8). They interpreted that having anchoring structures and the ability to organize and restructure knowledge with such foundational structures allowed students to manage to construct “complex cognitive structures with rich interiority” (p. 8). Meanwhile, concept maps produced by the least successful group of students suggested there were no organizational structures or common classifications retained in each individual’s concept maps throughout the semester. McGowen and Tall’s study highlights the fact that successful knowledge development is associated with the ability to organize knowledge structures around fundamental, basic ideas.
Baroody, Cibulsksis, Lai, and Li (2004) also argued developing understandings of “big idea—key ideas that underlie numerous concepts and procedures across topics” should be an integral part of students’ learning of mathematics (p. 254). As an example of such an idea, Baroody and colleagues mentioned that the idea of equal decomposition provides a basis for understanding unit principle, even number, division, fractions, measurement, and mean. The explanations the authors offered mainly centered around elementary mathematical ideas but, the idea of decomposition underlies many other mathematical ideas in secondary and collegiate mathematics as well.

For example, decomposition occurs in prime factorization of integers, in factorization of polynomials, in function decomposition for applying the chain rule, and in transforming a matrix into the row-echelon form. In all of these cases, decomposition allows us to see how an object is composed of more elementary components, which sometimes reveals essential characteristics of that object. Being able to see one overarching notion underlying many different mathematical ideas would help learners to value mathematics as a unified and coherent discipline (Coxford, 1995).

In the learning of collegiate mathematics, university students are taught abstract ideas that actually underpin many of the school mathematics concepts and procedures. For example, the notion of inverse in abstract algebra is a notion that is reflected in many different situations in school mathematics. Wasserman and Mamolo (2015) reported that, in Common Core State Standards, the notion of inverse underlies elementary school’s

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5 Unit principle means that “any number can be expressed as the sum of units (e.g., 5 = 1 + 1 + 1 + 1 + 1)” (Baroody, Cibulsksis, Lai, & Li, 2004, p. 254)
inverse operations, middle school’s transitioning to inverse elements, and high school’s transitioning to inverse functions.

Thus, it seems reasonable to investigate how university students come to codevelop their existing understandings by restructuring what they have previously learned and unifying some of their existing understandings under a big, overarching notion. The fourth and last category of transformative transitions is as follows:

| Unifying category: A learner makes a transformative transition by increasing the extent to which, in the learning of a new construct, seemingly disparate concepts are viewed as instantiations of an overarching idea. |

Because the unifying category also involves schema development that integrates a collection of mathematical objects and processes, I drew on Piaget and Garcia’s triad to describe how university students make transformative transitions in terms of unifying category. Following is an explanation of how the triad applies to the unifying category.

Suppose that $A_1, A_2, \ldots, A_n$ are the constructs in a learner’s mind that can be viewed as embodying a big, overarching idea from a knowledgeable outsider’s perspective. In an *intra-object level of unifying*, a learner focuses on each of $A_1, A_2, \ldots, A_n$ in isolation from each other and work with the constructs individually. In an *inter-object level of unifying*, a learner begins to coordinate and find some consistency or similarities between two or more constructs. S/he sees some aspects of two or more constructs can be explained by the same idea, but the coordination lacks holistic explanation or full-blown articulation of an overarching idea. In a *trans-object level of unifying*, a learner can see what is
essentially underlying $A_1$, $A_2$, …, $A_n$ and constructs a coherent schema that includes $A_1$, $A_2$, …, $A_n$ and the big idea that unifies all of $A_1$, $A_2$, …, $A_n$.

In Table 2.4, I exemplify how Piaget and Garcia’s triad was adapted to identify student’s making transformative transitions in terms of the *unifying* category using the mathematical notion of factorization.

Table 2.4. Levels in the unifying category and descriptions of each level with respect to understandings of factorization

<table>
<thead>
<tr>
<th>Levels in unifying category</th>
<th>Descriptions (with respect to understandings of factorization)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Intra-object level</strong></td>
<td>At the intra-object level, the student can provide a local and particular explanation about factorization. A student can demonstrate how to factor a whole number and how to factor a polynomial individually. However, these two ways may be incompatible with and in isolation from each other. A student at this level cannot coordinate the factorization processes across integers and polynomials. Rather, factoring integers and factoring polynomials are separate and unrelated entities.</td>
</tr>
<tr>
<td><strong>Inter-object level</strong></td>
<td>At the inter-object level, the student begins to coordinate factoring integers and factoring polynomials simultaneously and to explain some similarities or consistency between two or more entities. For example, s/he may see some consistency between prime factorization of integers and complete factorization of polynomials. A student at this level, however, cannot coordinate all of his/her constructs related to factorization in order to produce the desired results in the problems given. For example, s/he cannot see how factoring out a constant in a factorization of a polynomial gives a trivial factorization just as 1 does with a factorization of an integer.</td>
</tr>
<tr>
<td><strong>Trans-object level</strong></td>
<td>At the trans-object level, the student sees the same overarching idea (e.g., decomposition, reducible, irreducible, unit, uniqueness of complete factorization) underlies factorization of integers and</td>
</tr>
</tbody>
</table>
factorization of polynomials and so, constructs a coherent schema of his/her constructs related to factorization.

Summary of the categorical framework

In the preceding section, I identified four important categories to describe university students’ transformative transitions in their learning of collegiate mathematics based on my review of literature. In this section, I provide the definitions of four categories collectively and summarize the levels used in each category. Each of the four categories (listed in the following text) are intended to capture different ways university students might codevelop their existing mathematical understandings in the learning of collegiate mathematics. Four different qualities of mathematical understanding or knowledge suggested in the literature—depth, breadth, connectedness, coherence—gave rise to each of the four categories as follows.

Deepening category: A learner makes a transformative transition by increasing the depth level of existing understandings of a certain, single mathematical notion in the learning of a new construct.

Extending category: A learner makes a transformative transition by increasing the boundary of contexts in which a set of existing understandings are situated in the individual’s mind in the learning of a new construct.

Strengthening category: A learner makes a transformative transition by increasing the strength of the link between existing understandings of more than one mathematical construct in the learning of a new construct.

Unifying category: A learner makes a transformative transition by increasing the extent to which, in the learning of a new construct, seemingly disparate concepts are viewed as instantiations of an overarching idea.
Note that, in this framework, the deepening category captures how one’s understandings of a single mathematical notion such as function may deepen, whereas the extending, strengthening, and unifying categories capture how existing understandings of multiple notions may form a different relationship in one’s mind. Figure 2.6 depicts how each of the extending, strengthening, and unifying might be reflected in the change of the relationship between existing understandings in one’s mind.

![Diagram showing extending, strengthening, and unifying categories](image)

Figure 2.6. Extending, strengthening, and unifying: Different types of relationship change

To describe a qualitative leap in a transformative transition, levels for identifying student’s progress in each category were adapted from APOS theory (Arnon et al., 2014; Dubinsky, Dautermann, Leron, & Zazkis, 1994) and Piaget and Garcia’s triad
(1983/1989) (see Table 2.5). This set of categories with corresponding levels under each category provided a lens to analyze and describe transformative transitions in the current study.

Table 2.5. Four categories for transformative transitions and levels in each category

<table>
<thead>
<tr>
<th>Categories for transformative Transition</th>
<th>Levels (as descriptors of transformative transitions)</th>
<th>Levels adapted from</th>
</tr>
</thead>
<tbody>
<tr>
<td>Deepening</td>
<td>Action $\rightarrow$ Process $\rightarrow$ Object</td>
<td>APOS theory (Arnon et al., 2014; Dubinsky et al., 1994)</td>
</tr>
<tr>
<td>Extending</td>
<td>Intra $\rightarrow$ Inter $\rightarrow$ Trans</td>
<td>Piaget and Garcia’s triad (1983/1989)</td>
</tr>
<tr>
<td>Strengthening</td>
<td>Intra $\rightarrow$ Inter $\rightarrow$ Trans</td>
<td></td>
</tr>
<tr>
<td>Unifying</td>
<td>Intra $\rightarrow$ Inter $\rightarrow$ Trans</td>
<td></td>
</tr>
</tbody>
</table>

The categorical framework for transformative transitions in this study is a result of identifying four dimensions—deepening, extending, strengthening, and unifying—prominent in different theories/constructs related to the nature and development of one’s mathematical understanding. These four ways of conceptualizing transformative transitions provide a new perspective on how to characterize qualitative leaps in one’s mathematical understandings, in particular, in the context of learning collegiate mathematics.

This study also extends the lines of research that have used APOS theory and Piaget and Garcia’s triad. Existing studies (Baker, Cooley, & Trigueros, 2000; Clark et al., 1997; Cooley, Trigueros, & Baker, 2007) employed APOS theory and Piaget and Garcia’s triad as frameworks for interpreting and analyzing university students’ understandings as assessed at the moment. In these studies, in the context of task-based interviews, individuals’ understandings were interpreted in terms of the levels in APOS
theory and Piaget and Garcia’s triad. The current study goes beyond that by identifying indicators of some possible progress in students’ understandings with respect to specific dimensions. The current study employed both task-based clinical interviews and teaching interviews (which will be discussed in detail in the Methods chapter) not only to assess participants’ understandings at the moment but also to capture individual participants’ progress, if any, in a series of consecutive interviews along the levels in APO(S) theory and Piaget and Garcia’s triad.

In order to investigate growths in university students’ understandings through the lens of transformative transition, I considered what kind of environment is appropriate for observing students making transformative transitions and also ways in which new knowledge is constructed by many university students in collegiate mathematics courses. In the frequent use of a lecture-based approach and the need for content coverage in collegiate mathematics courses (Speer, Smith, & Horvath, 2010), mathematical definitions, propositions, and their proofs are typically introduced to university students as ready-made products (McCartney, 2012). However, as previously mentioned, from a Piagetian perspective, one’s mathematical knowledge develops through a qualitative leap that involves reconstructing and reorganizing of existing constructs of his/her own and, not through a simple accumulation of information or ready-made knowledge passed on by others. Recognizing the potential difficulties of making transformative transitions in the learning of collegiate mathematics, a body of literature that informs a way to frame a context for observing transformative transition was reviewed, and one approach using Abstraction-in-Context [AiC] framework (Dreyfus, Hershkowitz, & Schwarz, 2015) was chosen for this study. In what follows, I explain the framework itself first and why it was
a reasonable framework for the current study. Details on how it was used in this study are presented in the Methods chapter.

**Abstraction-in-context [AiC] framework**

The AiC framework is a framework for studying the process by which abstractions take place (Dreyfus, Hershkowitz, & Schwarz, 2015; Hershkowitz, Schwarz, & Dreyfus, 2001; Schwarz, Dreyfus, & Hershkowitz, 2009). In their initial development of the framework, Hershkowitz and colleagues (2001) defined abstraction as “an activity of vertically reorganizing previously constructed mathematics into a new mathematical structure” (p. 202). They used their empirical data from a number of case studies to identify students’ actions that seemed to be evidence of the abstraction process. They identified three observable epistemic actions undertaken by students in their process of engaging in a mathematical activity—recognizing (R), building-with (B) and constructing (C)—and noted that, through a chain of nested R, B, C-actions, their existing constructs are reorganized to produce a new construct. The definitions for each of recognizing (R), building-with (B), and constructing (C) are as follows:

- **Recognizing** refers to the learner seeing the relevance of a specific previous construct to the situation or problem at hand. *Building-with* comprises the use and combination of recognized constructs in order to achieve a localized goal such as the actualization of a strategy, a justification or the solution of a problem. *Constructing* consists of assembling and integrating previous constructs by vertical
mathematization to produce a new construct. (Dreyfus et al., 2015, p. 188, Emphases added)

Dreyfus and colleagues found that, when they analyzed a series of student’s actions that preceded construction of a new construct, students’ Building-with actions (B-actions) were nested in Constructing actions (C-actions), and Recognizing actions (R-actions) were nested in B-actions and C-actions. For example, a C-action on uniqueness of a factorization of polynomial may subsume a number of R- and B-actions on various constructs, such as prime factorization of a number, irreducible polynomial, unit, or associates. It is also possible that C-action on relativeness of irreducibility forms part of C-action on uniqueness of a factorization. Once the uniqueness is constructed in the subject’s mind, in later activities, R- and B-actions on the uniqueness may be nested in C-action on some higher level construct (e.g., unique factorization domain). This nested epistemic actions model (also called RBC-model) for the abstraction process is a main component of AiC framework.

According to their model, a learner’s previous constructs act as building blocks for constructing new knowledge and have to be used in his/her R-actions and B-actions in order for the learner to reorganize them into a new construct. At the first moment when the new construct emerges in the learner’s C-action, it may not be fully developed or may include some vagueness or contradictions from an expert’s perspective. Dreyfus and colleagues argued that the new construct will be gradually incorporated into the learner’s knowledge structure as the new construct gets to be recognized and built-with more and more in the further activities. A learner progresses in his/her abstraction process by engaging in further R-actions and B- actions with the new construct.
Dreyfus and colleagues noted the design of instruction or curriculum in AiC-related studies expresses “an intention of continuous transformation of constructs” as suggested in their nested epistemic actions model (p. 190). Hence, the design of the learning environment seems to be a crucial component in the studies involving AiC framework. That is, according to Dreyfus and colleagues, the existence of successions of activities in which a learner can engage in R-actions and B-actions with previous and emerging constructs is desirable for abstractions to take place. Also of note is that, in their model, a new construct emerges in a problem context in which existing understandings of a learner, as they are, are not sufficient for the problem solving or fully justifying the learner’s own claim. In such a context, existing understandings that were recalled and activated are assembled and reflected in a novel (to the subject) way in B-actions. This design principle and RBC model in AiC framework served as the basis for devising an instructional context in which I observed university students’ transformative transitions. The next section is a discussion of the fit between AiC framework and the current study’s focus on transformative transition. A section in the Methods chapter, entitled *Teaching interview [TI] tasks* on page 105, details the design of the instructional context.

**Why AiC framework?**

A key issue in the design stage of the current study was to provide an environment that could foster participants’ construction of transformative transitions. In collegiate mathematics learning, students may come to construct a variety of new
mathematical knowledge (Hodgson, 2001), but their previous mathematical understandings may remain unchanged and isolated from the new ones (Cofer, 2015; Lee, 2010; Ticknor, 2012). This type of learning would be interpreted as, from my perspective, simple transitions and not transformative transitions.

Although mathematics educators ultimately want their students to build coherent, holistic understandings of mathematics at every level, existing studies and anecdotal evidence (CBMS, 2012; Cofer, 2015; CUPM, 2015; Lee, 2010; Ticknor, 2012) suggest that university students find it difficult to do so in their learning of collegiate mathematics. Studies on collegiate students’ learning and understanding of abstract notions (Dubinsky et al., 1994; Hazzan, 1999, 2001; Larsen, 2013; Leron, Hazzan, & Zazkis, 1995; Mason, 1989) share the perspective that, when an abstract notion is introduced to the collegiate students in a formal and decontextualized manner from the beginning, students might experience significant difficulty in understanding the notion as an object that embodies mathematically important properties and relationships that may involve their previous constructs. Hazzan (1999, 2001), for example, noted that students were likely to try to find a way to deal with the abstract notion, when it seems to be devoid of meaning from their perspectives (Mason, 1989), by forcing it into a familiar context or a lower level such as by performing algorithmic actions or relying on prescribed procedures (also called reducing abstraction in Hazzan, 1999). Leron and Dubinsky (1995) also noted that university students had a great deal of difficulties in understanding abstract algebra concepts in a traditional setting. Based on their perspectives on learning (i.e., APOS theory), they employed an instructional strategy in which students were encouraged to experience a number of examples of a targeted
mathematical notion before getting introduced to a formal definition (see Dubinsky & Leron, 1994). Larsen (2013) and Leron, Hazzan and Zazkis (1995) also took issue with presenting abstract mathematical notions in a formal and decontextualized manner from the beginning. Accordingly, Leron and colleagues proposed introducing an abstract idea “through a naïve version” (p. 172); Larsen used the method of guided reinvention (Freudenthal, 1991)—which refers to a general heuristic of promoting the development of formal mathematical knowledge from students’ informal and intuitive knowledge—to guide students to reinvent the concept of isomorphism through informally exploring structural similarities between two group-operation tables. From the perspective of the current study, these studies collectively suggest that, when abstract notions are introduced in a formal and decontextualized manner, new knowledge may be assimilated by a student without any codevelopment of existing understandings—that is, through a simple transition.

Researchers (Cook, 2012; Larsen, 2004, 2013; Larsen & Lockwood, 2013; Leron, Hazzan, & Zazkis, 1995; Swinyard, 2011; Swinyard & Larsen, 2012; Tsamir & Dreyfus, 2002; Weber & Larsen, 2008) in collegiate mathematics learning seem to agree that the construction of abstract notions needs to be motivated by the ideas that are more comprehensible to students and that are less complex and easier to work with for students than the formal definitions. The guided reinvention approach employed in the studies of Larsen and colleagues and the approach of introducing a naïve version of abstract idea before a formal version (as in the study of Leron and colleagues) are supportive of this claim. AiC approach can be considered as one of such approaches in a sense that it
encourages students to start with the ideas that are accessible with their existing understandings.

In particular, the main premise of the AiC framework seems suitable for framing an approach to fostering university students’ transformative transitions. The reason is as follows. The AiC framework explains construction of new knowledge as the result of vertically reorganizing existing understandings. The RBC model in the AiC framework suggests that a learner’s existing understandings serve as building blocks for constructing new knowledge and the existing understandings have to be activated and reflected on in a chain of R- and B-actions in order for the new knowledge to be constructed in the learner’s mind. This inherent connection made between existing understandings and new ones is a distinctive feature of their RBC model for construction of new knowledge as well as a characteristic of the desired end state of transformative transitions. Because of this fit between the AiC framework and the current study’s focus on transformative transition, I drew on the AiC framework (specifically, its design principle and RBC model) to design an instructional context in which I could observe university students’ transformative transitions. In summary, the current study was built on the assumption that instructional context grounded in AiC framework that could guide university students to construct new knowledge by engaging in R-actions and B-actions on existing understandings, could support university students to make transformative transitions in their learning of collegiate mathematics. Details of how the AiC framework was used in the current study are explained in the Methods chapter.

In designing an AiC-grounded instructional context for observing transformative transition, factorization was chosen as a mathematical context of this study for multiple
reasons. In the following section, I provide explanations of why this mathematical context is a reasonable fit for the current study.

**Factorization in school and collegiate mathematics**

This study was intended to investigate growth in university students’ understandings—specifically, how they can advance their school mathematics understandings in the learning of collegiate mathematics—in an instructional context grounded in the AiC approach. Selecting an ideal mathematical context as a medium for achieving this goal, I considered the following three conditions. First, an ideal mathematical context has to be of considerable importance in the curricula of both school and collegiate mathematics. Second, an ideal mathematical context has to be rich in its connections to school mathematics concepts, procedures, and generalizations. Third, growth in one’s understanding in the mathematical context has to be observable within a *reasonably limited time frame*. A decision of limiting the number of observations makes sense because this study is an initial foray into studying growth in university students’ understandings with the specific types of qualitative leaps identified in the deepening, extending, strengthening, and unifying categories. In this section, I argue why meeting these conditions constitutes a reasonable criterion for selecting an ideal mathematical context and how “factorization” met the three conditions.

The first condition of having importance in the curricula of both school and collegiate mathematics is related to the need for establishing an empirical basis that could be of practical use in its later application. As this study is motivated by a practical issue
of the double discontinuity, what is learned from this study has to contribute ultimately to improving the instructional practice of collegiate mathematics. If a mathematical context is only tangentially dealt with in either school or collegiate mathematics curriculum, the results of the study using such a context may have little contribution to addressing the need of the field. Factorization as a mathematical context meets the first condition. Students learn and apply factorization across their elementary, secondary, and tertiary mathematics. Factorization is applied to the set of natural numbers at the elementary level, to the set of polynomials at the secondary level, and to integral domains at the tertiary level. Most of all, the mathematical importance of the construct of factorization cannot be overlooked, since factorization serves as a way of decomposing a mathematical object into a product of more fundamental, elementary objects within the structure. The decompositional nature explains why factorization provides a context for both the fundamental theorem of arithmetic and the fundamental theorem of algebra. In school mathematics, the decompositional role of factorization is especially useful in revealing roots of polynomial equations. In school algebra, as a way to solve a quadratic (or a higher degree) equation, students learn how to factor a polynomial expression and use the zero-product property to find the roots. Mathematics-intensive majors, by the time they take an abstract algebra or number theory class, are likely to have mastered skills of how to factor a polynomial but they are yet to learn much more about factorization by objectifying and studying factorization itself. Examples of such include the unique factorization theorem (which is discussed in great detail in the Methods chapter), the fundamental theorem of algebra, and the fundamental theorem of arithmetic. As an important mathematical context in both school and collegiate curricula, factorization
deserves to be a context for studying one’s growth in school mathematics understandings in the learning of collegiate mathematics.

The second condition of having rich connections to school mathematics concepts, procedures, and generalizations was needed to increase a chance to gather information-rich data. In order to observe various kinds of qualitative leaps in university students’ understandings, participants’ engagement in the mathematical tasks needed to be able to lay the groundwork for the qualitative leaps.

Factorization as a mathematical context could serve as a medium for designing such mathematical tasks. First of all, factorization as a procedure that applies to both integers and polynomials may be used to bridge the two structures, the set of integers and the set of polynomials. Factorization also relates to various constructs. For example, when a polynomial is to be completely factored over complex numbers, one may translate it into solving a corresponding polynomial equation, finding its roots, applying the quadratic formula, checking the conjugate pairs, rewriting the roots into a factored form, and so on. Relationships among the constructs of factorization, polynomial equation, roots, quadratic formula, and conjugate pairs may be illuminated from a different angle in the construction of a notably different perspective on factorization at the collegiate level.

Even if the first two conditions are met, a mathematical context that requires a substantial amount of time for the observation of growth in one’s understanding is beyond the scope of the current study. For example, the fundamental theorem of algebra as a mathematical context may be ideal if a study is implemented in a semester-long or a year-long course for mathematics-intensive majors. A text written by Fine and Rosenberger (2012) outlines such curriculum, and investigating students’ understanding
growth within such curriculum is, in fact, the kind of future direction that could be used to extend the current study. The third condition is, therefore, a methodological consideration that growth in one’s understanding in the mathematical context has to be observable within a reasonable time frame. Factorization as a mathematical context meets this third condition, and a detailed methodological design is explained in the Methods chapter.

The goal of this chapter was to explain theoretical and conceptual underpinnings—such as APOS theory, Piaget and Garcia’s triad, transformative transition, AiC framework, and factorization—that inform what needs to be considered for investigating the question proposed in the end of Chapter 1 (How do university students come to see school mathematics from an advanced viewpoint in their learning of collegiate mathematics?). Those underpinnings were used to frame the ways in which I observe, document, and analyze the targeted phenomenon of “coming to see school mathematics from an advanced viewpoint in the learning of collegiate mathematics.” In particular, a construct, transformative transition, with the four categories of deepening, extending, strengthening, and unifying, was developed as a way to describe qualitative growth in understanding that might indicate “coming to see school mathematics from an advanced viewpoint.” The levels adapted from the APOS theory and Piaget and Garcia’s triad in the transformative transition framework served as a way to describe growth in one’s school mathematics understandings. The AiC approach and factorization as a mathematical medium were used to specify the context of “learning of collegiate mathematics.” The specific research questions that reflect those theoretical and conceptual underpinnings are presented in the following section.
The research questions

The overarching research question of this study in a succinct version is as follows:

*Overarching research question (succinct version):* In the context of learning collegiate mathematics, how can the learners’ growth (and lack of growth) in their relevant school mathematics understandings be described and explained within the transformative transition framework?

It should be noted that this study is intended to explain not only growth in understandings but also lack of such growth, because both can potentially serve as an empirical basis for supporting university students to see school mathematics from an advanced viewpoint in their learning of collegiate mathematics. This overarching question can be further specified with details of “the context of learning collegiate mathematics” and the population of the “learners” of collegiate mathematics. As previously mentioned, the Abstraction-in-Context [AiC] framework (Dreyfus et al., 2015) was adopted to guide the design of an instructional context for the learning of collegiate mathematics in this study; the mathematical context of this study is factorization; the population of the current study is university students in mathematics-intensive majors. Hence the overarching research question in a detailed version is as follows:

*Overarching research question (detailed version):* When university students in mathematics-intensive majors engage in the learning of factorization at the collegiate level in an AiC-grounded instructional context, how can growth (and lack of growth) in their relevant school mathematics understandings be described and explained within the transformative transition framework?

Two research goals as identified in the overarching research question are: first, describing the growth in understandings within the transformative transition framework and, second, explaining the growth (and lack of growth) in the instructional context
implemented in the current study. To achieve the first goal, two specific research questions are addressed in this study as follows:

Research question (1): How can university students’ school mathematics understandings be described by the levels in each of the four categories of transformative transition—deepening, extending, strengthening, and unifying?

Research question (2): What transformative transitions do university students make in the four categories—deepening, extending, strengthening, and unifying—in an AiC-grounded instructional context?

This sequence of two questions seems reasonable, because the transformative transition framework is a nascent one with multiple categories and levels. The application of such a framework needs to be substantiated by careful empirical explanations for the categories and levels that constitute the framework. In answering the research question (1), I investigate how the levels in deepening, extending, strengthening, and unifying serve as a meaningful analytic lens for describing university students’ school mathematics understandings. In answering research question (2), I build on the levels of understandings identified in research question (1) and describe the growth in university students’ school mathematics understandings as advancement in levels, which is a transformative transition by its definition. Chapter 4 answers to the research question (1) and (2).

When the four categories with three levels capture some important growth in university students’ understandings (and possibly lack of such growth), a deeper analysis can be conducted to achieve the second goal of this study: to explain the growth and the lack of growth in the particular instructional context implemented in the current study. This goal is reflected in the following two research questions:
**Research question (3):** What are possibly relevant features of an AiC-grounded instructional context in which university students make transformative transitions?

**Research question (4):** When university students do not make transformative transitions in an AiC-grounded instructional context, what are obstacles to making those transitions?

In explaining the growth in university students’ school mathematics understandings, I situate and discuss the observed transformative transitions in the AiC-grounded instructional context. It should be noted my answer to research question (3) is proposed to provide possible explanations for what about the AiC-grounded instructional context seems to underlie the observed transformative transitions commonly; It is not intended to generate claims about a solid cause–effect relationship between the instruction and observed transformative transitions. To explain lack of growth in university students’ school mathematics understandings as indicated in research question (4), I consider settings in which university students do not make transformative transitions and locate possible reasons for seemingly not making transformative transitions. Answers to the research question (3) and (4) are discussed in Chapter 5. In the next chapter, I explain how I designed and implemented my study for addressing the four research questions proposed in this section.
Chapter 3 Method

As proposed in the previous chapter, the current study investigates the following research questions:

(1) How can university students’ school mathematics understandings be described by the levels in each of the four categories of transformative transition—deepening, extending, strengthening, and unifying?

(2) What transformative transitions do university students make in the four categories—deepening, extending, strengthening, and unifying—in an AiC-grounded instructional context?

(3) What are possibly relevant features of an AiC-grounded instructional context in which university students make transformative transitions?

(4) When university students do not make transformative transitions in an AiC-grounded instructional context, what are some obstacles to making those transitions?

The purpose of this study underlying the four research questions was to develop an empirical basis for describing and explaining, within the transformative transition framework, the growth (or lack of growth) in university students’ understandings of school mathematics in their learning of collegiate mathematics. To this end, the intention was to design a context in which I could infer participants’ understandings of relevant school and collegiate mathematics through the observation of their engagement in a variety of mathematical tasks. I also took into consideration successive R-, B-, C-actions as proposed by the AiC approach and applied the approach to participants’ engagement in
the learning of collegiate mathematics. The need for detailed and close observations of participants’ mathematical work led me to choose to use task-based interviews as a mode of inquiry. Those task-based interviews took two forms—specifically, teaching interview and clinical interview. In this chapter, I first discuss my design of the contexts of teaching interview and clinical interview for the purpose of this study. Second, I explain how instruments (including interview schedules and a prescreening survey for the selection of participants) were developed to provide insights and access to participants’ mathematical understandings. Third, I discuss the data collection process including the selection of participants. Finally, I explain how I analyzed participants’ understandings through the lens of the transformative transition framework in order to develop trustworthy descriptions of and explanations for the growth (or lack of growth) in university students’ understandings of school mathematics.

**A task-based interview context**

In mathematics education research, the task-based interview has become a mode of inquiry to look into students’ ways of engaging in mathematical thinking firsthand (Hunting, 1997). A teaching interview is a specific kind of task-based interview, and following is a description of how the general method of task-based interview informed the current study.

Because the current study was focused on how university students make transformative transitions, it was necessary to gather data while the students were making transformative transitions. Task-based interview settings allow researchers to have a
firsthand and in-depth experience with the ways in which interviewees engage in mathematical thinking or develop their understandings. As noted by Shaughnessy in his commentary on Erlwanger’s (1973) study, researchers can gather “a record of students’ actual thinking process” by conducting an interview (Carpenter, Dossey, & Koehler, 2004, p. 48).

Goldin (2003) had advice for researchers using task-based interviews. He advised researchers to be mindful of providing environments in which students can think freely in a task-based interview. In order to encourage subjects to think freely, I stated to my participants, in the beginning of every interview, that what matters is to share their thinking during the problem-solving processes not to provide right answers. As seen in the collected data, participants offered a variety of responses, and the variety suggests that participants were able to think freely about the open-ended questions during the interviews. I also indicated to them that my follow-up questions do not imply the correctness of their answers and that their purpose may be to help me (as an interviewer) to follow their logic better or to help them make further progress. Goldin also suggested that a researcher “maximize interaction with the external learning environment, providing a variety of external representational possibilities” (p. 281) as a way to support free thinking. Because of multiple representations that can be generated on a CAS calculator such as graphs, tables, and algebraic expressions, I had a CAS calculator available for

6 Although they referred to “a record of students’ actual thinking process,” one cannot be sure that an interview allows the documentation of students’ actual thinking process because one cannot have direct access to the students’ minds. However, data generated from an interview is taken as a type of evidence of students’ thinking processes.
them to use during the interviews. In case participants were not familiar with the CAS calculator provided, I demonstrated how to use the calculator and asked them to solve some calculation problems using the calculator and CAS commands of “SOLVE” and “FACTOR” before beginning the first interview.

The ways in which task-based interviews are used may differ in the extent to which teaching⁷ is involved in the interviews. A *clinical interview* is a task-based interview in which teaching is rarely involved. Zazkis and Hazzan (1999) noted that researchers employ clinical interviews “to discover intellectual phenomenon in learner’s approaches to problem situations, identify and describe underlying cognitive processes and determine child’s competence, that is, his or her capability to perform a certain task” (p. 430). The questions asked by the interviewer in a clinical interview serve the purpose of revealing students’ understanding of certain mathematical entities, discovering how students think about certain mathematical entities and why they think so, and gaining more evidence about the quality of the students’ thinking (Hunting, 1993). Thus, if a researcher employs the clinical interview method, s/he might be able to investigate students’ mathematical thinking with a minimal influence from the interviewer but it might be difficult to observe the process through which students construct new understandings and refine them further during the interview.

A modification of the task-based interview that allows the observation of students constructing new understandings is the *teaching interview*. In a teaching interview, the

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⁷ Teaching in this study means a series of actions with a particular intention for helping others understand some concepts or skills and engage in some processes.
interviewer asks questions with didactic intentions (Hershkowitz, Schwarz, & Dreyfus, 2001). I employed the method of teaching interview for the purpose of the current study. Hershkowitz and colleagues employed the teaching interview [TI] method for investigating a participant’s abstraction process. They described a TI as an interview in which the interviewer asks questions with the following didactic purposes: “(a) to cause [student] to explain what she was doing and why and (b) to induce her to reflect on what she was doing and thus possibly progress beyond the point she would have reached without the interviewer” (p. 204). The “teaching” in the TI method should not be interpreted as indicating that there is any lecture involved in the TI. The major strategy in the teaching interview, like that in other task-based interviews, is still questioning. The term “teaching” is used in order to indicate that the interviewer has some didactic intention in his/her questioning. In the following section, I discuss why the TI method was chosen and how it was employed in the current study.

An instructional context using Teaching interview [TI]

The AiC approach is designed to engage students in a series of R-actions, B-actions, and C-actions, culminating in students transforming their existing understandings. The didactic questions inherent in the teaching interview were designed to precipitate those actions. Moreover, in TIs, when a participant tries to address some of the didactic questions posed by an interviewer, the interviewer can be afforded the opportunity to investigate and witness firsthand the interviewee’s way of operationalizing his/her knowledge. When multiple interviews are conducted over time, such close
observations can serve as a reliable source from which an interviewer could infer the interviewee’s mathematical understandings and, furthermore, changes in his/her understandings. Since the purpose of this study is to investigate university students’ growth in their understandings, TI was a viable mode of inquiry in the current study.

To be specific, a teaching interview [TI] in the current study means an interview in which the interviewer engages the interviewee in a series of mathematical tasks with the purpose of directly experiencing and investigating what the interviewee does with what s/he knows and potentially how the interviewee develops his/her understandings over time. The interviewer in a TI plays two important roles—first, as a researcher, and second, as the more knowledgeable other for helping the interviewee’s process of reflecting on his/her existing understandings. The interviewer can fulfill the second role by posing the following types of questions: (a) clarifying or probing questions designed to investigate the interviewee’s actions and the underlying cognitive reasoning, (b) guiding questions designed to encourage the interviewee to reflect on his/her own actions or reasoning, and (c) redirecting questions designed to help the interviewee identify and resolve contradictions in his/her own thinking and advance his/her thinking. Examples of each of these types of questions are presented in the following paragraphs.

Because the interviewer plays a role that is similar to a teacher in asking those probing, guiding, and redirecting questions during TIs, any claim about participants’ growth in their understandings during the TIs needs to be carefully examined in light of what kinds of questions are asked in what ways by the interviewer. In the Results Chapters, my interpretation and analysis of participants’ responses is presented together with the questions that the interviewer asked prior to participants’ responses. In the
design stage of the TIs, the potentially heavy involvement of teaching during the interview raised the need for establishing the guidelines to which I needed to adhere as an interviewer. To address the need and to establish some credibility of the results by adhering to these guidelines, I share the guidelines for the TIs to which I adhered and provide exemplar questions under each guideline. As explained in the previous paragraph, the interviewer in a TI plays two important roles. For brevity’s sake, the two roles are now called researcher and teacher. Now, I provide the guidelines for each of the roles.

As a researcher in my teaching interview, the interviewer should

- Attend to and follow up apparent evidence of the interviewee’s transformative transition [or the phenomenon of interest of a given study], if noticed in the moment.
  - If the interviewee’s understanding seemed to suggest a higher level in terms of deepening, extending, strengthening, or unifying category than before, ask: **What did you think about when you concluded/did/said so?** [Guiding question]

- Suspend judgment and not assume that he or she understands what the interviewee is talking about.

- Probe the interviewee’s actions and the underlying cognitive reasoning.
  - If the interviewee brings up a new term, XXXX, ask: **What did you mean by XXXX? What is an example of XXXX?** [Clarifying or probing question]
o If the interviewee records a written inscription apparently intended to express something, ask: **What do you mean by what you’ve written here?** [Clarifying or probing question]

o If the interviewee proceeds without saying anything, ask: **Could you say out loud what you are doing/thinking?** [Clarifying or probing question]

o If the interviewee concludes or expresses XXXX without providing explanations, ask: **You indicated XXXX. Is there any reason for that? What made you think to do so?** [Clarifying or probing question]

As a teacher in my teaching interview, the interviewer should

- Allow the interviewee enough time to respond and explain in his/her own ways. Question himself or herself, asking: Am I giving the interviewee the opportunity to reveal his or her thinking?

- Help the interviewee identify and resolve (potential) contradictions in his/her own thinking and foster his/her thinking.

  o If the interviewee expresses XXXX (e.g., “(x² – 1)(x – 1) is factored completely”) and XXXX seems to have the potential to be problematic for making progress in the further activities, the interviewer could ask, for example: **How does it fit your description of factoring completely? How do we know that an expression is factored completely?** [Redirecting question]

- Allow the interviewee to have access to some resources when s/he needs them as a prerequisite.
o If the interviewee expresses that s/he does not know or remember what XXXX means (e.g., factoring completely), the interviewer could provide some examples/statement related to XXXX and ask, for example, **Here is an example/statement that is related to XXXX. Could you interpret this example/statement in terms of XXXX? [Redirecting question]**

- Encourage the interviewee to reflect on his/her own actions or reasoning.

  o If the interviewee does not seem to make further progress in the middle of problem solving and if it seems to be helpful to look back, the interviewer could ask, for example, **Could you revisit what you have done so far and tell me what you did or why you did so? [Guiding question]**

- Respond to the interviewee’s responses/questions in an open-ended fashion.

  o If the interviewee asks the interviewer whether XXXX is true (e.g., Suppose that the interviewee asks “Is 1*2*2*3 factored completely?”), ask for example: **How would you determine whether (the expression) is XXXX based on your description of XXXX? If your description does not give you enough information to determine whether (the expression) is XXXX, can you revise your description so that you can determine? [Guiding question]**

During the TIs, I followed these guidelines to limit my involvement as an interviewer. These guidelines provide background information about the TIs during which I observed and investigated university students’ making transformative transitions.
Clinical interview for PRE- and POST-TI interviews

As mentioned earlier, in comparison with a teaching interview, a clinical interview is a task-based interview in which an interviewer can observe cognitive processes used by participants in their problem solving, examine their understandings of certain mathematical entities, and gain more evidence about the quality of their mathematical understandings with minimal influence from the interviewer. I used a clinical interview method, not a teaching interview method, in my pre-TI and post-TI interviews in order to identify the levels of my participants’ understandings revealed in the problem solving (without access to resources such as textbooks) and eventually to compare their understandings before and after the TIs.

Thus, the follow-up questions I used in my pre- and post-TI interviews did not have a didactic intention and were asked for the purpose of clarifying and probing the interviewee’s actions and the underlying cognitive reasoning. Specifically, the following are the questions I used in pre- and post-TI interviews for clarifying and probing purposes:

- If the interviewee brings up a new term, XXXX, ask: **What did you mean by XXXX? What is an example of XXXX?**
- If the interviewee records a written inscription apparently intended to express something, ask: **What do you mean by what you’ve written here?**

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8 These questions are a subset of the questions that are used in a teaching interview.
If the interviewee proceeds without saying anything, ask: **Could you say out loud what you are doing/thinking?**

If the interviewee concludes or expresses XXXX without providing explanations, ask: **You indicated XXXX. Is there any reason for that?**

**What made you think to do so?**

The current section contains a discussion of two types of task-based interviews (teaching interview and clinical interview), provides justification for the choice of these as a mode of inquiry, and explains how the interview types were employed in the current study. While the interviews as modes of inquiry serve as observational contexts, the mathematical questions that are asked during the interviews would play a crucial role in generating relevant, information-rich data. The following section discusses the development of the interview schedules as well as other instruments developed for establishing the credibility of the current study.

### Instruments

There are five instruments developed in the current study. First, a prescreening survey was developed to select participants who are likely to produce relevant and information-rich data in their engagement during the interviews. Second, the teaching interview schedule was developed to reflect the AiC approach and engage participants in the learning of factorization at the collegiate level. Third, pre-TI and post-TI interview schedules were developed to observe participants’ understandings of relevant school mathematics before and after the TIs. Fourth, I developed two sets of supplementary
interview questions to investigate how participants perceived their doing of mathematics and to investigate external influences on their engagement during the interviews, respectively. In what follows, each of the five instruments is discussed in detail.

Prescreening survey

As discussed in Chapter 1, the target population of the current study was mathematics-intensive majors, which includes mathematics majors and secondary mathematics education majors. To select the participants who could provide the relevant information for addressing the research questions, I asked potential participants to fill out a mathematical prescreening survey (online). See Table 3.1 for the questions asked in the prescreening survey (for the specific role of each question, see Appendix A).

Table 3.1. Questions in a prescreening survey

<table>
<thead>
<tr>
<th>SCR-Q1: Which of the following expressions are polynomials in $x$?</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) $1 + x^2$  (b) $rac{1}{x}$  (c) $3$  (d) $x^{1000}$  (e) $\sin x$  (f) $x + 1$</td>
</tr>
<tr>
<td>(g) $\sqrt[4]{-2x + 3}$  (h) $\frac{2x-5}{x+6}$  (i) $\log(1 - 3x)$  (j) $e^{x+1}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>SCR-Q2: How would you define a polynomial?</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>SCR-Q3: Factor the following expressions, if possible.</th>
</tr>
</thead>
</table>
| $x^2 + 4x - 60$  
| $25 - x^2$  
| $2x^2 + 5x - 3$  |

<table>
<thead>
<tr>
<th>SCR-Q4: Find the roots of $x^2 + 5x - 3 = 0$.</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>SCR-Q5: Which of the following numbers are prime?</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1, 2, 3, 4, 5, 26, 27, 28, 29, 30$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>SCR-Q6:</th>
</tr>
</thead>
</table>
Which of the following numbers are …
integers?
- rational numbers?
- irrational numbers?
- real numbers?
- imaginary numbers?
- complex numbers?

(a) π  (b) \( \frac{\pi}{2} \)  (c) \( \sqrt{-2} \)  (d) \( \frac{3+\sqrt{-2}}{6} \)  (e) \( \frac{2}{6} \)  (f) \( e^2 \)  (g) \(-1\)

**SCR-Q7**: How are the integers and rational numbers different?

**SCR-Q8**: Did you learn the following concepts or theorems in your previous mathematics classes? If so, please list the concept or the theorem that you have learned. Also, please, if possible, state a definition of the concept and a statement of the theorem that you listed in your own words. PLEASE do not consult resources.

- Polynomial ring
- Irreducible polynomial
- Units (or invertibles)
- Associates
- Fundamental theorem of arithmetic
- Fundamental theorem of algebra
- Unique factorization theorem

**SCR-Q9**: When did you (or do you plan to) take MATH X (Basic Abstract Algebra) or MATH Y (Algebra for Teachers)? (*Actual course numbers instead of X and Y were included in the survey.)

**SCR-Q10**: What is your name and email address?

The following explains the justification for the prescreening survey and the three criteria applied for selecting ideal participants. Since the current study focused on changes in participants’ mathematical understandings, the attribute that mattered for the selection of my participants was the mathematical backgrounds that they would bring into the study. The mathematical backgrounds needed for carrying out the tasks during the interviews included (1) ability to distinguish polynomials from nonpolynomial
expressions, (2) awareness of number systems, and (3) proficiency with basic
factorization of polynomials and numbers. I collected information on potential
participants’ relevant mathematical backgrounds via their responses to SCR-Q1 through
SCR-Q7. Their responses were graded, and the percent correct on the items SCR-Q1
through SCR-Q7 was used to select ideal candidates with sufficient mathematical
background. To be specific, performance on items SCR-Q1 through SCR-Q7 at above
80% correct was the first criterion applied for identifying an ideal participant.

In addition, I needed my participants to have had little previous exposure to the
focal concepts/theorem of the TIs (i.e., irreducible polynomial, units, unique factorization
theorem) before coming to the interviews. In order to gauge their prior knowledge on
these without giving them information on what this study is going to be about, I listed in
SCR-Q8, seven topics that I selected from abstract algebra – only three of which were
focal topics in this study. I asked potential subjects to respond to whether they had been
introduced to these concepts previously and, if so, write the definition of the concept or
the statement of the theorem without consulting resources. I examined their responses
and, when they provided reasonably sophisticated definitions/statements for the focal
concepts/theorem, I excluded them from the pool of ideal participants (i.e., the second
criteria).

Lastly, an ideal candidate had to have no foreseeable exposure to the focal
concepts/theorem during the period of interviews. Because the topics (irreducible
polynomial, units, and unique factorization theorem) are likely to be covered in an
abstract algebra course, I strictly excluded the students who responded that they had
taken or were taking an abstract algebra course in SCR-Q9 (i.e., the third criteria).
Detailed recruitment processes are discussed in the section entitled *Participants* on page 122. The selected participants through the prescreening survey engaged in multiple task-based interviews. In the next section, I introduce a sequence of mathematical tasks designed for the teaching interviews and how they were developed.

**Teaching interview [TI] tasks**

Central to the design of TI tasks was a coordination between the AiC approach and the mathematical context of factorization. The current section builds on the two sections of Chapter 2, *Abstraction-in-context [AiC] framework* on page 78 and *Factorization in school and collegiate mathematics* on page 84, and explains how the mathematical tasks of TIs were developed. The first step toward my development of the TI tasks was an in-depth mathematical analysis of the Unique Factorization Theorem [UFT]—a terminal theorem about factorization. The following first contains a statement of the UFT and an explanation of the meaning of the UFT statement. Next is a discussion of three subconstructs of the UFT, which were used in identifying the order and the content of mathematical tasks of the TIs. The section concludes with a sequence of mathematical tasks designed for the TIs to reflect the AiC approach.

The UFT states that a complete factorization of a polynomial over an integral domain is essentially unique. Using symbolic notations, it can be re-stated as follows:

**Unique Factorization Theorem**: Let $f$ be a polynomial over an integral domain, $D$. Given any two complete factorizations of a polynomial $f$ over $D$ (say, $p_1p_2\ldots p_n$ and $q_1q_2\ldots q_m$), the number of irreducible factors in the
two factorizations are identical (hence, \( n = m \)) and, upon rearrangement, \( p_i \)
is an associate of \( q_i \) for all \( i \) between 1 and \( n=m \).

Of note here is that the TIs were not intended to conclude with the theorem (either by its name or its statement), let alone having a direct mention of the theorem during the TIs. Rather, the UFT was chosen as a content area that underlies the design of mathematical tasks. TIs were intended to communicate ideas relevant to the UFT not necessarily in the form of textbook-type language. Allowing participants to share their mathematical ideas in the language with which they feel most comfortable might provide an opportunity for gaining close access to participants’ understandings of factorization. A majority of tasks during the TIs were at least implicitly relevant to the UFT. For example, one of the questions during the TIs, “\textbf{CORE-Q7:} Are there multiple ways to factor completely the same polynomial given a fixed domain?” was asked to investigate participants’ understanding of complete factorization with respect to its uniqueness. Rather than focusing the analysis on whether participants think a factorization is unique or not, of importance during the TIs was participants’ reasoning—why they think there are multiple ways (or a unique way). Even if they say there are multiple ways, further explanation may reveal that they notice that different representations consist of essentially the same irreducibles that just differ by a constant-multiple or by the order in which they are written, reflecting a general understanding of the UFT. By designing mathematical tasks in which participants’ understandings can be observed and guided toward construction of the UFT during the TIs, key mathematical components of the UFT can be analyzed in-depth mathematically. Specifically, “integral domain,” “factors,” “factorization,”
“associate,” and “irreducible factor” are such key mathematical components and are explained in the following paragraphs.

First, a canonical definition of “integral domain” is that it is a nonzero commutative ring in which every product of nonzero elements is nonzero (Cuoco & Rotman, 2013, p. 192). Four (at least) instantiations of integral domain exist in school mathematics and they are the number systems, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$. Even if one may not be aware of the term “integral domain,” one can think about polynomials defined over $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$ as polynomials with the coefficients in the corresponding number system. What one may not be familiar with in understanding the UFT (based on solely school mathematics understandings) is the idea of fixing the number system to start with for factoring a polynomial. As stated in the first sentence in the UFT (“Let $f$ be a polynomial over an integral domain, $D$”) and in the following sentence (“two complete factorizations of a polynomial $f$ over $D$”), defining any factorization should be preceded by the consideration of what types of coefficients are allowed in the factorization.

Second, the following definition of “factor” (as a noun) is a commonly accepted definition in the field: “For any pair of $f(x)$, $g(x)$ of polynomials in $D[x]$, we say that $g(x)$ divides $f(x)$ if there is a polynomial $h(x)$ in $D[x]$ for which $f(x) = g(x)h(x)$. In this situation, $g(x)$ is a factor of $f(x)$” (Barbeau, 1989, p. 56). Hence, $x^2$ may be rewritten as $x^2 \cdot \frac{1}{x}$, but neither $x^3$ nor $\frac{1}{x}$ is a factor of $x^2$ since neither is a polynomial. To factor a polynomial over $D$ (i.e., using “factor” as a verb) means to write the polynomial as a product of its factors in $D[x]$. Factorization of a polynomial over $D$ means the act of factoring the polynomial over $D$ or a result obtained from the act of factoring (i.e., symbolic, written
representation). Thus, factorization is considered as decomposition of a given polynomial into a product of more fundamental, elementary polynomials (not rational or irrational expressions such as \( \frac{1}{x} \) or \( \sqrt{x} \)).

Third, the notion of “associate” in the UFT is to account for a trivial factorization that runs counter to establishing the simplicity of factorization. In order to understand “associate,” one needs to understand the concept of “unit.” Unit has been defined as “[a]n element that is a factor of every element of a structure” (Sawyer, 1959, p. 87) or an element of a structure that “has a multiplicative inverse in [the structure]” (Fraleigh, 1994, p. 280). In the set of polynomials over \( D \), a unit can be understood as a polynomial that is a factor of every polynomial over \( D \). For example, 2 is a unit in the set of polynomials over \( Q \), because for any polynomial \( f, f = 2 \cdot \left( \frac{f}{2} \right) \). To take care of this kind of trivial factorization (that is no longer useful in revealing essential building blocks of a given polynomial), a notion of “associate” is introduced: An associate of an element \( f \) in \( D[x] \) is an element equivalent to \( f \cdot u \) for some unit \( u \) in \( D[x] \). Hence, as long as “\( p_i \) is an associate of \( q_i \) for all \( i \)” as stated in the UFT, the two complete factorizations \( p_1p_2\ldots p_n \) and \( q_1q_2\ldots q_m \) are revealing essentially same information about the given polynomial \( f \).

Lastly, the notions of “unit” and “associate” are used to define an “irreducible” polynomial. A polynomial \( f \) over \( D \) is called irreducible if its only factors are units and its associates in \( D[x] \). In other words, the only ways to factor an irreducible polynomial (e.g., \( x-1 \) over \( Q \)) are trivial ones such as \( 2 \cdot \left( \frac{x-1}{2} \right) \). Hence, the “irreducible factors” \( (p_1, p_2, \ldots, p_n) \) in the UFT statement can be considered essential building blocks of the original polynomial \( f \).
In sum, the UFT that a complete factorization of a polynomial over an integral domain is essentially unique is a very subtle statement to understand and requires sophisticated examination of various aspects of complete factorization—for example, types of coefficients, meaning of factorization, how to deal with trivial factorization, and what to consider as irreducible. Hence, engaging in the UFT-related mathematical tasks could pose subtly challenging problems for mathematics-intensive majors, even if they are proficient at factorization at the secondary level. Also, the context could provide participants a chance to reflect on and, potentially, make transformative transitions in their existing understandings of factorization.

To carefully craft the mathematical tasks in ways that guide the construction of the UFT and encourage active reflection on relevant school mathematics understandings, I used the AiC approach. As previously mentioned, designing mathematical tasks grounded in the AiC was intended to give rise to a series of mathematical activities in such a way that subsequent activities build on previous ones and potentially transform the previous constructs in the learner’s mind, eventually arriving at construction of an idea that is new to the learner. For the successions of activities as suggested by the AiC approach, I first identified three subconstructs of the UFT to be sequentially constructed in the TIs. The three subconstructs are primeness, relativeness, and uniqueness of complete factorization.

To understand the UFT, the learner needs to understand: first, relativeness of a complete factorization in that whether a factorization is complete or not is relative to the domain over which a given polynomial is factored; second, primeness of a complete factorization in that each of the factors (p₁, p₂, …, pₙ) in the complete factorization
(p_1p_2…p_n) has to be irreducible over the domain; third, *uniqueness* of a complete factorization in that, although there may be different ways to rewrite p_1p_2…p_n into some other expression, q_1q_2…q_m, by reordering or by trivial factorization, the building blocks (p_1, p_2, …, p_n and q_1, q_2, …, q_m) are not fundamentally different upon one-to-one correspondence. In Figure 3.1, I present how the three subconstructs of the UFT may build on one’s understandings of factorization at the secondary level (the lower part in the figure).

![Diagram of Unique Factorization Theorem](image)

Figure 3.1. Mathematical analysis of the Unique Factorization Theorem for the design of teaching interviews

Participants’ existing understandings of complete factorization of polynomials and numbers at the secondary level could lay the groundwork for constructing the three
subconstructs of the UFT. Having identified the subconstructs of the UFT and the relevant school mathematics context, I generated a series of mathematical tasks for TIs in which interviewees can *recognize* and *build-with* their existing understandings toward their *construction* of the UFT, hence reflecting the RBC model (see Table 3.2; in Appendix B, each of the **CORE** questions was annotated in detail with potentially associated R-, B-, or C-actions). A summary of the purposes of the TI tasks and their relation to R-, B-, or C-actions is presented in the next paragraphs.

Table 3.2. Teaching interview tasks (see Appendix B for the complete interview schedules)

<table>
<thead>
<tr>
<th><strong>CORE-Q1:</strong> $x^2$ may be rewritten as $x^3 \cdot \frac{1}{x}$. What do you think of the statement that $x^2$ factors into $x^3 \cdot \frac{1}{x}$?</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>CORE-Q2:</strong> Students in high school algebra classes are often asked to ‘factor completely.’ What does the phrase ‘factor completely’ mean to you?</td>
</tr>
<tr>
<td><strong>CORE-Q3:</strong> According to your definition, please factor completely the following polynomials.</td>
</tr>
<tr>
<td>$P_1(x) = x^4 – 12x^2 + 36$</td>
</tr>
<tr>
<td>$P_2(x) = x^5 – x^2$</td>
</tr>
<tr>
<td>$P_3(x) = 4x + 16x^3$</td>
</tr>
<tr>
<td>$P_4(x) = 12$</td>
</tr>
<tr>
<td>$P_5(x) = 4x^2 + 2x – 20$</td>
</tr>
<tr>
<td>$P_6(x) = 2x^3 – 10x^2 + 8x$</td>
</tr>
</tbody>
</table>

(Q3-F1) (“F” stands for a follow-up question. *Some* follow up questions are not included in this table for the brevity’s sake; See Appendix B for details)

<table>
<thead>
<tr>
<th><strong>CORE-Q4:</strong> Please factor completely the same set of polynomials over each of the number systems, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ and record your results in the following table.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_1(x) = x^4 – 12x^2 + 36$</td>
</tr>
<tr>
<td>$P_2(x) = x^5 – x^2$</td>
</tr>
<tr>
<td>$P_3(x) = 4x + 16x^3$</td>
</tr>
<tr>
<td>$P_4(x) = 12$</td>
</tr>
<tr>
<td>$P_5(x) = 4x^2 + 2x – 20$</td>
</tr>
<tr>
<td>$P_6(x) = 2x^3 – 10x^2 + 8x$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Over $\mathbb{Z}$</th>
<th>Over $\mathbb{Q}$</th>
<th>Over $\mathbb{R}$</th>
<th>Over $\mathbb{C}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Over $\mathbb{Z}$</td>
<td>Over $\mathbb{Q}$</td>
<td>Over $\mathbb{R}$</td>
<td>Over $\mathbb{C}$</td>
</tr>
</tbody>
</table>
How would you describe what happened in this table in a general way? Or, how can you make some general statement about your answers in this table?

