DEVELOPMENT OF AN ADAPTIVE, MULTISCALE, NONLINEAR DISTURBANCE EQUATIONS SOLVER

A Dissertation in Aerospace Engineering by Amandeep Premi

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Abstract

This work presents the development of an incompressible, adaptive, multiscale, nonlinear disturbance equations solver in the finite-volume OpenFOAM(v4.1) framework. The nonlinear disturbance equations are derived from the Navier-Stokes equations by representing the primary flow variables as a sum of steady and time varying components and then subtracting the steady flow Navier-Stokes equations out of the resulting equations. The application of these equations to resolve the unsteady behavior of a fluid flow offers opportunities to save on substantial amounts of computational effort. These savings are made possible by reduction in the amount of spatial and temporal discretization that is required to solve the problem. However, these savings are only practical when certain conditions are met, which can vary based on the problem at hand. To encompass the domain of problems, where these savings are plausible, a generalized formulation of the nonlinear disturbance equations in terms of the equation formulation and numerical solution procedure is essential. In this work, the nonlinear disturbance equations are implemented in an incompressible format with both block-coupled as well as segregated solution strategies using collocated, finite-volume grids. A novel disturbance-pressure and disturbance-velocity coupling scheme is also developed for the segregated solver. The developed solver features multiscale capabilities with the addition of wavelet based runtime analysis. The wavelet analysis is restricted to a predefined number of scales to keep the computational-stencil small and local. This allows the analysis to be computationally inexpensive and compatible with finite-volume computational fluid dynamics (CFD) solvers. The multiscale architecture is utilized to enable adaptive mesh refinement with a physics based adaptable large eddy simulation (LES) type filter. The multiscale capabilities are also explored for detection of changes in the flow behavior, like the transition of the flow from laminar to turbulent. The application of the developed solver in the areas of laminar-flow stability and gust modelling is also explored.

This solver aims to implement a generalized form of incompressible nonlinear disturbance equations in both coupled/segregated form with adaptive multiresolu-
tion capabilities in an open-source finite-volume framework. This makes the current work a novel advancement in terms of concept and applicability. The generalized formulation makes it applicable to many unsteady-aerodynamics problems and the finite-volume framework with segregated solver makes it practical to apply it to real-world problems with variety of options for numerical solvers.
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List of Symbols

\( H^b_a \) Hankel function of b kind, of order a

\( J_a \) Bessel function of first kind, of order a

\( \bar{P} \) Mean pressure

\( Re \) Reynolds number = \( \frac{\rho \times \text{Velocity} \times d}{\mu} \)

\( \mathcal{U} \) Mean velocity in x-direction

\( \mathcal{V} \) Mean velocity in y-direction

\( \mathcal{W} \) Mean velocity in z-direction

\( Y^a_a \) Bessel function of second kind, of order a

\( d \) Characteristic dimension of the body

\( k \) Reduced frequency = \( \frac{\omega \times c}{2 \times \text{Velocity}} \)

\( \bar{p} \) Disturbance-pressure

\( s \) Reduced time = \( \frac{2 \times \text{Velocity} \times t}{c} \)

\( \tilde{u} \) Disturbance-velocity in x-direction

\( \tilde{v} \) Disturbance-velocity in y-direction

\( \tilde{w} \) Disturbance-velocity in z-direction

\( \rho \) Density of the fluid

\( \mu \) Dynamic viscosity of the fluid
\( \nu \)  Kinematic viscosity of the fluid

\( \omega \)  Frequency of oscillation in radians per second
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Dedication

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Chapter 1  
Introduction

Fluid dynamics encompasses the study of any kind of macroscopic motion of a fluid medium. It is an integral part of development for a variety of engineering applications, ranging from inkjet printheads to spacecrafts. The motion of fluids is a function of a number of forces acting on them, that can act in different ways depending on the flow conditions. A very small amplitude disturbance in a flow can cause changes at a macroscopic system scale in the mean flow behavior. Its imperative to understand these influences and fluid behaviors to improve the efficiency of the many engineering systems involving fluids. A great deal of fluid dynamics research has spawned a variety of models and theories to design efficient engineering systems. However, a big fraction of these models assume steady or quasi-steady flow variables. This steady or quasi-steady flow assumption can often lead to a level of inaccuracy in the solution that justifies higher fidelity methods. This has led to another field of research in unsteady fluid-dynamics, leading to different models to accommodate the effects of unsteady motion of the fluid. This work intends to build a generalized solver framework based on incompressible nonlinear disturbance equations to resolve unsteady flow. The current work builds on earlier concepts [13] and newer efforts [14] to remedy some of the issues relevant to currently available techniques for predicting the influence of unsteadiness in fluid behavior.

1.1 Motivation

In recent years, there has been a steep rise in concern regarding greenhouse gas emissions, which in turn has substantially increased the interest in more efficient
engineering systems. The impact of rising greenhouse gases in the environment is clearly seen by the observed rise in average temperature of the earth’s climate system. Most of the additional absorption of heat (due to the excess of greenhouse gases) is accounted for by the oceans and the melting polar icecaps, however, the increase of near-surface atmospheric temperature is hardly negligible, as can be seen in figure 1.1. The figure shows the global annual-mean, surface-air temperature change from 1880 to 2016, relative to the 1951-1980 mean [1].

![Global Mean Estimates based on Land Data only](image)

**Figure 1.1.** Global annual-mean, surface-air temperature deviation from the 1951-1980 mean. The solid black line is the global annual mean and the solid red line is the five-year LOWESS (locally weighted scatterplot smoothing) smoothed curve. The blue uncertainty bars (95% confidence limit) account only for incomplete spatial sampling [1].

Given the case of the aircraft industry, among the many others influenced by fluid systems, a lot of efforts are being directed towards developing aircraft with higher fuel-efficiency and improved air-traffic management. The International Civil Aviation Organization (ICAO) reports [2] that the current rate of improvements in aircraft technology and airline operations efficiency do not meet the required levels of fuel efficiency that need to be met to reach the efficiency goals set for the Aviation industry. This is very well illustrated in Figure 1.2 taken from [2], where different scenarios of rate of improvement of aircraft technology and airline’s operations efficiency are plotted to show the estimated $CO_2$ emissions by year. Clearly, the current rate (baseline in figure 1.2) of aircraft technology advancement needs to be improved even beyond the most optimistic rate scenario (lowest point
of the shaded orange region in figure 1.2) to be able to achieve the aspirational goal for $CO_2$ emissions minimization. The aspiration of reaching carbon neutral status, for the aircraft industry, by 2020 still falls short by about 1039 Metric tonnes of excess $CO_2$ emissions, as is indicated in the figure.

Increasing concerns about $CO_2$ levels due to air travel have made reducing emissions an environmental priority for both commercial airlines and the manufacturing industry alike. The regulation standards for noise and air quality by International Civil Aviation Organization (ICAO) have been progressively tightened, which is further motivating research for quieter, more fuel efficient future aircraft.

![Figure 1.2. Different scenarios of rate of improvement of aircraft technology and airline’s operational efficiency plotted against the goal in terms of $CO_2$ emissions by year [2].](image)

Amidst a plethora of ideas in the quest for a more efficient aircraft, the concepts of boundary-layer ingestion propulsion and reduction in skin friction drag through laminar-flow control stand out holding great potential for reductions in fuel burn [15–19]. Concepts like active-flow-control are among the ones holding highest technology readiness levels (TRL). Substantial efforts have been directed toward developing higher order modeling and simulation capabilities in areas of research
such as transition prediction, boundary layer ingestion, advanced airframe design etc., among many others [20]. This research effort aims at developing novel computational framework for prediction of unsteady flow scenarios, which can be applied to many of these areas of interest.

1.2 Background

The behavior of fluid flow is usually modeled through the Navier-Stokes equations. These equations do not have an analytic solution (other than some basic flow scenarios with simplifying assumptions). These equations are usually solved numerically at various levels of fidelity depending upon the level of discretization. It gets increasingly difficult to solve these equations, as the gamut of fluid behaviors/structures (spatial and/or temporal) scales with the Reynolds number. As the Reynolds number increases, the computational cost to accurately resolve all the flow features rises exponentially. Temporally, the equations are usually solved with steady/quasi-steady assumptions or by marching the solution forward in time with appropriate time discretization. Spatially, the different levels of fluid structures can be resolved depending on the scale at which the spatial domain is discretized. Depending on the flow scenario, different scales of these discretizations might hold significance, for example, in case of unsteady aerodynamics high temporal resolution becomes important and steady-flow models and assumptions tend to break down. Similarly, in flow scenarios where small spatial structures are relevant, high spatial resolution becomes important. These type of cases, where high temporal and spatial resolution is required, demand prohibitive amounts of computational resources. In terms of hardware, large supercomputing clusters are often used to resolve the Navier-Stokes (N-S) equations in high-resolution computational domains. In terms of software, high accuracy discretization schemes, matrix solution algorithms, boundary conditions etc. are required to run an accurate solution of the equations. The numerical solution of the Navier-Stokes equations without any approximating assumptions or additional modelling is referred to as Direct numerical simulation (DNS). It is too computationally prohibitive to simulate real-world flow scenarios of engineering interest in this framework. However, DNS is often used to understand basic flow physics and bolster our understanding of many fluid phenomena by running cases with smaller spatial domains of simpler geometries and for shorter
duration. There are several alternatives to DNS, such as Large-eddy simulation (LES), unsteady Reynolds-averaged Navier-Stokes (URANS) simulations, Reynolds-averaged Navier-Stokes (RANS) simulations, panel methods etc.. All these methods have some level of modelling assumptions built into them. These methods often employ empirical correlations for different modeling assumptions.

This work presents the incompressible, nonlinear disturbance equations (NLDE) as an alternative to some of the other high fidelity methods mentioned above for simulating unsteady flow behavior. These equations are derived from the Navier-Stokes (N-S) equations to represent the evolution of transient perturbations to the mean-flow. The NLDE can represent perturbations in the mean-flow at the same accuracy level that a N-S flow solver would achieve with much higher spatial resolution and/or higher-order numerical schemes. The advantage of running these perturbations separately is based on the trade-off between computational requirements of running NLDE along with the mean N-S equations versus running the N-S equations at a much higher spatial resolution. Generally speaking, simulating low amplitude, high frequency disturbances hold potential for computational savings by the use of NLDE. The other advantage comes from not having to deal with non-physical reflections and other numerical artifacts of the unsteady disturbance, due to boundary conditions, contaminating the whole solution. Solving the perturbations separately keeps the mean-flow away from any such numerical issues.

Morris et al. [13] implemented the generalized NLDEs and showed their potential and application in computational aeroacoustics. This formulation was implemented in a conservative form with an inviscid assumption for the disturbance. Since then the NLDEs have been explored in applications relevant to acoustics [21, 22] for highly accurate resolution of pressure perturbations on base flow solutions along with ship airwake interaction [23] and other attempts to incorporate them with RANS simulations [24]. A similar concept of separating the mean and fluctuating component of a flow for flow-stability computation has also been used in Direct Numerical Simulation (DNS) type simulations [6, 25, 26]. These implementations are often set in highly accurate finite difference frameworks. However, in the author’s knowledge no implementation of the incompressible NLDE’s in a finite-volume, segregated CFD solver framework has been reported. This work intends to fill that
gap as well as to incorporate multiscale and adaptive mesh refinement capabilities to the NLDE solver. To explore the applications of this method to different levels of disturbances, the solver has been applied to resolve small scale disturbances to the flow in the context of instabilities in laminar flow, along with larger scale disturbances to a flow in the forms of gusts. The ability to resolve the behavior of the flow at different levels of disturbances exhibits the versatility of the method.

1.3 Assumptions

The solver developed in this research effort assumes incompressibility. The other major assumption is that the flow variables, decomposed into steady and fluctuating parts, can be treated as two separate entities. The viscous dissipation of the disturbances is assumed to be affected only by the molecular (laminar) viscosity (turbulent viscosity is not added for the disturbances). The assumption that \( \tilde{u} = \tilde{v} = \tilde{w} \) (where \( \tilde{u}, \tilde{v}, \tilde{w} \) are the disturbance-velocities in the x, y and z direction respectively) is made in relevance to the multiscale analysis of the disturbances, that is, the magnitude of the disturbance-velocity vector is used to analyze the multiscale disturbance behavior in place of the individual disturbance-velocities. It might be argued that this assumption limits the multiscale resolution of the dynamics of the disturbance. However, the magnitude of the disturbance still gives an overall picture of the dynamics at different scales of the disturbance.

1.4 Layout of the Thesis

Chapter 2 of this dissertation presents the background information relevant to the derivation of the nonlinear disturbance equations. The concept of resolving the unsteady part of the flow variables separately has been implemented in different forms in the past. The different formulations and applications of this technique from other researchers are discussed in this chapter. The application of wavelets for multiscale analysis of perturbations to mean-flow is implemented in this work. This chapter presents some previous work in regard to wavelet techniques in multiresolution computational fluid dynamics. The capabilities of the NLDE solver developed in this work are demonstrated by application in two different areas. This chapter also provides some background information relevant
to these areas to better understand the candidacy of these cases (laminar flow instabilities and gusts) for the application of the solver. The details regarding the process of transition to turbulence are discussed as well as some current transition prediction schemes. The classical unsteady aerodynamics theories for gust response are discussed along with their limitations. The current techniques for gust response prediction and the improvements over the classical theories are also briefly reviewed.

Chapter 3 presents the implementation of the NLDE solver in OpenFOAM. The current setup in regards to solution strategy with the mean-flow equations is described. The details about the formulation of the proposed solver are also presented. This chapter details the block-coupled and segregated solution strategies used in the current work. An implicit disturbance-pressure equation for the coupled solver is developed in this chapter. This chapter also presents the disturbance pressure-velocity coupling equations for the segregated solution strategy. The validation and verification of the developed solver are presented in this chapter. The results from the developed NLDE solver are validated using results from linear stability theory. The numerical convergence of the method is verified using grid convergence studies and comparison of results from different implementations of the developed solver.

Chapter 4 details the incorporation of the multiscale framework for the NLDE solver. The validation results for identification of coherent structures with wavelet coefficients are presented in this chapter. The different wavelet coefficient criteria for adaptive meshing and flow behavior change detection are also discussed.

Chapter 5 presents the results from the application of the NLDE solver for resolution of laminar-flow instabilities and gust-response. The results are compared with results from other methods to show their validity. This chapter shows the versatility and the scope of applications of the proposed NLDE solver.

The final chapter, Chapter 6, concludes the dissertation and discusses contributions and future prospects of the work presented in this dissertation.
Chapter 2  |  Background

In this chapter, background information pertinent to the proposed nonlinear disturbance equations (NLDE) solver is presented. The disturbance equations and their significance are, hence, briefly described. Additionally, previously developed NLDE methods and research efforts are also reviewed. The next aspect of this work explores methods to develop a more efficient solution method while simultaneously extracting content and character from the NLDE predictions through multiscale analyses methods. Hence, theory relevant to wavelets in the context of multiscale modeling is also reviewed in this chapter. Lastly, this effort also involves the utilization of the NLDE solution. In this effort, two potential applications are explored that include: (1) transition to turbulence and (2) aerodynamic responses to gusts. This chapter reviews background pertinent to these application areas.

2.1 The Nonlinear Disturbance Equations

The equations describing the evolution of a disturbance in a base mean-flow are known as the disturbance equations. The disturbance equations are derived from the Navier-Stokes (N-S) equations by describing the primitive flow variables as a combination of a base mean flow component and a fluctuating component to represent the perturbation in the base flow, $\overline{U} + \tilde{u}$; $\overline{P} + \tilde{p}$ (also known as the Reynolds decomposition of the variables). Here the upper-case letters with an overbar denote the base flow mean variables and the lower-case tilde letters denote the perturbation to the base flow variables. Note that the pressure terms throughout this document are normalized with density (which is assumed constant). This form of the prim-
itive variables is then introduced into the Navier-Stokes equations, equation 2.1 (momentum) and equation 2.2 (continuity) (incompressible version is shown here).

\[
\frac{\partial U_i}{\partial t} = \nu \nabla^2 U_i - \frac{\partial P}{\partial x_i} - U_j \left( \frac{\partial U_i}{\partial x_j} \right) 
\] (2.1)

\[
\frac{\partial U_i}{\partial x_i} = 0 
\] (2.2)

The resulting equations, given by equations 2.3 and 2.4, are just a different representation of the same N-S equations. The mean-flow version of the equations 2.1 and 2.2, that is, the the primary flow variables being described as the average of the variables \((U; P, \text{steady state form})\) are then subtracted from equations 2.3 and 2.4.

\[
\frac{\partial (U_i + \hat{u}_i)}{\partial t} = \nu \nabla^2 (U_i + \hat{u}_i) - \frac{\partial (P + \hat{p})}{\partial x_i} - (U_j + \hat{u}_j) \left( \frac{\partial (U_i + \hat{u}_i)}{\partial x_j} \right) 
\] (2.3)

\[
\frac{\partial (U_i + \hat{u}_i)}{\partial x_i} = 0 
\] (2.4)

This yields a set of equations describing the evolution of the fluctuating component in the the mean flow, given by equations 2.5 and 2.6. These equations are known as the incompressible nonlinear disturbance equations (NLDE). The solution to the disturbance equations are the disturbances that the base flow can accommodate.

\[
\frac{\partial \hat{u}_i}{\partial t} = \nu \nabla^2 \hat{u}_i - \left( \frac{\partial \hat{p}}{\partial x_i} + (U_j + \hat{u}_j) \left( \frac{\partial \hat{u}_i}{\partial x_j} \right) + \hat{u}_j \left( \frac{\partial U_i}{\partial x_j} \right) \right) 
\] (2.5)
The nonlinear disturbance equations describe the evolution of the disturbances sustained by the mean base flow, which constitutes an initial/boundary value problem. These equations are useful in understanding the nature of flow instabilities and the propagation of fluctuations in the flow. The nonlinear term in Equation 2.5 ($\tilde{u}_j(\partial \tilde{u}_i / \partial x_j)$) describes the interaction of the disturbance with itself and other disturbances, which makes it especially important in describing nonlinear growth and resolving the secondary structures/instabilities. The information about the interaction of the disturbance with the base flow resides in the two terms with base flow velocity in them, i.e. $\bar{U}_j(\partial \tilde{u}_i / \partial x_j)$ and $\tilde{u}_j(\partial \bar{U}_i / \partial x_j)$. These terms describe the convective acceleration of the disturbance due to the base flow and the growth/decay of the disturbance owing to the momentum exchange with the base flow respectively. The viscous dissipation term ($\nu \nabla^2 \tilde{u}_i$) describes the decay of disturbances due to viscous diffusion.

These equations describe the complete evolution of the disturbances with respect to the Navier-Stokes equations. However, various approximations have been used in the past to make the solution more amenable to compute for a variety of problems. The most widely used approximation to these equations are the Linear Stability Equations (LSE), which spawn the Orr-Sommerfeld equation. The Orr-Sommerfeld equation forms an eigenvalue problem, which works well in resolving modal disturbances. The Orr-Sommerfeld equation assumes parallel flow (the mean flow does not change in the streamwise direction) and ignore the nonlinear term. Despite these assumptions, the solution to the equation combined with empirical correlations, does an acceptable job at predicting the evolution of disturbances and the eventual breakdown of the flow to turbulence. However, the Orr-Sommerfeld equation inherently lacks the capability of resolving secondary disturbances and the effect of complex non-parallel mean flows. In the case of transition prediction, the solution to that is usually a modification to empirical correlations to fit the required flow condition. In many cases, this necessitates some apriori prediction or knowledge of the instability scenario. More details about these methods are
discussed in later sections, where transition to turbulence is discussed in detail.

