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RIGIDITY FOR ABELIAN ACTIONS WITH RANK 1 FACTORS

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Abstract

Given the linear endomorphisms $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ on $\mathbb{T}^2$ and commuting perturbations f of A and g of B, we build a simultaneous smooth conjugacy between (f, g) and $(\hat{A}, B)$ where $\hat{A}(x, y) = (2x + \beta(y), 2y)$ for some smooth $\beta : \mathbb{T}^2 \to \mathbb{R}$. This shows rigidity of commuting maps with rank 1 factors beyond what was expected.

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Chapter 1 Introduction

Rigidity results are central in the study of dynamical systems. In general, when maps act in some way on a space, we want to know whether maps close to them in some sense, share their dynamical properties. The weakest form of this is topological conjugacy, and we are interested in finding conjugacies that preserve Hölder continuity, or even better, smooth structure. In particular, maps on tori are a rich field of study for such results.

In a classic result, Shub proved (1967) that homotopic expanding maps are topologically conjugate. In many cases this can be extended to Hölder conjugacy, but is not enough for smoothness.

Katok and Spatzier studied rigidity properties of $\mathbb{Z}^k$ and $\mathbb{R}^k$ hyperbolic actions. They proved Hölder and smooth rigidity results using cohomology and the Livsic theorem, as this paper intends to do.

Fisher, Kalinin, and Spatzier proved several results on rigidity of Anosov actions on tori and nilmanifolds [1]. Building on their work, in 2014, Wang and Rodriguez Hertz proved that C^∞ Anosov $\mathbb{Z}^r$ actions on tori, with no rank 1 factors are C^∞-conjugate to actions by automorphisms [2]. In this paper, we deal with particularactions of tori that do split into rank 1 factors.

Along the proof we will use in several instances the philosophy behind Livsic theorem that is that if we want a cocycle look in a particular form then we see this form happens along periodic orbits.

1.1 Setting

Let

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

We think of A and B as acting on $\mathbb{T}^2$. Observe that $AB = BA$.

Let f be a small $C^{1+\delta}$ perturbation of A and g a small perturbation of B (same differentiability) such that $f \circ g = g \circ f$.

To be more precise, let $\mathcal{U}$ be a neighborhood of A in the C^1 topology of functions $\mathbb{T}^2 \to \mathbb{T}^2$. We will assume that $f \in \mathcal{U}$ and the distance between f and A will be determined in the later proof. The neighborhood should be small enough for the functions f and A to be homotopic. We take a similar neighborhood of B.

Theorem 1. *There exists smooth $H : \mathbb{T}^2 \to \mathbb{T}^2$ (C^1) and $\beta : \mathbb{T}^2 \to S^1$ such that*

$$H \circ f \circ H^{-1} = \hat{f}, \quad \text{and} \quad H \circ g \circ H^{-1} = B$$

where

$$\hat{f}(x, y) = (2x + \beta(y), 2y).$$

Although we would like for $\hat{f}$ to equal A, this is not possible in general if we want to have a smooth H.

It is important to get rigidity in this case that though the matrix A does not give any difference to the x and y coordinates, the matrix B identifies the x direction very explicitly, though it does not determine a y direction.

Chapter 2 Preliminaries

Definition 1. *Let M be a smooth manifold with $f : M \to M$ a C^1 map on M. f is called an expanding endomorphism if the relation*

$$|(D_x f^n)v| > |v|$$

holds for some $n \in \mathbb{N}$ and for all $x \in M$ and $u \in T_x M$.

Expanding maps have several properties that will be useful for us. In particular, we will need the following theorem.

Theorem 2. *[3] If f is an expanding endomorphism of a closed manifold M, and the derivative $D_x f$ is a Hölder-continuous function of x, there is a unique f-invariant probability measure μ absolutely continuous with respect to λ, the Lebesgue measure.*

We also have the smoothness of the Radon-Nikodym derivative $\frac{d\mu}{d\lambda}$ as proven in [4] and [5].

Since f and A are close in the C^1 norm, and A is expanding - $(D_x Av) = Av = 2v$ - f is expanding as well. Thus, f has an invariant measure μ such that $\frac{d\mu}{d\lambda}$ is smooth.

Theorem 3. *(Moser) [6] Let M be a smooth compact orientable manifold and Ω_0 and Ω_1 be two volume forms on M with the same total volume: $\int_M \Omega_0 = \int_M \Omega_1$. Then there exists a diffeomorphism f such that $f^*\Omega_1 = \Omega_0$.*

So by conjugating with a smooth map we may assume without loss of generality that f preserves area λ.

We will also need another property of expanding endomorphisms.

Theorem 4. *[7] (Shub) Let*

$$f : M \to M$$

and

$$g : M \to M$$

be two homotopic expanding endomorphisms. Then f and g are topologically conjugate and this conjugacy is unique among continuous maps homotopic to the identity map.

Since f is a small perturbation of A and both are expanding, this theorem applies to them. Let h be this conjugacy. That is, let $h : \mathbb{T}^2 \to \mathbb{T}^2$ be a function, homotopic to identity such that

$$h \circ f = A \circ h$$

.

Unlike A, B is an invertible map

$$B^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

Let $H : \mathbb{T}^2 \to \mathbb{T}^2$ be defined by $H = B^{-1} \circ h \circ g$. Then,

$$H \circ f = B^{-1} \circ h \circ g \circ f = B^{-1} \circ h \circ f \circ g = B^{-1} \circ A \circ h \circ G$$

To make use of uniqueness, we need to show that this map is homotopic to the identity. Let $\tilde{H} : \mathbb{R}^2 \to \mathbb{R}^2$ be a lift of H to $\mathbb{R}^2$, and similarly $\tilde{h}$ be a lift of h.

To show that a H is homotopic to the identity it is enough to show that $\tilde{H}(x+n) = \tilde{H}(x)+n$ for every $x \in \mathbb{R}^2$ and $n \in \mathbb{Z}^2$. We know that $\tilde{h}(x+n) = \tilde{h}(x)+n$ and $\tilde{g}(x+n) = \tilde{g}(x) + Bn$ for every $x \in \mathbb{R}^2$ and $n \in \mathbb{Z}^2$ so

$$\tilde{H}(x+n) = B^{-1} \circ \tilde{h}(\tilde{g}(x) + Bn) = B^{-1}(\tilde{h}(\tilde{g}(x)) + Bn)$$

$$= B^{-1}(\tilde{h}(\tilde{g}(x))) + B^{-1}(Bn) = \tilde{H}(x) + n$$

$$= A \circ B^{-1} \circ \tilde{h} \circ G = A \circ \tilde{H}$$

using the fact that B^{-1} is linear.

This implies that, by uniqueness of h, that $H = h$. In other words, $h = B^{-1} \circ h \circ g$ or $B \circ h = h \circ g$.

And because h and B are invertible, $g = h^{-1} \circ B \circ h$ is also invertible.

We also have that h is Hölder continuous. The proof is as follows.

Proof. In this proof it will be useful, once again, to work in the universal cover of $\mathbb{T}^2$, $\mathbb{R}^2$. We take $\tilde{f} : \mathbb{R}^2 \to \mathbb{R}^2$ to be a lift of f to the real plane, and same with $\tilde{h} : \mathbb{R}^2 \to \mathbb{R}^2$. For convenience we will refer to the lifts by their regular names, f and h.