(Q4-F1) (Q4-F2) (not included in this table for the brevity’s sake; See Appendix B for details)

(Q4-F3) Follow-up based on student response to $P_4$ over $\mathbb{Q}$

(Case 1: When the interviewee wrote 12 under the cell of $P_4$ over $\mathbb{Q}$)
If a student claims that $2 \cdot 2 \cdot 3$ is also a complete factorization over $\mathbb{Q}$ because 2, 2, and 3 are all rational numbers, what would you say? (Delving deeper their understanding of relativity)

(Case 2: When the interviewee wrote $2 \cdot 2 \cdot 3$ under the cell of $P_4$ over $\mathbb{Q}$)
If a student claims that factorization of 12 over $\mathbb{Q}$ is meaningless because 12 can be factored over $\mathbb{Q}$ in infinitely many ways—such as $(1/2) \cdot 2 \cdot 2 \cdot 2 \cdot 3$, what would you say?

(Q4-F3-(a)) (End of questions in both Cases)
Now, have you ever heard of the term reducible or irreducible with respect to factorization? What does reducible or irreducible mean to you?

(Q4-F3-(b)) So, do you think 12 is reducible, irreducible, or neither of them over $\mathbb{Q}$? Why do you think so?

(Q4-F3-(c)) Mathematicians call the element such as 12 in $\mathbb{Q}$ a unit. That is, an element in a domain (such as $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$) that has its multiplicative inverse in that domain is called UNIT.

What are some examples of units in $\mathbb{Q}$? What about units in $\mathbb{Z}$?

(Q4-F3-(d))
Please select a unit in $\mathbb{Z}$. Is the unit you selected (as a polynomial) reducible, irreducible, or neither over $\mathbb{Z}$? Why do you think so?

What is the relationship among units, reducibles, and irreducibles?

(Q4-F4) Follow-up based on student response to $P_6$ over $\mathbb{Z}$
For example, when student had the following factorization: $P_6(x) = 2x^3 - 10x^2 + 8x = 2x(x-4)(x-1)$

(Q4-F4-(a)) Would you consider 2 as irreducible over $\mathbb{Z}$? If so, why? If not, why not?
How about over $\mathbb{Q}$?

(Q4-F4-(b)) Would you consider $2x$ as irreducible over $\mathbb{Z}$? If so, why? If not, why not?
How about over $\mathbb{Q}$?
Looking at this cell (P₆ over ℤ), please describe how reducible and irreducible may come into play.

**CORE-Q5:** What would you consider as an **irreducible polynomial** over each of the four domains, ℤ, ℚ, ℋ and ℂ?

**CORE-Q6:** How would you explain to your friend what **a complete factorization** of a polynomial over a domain D is? (*D is used to stand for ℤ, ℚ, ℋ or ℂ*).

**CORE-Q7:** Are there multiple ways to **factor completely** the same polynomial given a fixed domain?

**CORE-Q8:**
1) Make up a polynomial such that its irreducibility over ℚ is different from its irreducibility over ℋ.
2) Make up a polynomial such that its irreducibility over ℋ is different from its irreducibility over ℂ.
3) Make up a polynomial such that its irreducibility over ℤ is different from its irreducibility over ℚ.

**CORE-Q9:** Let c be an integer. Discuss the irreducibility of the polynomial, \( x^2 + c \), over ℤ, ℚ, ℋ and ℂ.

**CORE-Q10:** Let a, b, and c be integers. Discuss the irreducibility of the polynomial, \( ax^2 + bx + c \)
over ℤ, ℚ, ℋ and ℂ.

**CORE-Q11:** Examine the following statement: “If \( p(x) \) is irreducible over ℋ then \( p(x) = 0 \) has no roots in ℋ.” Is this statement true? If so, explain why it is true. If not, explain why it is not true.

**CORE-Q12:** Examine the following statement: “If \( p(x) = 0 \) has no roots in ℋ, then \( p(x) \) is irreducible over ℋ.” Is this statement true? If so, explain why it is true. If not, explain why it is not true.

The first four questions (CORE-Q1 to CORE-Q4) call for participants’ engaging in R-actions and B-actions on factorization and polynomial equations. Understanding of polynomial equations could be particularly useful in factoring an expression like \( P_7(x) = x^5 - x^2 \) over ℂ. The follow-up questions of CORE-Q4 engage participants to reflect on issues related to trivial factorization, the relationship between reducible and irreducible,
and the newly introduced definition of unit. Those follow-up questions potentially call for participants’ engaging in C-actions on subconstructs of the UFT (primeness, relativeness, uniqueness). Completing \textbf{CORE-Q1} to \textbf{CORE-Q4} would lay the groundwork for constructing the subconstructs of the UFT \textit{deliberately} in \textbf{CORE-Q5} to \textbf{CORE-Q7}. This strategy reflects the AiC approach in that subsequent activities are designed to build on previous ones and potentially transform the previous constructs in the learner’s mind, eventually arriving at construction of a new concept.

The next three questions (\textbf{CORE-Q5} to \textbf{CORE-Q7}) call for participants’ engaging in R-actions and B-actions on previously discussed constructs related to factorization (such as irreducible, existence of domain, issue of trivial factorization). Because the nature of the tasks (which asks for defining or determining a property), \textbf{CORE-Q5} to \textbf{CORE-Q7} call for \textit{deliberately} engaging in C-actions on subconstructs of the UFT (primeness, relativeness, uniqueness). Completing \textbf{CORE-Q5} to \textbf{CORE-Q7} would lay the groundwork for consolidating their understandings of the subconstructs of the UFT further in \textbf{CORE-Q8} to \textbf{CORE-Q12}. This strategy reflects an AiC approach in that this strategy may allow a new construct to be gradually incorporated into the learner’s cognitive structure as the new construct is available to be recognized and built-with more and more in the further activities. That is, participants’ newly constructed understandings could possibly be consolidated further in the subsequent R-actions and B-actions on the new constructs.

The next five questions (\textbf{CORE-Q8} to \textbf{CORE-Q12}) call for participants’ engaging in R-actions and B-actions on polynomial equation and factorization and also on subconstructs of the UFT (specifically, primeness, relativeness) that arose from the
previous activities. Beyond this point, the tasks ask the interviewee to apply what s/he had established so far and consolidate his/her understandings.

As mentioned earlier, one assumption in the current study was that this kind of instructional context following principles of AiC—in other words, engaging them in a series of R-, B-, C-actions which involve a transformation of existing understandings and development a new construct—would serve as a reasonable environment for observing transformative transitions. This assumption is closely examined in a result section in Chapter 5 entitled *Contextual features in making transformative transitions* on page 288.

**Pre- and post-TI interview tasks**

Tasks for pre-TI and post-TI interviews were designed to address the need for examining how participants’ existing school mathematics understandings relevant to the UFT context changed during or after their engagement in the TIs. For the direct comparison between participants’ responses before and after the TIs, pre-TI and post-TI interview tasks were created in pairs, with members of each pair of tasks designed to be identical or considerably similar. This section focuses on a discussion of how the tasks were developed and introduce a complete set of pre- and post-TI interview tasks.

Pre-TI and post-TI interview tasks were centered around two major constructs—factorization and polynomial equation. The two constructs were chosen because they are the main constructs to be recognized and built-with during the TIs for the construction of the UFT. In order to design specific tasks for pre- and post-TI interviews, I consulted resources such as *Center for Mathematics Education (CME) Project Algebra 1* (Cuoco,
Polynomials (Barbeau, 1989), and Learning Modern Algebra (Cuoco & Rotman, 2013) to look for the tasks that can reveal the levels of participants’ understandings of factorization and polynomial equation. As a result, some of the questions in these resources were adapted as my interview tasks. (The source for each of the borrowed or adapted questions is cited in Appendix C). Such examples of adapting existing questions can be found in **PRE/POST-Q1** and **PRE/POST-Q7** (see Table 3.3 for the questions).

Those interview questions are variations of the following question:

7. How can I factor any quadratic polynomial? (CME project algebra 1, p. 700).

**PRE/POST-Q1** and **PRE/POST-Q7** were asked to my participants because their responses could reveal (a) whether participants perceive factorization as a recipe-based action, a generic process, or an object (i.e., factorization) with certain properties, (b) whether they consider factorization or a polynomial in a localized context (e.g., only over integers) or in a broader context, and (c) whether they can coordinate various constructs in their polynomial equation schema, such as quadratic formula, roots, and factors. In total, nine pairs of questions were asked in the pre-TI and post-TI interviews. See Table 3.3 for the complete list of questions and Appendix C for more details.

Table 3.3. Pre- and post-TI interview tasks

| **PRE-Q1 (and POST-Q1)**: Can you factor \( ax^2+bx+c \) without knowing the specific values of \( a \), \( b \), and \( c \)? If so, how can you do so? If not, why do you think so? |
| **PRE-Q2 (and POST-Q2’)**: Find if possible \( a \), \( b \), and \( c \) for each of the following cases: |

---

9 The following **POST-Q2’** was prepared for participants who would have considered various contexts \( \mathbb{Z}[x], \mathbb{Q}[x], \mathbb{R}[x] \) and \( \mathbb{C}[x] \) as possibility in **PRE-Q2**. **POST-Q2’**: Find necessary and
(i) $ax^2+bx+c=0$ has one integer root and one rational root that is not integer;
(ii) $ax^2+bx+c=0$ has two rational roots that are not integers;
(iii) $ax^2+bx+c=0$ has two real roots that are not rational;
(iv) $ax^2+bx+c=0$ has one real root and one nonreal root;
(v) $ax^2+bx+c=0$ has no real roots.

PRE-Q3: Write a quadratic equation with the following characteristics: It has only rational coefficients and one of its roots is $1 + \sqrt{3}$.

POST-Q3: Write a quadratic equation which has $1 + \sqrt{3}$ as one of its roots.

PRE-Q4: A quadratic equation has only rational coefficients. If one root is irrational, is the other irrational or rational? Why is it so?

POST-Q4:
- If one of the roots of a quadratic equation is irrational, is the other irrational or rational? Why is it so?
- If one of the roots of a quadratic equation is nonreal, is the other nonreal or real? Why is it so?

PRE-Q5: Suppose that you have a quadratic equation $x^2+bx+c=0$, which has two roots $x_1$ and $x_2$. Find another quadratic equation whose roots are $2x_1$ and $2x_2$.

POST-Q5: Suppose that you have a quadratic equation $ax^2+bx+c=0$, which has two roots $x_1$ and $x_2$. Find another quadratic equation whose roots are $2x_1$ and $2x_2$.

PRE-Q6: Suppose that you have a quadratic equation $x^2+bx+c=0$, which has two roots $x_1$ and $x_2$. Find the roots of the following equation: $x^2–bx+c=0$.

POST-Q6: Suppose that you have a quadratic equation $ax^2+bx+c=0$, which has two roots $x_1$ and $x_2$. Find the roots of the following equation: $ax^2–bx+c=0$.

PRE-Q7: Can you factor any quadratic polynomial? If so, how can you do so? If not, why do you think so?

POST-Q7: Please explain your reasoning to the following questions.
(a) Can you factor any quadratic polynomial?
(b) Can you factor any quadratic polynomial uniquely?

sufficient conditions on the real numbers a, b, c, d for the equation $z^2 + (a+b\text{i})z + (c+d\text{i}) = 0$ to have exactly one real and one nonreal root.
(c) Can you factor any quadratic polynomial completely?
(d) Can you factor any quadratic polynomial completely and uniquely?

PRE-Q8 (and POST-Q8): According to the Common Core State Standards high school students need to *Understand that polynomials form a system analogous to the integers*. In what sense do you think they are analogous?  

PRE-Q9 (and POST-Q9): Please construct a concept map of YOUR ideas about polynomial in the context of school mathematics. Think of all of the things you associate with the concept of polynomial and construct a network among those ideas. (You may use the following ‘candidate’ ideas as you see them relevant. You do NOT need to use every idea listed and you may certainly use the ideas that are not listed.)

‘Candidate’ ideas:

<table>
<thead>
<tr>
<th>Polynomial</th>
<th>Factoring</th>
<th>Coefficient</th>
<th>Rational numbers</th>
</tr>
</thead>
<tbody>
<tr>
<td>Degree</td>
<td>Quadratic formula</td>
<td>Integers</td>
<td>Prime factorization</td>
</tr>
<tr>
<td>Equation solving</td>
<td>Discriminant</td>
<td>Complex numbers</td>
<td>Real numbers</td>
</tr>
<tr>
<td>Roots</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

*Follow-up: Please explain what you mean by each of the ideas in your concept map. Also please explain how and why the ideas are linked in your network.*

The instruments discussed heretofore are *mathematical* tasks and questions that were developed to investigate participants’ understandings in their doing of mathematics. In the following sections, I introduce two sets of supplementary interview questions that were developed to investigate (a) how participants perceive their doing of mathematics and (b) potential external influences on their doing of mathematics. Those supplementary questions were developed to address the need for establishing the credibility of this study.

---

10 Although the Common Core does not explicitly suggest polynomials in one variable in this standard statement, it seems reasonable to assume polynomials in one variable based on the fact that a notation such as \( p(x) \) was used in many other references to a polynomial in the Common Core document.
Incorporating retrospective questions in each of the interviews was a decision made based on a pilot study conducted in Spring 2015. The pilot study was intended to help identify potential methodological issues that might arise in the actual study and make relevant revisions or improvements in the interview questions. In the pilot study, the entire set of interview questions and tasks served as a tool for gaining observational access to how participants do mathematics; no questions were asked to gain access to how they perceive their own doing of mathematics. Hence, my observation of participants’ doing of mathematics was the sole data source from which I drew inferences about whether and how they had made a transformative transition in the pilot study. In drawing inferences about their transformative transitions, there was an unmet need for evidence of what the participants perceived as changes in their existing understandings. At that time, I realized that some retrospective questions could have been asked so that participants could reflect on their interviews and talk explicitly about the differences or similarities between what they newly learned (if any) and what they had previously known. In order to address this need by providing the opportunity for triangulation of data, a secondary set of data were obtained in the actual study by asking the retrospective questions in Table 3.4.

Table 3.4. Retrospective questions asked in the beginning and end of each interview

The following retrospective questions were asked at the END of each interview:

**END-Q1.** Before we end this interview, I would like to talk about today’s interview a little bit. If you were to describe what we did today (in terms of mathematics), how would you describe it?
END-Q2. What, if any, were the mathematical facts/concepts/procedures new to you?

END-Q3. Does what we did today remind you of any other mathematics you have done?

END-Q4. Is what you learned about [a big idea/topic for the interview; e.g., polynomial, factorization] or did today consistent with the way you thought about [a big idea/topic for the interview; e.g., polynomial, factorization] before? If so, how are they similar? If not, how are they different?

The following retrospective questions were asked at the BEGINNING of each interview (except for the first interview):

BEG-Q1. Before we begin this interview, I would like to talk about our last interview a little bit. If you were to describe what we did in our last interview (in terms of mathematics), how would you describe it?

BEG-Q2. What is [notion that was central in the previous interview; e.g., factorization, complete factorization, unit, irreducible polynomial] and how do you define it? Can you give me an example and nonexample of the [notion]?

END-Q2, 3, and 4 in Table 3.4 were asked to examine what participants explicitly perceived as new for them and as changes in their existing understandings; other questions including END-Q1, BEG-1, and BEG-2 were asked to help their reflection on their interviews in retrospect. The retrospective questions in the beginning of the interviews (BEG-1 and BEG-2) served two purposes: first, to boost their reflection process which can potentially contribute to producing rich responses to other retrospective questions; and second, to help them refresh their memory about what they had done in the previous interview and make their engagement in subsequent activities more fluent. The next section contains an introduction to another set of supplementary interview questions that were developed for investigating potential external influences on participants’ engagement during the interviews. Asking those questions also seemed to be crucial to establish the credibility of this study.
External influence questions

My pilot study also raised an important question that needed to be addressed for the credibility of my study: Could participants, in their making transformative transitions, be drawing on external resources and experiences other than their engagement in the TIs? For example, the courses participants were taking during the period of interviews could have covered a similar mathematical concept or procedure, and such learning experiences external to the engagement in the TIs might underlie the changes observed in their understandings during the interviews. If such external sources constituted a context in which my participants were making transformative transitions, it seems crucial to collect and analyze relevant information to understand what underlies their transformative transitions transparently. Hence, I collected data on potential sources of influence. At the end of the last interview, I asked the following questions in Table 3.5 to my participants:

Table 3.5. External influence questions asked in the end of the last interview

<table>
<thead>
<tr>
<th>EXIT-Q1</th>
<th>What mathematics courses or mathematics-intensive mathematics education courses are you taking this semester?</th>
</tr>
</thead>
<tbody>
<tr>
<td>EXIT-Q1.1</td>
<td>Did these courses cover the materials similar to or related to the ones in our interview during the period of our interviews? Can you share that part of the materials (e.g., class notes, textbook)?</td>
</tr>
<tr>
<td>EXIT-Q1.2</td>
<td>Did these courses affect the way that you think about the topics in our interview? If so, how did they affect?</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>EXIT-Q2</th>
<th>What outside resources (e.g., Google, books, colleague conversation), if any, did you consult about the topics that we went over during the period of our interviews?</th>
</tr>
</thead>
<tbody>
<tr>
<td>EXIT-Q2.2</td>
<td>Did these resources, if any, affect the way that you think about the topics in our interview? If so, how did they affect the way that you think about those topics?</td>
</tr>
</tbody>
</table>
Participants’ responses to these questions provided evidence consistent with a minimal influence from external experiences and resources in their engagement during the interviews. Their responses to the questions in Table 3.5 are discussed in detail in the section entitled *Data analysis* (see Table 3.9 on page 144).

The preceding five sections—(1) Prescreening survey, (2) Teaching interview [TI] tasks, (3) PRE- and POST-TI interview tasks, (4) Retrospective questions, and (5) External influence questions—centered on descriptions of the instruments developed for the current study as well as the purpose of the instrument as a tool for collecting relevant data for this study and for promoting trustworthy results. The next sections contain a discussion of how participants were recruited and selected and how data were collected from the participants.

**Participants**

The general population of the current study is mathematics-intensive majors—specifically, mathematics majors and secondary mathematics education majors. Hence, to recruit participants for the current study, I approached students in their second through fourth year whose intended majors were either a 4-year undergraduate mathematics major or a 4-year undergraduate secondary mathematics certification program at a large university. Participation was voluntary.

The recruitment process took place in two rounds. First, an invitation email (see Appendix D) was sent to the target group. To send out the invitation email, (a) I asked a coordinator of undergraduate advising in mathematics to forward the invitation email to
undergraduate students in the mathematics program, (b) I asked a mathematics education professor to forward the invitation email to undergraduate students in her class (e.g., MTHED 411), and (c) I asked a coordinator of the mathematics learning center of the university to forward the invitation email to undergraduate students who were working as peer-tutors in of the mathematics learning center. To ensure enough students to participate in the prescreening survey (see a previous section entitled *Prescreening survey* on page 102 for details), I made the prescreening survey available online and stated in the invitation email that “Two individuals among all participants who answered the prescreening survey will be selected by a random draw and get a prize (Starbucks gift card $25).” This advertisement seemed to be effective—in the first round of recruitment, I had 24 students who participated in the prescreening survey. After each prescreening survey, I applied the three criteria to the survey results for selecting ideal candidates. As previously discussed, to be qualified as an ideal participant, (a) his/her performance on items SCR-Q1 through SCR-Q7 needed to be at above 80% correct; (b) his/her responses to SCR-Q8 needed to evidence lack of sophisticated definitions/statements for the focal concepts/theorem; and (c) his/her responses to SCR-Q9 had to indicate s/he had not taken or was not taking an abstract algebra course at the time of interview. I checked all three criteria as a new survey result came in for each volunteer. As volunteers submitted completed surveys, follow-up emails were sent to ask for further participation to the volunteers who met the three criteria. In the follow-up email communication with qualified candidates, one volunteer did not respond to the follow-up email, and one withdrew his application for participation because it seemed impossible to schedule the
meeting times that were mutually available. As a result, four participants agreed to participate further in the first round.

A problem in this recruitment process was identified when I came to be aware of the majors of the first four participants in the email communication: All of the four participants were mathematics majors and none of them were secondary mathematics education majors. The mathematics education majors who participated in the prescreening survey had taken or were taking an abstract algebra course by the time of the recruitment and so they were excluded from the candidates list.

Hence, the second round of recruitment took place to recruit mathematics education majors. I visited the MTHED 411 class with instructor’s permission and advertised for participation by making a brief presentation during the break time. As a result, three secondary mathematics education majors participated in the prescreening survey in the second round of recruitment. Two of them met the criteria and agreed to participate further.

In sum, I successfully recruited and interviewed six participants in total. The participants were compensated with $10 in cash in return for their participation each hour. As described earlier, by the time of the interviews, four of those participants were majoring or had intended to major in mathematics; two of them were majoring in secondary education with emphasis in mathematics. None of those participants had yet taken and were not taking an abstract algebra course by the time of the interview. Table 3.6 shows the list of participants and their respective major areas of study.

Table 3.6. Participants and their major degree areas
<table>
<thead>
<tr>
<th>Pseudonym</th>
<th>Major degree area</th>
</tr>
</thead>
<tbody>
<tr>
<td>Calvin</td>
<td>Mathematics</td>
</tr>
<tr>
<td>Jason</td>
<td>Mathematics</td>
</tr>
<tr>
<td>Helen</td>
<td>Mathematics</td>
</tr>
<tr>
<td>Andy</td>
<td>Mathematics</td>
</tr>
<tr>
<td>Lucy</td>
<td>Secondary education with emphasis in mathematics</td>
</tr>
<tr>
<td>Sam</td>
<td>Secondary education with emphasis in mathematics</td>
</tr>
</tbody>
</table>

The order of their names in Table 3.6 reflect the order in which their first interviews occurred. In the following section, I discuss the data collection process in detail, which includes when and where the six participants engaged in the interviews, how the interviews were conducted and recorded, and what artifacts were generated and collected in the end of interviews.

**Data collection**

The data collection occurred during Fall 2016. The dates and times of interviews were decided based on mutual availability between the participant and myself. Each interview lasted approximately 60 minutes in length. A calendar view of dates of interviews is presented in Table 3.7.

Table 3.7. Participants and date of interviews

<table>
<thead>
<tr>
<th>Week</th>
<th>Tuesday</th>
<th>Thursday</th>
<th>Friday</th>
<th>Saturday</th>
<th>Sunday</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Calvin INT 1</td>
<td>Jason INT 2</td>
<td>Helen INT 1</td>
<td>Jason INT 3</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Calvin INT 2</td>
<td></td>
<td>Helen INT 2</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Jason INT 1</td>
<td></td>
<td>Calvin INT 3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>Calvin INT 4</td>
<td>Andy INT 1</td>
<td>Helen INT 3</td>
<td>Helen INT 4</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Calvin INT 5</td>
<td>Andy INT 2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td><strong>Jason INT 5</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Week 3  |  Calvin INT 6  
|---|---|
|  |  **Calvin INT 7**  
|  |  Andy INT 3  
|  |  Andy INT 4  
| **Helen INT 5**  |  Sam INT 1  
| **Helen INT 6**  |  **Helen INT 7**  

Week 4  |  Sam INT 3  
|---|---|
|  |  Andy INT 5  
| **Andy INT 6**  |  Lucy INT 1  

Week 5  |  Sam INT 4  
|---|---|
|  |  Sam INT 5  
|  |  Lucy INT 2  
| **Lucy INT 3**  |  Lucy INT 4  

Week 6  |  No interview  

Week 7  |  Lucy INT 5  
|---|---|
| **Lucy INT 6**  |  Sam INT 6  
|  |  Sam INT 7  

Week 8  |  Sam INT 8  
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Sam INT 9</strong></td>
<td></td>
</tr>
</tbody>
</table>

* Days of the interviews excluded Mondays and Wednesdays because of the interviewer’s teaching schedule.

In Table 3.7, the last interview with each of the participants is underlined/bolded and the number following “INT” indicates how many interviews were conducted for each participant. Although the same set of interview schedules were used throughout the six participants, Table 3.7 shows that the number of interviews per participant varied from five to nine. The variation in the number of interviews for each participant was due to the fact that the number of interviews was determined by when the participant completed the entire set of questions. To be specific, each participant engaged in one or two pre-TI interview(s), three to six TIs, and one post-TI interview. Most participants except for Sam completed all interview questions within five to seven interviews as expected in the design stage (the target number of total interviews was six). However, Sam, who was a very careful thinker, usually spent more time on each question than other participants. Upon his agreement, I was able to have more interviews with him to complete the series
of interviews. This seemed to have allowed him enough time to think through each question during the interviews. Also, Sam’s availability was limited due to his student-teaching, and so completing the entire set of interviews with him took place over several weeks. In sum, data collection included conducting a total of 40 interviews.

When multiple interviews were conducted on the same day, there were at least 30 minutes in between two consecutive interviews so that I, as an interviewer, could reflect on the previous interview (by memo writing) and prepare the subsequent interview. Before the interviews in each day, I reread the relevant portion of the interview schedules for the day and reviewed the guidelines for TIs (the guidelines were described earlier in the section An instructional context using Teaching interview [TI] on page 95) to remind myself of the rule to which an interviewer in TIs should adhere.

For all interviews, I used a designated interview room for the researchers in mathematics education program at Penn State. The access to the room allowed me to (a) prepare and wrap up each interview without time constraints, (b) set up multiple recording devices consistently for all 40 interviews, and (c) use them to securely audio- and video-record participants’ engagement during the interviews. Three recording devices were used as shown in Figure 3.2.
As illustrated in Figure 3.2, a camcorder (‘Recording 1’ in the figure) was used to capture their working zone (i.e., zone of the papers on which the participant writes or the calculator screens on which the participant works) more closely and clearly. This camera aimed directly down at the working zone (in a vertical direction), which was helpful to keep track of how their thoughts were developed as participants wrote over the top of the things that they had previously written. Another camcorder (‘Recording 2’ in the figure) was used to capture the participant’s working area and their gestures (but not their face) at a distance. The third device for data collection was an iPad pro with an application called ExplainEverything (2018). The application acts as a platform in which one can use
iPencil as a digital pen and iPad screen as a digital notebook. On the application, I preloaded interview questions so that participants could read them and write their answers (or other supplementary information) on the iPad as they engaged in the interviews. At the end of each interview, the same application (ExplainEverything) was used to create a video that simultaneously captured what participants were saying as they were writing. I used this third device for the purpose of efficient data analysis and presentation since it allowed screencapturing participants’ writings in-the-moment without overwritten texts or diagrams.

After each interview, all recordings from the three devices in Figure 3.2 were transferred to and saved in two external hard drives. The files created from the three devices included MOV files of videos, MP4 files of audios, and PDF files of the written work on the iPad. After the interview was over, when a participant used papers to respond to some questions, his/her written work was collected in a binder and scanned into a PDF file. When a participant used a calculator in an interview, the screencapture of the calculator was saved as a JPEG file. After each interview, I created a verbatim transcript of each interview within a week of the interview in one of the following two ways: (a) by personally transcribing the videorecording of an interview using a software program called F5 Transcription Pro (2018) and (b) by employing a professional transcriber11 and then editing the transcript while watching a videorecording of the interview. In sum, the following set of data was collected from each participant:

11 Hiring a professional transcriber was partially supported by the Heid, Blume, Zbiek mathematics education graduate endowment. I shared only the audiorecording of an interview (not videorecording) with the professional transcriber.
• Participant’s written responses to a prescreening survey

• Participant’s written work in pre-TI, TI, and post-TI interviews
  o Written work on iPad (exported from the ExplainEverything app)
  o Written work on papers (scanned files and paper copies)

• Participant’s work on a calculator (screencaptured JPEG files)

• Video recordings of pre-TI, TI, and post-TI interviews
  o Main video: A recording from the camera that aimed directly down at the working zone (Recording 1 in Figure 3.2)
  o Supplementary video 1: A recording from a diagonal angle
    (Recording 2 in Figure 3.2)
  o Supplementary video 2: iPad screenrecording (Recording 3 Figure 3.2)

• Audio recordings of pre-TI, TI, and post-TI interviews
  o Audio extracted from main video

• Transcripts of all interviews

In a program called F5 Transcription Pro (2018), each transcript was synchronized with the “Main video” file recorded from a vertical angle. Such synchronization allowed me to read the transcript and, simultaneously, watch the video of the corresponding portion. This way, I had access to the references of what the participants were physically pointing to when they used words such as “it,” “this one,” and “those roots.” Hence, transcripts were not annotated to specify those references. The main video occasionally did not capture some participants’ bodily gestures or their referencing behaviors. When this happened, I referred to secondary video sources to crosscheck. The transcripts with
synchronized videos served as a main source for the data analysis. The process of data analysis is discussed in the following section.

**Data analysis**

To develop credible descriptions of and explanations for the growth (or lack of growth) in participants’ understandings of school mathematics in their learning of collegiate mathematics, the records of interviews were analyzed in multiple passes. Each pass was guided by each of the four research questions. The analysis process is described in the next paragraphs.

The first pass of analysis consisted of identifying analysis units and coding them with relevant categories/levels of the transformative transition framework, which was intended to address the first research question as follows.

| How can university students’ school mathematics understandings be described by the levels in each of the four categories of transformative transition—deepening, extending, strengthening, and unifying? |

In the initial reading of a transcript, I began by determining the beginning and end of each of new interview task by when the question was first introduced and when the response was completed. I marked the beginning and end of each interview task using the question numbers such as **CORE-Q1** and **END-Q2**; when the question had multiple parts, I used the question numbers at the lowest hierarchical level such as **PRE-Q2 (iv)** and **Q4-F4-(a)**. Each participant’s entire response to each interview task became a potential analysis unit. In the potential analysis unit, I identified one or more
mathematical entities that the participant utilized in his/her response. Sometimes, participants’ responses were lengthy and had to be parsed by mathematical entities that were utilized in chronological order. Then, each partition became the actual analysis unit that would be examined through the lens of the transformative transition framework. In what follows, I first discuss how mathematical entities were identified in participants’ responses and how each analysis unit was coded according to the framework.

The mathematical analysis of the UFT in the designing stage afforded a range of mathematical entities that were expected to be utilized by participants in each interview task (e.g., factorization, polynomial equations, roots, number systems). In most responses, the mathematical entities utilized by participants were within the expected range. However, my search for mathematical entities in participant’s responses was not limited to the expected range. Whatever the participant had brought up as s/he saw relevant to the given interview task (e.g., \( \mathbb{C} \) as algebraic completion of \( \mathbb{R} \)) was also taken into consideration when identifying mathematical entities.

As might be expected, different individuals often utilized different mathematical entities in the context of the same task. For example, in their responses to PRE-Q2 (iv), some participants utilized their understandings of the quadratic formula to claim that having one real root and one nonreal root is impossible; some participants utilized their understandings of factorization to provide an example of having one real root and one nonreal root such as \((x - 1)(x - i) = x^2 - (1 + i)x + i = 0\). Even within one

\[ \text{PRE-Q2 (iv): Find (if possible) } a, b, \text{ and } c \text{ such that } ax^2 + bx + c = 0 \text{ has one real root and one nonreal root.} \]
participant’s response to one interview task, there sometimes existed a shift in focus and a change in utilized mathematical entities. When this happened, the response was partitioned into multiple analysis units.

Hence, each analysis unit was a subset of each participant’s response to each interview task, and the analysis unit consisted of several, consecutive sentences recorded in the transcript that could convey information about the participant’s mathematical understandings of the identified mathematical entity. On each analysis unit, I created analytic memos by considering and responding to the following questions:

- How does s/he treat XXXX (a mathematical entity) in this unit?
- Does the way s/he treats XXXX suggest any of action-level, process-level, and object-level understandings under the deepening category? [or Does the way s/he treats XXXX suggest any of intra-object level, inter-object level, and trans-object level understandings under each of the extending, strengthening, and unifying categories?]
- What is the evidence that suggests a particular level of understanding? To what extent is the evidence convincing?

The intent of the analytic memos was to provide a picture of the nature of participants’ understandings about the mathematical entity and, if applicable, decide into which level their understandings fall in each of the applicable categories—that is, deepening, extending, strengthening, and unifying. If the mathematical entity was factorization or polynomial equation (which are the major foci of this study), I analyzed the unit using the specific descriptions of each level/category developed in Chapter 2 (see Tables 2.1, 2.2, 2.3, and 2.4 for the descriptions); Otherwise, I followed general descriptions of the four
categories of the transformative transition framework, descriptions of levels in APOS theory (Arnon et al., 2014; Dubinsky, Dautermann, Leron, & Zazkis, 1994), and descriptions of levels in Piaget and Garcia’s (1983/1989) triad to code the analysis unit.

To deal with analysis units for which it was difficult to determine the level of understandings (e.g., whether it is action level or process level), I met with my advisor to solicit feedback on my difficulties/confusions involved in the coding scheme application. I revised the description of the analysis unit until we were able to reach agreement about the level.

After determining a relevant category/level for a given analysis unit, I marked the unit with a code of the assigned category/level. The codes that were used to mark each analysis unit had the following format: <category initial>_<level represented by a number>_<name of mathematical entity>. See Table 3.8 for the examples of code when a mathematical entity is factorization.

Table 3.8. Examples of codes according to my categorical framework for transformative transitions (when a mathematical entity is factorization)

<table>
<thead>
<tr>
<th>Code</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>D1_Factorization</td>
<td>Action-level understanding of factorization in the <strong>deepening</strong> category</td>
</tr>
<tr>
<td>D2_Factorization</td>
<td>Process-level understanding of factorization in the <strong>deepening</strong> category</td>
</tr>
<tr>
<td>D3_Factorization</td>
<td>Object-level understanding of factorization in the <strong>deepening</strong> category</td>
</tr>
<tr>
<td>E1_Factorization</td>
<td>Intra-object level understanding of factorization in the <strong>extending</strong> category</td>
</tr>
<tr>
<td>E2_Factorization</td>
<td>Inter-object level understanding of factorization in the <strong>extending</strong> category</td>
</tr>
<tr>
<td>E3_Factorization</td>
<td>Trans-object level understanding of factorization in the <strong>extending</strong> category</td>
</tr>
<tr>
<td>S1_Factorization</td>
<td>Intra-object level understanding of factorization in the <strong>strengthening</strong> category</td>
</tr>
<tr>
<td>S2_Factorization</td>
<td>Inter-object level understanding of factorization in the <strong>strengthening</strong> category</td>
</tr>
<tr>
<td>S3_Factorization</td>
<td>Trans-object level understanding of factorization in the <strong>strengthening</strong> category</td>
</tr>
<tr>
<td>U1_Factorization</td>
<td>Intra-object level understanding of factorization in the <strong>unifying</strong> category</td>
</tr>
<tr>
<td>U2_Factorization</td>
<td>Inter-object level understanding of factorization in the <strong>unifying</strong> category</td>
</tr>
<tr>
<td>U3_Factorization</td>
<td>Trans-object level understanding of factorization in the <strong>unifying</strong> category</td>
</tr>
</tbody>
</table>
Based on the analytic memos that belong to the same level and same category, I developed empirically-evidenced descriptions of each of the three levels in each of the four categories of transformative transition. The descriptions, which address the first research question, are provided in Chapter 4.

Each analysis unit was also color-coded by its category—to be specific, analysis units were highlighted in pink for deepening category, yellow for extending, blue for strengthening, and green for unifying. The color-codes in individual analysis units helped me, in subsequent passes, to trace a trajectory of participants’ understandings coded by the same category and yet by differing levels. Those analytic memos and codes constructed in the first pass of my data analysis were put together to build an integrated explanation in the second pass of data analysis.

To be specific, the second pass of analysis consisted of identifying threads of analysis units and describing specific changes in participants’ understandings in the four categories of transformative transition, which was intended to address the second research question as follows.

| What transformative transitions do university students make in the four categories—deepening, extending, strengthening, and unifying—in an AiC-grounded instructional context? |
| The goal of the second pass was to systematically analyze lengthy interview data (5 to 9 hours of interviews for each participant) and provide credible descriptions of the growth in their understandings in the course of the interviews using the transformative transition framework. To achieve this goal, I first constructed a chart (for each participant and each category) that listed the codes assigned in the first pass of data analysis chronologically. |
Those charts served as a quick reference to overview changes (if any) in the levels within each category for each participant. To construct those charts, I revisited the entire set of transcripts that had been annotated with analytic memos and a designated code (e.g., U2_Factorization) in the previous pass. As I was examining each participant’s interview transcripts, I recorded on a separate paper the codes assigned to each analysis unit and parenthetically the interview question number. The papers on which the codes were recorded were also color-coded in the same way as color-codes used in the previous pass (i.e., a pink paper for deepening category, yellow for extending, blue for strengthening, and green for unifying). For example, when an analysis unit in a participant’s response to PRE-Q8 was coded with U2_Factorization, I wrote down “U2_Factorization (PRE-Q8)” on a green paper because it belonged to the unifying category. When analysis units were coded by the same mathematical entity and by the same category, I recorded the codes on the same column in chronological order. Hence, each sequence of codes in the same column formed a cluster that provided snapshots of one’s mathematical understandings about the same mathematical entity over time. I call this sequence of codes (or, more broadly, the sequence of corresponding analysis units as well) a thread in the sense that the sequence of codes (or the analysis units) share a common category and mathematical entity. For example, a thread for a participant, Jason, related to a mathematical entity of

---

13 The term “thread” is often used to refer to a group of comments posted by multiple users on an Internet discussion forum that share a common subject. The way that “thread” is used in this study is similar to the normal use of the term in the sense that codes (or the analysis units) in a thread share a common subject; but what constitutes a thread in this study is all responses of or codes for a single participant.
factorization and the category of unifying consisted of the following sequence of codes (and parenthetically the interview question number):

U3_Factorization (PRE-Q8)
U3_Factorization (PRE-Q9)
U2_Factorization (CORE-Q3)
U2_Factorization (CORE-Q4)
U2_Factorization (Q4F3)
U3_Factorization (Q4F3)
U2_Factorization (Q4F3(a))
U3_Factorization (BEG-Q2_INT3)
U2_Factorization (BEG-Q2_INT3)
U3_Factorization (Q4F3(c, d) & Q4F4(a, b, c))
U3_Factorization (CORE-Q5)
U3_Factorization (CORE-Q6)
U3_Factorization (POST-Q8)
U3_Factorization (END-Q2_INT5)

Once threads were constructed for all participants and all categories, I began to determine whether any transformative transitions were apparently made by participants for each category—deepening, extending, strengthening, and unifying. As previously mentioned, a transformative transition involves a change in understandings from a lower level to a higher level (e.g., from action level to process level, from inter level to trans level). Hence, the first step toward identifying a transformative transition was to check how the numbers in the codes (which represents the levels) were changing across analysis units within a given thread. In the previous example of Jason’s thread, the codes changed back and forth between U2 and U3 and this indicates his understanding of factorization (as inferred from the analysis of each unit) was going back and forth between the inter-object level and the trans-object level in the unifying category.
After gathering such _sketchy_ information about how the levels had been changed in each thread, I checked the _details_ of the change by revisiting corresponding analysis units in the transcripts and the analytic memos written next to them. During this analysis stage, I examined the nature of the change in understandings across analysis units in each thread. In total, 13 threads provided rich information for understanding how a transformative transition might take place empirically. Based on these threads, I developed empirically-evidenced descriptions of transformative transitions in each category. The descriptions, which address the second research question, are provided in four sections in Chapter 4: _Extending observed in the course of interviews_ (on page 179), _Unifying observed in the course of interviews_ (on page 214), _Strengthening observed in the course of interviews_ (on page 242), and _Deepening observed in the course of interviews_ (on page 272).

An embedded goal in providing these descriptions was to communicate what the data revealed in a credible way. To achieve this goal, my interpretation of the events that occurred during the interviews was cross-checked with the interviewee’s own interpretation of the events (i.e., the actor’s point of view) through his/her responses to retrospective questions. Such cross-check provided me a chance to “evaluate the extent to which all evidence converges.” (Suter, 2012, p. 350). A method of cross-checking multiple data sources, also called _triangulation_, is a commonly recommended practice in qualitative research to support the credibility of a study (Denzin, 1970/2000; Maxwell, 1996; Suter 2012). As previously mentioned, an instrument composed of retrospective
questions (see footnote\(^{14}\) for some examples; see Table 3.4 for the entire set of retrospective questions) was designed for the purpose of providing the opportunity for triangulation of data. It should be noted, however, that the triangulation in this study is limited because of the lack of a third (and potentially more) data sources.

In those retrospective questions, participants were asked to talk explicitly about the differences or similarities between what they newly learned (if any) during the interviews and what they have previously known. Having this secondary source of data helped to add more credibility in drawing inferences about whether and how they had made a transformative transition, than if I had been limited to one and only source of data such as participants’ responses to mathematical tasks. Hence, in the second pass of my data analysis, participants’ responses to mathematical tasks during the interviews were triangulated with their own voice of how they used to think about relevant mathematical entities such as factorization. I provide an example of data triangulation in the next few paragraphs.

The example builds on Sam’s responses to **PRE/POST-Q8\(^{15}\)** and his retrospective statements about factorization. First, in **POST-Q8**, he said that “I think it is saying… that their students need to understand that the set of polynomials and the set of

\(^{14}\) **END-Q2.** What, if any, were the mathematical facts/concepts/procedures new to you?  
**END-Q3.** Does what we did today remind you of any other mathematics you have done?  
**END-Q4.** Is what you learned about [a big idea/topic for the interview; e.g., polynomial, factorization] or did today consistent with the way you thought about [a big idea/topic for the interview; e.g., polynomial, factorization] before? If so, how are they similar? If not, how are they different?  

\(^{15}\) **PRE-Q8 (and POST-Q8):** According to the Common Core State Standards high school students need to **Understand that polynomials form a system analogous to the integers**. In what sense do you think they are analogous?
integers kind of carry similar rules.” He explained how factorization similarly applies to both sets and used examples of irreducible elements from each set, 5 and \(x+4\), to show how the rule of reducibility applies to both. This response was in a marked contrast with his response to the same question during the pre-TI interview (i.e., PRE-Q8) in which he simply said “I’m not sure how to interpret that standard.”

The difference between Sam’s responses to PRE-Q8 and POST-Q8 suggests a possibility that his understanding of factorization of integers and factorization of polynomials might have gone through a transformative transition of unifying. However, the difference in his responses alone seems not sufficient to conclude the change was a transformative transition. There is another possibility that, in PRE-Q8, the factorization idea was not detected on his radar at that time or that Sam simply could not articulate the understanding even though he knew that factorization could be a mechanism that bridges the two structures, integers and polynomials.

When it was unclear whether the change in their responses was a matter of understanding or a matter of articulation, participants’ responses to retrospective questions were often useful to extrapolate the meaning of the data. For example, the following statements made by Sam illustrate an old way he had thought about factorization or what he perceived as a change in his own existing understandings:

- “I had only ever thought of factoring in terms of polynomials and not in terms of numbers”;  
- “Whenever I saw it like this [this refers to a problem, “How can you factor 60?”], the term ‘factor’ had a much different meaning to me than if I was given the quadratic equation”;
“It was weird to think about [factoring a number and factoring a polynomial altogether] for the first time.”

Consideration of both data sources—Sam’s responses to PRE/POST-Q8 and his retrospective statements—provided convincing evidence that he came to think of factorization differently from prior to his engagement in the interviews. Such converging evidence through triangulation of data allowed me to reach the following conclusion: Sam made a transformative transition, specifically unifying, by coming to view how an overarching idea such as reducibility underlies initially disparate ideas, factorization of polynomials and factorization of integers. Hence, the results provided in Chapter 4 build on my analysis of both data sources. The transformative transitions identified in the second pass of data analysis became the sources of analysis in the third pass.

The third pass of data analysis consisted of looking for and explaining contextual features of the AiC-grounded instructional context that seem to be closely related to making transformative transitions, which was intended to address the third research question as follows.

<table>
<thead>
<tr>
<th>What are possibly relevant features of an AiC-grounded instructional context in which university students make transformative transitions?</th>
</tr>
</thead>
</table>

As mentioned earlier, there were 13 threads in which transformative transitions were observed. In some of those threads, the transformative transitions that were evidenced by the events occurred distantly (e.g., one in the pre-TI and the other in the post-TI interview); in the other threads, the transformative transitions were observed in their continuous responses to a single mathematical task (e.g., they changed their answers
as meanings emerged). In the latter, I was able to follow up as an interviewer to understand what might have made them change their answers and how it might be related to the AiC-grounded instructional context. However, in most of the threads, it was not easy to identify precisely the moments in which participants made transformative transitions and analyze how it was possibly related to the interview context.

Hence, the third pass of data analysis involved returning to the entire set of transcripts to identify the analysis units that might be relevant to the transformative transition made within a given thread. In the analysis units, I looked for what it was that they engaged in during the moment of the interview context that could possibly explain the change in their understandings. As plausible explanations emerged for each thread, I then looked across the threads within the same category of transformative transitions (coding by the same color made the comparison across threads easier). I asked myself “In what ways are the threads in the [deepening] category similar or different? What patterns might explain the instances of [deepening]?” and the same questions for each of the extending, strengthening, and unifying categories. Constructing answers to those questions was an iterative process in that I repeatedly moved back and forth between the transcripts, plausible explanations obtained from each thread, and emerging patterns in those explanations across threads. This process is what might be best described as constant comparison method (Dye, Schatz, Rosenberg, & Coleman, 2000; Suter, 2012), in which insights emerge inductively as comparisons are made between different data bits. Using the constant comparison method, I drew inferences about the attributes of AiC-grounded instructional context that might be closely related to participants making
transformative transitions. The results are presented in the section of Chapter 5 entitled

*Contextual features in making transformative transitions* on page 288.

A reasonable question that needed to be addressed in the third pass of data analysis was: Are there other competing explanations that can be used to better understand the phenomenon of transformative transitions than contextual features of the AiC-grounded interviews? Experts in qualitative research methods (Maxwell, 1996; Suter, 2012) consider it crucial for the credibility of a study to explore and rule out alternative or competing explanations. In my study, the prescreening survey ensured that no participants had taken or were taking an abstract algebra course by the time of the interview. However, since participants were mathematics-intensive majors, they were taking several mathematics courses while they were participating in this study; and this might mean that what they learned from those mathematics courses could have possibly influenced their responses during the interviews and, furthermore, the transformative transitions they have made. For this reason, the instrument composed of external influence questions was designed (see the section entitled *External influence questions* on page 121) and the questions were asked at the end of the last interview. I provide a summary of participants’ responses to the external influence questions in Table 3.9. All responses essentially provided evidence consistent with a minimal influence of the mathematics courses they were taking on their engagement during the interviews.
Table 3.9. Participants’ responses to external influence questions

<table>
<thead>
<tr>
<th>Question</th>
<th>Participant’s responses</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>EXIT-Q1.</strong> What mathematics courses or mathematics-intensive mathematics education courses are you taking this semester? (see footnote (^{16}) for the course titles)</td>
<td>Calvin: MATH C, MATH D</td>
</tr>
<tr>
<td></td>
<td>Jason: MATH G</td>
</tr>
<tr>
<td></td>
<td>Helen: MATH A</td>
</tr>
<tr>
<td></td>
<td>Andy: MATH E</td>
</tr>
<tr>
<td></td>
<td>Lucy: MATH E</td>
</tr>
<tr>
<td></td>
<td>Sam: MATH B, MATH F (student-teaching)</td>
</tr>
<tr>
<td><strong>EXIT-Q1.1.</strong> Did these courses cover the materials similar to or related to the ones in our interview during the period of our interviews? Can you share that part of the materials (e.g., class notes, textbook)?</td>
<td>Calvin: (see text)</td>
</tr>
<tr>
<td></td>
<td>Jason: (see text)</td>
</tr>
<tr>
<td></td>
<td>Helen: No</td>
</tr>
<tr>
<td></td>
<td>Andy: (see text)</td>
</tr>
<tr>
<td></td>
<td>Lucy: (see text)</td>
</tr>
<tr>
<td></td>
<td>Sam: N/A</td>
</tr>
<tr>
<td><strong>EXIT-Q1.2.</strong> Did these courses affect the way that you think about the topics in our interview? If so, how did they affect?</td>
<td>Calvin: No</td>
</tr>
<tr>
<td></td>
<td>Jason: No</td>
</tr>
<tr>
<td></td>
<td>Helen: No</td>
</tr>
<tr>
<td></td>
<td>Andy: (see text)</td>
</tr>
<tr>
<td></td>
<td>Lucy: (see text)</td>
</tr>
<tr>
<td></td>
<td>Sam: N/A</td>
</tr>
<tr>
<td><strong>EXIT-Q2.</strong> What outside resources (e.g., Google, books, colleague conversation), if any, did you consult about the topics that we went over during the period of our interviews?</td>
<td>Calvin: None</td>
</tr>
<tr>
<td></td>
<td>Jason: None</td>
</tr>
<tr>
<td></td>
<td>Helen: None</td>
</tr>
<tr>
<td></td>
<td>Andy: None</td>
</tr>
<tr>
<td></td>
<td>Lucy: None</td>
</tr>
<tr>
<td></td>
<td>Sam: (see text)</td>
</tr>
<tr>
<td><strong>EXIT-Q2.2.</strong> Did these resources, if any, affect the way that you think about the topics in our interview? If so, how did they affect the way that you think about those topics?</td>
<td>Calvin: N/A</td>
</tr>
<tr>
<td></td>
<td>Jason: N/A</td>
</tr>
<tr>
<td></td>
<td>Helen: N/A</td>
</tr>
<tr>
<td></td>
<td>Andy: N/A</td>
</tr>
<tr>
<td></td>
<td>Lucy: N/A</td>
</tr>
<tr>
<td></td>
<td>Sam: No</td>
</tr>
</tbody>
</table>

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\(^{16}\) MATH A: Matrices  
MATH B: Ordinary differential equations  
MATH C: Ordinary and partial differential equations  
MATH D: Concepts of real analysis  
MATH E: Introduction to mathematical statistics  
MATH F: Linear algebra  
MATH G: Special topics (*The one Jason were taking in that semester covered complex numbers and the concept of ring.*)
• Calvin’s response to \textbf{EXIT Q1-1}: Calvin mentioned factoring skills of “mainly just degree 2 polynomials” were “a core skill that [he] needed” in his class, MATH C (Ordinary and partial differential equations). He added that, however, the course covered “nothing conceptual” such as “what it means to have coefficients in $\mathbb{C}$.”

• Jason’s response to \textbf{EXIT Q1-1}: Jason mentioned that, in the morning of the day of the last interview, his class (MATH G) covered how to solve a cubic equation and the cubic formula (See Figure 3.3). Since the interview questions involved solving polynomial equations, the cubic formula was relevant; but the relevance was only tangential and he did not use the formula during the interviews. Jason mentioned “we didn’t really talk too much about factoring [cubic expression] though” and “we didn’t really talk about this kind of thing \textit{this kind of thing seems to refer to what he had done in the interviews].”

\[
\sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{27}{27}}} - \frac{p}{2} = 0
\]

Figure 3.3. Jason’s written work for a solution of a reduced cubic equation

• Andy’s response to \textbf{EXIT Q1-1 and EXIT Q1-2}: Andy mentioned that, in his previous mathematics classes, “we’ve talked about multiplicative inverses of numbers… and how to factor” but he had “never heard of the term unit, and reducible, irreducible with polynomials before, in that explicit manner.” He also
indicated his mathematics courses did not affect the way he was thinking about the topics in the interviews; Instead, he mentioned how his prior experiences of factoring was different from how it was dealt with during the interviews as follows: “in classes I’ve taken throughout my life about factoring, we’re just taught to factor it a certain way… Just to kind of go off of that, to not like think for ourselves what it means to be reducible or irreducible.”

- Lucy’s response to EXIT Q1-1, EXIT Q1-2, and EXIT-Q2: Lucy mentioned, in her class, MATH B (Ordinary differential equations), she dealt with “a lot of factoring with complex numbers, but obviously not this in depth” as she did in the interviews. To the question whether the course affected the way that she thought about the topics in the interviews, she explained how “it blocked [her] thinking a little bit.” Because she was “used to the three cases [of root pairs] in differential equations” which were “two real, the repeated, and the imaginary,” it was difficult for her to think about cases such as a quadratic equation having one real root and one nonreal root in PRE-TI interview. In EXIT-Q2, she responded that after the first interview, she looked up on the Internet “what rational and irrational was just to make sure [she] knew exactly what they were” and “nothing after that.” She did not think the search affected the way she thought about the topics in the interviews.

The results of analyzing participants’ responses to the external influence questions suggest that the AiC-grounded interviews can be assumed to be the context in which transformative transitions took place. While the third pass of data analysis built on the threads in which transformative transitions were made, threads in which participants
remained at almost the same level and did not make transformative transitions were a major source for the last pass of data analysis.

The fourth pass of data analysis consisted of looking for and explaining possible obstacles to making transformative transitions, which was intended to address the fourth research question as follows.

| When university students do not make transformative transitions in an AiC-grounded instructional context, what are some obstacles to making those transitions? |

A constant comparison method was utilized for the fourth pass as for the third pass of the data analysis. I revisited the entire set of transcripts for those participants who did not make transformative transitions in each category. I looked for and documented what obstacles participants seemed to encounter when what seemed to be the opportunities for making transformative transitions were missed. The questions that I asked myself while identifying obstacles are the following: What may be the understanding they have that keeps them from making progress? What may be the tendency or belief they have that runs counter to transformative transitions? The analysis of the potential obstacles is presented in the section of Chapter 5 entitled Obstacles to making transformative transitions on page 311.

In all four passes of my data analysis, Sam’s data were the last ones to be analyzed. There are three reasons for this: (1) Chronologically Sam’s data were the last ones collected, (2) There was a need for saving one participant’s data until the end to determine whether the descriptions and explanations from the first five participants were stable upon the inclusion of the sixth participant’s data analysis result, and (3) Sam’s
interviews were the lengthiest, which meant the data contained the greatest volume of entries to be compared with the previous participants’ data.

The descriptions and explanations were considered as stable if the additional data is mostly consistent with existing explanations or reflecting existing explanations so that no more than minor changes are needed in the existing descriptions and explanations. Such a state in which new data no longer add new insight to existing explanations nor reveal new properties of each category or level is considered saturation in qualitative research (Fusch & Ness 2015; Seidman, 2006; Suter, 2012).