2.1.1 History

The idea of resolving the unsteady part of the flow separately has been around for a very long time. From the early days of hydrodynamic stability theory (see Appendix A) the theoretical analysis has been based on adding a small disturbance to the mean flow. Given that the results at the time were limited to mostly analytic relations, several simplifying assumptions were often made. The most successful formulation with these simplifying assumptions is the Orr-Sommerfeld equation [27,28]. The applications of which are still in use today.

With the advent of numerical techniques and the cheaper availability of computational resources to be able to solve the hydrodynamic equations, more complex situations could be modeled. DNS type simulations are often sought after to understand basic flow-physics and the nature of instabilities. Fasel [25] applied the same concept in a DNS type setting. He resolved the unsteady component of the flow separately from the mean flow component for applications where the unsteady component of the mean flow is of main interest. His application of the concept was limited to vorticity-stream function and velocity-vorticity formulations of the Navier-Stokes and disturbance equations discretized in a fully-implicit finite-difference solver. He used the method to resolve the flow instabilities in a boundary layer and plane Poiseuille flow. Fasel’s later research efforts (using the velocity-vorticity formulations) with Rist [6,26] showed the potential of this method for the study of the stability of laminar flows. Their results showed good agreement with linear stability theory and experimental results [29]. Hardin and Pope [30,31] used an incompressible mean-flow solver with stream function and pressure Poisson formulation for base flow calculations. They added perturbation terms with inviscid assumption and argued small changes in density for acoustic calculations. Their system of equations had 4 variables for acoustic calculations \((u', v', p', \rho')\) over the base incompressible flow solution. They employed a two-dimensional setup with separate grids for the mean-flow (viscous) and the acoustic (inviscid) calculations. Morris et al. [13] formulated a more generalized form of the disturbance equations.
This generalized formulation of NLDE was implemented in conservative form and derived from the compressible Navier-Stokes equations. This makes this form much more versatile in its formulation than other implementations. However, an inviscid assumption was made about the disturbances. Their solver used a steady RANS solution as input and resolved the NLDE in a finite-difference formulation. They showed the potential of the equations in their application to jet noise calculations and other computational aeroacoustics problems [21, 22]. Long [32] implemented the NLDE in a non-conservative form and showed their application to several test cases. Long’s formulations also had a simpler form than the traditional NLDE as implemented by Morris et al. [13]. Chyczewski et al. [33] applied the NLDE for turbulent wall-bounded shear flows. They used a flat-plate boundary-layer domain to show the application of the NLDE in predicting the boundary-layer turbulence statistics along with some channel flow simulations to establish wall models using NLDE (inconclusively). All these formulations were primarily finite-difference type formulations. However, Hansen et al. [34] applied the NLDE formulation in a finite-volume setting. Their work also attempted viscous disturbances and used several test cases to see the response. The approach was used to simulate unsteady, two-dimensional laminar flow around a cylinder. They obtained results with varying levels of accuracy for different base flow conditions.

Labourasse and Sagaut [24] employed an even more generalized form where they split the N-S equations into an ensemble-averaged and a fluctuating part in a multi-level RANS/LES coupling without any additional simplifications. Their RANS solution was calculated on a Cartesian finite-volume grid and the NLDE solution was evaluated using a finite-difference method. They used a zonally refined grid for the resolution of the disturbances. They simulated transitional boundary-layer on a two-dimensional turbine blade at a relatively low Reynolds number ($1.6 \times 10^5$ based on freestream velocity and chord). The NLDE were resolved in a small zone with a refined grid on the suction side of the turbine blade. This involved application of non-reflecting characteristic boundary conditions at the boundary of the NLDE subdomain. The fluctuation velocities were compared to classical LES and experimental results. Their results showed some promise, however, the fluctuations showed high variance with respect to experimental/classical LES results.
The proposed NLDE solver is formulated in a non-conservative, incompressible, finite volume implementation. The NLDE solver proposed in this work assumes incompressibility for the disturbances. However, viscous terms and mean-flow terms are not neglected. This implementation of the NLDE does not couple tightly with the base flow. The NLDE are one-way coupled to the base flow via explicitly defined source terms from the mean flowfield. This prevents the information from the disturbance solution to change the mean flow behavior. The proposed solver is implemented in a collocated finite-volume setting to enable its use in complex physical domains with unstructured grids. The solver is intended for efficient computation of disturbances. This work involves multiscale analysis of the disturbances using wavelets. The multiscale decomposition of the disturbances allows a deeper understanding of the dynamics of the disturbances with limited computational resources. The multiscale analysis of the NLDE solution is employed to enable adaptive mesh refinement (AMR) for an efficient resolution of the disturbances while avoiding zonal grid methodologies with complex interfacial boundary conditions. The following section provides background information about wavelets and their use in multiscale computational fluid dynamics.

2.2 Multiscale Analysis

Multiscale analysis refers to the analysis of a signal at different scales/frequencies. In the context of this work, the proposed model uses multiscale analysis of the disturbance solution to classify zones of high "activity" at different scales. The increase in variance of amplitude of the perturbation velocity at different scales constitutes relatively higher activity at one scale with respect to others. Spectral analysis to check the behavior of various frequencies/scales has been a common theme of many DNS simulations. However, Fourier analysis has its inherent limitations of the need for periodic, simple domains as well as high computational expense. To overcome the limitations due to Fourier spectral analysis, wavelets have been chosen. The following sections give background information on wavelets, which are an effective means of localized multiscale spectral analysis.
2.2.1 Wavelets

Wavelets provide an effective way of space and scale localization, which makes them ideally suited for adaptive algorithms. Wavelets have been used for signal and image processing under different nomenclature and basis functions, for some time. The pioneering work of Meyer and Mallat [35, 36] gave the first framework to construct mother wavelets in the setting of a multiresolution analysis. A mother wavelet, $\psi$ is a function used to generate dyadic wavelets/functions which form an orthogonal basis. Wavelet basis functions have compact support and can be localized in physical and scale spaces. The scale decomposition in wavelets is obtained by dilating and contracting the mother wavelet, $\psi$ which is translated in space to give spatial localization. Due to this space/scale localization property, wavelets are ideal candidates for solving a variety of problems in science and engineering. In the context of fluid dynamics, their application in turbulence analysis and synthesis has seen very wide acceptance in the field [37, 38]. The strong temporal/spatial intermittency, localized small structures in space/time, and large range of spatial scales in turbulence, make multiresolution schemes a strong candidate for turbulence simulations and analysis.

Wavelet based turbulence simulations are classified into wavelet based direct numerical simulations (WDNS) [39], coherent vortex simulation (CVS) [40], and stochastic coherent adaptive large-eddy simulation (SCALES) [41]. The WDNS approach uses wavelet based numerical methods to solve the wavelet-filtered Navier-Stokes equations with sufficiently small wavelet threshold. CVS is based on the idea of dividing the flowfield into coherent and non-coherent vortices. It is assumed that the non-coherent vortices do not contribute to the evolution of turbulence and need not be resolved. The SCALES technique builds on the CVS idea and instead of completely ignoring the incoherent part, it models the incoherent vortices with subgrid scale models. The idea of SCALES is similar to LES, however, due to the localization properties of wavelets the filter can be adaptive based on the flow solution at any resolution in space or wave-number domain [42]. Unlike LES, the distinction between resolved and unresolved motion is not limited to the size of structures, but depends on the energy content of the turbulent motions.
The capabilities of the adaptive SCALES concept can be very efficiently applied
to the laminar or transitional regime of the flow. The SCALES equations are
obtained by filtering the incompressible Navier-Stokes equations, and are qualita-
tively similar to the nonlinear disturbance equation. In the case of transitional
flows, the issue of tracking all the frequencies and then determining which ones
grow for transition to turbulence can be solved by defining an appropriate filters
based on parameters such as laminar kinetic energy [43], instability amplitude,
instability scale and external influence parameter. However, this involves adapting
the computational framework tailored for computation of quantities in the wavelet
domain. This can be resolved by utilizing wavelets coefficients as more of a sen-
sor for adaptive resolution and for multiscale setting of the disturbances, all the
while resolving the computation in the physical domain. This uses the available
framework without major changes and uses wavelets as an added capability/ex-
tension to the current framework. Several researchers [44–47] have utilized this
way of using wavelets in solving physical problems by using them as an extension
to conventional Navier-Stokes solvers. The aim of the proposed NLDE solver is
to achieve these results following on concepts very similar to those of these re-
searchers by using the multiscale nature of wavelet analysis for their sensory abilities.

2.2.1.1 Mathematics of Wavelet solution

A scalar field \( T(x) \) can be represented in terms of wavelet bases [39], as shown in
equation 2.7.

\[
T(x) = \sum_{i \in I^0_\psi} \bar{T}_i^0 \phi_i^0(x) + \sum_{j=0}^{\infty} \sum_{\mu=1}^{2^d-1} \sum_{i \in I^\mu_j} \bar{T}_i^{\mu,j} \psi_i^{\mu,j}(x) \tag{2.7}
\]

Here, \( \phi_i^0(x) \) and \( \bar{T}_i^0 \) are the scaling coefficients and the function at the coarsest
level \( j = 0 \), i.e. it gives an averaged field at the coarsest resolution. \( \psi_i^{\mu,j}(x) \) are the
wavelet coefficients different families, \( \mu \), at level of resolution, \( j \). \( \bar{T}_i^{\mu,j} \) represents the
details (fluctuations) in the \( T(x) \) field around the position of the wavelet \( \psi_i^{\mu,j}(x) \).
The subscript \( i \) denotes indices in d-dimensional space; i.e., \( i = (i_1, ..., i_d) \). \( I^0_\psi \), and
\( I^\mu_j \) are d-dimensional index sets, associated with scaling functions at zero level of
resolution and wavelets of family $\mu$ and level $j$. At a given level of resolution $j$, the scale (wave-number) of the wavelet is the same but is located at different spatial locations and the coefficient at each of these locations determines the amount of fluctuation in the field at the location for given ($j$) scale of fluctuations. The filtering procedure is accomplished by applying the wavelet-transform to the unfiltered velocity field, discarding the wavelet coefficients below a given threshold ($\epsilon$) and transforming back to the physical space. Figure 2.1 shows an example from [3] demonstrating the filtering technique. Figure 2.1(a) shows the unfiltered input velocity denoted by a hyperbolic function. Part (b) of the figure shows the different levels of decomposition ($j$) as different horizontal levels, with different density of wavelet application points ($i$). Part (c) of Figure 2.1 shows the filtered (with amplitude above a predefined threshold) wavelet coefficients highlighted in red (thicker points in grayscale). The locations of these coefficients denote the points where the solution shows high activity/change.

This kind of wavelet analysis where the function is solved in the wavelet space and then moved back to the physical domain is typical for a Galerkin type solution of a partial differential equation (PDE) using wavelets. However, solving the equations in wavelet coefficient space instead of physical space requires the transformation of boundary conditions as well. This can become complex depending on the domain. This problem can be solved by solving the PDE in a collocation type solution, as is implemented in the adaptive SCALES technique. The differential equations in a wavelet collocation method are solved in physical space on an adaptive computational grid. The grid adaptation in wavelet collocation methods is performed similarly to other wavelet-based methods. The only difference is that at the end of
each time step (or iteration) an additional wavelet transform is performed for the analysis of wavelet coefficients. References [48,49] demonstrate the application of adaptive wavelet collocation method for solving multi-dimensional, nonlinear PDEs. The same principles can be applied to solve the nonlinear disturbance equations. This work deals with the resolution of NLDE in physical domain. The wavelet technique is used for their multiresolution characteristics. This can be useful for analysis of the unsteady disturbance at different scales, coherent structure detection as well as for adaptive mesh refinement (AMR) framework. It is also expected that this framework can in future be easily adapted for turbulence prediction/detection [50,51].

To understand the applicability of the NLDE to turbulence prediction and studying instabilities within laminar flow, one must understand the process of transition and previous work conducted for the prediction of transition. The following sections give some background regarding the same. Following that, some background relevant to larger scale flow disturbances/gusts is provided.

### 2.3 Transition to Turbulence

Given the potential of laminar-flow-control technologies for reduction in skin friction drag, a lot of effort is being put to realize laminar-flow technologies into practice. To facilitate progress in the field of laminar-flow-control, computational prediction of the transition phenomenon from laminar to turbulent flow needs to be accurate. Recently, much progress has been made in the field of transition prediction in computational fluid dynamics (CFD) codes available publicly/commercially. The advances in the field of measurements and direct numerical simulations (DNS) of the Navier Stokes equations have also enabled a higher level of understanding about the instabilities in the boundary layers and their structures responsible for breakdown to turbulence. This has led to the understanding about the importance of both the magnitude as well as the frequency of external disturbances [12, 52–54]. However, current CFD transition models focus on only the magnitude of external disturbances and ignore the effects of the frequency content of the disturbance environment on transition. The current models are also limited to only a few types of transition scenarios, that can be predicted accurately, among the ones
listed in Figure 2.2 (usually path A - modal disturbances). Figure 2.2 shows the various paths that lead to the breakdown of laminar flow to turbulence. These different scenarios were first summarized by Morkovin [55]. The different paths in the figure (A,B,C,D and E) reflect the influence of different levels of external disturbance environment on the path a flow might follow to it’s breakdown. In this work, incompressible NLDE are proposed as a viable tool to study these different transition mechanisms. The application of the NLDE can enable one to resolve different types of instabilities and their behavior to a reasonable level without having to resort to DNS levels of accuracy and resolution. This ability gives the advantage of studying complex flow-instability phenomena with a computationally affordable simulation. The fact that it is implemented in a finite-volume CFD type framework, also gives it the ability to be easily applied to complex real world geometries. This capability is lacking in spectral DNS type simulations. The following sections give a brief overview of the transition phenomena and transition prediction schemes to understand the associated challenges with computational modeling of transition.

![Diagram of transition mechanisms](image)

**Figure 2.2.** Paths to turbulence in boundary layers [4].
The change of a fluid flow behavior from smooth laminar flow to seemingly chaotic turbulent flow is termed as "Transition". Transition, especially in boundary-layers, is an important aspect of fluid dynamics. It has a great impact on the design of efficient engineering systems. The first systematic research on this process was performed by Reynolds [56] during his pipe flow experiments. All the theoretical investigations in this matter are based on the notion that the process of transition is a consequence of the initial disturbances present in laminar flow that, depending upon their frequency and other conditions, grow, or decay, or stay neutral, and consequently determine if the boundary layer becomes turbulent or not. The following section describes the current understanding of the transition phenomenon. A brief history about the discovery of the process of transition is provided in Appendix A for the interested reader.

2.3.1 Transition Mechanism

There has been significant progress in the understanding of the transition process since the famous flat-plate experiment conducted by Schubauer and Skramstad [12]. However, a common trend in all stability theories for transition is the concept of initial disturbances in the flow, which can become unstable and change the flow to turbulent after a certain amplification level of the critical-frequency disturbance is achieved. For low freestream turbulence (FST) levels and predominately two-dimensional flows, transition in the boundary layer is mainly a consequence of the development of Tollmien-Schlichting (T-S) instabilities. At high turbulence intensity levels, the nonlinearity of the process increases. Wu [53] explains the different ways by which external freestream disturbances might influence the transition process in the boundary layer depending on their intensities. These different paths are summarized in Figure 2.2. As indicated by the figure, transition in the boundary layer is initiated through a receptivity process: a term first coined by Morkovin [55]. Receptivity, as defined by Reshotko [57], is the means by which an external disturbance enters a boundary layer. The growth of these disturbances inside the boundary-layer can vary depending on the characteristics of the disturbance. At low FST intensity levels (Turbulence intensity (T.I.) <0.1 percent), the external disturbances excite the normal modes of the boundary layer. These normal modes are the unstable eigenmodes that the laminar base flow can support. This is the
traditional path to transition (path A in Figure 2.2) where modal growth is the most significant, involving the familiar and well documented T-S, cross-flow (C-F) or Görtler mechanisms. At moderate FST intensity levels (0.1 percent $<$ T.I. $<$ 1 percent), the boundary-layer instabilities may be indirectly affected by the external disturbances, in terms of the linear and/or nonlinear growth of the instabilities due to interaction with new instability modes introduced in the boundary layer. The presence of transient growth of instabilities in this path to transition (paths B and C in Figure 2.2) and the interaction of various instability modes make it difficult to understand. Morkovin et al. [4] emphasized the importance of studying transient growth of disturbances that follow different paths to transition (summarized in Figure 2.2). Transient growth theory helps shed light on the occurrence of transition in conditions that can be subcritical to the T-S neutral curve. This is usually observed at moderate to high FST intensity levels. The linear and/or nonlinear interaction of different spectral modes of the same instability (non-orthogonal Squire and Orr-Sommerfeld modes, T-S modes of different spanwise wave number and frequencies), or between different kinds of instabilities (T-S and Klebanoff modes, C-F vortices and T-S waves), may enhance the instability amplification, or may even suppress it [58]. Transient growth is a consequence of non-orthogonality of the Squire and Orr-Sommerfeld eigenfunctions. It has a signature of algebraic growth followed by exponential decay. Transient growth is usually observed by the presence of streaky structured (optimal disturbances) instabilities in the boundary layer. These instabilities can be subcritical to the T-S neutral curve and show high amplification levels, causing transition to occur much earlier than by the normal modes. At high (FST) intensity levels (T.I. $>$ 1 percent), external disturbances may create secondary streaks of instability and take the highly nonlinear bypass route to transition (paths D and E in Figure 2.2). Low-frequency instabilities with low spanwise wavenumbers dominate the transition process at these FST levels. A more detailed description of transient growth theory and bypass transition can be found in Ref. [59].

The major factors that affect the transition phenomenon are the external freestream disturbances, pressure gradients, temperature, surface roughness and surface curvature. These factors usually work in conjunction with each other, which makes it harder to predict the occurrence of transition with high accuracy. Different
prediction methods take different approaches to take these factors into account while calculating the stability characteristics. The next section briefly describes some of these transition prediction methods.

2.3.2 Transition Prediction

Early efforts in transition prediction started from Reynold’s experiments [56] where he tried to come up with a parameter to predict the onset of turbulence in his pipe flow experiments. Following the same line of thought, many prediction methods [60–62] rely on experimental data to develop correlations to predict transition. Early efforts utilized an inviscid potential flow solution to feed the pressure gradients to a boundary layer solver. A boundary-layer solver would then calculate boundary layer development based on the prescribed pressure distribution using boundary-layer equations developed by Prandtl [63]. The correlation would be used to predict the behavior of boundary layer (laminar or turbulent) based on the calculated integral boundary-layer parameters. This amounted to an iterated solution between the corrected boundary-layer and potential flow solution until a converged solution is obtained.