Because f is homotopic to A, and h is homotopic to the identity, we can write $f(p) = 2p + a(p)$ and $h(p) = p + u(p)$ where $p = (x, y)$, and a and u are $\mathbb{Z}^2$-periodic. h is a solution to the conjugacy equation $h \circ f = A \circ h = 2h$, so this equation becomes

$$f + u \circ f = I + 2u$$

and hence

$$I + a + u \circ f = I + 2u$$

which becomes

$$\frac{a}{2} = u - \frac{1}{2} u \circ f.$$

, where I is the identity map. This last equation has as solution the series

$$u = \frac{1}{2} \sum_{n \geq 0} \frac{1}{2^n} a \circ f^n$$

This series absolutely converges because a is bounded, and it is easy to see that it satisfies the equation. It remains to prove that this function u is Hölder continuous. For that, we shall use the fact that f is Lipschitz (f is a small C^1-perturbation of A, therefore its derivative is bounded by $2 + \epsilon$). The function a is also Lipschitz because $a(p) = f(p) - 2p$. Also a is bounded because it is periodic and continuous. Let L be the maximum of the three bounds: the Lipschitz constants of f and a, as well as the bound of a. Then

$$|f(p) - f(q)| \leq L|p - q|,$$

$$|a(p) - a(q)| \leq L|p - q|,$$

and

$$|a(p)| \leq L.$$

for every $p, q \in \mathbb{R}^2$. This implies that

$$|f^n(p) - f^n(q)| \leq L^n |p - q|,$$

so letting θ be some number $0 < \theta < 1$ we have

$$|u(p) - u(q)| \leq \frac{1}{2} \sum_{n \geq 0} \frac{1}{2^n} |a(f^n(p)) - a(f^n(q))|$$

$$= \frac{1}{2} \sum_{n \geq 0} \frac{1}{2^n} |a(f^n(p)) - a(f^n(q))|^{\theta} |a(f^n(p)) - a(f^n(q))|^{1-\theta}$$

$$\leq \frac{1}{2} \sum_{n \geq 0} \frac{1}{2^n} |a(f^n(p)) - a(f^n(q))|^{\theta} L^{1-\theta} \leq \frac{1}{2} \sum_{n \geq 0} \frac{1}{2^n} |L^{\theta}||f^n(p) - f^n(q)|^{\theta} L^{1-\theta}$$

$$\leq \frac{1}{2} \sum_{n \geq 0} \frac{1}{2^n} L^{\theta} L^{n\theta} |p - q|^{\theta} L^{1-\theta} = (\frac{L}{2} \sum_{n \geq 0} \frac{L^{n\theta}}{2^n}) |p - q|^{\theta}$$

so, take θ small enough so that $L^{\theta} < 2$ and hence we get that

$$C = (\frac{L}{2} \sum_{n \geq 0} \frac{L^{n\theta}}{2^n}) < \infty$$

and hence h is Hölder.

□

Recall that the goal of this paper is to find a smooth conjugacy that takes (f, g) to $(\hat{f}, B)$, where $\tilde{f}$ is close to A. We have found a conjugacy, h, which has this property, but h is only Hölder continuous. The above discussion also shows why we relaxed the requirement for H to conjugate f to A - as expanding maps, they have a unique conjugacy. In the case that h is not smooth, we would be unable to proceed. However, g and B are not expanding, so it is possible to find another conjugacy that exactly takes g to B. We will be using h later on to build H.

We will need a few lemmas about the derivatives of f and g and Lebesgue measure.

Lemma 1. $g_* \lambda = \lambda$ *i.e.* $\det D_p g = 1$ *for every* $p \in \mathbb{T}^2$.

Proof. Let $g_*\lambda = \mu$ then μ is absolutely continuous with respect to Lebesgue and

$$f_*\mu = f_*g_*\lambda = g_*f_*\lambda = g_*\lambda = \mu.$$

So μ is also $f-$invariant and absolutely continuous, but λ is the only such measure so $g_*\lambda = \lambda$. □

Lemma 2. *Let $\alpha : \mathbb{T}^2 \to X$ be continuous. Then $\alpha \circ B = \alpha$ if and only if α is independent of the first variable: $\alpha(x, y) = \alpha(0, y)$ for all $x, y \in S^1$.*

Proof. $\alpha \circ B = \alpha$ is equivalent to $\alpha(x + y, y) = \alpha(x, y)$. Clearly this is true for any function α that only depends on the second coordinate. For the converse, notice that for y irrational, the set $\{ky : k \in \mathbb{Z}\}$ is dense in S^1, $\alpha(x, y) = \alpha(x + ky)$ for all k, so $\alpha(x, y) = \alpha(0, y)$ for irrational y. The result for rational y follows from continuity of α. □

Lemma 3. *$h_*\lambda = dx \times \nu$ for some B-invariant ν.*

Proof. Let $\mu = h_*\lambda$, Then μ is a fully supported, hence non-atomic $A-$ergodic measure, also invariant by B. Any such measure μ is of the form $dx \times \nu$ where ν is some probability measure - this is clear from looking at the dynamics of B, which leaves each horizontal circle invariant. □

Lemma 4. *If $g^k(p) = p$ then the D_pg^k has both eigenvalues with modulus 1.*

Proof. We know that $\det D_pg^k = 1$ so if the eigenvalues of D_pg^k do not have modulus 1, then p is a hyperbolic fixed point. The Stable Manifold Theorem then states that the stable manifold at p is of dimension 1, and in particular $W^s_g(p) \neq \{p\}$. Then $h(W^s_g(p)) = W^s_B(h(p)) \neq \{h(p)\}$ but for B, $W^s_B(x) = \{x\}$ for every x. Here $W^s_g(x) = \{y : d(g^nx, g^ny) \to 0 \text{ as } n \to \infty\}$.

2.1 Foliation by circles

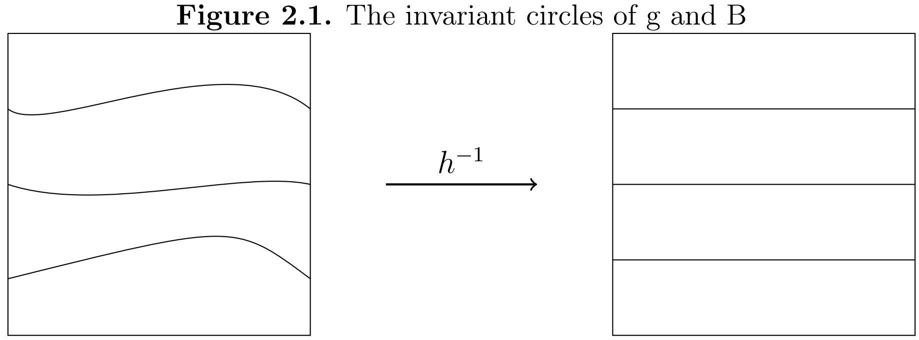
Since we have a Hölder continuous conjugacy $h = (h_1, h_2)$, we can use it to define the (Hölder) foliation by circles

$$\mathcal{C}(y) = h^{-1}(S^1 \times h_2(y)).$$

Note the dynamics of the map B. B keeps horizontal circles invariant, and acts as the rotation map $R_\theta : S^1 \to S^1$ where $R_\theta(x) = x+\theta$, on each circle $S^1 \times \{\theta\} \subseteq \mathbb{T}^2$. This conjugacy transfers that behavior onto the map g.

The analysis of these rotation maps is well-known. In particular, we have two distinct behaviors - for rational and irrational θ. For one, we will be using the classical result that irrational rotations of a circle are uniquely ergodic with respect to Lebesgue measure. In the following discussion, rational and irrational circles refer to the images of the circles with rational and irrational y-coordinates, respectively.

Figure 2.1. The invariant circles of g and B



$\mathbb{T}^2$ under g

$\mathbb{T}^2$ under B

Chapter 3
Matrix cocycles over abelian actions

The concept of cocycles is fundamental in the study of dynamical systems, particularly ones formed as skew products. Suppose $f : X \to X$ is a map; it can be thought of as a group, or more generally, semigroup action $T : \mathbb{Z} \times X \to X$, defined by $T(n, x) = f^n(x)$. A real-valued cocycle over this dynamical system is defined as a function $\alpha : \mathbb{Z} \times X \to \mathbb{R}$ that satisfies $\alpha(m+n, x) = \alpha(n, T(m, x)) + \alpha(m, x)$. In our case, dealing with dynamical systems over $\mathbb{Z}$ or $\mathbb{N}$, this is equivalent to defining a single function $\phi : X \to \mathbb{R}$. We can generate α from ϕ the same way we generate T from f. First, define $\alpha(1, x) = \phi(x)$. Then, the definition of cocycle can be expanded to

$$\alpha(n, x) = \alpha(n-1, T(1, x)) + \alpha(1, x) = \alpha(n-1, f(x)) + \phi(x)$$

$$= \alpha(n-2, f(f(x)) + \phi(f(x)) + \phi(x)$$

$$= \phi(f^{n-1}(x)) + \phi(f^{n-2}(x)) + \cdots + \phi(f(x)) + \phi(x)$$

.

