QUANTUM ENGINEERING OF FRACTIONAL HALL PHYSICS
IN AN ATOM-OPTICAL LAUGHLIN-PUMP OSCILLATOR

A Dissertation in
Physics
by
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Abstract

Quantum simulation in cold atomic physics has seen successes in many different fields and contexts of research. Simulation of gauge effects, often introduces new challenges, often originating from time-dependent potentials used to approximate new Hamiltonians. To overcome such obstacles, this thesis describes the innovation of a new method, named quantum Engineering by Amplified Stimulated Excitation (quEASE), for its use in an analogy between engineered coherent many-body states and the operation of a maser or a laser. By incorporating an ultra-cold atomic gas into an oscillator loop and driving the system into stable oscillation, a coherent mode of this many-body physical system can be engineered, and specific modes may be chosen through design of generalized filters used in the feedback loop. In particular, this thesis demonstrates an application of the quEASE method in the interrogation of Fractional Quantum Hall physics in cold atom systems, in which an optical pump oscillator is constructed based on an optical realization of Laughlin’s charge pump topology [1]. A pair of interferometers are coupled to each other by interacting with a common gas of atoms, in which time-reversal symmetry breaking modulation is transferred from one beam to another. The dynamics resemble essential features in electronic quantum Hall measurements, such as the transverse field and flux bound to particles. This experiment establishes a first direct transport measurement attempt in quantum simulation of fractional quantum Hall physics in cold atomic systems. A novel driven-oscillator scheme for probing the optical quantum pump oscillator is developed to overcome difficulties originating from weak coupling between light and atoms.
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Dedication

For My Parents Zhenya Liu and Jingui Dong
Chapter 1

Quantum Simulation of Many-Body Physics in Light-Atom Systems

Quantum Simulation has been a rapid developing field of study in recent years. Cold atom systems, because of their versatility and ease to manipulate, have been extensively studied as platforms of quantum simulation. This thesis demonstrates a new method of simulating quantum many body physics in light-atom systems and presents experimental results of applying this method in the study of quantum Hall physics. The first chapter is an introductory chapter, which contains three sections - section 1.1 discusses the breakthroughs and challenges in quantum simulation with cold atoms and introduces this new method, named as quantum Engineering by Amplified Stimulated Excitation (quEASE); section 1.2 discusses the discoveries of integer and fractional quantum Hall effects as well as the theoretic approaches to understand these phenomena; section 1.3 discusses the design of one innovative experiment, in which quEASE is applied in the study of quantum Hall physics.

1.1 Quantum Simulation with Synthetic Ground States and with quEASE

This section starts with subsection 1.1.1 as an overview of quantum simulation of many-body physical systems, in subsection 1.1.2 an analogy is established between coherent many-body states and classical oscillation, which leads to the idea of an innovative method of simulating quantum systems, which is discussed in subsection 1.1.3.
1.1.1 Quantum Simulation of Many-Body Physics

In this subsection, the development of quantum computation and quantum simulation is elaborated, and the challenges in simulating gauge fields in many-body problems are presented.

1.1.1.1 Quantum Computation for and Simulation of Quantum Systems

Many physical systems in the quantum limit, due to their complexity, are beyond the capabilities of the computational resources available to classical computing machines. As a result, their direct simulation requires approximations to limit the number of degrees-of-freedom in the system [2–4], simplify its time evolution [5,6], neglect the effects of intrinsically quantum behavior at certain scales [7,8], and stabilize the results against error [9–12]. In the 1970’s and 1980’s, it was recognized that (at least in principle) computing machines whose behavior, and by extension computing resources, scaled in a way similar to the systems under numerical study, could dramatically out-perform their classical counterparts. The formalization of this idea for quantum computation was most clearly envisioned by Richard Feynman in 1970’s [13], and today, a large number of theories and experiments are pursuing this idea.

However, the dream of generalized quantum computation of this sort is quite challenging, and today is codified by a series of criteria [14] that are being progressively attacked by experiments using laser-cooled ions, ultracold atoms, superconducting solid-state systems, and a range of other techniques. The status of these fields is steadily improving with time, but as yet, generalized quantum computation has not reached the point where the original dream of using them as numerical computation tools is an applied science.

For a somewhat more limited goal - that of simulating the behavior of complex quantum systems to understand coarse properties of their behavior - there is another strategy. With a sufficiently versatile quantum system with many degrees-of-freedom, one can attempt to "simulate" the behavior of natural systems. By introducing "the right ingredients" into such a generic quantum system, one can test which are truly essential to achieve a certain behavior of interest. For example, one might hope to reproduce the essential physics of high-temperature...
superconductivity in a system of cold-atoms [15,16], by introducing the appropriate interplay of interacting quantum bodies. It is helpful to think of this process as one of "engineering a Hamiltonian" for the versatile system to emulate the essential terms of the natural one. The list of potential examples is as long as one’s creativity, and the versatility of tools to manipulate the quantum system, permit.

This allows researchers to study the properties and dynamics of those extremely complex systems, by carrying out experiments on more simplified and straightforward physics systems, using 'simulation' to remove the challenging constraints of generalizing quantum computation steps. Among a large variety of generic quantum systems from which to construct simulators, cold atomic gases provide one of the more versatile platforms, due both to their easiness of preparation, and the wide variety of physical mechanisms available to control and manipulate them.

The quantum-degenerate limit for cold-atoms, as well as the realization of a large number of manipulation techniques, were achieved systematically over a period in the last four decades. Starting with the invention of laser cooling and trapping techniques in the 1970’s and 1980’s [17–23], atomic physicists have been able to 'freeze' the thermal motion of neutral atoms from a hot vapor at room temperature or higher, and those cold gases can be cooled as low as several nanokelvin, and even hundreds of picokelvin level using evaporative cooling. With the temperature sufficiently close to absolute zero and at sufficiently high density, a new state of matter can be achieved in this atomic gas. In 1995, Eric Cornell and Carl Wieman created the first Bose-Einstein Condensate with Rubidium-87 system [24], by carefully designed sequences of laser trapping and cooling and the application of evaporative cooling. In the same year, Wolfgang Ketterle's group characterized the properties of a condensate [25]. Time of flight images clearly depicted a macroscopic occupancy of ground state in the phase space, which is a clear evidence verifying the prediction made by Satyendra Nath Bose and Albert Einstein in 1924-25 [26–28]. Since this original achievement, gases of atoms in this quantum degenerate limit have been manipulated and probed in a large number of ways.

Starting in the early 2000’s, cold atomic gases have been systematically used in realizations of many different Hamiltonian systems, which are associated to a lot of complex systems that are difficult to understand or inaccessible in condensed matter physics, solid-state physics, and cosmological physics, etc. Today, atomic, molecular
and optical physicists are able to create many new states of matter and study their physics in the platform of cold atomic gases, thanks to the introduction of many innovative tools, including tools like Feshbach resonance [29–31] to tune the atomic interactions, optical lattices to create different types of periodic confinement [32,33], Spatially-noisy optical potentials to create disorder [34–36], and time-periodic driving of optical lattices and other parameters in the Hamiltonian to generate synthetic gauge fields [37–45] and other phenomena [46–50].

With the help of these new techniques, as well a growing list of new schemes and methods being proposed at a rapid pace, research in cold atoms has successfully simulated Bose Hubbard systems [32,51–56], superfluidity in a range of contexts [57–60], degenerate Fermi gases [29,61–64], the BEC-BCS cross-over [58,63,65–69], the Ising model Supersolid [70–74], topologically-ordered matter [75–79], and many other systems. Exciting progress has been made in many of these contexts, study collective excitations and non-equilibrium phenomena in many of these contexts. Exciting and probing these systems enjoy a wide variety of techniques themselves, including many time-dependent manipulations of the systems to perform spectroscopy [53,58,80–82], quantum quenches [83–87], and many other types of experiments. Topological defects, like vortices [88–90], solitons [91–94], and many other interesting collective effects have been created in different types of ordered phases, with even some work among them shown to be closely related to the evolution of the early universe [95–97] and even black hole physics [98–100].

Synthetic gauge fields for neutral atoms in particular (first described as such during pioneering experiments at NIST [101,102]), have opened up many opportunities to study solid-states physics. Neutral atoms don’t interact with electric and magnetic field in the same way that electrons in solid-state matters experience electromagnetism, since the atom lacks the charge of these systems, and effects like the Lorentz force of a charge particle in a magnetic field require some other origin. A simple technique to generate this uses the inertial effects of an "accelerating potential" in some form. For example, the techniques of periodically shaking or rotating an optical lattice can generate a new term in the single-particle Hamiltonian (giving a Coriolis force), which mimics the effect of a magnetic field in the appropriate (non-inertial) reference frame.

Such methods of creating artificial gauge fields allow realizations of a large variety of different types of Hamiltonian. Often their behavior can be understood in analogy
to quantum magnetism [73,103–105], describing the orbital degrees-of-freedom in the motion of cold atoms in an optical lattice effectively as spins [79,106–109].

One can also view the single- and many-body physics of such systems in analogy to natural systems of electrons, specifically seen in the integer quantum Hall regime [1,110] and spin-orbit coupling systems for atomic gases [42], and in the many-body effects expected for fractional quantum Hall physics [111,112]. Electrons in the strong transverse magnetic field are described by Landau levels and a macroscopic occupancy and non-trivial topology are in the core of this regime. A long list of laboratory schemes to create different artificial gauge fields have been attempted [41]. The pursuit of (particularly fractional) quantum Hall physics in the cold atomic systems has long been an impetus in this subfield of quantum simulation, as a means to observe long-sought fractionalized exchange effects for the first time in any physical system.

Many very serious attempts have been made, following different schemes such as cooling a rotating gas [39,113], selective dissipation processes [114], adiabatically varying a parameter [115], driving Berry phases in a dressed state [106], and using those methods, some exciting progresses have been achieved [116–120] toward creating the parent state in which elementary excitations with fractionalized exchange can be excited. However, one can see from the list of techniques above that many of these experiments begin with a common theme - either creating the essential element of a non-trivial gauge-field using an effectively accelerating (or at least time-dependent) manipulating field or opening the system of cold atoms to a bath created by an external field. The quantum mechanics of these systems is then either non-inertial, or open, both non-trivial extensions of non-relativistic quantum mechanics that require some additional consideration.

1.1.1.2 New Challenges in Quantum Simulation of Gauge-Fields in Many-Body Problems

During the many successes of experiments carried out in quantum simulation using cold atomic and other systems, a number and recurrent theme of new intellectual challenges have emerged concerning how to appropriately handle the non-inertial and time-dependent effects in such systems, and further in how to extend potential solutions into the context of many-body physics. The essential ingredient in these problems is apparent from the introduction above - engineering quantum
simulations with gauge-fields in them often extends the "Hamiltonian engineering" problem into time-dependent techniques, allowing energy to be exchanged between the manipulating fields and atoms, and forcing one to consider open systems in the quantum limit. In an alternate viewpoint, often the time-dependence can be removed by an appropriate transformation of frame, introducing non-inertial effects. The proper treatment of quantization in a non-inertial frame, however, is a poorly understood problem itself.

To understand this more clearly, we can consider a special class of problems - in the most common cases, the time-dependence used to create non-trivial gauge fields is periodic. The most well-known techniques for treating a time dependent Hamiltonian of this sort is to use perturbation theory, assuming the modulation is in some sense weak, and solving for its motion as a small perturbation of a well-defined quantum state. Usually one can solve for the energy levels of the single particle system (for example its band structure for a spatially periodic system), and define a ground state under such approximation. The effect of the time-dependent part is then to drive transitions between these states in time.

When the time dependent term becomes sufficiently large to be non-perturbative, one can apply a quantum-mechanical version of Bloch-Floquet theory to understand the time-dependence of the quantum state [121]. Here, one assumes the states are time-periodic over the same increment of time as the driving field, similar to the idea of a Bloch wave for electrons in the spatially-periodic potential of an ionic crystal. Like the latter have a quasi-momentum, the solutions to these classes of Hamiltonian are Floquet states characterized by their quasi-energy. This is attributed to the discrete temporal symmetry of the periodically driven system, similar to the discrete translational symmetry of a crystal. The result is an infinite ladder of such quasi-energy states, in analogy to the infinitely repeating zones of quasi-momentum for crystalline symmetry. The technique is mathematically and conceptually simple, but it has eliminated the comfortable concept of a quantum-mechanical ground state. The idea of generalized quantum simulation by first cooling a system into the ground-state of a controlled Hamiltonian is now lost, or at least blurred, in this first step.

Often the targeted ground state in cold atomic systems and condensed matter systems can be viewed as one state chosen from among the infinite ladder of Floquet series. The physical interpretation is that any one of these states is nearly identical
to the others, but having absorbed a different number of 'mechanical quanta' from
the driving field. In the example of atoms driven by laser light, the mechanical
quanta are simply photons, and the interpretation is simple. In more complex
driven systems, the driving field may not only be non-optical, by may be a coherent
'concert' of driving fields of different origin (for example, the dipole potential
of far-detuned optical lattice potentials, or quasi-static magnetic fields), and the
interpretation of the 'quanta' is somewhat subtler. It is easy in such situations
to encounter conceptual problems about the relevant physical phenomena - for
example, it is easy to discard the re-absorption of photons in the former example
from a bath under the premise that this field propagates rapidly away from an
initial disturbance. But if, for example, mechanical quanta are instead absorbed
from the slow periodic modulation of a quasi-static magnetic field, it is harder to
excuse an approximation that re-absorption is irrelevant.

The quantitative of understanding of heating also becomes particularly difficult
in these cases, as one must ascribe some aspect of coherence to these processes, as
well. For example, even when time-dependence can be completely absorbed by a
frame transformation - as, for example, in a rotating trap, these states can suffer
from stationary impurities, which in principle can lead to problems with heating
and dynamic instability. It is exceedingly difficult to quantify or even define heating
induced by such impurities. In quantum-simulation techniques where the time-
dependent manipulation is turned-on adiabatically to connect an easily-understood
quantum ground-state to a target (Floquet) eigen-state of the time-dependent
system, one is then often left with the nearly impossible task experimentally of
determining a temperature that is cold enough to establish the targeted many-body
states in the presence of this poorly-understood heating. Put in simpler language
- when the target quantum state of a system includes coherent motion, one must
find a way of distinguishing when incoherent motion is sufficiently strong to disrupt
the observable features of the state.

With this distinction of coherent and incoherent motion now blurred, the idea
of cooling a quantum system into a time-dependent ground state is lost, and one is
left applying a difficult framework of 'approximate ground state' and 'heating rate'
to many already challenging (many-body) problems. In this thesis, we will attempt
to approach the problem of quantum simulation of gauge-field systems from a
different line of attack both in the laboratory and on paper, taking the strong
time-periodic drive as a defining feature of the systems we study. To do that, we will appeal to a developed-language more closely in-tune with that defining feature - that of the physics of a driven or self-supporting oscillator like a laser or other electronic oscillator. Rather than discuss the "ground-state" of a closed system, we will describe the coarse dynamical state of the oscillator, and its transitions between them. In this language, our 'target ground states' are less like competing orders of a static condensed-matter system, and more like the competing modes of a laser.

The idea is perhaps not such a stretch for the cold-atom system - after all, it is not uncommon to make the analogy between a Bose Einstein condensate and a laser’s operation, as they both achieve a coherent state with a macroscopic occupancy of some 'single-particle' state. For the laser, this is many quanta (photons) in a single electromagnetic mode. For the simple un-driven condensate, it is many quanta (atoms) in the single particle ground-state (although it can differ from that due to interaction). To extend that idea to driven condensates, we simply allow that single atomic mode to be time-dependent. This analogy can be generalized from laser to other type of oscillators and one can adopt the knowledge and techniques from the well-studied field of classical oscillator and adapt them into the study of many-body physics. Therefore, one may create an excitation scheme using positive feedback like an electronic oscillator, rather than create an engineered ground state scheme in producing many-body state in cold atoms (discussed in detail in the next subsection). The advantages are also technical - we can now also build experiments with the same ideas in mind, and translate the problem of "designing a ground state" into the design of an appropriate "filter" into a feedback loop of some sort in the laboratory. If the filter selects out excitations of the type we want, and reinforces them with feedback, we have found a new strategy for quantum simulation.

1.1.2 The Analogy between Bose-Einstein Condensation and Classical Oscillation and Synchronization

To confront the challenges in quantum simulation, an innovative method is developed, originated from the analogy between coherent many-body states and classical oscillation. This subsection demonstrates how Bose-Einstein condensation, as well
as other coherent many-body states, are analogous to classical oscillation, i.e. a laser operation.

1.1.2.1 Condensation as a Thermodynamic Phase Transition in Static Potentials

As described in the previous section, the core idea of the work in this thesis lies in generalizing the similarity between an atomic BEC and laser (or other oscillator) operation. In truth, there are differences between a condensate formed in a Bose gas with massive particles and the collective modes of massless photons that interact with laser medium that are important to recognize. However, the two systems share one important feature - namely, that they both can be described quantum mechanically as coherent state with macroscopic occupancy of some mode. To justify this statement, we will first briefly review Bose Einstein Condensate from this perspective, defining such states mathematically in this section. In later sections, I will show how this concept can be generalized for more exotic interacting many-body states, like molecular BECs and BCS-states, and then to states relevant to gauge-field physics, namely specific states proposed to describe some fractional quantum Hall states.

From a generic perspective in condensed-matter physics, it is often said that Bose-Einstein can be considered as a kind of phase transition that breaks a particular type of U(1) symmetry, as correlations in the dynamics of an locally defined order-parameter take on a non-zero value over a macroscopic range. This is the language of the Landau-Ginzburg framework, used to describe nearly all transitions between macroscopic states in nature, both classical and quantum mechanical (with the appropriate extensions). This generic formalism is useful when one wants to consider its behavior near criticality, and as it responds to certain probes (like quenches), but is not particularly revealing of the underlying mechanisms for condensation.

Bose-Einstein condensate is a state of matter at extremely low temperature very close to absolute zero - as the interatomic spacing becomes comparable to the de Broglie wavelength of atoms, where quantum statistics starts playing an essential role, a large fraction of indistinguishable atoms tend to stay in the same quantum state. To make this definition precise, we can consider the condensed phase to be defined by the non-zero fraction of particles occupying a particular state (usually the ground-state of a static Hamiltonian) in the thermodynamic limit.
of an infinitely large system.