Reynolds [56] also had the idea that the stability of laminar flows could be calculated by integrating the equations of motion for small disturbances in the flow, but he could not come up with a theory to predict the stability characteristics. Lord Rayleigh, in a series of papers from 1880 to 1913 [64], studied the stability of laminar frictionless flows and came up with two major theorems. The first asserted that velocity profiles with a point of inflexion are unstable. The second stated that the velocity of propagation of neutral disturbances in the boundary layer is less than the maximum velocity of the mean flow (which is equal to the outer flow velocity). Both theorems lacked the consideration of viscosity, which was later added by Tollmien. The viscous correction explained the instability of velocity profiles without an inflexion point and the physically absurd singularity of frictionless flows, which occurs when propagation speed of disturbance in a boundary layer becomes equal to the mean flow velocity. The full linear stability equation, the Orr-Sommerfeld equation, was first solved by Tollmien to calculate
the neutral stability curve. This method is the basis for most widely accepted transition prediction schemes [65,66]. However, it also tends itself to a boundary layer solver type computational setup than a CFD compatible transition prediction method due to the non-local characteristics of the method. A boundary-layer solver calculates the behavior of parabolic boundary-layer equations which propagate information in one direction and the Orr-Sommerfeld method needs information relevant to upstream development of the disturbance to predict its evolution. This can be implemented in a boundary layer solver where the parabolic boundary layer equations are used to calculate boundary layer development in a space marching solution. In case of a CFD compatible transition prediction method, the prediction criteria needs to be locally defined. However, recent advancements in transition modelling in a CFD type setting has seen both correlation based [67] and stability equation based [68] methods. These methods are the current state of the art for transition modelling in a CFD type setting. Other techniques for calculation of stability characteristics of non-parallel flows have been developed and considered by many researchers [69,69,70]. In addition, others are using the Parabolized Stability Equations (PSE) [70] for analyzing both linear and nonlinear growth of instabilities due to receptivity and other external forcing functions. The PSE provide an efficient way of treating the evolution of disturbances in non-parallel base flows taking into account effects of external disturbances. However, the approach is still not as widely accepted as is the linear stability theory for predicting normal-mode transition (path A in Figure 2.2). Researchers have looked into direct numerical simulations (DNS) of the full Navier-Stokes equation for transition prediction [71]. However, the computational requirements for DNS are still extensive, even for moderately complex flows and geometries. Adjoint methods [72] have also been used for receptivity problems. These methods use a bi-orthogonal eigenfunction expansion in either temporal or spatial mode. For every eigensolution, an adjoint eigensolution with exactly equal and opposite frequency and wavenumber is assumed. The adjoint fields are easy to compute and represent the response of the corresponding source (external forcing function). The inner product of the source with the particular adjoint field gives the effective response of the source on the actual eigensolution. The adjoint technique has been applied to the PSE [73], the Navier Stokes equations, and other methods for finding the sensitivity of the flow to the forcing terms.
More advanced nonlinear dynamical systems concepts are also being applied to transition prediction. Landau had suggested that transition to turbulence might hold clues to solving turbulence. It is natural to consider the initial turbulence field as the final nonlinear cascade of instabilities in transition. Bifurcations are a usual characteristic of nonlinear dynamical systems. Such bifurcations have been observed in Rayleigh-Beénard and Taylor-Couette flows, which further encourages exploring the route to chaos theory as a possible means of analyzing transition to turbulence [74, 75]. However, this method needs an initial linear instability to build on, and hence, subcritical transition modes in problems like pipe-flow or plane-Couette flow do not have a straightforward solution. Nevertheless, techniques to overcome this limitation have been successfully applied [76]. Many other methods and theories have been proposed, along with empirical correlations to predict transition, some of which are used in conjunction with the above mentioned methods to accurately predict the complex phenomenon of transition. Following sections discuss some important methods in detail.

2.3.2.1 Linear Stability Theory

Rayleigh [64], Tollmien [77], Prandtl [63], Schlichting [78], Orr [27, 28], Sommerfeld [79], and others have played pivotal roles in the early mathematical theories on the stability of laminar flows to predict transition. The Orr-Sommerfeld equation, given by 2.8 (two-dimensional version), is the most accepted transition prediction method for low FST environments. The Orr-Sommerfeld equation is a small perturbation method, where a small disturbance is imposed on a laminar base flow and then their development is traced in space and/or time. It assumes incompressible, parallel-flow conditions. The full linear stability equation, the Orr-Sommerfeld equation, was first solved by Tollmien to calculate the neutral stability curve. The neutral stability curve marks the point of occurrence of instability, and from this point onwards the growth of the unstable disturbances can be traced further in the boundary layer. The point at which these frequencies cause transition is not definite and, in reality, would correspond to a certain critical amplitude of the instability beyond which it turns turbulent. In practice, it is difficult to find relations to predict this critical value, hence a degree of amplification is used to predict transition. This is the basis of the $e^n$ method [65, 66].
\[
\phi''' - 2\alpha^2 \phi'' + \alpha^4 \phi = iRe(\alpha U - \omega)(\phi'' - \alpha^2 \phi) - iRe\alpha U'' \phi
\] (2.8)

\(\phi\) = amplitude function, \(\alpha\) = dimensionless wave number, \(Re\) = Reynolds number

\(\omega\) = dimensionless frequency of the disturbance, \(U\) = mean-flow velocity in stream-wise direction.

The \(e^n\) method is a computationally efficient method, which solves the Orr-Sommerfeld equation and traces the critical frequencies for their growth. Based on a pre-specified \(n\) value, transition is assumed to occur when the amplitude of the critical frequency mode reaches \(e^n\) times the initial amplitude at the neutral curve. The \(e^n\) method does not include any receptivity or external forcing analyses. Other empirical methods, based on boundary layer shape parameters and Reynolds numbers, are still in use by some airfoil design codes and are considered inferior to the \(e^n\) method. However, techniques for evaluating the stability characteristics of more complex, non-parallel flows, with nonlinear disturbance growths and externally forced disturbances have also been developed [69–76,80].

### 2.3.2.2 CFD based Transition Prediction

CFD based transition modelling requires the prediction criterion to be strictly local with a finite stencil. This requirement restricts use of stability theory criterion to be directly applied in CFD frameworks. The current methods used in modelling transition in a CFD framework can be classified as local correlation based methods [67], stability theory derivatives [68] or direct resolution of fields (DNS) [71].

One of the most widely used transition model is the Langtry Menter [67] transition model. This model is based on \(\gamma - Re_\theta\) transport equations. It solves for intermittency - \(\gamma\), which defines the proportion of laminar to turbulent regime and a transitional momentum-thickness Reynolds number \(Re_\theta\), which defines the criterion for the flow to become turbulent. This model is developed to work in conjunction with the \(k - \omega\) SST turbulence model. The original model was developed to predict transition due to Tollmien-Schlichting induced transition, bypass transition and separation-induced transition. The model has also been modified to work in transition scenarios due to crossflow instabilities [81]. The model relies
on local correlations and empirical values calibrated to perform well at medium to high freestream turbulence environments. This model marked a milestone in making transition modeling more mainstream in the world of RANS CFD. It essentially developed on Abu-Ghannam and Shaw [61] to make a more CFD-compatible, transport equation version of the correlation technique used by Abu-Ghannam and Shaw. Abu-Ghannam and Shaw developed empirical correlation for transitional momentum-thickness Reynolds number as a function of Thwaite’s parameter and freestream turbulence-intensity. Their correlation was based on experimental results with moderate to high freestream turbulence-intensity.

On similar lines of thought, to make transition models more compatible with CFD framework, Coder and Maughmer [68] developed the amplification factor transport (AFT) Equation transition model. This model aimed at bringing the approximate-envelope method developed by Drela and Giles [82] into CFD framework. The method developed by Drela and Giles greatly simplifies the process of transition prediction through linear stability equations based $e^n$ method by tracking a single variable based on an envelope of frequencies than tracking every frequency. This makes it more computationally efficient than the full $e^n$ method. Coder and Maughmer [68] successfully implemented this technique into CFD framework through an amplification factor - $\tilde{n}$, transport equation given by equation 2.9:

$$\frac{\partial (\rho \tilde{n})}{\partial t} + \frac{\partial (\rho U_j \tilde{n})}{\partial x_j} = \rho \Omega F_{\text{crit}} F_{\text{growth}} \frac{d (\tilde{n})}{d \text{Re}_{\tilde{\varepsilon}_2}} + \frac{\partial}{\partial x_j} \left[ \frac{1}{\sigma_n} (\mu + \mu_t) \frac{\partial \tilde{n}}{\partial x_j} \right]$$

(2.9)

$F_{\text{crit}}$ - function describing the presence of unstable mode, $F_{\text{growth}}$ - function correlation for local shape factor, $\Omega$ - vorticity magnitude, $\text{Re}_{\tilde{\varepsilon}_2}$ - momentum-thickness Reynolds number, $\sigma_n$ - calibration constant.

This model was developed to work in conjunction with the Spalart-Allmaras [83] turbulence model. This model, being based on an envelope method, is primarily based on growth of Tollmien-Schillicting type instabilities. However, it has potential for application to more transition regimes. This model is another step towards using stability theory based correlations to be brought into the CFD framework.
Even though, both AFT [68] and $\gamma - Re_\theta$ [67] models provide accurate transition prediction (for certain transition scenarios) in CFD framework, they are still considered inferior to integral boundary layer (IBL) stability codes like Parabolized Stability Equations (PSE) and DNS in terms of resolution of actual physical phenomena related to instabilities and calculation of stability characteristics of the flow. The process of breakdown to turbulence, even at low freestream turbulence intensity (FSTI), involves eventual 3-dimensional, nonlinear, subharmonic breakdown. The effects of which are not modeled by linear stability theory (LST) based models. The rise in interest in laminar flow control technologies has caused a steep rise in research efforts for different flow control devices. The effect of sharp gradients in flow parameters on the evolution of instabilities in the boundary layer cannot be modeled accurately through LST derived models or correlations and needs direct resolution of the instability itself. As long as the distortion of the flow takes place over a length scale much longer than the characteristic length of the instability, LST/PSE are applicable. However, abrupt changes need special treatment [84]. DNS efforts to resolve such affects [85,86] have seen some success along with a local scattering approach [87–89].

Some efforts, like the DLR (Deutsches Zentrum für Luft- und Raumfahrt) TAU code [90], aim at coupling IBL (integral boundary layer) methods with RANS for more accurate and thorough resolution of stability characteristics of the boundary layer. However, it’s a loose coupling with both RANS and stability code being run in parallel and it is inherently limited by the linear stability theory assumptions (can be also limited to analysis of simple geometries). Other efforts, such as using Parabolized Stability Equations (PSE) [70] or full Navier Stokes solution (DNS) as in [6,71,91–93] among many other DNS setups, have been successful in resolving the actual physical phenomena related to the stability of the flow and the evolution of the instabilities. This research effort intends to fill a gap between the CFD transition models and DNS resolution of instabilities by resolving the instabilities at a coarser adaptive resolution. This is achieved through an adaptive LES type multiscale filtering of the domain and the solution of the unsteady/disturbance components separately. The generalized framework of the proposed solver with minimal assumptions/simplifications enables the capability to resolve different types
of instabilities and their growth. This can be very helpful in understanding complex transition scenarios with relatively inexpensive computational cost compared to DNS. The nonlinearity and the ability to resolve non-modal disturbances makes it very useful for complex flow scenarios and flow-control type setups.

2.4 Gust Modeling

Another application of resolving the unsteady flows involves gust modeling. The estimation of gust response i.e. the response of an aerodynamic body to unsteady oncoming flow, is a classic unsteady aerodynamics problem. The applications of gust response range from helicopter blade design to turbomachinery applications, wind farm design, boundary layer ingestion, along with myriad of flow situations where there is a time-dependent, relative motion of the aerodynamic body with respect to the oncoming flow field. This work will explore the application of the NLDE solver to the estimation of vertical gust response on an airfoil. However, there are no inherent assumptions to deter the use of the solver to explore other unsteady flow situations. The following section gives an overview of the theory and current techniques used for the calculation of the aerodynamic responses of gusts.

Early unsteady aerodynamics theories were based on incompressible, two-dimensional, thin airfoil theory assumptions. Wagner [94] obtained a solution for lift response due to transient step changes in angle of attack on a thin airfoil. These *indicial* responses can be used to model more complex unsteady flow situations via superposition of indicial responses using Duhamel’s integral. Küssner [95] calculated the indicial response of a flat plate due to a sharp edge gust in incompressible flow. Küssner’s solutions, known as the Küssner function are different from Wagner’s in that the gust penetration into the thin airfoil flowfield changes the equivalent angle of attack of only the part of the airfoil that has been penetrated as compared to Wagner where the angle of attack is assumed to change instantly for the whole airfoil. However, as the speed of travel of the gust with respect to the freestream increases both function can get the same aerodynamic response (Miles [96]). Another approximation to their solution was presented by Jones [97]. Theodorson [98] presented solutions for the aerodynamic response due to harmonic oscillations of an airfoil. The harmonic forcing in terms of angle of
attack is resolved into the aerodynamic response via a transfer function, now known as the Theodorson function, given by equation 2.10. Here the independent variable, \( k \), is the reduced frequency (frequency non-dimensionalized with half-chord and freestream velocity, \( k = (\omega \times c) / 2 \times \text{Velocity} \)). Theodorson function is plotted in Figure 2.3 for reference.

\[
C(k) = F + iG = \frac{H_1^2(k)}{H_1^2(k) + iH_0^2(k)}
\]  

(2.10)

where:

\[
F = \frac{J_1(J_1 + Y_0) + Y_1(Y_1 - J_0)}{(J_1 + Y_0)^2 + (Y_1 - J_0)^2}
\]

**Figure 2.3.** Theodorson function in the complex plane.
\[ G = \frac{Y_1 Y_0 + J_1 J_0}{(J_1 + Y_0)^2 + (Y_1 - J_0)^2} \]

Von Kármán and Sears [99] obtained solutions for harmonically varying gust interactions with a thin airfoil. Sears [100] later presented a practical form and application of their results. He also presented a solution for a sinusoidally varying vertical gust on a thin airfoil. The Sears transfer function is given by equation 2.11, which includes the Theodorson function in its definition for the sake of simplicity.

\[ \text{Figure 2.4. Comparison of Sears function with different reference points.} \]

Sears function can also be expressed without the Theodorson function, with the help of modified Bessel function of the second kind. The initial derivation of the function in [99] was performed for a chord, \( c = 2 \), which was later [100] extended to arbitrary chord, \( c \), by multiplying with factor \( = c/2 \). The forces were also calculated at mid-chord. Moving the reference point to the leading edge, which
becomes necessary when the gust interaction is to begin from the leading edge, changes the phase of the response. This can be seen in Figure 2.4, where the function is plotted with the reference point at mid-chord vs leading edge.

\[ S(k) = [J_0(k) - iJ_1(k)]C(k) + iJ_1(k) \]  \hspace{1cm} (2.11)

Both the Sears and Theodorson functions agree with each other for low reduced frequencies, as can be seen in Figure 2.5. However, at higher reduced frequencies the functions give different values. The Theodorson function becomes more insensitive to changes in reduced frequency as the reduced frequency increases compared to the Sears function which asymptotically decays with increase in reduced frequency. As \( k \to \infty \), Theodorson function magnitude \( |C(k)| \to 0.5 \) and Sears function amplitude \( |C(k)| \to 1/\sqrt{2\pi k} \). These theories are limited to the assumptions of potential flow and thin airfoil theory. To improve on these predictions, Goldstein and Atassi [101] studied the effects of gust distortion due to the airfoil flowfield and incorporated the effects in a modified Sears function. Atassi [102] also examined the effects of camber and angle of attack for a thin airfoil and attempted correction for thickness in work with other researchers [103,104]. At low frequencies these analytic relations give good results. However, at high frequencies, where the relevant turbulent length scales are comparable to the thickness of the airfoil, the thin airfoil assumptions are invalid and lead to overestimation of the lift [105–109].

Lockard and Morris [110] explored the effects of airfoil thickness, gust frequency and viscosity on radiated noise with a two-dimensional Navier-Stokes solver with a computational aeroacoustics (CAA) methodology. They used transport equations in non conservative form for gust velocities on a base mean flow solution. Their results are in agreement with Atassi et al.’s result. They found similar effects of thickness and angle of attack on airfoil loading response to gusts. The viscous effects were seen to increase the instability of the trailing edge wave disturbance leading to instability of the whole solution. Similar issues were reported with high reduced-frequency simulations. Gill et al. [111], recently conducted extensive analysis with a similar CAA methodology to gust response. They used the linearized Euler equations for calculation of unsteady gusts. They studied the effects of leading edge nose radius and thickness on noise radiated from a gust encounter. Their results verified the
Figure 2.5. Theodorson and Sears function for different reduced frequencies

earlier results about the importance of thickness and the distortion of gusts by the flowfield in the prediction of accurate gust response.

Given all these efforts, the importance of realistic flow resolution and accurate geometry representation are now understood to be very important. This has led to more and more research efforts geared towards modeling gust response based on full Navier-Stokes solution of the problem domain as well as experimental studies [112–117]. More recent works, such as Ghoreyshi et al. [118] have used time accurate, three-dimensional, compressible Navier-Stokes equation CFD solver to simulate basic gust cases for reduced order model generation. The application of these methods is computationally expensive and difficult to apply owing to boundary condition issues, numerical dissipation and dispersion issues
and spatial and temporal resolution requirements for high frequency cases. The application of DNS-LES to simulate gust response can be very computationally expensive. In the case of RANS simulations, the gust tends to dissipate quickly as it is seen as turbulence by the solver and it behaves accordingly. This work intends to explore the application of an NLDE solver developed in this work for gust response calculation. Given the generalized formulation, it overcomes the limitations of other techniques as well as being easier and cheaper to implement for higher frequency low intensity gusts than a full Navier-Stokes solver. In the future, it can be used to develop higher-fidelity, lower order models for gust response.

The efficient application of the proposed NLDE solver to various flow scenarios requires that the solver be implemented in a generalized framework. The proposed solver is implemented with multiple numerical strategies to inherit this capability. The development of the incompressible NLDE solver in OpenFOAM framework is discussed in the next chapter. The implementation of wavelet based multiscale capabilities is detailed in the chapter following the next.
In this chapter, several solution methods for the incompressible NLDE are developed, evaluated, and discussed. These methods are implemented within the OpenFOAM [119] framework. The results from the NLDE solver are then compared to analytic solutions from linear stability theory with the goal of providing verification of the numerical algorithm and implementation. Additionally, the results are evaluated for numerical convergence to verify that the method and numerical scheme are sufficient to capture the correct physics with a numerical mesh in line with the conventional CFD methods.

3.1 Disturbance Equations Solver

The NLDE solver developed in this work is intended to posses the capability to simulate a variety of unsteady-flow scenarios. The incompressible NLDE can be solved with different solution strategies to enable their application in different scenarios. In this work, two main approaches are used to couple the NLDE solver with the mean-flow N-S solver. The first approach involves computing a mean-flow using traditional RANS, which provides input for the NLDE solver. While the second co-simulates a URANS solution with the NLDE solution. These approaches are discussed in additional detail below.

The first NLDE simulation approach uses a steady mean-flow solution that couples with the NLDE solution to handle the unsteady aspects of the flow. This approach is relevant for many applications and is used for most (the results with adaptive mesh refinement [AMR] use secondary approach) of the results presented.
in this work. Thus, it is considered the general ('primary') variant of the NLDE solver. This approach is shown in Figure 3.1. The Figure shows the flow of the simulation setup where the converged mean-flow solution is obtained from a steady-state solver and the mean-flow quantities are fed in to the NLDE solver (highlighted with a bounding box). In the context of the OpenFOAM framework, the mean flowfield is obtained using the steady-state, incompressible flow solver in OpenFOAM, referred to as simpleFOAM [120]. The resulting flowfield is used to provide input for the NLDE solver, which is then used to calculate the behaviour of the disturbance variables. The converged solution (in Figure 3.1 disturbance variables without '*' denote the converged variables) is then fed to the multiscale solver (implementation details in Chapter 4) before the solution time is advanced. In this case, convergence of the disturbance variables is sought for each time step before the solution is marched further in time.