For invertible f, this equation will also hold for negative n.

Some functions ϕ are of particular interest. Suppose $\Phi : X \to \mathbb{R}$ is another function; then it can form a cocycle by letting

$$\phi(x) = \Phi(f(x)) - \Phi(x)$$

Indeed, this implies that

$$\alpha(n,x) = \sum_{i=0}^{n-1} \phi(f^i(x)) = \sum_{i=0}^{n-1} [\Phi(f^{i+1}(x)) - \Phi(f^i(x))] = \Phi(f^n(x)) - \Phi(x)$$

This relation is called the cohomological equation, and functions ϕ that satisfy it are known as coboundaries.

We will also need the concept of cocycles with values in groups other than $\mathbb{R}$, in particular $GL(2,\mathbb{R})$-valued cocycles. Our operation here will be multiplication.

$$\alpha(n+m,x) = \alpha(m, f^n(x))\alpha(n,x)$$

Once again we will refer to not just the function α, but also the function F, where

$$F(x) = \alpha(1,x)$$

, as a cocycle.

With this definition, we can, just as with the additive case, recover $\alpha(n,x)$ from $F(x)$. We have

$$\alpha(n,x) = \alpha(n-1, T(1,x))\alpha(1,x) = \alpha(n-1, f(x))F(x)$$

$$= \alpha(n-2, f(f(x))F(f(x))F(x)$$

$$= F(f^{n-1}(x))F(f^{n-2}(x)) \cdots F(f(x))F(x)$$

.

A linear cocycle can be thought of as an automorphism of a vector bundle over a manifold [8]. A common example we will be using is the derivative cocycle. Let M be a manifold with a trivial tangent bundle, for example T^2 or any other Lie group. Let $f : M \to M$ be a diffeomorphism. Then for each $p \in M$ the derivative $D_p f$ is a linear transformation in some group of automorphisms of the tangent space, where tangent spaces at different points are canonically identified by the bundle. This gives us the map $F : \mathbb{T}^2 \to GL(n,\mathbb{R})$.

In a dynamical system, understanding the behavior of the periodic points can allow us to understand the whole system, through the cohomological equation and Livsic's Theorem.

Theorem 5. *(Real Livsic) [6] Let M be a Riemannian manifold, $U \subset M$ open, $f : U \to M$ a smooth embedding, $\Lambda \subset U$ a compact topologically transitive hyperbolic set, and $\phi : \Lambda \to \mathbb{R}$ Hölder continuous. Suppose that for every $x \in \Lambda$ such that $f^n(x) = x$ we have $\sum_{i=0}^{n-1} \phi(f^i(x)) = 0$. Then there exists a continuous $\Phi : \Lambda \to \mathbb{R}$ such that $\phi = \Phi \circ f - \Phi$. Also Φ is unique up to an additive constant and Hölder with the same exponent as ϕ.*

We will also need a version of this theorem for multiplicative cocycles with values in $GL(2, \mathbb{R})$.

Theorem 6. *(Remark 2 after Theorem 3) [9] Let f be a cocycle over a hyperbolic dynamical system $T : M \to M$, defined by function $F : M \to \Gamma$ where Γ is a Lie group which admits a bivariant metric. This cocycle is a coboundary if for every periodic point x with $T^n(x) = x$, we have $F^{(n)}(x) = I$.*

Note that here cohomology takes the form

$$F(x) = G(T(x))G(x)^{-1}$$

This theorem in particular works for any abelian Γ. We will be using the theorem for several subgroups of $GL(2, \mathbb{R})$. Kalinin [10] proved a generalization to the nonabelian group $SL(2, \mathbb{R})$ but the full result is not needed in this paper.

We have defined cocycles for discrete dynamical systems - that is, $\mathbb{Z}$ or $\mathbb{N}$-actions. We will also need to look at pairs of commuting cocycles, which can be thought of as a single cocycle from a different semigroup - in this case $\mathbb{N}^2$. So, let us define the most general form of the cocycle.

Let Γ be a group. If S is a semigroup and $\alpha : S \to End(X)$ is an action, a $\Gamma-$cocycle is a map $\gamma : S \times X \to \Gamma$ satisfying the cocycle equation:

$$\gamma(s_1 s_2, x) = \gamma(s_1, \alpha(s_2)(x))\gamma(s_2, x).$$

We will be interested in the case $S = \mathbb{N}^2$ and various choices of Γ.

Now let us use this to define commuting cocycles. Let $A(\bar{n})$ be an $\mathbb{N}^2$-cocycle, which splits into the maps $F, G : \mathbb{T}^2 \to \Gamma$ where $F = A(1, 0)$ and $G = A(0, 1)$. Similarly take the map $\alpha : \mathbb{N}^2 \to End(\mathbb{T}^2)$ with $\alpha(1, 0) = f$ and $\alpha(0, 1) = g$, our commuting maps from Theorem 1.

The cocycle equation becomes

$$A(\bar{n}+\bar{m}) = A(\bar{n}) \circ \alpha(\bar{m})A(\bar{m})$$

Which for $\bar{n} = (1,0)$ and $\bar{m} = (0,1)$ and vice versa, becomes

$$G \circ fF = A(\bar{n}+\bar{m}) = F \circ gG$$

3.1 Solution to cocycle problem

Assume β is Hölder and $\beta \circ B - \beta = \gamma \circ A - \gamma$ for some continuous γ. Assume also that

$$\int \beta(x,y)dx = 0$$

for every y.

Proposition 1. *Assume β is as above, then there is u Hölder continuous such that $u - u \circ A = \beta$ and we also have $\gamma = u - u \circ B + \int \gamma d\lambda$.*

Proof. Let $\mathcal{P}(A)$ be the set of $A-$invariant probability measures. By Livsic theorem we only need to show that $\int \beta d\mu = 0$ for every $\mu \in \mathcal{P}(A)$ (In particular, every periodic orbit has an invariant atomic measure supported on it, for which $\int \beta d\mu = 0$ is equivalent to the hypothesis of the real-valued Livsic Theorem). Of course we only need to do it for a dense subset of measures.

Let $\mathcal{Q}$ be the set of all measures giving zero measure to rational circles.

Claim 1: $\mathcal{Q}$ is dense in the set of invariant measures.

Proof. We only need to approach measures supported on a periodic orbits since these are dense. Since A has a Markov partition (with 4 atoms) we can make the argument for the shift in 4 symbols. Take a periodic orbit of period n and consider the induced iterated partition by the 4^n cylinders of length n. So that every point in the periodic orbit belongs to a different cylinder. Let $\epsilon > 0$ and let μ_ϵ be the Bernoulli measure for σ^n (the $n-$th iterate of the shift) that gives weight $\frac{1-\epsilon}{n}$ to the cylinders intersecting the periodic orbit and weight $\frac{\epsilon}{4^n - n}$ to each of the other cylinders. This measure is fully supported and ergodic (for σ^n), hence the associated measure on the torus is also fully supported and ergodic (for B^n), hence gives weight 0 to rational circles. As $\epsilon \to 0$ this measure goes to the measure supported over the periodic orbit (this can be shown by measuring the size of the cylinders of length k around the points in the periodic orbit for k a large multiple of n but fixed and sending $\epsilon \to 0$). Now, μ_ϵ is invariant by σ^n and not σ. Take $\nu_\epsilon = \frac{1}{n}\sum_{k=0}^{n-1} \sigma_*^k \mu_\epsilon$, it is invariant by σ and also converges to the measure supported over the periodic orbit and is fully supported and the associated measure gives 0 weight to rational circles. □

Claim 2: $\int \beta d\mu = \int \beta dB_*\mu$ for every $A-$invariant measure μ.