This unique state of matter was first predicted in 1924 by Satyendra Nath Bose and Albert Einstein and the first experimental observation with cold atoms was made in 1995 by Eric Cornell and Carl Wieman at JILA, and almost simultaneously characterized by Wolfgang Ketterle of MIT, by simply cooling a large sample confined in a trap, until a sudden transition was observed in the coarse features of the sample [24, 25]. Figure 1.1 shows the classic visualization of the first creation and observation of atomic BEC in the laboratory - the suddenly changing coarse feature is apparent as a "condensate" of macroscopically many atoms occupying a nearly zero momentum single-particle state.

\[ N = \sum_{\epsilon} \frac{1}{z^{-1}e^{\beta\epsilon} - 1} \]  

**Figure 1.1.** The first atomic Bose-Einstein condensate created in experiments. Researchers from JILA published this iconic figure in the July 14, 1995 issue of Science magazine, reporting the first experiment to create an atomic Bose Einstein Condensate in an alkali vapor of Rubidium-87 cooled less than 170 billionths of a degree above absolute zero. This graphic shows a set of cross-sectional momentum-space density plots obtained by time-of-flight imaging, demonstrating non-zero occupancy of the zero-momentum ground state.

BEC is widely discussed in many undergraduate level statistical physics textbooks - here I will present a necessarily brief review on the concept to make explicit descriptions of more exotic forms of "condensed states" below. For an ideal non-interacting Bose gas occupying a box-like potential of volume \( V \), the total number \( N \) of particles can be found by summing over the occupancy of the single-particle state with a given energy \( \epsilon \) as
where $\beta = 1/kT$ with $k$ the Boltzmann constant and $T$ the temperature, and $z$ is the fugacity of the Bose gas, which is determined by the chemical potential $\mu$ as

$$z = \exp(\mu/kT).$$  \hfill (1.2)

In the thermodynamic limit, the volume $V$ of this gas is large, and the spectrum of the single-particle states is approximately continuous. One can replace the summation in equation (1.1) by integration, by making use of the asymptotic density of states in the non-relativistic limit

$$g(\varepsilon)\,d\varepsilon = (2\pi V/\hbar^3)(2m)^{3/2}\varepsilon^{1/2}\,d\varepsilon.$$  \hfill (1.3)

Using this, the summation can be taken to the continuum as a means to compute the density

$$\frac{N}{V} = \frac{2\pi}{\hbar^3}(2m)^{3/2}\int_0^\infty \frac{\varepsilon^{1/2}\,d\varepsilon}{z^{-1}\varepsilon^{1/2} - 1} + \frac{1}{V} \frac{z}{1 - z}. \hfill (1.4)$$

One may notice that the density of states $g(\varepsilon)$ above has a zero weight at $\varepsilon = 0$, which means the ground state contributes nothing to the statistical weight. This treatment is correct in the classical limit, but in quantum-mechanical treatment, each non-degenerate single-particle state should be assigned a statistical weight of unity. Following that principle, the ground state is taken out from the sum before the conversion from sum to integration. The last term in equation (1.4) captures the contribution from atoms in ground state in this way. It is valid now to take the lower integration limit of $\varepsilon = 0$, since $g(\varepsilon)|_{\varepsilon=0} = 0$ (a more rigorous justification can be found, for example, in R.K. Pathria [122]). Close to the classical limit where $z \ll 1$, this term is of the order $1/N$ and therefore negligible. When the value of $z$ is close to unity, this term has a scale that is comparable to $N/V$. Therefore, a macroscopic fraction of particles occupy the same single state $\varepsilon = 0$, and this phenomenon is the defining characteristic of Bose-Einstein Condensation.

We can rewrite equation (1.4), by replacing $z/(1 - z)$ by $N_0$, which denotes the number of particles in the ground state $\varepsilon = 0$, and we obtain a non-condensed density of atoms as

$$\frac{N - N_0}{V} = \frac{2\pi(2mkT)^{3/2}}{\hbar^3} \int_0^\infty \frac{x^{1/2}\,dx}{z^{-1}e^x - 1} = \frac{1}{\lambda^3} g_{3/2}(z), \hfill (1.5)$$
where \( \lambda = h/(2\pi mkT)^{1/2} \), and \( x = \beta \varepsilon \), while \( g_\nu(z) \) are Bose-Einstein functions defined as

\[
g_\nu(z) = \frac{1}{\Gamma(\nu)} \int_0^\infty \frac{x^{\nu-1}dx}{z^{-1}e^x - 1} = z + \frac{z^2}{2\nu} + \frac{z^3}{3\nu} + \ldots \quad (1.6)
\]

This is a function monotonically increasing with \( z \) at \( \nu = 3/2 \) and since \( z \) is bounded with an upper limit of unity, one obtains

\[
N_e = N - N_0 = V \frac{2\pi(2mkT)^{3/2}}{h^3} g_{3/2}(z) \leq V \frac{2\pi(2mkT)^{3/2}}{h^3} \zeta(3/2), \quad (1.7)
\]

where \( N_e \) represents the number of particles in all the excited states and \( \zeta(3/2) \approx 2.6124 \) is a Riemann zeta function. Equation (1.7) sets an upper limit for \( N_e \) - excited states can accommodate all of the particles if the total number of particles \( N \) does not exceed this limit. If temperature is low enough or the particle number large enough such that \( N > V \frac{2\pi(2mkT)^{3/2}}{h^3} \zeta(3/2) \), the remainder will be pushed to the ground state such that

\[
N_0 = N - V \frac{2\pi(2mkT)^{3/2}}{h^3} \zeta(3/2), \quad (1.8)
\]

A critical temperature can be obtained for fixed particle numbers in the system by setting \( N_0 = 0 \) and solving the equation above to find

\[
T_c = \frac{h^2}{2\pi mk} \left( \frac{N}{V} \zeta(3/2) \right)^{2/3}, \quad (1.9)
\]

below which, the system is a mixture of two different matter phases with \( N_e \) particles distributed over the excited states and \( N_0 \) particles accumulated in the ground state. Here we have \( N_e = N(T/T_c)^{3/2} \) and \( N_0 = N - N_e \).

So far, we have described the traditional picture of cooling a Bose gas into the ground state \( \epsilon = 0 \) of a static (box-like) potential. It is not usually discussed in textbook examples of this description, but one can view all the single particle quantum levels \( \epsilon \geq 0 \) as the modes that the Bose gas can take on, and that these modes are competing with each other in the thermodynamic limit. The reason is obvious - for a static potential, the state \( \epsilon = 0 \) always wins as the system is cooled, and the particles will never condense in another state. This is a consequence of having a well-defined time-independent and effectively closed system to consider thermodynamically in the grand-canonical picture.
It is natural to expect the condensate in our box might form in some generic set or superposition of states under slightly different conditions, like under the influence of a slowly varying force - after all, we would expect the condensate as whole simply moves under its influence. One can see how the picture above breaks down if one does so little as consider a box-like potential which is gently shaken in time. If we work in the laboratory frame, we might expect this perturbation drives transitions of atoms into excited states, and it is difficult to ascribe an equilibrium distribution like the thermal Bose-distribution assumed above. We could attempt to rectify this by transforming to a reference frame following the box, such that the walls are static again, but the inertial force in that frame is again a time-dependent drive. Naturally, there are a great number of tools to understanding phenomena like this, so we will consider one next.

1.1.2.2 Coherent States and Dynamics of the Order Parameter of a Condensate

To do this, we can try to appeal to the Landau-Ginzburg picture instead. As the temperature is lowered and condensation starts, we may also define an order-parameter and say that the long range coherence function of the matter field represented by it takes some non-zero value, which can be viewed as a phase transition which spontaneously breaks the $U(1)$ (or complex-phase) symmetry of the matter field. One quickly arrives at the Gross-Pitaevskii-Equation (GPE) [123], which describe the dynamics of this order parameter as a type of nonlinear wave equation describing the evolution of the condensed state. Under a periodically modulated force, like the example given in the last section, the order parameter’s time- (and space-) dependence can be described by this equation.

The GPE equation is very similar in spirit to the case in a laser that a single optical mode builds up from amongst all the different modes, ultimately achieving an electric field which is both large, and whose dynamics and structure are described by a non-linear optical wave equation determined by the large-signal properties (like gain-saturation) of the laser medium. In the case of laser operation, one acknowledges up-front the non-equilibrium nature of the process rather than use a thermodynamic equilibrium treatment as in the case of a Bose Einstein Condensate described in the previous section. The lasing state is understood as a steady state achieved by a balance between pumping and light emission. Below the lasing
threshold, the device simply amplifies spontaneous emission of light, while above threshold a macroscopic occupancy of one single mode is chosen, showing up as a coherent optical wave with phase coherence over the macroscopic distance of the lasing medium.

Therefore, this is also a phase transition that breaks a kind of $U(1)$ phase-symmetry for the electric field expressed as a phasor. But the laser example comes equipped with a canonical example of a complimentary treatment to the thermodynamic or steady-state conditions described above. Similar to the case for other types of oscillators, such as electronic oscillators built at lower frequencies, one can understand the lasing threshold using a small-signal analysis. If one considers the gain of a weak electromagnetic wave as it propagates across a cavity, the closed-loop transfer function demonstrates that a phase transition happens when some parameters are above threshold and the system takes a mode whose gain and phase response can make the closed-loop transfer function diverge. This divergence is again a signature of macroscopic build-up of a particular mode of oscillation, but it is ab-initio better-suited to the description of driven systems.

This analogy is a different physical perspective on viewing driven systems, and we will use it as inspiration for designing new attacks on quantum simulation. To provide a more serious justification and mathematical mapping between the two types of things, we will next try to build the analogy up from the microscopic physics by introducing the concept of a coherent state. To do this, we will try to work backward from the GPE equation somewhat, trying to treat it in a way similar to the analysis of a laser. The first step of this is to linearize the Gross-Pitaevskii picture, obtaining a kind of "small-signal analysis" of the atomic condensate. This is exactly the perspective taken in the standard treatment of elementary excitations of a condensate, usually obtained through a "Bogoliubov transformation" of its linearized dynamics [124]. The details of such an analysis can be found in many textbooks [125,126], but for the purposes of this discussion, can be viewed simply as a "dressing" of the atomic creation operators $\hat{a}^\dagger$ into a superposition with their Hermitian conjugates, the annihilation operators $\hat{a}$, and vice-versa.

In the case of a Bose Einstein condensate, following Bogoliubov ansatz of the order parameter, one can find that when undergoing spontaneous symmetry breaking, the condensate state is in or close to a coherent state $|\alpha\rangle$ defined by
\[ \hat{a}|\alpha\rangle = \alpha|\alpha\rangle, \quad (1.10) \]

\[ |\alpha|^2 = N_0 \] is the average number in the condensate. This coherent state can be derived as a displacement operator \( D(\alpha) \) applied to a vacuum state \(|0\rangle\).

\[ |\alpha\rangle = D(\alpha)|0\rangle = e^{\alpha \hat{a}^\dagger - \alpha^* \hat{a}}|0\rangle, \quad (1.11) \]

which represents a boost of the vacuum state without particles to one of an indefinite number - a superposition state of quantum states with different particle number.

In the Fock basis of states \(|n\rangle\) with definite number \(n\), this coherent state can be expanded as

\[ |\alpha\rangle = \sum_n e^{-|\alpha|^2/2} \frac{1}{\sqrt{n!}} \alpha^n |n\rangle. \quad (1.12) \]

According to the particle number states representation and the coherent state above, one can reason that above the BEC phase transition temperature, the Bose gas ground state is a statistical mixture of different particle number states which maximizes the entropy. Below transition temperature, the ground state approaches a pure coherent state as the temperature is lowered in which the entropy eventually vanishes at zero temperature. This coherent state is a linear superposition of particle number eigenstates, and its phase is localized by the interference between different particle number states \(|n\rangle\).

The same coherent state may be ascribed to the quantum-optical picture of a lasing state in an optical cavity [127]. In this case, the Fock-states \(|n\rangle\) represent states of definite photon number inside the cavity.

### 1.1.2.3 Coherent States of Other Many-body Systems

One might guess that the coherent state picture of a many-body physical system is not limited to just the Bose Einstein Condensate and laser, but also may appear in many other systems. In fact, the concept is simply one of macroscopic occupancy, combined with the idea that the phase and amplitude of a "most-nearly classical" macroscopic wave should come naturally imbalanced in uncertainty.

We could attempt to apply the same idea to the "particles" comprising any many-body system - making the ansatz that the majority, or the typical, many-body
ground state is a macroscopic occupation of "something." The "something" need not be the bare particles we made the system from before considering their interaction with each other, or with some other field. As Phillip Anderson said, "more is different" [128] - a complicated many-body system is generally characterized by its long-range correlations induced by such effects, and there is no reason to believe the "small-signal" or linearized analysis perturbs about such a simple state. In fact, this coarse state of the system can result in characteristically different small-signal behavior even with small changes in the parameters of the microscopic system, and this is precisely what one means by a phase-transition. Coherent states are widely applied in describing the collective modes of such systems, and can often be used to describe the quantum mechanics of small deviations from their ground states.

For example, one might consider the question of whether a system of fermionic particles can form a similar type of coherent state. Naturally, this is not possible if the coherent state is built from fermionic operators $\hat{c}^\dagger$ like the bosonic $\hat{a}^\dagger$ above, since the fermionic operators square to zero by the odd fermionic exchange symmetry. Nevertheless, if a strong attractive interaction is present between fermions, we might expect they form bound, paired, particles of integral spin which themselves act as bosons and condense into a coherent state. Naturally this can happen, and forms a basis of description for a molecular condensate, which has been observed with cold atoms [65,66]. In this case, the appropriate bosonic operator $\hat{a}^\dagger = \hat{c}_i^\dagger \hat{c}_j^\dagger$, where the states $i \neq j$ represent the molecular bound state.

One might also ask if the same type of pairing can happen in other ways. Of course, this too has a long history, and there exist another set of coherent states smoothly connected to the molecular states above. In the context of superconductivity, the Bardeen-Cooper-Schrieffer (BCS) theory describes a coherent state formed by Cooper pairs of fermionic electrons condensing into a boson-like superfluid state. A Cooper pair creation operator at momentum $k$ can be defined as $P_k^+ = c^+_k \uparrow c^+_\bar{k} \downarrow$. The BCS ground state can be built-up by applying this creation operator on the vacuum state

$$|\Psi_{BCS}\rangle = \text{const } e^{\sum_{k} \alpha_k P_k^+} |0\rangle,$$  \hspace{1cm} (1.13)

where $\alpha_k$ are complex and the ground state energy is minimized by adjusting them.

Since electrons are fermions that obey Pauli’s exclusion rule, multiple occupation of a Cooper pair state is not allowed. Hence, all powers from 2 and beyond of $P_k^+$
applied to are zero, and the BCS wavefunction can be written as a product state as

$$|\Psi_{BCS}\rangle = \text{const} \prod_{k=k_1}^{k_M} (1 + \alpha_k P_k^+)|\phi_k(0)\rangle,$$

(1.14)

where $|\phi_k(0)\rangle$ is the empty Cooper pair state involving $k$ and $-k$, and we have $|0\rangle = \prod_{k=k_1}^{k_M} |\phi_k(0)\rangle$.

Here $|\phi_k(0)\rangle$ and $|\phi_k(1)\rangle$ represent empty and occupied Cooper pair states which form a complete and orthonormal set. This BCS state shows that at low enough temperature and the presence of attractive interaction between electrons, electrons from Cooper pairs and behave like bosonic particles. Below a transition temperature, a macroscopic occupancy of a BCS state occurs and electrons flow in the system as a whole, which results in low temperature superconductivity. In fact, the coherent state of Cooper pairs is the basis for the Landau-Ginzburg picture of superfluidity and was one of the first applications of a phenomenological description of a phase-transition for a quantum-degenerate system [129].

1.1.2.4 Coherent States in Systems with Topological Order

In fact, the existence of so many states describable by a locally coherent quantum state leads one to wonder whether in fact all quantum many-body states at zero temperature could be described in this way. After all, if all many-body systems in nature follow this framework, a great number of physical phenomena could be described solely on the basis of this similarity, differing only by the microscopic nature of 'what exactly is condensing.'

To challenge this notion, one can attempt to choose as an example a state which in some basis, seems not to follow this model at all. A good example of this, and one we will expand on greatly below, is the state of a two-dimensional gas of fermionic particles in a strong magnetic field, which is known to enter into a series of fractional quantum Hall (FQH) states as a function of the strength of the magnetic field. The discovery of these states in the 1980’s threw a curve-ball at the Landau-Ginzburg framework, after it became understood that there was no 'local' state or order definable for the many-body state. In fact, to detect the presence of order in these systems, one must look over macroscopic distances.

The composite particle picture in the explanation of fractional quantum Hall
effect [130] gives us one intellectual tool to approach the possibility of a coherent
description of such states. In this picture, unlike the one above, particles
bind not to each other, but to a subtler excitation of the many-body system. The
existence of a strong magnetic field and charged particles with rapid microscopic
motion allows one to guess a kind of "binding" between particles and vortex-like
motion of the fluid they comprise. One might venture to guess that if the fractional
Hall states might be described in such simple language, there might also exist a
coherent-state like description of even these unique states.

In fact, this is certainly true for at least some of the FQH states. The explicit
construction of a Landau-Ginzburg state for fractional Hall systems requires some
rethinking of the few-and many-body arguments of molecular condensates and
Cooper-pairing above. It was described eloquently by Shoucheng Zhang and his
collaborators to approach FQH state through the Chern-Simons-Landau-Ginzburg
(CSLG) theory [131].

This idea is capable of capturing phenomenological aspects of the FQH states
with filling factors (the ratio of vortex to particle number) of \( \nu = 1/(2k + 1) \), where
\( k = 1, 2, 3, \ldots \). The simple physical picture constructed from this theory is to map an
interactive two-dimensional electron gas in external magnetic field into a bosonic
fluid under a special kind of gauge transformation.