A second variant is developed for two-way coupled, transient analyses, where the disturbance equations are solved alongside the mean-flow equations. This solver setup is shown in Figure 3.2. The Figure shows the simulation flow for such setup, where the temporally converged mean-flow velocities from a URANS solver are used to couple to the NLDE solution at each time-level. Here, the disturbance
and mean-flow variables are advanced together in time. This solution strategy is essential when any type of two-way coupling of the mean-flow and NLDE solver is used, where the mean-flow solution can be affected by the NLDE solution (or the multiscale analysis). In the context of this work, application of the multiscale framework for AMR or turbulence detection would require such a setup. This solution strategy is more complex to handle. It requires both the solvers (mean-flow

Figure 3.2. Secondary version of the NLDE solver.
and NLDE) to converge with each-other (two-way coupling convergence, as shown in Figure 3.2) before the combined solution can be advanced forward in time. Because of this, such an approach is only suggested as demanded by physical demands of the simulated conditions. In terms of the mean-flow solver, several transient solvers for incompressible flows are available in OpenFOAM and are utilized in this effort.

In solving the incompressible NLDE, there are four equations with three disturbance-velocity variables, a disturbance-pressure variable, and terms associated with the mean-flow velocity fields. In the present work, the mean-flow solution is considered as either a one-way coupled or loosely coupled solution. Hence, the mean-flow gradient terms (see $\tilde{u}_j \partial \tilde{U}_i / \partial x_j$ from Equation 3.1) are treated as a source term in the NLDE solver. The mean-flow convection term ($\U_j \partial \tilde{u}_i / \partial x_j$ from Equation 3.1) is solved alongside the nonlinear term ($\tilde{u}_j \partial \tilde{u}_i / \partial x_j$ from Equation 3.1). In doing this, an explicit coupling between the mean-flow and the NLDE is achieved. Such a one-way coupled approach is more amenable to cases where the mean-flow is considered steady and is solved beforehand, however, some cases demand that both systems of equations be run in parallel. This situation requires extra care owing to the unsteady or potentially unresolved (in case of future two-way turbulence model coupling) state of the mean-flow equations. This type of situation requires a tighter coupling between the two systems of equations and/or high levels of under-relaxation. The secondary variant of the solver deals with this issue. A disturbance-pressure Poisson equation is derived for this case in a segregated solver setting to enable a tighter coupling with the base N-S equations. The details about numerical solution of the incompressible NLDE developed in this work are presented in the following section.

$$\frac{\partial \tilde{u}_i}{\partial t} = \nu \nabla^2 \tilde{u}_i - \left( \frac{\partial \tilde{p}}{\partial x_i} + (\U_j + \tilde{u}_j) \left( \frac{\partial \tilde{u}_i}{\partial x_j} \right) + \tilde{u}_j \left( \frac{\partial \U_i}{\partial x_j} \right) \right)$$

(3.1)

$$\frac{\partial \tilde{u}_i}{\partial x_i} = 0$$

(3.2)

where $i$ is the Einstein summation index for the three spatial dimensions, $\U$ is the mean-flow velocity, $\tilde{u}$ is the disturbance velocity, $\tilde{p}$ is the disturbance-pressure
and $\nu$ is the kinematic molecular viscosity.

### 3.1.1 Numerical Method

The incompressible NLDE solver developed in this work is intended for efficient solution of unsteady flows on complex real-world geometries. To accomplish this, the incompressible NLDE are solved on a finite-volume collocated grid. The development and implementation of the NLDE on a collocated grid with different solution strategies is discussed in this section.

The implementation of the NLDE on a collocated grid is accomplished through Rhie-Chow [121] interpolation to avoid non-physical pressure oscillations in the solution. Figures 3.3 and 3.4 show a one dimensional finite-volume cell with its neighbors to demonstrate a collocated versus a staggered grid. Figure 3.4 shows a staggered grid setup, where pressure and velocity are calculated on two separate grids to remedy pressure oscillations in the solution which can occur due to the nature of the discretized form of the equations. In a collocated grid, shown in Figure 3.3, the pressure and velocity are calculated on the same grid by implementing Rhie-Chow interpolation. This enables one to use unstructured and more complex grids. The use of collocated grid also decreases the memory-overhead and implementation-complexity of having two grids.

**Figure 3.3.** A finite-volume collocated grid cell with neighbors. Here, $P$ is the pressure, $U$ is the velocity and $i$ is the cell index.

**Figure 3.4.** A finite-volume staggered grid cell with neighbors. The green grid corresponds to the velocity grid and the red grid corresponds to the pressure grid. Here, $P$ is the pressure, $U$ is the velocity and $i$ is the cell index.
Figure 3.5. A two-dimensional, finite-volume collocated grid cell, P with control area (volume for three-dimensional generalization) $\Omega_P$ and neighbors, N comprising of cells, T, R, B and L and the corresponding shared faces, t, r, b, and l respectively.

Now, consider the two-dimensional, collocated grid shown in Figure 3.5. The NLDE given by Equation 3.1 is discretized on this domain. The cell, P is surrounded by its neighbours, N consisting of the cells T, R, B, and L with shared faces, t, r, b and l respectively. It should be noted that the two-dimensional domain is shown only for the ease of representation, the NLDE solver is implemented in three-dimensional framework. Integrating equation 3.1 in the control volume $\Omega_P$ of cell P for spatial discretization (assuming that the variables do not change within the control volume), Equation 3.3 is obtained.

$$\iint_{\Omega_P} \frac{\partial \tilde{u}}{\partial t} = \iint_{\Omega_P} [\nu \nabla^2 \tilde{u} - (\frac{\partial \tilde{p}}{\partial x} + (\bar{U} + \tilde{u}) (\frac{\partial \tilde{u}}{\partial x} ) + \tilde{u} (\frac{\partial \bar{U}}{\partial x} )]. \quad (3.3)$$

Then,

$$\frac{\partial \tilde{u}}{\partial t} \Omega_P = \iiint_{\Omega_P} [\nabla.(\nu \nabla \tilde{u} - (\bar{U} + \tilde{u}).(\tilde{u}) )] - \nabla \tilde{p} \Omega_P - \tilde{u} \nabla \bar{U} \Omega_P. \quad (3.4)$$

Using Gauss’s divergence theorem, which for a vector field, $F$, through a volume, $V$, enclosed by surface, $S$, states:

$$\iiint_V (\nabla \cdot F) dV = \iint_S (F \cdot n) dS$$
where $n$ is the vector normal to the surface $S$,

leads to,

$$
\frac{\partial \tilde{u}}{\partial t} \Omega_P = \iint_S (\nu \nabla \tilde{u} - \bar{U} \tilde{u} - \tilde{u} \tilde{u}) \cdot n dS - \nabla \tilde{p} \Omega_P - \tilde{u} \nabla \bar{U} \Omega_P, \tag{3.5}
$$

or,

$$
\frac{\partial \tilde{u}}{\partial t} \Omega_P = \iint_S (\nu \nabla \tilde{u} - \bar{U} \tilde{u} - \tilde{u} \tilde{u}) \cdot dS - \nabla \tilde{p} \Omega_P - \tilde{u} \nabla \bar{U} \Omega_P, \tag{3.6}
$$

where $dS = n.dS$ is the surface normal vector for cell $P$.

Evaluating the surface integral by summation over all the faces $(f)$ enclosing the control volume at node $P$ (capital letters denote nodes and small letters denote faces)

$$
\frac{\partial \tilde{u}}{\partial t} \Omega_P = \sum_{f = \Omega_P} (\nu \nabla \tilde{u} - \bar{U} \tilde{u} - \tilde{u} \tilde{u})_f \cdot S_f - \nabla \tilde{p} \Omega_P - \tilde{u} \nabla \bar{U} \Omega_P. \tag{3.7}
$$

Now, integrating w.r.t time to evaluate the solution at time, $t$ using first order implicit time discretization (for demonstration purposes only, the actual schemes used for the test cases vary).

$$
\int_{t - \Delta t}^{t} \frac{\partial \tilde{u}}{\partial t} \Omega_P = \int_{t - \Delta t}^{t} \left\{ \sum_{f = \Omega_P} (\nu \nabla \tilde{u} - \bar{U} \tilde{u} - \tilde{u} \tilde{u})_f \cdot S_f - \nabla \tilde{p} \Omega_P - \tilde{u} \nabla \bar{U} \Omega_P \right\} \text{ R.H.S.} \tag{3.8}
$$

Rewriting in finite difference form,
\[
\frac{(\tilde{u}_P^t - \tilde{u}_P^{t-\Delta t})\Omega_P}{\Delta t} = \left[ gR.H.S^t + (1 - g)R.H.S^{t-\Delta t} \right],
\]

(3.9)

where \( g \) is a weighting factor between 0 and 1, \( g = 1 \) for a fully implicit scheme as is used in this work. Collecting coefficient terms for the node in question \( (P) \) and neighboring faces \( (f) \) and expressing it in coefficient form:

\[
a_P^t \tilde{u}_P^t + \sum_{f=\Omega_P} a_f^t \tilde{u}_N^t + \nabla \bar{p}_P \Omega_P = \text{Explicit source terms},
\]

(3.10)

where the explicit source terms constitute the mean-flow gradient terms (see \( \tilde{u}_j \partial \bar{U}_i / \partial x_j \) from Equation 3.1) and any external forces, momentum sources or sinks for generalized formulation of the solver to accommodate different conditions.

This can be rewritten as:

\[
\tilde{u}_P^t + \frac{\sum_{f=\Omega_P} a_f^t \tilde{u}_N^t + \nabla \bar{p}_P \Omega_P}{a_P^t} = \frac{b_P}{\text{Source terms}/a_P^t}.
\]

(3.11)

Simplifying further,

\[
\tilde{u}_P^t = b_P - \frac{\sum_{f=\Omega_P} a_f^t \tilde{u}_N^t}{a_P^t} - \nabla \bar{p}_P \frac{\Omega_P}{a_P^t},
\]

(3.12)

accumulating the terms on the right hand side for a more compact notation,

\[
\tilde{u}_P^t = R_P^t - D_P^t \nabla \bar{p}_P.
\]

(3.13)
This gives an explicit equation for disturbance-velocities at the cell P derived from the disturbance momentum equation (Equation 3.1). These disturbance-velocities do not satisfy continuity (Equation 3.2). To satisfy Equation 3.2 and solve for disturbance-pressure (whilst using Rhie-Chow interpolation for collocated grids), two different approaches can be taken depending upon the numerical solution strategy. The system of equations can either be solved together in a coupled manner or sequentially in a segregated manner for each variable. Both the methods have their advantages and weaknesses. In this work, both solution strategies are developed and implemented. The following sections give details about the two solution strategies.

3.1.1.1 Block-Coupled Solver

A block-coupled NLDE solver is formulated with an implicitly coupled disturbance-pressure equation, to be solved alongside the disturbance-velocity equations. This pressure equation is derived using Rhie-Chow [121] interpolation, details of which are discussed in this section.

In a block-coupled solution strategy, all the variables are solved together in the same matrix solver system with an implicit coupling between the variables. The disturbance-pressure equation is solved in the same matrix system as the disturbance-velocities. However, Equation 3.2 does not contain any disturbance-pressure terms and it serves as the forth equation in the four incompressible NLDE system of equations for the four primary variables (3 disturbance-velocities and 1 disturbance-pressure). Hence, an implicit disturbance-pressure equation is derived to implement the incompressible NLDE system of equations in a coupled solver framework. This has been accomplished in this work for NLDE equations following ideas from Darwish et al. [122], who developed a fully-coupled, incompressible, steady-state, Navier-Stokes solver.

To derive the disturbance-pressure equation and implement the Rhie-Chow interpolation, the discretized momentum equation for the NLDE (Equation 3.13) is
interpolated to the faces (from the cell center, P). The face velocities are then used to satisfy the continuity equation (Equation 3.2). In the case of NLDE equations, equation 3.13 is transferred to faces \( f \) between the cell \( P \) and neighbors \( N \). The equation for the disturbance-velocity at face \( f \) can be written as:

\[
\tilde{u}_f^t = \mathcal{R}_f^t - \mathcal{D}_f^t \nabla \tilde{p}_f^t
\]

(3.14)

where overbar indicates an interpolated value and,

\[
\mathcal{R}_f^t = g_f R_P^t + (1 - g_f) R_N^t,
\]

(3.15)

\[
\mathcal{D}_f^t = g_f D_P^t + (1 - g_f) D_N^t,
\]

(3.16)

where \( g_f \) is the weighting factor for interpolation. For a uniform grid and linear interpolation, \( g_f = 0.5 \). Using equations 3.13, 3.15 and 3.16, equation 3.17 can be derived.

\[
\tilde{u}_f^t = \mathcal{R}_f^t - \mathcal{D}_f^t (\nabla \tilde{p}_f^t - \nabla \tilde{p}_f)
\]

(3.17)

To implement the continuity equation, Gauss’s divergence theorem can be applied to equation 3.2 in a control volume at \( P \), to obtain,

\[
\sum_{f=\Omega_P} \tilde{u}_f^t \cdot S_f = 0.
\]

(3.18)

From equation 3.17 this can be rewritten as:
\[
\sum_{f=1}^{N_p} (\bar{u}_f^I - \bar{D}_f^I (\nabla \bar{p}_f^I - \nabla \bar{p}_f^I)).S_f = 0.
\] (3.19)

Equation 3.19 is an implicit disturbance-pressure equation to couple with the disturbance velocity. This equation serves to build the block-coupled matrix for the coupled NLDE solver. These equations are then solved together in the final matrix to obtain the disturbance pressure and velocities. For a block-coupled matrix, each term in the coefficient matrix, \( A \), for a system of equations of the form \( Ax = b \), is a matrix itself. This matrix describes the connection between the disturbance velocity components and the disturbance pressure. Each term in the variable vector, \( x \), is a vector itself containing the disturbance velocities and pressure. This can be understood more clearly by looking at the matrix structure shown below.

Block-coupled matrix structure:

\[
\begin{bmatrix}
A_{11} & A_{12} & \ldots & \ldots & A_{1N} \\
A_{21} & A_{22} & \ldots & \ldots & \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
A_{N1} & A_{N2} & \ldots & A_{NN}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_N
\end{bmatrix}
= \begin{bmatrix}
b_1 \\
b_2 \\
\vdots \\
b_N
\end{bmatrix}
\]

source terms

\[
A_{ij} = \begin{bmatrix}
a_{\bar{u}\bar{u}} & a_{\bar{u}\bar{v}} & a_{\bar{u}\bar{w}} & a_{\bar{u}\bar{p}} \\
& a_{\bar{v}\bar{v}} & a_{\bar{v}\bar{w}} & a_{\bar{v}\bar{p}} \\
& & a_{\bar{w}\bar{w}} & a_{\bar{w}\bar{p}} \\
& & & a_{\bar{p}\bar{p}}
\end{bmatrix}_{ij}
\]

here \( i, j \) are the matrix indices.

Each of the scalar terms in \( A_{ij} \) define the interrelationship (influence) of the
variables with each other and is obtained from the discretization scheme coefficients.

\[
x_i = \begin{bmatrix} u_i \\
                         v_i \\
                          w_i \\
                         p_i \end{bmatrix}
\]

A block-coupled solver obtained this way is more robust with improved convergence qualities in comparison to a segregated solver. This can be especially helpful in case of the NLDE solver, since the perturbations can be very sensitive to initial and boundary conditions. However, this improved robustness quality may disappear with highly fluctuating variables (as can be the case with disturbance variables). Due to the solution of all the variables in the same matrix system, the memory and time requirement (even though time to convergence might be less for a coupled solver) per time step is higher for a block coupled solver than a segregated/sequential solver. The lower memory requirement for segregated solvers also makes them attractive in terms of potential for highly efficient and massive parallelization on graphics processing units (where memory per core is limited). The block-coupled matrix structure (matrix conditioning) can also lead to challenging matrix solution procedures. This requires specialized linear matrix solvers. On the other hand, sequential solvers have a huge variety of linear solver systems available for any type of matrix conditioning. Additionally, the current mainstream distribution of OpenFOAM does not come bundled with coupled solver libraries and tools. Only the community run experimental distribution - foam-extend, is available with a coupled solver and libraries to develop a block-coupled solver in OpenFOAM framework. The support, reliability and documentation for this distribution is very limited. This justifies the development of segregated solution procedure for the NLDE solver, for cases where the coupled solver is not the best solution. The following section discusses the development and implementation of the segregated solution strategy for the NLDE solver.
Solve momentum equation for intermediate velocity

\[ \tilde{u}_t^* \]

Solve pressure Poisson equation

\[ \tilde{p}_t^* \]

Corrected velocity

\[ \tilde{u}_t^{**}, \tilde{p}_t^* \]

Converged?

\[
\begin{align*}
\text{yes} & : \tilde{u}_t, \tilde{p}_t \\
\text{no} & : t = t + \Delta t
\end{align*}
\]

**Figure 3.6.** The PISO (Pressure-Implicit with Splitting of Operators) \[5\] coupling algorithm.

### 3.1.1.2 Segregated Solver

A segregated solution approach involves the calculation of each variable separately. In this section, the segregated solution approach for the NLDE is explored with different solution algorithms for the disturbance pressure-velocity coupling. These approaches are implemented in the solver and have been presented here for comparison. The segregated, disturbance pressure-velocity coupling for the NLDE equations is implemented following PISO (Pressure-Implicit with Splitting of Operators) \[5\], PIMPLE (PISO + SIMPLE \[123\]) \[120\] and a modified PIMPLE algorithm.

The coupling of the disturbance pressure-velocity for the segregated solver follows similar steps as pressure-velocity coupling in incompressible Navier-Stokes equations. The incompressible pressure-velocity systems contain two coupling setups, 1) the convective nonlinear flux term coupling (\( \tilde{u}_j \partial \tilde{u}_i / \partial x_j \) and mean-flow gradient term) and 2) the pressure-velocity coupling. In cases where small Courant-Friedrichs-Lewy (\( CFL = \tilde{u} \Delta t / \Delta x < 1 \)) number is used, the convective term (\( \tilde{u}_j \)) in the nonlinear term (\( \tilde{u}_j \partial \tilde{u}_i / \partial x_j \) and mean-flow gradient term) can be used from the
previous time-level. This type of solver setup can be employed with a PISO type coupling for the pressure and velocity coupling. The PISO pressure-velocity coupling algorithm is shown in Figure 3.6. The figure shows the step by step implementation of the algorithm in context of the NLDE solver.

To begin the calculation of a PISO coupling loop, first equation 3.20 is constructed into matrix formulation. This equation is then solved with source terms and disturbance-pressure gradient term (and the convective disturbance-velocity terms from the previous time/iteration) for the intermediate disturbance-velocity ($\tilde{u}^*$) predictor step. This formulation can be expressed in general discretized form as equation 3.21:

$$\frac{\partial \tilde{u}^*_i}{\partial t} + (\bar{U}_j + \tilde{u}_o^j)\left(\frac{\partial \tilde{u}^*_i}{\partial x_j}\right) + \tilde{u}_o^j\left(\frac{\partial \bar{U}_i}{\partial x_j}\right) - \nu \nabla^2 \tilde{u}^*_i + \frac{\partial \tilde{p}^o}{\partial x_i},$$  \hspace{1cm} (3.20)

$$a_P \tilde{u}^*_P + a_N \tilde{u}^*_N = S - \frac{\partial \tilde{p}^o}{\partial x_i},$$  \hspace{1cm} (3.21)

where the exponent $*$ refers to the intermediate disturbance-velocity and the exponent ‘$o$’ refers to the value from the previous time step. The coefficient matrices $a_P$ and $a_N$ denote the discretization coefficients for the cell in question, P and its neighboring cells, N respectively. S denotes the source terms matrix. The mean velocity gradient term ($\tilde{u}_o^j\partial \bar{U}_i/\partial x_j$) is a part of S, the source term matrix, in this momentum predictor step. It is important to note that the terms with the convective disturbance-velocity in the momentum predictor step are calculated from the disturbance-velocity of a previous time-step in case of PISO coupling strategy.