This follows from existence of γ.

$$\int \beta d\mu - \int \beta dB_*\mu = \int \beta - \beta \circ B d\mu = \int \gamma - \gamma \circ A d\mu =$$

$$\int \gamma d\mu - \int \gamma dA_*\mu = 0$$

Given μ define

$$\mu_n = \frac{1}{n}\sum_{k=0}^{n-1} B_*^k \mu$$

Claim 3: If $\mu \in \mathcal{Q}$ and ν is a limit of the sequence μ_n then $\nu \in \mathcal{Q}$ and there is a measure η such that $\nu = dx \times \eta$.

ν gives 0 measure to rational circles since the dynamics of B leave all horizontal circles invariant. Unique ergodicity of irrrational circles then gives the proof.

Finally

$$\int \beta d\mu = \int \beta d\mu_n = \int \beta d\nu = \int \beta(x,y) dx \times \eta = \int \left(\int \beta(x,y)dx \right) d\eta = 0$$

if $\mu \in \mathcal{Q}$.

Now uniqueness of solution of the cohomological equation up to a constant tells us that $\gamma = u - u \circ B + \int \gamma d\lambda$.

To prove this, let $\delta = u - u \circ B$. We have

$$\delta \circ A - \delta = (u - u \circ B) \circ A - (u - u \circ B)$$

$$= (u - u \circ A) \circ B - (u - u \circ A) = \beta \circ B - \beta$$

So since $\int \delta d\lambda = 0$ and δ satisfies the same cohomological equation as γ, for the function $\beta \circ B - \beta$, we must have $\delta = \gamma$.

□

Lemma 5. *Given any function $b : \mathbb{T}^2 \to \mathbb{R}$ there are functions $\hat{b}$ and b_0 such that*

$$b = b_0 + \hat{b}$$

where

$$\hat{b} \circ B = \hat{b}$$

and

$$\int b_0(x_1, x_2) dx_1 = 0$$

for every x_2.

Proof. Let $\hat{b}(x_1, x_2) = \int b(x_1, x_2) dx_1$. Notice that $\hat{b}$ is invariant on horizontal circles, which forces $\hat{b} \circ B = \hat{b}$. Also

$$\int b_0(x_1, x_2) dx_1 = \int (b - \hat{b})(x_1, x_2) dx_1 = 0$$

□

Chapter 4
Conjugacy for the derivative cocycle

4.1 Trivializing the derivative cocycle

Our strategy is to first bring the derivative maps to a more convenient upper-triangular form, by a cocycle-type relation.

Main Proposition 1. *There is a continuous $C : \mathbb{T}^2 \to SL(2, \mathbb{R})$ such that*

$$C(f(x))^{-1} D_x f C(x) = a(x) \begin{pmatrix} 1 & \hat{b}(x)+b_0(x) \\ 0 & 1 \end{pmatrix} \quad (4.1)$$

$$C(g(x))^{-1} D_x g C(x) = \begin{pmatrix} 1 & 1+c(x) \\ 0 & 1 \end{pmatrix} \quad (4.2)$$

where

$$a = 2\frac{\kappa}{\kappa \circ f}$$

for some Hölder continuous function $\kappa > 0$, $\kappa \circ g = \kappa$. Also

$$b_0 \circ g - b_0 = c \circ f - c$$

$$\hat{b} \circ g = \hat{b}, \qquad \int (b_0 \circ h^{-1})(x_1, x_2) dx_1 = 0$$

and

$$\int (c \circ h^{-1})(x_1, x_2) dx_1 = 0$$

for every x_2.

In fact, ideally we would have $a(x) = 2$ for all x, but because of a Livsic periodic orbit argument,

$$a = 2\frac{\kappa}{\kappa \circ f}$$

will suffice. Note that the last part follows automatically from Lemma 5.

The proof of Main Proposition 1 will happen in the next 3 sections.

4.1.1 KS trivialization

We will be heavily using a result by Kalinin and Sadovskaya, applied to the function f. The theorem only works for diffeomorphisms, so since f is not invertible, we cannot apply it as stated. So we will need to look at an invertible shift map that is conjugate to f, and apply the result back to f.

First, let us define the objects we will apply the theorem to.

Let

$$X_f = \{\underline{x} \in (\mathbb{T}^2)^{\mathbb{Z}} : f(x_i) = x_{i+1}\},$$

$\sigma : X_f \to X_f$ be the shift and $p : X_f \to \mathbb{T}^2$ be $p(\underline{x}) = x_0$. Observe that $p \circ \sigma = f \circ p$.

The map σ over X_f has a hyperbolic structure.

Notice that the fibers of this projection map form the local stable manifolds of the shift map $W^s_{loc}(\underline{x}) = p^{-1}(x_0)$.

The unstable manifolds $W^u(\underline{x})$ for $\underline{x} \in X_f$ that remain will have dimension 2 (they project down to $\mathbb{T}^2$, f is everywhere an expanding map) and can be canonically identified because the tangent bundle of the torus is trivial. So we can define the derivative cocycle.

The derivative cocycle over the shift map is

$$D\sigma(\underline{x}) = D_{p(\underline{x})} f$$

Recall that for a function F defining a cocycle over f, $F^{(n)}$ is defined by

$$F^{(n)}_x = F_{f^{n-1}x} \circ \cdots \circ F_{fx} \circ F_x$$

In the following theorem we define the Lyapunov exponents λ_+ and λ_- as follows.

$$\lambda_+(\underline{x}) = \lim_{n\to\infty} \frac{1}{n} \log \|D\sigma_{\underline{x}}^{(n)}\|$$

$$\lambda_-(\underline{x}) = \lim_{n\to\infty} \frac{1}{n} \log \|(D\sigma_{\underline{x}}^{(n)})^{-1}\|^{-1}$$

Theorem 7. *[8]*

Let $\sigma : X_f \to X_f$ and suppose that for every σ-periodic point $\underline{x}$ the Lyapunov exponents for $D\sigma$ of the orbit of $\underline{x}$ satisfies $\lambda_+(\underline{x}) = \lambda_-(\underline{x})$. Then there exist a flag of β-Hölder $D\sigma-$ invariant sub-bundles

$$\{0\} = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 = \mathbb{R}^2$$

and β-Hölder Riemannian metrics on the factor bundles $\mathcal{E}_i/\mathcal{E}_{i-1}$, $i = 1, 2$ so that for some positive β-Hölder function $\phi_i : M \to \mathbb{R}$ the factor-cocycles induced by the cocycle $\phi_i D\sigma$ on $\mathcal{E}_i/\mathcal{E}_{i-1}$ are isometries.

The eigenvalues of $D\sigma$ are thus the same as the eigenvalues of Df, which we will show are equal at periodic orbits. Thus this theorem applies to the space X_f.

4.1.2 Trivialization revisited

Our goal in the Main Proposition is to find a certain linear endomorphism in each tangent space; this is equivalent to specifying a subbundle $\mathcal{E}_1$.

Notice that $\mathcal{E}_1$ could be 0- or 2-dimensional in which case the cocycle $D\sigma$ is conformal, or it could be 1-dimensional. In that case, we have an invariant line for the matrix $C(x)$ which means that it can be conjugated to an upper-triangular matrix. This and the independence from x gives us a way of stating Theorem 7 in a way that will be useful to prove the Main Proposition.