In the first and second passes of data analysis, adding Sam’s data to the existing set of analyses gave diminishing returns in terms of developing empirical descriptions of levels of each category in general. In most of the levels exhibited by Sam’s data, analysis of Sam’s data reflected what was already learned from the analysis of the other five participants’ data; For the action level of the deepening category, however, analysis of Sam’s data contributed to a richer description of the level. This was because, while Sam demonstrated both action and process conceptions of factorization informing both levels’ empirical instantiations, other participants rarely demonstrated action conceptions of factorization during the interviews. Hence, the additional data from Sam’s interviews were helpful for enriching description of action level of the deepening category, but overall the results obtained from the first and second passes of data analysis seemed to stay stable upon the inclusion of Sam’s data.

In the third and fourth passes of data analysis, many of the emerging explanations drawn from the analysis of the first five participants’ data were reflected in and empirically supported by additional data entries from Sam’s interviews. As a result, in
response to the third research question, four contextual features of the AiC-grounded instructional context were identified (those are presented in Chapter 5) and three of the four were partially supported by Sam’s data. In response to the fourth research question, four obstacles to making transformative transitions were identified (also presented in Chapter 5) and three of the four were also partially supported by Sam’s data. No additional insight was drawn from the analysis of Sam’s data in the third and fourth passes of data analysis, thereby suggesting saturation of analysis.

In sum, the current section illustrated my data analysis process and explained some of the strategies I employed during the analysis as means of addressing proposed research questions in a credible way. Those strategies included (1) implementing data triangulation, (2) exploring competing explanations, and (3) confirming saturation of analysis. In the following section, I discuss more strategies that were adopted as means of providing evidence in support of trustworthiness of this study—specifically, (4) monitoring and disclosing researcher’s subjectivity, (5) testing intercoder agreement, (6) allowing audit trail, and (7) providing thick, rich descriptions.

**Trustworthiness of this study**

Since a researcher’s knowledge, experiences, perspectives, assumptions, and insights serve as a basis for a research study at the most fundamental yet implicit level, examining the subjectivity of a researcher probably needs to be a part of discussing trustworthiness of his/her study. In this section, I first talk about my preparation and qualification for becoming a sole researcher of the current study in order to help readers
understand me as a researcher. I then explain how the issue of researcher’s subjectivity was addressed in this study by consistently monitoring my own perspectives, assumptions, and potential biases.

The topic of this study is rooted in my long-standing curiosity with respect to how collegiate mathematics can be possibly related to school mathematics and my doubts regarding the role that collegiate mathematics courses play in preparation of mathematics teachers. As I was exposed to more research studies in my graduate programs, I found empirical research-based evidence that it is highly likely that university students’ knowledge acquired in collegiate mathematics is isolated from their knowledge acquired in school mathematics, and this issue became a sincere educational research problem to me.

Throughout my seven years of doctoral study, I have been deeply and personally committed to the work of addressing this educational problem in various ways. First, I studied, found, and wrote articles about specific collegiate mathematical content that allows mathematics teachers to look at underlying/fundamental characteristics of school mathematics concepts, and thus, to utilize the collegiate mathematical content for their teaching. During my graduate coursework, I conducted an interview project with prospective teachers in order to understand their understandings of advanced mathematical concepts as related to hypothetical teaching contexts. I also conducted a studying-teaching project by observing a mathematics class for teachers. What I learned from the observation helped me to develop my own perspectives on what role collegiate mathematics might play in advancing school mathematics understandings. My dissertation study builds on these threefold efforts—mathematical study, empirical
interview study, and teaching-observation study—and these efforts definitely made me better prepared in terms of practical or theoretical considerations for my study and the development of my research instruments.

My teaching experiences at the collegiate level also helped me to grow as a careful researcher in investigating university students’ mathematical understandings. By interacting with students in both mathematics education and mathematics courses on a daily basis for several years, I have been developing my mind to better understand student thinking, my skills in communicating with students, and my tendency to pay careful attention to what students are saying. I believe my accumulated teaching experiences were especially useful in the data analysis phase for understanding and interpreting the phenomenon reflected in my data. Regardless of the degree of preparedness and qualification of a researcher, the subjectivity of the researcher that forms the basis for a qualitative study needs to be thoroughly examined.

Disclosing and consistently monitoring researchers’ own perspectives, assumptions, and potential biases is an important practice in qualitative research (Creswell & Miller, 2000; Maxwell, 1996; Suter, 2012). In Chapters 1 and 2, I already described my entering beliefs and perspectives on some of the issues in my study. Put simply, (a) I consider interconnectedness among mathematical ideas as an essential feature of mathematics; (b) I believe learning of collegiate mathematics could provide a context for advancing university students’ school mathematics understandings; and (c) I think learning through making transformative transitions should be an important part of university students’ learning of collegiate mathematics.
During the data collection phase, I was mindful of whether my “hope” for seeing participants making transformative transitions came into play and whether my interpretation of participant’s actions/words involved imposing my lens or meaning on the data without closely examining the meaning that the participant attached to the actions/words. To ensure a careful and thoughtful examination of participants’ actions/words during the interviews and minimize the risk of potential bias, I closely followed my interview schedules and guidelines (e.g., Suspend my judgment – Do not hastily assume that I know what s/he is talking about – Ask her/him to talk more in order to really understand and follow up her/his thinking). Before each of the interviews, I read these guidelines and reminded myself of the need for monitoring myself throughout the interview. During the interviews, I asked myself consistently “am I giving them enough chance to reveal their understandings? Did they really mean it or do I want to interpret it that way?”

My educational background may also have an impact on the way that I interpret and follow up on my participants’ mathematical thinking. Before I came to the United States for my master’s degree, I was educated in South Korea. Thus, my own experience with learning mathematics might be different from my participants’ experiences. I learned mathematics in the environment in which teachers and assessments both emphasized formal ways of thinking (e.g., thinking through formal definitions, proving each formula when it is introduced). Since then, I have developed the tendency to think about mathematics in a formal way. This tendency might have prevented me from seeing the promise in certain idiosyncratic directions that my participants had taken and from gathering empirical evidence for the participant’s making transformative transitions. To
guard against this potential risk, I crafted my interview schedules so that they could be ready for multiple directions to be taken by the participants (see Appendix B for examples of the multiple directions, e.g., CASE A, B, and C in Q4-F3-(b)).

During the data analysis phase, I monitored my subjectivity by meeting with my advisor regularly and soliciting feedback on my interpretation of the data. I discussed my coding results with samples of data and, in a later phase, emerging themes or explanations on coded data. My advisor challenged the inferences drawn from my data analysis and the validity of my logic used in constructing the arguments; we discussed alternative explanations until we reached agreement.

Considering others’ perspectives in analysis of the same phenomenon is always a useful strategy for monitoring a researcher’s subjectivity (Maxwell, 1996). In support of reliability of coding/rating schemes of qualitative data, researchers provide evidence that reasonably similar results of coding/rating would be obtained if the same data set was analyzed by other skilled researchers. To be specific, researchers can support the reliability of their findings by testing “the degree of consistency with which instances are assigned to the same category by different observers” (Silverman, 2000, p. 188). In the current study, the coding scheme of applying the transformative transition framework was tested for its reliability, and intercoder agreement was measured.

To measure the intercoder agreement, I first trained a group of fellow graduate students and faculty in the coding scheme in mathematics education seminar meetings (6-hour length in total). I then asked them to determine levels of participants’ understandings in portions of several different transcripts. As a result, there was a considerable agreement (83 %) between my coding and their coding decisions. The
literature on interrater agreement (e.g., Graham, Milanowski, & Miller, 2012; Hartmann, 1977; Stemler, 2004) suggests that the acceptable percentage of agreement varies from 75% to 90%. Hence, the intercoder agreement of 83% in this study seems to be in support of the reliability of the coding methods. The detailed information about training and coding processes is provided in Appendix F.

Another strategy I employed to enhance the credibility of this study was to make the data and method as transparent as possible to the readers. Methodological experts call this notion of providing detailed and thorough explanations of how the data were collected and analyzed an audit trail (Creswell & Miller, 2000; Thompson, 2014). Qualitative researchers are also recommended to chronicle the evolution of their thinking and document rationale for all choices and decisions made during the research process. In this study, I documented processes and procedures involved in collecting and interpreting data (e.g., teaching interview guidelines, detailed interview schedules with multiple possible routes according to participants’ thought trajectory, results interpreted with participants’ written work and quotes) so that readers can track the processes and procedures. I also tried to document and report the rationale for my choices and decisions regarding the selection of participants, design of mathematical tasks, and guidelines for the teaching interview. This hopefully makes it plausible for the readers to determine a degree of the reliability of this study.

While an audit trail is a strategy in sharing research processes, thick, rich descriptions is a strategy of sharing research products with in-depth details. By providing thick and rich descriptions of evidence in use and the logic of arguments, researchers open up for readers the possibility of inspecting the credibility of the account and the
applicability of the results to other similar contexts (Creswell & Miller, 2000). Also, thick and rich descriptions allow readers to make decisions about the robustness of the researcher’s arguments (Seidman, 2006). Hence, my goal in presenting the findings was to do so with clear illustrations of the instances from which my inferences were drawn and detailed evidence that led me to the conclusions I reached.

In the following two Results Chapters, I share insights I gained from my data analysis through thick, rich descriptions. The findings are presented with supporting evidence such as excerpts of relevant portions of annotated transcripts and screencaptures of participants’ written work. Descriptions of participants’ transformative transitions are situated within the interview context, and activities or interviewers’ prompting questions surrounding the moment of change are illustrated in the next two chapters.
Chapter 4 Transformative transitions: Levels and growth

The ultimate goal of this study was to describe levels of university students’ transformative transitions and, where they exist, to explain the transformative transitions in their learning of collegiate mathematics within an interview context grounded in the Abstraction-in-Context [AiC] approach. To this end, four research questions were proposed as follows:

1. How can university students’ school mathematics understandings be described by the levels in each of the four categories of transformative transition—deepening, extending, strengthening, and unifying?

2. What transformative transitions do university students make in the four categories—deepening, extending, strengthening, and unifying—in an AiC-grounded instructional context?

3. What are possibly relevant features of an AiC-grounded instructional context in which university students make transformative transitions?

4. When university students do not make transformative transitions in an AiC-grounded instructional context, what are some obstacles to making those transitions?

The current chapter focuses on the first two research questions and presents findings from the analysis of participants’ responses that revealed transformative transitions in the four categories. Each of the four subsections in this chapter (Extending, Unifying, Strengthening, and Deepening) begins with an overview of three levels in each category, describes participants’ understandings that represent each of the three levels, and finally
illustrates transformative transitions (or advances in levels) observed in the course of interviews. The next chapter builds on the findings discussed in Chapter 4, and situates and discusses the observed transformative transitions in the AiC-grounded instructional context (Research Question 3). Chapter 5 also provides possible explanations for the situations in which participants did not make transformative transitions and identifies obstacles to making transformative transitions (Research Question 4).

**Extending**

The current section focuses on elaborating the extending category using empirical data and provides detailed analysis of participants’ understandings with respect to Piaget and Garcia’s Triad (1983/1989). As discussed in chapter 2, a learner might reorganize his/her existing understandings by extending the boundary of contexts in which a known (to the subject) construct is recognized, considered, or utilized. Here a context means a set of situations or circumstances that form the setting in which the subject applies the recognized construct. The boundary of contexts is considered extended as the subject comes to understand and utilize the construct in terms of increasingly various contexts. The process of extending the boundary of contexts can be understood from the perspective of schema development from intra-object level, to inter-object level, and to trans-object levels as suggested in Piaget and Garcia’s triad. I provide a brief description
of each level in the following paragraphs using a central construct in the current study, factorization of a polynomial.\textsuperscript{17}

The most primitive and basic level of the extending process as it applies to factorization of a polynomial is having only one context in terms of which factorization of polynomial is considered. For example, in school mathematics, factorization is typically proposed in the context of polynomials with integer coefficients (say, $\mathbb{Z}[x]$). A given polynomial is a member of $\mathbb{Z}[x]$ and, when factored, it has to be expressed as a product of other polynomials which are also members of $\mathbb{Z}[x]$. When a student repeats performing and considering factorization in such a particular and localized (from outsider’s perspective) context, the student may assume the particular context is the only context in which factorization can be considered. Hence, even if s/he is aware of broader number systems such as $\mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$, it may not be natural for the student to think of different and additional contexts for factorization such as $\mathbb{Q}[x], \mathbb{R}[x]$ and $\mathbb{C}[x]$. The student might rather resort to the context with which s/he is most familiar and that s/he has seen as a most typical type. For example, the student might think $x^2 + 4x + 6$ cannot be factored because it cannot be expressed as a product of linear expressions with integer coefficients. Such a tendency to consider and apply a construct only within a particular and localized context (sometimes, not even recognizing the boundary of the context that

\footnotesize{\textsuperscript{17} This study focuses on a \textit{polynomial} rather than a \textit{polynomial function}. A \textit{polynomial} is considered as an element in a polynomial ring, whereas a \textit{polynomial function} is considered as a relation satisfying the characteristics of a function. Factorization of a polynomial reveals building blocks (which are elements of the same polynomial ring) that compose the given polynomial. Also, polynomials are \textit{single-variable} throughout in this document.}
the subject is limiting) is an indicator of *intra-object level understanding* of factorization with respect to the extending category. The empirical data that represent intra-object level will be provided in detail in the subsection entitled *Intra-object level understanding in the extending category* [E1] on page 160.

As the learner begins to explore the possibility of considering or applying factorization within other contexts, such as $\mathbb{Q}[x]$, $\mathbb{E}[x]$ and $\mathbb{C}[x]$, the boundary of contexts in which s/he situates his/her understanding of factorization may be extended. Such extension might require significant coordination among all components of the schema of factorization that are utilized at the moment. The intermediate step of beginning (or being able) to examine how the existing understandings related to the familiar (to the subject) situation of factorization can be extended and applied to an unfamiliar or atypical (to the subject) situation but failing to coordinate all relevant components is an indicator of *inter-object level understanding* of factorization with respect to the extending category. The empirical data that represent inter-object level will be provided in detail in the subsection entitled *Inter-object level understanding in the extending category* [E2] on page 165.

Finally, at the *trans-object level*, the subject fully coordinates components related to factorization and various contexts for factorization, in an activated, coherent schema.

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18 The code E1, as the combination of <category initial><level represented by a number>, is an abbreviation of intra-object level understanding in the extending category. To refer to a section more efficiently and differentiate explicitly the three categories using Piaget and Garcia’s triad, a section title using the abbreviation, for example, “section E1” will be used in place of “section entitled *Intra-object level understanding in the extending category*”; the full title will be used when it is introduced first time.
At this level, the subject demonstrates two competencies: first, an awareness of all the relevant (to the given problem) contexts for factorization and also the boundary of contexts within which s/he considered factorization, and second, an ability to examine how the choice of different contexts might affect the way factorization operates. The empirical data that represent trans-object level will be provided in detail in the subsection entitled *Trans-object level understanding in the extending category* [E3] on page 172. In the following subsections, I illustrate in detail participants’ understandings that suggest intra-object, inter-object, and trans-object levels [E1, E2, and E3], respectively, and then describe increase in the levels in extending category as a growth in one’s existing understandings.

**Intra-object level understandings in the extending category** [E1]

During PRE-TI interviews and beginning of TIs, some participants’ responses revealed a tendency to consider and apply factorization only in a particular context, \( \mathbb{Z}[x] \) (polynomials with integer coefficients). Sam, especially, expressed difficulty with thinking of how to factor a polynomial beyond the context of \( \mathbb{Z}[x] \) in the beginning of the interviews. His responses and illustrations of his answers indicated his implicit assumption that, in his factorization, the coefficients always have to be integers.
Specifically, Sam’s responses to questions **PRE-Q1** and **PRE-Q7** revealed his intra-object level understanding [E1] with respect to factorization. In **PRE-Q1**, Sam answered “you need the values of $a$, $b$, and $c$ to sort of determine how you can factor something” and illustrated his point using examples, one that is factorable ($x^2 + 4x + 4$) and the other that is not factorable ($x^2 + 4x + 6$). He explained why the latter is not factorable as follows: “There are no factors of 6 that will also add to positive 4 because I am thinking of the pairs 1 and 6, and 2 and 3. And none of those can be manipulated to add to 4. So that would not be factorable.” His explanation implies that he was assuming only integer coefficients are possible in potential linear factors of $x^2 + 4x + 6$.

In his response to **PRE-Q7**, the reason why Sam tended to consider factorization only within $\mathbb{Z}[x]$ became clearer. Contradicting his previous answer in **PRE-Q1**, he answered “I do think that any quadratic polynomial can be factored.” When asked to elaborate further, he explained even though he thought any quadratic polynomial can be factored, he did not know how to do so; he said “I just personally don’t know how I would go about it [“it” seems to refer to factoring $ax^2 + bx + c$ without knowing the specific values of $a$, $b$, and $c.”]” Sam thus articulated that he found it difficult to apply his understanding of factorization to another context that may include noninteger or irrational coefficients. Moreover, he illustrated why an expression such as $x^2 + 4x + 6$ (which he originally said was non-factorable in **PRE-Q1**) is a difficult case for him to factor. He

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19 **PRE-Q1**: Can you factor $ax^2 + bx + c$ without knowing the specific values of $a$, $b$, and $c$? If so, how can you do so? If not, why do you think so?

20 **PRE-Q7**: Can you factor any quadratic polynomial? If so, how can you do so? If not, why do you think so?
stated “You just may get irrational solutions, which is why, for something like this,
\[\text{referring to } x^2 + 4x + 6 \text{ you can’t just go straight. [Here “irrational solutions” appear to be the solutions to the equation that Sam was seemingly thinking about solving: } x^2 + 4x + 6 = 0.\]” Interestingly, he pointed out that one needs to use the quadratic formula to find the solutions but never connected the quadratic formula back to the factorization of either \(x^2 + 4x + 6\) or \(ax^2 + bx + c\).

Sam’s tendency to stay within \(\mathbb{Z}[x]\) was also evident in his initial definition of complete factorization and his work on \textbf{CORE-Q3}\(^{21}\). See the following equations that Sam wrote in his response to \textbf{CORE-Q3}.

\[
\begin{align*}
P_1(x) &= x^4 - 12x^2 + 36 = (x^2 - 6)(x^2 + 6) \\
P_2(x) &= x^5 - x^2 = x^2(x - 1)(x^2 + x + 1) \\
P_3(x) &= 4x + 16x^3 = 4x(1 + 4x^2) \\
P_5(x) &= 4x^2 + 2x - 20 = 2(x - 2)(2x + 5) \\
P_6(x) &= 2x^3 - 10x^2 + 8x = 2x(x - 1)(x - 4)
\end{align*}
\]

For Sam, “to factor completely” meant, “to take a number or expression and split it into factors that cannot be reduced any further.” Although he did not specify in terms of which domain he would consider complete factorization, his work on \textbf{CORE-Q3} notably

\(^{21}\) \textbf{CORE-Q3}: According to your definition, please factor completely the following polynomials.
\[
\begin{align*}
P_1(x) &= x^4 - 12x^2 + 36 \\
P_2(x) &= x^5 - x^2 \\
P_3(x) &= 4x + 16x^3 \\
P_4(x) &= 12 \\
P_5(x) &= 4x^2 + 2x - 20 \\
P_6(x) &= 2x^3 - 10x^2 + 8x
\end{align*}
\]
revealed his implicit assumption that, in his factorization, the coefficients always have to be integers.

Other participants’ implied assumption with respect to the context that determines the type of coefficients was also revealed in their consideration of the quadratic formula. In PRE-Q2 (iv), participants were asked to find (if possible) $a$, $b$, and $c$ such that $ax^2 + bx + c = 0$ has one real root and one nonreal root. The question did not specify any condition in terms of $a$, $b$, and $c$ values. Three of the participants (Helen, Lucy, and Calvin) concluded immediately that the case of having one real root and one nonreal root is impossible in consideration of quadratic formula as follows:

- **“That can’t happen**... If one real root, this one [referring to $b^2 - 4ac$ in the quadratic formula] has to be greater than or equal to 0. For nonreal root, this one [referring to $b^2 - 4ac$] has to be less than 0... Two of them can’t happen at the same time.” (Helen INT1)

- **“I don’t believe that you could do that** because if you’re taking the roots, then the discriminant is the same, and if your discriminant is negative for one root, it’s negative for the other root, so I don’t believe that you could have a real root and a nonreal root.” (Lucy INT1)

- **“$b^2 - 4ac$ determines the type of root you have. In order to have one real and one nonreal... $b^2 - 4ac$ to be greater than or equal to 0. This is for a real [root]. For a complex [root] you’d need $b^2 - 4ac$ to be less than 0, which is somewhat of a contradiction because $b$ and $a$ and $c$ never change... This can never hold true.”** (Calvin INT1)
The responses of Helen, Lucy, and Calvin suggest that they thought $\sqrt{b^2 - 4ac}$ would be the only place in the quadratic formula from which an imaginary number may be obtained. But this thought process is valid only if the context of applying the quadratic formula is limited to when the coefficients are real numbers, and none of them explicitly mentioned such a necessary condition for their statements. Regardless of whether they had recognized that their statements are true only within $\mathbb{R}[x]$, it seemed clear they were considering the quadratic formula within a particular and localized context, suggesting intra-object level understanding [E1].

However, if one can consider the quadratic formula in various contexts including $\mathbb{C}[x]$, the case of having one real root and one nonreal root is a clear possibility. For example, Jason in PRE-Q2 (iv) specified in which context the case is possible and in which context it is not possible. Jason answered to PRE-Q2 (iv) as follows:

It has a real root and a nonreal root. Well, so, in order for this to happen, we can’t have real coefficients. Because if we have real coefficients for $a$, $b$, and $c$, then complex roots come in pairs. So in order for this to happen, we can just say $(x - 1)(x - i)$. Or 1 and $i$ are the roots. And that’s going to be $x^2 - (1 + i)x + i$.

Jason’s response suggests that he was aware of different contexts for applying the quadratic formula (that is, real coefficients and nonreal coefficients) and was able to examine how the choice of context bears upon the possibility of the case of having one
real root and one nonreal root. Evidently, polynomials with noninteger coefficients such as $\mathbb{Q}[x]$, $\mathbb{R}[x]$ and $\mathbb{C}[x]$ served to provide an unfamiliar and atypical context for most of the participants (except for Jason) to consider factorization and the quadratic formula. In the following section, I discuss the processes through which they struggled to extend their boundaries of contexts in terms of which factorization and the quadratic formula are considered.

**Inter-object level understandings in the extending category [E2]**

Inter-object level understandings [E2] are characterized by the subject’s effort toward and yet, some difficulty with transferring a familiar concept/procedure to a novel setting in which the concept/procedure was not originally proposed and coordinating all relevant components involved in the extension of the boundary of contexts. The formation of inter-object level understandings [E2] is a process in which the tendency of assuming a particular and localized context (which characterizes E1) is challenged and possibilities beyond the existing boundary are explored by the cognizing subject. The current section illustrates participants’ responses that suggest inter-object level understandings.

For Sam, considering and applying factorization within a less familiar (to him) context of $\mathbb{R}[x]$ seemed to be a considerable source of difficulty. As aforementioned in section E1, Sam demonstrated proficiency in factoring polynomials within a particular
context of \( \mathbb{Z}[x] \). In a following question (\textbf{CORE-Q4}), Sam was asked to take the same set of polynomials and factor them completely over each of the number systems, \( \mathbb{Z} \), \( \mathbb{Q} \), \( \mathbb{R} \) and \( \mathbb{C} \). First, with \( P_1(x) = x^4 - 12x^2 + 36 \), Sam noticed he could see \( x^2 - 6 \) as a difference of squares and used \( x^2 - 6 = (x + \sqrt{6})(x - \sqrt{6}) \) to conclude that the complete factorization of \( P_1(x) \) is \( (x + \sqrt{6})^2(x - \sqrt{6})^2 \) over \( \mathbb{R} \). Then his observation of a difference of squares began to affect even linear expressions. For example, in his transition from over \( \mathbb{Q} \) to over \( \mathbb{R} \), Sam rewrote \( x - 1 \) into \( (\sqrt{x} + 1)(\sqrt{x} - 1) \). See Figure 4.1 (the last two columns) for his initial answers to \textbf{CORE-Q4} over \( \mathbb{Q} \) and \( \mathbb{R} \).

![Figure 4.1. Sam’s initial work on \textbf{CORE-Q4} (over \( \mathbb{Q} \) and \( \mathbb{R} \) only)](image)

Noticing that the difference of squares can be potentially applied endlessly (e.g., \( \sqrt{x} - 1 = (\sqrt{x} + 1)(\sqrt{x} - 1) \)), Sam even commented “that leads me to think, like, is there ever such thing as completely factoring something?” Eventually the interviewer brought his attention to the definition of polynomial by asking “does your normal way of factoring

\[\text{\textbf{CORE-Q4}: Please factor completely the same set of polynomials over each of the number systems, } \mathbb{Z}, \mathbb{Q}, \mathbb{R} \text{ and } \mathbb{C} \text{ and record your results in the table. How would you describe what happened in this table in a general way? Or, how can you make some general statement about your answers in this table?}\]
have anything to do with the definition of polynomial?” The question led him to notice
the irrational expressions such as $\sqrt{x} - 1$ are not polynomials and to change his definition
of complete factorization to reflect factorization results in a product of polynomial
factors. It seemed that, when Sam tried to consider factorization within a less familiar
context ($\mathbb{R}[x]$) and extend the boundary of contexts for factorization, he had probably
conflated allowing irrational coefficients with allowing irrational expressions such as $\sqrt{x}$,
suggesting his inter-object level understandings [E2]. As it is seen in Sam’s case, as
previously described, the extending process may require significant coordination, as a
new context likely brings in more components to coordinate.

Participants’ efforts toward understanding and applying the quadratic formula
within a less familiar context of $\mathbb{C}[x]$ also involved some struggle and required significant
coordination. For example, during POST-TI interviews Andy was given a problem in
which he had to consider a quadratic equation over complex coefficients as follows:

**POST-Q2':** Find necessary and sufficient conditions on the real numbers

$a, b, c, d$ for the equation $z^2 + (a + bi)z + (c + di) = 0$ to have

exactly one real and one nonreal root (Barbeau, 1989, p.40).23

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23 **POST-Q2'** was prepared for only two participants, Andy and Jason, who had already
considered various contexts $\mathbb{Z}[x], \mathbb{Q}[x], \mathbb{R}[x]$ and $\mathbb{C}[x]$ as possibility in **PRE-Q2**. In **PRE-Q2 (iv)**,
Andy provided $(x - 1)(x + i)$ as the case of having one real root and one nonreal root. (*The
answer to POST-Q2' is $d^2 = b(ad - bc)$*)
Andy carefully examined whether and how the given quadratic equation could have one real and one nonreal root. He used the quadratic formula for the given equation to reason through the given statement as seen in Figure 4.2.

$$X = \frac{-a + bi \pm \sqrt{(a+bi)^2 - 4(c+di)}}{2}$$

Figure 4.2. Andy’s work on POST-Q2'

At first, Andy conjectured having one real and one nonreal root might be impossible, thinking that the radicand, \((a + bi)^2 - 4(c + di)\), would be either positive or negative (*this is not necessarily true because of the imaginary part). Here the logic Andy applied to the radicand (or the expression in the position of used-to-be discriminant) seemed to be what he was familiar with in the context of \(\mathbb{R}[x]\), but he did not seem to notice it was no longer valid in the context of \(\mathbb{C}[x]\). Andy’s exploration on the possibility of having one real and one nonreal root continued as follows, suggesting Andy was going through a coordination process.

I don’t know if it’s possible... I’m trying to think. It looks like it might - because usually right here [referring to the former \(bi\) in his quadratic formula in Figure 4.2] you don’t have that \(bi\). It’s just an integer there...

Now that we have that \(bi\) in front, whenever we add it [referring to \(bi\)] to this number (referring to \(\sqrt{(a + bi)^2 - 4(c + di)}\) in his quadratic formula in Figure 4.2) there’s a chance that it could kind of cancel out...
each other... So... for real number... this $bi$ has to cancel something going on here [referring to $\sqrt{(a + bi)^2 - 4(c + di)}$]... I’m trying to think how I can show the conditions for those numbers. So I guess we can at least expand some of these things. The 2 at the bottom doesn’t even matter.

Andy finally could not figure out the conditions for having one real and one nonreal root. In EXIT Q1-124, Andy noted how unfamiliar this context of $\mathbb{C}[x]$ was for him by stating “I’ve never seen like in a class where we’ve used a polynomial with a nonreal coefficient in a quadratic form. I’ve seen it whenever you break it up into the factor, you have like $x$ plus or minus some kind of complex thing, but you usually never go back to the quadratic format of it.” Even if the context was new to him, the change in his answers (from initially saying the case of having one real and one nonreal root is impossible, to discovering how it might be possible) suggests Andy was gradually extending the boundary of contexts in terms of which he could consider the quadratic formula.

The difficulty with coordinating the role of discriminant in the quadratic formula in the context of $\mathbb{C}[x]$ was also evidenced by Sam’s reasoning with an equation $(x - \pi)(x + i) = 0$. In his response to POST-Q425, Sam immediately answered the other root can be either nonreal or real. Sam provided $(x - \pi)(x + i) = 0$ as an example of having one real and one nonreal root. Then the interviewer asked Sam to explain how

24 EXIT Q1-1. Did [mathematics courses you are currently taking] cover the materials similar to or related to the ones in our interview during the period of our interviews?
25 POST-Q4: If one of the roots of a quadratic equation is nonreal, is the other nonreal or real? Why is it so?
the case of having one real and one nonreal root makes sense using the quadratic formula. At that point, Sam noticed that, from the quadratic formula, if one of the roots of a quadratic equation is nonreal, then it means the discriminant is negative, and so the other root has to be nonreal as well (of course, this reasoning is valid if one is assuming a subset of $\mathbb{R}[x]$, but not in $\mathbb{C}[x]$). To address the contradiction between the existence of an example he had provided himself ($(x - \pi)(x + i) = 0$) and his observation of nonreal roots occurring in a pair from the quadratic formula, Sam checked if the quadratic formula on $(x - \pi)(x + i) = 0$ still gives the roots $\pi$ of and $-i$ (see Figure 4.3).

$$
\frac{\pm \sqrt{(\pi - i)^2 - 4(-\pi)(i)}}{2}
= \frac{\pm \sqrt{\pi^2 - 2\pi i + i^2 + 4\pi i}}{2}
= \frac{\pm \sqrt{\pi^2 - 2\pi i + i^2 + 2\pi i}}{2}
= \frac{\pm \sqrt{(\pi + i)^2}}{2}
= \frac{\pm \pi + \sqrt{2}i}{2}
$$

Figure 4.3. Sam checking the quadratic formula with $(x - \pi)(x + i) = 0$

Even after he checked how the quadratic formula granted one real and one nonreal root, Sam still appeared to find the result confusing; he said “I would have to take some time to think about that... I don’t know. I am not entirely sure.” Sam did not seem to notice that, when coefficients are nonreal, $\sqrt{b^2 - 4ac}$ is not the only source of the imaginary
part of the quadratic formula. As Sam indicated he had not ever worked with polynomials with complex coefficients before the interviews, his unfamiliarity with complex coefficients (or even, maybe, with complex numbers themselves) could have made it difficult for him to address the contradiction he noticed.

The preceding illustrations from Sam’s and Andy’s work exemplify inter-object level understandings [E2] in the sense that: (a) Sam and Andy became aware of a new context in which the quadratic formula may be utilized and could consider possibilities beyond their routine thought process (or at least question their own rule that the discriminant determines the types of roots), but (b) analyzing or understanding characteristics of the quadratic formula within the new context was not completely successful to produce the desired results in the given problem. Full-blown coordination (e.g., knowing from where the contradiction arose) seemed to require understanding how elements forming the new context (e.g., complex numbers) behave under various transformations. For example, like Andy, Jason also indicated he had never dealt with polynomials with complex coefficients before the interviews; however, he reasoned through POST-Q2\textsuperscript{26} successfully using his construct of complex numbers to find the condition for having one real and one nonreal root. Jason’s case will be discussed in detail in the next section. In what follows, I illustrate trans-object level understandings [E3] in which participants fully coordinate components within a less familiar context in

\footnotesize

\textsuperscript{26} \textbf{POST-Q2':} Find necessary and sufficient conditions on the real numbers \(a, b, c, d\) for the equation \(z^2 + (a + bi)z + (c + di) = 0\) to have exactly one real and one nonreal root (Barbeau, 1989, p.40).
the extension of boundary of contexts and demonstrate awareness of all the relevant contexts and an ability to examine distinct roles of different contexts.

Trans-object level understandings in the extending category [E3]

As in previous observations about participants’ reasoning with the quadratic formula within $\mathbb{C}[x]$, it is worth noting how Jason analyzed the case of having one real and one nonreal root in POST-Q2'. First of all, $\mathbb{C}[x]$ was a less familiar context for Jason (as it was for other participants) as he noted in one of retrospective questions. He stated, “I haven’t really done anything with polynomials with complex coefficients. Like I’ve heard about them, and it’s not like a super strange idea. It’s just that I’ve never dealt with them before.” His full, written solution is provided in Figure 4.4. Because his written solution does not necessarily include all transitional thoughts he had expressed verbally, Figure 4.4 will be followed by detailed explanation of his solution and analysis of it.
To begin with, Jason first applied the quadratic formula to \( z^2 + (a + bi)z + (c + di) = 0 \) and obtained the roots, \( z = \frac{-a - bi \pm \sqrt{a^2 - b^2 + 2abi - 4c - 4di}}{2} \). He noted, to have
one real and one nonreal root, Im\(\sqrt{a^2 - b^2 + 2abi - 4c - 4di}\) has to equal either \(bi\) or \(-bi\) so that the imaginary parts in the quadratic formula cancel out in one of its roots. Jason’s use of the conventional notation for the imaginary part of a complex number, \(\text{Im}(\cdot)\), indicates, to some extent, his familiarity with complex numbers. He also noted, if \(b = 0\), “there is nothing to cancel out” so it is impossible to have one real and one nonreal root. Jason then set
\[
\sqrt{a^2 - b^2 + 2abi - 4c - 4di} = p + qi
\]
for \(p\) and \(q\) that are real numbers and reflected his previous observation that, to obtain one real root, \(q = b\) or \(q = -b\) (in both case, \(q^2 = b^2\)). By squaring both sides of (eqn. 1) he obtained
\[
p^2 - q^2 + 2pqi = a^2 - b^2 - 4c + i(2ab - 4d)
\]
Because \(q^2 = b^2\), he simplified (eqn. 2) into
\[
p^2 + 2pqi = a^2 - 4c + i(2ab - 4d)
\]
Now by comparing the real and imaginary parts of both sides of (eqn. 3), he obtained
\[
p^2 = a^2 - 4c
\]
and
\[
2pq = 2ab - 4d
\]
At this point, Jason solved (eqn. 5) for \(p\) to get \(p = a - \frac{2d}{b}\) (assuming the case of \(q = b^{27}\)), in order to isolate \(p\). Squaring both sides of \(p = a - \frac{2d}{b}\) he obtained \(p^2 = a^2 +\)

\[\text{-------------------}\]

\[27\] Jason did not check the case of \(q = -b\) but the case also gives the same result. However, there was no indication that Jason was aware that they would produce the same result.
\[
\frac{4d^2}{b^2} - \frac{4ad}{b} \quad \text{which he used to substitute for } p^2 \text{ in (eqn. 4) to find the relationship between } a, b, c, \text{ and } d. \text{ He finally obtained } a^2 + \frac{4d^2}{b^2} - \frac{4ad}{b} = a^2 - 4c \text{ and, after simplification, } d^2 - adb = -cb^2, \text{ thereby successfully producing the desired result in the problem.}
\]

Whereas Andy and Sam expressed difficulty with dealing with the quadratic formula applied in \( \mathbb{C}[x] \) (as discussed in section E2), Jason immediately noted in his application of the quadratic formula, \[-\frac{a-bi\pm\sqrt{a^2-b^2+2abi-4c-4d}}{2}, \text{ where a source might be for nonreal components and, to further his reasoning, separated the expression,} \]

\[
\sqrt{a^2 - b^2 + 2abi - 4c - 4d}i, \text{ into the sum of the real and imaginary parts. Even though Jason indicated } \mathbb{C}[x] \text{ was a less familiar context for him, he seemed to have little (if any) difficulty utilizing the quadratic formula the new context and coordinating all necessary components. In particular, Jason’s immediate recognition that} \]

\[
\sqrt{a^2 - b^2 + 2abi - 4c - 4d}i \text{ is a complex number with real and imaginary parts (in which both parts may be nonzero) was a marked contrast with other participants’ assumption that the square root of an expression would result in either a real or imaginary number. Jason’s effective transfer of a known construct, quadratic formula, to a less familiar context } \mathbb{C}[x] \text{ seemed to have benefited from Jason’s insight into or familiarity with how complex numbers behave under certain transformations (such as taking the square root). Also, his prior understanding of complex numbers seemed to help him account for how the change of context from the familiar, } \mathbb{R}[x] \text{ or its subset, to the unfamiliar, } \mathbb{C}[x], \text{ might affect some essential features of the quadratic formula (e.g., the discriminant determining the types of roots, the square root part being the only source of} \]
an imaginary number). As demonstrated in Jason’s work, awareness of various relevant contexts for a construct and ability to account for how the choice of different contexts might affect the way the construct is applied are two key features of trans-object level [E3]. In what follows, Andy’s and Jason’s approaches to PRE-Q1, CORE-Q3, and POST-Q1 exemplify how various contexts for factorization may be explicitly recognized and accounted for.

In PRE-Q1, Andy and Jason began their responses by pointing out that their answers would be different depending on the context of factorization. Andy, for example, stated that “Are we factoring into real numbers or nonreal numbers?” The interviewer prompted him to consider both cases and Andy explained his reasoning as follows: “With real number, we have to know $a$, $b$, and $c$… So, to have factoring with real numbers and to have real values of $x$, we would have to know if $a$, $b$, and $c$ follow that $\sqrt{b^2 - 4ac}$ is a positive or a negative number.” Similarly, Jason also noted his answer to PRE-Q1 would be different depending on the context (over reals versus over complex numbers) as follows:

**Over reals, no** because you could have a quadratic where its roots are complex and then you’re not going to have any roots to factor into... But if we do it **over the complex numbers**, then we know that there must exist

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28 **PRE-Q1**: Can you factor $ax^2 + bx + c$ without knowing the specific values of $a$, $b$, and $c$? If so, how can you do so? If not, why do you think so?
two roots. So we can factor it into its roots \( x \) minus say \( z_1 \), \( x \) minus \( z_2 \).

But those roots might be of some really ugly form or something like that.

Both Andy and Jason seemed to be sensitive to the context allowed (or assumed) in their responses and were able to explain why it is not always possible to factor \( ax^2 + bx + c \) over reals and, in contrast, why it is always possible to do so over complex numbers.

This tendency to consider the context in which they situate or apply factorization was also observed in their responses to \textbf{CORE-Q3}\textsuperscript{29}. For example, Andy was applying his definition of complete factorization to polynomials in \textbf{CORE-Q3}. Although his initial definition of complete factorization did not specify the types of coefficients (or the boundary of contexts), Andy almost immediately noted, when trying to factor \( P_1(x) = x^4 - 12x^2 + 36 \), not specifying the coefficients may be an issue in applying the definition to the given polynomials. He stated “my definition doesn’t mention whether they have to be real or rational or irrational numbers… So my definition isn’t exactly complete, I guess.” Andy then decided to factor all expressions completely over \( \mathbb{C} \) and also pointed out intermediate answers as complete factorization over other domains, which suggests his awareness of relevant contexts for considering factorization.

\textsuperscript{29} \textbf{CORE-Q3:} According to your definition, please factor completely the following polynomials.
\[
\begin{align*}
P_1(x) &= x^4 - 12x^2 + 36 \\
P_2(x) &= x^5 - x^2 \\
P_3(x) &= 4x + 16x^3 \\
P_4(x) &= 12 \\
P_5(x) &= 4x^2 + 2x - 20 \\
P_6(x) &= 2x^3 - 10x^2 + 8x
\end{align*}
\]
Jason also commented on how his complete factorization of \( P_2(x) = x^5 - x^2 \) might differ depending on the coefficients. First, he noted that “this [referring to his factorization, \( P_2(x) = x^5 - x^2 = x^2(x-1) (x^2 + x + 1) \) is completely factorized over the reals” and also “I’d say throughout most of high school, this is where people would stop. But it’s possible to go further.” Jason’s comment on the usual practice of factorization in high school implies that for him, the distinction between two contexts (\( \mathbb{R}[x] \) and \( \mathbb{C}[x] \)) had to do with the boundary between high school mathematics and the mathematics beyond. Recognizing both possibilities explicitly, he provided answers for each context: \( x^2(x-1) (x^2 + x + 1) \) as complete factorization over the reals and \( x^2 \cdot (x - 1) \cdot (x - \frac{-1+i\sqrt{3}}{2})(x - \frac{-1-i\sqrt{3}}{2}) \) as complete factorization over the complex numbers. Such sensitivity to possible contexts in consideration of factorization illustrated in the preceding cases of Andy and Jason is in contrast with Sam’s response to CORE-Q3, in which he implicitly assumed integer coefficients with no verbal or written indication (details of Sam’s case was discussed in section E1).

Jason’s and Andy’s responses to POST-Q1 (which is the same question as PRE-Q1) indicated even more detailed consideration of contexts (that is, \( \mathbb{Z}[x] \), \( \mathbb{Q}[x] \), \( \mathbb{R}[x] \), and \( \mathbb{C}[x] \)) and explanation for under which condition factorization is possible in each of the contexts. For example, Jason in POST-Q1 responded “assuming we’re working over a most general set, so like over \( \mathbb{C} \), we can always factor it” using the roots obtained from the quadratic formula. He continued by stating “This works over the other sets as well over \( \mathbb{R}, \mathbb{Q}, \) and \( \mathbb{Z} \) as long as all of these sort of numbers [referring to \( \frac{-b+\sqrt{b^2-4ac}}{2a} \) and
\[
\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\] are in those sets. So like if this square root turns out to be an imaginary number then it only works over \( \mathbb{C} \), but if it’s real then this works over \( \mathbb{R} \) and if everything is rational it works over \( \mathbb{Q} \) and that kind of thing.”

Illustrations of Jason’s and Andy’s work in the current section represent trans-object level understandings [E3]. Their illustrations reflect their sensitivity to relevant (to the given problem) contexts for a construct under consideration (either factorization or quadratic formula). Furthermore, Jason and Andy explicitly accounted for how different contexts impact the way the construct may be utilized. Such a comprehensive view of how components all coordinate when a schema is activated is an indicator of the most advanced level in the extending process [E3]. In the next section, I build on the explanations of intra-object, inter-object, and trans-object levels provided in the preceding three sections to describe how participants extend their boundaries of the contexts in their consideration of factorization or quadratic formula.

**Extending observed in the course of interviews**

The current section focuses on exploring the second research question with respect to extending—what transformative transitions, especially extending, do university students make in the course of interviews? Overall, participants seemed to be familiar with the four number systems, \( \mathbb{Z}, \mathbb{Q}, \mathbb{R} \) and \( \mathbb{C} \) when it comes to consideration of roots of a polynomial equation; however, it did not seem natural for them to consider and apply the construct of factorization or quadratic formula within a context such as \( \mathbb{R}[x] \) or \( \mathbb{C}[x] \). On the contrary, it seemed significantly challenging.
In what follows, a diagram is presented in Figure 4.5 to provide an overview of how participants’ understandings of factorization or quadratic formula were advanced in (and distributed among) the three levels of E1, E2, and E3. The diagram suggests prevalent levels within each group of interviews (among PRE-TI, TIs, and POST-TI), shows a trend in or trajectory of each participant’s understandings in the course of interviews. The diagram also serves as a supplement to the narrative in the current section. Detailed directions for interpreting the diagram is presented in a list of bullet points following Figure 4.5.

Figure 4.5. Participants’ understandings of factorization or quadratic formula with respect to extending

- As shown in the small box on the right of the diagram, six participants are assigned different types of bullets distinguished by shapes (circle and square) and colors (black, white, and gray). The vertical axis indicates different levels in the extending category. The horizontal axis indicates in which group of interviews (among PRE-TI, TIs, and POST-TI) the observation occurred.
• A bullet on the graph with coordinates (X, Y) represents an observation that the specified individual demonstrated his/her understanding(s) at level Y in the interview group X. For example, a black, circular bullet plotted at (PRE-TI, E3) indicates that Jason demonstrated his trans-object level understandings in the extending category during his PRE-TI interview.

• The existence of a bullet on the graph means the specified individual’s understandings at the corresponding level was observed at least once during the corresponding interview(s); this diagram does not provide information about the frequency of observations but is used as an indicator of growth in level.

• The vertical arrow represents a transformative transition between levels in a relatively short time frame (e.g., within a one-hour interview). For example, during Post TI, the arrow connecting the white squares reflects the fact that Lucy exhibited her understandings of factorization and the quadratic formula at both the intra-object and the inter-object levels during that interview. No arrow was used in the diagram to represent other transformative transitions that occurred over a lengthier period (e.g., across PRE-TI, TIs, and POST-TI).

• One caveat in reading this diagram is that, for each individual, more than one bullet may coexist within a group of interviews (among PRE-TI, TIs, and POST-TI) without being connected by vertical arrows. That is, the coexistence of different levels did not always mean a transformative transition, because the levels may have occurred for different mathematical constructs. Witnessing an individual’s understanding of the same mathematical entity at different levels seemed natural, because the interview questions were intended to shed light on
participants’ understandings of factorization and polynomial equations from multiple angles. A wide range of interview questions seemed to have allowed observing various parts of one’s understanding of, say, factorization being activated and utilized at different levels. For example, Sam’s data were coded at both E1_Factorization (CORE-Q3) and E2_Factorization (CORE-Q4) during TIs, both of which were discussed in sections E1 and E2. In the diagram, this coding was reflected in two black square dots plotted at both E1 and E2 during TIs; but there is no vertical arrow between them. No arrow was used because there was no significant evidence that suggested a transformative transition when comparing the two analysis units.

Now that the rules of how to read the diagram have been stated, it may be meaningful to draw some key information from the diagram in Figure 4.5.

First, with respect to the prevalent levels within each group of interviews, it is noteworthy that five of the six participants (except for Helen) demonstrated understandings at E2 or E3 during POST-TI interviews, whereas only two participants (Jason and Andy) did so during PRE-TI interviews. This general trend of shifting towards a higher level might indicate that, as a result of the teaching interviews and of participating in the pre-TI interview, participants became aware of increasingly various contexts to consider when they make a certain conclusion or explain a statement related to factorization or the quadratic formula in the given problems.

Second, it is possible to trace the trajectory of each participant’s understandings within the extending category using the diagram in Figure 4.5. For example, Jason’s data were coded at E3 throughout all three groups of interviews, meaning Jason maintained
trans-object level in the extending category throughout the interviews. For Sam, the position of each black square dot on the diagram tells that his data were coded at E1 during PRE-TI; E1 and E2 during TI; and E2 during POST-TI. The gradual change from E1 to E2 over time suggests a possibility of transformative transition in Sam’s understanding of factorization.

Detailed illustration of Sam’s data that evidences the growth from E1 to E2 is provided in the current section as an example of a transformative transition. In addition to Sam’s responses, Calvin’s and Lucy’s responses also suggested their boundaries of contexts for factorization or quadratic formula became broader than the boundaries they had previously assumed (as indicated by the vertical arrows in the diagram). Hence, the current section provides the analysis of Sam’s, Calvin’s, and Lucy’s understandings as descriptions of transformative transition in the extending category.

As discussed in section E1, Sam initially demonstrated a tendency to assume implicitly a particular and localized context of $\mathbb{Z}[x]$ in consideration of factorization in his responses to PRE-Q1, PRE-Q7, and CORE-Q3. During TIs, Sam continued to coordinate various constructs (such as irrational numbers and irrational expressions in $x$) involved in consideration of factorization within contexts of $\mathbb{R}[x]$ or $\mathbb{C}[x]$ (discussed in detail in section E2). During POST-TI interviews, Sam’s responses suggested his sensitivity to possible contexts for factorization relevant to the given question, but he still expressed difficulty with considering factorization within $\mathbb{C}[x]$, which indicates his inter-object level understandings [E2].
For example, in POST-Q7\textsuperscript{30} Sam responded, “it would depend on what domain we are in. So there are some quadratic polynomials that can be factored in the set of integers. There are also some that cannot, if you are staying in the domain of the integers... So to me that question would depend on the domain.” Sam’s response reflects his recognition of various contexts in terms of which this question may be considered, which is in contrast with his responses to PRE-Q1 and PRE-Q7 where he implicitly assumed the context to be polynomials with integer coefficients.

However, Sam still found it difficult to actually factor $ax^2 + bx + c$ over $\mathbb{C}$. In POST-Q1\textsuperscript{31}, he explored how he could factor $ax^2 + bx + c$ when it has nonreal roots as follows: “$b^2 - 4ac$ is negative, that means $4ac$ is greater than $b^2$. Um... I don’t think it got me anywhere... I am trying to think of a way without knowing these values [$a$, $b$, and $c$]. I can’t necessary see how to factor it without knowing $a$, $b$, and $c$.” Overall, Sam seemed to be able to determine whether a quadratic expression can be factored or not depending on a given context but not be able to actually factor the expression $ax^2 + bx + c$ without knowing the values of $a$, $b$, and $c$. Hence, the boundary of contexts was extended in his consideration of factorization but what he could do within the broader boundary seemed to be still limited or not fully coordinated [suggesting the growth from E1 to E2, but not to E3].

Extension of boundaries of available contexts was also evidenced in Calvin’s and Lucy’s reasoning in responses to questions in POST-TI interviews. As discussed in

\textsuperscript{30} POST-Q7: Can you factor any quadratic polynomial?
\textsuperscript{31} POST-Q1: Can you factor $ax^2 + bx + c$ without knowing the specific values of $a$, $b$, and $c$? If so, how can you do so? If not, why do you think so?
section E1, roots occurring in a conjugate relationship by the quadratic formula (e.g.,
$1 + \sqrt{2}$ and $1 - \sqrt{2}$, $\frac{1}{2} + \frac{i\sqrt{2}}{2}$ and $\frac{1}{2} - \frac{i\sqrt{2}}{2}$) seemed to be strongly established in some of the participants’ thinking. Calvin and Lucy, for example, responded in PRE-Q2 (iv) that a quadratic equation having one real and one nonreal root is impossible. To be specific, they said “This can never hold true” and “I don’t believe that you could do that” respectively (see section E1 for more details). However, in their answers to POST-Q2 (iv)$^{32}$ and POST-Q4$^{33}$, which did not restrict the types of coefficients, Calvin and Lucy discussed other possibilities and demonstrated awareness of a broader context and how it might affect the conjugate pair idea they originally had.

For example, in POST-Q2 (iv), Calvin first noted “I don’t believe (iv) is possible because complex roots come in pairs so to speak. Intuitively that was my gut instinct.” Almost immediately after his initial response, Calvin corrected his thought by saying,

Oh, no I guess (iv) is possible if you consider one real root—we’re just going to call it negative one. And then we’re just taking $i$ for a nonreal root. Then you get $(x + 1)(x - i)$ if we’re letting our coefficients be in any number system. So in $\mathbb{C}$ this makes sense but in $\mathbb{R}$ it doesn’t. In $\mathbb{Q}$ and $\mathbb{Z}$ it doesn’t.

$^{32}$ POST-Q2 (iv): Find (if possible) $a$, $b$, and $c$ such that $ax^2 + bx + c = 0$ has one real root and one nonreal root.

$^{33}$ POST-Q4: If one of the roots of a quadratic equation is nonreal, is the other nonreal or real? Why is it so?
When the interviewer followed up by asking the reason for the change he made in his answer, Calvin explained as follows:

The conjugates... This kind of relates to this example that we have in class a ton [it was not specified which class he was referring to], and I guess in class we’re working strictly... a, b, and c have to be reals.
When you have [pointing to the example he came up with, $(x + 1)(x - i)$] one real root and one nonreal root you kind of end up with $b$ and $c$ not being reals. But I don’t see why this isn’t a valid definition of a polynomial in $\mathbb{C}$ if you just combine the terms.

As Calvin mentioned, quadratic equations that he had dealt with in class were exclusively within the context of $\mathbb{R}[x]$ and it could have initially led him to conclude that “complex numbers come in pairs,” which is mathematically correct in that particular context. In comparison with Calvin’s response to PRE-Q2, there are two differences: (a) Whereas Calvin in PRE-Q2 did not specify (or even mention) the types of coefficients he assumed for his claim “this can never hold true”, in POST-Q2 he made it explicit that the case of having one real and one nonreal root is impossible in $\mathbb{R}[x]$, $\mathbb{Q}[x]$, and $\mathbb{Z}[x]$, thereby suggesting awareness of (or sensitivity to) the context in terms of the construct in consideration (in this case, the quadratic formula), and (b) Calvin was able to produce the desired result in the given problem by considering a broader context, $\mathbb{C}[x]$, noting the conjugate pair idea may not be valid in the realm of polynomials with complex
coefficients. The change in Calvin’s responses suggests he was able to think of the same problem in a less familiar context for him, albeit not immediately. In his schema activated at that moment, at least four different contexts, \( \mathbb{Z}[x] \), \( \mathbb{Q}[x] \), \( \mathbb{R}[x] \), and \( \mathbb{C}[x] \), were included in the boundary within which Calvin considered the quadratic formula, suggesting the extension of the boundary of contexts compared to his PRE-TI interview.

In the same question, POST-Q2 (iv), Lucy also provided an answer different from her initial response to PRE-Q2 (iv). After reading the question, Lucy responded as follows after long pauses: “One real root and one nonreal root. So. Mmm... [pause for 11 seconds] I don’t know. [pause for 8 seconds] I guess this is possible but you’re going to have a nonreal equation here.” She then provided an example, \((x + 2)(x + 2i)\). When the interviewer asked to explain the reason for pauses in the beginning, Lucy explained she had to “step back” for the following reason:

Usually, you don’t see quadratics written like this [referring to the expanded form of her example \(x^2 + 2x + 2ix + 4i\)] so I wasn’t sure if it was acceptable for me to rewrite this [referring to \((x + 2)(x + 2i)\)] as this [referring to \(x^2 + 2x + 2ix + 4i\)]. So that’s why I took a step back because usually if you have an imaginary root you usually have a double imaginary so it gets rid of the \(i\)s. What I’m used to seeing is always \(i\) squared go to \(-1\) and you’re done with it, not it’s written in quadratics like this [pointing to the expanded form of her example \(x^2 + 2x + 2ix + 4i\)]. That’s why I was taken aback by it.
The cancellation of $i$s under addition and multiplication that occurs in the set of conjugate pairs seemed to be what Lucy was accustomed to seeing in quadratic equations. The fact that her example, $(x + 2)(x + 2i) = 0$, had roots that are not in a conjugate relationship probably made her question her example because it was different from what she was accustomed to seeing. As mentioned in section E1, in PRE-Q2 (iv) Lucy used the quadratic formula to explain why the roots had to occur in a conjugate pair without explanation on any condition. The change in Lucy’s responses from PRE-Q2 (iv) to POST-Q2 (iv) suggests a coordination process, characteristic of the inter-object level of understanding. That is, Lucy began to notice how the idea of conjugate pairs derived from the quadratic formula within a familiar context may not directly transfer to an unfamiliar situation such as in a “nonreal equation” (her words).