The intermediate disturbance-velocities calculated in this step are not divergence free (i.e. they do not satisfy equation 3.2). To remedy this situation and obtain the first round of corrected disturbance-velocities, equation 3.22 is obtained.
Here, \( u^{**} \) is the first iteration of the pressure corrected disturbance velocity and \( p^* \) is the intermediate disturbance pressure. The intermediate disturbance pressure \( (p^*) \) is calculated using equation 3.23 to calculate the first corrected disturbance velocity, which satisfies the divergence free criterion of equation 3.2.

\[
\bar{u}_{P}^{**} = a_{P}^{-1}(S - a_{N}\bar{u}_{N}^{*}) - a_{P}^{-1}\frac{\partial\tilde{p}^{*}}{\partial x_{i}}
\] (3.22)

Equation 3.23 is obtained by applying the divergence operator to the disturbance-velocity corrector step, as shown in equation 3.24. The left hand side of equation 3.24 equals zero, owing to equation 3.2. This gives us the first iteration of the pressure-corrected, divergence-free, disturbance velocities. These steps are iterated till a desired number of iteration steps or prescribed convergence criteria are met.

\[
\nabla^{2}(a_{P}^{-1}\tilde{p}^{*}) = \nabla.(a_{P}^{-1}(S - a_{N}\bar{u}_{N}^{*}))
\] (3.23)

In cases where tighter coupling between the variables and/or the ability to use \( CFL > 1 \) is required, the PIMPLE [120] coupling algorithm should be employed. In the PIMPLE coupling strategy, both the nonlinear, convective disturbance-velocity terms as well as the disturbance-pressure and velocity coupling terms are updated and iterated till the desired convergence levels are attained at every time-step of the solution. The PIMPLE algorithm can be thought of an extension of the PISO algorithm, so the full implementation of PIMPLE algorithm will not be presented here in detail. A modified PIMPLE algorithm is also implemented and will be discussed after the regular PIMPLE implementation has been described.

In case of PIMPLE algorithm, the final pressure-corrected disturbance-velocities from the PISO loop are reiterated through the momentum corrector and the pressure corrector steps with updated convective disturbance-velocity terms and
under-relaxation factors. This coupling strategy is depicted in Figure 3.7. The figure shows the 'outer correction loop' that is performed in addition to the PISO loop. For each outer loop pressure-corrected velocities are iterated until the convergence criteria or the prespecified number of iteration loops is met. The outer correction loop updates the convective disturbance-velocity terms.

Another alternative coupling method has been developed where a disturbance-
pressure is calculated for every first outer iteration loop (i.e. the first calculation of intermediate disturbance-velocity for every time step). This method is added as an extra step to the PIMPLE coupling strategy and is shown with dashed lines in Figure 3.7. This initial disturbance-pressure guess encourages a quicker convergence as well as a tighter coupling between the disturbance variables and the mean-flow velocity as well. This should be especially helpful in stiffer cases involving a tighter coupling of the variables and when the base flow and disturbance equations are run simultaneously (required for two-way coupling and adaptive meshing). This disturbance-pressure equation is derived while satisfying both equations 2.2 (the Navier-Stokes continuity equation) and 3.2. The derivation for the disturbance-pressure correction equation is as follows.

Taking divergence of both sides of equation 3.1:

\[
\nabla \cdot (\frac{\partial \tilde{u}_i}{\partial t}) = \nabla \cdot (\nu \nabla^2 \tilde{u}_i - (\nabla \tilde{p} + (\tilde{U}_j + \tilde{u}_j)(\nabla \tilde{u}_i) + \tilde{u}_j(\nabla \tilde{U}_i)))
\]

(3.25)

Using the identity for curl of a curl, \((\nabla \times (\nabla \times \mathbf{A}) = \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A})\),

\[
\nabla^2 \tilde{u} = \nabla (\nabla \cdot \tilde{u}) - \nabla \times (\nabla \times \tilde{u})
\]

(3.26)

Taking the divergence of both sides and using the vector calculus identity \(\nabla.(\nabla \times \mathbf{A}) = 0\) gives,

\[
\nabla \cdot \nabla^2 \tilde{u} = \underbrace{\nabla^2 (\nabla \cdot \tilde{u})}_{=0 \text{ from eqn. 3.2}} - \underbrace{\nabla.(\nabla \times (\nabla \times \tilde{u}))}_{=0},
\]

(3.27)

so that,
\[ \nabla \cdot \nabla^2 \tilde{u} = 0. \]  

(3.28)

\[ \frac{\partial \nabla \cdot \tilde{u}}{\partial t} = - \nu \nabla^2 \tilde{u} - (\nabla^2 \tilde{p} + \nabla \cdot (\tilde{U} + \tilde{u}) (\nabla \tilde{u})) + \nabla \cdot (\tilde{u} (\nabla U)) \]  

(3.29)

Using equations 2.2 (the Navier-Stokes continuity equation), 3.2 and 3.28

\[ \nabla^2 \tilde{p} = -2(\nabla \tilde{U} : \nabla \tilde{u}^T) - \frac{\nabla \tilde{u} : \nabla \tilde{u}^T}{\text{nonlinear contribution}} \]  

(3.30)

Equation 3.30 is used as an additional pressure corrector for tighter coupling of the base flow equations with the NLDE equations to form a modified PIMPLE algorithm. The development of these different coupling algorithms as well as the different solver solution strategies present a wide variety of options for the application of the developed NLDE solver to different types of unsteady flow scenarios. The following sections provide validation and verification of the results from the NLDE solver.

### 3.1.2 Discretization

The discretization schemes used for the solution of the disturbance variables are selected from the available numerical schemes in the OpenFOAM framework. The majority of all the cases presented in this work have been implemented using second-order unbounded Gaussian integration schemes with linear interpolation for spatial discretization. Temporal discretization has been implemented using a second-order "backward" discretization scheme. This is a second-order implicit, potentially unbounded, transient scheme.

### 3.2 Validation

Validation and verification are particularly critical and challenging in the study of the stability of laminar flows. Reed and Saric [124] list a variety of issues with CFD validation of stability and transition of laminar flows. The challenges associated with the CFD analysis of these laminar flow instabilities make it a great
candidate for NLDE solver evaluation. To validate the disturbance equation solver implemented in this work, a laminar flow stability case is tested. The following sections detail the steps.

3.2.1 Comparison with Linear Stability theory

The results of the nonlinear disturbance equation solver are compared with linear stability theory results of a Blasius boundary layer profile. The validation is performed on the same lines as the method used by Rist and Fasel [6], who used this technique to validate their version of a finite-difference, velocity-vorticity NLDE solver. The results from their solver were validated using results from linear stability theory.

\[ \text{Figure 3.8. Figure taken from [6], (a) Computational domain used for LST validation, (b) normal disturbance velocity distribution over the suction/blowing strip} \]

The test case shown in Figure 3.8 (a) (taken from [6]) shows the domain and scenario used to validate the NLDE against the LST results for a Blasius solution. However, for this validation study a two-dimensional version of the domain is used. In this work, the inlet boundary-condition of the domain is defined using a prescribed Blasius boundary-layer at a Reynolds number \( (Re = U_\infty (x - x_0) / \nu) \) of \( 10^5 \). The length of the domain in the streamwise direction is equal to 4 metres and in the normal direction is equal to 1 meter. The domain is discretized with...
260 points in the streamwise direction and 110 points in the wall-normal direction. The top boundary of the domain is described using freestream conditions, bottom boundary uses a no-slip condition to simulate a flat-plate and the outlet of the domain uses zero-gradient boundary conditions. The mean-flow calculations are performed using an incompressible, steady, laminar-flow solver - simpleFOAM. The steady-state solution of the developed Blasius boundary-layer in the domain is then used for disturbance calculations. The disturbance is introduced in the boundary-layer using a suction/blowing slit, as shown in Figure 3.8 (a) between $x_1$ and $x_2$. The distribution of the disturbances shown in Figure 3.8 (b), were used by Rist and Fasel to define the disturbances at the slit. In this work, the disturbance is defined by Equation 3.31 in terms of the wall-normal disturbance-velocity. The distribution of the wall-normal disturbance-velocity over a "suction/blowing" strip is defined by the function $f(x)$ in Equation 3.31. The function is described by a simple sine function with a wavelength equal to the width of the slit ($x_2 - x_1$). The slit is defined between $x_1 = 0.95$ to $x_2 = 1.0$.

$$\tilde{v}(x, z, t) = |v| f(x) \sin(\beta t)$$  \hspace{1cm} (3.31)

where $|v| = 10^{-5}m/s$ is the amplitude and $\beta$ is the frequency of the two-dimensional input disturbance. $f(x)$ is the function to define the vertical velocity distribution for suction/blowing type boundary condition at the suction slit, as shown in 3.8 (b).

The results obtained from this numerical setup of the NLDE solver are compared with results from LST for validation. To understand the LST results the following section details the steps taken to obtain the LST results for comparison.

3.2.1.1 Stability Calculations from the Orr-Sommerfeld Equations

The calculation of the theoretical results for stability of a laminar Blasius flow profile was performed using the Orr-Sommerfeld equation (recall description from Chapter 2). This section details the method used to obtain solutions for the Orr-Sommerfeld equation. The stability solution obtained from this method is used to validate the
results from the NLDE solver.

The Orr-Sommerfeld equation is derived from the nonlinear stability equations (3.1, 3.2). The simplifying assumptions of parallel flow ($\bar{U} \approx U(y)$, $V \approx 0$ & $W \approx 0$) and linearity (ignoring nonlinear terms) lead to the set of equations 3.32. The equations shown here assume incompressible flow.

\[
\frac{\partial \hat{u}}{\partial x} + \frac{\partial \hat{v}}{\partial y} + \frac{\partial \hat{w}}{\partial z} = 0, \\
\frac{\partial \hat{u}}{\partial t} + \bar{U} \frac{\partial \hat{u}}{\partial x} + \hat{v} \frac{\partial U}{\partial y} = -\frac{\partial \hat{p}}{\partial x} + \frac{1}{Re} \nabla^2 \hat{u}, \\
\frac{\partial \hat{v}}{\partial t} + \bar{U} \frac{\partial \hat{v}}{\partial x} = -\frac{\partial \hat{p}}{\partial y} + \frac{1}{Re} \nabla^2 \hat{v}, \\
\frac{\partial \hat{w}}{\partial t} + \bar{U} \frac{\partial \hat{w}}{\partial x} = -\frac{\partial \hat{p}}{\partial z} + \frac{1}{Re} \nabla^2 \hat{w}.
\] (3.32)

After eliminating the pressure terms and prescribing wave-like solutions of the form $\psi = \phi(y)e^{i(\alpha x + \beta z - \omega t)}$, where $\alpha, \beta$ denote the streamwise and spanwise wavenumbers and $\omega$ denotes the frequency of the disturbance, the Orr-Sommerfeld equation can be represented in velocity or velocity-vorticity formulation. In either case, the generalized version of the equation is given by equation 3.33. This equation presents itself as an eigenvalue problem with the wave speed, $c = w/\alpha$ as the eigenvalues of the system of equations. Using the wavelike form of the solutions or taking Fourier transform of the equations in 3.32, the derivative matrices ($\mathcal{D}$) for the amplitude function can be discretized to build the matrices for the linear algebra form.

\[
\left( \bar{U} - c \right) \left( \mathcal{D}^2 - k^2 \right) - \mathcal{U}'' + \frac{i}{\alpha Re*} \left( \mathcal{D}^2 - k^2 \right)^2 = 0
\] (3.33)

where $\mathcal{D}$ is derivative matrix in spectral domain, $k = \sqrt{\alpha^2 + \beta^2}$, $c = w/\alpha$, ’ denotes differentiation with respect to wall-normal direction and $Re*$ is the Reynolds number based on displacement thickness, $\delta^*$. 53
In this work, the stability code was implemented in Python programming language with the help of libraries in the *numpy* [125] and *scipy* [126] packages. The choice of using python was made for easier parallel implementation of the code.

The system of equations is discretized on a grid of Gauss-Lobatto collocation points using the Chebyshev method. The $N_{\text{cheb}} + 1$ grid points are defined as:

$$
\eta_{\text{cheb}} = \cos\left(\frac{\pi i}{N_{\text{cheb}}}\right),
$$

$$
0 \leq i \leq N_{\text{cheb}},
$$

where $-1 \leq \eta_{\text{cheb}} \leq 1$. This distribution of points is such that the points are clustered close together near the boundaries of the domain (i.e. $\eta_{\text{cheb}} = \pm 1$). The first and second-order derivative matrices in the spectral domain, $D^1$ and $D^2$ respectively are constructed using the analytic relations from Peyret [127]. The third and fourth-order spectral derivative matrices, $D^3$ and $D^4$ can be obtained through matrix multiplication of the $D^1$ and $D^2$ matrices. This can, however, lead to amplification of numerical error in certain cases. The terms of the first two Chebyshev operators, $D^1$ and $D^2$ are defined using relations give in equations 3.34 and 3.35.

$$
D_{ij}^1 = \begin{cases} 
 b_i(-1)^{i+j} & \text{for } i \neq j, \\
 \frac{b_j(\eta_{\text{cheb}} - \eta_{\text{cheb}})}{2(1 - \eta_{\text{cheb}}^2)} & \text{for } i = j, 
\end{cases}
$$

$$
D_{00}^1 = -D_{N_{\text{cheb}}N_{\text{cheb}}}^1 = \frac{2N_{\text{cheb}}^2 + 1}{6},
$$

$$
b_0 = b_{N_{\text{cheb}}} = 2,
$$

$$
b_i = 1, \text{ for } 1 \leq i \leq (N_{\text{cheb}} - 1).
$$

where $i, j$ are the row/column indices of the matrix, which range from 0 to $N_{\text{cheb}}$. $b$ is a coefficient for near boundary, $\eta_{\text{cheb}}$ are the Gauss-Lobatto collocation points and $N_{\text{cheb}} + 1$ is the total number of points. The terms of $D^2$ are defined using the relations:
\[ D_{ij}^2 = \begin{cases} 
(\frac{(-1)^i j (\eta_{\text{cheb}})^2 + \eta_{\text{cheb}} \eta_{\text{cheb}} - 2}{b_j (1 - \eta_{\text{cheb}}^2) (\eta_{\text{cheb}} - \eta_{\text{cheb}})^2}} & \text{for } i \neq j, \\
\frac{- (N_{\text{cheb}}^2 - 1) (1 - \eta_{\text{cheb}}^2) + 3}{3 (1 - \eta_{\text{cheb}}^2)} & \text{for } i = j, 
\end{cases} \]

\[ D_{0j}^2 = \frac{-2 (-1)^j (2 N_{\text{cheb}}^2 + 1)(1 - \eta_{\text{cheb}}) - 6}{3 b_j (1 - \eta_{\text{cheb}})^2}, \]  

\[ D_{Nj}^2 = \frac{-2 (-1)^{i+N_{\text{cheb}}} (2 N_{\text{cheb}}^2 + 1)(1 + \eta_{\text{cheb}}) - 6}{3 b_j (1 + \eta_{\text{cheb}})^2}, \]

\[ D_{00}^2 = D_{N_{\text{cheb}},N_{\text{cheb}}}^2 = \frac{2 N_{\text{cheb}}^4 + 1}{15}. \]  

To avoid interpolation errors, the Blasius equation solution is calculated on same distribution of points. However, the domain had to be scaled from \(-1 \leq \eta_{\text{cheb}} \leq 1\) to \(0 \leq \eta \leq \eta_{\text{max}}\), where \(\eta\) is the Blasius length scaling parameter and \(\eta_{\text{max}}\) is the maximum value of the parameter (assumed as a cutoff approximation for theoretical \(\eta \approx \infty\)). The solution was found to be slightly sensitive to the value of \(\eta_{\text{max}}\), which is consistent with the findings of Theofilis [128]. An algebraic scaling relation is used for the domain, given by:

\[ \eta = \frac{\eta_{\text{max}} (1 - \eta_{\text{cheb}})}{2}. \]  

The scaling for the Chebyshev derivative matrices is derived using the same scaling relation. The new derivative matrix is defined as:

\[ D_{m}^m = \frac{\partial \eta_{\text{cheb}}}{\partial \eta} D_{\eta_{\text{cheb}}}^m. \]  

55
which gives:

\[
D_1^{\eta} = \frac{-2D_{\eta,\text{cheb}}}{\eta_{max}}, \\
D_2^{\eta} = \frac{4D_{\eta,\text{cheb}}}{\eta_{max}^2}, \\
D_4^{\eta} = \frac{16D_{\eta,\text{cheb}}}{\eta_{max}^4}.
\]

(3.38)

The scaled domain was used for the Blasius calculations. The results are compared with Ganapol’s [7] results in Figure 3.9. The Figure shows the collocation points of calculations showing their distribution and the comparison of the results with Ganapol’s solution.

![Blasius flow solution](image)

**Figure 3.9.** The Blasius flow solution is calculated on the scaled Gauss-Lobatto’s collocation points and compared with Ganapol’s [7] calculations.

The scaled velocity profile, velocity derivatives and the Chebyshev derivatives are used to form the Orr-Sommerfeld equation into a generalized eigenvalue problem. To resolve temporal stability of the profile, a real \( \alpha \) wavenumber is used as input to the equations and complex eigenvalues of wave speed, \( c (= w/\alpha) \) are obtained. The
real part of the eigenvalue can be related to the frequency of the disturbance and
the imaginary part denotes the growth/decay of the disturbance. The eigenvalue
with the most positive imaginary part is the most unstable eigenmode of the finite
number of discrete eigenvalues. The eigenvalues for semi-finite boundary layer flows
like the Blasius flow has the most unstable eigenmode close to the wall. This causes
a slower wave speed ($c_r \to 0$, where $c_r$ is the real part of the complex wave-speed)
than the eigenmodes closer to the freestream ($c_r \to 1$). An example of such a case is
shown in Figure 3.10. The figure shows the temporal stability characteristics of the
Blasius profile at a Reynolds number (based on displacement-thickness) of 500 and
$\alpha = 0.25$. This case has one unstable eigenmode, called the Tollmien-Schlichting
eigenmode. This mode is highlighted in the figure with a square surrounding it.
This eigenmode has the most positive (maximum growth rate) $c_i$ (the imaginary
part of the complex wave-speed) for a wall mode ($c_r \to 0$) in Figure 3.10.

![Figure 3.10. The wave speed, c eigenvalues from temporal stability of Blasius boundary-
layer profile with the most unstable eigenmode highlighted with a square.](image)

To resolve the spatial stability of the profile, a mapping procedure is employed
to avoid solving a polynomial eigenvalue problem. In this method the eigenvalues
are obtained for a range of complex $\alpha$ and the most unstable eigenvalue is stored.
These eigenvalues are then mapped onto the complex eigenvalue plane and the
interpolated values of $\alpha$ for $w_i = c_i = 0$ are plotted back on to complex $\alpha$ plane to
find the boundary between the spatially stable and unstable solutions. This is then correlated for the most negative value of the $\alpha_i$ at $w_i = c_i = 0$. This relates to the most spatially unstable eigenmode at frequency $w_r$ with a growth rate proportional to $\alpha_i$. However, this method still uses the temporal eigenvalues to interpolate the most unstable spatial mode. This might account for some discrepancy in the comparison with the results of the NLDE solver, which are the result of a purely spatial analysis. To improve these calculations of spatial stability, companion matrix method [129] should be used. This method was developed by Bridges and Morris for efficient solution of the spatial stability problem in spectral domain.