Proposition 2. *(Consequence of 3.9 in [8]) If for every periodic orbit $f^n(p) = p$ the eigenvalues of $D_p f^n$ have same modulus, then there is a (Hölder) continuous $C : \mathbb{T}^2 \to GL(2, \mathbb{R})$ such that either*

1. *$C(f(x))^{-1} D_x f C(x)$ is conformal for every x,*

2. *$C(f(x))^{-1} D_x f C(x)$ is upper triangular for every x,*

The difference between Theorem 7 and Proposition 2 is that the statement in Theorem 7 says that the bundles and the Riemannian metric depends on the point $\underline{x} \in X_f$ rather than on the point in $\mathbb{T}^2$ what is expected in Proposition 2.

We have the map $p : X_f \to \mathbb{T}^2$, we need to show that these objects are constant on preimages of p, i.e. they are coherent along $W^s_{loc}(\underline{x})$ for $u_x \in X_f$. To this end we use Proposition 4.4 in [8] and the argument in the proof of Theorem 3.3 in [8].

4.2 Beginning of proof of Main Proposition 1

In this section we will start the proof of the Main Proposition that will take the next section also.

To prove Main Proposition 1 we first show that eigenvalues along periodic orbits are equal, thus we can apply Proposition 2. Then we show how Proposition 2 can be transformed into Main Proposition 1.

Lemma 6. *For every periodic orbit $f^n(p) = p$ the eigenvalues of $D_p f^n$ have same modulus.*

Proof. Let k be such that $g^k(p) = p$. Then

$$D_p f^n D_p g^k = D_p g^k D_p f^n.$$

By Lemma 4 the eigenvalues of $D_p g^k$ are both of modulus 1. If the eigenvalues of $D_p f^n$ are not of the same modulus then $D_p g^k$ is diagonalizable over $\mathbb{R}$ and hence $D_p g^k = \pm I$, by taking $2k$ we may assume that $D_p g^k = I$.

We will need the following linearization theorem by Hartman.

Theorem 8. *(Main Theorem in [11]) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a C^2 diffeomorphism such that $T(0) = 0$ and assume $D_0 T = L$ is a contraction then there is $H : U_0 \to \mathbb{R}^2$ (U_0 and neighborhood of 0) a $C^{1+\alpha}$ diffeomorphism such that $H_0 \circ T = L \circ H_0$. Moreover $H_0(0) = 0$ and $D_0 H_0 = I$. (α can be computed explicitly).*

Continuation of proof of Lemma 6 assuming f is a small perturbation of A.

Let $\lambda_2 < \lambda_1 < 1$ be the eigenvalues of $(D_p f^n)^{-1}$, if f is close enough to A we have that $\lambda_1^{1+\delta} < \lambda_2$ for some small δ which can be taken smaller that α (from

Theorem 8) . Let $T = \psi^{-1} \circ f^{-n} \circ \psi$ where ψ is a neighborhood chart such that $\psi(0) = 0$. Take H_0 fromTheorem 8 and let

$$G = H_0 \circ \psi^{-1} \circ g^k \circ \psi \circ H_0^{-1}.$$

Then $G \circ L = L \circ G$. Let $G = (G_1, G_2)$ and observe that $D_0 G = I$ since we are assuming $D_p g^{2k} = I$. Let $\alpha = (a, b)$ then

$$\lambda_i^n \partial^\alpha G_i = \lambda_1^{an} \lambda_2^{bn} \partial^\alpha G_i \circ L^n$$

Applying this to the case $\alpha = (1, 0)$ and $\alpha = (0, 1)$ we get that $G_1(x, y) = x$ for every x and $G_2(x, y) = y + u(x)$ for some $C^{1+\alpha}$ function u. Moreover

$$\lambda_2 u(x) = u(\lambda_1 x).$$

Taking derivatives we obtain that

$$\sigma u'(x) = u'(\lambda_1 x)$$

where $\sigma = \frac{\lambda_2}{\lambda_1}$. So

$$\left(\frac{\sigma}{\lambda_1^\alpha}\right)^n \frac{u'(x)}{x^\alpha} = \frac{u'(\lambda_1^n x)}{(\lambda_1^n x)^\alpha}.$$

Now

$$\frac{\sigma}{\lambda_1^\alpha} = \frac{\lambda_2}{\lambda_1^{1+\alpha}} > 1$$

and the right hand side is uniformly bounded since u' is $\alpha-$Hölder hence $u'(x) \equiv 0$. Since $u(0) = 0$ then $u \equiv 0$ and hence G is the identity map. This implies g^k is identity map which implies B^k is identity map and this is a contradiction.

The next lemma gives us more information of the derivative cocycle along a periodic orbit.

Lemma 7. *Let p be a periodic orbit for f, $f^n(p) = p$ then p is also periodic for g, say $g^k(p) = p$, then upto a matrix conjugacy $D_p g^k$ and $D_p f^n$ are upper triangular. Moreover $D_p g^k$ has eigenvalues 1 and is not the identity map and $D_p f^n$ has equal eigenvalues.*

Proof. We continue with the notation of the proof of Lemma 6 above. We have

$G \circ L = L \circ G$. The same argument above gives that G is a linear map. Since G is locally (smoothly) conjugated to g^k and g^k is locally (continuously) conjugated to B^k we get that G is localy (continuously) conjugated to B^k. Call $\hat{H}$ this conjugacy. Then $\hat{H}(Fix(B^k)) = Fix(G)$. Since $Fix(G)$ is the eigenspace of 1 for linear maps, depending on its dimension, the fixed point set is either a point, open or the matrix is conjugated to an upper triangular matrix. Since the fixed point set of B^k is a line, the fix point set of G cannot be open or a point and hence G is conjugated to an upper triangular matrix. Finally since L commutes with G, L has to be conjugated to an upper triangular matrix at the same time as G. Notice that the determinant of G is 1 and being conjugated to an upper triangular non-identity matrix this gives the rest of the proof. On the other hand, $D_p f^n$ being conjugate to upper triangular and having eigenvalues of the same modulus implies it has equal (real) eigenvalues.

□

4.2.1 Use of Theorem 7

Lemma 6 allows to enter in the hypothesis of Theorem 7 and get that there is $M : \mathbb{T}^2 \to SL(2, \mathbb{R})$ such that either

$$F(x) := (M(f(x)))(D_x f)(M(x))^{-1} = \begin{pmatrix} a(x) & b(x) \\ 0 & d(x) \end{pmatrix} \tag{4.3}$$

or

$$F(x) := (M(f(x)))(D_x f)(M(x))^{-1} = \begin{pmatrix} a(x) & -b(x) \\ b(x) & a(x) \end{pmatrix} \tag{4.4}$$

Define

$$G(x) := (M(g(x)))(D_x g)(M(x))^{-1}$$

and notice that $\det G(x) = 1$ for every x.

4.3 Continuing the proof of Main Proposition

4.3.1 Case 4.4

In this section we will show that in case 4.4, we can further conjugate the derivative cocycle to get $b = 0$ and hence we can assume we are in case 4.3.

Write $F(x) = \rho(x)\Theta(x)$ where $\rho : \mathbb{T}^2 \to \mathbb{R}^+$ and $\Theta : \mathbb{T}^2 \to SO(2)$. Let p be a periodic orbit, $f^n(p) = p$. Then by Lemma 7 we know that $D_p f^n$ is conjugated to an upper triangular matrix, hence $F^{(n)}(p)$ is conjugated to an upper triangular matrix and hence, being conformal we have that it has to be diagonal. Hence $\Theta^{(n)}(p) = I$, so by the Livsic theorem, we can conjugate the cocycle Θ to the identity cocycle and hence F can be conjugated to be a multiple of the identity, i.e. $b = 0$.

4.3.2 Case 4.3

In this section we will show that in case 4.3, we can further conjugate the derivative cocycle to get $a = c \equiv 2$.

The first step is to make $a = c$.