One may start with a microscopic Hamiltonian of the two-dimensional electron
gas, held in a magnetic field described by the vector potential \( \vec{A} \),

\[
\mathcal{H} = \frac{1}{2m} \sum_i \left[ p_i - \frac{e}{c} A(x_i) \right]^2 + \sum_i e A_0(x_i) + \sum_{i<j} \frac{e^2}{|x_i - x_j|},
\]

(1.15)

here \( i = 1, 2, \ldots, N \) denotes the labels of electrons, and \( A_0 \) is the scalar potential
of any externally imposed electric field. Then the Schrödinger equation governing
the electronic wave-function is

\[
\mathcal{H} \Psi(x_1, \ldots x_N) = E \Psi(x_1, \ldots x_N),
\]

(1.16)

where \( \Psi(x_1, \ldots x_N) \) is a totally antisymmetric space wave function describing the
electron system, with electron coordinates \( x_i \). One might physically reason that a
moving electron, particularly one with circulation in its velocity field, might generate
a microscopic magnetic field of its own. That idea might serve as inspiration for a
simple trick - one can introduce a gauge transformation by adding a new vector
\( \mathbf{a}(x_i) \) potential along with the existing vector potential \( \mathbf{A} \):

\[
\mathbf{a}(x_i) = \frac{\phi_0}{2\pi} \theta \sum_{j \neq i} \nabla \alpha_{ij}, \tag{1.17}
\]

where \( \phi_0 = \frac{hc}{e} \) is the unit of flux quantum, and \( \alpha_{ij} \) is the angle between a vector from particle \( i \) to particle \( j \) and a reference direction. A new Schrödinger equation can be written down with this new Hamiltonian, taking \( \vec{A} \rightarrow \vec{A} + \vec{a} \)

\[
\mathcal{H}' \phi(x_1, ..., x_N) = E' \phi(x_1, ..., x_N) \tag{1.18}
\]

Here, \( \phi(x_1, ..., x_N) \) now represents a bosonic wave function, which is totally symmetric under the exchange of coordinates. It is easy to prove that these two different behaviors - anti-symmetric exchange in the fermionic \( \Psi \) from (1.16) and symmetric exchange from the bosonic \( \Phi \) (1.18) are consistent when

\[
\theta = (2k + 1)\pi, \quad k = 0, 1, 2, ...
\]

because the effect on the orientation of angles \( \alpha \) used to form the statistical gauge field (1.49) have an odd parity effect under exchange.

Using this, a problem with fermions in external field is mapped into a problem with bosons in external field plus the statistical field \( \mathbf{a} \). Such gauge interaction can be derived from a Chern-Simons action term, which also gives rise to the equations describing the dynamics of this field, analogous to the continuity equation of electromagnetic field determined by Maxwell’s equation. A mean field solution of the CSLG action in the presence of an external magnetic field describes a boson superfluid state which fixes the filling factor at

\[
\nu = \frac{\pi}{2\chi + 1}.
\]

Such phase transition happens because when equation (1.20) is satisfied, the statistical field and external field cancel each other. This superfluid phase of the boson field corresponds to the quantum Hall phase of the original electron system and such formalism can be viewed as mapping electrons to bosons carrying the same charge and bound with \( (2k + 1)\phi_0 \) flux quantum. Then the FQH state is a coherent state of the boson field when external magnetic field cancels the flux quantum.
Zhang’s composite particle picture is very similar in spirit to the composite fermion theory constructed by Jain [130]. Although both theories originated from the context of the FQHE, the composite fermion picture captures most values of the filling factor $\nu$ observed in experiments while CSLG only provides wave function for the $\nu = 1/(2k + 1)$ class. Composite fermion theory has also gone beyond the context of FQHE and find its applications in more physical systems. It’s far more difficult to identify the coherence in the composite fermion pictures but it’s appropriate to argue that some classes of FQH states can be expressed in coherent states. We will come back to the concept of composite fermions in later sections on the context of fractional quantum Hall effect.

At this point, we have shown several examples of coherent states in time-independent many-body physical systems and undoubtedly there are many more examples than we could introduce here. Together, these cases should provide some simple anecdotal evidence that one may be able to take advantage of the coherence of these many-body physical states, and describe many in the same language we more often use in the description of a laser, which while also described by a coherent state, lacks the concept of a "ground state" that the many-body examples possess in an appropriate frame. The final example is particularly important for several reasons - one is for the reason we laid out at the beginning, it is simply at the outset challenging to see how it could follow the Landau-Ginzburg framework.

But another we can see from how the states were put together - there was an aspect of motion in the state of individual particles in its ground state. When one steps back a bit from this, there is a simple kind of insight - motion in this state exhibits a kind of "self-synchronized" behavior. Without the statistical gauge-transformation that turned a fermionic wave-function into one that was effectively bosonic, we had to think of the internal "vortex-like" motion of the fluid has bound itself to the particle-like excitations away from its mean-field behavior. That is a very complicated internal mechanical synchronization problem. By transforming to a kind of many-body "reference-frame" by introducing the statistical transformation, we apply a simpler picture of condensation into a coherent state.
1.1.2.5 Synchronization of Coupled Oscillators - the Adler and Kuramoto Models

This idea of 'synchronization' is not so familiar in the context of many-body theory. In fact, one can read textbooks on the subject and never see the word in print. The situation is very different, however, if one opens a textbook on laser physics or electronic oscillators, where the problem of a non-linear oscillator driven by a periodic force is a defining type of problem. For classical electronic oscillators, this problem was studied in detail by Adler in the 1960's [132], and for laser oscillators, becomes of central importance under the heading of injection-locking phenomena [133–136]. Our application of these ideas to many-body quantum systems, pervasive in condensed matter physics, will be somewhat transitive from that perspective - we will be applying the non-equilibrium notions inherent to these fields to the more commonly thermodynamically-motivated ideas in cold quantum matter.

A great way to excuse this is to point out it has often been done before in loose language by experimental atomic physicists when confronted with the task of describing the physics of Bose condensation of atoms while working in a laboratory reality dominated by laser physics. It is often a source of cultural disconnect between those familiar with the 'traditional thermodynamic' story presented above, with a more mechanistic picture of why Bose-condensation is favored at low temperatures and high-density. We hinted at it coarsely above - as the temperature of a many-body atomic gas is cooled, the coherent de-Broglie wavepacket extends over longer distances, spanning the inter-particle separation, and allowing individual atoms to 'sync-up,' forming an extended coherent matter wave. The language is nice, but there is not an obvious manifestation of that picture present in the thermodynamic argument predicting macroscopic occupation of a ground state.

The coherence in Bose Einstein condensate and other many-body physical systems naturally guide one to relate those coherent states to a more generalized concept - the idea of many classical oscillators driving one another. In laboratory systems, even some of the lasers used to construct the experiment described later in this thesis, coupled oscillators are known to synchronize to each other. This is obviously demonstrated daily in quite a number of physical systems, such as two pendula hanging from the same beam [137], small numbers of injection locked lasers [138–141], and even chirping crickets [142], clapping hands in the audience at
an orchestra concert [143], neural networks [144], flame dynamics [145] and some simpler physics systems such as coupled arrays of Josephson Junctions [146].

Synchronization of coupled oscillators is a well-studied phenomenon which has its importance in both theoretical studies and technical applications. For just a single driven, or two coupled non-linear oscillators, different behaviors can occur as energy and coupling strength vary. Many of the concepts in this region are well understood, for example, the 'locking range' over which one oscillator will stay synchronized with another as their isolated oscillation frequencies are varied is easily described by the so-called 'Arnold tongue' [147], and sub-harmonic locking is described by the devil’s staircase [148]. Several of these ideas will be important in the experiments below, but here we want to connect the concept of oscillator-locking to the many-body coherence problem above. For this, we want to consider the many particles of a many-body system as individual oscillators. In the thermodynamic limit - essential for our definition of condensation in the thermodynamic treatment - the number of such oscillators is large.

A large set of oscillators weakly coupled to each other can be characterized by the famous Kuramoto model [149,150], and one can identify a smooth connection between Kuramoto model and the Bose Einstein condensation using the 'experimentalists’ loose language' above. One can also view a multimode laser medium as a large set of coupled oscillators corresponding to inhomogeneously polarized regions of a gain medium, and a lot of phenomenon such as mode competing can be described in the language of Kuramoto model. Oscillator-locking in laser physics and elsewhere is inherently a non-linear phenomenon, and from that perspective, even the mechanical aspects are occasionally challenging.

In that light, it is worth a brief review of the Kuramoto model and its synchronization phenomena. Usually the Kuramoto model is defined by an equation of motion for the "angular variable" of an oscillator following the equation of motion

$$\frac{d\theta_i}{dt} = \omega_i + \frac{K}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i), \quad i = 1 \ldots N. \quad (1.21)$$

The angular variable $\theta_i$ can represent the angular orientation of a pendulum, the phase of motion of an oscillator, or the phase of the order parameter in a $U(1)$-broken symmetry system like a condensate. Notice that without the second term on the right side, the motion of this angle is simply one with a fixed angular
velocity $\omega_i$, which defines the natural frequencies of motion of a set of decoupled oscillators. For $K \neq 0$, the motion is both coupled and non-linear. In detail, the equations above describe the phase dynamic of the $i$th oscillator in an all-to-all coupling oscillator network in which the coupling between two oscillators depends on their relative phase through a sinusoidal function of their phase difference.

A few assumptions are often made (and can easily be lifted) in this model, such as identical or near identical oscillator frequencies $\omega_i$ and weak couplings $K/N \ll \omega_i$. The model is exactly solvable in some cases by using certain transformations [149,151], and a mean-field-like picture can be constructed to describe its coarse behavior.

The latter approach is useful for understanding transitions between different types of behavior. There are two important regions of dynamics of the Kuramoto model, namely an incoherent state at small $K$ and a coherent state for stronger coupling strengths $K$. In the low $K$ limit, there’s no coherence and all oscillators drift randomly with independent phases and frequencies $\omega_i$; when $K$ is sufficiently strong, this system can achieve a fully synchronized state, in which all the oscillators share a common frequency; there is also an intermediate region in which the system is partially coherent, namely some oscillators synchronize and others stay in the incoherent state.

This phase transform behavior should sound a bit similar to the condensation of a Bose gas. With little effort, one can connect Kuramoto model to a certain class of Hamiltonian systems, which characterizes a class of classical systems including Bose-Einstein condensates [152].

One may also relate Kuramoto model to a multi-mode laser medium where mode locking can happen. Particles in the laser medium have inhomogeneous gain profiles and transfer functions, leading to natural oscillators that tend to oscillate around their own natural frequency when coupling is negligible. Macroscopically coherent states happen when a macroscopic portion of the particles are frequency-locked together and lasing at the same mode. This provides another viewpoint of correlating coupled oscillator systems to physical systems with coherent states. With that analogy in mind, one can look for an alternative way of treating the difficulties in the field of quantum simulation, which forms the kernel of this thesis, discussed in the next subsection.

In this way, we can form a more quantitatively explicit connection between locked-
oscillator dynamics and the creation of coherent many-body states in quantum simulation experiments. One should note that it is not as if we have given the means by which to solve an impossibly difficult many-body problem which was otherwise intractable. For example, the mode-competition problem in a generally non-linear lasing medium is also often impossibly difficult to solve from first principles. In that light, we have only taken one difficult non-linear problem - that of a time-dependent many-body system - and replaced it with another. What we have gained, however, is a new "experimentalist’s perspective" on how to approach the manufacture of many-body states using driven oscillator systems.

1.1.3 Engineering Many-Body States by Driving Coherent Oscillator Dynamics- quEASE

The traditional means for quantum simulation generally begin with a step which would be described in the more general context of quantum computation as "initialization." Here, one generally cools a versatile quantum system, such as a condensate, into an easily understood quantum ground state of a time-independent system using known cooling techniques. Depending on the simulation target, one may start with bosonic or fermionic constituents, and perhaps confined in simple geometries like a bulk trap or an optical lattice. From this well-defined initial state, one adiabatically changes the applied fields and potentials toward an engineered target Hamiltonian. Typically, it is difficult to engineer continuous cooling processes that continue to be effective through the adiabatic change of Hamiltonian, and as a result, it is necessary to keep the heating mechanisms below some level deemed sufficient to stay near the ground state.

Unfortunately, it is often very difficult to place an ab-initio theoretical bound for this heating rate, and moreover, it is also often difficult in practice to reliably measure these heating rates. Further, as described in previous sections, in many instances of quantum simulation - particularly those involving the simulation of gauge field dynamics - a number of obstacles must be overcome in engineering a targeted quantum many-body state. These take the form of both theoretical and technical difficulties, primarily involving the proper thermodynamic treatment of driven (time-dependent) quantum systems.

The analogy built above between coherent state descriptions of many-body
ground states and the character of oscillation of a single oscillator (or a set of coupled oscillators) offers a different path by which experiments may create unique many-body states of quantum matter in a potentially more robust way. Since the idea of such a process builds up a quantum state by amplifying quantum noise (much in the way a laser does), we will refer to this method of quantum simulation as "quantum Engineering by Amplified Stimulated Excitation" (quEASE). A sketch of the conventional cooling method is illustrated in figure 1.2a for comparison - as illustrated in figure 1.2b, rather than cool into a defined ground state, the quEASE method builds an oscillator feedback loop containing the quantum system under study.

The idea of quEASE is quite simple - like any servo, the system is probed, filtered, and a signal is fed back to manipulate the system in order to control its behavior. Just like any other systems with feedback control, its coarse behavior is determined by the stability conditions for the servo loop. The relevant parameter governing stability is the closed-loop transfer function, including the response of the measurement, actuator and filter between them. By adjusting the closed-loop gain via these three components to make the loop oscillate, a coherent state is built up in the physical system when the overall small-signal transfer function diverges for some mode, and system as a whole responds by increasing the amplitude associated with this mode into the large-signal regime, where non-linear effects begin to dominate. The oscillating mode now plays the role of a ground-state in a static system, but its description is natural for the time-dependent problem - for example, we have no problem with the approximate treatment of a rotating-wave approximation, or the thermodynamically confusing prospect of an infinite Floquet-ladder of quasi-energy states.

In order to understand the basic oscillation mode of this feedback loop, one can be guided by the simple ideas familiar from the small-signal analysis of a conventional electronic oscillator or the regenerative optical resonator in a laser. The analysis of such systems is often somewhat ad hoc, but can be viewed in a loose sense as the summation of a signal after an infinite number of passes through a medium with a defined small-signal gain in a single pass of $\tilde{g}_{\text{open}}$. In a conventional oscillator, the closed-loop gain can then be described as the geometric sum

$$\tilde{g}(\omega) = \sum_{n=1}^{\infty} \tilde{g}_{\text{open}}^n = \frac{1}{1 - \tilde{g}_{\text{open}}(\omega)},$$  (1.22)
Figure 1.2. Conventional quantum simulation methods by engineered ground state and cooling and the proposed quEASE scheme of quantum simulation in cold atoms. (a) In conventional quantum simulation, the atomic gas is prepared in some type of engineered environment in order to create a specific Hamiltonian, and cooled into its many-body ground state. (b) Quantum Engineering by Amplified Stimulated Excitation (quEASE), is more similar to driving an oscillation in a feedback loop, or pumping a laser medium to emit coherent light, and creates a many-body coherent states by driving the system into oscillation. The selection of a specific coherent state is done by design of specialized filters, amplifiers or detectors.

where \( \tilde{g} \) is typically written as a complex gain describing both the amplitude and phase response of the medium. In this picture, \( \tilde{g}_{\text{open}}(\omega) \) is the open loop gain for the entirety of the loop, which depends on all the details of the feedback loop, including measurement, filter and actuator. Oscillation occurs when \( |\tilde{g}_{\text{open}}(\omega)| = 1 \), with a phase response approaching 0°, under which condition \( \tilde{g}(\omega) \) diverges.

The open loop transfer function can most easily be altered by inserting a designed filter into the feedback loop. For only a single relevant mode, the gain is simply a complex number - its amplitude profile against frequency can be shaped and the phase relation adjusted by the design of a simple electronic filter, and thus
the oscillation mode, the threshold and system behavior around the threshold can be varied. Analysis of electronic oscillators are ubiquitous in any electronics or control theory textbook [153], and often characterize the small-signal gain dependence with Bode plots.

In the case of regenerative optical resonators, i.e., lasers, a single mode may be selected by limiting the gain to be substantial only for a single optical mode. In these cases, optical feedback is limited using spatial mode selector, such as an intra-cavity aperture. This mode selector can also be considered as a "mode filter", which changes the transfer function of the extended laser medium. Following those existing ideas for the laser system, we are naturally led to consider a filter for many-body quantum systems to choose a specific targeted coherent state in the quEASE scheme. As the oscillation mode eventually depends on both the spectral shaping of the gain and mode-dependent gain introduced by a mode filter, one obtains control over the target state by adjusting either the filter inside the feedback loop or the mode-filtration scheme. Alternatively, one can consider making mode-selective measurement schemes - for example, if one is interested in creating many-body states with a specified value of a particular response function (e.g., the magnetization), one can produce an oscillator specifically tailored to measure and feedback to the system based on its magnetic response functions.

At this point, we have described just the basic elements of the quEASE method for engineering many-body quantum states. Naturally, one can envision a number of refinements on this simple idea of an oscillator with a designed feedback loop, such as making the filter actively tunable, stimulating the system with an external signal, or including multiple loops coupled to the original one through the physical system under study. With some combination of these techniques we can not only arrange to create a specific many-body state, but also interrogate it to know that we have been successful. Many of these ideas ultimately will be applied in the actual design of the experiments with cold atom described in the remainder of this thesis. For now, we will concentrate on applying the simplest version of this design to an important target ground state using a very specially designed filter. The target state will be an analog to those found in fractional quantum Hall states of electrons, and the filter we will use to achieve this will sensitive only to specialized "topologically interesting" response functions, analogous to the off-diagonal Hall resistivity of the electron systems.
Before we get to the detail of the experiments, it is worth broadening our introduction to these types of systems, and considering the capacity of this innovative method of quantum simulation of specifically these states. As described in the previous section, the composite fermion system models the many-electron physical system with certain fractional Hall topologies well. In next section, we will have a brief review of the Integer and Fractional Quantum Hall physics, specifically with an eye toward designing an appropriate "mode selecting" filter that keys-in on building-up specifically the states we are most interested in using the quEASE method.