### 3.2.1.2 Results

The results from the Orr-Sommerfeld solver discussed in the previous section are used to validate the results from the NLDE solver. For validation of the Blasius flow results, a two-dimensional disturbance was input with $\beta = 10$ in equation 3.31 and the results were compared at $x = 2.0$ with LST eigenfunctions as shown in Figure 3.11. Figure 3.11 (a) shows the Blasius boundary-layer profile that was used for prescribing the inlet velocity profile for the NLDE domain. Figure 3.11 (b) shows the eigenvalues obtained from linear stability theory. Here, the horizontal axis shows the real value of the wave speed and the vertical axis shows the imaginary value of the wave speed. The eigenvalues plotted on this graph shows the possible solutions of disturbance for a non-zero wavenumber. These eigenvalues can be thought of as distributed in three main branches, which are labeled as the A ($c_r \to 0$), P ($c_r \to 1$) and S ($c_r \simeq 0.9$) branch, given by Mack [130]. The most amplified wave is found on the the A branch of this diagram as the value with the most positive $c_i$. The eigenfunctions for this eigenvalue are then compared with the NLDE results. These are shown in Figure 3.11 (c) and (d). The normalized wall-normal and streamwise disturbance-velocities are compared with the eigenfunctions of the most unstable eigenvalue from linear stability theory and the results show good agreement. The general qualitative behavior of the results is identical to the results from linear stability theory. However, the secondary peak of the streamwise disturbance and the slope of the wall-normal disturbance show some disagreement between the two results. A change in slope of the wall-normal disturbance distribution close to the wall is also seen in the NLDE results, which is absent in the LST results. There
are a number of factors which may be responsible for the observed discrepancies. Some of these factors include absence of true spatial stability solution of the LST, presence of nonlinear and non-parallel influences in the NLDE results, development of the Blasius profile in the NLDE solver to a slightly different profile than used for LST calculations and interpolation errors in the mapping method for solution of LST. Despite these influences, the results are able to capture the disturbance behavior well.
in order to ensure that the method remains accurate on a conventional mesh with the numerical schemes used here, a mesh sensitivity study is performed. The study is evaluated for the same flat plate geometry as the Blasius results. The base grid used for validation with the LSE results is refined and coarsened to perform the same validation study on the subsequent grids. The finer grid is obtained by doubling the number of points in both direction and the coarse grid is obtained by halving the number of points in both directions. The results can be seen in figure 3.12. The results indicate a grid convergence with the base mesh itself, given that the results from the base and refined mesh show almost no difference between them. The coarse grid shows some difference in amplitude of the perturbation distribution, however, the difference is small and similar qualitative behavior is exhibited in both the results. This result gives confidence in the resolution requirements of the NLDE solver for disturbance resolution. The fact that even the lowest resolution mesh is still able to resolve the qualitative behavior and the highest resolution mesh did not add any extra information to the base mesh solution, confirms the low spatial-resolution requirements of the NLDE solver to resolve disturbances in
3.2.3 Coupling Algorithm Sensitivity

To evaluate the implementation of the different coupling algorithms, results from the different methods are compared on the same flat plate geometry as the Blasius results. The base grid used for the validation test is used here as well. Similar convergence levels for each time step were prescribed for all the solvers. The results can be seen in figure 3.13. The results presented in the figure indicate that all implementations of the solver agree well with each other. The coupled solver does show slightly different behavior. However, the difference is small and similar qualitative behavior is exhibited in the results. The figure also indicates that the results from the modified PIMPLE (PIMPLE-R in the figure) implementation fall between the other segregated solvers and the coupled solver in the secondary peak region of the disturbance.
3.3 Conclusions

The results from the Blasius validation case show promise with the accuracy of the calculated disturbance behavior. The same case has been run with a non-critical frequency to observe the response of the solution. Figure 3.14 shows the decay of the non-critical frequency compared to the growth of the critical frequency. This result shows that the solution is able to resolve the receptiveness of the boundary layer to different frequencies at a resolution much lower than used in DNS simulations. The ability of the solver to resolve the evolution of a disturbance with accuracy (compared to linear theory) demonstrates the validity of its solutions. The efficiency of the solver in resolving disturbances can be further improved with adaptive mesh refinement (AMR). The next chapter discusses the implementation of multiscale capabilities in the NLDE solver. The chapter also discusses the potential of extracting and understanding more about the dynamics of the disturbances and implementing efficient AMR using wavelets.
Figure 3.14. $\tilde{u}$ amplitude along $\xi = 0.67$, for unstable (top) vs stable frequency (bottom) of the input disturbance
This chapter discusses the implementation of a multiscale strategy intended to both enable the NLDE solver to become more efficient and extract key information. In terms of extracting information, this multiscale analysis can be used to understand the different scales/frequencies, their magnitude, and their relation with each other. Using this information, a deeper understanding of the disturbance propagation at different scales can be understood. In this work, multiscale methods are utilized for: (1) detection of coherent structures in the disturbance field, (2) adaptive mesh refinement sensor and (3) detection of changes in the behavior of the disturbance (eg. transition mechanism). In this work, wavelets are used to analyze the different scales resolved in the NLDE solver to provide the aforementioned benefits. The following sections present the implementation strategy and some results relevant to the multiscale part of the solver.

4.1 Wavelet Analysis

4.1.1 Implementation

In the past couple of decades, wavelets have been utilized in a number of areas within the field encompassing computational fluid dynamics. These applications can be classified into three main categories [38] (1) pure wavelet based solvers (where the partial differential equations are discretized in the wavelet domain), (2) wavelet based multiresolution methods (where wavelets are used to perform multiscale analysis) and (3) wavelet optimized methods (where wavelets are used as
Figure 4.1. The Haar wavelet [8].

sensors for adaptive meshes or compression strategies). The current implementation of the wavelet analysis in the NLDE solver falls somewhere in between the second and third category. More details about the past implementations in computational fluid dynamics and background information regarding wavelets has been provided in Chapter 2.

In this work, the multiscale analysis is performed at every time step on the converged solution of the disturbance velocities from the NLDE solver. The details of the sequence of steps is provided in the flowchart in Figure 3.7. The magnitude of the converged disturbance velocities is used for convolution with the wavelet basis functions. The wavelets analyses are performed along the Cartesian streamwise mesh direction, using data from each of the cell centers and its neighbors. The boundaries of the domain are treated using Neumann boundary conditions.

The primary wavelet used in this work is the Haar wavelet [131]. Note that Daubechies D4 wavelet [132] was also implemented and explored but is not used in this work to save on computational expense and complexity. The results from the Daubechies wavelet are similar to the Haar wavelet and hence will not be discussed.
in this work. The Haar wavelet, shown in figure 4.1, was first proposed by Alfréd Haar in 1910 [131]. For a Haar wavelet, only two discrete levels of scaling coefficient (field being analyzed or previous level average coefficients of the field) are required for each computation. On the other hand, the Daubechies D4 wavelet (shown in figure 4.2) requires a four point scaling coefficient stencil (within the CFD domain) for the calculation of each wavelet coefficient. This means a larger overall stencil and more operations per cell. The Haar wavelet is chosen for its compact support, which enables a small computational molecule, and its ability to detect sudden fluctuations. The application of the Haar wavelet involves algebraic operations, which also saves on computation time (depending on architecture, processor generation and a multitude of other factors) compared to convolution operations (required for most other wavelets).

The NLDE solver employs three computational stencils for the three levels of the Haar wavelet transform used for all the calculations presented in this work. The stencils, for a cell \( i \), at the three different levels are shown in figure 4.3. These stencils are applied on every cell in the domain. This type of wavelet-transform is called the stationary transform since the grid is not downsampled (eliminating
adjacent cells for coefficient calculation) at every level. This helps in translation invariance of the coefficients, however, it also entails redundant oversampling of the flowfield at every point (due to overlapping stencils from adjacent cells). The detail coefficients of each level signify the relative level of detail or disturbance (high frequency content) present at that level. The high frequency content of the flowfield is identified with the smallest wavelets/first detail coefficients ($c_0$ in figure 4.3). The largest wavelet (from $i - 2$ to $i + 3$ in figure 4.3 - coefficient $c_2$) identifies the larger scale/low frequency content of the flowfield. In the case of the NLDE solver, the flowfield being analyzed is the disturbance flowfield. Therefore, each level of the scaling and wavelet detail coefficients denote disturbances of different scales/frequencies. Each cell center in the NLDE solver is applied with the three levels of wavelet decomposition, shown in figure 4.3. Up to five levels of wavelet decomposition were explored. However, three levels of wavelet decomposition were found to be sufficient for the current application.

The Haar wavelet can be very efficiently applied in a matrix structure, as is shown in equations 4.1, 4.2 and 4.3. Every level requires one matrix multiplication each for the scaling and detail coefficients. However, these matrices are sparsely populated with just two non-zero elements in each row (Haar wavelet - two point function). This makes it very computationally inexpensive to calculate these coefficients. The elements at the boundaries (encircled in equations 4.1, 4.2 and 4.3)
need special treatment. In this work, the boundaries are treated with upwind (for inlet boundaries) or backward difference (for outlet type boundaries) type of stencil and the computation is performed via for-loops instead of matrices for the ease of debugging. The matrix implementation, however, is also in place and is expected to be used for future simulations.

The calculation of the first level of wavelet coefficients is performed using the magnitude of disturbance velocity for the field being analyzed. This calculation yields the high frequency wavelet detail coefficients ($c_0$), as shown below. Similar calculation, without the negative sign, is done to obtain the scaling coefficients (averaging field, $a_0$). This scaling coefficient field, in essence, provides the first level of averaged/smoothed signal. Every level of wavelet decomposition of a signal is made up of these two coefficients. The original signal/field can be reconstructed by combining the two coefficient fields at each level.

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 & 0 & \ldots & \ldots & 0 \\ 0 & 1 & -1 & 0 & \ldots & 0 \\ 0 & \ldots & \ldots & \ldots & \ldots & 0 \\ 0 & \ldots & \ldots & 1 & -1 & 0 \\ 0 & \ldots & \ldots & \ldots & \ldots & 1 \end{bmatrix} \begin{bmatrix} |\tilde{u}|_1 \\ |\tilde{u}|_2 \\ \ldots \\ |\tilde{u}|_{N-1} \\ |\tilde{u}|_N \end{bmatrix} = \begin{bmatrix} c0_1 \\ c0_2 \\ \ldots \\ \ldots \\ c0_{N-1} \\ c0_N \end{bmatrix}$$

(4.1)

The first level smoothed signal, i.e. scaling coefficient field ($a_0$) is used to calculate the second level detail ($c_1$) and scaling coefficients ($a_1$).
The second level smoothed signal, i.e. scaling coefficient field \( (a_1) \) is used to calculate the third level detail coefficients \( (c_2) \) and scaling coefficients \( (a_2) \).
4.1.2 Analysis

The wavelet decomposition of the disturbance velocities from the NLDE solver is used to better understand the disturbance structure and evolution. The wavelet coefficients are used to detect the presence of coherent structures in the disturbance. This information is then used for mesh refinement around the coherent structures. This helps in better resolution of the coherent structures, which can aid in better understanding of the growth and evolution of these coherent structures. The multiresolution information (wavelet coefficients at different levels) is also used to detect regions of turbulent type behavior. This information can be further developed in the future to enable more intelligent mesh refinement schemes and two-way coupling with base flow.

The coherent structures in the disturbance field are differentiated by separating the noise or the high frequency components from the disturbance signal. This highlights the application of the multiscale strategy in identifying the coherent structures in the disturbances. The large scale wavelet coefficients ($c_2$) are used to detect the coherent structures in the disturbance flowfield. Figure 4.4 shows the application of the wavelet decomposition on a three-dimensional flat plate domain. The figure shows the location of high values of $c_2$ in the domain. This information can be used for local refinement of the mesh to better resolve the coherent structures of the disturbance. Figure 4.5 shows the disturbance velocity contours on the mesh, where the wavelet analysis was applied. The left part of the figure shows the baseline mesh with the initiation of the disturbance. The disturbance structure is a two-dimensional vortex, which is detected by the large scale coefficient of the wavelet analysis, shown in figure 4.4. The right part of figure 4.5 shows the mesh after the wavelet criterion is applied for mesh refinement.

Traditionally, threshold based criteria are used for adaptive mesh refinement using wavelets. However, other truly autonomous techniques have also been proposed in the past [44,133]. A number of criteria were explored in this work to attain semi to fully autonomous operation to avoid dependence on finding threshold values. The results shown in Figures 4.4 were governed by the value:
The regions with values higher than 0.33 for $c_3$ were set to be refined by the solver. This criterion works well in identifying areas with large coherent structures and is seen to develop well with the solution. The quantity, $c_3$ is meant to find regions where the large scale coefficients are more than 1.33 times the smallest scale coefficients. The rationale behind the value of 0.33 being that an approximately equal distribution of the disturbances among three wavelet coefficients would mean absence of a substantial large scale disturbance, to be a coherent structure. Therefore, a coefficient bigger than at least a third of the average value of an incoherent three coefficient field would indicate coherence.

Another application of the multiscale architecture, explored in this work, is in detecting changes in flow behavior. Figure 4.6 shows a rectangular domain with large-scale, low-frequency sinusoidal velocity in the streamwise direction (from left to right) with a small region of Gaussian distributed high-variance/noisy velocity field. On the right side of the figure, a line plot of the velocity along the center-line (from left to right) is also shown. These fields are fabricated velocity regions and are not resolved due to any flowfield. The purpose of this field is to explore the appli-
Figure 4.5. Detection of coherent structures and adaptive meshing using wavelet coefficients. Left: the disturbance velocity on the base mesh, right: the disturbance velocity on the refined mesh.

cation of wavelet based multiscale strategy to detect regions of noisy, turbulent-like regions. The increase in the general magnitude of velocity in the turbulent-type region is implemented to ensure that the dip in coefficient values (shown in figure 4.7) is not a mere mirroring of the base flowfield. Figure 4.7 shows the results of the wavelet analysis on the domain. The left side of the figure shows the distribution of three-level wavelet coefficients along the centerline of the domain. The right side shows the criterion coefficient, $c_3$ (generic name given to criteria in this work, not the same criterion as one mentioned earlier for mesh refinement) along the same centerline. The criterion coefficient clearly demonstrates the difference between the regions. The definition of $c_3$, as implemented in this case, is:

$$c_3 = \frac{c_2}{c_0 + c_1 + c_2}$$

This criteria was set to explore differences between regions, where the flowfield
has different distribution of disturbances among different scales/frequencies. It is assumed that regions with similar levels of activity at different scales/frequencies of disturbances are more turbulent-like (inferring analogy from the characteristic mixing and energy transfer to different scales in turbulent flow) than regions where only some disturbance scales/frequencies have high activity levels (laminar flow type behavior).
4.1.3 Results

The current implementation of the wavelet strategy in the NLDE solver is seen to successfully detect coherent structures and changes in flow behavior/topology. Traditionally, the fast wavelet transform lends itself to a $O(N)$ algorithm to compute $N$ wavelet coefficients in its matrix type implementation. However, the current implementation sees more use in the for-loop type implementation due to the solver being in developmental phases. The computation time is still negligible compared to the solution time. The time required for the wavelet analysis, on five cores of AMD Phenom II X6 1045T processor with 8 Gigabytes of random access memory (RAM) for a computational mesh with 2,022,405 cells, is less than 0.5 seconds of clock time per time step. For the same mesh parameters, the solution of the flow equations was over 2100 seconds for each time step (time-accurate simulation with pimple implementation runtime for both N-S and NLDE solver run together to convergence on each time step). Hence, the time taken by the wavelet analyses can be considered negligible with respect to the NLDE solution time and is worth considering in the context of a CFD simulation.

The implementation of the multiscale strategy concludes the development of the proposed NLDE solver for this work. To explore the application of the developed solver, the NLDE solver is applied to two application cases. The results from these cases are discussed in the following chapter.
Chapter 5  
Results

This chapter presents the results from some application cases of the NLDE solver. The NLDE solver, coupled with the wavelet multiscale analysis method is applied to several flow scenarios. Such scenarios are deemed good candidates for the NLDE solver based on their high computational demands and complexity. The flow scenarios considered here for analysis are the stability of laminar flows and the aerodynamic response of an airfoil to low-amplitude, high-frequency gusts. These cases require very high computational resources to be resolved well with conventional N-S based CFD solvers.

The resolution of laminar flow instabilities, their development and breakdown to turbulence demands considerable spatial and temporal resolution. Using a conventional CFD solver, the low amplitude of the instabilities with respect to the mean flow tends to dissipate the instabilities unless very high spatial resolution is employed. The unsteady and evolving nature of the instabilities necessitates relatively high temporal resolution as well. The unsteady nature of the problem also gives rise to issues related to the definition of boundary conditions for accurate representation of the flow. For these reasons, supercomputing clusters with specialized flow solvers and schemes are often used to resolve these flow scenarios and the simulations are limited to canonical cases.

The resolution of these laminar flow instabilities in a NLDE type setting can alleviate the issues regarding high spatial resolution and numerical conditioning. The application of the wavelet based multiresolution framework can provide important information regarding the different scales present in the simulation. Such
information can help to better understand and process the physical phenomena as well as enhance the spatial resolution when required. The multiscale framework can also help to detect regions where the flow turns turbulent. This can be helpful in regards to more reliable and efficient simulations, as well as future two-way coupling of the NLDE solution to the base flow with a turbulence model. The spatial-resolution enhancement is also expected to aid in avoiding excessive meshing of the domain and resolving NLDEs in the turbulent regions, further reducing the computational cost.

In the case of low-amplitude high-frequency gusts, turbulence models for RANS tend to dissipate gusts. This necessitates LES or DNS methods for gust simulations, which increases the computational cost of the simulation by many orders of magnitude. The high-frequency (leading to small spatial wavelength) and low-amplitude of the gust demands high spatial resolution of the computational domain around the gust and its propagation path. The high frequency also leads to the requirement of high temporal resolution of the solution. Owing to these difficulties, the analysis of aerodynamic gust response presents itself as an ideal case for the NLDE solver, where the gust can be resolved separately from the mean flow. The mean flow can then be run with steady or quasi-steady assumptions. The spatial resolution requirements for the simulation are lowered due to the separate resolution of the gust on the relatively coarse grid of the base flow. By resolving the small amplitude disturbance/gust on the same mesh but in a NLDE setting, the effective spatial resolution of the solution is increased. The application of the NLDE solver with multiresolution capabilities can also help in resolving the effect of different frequencies/scales on the aerodynamics response.