Lemma 8. *Let $t(x) = \frac{a(x)}{c(x)}$,then there is a function $\eta : \mathbb{T}^2 \to \mathbb{R}^+$ such that*

$$t(x) = \frac{\eta(x)}{\eta(f(x))}.$$

Proof. By Livsic's Theorem applied to the multiplicative group $\mathbb{R}^+$ we only need to show that $t^{(n)}(p) = 1$ whenever $f^n(p) = p$. Notice that $t^{(n)}(p)$ equals the quotient of the eigenvalues of $D_p f^n$ and by Lemma 7 these eignevalues coincide, so $t^{(n)}(p) = 1$. □

Now considering the new cocycle after conjugating by

$$\begin{pmatrix} \sqrt{\eta(x)} & 0 \\ 0 & \frac{1}{\sqrt{\eta(x)}} \end{pmatrix}$$

we get that the matrix

$$\begin{pmatrix} \sqrt{\eta(f(x))} & 0 \\ 0 & \frac{1}{\sqrt{\eta(f(x))}} \end{pmatrix} F(x) \begin{pmatrix} \sqrt{\eta(x)} & 0 \\ 0 & \frac{1}{\sqrt{\eta(x)}} \end{pmatrix}^{-1}$$

has the elements in the diagonal identical. So we may assume that $a = c$.

Let us write

$$F(x) = a(x) \begin{pmatrix} 1 & b(x) \\ 0 & 1 \end{pmatrix} =: a(x)F_0(x)$$

and

$$G(x) = \begin{pmatrix} \alpha(x) & \beta(x) \\ \gamma(x) & \delta(x) \end{pmatrix}.$$

By the cocycle property from the discussion of $\mathbb{N}^2$-actions and cocycles, $F_0 \circ gG = G \circ fF_0$. We get that

1. $\alpha + b \circ g\gamma = \alpha \circ f$
2. $\beta + b \circ g\delta = \alpha \circ fb + \beta \circ f$
3. $\gamma = \gamma \circ f$
4. $\delta = \gamma \circ fb + \delta \circ f$.

Equality 3 gives that γ is constant γ_0, we want to show that indeed it is 0.

Lemma 9. *Either $\gamma_0 = 0$ or by conjugating further the cocycle we can assume $F_0 = I$ and $G(x) \equiv G_0$ is constant and further conjugate to keep $F_0 = I$ and G_0 upper triangular with 1 in the diagonal.*

Proof. If $\gamma_0 \neq 0$ we have that equality 1 gives us that

$$b = \frac{\alpha \circ g^{-1}}{\gamma_0} \circ f - \frac{\alpha \circ g^{-1}}{\gamma_0}$$

and by conjugating F by the matrix

$$\begin{pmatrix} 1 & -\frac{\alpha \circ g^{-1}}{\gamma_0} \\ 0 & 1 \end{pmatrix}$$

we get that the new $F_0 = I$ and the cocycle equation becomes $G \circ A = G$ and hence $G = G_0$. Finally, since G is conjugated to upper triangular for the fixed points of f we get the Lemma. □

So we can assume $\gamma \equiv \gamma_0 = 0$.

So equalities 1, 2, 3 and 4 become:

1. $\alpha = \alpha \circ f$
2. $\beta + b \circ g\delta = \alpha \circ fb + \beta \circ f$
3. $\gamma = 0$
4. $\delta = \delta \circ f$.

Equalities 1 and 2 tell us that α and δ are constant and since G has both eigenvalues 1 for the fixed points of f this constant is 1, i.e. $\alpha \equiv \delta \equiv 1$.

4.4 End of proof of Main Proposition

Let us summarize what we got so far in the following Lemma:

Lemma 10. *There is $C : \mathbb{T}^2 \to SL(2, \mathbb{R})$ such that*

$$(C(f(x)))(D_x g)(C(x))^{-1} := G(x) = \begin{pmatrix} 1 & \beta(x) \\ 0 & 1 \end{pmatrix}$$

and

$$(C(f(x)))(D_x f)(C(x))^{-1} := F(x) = a(x) \begin{pmatrix} 1 & b(x) \\ 0 & 1 \end{pmatrix}$$

where

$$a \circ g = a \tag{4.5}$$

$$b \circ g - b = \beta \circ f - \beta \tag{4.6}$$

4.4.1 Trivializing the cocycle a.

Lemma 11. *If $f^n(p) = p$ then $a^{(n)}(p) = 2^n$.*

Proof. Again we shall follow ideas in the proof of Lemma 6. So, assume that $f^n(p) = p$ and $g^k(p) = p$. By taking an appropiate lifting to the universal covering, we have that $\widetilde{f^n}(\tilde{p}) = \widetilde{g^k}(\tilde{p}) = \tilde{p}$, moreover, $\widetilde{f^n}$ is a global repellor, so can apply again Theorem 8 and have a linealization $H_0 : \mathbb{R}^2 \to \mathbb{R}^2$ such that $H_0(\tilde{p}) = 0$ and

$$H_0 \circ \widetilde{f^n} \circ (H_0)^{-1} = F^{(n)}(p)$$

and

$$H_0 \circ \widetilde{g^k} \circ (H_0)^{-1} = G^{(k)}(p)$$

We know that $Fix(G^{(k)}(p))$ is the horizontal line $[x_2 = 0]$. Hence $H_0(FixG^{(k)(p)}) = Fix(\widetilde{g^k})$ is a smooth curve $(C^{1+\alpha})$. On the other hand, $h \circ \widetilde{g^k} \circ h^{-1} = B^k + c_p$ for some $c_p \in \mathbb{R}^2$, hence we have that $Fix(B^k + c_p)$ is a horizontal line, hence $h^{-1}(Fix(B^k + c_p)) = Fix(\widetilde{g^k})$ is the lift of the circle $\mathcal{C}(p)$.

So, the lift of $\mathcal{C}(p)$ is a smooth curve, hence $\mathcal{C}(p)$ is smooth $(C^{1+\alpha})$.

Let p_0 was of least period in $\mathcal{C}(p)$ and N be its period. Hence $\mathcal{C}(p) = \mathcal{C}(p_0)$ and $f^N(\mathcal{C}(p)) = \mathcal{C}(p)$. Let us identify the $C^{1+\alpha}$ circle $\mathcal{C}(p)$ with S^1 by parametrizing by arc length, call $\gamma(t)$ the parametrization (assume length is 1 without loss of generality), then we have a $C^{1+\alpha}$ map $\phi : S^1 \to S^1$, that is $\phi = f^N|\mathcal{C}(p)$.

Let $X(x)$ be a Hölder continuous unit tangent vector to $\mathcal{C}(p)$ through x. And let $Z(x) = C(x)X(x)$. Then

$$a^{(N)}(x) \begin{pmatrix} 1 & b^{(N)}(x) \\ 0 & 1 \end{pmatrix} Z(x) = \lambda(x) Z(f^N(x))$$

on $\mathcal{C}(p)$ where

$$\lambda(\gamma(t)) := \phi'(t).$$

By equation 4.5 we have that $a^{(N)} \circ g = a^{(N)}$ and hence $a^{(N)}$ is constant along the circle $\mathcal{C}(p)$, let us call this constant a_0, so that $a^{(N)} \equiv a_0$. So, in particular $a^{(Nr)}(x) = a_0^r$ on $\mathcal{C}(p)$.

Let $q \in \mathcal{C}(p)$ be periodic, say $f^{rN}(q) = q$ then we get that

$$a^{(rN)}(q) \begin{pmatrix} 1 & b^{(rN)}(q) \\ 0 & 1 \end{pmatrix} Z(q) = \lambda^{(r)}(q) Z(q)$$

and since $Z(q) \neq 0$ we have that $a^{(rN)}(q) = a_0^r = \lambda^{(r)}(q)$. So, for a periodic point $\phi^r(t) = t$, $(\phi^r)'(t) = a_0^r$. By Livsic Theorem ϕ' is multiplicatively cohomologous to constant a_0, i.e.

$$\phi' = a_0 \frac{w}{w \circ \phi}$$

for some positive Hölder continuous function w. We may assume that

$$\int_{S^1} w dt = 1$$

and hence there is $q : S^1 \to S^1$, a diffeomorphisms such that $q' = w$, so that

$$(q \circ \phi)' = (a_0 q)'$$

and hence, for some constant c_0,

$$q \circ \phi = a_0 q + c_0.$$

This implies that

$$a^{(N)}|\mathcal{C}(p) = a_0 = deg(\phi) = deg(f^N|\mathcal{C}(p)) = 2^N$$

and hence $a^{(n)}(p) = (a^{(N)})^l = (2^N)^l = 2^n$, where $n = lN$. □

Finally, we can use Livsic Theorem and get that there is a Hölder continuous function $\kappa > 0$ such that

$$a = 2\frac{\kappa}{\kappa \circ f}$$

notice that since $a \circ g = a$, we obtain that $\kappa = \kappa \circ g$.