1.2 Integer and Fractional Quantum Hall Effect

This sections introduces the discoveries of integer and fractional quantum Hall effects and presents an overview of theoretical approaches to understand the origins of these phenomena. Subsection 1.2.1 introduces the experiments in which Hall conductance of electron gas were observed to quantize to integer and fractional values; Subsection 1.2.2 and 1.2.3 discuss theories of explaining the microscopic origin of quantum Hall effects.

1.2.1 Quantum Hall Effects

The discovery of quantum Hall effects were a major breakthrough and Laughlin’s quantum pump picture provided an elegant explanation of its microscopic origin, which is discussed in this subsection. The discovery of fractional quantum Hall effect is also introduced here.

1.2.1.1 Integer Quantum Hall Effect

The experimental observation of the exact quantization of resistance in integer quantum Hall systems (IQH) - 2-dimensional electron systems - was first achieved by Klaus von Klitzing in 1980 [110], and fractionally quantized Hall conductance was first observed shortly afterward by Daniel Tsui, Horst Störmer and Arthur Gossard in 1982 [111]. Since then, quantum Hall physics has attracted the devotion of generations of theorists and experimentalists alike to the study of this class of exotic phenomenon. The work that has been devoted into this field of study has not only
deepened our understanding of phase transitions in 2-D materials in low temperature and strong magnetic field, but also extended into other physical regimes and established new branches of research fields. The quantum Hall effects, both integer and fractional, have shown their broad significance due to fundamentally new effects in connection to basic quantum field theory \[131,154,155\], precision measurement \[156–158\], and the importance of topology in defining ordered matter \[159–162\]. Klaus von Klitzing was awarded the 1985 Nobel Prize for the discovery of the integer quantum Hall effect, and the 1998 Nobel Prize was shared Robert Laughlin, Daniel Tsui and Horst Störmer for the discovery and theoretical explanation of the fractional quantum Hall effect.

The first experimental realization of the Integer Quantum Hall effect is illustrated in figure 1.3a. The measurement was carried out in a two-dimensional electron gas in a silicon-based material. When the temperature is sufficiently low and magnetic field is strong, the Hall conductance of the electron system takes on the quantized values

\[
\sigma = \nu e^2/h, \quad \nu = 1, 2, 3, \ldots
\]  

(1.23)

Here, $e$ is the electron charge, and $h$ is the Planck constant. The integer quantization of the filling factor $\nu$ is shown as Hall plateaus in the plot of Hall conductance/resistance. The plateaus are exact and robust, and do not depend sensitively on the microscopic details of the system. Subsequent works have detected exact quantization of Hall conductance in many more systems such as other Gallium Arsenide heterostructures \[158,163\], graphene \[164\], and magnesium zinc oxide \[165\].

Today, a majority of integer quantum hall experiments are performed in the gallium arsenide materials. The quantum Hall effect has importance in precision measurement since it defines a new standard of electrical resistivity. It also provides a precise measurement of the fine structure constant $\alpha$, a physical quantity of primary importance in quantum electrodynamics, and more broadly in quantum mechanics at large. It is perhaps somewhat surprising that such precise measurements can be carried out in the intrinsically disordered environment of a solid-state crystalline system, and ironically, tend to present more robustly quantized Hall plateaus in the presence of disorder than in its absence.

There are really two means by which to explain why this occurs - one through the existence of localized states centered on crystalline defects, and a second appealing
Figure 1.3. Quantization of Hall conductance in the 2-D electron gas and Laughlin’s charge pump argument. (a) The plot presented shows the Hall resistance and longitudinal resistivity as functions of external magnetic field, which was experimentally measured by Klaus von Klitzing in a GaAsAl$_x$Ga$_{1-x}$As heterostructure (Figure adapted from Klitzing Nobel Prize Lecture Dec. 9, 1985). The temperature of the electronic system is about 8 mK. As the strength of the magnetic field increases, the Hall resistance $\rho_{xy}$ increases step-like, and shows plateaus where the longitudinal resistance $\rho_{xx}$ disappears. Here, the important quantity is $h/e^2 = 25812.807557(18) \Omega$, the von Klitzing constant named after his discovery of exact quantization. Quantization is demonstrated by the Hall conductance taking multiples of $e^2/h$, or the Hall resistance taking values of $h/\nu e^2$, where $\nu = 1, 2, 3, 4, \ldots$. In this plot, one can clearly identify those plateaus corresponding to $\nu = 3, 4, 5, 6, 7, 8, 9$. (b) This sketch from Robert Laughlin illustrates his charge pump argument as an explanation of the origin of exact quantization. In this thought experiment, a strong magnetic field $B$ is perpendicular to a 2-D sample. A gauge transformation of adding a constant vector potential, which has no physical meaning otherwise, acquires one if the sample is wrapped into a loop in the direction that the current $I$ flows. As the magnetic flux $\phi$ adiabatically increases by one flux quantum $\Delta \phi = hc/e$, one electron per Landau level is transferred from one edge to the other, depicted in the diagram, as an arrow from the hollow circle to the solid dot. Laughlin’s argument successfully explained the exact quantization at the microscopic level, however, as a macroscopic quantity, the Hall conductance requires the calculation of expectation values defined by fields across the entire sample. The explanation is robust to deformations of the loop, implying the quantization is robust against non-topological manipulations; this was understood later with the introduction of topological numbers. (Figure adapted from Robert B. Laughlin’s Nobel lecture Dec. 8, 1998)
to some elegant ideas from topology. In later sections, we will use the topological insight as motivation to develop a 'topological mode filter' in a quEASE scheme to generate quantum Hall states in atoms.

Electrons, confined in a two-dimensional strip of material and subjected to a magnetic field, follow cyclotron orbits. In the quantum mechanical regime, these orbits are quantized in the sense that their energy takes discrete values, known as Landau levels. When the magnetic field is strong enough, these Landau levels are so degenerate that all the electrons occupy only a few Landau levels. In a quantum Hall experiment, plateauing of $\sigma_{xy}$ occurs when there is no dissipation and $\sigma_{xx}$ vanishes. Without appealing to topology, one can understand the existence of plateaus in the resistivity as arising from magnetic field values for which the chemical potential (in a non-interacting system, the Fermi-energy) sits in the gap between landau levels corresponding to integral cyclotron motion [166].

From this simple theory, the exact quantization would seem to require a precise match between the carrier density and magnetic field strength. This would seem to argue that a Hall plateau should have 'zero width' in magnetic field. However, robust quantization of the Hall conductance over a wide range of magnetic field strength value has been observed, even when the systems have impurities that are not quite under control. In fact, weak disorder is essential for the observation of quantum Hall effect in real experiments - disorder introduces localized states with energies intermediate between Landau levels. Without them, it is meaningless to consider the case where the chemical potential resides between levels, and with them, plateaus are broadened to finite widths.

There is another side to this story, drawing on less familiar topics in topology, but compelling as an explanation of this same robustness. Using topology, one can take the effects from both disorder and gauge invariance into account. This explanation was first emphasized by Robert Laughlin, and is illustrated in fig 1.3(b). This takes the form of a gedanken experiment - one considering how electromagnetic flux winds through a Hall sample (and the associated wire loops necessary to measure voltage across it) often referred to as 'Laughlin's charge pump.'

By treating a two-dimensional sample as a ribbon wrapped up into a loop and magnetic flux $\phi$ penetrating through the loop at normal direction, one can derive the total current as the adiabatic derivative of total energy with respect to magnetic flux from Faraday's law of induction.
\[ I = \frac{\partial E}{\partial \phi} \quad (1.24) \]

As the magnetic flux varies by a single quantum \( h/e \), the total energy can change only through re-distribution of the electrons in their Landau levels since the Hamiltonian is adiabatically connected to original one through a gauge transformation. This results in a net effect that for every flux quantum, one electron in landau level \( n \) is pumped from the left of the sample to the right in fig 1.3b. As the potential drop between two sides is \( V \), we have the Hall current as

\[ I = \frac{neV}{\Delta \phi} = \frac{e^2V}{h}, \quad n = 1, 2, 3, \ldots \quad (1.25) \]

and therefore, the Hall resistivity is easily obtained as \( R_{Hall} = ne^2/h \). The topological aspect of this argument is clear when one considers the redistribution of flux through this process - by transporting one or more magnetic flux quanta across the ribbon in a "pumping process", a single electron’s guiding center is moved from one edge of the strip to the other; in doing so, the phase winding in its wave-function is mapped from one edge of the strip to the other, and the magnetic flux tied to it is transported into the interior of the loop. We can therefore see that Laughlin’s charge pump provides a way of understanding the quantization of Hall conductance from the adiabatic manipulation of the wave-function of a single electron. It is easy to see that if the geometry of this system is smoothly manipulated without change of topology, the pumping process is unchanged.

Today, it is understood that the integers in the quantum Hall experiments are examples of topological quantum numbers, which are known more generally as Chern (or TKNN) numbers \([159,167]\). They are examples of "topological invariants" - numbers which are defined by the global topological properties of a medium or state, but insensitive to local details. As a result, transitions between states corresponding to different topological numbers require a global reconstruction of the wave-function, and special relations can be found between probability currents on the edges of the system and properties in the bulk. Today, there are several other condensed matter systems with well-understood topological invariants \([168,169]\), so the idea extends well beyond explanation of the Hall effect. Similar arguments can also be applied to periodic systems in the extreme limit that the magnetic flux quanta per lattice plaquette becomes comparable to one. This case is described by
the so-called 'Hofstadter butterfly' [170] model and can also be used to introduce topological numbers.

With this argument, we have only justified quantization in the integer quantum Hall effect and other single-particle or non-interacting systems. The full explanation of the fractional quantum Hall effect requires substantially more elaborate arguments applied to the many-body wave-function, for which we gave some flavor in our discussion of the Landau-Ginzburg picture applied to fractional quantum Hall states in a previous section. In the fractional case, topology plays a subtler role in defining the transport properties - the 'statistical gauge field' described previously introduces a non-trivial modification to the story of Laughlin's charge pump above.

1.2.1.2 Fractional Quantum Hall Effect

Just two years after the experimental observation of the integer quantum Hall effect, the more exotic fractional quantum Hall effect phenomenon was discovered. In two-dimensional systems of electrons formed in extremely pure gallium arsenide heterostructures created with modulation-doping, Tsui and Störmer found the Hall conductance, in units of $e^2/h$, can be quantized to rational fractions instead of simply the integers as above [111]. As shown in figure 1.4, samples subjected to strong magnetic fields and held at low temperature reveal plateau structures similar to those in the integer quantum Hall measurements, but at the many more possible fractions formed by integer numbers. We might replace the integer $n$ in the integer Hall effect, and label these plateaus by a new rational index $\nu$. The clearest fractional plateau in early measurements corresponded to $\nu = 1/3$, but very quickly many more plateaus became visible as sample qualities improved. Similar to the case for the integer Hall system, it is sensible to consider the index $\nu$ as a 'filling factor,' i.e. the ratio of electrons to magnetic flux quanta. Among the filling factors observed, $\nu = \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, etc$ and $\nu = \frac{2}{3}, \frac{3}{5}, \frac{4}{7}, etc$ are the principally visible series of fractions in experiment. The description of these states presented a clear theoretical challenge after their unanticipated observation, and much effort was devoted to this description.

Unlike the integer quantum Hall effect, which can be understood with a single particle wave-function, there is no known way that the FQHE cannot be understood through the behavior of individual electrons in a magnetic field, and thus the effort to understand these states concentrated on the collective effects introduced by the
Figure 1.4. Measurement of the fractional quantum Hall effect in a 2-D electron gas system. Here, the Hall resistance $\rho_{xy}$ and longitudinal resistivity $\rho_{xx}$ are plotted as functions of magnetic field $B$ at four different temperatures, 0.48K, 1.00K, 1.65K, 4.15K. A GaAsAl$_{0.3}$Ga$_{0.7}$As sample is cut into the shape for conventional Hall conductance measurement shown in the inset. $\rho_{xy}$ is in the unit of Klitzing constant, and clear integer quantization can be seen on the left side of the plot, as Hall resistance takes value corresponding to integer filling factors while $\rho_{xx}$ vanishes. The top scale shows the filling-factor $\nu$ defined as $\nu = nh/eB$. At the extreme quantum limit as the magnetic field $B$ is around 150kG and its temperature below 1K, a new plateau emerges at $\nu = 1/3$, referring that the Hall resistance quantized to a rational fractional value as $\rho_{xy} = 3h/e^2$. This corresponds to $1/3$ of an electron's charge pumped per magnetic flux quanta in Laughlin's flux winding experiment. More and more rational filling factors have been discovered in following experiments. Understanding such emergent quantum phases requires one takes into account the long range interaction and entanglement of electrons in their ground state. (Figure adapted from [111])
interaction between electrons and its effect on the many-body wave-function. With a strong single-particle degeneracy (like that introduced by the collapse of the many individual states inside a single Landau level) and sufficiently strong interparticle interactions, one would anticipate these states are strongly correlated.

Moreover, combined with the topological properties inherent in the integer system, one might guess that the fractional Hall states corresponding to different plateaus are describable by the topological aspects of the many-body wave-function. For many states, these ideas turn out to be true, and many of the fractional Hall states are known today to represent a new class of many-body states that conform ideas in "topological field theory" [171]. There are many theoretical concepts dedicated to understanding various states observed in experiments, some following this idea very closely, and others introducing a range of new tools to understand these states. Major steps in the development of theory of fractional quantum Hall effect were marked by Laughlin’s trial wavefunction [112] for the $\nu = 1/3$ state and quasiparticles, Bertrand Halperin’s fractional exchange statistics of quasiparticles [172], the Moore-Reade states [173], Duncan Haldane’s hierarchy of states [174] and Jainendra Jain’s composite fermions [130], which will be discussed shortly in next subsection. The clearest picture of states without allusion to the theoretical ideas proposed in these descriptions can be obtained by direct numerical diagonalization of model Hamiltonians [175]. Today, the understanding of these candidate states is very advanced, but to some extent the unambiguous association of these wave-functions with different experimentally observed plateaus is still an on-going process, particularly for some more exotic states and filling fraction.

From the totality of these theoretical constructs, the impact of discovering the fractional quantum Hall effect is clear - these phases of matter challenge the paradigm of Landau’s (local) symmetry breaking theory [166]. Precisely because these new topological invariants are global properties of the state, it is now impossible to introduce an order parameter defined entirely by local properties of the microscopic matter. Since this time "topological order" has begun to play an essential role of the understanding of some new states of matter. Because of this impact, the 1998 Nobel Prize of physics was awarded to Tsui, Störmer, and Laughlin for the discovery and explanation of FQHE, though credit should be extended to the full community described above.
1.2.2 Approach to the Understanding of Fractional Quantum Hall Effect

To understand the origin of fractional quantum Hall effect, a lot of theoretical models have been proposed. This subsection provides an overview of these approaches.

1.2.2.1 Laughlin’s Quasi-Particle Wavefunction

Laughlin’s argument of a flux-winding gedanken experiment provides a straightforward picture for demonstrating the exact quantization in the integer quantum Hall effect. This phenomenological method is also sufficient to explain the fractional filling factors observed by Tsui and Störmer, with the aid of a guess for the many-body wave-function.

Unlike the single particle state in the IQHE, which can be adiabatically deformed to a non-interacting electron state, fractional quantum Hall states reveal an entirely new state of matter, in which interaction plays an essential role. Based on the fact that the only qualitative difference between IQHE and FQHE is the quantum of the plateau observed in measurement, Laughlin suggested that - similar to IQHE - the argument concerning the deformability of the wave-function and the exact quantization should still hold in the FQHE. Therefore, in some way, the fractional quantum Hall state must be possible to describe as a state with some type of fractional filling factor of a Landau level.

There are also some simple mathematical tools one can use to guide the construction of a candidate wave-function. In two-dimensions, it is convenient to express the two rectilinear coordinates of a particle \( x, y \) in a simpler form as complex numbers \( z = x + iy \) and its conjugate \( z^* \). This step is a bit more convenient than just its notational simplicity - for example, any harmonic function of these coordinates (one satisfying Laplace’s equation in two-dimensions) is analytic, and can only depend formally on \( z \) and not its conjugate. Restricting the wave-functions in this way limits the choices considerably, leaving the only choices harmonic in all particle’s coordinates to be expressible as a power-series in the \( z_i \) for all the particles. Within that subset, one might work to find strongly correlated states, where the probability to find two particles in the same position is low, and thereby reduce the interaction energy. Finally, electrons are fermionic, so the wave-function should have odd exchange symmetry. Within those limitations, Laughlin proposed the prototype...
ground state for the $\nu = 1/3$ FQHE as

$$\Psi_m(z_1, ..., z_N) = \left( \prod_{j,k} (z_j - z_k)^m \right) \times \exp\left[ -\frac{1}{4l^2} \sum_j |z_j|^2 \right],$$

(1.26)

where $m = 3$ is an odd integer. Aside from the gaussian envelope with width $l$ modulating the total density, the wave function is analytic in all of the coordinates $z_j = x_j + iy_j$ for the location of the $j$th particle expressed as a complex number.

This ground state was conceived as a variational ground state of the Hamiltonian

$$\mathcal{H} = \sum_j \left\{ \frac{1}{2m} \left( \frac{\hbar^2}{i} \nabla_j - \frac{e}{c} A(r_j) \right)^2 + V_{\text{ion}}(r_j) \right\} + \sum_{j<k} v(r_j - r_k),$$

(1.27)

where the magnetic field is introduced through the vector potential

$$\vec{A}(\vec{r}) = \frac{B}{2} (xy - yx) \quad \text{and} \quad V_{\text{ion}} = -\rho \int_{\text{sample}} v(\vec{r} - \vec{r}') d\vec{r}'.$$ 

(1.28)

The potential $V_{\text{ion}}$ presented by the crystal ions balances the natural coulomb interaction between electrons $v(r) = e^2/r$ by a background charge density $\rho$ so that the sample is stable. One can work out that the electron density must be $1/2\pi ml^2$ from the local electrical charge neutrality.