In a related problem, POST-Q4, Lucy’s response suggested she had not fully coordinated the ideas brought in for examining the case of having one real and one nonreal root [E2]. In POST-Q4, Lucy answered if one of the roots of a quadratic equation is nonreal, the other root could be either real or nonreal referring to the example she had created in POST-Q2 (iv); Lucy’s example was $(x + 2)(x + 2i)$. Then the interviewer asked Lucy to explain how the quadratic formula giving a conjugate pair could be applied to the case of having one real and one nonreal root. Lucy commented, even though it is easy to devise an example of a quadratic equation using roots of given

\[ \text{POST-Q4: If one of the roots of a quadratic equation is nonreal, is the other nonreal or real? Why is it so?} \]
types she was “not sure how it [the case of having one real and one nonreal root] works with the quadratic formula” and noted “I am not getting how this comes into play. I guess it will just be really ugly numbers if I do this [the quadratic formula].” Although Lucy could not fully explain how the quadratic formula may produce nonconjugate pairs within \( \mathbb{C}[x] \), the change from thinking the quadratic formula is intrinsically impossible to give one real and one nonreal root to the belief that obtaining a conjugate pair depends on the types of equations (that is, equations with real coefficients versus those with nonreal coefficients) evidences the boundary of contexts was extended for Lucy and she came to be aware of a larger context in which the quadratic formula may be considered.

It is worth noting that the preceding episodes of Sam, Calvin, and Lucy also suggest participants’ prior understandings with respect to the different number systems including \( \mathbb{R} \) and \( \mathbb{C} \) may not necessarily result in awareness of derived contexts such as \( \mathbb{R}[x] \), and \( \mathbb{C}[x] \) that need to be considered when applying factorization or the quadratic formula and thinking about their properties. In fact, when selecting participants, one of the prescreening questions asked to determine to which number system(s) a set of various complex numbers each belong. In the question, all six participants demonstrated they were aware of different number systems and able to distinguish among them. Participants’ ability to recognize various number systems contributed to their exploration of irreducibility over different domains and construction of an abstract definition of complete factorization over a generic domain \( D \) in the course of interviews. Such activities with which participants were engaged during TIs seemed to help them to think about the concept familiar to them (such as factorization and quadratic formula) within a
less familiar context (such as \( \mathbb{F}[x] \) and \( \mathbb{C}[x] \)), which meant, for some participants, thinking “outside the box.” A detailed discussion of the interview context as it relates to participants’ transformative transitions in the extending category is provided in Chapter 5, under section A sequence of tasks in the order of defining-applying-describing on page 290.

**Unifying**

In this section, I elaborate the process of *unifying* using the data collected and analyze participants’ understandings with respect to three levels of unifying process through the lens of Piaget and Garcia’s triad—intra-object, inter-object, and trans-object levels. As defined in chapter 2, one can reorganize his/her existing understanding of mathematics by *unifying* more than one construct (that were previously unrelated and disparate in the subject’s mind) under a certain overarching mathematical notion in the learning of a new construct. One’s schema development through unifying can be described as a growth in one’s understanding from intra-object, to inter-object, and to trans-object level.

The context of the interview, which was centered on the unique factorization theorem, allowed me to observe how participants’ understandings related to factorization of integers and factorization of polynomials are unified through the intra-, inter-, and trans-object level or remain unchanged at a certain level after the teaching interviews [TIs]. During the interviews (including all TIs and PRE- and POST-TI interviews), I asked several questions about factorization of expressions and relationship between
integers and polynomials and asked them to draw concept maps that involve constructs such as polynomial, factoring, integers, and prime factorization. In the following paragraphs, I explain the unifying process using these constructs to provide backgrounds for the interpretation and analysis of my participants’ responses.

From a knowledgeable outsider’s perspective, a decomposition aspect of factorization can be seen as an overarching notion underlying many different mathematical procedures, such as factoring a whole number, factoring a polynomial, and factorization of a matrix which underlies Gaussian elimination (also known as LU-factorization, see Allen (2003 p. 201)). However, a learner might have disconnected conceptions on factorization of integers and factorization of polynomials and work with them in an entirely different mindset without recognizing an overarching notion underlying them. Such a tendency to work with them in isolation from each other indicates intra-object level understanding of factorization. The empirical data that represent intra-object level will be provided in detail in the subsection entitled Intra-object level understanding in the unifying category [U1] on page 193.

As the learner begins to coordinate and find similarities between the two factorizations (and, even further, among decompositions of other objects such as functions under composition and matrices under multiplication35), s/he comes to

35 Decomposition of an object can be considered as a process of re-presenting the object as a combination* of more elementary objects within a structure, which reveals essential characteristics of the given object (*the combination method would be determined by the operation used for the target characteristic.) Decomposition in this way underlies factorizations of polynomials, factorizations of integers, function decomposition (e.g., for applying the chain rule), and transforming a matrix into the row-echelon form.
understand that some aspects of several constructs can be explained by the same, overarching idea, but it may be still difficult to coordinate all relevant constructs related to factorization in order to produce the desired results in the problem given. This intermediate step of starting to see some similarities but not being able to coordinate them fully indicates inter-object level understanding of factorization. The empirical data that represent inter-object level will be provided in detail in the subsection entitled Inter-object level understanding in the unifying category [U2] on page 199.

Finally, at the trans-object level, when the learner attempts to compare sets of polynomials and integers or to describe his/her meaning of factorization, s/he demonstrates a coherent schema of his/her constructs related to factorization by building his/her explanations on specific overarching ideas which underlie factorization of integers and factorization of polynomials. There exist several overarching ideas that serve as connectors of the two constructs: decomposition, reducible, irreducible, unit, uniqueness of complete factorization, and so on. For example, if one can understand how the idea of unit underlies the factorization in general, s/he can explain how and why the factorization of 12 in different polynomial rings, \( \mathbb{Z}[x] \) and \( \mathbb{Q}[x] \), looks different. In \( \mathbb{Z}[x] \), 12 is not a unit (because it does not have its multiplication inverse) and it can be represented as a product of other irreducible elements, namely, \( 2 \cdot 2 \cdot 3 \). In \( \mathbb{Q}[x] \), 12 is a unit (because its multiplicative inverse, \( \frac{1}{12} \), exists in the set) and that means there are infinitely many ways to represent it as a product of other elements in the set. Hence, it makes sense to define the factorization of 12 in \( \mathbb{Q}[x] \) to be itself, 12. The empirical data that represent trans-object level will be provided in detail in the subsection entitled Trans-object level
understanding in the unifying category [U3] on page 209. In the following subsections, I illustrate in detail data-informed view on intra-object, inter-object, and trans-object levels, respectively, and then describe unifying as a growth in one’s understanding from intra-object, to inter-object, and to trans-object level.

**Intra-object level understandings in the unifying category [U1]**

In the PRE-TI interview questions that were designed to measure participants’ level of understandings with respect to the three levels of unifying process, common phenomena were observed in their mathematical responses: the difficulty of identifying similarities between two highly analogous structures (specifically, the set of integers and the set of polynomials), and the tendency to consider factorization of integers and factorization of polynomials in isolation from each other. In PRE-Q8 participants were asked to interpret and make sense of a statement in the Common Core State Standards, “polynomials form a system analogous to the integers.” In their attempts to relate polynomials and integers, the majority of participants (Sam, Lucy, Calvin, Andy, and Helen) indicated that it made little sense to them that polynomials and integers form analogous systems. (See their statements in the following paragraph.)

- “I’m not really sure what that means. I’m not sure how to interpret that standard.” (Sam INT2)

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36 PRE-Q8: According to the Common Core State Standards high school students need to *Understand that polynomials form a system analogous to the integers*. In what sense do you think they are analogous?
• “I mean… I guess with just like, $x^2$, you know you’re going to have two roots. And like $x^3$, you know at max you’re going to have three roots. So whatever the highest degree is the highest amount of roots you could possibly have. Maybe that’s what they’re trying to say. I don’t know. I’m just trying to figure out what this means.” (Lucy INT1)

• “In what sense are they analogous? Integers, just relating it to the power, the exponent has to be an integer in order to be a polynomial - Your coefficients can really be anything. Your roots can be anything depending on your coefficients, your powers.” (Calvin INT2)

• “I don’t know how that would be analogous to integers in that sense because you would have a rational coefficient. You would have rational coefficients and you’d have rational roots with that, so I don’t know how that relates to integers because it doesn’t have much to do with integers in the piece. I don’t know in what way they’re trying to show that polynomials are analogous to integers. . . . I mean there’s the degrees of polynomials are always integers, so maybe in that sense they’re similar because every polynomial has form $a_0 + a_1x^1 + \cdots + a_nx^n$. Every polynomial has that form by definition of a polynomial where the $n$’s are all integers. You can’t have an $x$ raised to the negative- they’re all strictly positive integers also.” (Andy INT1)

• “In my opinion I don’t think it’s true though. . . . In this case I think they limit it to integer but I think polynomials they can be anything. . . . [High school students] should know that in the future when they learn it’s going
to be more complicated than integer such as like rational or irrational. It
still can form polynomials. It can’t be the **power** but the **number in front**
can something not integer [sic].” (Helen INT2)

Even though these five participants knew the meanings of polynomial and integer, and showed their ability to efficiently operate factorization on them separately (as evidenced by their responses to other interview questions), they did not seem to have constructed an overarching structure to relate the two sets or have reflected on what aspects of the two sets can be considered similar or analogous. Instead, expressing some level of uncertainty in their answers, they identified aspects of a polynomial (e.g., coefficient, exponent, degree, roots, number of roots) in which integers might appear. These surface-level connections (e.g., connections between numeric inscriptions) between the set of integers and the set of polynomials suggest they have a tendency to focus on the set of integers and the set of polynomials in isolation from each other (in other words, intra-object level understanding).

A similar pattern was observed in their responses to **PRE-Q9** in which participants were asked to draw a concept map of their own ideas about polynomials in the context of school mathematics. Once they completed the initial construction of a concept map, they were asked to include in their concept map additional ideas such as factoring, integers, and prime factorization (if initially not included). Their responses revealed some participants were unable to relate prime factorization with factoring of a

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**PRE-Q9**: Please construct a concept map of YOUR ideas about polynomial in the context of school mathematics. (A full version of this question can be found in Appendix C.)
polynomial. A dominantly held view among participants seemed to be that prime factorization applies to a number whereas factoring applies to a polynomial so it does not make sense to connect prime factorization with polynomials.

For example, after Andy constructed his concept map of polynomial as shown in Figure 4.6, he was asked to include additional ideas including prime factorization. I asked “is there any connection in your mind between polynomial and prime factorization?” Andy responded “Polynomial and prime factorization. [pause] I don’t think so. There might be. I don’t know if I’m familiar with that or not. Are there polynomials related to prime factorization? Not that I’m aware of. I don’t know any connection between those. At least I don’t remember there being any connection between those.” It seemed that Andy tried to find a direct relation between particular polynomials and prime factorization, not considering the decomposition process underlying prime factorization can be applied to polynomials as well.
Even when participants attempted to connect prime factorization to their concept maps, some attempt remained unsatisfactory to the participant. For example, when Sam was asked to include “prime factorization” in his concept map, he said “prime factorization, that’s definitely not something that I’d thought about. I guess that can kind of relate to factoring, because of that $c$ value.” (The $c$ value here refers to a constant term in a quadratic polynomial.) Then he connected the two—factoring and prime factorization—in his concept map (see Figure 4.7) and elaborated on how he might possibly connect them: “Factoring, if we have something like $x^2 + bx + c$, to factor this you have to look at your $c$ value and find two factors of $c$ that multiply to get $c$ and add to get to $b$. There was one specific case where if $c = 6$ and $b = 5$. I think that would work. Then $x^2 + 5x + 6$. The prime factorization of 6 is 2 and 3, and that would also be the
values that add to 5 that you are looking for. You’d have $x + 2$ and $x + 3$. That would be the factored form $x^2 + 5x + 6$. In that case, the prime factorization kind of came into play, but I don’t think that would be the case for every scenario.”

Figure 4.7. A portion of Sam’s concept map of polynomial (PRE-TI)

As stated in his last comment, Sam correctly pointed out that the connection he has made worked for only limited cases. It is still interesting that he connected the two ideas in a way that prime factorization becomes a part of the steps that need to be performed in factorization of a polynomial. While he was factoring the polynomial $x^2 + 5x + 6$ into two irreducible factors, it did not seem to occur to him that the factorization is essentially the same as the prime factorization in that both arrive at the product of irreducible factors (that no longer have nontrivial factorization). It seemed that Sam considered factoring integers and factoring polynomials as separate and unrelated entities, which suggests intra-object level understanding.

In POST-TI interviews, some participants commented on the way that they had previously thought about factoring. Sam mentioned “I had only ever thought of factoring
in terms of polynomials and not in terms of numbers.” Lucy, in her explanation of similarities between integers and polynomials, stated “They both can be factored, like how we did the example 12 with the 2 times 2 times 3. Same thing with $x^2 - 1$. Both of them can factor, but sometimes people, including myself, don’t look at 12 and say oh it can factor. I think that’s why they built that bridge, or they don’t build the bridge rather, to numbers versus polynomials because you just see a number and you’re like oh that’s fine. It actually can be completely factored as well as polynomials can be.” Such conception of factoring that is restricted to the set of polynomials and somewhat compartmentalized seems to have closed the door on recognizing an overarching notion underlying both factorization of integers and factorization of polynomials during PRE-TI interviews. While absent in the above cases, the recognition of this overarching notion (partially or fully) is indicative of the next levels in the unifying process. In the following section, I discuss the second level in the unifying process, which is inter-object level understanding.

**Inter-object level understandings in the unifying category [U2]**

In the episodes of this section, participants try to coordinate factorization of different expressions (e.g., integers and nonconstant polynomials) and explain an underlying phenomenon across different cases. These episodes are also characterized by the subject’s partially successful attempts in explaining factorization across different expressions consistently and in using an overarching idea for that explanation. This kind
of coordination lacking holistic explanation or fully developed articulation of an overarching idea is considered an indicator of the inter-object level.

During TIs, participants were asked to define complete factorization so that the definition consistently applies to and explains complete factorization of polynomials over various domains. In their coordination, participants readily realized that complete factorization of an expression is characterized by irreducibility (their verbalization might vary; they used phrases like “no-longer factorable” or “cannot be factored further”). This observation made by participants indicates that some aspect of the complete factorization was understood as what underlies different examples of complete factorization and an overarching notion of irreducibility was used to explain those examples globally. However, most participants were not able to coordinate other aspects fully, especially the role of unit across different factorizations.

For example, Andy struggled to explain how his generic description of complete factorization might apply to case of units. Initially, he stated “a polynomial is irreducible over \( D \) if it cannot be factored into other polynomials which are still contained in \( D \) [meaning the coefficients of the polynomials are elements in \( D \)]” and used it to define complete factorization as follows: “a polynomial is completely factorized when all of the factors are irreducible or units.” According to this definition, it was still unclear whether, for example, \( 2x + 1 \) should be considered irreducible over \( \mathbb{Q} \) because \( 2x + 1 \) can be factored further into \( 2(x + \frac{1}{2}) \). Andy was asked to determine how he would go about this sort of case.
In formal mathematics, units are considered neither reducible nor irreducible. Then an irreducible element can be defined as an element \( f \) whose only factorization over a given domain \( D \) is trivial factorization, which would result in \( u \cdot \frac{f}{u} \) where \( u \) is a unit in \( D \). Then \( 2x + 1 \) is considered irreducible over \( \mathbb{Q} \), thus precluding redundant and trivial factorization such as \( 2(x + \frac{1}{2}) \) from disqualifying it as complete factorization.

To Andy, however, the unit idea seemed to be more of a barrier than a help. Andy struggled with the question of whether units are reducible or irreducible and how units are related to being reducible and irreducible in polynomials. He claimed the unit idea “makes defining if something is reducible or irreducible kind of difficult.” Coordinating the unit idea (which was newly introduced to him during the TI) with his previously constructed ideas of reducible and irreducible in order to explain factorizations consistently seemed to be (for him) only a superficial matter of how to write the form (rather than a mathematical matter of finding a consistent overarching explanation for all cases). When asked to determine how to view \( 2x + 1 \) (i.e., whether it is irreducible or not), Andy responded as follows:

I don’t know which one I would want to use, whether I want to keep it like this [this refers to \( 2x + 1 \)] and say \( 2x + 1 \) is irreducible, or be able to say once we factor out whatever makes the coefficient 1 in front of the \( x \), that this [this refers to \( x + \frac{1}{2} \)] is now the irreducible form of it. I don’t know which one is the better way to do it, because it’s the same thing both ways,
kind of, just written in a different form. It’s just pretty much just semantics of what you want to define as [irreducible].

He continued by deciding, if it was over $\mathbb{Q}$, he would factor $2x + 1$ further to get $2(x + \frac{1}{2})$ and, if it was over $\mathbb{Z}$, he would stop at $2x + 1$. Andy did not provide any particular reasoning behind this choice, but the choice seemed to indicate his use of mathematical understandings that were available to him in that moment; that is, the role of unit in defining irreducibility or complete factorization across different cases was not accessible to him.

Table 4.1 (reproduced from his written work) reflects his choice of how to treat the construct of unit and shows Andy’s complete factorization of each of the polynomials in the first row over different domains. His initial responses to some of them had evolved as he refined his definition of complete factorization and chose how to handle units. What is presented in the table is his final answer. As presented in the table, he made the leading coefficients 1 for all the factors in complete factorization except for $P_3(x)$ and $P_5(x)$ over $\mathbb{Z}$.

Table 4.1. Andy’s final answer to **CORE-Q4** reflecting his own definition of complete factorization

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**CORE-Q4:** Please factor completely the same set of polynomials over each of the number systems, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ and record your results in the table. How would you describe what happened in this table in a general way? Or, how can you make some general statement about your answers in this table?
In the end of his work on CORE-Q4, Andy was asked to provide a comprehensive definition of complete factorization over the four domains ($\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$) and Andy defined it to be factorization that satisfies the following three conditions.\(^{39}\)

a) In complete factorization, each polynomial factor cannot be factored further without leaving the domain. For example, $x^2 - 6$ cannot be factored further over $\mathbb{Z}$.

b) In complete factorization, the highest degree term in each polynomial factor has coefficient 1. (*And this applies to the domain $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ only, not $\mathbb{Z}$)

c) In complete factorization, an integer polynomial is factored by its prime factorization.

The interviewer, at that point, prompted by asking Andy to consider what if the leading coefficient of a given polynomial is noninteger (e.g., a fraction or an imaginary number).

To accommodate these cases, Andy included additional conditions as follows:

d) In complete factorization, a fraction is written in simplest form.

\(^{39}\) The descriptions are syntheses of Andy’s written and verbalized answers.
e) In complete factorization, a complex number has to be written in the form $1 + bi$.

The very fact that Andy had to add new conditions/descriptions such as (d) and (e) upon each observation of new cases in order to define complete factorization indicates only partially successful coordination in his factorization schema across different cases (i.e., his actions were indicative of inter-object level understandings). The concept of unit could have been useful in reaching the level of full-scale coordination and explaining complete factorization of different cases consistently, but Andy did not find the unit idea helpful in defining complete factorization or irreducible element.

In fact, understanding how the unit idea is related to reducible and irreducible ideas and how it comes into play in the larger context of complete factorization seemed to be challenging for all participants. Only one of the six participants, Jason, eventually coordinated factorization across different cases using those abstract ideas of unit, irreducible, and reducible that transcend specific domains of coefficients. In what follows, I present Jason’s in-progress coordination with respect to the unit idea in order to focus on characterizing inter-object level, U2. In the next section on U3, I also present Jason’s subsequent fully developed coordination.

Jason, after his initial responses to the chart in CORE-Q4, defined a reducible polynomial as follows: A polynomial is considered reducible if it can be factored into two polynomials of lesser degree. He then defined an irreducible polynomial as a polynomial that is not reducible. (At this point, Jason was not yet introduced to the notion of unit.) The interviewer then posed a question: “According to your definition, is it fair to say that when you multiply two irreducibles, you get reducible?” Jason rephrased it and
wrote it down as a sentence “the product of two irreducible polynomials is a reducible polynomial.” He seemed to be confident in saying the statement is true.

To probe his thinking, the interviewer pointed to his response to the factorization of $P_3(x) = 4x + 16x^3$ over $\mathbb{Q}$ (which was $4x \cdot (1 + 4x^2)$) and asked “which part [of $4x \cdot (1 + 4x^2)$] is irreducible according to your definition?” This question appeared to prompt Jason to identify a problem with saying the statement is true. The conversation continued as follows:

Jason: Basically what I’m thinking right now is that... with regards to this statement, we generally consider constants to be polynomials so like 4 is a polynomial, right? And then we have like, ... $x$ is a polynomial. So then we have $4x$ and that’s a polynomial. And according to this [this refers to the statement “the product of two irreducible polynomials is a reducible polynomial”] we would consider that [that refers to $4x$] to be reducible... But at the same time you could also write $16x$ over 4 so this sort of makes me think that there is a bit more detail we have to go into here in regards to the constants.

I: Mm hmm. So how do you go about it then?

Jason: Um... I guess there are two approaches, one of which is probably better than the other. The worst one would be basically not worrying about constants as polynomials because they don’t, you know, impact the overall structure that much.

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40 In the transcript, “I” refers to the interviewer. In subsequent excerpts, three un-spaced dots (…) were used for a short pause or an unfinished sentence.
But probably the better way to do it would be to ask that, if we have our polynomial as a product of other polynomials. Then we ask the, the leading coefficient of each one is 1... So if we request that the leading term of each factor is 1, then... that \([that \ refers \ to \ 4x \cdot (1 + 4x^2) \ over \ \mathbb{Q}]\) would be then \(16x \cdot (x^2 + \frac{1}{4})\) and that would, I guess, be that factorization there.

Up to this point, Jason’s approach to dealing with a leading coefficient of a given polynomial that is not 1 does not seem to differ from Andy’s and he was, like Andy, \textit{trying to coordinate} different components of complete factorization as consistently and comprehensively as possible, suggesting inter-object level understandings. Jason’s remark of “there is a bit more detail we have to go into here” indicates he recognized a need for more careful definition of reducible/irreducible polynomial or for some conditions on the statement itself (i.e., the product of two irreducible polynomials is a reducible polynomial). Soon after, Jason was introduced to the notion of unit in \textbf{Q4-F3-(c)}.

\textbf{(Q4-F3-(c))} Mathematicians call the element such as 12 in \(\mathbb{Q}\) a \textit{unit}. That is, an element in a domain (such as \(\mathbb{Z}, \mathbb{Q}, \mathbb{R}\) and \(\mathbb{C}\)) that has its multiplicative inverse in that domain is called \textbf{UNIT}.

Jason’s first response after reading the definition was “Ooh. That’s interesting. I like that definition.” In his explanation to why he liked the new (to him) definition we can observe
in-progress coordination of how this new idea of unit is taking its place in his schema of factorization. He explained,

Because basically one of the problems we were running into earlier was like the fact that $2^2 \cdot 3$ equals $2^3 \cdot 3 \cdot \frac{1}{2}$. Two and $\frac{1}{2}$ are multiplicative inverses so really we could have written times 1— which doesn’t change anything if you break it into those [those refers to $2^3 \cdot 3 \cdot \frac{1}{2}$]. But I think what this [this refers to the definition] is basically saying is that you kind of ignore whole things that have their multiplicative inverses in there so you can’t really do this trick to it [“this trick” refers to rewriting $2^2 \cdot 3$ into $2^3 \cdot 3 \cdot \frac{1}{2}$].

It seemed that Jason found the unit idea to be a help in solving the problem he had previously noticed in dealing with a constant multiplied in front of a polynomial (or the leading coefficient). Jason continued to explore whether units are considered reducible, irreducible, or neither and, through observation of a few examples (such as $1 \cdot 1 = 1$, $(-1)\cdot (-1) = 1$), remarked “a constant sort of messes that up a little bit. How would you rationalize not thinking about that?” Eventually, he came up with a reasonable solution in saying “perhaps instead of saying units are irreducible we could say that multiplication by a unit does not impact reducibility, perhaps?” Here, the word “perhaps” indicates uncertainty from his perspective at that point and indicates that the coordination...
process was still on-going. He further examined his logic by revisiting the definition of unit as follows:

Um let’s see here. I guess going back to sort of what the units meant. That means that they have multiplicative inverses in the set. So what that means is we can write, so say we have a generic unit, say we represent that as \( u \). And then we have its inverse, which is also in the set. And then we can write any number \( a \) as \( a u u^{-1} \). So there’s no point in saying that \( a \) is reducible if we can write it as \( a u u^{-1} \) because we can do that with any number.

This led him to finally conclude “

**multiplying by units doesn’t change reducibility**

and, that way, address the trouble caused by consideration of a unit-multiple of an irreducible polynomial. Hence, we may consider he had completed the coordination process. It is noticeable that, during the coordination process, Jason tried to explain this new way of factoring a polynomial while maintaining an alignment with his own definitions. That is, he developed a way to see how ideas of reducible, irreducible, and complete factorization all fit together by using the new (to him) concept of unit rather than ignoring those cases. By doing so, Jason have approached the more advanced level in the unifying process, trans-object level understanding, which I discuss in the following section.
Trans-object level understandings in the unifying category [U3]

Continuing with Jason’s case, we can see how the coordination process of a factorization schema with an emerging overarching idea (such as unit) at the inter-object level resulted in the subject’s full-scale articulation of factorization across cases with the overarching idea. By taking what he had concluded in the previous interview (i.e., conclusion that “multiplying by units doesn’t change reducibility”), Jason first revised his definition of an irreducible polynomial into “a polynomial that cannot be written as the product of other polynomials (units not considered for ‘other polynomials’).” He also provided examples according to his definition, explaining \((x - 4)\) is irreducible over \(\mathbb{Z}\), but \(2(x - 4)\) is not irreducible because it can be written as product of other polynomials—2 and \((x - 4)\)—and neither is a unit in \(\mathbb{Z}\). Meanwhile, over \(\mathbb{Q}\), \(2(x - 4)\) is considered irreducible, and \(2(x - 4)^2\) is reducible because it can be written as a product of other polynomials that are not units in \(\mathbb{Q}\).

He then defined complete factorization of a polynomial over a domain \(D\) as “a factorization into polynomials, all in \(D\), such that each factor is irreducible.” He was asked to take the refined definition and explain some of the complete factorization he produced in CORE-Q4 table. He picked \(P_3(x) = 4x + 16x^3\) and \(P_4(x) = 12\) and successfully explained how his generic definition of complete factorization over a domain \(D\) applies to all cases of factorization across \(\mathbb{Z}, \mathbb{Q}, \mathbb{R}\) and \(\mathbb{C}\) (indicating trans-object level) as shown below:
For here \( P_3(x) = 4x + 16x^3 \) over \( \mathbb{R} \), we have it as the product of \( 4x \) and \( 1 + 4x^2 \) and so up to multiplication by units, you know, like 4 is a unit so these are still basically irreducible polynomials over \( \mathbb{R} \). Um but then, of course, over \( \mathbb{C} \) we can factor even more. And specifically, I want to go back to this example \( P_4(x) = 12 \). Um, so then over \( \mathbb{C} \), 12 is a unit so you sort of leave it as 12 because it’s just, it’s just a unit. Um, you can’t really reduce it in a nontrivial way. Um same thing with over \( \mathbb{R} \) and \( \mathbb{Q} \). But with this definition now over \( \mathbb{Z} \), because 12 is not a unit in \( \mathbb{Z} \), you would have to write \( 2^2 \cdot 3 \). Um those are the irreducibles over \( \mathbb{Z} \).

After all, Jason was the only participant who had a different complete factorization of 12 over \( \mathbb{Z} \), \( \mathbb{Q} \), \( \mathbb{R} \) and \( \mathbb{C} \) based on his abstract definition of complete factorization (i.e., \( 2^2 \cdot 3 \) over \( \mathbb{Z} \) but 12 over \( \mathbb{Q} \), \( \mathbb{R} \) and \( \mathbb{C} \)). All other participants had \( 2^2 \cdot 3 \) over all \( \mathbb{Z} \), \( \mathbb{Q} \), \( \mathbb{R} \) and \( \mathbb{C} \). This may indicate the unit idea was not fully coordinated in the factorization schemas of all participants except Jason.

Also of note is that Jason’s definition of complete factorization still applies to the mechanism of prime factorization with integers, if we consider prime integers as irreducible and composite integers as reducible. Jason apparently seemed to appreciate the similarity (or the essential sameness) between the two systems of integers and
polynomials. In POST-Q8, in order to explain how polynomials form a system analogous to the integers, he juxtaposed the two systems in a chart (see Figure 4.8) stating “they are analogous in the sense that um you can sort of work with them in terms of factoring... we have sort of reducible and irreducible for both polynomials and integers.”

As shown in Figure 4.8, he listed three types of elements in each system (i.e., Reducible, Irreducible, and Unit) and wrote down examples of each type. He also juxtaposed the unit idea in both systems, having 1 as an example of units in \( \mathbb{Z} \), and continued by pointing out “here [“here” refers to the cell in the cross-section of unit and polynomials] our units are if you’re over, say \( \mathbb{C} \), then it’s any nonzero constant multiple. So the \( a \) in front usually.”

Thus, he seemed to have grasped how different types of elements are related under

\[\begin{array}{|c|c|c|}
\hline
\text{Polynomials} & \text{Integers} \\
\hline
\text{Reducible} & x^2 - 1 & 15 \\
\hline
\text{Irreducible} & (x-1)(x+1) & 3.5 \\
\hline
\text{Unit} & \alpha & 1 \\
\hline
\end{array}\]

Figure 4.8. Jason’s work on POST-Q8

\[4^1\text{ POST-Q8: According to the Common Core State Standards high school students need to Understand that polynomials form a system analogous to the integers. In what sense do you think they are analogous?}\]
factorization regardless of the systems and how these relationships can be explained by those overarching notions, such as reducible, irreducible, and unit.

While Jason appeared to be open to the concept of unit as way to advance his schema, the unit concept is not necessary to obtain trans-object level of understanding. In what follows, I present an example of trans-object level in which the participant did not involve a unit concept in his explanations. A unit concept may not be necessary to obtain trans-object level as long as the concept does not come into play in reaching a successful coordination status under the given problem situation. If mathematical ideas that are activated in one’s schema at the moment are coordinated enough to successfully address the problem and provide a coherent and overarching explanation that transcends different cases, the subject’s understanding may be viewed as trans-object level. For example, Jason in PRE-Q8\(^42\) (which took place before all TIs) successfully reasoned that integers and polynomials are analogous systems, without using the unit idea. Jason was the only participant who presented an adequate answer to PRE-Q8. Other participants’ responses to the same question were described in section U1.

To be specific, in PRE-Q8, potentially influenced by the fact that some of the preceding questions focused on factoring during the PRE-TI interview, Jason pointed out “[the Common Core] is probably talking about the factoring.” He continued his reasoning as follows.

\(^42\) Note this question is same as POST-Q8, which is “According to the Common Core State Standards high school students need to Understand that polynomials form a system analogous to the integers. In what sense do you think they are analogous?”
So, starting with integers, um. We have a **nice theorem** that says that we can factor an integer into powers of primes. That’s \( p_1^{n_1} p_2^{n_2} \) etc., to like \( p_k^{n_k} \). And we can do that for any integer in a unique way. Um, **similarly for polynomials**, we can say that \( a_n x^n + a_{n-1} x^{n-1} \) all the way down to \( a_1 x^1 + a_0 \). And that equals \( x \) minus, uh, what do I think I want to use for root? Say \( r_0 \). Actually this is a degree and so it’s going to have . . . 

\((x - r_1) \ldots \) to \((x - r_n)\). So they’re sort of similar.

Here Jason demonstrated how the decomposition idea underlies both systems of integers and polynomials, which successfully addressed the problem given. His reference to “a nice theorem” seems to suggest that he probably was reasoning about the fundamental theorem of arithmetic and, later on, even the fundamental theorem of algebra. Although the objects in the two systems might look different (e.g., expressions with a variable \( x \) versus numbers only), Jason did not appear to orient to the appearance of the systems and was more focused on the similarities in the mechanism of factorization and its role in revealing similarities between the two systems of integers and polynomials. Other participants have also demonstrated trans-object level [U3] and those examples are developed in the next section entitled **Unifying observed in the course of interviews**.

Up to this point, I presented a data-informed view of intra-object, inter-object, and trans-object levels, respectively, by capturing relevant moments during the participants’ PRE-TI, TI, and POST-TI interviews. In the next section, I finally describe the act of unifying as advancement in levels, through U1, U2, and U3.
Unifying observed in the course of interviews

Now that individual levels in the unifying process have been developed, an analysis of the data can be investigated with regard to the research question: What transformative transitions, especially unifying, do university students make? That is, what reorganizational acts of unifying are observed in the course of interviews? In general, participants had difficulty in reaching trans-object level when the context of the problem involved (or had to involve) the notion of unit. In other problem contexts, most participants were successful at explaining factorization across different cases using overarching ideas such as decomposition, reducible, and irreducible during and after TIs.

The diagram presented in Figure 4.9 provides an overview of how participants’ understandings of factorization were advanced in (and distributed among) the three levels of U1, U2, and, U3. Detailed directions for interpreting the diagram can be found in a list of bullet points in the section Extending observed in the course of interviews on page 179.

![Diagram](image)

Figure 4.9. Participants’ understandings of factorization with respect to unifying
As illustrated in the diagram, there seems to be a general pattern of shifting towards a higher level—specifically, five of the six participants (except for Helen) demonstrated understandings at U2 or U3 during POST-TI interviews, whereas only two participants (Jason and Calvin) did so during PRE-TI interviews. This trend of shifting from U1 to U3 implies that participants might have come to view previously disparate ideas (e.g., factorization of integers and factorization of polynomials) as instantiations of an overarching idea (e.g., decomposition) as a result of engaging in the teaching interviews and, perhaps, by participating in the pre-TI interview.

Tracing the levels in the diagram for each individual also shows some possible transformative transitions. For example, Sam’s data were represented by the points at (U1, PRE-TI), (U2, TI), and (U3, POST-TI), suggesting a transition from U1 to U2 and to U3 over time. Also, the vertical arrows—one from (U2, TI) to (U3, TI) for Jason, the other from (U1, POST-TI) to (U3, POST-TI) for Andy—indicate transformative transitions which took place in a relatively short time frame (e.g., within one interview, or even within one interview question). The current section provides descriptions of transformative transition in the unifying category by building on Jason’s, Andy’s, and Sam’s cases.

Although, at a glance, the unifying process might be viewed as a linear process from U1 to U2 to U3, the data revealed that the process is not necessarily linear. Jason’s understanding of factorization, for example, shifted from U3 to U2, then back to U3. Each of these instances was described in detail in previous sections; however, presenting them in sequence can help contextualize the unifying process. A brief summary follows:
(i) Jason’s understanding of factorization began at U3: In PRE-Q8, Jason demonstrated his unified understanding of factorization as decomposition, which he applied to the set of integers and the set of polynomials, in order to explain how polynomials form a system analogous to integers. (For more details, refer to section U3.)

(ii) Jason’s understanding of factorization shifted back to U2: During TIs, in CORE-Q4 and its follow-up questions, Jason had to deal with polynomials with a leading coefficient \( \neq 1 \) and found glitches in his definition of irreducible and in thinking about the statement that he assumed to be true (i.e., the product of two irreducible polynomials is a reducible polynomial). He attempted to resolve this issue but was not successful at fully coordinating them until he figured out the idea of unit plays a crucial role in avoiding trivial factorization. (For more details, refer to section U2.)

(iii) Jason’s understanding of factorization shifted to U3: During TIs and in POST-Q8, he concluded “multiplying by units doesn’t change reducibility” and observed if a polynomial cannot be reduced in a nontrivial way, it needs to be considered irreducible. This allowed him to explain factorization across different expressions in a unified and coherent way. (For more details, refer to section U3.)

Piaget’s constructivist approach to a learner’s schema development explains this phenomenon of stepping-back-and-forward as a growth in one’s understanding. It is possible when a learner encounters a new experience, the subject struggles to assimilate the new experience and accommodate the existing schema to reach equilibrium status. It seemed that, for Jason, explaining factorization of polynomials with a leading coefficient...
≠ 1 and using the concepts of reducible or irreducible in that explanation was an unfamiliar experience and, through this experience, he further developed his existing schema of factorization with a new component of unit concept. This kind of schema development is a characteristic of the unifying process because the additional component to the schema (i.e., unit idea) allowed him to use it as an overarching idea to see the sameness in complete factorization across different expressions. Hence, the shifts from U3 to U2 to U3 can be considered ultimately a unifying process.

Whereas Jason’s unifying process took almost one interview for him to get to a solid conclusion, some unifying processes seemed to happen immediately. For example, an immediate shift from U1 to U3 was observed in Andy’s response to \textbf{POST-Q8} (the Common Core question). The following excerpts show how Andy arrived at a conclusion that “polynomials are, it’s a system where they’re composed of smaller polynomials just like integers.”

Andy: We’ve talked about this before and I had no idea what that meant \textit{[this refers to the question in POST-Q8]}.

I: How about now?

Andy: How about now? Understand that polynomials form a system. \textit{[Andy reads the problem in a whisper]} It still doesn’t make sense to me.

I: Can we think back to what we did so far in our past five interviews?

Andy: Well we talked about factoring, what it means to be completely factored, and over certain domains how things change. So I guess polynomials form a system, and now it’s the integers in the sense that some polynomials can be factored,
whereas some integers can be factored. And some polynomials are composite
the same as some integers are composite. So you can have like prime polynomials
like you have prime integers. You have composite polynomials; you have
composite integers. So polynomials are—it’s a system where they’re composed
of smaller polynomials just like integers. There are numbers that are composed
of smaller integers. I guess in that sense—they’re analogous in that sense.

Andy’s explanation about the analogous nature between the set of integers and the
set of polynomials centered on overarching ideas such as decomposition, reducible, and
irreducible. With the ideas activated at that moment (i.e., polynomials, integers, factoring,
reducible, irreducible, etc.), Andy seemed to provide a fully developed articulation about
relationship among those ideas, especially about how they are analogous (suggesting
trans-object level). However, his initial response, “still doesn’t make sense”, indicates
that what came to his mind first was no association between them (suggesting intra-object
level).

As discussed in Chapter 2, one’s understanding, from an observer’s perspective,
is only inferable from his/her articulation of the insights into the given problem in the
moment. Hence, Andy’s understandings as inferred from his articulation in POST-Q8
does suggest a shift from U1 to U3. However, one might wonder to what extent such an
immediate shift from U1 to U3 that skips an intermediate level of U2 provides evidence
of a shift in his actual cognitive structure (which is not directly accessible). A possible
explanation is that the sudden shift from U1 to U3 in Andy’s responses could have been
the matter of recognizing the relevance between factorization and the problem context of
**POST-Q8**, rather than the function of an actual change in his cognitive structure relevant to factorization. Reflection on the past TIs (prompted by the interviewer) could have triggered him to bring in previously established relationships in his mind between factorization of integers and factorization of polynomials.

While Andy’s data only *tentatively* suggested a growth of his understanding from U1 to U3, other participants’ data suggested more convincing evidence that they came to develop a decomposition view in both factorization of integers and factorization of polynomials (as an overarching perspective). For example, the comparison of Sam’s responses to PRE/POST-Q8 and the data triangulation afforded by his retrospective statements strongly suggested a growth in his understandings. While Sam responded to PRE-Q8 that he is “not sure how to interpret that standard,” his response to POST-Q8 reflected his overarching understanding of factorization that applies to both the set of integers and the set of polynomials. In POST-Q8, he responded, “I think it is saying… that their students need to understand that the set of polynomials and the set of integers kind of carry similar rules.” He used examples of irreducible elements from each set, 5 and \( x + 4 \), to show how the rule of reducibility applies to both. He also explained how factorization similarly applies to both sets as follows: “whenever I see an expression and it gets factored, ... I am going to see now that it was split up into two or possibly more things that can be multiplied together to get back your original expression.” This

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43 PRE-Q8 (and POST-Q8): According to the Common Core State Standards high school students need to *Understand that polynomials form a system analogous to the integers*. In what sense do you think they are analogous?
seemingly basic decomposition view that might be helpful to obtain trans-object level, however, may not naturally develop in students.

Several retrospective statements made by Sam (as follows in the bullet points) illustrated an old way he had thought about factorization and suggested his previously held view that separates factorization of polynomials from factorization of integers.

- “I had only ever thought of factoring in terms of polynomials and not in terms of numbers”;
- “It was weird to think about [factoring a number and factoring a polynomial altogether] for the first time, I guess, but I’m actually seeing more similarities between the two now than differences, now that I think about it”;
- “Whenever I saw it like this [this refers to a problem, “How can you factor 60?”], the term ‘factor’ had a much different meaning to me than if I was given the quadratic equation.”

This indicates his understanding of factorization was at intra-object level originally. He explained the “difference” had resided in the ways the elements “look” in each system respectively: “whenever you’re trying to factor a number, you’re just looking for other numbers, really, that go into it. Whenever you have an equation or an expression like $x^2 + bx + c$, you’re looking for two other expressions that, when multiplied, go into the expression.”

It seems possible, in school mathematics, students come to activate different schemas of factorization depending on the appearance/inscriptions of problems that students are solving. The series of questions during the TIs influenced participants to take
a much deeper look into factoring and define it more carefully so that it explains all cases consistently and comprehensively. As such, the questioning seemed to have provided a chance for them to develop a coherent and unified approach to factoring in general. A detailed discussion of the interview context as it relates to participants’ transformative transitions in the unifying category is provided in Chapter 5, under section *An unusual or unexpected encounter with a problem beyond their routines or assumptions* on page 297.

**Strengthening**

The current section focuses on the process of strengthening and provides analysis of participants’ schemas of polynomial equation with respect to three levels of strengthening process. As defined in chapter 2, a learner can reorganize his/her existing understandings of mathematics by *strengthening the link between existing understandings of more than one mathematical notion* in the learning of a new construct. Furthermore, the strength of a link in one’s mind might vary (also explained in chapter 2). To reiterate, there are three levels of strength. First, two ideas might have no direct link or be weakly connected through a secondary link. Second, ideas might be linked in only one direction or, even if the ideas are linked bidirectionally, some part of the link might be loose or not fully coordinated. Third, ideas might be linked bidirectionally where all relevant components are fully coordinated in one’s mind. These three qualitatively different levels of the strength of a link can be seen as an adaptation of Piaget and Garcia’s triad—*intra-object, inter-object, and trans-object levels*—to this particular category of strengthening.
By the time a university student is ready to take upper level mathematics classes (e.g., an abstract algebra course), it is reasonable to assume that the student’s schema of *polynomial equation* might be full of various mathematical concepts, procedures, and relationships between them. For example, one’s schema of *polynomial equation* might be composed of one’s understanding of relationships between (1) a polynomial equation and its root(s), (2) a polynomial function and its zero(s), (3) a polynomial function and the \(x\)-intercept(s) of its graph, and (4) a quadratic equation and the quadratic formula (or the discriminant). Here the polynomial equation/function may take a standard form or a factored form. During the interviews of this study, a number of questions related to polynomial equations were asked not only to examine the strength of the links participants activate and utilize in the problem context, but also potentially to help them to strengthen the links between their existing understandings in the course of interviews.

To provide background for the interpretation and analysis of participants’ responses, I exemplify the strengthening process briefly using relationships between a generic polynomial equation over \(\mathbb{C}\) and its roots.

From a knowledgeable outsider’s perspective, the fact that \(a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = a_n (x - r_1) \cdots (x - r_n)\) might be sufficient to tell that the polynomial equation, \(a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0\), has a set of roots, \(\{r_1, r_2, \cdots, r_n\}\). Conversely, one might be able to say that, if an \(n\)-th degree polynomial equation has the set of roots, \(\{r_1, r_2, \cdots, r_n\}\), then the polynomial is of the form, \(a(x - r_1) \cdots (x - r_n)\) for some nonzero \(a\). Furthermore, one might be able to say that, if \(r\) is a zero of a polynomial
function, then \((x - r)\) is a factor of the polynomial. Such link between the polynomial (in either standard form or factored form) and its roots can be seen as bidirectional.

However, a learner might not necessarily see or operationalize the link immediately when it is appropriate to do so. For example, a student might not utilize the fact that \(x = a\) is a root of \(x^3 - a^3 = 0\) in order to think about why the factoring formula, \(x^3 - a^3 = (x - a)(x^2 + ax + a^2)\), makes sense. If his/her schema of polynomial equations activated in the moment does not bring in a link between relevant constructs when it is appropriate, and s/he demonstrates a tendency to focus on a construct in isolation from another relevant construct, the student’s understandings related to polynomial equations at that particular moment are considered to be at *intra-object level*. Even if the student had made the link between a polynomial equation and its roots previously in a different context, the fact that it is not brought up immediately and consistently would suggest weakness of the link. The empirical data that represent the intra-object level will be provided in detail in the subsection entitled *Intra-object level understanding in the strengthening category [S1]* on page 224.

The intermediate step of beginning (or being able) to relate and coordinate two or more constructs simultaneously but failing to coordinate some component is an indicator of the *inter-object level*. Especially, considering directionality of a link between two constructs, one might demonstrate a link that is only unidirectional (e.g., being able to find the roots given a quadratic polynomial, but not being able to find a quadratic polynomial given a pair of roots). It is also possible to demonstrate a bidirectional link but still fail to coordinate some component in making the link. Specific empirical data
that represent the inter-object level will be provided in detail in the subsection entitled *Inter-object level understanding in the strengthening category [S2]* on page 229.

Lastly, at the most advanced level, a student can operationalize a link between constructs in his/her schema of polynomial equation (as it is relevant and appropriate in order to solve the given problem) and be able to coordinate all necessary components. With the fully developed coordination and bidirectional connection, the relevant schema activated at the moment can be viewed coherently and the understanding is considered to be at the *trans-object level*. The empirical data that represent the trans-object level will be provided in detail in the subsection entitled *Trans-object level understanding in the strengthening category [S3]* on page 238. In what follows, I analyze participants’ understandings that belong to each of the three levels [S1, S2, S3] and describe the strengthening process using examples of growth in one’s understanding from S1 to S2, and from S2 to S3.

**Intra-object level understandings in the strengthening category [S1]**

Intra-object level understandings with respect to the link between a polynomial equation and its roots were rarely observed throughout the interviews and across participants. Since participants of the current study were mathematics-intensive majors, their cumulative mathematical experiences by the time of the interviews likely involved quadratic equations and their roots, if not polynomial equations of a higher degree and their roots. It is common practice for students in school algebra to take, for example, a quadratic equation and solve for the variable of the equation. By applying factorization,
factoring formulas, zero product property, or quadratic formula, students link a given polynomial equation to its roots.

Most participants, as expected, were able to readily utilize a link between roots and the corresponding polynomial equation in many cases. For example, in PRE-Q6, Helen immediately interpreted “a quadratic equation has two roots $s$ and $t$” as $(x - s)(x - t) = 0$. Interestingly, however, neither Sam nor Helen determined that the link would be relevant and useful in one particular case. In the following paragraphs, I focus on these cases of not utilizing the link when it is most appropriate to do so and present it as an example of intra-object level understanding.

These episodes came from participants’ responses to how to derive or make sense of the factorization of $P_2(x) = x^5 - x^2$. After factoring out $x^2$ and getting $x^2(x^3 - 1)$, every participant mentioned or implicitly referred to the formula for a difference of cubes. Their statements are presented below:

- “I’m trying to recall that formula.” (Sam)
- “I actually got stuck. I know there’s a way to do it, to split it up, but I forgot.” (Helen)
- “Difference of two cubes… I know there’s a way to do it… I don’t remember how to factor this.” (Andy)
- “I forgot how to factor these.” (Calvin)
- “It is probably something where you can break it into like $x - 1, x + 1$, something like that.” (Lucy)
- “Bring out the $x^2$ and the $x^3-1$. And that’s the difference of cubes.” (Jason)
After the recognition of the formula, they diverged in their approaches to factoring $x^3 - 1$. Some tried to recall the factoring formula and others derived the factorization using some relevant links. To give a short example of successfully utilizing a link, Andy figured out the factorization of $x^3 - 1$ by building on the relationship between the function $p(x) = x^3 - 1$ and the $x$-intercept of the graph of the function. He stated that “since we know 1 is a root [from the graph], we can factor this into $(x - 1)$ times something” and then continued to figure out the rest (i.e., $x^2 + x + 1$) using long-division of polynomials.

However, it did not occur to Sam and Helen to think of such a link between a polynomial function and its zeros in their factorization of $x^3 - 1$. To be specific, Sam derived its factorization using guess-and-check methods. He first guessed that the formula would be $(x - 1) (x^2 - x + 1)$, which he confirmed was not equivalent to $x^3 - 1$. He then guessed that the formula would be $(x - 1) (x^2 + x + 1)$ and checked that this factorization is equivalent to $x^3 - 1$. Because in both guesses he had $(x - 1)$ as its first factor, the interviewer probed his thinking by asking “how did you know to start with $(x - 1)$? Are they from recall?” Sam’s answer below seems to suggest he focused solely on the formula itself rather than bringing in relevant links to justify his choice of the first factor $(x - 1)$:

I will be honest. Yes, it was from recall. There’s, like, half complete guess and half recall. I don’t know. I just. Basically... I knew that ... the first factor had two terms in it and the second factor was going to have three terms. And I was pretty sure that, for difference of cubes, this was a
subtraction \(\text{[this refers to } (x-1)\text{]}\) and these were the two signs \(\text{[these refer to the plus signs in the following expression } (x^2 + x + 1)\text{]}\) that I was unsure about. But I knew for this term, \(\text{[this term refers to } x^2 \text{ within } (x^2 + x + 1)\text{]}\) you have to square your \(a\) value so in the general case you have \(a^3 - b^3\) then equals \((a-b)(a^2 + ab + b^2)\). So, the \(a\) value in this case is \(x\) and you square that to get \(x^2\), and again, I just guessed both of the sign \(\text{[both of the sign refers to the plus signs in the following expression } (x^2 + x + 1)\text{]}\) but then the second term, I knew would be \(x\) times one and that gave me \(x\). And then the third term would be one times one, one squared which is one.

Sam’s response suggests that his schema activated at that moment was more focused on what the forms of the formula were or how they looked like in his memory. Thus, he did not operationalize a link between polynomial function \(p(x) = x^3 - 1 = (x-1)(x^2 + x + 1)\) and a zero of it (i.e., \(x = 1\)), even if it was appropriate to address the question asked (which suggests intra-object level understanding [S1]).

Helen, who noted she “forgot” how to “split up” \((x^3-1)\), also managed to come up with an answer \((x-1)(x^2 + x + 1)\). She first started by dividing \((x^3-1)\) by \((x-1)\) and said “I’ll see if this works.” The division was completed with a remainder of 0 and she was confident with her factorization. At that point, the interviewer asked Helen how she knew to begin with \((x-1)\). The following excerpts show she might have been drilled
in the practice of dividing \((x^n - 1)\) by \((x - 1)\) but she had difficulty in explaining why it makes sense to do so.

I: I’m wondering how come you were able to pick \((x - 1)\) to begin with.

Helen: Because I remembered long ago, it was like in middle school, that if I see the term \(x\) [to the] something minus one, I will pick this \([this\ refers\ to\ (x - 1)]\) for the first one, and then to start with, and if it doesn’t work out, I will pick something else, but always start with something like this \([this\ refers\ to\ (x - 1)]\)... [Helen shows that \((x^5 - 1)\) is divisible by \((x - 1)\) as an example.]

I: Is there a way to make sense of why \((x - 1)\) works always?

Helen: I don’t know. Um. Honestly, I don’t know why. It just works. I was so little, so I can’t ask why. I will think of a reasonable reason and tell you next time.

Helen seemed to be following a rule—when you have \((x^n - 1)\), always divide it by \((x - 1)\)—but her saying “I don’t know why. It just works” suggests that she had not established a link between the polynomial \((x^n - 1)\) and one of its zeros, \(x = 1\), in her schema.

It is worth noting that Sam and Helen both were able to readily utilize the link between roots and polynomials when it comes to a quadratic equation in both directions (from a pair of numeric roots to a quadratic equation whose roots are the given pair; and also from a quadratic equation to its roots). It is possible that, for them, the link is evident only for quadratic expressions but not as evident for a polynomial of a higher degree.
Although they were able to figure out the factorization of $x^3 - 1 = (x - 1)(x^2 + x + 1)$ based on their vague recollection of a factoring formula/rule, focusing on the formula/rule itself might have suppressed an understanding that zeros are connected to the original polynomial function. This demonstrates that their understandings are at the intra-object level [S1]. While the current section (i.e., section S1) described episodes in which participants were not operationalizing a link between constructs related to polynomial equation when it is appropriate to do so, the following section (i.e., section S2) focuses on the instances in which participants attempted to utilize a relevant link but it was ineffective in producing the desired results because either the link was dominantly unidirectional or the link was partially but not fully coordinated.

**Inter-object level understandings in the strengthening category [S2]**

As mentioned in section S1, participants were able to readily utilize a link between a quadratic equation and its roots in most cases. However, in some cases, the link seemed to be only unidirectional (i.e., *from* a quadratic equation *to* its roots) and relating them in the opposite direction (i.e., *from* a set of roots *to* a quadratic equation with the given set of roots) seemed to be missing. This phenomenon is an indicator of inter-object level understanding.

For instance, when Calvin was asked to find a quadratic equation $ax^2 + bx + c = 0$ whose set of roots satisfy certain conditions in the problem given, he did not seem to be able to directly connect a pair of given roots to a polynomial equation. To be specific,
in **PRE-Q2** (i), Calvin was asked to find $a$, $b$, and $c$ values such that $ax^2 + bx + c = 0$ has one integer root and one rational root that is not integer. For a student who can utilize a direct link *from* roots *to* a polynomial equation, it would be possible to take any integer, say 1, and any rational number that is not integer, say $\frac{1}{2}$, and consider a quadratic equation whose roots are 1 and $\frac{1}{2}$, which can be $(x - 1)\left(x - \frac{1}{2}\right) = x^2 - \frac{3}{2}x + \frac{1}{2} = 0$ or any other constant-multiple of it such as $2x^2 - 3x + 1 = 0$.