This finishes the proof of the Main Proposition.

4.5 Putting things together

Proposition 3. *In Main Proposition 1 we can assume that* $b_0 = c = 0$*. Moreover, by conjugating by* $\begin{pmatrix} \frac{1}{\kappa(x)} & 0 \\ 0 & \frac{1}{\kappa(x)} \end{pmatrix}$ *we obtain that*

$$C(f(x))D_xf(C(x))^{-1} = 2\begin{pmatrix} 1 & \hat{b}(x) \\ 0 & 1 \end{pmatrix}$$

and

$$C(g(x))D_xg(C(x))^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Proof. By section 3.1, $\beta = b_0 \circ h^{-1}$ and $\gamma = c \circ h^{-1}$ are in the hypothesis of Proposition 1 and hence are cohomologous to 0. Taking $U = u \circ h$ and conjugating by the matrix

$$\begin{pmatrix} 1 & U(\bar{x}) \\ 0 & 1 \end{pmatrix}$$

in equations 4.1 and 4.2 we get the proof. □

Chapter 5
Building the Conjugacy

The Main Proposition gives us a conjugacy relation for the derivative maps of f and g, so what remains is to use it to find a conjugacy for f and g proper.

Let us define the $1-$form

$$\nu_x(v) = \pi_2 C(x) v$$

where π_2 is projection onto the second coordinate in $\mathbb{R}^2$ and $C(x)$ is given by Proposition 3. We want to find a C^1 $0-$form ϕ such that $d\phi = \nu$. Then indeed ϕ will be equal to h_2 (upto a constant) and hence h_2 will be C^1 (and even $C^{1+\alpha}$).

Since ν is only Hölder, and not necessarily differentiable we cannot check in the easy way that it is closed. Even if closed it will not be exact (on the torus), but in any event we only will integrate it in the universal covering, which is contractible.

Instead of showing it is closed we will show that the integral over any closed curve that is homotopically trivial is 0. Once we know this we go to define the primitive by integrating along paths. The fact that the definition is independent of path (in the universal covering) follows automatically.

So let us lift all the objects to $\mathbb{R}^2$ (we will not change their names) assume that $f(0) = 0$ and observe that $f_* \nu = 2\nu$.

Proposition 4. *If γ is a C^1 closed curve in $\mathbb{R}^2$ then $\int_\gamma \nu = 0$.*

Proof. Since we are in the universal covering we can take negative iterates of f and we get $f^{-n}(x) \to 0$ for every x. We also know that there is $\lambda < 1$ such that $|f^{-n}(x)| \leq \lambda^n |x|$. Even more is true. By Proposition 3 we know that there is $C > 0$

such that

$$\|Df^{-n}\| = \|(Df^n)^{-1}\| \leq C2^{-n}n$$

for some constant $C > 0$ and $n \geq 1$. Hence, by the mean value theorem we get that

$$|f^{-n}(x)| \leq C2^{-n}n|x|$$

for every $x \in \mathbb{R}^2$. Finally, since ν is Hölder continuous, there are $C > 0$ and $\alpha > 0$ such that for every vector $v \in \mathbb{R}^2$ of norm 1,

$$|\nu_p(v) - \nu_q(v)| \leq C|p-q|^\alpha$$

if $|p - q| \leq 1$.

Let $\gamma : [0,1] \to \mathbb{R}^2$ such that $\gamma(0) = \gamma(1)$. Hence

$$\begin{aligned} \int_\gamma \nu &= 2^n \int_{f^{-n}\gamma} f_*^{-n}\nu = 2^n \int_0^1 \nu_{f^{-n}(\gamma(t))}\left(Df^{-n}\dot\gamma(t)\right) dt \\ &= 2^n \int_0^1 \left[\nu_{f^{-n}(\gamma(t))}\left(Df^{-n}\dot\gamma(t)\right) - \nu_0\left(Df^{-n}\dot\gamma(t)\right)\right] dt \end{aligned}$$

here we used that the constant 1−form ν_0 is exact. So taking absolute values

$$\begin{aligned} \left|\int_\gamma \nu\right| &\leq 2^n \int_0^1 \left|\nu_{f^{-n}(\gamma(t))}\left(Df^{-n}\dot\gamma(t)\right) - \nu_0\left(Df^{-n}\dot\gamma(t)\right)\right| dt \\ &\leq 2^n \int_0^1 \left|\left[\nu_{f^{-n}(\gamma(t))} - \nu_0\right](v_n(t))\left|Df^{-n}\dot\gamma(t)\right|\right| dt \end{aligned}$$

where $v_n(t) = \frac{Df^{-n}\dot\gamma(t)}{|Df^{-n}\dot\gamma(t)|}$ Finally, this last term can be bounded by

$$\begin{aligned} & 2^n \int_0^1 \left|\left[\nu_{f^{-n}(\gamma(t))} - \nu_0\right](v_n(t))\left|Df^{-n}\dot\gamma(t)\right|\right| dt \\ \leq\ & 2^n \int_0^1 C|f^{-n}(\gamma(t))|^\alpha C2^{-n}n|\dot\gamma(t)|dt \end{aligned}$$

Let C be such that $|\gamma(t)| \leq C$ and $|\dot\gamma(t)| \leq C$ for every t then we get that

$$\left|\int_\gamma \nu\right| \leq C^2 2^n (2^{-n}n)^\alpha 2^{-n}n = C^2 2^{-\alpha n} n^{1+\alpha}.$$

Since the right hand side goes to 0 we get that the left hand side is 0. □

We now know that a line integral of the 1-form ν is path-independent. Therefore we can define the integral $\phi(p) = \int_0^p \nu$, using for example the line segment from 0 to p. Since ν is Hölder we get that ϕ is $C^{1+\alpha}$.

Since $f_*\nu = 2\nu$ and $g_*\nu = \nu$, we get that $\phi(f(p)) = 2\phi(p)$ and $\phi(g(p)) = \phi(p)$ for every p (since $\phi(0) = 0$) and hence $\phi \circ f = 2\phi$ and $\phi \circ g = \phi$.

On the other hand ν is a lift of a form on the torus hence ν is $\mathbb{Z}^2-$equivariant, which implies that $\phi(p+\bar{n}) - \phi(p) = c(\bar{n})$ for every p, for some $c(\bar{n}) \in \mathbb{R}^2$. It follows from the definition that $c : \mathbb{Z}^2 \to \mathbb{R}$ is a homomorphism.