This wave-function also exhibits an energy gap between the lowest excited state and the ground state. In some sense, one may consider this energy gap as the origin of the fractional plateau - in the winding experiment, as one flux quantum is adiabatically pumped through the loop representing the two-dimensional electron system, electrons re-distribute themselves to reset the energy and the existence of this energy gap forced a net effect of a fraction of electrons per Landau level are transported through the sample.

Laughlin’s quasiparticle wavefunction can easily be extended to a set of fractional filling factors $\nu = 1/m$ with $m = 3, 5, 7, ...$, but it is unclear how one might obtain the other tens of filling factors observed in various experiments soon after the original discovery of the first fractionalized plateaus in this way.

Considerable effort has since been devoted to understanding the FQHE over the full range of rational fractions observed in experiment. Many such attempts are inspired by mathematical insight into the choice of polynomial pre-factor, and others contain some physical insight into the mechanisms that couple the magnetic vector potential to particles. Some sets, for example, particularly concerning the $\nu = 5/2$
plateau [176], are still not fully resolved and the microscopic origin of FQHE is still not fully understood. Today, this is a major research topic in condensed matter physics.

### 1.2.2.2 Effective Charge and Statistics of Excitations in Laughlin’s Wave-Function

Before proceeding to a more robust model of a larger set of the fractional Hall sequences, it is worth elaborating on some of the unique features of the fractionalized phases that can be seen easily from Laughlin’s wave-function.

A simple lesson can be learned by poking a hole in Laughlin’s state to represent a kind of defect. Since a hole of this type acts as a quasiparticle excitation, it is customary to call the state obtained this way a "quasi-hole" state. The natural construction is to assume particles orbit this hole the way they would any particle, and write the state as

$$
\Psi_m(z_1, \ldots, z_N) = (z_j - z_h) \prod_{j,k} (z_j - z_k)^m \times \exp\left[-\frac{1}{4l^2} \sum_j |z_j|^2\right],
$$

where $z_h$ is the location of the hole.

One can consider the dynamics of the hole’s location under the evolution suggested by the model Hamiltonian above - Laughlin’s quasi-hole state suggests that the quasiparticle and quasihole excitations act under the influence of a field as particles with fractions of elementary charge $e^* = e/m$. This is interesting, as fractionally-charged objects are uncommon among elementary excitations in Landau-Ginzburg matter.

Perhaps more surprising is how these excitations behave when two are exchanged. One can show that elementary excitations in many-body systems composed of fundamental bosons or fermions in Landau-Ginzburg systems act themselves as either fermions or bosons [177]. Converting from one to the other can occur due to pairing as described above, but it is difficult to see how any other statistical behavior could occur, as all such objects, quantum mechanically, can be expressed with creation operators as integral powers of their constituents’.

Quasi-hole and -particle excitation in FQHE systems are not paired to one another, so much as they seem paired to the magnetic flux (or the circulation in
their mean field). The question as to how they behave under exchange is a bit subtler, as one must envision an adiabatic process under which their positions, which define a global texture of quantum entanglement in the wave-function, along with the position of the hole. In loose language, we might view the exchange process as dragging a statistical flux quantum around with the hole, and leaving the state of the full system, hole and flux, in a new state. The Berry-phase associated with this process exhibits fractionalized values \([162,178]\), and thus the exchange phase is neither the \((2k + 1)\pi\) or \(2k\pi\) we associate with fermions and bosons, respectively, for integral \(k\).

Those fractionally charged quasiparticles are named anyons due to this new behavior. They are exclusive present in two dimension - the exchange path taken by quantum mechanical particles in three dimensions can always be deformed to an infinitesimal loop, leaving the topological contribution to this phase always zero, and implying particles in three dimensions are always either bosons or fermions. In two dimensions, the exchange paths cannot be deformed to a point without one particle crossing through the other - the topological contribution to the phase is thus not limited in this way to zero.

This idea of anyon statistics was further generalized starting from the behavior of fractionally charged quasiparticles in Laughlin parent states, principally by Bertrand Halperin, and demonstrated more explicitly in calculations by Daniel Arovas, John Schrieffer, and Franck Wilczek [179]. Systems supporting anyons show a new type of adiabatic manipulation of their wave-functions, undergoing "braiding" operations under pairwise exchange of excitations. In some cases, the quantum-state obtained by this manipulation is degenerate with the original, and thus the manifold of degenerate ground states can be described using the braid group in two dimensions. This feature divides anyonic systems by their braiding behaviors into abelian anyons and non-abelian anyons. Non-Abelian anyons have attracted a lot of attention since researchers have found their application in quantum computation, effectively encoding information into a ground-state by a series of these braiding operations.
1.2.3 The Composite Fermion Picture and Inspiration for Experiments with Cold Atoms

We have seen that Laughlin’s wave-function was only obviously applicable when the Landau level filling factor $\nu = 1/m$. The rich family of states represented by fractionalized phases in the FQHE has attracted a long list of intelligent minds to further elaborate choices of many-body states.

Starting from Laughlin’s wave-function, many attempts to generate further states rely on a technique of generating 'hierarchy states' by using Laughlin’s proposition of states labeled by choice of $m$ to generate a new series of states. The basic physical idea was to form new states at different filling factors by condensing quasi-particle excitations into their own Laughlin-like states.

There are constraints on the new states and their fillings imposed by the fractional statistics of the quasi-particles used to construct the state. For example, $\nu = 2/5$ and $2/7$ states can be generated from $\nu = 1/3$ in that sense, and further sets of states can be constructed from those new states. Construction can continue in this way and therefore a full hierarchy of states of odd-denominators filling fractions can be derived [112].

The most successful theoretical work along a similar arc, which has now been widely accepted, was the so-called "composite fermion" picture introduced by Jain [130]. The discussion of this idea will be on the next subsection and it is closely related to the design of the experiments undertaken in this thesis.

1.2.3.1 Composite Fermions and Generalizations

To explain a larger subset of the fractional quantum Hall states, Jain proposed a process by which to construct states somewhat different from the hierarchies described above, now known as the composite fermion picture [180]. The idea of this is shown in figure 1.5 - when the magnetic field is stronger than necessary for the integer Hall effect at $\nu = 1$ to occur, the degeneracy of lowest Landau level is high enough that all the electrons can be accommodated in that level. Within this level, the kinetic energy is constant, and it is left to interactions to break the degeneracy.

The Coulomb repulsion between electrons is typically screened in an ionic crystalline lattice, and the effective interaction can be modeled with a variety of
forms [181]. It is not so critical which of these one chooses for the purposes of this discussion, as later we will concern ourselves with neutral atoms anyway. If we were to admit a simple Coulomb interaction between electrons, it would form the only relevant energy scale in the system, and the full many-body Hamiltonian for particles within a Landau level could be simply written in the form

$$\mathcal{H} = \frac{1}{2} \sum_{j \neq k} \frac{1}{4\pi\epsilon_0} \frac{e^2}{|z_j - z_k|}. \quad (1.30)$$

Finding the exact solution of the many-particle Schrödinger equation with this Hamiltonian is difficult. It isn’t obvious how to use perturbation theory the way one often can for interacting systems, since the interaction strength isn’t safely smaller than anything relevant without a kinetic energy term.

Even without perturbation theory to rely on, we can use the generic concept of looking for a "good particle" to build many-body states from, familiar from weakly interacting (perturbative) systems like Fermi-liquids as interaction-dressed quasi-particles, or from strongly interacting (non-perturbative) systems supporting soliton-like excitations. It is natural to look for some basic particle-like building block, which might be expected to maintain its form as it propagates, and which interacts weakly with others of its kind.

The composite-fermion picture constructs a specific version of this "smallest building block" specifically to resolve this problem of macroscopic degeneracy of a Landau level. In fact, if the particles are chosen well, the ground-state of an effectively non-interacting set of these building blocks may no longer be degenerate, and can be stable for weak interactions added, in principle, perturbatively. Since it is clear that magnetic flux plays a critical role, one could venture that this building block is best chosen as a combination of a bare electron and some number, say 2p of flux quanta. In Jain’s construction, these are referred to as "composite fermions," as they still behave odd under exchange. If the total flux absorbed into the bare fermions in this way does not match the applied magnetic flux, the composite will move in a residual magnetic field.

We might expect that the composite fermions in this residual field themselves show an integer quantum Hall effect, forming a set of effective Landau levels of their own. In this way, inside the lowest landau level, we might expect a series of states defined by an effective filling factor higher than one. There is still degeneracy
in these states, but it is less degenerate than we started with above.

**Figure 1.5.** The Composite-Fermion approach to understanding FQHE states. Strongly interaction electrons in a strong magnetic field $B$ can be seen as weakly-interacting composite fermions in a weaker effective magnetic field, where the lost magnetic flux are bound to electrons, as each electron captures an even number of flux quanta. This technique transforms a complicated problem of strongly coupled electrons to a simpler one of weakly/non-interacting composite fermions. By analyzing the composite fermion Landau levels and composite Fermi sea, one can identify a list of effective filling factors which have a one-to-one correspondence with many of the filling factors that have been observed in different systems. Although the composite Fermion idea originated from the fractional quantum Hall effect, it has been extended beyond it and understood to play an essential role in understanding a larger variety of emergent quantum phenomena. (Figure adapted from [181])

If we are to turn this insight into a kind of wave-function, we should start from the idea that the particles have captured $2p$ flux quanta before forming a Fermi-sea. Like a non-interacting fermi gas, the sea can be described by a Slater determinant wave-function $\Phi$ assuring the odd exchange symmetry, and the flux-capture can be represented by a separate polynomial factor representing the flux absorption

$$\Psi^\times (z_1, ... , z_N) = \Phi^\times \prod_{j,k} (z_j - z_k)^{2p},$$

(1.31)

By using the composite fermion theory, we reasoned the strongly interacting electrons in a strong magnetic field transform into weakly interacting composite fermions in a weaker effective magnetic field, screened by an amount equal to what was absorbed,

$$B^\times = B - 2p\rho \phi_0,$$

(1.32)

where $B$ is the external magnetic field, $\rho$ is the electron density of a two-
dimensional system, and $\phi_0 = h/e$ is the elementary quantum of magnetic flux. The effective filling factor of the composite fermions can be calculated in the similar way as the integer quantum Hall effect by $\nu^* = \rho \phi_0/|B^*|$. The filling factor $\nu$ for bare electrons is now simply related to that of composite fermions by

$$\nu = \frac{\nu^*}{2p \nu^* \pm 1}. \quad (1.33)$$

where the minus sign corresponds to situations when $B^*$ points to the opposite direction as $B$.

Using the composite fermion picture, we have written the microscopic wave-function of the system using an effective wave-function for weakly interacting composite fermions. The fractional quantum Hall effect for electrons can then be interpreted as the integer quantum Hall effect of the composite fermions, when their filling factor $\nu^*$ is integral. This not only predicts a hierarchy of fractionalized states, but also unifies the FQHE and IQHE into a common language - when one electron absorbs an even number $2p$ of flux quantum and form a composite fermion, the composite fermions form a Fermi sea when $B^*$ vanishes and composite-fermion Landau levels when $B^* \neq 0$.

We can also capture Laughlin’s wave-functions above through the choice $\nu^* = 1$, for which the filling factor $\nu = 1/(2p + 1)$ for an aligned absorbed field - in this case, the composite fermion wave-function reduces exactly to the form given by Laughlin. Composite fermion theory can also be beyond the quantum Hall effect, for example, when the filling factor of the electron is $\nu = 1/2$, there is no quantum Hall effect, but in the composite fermion language the magnetic field $B^* = 0$, and this corresponds to a perfect composite Fermi sea.

1.2.3.2 Quantum Hall Physics with Cold Atoms

On its face, it may seem that the integer and fractional quantum Hall effects are phenomena unique to the behavior of charged particles in strong magnetic fields. After all, we have seen from the composite fermion picture that it may be viewed as driven by the pairing between electrons and flux quanta of the magnetic field - without these two components, and the coupling introduced between them by the Lorentz force, how could one observe similar behavior? A hint of the fact that similar physics can be extended to charge-neutral particles can be found by
comparing the similarity of the Lorentz force, which is oriented perpendicular and proportional to a charged particle’s velocity, and the Coriolis force, which for a particle in a rotating reference frame is related to the velocity in the same way. This suggests that the rotational vector defining such a frame plays a similar role for neutral particles as a magnetic field does for charged particles.

This idea, applied to cold and neutral atoms held in a rotating trap, was first proposed as a means to observe quantum Hall effects, both integer and fractional, by Wilkin, Cooper, and Gunn [182], in 1999. There are several ways to understand the fractional Hall limit for ultracold neutral atoms. As described above, bosonic atomic gases naturally condense in the quantum degenerate limit, forming a superfluid. Taken into rotation, superfluids develop quantized vorticity [183,184], which has been seen in many experiments [185] with large superfluids consisting of tens of thousands of particles or more. The quantum Hall limit is approached when the number of such vortices (each representing $N \hbar$ of angular momentum) approaches the number $N$ of particles in the superfluid gas. Alternatively, this implies the total amount of angular momentum in the gas is of order $N^2 \hbar$.

Naturally, this picture of quantum Hall physics differs from the electron system not only in the lack of charge for its basic components, but also by their statistical character - atoms with bosonic exchange instead of fermionic electrons. Nevertheless, the concept of flux-binding or -attachment can be viewed similarly, associating the vortex-number of the cold-atom system with the magnetic flux of the electron system. The "filling factor" can then be defined as the number of atoms per vortex, and we are free to consider whether it is possible to describe a set of interacting bosonic particles in this rotating reference frame better as a set of weakly interacting composite particles of atoms with attached vortices.

It is also natural to ask whether in this analogy it also makes sense to describe the single-particle eigenstates using Landau levels, and perhaps more to the point, whether these levels exhibit the same type of macroscopic degeneracy. In fact, for a particle which is also harmonically trapped, it is easy to see that this too is possible. In going to the non-inertial reference frame, it is necessary to substitute the kinetic momentum $\vec{p}$ of a particle with $\vec{p} + m \vec{\Omega} \times \vec{r}$ in the frame rotating with rotation vector $\vec{\Omega}$ (with magnitude equal to rotation rate, and along the axis of rotation). In the new frame, the single-particle energy is simply
\[ H_{sp} = (\vec{p} + m\vec{\Omega} \times \vec{r})^2 / 2m + m\omega^2 r^2 / 2 \]  

(1.34)

where \( \omega \) is the harmonic trap frequency. In the limit that the frame rotates near the trap frequency, \( \Omega \sim \omega \), the centrifugal energy balances against the trap energy, allowing particles to become nearly deconfined. Here, the cross-term in the first term of \( H_{sp} \) dominates, and the kinetic energy is approximately

\[ H_{sp} \approx (1 - \Omega/\omega) L_\Omega \]  

(1.35)

which exhibits a macroscopic degeneracy (just like the electrons in a magnetic field) when \( \Omega = \omega \). In fact, if we were to follow the eigenstates of a two-dimensional harmonic oscillator into the rotating frame as a function of rotation rate, we would see many such collapsed bands of states spaced in energy by \( 2\hbar\omega \), representing a series of Landau levels analogous to those of charged particles in the magnetic field. The analogy is perfect only in this 'centrifugal limit' where \( \Omega = \omega \).

Finally, to complete the analogy, we need for particles to interact. Above, we were deliberately agnostic about the form of interaction, choosing for the sake of concreteness a Coulomb-like potential between electrons. For cold atoms, with no additional fields applied, the interactions arise from van der Waals forces, and can be characterized at ultralow temperatures by the s-wave scattering length. For the atoms typically used, it is a repulsive interaction.

With all of these pieces of the analogy in hand, one might expect the physics of the fractional Hall problem with composite fermions in a magnetic field to carry-over into the physics of composite bosons in a rotating and harmonically trapped superfluid near this centrifugal limit. Indeed, this should be the case [182, 184, 186, 187], and one might expect states of even exchange symmetry similar to those of odd above, to appear. Since this analogy was pointed out, considerable experimental effort [88, 183, 188] has been put forward to realize it in the laboratory, and to drive experiments observing classical vorticity [60, 89, 90, 113, 189, 190] into the Hall limit of high-vorticity [37, 191, 192].

Rather than increase the number of vortices, one can also reach the Hall regime by fixing vortex number and reducing the particle number, even down to just a handful of atoms. This is the approach taken by Gemelke, et. al. [39], and achieved the highest filling factors to date [41]. Illustrated in figure 1.6, clusters of rotating...
ultra-cold atoms are created by loading bosonic atoms into an optical lattice, which have local rotations on each site, produced by modulating the optical phase of the lattice beams. There is also a pair of counter propagating beams producing a standing wave, which creates a two-dimensional confinement. Theoretical work has proved that those rotating clusters can be induced into quantum ground states with strong coupling and long-range correlation, that can be mapped into some fractional quantum Hall states. One may understand it as the transformation into rotating frame gives rise to the analogous magnetic field which can be somehow quantized, and the quantized flux quantum is bound to atoms forming composite particles showing fractional statistics.

![Image](image.png)

**Figure 1.6.** Rotating few-body quantum Hall experiment by Gemelke, et. al. [39], in which an artificial magnetic field is created at the rotating minima of the optical lattice potential - atoms are prepared in BEC then loaded into 2-D trap formed by superposition of optical lattice and a standing wave. The trap minima are rotated about their own center, as created by phase modulation of the lattice beams. By doing so, strong correlations were observed, providing some evidence of entering into the FQH regime.

Experiments of these types face a number of difficult challenges. First, the relatively weak strength of interactions means that the approach toward the centrifugal limit must be made very close to achieve significant mixing of single-particle levels and produce strong correlations. Working this close to the centrifugal limit
introduces strong heating effects, which limit the temperatures achievable using these schemes - this is exacerbated by the fact that few viable cooling mechanisms have been identified that can work in concert with the rotating trap. Second, and experimentally least well-addressed, is that "smoking gun" signatures of success are not as easily found as in the electronic analogs. The Hall resistivity, which forms the classic signature in electron systems, relies on two basic mechanisms which are lacking in the cold-atom system - a linear response-function measurement (conductivity for the electronic QHE), and the presence of disorder. For the cold atom systems, the applied potentials are extremely smooth, and lack the disorder that was essential in the electronic systems to widen and stabilize the plateaus in the resistance measurements. Moreover, transport in such systems is ballistic, rather than describable using resistivity, or its analog in a neutral fluid, viscosity.