However, for Calvin, this type of link was not readily available while he was considering **PRE-Q2**; instead, he used the link *from* an equation *to* its roots. Calvin chose to begin with the given quadratic equation, $ax^2 + bx + c = 0$, factored out $a$, and applied the quadratic formula for the equation $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$. Figure 4.10 shows Calvin’s approach to this problem. Then he noticed, for the roots to be an integer and a rational number, the expression underneath the square root had to be a perfect square (which he set equal to $s^2$ ) and finally concluded the roots are $\frac{-b + sa}{2a}$ and $\frac{-b - sa}{2a}$. However, he could not figure out which combination of $s$, $a$, and $b$ would give him one integer root and one rational root that is not integer.

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**PRE-Q2**: Find if possible $a$, $b$, and $c$ for each of the following cases:

(i) $ax^2 + bx + c = 0$ has one integer root and one rational root that is not integer.
In Calvin’s approach, it is notable that he was very proficient in operationalizing the link from the quadratic equation \((x^2 + \frac{b}{a}x + \frac{c}{a} = 0)\) to its roots \((\frac{-b + sa}{2a} \quad \text{and} \quad \frac{-b - sa}{2a})\) but, even when the link was not sufficient to address the problem, the opposite directional link was not activated in his schema. That is, it did not seem evident for him that he can begin with a set of specific numeric roots that satisfy the given condition and, as a result, obtain the values of \(a\), \(b\), and \(c\). It is also important to note that Calvin did not readily realize \( ax^2 + bx + c = 0 \) and \( x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \) have the same set of roots. Otherwise, he would have applied the quadratic formula to the former equation, not the latter. In fact, Calvin’s response in a later interview (INT 6) suggested that the fact that multiplication or division by a scalar does not change the set of roots of a given quadratic equation had not occurred to him until that interview. It appeared that he had to engage in a thought.
process to validate this fact for himself. Calvin’s thought process will be discussed in detail in the section entitled *Strengthening observed in the course of interviews* on page 242.

Sam, in **PRE-Q5**, also demonstrated a tendency to begin with a quadratic equation to obtain its roots even when a question gave a pair of roots and asked to find a quadratic equation whose roots are the given pair (see Figure 4.11 for the statement of the problem and Sam’s work on it).

![Figure 4.11. Sam’s answer to PRE-Q5 showing unidirectional link](image)

In order to find a quadratic equation whose roots are 2s and 2t, Sam first considered an unknown quadratic equation, $a'x^2 + b'x + c' = 0$, and substituted 2s in for $x$. After he wrote down the substituted result, $a'(2s)^2 + b'(2s) + c' = 0$, he attempted to factor the left-hand side of the equation but it was not successful (see Figure 4.11 for his attempt). At that point, he mentioned “I don’t like that route that I took anymore” and took another
direction, applying the quadratic formula to \( a'x^2 + b'x + c' = 0 \). His second approach using the quadratic formula was not successful either.

Sam’s attempts show he had to begin with setting up a quadratic equation, \( a'x^2 + b'x + c' = 0 \), to reason through the problem. Although he utilized the link from the root, \( 2s \), to the equation by writing \( a'(2s)^2 + b'(2s) + c' = 0 \), the link was not as strong as using both of the roots, \( 2s \) and \( 2t \), to obtain an equation such as \( (x - 2s)(x - 2t) = 0 \). Sam’s second approach suggests he was proficient in linking from an equation to its roots, that is, in one direction.

In both Calvin’s and Sam’s approaches described in the preceding paragraphs, it was evident that the link between a quadratic equation and its roots was dominantly unidirectional even when the opposite directional link was more appropriate (from the observer’s perspective) to produce the desired results (which exemplifies the inter-object level, S2). Both of them did not see that, given two roots \( \alpha \) and \( \beta \), a quadratic polynomial can be obtained using its factored form (i.e., \( a(x - \alpha)(x - \beta) \) for some constant \( a \)) in the problem contexts.

Interestingly, Sam and Calvin were able to utilize a bidirectional link between roots and a quadratic equation in the context of factoring a quadratic equation with numeric coefficients (i.e., not letter coefficients such as \( a, b, \) and \( c \)). However, the bidirectional link was not always fully coordinated. In what follows, I exemplify another type of the inter-object level, S2, by discussing how participants demonstrated a bidirectional link when they factored a quadratic expression, \( 1 + 4x^2 = 0 \), over \( \mathbb{C} \) and how their bidirectional link suggested a lack of full coordination.
A common phenomenon observed in many participants’ responses was that they appeared to believe that knowing a pair of roots of a quadratic equation would be sufficient to determine the original quadratic expression in its factored form uniquely. For example, four of the six participants were certain that the fact that the roots of $1 + 4x^2 = 0$ are $\pm \frac{1}{2}i$ was sufficient to (incorrectly) determine that the factorization of $1 + 4x^2$ is $(x - \frac{1}{2}i)(x + \frac{1}{2}i)$. That is, their responses exposed a taken-for-granted assumption that the leading coefficient has to be 1. The coordination of the leading coefficient’s scalar role (e.g., multiplication by 4 in $4\left(x - \frac{1}{2}i\right)\left(x + \frac{1}{2}i\right) = 1 + 4x^2$) was missing in the link they made between a set of roots and a polynomial equation.

The exposure of this assumption resulted from the questions during TIs. Specifically, participants were asked to factor $p_3(x) = 4x + 16x^3$ completely over $\mathbb{C}$ and, after factoring out 4x to get $4x(1 + 4x^2)$, four participants—that is, Sam, Lucy, Andy, and Calvin—solved the equation $1 + 4x^2 = 0$ for $x$ and provided $4x(x - \frac{1}{2}i)(x + \frac{1}{2}i)$ as their complete factorization of $p_3(x)$ over $\mathbb{C}$. In their solutions, it appears that they utilized a bidirectional link because they first began with an equation, $1 + 4x^2 = 0$, and obtained its roots, $\pm \frac{1}{2}i$, and then connected them in the opposite direction (i.e., began with the roots, $\pm \frac{1}{2}i$, and concluded that the factored form of the original expression is $(x - \frac{1}{2}i)(x + \frac{1}{2}i) = 0$). The second link from the roots to the equation is incomplete because of the lack of coordination of the role played by the leading coefficient with respect to factorization.
Three of the participants (Andy, Lucy, and Sam) did not notice the mismatch between $4x + 16x^3$ and their answers, which were $x + 4x^3$ when fully expanded. Calvin was the only individual who noticed the inconsistency between the intermediate factored form $4x(1 + 4x^2)$ and his answer $4x(x - \frac{1}{2}i)(x + \frac{1}{2}i)$. At this point, he said “I don’t think this works... I need $4x^2$ somewhere, and I completely dropped that.” Then Calvin began to reexamine which part of his reasoning caused the mismatch. Notably, his reexamination centered only on the first link (from the equation to the roots) and he did not seem to question the second link (from the roots to the equation).

Calvin tried three different methods to check if he had the correct set of roots of the equation, $1 + 4x^2 = 0$, first by taking the square root, second by applying the quadratic formula, and third by substituting one of the roots for $x$ in the original equation (see Figure 4.12 (a), (b), and (c), respectively).

![Figure 4.12](image)

Figure 4.12. Calvin’s approaches for obtaining or confirming roots of the equation, $1 + 4x^2 = 0$

Calvin registered confusion because when he expanded the supposedly correct factorization $(x - \frac{1}{2}i)(x + \frac{1}{2}i)$ (which he had confirmed using three different methods) it
did not return $1 + 4x^2$. After spending a considerable amount of time trying to coordinate the inconsistency, he did not figure out the scalar role played by the leading coefficient 4 and the fact that there could be infinitely many quadratic equations whose roots are $\pm \frac{1}{2}i$.

Instead, he concluded as follows:

My tentative conclusion would be... I don’t want to commit [to] that. It would be that this [“this” refers to $(x - \frac{1}{2}i)(x + \frac{1}{2}i)4x$] would be the complete factorization over complex [numbers]. And the way that I’m used to thinking about it with multiplying the roots together to get you back to your starting function—it’s only consistent over $\mathbb{R}$. That would be my general conclusion. [Calvin INT 3 32:06-]

Throughout Calvin’s effort to address the mismatch, the procedure of using roots (in this case, $\pm \frac{1}{2}i$) to determine the factored form of a given expression (in this case, $(x - \frac{1}{2}i)(x + \frac{1}{2}i)$ as factorization of $1 + 4x^2$) was never revisited or questioned by him.

Three other participants, Sam, Lucy, and Andy, also seemed to be confident in their answer (i.e., $p_3(x) = 4x + 16x^3 = 4x(x - \frac{1}{2}i)(x + \frac{1}{2}i)$) and think that having the roots (say, $\alpha$ and $\beta$) of a quadratic equation meant they had the factorization of $(x - \alpha)(x - \beta)$ rather than be open to the possibility of $a(x - \alpha)(x - \beta)$ for some $a$ that is not 1.
This incomplete link from roots to a quadratic equation was also evidenced by Helen’s reasoning in POST-Q6. Helen readily interpreted the phrase “a quadratic equation $ax^2 + bx + c = 0$, which has two roots $s$ and $t$” by setting up the following equation: $(x - s)(x - t) = ax^2 + bx + c$. She then expanded the left-hand side to get $x^2 - (s + t)x + st = ax^2 + bx + c$ and said “So $a$ will be 1, $(s + t)$ will be minus $b$, and $st$ [will be] equal to $c$.” The underlying assumption she had made appeared to be $(x - s)(x - t)$ is the quadratic equation which has two roots $s$ and $t$, not realizing there are infinitely many quadratic equations which has the same set of roots. Hence, even though Helen was able to utilize the link in both directions (from roots to an equation, from an equation to roots), the link cannot be considered fully coordinated.

It is possible that assuming the leading coefficient is 1 (consciously or subconsciously) had always worked for them, and as such their comfort with the routine of applying roots to factorization could have made them solidify this link from roots to a quadratic equation without a careful examination. Even after TIs in which they intensively dealt with polynomials with a leading coefficient $\neq 1$ and thought about the role of units in complete factorization, Helen, and Lucy maintained and exhibited their habit of assuming leading coefficient is 1 in their POST-TI interviews. Fortunately, Calvin noticed multiplying by “the magic 4” might solve the problem in the end of the same interview (INT 3 54:34) and later on, in INT 6, convinced himself that multiplying or dividing a polynomial equation by a scalar does not change the set of roots of the

\[ ax^2 - bx + c = 0 \]

\[ POST-Q6: \text{Suppose that you have a quadratic equation } ax^2 + bx + c = 0, \text{ which has two roots } s \text{ and } t. \text{ Find the roots of the following equation: } \]

\[ ax^2 - bx + c = 0 \]
polynomial equation. Sam also went through a similar coordination process when he noticed the difference between $p_5(x) = 4x^2 + 2x - 10$ and his factorization $2(x - 2)(x + \frac{5}{2})$. Calvin’s and Sam’s strengthening process from S2 to S3 will be discussed in detail in the section entitled *Strengthening observed in the course of interviews* on page 242.

The preceding illustrations from Calvin’s, Sam’s, and Helen’s work represent inter-object level understandings [S2] that are characterized by a dominantly unidirectional link or a bidirectional link that is not fully coordinated. The lack of coordination was evidenced by: the participants’ responses when they incorrectly assumed that a pair of roots *uniquely* determines a quadratic equation by setting the leading coefficient equal to 1; and the participants’ failure to see (perhaps because of the assumption) that multiplication or division by a scalar does not change the set of roots of a given quadratic equation. In what follows, I present an analysis of instances in which participants demonstrated fully developed coordination with respect to a bidirectional link between a quadratic equation and its roots.

**Trans-object level understandings in the strengthening category [S3]**

A student’s ability to utilize a link between constructs in his/her schema and to coordinate all necessary components as they are relevant and appropriate in the problem context is evidence that suggests trans-object level understandings [S3]. In light of this perspective, a student who understands the effect of an assumption (such as assuming the
leading coefficient is 1) when utilizing a link between a quadratic equation and its roots can be considered to have trans-object level understandings.

Although some participants’ actions across distinct tasks in the interviews revealed they often applied an assumption incorrectly when utilizing a link between a quadratic equation and its roots, Andy seemed to be mindful of an assumption (i.e., the leading coefficient is 1) in his work in POST-Q6. Andy was able to fully coordinate how the assumption is related to the link he was utilizing and explain why consideration of a general quadratic equation $ax^2 + bx + c = 0$ may be reduced/translated to consideration of $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$ in a given problem context. To be specific, in POST-Q6, Andy presented $(x - s)(x - t)$ as the case of when $a = 1$ when interpreting the phrase “a quadratic equation $ax^2 + bx + c = 0$, which has two roots $s$ and $t$.” Andy successfully completed this problem with this particular case assuming $a = 1$. The interviewer probed further by asking “would the answer be the same if $a$ is not 1?” He immediately answered “It think it will be the same” and explained the reason as follows:

Because… you could turn it [referring to $ax^2 + bx + c$] into, no matter what the $a$ is, into this form [referring to $(x - s)(x - t)$] just by factoring out the $a$… the $a$ is just kind of a multiple of the whole thing.

Because you have like $ax^2 + bx + c$, that’s the same thing as a times $x^2 + \frac{b}{a}x + \frac{c}{a}$ [writing down what is in Figure 4.13]. Because this

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46 POST-Q6: Suppose that you have a quadratic equation $ax^2 + bx + c = 0$, which has two roots $s$ and $t$. Find the roots of the following equation: $ax^2 - bx + c = 0$
equation\textsuperscript{47} [referring to $x^2 + \frac{b}{a}x + \frac{c}{a}$] has the same roots as this equation [referring to $ax^2 + bx + c$]. But the $a$ is a 1 [referring to the leading coefficient of $x^2 + \frac{b}{a}x + \frac{c}{a}$]. So I think the number of $a$ doesn’t really affect what happens... Because this [referring to $\frac{b}{a}$] could be your new $b$...

and this [referring to $\frac{c}{a}$] is your new $c$, and you factor it the same way

[“it” refers to $x^2 + \frac{b}{a}x + \frac{c}{a}$]... Now this whole thing [referring to $x^2 + \frac{b}{a}x + \frac{c}{a}$] is kind of multiplied by $a$ though. It still has the same roots. It’s just the factoring is a little bit different.

Figure 4.13. Written work accompanying Andy’s explanation of why assuming $a = 1$ is acceptable

Andy’s explanation implies that, even if $a$ is not equal to 1, after factoring $a$ out, he would get a new quadratic equation $x^2 + b’x + c’ = 0$ for which the same set of roots \{s, t\} applies, and so he would still be able to apply the same solution strategy. Notably, Andy’s being aware that $a = 1$ is one of all possible cases and being able to articulate why

\textsuperscript{47} Andy used the term equation to refer to the expression, $ax^2 + bx + c$, and also the expression, $x^2 + \frac{b}{a}x + \frac{c}{a}$.
assuming $a = 1$ is acceptable in the problem context suggests a qualitatively different
level of mathematical understanding when compared with other participants who
assumed $a = 1$ is the only case (which is a distinguishing feature of S3 from S2).

One participant, Jason, consistently demonstrated more advanced coordination of
the leading coefficient in his use of the link between a quadratic equation and its roots
across all interviews. For instance, in his work in POST-Q1\(^{48}\) and POST-Q5\(^{49}\), he
included and kept the leading coefficient $a$ throughout his solutions. In POST-Q1, he
responded if “working over a most general set, so like over $\mathbb{C}$” he could always factor a
given quadratic expression $ax^2 + bx + c$ into $a(x - \frac{-b+\sqrt{b^2-4ac}}{2a})(x - \frac{-b-\sqrt{b^2-4ac}}{2a})$.

Jason’s work suggested his immediate use of a bidirectional link between a polynomial
function and its zeros (in this case, $ax^2 + bx + c$ and $-\frac{b\pm\sqrt{b^2-4ac}}{2a}$) and translation of the
given polynomial function into its factored form using the zeros. Here it is important to
note Jason effectively addressed all possible cases by not making any assumption about
the leading coefficient $a$ (unlike what many other participants did). Such use of
bidirectional link with full-blown coordination of the leading coefficient $a$ throughout the
solution was also observed in Jason’s work in POST-Q5 as shown in Figure 4.14.

\(^{48}\) POST-Q1: Can you factor $ax^2 + bx + c$ without knowing the specific values of $a$, $b$, and $c$? If
so, how can you do so? If not, why do you think so?

\(^{49}\) POST-Q5: Suppose that you have a quadratic equation $ax^2 + bx + c = 0$, which has two roots
$s$ and $t$. Find another quadratic equation whose roots are $2s$ and $2t$. 
Building on the illustrations of each of the three levels of strengthening—that is, intra-object, inter-object, and trans-object levels—provided in the current and the preceding two sections, I discuss the process in which participants’ responses suggested changes in the levels during the course of interviews and present them as illustrations of strengthening.

**Strengthening observed in the course of interviews**

In the current section, I focus on how participants’ existing understandings in their schemas of polynomial equation changed in the course of interviews, exploring the second research question with respect to strengthening—What transformative transitions, especially strengthening, do university students make in the course of interviews? Overall, participants seemed able to readily utilize a link between a polynomial equation
and its roots bidirectionally when the polynomial equation is *quadratic, with numeric coefficients*, especially when the *leading coefficient is 1*. For example, all of the six participants immediately and correctly factored $x^2 + x + 1$ into $(x - \frac{-1 + \sqrt{3}i}{2})(x - \frac{-1 - \sqrt{3}i}{2})$ during their factorization of $p_2(x) = x^5 - x^2$ over $\mathbb{C}$. However, it turns out that the strength of the link between a polynomial and roots varied across participants when they encountered a polynomial with more features to coordinate, such as a polynomial with a leading coefficient not equal to 1 (e.g., $1 + 4x^2$ or $4x^2 + 2x - 10$), a polynomial with letter-coefficients (e.g., $ax^2 + bx + c$), and a polynomial with a higher degree (e.g., $x^3 - 1$). For some, like Jason, a link between a polynomial and roots stayed stable and effective even when they needed to coordinate more components. For others, the link they were accustomed to making did not transfer successfully when encountering unfamiliar situations, and so, for example, they had to reexamine the existing link to understand the effect of factoring out the leading coefficient $a$ on the roots of a given polynomial equation. The current section highlights such a coordination process using Calvin’s and Sam’s cases of strengthening from S2 to S3.

A diagram\(^{50}\) in Figure 4.15 supplements the overview provided in the preceding paragraph. The diagram indicates how participants’ understandings of polynomial equation were advanced in (and distributed among) the three levels of S1, S2, and S3.

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\(^{50}\) Detailed directions for interpreting the diagram can be found in a list of bullet points in the section *Extending observed in the course of interviews* on page 175.
First of all, when comparing the dots in POST-TI and PRE-TI, it is notable that four of the six participants demonstrated understandings at S3 during POST-TI interviews, whereas only one participant did so during PRE-TI interviews. This general pattern may be attributed to their exposure to and exploration of a variety of polynomial expressions during the teaching interviews. When looking into the trajectory of individual’s understandings, analysis of Sam’s data revealed a transformative transition from S2 to S3 during a teaching interview (as indicated by the vertical arrow in the diagram). Sam’s case is discussed in detail in the current section as an illustration of transformative transition in the strengthening category.

One additional direction for reading this diagram is the meaning of the zigzag arrow. The zig-zag arrow for Calvin’s shift from S2 to S3 indicates that the shift occurred over a lengthier period than one-hour single interview (to be specific, INT 3 around 32:06 \(\rightarrow\) INT 3 around 54:34 \(\rightarrow\) INT 6 around 51:15). Calvin’s several attempts which
eventually resulted in a transformative transition from S2 to S3 are illustrated first in the current section, before the discussion of Sam’s data.

As mentioned in section S2, Calvin seemed to think that, given an equation $1 + 4x^2 = 0$, having the roots of it (i.e., $\pm \frac{1}{2} i$) meant he had the factorization $1 + 4x^2 = \left(x - \frac{1}{2} i\right)\left(x + \frac{1}{2} i\right)$. To briefly summarize, Calvin, as soon as he noticed the mismatch between the original expression $1 + 4x^2$ and his factored form $\left(x - \frac{1}{2} i\right)\left(x + \frac{1}{2} i\right)$, attempted multiple methods to reexamine the first link he had utilized (i.e., the link starting from the equation and leading to the roots) but the opposite directional link (starting from the roots and leading to the equation) was never revisited. The conversation ended without coordination of the second link; Calvin concluded that things may behave differently over $\mathbb{C}$ than over $\mathbb{R}$ and that the difference could be the reason why the expanded form of $\left(x - \frac{1}{2} i\right)\left(x + \frac{1}{2} i\right)$ does not match with $1 + 4x^2$ [INT 3 32:06].

Later, in the same interview [INT 3 54:34], Calvin was given an opportunity to revisit his factorization of $p_3(x) = 4x + 16x^3$ over different domains, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$, to explain how his definition of complete factorization (see Figure 4.16) might apply to his factorization examples in the factorization table that appeared in CORE-Q4. Calvin originally answered $4x(1 + 4x^2)$ over $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ and $4x(x - \frac{1}{2} i)(x + \frac{1}{2} i)$ over $\mathbb{C}$ in his table.
Figure 4.16. Calvin’s working definition of complete factorization over \( \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \text{and} \mathbb{C} \)

The last part of his definition (“coefficient of \( x \) being 1 if able” in Figure 4.16) was meant to say that in his complete factorization the leading coefficient of each linear or quadratic factor had to be 1 if it was possible in the context of factorization. In order to follow this criteria, he revisited his factorization \( 4x(1 + 4x^2) \) over \( \mathbb{Q} \) and also over \( \mathbb{R} \) and changed his answer into \( 4x \cdot 4(\frac{1}{4} + x^2) \). He continued examining his answer \( 4x(x - \frac{1}{2}i)(x + \frac{1}{2}i) \) over \( \mathbb{C} \) and the conversation followed as in the excepts below.

Calvin: I think I like this one most, this one is fine [this one refers to \( 4x(x - \frac{1}{2}i)(x + \frac{1}{2}i) \) over \( \mathbb{C} \)] ... Oh wait, am I missing something? No, I think this is fine. I still don’t like this because I am used to just ... I don’t know. That’s weird because if we do it with this one we get that same power [this one still refers to \( 4x(x - \frac{1}{2}i)(x + \frac{1}{2}i) \) and “do it” probably refers to expanding]. **Something is wrong ...**

**We get the same power, but we don’t get the same coefficient** [referring to the original expression \( p_3(x) = 4x + 16x^3 \)].

I: So can you examine what you’ve done, if there is a way to make ...?
Calvin: Make them consistent?
I: Yeah.

Calvin: If we FOIL this [referring to an algorithm for expanding \((x - \frac{1}{2}i)(x + \frac{1}{2}i)\)] we get \(x^2 + \frac{1}{4}\), because you get minus \(\frac{1}{4}i^2\), so it’s \(x^2 + \frac{1}{4}\). I guess if we multiply that by four then we get our coefficients to be correct again. So I think I might multiply by a magic four, and I’m FOIL-ing it in my head and then you get \(x^2 + \frac{1}{4}\). If you multiply that by four you get \(4x^2 + 1\) and then when you multiply that \(\text{[referring to } 4x\text{]}\) you get \(16x^3 + 4x\). I am just going to ... I don’t know where I dropped that [that seems to refer to 4]. I might have dropped that somewhere along the line.

The fact that Calvin referred to the solution as “magic four” and did not recognize the “magic four” as the same as the leading coefficient he factored out over \(\mathbb{Q}\) and over \(\mathbb{R}\) suggests that coordination process was still not complete. In fact, his revisit to factorization of \(p_3(x) = 4x + 16x^3\) occurred again in a later interview [INT 6 51:15] in order to incorporate the concept of unit. His responses suggested that he was still not clear about the preservation of roots of a polynomial equation under the multiplication or division by a constant (which suggests that his understanding was at the inter-object level). In INT 6, Calvin was asked whether factoring out a unit from a complete factorization over \(\mathbb{Q}\) would be still accepted as another complete factorization and also along the same line, if both of \(4x(1 + 4x^2)\) and \(4x \cdot 4(\frac{1}{4} + x^2)\) are considered as
complete factorization of over $\mathbb{C}$. He then registerd a lack of support for this idea, saying “maybe you don’t want to divide out by a coefficient. Maybe that really screws things up.” Then he used an example trying to explain his reasoning (see Figure 4.17) and realized the result was different from his initial expectation: “If you have $2x + 1$, you have an $x$-intercept of ... **No wait, hold on. What?** You just divide out by 2 and you end up with $x + \frac{1}{2}$. **No, it still works!**”

![Figure 4.17. Calvin confirming a root is preserved when factoring out a constant](image)

After this observation, he explained “I was thinking, yeah, somehow these [drawing the arrow in Figure 4.17] have different $x$-intercepts, but they clearly do” and wrote “doesn’t matter up to [coefficient].” In his exploration of the effect of units in complete factorization, he finally coordinated the piece that he could not explain earlier while utilizing a link between a polynomial equation and its roots. He finally accepted the fact that the set of roots of a given polynomial equation is preserved under multiplication by a (nonzero) coefficient/unit/scalar (e.g., $1 + 4x^2 = 0$ and $\frac{1}{4} + x^2 = 0$ have the same roots).
The reason that Calvin’s coordination (or the strengthening from S2 to S3) required a lengthy period of time could be possibly attributed to his conflated use of two constructs, roots and factors. As seen in Figure 4.16, he defined factoring completely over \( \mathbb{Z}, \mathbb{Q}, \mathbb{R} \) and \( \mathbb{C} \) as “to write the polynomial as a product of its roots, with coefficients in the corresponding number system” and what he seemed to mean by “roots” in his definition was irreducible factors of a given polynomial. With such association without clear distinction between the two constructs, he could have thought changing the form of factorization might change the \( x \)-intercepts. Hence, the process of strengthening may involve or benefit from reflecting on how ideas in the schema are distinctly linked. In Calvin’s case, strengthening his link between a polynomial equation and its roots involved his exploring and examining how exactly factoring out a unit from a polynomial equation is related to the roots of the polynomial equation.

Similarly, Sam’s work for understanding the difference between \( p_5(x) = 4x^2 + 2x - 10 \) and his factorization of it \( 2(x - 2)(x + \frac{5}{2}) \) revealed an interesting transition from S2 to S3 in his reasoning. However, while Calvin focused on how roots of an equation are preserved when factoring out a unit, Sam noticed that there may be more than one equation that has the same root in resolving the issue.

Initially for his factorization of \( p_5(x) = 4x^2 + 2x - 10 \), Sam first factored out 2 and applied the quadratic formula to \( 2x^2 + x - 5 = 0 \) to obtain the roots (i.e., \( 2 \) and \( -\frac{5}{2} \)), utilizing a link from an equation to its roots. Then he used the roots to factor \( p_5(x) = 4x^2 + 2x - 10 \) into \( 2(x - 2)(x + \frac{5}{2}) \), utilizing a link from the roots to the original equation but not fully coordinating the leading coefficient 2 in the equation he used for
the quadratic formula, $2x^2 + x - 5 = 0$. This suggests again an assumption that a pair of roots determines the related quadratic equation [S2].

The interviewer asked Sam to show how his factorization satisfies his definition of complete factorization, which was “to take a number or expression and split it into polynomial factors that cannot be reduced any further.” In the process, Sam checked the expanded form of his factorization, noticed the difference between the original $4x^2 + 2x - 10$ and the expanded form of his answer $2x^2 + x - 5$, and said “trying to figure out why. I’m not sure why.” Sam then began to use another method (see Figure 4.18) and obtained the correct factorization.

![Figure 4.18. Sam applying another method for checking the factorization of P5](image)

After his successful second attempt, he compared his current answer $2(x - 2)(x + 5)$ with his previous answer $2(x - 2)(x + \frac{5}{2})$ as follows:

I used a different method instead of using the quadratic formula. I think

I’m seeing... because this [referring to $(x + \frac{5}{2})$ in his original factorization] being a zero or a factor, we could set it up like this [writing
$x + \frac{5}{2} = 0$ and get $x = -\frac{5}{2}$, which was one of the values of $x$ we got in quadratic formula. But with this [referring to $(2x + 5)$ in his new factorization], it’s just a different way to get $-\frac{5}{2}$. So $2x + 5 = 0$ would be $2x = -5$ which is going to be $x = -\frac{5}{2}$.

In his comparison between two factorizations, Sam seemed to have noticed that two different equations may result in the same root. Then he continued comparing two methods (one using the quadratic formula, the other using factoring by grouping) as follows:

I think this [referring to his new method] works better than the quadratic formula if you can use it... I mean, the quadratic formula is okay if you’re trying to find the roots but this [referring to his new method] is probably better suited to help you rewrite expressions in factored form because, the problem I ran into with one of the roots being $-\frac{5}{2}$, there are at least now two different ways we could have wrote that here [referring to the position of a factor $x + \frac{5}{2}$ in the factorization $2(x - 2)(x + \frac{5}{2})$]. I wrote it the first time $x + \frac{5}{2}$ whereas it actually needed to be $2x + 5$.

As shown in his explanation, Sam indicated that having the root equal to $-\frac{5}{2}$ does not necessarily mean that the factor is $(x + \frac{5}{2})$ instead of $(2x + 5)$ and that there could be
several linear equations that have the same root. For that reason, Sam seemed to think that the quadratic formula alone is not enough to determine complete factorization of a given polynomial, thereby addressing the problem he previously had in making the link from a set of roots to its original equation [S3].

In both Calvin’s and Sam’s work illustrated in preceding paragraphs, we saw how they dealt with unexpected results they had obtained from applying a routine (to them) procedure of factoring (i.e., relating roots with a factored form). In their procedures, they utilized a bidirectional link between a quadratic equation and the roots of the equation, but initially they could not see the role of the leading coefficient in their factorization. Notably, in both cases, this lack of coordination was resolved in the context of addressing more conceptual questions such as “How does your definition apply to your complete factorization?” and “Does your definition accept both of these as complete factorization?” These questions were asked during TIs to help them construct a robust and abstract definition of their own that is suited for the collegiate level but also made them explore the entire process of factorization in which they had engaged. Such conceptual questions appeared to help them to identify the entities they are relating and to establish a fully developed link between them. A detailed discussion of the interview context as it relates to participants’ transformative transitions in the strengthening category is provided in Chapter 5, under section A task with a variety of instantiations of a single entity to reflect on (page 301).
Deepening

The main purpose of this section is to illustrate participants’ understandings of factorization at different levels—action, process, and object levels (adopted from APOS theory)—and describe deepening as a growth in one’s understanding from action level, to process level, and to object level. In this section, I also use the term “conception” (Arnon et al., 2014, p. 18) to be consistent with other APOS literature and to refer to an intrapersonal and individual understanding that is possibly idiosyncratic.

As discussed in Chapter 2, an action conception [D1] is considered to be the initial level of a deepening mechanism to be transformed later into a process conception and further into an object conception, which indicates the growth in depth of an individual’s understandings. It is important to clarify how an action conception can be characterized, especially in comparison to a process conception. In general, an action conception is characterized by the individual’s understanding a mathematical idea as a repeatable mental or physical action to be performed in response to external guidance, cues, or stimuli (Arnon et al., 2014; Asiala et al, 1996). Applying this characterization to the understanding of factorization, a subject with an action conception of factorization might need to have an explicit formula or procedural guidance to execute the action of factorization. It is also possible that s/he considers factorization as a set of repeatable steps to be carried out in response to external stimuli, such as a specific type of expression (e.g., $x^2 - 5x + 6$). With a specific expression such as $x^2 - 5x + 6$, one can think of factorization as a set of steps: first, looking for factor pairs of the constant term, 6; second, choosing the factor pair (in this case, -2 and -3) that adds to the linear term’s
coefficient, -5; and third, rewriting the given expression as a product of two linear expressions using the factor pair found, \((x - 3)(x - 2)\). For a subject with an action conception of factorization, this procedure may be cued step by step, which means s/he may need to see the result of a certain step to decide what to do in the next step. In contrast, a situation of factoring unspecified expressions such as \(ax^2 + bx + c\) or describing how to factor a quadratic expression in general can be a source of difficulty because no explicit cues are available to guide the steps to take as in a quadratic expression with numeric coefficients. The empirical data that represent action conceptions (or action level understandings) will be provided in detail in the subsection entitled Action level understandings or action conceptions [D1] on page 256.

A subject with a process conception [D2] no longer has to rely on external guidance, cues, or stimuli. APOS theorists describe a process conception in contrast with an action conception as follows:

When an action is repeated, and the individual reflects upon it, it may be interiorized into a process. That is, an internal construction is made that performs the same action, but now, not necessarily directed by external stimuli. And individual who has a process conception of a transformation can reflect on, describe, or even reverse the steps of the transformation without actually performing those steps. In contrast to an action, a process is perceived by the individual as being internal, and under one’s control, rather than as something one does in response to external cues. (Asiala et al., 1996, p. 7)
Also, a student with a process conception is able to “imagine carrying out the steps without necessarily having to perform each one explicitly” through the interiorization of the action and has a way to describe the action in a general way (Arnon et al., 2014, p. 20). Hence, a student with a process conception of factorization would be able to conceive (or imagine) a resultant factorization of a polynomial without the need of explicit calculation and have a way to describe how s/he might factor an arbitrary, unspecified expression such as \( ax^2 + bx + c \). The empirical data that represent process conceptions (or process level understandings) will be provided in detail in the subsection entitled *Process level understandings or process conceptions* [D2] on page 263.

Lastly, an object conception [D3] is characterized by the individual’s ability to treat the mathematical idea as an object and to apply an action on the object without the need of evoking a relevant process or action. For example, when it comes to the idea of factorization, an object conception is demonstrated through one’s ability to apply other actions to the object of factorization: a) *impose* certain conditions to factorization to define complete factorization, b) *determine* that a certain property such as uniqueness is associated with complete factorization, and c) *characterize* factorization over one domain in comparison to factorization over another without the need of going through the process of factorization. The empirical data that represent object conceptions (or object level understandings) are provided in detail in the subsection entitled *Object-level understandings or object conceptions* [D3] on page 267.

Depending on the level of deepening (action, process, and object), different aspects of factorization will be highlighted in participants’ responses. In the first two subsections which are about action and process conceptions, the word “factorization” is
used mainly as transformation of factoring a polynomial. This is because participants with action or process conceptions tend to focus on procedural aspects of factorization (e.g., what they have to do at each step, which procedures are to be applied at each step of the transformation, or how to apply a formula). In the last subsection, which is about object conceptions, factorization mainly refers to a result obtained by the transformation of factoring. At an object level, students see factorization (which was originally procedural and dynamic at a process level) as a conceptual entity to which certain actions can be applied. Under each subsection, I provide empirical illustrations of action/process/object conceptions in the context of factorization and describe deepening as a growth in one’s understanding from action, to process, and to object level.

**Action level understandings or action conceptions [D1]**

The current section provides an analysis and illustrations of Sam’s work, which suggests his action conception of factorization. In subsequent paragraphs, Sam’s conception of factorization will be discussed in light of characterizations of an action conception: understanding factorization as a set of repeatable steps to be carried out in response to a specific expression or as an application of a factoring formula if available.

When given a polynomial with numeric coefficients such as $x^4 - 12x^2 + 36$ and $2x^3 - 10x^2 + 8x$, Sam demonstrated fluency in carrying out steps needed to factor the polynomial. However, given an expression such as $ax^2 + bx + c$ with the values of $a$, $b$, and $c$ unknown, he expressed difficulty in factoring the expression. To be specific, in
response to PRE-Q1\textsuperscript{51}, Sam explained how he would normally apply a method of factorization given a specific polynomial but registered a lack of confidence in dealing with $ax^2 + bx + c$ with unknown values for $a$, $b$, and $c$. The following excerpts show his responses to PRE-Q1.

Sam: Well… I guess… Right off the bat I would say I wouldn’t think so. Because based on the values of $a$, $b$, and $c$. You need the values of $a$, $b$, and $c$ to sort of determine how you can factor something.

I: Can you tell me more about that? Maybe with examples?

Sam: Yeah. So maybe some example would be like if $a = 1$, $b = 4$, $c = 4$ and you can have $x^2 + 4x + 4$. And that can factor into $x + 2$ and $x + 2$.

I: How did you know that it can be factored that way?

Sam: Um. I guess. The way I was always taught is if the leading coefficient is 1, then you can separate the $x^2$ into $x$ and $x$ and, at that point, I look at the $c$ value and I look for factors of 4. I know that 2 multiplied by 2 would give you 4. But you need to find the pair that also adds to the $b$ value. But you know that $2 + 2$ is 4. So that is how I factored that. I guess if I look at the $a$, $b$, $c$, arbitrary values now, I am really not seeing the way to factor that without actual values for $a$, $b$, $c$.

\textsuperscript{51} PRE-Q1: Can you factor $ax^2 + bx + c$ without knowing the specific values of $a$, $b$, and $c$? If so, how can you do so? If not, why do you think so?
To be able to perform factorization, Sam seemed to need a specific expression in front of him for applying the steps of factorization (the steps are bolded in the excerpts). Sam’s treatment of factorization as a sequence of repeatable steps to be carried out highlights his action conception of factorization. Also, Sam’s difficulty with conceiving/imagining a resultant factorization of $ax^2 + bx + c$ when the values of $a$, $b$, and $c$ are not specifically given suggests that he had not yet fully reached a process level. To qualify as a process level, the understanding should be applicable to a variety of situations (Dubinsky et al., 1994), suggesting the individual has both a general approach and comprehensive understanding that are not specific to each problem given (Arnon et al., 2014). Hence, the process of generalization appears to be a part of what is required for one to construct a process conception.

Sam’s work in PRE-Q7 further evidenced Sam’s conception of factorization at an action level rather than at a process level. Although PRE-Q7 essentially asked the same thing as PRE-Q1 (except for $ax^2 + bx + c$ changed into “any quadratic polynomial”), Sam responded to PRE-Q7 that “any quadratic polynomial is factorable.” At that point, the interviewer asked him to compare his responses to PRE-Q1 and PRE-Q7. As shown in the following excerpts, Sam elaborated that, although he believed that any quadratic polynomial is factorable, he was not sure how to go about factoring $ax^2 + bx + c$ without knowing the specific values of $a$, $b$, and $c$.

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52 PRE-Q7: Can you factor any quadratic polynomial? If so, how can you do so? If not, why do you think so?
Sam: The issue I had [in PRE-Q1] was that, with the values $a$, $b$ and $c$ being unknown, I wasn’t able to factor that, but I do think that for any numbers that you plug in, there is a way to factor it. You just may get irrational solutions, which is why, for something like this, [this refers to an expression he has written previously $x^2 + 4x + 6$] you can’t just go straight. For that step, you have to use the quadratic formula to find the solutions. I do think that any quadratic polynomial can be factored.

I: Oh, I see. Oh. Can you repeat what was your answer for this one? [this one refers to PRE-Q1]

Sam: For this one?

I: Yeah.

Sam: Okay. Whenever it asks me if I can factor $ax^2 + bx + c$, without knowing specific values of $a$, $b$ and $c$, it’s not that I don’t think this is factorable. I do. I just personally don’t know how I would go about it. I was trying to think of a way to write this [this refers to $ax^2 + bx + c$] in this type of form [this type of form refers to $(x + 2)(x + 2)$]. I wasn’t able to do that initially, but I do think you can factor that for any values of $a$, $b$ and $c$.

I: Thank you for the clarification. Let’s move back to PRE-Q7.

Sam: Okay. Okay.

I: Your answer is yes?

Sam: Right. If so, how can you do so? This would have been where I was stumped in the first place, was how you can go about doing it. I’m really not sure how you could do it.
Sam’s response that “I just personally don’t know how I would go about it” is interesting because what he mentioned a few lines ahead (i.e., “for something like this, you can’t just go straight. For that step, you have to use the quadratic formula to find the solutions”) is exactly how one might go about it when specific coefficients are not given. One possible explanation for Sam’s not using the quadratic formula in factorization of $ax^2 + bx + c$ might be that Sam’s previous conception of factorization as a set of repeatable steps was so strongly established in his schema that he might have thought he needed to find a similar set of steps for the cases when noninteger coefficients are needed for the factorization.

In PRE-Q9, Sam’s illustration of his approach to factoring also suggested he treated factorization as performing a set of repeatable steps. To be specific, when Sam was asked to construct his concept map around polynomial and factoring, he described two types of factorization he was taught (see Figure 4.19 for his representation of the two cases in his concept map). He stated, “Factoring. Oh, I see factoring of two different types, one where the leading coefficient is 1, and then the leading coefficient does not equal 1, because I was always taught two different methods of factoring in those specific cases.” The way that he described factoring by dividing the cases and associating the division to two different methods suggests his understanding of factorization was focused on mechanical aspects of repeatable action.
With another polynomial $P_3(x) = 4x + 16x^3$, Sam also made a statement that suggests an action conception of factorization. During an interview, Sam factored $P_3(x) = 4x + 16x^3$ into $4x(1 + 4x^2)$ and determined that it cannot be factored further. The following short excerpts show why he thought it could not be factored further.

Sam: Okay, so that point, I don’t think that’s factorable any further.

I: How did you know that?

Sam: I think I recall that there is no formula for sum of squares to factor. That’s what I’m thinking anyway. I guess that would be the - that is the factored form $[that~refers~to~4x(1 + 4x^2)]$.

For him, nonexistence of a formula for factoring “sum of squares” seemed to sufficiently explain why the polynomial $(1 + 4x^2)$ could not be factored any further. His association between factorization and a search for an applicable formula suggests his lack of internal
control over the factorization situation and reliance on external guidance such as an explicit formula (which suggests an action conception).

Throughout all nine interviews, Sam performed factorization proficiently when given specific expressions. He seemed to treat factorization as a set of repeatable steps to be carried out in response to a specific external stimulus such as $x^2 + 4x + 4$ or an application of a factoring formula if available. However, as shown in preceding episodes, without specified coefficients he registered a lack of confidence in how to factor an expression.

Sam’s difficulty with factoring $ax^2 + bx + c$ and need for specific and explicit expressions for factorization are consistent with the characterization of action conception in APOS theory. Asiala and colleagues (1996) explain that when students have action conceptions of functions, they might find it difficult to interpret a function with no explicit function rule such as a sine function. To understand a function without an explicit function rule, students would need a process conception of function, in which they can imagine the process of associating an input with a corresponding sine value.

The findings in this section show that some mathematics-intensive majors may have difficulty with considering factorization at a process level, although they may have mastered factorization at an action level and be able to perform factorization of a given polynomial successfully. In the following section, I discuss instances in which participants demonstrated a process conception of factorization [D2], which involves more than applying formulas and performing mechanical procedures for factorization.
Process level understandings or process conceptions [D2]

As an *action* of performing factorization or applying a factoring formula is interiorized into a *process* by a subject, his/her reliance on *external* cues that lead him/her to certain directions is reduced and replaced by *internal* control over factorization situations. One with a process conception of factorization can imagine the resultant factorization (rough or precise) in her/his mind without actually performing the action. Moreover, mechanical steps for performing factorization receive less attention by the subject; steps in the middle may be skipped or even reversed. With a process conception of factorization, one’s approach to factorization is comprehensive and more general than case-by-case approaches with an action conception.

The episodes described in the current section highlight participants’ being able to conceive the resultant factorization of a polynomial without the need of explicit calculation or the intermediate steps in the calculation. First, Calvin in his response to PRE-Q7 demonstrated a process conception of factorization with $ax^2 + bx + c$. In contrast with Sam, who needed explicit values for $a$, $b$, and $c$ to consider the factorization of $ax^2 + bx + c$ (as discussed in section D1), Calvin did not seem to have to rely on the values for $a$, $b$, and $c$ to factor the expression. Calvin first wrote $ax^2 + bx + c = 0$ and noted that “this [pointing to the left-hand side of the equation] is any polynomial, any quadratic polynomial,” meaning the values of $a$, $b$, and $c$ are not fixed. Calvin then

53 PRE-Q7: Can you factor *any* quadratic polynomial? If so, how can you do so? If not, why do you think so?
factored out the leading coefficient to get \( a \left( x^2 + \frac{b}{a} x + \frac{c}{a} \right) = 0 \). At this point, without having to calculate the roots of \( x^2 + \frac{b}{a} x + \frac{c}{a} = 0 \), Calvin continued by assuming the roots are \( s \) and \( t \) and proposed \( a(x - s)(x - t) = 0 \) as a factorization of \( ax^2 + bx + c = 0 \), thereby concluding any quadratic polynomial can be factored. He omitted the intermediate step of calculating the roots of \( x^2 + \frac{b}{a} x + \frac{c}{a} = 0 \), and he seemed to have done so intentionally as he noted, if using the quadratic formula on \( x^2 + \frac{b}{a} x + \frac{c}{a} = 0 \), “it would be messy with a bunch of fractions and radicals.” The fact that Calvin was successfully able to imagine the resultant factorization while skipping the intermediate steps in the of factorization suggests his process conception of factorization.

Calvin’s process conception of factorization was also revealed in his approach to factorization of \( x^3 - 1 \). In their factorization of \( P_2(x) = x^5 - x^2 \), several participants treated factorization of \( x^3 - 1 \) as applying the well-known formula for a difference of cubes, \( a^3 - b^3 = (a - b)(a^2 + ab + b^2) \) or \( x^3 - 1 = (x - 1)(x^2 + x + 1) \) and, for two of the participants (Sam and Helen), applying the formula seemed to be the only way available for them to treat the factorization. However, for Calvin (who initially stated “I forget how to factor these”), the formula for a difference of cubes did not seem to be needed; Calvin seemed to have an internal control over the factorization by treating factorization as a generic process (suggesting a process conception).

After stating “I forget how to factor these,” Calvin continued by saying “we have a root of one” and wrote \( x^3 - 1 = (x - 1)(x^2 \quad \quad ) \) with the blanks in the second factor of the right-hand side. He then filled in the blank with the constant term of the second factor, \( x^3 - 1 = (x - 1)(x^2 + 1) \), experimented (with silence; details
described in this paragraph) with $+x$ and $-x$ for the middle term in the second factor, and concluded $x^3 - 1 = (x - 1)(x^2 + x + 1)$. When the interviewer asked him to explain how he arrived at the factorization, Calvin illustrated each one of the steps he took: first, “We know $x = 1$ is a root. It solves this equation [referring to $x^3 - 1$] equaling zero, which has been represented as $x - 1$”; second, “$x^3 - 1$, that’s what we want to get to. We’re trying to rewrite that. We need an $x^3$ term. So we know that the first term [in the second factor] is going to have to be $x^2$”; third, “I know my last term [in the second factor] has to be a plus 1. The only term without $x$’s is this minus 1 so far [this minus 1 refers to minus 1 in $(x - 1)$]. Then in order to get a minus 1 in this polynomial [this polynomial refers to $x^3 - 1$ we need a plus 1”; lastly, “Then I kind of just guessed that $+x$ would work here. Then we get [expanding $(x - 1)(x^2 + x + 1)$] $x^3 + x^2 + x - x^2 - x - 1$. Then these all cancel and we get $x^3 - 1$.” In the illustration of his own work, it was notable that Calvin was able to reasonably envision or expect what the resultant factorization is (roughly) going to be throughout the factorization process and simply needed to determine the details (such as coefficients) to get to a final answer. Such level of internal control over the factorization situation (that does not rely on an external formula) is an indicator of a process conception, which is one level more advanced than an action conception.

An action of applying a factoring formula seemed to possibly develop into a mentally internalized process of recognizing a structure of an expression and conceiving the resultant factorization of a polynomial. For example, in his factorization of $P_3(x) = 4x + 16x^3$, Jason used a structure of the expression $1 + 4x^2$ in order to conclude with
$4x(2x + i)(2x - i)$ as the complete factorization of $P_3$. The following excerpts from Jason’s work show that Jason recognized $1 + 4x^2$ as a difference of squares, which led to the factorization without the need of explicit calculation.

$P_3$. We have $4x + 16x^3$. So you can bring out $4x$. You get $1 + 4x^2$. Um, again that’s factorized over the reals. But we can bring it further and say $4x$ times…

**that is in the sense… [that refers to $1 + 4x^2$]** Let’s see. That is in a way a **difference of squares**. ‘Cause then you have $(2x + i)(2x - i)$. [Jason silently checks his answer, $(2x + i)(2x - i)$, for about 2 seconds—showing a distribution-like behavior with his pen. He does not write anything.] Yeah. Okay.

Yeah. So that, and that would be factorized over, um, complex numbers.

Jason appeared to have built on the equivalence between $4x^2 + 1$ and $(2x)^2 - i^2$ (this step was not explicitly explained by Jason but could be inferred) to recognize $4x^2 + 1$ as “a difference of squares” and such recognition of the structure seemed to be fluidly and immediately followed by the resultant factorization, $(2x + i)(2x - i)$, or the product of the sum and difference of what is being squared.

The preceding illustrations from Calvin’s and Jason’s work represent a process conception of factorization [D2]. Specifically, they were able to successfully treat factorization situations without the reliance on external, guiding instructions or explicit steps. Instead, Calvin and Jason were able to imagine a resultant factorization (or at least some picture/expectation of what the result is going to look like) and complete the
process by omitting some intermediate steps. At the most advanced level in deepening, one can think of factorization as an entity (an object) without having to evoke a relevant process/action of factorization or a specific example of factorization. Furthermore, at this advanced level, an individual is able to apply an action on factorization. In what follows, I present analysis of episodes in which Jason demonstrated this advanced level, that is, an object conception of factorization.

**Object level understandings or object conceptions [D3]**

The current section focuses on Jason’s object conception of factorization revealed in CORE-Q2, CORE-Q6, and POST-Q7 and, in so doing, contrasts Jason’s work with the work of other participants who demonstrated a process conception of factorization in the same set of problems. In general, little evidence was found that participants other than Jason had formed an object conception of factorization. Given interview questions in which they had to apply some actions on factorization (e.g., defining, characterizing, comparing), other participants tended to build with their process conception of factorization or to reason from specific examples of factorization.

Throughout TIs, participants were asked to define complete factorization twice: first in CORE-Q2\(^{54}\) so that they could use their own definition of complete factorization

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\(^{54}\) **CORE-Q2:** Students in high school algebra classes are often asked to ‘factor completely.’ What do you think the phrase ‘factor completely’ means?
to engage with the task in CORE-Q3, and second in CORE-Q6 as a summary of their exploration on the notion of irreducible, reducible, and unit. Jason’s responses are distinguished from other participants’ responses to CORE-Q2 and CORE-Q6. While Jason defined complete factorization as *a factorization* that satisfies a set of particular characteristics, other participants defined complete factorization solely using a process involved in performing complete factorization. In the next few paragraphs, the distinction between Jason’s and others’ responses is discussed in detail. Here, because participants constructed their own definitions without referencing any external texts, their definitions can be considered close approximation of the way they think about what it means to factor completely.

To be specific, Jason’s written definition of “factoring completely” in CORE-Q2 reads as follows: “*A factorization* of an object to a target set *such that* none of the components can be factored anymore.” In CORE-Q6, building on the notion of irreducible, he defined complete factorization as “*A factorization* into polynomials, all in D, *such that* each factor is irreducible.” The fact that Jason imposed conditions on factorization (e.g., each factor is irreducible) and defined complete factorization as a

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**CORE-Q3**: According to your definition, please factor completely the following polynomials.
- \( P_1(x) = x^4 - 12x^2 + 36 \)
- \( P_2(x) = x^5 - x^2 \)
- \( P_3(x) = 4x + 16x^3 \)
- \( P_4(x) = 12 \)
- \( P_5(x) = 4x^2 + 2x - 20 \)
- \( P_6(x) = 2x^3 - 10x^2 + 8x \)

**CORE-Q6**: How would you explain to your friend what *a complete factorization* of a polynomial over a domain D is? (*D is used to stand for \( \mathbb{Z}, \mathbb{Q}, \mathbb{R} \) or \( \mathbb{C} \)).
particular type of factorization is an indicator that Jason was able to treat factorization as an entity on which to apply another action (e.g., imposing conditions, determining characteristics of factorization that need to be satisfied to be considered complete).

On the other hand, the other participants’ definitions of complete factorization suggested their process conception of factorization. For example, Calvin wrote, in CORE-Q2, "Factoring completely over (\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}) is to rewrite the polynomial as a product of its roots with coefficients in the corresponding number system" and, in CORE-Q6, “a polynomial is completely factored over D if it is written as a product of irreducible or (tentative: unit) terms.”

Also, Sam wrote, in CORE-Q2, “Factor completely: To take a number or expression and split it into polynomial factors that cannot be reduced any further” and, in CORE-Q6, “to take a number or expression and split it into polynomial factors that are irreducible without leaving the domain.” Calvin’s and Sam’s definitions highlight an action applied on a given polynomial (e.g., rewriting it as a product form, splitting it into polynomial factors) that needs to be performed to obtain complete factorization, which is in contrast with Jason’s definition that suggested an action applied on factorization itself.

Jason’s object conception of factorization was further evidenced in his response to POST-Q7, in which participants were asked to think about completeness and uniqueness of factorization and the relationship between completeness and uniqueness.

\[57\] POST-Q7: Please explain your reasoning to the following questions.
(a) Can you factor any quadratic polynomial?
(b) Can you factor any quadratic polynomial uniquely
(c) Can you factor any quadratic polynomial completely?
(d) Can you factor any quadratic polynomial completely and uniquely?
Because the question itself did not entail any specific polynomial, it provided a chance for the interviewer to observe and investigate whether participants could treat factorization without the need of specific polynomials in front of them to reason through.

After answering (a) through (d) in POST-Q7, Jason was asked to talk about the difference between question (b) and question (d). In his comparison between (b) and (d), he first noted “I guess (b) doesn’t ask that it be a complete factorization but I feel like that the complete factorization sort of helps define the unique factorization that we’re looking for. So they’re definitely very related.” As such, Jason appeared to have no need to resort to specific examples as he elaborated this difference:

Given a factorization, that defines a unique polynomial. But a polynomial can be broken down into multiple factorizations. But if you impose conditions on the factorization then you can sort of restrict it to a single one. So if this is factorizations [drawing the set diagram at the upper left corner of Figure 4.20] and this is polynomials [drawing the set diagram at the upper right corner of Figure 4.20], then there are lots of these that all map to the same one [Jason draws several dots in the set of factorizations while speaking “lots of these” and draws arrows to a single dot in the set of polynomials].
Figure 4.20. Jason’s diagram describing the uniqueness of complete factorization (his original work on the top, and a re-presentation of his diagram on the bottom)

Um but if you just look to like complete ones, like if you put a restriction on it, like there’s only one there [Jason selects one dot in the set of factorizations, circles around it, and writes “complete” as shown in Figure 4.20]. So I guess if we’re not allowed to put restrictions on these [pointing to question (b)] then I would say uh no, because you can write, say, $x^2 - 1$ as just $x^2 - 1$ or you can write it as $(x - 1)(x + 1)$. So I guess that [that refers to factorization] wouldn’t be necessarily unique, because there’s more than one way of writing it down, but in (d) if we
impose complete factorization then we’re not allowed to do that, and
there’s only one factorization.