Claim 1. *$c(n_1, n_2) = an_2$ for some $a \in \mathbb{R}$.*

Proof. We know that c is a linear function on $\mathbb{Z}^2$, therefore it can be linearly extended to a function $c : \mathbb{R}^2 \to \mathbb{R}$. Let $\hat{\phi} = \phi - c$; then $\hat{\phi}(p + \bar{n}) = \hat{\phi}(p)$ for every $\bar{n} \in \mathbb{Z}^2$. Since $\phi \circ g = \phi$ this implies that $c \circ g + \hat{\phi} \circ g = c + \hat{\phi}$ and hence

$$c \circ B + c \circ \hat{g} + \hat{\phi} \circ g = c + \hat{\phi}.$$

Now $\hat{\phi}$ and $\hat{g}$ are bounded since they are $\mathbb{Z}^2$ periodic. Hence $c \circ B - c$ has to be linear and bounded hence $c \circ B - c = 0$. So $c \circ B = c$, i.e. $c(x + y, y) = c(x, y)$ for every x, y hence $c(y, 0) = 0$ for every y, i.e. $c(x, y) = ay$ for some y. □

Claim 2. *$a \neq 0$.*

Proof. If $a = 0$ then ϕ is continuous and $\mathbb{Z}^2$-periodic, and therefore bounded. On the other hand, $\phi \circ f = 2\phi$. So if $\sup |\phi| = K$ we can find some x such that $\phi(x) = K$, which will imply $\phi(f(x)) = 2\phi(x) = 2K \leq K$, which can only happen if $K = 0$. This implies that ϕ is identically 0, and therefore $\nu = d\phi$ is identically 0. However, for each x, $C(x)$ is an invertible map and has a dimension-2 image, so $\nu_x(v) = \pi_2(C(x)v)$ cannot be identically 0. □

Dividing by a we may assume that $a = 1$ (we can modify C accordingly if needed).

Hence we have that ϕ induces a map $\phi : \mathbb{T}^2 \to S^1$ homotopic to projection onto second coordinate and solving equation $\phi \circ f = 2\phi$. Now, h_2 is the only solution to this equation, hence $h_2 = \phi$ is $C^{1+\theta}$.

Lemma 12. *h_2 is as smooth as f.*

Proof. Let us write $f^{-1} = \frac{1}{2}Id + \frac{1}{2}\eta$, $\eta = (\eta_1, \eta_2)$ (observe that η is a priori not periodic). We have that $d\eta_2(0) = 0$ and hence $|\eta_2(\bar{x})| \le C|\bar{x}|^2$. Let

$$\hat{\psi} = \sum_{n\ge 0} 2^n \eta_2 \circ f^{-n}.$$

Since

$$\|f^{-n}(x)\| \le Cn2^{-n}\|x\|$$

and $|\eta_2(\bar{x})| \le C|\bar{x}|^2$ we get that the series is convergent. Moreover, since f^{-1} is a contraction, we can see that taking further derivatives only improves the convergence and hence $\hat{\psi}$ is as smooth as f.

Let $\psi(x_1, x_2) = x_2 + \hat{\psi}(x_1, x_2)$. Then

$$\psi \circ f^{-1}(x_1, x_2) = \frac{1}{2}x_2 + \frac{1}{2}\eta_2(x_1, x_2) + \hat{\psi} \circ f^{-1}(x_1, x_2) =$$

$$\frac{1}{2}x_2 + \frac{1}{2}\eta_2(x_1, x_2) + \sum_{n\ge 0} 2^n \eta_2 \circ f^{-n-1} =$$

$$\frac{1}{2}x_2 + \frac{1}{2}\eta_2(x_1, x_2) + \frac{1}{2}\sum_{n\ge 1} 2^n \eta_2 \circ f^{-n} =$$

$$= \frac{1}{2}(x_2 + \hat{\psi}(x_1, x_2)) = \frac{1}{2}\psi(x_1, x_2)$$

.

This means that $\psi \circ f^{-1} = \frac{1}{2}\psi$ or equivalently $\psi \circ f = 2\psi$.

Claim 3. *The equation $H \circ f = 2H$, $H \in C^{1+\theta}$, $D_0H(v_1, v_2) = v_2$ has at most 1 solution.*

Proof. Assume H_1 and H_2 are solutions to the equation, then $\eta = H_1 - H_2$ is also a solution with $D_0\eta = 0$ and $\eta \circ f^{-n} = 2^{-n}\eta$, hence

$$\eta(\bar{x}) = 2^n \eta(f^{-n}(\bar{x}))$$

Now $|\eta(\bar{x})| \le C|\bar{x}|^{1+\theta}$, hence

$$|\eta(\bar{x})| \le 2^n C|f^{-n}(\bar{x})|^{1+\theta} \le C2^n(2^{-n}n)^{1+\theta}|\bar{x}| \to 0$$

□

We know that h_2 is a Hölder-continuous solution to the conjugacy, and ψ is a smooth solution to the same. From the previous claim we know such a solution must be unique and so the two functions must be the same. Hence h_2 is as smooth as f. □

Now, recall our goal.

We want to find $H : \mathbb{T}^2 \to \mathbb{T}^2$ and $\beta : \mathbb{T}^2 \to S^1$ such that

$$H \circ f \circ H^{-1} = \hat{f}, \quad \text{and} \quad H \circ g \circ H^{-1} = B$$

where

$$\hat{f}(x, y) = (2x + \beta(y), 2y).$$

Let $R(x, y) = (x, h_2(x, y))$. Assuming f is close to A we have that h is a diffeomorphism. Let $L(x, y) = (\alpha(x, y), y)$. We are looking for $H = L \circ R$ such that H preserves area and $H(0) = 0$. This means that $\det(DH) = 1$ which is equivalent to

$$\det DL \circ R = (\det DR)^{-1} = \left(\frac{\partial h_2}{\partial y}\right)^{-1}$$

or equivalently, that

$$\frac{\partial \alpha}{\partial x} = \left(\frac{\partial h_2}{\partial y}\right)^{-1} \circ R^{-1}.$$

This equation can be easily solved integrating with respect to x variable. Observe that

$$H(x, y) = L(x, h_2(x, y)) = (\alpha(x, h_2(x, y)), h_2(x, y)) = (H_1(x, y), h_2(x, y))$$

for $H_1(x, y) = \alpha(x, h_2(x, y))$.

Let $\hat{A} = H \circ f \circ H^{-1}$ and $\hat{B} = H \circ g \circ H^{-1}$. Since H preserves area both $\hat{A}$ and $\hat{B}$ preserve area. Moreover, by the properties of h_2 we get that

$$\hat{A}(x, y) = (2x + 2r(x, y), 2y)$$

and

$$\hat{B}(x, y) = (x + y + s(x, y), y).$$

for some $r(x,y)$ and $s(x,y)$.

Since $\hat{B}$ preserves area, it follows that $s(x,y) = s(y)$. Now the commutativity of $\hat{A}$ and $\hat{B}$ gives that

$$(2x + 2r(x,y) + 2y + s(2y), 2y) = 2x + 2y + 2s(y) + 2r \circ \hat{B}(x,y), 2y)$$

and hence

$$s(y) - \frac{1}{2}s(2y) = r(x,y) - r \circ \hat{B}(x,y).$$

Let us integrate the right hand side with respect to the x variable, then

$$\int r \circ \hat{B}(x,y)dx = \int r(x + y + s(y), y)dx = \int r(x,y)dx$$

hence the integral of the right hand side with respect to dx is 0 and the integral of the right hand side is $s(y) - \frac{1}{2}s(2y) = 0$ and hence $s(y) = 0$ for every y.

This implies that $\hat{B} = B$ and $r \circ B = r$, since we know that $B-$invariant function are exactly the ones that are independent of the x variable we are done.

This follows from the fact that $r(B(x,y)) = r(x+y,y) = r(x,y)$ implies that for any irrational y, $r(x,y) = r(x+ky)$ which forms a dense set in the circle. This means that r is independent of x on every irrational circle, and they are dense in $\mathbb{T}^2$, so by continuity of r it is independent of the first coordinate.

$$\hat{A}(x,y) = (2x + r(y), 2y)$$
$$\hat{B}(x,y) = B(x,y) = (x+y,y).$$

r has the desired properties of β from Theorem 1.

5.1 Further Research

The methods of this paper should be adaptable, with some variation, to other pairs of commuting maps on the torus, for example

$$A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

The map A is expanding in all directions like before but does not have equal eigenvalues.

The new map B, unlike the old map $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, does not have the dynamics of a twisted extension (split into horizontal circles with rotations) so we will not be able to use the same method exactly. However, the same Livsic techniques should work in this case.

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Vita

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