While these works have pioneered development of quantum simulation into the domain of quantum Hall physics, the challenges described above have prevented clear observation of many of its hallmark signatures. The difficulty in studying fraction quantum Hall physics in this direction is that the many-body states are hard to directly characterize - the measurements that have been done are not directly analogous to the transport measurements in electron systems, and can only indirectly be linked to the structure of the states, rather than the novel transport we are familiar with. Here, I will present the quEASE method as both a direct replacement for the methods described above for producing a QHE state, and simultaneously as an attempt at the first transport-based measurement of fractional quantum Hall effects in a cold atomic system.

Another way of looking at this is to say that - despite the close analogies between Lorentz and Coriolis forces, Landau levels and the energy structure of a rotating trap, interacting electrons and colliding atoms - the analogy to the fractional quantum Hall system is still incomplete. We need to somehow finish by introducing the measurement process itself, best illustrated in Laughlin’s charge pump argument, and the underlying features it was built on - namely, disorder to stabilize and widen the plateaus in some analog of electrical Hall resistance. The idea and design of experiment are presented in next section.
1.3 Design of Atom-Optical Experiment as Realizations of Laughlin’s Quantum Pump

In the previous two sections, we introduced and reviewed two important concepts, first introducing a new attack on quantum simulation using the so-called quEASE method of building up coherent excitations in a feedback arrangement incorporating a quantum degenerate atomic gas, and later reviewing the physics and simulation of the fractional quantum Hall effect using cold atoms. In this section, I will describe how these two ideas can be put together to produce a new method of attack on generating states in cold atomic matter that emulate as many features of quantum Hall experiments as possible, by using quEASE experiment designs. The design of this analogous apparatus in atom-light system is discussed in detail in subsection 1.3.1. Subsection 1.3.2 and 1.3.3 discuss two different phases in the construction of the complete experiment.

1.3.1 The Atom-Optical Laughlin Pump

The goal of this experiment is to construct an apparatus that is a full analogy to the experiments of measuring Hall conductance in electron gas systems, and each aspect of this analogy is discussed in this subsection.

1.3.1.1 Optical Interferometry with Laughlin’s Pump Topology

To begin this section, we will first establish a basic observation about Laughlin’s charge pump argument and the topology of the measurement apparatus in which it occurs. In order to measure the transverse conductivity of an electronic Hall system, we first note that two "loops of wire" are necessary - one in which a current is excited, and a second in order to measure the transverse voltage difference sustained by the flowing charge carriers in the external magnetic field. Since both loops must sustain some (at least intermittent) current and contain some potential energy drop across portions of the loop to complete the measurement, we will not make too much of the distinction that one "supports a current" and another "provides means to measure the potential," and treat the two more-or-less symmetrically. Within each wire, current flow may be considered to be carried by bare electrons in a simple conducting metal, before entering the Hall bar where the carriers, in
the composite fermion language, are 'dressed' by magnetic flux quanta. We can consider these charge carriers in a classical limit, in which we think of the electronic current as a continuous quantity defined by the flux of a continuously-valued charge density, or in the quantum limit, as composed of discrete electrons, which are in turn represented by a wave-function whose complex-magnitude reflects a quantity proportional to the classical charge density.

Our goal in this section is to introduce an atom-optical analog to this system. To do so, we will start from the same geometry of two loops intersecting at a quantum-mechanical medium - in this case, the loops will form the arms of two Sagnac interferometers, and the quantum-mechanical medium will be an ultracold atomic gas, which is coupled weakly to light in the Sagnac interferometers by its polarizability. Within the arms, photons will flow in a way analogous to the bare electrons of the Hall measurement, and within the volume of the vapor, we may alternatively consider the excitations as photons dressed by excitation of the dielectric medium, or as atoms weakly dressed (excited) by the light.

In this way, one naturally arrives at an apparatus with geometry that is similar to the conventional quantum Hall effect measurement, which is best described by Laughlin’s charge pump thought-experiment. The design of this experiment incorporates both ideas of a direct transport measurement and an oscillator scheme for coherent many-body physics. The scheme of the experiment is illustrated in figure 1.7, and in the next sections, we will describe the incorporation of atomic vapor-induced effects that emulate the activity of a Hall medium on its charge carriers.

1.3.1.2 Spin-Orbit Coupling and Non-Reciprocal Optical Interactions in Atomic Vapors

To create a measurement that is as similar to the transport measurement in the electron system, we can first identify that the origin of coupling between the two arms of the electronic version is the transverse force created by the Lorentz force on charged carriers moving at non-zero velocity. Rather than rely, as we did in the previous section, on the Coriolis effect to mimic the Lorentz force, here we will introduce a different mechanism, spin-orbit coupling in an atomic gas, to make the analogy.

The "spin" of an atomic gas is represented by the hyperfine spin structure
introduced by the electron-nuclear spin coupling of the atom in its electronic ground state and by nuclear-spin’s coupling to the electron’s total angular momentum in the electronic excited states. When light is applied to an atom, these states are coupled together selectively according to the polarization of the optical field. The local conservation of angular momentum introduces selection rules which permit this coupling to take non-zero values only when the angular-momentum-one photon changes the atomic spin projection along a given axis by the photon spin’s projection, of magnitude one or zero. In a two-photon process, a single photon is both absorbed and emitted, coupling immediately to a virtual state - in this case the atom can transition by as much as two units of angular momentum along a given axis. In spin-orbit coupling, the center-of-mass state of the atom is altered through its recoil in this process, and the forces acting on the center-of-mass due to the optical coupling depend on the spin-state of the atom in its electronic ground state manifold. The generation of these effects can be done in various ways - a review of this larger topic may be found in ref. [41].

Rather than be very explicit about the induction of a particular type of spin-orbit coupling, we will instead approach the problem from a standpoint more closely tied to the basic effect we are trying to induce - a Lorentz-like force (a transverse force controlled by the particle velocity) governed by an external field in the form of a pseudo-vector like the magnetic field or rotation vector of a rotating frame. Note that under either time-reversal or a parity-flip, the introduction of either form of pseudo-vector breaks a symmetry, which is necessary to produce a force of this nature. We will take this broken time- or parity- reversal symmetry as a defining characteristic, and look for optical mechanisms in a cold atom vapor acting as a dielectric that can achieve the same.

Optical systems breaking time- and parity- reversal symmetry are rare in nature and are generically referred to as "non-reciprocal." Media exhibiting the Faraday effect are a prime example of this, and are often built from specialized materials place into strong magnetic fields. In this experiment, we will use this effect in multiple ways, inducing it in ordinary glasses used to construct optical fiber, as well as inducing the effect in a vapor of laser-cooled atoms. It is the latter application that allows us to introduce a Lorentz-like force, in which a Faraday-medium created by "itinerant" (unbound) polarizable atoms couples to the polarization of the light in a non-reciprocal way. In the next section, I will describe the ways in which the
optical Faraday effects introduced into a Sagnac-geometry interferometer (like either of the intersecting pair we introduced in the last section) create a very sensitive, and robust, detector for certain features in its interior.

1.3.1.3 Sagnac Interferometry and Optical Hall Effects

Before we get into the details of the design of this experiment, here’s a little reminder of Sagnac interferometer and optical Faraday effect.

Interferometers, in all forms, compare the phase of a wave accrued along two paths, or arms, by splitting and recombining a single wave, allowing interference to convert the relative phase into an amplitude. They are sensitive monitors of the medium in each of the arms, as often even a small variation in that medium can accrue an effective path length difference equivalent to a single wavelength of the excitation. There are several geometries of interferometer distinguished by the geometry of beam-splitters and any additional routing components like reflectors, and many are familiar, such as the Mach-Zender, the Michelson, Fizeau and Fabry-Perot.

A Sagnac interferometer is unique in forming its 'arms' by using the same optical path in reverse in one arm as the other. This is often achieved by sending one beam through a beam-splitter, and routing the two outputs back into one another in a ring-shaped geometry. In this way, two beams propagate in the same path but opposite direction. If the Sagnac interferometer is well aligned and the entrance light forms a parallel bundle of rays, the two arms have identical optical path length since they share exactly the same components, and the only difference is the order that light passes through optics. Any disturbance to individual components inside the interferometer will affect both arms by exactly the same amount, thus has no effect to the relative phase between those two arms. Moreover, any medium placed in its interior (such as an atomic vapor) is seen by both beams, albeit in reversed directions. Thus, for example, the interference of the arms is unchanged as the index of refraction (assuming it to be given by a typical dielectric) is varied.

An interference pattern is formed as two arms exit the ring and its phase is only sensitive to features that can break the reciprocity. There are a number of possible origins for broken reciprocity - in addition to the Faraday effect mentioned above, one can consider simple origins, such as a rotation to the whole interferometer apparatus. Due to this robustness, Sagnac interferometry is extensively used in
inertial guidance systems such as GPS, serving as a ring laser gyroscope to account for the rotation of the Earth.

Of course, another candidate non-reciprocal effect that can alter the Sagnac phase is the optical Faraday effect, though its action is a bit subtler than rotation of the interferometer was. To understand this effect in a more elegant setting, we will first reconsider the Sagnac interferometer with the beam-splitter replaced with a polarizing beam-splitter, such that the two counter-propagating waves also have crossed linear polarizations. From this, we will also introduce two quarter-wave plates to convert each linear polarization to circular - since the two polarizations must remain orthogonal, we will have one of each handedness, where the handedness is defined by comparing the rotation of the electric field to the propagation direction of the light. The result is a ring-shaped optical path with counter-propagating beams, where at any position in its interior, the two beams' electric field rotates in time in the same direction. One beam can be 'taken into the other' by simultaneously flipping its propagation vector and its handedness.

We wish to understand how a Faraday medium in the interior of an interferometer of this type can affect its output. In a Faraday medium, light interacts with an external magnetic field applied to the dielectric medium through "gyro-electromagnetism," obtained when the dielectric permittivity tensor is diagonal. The result is a rotation of the plane of polarization if the beam is linearly polarized, described as:

\[ \beta = \nu Bd, \tag{1.36} \]

where \( \beta \) is the angle of rotation, \( \nu \) is the Verdet constant for the material, \( B \) is the magnetic flux density in the direction of light propagation, and \( d \) is the length of the path where light and magnetic field interact. Here \( \nu \) is an empirical value and has dependence on temperature and wavelength.

Alkali vapors, like many cold atomic gases are, can exhibit Verdet constants that are a few orders magnitude higher than other bulk materials [193]. It is easy to see why, if one considers that an optically pumped medium (one in which all atoms are populated into a "stretched" state of maximal hyperfine spin) decouples from one circular polarization. If two circularly polarized beams with opposite handedness propagate through such medium, one can observe a slight difference in the speed of propagation, and therefore a relative phase shift between beams coming out from
the medium. This version of the Faraday effect or Faraday rotation is widely used in optical technology and laser applications, such as magnetic field sensors [194], optical isolators [195–197] and optical circulators [195]. In our application, the extremely high Verdet constant means that an atomic vapor can exhibit a large phase difference in the Sagnac output with a very modest applied magnetic field.

Ultimately, it is more than the Faraday effect that we will exploit - to generate Hall-like behavior, we will also need to consider what happens when our Faraday medium (atoms) is put into motion under an external influence. We will address this question specifically from the context of our 'Laughlin-pump' geometry next.

1.3.1.4 Induced Faraday Effects and Cross-Interferometer Gain in the Laughlin-Pump Geometry

In the Laughlin-Pump geometry, two Sagnac interferometers intersect at the location of an atomic vapor. Taking a cue from the electronic system, a transport-like measurement scheme can be constructed in which a modulation induced in the optical fields of one of these interferometers causes a change in the characteristics of the atomic vapor, in turn causing a modulation of the optical phases of the second interferometer through the vapor's effect as a dielectric. In this way, the dielectric mediates an interaction between the two interferometers, and one can define a 'gain' for the system relating the small variation of the "output interferometer" due to the modulation of the "input interferometer." In the Sagnac geometry, the strengths of both modulations are most naturally defined by the relative phase accumulated by beams counter-propagating in their arms.

In the previous sections, the non-reciprocal effect of an atomic vapor was discussed through the optical Faraday effect, resulting in a rotation of the polarization of an optical wave due to the presence of an external magnetic field. To understand the effect of the input interferometer, we will now, in some sense, describe the same type of process in reverse, explaining how the "external magnetic field" can be effectively modulated by controlling the properties of the input optical interferometer.

That such an effect can occur is clear when one considers the spin-dependent potential created by far-off-resonant light interacting with an atomic vapor. When a far-detuned optical field contains spatial variation of its polarization vector, the largely scalar (spin-independent) potential formed by the optical dipole force is
accompanied by smaller spin-dependent terms depending on the local polarization fields. Depending on the strength of the detuning, this can give rise to dominant effects that are either vector, or tensor, in the atomic spin [198,199]. The vector portion of the light-shift can be understood as an effective magnetic field - one that causes a Zeeman-like shift of the magnetic sublevels of the atom similar to the effect of a true physical magnetic field. In this way, the external magnetic field is augmented by the spatially-varying optically-induced effective field, and can modulated in time, as well. Through the Faraday-effect, the output interferometer phase can be modulated by it.

For two counter-propagating beams with a complex polarization vector $\vec{\epsilon}$, the effective magnetic field has the magnitude [41]

$$B_{vls} = \frac{iu_v(\vec{\epsilon} \times \vec{\epsilon}^*)}{\mu_B g_J}$$

and it is apparent from its form that it arises from non-reciprocal optical geometries. Here '$vls$' denotes the vector light shift, $u_v$ is the vector polarizability, $\vec{\epsilon}$ ($\vec{\epsilon}^*$) is the polarization vector (and its conjugate), $\mu_B$ is Bohr magneton, and $g_J$ is the Lande factor. Inducing a modulation of this field requires the differential modulation of phases in the counter-propagating arms of the input Sagnac interferometer, which can be accomplished by introducing another medium supporting Faraday effect physically displaced from the vapor, and modulating a magnetic field there. In this experiment, this was arranged by driving a Faraday response from the glass forming the delivery system for light in the input interferometer.

1.3.1.5 Trapping and Laser-Cooling in the Laughlin-Pump

For any experiment of this type to work, the atomic vapor must somehow be confined and held long enough to use it as a cold dielectric. This requires some type of confining potential, and this is most easily provided by the light in one or both of the optical interferometers. Further, it is necessary to also provide some form at least pre-cooling, or even concurrent cooling mechanism to keep atoms at velocities corresponding to a photon recoil or below. To arrange for this, the experiment was designed with an input interferometer formed by far-detuned light at a wavelength of 1064nm, providing a largely scalar potential with a weak spatially-varying vector light shift. The output interferometer was formed with near-detuned light, tuned
to the $F = 1 \to F' = 0$ transition of the atoms. Together, these two optical fields form a system that resembles that used in Raman-sideband cooling, and provides a slow cooling mechanism that simultaneously traps and retains the atoms in the interaction volume. Thus, the effects coupling interferometers in the Laughlin-Pump geometry described above occur in this dissipative context.

The nearly-detuned output interferometer laser light acts as an optical pumping beam in the cooling process, so we will alternately refer to it as the output interferometer light when we consider its role in the measurement process, and as a pumping beam when we are emphasizing its role in the dissipative cooling process.

1.3.1.6 Diffusive Transport and Optically-Induced Disorder in Atomic Media

In our analogy to the electronic Hall systems, we noted the importance of extending the analogy fully into the regime in which transport phenomena, now more clearly defined by the cross-interferometer gain of the Laughlin-pump, are of a similar type (diffusive) to that in a weakly disordered solid-state environment. In cold atom experiments, the applied potentials are most often either smooth on the microscopic scale, as would be the case in an optical tweezer or dipole trap, or regularly ordered as in an optical lattice formed by interfering laser beams. In both cases, transport, as defined by the atomic current induced by a force, is dominantly 'ballistic,' in that the force results in an acceleration defined by a mass (roughly the atomic mass in a smooth potential and the effective mass in an optical lattice potential). Disordered potentials, like those assumed in the Drude model [200] of electrical conductivity, lead to a diffusive transport, and a current which is (in steady-state) proportional to the applied potential difference with a constant of proportionality, the conductivity, determined by the disordered medium. We would like to create the same for the Laughlin-pump. Since the majority of the scalar potential is provided by the far-detuned light in the input interferometer, we will choose to create a disordered optical field with its configuration.

To do this, the YAG laser intensity at the atomic medium is made speckle-like, by interfering many modes with randomly chosen phases inside the Sagnac interferometer. This can be easily accomplished by inserting multimode fibers in the Sagnac arms to deliver light to the location of the atomic vapor. This creates disordered scalar and vector dipole potential that traps cold neutral atoms.
1.3.1.7 Inducing Non-Reciprocal Modulation in an Optically Disordered Potential

To create a modulated effective magnetic field, a non-reciprocal modulation of the input interferometer beams must be produced. To do this, we induce a Faraday effect in the glass of the multimode fiber used to deliver the far-detuned light.

Part of the multi-mode fiber passes through the interior of a solenoid with thousands of turns of copper wire. An external electrical current source is applied to the solenoid, generating a nearly tesla-scale magnetic field inside the solenoid, which lies nearly parallel with the propagation axis of the optical fiber. The relatively weak Verdet constant of optical glass, combined with this large magnetic field, induces a polarization-dependent phase shift which breaks optical reciprocity. For a driving current harmonic in time, the effective magnetic field induced in the atomic vapor is also time-periodic, though locally varying according to the interference of beams.

1.3.1.8 Putting Together the Full Functional Behavior of the Optical Laughlin-Pump

At this point, we have put together all of the necessary elements to describe the Optical Laughlin Pump, and have extended the analogies present in the atomic Hall experiments all the way to the measurement apparatus, and even duplicated the continuous cooling present in electronic systems by the presence of a dissipative cooling mechanism.