The results are discussed here with respect to the present and potential capability of the solver for the two application cases. The current stable version of the solver is one-way coupled with the base flow solver. This leaves room for improvement in the computational efficiency of the solver and its capabilities. The application case for the gust response is resolved for a single frequency gusts to enable comparison with theory. This obviates the need for multiresolution capabilities of the solver in the gust cases and will not be discussed here. The following sections detail the application cases and their results.
5.1 Stability of Laminar Flow

Chapter-3 showed the capabilities of the NLDE solver in resolving modal flow instabilities. However, the real advantage of the solver lies in its application to more complex flow scenarios with non-modal instabilities. These types of scenario are expected to involve highly complex real-world geometries and complex flow-scenarios with non-linear, interacting instabilities. Such interactions are the basis of many flow-control devices, where the interactions of the primary disturbance with actively or passively generated disturbances are of importance. One example of such a flow-control device is the application of micron-sized, discrete roughness elements as a means to control transition due to crossflow instabilities. The lure of this type of flow control is its passive nature, which means that no power is spent on running the device. However, even after years of research on the subject (numerical and experimental [134–139]), recent flight-tests [140] concluded that further investigation using higher-order computational methods (DNS) are needed to better understand the process. To study the behavior of interacting instabilities, the NLDE solver is applied to the case of the interaction of primary, two-dimensional (T-S) disturbances with three-dimensional (spanwise) disturbances. This type of interaction can be studied using the case of the K-type transition.

The K-type and H-type transition scenarios involve the interaction of spanwise waves of subharmonic (H-type) or fundamental (K-type) frequency with the two-dimensional primary disturbance (T-S waves). Both of these transition mechanisms were discovered as part of wind-tunnel experiments with forced disturbances [29,54]. Since then, they have been the focus of several numerical studies [6,10,26,141]. The numerical studies conducted in this regard are usually DNS simulations [6,10,26], however some theoretical approaches have also been seen in the past [141]. K-type transition occurs when the fundamental two-dimensional disturbance interacts with an oblique wave of the same frequency. This leads to non-linear growth and generation of secondary structures, known as Λ vortices. The non-linear growth leads to an early breakdown of the laminar flow to turbulence. Figure 5.1 shows these structures. The Figure shows results from DNS of K-type (Left - aligned vortices) and H-type (Right - staggered vortices) conducted by Sayadi et al. [142]. These types of DNS simulation require very high computational resources. The
Figure 5.1. Development of A vortices seen in Sayadi et al. [10] simulation of K-type transition scenario (left) and H-type transition (right). Iso-surfaces of second invariant of the velocity gradient tensor colored by contours of streamwise velocity are shown here, taken from [10].

Simulation by Sayadi et al. was run on BlueGene [143] supercomputing clusters using 32K cores. The simulation time for each flow throughout was about 2 million computational hours [144]. However, it must be noted that the simulation also included extensive resolution of turbulent scales in the simulation, which are often the more computationally expensive aspect. In the present effort, the NLDE solver is used to predict the K-type transition scenario and the results are discussed in the following section.

5.1.1 Computational setup

In this simulation, the block-coupled NLDE solver was used. The domain used for the simulation of the K-type scenario is sketched in Figure 5.2. The computational domain extends 8.6*0.92*0.606 length units in the $x$, $y$ and $z$ directions respectively. For the K-type simulation, the domain was discretized spatially with $260 \times 70 \times 40$ cells in the $x$, $y$ and $z$ directions respectively, which totals to 728,000 cells. The cell dimension ratios were kept uniform in the $x$ and $z$ directions. The $y$ cells were expanded such that the last cell was sixty times bigger in the normal direction than the first cell (at the wall). For the results of the K-type simulation, the temporal resolution was 1/50th of the period of the of the primary waveform. The inlet velocity profile (on the plane of $x = x_0$) was prescribed as a Blasius velocity profile for the simulation. The inlet Reynolds number based on the distance from the
Figure 5.2. The simulation is conducted in the domain highlighted by dashed lines. The domain starts from a location downstream of the leading edge at $x_0$. The suction-blowing disturbance strip extends from $x_1$ to $x_2$. The disturbance is specified over the strip in terms of the vertical disturbance velocity denoted by
\[
\tilde{v} = A_1 \sin(\phi x) \sin(\omega_1 t) + A_2 \sin(\phi x) \cos(\lambda z) \sin(\omega_2 t).
\]  
$A_1, A_2$ are the amplitudes of the two-dimensional and spanwise wave and $\omega_1, \omega_2$ is the frequencies of the respective disturbances. The functions in red and green define the distribution of the respective colored disturbance on the disturbance strip, as shown in the Figure.

leading edge, $Re_{x_0}$ was equal to 2500. In addition, as indicated in green in Figure 5.2, a disturbance was introduced into the system via a disturbance strip. The disturbance strip, in the form of a variable suction-blowing velocity field, extended from $x_1$ to $x_2$. Following on similar lines as Fasel [26], Rist [6] and Huai et al. [145], the disturbance is defined by equation 5.1.

\[
\tilde{v} = A_1 \sin(\phi x) \sin(\omega_1 t) + A_2 \sin(\phi x) \cos(\lambda z) \cos(\omega_2 t) 
\]  
(5.1)

Here, $A_1$ and $A_2$ denote the amplitudes of the two disturbances, $\phi$ is the frequency in radians for the suction/blowing type boundary condition such that the wavelength of $\sin(\phi x)$ is equal to the width of the disturbance strip. $\lambda$ is the wavelength of the spanwise wave and $\omega_1, \omega_2$ are the frequencies of the primary and oblique disturbances respectively. For K-type transition $\omega_1 = \omega_2$ in contrast to H-type transition, where the spanwise wave is subharmonic and set as $\omega_2 = \omega_1/2$. The frequency of the primary disturbance $\omega_1$, is set according to the non-dimensional fre-
Figure 5.3. NLDE results from K-Type laminar flow transition scenario. Iso-volumes of $\lambda_2$ colored by mean streamwise velocity are shown during the initial stages of the transition process before the breakdown of the instabilities. The development of secondary structure from the interaction of the two instabilities (left) and the development of horseshoe vortices from the secondary structures (right) is shown in this figure.

Frequency $F = \omega_1 \nu / U_\infty^2$ set as $1.24 \times 10^{-4}$. These non-dimensional values are based on the critical frequencies in the neutral stability diagram, as explained in [26] and [10].

The present NLDE simulations were run on a 5th generation Hewlett-Packard ProLiant DL585 rack-mounted computational node. The node houses 16 processing cores (four quad-core AMD Opteron 8356 processors) running at 2.3 GigaHertz with 64 Gigabytes of random access memory. The algebraic multigrid method (AMG) was used for the linear system solution. The overall wall-clock time was $\approx 75$ hours (around 1200 CPU hours) for the base solver to generate the mean solution and the NLDE solver to complete 10 convective length periods of the streamwise disturbance (complete flow through). Note that these CPU times also include saving of the entire flowfield every other time step for visualization purposes, which has a non-negligible impact on the CPU time. Nevertheless, the resources required by the NLDE solver are orders of magnitude lower than the present state-of-the-art demanding DNS. The time taken by the NLDE simulation of K type transition is upto three orders of magnitude lower than a comparable DNS simulation.

5.1.2 Discussion of results

Figure 5.3 shows the results from a flat plate simulation of a K-type [29] transition scenario. This scenario results in $\Lambda$ shaped vortices which eventually give rise to
horseshoe shaped structures, leading up to transition to turbulence. Similar results were simulated by Sayadi et al. [10], shown in Figure 5.1. Their results, however, used a computational domain with $4096 \times 550 \times 512$ cells, which amounts roughly 1.15 billion cells.

The initial evolution of the disturbance in the laminar region can be seen in Figure 5.3. The interaction of the two disturbances appears to give rise to secondary structures. The results from the NLDE solver show resolution of the disturbance and the interaction of the two wavenumbers. This results in the formation of \( \Lambda \) shaped vortices, which give rise to horseshoe vortices. These horseshoe shaped structures, shown in Figure 5.4, evolve into turbulent spots eventually leading to complete breakdown into turbulence. Figure 5.6 shows the breakdown region with traces of horseshoe type structures evolving into fully turbulent region. This is consistent with the findings of Sayadi et al. [10]. The complete phenomenon can be observed in Figure 5.5. The figure shows the complete evolution of the disturbances into breakdown to turbulence.

Figure 5.7 shows the skin-friction coefficient over the flat plate. The result was obtained by calculating the skin-friction of the combined mean flow and disturbance velocities. The results were spatially averaged over the span of the flat plate and
temporally averaged over one period of the primary disturbance. The figure also shows the laminar correlation for skin friction coefficient of a Blasius boundary layer and turbulent correlation from White [11]. This Figure shows that the skin friction change due to transition of the flow can be predicted by the NLDE solver despite the laminar run of the base flow. The breakdown region of these horseshoe vortices coincides with the spike in skin friction coefficient, shown in Figure 5.8. The figure shows the collocation of the legs of a passing horseshoe vortex with the spikes in skin friction in a sliced plane of the domain. The skin friction coefficient calculated during different times during the simulation are plotted in Figure 5.9 to see the evolution of skin-friction coefficient as the hairpin vortex convects into the breakdown region. The initial spike in the skin friction coefficient is also consistent with other reports [10]. Towards the end of the domain the skin-friction coefficient is seen to decline, which is inconsistent with physical behavior (since the boundary layer after the breakdown is expected to develop into full turbulent boundary layer). This can be attributed to the lack of two way coupling of the NLDE solver with the mean-flow and to boundary-condition effects. The occurrence of issues near the boundaries of the domain in unsteady simulations is consistent with findings of other researchers in DNS simulations [6,10,26]. The non-physical reflection of the unsteady flow from the boundaries owing to inappropriate boundary-conditions can occur. This is usually treated by applying specialized boundary-conditions or by allowing extra 'sponge-region' near the boundaries where the solution is artificially decayed to avoid reflections. The NLDE results show dampening of the skin friction and the disturbances at the exit boundary of the domain. However, the results within the domain show good resolution and merit further application of the solver to more complex transitional scenarios.

To compare with the results of Rist and Fasel [6], another simulation with parameters similar to their simulation was run. This simulation was run without the visualization data to increase the speed of computation. The results from the simulation were averaged in the spanwise direction and Fourier analyzed in time at $y/\delta = 0.26$, to compare with the numerical results of Rist and Fasel. The magnitude of the spanwise-averaged, disturbance-velocity was obtained at multiple locations (downstream of the disturbance strip) for Fourier analysis. The amplitude of the frequency equal to the primary disturbance frequency was extracted from the
Figure 5.5. NLDE results from K-Type, laminar-flow transition scenario. Iso-volumes of the $\lambda_2$ criterion (calculated for total velocity, $U + \bar{u}$) colored by mean streamwise velocity are shown. From top left: the two-dimensional disturbance being modulated by the spanwise disturbance into three-dimensional structures, top right: development of the $\Lambda$ shaped vortices. Middle left: development of horseshoe vortices from $\Lambda$ shaped vortices, middle right: breakdown of the horseshoe vortices into turbulent spots. Bottom left: merging of turbulent spots to form turbulent flow, bottom right: complete disturbance development.
Figure 5.6. NLDE results from K-Type laminar flow transition scenario. The breakdown of the horseshoe vortices to turbulent flow is seen in this figure. The turbulent region is seen to have very slight characteristics of horseshoe vortices in it.

Figure 5.7. Skin friction coefficient plotted with Reynolds number (based on streamwise distance from the leading edge) for the K-type transition scenario. Laminar (Blasius) and turbulent (White [11]) correlations are plotted for reference. The NLDE results are spatially averaged across the span of the flat plate and time averaged for one full period of the primary disturbance.
Figure 5.8. Spikes in the skin-friction coefficient, collocated with legs of a passing horseshoe vortex are shown indicating the role of the legs of the vortices in increasing skin-friction on the wall.

Fourier results for the comparison in Figure 5.10. Figure 5.10 shows the comparison of the growth of the primary disturbance with Reynolds number (based on the distance from the leading edge). The log of the disturbance amplitude as a percentage of the freestream velocity is compared. The results show good agreement with the numerical results of Rist and Fasel. There is a difference in the results closer to the disturbance strip location. This is due to slightly downstream location of the disturbance strip in the current setup than in Rist and Fasel’s simulation. However, downstream growth of the disturbance is found to be in good agreement.

5.1.2.1 Multiresolution Analysis

The results from the K-type transition scenario are analyzed with the multiresolution part of the solver. Figures 5.11 and 5.12 show the results from two time steps of the simulation. The upper half of Figure 5.11 shows the iso-volumes of large-scale wavelet detail coefficient (c2) and the lower half shows the small-scale wavelet detail coefficient (c0). Both sets of volumes are formed for the same absolute range of values. This shows an early time step in the simulation where the Λ vortices have
Figure 5.9. Skin friction coefficient plotted with $x$-based Reynolds number for the K-type transition scenario. Laminar (Blasius) and turbulent (White [11]) correlations are plotted for reference. The NLDE results at three different times during the simulation are plotted. The spikes in the skin friction coefficient are seen to move towards the end of the domain. This is attributed to passing of the horseshoe vortices as indicated in Figure 5.8.

Figure 5.10. The log of disturbance amplitude as a percentage of the freestream velocity is plotted against Reynolds number based on distance from the leading edge.
started forming into horseshoe vortices. It can be seen that the large scale (low frequency) coefficient identifies the disturbance structures in the flow. The small scale (high frequency) coefficient only shows some magnitude around the edges of the laminar domain where horseshoe structures are formed. This is consistent with the results in Chapter 4, where large scale sinusoidally varying velocities were seen to have high large scale coefficient and the turbulent region showed an equivalent distribution of scale coefficients.

Figure 5.12 shows the iso-volumes of the large-scale coefficient (top) and the small-scale wavelet detail coefficient (bottom). Both the volumes are formed for the same absolute range of values. This figure shows a time step in the simulation after the breakdown to turbulence has occurred. It can be seen that the large scale (low frequency) coefficient identifies the disturbance structures in the flow before and after the horseshoe vortices are formed. The turbulent spots after the formation of the horseshoe vortices are also clearly seen with the presence of large-scale coefficients. The small scale (high frequency) coefficient only show structures at and after the formation of the horseshoe vortices. The absence of high frequency components in the pre-transitional region bolsters the confidence in the application of wavelet based multiresolution analysis in the differentiation of regions with laminar/turbulent coherent structures.

The capability of the solver in resolving complex laminar-flow disturbances within reasonable computational resources is well demonstrated with this example. Further improvements can be achieved in the resolution of the disturbances by enabling adaptive meshing of the solution. The following section explores another application area for the NLDE. The scale of the disturbance in the aerodynamic responses of gusts can be much bigger than the laminar-flow stability problem. The application of the NLDE solver to the prediction of gust-response is explored in the following sections.
Figure 5.11. Top: Iso-volumes of large-scale, wavelet detail coefficient before breakdown to turbulence. Bottom: Iso-volumes of small-scale wavelet detail coefficient before breakdown to turbulence. Both are developed with the same absolute range of values.
Figure 5.12. Top: Iso-volumes of large-scale, wavelet detail coefficient after breakdown to turbulence. Bottom: Iso-volumes of small-scale wavelet detail coefficient after breakdown to turbulence. Both are developed with the same absolute range of values.
5.2 Gust Response

The application of URANS methods to flow scenarios with high-frequency gusts can be a challenge, as the turbulence models result in excessive dissipation of the gust. This results in a highly attenuated aerodynamic response. The application of the NLDE solver to remedy this scenario is explored in this section. The NLDE solver is applied here for low-amplitude, high-frequency gusts, which can be especially challenging to resolve with reasonable computational resources. The cases discussed here were run for a NACA0018 symmetrical airfoil at $Re = 300,000$ and $\alpha = 2^\circ$ and a NACA0012 symmetrical airfoil at $Re = 1,000,000$ and $\alpha = 0^\circ$.

![Figure 5.13. Airfoil (chordlength = $c$) in a steady velocity, $\bar{U}$ is simulated in an incompressible steady solver and the resulting flowfield is used as input to the NLDE solver with a sinusoidal vertical gust (disturbance) $\tilde{v}$. The gust is propagated towards the airfoil due to the base flowfield from $2c$ distance ahead of the airfoil leading-edge.](image)

5.2.1 Computational Setup

The block-coupled version of the NLDE solver was used for this simulation. Figure 5.13 shows the initial setup for the NLDE solver. The figure shows the prescribed sinusoidal, vertical, gust velocity, $\tilde{v}$ in the base flowfield with the mean freestream velocity, $\bar{U}$. The converged results from the base-flow solver were used in the NLDE solver, where the sinusoidally-varying vertical gust (disturbance) was propagated towards the airfoil from $2c$ ($c$ is the chord-length of the airfoil) distance ahead of the leading-edge of the airfoil. The computational domain extends $10c$ in the front and $15c$ behind the airfoil. The top and bottom extents of the domain are $25c$ apart.
with the airfoil in the middle. The computational domain is two-dimensional with 455,000 cells for the NACA0018 case and 275,000 cells for the NACA0012 airfoil. In the simulations with NACA0018 airfoil, a steady, incompressible, fully-turbulent base-flow solver was used to obtain the mean flow quantities for the NLDE solver. The simulations with the NACA0012 airfoil were run with both laminar and turbulent base flow solvers. The temporal resolution in the case of the NLDE simulations was kept high to capture the unsteady aerodynamic response of the airfoil and higher harmonics (relative to the gust frequency) response. In case of the highest frequency gust, the temporal discretization was such that the one cycle of the input gust was discretized 650 times in time. For the lowest frequency gust, the temporal discretization was approximately 530 times per one cycle of the gust. The gust is defined as a more localized perturbation in the flow with one single cycle of the sine wave extending two chord-lengths in height from each side of the airfoil, in place of a true "gust front", which stretches to the full extent of the domain’s vertical direction. It was also assumed to be more realistic for a small scale gust, when applied in such localized manner. The amplitude of the gust was prescribed to be 5% of the mean freestream velocity for all the cases discussed here except the case where effect of amplitude is explored. The gust travels at freestream velocity towards the airfoil due to the base flowfield.

5.2.2 Discussion of Results

5.2.2.1 Frequency Domain

Figures 5.14 and 5.15 show the gust response amplitudes compared between results from the NLDE solver, Sears function, Theodorson function (for reference) and the thickness-corrected Sears function from Lysak et al. [107](Lysak’s correction is meant for airfoils with an elliptical leading edge). The $\Delta c_l$ results from the NLDE solver are time dependent and the maximum $\Delta c_l$ from the NLDE results is used in each case for comparison. The lift coefficient results from the NLDE solver are normalized with $2\pi c_0/U_\infty$ to compare with the analytic response. Figure 5.14 shows the comparison for the NACA0018 airfoil at $Re = 300,000$ and $\alpha = 2^\circ$. The results at $k = 1.182$ in Figure 5.14 are from multiple runs for different amplitude levels of the gust. The effect of the different gust amplitudes is discussed in a following
The lift response shown in this figure shows good agreement with the theory at high reduced frequencies ($k = 2.364$ & $k = 4.75$). At lower reduced frequencies, the results start to deviate away from the theory. At $k = 0.591$, the response shows much higher attenuation than the prediction from theoretical methods. Figure 5.15 shows the same comparison for a NACA0012 airfoil at $Re = 1,000,000$ and $\alpha = 0^\circ$. There are two sets of results shown in this figure with different base flow conditions, laminar and turbulent. For each of these cases, two different $\Delta c_l$ are presented. In one case the maximum $\Delta c_l$ is used for comparison, the other case uses the mean of the two extremes (maximum $\Delta c_l$ and absolute value of minimum $\Delta c_l$) for the comparison. Both $\Delta c_l$ values are again normalized with $2\pi c\bar{\nu}/U_\infty$. The positive value of difference between the maximum and the averaged $\Delta c_l$ values confirms that the amplitude of the minimum $\Delta c_l$ value is lower than the maximum. This is seen to be consistent for different frequencies and although not shown (can be observed in the time domain results presented in the following section), the same behavior was also observed in the NACA0018 cases. This argues for the physical nature of
Figure 5.15. NLDE results compared with analytic models for a NACA0012 airfoil at \( Re = 1,000,000 \) and \( \alpha = 0^\circ \). Circles denote simulations ran with turbulent base flow and '+' denote simulations with laminar base flow solution. Black symbols denote response function amplitudes calculated with average amplitude of the lift coefficient response and red symbols denote the response function amplitudes calculated with the maximum amplitude of the lift coefficient response.