His diagram (in Figure 4.20) represents a mapping between polynomials and
factorizations and highlights the uniqueness of complete factorization. Notably, Jason
simply used dots to represent factorizations as if they are elements in a certain set. This
suggests Jason was able to treat factorization as an object on which to apply an action
(e.g., mapping) without the need to evoke its process (suggesting an object conception).
Also, Jason treated complete factorization as “condition-imposed” factorization that is
unique among other multiple factorizations and did not have to revisit the process of
factorization in order to determine uniqueness of complete factorization. Jason used a
specific example sparingly throughout his explanation and, when using a specific
example ($x^2 - 1$), he appeared to do so only to further explain his points to the
interviewer, not to help his thinking. In sum, Jason seemed to be effectively operating at
the object level without having to step back to the process level to work on POST-Q7.

Building on the illustrations of three levels of deepening—action, process, and
object levels—I will now discuss the deepening observed during the course of interviews
that suggested changes of levels from D2 to D3.

**Deepening observed in the course of interviews**

In the current section, I discuss how participants’ conceptions of factorization
changed in the course of interviews, exploring the second research question with respect
to deepening: What transformative transitions, especially deepening, do university students make in the course of interviews? In general, participants rarely demonstrated object conceptions of factorization, except for Jason, which means the transition from D2 to D3 was also rarely observed. The lack of witnessing object conceptions is also shown in the diagram\(^{58}\) in Figure 4.21.

![Diagram](image)

**Figure 4.21.** Participants’ understandings of factorization with respect to deepening

Looking over how participants’ understandings of factorization are distributed among the three levels in the diagram, it is evident that all six participants demonstrated their process conceptions of factorization throughout the interviews, while a few participants also demonstrated either action conceptions or object conceptions. However, the analysis of two participants’ data (Calvin and Jason) revealed transformative transitions from D2 to D3 during POST-TI (as indicated by the two vertical arrows in the

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\(^{58}\) Detailed directions for interpreting the diagram can be found in a list of bullet points in the section *Extending observed in the course of interviews* on page 175.
diagram). The current section focuses on those two cases of Calvin and Jason to provide descriptions of transformative transition in the deepening category.

In POST-Q7, most participants seemed to need to evoke their process conceptions of factorization in order to determine whether a certain property (e.g., uniqueness) can be associated with factorization or complete factorization. However, for one participant, Calvin, exploring uniqueness or completeness of factorization at a process level seemed to give rise to an emerging object conception of factorization [which suggests in-development deepening from D2 to D3].

In response to both POST-Q7 part (b) and part (d), Calvin initially responded “yes, you can” without any hesitation. The interviewer asked him to explain if there exists any difference between part (b) and part (d) and, at that point, Calvin began to reason about how “factoring completely” might be related to the uniqueness of factorization. Calvin chose to use a cubic polynomial, \(60x^3\), in \(\mathbb{Q}[x]\) to think about the difference between part (b) and part (d). Calvin proposed \(x \cdot x \cdot x\) as the complete factorization of \(x^3\) and concluded “I think maybe if you completely factor it, it ends up being unique.” Then he continued as follows:

\[\text{\underline{POST-Q7}}\text{: Please explain your reasoning to the following questions.}
\text{(a) Can you factor any quadratic polynomial?}
\text{(b) Can you factor any quadratic polynomial \textit{uniquely}?}
\text{(c) Can you factor any quadratic polynomial \textit{completely}?}
\text{(d) Can you factor any quadratic polynomial \textit{completely} and \textit{uniquely}?}
\]

\(60\) Calvin did not articulate the reason why he chose a cubic expression even though the questions were restricted to quadratic polynomial.
An incomplete factorization would be $x^2 \cdot x \ldots$ I guess it depends on your definition of factoring because if you allow coefficients and units then you could have nonunique steps in your [factorization] because if you had like $\frac{1}{3}x \cdot 3x^2 \ldots$ [Calvin also writes $\frac{1}{5}x \cdot 5x^2$ without speaking] Oh, I guess… this is an invalid factorization. Because we’re multiplying by units and you can’t just factor out by a unit and just say that’s a different factorization.

At that point, Calvin still seemed uncertain how to conclude his response to part (b) in comparison to part (d). It is interesting that even though Calvin obtained two different factorizations $x \cdot x \cdot x$ and $x^2 \cdot x$ (one complete and the other incomplete), he did not recognize these as examples to refute the statement under examination “a polynomial can be factored uniquely” in part (b). One possibility is that Calvin did count the complete factorization ($x \cdot x \cdot x$) as one of factorizations of $x^3$; it might be that factorization is understood as a process that is different from the process of complete factorization by Calvin. Such differentiation (rather than seeing complete factorization as a factorization) suggests a lack of an object conception of factorization. As discussed earlier in section D3, an indicator of having an object conception of factorization is the capability to impose conditions on factorization and define complete factorization as a particular type of factorization. Instead, Calvin’s statement that “you could have nonunique steps in your [factorization]” indicates he was rather more focused on the process of factorization in his search for nonunique factorizations. A tentative conclusion he made after this
exploration (shown in the following excerpts), however, seemed to suggest Calvin was not exclusively operating at a process level.

I think in your complete factorization they are unique but... maybe *if you factor it uniquely, then it’s completely factored?* I mean this last statement [*referring to part (d)*] by itself is true, but *if you factor something uniquely, is there an implied completeness to that?* Maybe no, maybe there’s not an implied completeness to it.

Calvin’s tentative conclusions that completeness of factorization implies uniqueness but that uniqueness of factorization does not imply completeness, were not fully justified. However, two observations that follow suggest that Calvin was operating at an object level: a) the way Calvin treated factorization in his logic without relying on factorization process/action, and b) that he came to a general conclusion with respect to factorization in an if–then statement. There are two possible explanations for the change in Calvin’s (demonstrated) conception of factorization in the preceding episode, which first highlighted a process conception, and then an object conception—a transition from D2 to D3.

First, it is possible to understand this phenomenon from the perspective of deepening from D2 to D3. For Calvin, exploring the difference between part (b) and part (d) of POST-Q7 at a process level could have influenced him to attend more closely to determining whether a property of uniqueness could be associated with factorization, giving rise to an emerging object conception of factorization.
For the second possible explanation, it is important to note that an individual who once constructed an object conception may also demonstrate his/her process conception when the situation is perceived as necessitating evocation of the process. According to APOS theorists, one may de-encapsulate “Object back to its underlying Process and reconstruct her or his Process conception in order to assimilate the new context” (Arnon et al., 2014, p. 176). Hence, they argue the reconstruction of a mathematical concept involves “a nonlinear progression through the Action-Process-Object sequence” (p. 176). Then it is possible Calvin had at some point (not during the course of interviews) constructed an object conception of factorization and, during the interview, he perceived the situation involving POST-Q7 (b) and (d) as a new context to assimilate using its underlying process of factorization. As a result, Calvin could have reconstructed his conception of factorization at an object level by relating two ways of characterizing factorization—i.e., uniqueness and completeness.

The progression from object, to process, and back to object [D3 → D2 → D3], where the later object conception is obtained from “encapsulating richer processes” than the prior object conception (Dubinsky, 1997, p. 98), seemed to have occurred in Jason’s reasoning in POST-Q7 as well. Jason’s episode has to do with how to conceive “identity factorization,” that is, how to treat a case in which the form of an expression does not change before and after the factorization. In school mathematics, it is a common practice to say, for example, $x^2 + 1$ cannot be factored over $\mathbb{R}$ but it can be factored over $\mathbb{C}$. In collegiate mathematics, however, factorization of a polynomial is considered as expressing the given polynomial as a product of its irreducible factor(s); and the product can have only one factor (Cuoco & Rotman, p. 236). Since $x^2 + 1$ is an irreducible factor
of itself over $\mathbb{R}$, $x^2 + 1$ is a factorization of itself over $\mathbb{R}$ (or we may call it an *identity factorization*). Hence, depending on which process of factorization (e.g., breaking down an expression in school mathematics, versus expressing it as a product of its irreducible factors in collegiate mathematics) is encapsulated to give rise to its object conception, a factorization of $x^2 + 1$ over $\mathbb{R}$ may or may not be deemed to exist.

Jason had demonstrated his object conception of factorization in other instances prior to POST-Q7 but, in his initial response to POST-Q7, he appeared to treat factorization as a process—specifically, a process that transforms a given expression into a *different* form (hence excluding the identity factorization). However, his response gradually changed, without any prompt from the interviewer, to accept identity factorization. The change in his response seemed to reflect his definition of complete factorization established in TIs, which was “a factorization into polynomials, all in $\mathbb{D}$, such that each factor is irreducible.” Jason’s responses to POST-Q7 part (a) and part (c) are provided in the following excerpts. Jason began by answering to part (a), “Can you factor any quadratic polynomial?” as follows.

Jason: Yes, if you pick a good set like $\mathbb{C}$ because then your roots are definitely going to be there. **If you don’t pick $\mathbb{C}$, then it’s possible that you can’t factor it.** [Jason answers to part (b)—not included in this excerpt—and moves onto part (c)] Can you factor any polynomial completely? Um so let’s say, uh **depending on the set you pick**, like your domain, then yeah because you could factor into like $a$, $x$ minus the first root, $x$ minus the second root [writing $a(x - r_1)(x - r_2)$], um and
that’s considered complete factorization… [pause] Um and I’d say actually, over any domain you can do this as well [this seems to refer to complete factorization]. It might not be into these two things [two things refer to \((x - r_1)\) and \((x - r_2)\)]. It might still be a quadratic term depending on what these roots are. And that would be sort of a complete factorization.

In short, Jason began by holding a view that some polynomials cannot be factored depending on the domain, but after a short pause, changed his answer to “actually, over any domain you can do this as well.” The interviewer followed up his reasoning, as shown below.

I: Mm hmm. So you first mentioned that it depends on the domain and then you, I think, changed your answer. What were you thinking when you changed your thoughts?

Jason: So like over certain domains you can’t break it down any more. So like \(x^2 + 1\) over reals, you can’t really factor that any more. But I guess you could say it’s already factored completely because it’s irreducible [it refers to \(x^2 + 1\)]. Um, so then it’s sort of you can always do it completely… So what I was thinking here [referring to his written answer to part (a) “Yes, if you pick a good set (like \(\emptyset\)”) is that you have to break it down into smaller parts and, if you just leave it as it is, then it sort of doesn’t count [as factorization]. But then here [referring to the expression \(x^2 + 1\)] I was thinking it does count [as factorization]. You sort
of have it [*it seems to refer to factorization*] there, so I was thinking it’s already sort of factored…

I: So, is it then, if we consider factorization as some sort of a function, can we apply it to every single polynomial?

Jason: *Um I would say that you can apply it [*it refers to factorization*] to all polynomials. It’s just for some of them the output would be the input.* Um so you could say FR [writing $F_{\mathbb{R}}$; see Figure 4.22] That’s a factorization function over the domain reals. [*Jason writes $F_{\mathbb{R}}(x^2 + 1) = x^2 + 1$*] Uh it’d be like, *it’d act just as the identity does on them.*

![factorization function](image)

Figure 4.22. Jason’s representation of factorization as a function $F_{\mathbb{R}}$

It is notable that Jason was a participant who had demonstrated an object conception of factorization prior to this point of the interview (for example when defining complete factorization, as explained in section D3). However, Jason utilized a process conception of factorization in his initial response to POST-Q7 part (a) and part (c). It is possible that the way questions were phrased (using the verb “factor” that potentially emphasizes a process of factorization) led Jason to perceive the problem context as more appropriate to utilize his process conception of factorization rather than an object conception.
Particularly, Jason’s response to **PRE-Q7**\(^{61}\) was consistent with his initial response to **POST-Q7** (a) that “it depends on the domain.”

Interestingly, however, Jason reversed his former answer in the sequence of responses to **POST-Q7** by stating “actually, over any domain you can do [complete factorization].” It seemed Jason had moved his focus away from “break[ing] it down into smaller parts” (i.e., a process conception of factorization) to accept “leav[ing] it as it is” as a valid factorization. The analogy with a function, \(F_\mathbb{R}\), in the end of the preceding excerpts also highlights his conception of factorization as an object represented by an entity.

The gradual transition from D2 to D3 observed in Jason’s conception of factorization was further evidenced in his response to **END-Q4**\(^{62}\) in the last interview (INT5). When Jason was asked to compare his way of thinking about factorization before and after the whole interviews, he responded as follows:

> I guess before, any time I was doing any factoring, it was more just to get the job done. Like basically get it to the point where you sort of understand the structure. You don’t really like, you care about factoring it completely but you never really thought about what it means to factor it completely. Um whereas with these [reference for “these”]

---

\(^{61}\) **PRE-Q7**: Can you factor any quadratic polynomial? If so, how can you do so? If not, why do you think so?

\(^{62}\) **END-Q4**: Is what you learned about factorization today consistent with the way you thought about factorization before? If so, how are they similar? If not, how are they different?
unclear; possibly “these” refers to the interview questions], I was sort of forced to think about what that actually means and what sort of form you want it to be in, in order for it to be that [“that” seems to refer to complete factorization and “it” seems to refer to a polynomial].

Jason’s own description of the change in the way that he thought about factorization seems to be consistent with the analysis of the preceding episode that revealed a progression from a process conception of factorization to an object conception of factorization. Although Jason had started with already sophisticated understandings of factorization from the beginning the interviews, some problems he encountered during the interviews (such as POST-Q7) could have helped him to re-examine his prior understandings of factorization from a different and probably higher standpoint, thereby reconstructing his existing conceptions of factorization.

As Dubinsky (1997) noted, “any action, process, or object can be reconstructed, as a result of experiencing new problem situations on a higher plane, interiorizing more sophisticated actions and encapsulating richer processes” (p. 98). Borrowing Dubinsky’s language, one can say that Calvin’s and Jason’s episodes illustrated in the current section can not only be characterized as a transition from a process conception [D2] to an object conception [D3] but also as “encapsulation of richer processes,” which include consideration of uniqueness/nonuniqueness of factorization and that of identity factorization.
Summary

This chapter presented the major findings of this study, offering an empirical elaboration of the categorical framework of transformative transition. Findings were organized according to the four categories—extending, unifying, strengthening, and deepening. Under each category, each of the three levels (intra, inter, trans levels or action, process, object levels) was substantiated by analysis and illustrations of participants’ understandings of factorization and polynomial equations. Also, the last subsection for each category presented descriptions of transformative transitions made by participants.

Empirical findings presented in this chapter support the use of the categorical framework of transformative transition as an analytical lens for investigating university students’ school mathematics understanding and its growth in their learning of collegiate mathematics. Application of the framework according to the four categories and three levels under each category makes it possible to make sense of the intricacies of one’s mathematical understanding and its growth in an organized way.

To be specific, the findings reveal that university students’ school mathematics understandings—specifically those of factorization and polynomial equations—may exist at different levels in different forms. The following bullets exemplify some differences in participants’ understandings of factorization and polynomial equations. (All of these were described in much greater detail in the preceding four sections.)

- For some participants, the context of factorization of a polynomial was implicitly assumed to be over \( \mathbb{Z} \), and for some others, several contexts of
factorization were considered for making context-relevant mathematical conclusions about factorization (see the section *Extending* on page 157 for more details).

- For some participants, factorization of integers and factorization of polynomials appeared to have no bearing on each other, and for some others, both seemed to be understood essentially the same in that both reveal the structure of the given element in each set (see the section *Unifying* on page 190 for more details).

- For some participants, utilizing a link between a polynomial equation and its roots bidirectionally appeared to be challenging, and for some others, a bidirectional link seemed to be strongly established in their minds (see the section *Strengthening* on page 221 for more details).

- For some participants, factorization was mainly treated as a set of steps to be performed to get an answer, and for some others, factorization was treated as if it is an object to which certain conditions can be imposed without needing to evoke any process related to it (see the section *Deepening* on page 253 for more details).

Not only did the framework afford the characterization of different levels of one’s understandings, but it also offered different ways of characterizing one’s growth in school mathematics understandings. The four categories in the framework seemed useful in that characterization. By tracing a trajectory of each participant’s understandings
within each thread\textsuperscript{63}, I have documented and provided some evidence of advancing in levels in each of the four preceding sections.

To be specific, most participants (except for Helen) were observed to change or enrich their prior understandings of factorization and polynomial equations through extending, unifying, strengthening, and deepening in the course of interviews. For example, Jason’s transformative transitions described in sections *Unifying observed in the course of interviews* (page 214) and *Deepening observed in the course of interviews* (page 272) showed a nonlinear progression of \( U_3 \rightarrow U_2 \rightarrow U_3 \) and \( D_3 \rightarrow D_2 \rightarrow D_3 \), in which the later \( U_3 \) and \( D_3 \) were advanced versions of the previous \( U_3 \) and \( D_3 \). This implies that school mathematics understandings that were once constructed at a trans-object level or as an object conception may be still enriched further as new problem contexts are encountered in students’ collegiate mathematics experiences (such as dealing with trivial factorization or identity factorization). This observation was consistent with Piaget’s constructivist interpretation of a cyclic development of knowledge through (a) initial equilibrium status, (b) a state of disequilibration caused by the cognizing subject’s dissatisfaction in the coordination between prior knowledge and incoming information, and (c) reaching a more stable equilibrium. Some other participants, who had lacked in trans-object level understandings or object conceptions during PRE-TI, were also observed to have advanced in levels during POST-TI. The transformative transitions

\textsuperscript{63} The notion of thread is defined and explained in detail in the Methods chapter. Simply put, a thread is a sequence of codes (or, more broadly, the sequence of corresponding analysis units) that provides snapshots of one’s mathematical understandings about the same mathematical entity over time.
included, for example, increasing the boundary of contexts for considering and situating factorization of polynomials (see the section *Extending observed in the course of interviews* on page 179 for details) and coming to establish a bidirectional link between a polynomial equation and its roots (see the section *Strengthening observed in the course of interviews* on page 242 for details).

These findings are in contrast to the trends suggested by Cofer (2015) and Ticknor (2012). In both studies, school mathematics understandings of participants appeared to remain unchanged and unaffected by their learning of collegiate mathematics. In the current study, several instances of growth in participants’ school mathematics understandings were witnessed while they were engaging in the learning of factorization at the collegiate level. The findings of the current study, however, should be interpreted with caution because of its particular design component—that is, AiC-grounded teaching interviews (and this is one of the foci in the next chapter). Whereas the current study was conducted with such a particular instructional component, Cofer (2015) and Ticknor (2012) were conducted in an actual classroom setting with no experimental components. Despite this caution, the current findings are encouraging because such an occurrence of growth itself provides evidence that university students’ school mathematics understandings *can* be reorganized and restructured while they are learning collegiate mathematics.

At this point, it seems reasonable to consider the following questions: To what can this growth be attributed? Also, to what can a lack of growth be attributed? Considering those questions might add a new dimension of understanding with respect to
the findings in Chapter 4. In fact, these questions each underlies the third and fourth research questions of this study as follows.

- What are possibly relevant features of an AiC-grounded instructional context in which university students seem to make transformative transitions?

- When university students do not make transformative transitions in an AiC-grounded instructional context, what are some obstacles to making those transitions?

Chapter 5 provides a discussion of possible answers to those questions.
Chapter 5 Transformative transitions: Context and obstacles

In Chapter 4, some growth in participants’ school mathematics understandings was described as advance in levels within the four categories of the transformative transition framework. Of note in interpreting the findings of this study was that the growth was observed in a particular setting, which is the AiC-grounded instructional context. Chapter 5 provides a discussion of the context in which the growth was observed and identifies possible contextual features to which the growth might be attributed. The current chapter also discusses some obstacles that seemed to runs counter to making transformative transitions when a lack of growth was observed. Hence, Chapter 5 is composed of two sections: Contextual features in making transformative transitions and Obstacles to making transformative transitions.

Contextual features in making transformative transitions

What are some features of an AiC-grounded instructional context in which university students make transformative transitions? In this section, I address this question by investigating patterns in the threads of the levels of transformative transitions in the same category in terms of the context in which the transformative transitions were made and situated. Therefore, this section focuses on presentation of results based on analysis across threads, whereas Chapter 4 presented results obtained from analysis of each thread.

Since sections in Chapter 2 and Chapter 3 (entitled Abstraction-in-context [AiC] framework on page 78 and Teaching interview [TI] tasks on page 105) included extensive
description of the AiC approach, the description of AiC and TI in this section is limited to the gist of general principles of the AiC approach that was implemented in the teaching interviews [TIs]. This section then discusses four specific contextual features of the AiC-grounded TIs that seemed to be underlying and closely related to transformative transitions made by some participants.

As discussed in the previous sections on the AiC, the tasks in the TIs were devised using the AiC approach for the purpose of observing participants’ transformative transitions in the current study. In short, the principles of the AiC approach suggest that instruction based on the AiC intend “continuous transformation of constructs” (Dreyfus, Hershkowitz, & Schwarz, 2015, p. 190) that involves the learner’s engagement in the chain of Recognizing, Building-with, and Constructing actions (also called as R-, B-, and C-actions). In the instruction based on the AiC approach, the design was that the learner’s prior understandings would be activated in his/her R- and B-actions and serve as building blocks for a targeted C-action. According to Dreyfus and colleagues, C-action “consists of assembling and integrating previous constructs by vertical mathematization to produce a new construct” (p. 188) and such vertical reorganization takes place when a problem context is accessible via existing understandings and yet, not fully addressable with only existing understandings. That is, when existing understandings of the subject, as they are, are not sufficient for addressing the problem or for justifying the subject’s own claims, the context is likely to give rise to a targeted C-action. The tasks in the teaching interviews were designed based on the principles of the AiC approach described in this paragraph.
Analysis across threads that contained transformative transitions revealed engaging in the sequence of tasks grounded in the AiC approach could have provided a fertile environment for some participants to reorganize existing understandings either to produce a new construct or to incorporate a new construct in their schemas. A reorganization process inherent in their nested RBC actions seemed to underlie the growth in participants’ school mathematics understandings in four different ways, extending, unifying, strengthening, and deepening, although there is no assumption that these categories comprise all of the areas of growth. To discuss possible relationships between contextual features of AiC-grounded interviews and participants’ making transformative transitions, I draw on the instances of transformative transitions discussed in Chapter 4.

A sequence of tasks in the order of defining-applying-describing

The first contextual feature to which some participants’ transformative transitions appeared to be attributed is as follows:

A context in which a participant defines a construct in his/her own language, applies it to a variety of examples, and describes how his/her definition consistently applies to all examples seems to be closely related to some participants’ making a transformative transition of extending.

In this section I first provide a summary of some participants’ transformative transitions analysis from which the contextual feature arose. Then I explain how the sequence of defining-applying-describing allowed participants to engage in R-actions, B-actions, and
C-actions as in the AiC approach and how such engagement seemed to be closely related to their transformative transitions observed.

Analysis across three threads that contained transformative transitions in the extending category resulted in the tentative conclusion that this sequence of defining-applying-describing could have been related to the transitions they had demonstrated during the interviews. Specifically, Lucy, Calvin, and Sam demonstrated that they had increased the boundary of contexts in which they could consider factorization in several POST-TI interview questions (POST-Q1\textsuperscript{64}, POST-Q2 (iv)\textsuperscript{65}, POST-Q4\textsuperscript{66}, and POST-Q7\textsuperscript{67}). Those transformative transitions were discussed in detail in section Extending observed in the course of interviews on page 179.

In short, a comparison between their PRE- and POST-TI interviews revealed these participants had become aware of increasingly various contexts for considering factorization and to demonstrate sensitivity to contexts in their making a certain claim/conclusion about factorization. For example, Calvin in his response to PRE-Q2 (iv) (the problem is identical to POST-Q2 (iv)) said “This can never hold true” but in POST-Q2 (iv), he was able to provide an example of such case: $(x + 1)(x - i)$. Calvin

\textsuperscript{64} POST-Q1: Can you factor $ax^2 + bx + c$ without knowing the specific values of $a$, $b$, and $c$? If so, how can you do so? If not, why do you think so?

\textsuperscript{65} POST-Q2 (iv): Find (if possible) $a$, $b$, and $c$ such that $ax^2 + bx + c = 0$ has one real root and one nonreal root.

\textsuperscript{66} POST-Q4: If one of the roots of a quadratic equation is nonreal, is the other nonreal or real? Why is it so?

\textsuperscript{67} POST-Q7: Please explain your reasoning to the following questions.
(a) Can you factor any quadratic polynomial?
(b) Can you factor any quadratic polynomial uniquely?
(c) Can you factor any quadratic polynomial completely?
(d) Can you factor any quadratic polynomial completely and uniquely?
also articulated that “in ℂ, [having one real root and one nonreal root] makes sense, but in ℜ, it doesn’t. In ℚ and ℤ, it doesn’t.” Calvin’s mentioning the four domains ℤ, ℚ, ℜ and ℂ probably has to do with the fact that the four domains were used during the TIs for constructing a notion of complete factorization over a general domain D. However, no tasks in TIs used a quadratic polynomial with one real root and one nonreal root, and so the change in their responses (from “never hold true” to “makes sense” in this case) cannot be simply considered as what they learned during the TIs.

As I examined their threads in the extending category looking for some possible explanations for the change, I noticed a pattern that Sam, Lucy, and Calvin all encountered a moment in which their own definitions of complete factorization were not sufficient to explain their actualization of complete factorization over different domains. This moment occurred in a sequence of tasks in which they were (1) defining, (2) applying the definition, and (3) describing the application of the definition. This sequence was employed for three constructs (complete factorization, reducible polynomial, and irreducible polynomial) during the teaching interviews. The sequence of defining-applying-describing tasks relevant to this section is presented in Table 5.1.

Table 5.1. A sequence of tasks which engages participants in defining, applying the definition, and describing the application of definition

<table>
<thead>
<tr>
<th>Task type</th>
<th>Tasks</th>
</tr>
</thead>
<tbody>
<tr>
<td>Defining</td>
<td><strong>CORE-Q2</strong>: Students in high school algebra classes are often asked to ‘factor completely.’ What do you think the phrase ‘factor completely’ means?</td>
</tr>
</tbody>
</table>
| Applying the definition | **CORE-Q3**: According to your definition, please factor completely the following polynomials.  
  \[ P_1(x) = x^4 - 12x^2 + 36 \]  
  \[ P_2(x) = x^5 - x^2 \]  |
CORE-Q4: Please factor completely the same set of polynomials over each of the number systems, \( \mathbb{Z} \), \( \mathbb{Q} \), \( \mathbb{R} \) and \( \mathbb{C} \) and record your results in the following table.

Follow up:
How does [an expression in participant’s answers in the table] fit your description of factoring completely?

<table>
<thead>
<tr>
<th>( p_1(x) = x^4 - 12x^2 + 36 )</th>
<th>( p_2(x) = x^5 - x^2 )</th>
<th>( p_3(x) = 4x + 16x^3 )</th>
<th>( p_4(x) = 4x + 2x - 20 )</th>
<th>( p_5(x) = 2x^3 - 10x^2 + 8x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Over ( \mathbb{Z} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Over ( \mathbb{Q} )</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Over ( \mathbb{R} )</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Over ( \mathbb{C} )</td>
<td></td>
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<td></td>
</tr>
</tbody>
</table>

The sequence of defining-applying-describing appeared to have allowed Sam, Lucy, and Calvin engage in R-, B-, and C-actions as in the AiC approach in the following ways. In CORE-Q2, all three participants were able to recognize ‘factor completely’ as something familiar to them (i.e., R-action) and were able to define (or describe the meaning of) the construct with relative ease based on their prior understandings (i.e., B-action). At this point, participants’ definitions (see Table 5.2) lacked recognition of a context in which complete factorization was being considered.
Table 5.2. Participants’ initial description of the meaning of “factor completely” in
CORE-Q2

<table>
<thead>
<tr>
<th>Participant</th>
<th>Participant’s description of the meaning of “factor completely”</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sam</td>
<td>To take a number or expression and split it into factors that cannot be reduced any further</td>
</tr>
<tr>
<td>Lucy</td>
<td>To decompose into smallest parts</td>
</tr>
<tr>
<td>Calvin</td>
<td>Rewriting the polynomial as a product of its roots</td>
</tr>
</tbody>
</table>

In CORE-Q3, Sam, Lucy, and Calvin were asked to apply their own definitions from CORE-Q2 to six different polynomials (i.e., B-action). Although their definitions did not articulate over which domain the complete factorization is defined, Sam’s and Lucy’s responses in other tasks revealed their implicit assumption that their complete factorization was defined over \( \mathbb{Z} \); and Calvin’s responses revealed his implicit assumption over \( \mathbb{R} \). Up to this point, the absence of a context in their own definitions seemed to go unnoticed by the three participants, and their tendency of assuming a particular context in their consideration of factorization appeared to persist.

In CORE-Q4, participants encountered four different contexts (over \( \mathbb{Z}, \mathbb{Q}, \mathbb{R} \) and \( \mathbb{C} \)) in which they were asked to consider complete factorization. At this point, going through all four domains seemed to require minor (in Lucy’s case) to major (in Calvin’s and Sam’s cases) adjustment in their previous conceptions of complete factorization because, as described previously, complete factorization for them was meant to be defined in a particular context. For example, Calvin registered confusion when he tried to factor \( P_1(x) \) completely over \( \mathbb{Z} \), by comparing the situation with his previous way of factoring completely over \( \mathbb{R} \) as follows:
Calvin: I have $(x^2 - 6)^2$ [referring to his factorization of $P_1(x) = (x^2 - 6)^2$ = $(x - \sqrt{6})^2(x + \sqrt{6})^2$ in CORE-Q3]. It is a factorization of this larger polynomial $x^4 - 12x^2 + 36$. But if you go one step further it becomes irrational. I’m not sure what step I want to stop at, if I want to stop at this step [referring to $(x^2 - 6)^2$], or just the normal polynomial step [referring to $x^4 - 12x^2 + 36$].

I: I want to ask, what would it mean to you to factor completely over $\mathbb{Z}$?

Calvin: I guess if the roots can be represented with integers, whereas with the expression $x^2 - 6$ technically, no... **All of this is weird. I’m thinking over $\mathbb{R}$, but it’s not.**

**It’s over $\mathbb{Z}$**. I think it’s a tentative $(x^2 - 6)^2$... You can factor $P_1$ as $(x^2 - 6)^2$.

Six is an integer so your roots are $x^2$ is equal to six. I don’t know. I’ll go with it for now, and then if I see some inconsistency over it, I’ll come back and change it.

Calvin’s saying “All of this is weird. I’m thinking over $\mathbb{R}$, but it’s not, it’s over $\mathbb{Z}$” indicates that his previous way of thinking of complete factorization as “rewriting the polynomial as a product of its roots” with an implicit assumption over $\mathbb{R}$ was not sufficient to deal with a new (to him) context, $\mathbb{Z}[x]$. As he completed all 24 cells in the table associated to CORE-Q4 with the goal of maintaining consistency in his complete factorizations, Calvin finally reached construction of new definition of complete factorization as follows: “Factoring completely over $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ is to write the
polynomial as a product of its roots, with coefficients in the corresponding number system.”

In his written definition above, a targeted C-action in the sequence of CORE-Q2 to Q4 (which was construction of the relativity of complete factorization) was manifested when he indicated “with coefficients in the corresponding number system.” One possible way to bridge Calvin’s engagement in this sequence of tasks during TIs and his transformative transition demonstrated in POST-TI interviews is as follows: Calvin’s C-action on the relativity of complete factorization could have involved reorganization in his schema of factorization by extending the boundary of contexts in which he could consider factorization (initially strictly over $\mathbb{R}$, to gradually include other contexts such as over $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{C}$). Such an extended boundary could have allowed him to consider a quadratic polynomial such as $(x + 1)(x - i)$ as a possibility for his response to POST-Q2 (iv).

Sam and Lucy also explored different possibilities of complete factorization of the six polynomials depending on the domain of choice in CORE-Q4. As a follow-up question, they were also asked to devise definition of complete factorization as reflected in their responses in the table and to articulate how their definitions consistently applied to all factorizations in the table. In this last task of the sequence, Lucy and Sam also engaged in a targeted C-action on the relativity of complete factorization by indicating, for example, that the factorization has to be completed “without leaving the domain.”

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68 Calvin seemed to use “roots” to refer to both irreducible factors of a given polynomial and roots of the corresponding polynomial equations in other occasions.
In this sequence of (1) defining, (2) applying the definition, and (3) describing the application of definition, it should be noted that the sequence started with establishing their own definitions, which reflects beginning with R- and B-actions in RBC model. For Lucy, Calvin, and Sam, these R- and B-actions throughout their engagement in CORE-Q2 to Q4 seemed to have served as building blocks for a targeted C-action on relativeness of complete factorization. Also having a variety of polynomials whose irreducibility varies across different domains and pinning down what they meant by complete factorization that could explain a variety of examples they produced (i.e., the third step in the sequence) could have been a suitable context for them to build a context-sensitive construct of complete factorization. By engaging in this sequence of tasks, not only did participants seem to become aware of multiple relevant contexts for factorization but they were also able to demonstrate their understandings of how the choice of different contexts might affect the way factorization operates in their POST-TI interviews.

An unusual or unexpected encounter with a problem beyond their routines or assumptions

The second contextual feature that arose from the analysis of participants’ transformative transitions in the unifying category is as follows:

A context in which a participant encounters a possibility beyond his/her routine or examines unexpected cases that cannot be explained by his/her prior understandings seems to be closely related to some participants’ making a transformative transition of unifying.
This section first provides a summary of some participants’ transformative transitions in the unifying category, discusses how the context of making unusual/unexpected observations reflected the AiC approach, and concludes with how such a context might help explaining their transformative transitions that were observed.

Analysis across threads that contain transformative transitions in the unifying category resulted in the tentative conclusion that unusual or unexpected encounters with problems beyond their routines or assumptions could have been related to the transitions they had demonstrated during the interviews. Specifically, Calvin, Lucy, Sam, and Andy who demonstrated their intra-object level understandings [U1] in PRE-Q8\textsuperscript{69} and PRE-Q9\textsuperscript{70} were able to explain factorization across different cases using overarching concepts such as decomposition, reducible, and irreducible during TIs and in POST-Q8 and POST-Q9. Those transformative transitions were discussed in detail in sections U1 (page 193), U3 (page 209), and Unifying observed in the course of interviews (page 214).

Among the four, three participants (Calvin, Lucy, and Sam) seemed to initially hold a view that separates factorization of polynomials from factorization of integers and did not demonstrate a decomposition view that could explain and underlie both factorization of integers and factorization of polynomials during their PRE-TI interviews. In their POST-

\[69 \text{PRE-Q8: According to the Common Core State Standards high school students need to Understand that polynomials form a system analogous to the integers. In what sense do you think they are analogous?}\]

\[70 \text{PRE-Q9: Please construct a concept map of YOUR ideas about polynomial in the context of school mathematics. (A full version of this question can be found in Appendix C.)}\]
TI interviews, however, their responses suggested that they were able to see both integers and polynomials could be factored into (“split into,” “decomposed into,” “written as a product of” in their words) more basic components such as primes and irreducibles, thereby indicating a transformative transition in the unifying category.

In my analysis of their threads in the unifying category looking for some explanations for the change, I identified the pattern that Calvin, Lucy, and Sam all seemed to find their encounters with \( p_4(x) = 12 \) in CORE-Q3 and CORE-Q4 somewhat unexpected. Their encounters being “unexpected” was evidenced by the nature of their reactions upon their encounter with \( p_4(x) = 12 \) or by their self-reports during the retrospective questions. For example, when Calvin encountered \( p_4(x) = 12 \) in CORE-Q3, he said “this doesn’t really have any roots… I would defer this one for now if that’s fine. The \( p_4(x) \) is equal to 12. I’m just going to leave it. I can’t really do any manipulation on it. There’s no zeros.” In his second encounter with \( p_4(x) = 12 \) in CORE-Q4, he again registered a lack of confidence by stating “\( p_4(x) \), oh Lord. It doesn’t have roots” and concluded “I guess the complete factorization of it would be 12 in all four systems.” It seemed that, for Calvin who treated complete factorization as “rewriting the polynomial as a product of its roots,” the expression \( p_4(x) = 12 \) was beyond his routine procedure of factoring a polynomial. Notably, Calvin in WARM-UP-Q1 (How can you factor 60 completely?) responded that 60 = \( 2^2 \cdot 3 \cdot 5 \), which means if he were asked to factor 12 completely in a standalone question, without being in the group of polynomials as in CORE-Q3 or Q4, he probably would have answered 12 = \( 2^2 \cdot 3 \). In Calvin’s schema, it seemed possible that different procedures of factorization were
evoked when 12 was viewed as an integer (into $2^2\cdot3$) and when viewed as a polynomial (into 12 as is).

Sam and Lucy also commented that the way that they had previously thought about factoring did not necessarily involve an expression such as $p_4(x) = 12$. Sam in one of retrospective questions stated, “I had only ever thought of factoring in terms of polynomials and not in terms of numbers” and “It was weird to think about [factoring a number and factoring a polynomial altogether] for the first time.” Lucy also mentioned, “people, including myself, don’t look at 12 and say oh it can factor” in her response to POST-Q8.

Such an unusual or unexpected encounter with an expression $p_4(x) = 12$ in their B-actions on factorization may be viewed as a necessary step for a targeted C-action (in this case, C-action on the primeness of complete factorization) from the AiC perspectives. As previously mentioned, C-action takes place through a vertical reorganization in which the problem situation cannot be fully addressed using only existing understandings. For example, in Calvin’s case, his original definition of complete factorization ("rewriting the polynomial as a product of its roots") was not sufficient to account for the case of $p_4(x) = 12$. In follow-up questions to CORE-Q4, Calvin was asked to consider the notion of irreducible in his factorization, which redirected his focus from roots to factors that make up complete factorization. In CORE-Q6\(^71\), he eventually constructed his final definition of complete factorization that could

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\(^71\) **CORE-Q6**: How would you explain to your friend what a complete factorization of a polynomial over a domain $D$ is? (*$D$ is used to stand for $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$).*
be also applied to the case of \( p_4(x) = 12 \) as follows: “A polynomial is completely factored over \( D \) if it is written as a product of irreducible or (tentative: unit) terms over \( D \).”

All three participants, Calvin, Sam, and Lucy, by incorporating the concept of irreducible in their definition of complete factorization, constructed their definitions of complete factorization in CORE-Q6 that manifested their understanding of the primeness of complete factorization. One possible way to relate their construction of the primeness that embraced initially unexpected observation of \( p_4(x) = 12 \) and their transformative transition demonstrated in POST-TI interviews follows. Their C-action on the primeness of complete factorization could have involved reorganization in their schemas of factorization by including the unexpected case such as \( p_4(x) = 12 \), which eventually gave rise to an overarching perspective on both factorization of integers and factorization of polynomials. Such an overarching perspective could have allowed them to examine the statement in POST-Q8 (polynomials form a system analogous to the integers) from a different perspective when compared to their initial interpretation of the same statement in the PRE-TI interview, thereby providing a venue for them to make transformative transitions in the unifying category.

**A task with a variety of instantiations of a single entity to reflect on**

The third contextual feature that arose from the analysis of participants’ transformative transitions applied to the strengthening category as follows:
A context in which a participant encounters a variety of instantiations of a single entity (e.g., a variety of polynomial expressions) to reflect on seems to help reveal potential glitches in his/her understandings. Such a context, when the glitch is noticed and appropriately addressed, seems to be closely related to some participants’ making a transformative transition of strengthening.

This section first presents a summary of Calvin’s and Sam’s transformative transitions in the strengthening category and concludes with how the context of having a variety of instantiations of a single entity to reflect on (as a context for enacting RBC actions) might be closely related to their transformative transitions that were observed.

As discussed in detail in section Strengthening observed in the course of interviews (page 242), Calvin and Sam both made a transformative transition in the strengthening category by reexamining the link between a polynomial equation and its roots in their schema of polynomial equation (in short, a root–equation link). Calvin in his factorization of $P_3(x) = 4x + 16x^3$ factored out $4x$ and related the roots of $1 + 4x^2 = 0$ to the original polynomial, thereby obtaining $4x \left( x - \frac{1}{2}i \right) \left( x + \frac{1}{2}i \right)$ as complete factorization of $P_3(x)$. Calvin immediately noticed that his answer was not equivalent to $P_3(x)$, but it took him two separate interviews (INT 3 and INT 6) to address completely the gap between his answer and the original expression. Even after he noticed that they differ by a factor of 4, he registered a confusion, which turned out to be the reflection of his previous way of thinking that factoring out a leading coefficient might “screw things up” and change the roots of an equation. He finally resolved this issue by checking
simple examples, $2x + 1 = 0$ and $2(x + \frac{1}{2}) = 0$, having the same root, and convinced himself that $16x \left( x - \frac{1}{2}i \right) \left( x + \frac{1}{2}i \right)$ is complete factorization of $P_3(x)$.

Factoring $P_5(x) = 4x^2 + 2x - 20$ caused a similar issue for Sam. After factoring out 2, Sam applied the quadratic formula to $2x^2 + x - 5 = 0$, related the roots (i.e., 2 and $-\frac{5}{2}$) to the original polynomial, and obtained $2(x - 2)(x + \frac{5}{2})$ as complete factorization of $P_5(x)$. Once Sam noticed the inconsistency between $P_5(x)$ and the expanded form of his answer $2x^2 + x - 5$, he stated “trying to figure out why. I’m not sure why.” Sam then began to factor $P_5(x)$ again using a grouping method, which led him to observe that different equations may result in the same set of roots. In both Calvin’s and Sam’s cases, the root–equation link was originally not fully coordinated to address the problem situation, but they eventually addressed the issue by understanding the role of the leading coefficient in linking a polynomial equation and its roots.

In relating those transformative transitions with the teaching interview context, it was notable that the same kind of glitch was noticed by Sam and Calvin in dealing with two different expressions. For Sam, dealing with $P_5(x) = 4x^2 + 2x - 20$ and for Calvin, dealing with $P_3(x) = 4x + 16x^3$ each revealed their implicit assumption that the leading coefficient has to be 1, when they took a set of roots, say $s$ and $t$, of $P(x) = 0$ and related them back to the original polynomial as $P(x) = (x - s)(x - t)$. In fact, Sam applied this assumption to obtain the same answer as Calvin’s original answer to $P_3(x)$, but the mismatch between his answer and $P_3(x)$ went unnoticed by Sam. This implies that different individuals may notice a glitch in their understandings in dealing with different entities (in this case, $P_3(x)$ and $P_5(x)$) and that having a variety of instantiations of
polynomial on which a learner can reflect might increase a chance for him/her to examine his/her own thinking. When such a glitch is noticed by participants and appropriately addressed, the context itself seemed to have acted as a precursor of their strengthening process.

As previously mentioned, the variety of polynomials in \textbf{CORE-Q3} and \textbf{Q4} (see Table 5.3 for how they differ under different criteria) was intended to provide a venue for constructing a definition of complete factorization over general domain \(D\) as comprehensively as possible.

Table 5.3. A variety of polynomials in \textbf{CORE-Q3} and \textbf{CORE-Q4} when compared under five different criteria

<table>
<thead>
<tr>
<th>Polynomial</th>
<th>Degree</th>
<th>Leading coefficient</th>
<th>Applicable formula(s) (*not necessary)</th>
<th>Root types of (P(x) = 0)</th>
<th>Complete factorization takes a different form between (\mathbb{Q}[x]) and (\mathbb{R}[x])</th>
</tr>
</thead>
<tbody>
<tr>
<td>(P_1(x) = x^4 - 12x^2 + 36)</td>
<td>4</td>
<td>1</td>
<td>Difference of squares formula for (x^2 - 6); Quadratic formula for (x^2 - 6 = 0)</td>
<td>Irrational roots</td>
<td>(\mathbb{Q}[x]) and (\mathbb{R}[x])</td>
</tr>
<tr>
<td>(P_2(x) = x^5 - x^2)</td>
<td>5</td>
<td>1</td>
<td>Difference of cubes formula for (x^3 - 1); Quadratic formula for (x^2 + x + 1 = 0)</td>
<td>Integer roots and nonreal roots</td>
<td>(\mathbb{R}[x]) and (\mathbb{C}[x])</td>
</tr>
<tr>
<td>(P_3(x) = 4x + 16x^3)</td>
<td>3</td>
<td>16</td>
<td>Quadratic formula for (4x^2 + 1 = 0); Difference of squares formula for (4x^2 + 1)</td>
<td>An integer root and imaginary roots</td>
<td>(\mathbb{Z}[x]) and (\mathbb{Q}[x];) (\mathbb{R}[x]) and (\mathbb{C}[x])</td>
</tr>
<tr>
<td>(P_4(x) = 12)</td>
<td>0</td>
<td>12</td>
<td>-</td>
<td>No root</td>
<td>(\mathbb{Z}[x]) and (\mathbb{Q}[x])</td>
</tr>
<tr>
<td>(P_5(x) = 4x^2 + 2x - 20)</td>
<td>2</td>
<td>4</td>
<td>Quadratic formula for (2x^2 + x - 10 = 0)</td>
<td>An integer root and a noninteger rational root</td>
<td>(\mathbb{Z}[x]) and (\mathbb{Q}[x])</td>
</tr>
</tbody>
</table>
As shown in the table, the six polynomials vary in terms of their degrees, leading coefficients, applicable formulas (if any), and root types. This variety was intended to help participants gradually enact their R- and B-actions on *multiple aspects* of polynomials in their factorization and, as a result, C-actions on three subconstructs of the UFT (relativeness, primeness, and uniqueness), which otherwise possibly could have been dismissed. For example, if all the polynomials had the leading coefficient of 1, an issue of trivial factorization in defining complete factorization could have been less salient for some participants and not addressed in their definition of complete factorization. Especially for the construction of *uniqueness* of complete factorization, one needs to examine the role of the leading coefficient in allowing multiple forms of factorization. Because those *multiple* forms are caused by trivial factorizations with multiplication and division by a unit, a natural question arose: Which should be considered as complete factorization of $P_3(x) = 4x + 16x^3$ between, for example, $4x(1 + 4x^2)$ and $16x\left(\frac{1}{4} + x^2\right)$ over $\mathbb{F}$? One possible solution to this question is that both may be considered complete factorization but, up to multiple of a unit, complete factorization is essentially unique. In order to help participants to engage in a targeted C-action on his kind of uniqueness, four out of six polynomials were chosen to have the leading coefficient not equal to 1.

In particular, Calvin’s transformative transition (i.e., realizing factoring out the leading coefficient does not change the set of roots) occurred in the context of exploring
the role of units in defining complete factorization. As described in detail in the section *Strengthening observed in the course of interviews* (page 242), Calvin was applying his definition of complete factorization to each of his answers in the table in **CORE-Q4**, and the very question that initiated his exploration of two examples, $2x + 1 = 0$ and $2(x + \frac{1}{2}) = 0$, was “Does your definition accept both of these [referring to two different factorizations provided by Calvin in one of the cells in the table] complete factorizations?” This interviewer’s question, which was originally directed to the uniqueness of complete factorization, seemed to function as a venue in which Calvin realized that factoring out the leading coefficient does not change the set of roots of a polynomial equation.

To summarize, a byproduct of having a variety of polynomials for targeted R-, B-, C-actions during the teaching interviews seemed to be that participants had more venues in which to observe a potential glitch in his/her understandings. For Sam and Calvin, such observation of a glitch led them to reexamine an assumption they seemed to be accustomed to taking for granted in their utilization of a link between polynomial equation and its roots; and the observation seemed to create momentum for their transformative transitions in the strengthening category.

**A task for determining the truth of a general statement regarding a construct**

The fourth contextual feature that arose from the analysis of participants’ transformative transitions in the deepening category is as follows:
A context in which a participant determines truth of a general statement about a construct seems to be closely related to some participants’ making a transformative transition of deepening.

In this section, I summarize Jason’s and Calvin’s transformative transitions from which this fourth contextual feature arose. Then the context of determining the truth of a general statement, which involves R- and B-actions on a process conception, is discussed in relation to their transformative transitions observed in the deepening category.

As discussed in detail in the section Deepening observed in the course of interviews (page 272), Jason and Calvin both made transformative transitions—from a process conception to an object conception of factorization—while they were working on POST-Q7. Jason, in part (a) and (c) of POST-Q7, started by thinking that some quadratic polynomials cannot be factored or factored completely. In this initial reasoning, Jason seemed to be operating on his process conception of factorization as suggested by his illustration such as “you have to break it down into smaller parts.” However, Jason almost immediately changed his initial response without any prompt by the interviewer. He appeared to move his focus away from a process of factorization (or “break[ing] it down into smaller parts”) and attend to the question of whether complete factorization would exist for any quadratic polynomial. Consequently, Jason concluded to accept the

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POST-Q7: Please explain your reasoning to the following questions.
(a) Can you factor any quadratic polynomial?
(b) Can you factor any quadratic polynomial uniquely?
(c) Can you factor any quadratic polynomial completely?
(d) Can you factor any quadratic polynomial completely and uniquely?
case of “leav[ing] it as is” as complete factorization. He provided $x^2 + 1$ over $\mathbb{R}$ as an example of an expression the complete factorization of which is going to be itself. Jason also used a notation such as $F_\mathbb{R}(x^2 + 1) = x^2 + 1$ to represent the identity-like factorization. This suggests that Jason was able to treat factorization as if it is an object that could be represented by an entity, $F_\mathbb{R}$.

Calvin also seemed to utilize his process conception of factorization when comparing part (b) and (d) of POST-Q7 initially. After comparison, however, Calvin seemed to be operating at an object level. Calvin concluded that “this last statement [referring to part (d)] is true” and posed a question to himself with respect to part (b) in a conditional statement form: “if you factor something uniquely, is there an implied completeness to that?” Although the question was not further explored nor justified by Calvin, Calvin appeared to be trying to associate a certain characteristic (e.g., completeness) to factorization in his questioning. This act of associating a characteristic to factorization without evoking its process/action conception can be viewed as an indicator of an object conception.

Common to both participants’ reasoning is that Jason and Calvin both initially utilized their process conceptions of factorization but further comparison between different parts of POST-Q7 led them to naturally attend to some ontological questions such as whether complete factorization exists for any quadratic polynomial and whether there exists “an implied completeness to [unique factorization].” In fact, the four-part questions in POST-Q7 that were examined by Jason and Calvin can be translated into the following four true/false statements:

(a) Any quadratic polynomial can be factored. (T/F)
(b) Any quadratic polynomial can be factored uniquely. (T/F)

(c) Any quadratic polynomial can be factored completely. (T/F)

(d) Any quadratic polynomial can be factored completely and uniquely. (T/F)

Perhaps, reasoning through and determining truth of such a general statement that applies to “any quadratic polynomial” (not only to a specific polynomial) could have involved a shift in focus from factorization as a process to factorization as an ontological entity. For example, when Jason was reasoning through part (c), the question could have been translated in his mind into “does complete factorization always exist for any quadratic polynomial?” He did not verbalize such question explicitly, but he did make a relevant comment such as “it’s sort of you can always do it completely [it seems to refer to factoring a quadratic polynomial]” in his explanation.

In sum, this context of determining truth of a general statement seemed to have provided a venue for observing some participants’ shift from a process conception to an object conception of factorization. A possible hypothesis is that, for some university students, determining truth of a general statement may lead them to reexamine their relevant existing understandings from a notably different perspective. A norm that had previously been accepted without a doubt when factoring individual polynomial expressions (such as a resultant factorization being a product of lesser degree polynomials) may be questioned and given a second thought in determining truth of a general statement. While the question is being addressed, factorization may become an object under consideration (e.g., does factorization always exist?); when the question is addressed, the student may leave with an enriched conception of factorization.
Summary of contextual features

In the preceding four sections, the particularity of the AiC-grounded interview context was discussed in detail to shed light on participants’ transformative transitions observed in the current study. The goal was to identify and explain contextual features that seemed to underlie the observed transformative transitions commonly. Four contextual features were identified as follows:

i) A sequence of tasks in the order of defining-applying-describing,

ii) An unusual or unexpected encounter with a problem beyond their routines or assumptions,

iii) A task with a variety of instantiations of a single entity to reflect on, and

iv) A task for determining the truth of a general statement regarding a construct.

Collectively, the four features offer some strategies for designing the teaching of collegiate mathematics that both builds on and advances students’ prior understandings. Evidence suggested that, while participants were engaging in those contexts, some of them have constructed and consolidated the three subconstructs of the UFT (relativeness, primeness, uniqueness) and furthermore, have advanced their school mathematics understandings in four different ways, extending, unifying, strengthening, and deepening.

Underlying their transformative transitions in all four contexts seemed to be their observation of and reflection on some assumptions and norms that had been exercised and established in their school mathematics. Participants’ prior assumptions and norms discussed in the preceding sections include, for example, assuming factorization over \( \mathbb{Z} \) and \( \mathbb{Z} \) only, separating polynomial factorization from number factorization, assuming
leading coefficient is 1 when translating roots to factors, thinking factorization always results in a product of lesser degree polynomials. These assumptions and norms were reexamined by the participants from different angles in the four contexts, and as a result, their prior understandings were reconstructed and interrelated to form an enhanced understanding. In Freudenthal’s words, the phenomenon can be described as: “mathematics exercised on a lower level becomes mathematics observed on the higher level” (1978/2004, p. 61).

Although the four contexts seemed to have facilitated some participants’ transformative transitions, those contexts seemed not sufficient to ensure transformative transitions. Some other participants remained at the same level of understandings or had some difficulty in advancing in levels while engaging in the interviews. Hence, the next section provides an analysis of threads in which no transformative transitions were made and a discussion of what might run counter to making transformative transitions.

**Obstacles to making transformative transitions**

The current section focuses on the obstacles that participants encountered when the opportunities for making transformative transitions were missed. Four obstacles, which seemed to keep participants from making progress, were identified from the analysis of data as follows:

i) Tendency to overlook the mathematical entities under consideration

ii) Conflated use of different concepts or procedures

iii) Compartmentalization accompanied by reliance on formulas
iv) Utilitarian/instrumentalist view

These obstacles can be understood as the tendency or belief university students might have that runs counter to transformative transitions. In what follows, the four obstacles are discussed in detail and proposed as possible explanations for a lack of growth in some participants’ understandings.

**Tendency to overlook the mathematical entities under consideration**

Among participants whose understanding of factorization stayed at the inter-object level in the unifying category (U2) during the TIs, a pattern was observed that they had the tendency to overlook the mathematical entities under consideration. More specifically, in their association of a mathematical property with a mathematical entity, they seemed not to retain a focus on the entity with which they began, thereby allowing entities under consideration to change. From an outsider’s perspective, it seemed that participants detached a property from a mathematical entity under consideration and re-attached the property to another entity. This phenomenon of failing to recognize and oversee the mathematical entities under consideration is discussed in detail in the current section building on the case of associating a property of irreducibility to a polynomial.