In summary, as atoms are perturbed by non-reciprocal modulation of the input interferometer, the output interferometer affected by the dynamics of atoms induced through modulation of the local effective magnetic field. In turn, atoms modulate the output interferometer through optical Faraday effect, which breaks time reversal symmetry, and thus we expect the output Sagnac phase will be changed due to the dynamics of atoms. A measurement of the Sagnac phase of the optical pumping beam is essentially a measurement of the response function of this apparatus in terms of the non-reciprocal optical effect.

A key technical idea underlying this picture of two intersecting optical loops is that one beam can only be affected by the other through atom-light interactions. Technically in the laboratory, this is a robust assumption - the presence of other
materials that non-negligibly couple the far-detuned light of the input interferometer at 1064nm and the output interferometer at 780nm is unlikely, as this requires a sufficiently strong optical non-linearity of some kind. Such materials are somewhat rare in nature and are doubly-rare to also act in optically-non-reciprocal ways. In short, without the influence of the atoms, the cross-interferometer gain of this system can easily be made immeasurably small.

By driving the optical fiber in the input Sagnac loop, we can induce a Faraday effect in the output loop, with good confidence that the coupling is mediated solely by the atomic vapor, and not the surrounding materials. Since the Sagnac geometry of the output interferometer is uniquely sensitive to non-reciprocal effects, we can exclude by common-mode most other disturbances to the system, for example the acoustic vibration of mirror mounts or other optics in the assembly.

Since the output interferometer response is this selective, we can use a very weak modulation of the input interferometer. Since the disturbance to the atoms is small, it can be made largely non-destructive to the sample, and with in-born dissipative cooling, can be run continuously.

With the piecewise description in hand of how this apparatus induces and measures an effect through a cross-interferometer gain described by effective magnetic fields and Faraday effects, we can now attempt to step back a bit and describe the function of the apparatus in a slightly different way.

One can view the cross-coupling of the interferometers as a kind of wave-mixing process in non-linear optics. It is customary to describe non-linear optics in terms of the optical susceptibility, and to obtain an interaction between two fields of this type, one must introduce off-diagonal components of the $\chi^{(3)}$ tensor that describes the third-order susceptibility of the medium - representing the change in optical-frequency susceptibility proportional to the third power of the optical electric fields. We will take a closer look from this perspective in later sections.

### 1.3.1.9 Small-Signal Gain and AC Hall Measurements

We should also note that when we introduced the electronic Laughlin charge pump idea above, we described it as a "DC measurement" in which the potential and current were measured as static quantities, and the resistance defined with their ratio. In describing the Optical Laughlin Pump above, however, we subtly changed the context by inducing small periodic driving of the input interferometer phase
and measured the same small response of the output. This "small-signal" response is more similar to a measurement of the differential conductance of a Hall system, as would be obtained by dithering the current in a Hall bar and measuring the induced dither on the transverse potential.

A closer analogy can be made with experiments by recognizing that the bulk of experimental electronic Hall measurements are also made in an 'AC' manner, and for small signals. The differential conductance $\sigma = dI/dV$ is often a more technically robust quantity to extract in a laboratory, as it rejects slowly-varying influences, like that encountered from thermocouple-like effects in the connections of dissimilar metals at varying sample temperature, or other offsets like would occur from a leakage current.

This can be taken to an extreme in the optical Hall effect. The optical Hall effect is an AC-version of the Hall effect at optical frequencies, where the magnetic field induces dielectric displacement at an optical wavelength, transverse and longitudinal to the incident electric field. There is a moderately long history in the investigation of optical Hall effect, especially in the quantum Hall regime. As mentioned in the previous section, a sufficient amount of disorder is essential for the quantum Hall plateaus to occur in the static Hall measurement. Since the localization length has to be smaller than the sample size or the inelastic scattering length [201–207]. Whether an AC field can delocalize the quantum states of the electrons, and at what frequency the quantum Hall effect starts to breakdown remain inconclusively answered questions.

The effect in the optical frequency regime is interesting, not only because the lower frequency behavior has been studied and largely understood, but more importantly because the energy scale that corresponds to the optical frequency can be comparable to the Landau level spacing, e.g. $\hbar \omega = 10^{-2}$eV when $\omega = 10^{12}$Hz, which is exactly the spacing between Landau levels when magnetic field is about 10T. An optical Hall conductivity $\sigma_{xy}(\omega)$ can be defined by the Faraday rotation angle of a linearly polarized beam bouncing off a sample. Some theoretical predictions have been made and experimentalists have observed in two-dimensional electron gases that the optical Hall effect preserves plateau structure, which can be attributed to the topological nature of the electron’s wave-function.

There is some good sense in discussing the new physics opened by optical Hall effects in the context of the cold atomic analogy. With disorder fully under a
cold atomic physicist’s control, the localization length can be easily independently varied, and compared to the AC frequencies applied. From this context, it is clear the atomic system has more than just an analogy to exploit, but is likely taken into newer contexts which are somewhat difficult to tackle in the electronic systems.

1.3.1.10 Making the Hall Analogy More Explicit - the Vector Light Shift and Hall Fields

Having built-up the Optical Laughlin Pump measurement scheme systematically in previous sections, it is now worth revisiting the original analogy between Hall physics for charged electrons and the neutral atom system. The core of that analogy laid in finding a transverse force for neutral particles that emulated the Lorentz force - perpendicular to and proportional in magnitude to the particle velocity. The original example given for this was a rotating trap or reference frame, but it is clear from the construction of the Optical Laughlin Pump that there is no explicitly rotating geometry or frame to apply. Thus, if our analogous Lorentz-like force is present, it must take another form. We gave just a hint of this in a previous section by introducing the ’spin-orbit’ coupling introduced by laser light for neutral atoms - in this section, we will try to make the effect more explicit.

Like the case for a charged particle in a magnetic field, and for a neutral spin-less particle in a rotating reference frame, the effect of the spin-orbit coupled atom is the presence of an induced vector potential for single atoms. The single-particle energy for a spin-coupled atom can be written explicitly as

\[
\mathcal{H}_{soc} = \frac{\hbar^2}{2m} \sum_{i=1,2,3} (k_i - \sum_j A_{ij}\sigma_j)^2
\]  (1.38)

where \(k_i\) is the atomic center-of-mass \(k\)-vector, \(\sigma_i\) is the spin-operator along the \(i^{th}\) direction, and \(A_{ij}\) is the corresponding \(i^{th}\) component of the vector potential depending on the \(j^{th}\) spin-component. Like the case for charged particles, not all configurations of vector potential represent the case for the Hall effect of a uniform and static magnetic field - this field should have a non-zero curvature, as in the scalar case the magnetic field is given by the curl of the vector potential. The spin-orbit version can be considerably richer, and more complex, than the scalar version, owing to the tensor-operator structure.

If we interpret \(v_i = \hbar k_i/m\) as a velocity, we can see that the portion of the
energy first-order in the spin operators is proportional to it. Associating this, and the term quadratic in the spin-operator with a potential energy, we can write the Lagrangian for the system as the difference of kinetic and potential energy,

\[ \mathcal{L}_{soc} = \sum_i m v_i^2/2 + \frac{\hbar}{2} \sum_{ij} v_i A_{ij} \sigma_j - \frac{\hbar^2}{2m} \sum_{ijk} A_{ij} \sigma_j A_{ik} \sigma_k \]  

(1.39)

Writing the velocity-dependent part of the total force \(-\nabla \mathcal{L}_{soc}\) as

\[ F_k^{(v)} = (\hbar/2) \sum_{ij} A_{ij,k} v_i \sigma_j \]  

(1.40)

we see that there is a Lorentz-like force term in the spin-orbit coupled problem, determined by \(A_{ij,k} = \partial A_{ij}/\partial x_k\). For the force to be transverse to velocity, it is necessary that \(A_{ij,i} = 0\) for all \(i\), and would be sufficient that \(A_{ij,k}\) be proportional to the Levi-Cevita unit tensor \(\epsilon_{ijk}\). While more complex in that this force is spin-dependent, it remains similar to the Lorentz or Coriolis forces in the essential way that it is a velocity-dependent non-dissipative force - it transfers no energy to the particles due to the orthogonality of the velocity and force. In this way, we have more explicitly recovered the core piece of the original analogy to the electronic Hall problem.

It remains to define the components of the vector potential \(A_{ij}\) in terms of the optical fields. Decomposing the spin operator product in a basis formed by the identity and higher products

\[ \sigma_j \sigma_k = \sum_n C^{(n,j,k)}_{i(1)\ldots i(n)} \prod_{m=1}^{n} \sigma_{i(m)}, \]  

(1.41)

the final term of the Lagrangian becomes at first order in spin operators

\[ -\frac{\hbar^2}{2m} \sum_{ijkl} A_{ij} A_{ik} C^{(1,j,k)}_{\ell} \sigma_{\ell}. \]  

(1.42)

The effective magnetic field from the vector light shift can then be read off from this term as

\[ B_{\ell}^{vls} = \frac{\hbar^2}{2m} \text{Tr}(AC_{\ell}^{(1)} A^T) \]  

(1.43)

and therefore, the vector light shift can be related to the vector field appearing
in the Lorentz-like force law through the vector potential $A$. Taking the spatial derivative, rearranging and equating terms,

$$\partial_m B^vls = \frac{\hbar^2}{2m} \sum_{ij} A_{ij} A_{ik,m} [C^1_{ij} + C^1_{ij,k}]$$

(1.44)

To relate the Hall field with the vector light shift, we consider the motion of an atom over one period $T$ of an external driving field (later we will take this to be the AC modulation of our input interferometer). Assuming that the motion of atoms is periodic over this same time-scale, we can define an orbit $x_j[t] = x_j[t+T]$ in space and $\chi_k[t] = \chi_k[t+T]$ in spin space. In steady-state, if we assume the particle has an energy gain $\Delta E$ per period,

$$\Delta E = \int_{t_0}^{t_0+T} dt \sum_k \chi^T F_{k} \dot{\chi}_k = \oint_{\gamma} \sum_k \langle F_k \rangle_{\chi} \, dx_k$$

(1.45)

where we express the expectation value with respect to the spin-orbit $\chi[t]$, and the physical path of the orbit as $\gamma$. If we take the Lorentz-like force above to define an effective Hall-type field for the orbit $\gamma$,

$$\langle F_{k}^{(v)} \rangle_{\chi} = \langle h/2 \rangle \sum_{ij} A_{ij,k} v_i \langle \sigma_j \rangle_{\chi}.$$  

(1.46)

Note that the energy change would be manifestly zero if $\chi[t]$ were constant through the orbit, because the path integration would be taken as a closed loop over a total derivative.

By combining equation 1.44, 1.45, and 1.46 together, one can realize that the gradient of the effective magnetic field from the vector light shift gives rise to a Lorentz-like force onto atoms, which emulates the external magnetic field for Hall measurement.

1.3.1.11 Closing the Loop on Laughlin’s Charge Pump

Before finishing the design of our quEASE method for exciting Hall states, we have one last step to perform - closing the loop on our quantum system. Returning to the schema for Laughlin’s charge pump argument, this step is analogous to using the measurement of the potential developed across the hall ribbon to modulate the current in it. Since we originally introduced the charge pump apparatus, we
have changed perspective slightly on this, separating the small-signal modulation of
current and voltage from their large-signal parts. If we consider each of the original
pump variables $V = \bar{V} + v$ and $I = \bar{I} + i$, we can see that the large signal parts
$\bar{V}, \bar{I}$ can now be considered as the "DC-operating point" of an electronic oscillator,
while the oscillation wave-form and stability conditions can be analyzed from the
small-signal parts $v, i$. If we play the same trick with the magnetic field $B = \bar{B} + b$
and flux $\Phi = \bar{\Phi} + \phi$, we can see that the (fractional) Hall state is determined by
the operating point defining $\bar{B}$. Like any non-linear system with feedback, the
operating point can be changed by changing the small-signal stability conditions.

In the case of Laughlin’s charge pump, changing the large signal response
determined by the value of $\bar{B}$ requires that the flux penetrating the area of the
ribbon be made macroscopically larger or smaller. One can accomplish this by
moving some of the flux $\bar{\Phi}$ from the annulus interior into the area of the ribbon,
but the flux there has a one-to-one relation with the large-signal current $\bar{I}$, so the
choice of Hall phase is linked to the large-signal current.

In general, the large-signal operating point of a non-linear system is not so easy
to predict without a detailed understanding of the non-linear parts of the problem -
the closed-loop small-signal gain can diverge for multiple choices of the large-signal
operating point. This is the case, for example, in a multi-mode laser - as the
overall gain of the system is increased, the open-loop small-signal gains can either
saturate in multiple locations at unity, representing multi-mode oscillation, or the
competition for energy between modes can cause a single mode to dominate, forming
a single-frequency, single-mode oscillator. In the latter case, if the parameters
of the oscillator are smoothly and adiabatically varied and returned to the same
point, the operating point will smoothly follow, returning to the same point as
well. However, if the parameters are diabatically varied, the operating point can be
changed - by the argument above, this must correspond to a macroscopic capture
of flux and change of current. For a macroscopic and homogeneous medium, the
small-signal gain moves from a region where it is nearly zero to one where it is
divergently large at this point, as the differential conductance diverges. In this
way, an oscillator’s small-signal gain must pass through unity near these transition
points, forming an unstable mode of oscillation at each. Below, we will exploit this
behavior to attempt to probe the Hall phases in the Optical Laughlin Pump.
1.3.1.12 Closing the Loop on the Optical Laughlin Pump - an Example of a Topological Mode Filter

The Optical Laughlin Pump can be closed in an analogous manner, forming an oscillator by feeding the output Sagnac phase back to the solenoid drive for the input interferometer fiber to induce a Faraday phase shift in the glass. This modulates the vector light shift at the vapor, in turn causing a Faraday shift on the output Sagnac phase, and closing the loop. The large signal response, or operating point, of this system is in some part determined by the dynamics of the atomic vapor. Depending on the characteristic of the excited real- and spin-space orbits, the small-signal cross-interferometer gain is different, and by closing the loop in different ways, one might expect the large-signal response, or dynamical phase, to differ.

As the cross-interferometer gain is relatively small for an optically thin vapor, one naturally expects to need to introduce some amplification scheme to enhance the signal such that the small signal open-loop gains approach unity. One can also introduce some control over the type of dynamics excited by introducing a spectral filter into the loop. By choosing the window of frequencies over which this filter is appreciable to coincide with a particular timescale for a process in the trapped vapor, oscillation modes can be chosen to be relevant only with these processes.

The idea of controlling the oscillation using filters brings us back to a point we made during the introduction of the quEASE method, now with a much clearer context. There, we suggested that aside from spectral filters, one can also introduce spatial 'mode filters' similar to the operation of single-mode lasers. We can now understand better the design of the double-Sagnac interferometer system from a different context - the Optical Laughlin Pump measures a response function which is clearly topological in nature. Since the response function is part of the loop’s gain, we can consider its topological sensitivity as a kind of 'topological filter' built into the loop. In this way, we can discriminate behavior which is uniquely changing the topological indices of the phase, rather than responding to transitions of other, more local, types.

Using this method to probe a vapor’s response has several advantages beyond using filters to conveniently select-out the modes we are interested in. For example, one might expect the use of an unstable oscillator allows signal to be fished-out from a reasonably noisy floor of excitation. In this way, one does not need to adiabatically manufacture a topological state with high-fidelity, but rather grows
one from small seeds of noise as a coherent oscillation built as an instability in another state.

1.3.1.13 Overcoming Gain Noise in quEASE Experiments

If we stopped at the story above for building an quEASE simulation of topological phases, it would sound like a good and simple story. The laboratory realization, however, is another challenging story, and certain features of the problem need to be taken into account. Clearly, a defining characteristic of this method is that the final dynamical state of the quEASE oscillator is controlled by the open-loop gain of the system.

In the case of this experiment, we rely on the oscillation to drive the atom-light system into a many-body state with topological order by inserting some filters to select certain topology. As shown in figure 1.7, a photo detector is monitoring the exit port of the optical pumping Sagnac. This photo detector is watching the interference fringe and converting the optical intensity profile, i.e. the Sagnac phase, into an error signal in the form of electrical voltage. Then the voltage signal is passing through a filter and being fed to the solenoid which drives the magnetic field along the direction of YAG fiber. To work in the weakly dissipative regime consistent with continuous laser-cooling of the vapor, the light levels in the output Sagnac interferometer must be kept at levels of approximately 1-100pW. At such low light-levels, one is forced to work with highly sensitive photo-detectors, typically either heterodyne-detectors (as ultimately used here) or avalanche-style or photo-multiplier detectors with an in-born physical gain mechanism. These methods exchange gain-stability in order to achieve simply very high gain, and suffer from large levels of multiplicative noise.

For the quEASE method, which is intrinsically sensitive to gain, this causes a non-trivial problem. If we rely on detecting onsets of oscillation, which occur at the first point of unity small-signal gain, the multiplicative gain noise would require very long integration times to recover meaningful data. Instead, we can attempt to overcome this problem, finessing the small signal variation problem by deliberately introducing very high small-signal gains everywhere, and interrogating the competition between large-signal operating points. Since this is somewhat difficult to understand from small-signal analysis, and large-signal behavior is intrinsically difficult to understand on face, we will resort to using an analogy to
the operation of a laser. There, one can imagine operating a laser at pump levels far above threshold, and varying the position or size of a mode-selecting filter. By observing the points at which its lasing mode suddenly changes, one learns about the structure of the resonator’s modes. Of course, one could do the same with a tunable spectral filter, studying the points at which mode-hops occur as an etalon is tuned.

Below, we will elaborate on this analogy, introducing, instead of a varying mode- or spectral-filter, a varied external driving force to the oscillator.

1.3.1.14 Choosing a Scale for the Laughlin Oscillator Period

Closing the loop on Laughlin’s charge pump created a situation where oscillation occurs preferentially at a large-signal operating point poised between two phases of the Hall system. We can reason that the characteristics of the oscillating mode are determined by differences of these phases. By appealing to the composite-fermion picture, we can visualize this as modulation between two different values of absorbed flux and/or composite-fermion landau level, either composing the charge-carriers with different numbers of flux quanta, and/or forming orbits about a different number.