This effect rather than numerical. This can be explained by the inability of the aerodynamic response to react at the same pace as the oncoming gust frequency. Since the initial portion of the gust changes the lift in one direction, the reversal of the force direction is not caught up by the aerodynamic response of the airfoil as rapidly as the direction of the oncoming gust is changed. However, this hypothesis relies on the frequency of the gust. This should mean that with increasing frequency, the aerodynamic response should have higher discrepancy between the two extremes of the response to a sinusoidal gust. This can be verified from Figure 5.16, where the higher frequency results are seen to have higher relative discrepancy between the two extremes. The figure shows the percentage difference in the two extremes relative to the maximum \( \Delta c_l \) for different reduced frequencies. It is also observed that the behavior of the NLDE solver in the results in Figure 5.15 is similar to the NACA0018 cases. The lower reduced frequencies show higher damping in the lift response than is predicted by the theory. The consistency of this behavior in both
Reduced frequency $k$ 

Figure 5.16. Relative percentage difference in $\Delta c_l$ values for different reduced frequencies for NACA0012 airfoil at $Re = 1,000,000$ and $\alpha = 0^\circ$.

Figures 5.14 and 5.15 suggests that this effect is not attributed to Reynolds number, thickness or angle of attack. This can be attributed to the breakdown of the NLDE assumption at lower frequencies, that the steady and unsteady part of the flows can be treated separately with only one-way coupling between them. The peculiarities of the aerodynamic response seen in this section can be better understood with their inspection in the time domain. The following section sheds more light into the topic.

5.2.2.2 Time Domain

The response of an airfoil to the time dependent sinusoidal gust can be better understood with the help of time domain results showing the aerodynamic response of the airfoil with time. This section presents the time domain results in terms of reduced time, $s$ where time, $t$ is non-dimensionalized with half the chordlength of the airfoil and the freestream velocity using the relation $s = U_\infty t/(c/2)$. The comparisons shown in this section are performed with alignment of the first peak of the response since the actual time of the impact can be different for different cases.
and is hard to ascertain. This is due to the influence of the gust starting much before the actual gust front is encountered. The results discussed in this section are for a 5% amplitude gust only. Figures 5.17 and 5.18 show the response of the NACA0018 airfoil (at $Re = 300,000$, $\alpha = 2^\circ$) at two different reduced-frequencies. The $c_l$ response of the airfoil is compared with theoretical predictions. Figure 5.17 shows the response of the airfoil at $k \simeq 0.06$ compared with Sears function. Figure 5.18 shows the response of the airfoil at $k \simeq 2.4$ compared with Sears function and Sears function with Lysak’s thickness correction. The general response of the airfoil in both the cases follows the shape of the oncoming gust. However, the amplitude of the response is less than is predicted by the theory and the frequency response of the airfoil is lower than the input-gust frequency. The amplitude of the first half of the response is seen to match closely to Lysak’s corrected Sears function in case of the higher frequency case, shown in figure 5.18. This confirms that part of the effect can be due to the finite thickness of the airfoil. The NLDE solver also encounters the boundary layer of the base flow (along with laminar viscous terms in the NLDE) introducing viscous effects into the response, which is missing in most gust modeling techniques due to their potential flow formulation. This may explain the lagged behavior (apparent decrease in frequency of the response at higher gust frequencies) of the NLDE results as compared to the analytic models as well as some contribution to the decrease in maximum amplitude of the response due to viscous dissipation. The effect of the base flowfield on the propagating gust results in slight distortion of the gust and this can be seen in all the time domain results where the gust response is seen to start before and after the gust front actually hits the airfoil. Figures 5.19, 5.20 and 5.21 show the response of the NACA0012 airfoil (at $Re = 1,000,000$, $\alpha = 0^\circ$, turbulent base flow) in terms of $\Delta c_l, \Delta c_d$ and $\Delta c_m$, coefficients at different reduced frequencies. The difference between the magnitude of the two peaks is clearly visible in all the results in these figures and the conclusion from the frequency domain results are further bolstered. The decrease in the drag was mostly attributed to higher increase in favorable (lower) disturbance-pressure in the front (close to the leading-edge of the airfoil) of the airfoil on one surface of the airfoil than the increase in adverse disturbance-pressure on the front of the other surface. The slight increase in skin friction drag was offset by the decrease in pressure drag. The drag is seen to rise during the change of direction of the gust. This effect is more pronounced at the higher frequencies where this change
happens more rapidly. This can be seen in Figure 5.20 where the higher frequency results show a higher increase in $\Delta c_d$ than the lower frequencies. During this phase, both surfaces show a buildup of adverse pressure gradient due to the disturbance pressure towards the front of the airfoil.

Figures 5.22 and 5.23 compare the response of the NACA0012 airfoil at two different reduced frequencies for laminar versus turbulent base flow. The frequency of the input gust is not the critical frequency of the stability characteristics of the boundary layer of the base flow conditions. The general behavior is seen to be similar in both the cases. However, this is expected to change if the frequency of the oncoming gust were aligned with the critical frequencies of the laminar boundary layer of the base flow solution. As was observed in the results from the laminar flow instabilities section, the critical frequency would trigger a rapid increase in skin friction at the wall due to rise in disturbance magnitude. This was not seen to happen for the gust frequencies that were tested.

5.2.2.3 Gust Amplitude

The effect of different gust amplitudes was explored for the NACA0018 airfoil at $Re = 300,000$, $\alpha = 2^\circ$ with turbulent base flow conditions. The cases were run for gusts of 0.5, 5 and 50% of the freestream velocity at a reduced frequency of $k \approx 1.2$. The results of the gust response can be seen in Figure 5.24. The response here is quantified in terms of the normalized $\Delta c_l$ (divided by $2\pi c \bar{v}/U_\infty$). The response is seen to attenuate as higher gust amplitudes are considered. Part of this effect can be attributed to the inability of the flow to respond rapidly to the gust. For higher gust amplitudes, the response needs to adjust even quicker for higher $\Delta c_l$ at the same reduced frequency. The high amplitude of the gust leads to higher skin friction and potential for separation as well. The higher dissipation due to higher amplitudes and the potential for separation can be the reason behind the decayed response of the airfoil. The case with the highest amplitude (50% of the mean freestream velocity) gust passing over the airfoil has some equivalence to high amplitude pitching of the airfoil. Similar situations arise in many practical scenarios such as the high frequency pitching of the blade on a helicopter. The response from the NLDE results for this case show dynamic stall type behavior. However, it is hard to interpret the physical significance of this result since the
Figure 5.17. $\Delta c_l$ response comparison from NLDE and Sears function for a NACA0018 airfoil at $Re = 300,000$ and $\alpha = 2^\circ$ encountering a sinusoidally varying vertical gust of 5% amplitude (relative to mean freestream velocity) with a reduced frequency, $k = 0.591$

base flow field is still attached. The instantaneous snapshots of the case, shown in Figure 5.25 and 5.26, show $\lambda_2$ criterion contours to highlight the separated flow vortices passing over the airfoil as the first half of the sinusoidal gust passes over the airfoil. Figure 5.25 shows the airfoil with background total flowfield (mean + gust) overlaid with $\lambda_2$ contours along with the instantaneous normalized $C_p$ (calculated using base pressure field combined with disturbance pressure field, normalized with the base dynamic pressure). Figure 5.26 shows the same instant with the same flowfield combined with base and total (mean + gust) wall shear stresses on the right. The wall shear stress for both upper and lower surface are shown, with and without the effect of the gust. This type of case where the disturbance variables show separation type behavior can be included in case where the assumption of linear superposition of the base flow and disturbance breakdown. This type of case might be considered more physically relevant if the NLDE solver was two-way coupled with the base solver.
Figure 5.18. $\Delta c_l$ response comparison from NLDE and Sears function for a NACA0018 airfoil at $Re = 300,000$ and $\alpha = 2^\circ$ encountering a sinusoidally varying vertical gust of 5% amplitude (relative to mean freestream velocity) with a reduced frequency, $k = 2.364$.

Figure 5.19. Comparison of $\Delta c_l$ for a NACA0012 airfoil at $Re = 1,000,000$ and $\alpha = 0^\circ$ at different reduced frequencies for sinusoidal gusts of 5% amplitude.
Figure 5.20. Comparison of $\Delta c_d$ for a NACA0012 airfoil at $Re = 1,000,000$ and $\alpha = 0^\circ$ at different reduced frequencies for sinusoidal gusts of 5% amplitude.

Figure 5.21. Comparison of $\Delta c_m$ for a NACA0012 airfoil at $Re = 1,000,000$ and $\alpha = 0^\circ$ at different reduced frequencies for sinusoidal gusts of 5% amplitude.
Figure 5.22. Comparison of $\Delta c_l$ for a NACA0012 airfoil at $Re = 1,000,000$ and $\alpha = 0^\circ$ at different reduced frequencies for sinusoidal gusts of 5% amplitude. The comparison is done for laminar and turbulent base-flow conditions.

Figure 5.23. Comparison of $\Delta c_d$ for a NACA0012 airfoil at $Re = 1,000,000$ and $\alpha = 0^\circ$ at different reduced frequencies for sinusoidal gusts of 5% amplitude. The comparison is done for laminar and turbulent base-flow conditions.
Figure 5.24. NLDE results for sinusoidally varying vertical gusts of different amplitudes at $k = 1.182$ for a NACA0018 airfoil at $Re = 300,000$ and $\alpha = 2^\circ$.

The results from the two application cases show merit to acknowledge the developed solver as a viable solution to calculate these unsteady flow scenarios. The following chapter presents the conclusions from this research effort and potential future work relevant to further this research.
Figure 5.25. NACA0018 airfoil at $Re = 300,000$ and $\alpha = 2^\circ$ encountering a sinusoidally varying vertical gust of 50% amplitude. Left: total velocity magnitude flowfield overlaid with contours of $\lambda_2$ criterion, Right: Instantaneous normalized $C_p$ on the airfoil.

Figure 5.26. NACA0018 airfoil at $Re = 300,000$ and $\alpha = 2^\circ$ encountering a sinusoidally varying vertical gust of 50% amplitude. Left: total velocity magnitude flowfield overlaid with contours of $\lambda_2$ criterion, Right: Instantaneous wall shear stress coefficient on both the surfaces of the airfoil with (blue) and without (red) the gust.
Chapter 6  
Concluding Remarks

A multiscale, adaptive, nonlinear disturbance equations solver was developed in this work. The solver was aimed at providing an alternative method to resolve unsteady fluid mechanisms in a computationally affordable manner. The solver was implemented in a finite-volume, open-source framework to enable the capability of computation in complex domains and ease of expansion of capabilities respectively. Multiscale capabilities using wavelets were also added to the solver. In this chapter, the conclusions, contributions and potential future work related to this research effort are summarized.

The NLDE solver developed in this research effort was validated using results from the Orr-Sommerfeld equation. The results from the solver agree well with the Orr-Sommerfeld results confirming the validity of the NLDE results. The finite-volume formulation used in this work employed Rhie-Chow interpolation. The addition of the interpolation allows the use of collocated grids. The agreement of the results from the NLDE solver with theory as well as good resolution of disturbance structures in the K-type transition simulation exhibit the applicability of Rhie-Chow interpolation to the solver. It confirms that the error due to the interpolation does not render the results unworthy. The multiscale framework implemented in the solver was shown to perform well in locating the disturbance structures for adaptive mesh refinement (AMR). The framework was also shown to have the potential for detecting turbulent regions. This type of capability can be helpful for future two-way coupling of the solver. This confirms the value of multiscale modeling for improving the computational efficiency of the model and the potential for transition to turbulence modeling.
The application of the NLDE solver for laminar-flow stability and gust modeling was explored. The potential of the solver in resolving complex transition phenomena was explored with K-type transition simulation. The solver performed well in resolving the behavior of interacting instabilities, their evolution, secondary structures as well as breakdown to turbulence. The solver showed potential as an alternative to DNS calculations for this type of simulations with much lower computational requirements. The results from the solver in resolving an airfoil’s response to sinusoidal vertical gusts showed good agreement with Sears function (and Sears functions with Lysak’s correction). However, the NLDE results are not limited by thin-airfoil theory assumptions, thus the results carry more information than the classical theoretical results. In both the application cases, however, the limiting cases were also seen where the solver is not able to match results very well with other methods. In case of transition, the solver was unable to resolve the turbulent region well. This shortcoming was assumed to emanate from lack of two-way coupling between the mean-flow and the NLDE solver, the assumption of linear superposition of the mean and fluctuating components of the flow (Reynolds decomposition) and from the lack of non-intrusive boundary conditions. In case of gust modeling, the low-frequency and high-amplitude gusts were seen to produce results that did not match results from the theoretical models as well as the high-frequency/low-amplitude cases. This can again be attributed to the assumption of Reynolds decomposition.

### 6.1 Contributions

This research effort has presented the development of a novel incompressible NLDE solver in both coupled and segregated solver frameworks. The solver developed in this work has also shown the applicability of Rhie-Chow interpolation with novel disturbance pressure-velocity coupling to study the evolution of unsteady disturbances. The multiscale capabilities developed for the solver were shown to detect coherent structures for adaptive mesh refinement purposes. The solver was successfully applied to solve cases in the areas of gust modeling and laminar flow stability.
6.2 Future Work

The current model provides a proof of concept to develop this solver further. The mean-flow solvers utilized in this work assumed incompressibility. The application of the incompressible NLDE solver with compressible mean-flow cases as well as compressible versions of both solvers should be explored in the future. The current disturbance framework is anisotropic, however, the multiscale decomposition is based on the assumptions of isotropy and a one-dimensional computational stencil for wavelets is used. This limitation is very easily removed and should be pursued in the future. The turbulent zone can also benefit from a scale dependent turbulent model, since the NLDE solver provides the disturbance information which was not available previously. This can be explored for the development of an adaptive turbulence model enabling two-way coupling with the base-flow solution. The influence of mesh refinement/movement on the physical significance of the scale based criterion (AMR or transition) also needs to be explored in the future. The well-known compression abilities of wavelets can be used to reduce memory overhead for the coupled solver. The reduced memory requirements of the NLDE solver, especially in case of the segregated solver, can be advantageous in using graphics processing units (GPUs) for computation. This can further reduce the energy required to run computationally expensive flow simulations. With the reduction in computational effort and the finite-volume implementation, the solver can be employed to run complex, full-scale, real-world geometries. This can be very helpful in understanding flow-phenomena in real-world scenarios and should be pursued in the future. The wavelet method also has the potential of being applied in the time domain for adaptive temporal resolution as well.

In terms of applications, several suggestions to improve the prediction accuracy can be implemented. The use of wavelet based criterion for turbulence detection can be used for two-way coupling of the NLDE solver with the mean-flow solver. The Reynolds-decomposition of the primary flow variables can also be modified to introduce some nonlinearity and tighter coupling between the NLDE and the mean-flow solver. The multiscale criterion developed in this work may be directly
applied to the instantaneous flow variables in place of the decomposed variables to implement turbulence detection or AMR capabilities. The application of the solver to study complex transition delay techniques should be pursued. This can be very useful to understand and explore laminar-flow-control technologies. In case of gust modelling, a distribution of frequencies, in place of a single frequency gust should be explored. This would help simulate more realistic scenarios for wake interaction and turbulent boundary-layer ingestion. The research from this work will be useful in extending these capabilities in the future versions of the solver.

In summary, a multiscale, adaptive, nonlinear disturbance equation solver was developed in this research effort. The NLDE solver, implemented in a finite-volume setting with both segregated and coupled solver, paves the way to a variety of numerical solvers and different grid types to be easily employed. This can enable high-fidelity disturbance calculations in complex physical domains within reasonable computational cost. The NLDE solver inherits multiresolution capabilities of wavelets and adapts them for NLDE calculations. The lower spatial resolution requirements of the nonlinear disturbance equations and adaptive wavelet filters provide high computational efficiency along with high-fidelity details of unsteady flowfields. The multiscale framework is seen to effectively identify regions to increase the spatial resolution and well as identify regions of breakdown of the instabilities. This capability enables the solver to predict complex physical phenomena without any dependence on empirical correlations. The results compare well with theory and the solver is able to predict different levels of disturbances with accuracy, as is evident from the two different application cases explored in this work.
Appendix  
The History of Transition

1 Early Transition discovery

In 1883, Osborne Reynolds became the first person to experimentally demonstrate the presence of “direct” and “sinuous” states of flow [56], which are now known as laminar and turbulent flow respectively. He was also the first person to acknowledge the effect of initial freestream disturbances and surface roughness on the process of transition from a laminar to a turbulent state. Reynolds concluded that once the flow reaches a critical value of the combination of certain parameters (now known as critical Reynolds number), the orderly pattern of laminar flow ceases to exist and the flow becomes turbulent, which is characterized by significant randomness and disorder as compared to laminar flow.

Later, in the early 20th century, Prandtl [63] studied transition in boundary layers. Prandtl [63] emphasized the role of viscosity in the theories describing the stability of the flow. He realized that the neglect of viscosity was the reason for the discrepancy between Rayleigh's stability theory [64] and Reynolds's experiments [56]. Following the same train of thought, in 1929 Tollmien [77] solved the viscous stability equation (Orr-Sommerfeld equation [27, 28, 79]) for a flat plate. The major assumption in the solution was that of parallel flow, which ignored the variations in mean flow in the streamwise direction. Solving the equation under this assumption, Tollmien found the neutral boundary for obtaining the critical Reynolds number, but the results did not match well with experimental values of actual transition locations. Schlichting [78] realized that a certain degree of
amplification of the instability must be reached before the flow becomes turbulent, rather than just at the point of instability (points on the neutral curve). He calculated the magnitude of the instability parameters in the interior of the neutral stability curve, i.e., in the unstable region, for a flat plate and tried to match the experimental results for transition.

Even with all of the attempts to explain transition, there was little acceptance
of these theories until the historic experiment of Schubauer and Skramstad [12]
demonstrated the presence of the instabilities and their amplification towards the
point of transition. The high turbulence level of wind tunnels at that time was the
major reason why the instabilities were not observed earlier, as well as the fact that
there was little agreement between the theory and experimental results. The low
turbulence intensity levels of the tunnel used by Schubauer and Skramstad made it
possible to see the emergence of instabilities in the flow with the help of hot wire
anemometry, as shown in figure .1.
Bibliography


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Biography

Amandeep Premi was born in Chandigarh, India. He obtained his bachelor’s degree in Mechanical Engineering from the Delhi Technological University (formerly Delhi College of Engineering) in 2007. Following that, he worked in a thermal power plant design firm in New Delhi, India before he decided to pursue his graduate studies. He obtained a master’s degree in Aerospace Engineering from The Pennsylvania State University in 2011. His work was focused on experimental aerodynamics and turbulence characterization of wind-tunnel facilities. He continued his research interests in transition to turbulence using computational methods for his later work. Amandeep also worked as a consultant for wind-tunnel characterization and development with Skidmore, Owings and Merrill LLP (2015).

Amandeep is a competitive weightlifter and has competed in several USAW (USA Weightlifting) sanctioned weightlifting meets. He is also an avid hackathon participant and has participated in many hackathons during his graduate school life.

Selected Publications