Associating a property of irreducibility to a polynomial implies that, given a polynomial, one can determine whether the polynomial is irreducible and furthermore describe what an irreducible polynomial means in general. An irreducible polynomial is defined as a nonunit polynomial that has no factors except trivial ones within the set of polynomials; Here a trivial factor of a polynomial, $p$, means a unit or a unit-multiple of
itself. Hence, the factorization $2 \cdot \left(\frac{x-1}{2}\right)$ does not prevent $x - 1$ from being an irreducible polynomial over $\mathbb{Q}$. In other words, if a polynomial $f$ is irreducible, whenever $f$ is written as the product of two polynomials $p$ and $q$ (i.e., $f = p \cdot q$), either $p$ or $q$ has to be a unit. Obviously, a polynomial cannot be irreducible and reducible at the same time. However, for some participants (Helen, Calvin, and Sam), equivalent polynomials, depending on the way they are written, were considered sometimes reducible, other times irreducible.

Helen’s case will be illustrated in detail as a representative among the three participants. An interesting statement made by Helen that is worth investigating is “4 is reducible but $2^2$ is irreducible.” To understand her logic behind this statement, I first introduce her initial definition of reducible and irreducible polynomials. Helen verbally described the difference between reducible and irreducible polynomials as follows:

“reducible would be if you are able to factor it [into] smaller terms. Irreducible means, you hit a certain level, like the smallest factor. You can’t reduce it any more—so irreducible.” In her written definition, she seemed to be more specific in terms of which entity she was referring to in defining reducible and irreducible polynomials. The following is a reproduction of Helen’s written definitions of reducible and irreducible.

**Reducible:** A term [that] can be factored to the smallest factors

**Irreducible:** A small term which cannot be reduced further in terms of multiplication

**Examples:** $(1 + 4x^2)$ is reducible over $\mathbb{C}$. $(x + \frac{1}{2}i)$ and $(x - \frac{1}{2}i)$ are each irreducible over $\mathbb{C}$. 
Helen’s use of “term” in her written definition along with her examples seemed to suggest she was able to associate the property of irreducibility to a polynomial. However, in her response to [core-q9] in a later interview, Helen seemed to be no longer referring to the same entity when she discussed irreducibility. Helen was examining the irreducibility of \( x^2 + c \) when \( c = 0 \) as follows.

Helen: For the case \( c = 0 \), \( x^2 \) can be, is reducible to \( x \) times… Oh, it is irreducible. It \([\text{referring to } x^2]\) can be split up to \( x \) times \( x \) but then these two \([\text{referring to } x \text{ and } x]\) are the same. So it \([\text{referring to } x^2]\) would be irreducible over \( \mathbb{Z}, \mathbb{Q}, \mathbb{R} \) and including \( \mathbb{C} \). Because there is not much that I can do.

I: Is it consistent with your definition? \([\text{pointing to Helen’s written definition of “irreducible”}]\) A small term which cannot be reduced further.

Helen: Mm-hmm.

I: You mean \( x^2 \) satisfies this description?

Helen: Mm-hmm. So the case that was irreducible \([\text{writing } (x - 1)^2]\)… Um. Even though when you look at it as a whole, it can be like \( (x - 1) \) times \( (x - 1) \), since these two \([\text{referring to } (x - 1) \text{ and } (x - 1)]\) are the same exact term, it wouldn’t

\[\text{------------------------}\]

\[73\] Helen’s use of “term” is obviously different from a formal definition of “term” in mathematics.

\[74\] CORE-Q9: Let \( c \) be an integer. Discuss the irreducibility of the polynomial, \( x^2 + c \), over \( \mathbb{Z}, \mathbb{Q}, \mathbb{R} \) and \( \mathbb{C} \).
be considered two different, like $2^2$. So this \textit{referring to $(x - 1)^2$} is irreducible just like $2^2$ is irreducible.

I: Oh, so $2^2$ is irreducible too?

Helen: Yeah. \textbf{4 is reducible but $2^2$ is irreducible.}

The fact that equivalent polynomials (e.g., 4 and $2^2$; $(x^2 - 2x + 1)$ and $(x - 1)^2$) were considered sometimes reducible, other times irreducible suggests that Helen overlooked the mathematics entity (i.e., a polynomial itself) under consideration. Helen seemed to re-associate the property of irreducibility to a form of a polynomial (for example, whether it is written in an expanded form or in a complete square form). Also, her recognition that $(x - 1)$ is an irreducible polynomial could have possibly led her to attach the property to another entity $(x - 1)^2$ that looked similar without noticing the entity under consideration has changed.

Similar phenomena were observed in Calvin’s and Sam’s responses. Calvin described “an irreducible polynomial is written as its complete factorization over a certain number system” and exemplified his point by providing $(x^2 - 6)^2$ as an irreducible polynomial over $\mathbb{Q}$ but $x^4 - 12x^2 + 36$ as a reducible polynomial over $\mathbb{Q}$. When the interviewer asked if it is okay to have two equivalent polynomials and call one of them reducible and the other irreducible, Calvin did not seem to find it problematic. He concluded “I would say it’s reducible if it can be factored more. So this \textit{referring to} $x^4 - 12x^2 + 36$ can be factored more. And irreducible means it cannot be factored more. In $\mathbb{Q}$, it \textit{referring to $(x^2 - 6)^2$} cannot be factored more, so it’s irreducible. This
[referring to $x^4 - 12x^2 + 36$] is reducible.” In similar fashion, Sam also noted

$(4x + 16x^3)$ is reducible over $\mathbb{R}$ “because you can factor into two other expressions” but

$4x(1 + 4x^2)$ is irreducible over $\mathbb{R}$ “because I can’t factor either of these expressions

[referring to $4x$ and $(1 + 4x^2)$] without using units and staying in the real domain.”

All three participants (Helen, Calvin, and Sam), at some point of their interviews, were asked by the interviewer to focus on a polynomial itself (regardless of how it is written) than on its forms when considering irreducibility. However, their tendency to overlook the entity under consideration (i.e., a polynomial) seemed to be overpowering and they kept returning to associate irreducibility to the forms of a polynomial. With such a tendency, it is reasonable to expect a statement such as “the product of two irreducible polynomials is a reducible polynomial” (heretofore called the statement* in this section) would be interpreted as false right away. It is because, as long as the product of two irreducible polynomials is written in a factored form, it would be still considered irreducible for these participants. Hence, their reflection upon the statement* did not naturally bring up the issue of trivial factorization, nor did it provide a rationale behind the unit concept.

In contrast, Jason associated a property of irreducibility to a polynomial (not to a form of a polynomial); and so, to him, the statement* was initially believed to be true. However, upon his discovery of trivial factorization such as $4 \cdot x$ and the issue of whether $4 \cdot x$ should be considered reducible and irreducible, Jason found the statement required further examination. Having recognized such an issue, Jason found the unit idea as a help in addressing the issue and concluded: “multiplying by units doesn’t change reducibility.”
(See sections U2 (page 199) and U3 (page 209) for Jason’s investigation of the statement and the consequent establishment of his conception of unit.)

Hence, Jason’s examination of the statement* can be viewed as having provided a rationale for him to consider a third category (i.e., the unit concept) in addition to categories of reducible and irreducible polynomials. Using all three concepts of reducible, irreducible, and unit, Jason was able to successfully provide a comprehensive, overarching explanation of complete factorization across different examples including various polynomials and integers. In fact, Jason was the only participant who made a transformative transition from U2 to U3 during the TIs. All other participants had some difficulty in coordinating the unit idea with their existing factorization schema, thereby staying at the inter-object level in the unifying category (U2) during the TIs.

For some of them (Helen, Calvin, and Sam), the tendency to overlook the mathematical entities under consideration might explain their difficulty in coordination of the unit idea because it could have gotten in the way of figuring out the role played by a unit element. As previously mentioned, when a single polynomial can be considered as both reducible and irreducible, one’s exploration of the statement* would not enlighten the need of a third category of unit in addition to categories of reducible and irreducible polynomials. Hence, introducing a unit concept could have been perceived only artificially by those participants. In sum, the tendency to overlook the mathematical entity under consideration seemed to act as an obstacle to a potential transformative transition in the unifying category and to interrupt their investigation of how the concepts of reducible, irreducible, and unit can be used to explain factorization cross different polynomial sets and the set of integers.
Conflated use of different concepts or procedures

An obstacle that seemed to keep some participants from strengthening the link between two mathematical constructs was their conflated use of related terminologies or procedures. Somewhat ironically, having a too tight link between two constructs without a clear distinction seemed to run counter to a strengthening process. Some participants’ use of vocabulary (e.g., between “factors” and “roots”) and their reasoning (e.g., factorization based on finding roots) suggested they were not necessarily making important distinctions they should have been making between different mathematical entities. Often times, their procedural knowledge of translating a factorization problem into a finding-roots problem seemed to preempt their conceptual thinking (e.g., what factorization means; how roots and polynomial equations are related), which is likely to have made it more difficult to establish a fully coordinated bidirectional link between roots and polynomial equations.

In fact, some participants’ tendency to treat factorization of a quadratic polynomial as undetachable from finding the roots of the corresponding quadratic equation was discussed in detail in section S2 (page 229). Two of them (Calvin and Sam) eventually made the transition from S2 to S3 with some difficulties, as discussed in the section entitled Strengthening observed in the course of interviews (page 242). For others (Helen and Lucy), their tendency of assuming (consciously or subconsciously) that roots determine a single polynomial equation still persisted in their POST-TI interviews. The current section focuses on what might explain and underlie the difficulties in and missed opportunities for moving forward from S2 to S3.
A procedure of factoring a polynomial by finding its roots involves multiple constructs—e.g., polynomial as a function, a polynomial equation, roots of an equation, factors of an expression, factorization of a polynomial. The diagram in Figure 5.1 depicts an example of how one might utilize and build on these constructs in the factorization procedure. The procedure starts with the upper left corner with a polynomial, with continued transition to the corresponding polynomial equation, to the roots of the equation, to the factors of the given polynomial, to a potential polynomial equation with the set of roots found, and finally to the factorization of the given polynomial.

**Figure 5.1. Procedural connection with conceptual distinction**

When the procedure of factorization is rushed without making a sharp distinction between constructs and skipping intermediate, necessary constructs (for example, see Figure 5.2), not only might the resultant factorization be inequivalent but also opportunities for coordinating these multiple constructs and establishing a conceptual link
between them (i.e., a part of strengthening process) would be unlikely to take place and likely to be missed.

![Diagram showing polynomial factorization and equation roots]

Figure 5.2. Procedural connection without proper conceptual considerations

Complete factorization of $P_3(x) = 4x + 16x^3$ was a representative case in which participants (Calvin, Sam, and Lucy) utilized an immediate procedural link between factorization and finding roots; and their reasoning seemed to be very close to what was depicted in Figure 5.2. A common behavior was observed that they immediately associated the factorization problem with strategies for solving an equation. As shown in Figure 5.3–Figure 5.5, Calvin, Sam, and Lucy all identified the roots of a quadratic equation, $1 + 4x^2 = 0$ by applying the quadratic formula or by taking square roots of both sides of an equation, in order to factor $P_3(x) = 4x + 16x^3$. This phenomenon is noteworthy because nowhere in the problem statement is one asked to solve an equation. The problem (i.e., **CORE-Q3**) stated “Please factor completely the following polynomials” and one of the polynomials was $P_3(x) = 4x + 16x^3$. 

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**Polynomial, $p(x)$**
(e.g., $1 + 4x^2$)

**Not equivalent**

**Factorization of the polynomial, $p(x)$**
(e.g., $(x - \frac{i}{2})(x + \frac{i}{2})$)

**Polynomial equation, $p(x) = 0$, often not identified explicitly**

**Roots of an equation**
(e.g., $x = \frac{i}{2} \text{ or } x = -\frac{i}{2}$)

**Immediate transition; Intermediate transitional concepts not considered**
(e.g., from roots, to factors, to an equation, and to the original polynomial)
Figure 5.3. Calvin’s factorization of $P_3(x) = 4x + 16x^3$ using roots

Figure 5.4. Sam’s factorization of $P_3(x) = 4x + 16x^3$ using roots

Figure 5.5. Lucy’s factorization of $P_3(x) = 4x + 16x^3$ using roots

As seen in Figure 5.3–Figure 5.5, the three participants provided $4x(x - \frac{1}{2}i)(x + \frac{1}{2}i)$ as their complete factorization of $P_3$ over $\mathbb{C}$, which is, in fact, not equivalent to $P_3(x) = 4x + 16x^3$. Lucy and Sam did not seem to notice the discrepancy between
4x^2 + 16x^3 and their answers, which were equivalent to x + 4x^3 when fully expanded. They seemed to think a set of roots definitely determines the factorization of a given polynomial. Even upon the interviewer’s follow-up question such as “How can you confirm that it is a complete factorization?” they seemed to be confident in their answers and the discrepancy was not brought to their attention at all. They did not revisit or question the process of transforming the roots into the factored form and its relation to the factorization. Notably, in their solutions, Sam and Lucy did not specify which equation they were solving. Their thought process in this solution (as well as the written and verbalized solutions) could have been missing important constructs (e.g., equation) and distinction between constructs (e.g., roots and factors), thereby not allowing a chance to explore a needed construct for strengthening process (e.g., role of unit in factorization).

Calvin was the only one who noticed the inconsistency between the intermediate factored form 4x(1 + 4x^2) and his answer 4x(x = \frac{1}{2}i)(x = \frac{1}{2}i). To briefly reiterate Calvin’s approach (for more details see section S2 on page 229), Calvin checked whether he had a correct set of roots using three different methods: first by obtaining the roots by taking the square root of both sides of x^2 = -\frac{1}{4}, second by applying the quadratic formula, and third by evaluating the original polynomial function at one of the roots he found. At that moment, Calvin was not able to identify why his answer did not match the original polynomial. As discussed in the section entitled *Strengthening observed in the course of interviews* (page 242), it was a particularly long journey for Calvin to address the issue of mismatch between the two; the process required three distinct intervals of time during two separate interviews (INT 3 and INT 6). One might wonder why it was so
difficult for Calvin to address the discrepancy he noticed. It might be worth considering that Calvin chose to confirm his roots using three different methods without looking for other possible explanations pertaining to the relationship between roots of a polynomial equation $p(x) = 0$ and factorization of a polynomial $p(x)$. This might suggest that Calvin understood factorization and finding roots as interchangeable without distinction.

Calvin’s conflated use of the terminologies, roots and factors, might be confirming evidence that suggests interchangeable nature of the two constructs in his mind without his needing distinction between the two constructs. Several instances were observed in which Calvin used “roots” in place of “factors” in his verbal and written explanations. For example, in his definition of complete factorization, he wrote “Factoring completely over ($\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$) is to rewrite the polynomial as a product of its roots, with coefficients in the corresponding number system.” In another interview, to explain his anticipation that expanding his answer $4(x - \frac{1}{2}i)(x + \frac{1}{2}i)$ would result in the original polynomial $4x + 16x^3$, he said “The way that I’m used to thinking about it with multiplying the roots together to get you back to your starting function—it’s only consistent over $\mathbb{R}$.” The fact that Calvin used “roots” when he probably meant “factors” in those statements perhaps suggests those two were interchangeable entities for him, which can potentially explain why “finding roots” was the central focus for him when he was “factoring a polynomial” and investigating the mismatch between his factorization and a given polynomial.

Of course, factorizations, roots, polynomials, and equations are closely related mathematical constructs, but one needs to be able to distinguish these constructs clearly
in order to be able to establish sophisticated relations among them. As shown in participants’ approach to factorization of $4x + 16x^3$ in this section, the procedure they applied for the factorization consisted of an immediate transition from a polynomial to finding roots, say, $r$ and $s$ (often without any indication of what equation defined the roots), and to a product of linear factors, $(x - r)(x - s)$. In this procedure, important distinctions between roots ($r$ and $s$, in this case) and factors ($x - r$, $x - s$), units, and unit-multiples of $(x - r)$ and $(x - s)$ of a given equation were rarely made. Furthermore, for Calvin, roots and factors seemed to be used interchangeably. Such conflated use of different concepts of procedures seemed to make it difficult for participants to conceptually explore how, for example, roots and polynomial equations are related by exploring the role of units in factorization, thereby acting as an obstacle to a strengthening process.

**Compartmentalization accompanied by reliance on formulas**

When some participants’ (Sam and Helen) understandings persisted at the same level of strengthening, in particular at the intra-object level [S1], it seemed that their tendency to compartmentalize factorization situations suppressed a potentially possible strengthening process. Here the word *compartmentalization* was used to refer to a tendency to assign and utilize a particular strategy for only a particular group of factorization situations, even when the strategy is compatible and relevant to other groups of factorization situations. A similar phenomenon was documented in children’s understanding of fractions and described as “compartmentation” in Vinner, Hershkowitz,
and Bruckheimer (1981). Vinner and his colleagues used an example of children’s strategy of adding fractions in an incorrect way: $1/2 + 1/4 = 2/6$ (i.e., add numerators and denominators separately to obtain the result). Here, children’s prior knowledge of fractions such as $2/6 = 1/3$ (equivalent fractions) and $1/3 < 1/2$ (a quantitative comparison) could be used to see the contradiction in adding $1/2$ and $1/4$ to get a smaller quantity $2/6$. However, in utilizing their own rule of adding fractions, children may be focusing only on the rule being applied at the moment and not necessarily see the relevance of their prior knowledge to the situation that might be useful to check the result. This phenomenon of “not using [the relevant prior knowledge] or seeing its relevance to their present context” was named compartmentation\(^7\) by Vinner and his colleagues (p. 73).

This obstacle that I call compartmentalization accompanied by reliance on formulas, was initially brought to my attention when Sam and Helen tried to restore and make sense of a formula for a difference of cubes. In what follows, I first discuss how Sam and Helen missed the opportunities for the strengthening and provide compartmentalization as a possible explanation.

Sam’s and Helen’s work on factorization of $x^3 - 1$ (which was described in detail in section S1 on page 224) revealed that Sam and Helen both could not initially recall a relevant formula; however, Sam eventually developed $(a^3 - b^3) = (a - b)(a^2 + ab + b^2)$ and $(x^3 - 1) = (x - 1)(x^2 + x + 1)$ using guess-and-check method; and Helen

\(^7\) *Compartmentalization* in this study and *compartmentation* in Vinner, Hershkowitz, and Bruckheimer (1981) are considered synonyms.
completed the factorization building on her recollection that \((x^n - 1)\) is always divisible by \((x - 1)\). Interestingly, although they were able to reproduce the formula based on their vague recollection, explaining why the formula makes sense was a source of difficulty for them. As discussed in section S1, when the interviewer asked “how did you know to start with \((x - 1)\)?” and “I am wondering how come you were able to pick \((x - 1)\) to begin with” respectively, they did not provide a satisfactory explanation; instead, both stated explicitly “I don’t know” in their answers. To put it simply, Sam and Helen knew “how” to obtain and apply the formula but did not understand “why” the formula makes sense. One way to explain the “why” part is using the fact that, when \(x = a\) is a root of \(p(x) = 0\), \((x - a)\) is a factor of \(p(x)\) (namely, the factor theorem). To be specific, since the given expression \((x^3 - 1)\) returns 0 when evaluated at \(x = 1\), one may conclude \(x = 1\) is a root of \(x^3 - 1 = 0\). This explains why \((x - 1)\) has to be a factor of \((x^3 - 1)\). As such, understanding the “why” part requires examining the factorization in relation to other relevant constructs such as roots and polynomial equations, thereby providing a chance to strengthening the link between activated constructs.

For Sam and Helen, however, the interviewer’s question on the “why” did not lead them into examining relevant constructs, such as roots or equations (and hence no strengthening process). In particular, the interviewer’s question that focused their attention to the factor \((x - 1)\) in \((x^3 - 1) = (x - 1)(x^2 + x + 1)\) did not lead them to consider the root implied by the factor \((x - 1)\) at all. It seemed that the expression in the form of a difference of cubes was so tightly associated with a particular formula that they did not realize they did not have to rely on the formula. Also of note is that Sam and
Helen were able to utilize the link between a polynomial equation and roots (as in the factor theorem) when they were dealing with quadratic expressions. For example, in **PRE-Q6**, Helen readily interpreted “a quadratic equation has two roots $s$ and $t$” as $(x - s)(x - t) = 0$. One might wonder why this link that seemed so easily accessible to them with some quadratic expressions was no longer available for them when addressing the cubic expression, $(x^3 - 1)$.

What would account for the missed opportunities for strengthening the link between a polynomial equation and its roots with a polynomial, $x^3 - 1$? One explanation for this phenomenon might be a tendency to compartmentalize factorization situations. Two realms—the realm of quadratic expressions and the realm of formula-type expressions—appeared to be separate for them and hence ruled by different methods. Perhaps, when they were dealing with formula-type expressions (e.g., sum and difference of cubes such as $x^3 - 1$; even more generally, sum and difference of powers\(^{76}\) such as $x^n - 1$), their reliance on and memorization of the formula could have overpowered or suppressed their understanding of a potentially useful link between a polynomial equation and its roots that was established in the other realm.

Their self-reported learning experiences with respect to factorization provided insight into the type of practices in school mathematics that could have contributed to forming the tendency of compartmentalization. My hypothesis is that their drilled practice in factorization may have led them to develop a tendency to compartmentalize

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\(^{76}\) Helen’s school mathematics experiences consisted of education in the U.S. and also an Asian country according to her self-report during the interviews. The factorization of $x^n - 1$ was, according to her, what she learned in the Asian country.
factorization situations, which then might serve as an obstacle to a strengthening process. For example, in one of retrospective questions (“Did what we did today remind you of any other math that you’ve done?”), Sam described how he was accustomed to separate his thinking of quadratic expressions with leading coefficient of 1 and the ones with leading coefficient not equal to 1 as follows:

I do think back to whenever I was initially learning how to factor expressions. Now I’m curious. Is there going to be as much of a difference now whenever the leading coefficient is 1 and whenever it’s not 1? Before this interview today, I completely saw them as almost like two different animals to try and handle. I saw them as two completely different things, because I was told, given two different processes to attack those two things.

For Sam, applying “different processes to attack [different types of expressions]” might have been a norm, and the cubic expressions such as \(x^3 - 1\) could have been considered as a separate case on which he would no longer apply the root–equation relationship. On the same retrospective question, Helen also described how she developed her factorization skills in her middle school as follows:

During my middle school, I was just accepting it [it seems to refer to a factorization formula], and keep doing it, but not being mindful of what I’m doing. Going with the feeling. If you remember, the other day, when I
was trying to do \((x^3 - 1)\), you asked me why I picked \((x - 1)\). That was just because I go with my feeling. Now… I have to ask myself “Why am I doing it?”

Helen’s illustration of her experiences in the middle school suggests that she was probably drilled in practicing application of the formula enough to be able to “go with her feeling.” She further stated “They [“they” seems to refer to the mathematics teachers she had in her middle school] had like seven equations [writing \((a \pm b)^2\) and \(a^2 - b^2\)] that I had to remember. That’s why I kind of memorized stuff without being explained why.” Her mastery of the “seven equations” or factorization formulas without reflecting on why they make sense could have led her to compartmentalize factorization situations into the ones where she could apply the formulas mindlessly and the ones where she needs to be perhaps more “mindful of” what needs to be done (including the root–equation relationship).

From Sam’s and Helen’s self-reported learning experiences, one might reasonably infer that compartmentalization accompanied by reliance on formulas could lead one to be short-sighted on how different constructs are related. Such a tendency of compartmentalization and of seeing formulas as a necessity rather than a choice could have preempted their potential willingness or capacity to search for and utilize the link between existing understandings, thereby playing against a strengthening process.
Utilitarian/instrumentalist view

This section focuses on analysis of Helen’s responses that revealed lack of change in the levels of her understandings (specifically, from E1 to E1). In particular, Helen showed little change in her response to an identical question asked before and after the teaching interviews—PRE-Q2 (iv) and POST-Q2 (iv). In what follows, I present an analysis and illustrations of such lack of change in Helen’s understandings and suggest her demonstrated view of mathematics as an obstacle to an extending process. (Helen was an international student whose native language was not English; Excerpts from her interviews may include grammatically nonstandard sentences.)

Unlike all other participants who were able to provide an example of $ax^2 + bx + c = 0$ such that it has one real root and one nonreal root in POST-Q2 (iv), Helen’s response to POST-Q2 (iv) was almost identical to her response to PRE-Q2 (iv). Even though the question did not impose any condition on the $a$, $b$, and $c$ values (e.g., $a$, $b$, and $c$ are integers), Helen seemed to implicitly assume $a$, $b$, and $c$ to be real numbers in her thinking that having one real root and one nonreal root is impossible. See the following paragraphs for her responses:

- Helen’s response to PRE-Q2 (iv): “That can’t happen... If one real root, this one [referring to $b^2 - 4ac$ in the quadratic formula] has to be greater than or equal to 0. For nonreal root, this one [referring to $b^2 - 4ac$] has to be less than 0... Two of them can’t happen at the same time.”

77 PRE/POST-Q2 (iv): Find (if possible) $a$, $b$, and $c$ such that $ax^2 + bx + c = 0$ has one real root and one nonreal root.
Helen’s response to **POST-Q2 (iv)**: “It is not possible, right? It is because when we consider \( b^2 - 4ac \). In order to make one real root and one imaginary root, it has to be like \( b^2 - 4ac \) is greater than 0, and \( b^2 - 4ac \) is less than 0. But, you have the same \( b \), the same \( a \), the same \( c \), so this case [(iv)] is impossible.”

Her consistent answers in **PRE/POST-Q2 (iv)** indicate that Helen did not come to think about or account for \( \mathbb{C}[x] \) as a possible context when considering the conjugate pair idea, even though the majority of the questions during the teaching interviews led her to explore factorization and polynomials over various domains including complex numbers. Such resistance to accepting a new context \( \mathbb{C}[x] \) was particularly strong in Helen’s reasoning when compared to other participants. As discussed in detail in the section entitled *Extending observed in the course of interviews* (page 179), other participants who had believed that having one real and one nonreal root is impossible in **PRE-Q2 (iv)** such as Lucy and Calvin demonstrated awareness of a broader context in their consideration of the conjugate pair idea in **POST-Q2 (iv)** after the teaching interviews, thereby extending the boundary of contexts.

In search for a reasonable explanation for how a sequence of interviews (7 interviews in total) may have had such minimal impact on Helen’s thought about **PRE/POST-Q2 (iv)**, I found a pattern in the way she treated complex numbers. It appeared complex numbers are of little significance to Helen who was majoring in mathematics from an engineering perspective (Helen was pursuing concurrent majors in mathematics and engineering at the time of interview); and her language in multiple
instances suggested her conscious, intended exclusion of \( \mathbb{C}[x] \) in her consideration of factorization.

For example, when Helen explained her claim that not every quadratic polynomial is factorable, she stated “In some case it can’t be [factored] if you get imaginary numbers [she seems to imply imaginary roots]. It’s a pain if you try to factor it. I don’t try... I guess in some case, if you have a certain level of math you can factor it. For basic ones, you don’t even bother to think that exists [that seems to refer to imaginary numbers].” She appeared to rule out complex numbers in her consideration of factorization, and her stance on complex numbers became much clearer in her response to a follow-up question on CORE-Q3, which was abbreviated into Q3-F1.\(^{78}\) Her response to Q3-F1 (presented in the following excerpts) demonstrated her intention to limit factorization to be within the context of \( \mathbb{R}[x] \).

Helen: This [referring to \( \frac{-1+\sqrt{3}i}{2} \) in the question] is an imaginary number. So, it’s not real. It doesn’t count.

I: It doesn’t count?

Helen: Yeah. Because you only take real numbers. Based on the definition of the completely factored...

\(^{78}\) (Q3-F1) (Case 2) Follow-up based on \( P_2 \): What do you think about the factorization of \( P_2(x) \) being \( x^2(x - 1)(x - \frac{-1+\sqrt{3}i}{2})(x - \frac{-1-\sqrt{3}i}{2}) \) instead of \( x^2(x - 1)(x^2 + x + 1) \)? What do you consider to be ‘completely factored’?
I: Which is right here. [The interviewer takes out Helen’s written definition of complete factorization. Helen had written “To factor polynomials completely \[ \rightarrow \] Split the variables into the smallest factors”]

Helen: There, **smallest factor doesn’t mean you have to split into imaginary numbers**, because imaginary number doesn’t fall into the factoring of a polynomial anyway.

Helen further explained her reasoning by contrasting a factorization problem with an equation-solving problem. She appeared to accept complex numbers when solving an equation and not to consider complex numbers when factoring a polynomial as follows:

Helen: If you want me to solve this [referring to \( x^2(x-1)(x^2 + x + 1) \)], if you want me to make this equal to 0, so then I’m going to have \( x = 0 \), \( x = 1 \), and \( x \) equals this thing [referring to \( \frac{-1+\sqrt{3}i}{2} \) and \( \frac{-1-\sqrt{3}i}{2} \)]. But it’s not real. **Here** [referring to her written definition of complete factorization] we’re dealing with real numbers, not imaginary, complex numbers.

I: Is there a reason we should consider only real numbers?

Helen: Yeah, because **most of the stuffs that we’re dealing with, besides some kind of electric, like, EE stuff** [EE refers to electrical engineering], they deal with complex numbers. But **most of us deal with real numbers**, because it wouldn’t make sense if we were like... I don’t know, let me think of it... [long pause] I think there should be a reason why they don’t introduce complex numbers until later on, when we get introduced to calculus. There’s a reason. I’ve never thought
about it, to be honest with you, until now. So, if they did not teach complex numbers back then, which means **there’s no use for this**. Also, on my calculator, whenever we deal with complex numbers, it appears, no solution, so there shouldn’t be a solution.

In the preceding excerpts, Helen’s view of complex numbers—that is, that we do not deal with complex numbers but for exceptional occasions such as in electrical engineering, and complex numbers are probably of no use—suggests that she was oriented to application aspects of mathematical entities. Her application-oriented view could have contributed to her resistance to accepting $\mathbb{C}[x]$ as a possible context in her response to POST-Q2 (iv) and run counter to extending the boundary of contexts in her consideration of factorization.

In fact, her tendency to describe and judge mathematical entities based on utilitarian values was observed on multiple occasions. Ernest (1988) proposed that such a view of mathematics can be categorized as an “instrumentalist view,” which he considered to be at the lowest level in a hierarchy of three philosophies of mathematics. To be specific, Ernest (1988) proposed three philosophies of mathematics as follows: “the instrumentalist view that …mathematics is a set of unrelated but utilitarian rules and facts”, “the Platonist view of mathematics as a static but unified body of certain knowledge”, and “the problem solving view of mathematics as a dynamic, continually expanding field of human creation and invention” (para. 7). Although no explicit question was asked for Helen to discuss her view of mathematics during the interviews, her comments related to multiple mathematical entities (as well as complex numbers)
suggested Helen’s view, among Ernest’s three views, can be best described by an instrumentalist view. In the following paragraphs, I provide examples of how she described prime numbers and differential equations from her utilitarian perspective as an engineering/mathematics major.

In PRE-Q9, Helen was asked to include in her concept map of polynomial additional ideas including prime factorization. In her discussion of prime factorization, she shared (without any prompt from the interviewer’s end) how she viewed prime numbers. She stated:

I didn’t think that prime number was important to be honest, but it was good to know… It’s always good to keep something that you think is useless, but just in case I have to talk to someone and I can be knowledgeable [in the] talk. The whole time of my life, it [it refers to prime number] was some kind of useless.

Helen then continued to describe the “talk” in which she was able to use her knowledge of prime numbers as follows:

79 During the interviews, she seemed to perceive herself more as an engineering major than as a mathematics major even though officially her undergraduate program was concurrent majors in mathematics and engineering.

80 PRE-Q9: Please construct a concept map of YOUR ideas about polynomial in the context of school mathematics. (A full version of this question can be found in Appendix C.)
The question was... from 1 to 1,000 how many prime numbers are going to be in there? It’s kind of a trick in that class [that class refers a course titled “Discrete mathematics’’], but until that class I didn’t mention about any prime thing in my life. Because I’m an engineer. I only deal with things like, “this is real.” You know a real-life thing… If I wasn’t doing math minor, I wouldn’t care about prime number to be honest.

Her last comment suggests she would place little value on something that is considered not real or useful from her stance, thereby demonstrating an instrumentalist view.

Continuing her course-taking experience (as a side talk), Helen also shared a journey of how she came to see value in differential equations as an engineering major.

There was one class that I kept asking the professor, it was the ODE class [a course titled “Ordinary Differential Equations’’] and at that point, every day I kept asking her, “why do I need to know about this?” I don’t see any application. I think it would be a good idea if the teacher or professor should tell the student what the application is… [omitted]… I thought it [it refers the ODE class] was useless for my major until recently when I take a lot of higher level of my major classes. I realize that [differential equation] is really important for my major.

81 Even though she stated “math minor,” Helen was pursuing concurrent majors in mathematics and engineering. In a later interview, she clarified that mathematics and engineering were equally weighted majors in her undergraduate program (i.e., she was not “minoring” in mathematics).
In her description of prime numbers and differential equations (including what is not presented in the preceding excerpts), “real-life,” “application,” “use,” and “useless” appeared frequently. The high frequency of those terms in her own description of how she viewed various mathematical entities seemed to be an indicator that Helen’s encounter with mathematical contexts (including the teaching interviews) could have been judged based on her own utilitarian values. The opportunities for growth (from a Platonist perspective) through the teaching interviews could have been perceived by Helen as having little utilitarian values, hence resulting in lack of change in the levels of her understandings (specifically, from E1 to E1).

**Summary**

This chapter presented some explanations for the growth and the lack of growth in participants’ understandings within the particular instructional context of this study, providing interpretive insights into the findings presented in Chapter 4. Findings in Chapter 5 were organized into two sections, *Contextual features in making transformative transitions* and *Obstacles to making transformative transitions*. Each section portrayed (a) four contextual features that appeared to be fruitful and supportive in witnessing participants’ making transformative transitions and (b) four tendencies demonstrated by participants that seemed to act as obstacles to making transformative transitions. Those contextual features and obstacles are listed in Table 5.4.

Table 5.4. List of contextual features and obstacles identified in this study
A synthesis of what has been discussed in the foregoing is given in the next few paragraphs. As displayed in several places throughout this dissertation, this study was guided by a principle that values learners’ developing advanced perspectives on their prior mathematics knowledge in their learning of collegiate mathematics (as recommended in CBMS, 2012; CUPM, 2004, 2015). Empirical findings presented in this chapter suggest that such development of advanced perspectives, when interpreted through the lens of transformative transition framework, might need to be understood as the dynamic interaction of learners’ idiosyncrasies and instructional contexts. That is, a learner’s responsiveness to a contextual feature of instruction (such as the ones in the second column of Table 5.4) may depend on what kinds of tendency a learner has (such as the ones in the third column of Table 5.4) and how strong the tendency is; as a result,
the instructional context may or may not be sufficient to create momentum for the learner’s transformative transition.

For example, one participant, Jason, showed none of the four tendencies that are identified as possible obstacles to making transformative transitions. For Jason, determining truth of a general statement such as “the product of two irreducible polynomials is a reducible polynomial” (i.e., contextual feature (iv)) seemed to create a venue for appreciating the need and role of unit concept in dealing with trivial factorization. In subsequent interview tasks, Jason used the unit concept in explaining the sameness in complete factorization across different expressions, suggesting a transformative transition in the unifying category.

However, when an individual’s tendency (such as the ones in Table 5.4) is strong, it seemed to override a contextual feature (such as the ones in Table 5.4). Helen, who seemed to hold a relatively strong utilitarian view (i.e., obstacle (iv)), demonstrated a tendency to exclude \([x]\) from possible contexts in her consideration of factorization because it was believed “there’s no use for [complex numbers].” Helen’s utilitarian view seemed to persist despite a sequence of tasks in the order of defining-applying-describing (i.e., the contextual feature (i)), which seemed to be supportive in some other participants’ making a transformative transition in the extending category.

Another combination of contextual features and obstacles is as follows: An individual with a demonstrated obstacle may still respond to a contextual feature and make a transformative transition despite the obstacle. Calvin, who demonstrated a tendency to conflate use of roots and factors (i.e., obstacle (ii)), did seem to experience significant difficulty in understanding the mismatch between a polynomial expression
and his factorization of the polynomial expression. However, Calvin reacted to the
contextual feature (iii), made multiple and intentional efforts in understanding the
perceived mismatch, and eventually addressed the source of mismatch, thereby
completing a transformative transition in the strengthening category.

Overall, being confined to a particular potentially hindering frame or tendency
(such as the ones identified as obstacles in this study) may result in little-to-no
responsiveness to an instructional context that is intended to support university students’
making transformative transitions. Interestingly, the four obstacles identified in this study
may be rooted in participants’ school mathematics experiences. Deiser and Reiss (2014)
claimed in school mathematics, “mathematics is experienced by students as a subject
which is dominated by calculation and guided by recipes” (p. 52). If calculation or getting
the right answer played a dominant role in their school mathematics, some participants’
tendencies—such as overlooking mathematical entities under consideration, conflated use
of different concepts, and compartmentalization—seemed to make sense because there
may have been no need for challenging that tendency in their school mathematics. What
they learned in school mathematics indeed served as resources for them during the
teaching interviews, but the tendency they have developed in their school mathematics,
ironically, may also act as barriers to emerging understandings at the collegiate level. In
the final chapter, I conclude with an interpretation of this study at a deeper level by
presenting a holistic view of it in relation to the current knowledge of the field and
discuss the implications and limitations of the current study.
Chapter 6 Discussion

The purpose of this study was to describe and explain university students’ growth (and lack of growth) in their school mathematics understandings in the context of learning relevant collegiate mathematics, through the lens of a transformative transition framework. To this end, the following four research questions were addressed in Chapter 4 (RQ 1 and 2) and Chapter 5 (RQ 3 and 4):

(RQ 1) How can university students’ school mathematics understandings be described by the levels in each of the four categories of transformative transition—deepening, extending, strengthening, and unifying?

(RQ 2) What transformative transitions do university students make in the four categories—deepening, extending, strengthening, and unifying—in an AiC-grounded instructional context?

(RQ 3) What are possibly relevant features of an AiC-grounded instructional context in which university students make transformative transitions?

(RQ 4) When university students do not make transformative transitions in an AiC-grounded instructional context, what are some obstacles to making those transitions?

This chapter discusses conclusions drawn from the findings presented in Chapter 4 and 5 and accounts for how the findings collectively inform the central problem of this study—the double discontinuity (Klein, 1908/1924). The conclusions section is followed by implications and contributions of the current study in which the findings and conclusions are discussed in light of past research, other theories, and the practice of teaching.
collegiate mathematics. This chapter concludes with a discussion of the limitations of this study and future research directions.

Conclusions

Given the central role of the problem of the double discontinuity in motivating this study, it seems worthwhile to revisit some recommendations made by CBMS (2001) regarding mathematics majors courses and capstone courses for Prospective Secondary Mathematics Teachers [PSMTs]: (a) PSMTs need to be supported to “make insightful connections between the advanced mathematics they are learning and the high school mathematics they will be teaching”; and (b) PSMTs need to be given opportunities in which “conceptual difficulties, fundamental ideas, and techniques of high school mathematics are examined from an advanced standpoint” (p. 39). Those recommendations, which were attributed to recognition of the double discontinuity problem, illustrate what mathematics educators might consider desirable. However, as explained in Chapter 1, empirical and anecdotal evidence (CBMS, 2012; Cofer, 2015; CUPM, 2015; Lee, 2010; Ticknor, 2012) suggests that such desirable situations might be difficult to take place naturally and may require significant innovation in the instruction of collegiate mathematics.

Because of the anticipated difficulty of witnessing such desirable situations, this study employed special tools—the teaching interviews [TIs] and the Abstraction-in-Context [AiC] approach—for designing instruction of collegiate mathematics in a one-on-one interview setting in which participants’ relevant school mathematics
understandings are likely to be activated and reflected on. The intention was to use TIs as a rough approximation of collegiate mathematics lessons and to gain insights into how to support university students in actual mathematics courses. An underlying assumption in this design was that a solution to the double discontinuity problem might be able to be found in *components of instruction of collegiate mathematics*. This assumption seems to be consistent with how the field has been approaching to the double discontinuity issue.

In CBMS (2001, 2012), for example, authors proclaimed a need for changing instruction of core collegiate mathematics courses to help students to see some aspects of school mathematics “from an advanced standpoint” and for designing and offering special mathematics courses for prospective teachers in order to address the problem of the double discontinuity.

In what follows, I first discuss the findings that support this assumption that a solution to the double discontinuity problem can be found in *components of instruction of collegiate mathematics*. I then discuss the findings that may add a new dimension of understanding the issue of double discontinuity by claiming that some *components external to collegiate mathematics instruction*, such as students’ individual tendencies, should also be considered when supporting students to see school mathematics from advanced viewpoint. I argue that addressing the problem of the double discontinuity requires consideration of mathematics education at all levels from K–12 as well as at the collegiate level.

Corroborating the assumption (as stated in the preceding paragraphs) that exists currently in the field, the results of this study suggested the following: University students can reexamine and advance their understandings of fundamental ideas or
techniques of school mathematics in their learning of collegiate mathematics and, when they do so, *some features of an instructional context seem to have potential contributions to the advancement in levels of their understandings*. As discussed in Chapter 4, five out of six participants were observed to enrich their prior understandings of factorization and polynomial equations through transformative transitions in the course of interviews.

Chapter 5 built on this observation and presented potentially fruitful contextual features of instruction for creating momentum for transformative transitions. Those included: (i) A sequence of tasks in the order of defining-applying-describing; (ii) An unusual or unexpected encounter with a problem beyond their routines or assumptions; (iii) A task with a variety of instantiations of a single entity to reflect on; and (iv) A task for determining the truth of a general statement regarding a construct. A conclusion that can be drawn from these findings is that it may be important to consider those contextual features—or more broadly, *how* things are presented as well as *what* is presented in collegiate mathematics instruction—in the endeavor of addressing the problem of the double discontinuity.

The four contextual features, which are rooted in the AiC approach, can be considered reflective of teaching practice that supports students’ *constructive* learning. The AiC approach was chosen because it was viewed in this study that growth of existing understandings involving the connection between school and collegiate mathematics would require constructive acts by the learner. The fact that the four contextual features rooted in the perspective of constructive learning were identified as potentially fruitful might suggest the importance of an instructor’s stance on how mathematics is learned and how to teach mathematics to support such learning. For some collegiate mathematics
instructors and researchers—who hope to address the double discontinuity problem in their classes and yet mainly use an instructional approach of passing on to students mathematical definitions, propositions, and their proofs as ready-made products—considering the four contextual features may provide a chance for extending their thinking about how to support students’ growth in their existing understandings.

Instructors of collegiate mathematics might consider, for example, “a sequence of tasks in the order of defining-applying-describing” to provide their students with opportunities to reexamine their prior knowledge from a different angle.

More specifically, suppose a polar curve integration is to be introduced in a second or third course of college calculus. Students in this course may be assumed to have learned a definite integral in their first course of college calculus; if mathematics-intensive majors, it is also likely that such learning took place in their high school mathematics. An instructor may begin by asking students to define a definite integral that they already learned in high school or in a previous calculus class and to apply the definition to an area surrounded by a polar curve and its lower and upper bounds (e.g., $r = \sin \theta$ from $\theta = 0$ to $\theta = \pi/2$; see the graph on the left of Figure 6.1).
Students’ typical definition of a definite integral would probably be based on the Cartesian coordinates and may use a limit of a Riemann sum. If they apply their definition of a definite integral to a polar curve defined on the polar coordinates \((r, \theta)\) by simply replacing the variables, i.e., \(x\) with \(r\), and \(y\) with \(\theta\), their attempts to describe the resulting area using their definition would not be successful. In case of \(r = \sin \theta\) from \(\theta = 0\) to \(\theta = \pi/2\), the actual area surrounded by a polar curve and its lower and upper bounds can be obtained by calculating the area of a half-circle, which is \(\pi/8\); however, the definite integral obtained by simply replacing \(x\) with \(r\), and \(y\) with \(\theta\), would result in 1 (see Figure 6.2).
Their assumption on universal applicability of the definite integral definition might be reexamined at that point. Such reexamination then can be followed by extending the boundary of contexts for considering infinitesimal areas in the definition of a definite integral—from considering only rectangular areas on the Cartesian coordinates to including circular sector areas as well on the polar coordinates. An observation of mismatch between an expected answer based on a certain assumption and the actual answer may result in a reorganizational impact on a student’s cognitive system.

Such a reorganizational impact seemed to have been observed in some participants’ data in this study. As described in Chapter 4, some participants, in the beginning of interviews, had demonstrated assumptions that seemingly had been established in their school mathematics (e.g., assuming leading coefficient is 1 when translating roots to factors, separating polynomial factorization from number
factorization, thinking factorization always results in a product of lesser degree polynomials). These assumptions appeared to be reexamined by the participants from a different and advanced standpoint while engaging in the TIs with the four contextual features identified in this study.

Although those contextual features might be worth considering in the design of instruction, the findings also suggested that there exists a tension between instructional contexts and individual tendencies. As discussed in the section Obstacles to making transformative transitions in Chapter 5, the instructional context did not necessarily create momentum for making a transformative transition for some participants, especially when they have the tendencies listed as follows, and that the tendencies occasionally seemed to overrule the instructional context.

(i) Tendency to overlook the mathematical entities under consideration;

(ii) Conflated use of different concepts or procedures;

(iii) Compartmentalization accompanied by reliance on formulas;

(iv) Utilitarian/instrumentalist view

A conclusion that can be drawn from this finding is that a solution to the double discontinuity problem may not be found solely in components of collegiate mathematics instruction. Casting the onus for the double discontinuity problem solely on the particular design of undergraduate programs/courses might be misplaced. Possible causes might be not only a program/course design but also factors directly related to students’ personal tendencies and their school mathematics education to which the tendencies might be attributed.
The findings of this study help researchers to better understand the intricacies of the problem of the double discontinuity by considering the learning of collegiate mathematics as a function of learner’s idiosyncrasy and instructional features interacting with each other. A source of the double discontinuity problem may be found in individual learners’ idiosyncratic tendencies that are displayed in their doing of mathematics. The findings of this study that draw attention to individual learners’ tendencies extend the existing literature which identified a source of the double discontinuity problem as residing in what they do not know. Reporting their attempts to redesign courses for PSMTs, Deiser and Reiss (2014) noted “the mathematical knowledge acquired in secondary schools does not necessarily constitute a reliable foundation for university mathematics” (p. 60); similarly, Winsløw and Grønbæk (2014) noted “the knowledge that they [students] are supposed to use [in their capstone course] is not always what is most familiar to them” (p. 23). However, the current study suggests an additional source of the problem, which may be located in how students do mathematics and the tendency they have developed in their K–12 mathematical experiences. Hence, addressing the problem of the double discontinuity would need to take into consideration mathematics education at all levels from K–12 as well as at the collegiate level. Some implications for practitioners, as well as for researchers, that are drawn from the current study are discussed in the next section.
Implications and contributions

It has been argued that educators of collegiate mathematics need better (than current) ways of supporting students to overcome the issue of the double discontinuity in their learning of collegiate mathematics in order to improve the practice of teaching collegiate mathematics (CUPM, 2015). The results of this study have several implications in this regard for educators of collegiate mathematics. Those implications are twofold: (a) the contextual features identified in this study may be employed as potentially fruitful strategies for designing the teaching of collegiate mathematics, and (b) the students’ tendencies identified in this study may need to be given attention to and challenged explicitly and consistently by instructors throughout their instruction.

As discussed in Chapter 5, this study produced an empirical basis for identifying potential ways to support university students to make transformative transitions and, as a result, four contextual features were suggested as contributing to some participants’ making transformative transitions. Although the setting of the instructional context in this study was one-to-one, the generic features of the context may have the potential to be incorporated in a classroom setting. For example, as described in the previous section, the first contextual feature (i.e., a sequence of tasks in the order of defining-applying-describing) might be used in an introductory lesson of a polar curve integration. Although students’ starting with establishing their own definition might be a relatively uncommon practice in the education of collegiate mathematics, there seemed to exist benefits in doing so. Students, by applying their own definition to a variety of relevant examples, may come away with a better sense of what meanings they used to assign to the term
being defined and identify missing but essential conditions that need to be considered for their definition to be a comprehensive one.

Given the interconnected nature of mathematics, there might exist potentially rich and various opportunities for instructors of collegiate mathematics to implement those contextual features. Instructors may take the contextual features identified in this study into consideration when, for example, sequencing multiple tasks, choosing expressions/examples of a mathematical entity, and posing questions. Notably, the fourth contextual feature (i.e., a task for determining the truth of a general statement regarding a construct) might be helpful in thinking about the kinds of questions that can be posed to help students encapsulate their process conception of a mathematical entity into an objection conception.

An example of an inverse function might clarify the point. Typically, university students’ prior mathematical experiences with inverse functions in school mathematics are action- or process-oriented; that is, in thinking about inverse functions, students focus on reversing the relation defined by the original function or switching the variables $x$ and $y$ in a function rule to obtain a function rule of the inverse function of a given function. When introducing the concept of inverse function, an instructor might consider asking a question such as “Is it possible to find an inverse function for any given function?” rather than presenting a formal definition of inverse function and telling that the original function needs to be injective right away. When prompted by the question, students may realize that their usual way of treating inverse functions is no longer sufficient to determine yes or no in response to the question, because there is no particular function for which its inverse function needs to be found and to which an appropriate process can be
applied. If students find it difficult to answer this question, multiple instantiations of a function (e.g., function rules such as \( y = 2x + 1 \), \( y = 2^x \), \( y = \sin x \), some graphs, and some tabular representations of other functions) may be given so that they can try to find the inverse function of each. By grouping the functions for which they can find inverse functions and separating them from the functions for which they cannot find inverse functions, students are forced to impose specific characteristics to each group (e.g., injective versus noninjective), thereby treating the concept of inverse function as an object. The empirical findings of this study as well as the application of a contextual feature to the case of inverse function in this paragraph were intended to help instructors to enrich their repertoire of teaching strategies for supporting university students’ development of coherent and connected mathematical knowledge in their collegiate learning experiences.

Although some contextual features identified in this study might be helpful in designing a lesson, the findings regarding *potentially hindering tendencies interacting with instructional contexts* also help better understand why it might be difficult for some university students to come to see school mathematics from an advanced standpoint in their collegiate mathematics learning experiences. Underlying the dynamic interaction of learners’ idiosyncratic tendencies and instructional contexts may be the difference in foci between school mathematics (i.e., where the tendencies began to form) and collegiate mathematics (i.e., where the instructional contexts are built on). Deiser and Reiss (2014) noted “calculation [which] has been the constituting aspect of mathematics” in school mathematics is “losing its dominant role and is replaced by a plethora of definitions, theorems, and proofs” at the university level (p. 52). Perhaps, successful transition from
school mathematics to collegiate mathematics might require university students to break out of their familiar frame (e.g., applying a mathematical procedure to get the correct answer, not necessarily considering what that means or why that makes sense) and draw on some unfamiliar practices (e.g., defining, describing, reflecting on their own definitions, true/false conceptual questioning). While interacting with university students through classroom conversation or grading students’ assignments, quizzes, and exams, instructors may pay attention to some of the tendencies identified in this study in their students’ work (written or spoken) and challenge those tendencies explicitly and consistently. Regarding a utilitarian/instrumentalist view that may hinder university students’ potential growth in their understandings, different views of mathematics might need to be a focal topic of discussion in, for example, the first-year seminar for mathematics-intensive majors.

Although some suggestions (as the ones in the preceding paragraph) are relevant to only collegiate mathematics instruction, this study has practical implications for mathematics educators at both collegiate and secondary levels as the tendencies identified in this study may have stemmed partially from participants’ school mathematics experiences. Concerning the first two tendencies identified in this study—(i) Tendency to overlook the mathematical entities under consideration and (ii) Conflated use of different concepts or procedures—it seems desirable that students are encouraged to explore and keep in mind what the entities are on which they are operating. For example, when students operationalize factorization on “something,” they need to be able to identify whether that “something” is an expression, an equation, a function, a polynomial, or some other entity. While factorization is being applied, they may need to differentiate
when to connect expressions with equal signs and when not to. When factorization is completed, they can be asked to explain the results in their language first; a teacher can then examine whether the student can distinguish among, for example, roots, factors, and terms. It may be that, when school mathematics instruction is focused on calculation only or getting the right answer, students are not given enough chances to pay attention to mathematical entities on which the calculation is operationalized and how to refer to and differentiate those entities.

In regard to the third tendency identified in this study (i.e., Compartmentalization accompanied by reliance on formulas), it seems reasonable to suggest moving away from rote memorization and instrumental understandings (Skemp, 1976/2006). Criticizing rote memorization is not a new idea in the field of mathematics education, but I recommend instructors (and their students as well) make a careful distinction between repeated practice for the sake of memorization and repetition that involves reflective thinking and questioning each time. Reflective thinking and questioning may be promoted by asking questions such as “What does it mean? Why does it make sense?” regularly. Harel and Sowder’s (2005) notion of “repeated reasoning” is consistent with the latter type of practice. While reflective repetition is crucial in interiorizing and internalizing a process under consideration (as claimed by APOS theorists), rote memorization without reflective thinking might only increase a learner’s reliance on repeatable actions or formulas. By carefully modeling and encouraging reflective repetition, teachers may help students come to see mathematical formulas as a choice rather than a necessity and to resist a compartmentalization mode such as the ones observed in the current study.
In addition to implications for practitioners, this study also contributes to the field’s growing understanding of the complexity of students’ understandings at the collegiate level—specifically, the intricate ways in which students reason through mathematical tasks that bridge school mathematics and collegiate mathematics. The lens that was used to analyze the complexity of students’ understandings was the categorical framework of transformative transition, which I believe is a major contribution of this study. The four categories of transformative transition were developed from the mathematics education research literature on knowledge construction and development in order to capture different kinds of qualitative leaps that might take place in students’ existing understandings. By documenting and reporting, in all four categories, advancement in levels of transformative transition that reflects meaningful growth in school mathematics understandings, this study attempted to advance the discussion of the double discontinuity problem that had been centered on deficits in students’ understandings.

The findings of this study also help to shed light on the important role that various theoretical frameworks play in the process of analyzing students’ understandings. Adopting the two existing frameworks (APOS theory and Piaget and Garcia’s triad) and the levels in the frameworks (action/process/object and intra/inter/trans) seems to have allowed the transformative transition framework to describe the qualitative nature of students’ understandings in an organized way. The coding scheme of applying the levels to the participants’ understandings of factorization and polynomial equations was tested for its intercoder agreement. The result of the intercoder agreement exercise (83 %) seemed to support the reliability of the coding methods. Hence, the findings also extend
the literature on APOS theory and Piaget and Garcia’s triad by exemplifying the levels and advancement in levels empirically and by contextualizing them in a mathematical topic area that has rarely been explored at the collegiate level.

The empirical elaboration of the categorical framework of transformative transition provided in this study can offer mathematics educators insights into how to interpret and organize their experiences of university students’ learning and how to support university students to see school mathematics from an advanced viewpoint.

**Limitations and future studies**

Although the categorical framework of transformative transition and the findings of this study might have some contributions to educators’ and researchers’ making sense of a learner’s mathematical understanding and analyzing its intricacies in an organized way, it is important to consider some limitations that might exist when applying the framework and interpreting the results. The first limitation has to do with the applicability of the framework beyond the mathematical topics of this study, factorization and polynomial equations. Although the framework was derived from the literature on mathematical knowledge in general (not from the literature about those specific topics), one might reasonably question whether the categorical framework is applicable universally regardless of the choice of mathematical topics. The question of how effective this framework might be in capturing the nature of one’s mathematical understanding and its growth in general is not yet addressed. Hence, as a developer of the framework, I intend to continue mathematical investigations further and attempt a similar
study using other mathematical topic areas. Also, the framework has a limitation in that the strategy in my literature review for generating the list of four categories was not designed to be comprehensive. It seems possible additional categories (other than extending, unifying, strengthening, and deepening) might arise empirically, as the framework is tested across a broad range of topic areas.

The second limitation concerns the subjectivity in a researcher’s application of the framework in his/her interpretation of data. As discussed earlier in the illustration of Andy’s shift from U1 to U3 in Chapter 4, a claim of a shift from one level to another (e.g., from U1 to U3) needs to be examined with caution when the claim solely builds on observation of a participant’s written or verbal articulation. There seemed to exist different levels of evidence for determining whether the shift in articulation suggested a shift in his/her actual cognitive structure. Hence, describing a shift in levels should accompany a careful illustration of evidence, and the interpreter should attend to how substantial the evidence is (using triangulation of data, if possible) to suggest a shift in the cognizing subject’s actual cognitive system objectively. However, a certain degree of subjectivity seems inevitable when a single researcher attempts to draw inferences from a participant’s articulation and make sense of the participant’s mathematical understanding and its growth. Therefore, a group of researchers may be more appropriate in the analysis stage (Maxwell, 1996; Suter, 2012).

In fact, having more than one researcher could also resolve some methodological limitations of this study, allowing separation of a researcher playing a teacher’s role and a researcher playing an observer’s role. The current study collected data by conducting teaching interviews [TIs] in which I played a dual role as a teacher and an observer. This
might raise a question as to whether the interviewer as a teacher led the participant into specific directions to observe what s/he hoped to observe. To guard against this potential risk, I documented guidelines of TI to which I adhered as an interviewer. Regardless of this, however, the separation of a teacher and an observer seems to be needed in a future study to test how likely transformative transitions are to take place in non-TI settings. A possible avenue for future research is to examine to what extent the contextual features (identified in this study as supportive in witnessing participants’ transformative transitions) might be transferable in authentic settings of teaching and learning of collegiate mathematics.

Another limitation of this study that has to do with a small number of participants and the prescreening process is the issue of external generalizability—the generalizability beyond the setting or group studied (Maxwell, 1996, p. 97). This study was designed to take a first step toward establishing an empirical basis for transformative transitions, and it must be stressed that this study was not intended to make a general claim about the population of mathematics-intensive majors. Because I wanted to understand in depth how individual students’ existing understandings grow, I had multiple interviews with each individual in a small group of participants—a total of 40 interviews with six participants—rather than, for example, 40 interviews with 20 participants. Also, the prescreening process narrowed down the spectrum of the participants, which did not allow a comprehensive representation of mathematics-intensive majors’ understandings of factorization and polynomial equations. Hence, another avenue for a future study is to adapt the framework of transformative transitions to conditions of a large amount of data and a wide spectrum of students and to explore whether the framework can serve as an
assessment tool for student understandings in various dimensions of deepening, extending, strengthening, and unifying.

Having discussed the limitations of the current study and some reasonable next steps guided by the limitations, new ideas that might be explored and researched further are given next. One future direction that might extend the current research is to explore the nature of the obstacles to making transformative transitions that were identified in this study (e.g., tendency to overlook the mathematical entities under consideration, conflated use of different concepts or procedures). A future study might examine the prevalence of those obstacles in mathematics-majors’ understandings and generate some strategies for helping them to eliminate those obstacles.

Lastly, this study has taken only a small, but important, step toward addressing the big issue of the double discontinuity. Much work remains to be done to shed light on prospective secondary mathematics teachers’ transition from the learning of collegiate mathematics to the teaching of school mathematics. Even if a prospective secondary mathematics teacher made transformative transitions and came to see school mathematics from an advanced viewpoint in their learning of collegiate mathematics, it remains to be seen to what extent and in what ways his/her teaching of relevant school mathematics might reflect and benefit from such formation of the advanced viewpoint. Possible sub-questions might include: Would s/he be more careful in his/her use of vocabulary (e.g., roots and factors) when teaching relevant areas? Would s/he be making a clear distinction between relevant constructs when linking them? Would s/he be more proactive in leveraging students’ prior understandings when introducing new mathematical concepts or procedures (e.g., introducing factorization of polynomials building on students’ prior
knowledge of factorization of numbers)? Would s/he be foreshadowing what is going to be learned by the students in years to come when it is appropriate to do so (e.g., fundamental theorem of algebra)? By investigating the ways in which teaching of school mathematics may benefit from having an advanced viewpoint on school mathematics, a key question in the double discontinuity problem would be able to be addressed.
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Appendix A: Prescreening Survey Questions

Goals of this prescreening survey
To identify participants who are likely to be able to carry out the tasks during the Teaching Interviews [TIs] with their existing understandings and who have not learned target topics or have learned but not familiar with the topics yet.

Who do I want to recruit?
Ideal participants are those whose answers to SCR-Q1 to SCR-Q6 in the following table suggest that they have relevant background knowledge needed for engaging in the TIs and whose answers to SCR-Q7 suggest they have not learned target topics or have learned but not familiar with the topics yet. The actual survey can be found in the following link:
https://docs.google.com/forms/d/e/1FAIpQLScXRt089UeGgs3sIWXqeS8sEDPFHWHb
TTrsAeyAvtaARFFdw/viewform

Questions and purpose of each question

<table>
<thead>
<tr>
<th>Question</th>
<th>Purpose of the question</th>
</tr>
</thead>
<tbody>
<tr>
<td>SCR-Q1: Which of the following expressions are polynomials in $x$?</td>
<td>I am asking this question to find out whether they can distinguish polynomials from nonpolynomials. Their ability to recognize an expression as a polynomial is necessary when they perform tasks in which they are asked to make up polynomials with certain properties in the TIs.</td>
</tr>
<tr>
<td>(a) $1 + x^2$</td>
<td></td>
</tr>
<tr>
<td>(b) $\frac{1}{x}$</td>
<td></td>
</tr>
<tr>
<td>(c) $3$</td>
<td></td>
</tr>
<tr>
<td>(d) $x^{1000}$</td>
<td></td>
</tr>
<tr>
<td>(e) $\sin x$</td>
<td></td>
</tr>
<tr>
<td>(f) $x + 1$</td>
<td></td>
</tr>
<tr>
<td>(g) $\sqrt{-2x + 3}$</td>
<td></td>
</tr>
<tr>
<td>(h) $\frac{2x - 5}{x + 6}$</td>
<td></td>
</tr>
<tr>
<td>(i) $\log(1 - 3x)$</td>
<td></td>
</tr>
<tr>
<td>(j) $e^{x+1}$</td>
<td></td>
</tr>
<tr>
<td>SCR-Q2: How would you define a polynomial?</td>
<td>I am asking this question to find out students’ concept definition of a polynomial. They will need to use their concept definition of polynomial to make up polynomials with certain properties and define a complete factorization of a polynomial in the TIs.</td>
</tr>
</tbody>
</table>
**SCR-Q3:** Factor the following expressions, if possible.

\[
\begin{align*}
&x^2 + 4x - 60 \\ &25 - x^2 \\ &2x^2 + 5x - 3
\end{align*}
\]

I am asking this question to find out whether they can factor some basic polynomials. They will need to utilize this ability to factor more complicated polynomials in the TIs.

**SCR-Q4:** Find the roots of \(x^2 + 5x - 3 = 0\).

I am asking this question to find out whether they can apply the quadratic formula to find the roots of a quadratic equation. They will need to utilize this ability to think about the relationship between a quadratic equation and its roots and also to factor polynomials with noninteger roots in the TIs.

**SCR-Q5:** Which of the following numbers are prime?

1, 2, 3, 4, 5, 26, 27, 28, 29, 30

I am asking this question to find out whether they can distinguish primes from nonprimes. Their ability to recognize a number as a prime is necessary when they are asked to factor a constant polynomial over \(\mathbb{Z}\) in the TIs.

**SCR-Q6:** Which of the following numbers are …

- integers?
- rational numbers?
- irrational numbers?
- real numbers?
- imaginary numbers?
- complex numbers?

(a) \(\pi\)
(b) \(\frac{\pi}{2}\)
(c) \(\sqrt{-2}\)
(d) \(\frac{3+\sqrt{-2}}{6}\)
(e) \(\frac{2}{6}\)
(f) \(e^2\)
(g) \(-1\)

I am asking these questions to find out whether they can distinguish number systems. Their ability to recognize number systems is necessary to perform tasks in which they have to factor a polynomial over different number systems and they have to consider roots of a polynomial equation in a certain number system.

**SCR-Q7:** How are the integers and rational numbers different?
<table>
<thead>
<tr>
<th>Question (Q)</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>SCR-Q8:</strong></td>
<td>Did you learn the following concepts or theorems in your previous mathematics classes? If so, please list the concept or the theorem that you have learned. Also, please, if possible, state a definition of the concept and a statement of the theorem that you listed in your own words. PLEASE do not consult resources.</td>
</tr>
<tr>
<td></td>
<td>• Polynomial ring</td>
</tr>
<tr>
<td></td>
<td>• Irreducible polynomial</td>
</tr>
<tr>
<td></td>
<td>• Units (or invertibles)</td>
</tr>
<tr>
<td></td>
<td>• Associates</td>
</tr>
<tr>
<td></td>
<td>• Fundamental theorem of arithmetic</td>
</tr>
<tr>
<td></td>
<td>• Fundamental theorem of algebra</td>
</tr>
<tr>
<td></td>
<td>• Unique factorization theorem</td>
</tr>
<tr>
<td><strong>SCR-Q9:</strong></td>
<td>When did you (or do you plan to) take MATH435 (Basic Abstract Algebra) or MATH470 (Algebra for Teachers)?</td>
</tr>
<tr>
<td><strong>SCR-Q10:</strong></td>
<td>What is your name and email address?</td>
</tr>
</tbody>
</table>

I am asking this question to find out whether they have learned and are already familiar with these concepts/theorems to be developed in the TIs.

I am asking this question to find out whether they have taken or are taking an abstract algebra course.

I am asking this question to draw lots and to let them know the result of this pre-screening survey.
Appendix B: Teaching Interview Questions

(Anticipated interview hours per subject: 4 to 6 hours in total)

Goal of this teaching interview
To engage participants in R-, B-, C- actions for constructing and consolidating their understanding of the UFT; Also, to examine how their existing understandings of factorization and polynomial equation are changing and to observe what transformative transitions were made, if any, during the TIs

Questions and purpose of each question

PART 0

Before beginning CORE-questions, I will ask two WARM.UP questions about pre-collegiate constructs such as prime factorization and its uniqueness. The questions are as follows:

**WARM.UP-Q1**: How can you factor 60?
(follow-up: How can you factor 60 completely? How do you know you factored it completely? If someone else claims $1*2*2*3*5$ is a complete factorization of 60, what would you say?)

**WARM.UP-Q2**: How can you factor 60 over rational numbers?
(follow-up: What would it mean if you factor 60 into the product of rational numbers, instead of the product of positive integers? How is the way you factored 60 in **WARM.UP-Q1** different from the way you factor 60 over rational numbers?)

I am asking these questions so that I can better understand where my participants are in terms of understanding prime factorization of numbers and its uniqueness and how they might go from there to constructing UFT of a polynomial. Similar questions will be asked in Q4-F2 and Q4-F3 later, but, in the context of polynomial factorization.
PART 1

- The first four questions in the TIs (CORE-Q1 – CORE-Q4) call for participants’ engaging in R- and B- actions on factorization or polynomial equation and, potentially, for engaging in C- actions on subconstructs of the UFT (primeness, relativeness, uniqueness). Completing CORE-Q1 – CORE-Q4 would lay the groundwork for constructing the subconstructs of the UFT (primeness, relativeness, uniqueness) deliberately in CORE-Q5 – CORE-Q7. This strategy reflects AiC approach in that subsequent activities are designed to build on previous ones and potentially transform the previous constructs in the learner’s mind, eventually arriving at construction of a new concept.

- I anticipate approximately two interview-hours will be needed for completing CORE-Q1 – CORE-Q4. There are in total 15 questions (including prestructured follow-up questions) within CORE-Q1 – CORE-Q4:

<table>
<thead>
<tr>
<th>Question</th>
<th>Purpose of Question; Some potential responses and follow-up on them</th>
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<tbody>
<tr>
<td>CORE-Q1</td>
<td>*In this column R stands for recognizing action; B stands for building-with action; C stands for constructing action in RBC model of AiC framework.</td>
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<td>CORE-Q2</td>
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<td>CORE-Q3</td>
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<td>Q3-F1</td>
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<td>CORE-Q4</td>
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<td>Q4-F1</td>
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<td>Q4-F2</td>
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<td>Q4-F3</td>
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<td>Q4-F3-(a)</td>
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<td>Q4-F3-(b)</td>
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<td>Q4-F3-(c)</td>
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<td>Q4-F3-(d)</td>
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<td>Q4-F4-(a)</td>
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<td>Q4-F4-(b)</td>
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<td>Q4-F4-(c)</td>
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</table>

CORE-Q1: $x^2$ may be rewritten as $x^3 \cdot \frac{1}{x}$. What do you think of the statement that $x^2$ factors into $x^3 \cdot \frac{1}{x}$?

*In this column R stands for recognizing action; B stands for building-with action; C stands for constructing action in RBC model of AiC framework.

[Purpose] I am asking this question to engage them in R-, B-actions on factorization, R-, B-actions on polynomial, and potentially in C-action on existence of the domain $D[x]$ on which factorization operates. A secondary purpose of asking this question is to collect evidence, based on what they have done before (e.g., in pre-TI interviews), of their making transformative transitions with respect to factorization and polynomial.
Interviewee may justify his or her claim by building with various constructs.

- Interviewee might **disagree** with the statement by building with the notion of polynomials—e.g., $1/x$ is not a polynomial and factorization is defined in a set of polynomials.

- Interviewee might **disagree** with the statement by building with the notion of polynomial degrees—e.g., the degrees of factors of a polynomial are usually smaller than the polynomial’s degree, so $x^3$ cannot be a factor of $x^2$.

- Interviewee might **disagree** with the statement by building with a usual procedure of factoring $x^2$—e.g., I would normally factor it into $x$ times $x$.

  Follow up: What do the term ‘to factor’ mean to you? [Assigned meaning will be reflected on the CORE-Q2.]

Interviewee might **agree** with the statement by recognizing

$$x^2 = x^3 \cdot \frac{1}{x}.$$  

Follow up: What does the term ‘to factor’ mean to you? How does that meaning come into play when you ‘factor’ $x^2$? How would you describe your normal way of factoring? [Assigned description/meaning to the ‘normal’ way of factoring will be reflected on the CORE-Q2.]

---

**CORE-Q2**: Students in high school algebra classes are often asked to

[**Purpose**] I am asking this question to engage them in R-, B-actions on factorization, R-, B-

---

82 “Building-with comprises the use and combination of recognized constructs in order to achieve a localized goal such as the actualization of a strategy, a justification or the solution of a problem.” (Dreyfus et al., 2015, p. 188)
‘factor completely.’ What does the phrase ‘factor completely’ mean to you?

actions on polynomial, and potentially in C-action on primeness of the factors in a complete factorization of a polynomial.

A secondary purpose of asking this question is to collect evidence, based on what they have done before, of their making transformative transitions with respect to factorization and polynomial.

Here, interviewee might start with some initial, vague notion such as the idea that to factor completely means to factor until you cannot factor further. This will be revisited in later activities–for constructing the relative of the primeness and the definition of irreducibles.

Follow up: How is the direction to ‘factor completely’ different from or similar to the direction to ‘factor’–as you defined in the previous CORE-Q1? (Show what the subject has written for what it means ‘to factor’ in response to CORE-Q1.)

Follow up when they have difficulties:

General principles of Teaching

Interviews will be applied. (e.g., What is an example of factoring? --- then--- Is that factored completely? If I factor $(x^3+x^2-x-1)$ into $(x^2-1)(x-1)$, then is that factored completely?)

**CORE-Q3:** According to your definition, please factor completely the following polynomials.

\[
\begin{align*}
P_1(x) &= x^4-12x^2+36 \\
P_2(x) &= x^2-x^2 \\
P_3(x) &= 4x+16x^3 \\
P_4(x) &= 12 \\
P_5(x) &= 4x^2+2x-20 \\
P_6(x) &= 2x^3-10x^2+8x
\end{align*}
\]

[Purpose] I am asking this question to engage them in R-, B-actions on factorization, R-, B-actions on polynomial, R-, B-actions on polynomial equation and potentially in C-action on primeness of the factors in a complete factorization of a polynomial.

A secondary purpose of asking this question is to collect evidence, based on what they have done before, of their making transformative transitions with respect to factorization and polynomial equation.
Interviewee may ask which domain to factor completely over. If this is the case, proceed to CORE-Q4.

Interviewee may assume they factor over $\mathbb{Z}$ or $\mathbb{Q}$. If this is the case, asked the Q3-F1.

**Q3-F1**  
*(Case 1) Follow-up based on $P_1$ over $\mathbb{R}$:*  
What do you think about the factorization of $P_1(x)$ being $(x - \sqrt{6})^2(x + \sqrt{6})^2$ instead of $(x^2 - 6)^2$? What do you consider to be ‘completely factored’?

*(Case 2) Follow-up based on $P_2$:*  
What do you think about the factorization of $P_2(x)$ being $x^2(x - 1)(x - \frac{-1+\sqrt{3}i}{2})(x - \frac{-1-\sqrt{3}i}{2})$ instead of $x^2(x - 1)(x^2 + x + 1)$? What do you consider to be ‘completely factored’?

**Purpose**  
I am asking this question to engage them in R-, B-actions on factorization, and potentially in C-action on relativity of the primness in that primness of a certain polynomial depends on the domain on which the factorization is defined.  
A secondary purpose of asking this question is to collect evidence, based on what they have done before, of their making transformative transitions with respect to factorization.

Interviewee may see the equivalence between the two expressions (it is likely if they passed prescreening). The issue is if they recognize what makes the factorizations different.

Interviewee might say that ‘completely factored’ depends on which number system the coefficients are allowed to be.  
If Interviewee insists on a particular number system such as $\mathbb{Z}$ or $\mathbb{Q}$ for thinking about complete factorization, I need to ask redirecting questions.

- e.g., Now let’s suppose that the coefficients can be any real numbers. Then what would it mean to factor completely when the coefficients are allowed to be any real numbers?

Now, we say that we factor over reals when allowing the coefficients to be any real number in the factorization. Similarly, we say that we factor over complex numbers when allowing the
coefficients to be any complex number in the factorization.
Now I want you to factor the same set of polynomials over different number systems. (Ask whether the symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are familiar to them. If not, explain what they represent.)

CORE-Q4: Please factor completely the same set of polynomials over each of the number systems, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ and record your results in the following table.

<table>
<thead>
<tr>
<th>Over $\mathbb{Z}$</th>
<th>Over $\mathbb{Q}$</th>
<th>Over $\mathbb{R}$</th>
<th>Over $\mathbb{C}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_1(x) = \frac{x^4 - 12x^2 + 36}{x^2}$</td>
<td>$P_2(x) = \frac{4x^4 + 16}{x^3}$</td>
<td>$P_3(x) = \frac{4x^4}{x^2}$</td>
<td>$P_4(x) = \frac{2x^4 - 20}{10x^2 + 8x}$</td>
</tr>
</tbody>
</table>

How would you describe what happened in this table in a general way? Or, how can you make some general statement about your answers in this table?

[Purpose] I am asking this question to engage them in $R$-, $B$-actions on factorization, $R$-, $B$-actions on polynomial, $R$-, $B$-actions on polynomial equation and potentially in $C$-action on relativeness of the primness, irreducibility, uniqueness and units.

A secondary purpose of asking this question is to collect evidence, based on what they have done before, of their making transformative transitions with respect to factorization, polynomial, and polynomial equation.

The interviewee might explain that the complete factorizations are the same in some cases and different in other cases (e.g., when we have a larger number system, we can have more zeros than in a smaller number system—which makes the complete factorization different in the two number systems). When participants point out the complete factorizations are the same in some cases and different in other cases, ask them: How can you characterize and distinguish those that will be the same
When Interviewee experiences difficulty in discovering/explaining this, I need to ask guiding or redirecting questions.

(Q4-F1) Follow-up based on student response to $P_2$ over $\mathbb{R}$: For example, $P_2(x) = x^5 - x^2 = x^2(x-1)(x^2 + x + 1)$

A student claims that $x-1$ is not factored completely over $\mathbb{R}$ because there is a factorization $x - 1 = (\sqrt{x} - 1)(\sqrt{x} + 1)$. What do you think of this student’s claim?

[Purpose] I am asking this question to engage them in R-, B-actions on factorization, R-, B-actions on polynomial, and potentially in C-action on existence of the domain $D[x]$ on which factorization operates. A secondary purpose of asking this question is to collect evidence, based on what they have done before, of their making transformative transitions with respect to factorization and polynomial.

This question is similar to CORE-Q1 but has a role of consolidating what could have been discussed in CORE-Q1 as well as combining the discussion with the notion of ‘factorization over $\mathbb{R}$.’ Interviewee might disagree with the statement by building-with the notion of polynomial—e.g., $\sqrt{x} - 1$ and $\sqrt{x} + 1$ are not polynomials.

If the interviewee agrees with the statement, ask why and follow up on it using redirecting questions. The question might be like: How does this fit your description of ‘factoring completely over $\mathbb{R}$’? How is this factorization similar to or different from the way that you would normally factor $x-1$?

Source: “A student claims that $x-1$ is not irreducible in $\mathbb{Q}[x]$ because there is a factorization $x - 1 = (\sqrt{x} - 1)(\sqrt{x} + 1)$. Explain the error of his ways.” (Cuoco & Rotman, 2013, p. 243)
(Q4-F2)
Follow-up based on student response to $P_4$ over $\mathbb{Z}$: For example, $P_4(x) = 12 = 2 \times 2 \times 3$

If a student proposed $1 \times 2 \times 2 \times 3$ as a complete factorization of 12 over $\mathbb{Z}$, what would you say?

[Purpose] I am asking this question to engage them in R-, B-actions on factorization, R-, B-actions on prime numbers, and potentially in C-action on unit in $\mathbb{Z}$ and its relation to the uniqueness of the complete factorization. A secondary purpose of asking this question is to collect evidence, based on what they have done before, of their making transformative transitions with respect to factorization.

This question asks the interviewee to compare his/her complete factorization of 12 over $\mathbb{Z}$ (which is likely to be $2 \times 2 \times 3$) and a different way proposed in the question statement ($1 \times 2 \times 2 \times 3$).

Interviewee might point out that $1 \times 2 \times 2 \times 3$ is a trivial factorization and doesn’t give any new information about what 12 is composed of. So, the interviewee might say $2 \times 2 \times 3$ is a better complete factorization. Other explanation might be based on simplicity, parsimony, avoiding redundancy, or avoiding infinitely many ways to have a prime factorization.

If the description of ‘factoring completely over $\mathbb{Z}$’ from CORE-Q2 has not been brought up yet, draw his/her attention to his/her own definition/description of ‘factoring completely’ over $\mathbb{Z}$ from CORE-Q2 by asking,

How does this (i.e., $1 \times 2 \times 2 \times 3$) fit your description of ‘factoring completely’ over $\mathbb{Z}$? How is this factorization similar to or different from the way that you would normally factor completely 12 over $\mathbb{Z}$?

It is possible that their initial description/definition does not provide enough criteria for determining whether $1 \times 2 \times 2 \times 3$ is to be considered a complete factorization. I might need redirecting questions at this point as follows:
Can you explain how your original description provides enough information/criteria to determine whether $1\times2\times2\times3$ is a complete factorization?

→ They might say yes but ask further. Ask them to explain how it does. Focus on how they treat 1—i.e., do they consider or exclude 1 as a prime factor in a complete factorization?

If someone else factors $P_4(x) = 12$ completely over $\mathbb{Z}$, could he or she correctly come up with a different answer than yours ($2\times2\times3$)?

→ This question touches on the uniqueness and potentially the idea that 1 is not considered prime.

(Q4–F3)

Follow-up based on student response to $P_4$ over $\mathbb{Q}$:

(Case 1: When the interviewee wrote 12 under the cell of $P_4$ over $\mathbb{Q}$)
If a student claims that $2\times2\times3$ is also a complete factorization over $\mathbb{Q}$ because 2, 2, and 3 are all rational numbers, what would you say? (Delving deeper their understanding of relationalness)

(Case 2: When the interviewee wrote $2\times2\times3$ under the cell of $P_4$ over $\mathbb{Q}$)
If a student claims that factorization of 12 over $\mathbb{Q}$ is

[Purpose] I am asking these questions to engage them in R-, B-actions on factorization, and potentially in C-action on unit and its relation to the uniqueness of the complete factorization.

A secondary purpose of asking this question is to collect evidence, based on what they have done before, of their making transformative transitions with respect to factorization.

Case 1: Interviewee might point out that 2 is a product of its factors in $\mathbb{Q}[x]$, so as 3, and so as any other constants in $\mathbb{Q}[x]$. Thus, 12 is a complete factorization as it is, or there is no point of considering factorization of 12.

Otherwise, ask redirecting questions by asking them to compare the student’s claim with their definition/description of factoring completely over $\mathbb{Q}$.

Case 2: Interviewee might agree or disagree for his/her own reasons. Listen to their reasoning
meaningless because 12 can be factored over \( \mathbb{Q} \) in infinitely many ways—such as \((1/2)\cdot 2\cdot 2\cdot 2\cdot 3\), what would you say?

(Q4-F3-(a))

(End of questions in both Cases)

Now, have you ever heard of the term reducible or irreducible with respect to factorization? What does reducible or irreducible mean to you?

(Q4-F3-(b))

So, do you think 12 is reducible, irreducible, or neither of them over \( \mathbb{Q} \)? Why do you think so?

and, in so doing, guide them towards refining the meaning of factoring completely over \( \mathbb{Q} \).

(Q4-F3-(a))

First discuss what is meant by ‘reducible’ or ‘irreducible.’ It is usually the case that the terms are interpreted in generic ways (but not in a well-defined way; STAGE 1).

Using the factors in the polynomial table from CORE-Q4 might be a good way to start. Pick some examples and ask:

Would you consider this (e.g., \( x^3-1 \)) reducible or irreducible over \( \mathbb{Q} \)? How about others? ---- then, what do you think reducible or irreducible over \( \mathbb{Q} \) means?

How are these notions of irreducible or reducible related to your description of factoring completely over \( \mathbb{Q} \)?

(Q4-F3-(b))
(Case A) Interviewee might say 12 is reducible because it can be written as a product of its factors in a given domain: STAGE 2.

This could be problematic in the later activities. Ask guiding or redirecting questions as follows:

Does it go by YOUR definition of reducible?
(They might re-examine their definition or they might still say yes.)

If someone else factors $P_5(x) = 4x^2+2x–20$ completely over $\mathbb{Q}$ given that elements such as 12 are considered to be reducible, could he or she correctly come up with a different answer than yours?
(\(\rightarrow\) This question touches on the uniqueness and potentially the idea that units are not considered reducible.)

(Case B) Interviewee might say 12 is irreducible because it doesn’t need to be factored further in a complete factorization: STAGE 2. This could be problematic in the later activities. Ask guiding or redirecting questions as follows:

Does it go by YOUR definition of irreducible?
(They might re-examine their definition or they might still say yes.)

If someone else factors $P_5(x) = 4x^2+2x–20$ completely over $\mathbb{Q}$ given that
Mathematicians call the element such as 12 in $\mathbb{Q}$ a *unit*. That is, (Write) an element in a domain (such as $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$) that has its multiplicative inverse in that domain is called UNIT.

What are some examples of units in $\mathbb{Q}$? What about units in $\mathbb{Z}$?

Please select a unit in $\mathbb{Z}$. Is the unit you selected (as a polynomial) reducible, irreducible, or neither over $\mathbb{Z}$? Why do you think so?

What is the relationship among units, reducibles, and irreducibles?

(Q4-F3-(d))

STAGE 3: Boundaries for the notion of irreducible or reducible are clearer than STAGE 1.
### Purpose
I am asking these questions to engage them in R-, B-actions on factorization, and potentially in C-action on irreducibility (primeness), relatedness of the primeness, and unit and its relation to the uniqueness of the complete factorization.
A secondary purpose of asking this question is to collect evidence, based on what they have done before, of their making transformative transitions with respect to factorization.

### Follow-up based on student response to P₆ over ℤ:
For example,

\[ P₆(x) = 2x³ - 10x² + 8x = 2x(4)(x-1) \]

### (Q4-F4-(a))
Would you consider 2 as irreducible over ℤ? If so, why? If not, why not? How about over ℚ?

### (Q4-F4-(b))
Would you consider 2x as irreducible over ℤ? If so, why? If not, why not? How about over ℚ?

### (Q4-F4-(a))
2 is irreducible over ℤ and a unit over ℚ.

Interviewee might point out 2 is not a unit in ℤ but a unit in ℚ (from what s/he came to know in the previous activity). Thus, 2 is an irreducible polynomial over ℤ but not over ℚ.

If the interviewee thinks 2 is not irreducible over ℤ, ask a redirecting question:

How was 2 treated in other factorizations such as a factorization of \( P₄(2*2*3) \) over ℤ?

### (Q4-F4-(b))
2x is reducible over ℤ and irreducible over ℚ.

Interviewee might say 2x can be factored into two irreducibles, 2 and x, over ℤ and so 2x is reducible over ℤ. Also, s/he might say since 2 is a unit in ℚ, 2x is considered to be irreducible.

**STAGE 4**: Interviewee might say that 2x is reducible over both ℤ and ℚ—that is, over ℤ, 2x is reducible into two irreducibles and, over ℚ, 2x is reducible into one irreducible and one unit. This reasoning entails a contradiction in it. Ask a redirection question:

It seems to be that you would consider a product of one irreducible and one...
(Q4-F4-(c))
Looking at this cell (P₆ over ℤ), please describe how reducible and irreducible may come into play.

unit as reducible. Let’s consider a product of one irreducible and one unit in the polynomials over ℤ: 2 and 1. Their product is 2. Then is 2 irreducible or reducible? ---- then---- how would you revise your original thought?

PART 2

• Next three questions (CORE-Q5 – CORE-Q7) call for participants’ engaging in R- and B- actions on factorization, or polynomial and for engaging in C- actions on subconstructs of the UFT (primeness, relativeness, uniqueness) deliberately. Completing CORE-Q5 – CORE-Q7 would lay the groundwork for consolidating their understandings of the subconstructs of the UFT (primeness, relativeness, uniqueness) further in CORE-Q8 – CORE-Q12. This strategy reflects AiC approach in that this strategy may allow a new construct to be gradually incorporated into the learner’s knowledge structure as the new construct gets to be recognized and built-with more and more in the further activities. That is, participants’ newly constructed knowledge may be consolidated further in the subsequent R- and B- actions on the new constructs.

• I anticipate approximately two interview-hours will be needed for completing all the rest: CORE-Q5 – CORE-Q12.

| Question | Purpose of Question; Some potential responses and follow-up on them |
CORE-Q5: What would you consider as an irreducible polynomial over each of the four domains, \( \mathbb{Z} \), \( \mathbb{Q} \), \( \mathbb{R} \) and \( \mathbb{C} \)?

**Purpose** I am asking this question to engage them in \( R \)-, \( B \)-actions on factorization, \( R \)-, \( B \)-actions on polynomial, and deliberately in \( C \)-action on irreducibility.

A secondary purpose of asking this question is to collect evidence, based on what they have done before, of their making transformative transitions with respect to factorization and polynomial.

Interviewee might say that an irreducible polynomial over \( D \) is a nonunit polynomial such that the only factors are units and unit-multiples of itself over \( D \).

STAGE 5

When they have difficulties in defining/describing, ask guiding questions first using the examples from CORE-Q4 table:

Could you revisit what you have done so far and tell me what you did or why you did so?

How would you determine whether (the expression) is irreducible? What descriptions can be used to determine whether something is irreducible? If your description does not give you enough information to determine whether (the expression) is irreducible, can you revise your description so that you can determine?

CORE-Q6: How would you explain to your friend what a complete factorization of a polynomial over a domain \( D \)?

**Purpose** I am asking this question to engage them in \( R \)-, \( B \)-actions on factorization, \( R \)-, \( B \)-actions on polynomial, and deliberately in \( C \)-action on existence, primeness, and relativeness.
is? (*D is used to stand for \( \mathbb{Z} \), \( \mathbb{Q} \), \( \mathbb{R} \) or \( \mathbb{C} \)).

<table>
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<tr>
<th>CORE-Q7: Are there multiple ways to factor completely the same polynomial given a fixed domain?</th>
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<tbody>
<tr>
<td>[Purpose] I am asking this question to engage them in R-, B-actions on factorization, R-, B-actions on polynomial, and deliberately in C-action on uniqueness. A secondary purpose of asking this question is to collect evidence, based on what they have done before, of their making transformative transitions with respect to factorization and polynomial.</td>
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Interviewee might point out a complete factorization can be written differently for the same polynomial. For example, for \( P_5(x) = 4x^2 + 2x - 20 \), the following is how complete factorizations might ‘look’ different:
2(2x+5)(x−2), (2x+5)(2x−4), (4x+10)(x−2), (2x−4)(2x+5), etc. I might want to ask them:

Can you make some general statements about these different factorizations?
What are the relationships between the factors in different-looking factorizations?

Then, given a certain domain, is a complete factorization of a polynomial unique or not unique? In what sense, is it unique or not unique?

It doesn’t seem important for them to think it is unique (because the representations are different!). Rather, what is important is to listen to their reasoning—why they think it is unique or not unique. Even if they say it is not unique, it may be that they notice different representations consist of essentially the same irreducibles that are just differ by units.

PART 3

• Next five questions (CORE-Q8 – CORE-Q12) call for participants’ engaging in R- and B- actions on polynomial equation, or polynomial and also on subconstructs of the UFT (primeness, relativeness) which arose from the previous activities. Beyond this point, the tasks ask the interviewee to apply what s/he has established so far and consolidate her/his understandings.
• It is possible that they use the examples in the previous tasks for thinking about the tasks here.
• As follow-up questions to the participants’ responses, I will use the basic principles for teaching interview in this section. I will be asking guiding/redirectioning questions when students have difficulties. The questions most likely to have them reveal their current thinking (e.g., What would be the examples of XXXX?) and reflect on what they did with the polynomials in previous tasks and how they defined core constructs before (e.g., How does this fit with what you have done in XXXX? How does this fit with your description/definition of XXXX? What can you do to verify that YYYY is XXXX?).
<table>
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<tr>
<th>Question</th>
<th>Purpose of Question; Some potential responses and follow-up on them</th>
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</table>
| **CORE-Q8:**
1) Make up a polynomial such that its irreducibility over \( \mathbb{Q} \) is different from its irreducibility over \( \mathbb{R} \).
2) Make up a polynomial such that its irreducibility over \( \mathbb{R} \) is different from its irreducibility over \( \mathbb{C} \).
3) Make up a polynomial such that its irreducibility over \( \mathbb{Z} \) is different from its irreducibility over \( \mathbb{Q} \). *(none of such nonconstant polynomial exists if using integer coefficients)*

*(Encourage quadratic trinomials! They tend to try binomials)*

**CORE-Q9:** Let \( c \) be an integer. Discuss the irreducibility of the polynomial, \( x^2 + c \), over \( \mathbb{Z}, \mathbb{Q}, \mathbb{R} \) and \( \mathbb{C} \).

**CORE-Q10:** Let \( a, b, \) and \( c \) be integers. Discuss the irreducibility of the polynomial, \( ax^2 + bx + c \) over \( \mathbb{Z}, \mathbb{Q}, \mathbb{R} \) and \( \mathbb{C} \).

**CORE-Q11:** Examine the following statement: "If \( p(x) \) is
<table>
<thead>
<tr>
<th>irreducible over $\mathbb{R}$ then $p(x) = 0$ has no roots in $\mathbb{R}$. Is this statement true? If so, explain why it is true. If not, explain why it is not true.</th>
<th>actions on polynomial equation, $R$-, $B$-actions on irreducibility and $R$-, $B$-actions on relatedness. A secondary purpose of asking this question is to collect evidence, based on what they have done before, of their making transformative transitions with respect to polynomial and polynomial equation.</th>
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<tr>
<td><strong>CORE-Q12:</strong> Examine the following statement: “If $p(x) = 0$ has no roots in $\mathbb{R}$, then $p(x)$ is irreducible over $\mathbb{R}$. Is this statement true? If so, explain why it is true. If not, explain why it is not true.</td>
<td>(true)</td>
</tr>
<tr>
<td>[Purpose] I am asking this question to engage them in $R$-, $B$-actions on polynomial, $R$-, $B$-actions on irreducibility and $R$-, $B$-actions on relatedness. A secondary purpose of asking this question is to collect evidence, based on what they have done before, of their making transformative transitions with respect to polynomial and polynomial equation.</td>
<td></td>
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<tr>
<td>(This is true only if it is of second or third degrees. $x^4 + 3x^2 + 2$ is reducible but has no zeros in $\mathbb{R}$.)</td>
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</table>
Appendix C: Pre- and Post-TI Interview Questions

(Anticipated interview hours per subject: 1 to 2 hours for each)

Goal of this pair of pre- and post-TI (clinical) interviews
To examine how their existing understandings of factorization and polynomial equation have changed (or not changed) after their engagement in the TI and to observe what transformative transitions were made, if any

Which category of transformative transitions do I expect to occur?
Expected transformative transitions that may potentially occur as a result of engaging in the TIs are specifically as follows:
- Deepening–understanding of factorization (for description of levels, see Table 2.1),
- Extending–understanding of factorization (for description of levels, see Table 2.2),
- Strengthening–understanding of polynomial equation (for description of levels, see Table 2.3),
- Unifying–understanding of factorization (for description of levels, see Table 2.4).

However, the analysis of data would not be limited to these five because participants might utilize some other constructs in their existing understandings in responding to these questions and make some transformative transitions with respect to the constructs.

Questions and purpose of each question
Questions starting with “PRE” will be asked during the pre-TI interview and the ones starting with “POST” will be asked during the post-TI interview.

<table>
<thead>
<tr>
<th>Questions</th>
<th>Purpose of the questions</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>PRE-Q1 (and POST-Q1):</strong> Can you factor $ax^2+bx+c$ without knowing the specific values of a, b, and c? If so, how can you do so? If not, why do you think so?</td>
<td>I am asking this question before and after the TIs so that I can compare their responses (before vs. after) and observe what transformative transitions were made, if any. Expected transformative</td>
</tr>
</tbody>
</table>
transitions are the following:

- Deepening—understanding of factorization,
- Extending—understanding of factorization,
- Strengthening—understanding of polynomial equation.

Source:
“7. How can I factor any quadratic polynomial?”
(Cuoco, 2009, p. 700)

PRE-Q2 (and POST-Q2): Find if possible a, b, and c for each of the following cases:
(i) \(ax^2+bx+c=0\) has one integer root and one rational root that is not integer;
(ii) \(ax^2+bx+c=0\) has two rational roots that are not integers;
(iii) \(ax^2+bx+c=0\) has two real roots that are not rational;
(iv) \(ax^2+bx+c=0\) has one real root and one nonreal root;
(v) \(ax^2+bx+c=0\) has no real roots.

POST-Q2’: Find necessary and sufficient conditions on the real numbers a, b, c, d for the equation
\[z^2 + (a+bi)z + (c+di) = 0\]
to have exactly one real and one nonreal root.

I am asking this question before and after the TIs so that I can compare their responses (before vs. after) and observe what transformative transitions were made, if any. Expected transformative transitions are the following:

- Extending—understanding of factorization,
- Strengthening—understanding of polynomial equation.

Source 1:
“2. Find a value of \(k\) such that \(2x^2–3x+k\) has each solution set.
a. two real-number solutions
b. one real-number solution
c. no real-number solutions”
(Cuoco, 2009, p. 700)
### Source 2:
“8. Find necessary and sufficient conditions on the real numbers a, b, c, d for the equation  
\[ z^2 + (a+bi)z + (c+di) = 0 \]
to have exactly one real and one nonreal root.”  
(Barbeau, 1989, p. 40)

### PRE-Q3: Write a quadratic equation with the following characteristics: It has only rational coefficients and one of its roots is  
\[ 1 + \sqrt{3}. \]

### POST-Q3: Write a quadratic equation which has  
\[ 1 + \sqrt{3} \]
as one of its roots.

### PRE-Q4: A quadratic equation has only rational coefficients. If one root is irrational, is the other irrational or rational? Why is it so?

### POST-Q4:
- If one of the roots of a quadratic equation is irrational, is the other irrational or rational? Why is it so?
- If one of the roots of a quadratic equation is nonreal, is the other nonreal or real? Why is it so?

### PRE-Q5: Suppose that you have a quadratic equation  
\[ x^2+bx+c=0, \]
which has two roots \( x_1 \) and \( x_2 \). Find another quadratic equation whose roots are \( 2x_1 \) and \( 2x_2 \).

### POST-Q5: Suppose that you have a quadratic equation  
\[ ax^2+bx+c=0, \]
which has two roots \( x_1 \) and \( x_2 \). Find another quadratic equation whose roots are \( 2x_1 \) and \( 2x_2 \).

### PRE-Q6: Suppose that you have a quadratic equation  
\[ x^2+bx+c=0, \]
which has two roots \( x_1 \) and \( x_2 \). Find the roots of the following equation:  
\[ x^2–bx+c=0. \]
**POST-Q6:** Suppose that you have a quadratic equation $ax^2+bx+c=0$, which has two roots $x_1$ and $x_2$. Find the roots of the following equation:

$$ax^2–bx+c=0.$$  

**PRE-Q7:** Can you factor *any* quadratic polynomial? If so, how can you do so? If not, why do you think so?

**POST-Q7:** Please explain your reasoning to the following questions.

(a) Can you factor any quadratic polynomial?
(b) Can you factor any quadratic polynomial *uniquely*?
(c) Can you factor any quadratic polynomial *completely*?
(d) Can you factor any quadratic polynomial *completely* and *uniquely*?

This question is similar to PRE-Q1 but asks the same idea without symbolic representations. Participants’ answers to this question can supplement their answers to PRE-Q1 (or POST-Q1). Also, by asking this question before and after the TIs, I can compare their responses (before vs. after) and observe what transformative transitions were made, if any. Expected transformative transitions are the following:

- Deepening—understanding of factorization,
- Extending—understanding of factorization,
- Strengthening—understanding of polynomial equation.

Source:

**PRE-Q8 (and POST-Q8):** According to the Common Core State Standards high school students need to *Understand that polynomials form a system analogous to the integers.* In what sense do you think they are analogous?

<table>
<thead>
<tr>
<th>I am asking this question before and after the TIs so that I can compare their responses (before vs. after) and observe what transformative transitions were made, if any. Expected transformative transitions are the following:</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Unifying–understanding of factorization</td>
</tr>
</tbody>
</table>

Source: “… displaying parallels between integers and polynomials. For polynomials over a field, we have, so far * extended the notion of divisibility * generalized “prime” to “irreducible” * shown that factorizations into irreducibles exist * established a division algorithm * shown that the gcd of two polynomials exists and is unique * shown that the gcd of two polynomials is a linear combination of them.” (Cuoco & Rotman, 2013, p. 245) |

<table>
<thead>
<tr>
<th><strong>PRE-Q9 (and POST-Q9):</strong> Please construct a concept map of YOUR ideas about polynomial in the context of school mathematics. Think of all of the things you associate with the concept of polynomial and construct a network among those ideas. (You may use the following ‘candidate’ ideas as you see them relevant. You do NOT need to use every idea listed and you may certainly use the ideas that are not listed.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>I am asking this question before and after the TIs so that I can compare their responses (before vs. after) and observe what transformative transitions were made, if any. Expected transformative</td>
</tr>
</tbody>
</table>
‘Candidate’ ideas:

<table>
<thead>
<tr>
<th>Polynomial</th>
<th>Factoring</th>
<th>Coefficient</th>
<th>Rational numbers</th>
</tr>
</thead>
<tbody>
<tr>
<td>Degree</td>
<td>Quadratic formula</td>
<td>Integers</td>
<td>Prime factorization</td>
</tr>
<tr>
<td>Equation solving</td>
<td>Discriminant</td>
<td>Complex numbers</td>
<td>Real numbers</td>
</tr>
<tr>
<td>Roots</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Follow-up: Please explain what you mean by each of the ideas in your concept map. Also please explain how and why the ideas are linked in your network.

Transitions are the following:

- Deepening—understanding of factorization,
- Extending—understanding of factorization,
- Strengthening—understanding of polynomial equation,
- Unifying—understanding of factorization.
Appendix D: Invitation email – Request for participation

**EMAIL SUBJECT:**

Research Participation Invitation: University Students’ Making Meaningful Transitions in Their School Mathematics Understandings within an Abstraction-in-Context Approach

**EMAIL MESSAGE BODY:**

Hello,

I asked Dr (or Ms or Mr). ___________ to forward this email to you. My name is Younhee Lee, and I am a PhD candidate in the College of Education. For my dissertation study at the Penn State, I am conducting research on how individuals at the collegiate level think about mathematical ideas related to school mathematics.

For my dissertation study, I am looking for volunteers over the age of 18 to participate in six individual interviews during Fall 2016. Each of six interviews would last less than 60 minutes, during which I would be asking you to think about and share your ideas about some mathematical topics – e.g., polynomials and factoring. None of the questions will deal with any personal or sensitive information. Upon the completion of six interviews, you will be offered $60 in cash in return for your participation. When it seems useful or necessary to conduct an additional interview, I may ask you to participate in an additional interview and compensate your participation accordingly (i.e., $10 for each additional hour or percentage thereof if the additional interview is less than 60-minute long).

The interview will be audio- and video-recorded but the video will capture only the papers on which you write and the computer/calculator screens on which you work; your face will not be captured on the video recording. The data collected from your participation will be kept confidential and it will have no effect on your course grade. I am looking for about six to eight volunteers, so if you do volunteer, I cannot guarantee that I will end up interviewing you.

There exists a prescreening survey in this study, so if you are interested in participating in this study, please complete the prescreening survey in the following link without consulting resources: [https://docs.google.com/forms/d/e/1FAIpQLScXRTt089UeGgs3sIWXqeS8sEDPFIWHBx7TrsAeyAvtaARFFdw/viewform](https://docs.google.com/forms/d/e/1FAIpQLScXRTt089UeGgs3sIWXqeS8sEDPFIWHBx7TrsAeyAvtaARFFdw/viewform)

The survey would take no longer than 15 minutes. You will be notified via email if you are selected for the interviews and will be asked to schedule meetings for the interviews.
Two individuals among all participants who answered the prescreening survey will be selected by a random draw and get a prize (Starbucks gift card $25). The winners will be notified by email. The drawing may occur several weeks after your participation.

If you think you are interested or if you have any questions, please email or call me (yul182@psu.edu; 814-777-8382). Thanks!
Younhee Lee
Appendix E: Summary Explanation of Research

The following statement is what will be given to the participants in the beginning of their first, pre-TI interview.

SUMMARY EXPLANATION OF RESEARCH
The Pennsylvania State University

Title of Project: University Students’ Making Meaningful Transitions in Their School Mathematics Understandings within an Abstraction-in-Context Approach

Principal Investigator: Younhee Lee
Telephone Number: (814)-777-8382
Advisor: Dr. M. Kathleen Heid
Advisor Telephone Number: (814)-865-2226

You are being invited to volunteer to participate in a research study at Penn State. This summary explains information about this research.

- My primary objective is to investigate how individuals at the collegiate level construct their new mathematical knowledge and reconstruct school mathematics understandings concurrently.
- You will be asked to think about and share your ideas (what you think, what you notice, or why you think so) about the topics to be covered in a series of interviews (e.g., polynomials and factoring). The interview questions consist of only mathematical tasks. The interview will be audio-recorded and video-recorded. The video will capture only the papers on which you write and the computer/calculator screens on which you work; your face will not be recorded.
- The data collected from your participation in this research is confidential. The data will be stored and secured in an encrypted external hard drive and in an encrypted back-up copy. Only the principal investigator and the advisor (Dr. Heid) will have access to the audio, video, and written data files. A professional transcriptionist might have access to the audio data files for transcription job but the audio data would not include your name or identifying information. Nobody will have access to your real identity other than the principal investigator and the advisor. All recordings and transcripts will be destroyed ten years after the end of the research project (in May of 2026).
- In the event of a publication or presentation resulting from the research, no personally identifiable information will be shared. Your data will be assigned a pseudonym; your real name will not appear as part of any file name or be used in any presentation or publication.

If you have questions or concerns, please contact Younhee Lee at 814-777-8382. If you have questions regarding your rights as a research subject or concerns regarding your privacy, please contact the Office for Research Protections at 814-865-1775.
Your participation is voluntary and you may decide to stop at any time. You do not have to answer any questions that you do not want to answer.

Your participation implies your voluntary consent to participate in the research.
Appendix F: Intercoder agreement—Training/coding process

1. Coders
   a. The dissertation advisor and five doctoral students in mathematics education. Three of doctoral students were at their dissertation phases.
   b. The coders were the members in MTHED 590, Mathematics Education Colloquium, in 2017 Spring.

2. Date/Time of meetings
   a. 2/28/2017 (2-5pm)
   b. 3/21/2017 (2-5pm)

3. Location of meetings
   a. 206 Chambers
   b. One member participated via virtual meeting.

4. The objectives
   a. To report the extent to which the identification of the levels under each of the categories coincides among coders
   b. The acceptable level of agreement for the current study is set to be 80%. In order to determine how much agreement among raters would be considered acceptable, I consulted the literature on interrater reliability (Hartmann, 1977; Graham, Milanowski, & Miller, 2012; Stemler, 2004). The acceptable percentage of agreement found in the literature varies from 75% to 90%.

5. The training process
   a. One week prior to the first meeting, I requested the coders to read the descriptions of categories (Deepening, Broadening/Extending, Strengthening, and Unifying) and levels before they came to the meeting.
   b. During the first meeting on 2/28/2017, I explained categories and provided a general description of three levels in each of the four categories.
   c. I provided 4 episodes (Episode #1-4) which had been taken from actual interviews. The 4 episodes were given along with explanations for their identification of categories and levels. Coders had a chance to ask questions about coding scheme and we discussed and resolved questions as a group.
   d. During the second meeting on 3/21/2017, I provided more detailed descriptions of three levels in each of the four categories. Considering the 3-week time gap between the first and second meetings, I revisited Episode #1 and #3 that we had covered in our first meeting. To be specific, the coders were first asked to individually code Episode #1. I led a discussion concerning the coding of the episode. Reasons for differences were discussed. Then we moved on Episode #3 and repeated the process.
   e. I provided 4 additional episodes (Episode #5-8) which also had been taken from actual interviews. The coders were then asked to individually code the 4
additional episodes. I led a discussion concerning the coding of the episodes. Reasons for differences were discussed. I checked how many agreed with my identification. Goal was to arrive at acceptable agreement with discussion (i.e., 5 people out of 6 members agreed with my identification) so that coders are ready for the final agreement measure.

6. The final agreement measure process
   a. I decided who would constitute the final coding team. The member who participated via virtual meeting could not participate in the beginning sessions of both meetings because of technical difficulties. For that reason, his rating was not counted toward the agreement calculation.
   b. I provided 6 additional episodes (Episode #9-14) which also had been taken from actual interviews. The coders were then asked to individually code the 6 additional episodes.

7. The level of agreement
   a. The average level of agreement was 83.3%. The following table contains the details about the final agreement measure.

<table>
<thead>
<tr>
<th></th>
<th>Episode 9</th>
<th>Episode 10</th>
<th>Episode 11</th>
<th>Episode 12</th>
<th>Episode 13</th>
<th>Episode 14</th>
<th>Percentage of Agreement</th>
</tr>
</thead>
<tbody>
<tr>
<td>The researcher</td>
<td>D2</td>
<td>S3</td>
<td>S2</td>
<td>U2</td>
<td>B3</td>
<td>D1</td>
<td>N/A</td>
</tr>
<tr>
<td>Rater 1</td>
<td>D1 *</td>
<td>S3</td>
<td>S2</td>
<td>U2</td>
<td>B3</td>
<td>D1</td>
<td>83.3%</td>
</tr>
<tr>
<td>Rater 2</td>
<td>D2</td>
<td>S3</td>
<td>S1 *</td>
<td>U2</td>
<td>B3</td>
<td>D1</td>
<td>83.3%</td>
</tr>
<tr>
<td>Rater 3</td>
<td>D2</td>
<td>S3</td>
<td>S2</td>
<td>U2</td>
<td>B3</td>
<td>D1</td>
<td>100%</td>
</tr>
<tr>
<td>Rater 4</td>
<td>D2</td>
<td>S2 *</td>
<td>S1 *</td>
<td>U2</td>
<td>B3</td>
<td>D1</td>
<td>66.7%</td>
</tr>
<tr>
<td>Rater 5</td>
<td>D1 *</td>
<td>S3</td>
<td>S2</td>
<td>U2</td>
<td>B3</td>
<td>D1</td>
<td>83.3%</td>
</tr>
</tbody>
</table>

   **Average Percentage of Agreement**  **83.3%**

8. Conclusion
The identification of the levels under each of the categories seemed reliable as the level of agreement (83.3%) is at an acceptable range (> 80%).
VITA

Younhee Lee

Education
Ph.D.  Curriculum and Instruction (Mathematics Education), The Pennsylvania State University, December 2018
M.A.  Mathematics, The Pennsylvania State University, December 2014
M.A.  Teaching and Learning (Mathematics Education), The University of Iowa, June 2011
B.S.  Mathematics Education, Seoul National University, February 2009

Academic and Professional Experience
Lecturer, Department of Mathematics, The Pennsylvania State University, 2017-2018
Graduate assistant, Department of Curriculum and Instruction, The Pennsylvania State University, 2011-2017
Graduate assistant, Department of Teaching and Learning, The University of Iowa, 2010-2011
Mathematics teacher, Samsung High School, 2009

Selected Publications

Selected Presentations