There is an interesting question regarding the optimum frequency to allow for this dynamic - naturally, we can insert a spectral filter with any passband we choose, allowing oscillation to occur wherever we like. Making an analogy to the localized order of a traditional Landau-Ginzburg system, we might expect that as an oscillation occurs, the Optical Laughlin Pump periodically moves to either side of a transition. After crossing the transition, small domains of the newly stabilized phase grow in size, reaching a length-scale determined by the growth rates and period of oscillation. Remembering that the argument for plateaus in the transport properties relied critically on the presence of localized states and disorder, it is natural to choose this time-scale to be comparable to a ‘localization time’.

In disordered systems, a natural choice for this scale is defined by the mobility edge [208], an energy scale above which excitations are delocalized. For cold atom systems, this occurs a reasonable factor larger than the characteristic tunneling time-scales, which for the experiment below are on the order of hundreds of microseconds to milliseconds.
1.3.1.15 Probing an Oscillator by Injection-Locking

Above, we proposed interrogating the closed-loop optical Laughlin pump’s large signal operating point by adiabatically varying a parameter (like a mode filter in a laser) to interrogate the points at which it diabatically 'hopped' between modes or phases. It is difficult to tune the parameters of the measurement apparatus in the laboratory, so the natural place to look is toward slow variation of the spectral filters in the feedback loop. We still need to observe something change, and the natural parameter to watch is the small-signal gain itself. There is an elegant way to manage both of these ideas at the same time, and simultaneously keep the precision of the method good.

On an electronics bench, a tunable spectral filter can be created by forming a driven active filter, converting a reference periodic signal into bandpass filter for other signals, centered at the reference frequency. A byproduct is that the filter bandpass can be dynamically modulated by simply sweeping the frequency of the reference signal. A filter of this design allows us to pull the natural oscillation frequency around at will, and gives us an 'adiabatic parameter' to vary.

We decided the appropriate place to look for a sudden change as the oscillator’s frequency is varied in the small-signal gain itself. It was a bit duplicitous, as we already noted the technical complication of multiplicative gain noise, which would make this somewhat challenging to probe. Nevertheless, we could imagine stimulating the system with a small oscillating 'Hall current' or input interferometer dither and measuring the response. Because the loop is closed and the gain large, this will not be a 'small-signal gain measurement.' In fact, it resembles far better the situation of an injection-locked laser, in which a weak coherent signal is sent into a resonant cavity. If the injected signal is sufficiently close to the freely oscillating frequency of the system, we expect the oscillation synchronizes, or injection-locks, to the injection signal’s frequency. The dynamics of injection-locking are well-understood, and are perhaps one of the best understood corners of oscillator dynamics in the strongly non-linear regime. The problem was studied by Adler [132] in the mid-twentieth century in the context of electronic oscillators, and is widely applied to the study of injection-locked laser systems. Based on a simple mode of a driven oscillator, Adler developed relations predicting the structure of a "wedge-"like structure in the space of frequency and drive strength where an oscillator will synchronize, or lock phase, with the driving force at long times.
1.3.1.16  Dynamical Phase Measurement in Injection-Locked Oscillators

In the last section, we introduced the idea of an actively driven filter whose passband was driven by an external signal which controls its center, and whose bandwidth is determined by fixed values of several passive components. To probe the system, we also introduced the concept of an oscillator driven near its natural resonance frequency, and suggested that probing dynamics near Adler’s wedge provides some probe of its dynamics. To be explicit, we could imagine following a procedure as a four-step process in Adler’s diagram, starting by driving the oscillator below resonance, increasing the frequency at fixed amplitude until unlocking, reducing amplitude at fixed frequency, decreasing the frequency to relock, passing through resonance and past until unlocking again, and increasing amplitude to return to the starting point. To quantify its behavior, one could compare the relative phase accumulated along a well-defined path between the oscillator and the driving force as a sensitive measure of the location of these boundaries. Provided one stops somewhere in the locking region described by Adler’s wedge, presumably this phase difference should be an integer multiple of $2\pi$, equal to the number of times the oscillator’s phase has "slipped" as it went out of lock. The precise pathway is not important for this fact to be true, and we might even consider measuring the variation of the phase slippage as the path is deformed as an effective probe of the oscillator.

Adler’s treatment of the locked oscillator, and models that built on it later, can give us far better intuition into this problem, however. From them, we can see that the integral phase-slippage is just one possible behavior, and that depending on the non-linear parts of the problem, the behavior is far richer. In fact, it is easy to see that synchronization cannot occur between two or more oscillators without a non-linear coupling between them - without it, we could always transform to normal modes where the oscillators are independent. In the many-coupled-oscillators version of this problem, named the Kuramoto model, it is customary to propose a model with a Hamiltonian in phase (angle) and energy (‘action’) variables of the oscillators. In this first part of its consideration, we will envision just two oscillators, representing the Laughlin pump oscillator as a whole, and our probing frequency. For any number $N$, the Kuramoto dynamics are written as a system of equations
\[ \dot{\theta}_i = \omega_i + \sum_{j \neq i} \Gamma(\theta_j - \theta_i) \quad (1.47) \]

the non-linear term coupling oscillators together according to \( \Gamma \) is determined by the physical mechanism coupling them together, and cannot occur without a base non-linearity somewhere in the oscillator’s feedback loop. A simple choice is \( \Gamma(\Delta \phi) = K \sin(\Delta \phi) \), and with \( N = 2 \) is the simplest model by which to infer Adler’s wedge. It is clear that for the right sign of \( K \), the dynamics drive the system toward a lock in which \( \Delta \phi = 2n\pi \) for some integer \( n \). For real-world systems, this is not the only choice, and one can also find, for example, \( \Gamma(\Delta \phi) = K \sin(p \Delta \phi) \), in which case the system can lock to \( \Delta \phi = 2n\pi/p \), rational fractions \( n/p \) of \( 2\pi \).

Driving systems like these near the locking boundaries, one would expect to see this reflected in the relative phase measured at different portions of a driving pathway like the one described above. The fractions observed reflect the non-linear terms in its dynamics.

To perform an experiment like this, one needs a device that can accurately drive the system diabatically about its natural oscillation frequency, and record the relative phase dynamics. With control of both the spectral filter used in the oscillator loop, and the driving force, one can accurately pull them around one another as described above on the amplitude and frequency parameter space. Since this is manifestly time-dependent drive, we can simplify things a bit by describing the same operation on the complex frequency plane, treating the amplitude modulation as an exponentially-growing or shrinking parameter at all moments in the waveform.

To produce a scheme for performing this measurement in the lab accurately, one can generate both of the signals from a common digital signal processing device. A field-programmable gate array (FPGA) is adopted to generate a signal to control the center frequency of the active filter. This signal is carefully designed to be a complex waveform, i.e. its amplitude and frequency both varying periodically and effectively it can be described by a complex frequency:

\[ s(t) = V_0 e^{i\tilde{\omega}t}, \quad \tilde{\omega} = \omega_0 + \omega_r \sin(i\Omega t) + i\omega_i \cos(i\Omega t), \quad (1.48) \]

where \( V_0 \) is a real amplitude of voltage, \( t \) is time, and \( \tilde{\omega} \) is the complex frequency moving along an ellipse on the complex plane, which centers at \( (\omega_0, 0) \) and has semi-major and semi-minor axes to be \( \omega_r \) and \( \omega_i \). This complex frequency is
moving along this ellipse periodically, characterized by the frequency $\Omega$. All these parameters can be easily adjusted in FPGA. The FPGA not only generates the control signal of the active filter, but also creates a complex conjugate of the first signal, which is injected into the oscillation loop and is summed together with the error signal from optical pumping Sagnac phase measurement. By carefully choosing the parameters of the complex ellipse, we can move the center control signal and the injection signal close enough.

The idea behind this scheme is that if the atom-light system builds up an oscillation, it will show up as a pole in the complex plane somewhere close to the center control, and its actual location depends on the system closed-loop transfer function. As the injection signal is close enough to the pole, the oscillation of the system can be pulled by the injection signal, and in a narrow region of parameter space, the oscillation can be locked up to the injection.

One wants to choose the orbit frequency $\Omega$ small enough that the motion of the complex frequency along the elliptical orbit is mostly adiabatic. The adiabatic variation of a parameter naturally leads one to Berry’s geometric phase, which describes the phase accrued by the system under a fully adiabatic periodic motion. The Berry phase can be observed as an accumulated phase difference between the system’s response and the injection signal, as the complex frequencies have gone around the elliptical orbit once. As we can see, this Berry’s phase depends on the structure of the poles contained in the orbit and one can connect the value of this phase slip to certain topological order of the many-body state that the system is pumped into.

It is natural to postulate that if the atom-light system is driven into a topological state relative to integer or fractional quantum Hall state, this phase slip measurement can show some correspondence to the value of the Landau Level filling factors. The idea of this injected oscillation will be discussed in more detailed way in the next section.

1.3.1.17 Stepping Back from the Driven Laughlin Oscillator to Understand its Operation

We have now discussed the essential parts of the design of this experiment. We have created a pump-and-probe process capable of directly measuring the transport properties of a topological many-body state in a light-atom system, with a geometry
analogous to Laughlin’s charge pump picture. The system is embedded in a driven oscillator, which is designed for the purpose of exciting the system into a coherent many-body state chosen by a kind of topological filter built into the measurement process, and probing the inherent non-linearities in the many-body phase.

At the microscopic level, we are interested in cases that the light-atom system carries non-dissipative forces (described by the spin-orbit coupling above) that are similar to the Lorentz force experienced by electrons in a magnetic field. The electrons in the solid-state system experience the Lorentz force acting perpendicular to their motion, forming closed semi-classical orbits with radii increasing with the Landau level. Inside each of these orbits, at least on the average, must be some number $n$ of magnetic flux quanta labeling the Landau level. Hall measurements are done under an equilibrium condition, where a current, represented by the drift of the center of these orbits, establishes a static transverse electric field by redistribution of charges to balance the convective modification of the Lorentz force. In the fractionalized version, according to the composite fermion picture, some other number, $p$ of fluxes are attached to each fermion, and another number $\nu^*$ are contained in its orbit.

If one were to view the 'dynamics' of motion in this ground-state, in loose language, one would see composite fermions winding $p$ flux around $\nu^*$ others once every period of their effective cyclotron motion. The passage from one fractionalized state to another modifies these two numbers, liberating some number of fluxes from the composite particle and entraining a new number inside the cyclotron orbit. If the same number is liberated as entrained, no new flux need to be added to the sample area, but if these differ, that flux must be taken from inside Laughlin’s ribbon, and to do that, the macroscopic value of the current must be modified. This is the mechanism that gives our oscillator a large gain, and these are the transitions we are most likely to measure in our apparatus.

Thus if we want to understand the light-atom system more clearly, we want to understand the dynamics of this flux more intuitively and translate the physical picture into a language more natural to optics and dielectric media like an atomic vapor. A simple first step is to recognize that charged particles interact with magnetic flux in a way that centrally involves their angular momentum. For a neutral atom system then, one driven by induced polarization from an optical field, we should be concerned with the transfer of angular momentum between
fields the optical and atomic fields. The light can carry angular momentum in two ways - orbital by the optical phase profile across its wavefront, and spin, through its polarization. The atoms are similar, carrying both an intrinsic spin (strongly coupled to the optical polarization), and orbital motion.

If we step back a bit from the coupled Sagnac interferometers, we can view the movement of the Sagnac phase as a kind of change in its angular momentum. This requires 'looking from above' at the interferometer and recognizing that if the phase-evolution along the beam path as one traces around counter-clockwise is not equal to that tracing counter-clockwise, this is the only condition under which the Sagnac interferometer (fringe) phase is non-zero. If the interferometer were moving while one traced, the optical phases would naturally be different, so certainly it would respond to a change of angular momentum of the apparatus as a whole, and this is simply describing a ring-gyroscope. We could apply a similar idea if the index-of-refraction profile presented by something in its interior was rotating, presenting a net angular momentum from the polarization field induced in its volume. We can consider extending our view of these fields far into the wings of the beams propagating in the Sagnac, all the way throughout its interior. In order for the Sagnac output phase to increase by one fringe, a 'defect' would need to enter into this Sagnac interior, representing a point at which an arbitrarily small loop about this point would see the optical phase increase by $2\pi$. Of course, if we zoom out and look at a small incline, we will realize this point is really a line in three-dimensions, around which the optical phase winds.

Since the total angular momentum of the optical field is conserved, this line must not be destroyed or broken as the system evolves. To pass through an intermediate Sagnac phase, this 'vortex line' must then be removed from the interior through some process which accomplishes this adiabatically. Of course we could say the same about either interferometer, so we are free to think about the 'looping' of optical vorticity around the axis of each, and consider the modulation of Sagnac phase in either as a kind of topological process of moving these vortex lines across the axes.

In order to have cross-interferometer gain, we are interested in where the passage of vorticity out of one interferometer is guaranteed to be accompanied by passage into the other. The natural place for this to occur is in the atomic vapor where they intersect, and the natural processes involve the temporary transfer of angular
momentum, orbital and/or spin, into the gas of atoms. There are many ways in which this can occur, and unlike traditional treatments of a dielectric medium, the atomic vapor can easily begin to move with internal (orbital) motion. The dielectric response is then much richer in its ability to carry angular momentum, and this is precisely the kind of dynamic we are looking to probe. If the response is to be Hall like, we might expect that the typical atom both absorbs and circulates around integral optical orbital and spin angular momentum flux. We are sensitive to both in the interferometer by both its contrast and its phase, but the precise amount of flux transferred from one loop to the other is dictated by the states in the atomic vapor.

1.3.2 Pole Winding Interrogation of Oscillators - A Simple Driven Electronic Oscillator

Since the design of this experiment is somewhat complex, one can approach it through a few different phases of development. In each step, one or more fundamental aspects of the design will be verified. In the following subsections, the apparatus in each phase will be discussed in detail. The majority of the apparatus is devoted to the laser-cooling and trapping of neutral atoms (Rubidium-87) to form the atomic vapor, and in the optical context described above. This apparatus will be described in chapter 2 of this thesis. A considerably smaller, but technically subtle, portion of the apparatus is formed by the precisely driven optical version of a Laughlin oscillator, which is described in the remainder of this chapter. In this section, we will begin with a simple electronic oscillator, incorporating non-linearity in order to verify the presence of non-integral phase-slipping by rational fractions of $2\pi$.

1.3.2.1 Driven Oscillator Scheme and Active Filter Design

To verify the principle of driven oscillators and the Adler and Kuramoto models, a pedagogical design of an electronic oscillator was first developed. Presented in figure 1.8, the initial phase of the apparatus design included a two-frequency synthesizer made from an FPGA, an active filter controlling passband based on one signal from the synthesizer, a non-linear gain control module made from another FPGA, a solenoid generating magnetic field, and a pick-up coil generating signal
which is fed back to the active filter. There are one impedance buffer and an audio amplifier between the non-linear module and the solenoid, for the purpose of optimally driving the solenoid around the oscillation frequency.

As discussed in previous section, the two frequency synthesizer is an FPGA generating two analog signals using digital signal processing and analog-to-digital conversion, producing two signals that move along orbits in the complex plane, which is centered at a point on the real axis, i.e. a real frequency. One of these two signals is used to define the center of the pass-band of an active filter, and the other is the injection signal used to excite the system. We developed a programming method of generating complex waveforms using the FPGA with limited computational resources, which enables sufficiently good resolution for the waveform to be used in experiments. The detail of this methodology is discussed in Appendix A.

The active filter is an innovative design which allows an adjustable narrow pass-band. It follows a 'mix-up-mix-down' scheme, incorporating a signal in one input, which we will call the center-frequency control signal, and which defines the center of the pass-band for the other input, which we will use to filter the oscillator system response inside its loop. Using a mixer, the two inputs are first beat such that their lower sideband is near to DC. Passing it through a passive low-pass filter before mixing the result back up with the control signal input allows the low-frequency filter’s passband to be mapped into a band-pass filter centered at the frequency of the control signal. In this way, an extremely high-Q bandpass filter can be constructed precisely at the control frequency. The details of this active filter will be discussed in next chapter, along with its electrical circuit diagram.

Using the active filter, one of the two outputs of the FPGA can be used to define an oscillation pole for the closed-loop response of an electronic oscillator. To anticipate the eventual drive of the Faraday effect in an optical fiber, we will define the electronic oscillator circuit using the inductance of a high-power magnetic coil and additional capacitance to form a resonator. Since the magnetic coil necessary generates a sizable magnetic field, we can close the loop by sensing the field using a small pick-up loop, feeding its output back through the filter and into the drive of the coil again.

In order to understand how this driven oscillator works, we can ignore the injection signal and the non-linear gain stage for a moment. The loop is then simply 'solenoid - pickup coil - filter - amplifier", which has a closed-loop response mainly
defined by the active filter. The active filter provides a transfer function similar to a LC resonator, which forms a 2-pole roll-off in the Bode plot on either side of the resonance. In the experiment, we set the pass-band center to be around 820Hz and the bandwidth to be 100Hz. This center frequency is chosen to avoid, in the ultimate apparatus, both atomic vibrational levels at tens of kilohertz and lower frequency harmonics of 60Hz where laboratory noise worsens.

The oscillation of this simple loop builds up from noise, occurring several Hertz away from the center frequency of the active filter, according to the small-signal condition for oscillation. Depending on the linear gain coefficient, the oscillation can happen either below or above the center frequency defined by the filter. This is easy to understand from the feedback theory, as oscillation happens when the open-loop gain reaches unity while having a 180° phase shift from the input signal. With that in mind, we can fine-tune the active filter so that the oscillation happens sufficiently close to the center frequency, which will be very important once injection is added in.

When this electronic system starts to oscillate, an AC magnetic field with amplitude of a few millitesla is generated in the solenoid and the pick-up coil, with only three windings, will pick up an in phase signal with amplitude on the order of 100 mV. We can easily shift the oscillation frequency, either by changing the gain coefficient in the active filter, or by changing the center frequency of the active filter through re-programming the two-frequency synthesizer.

Once the driven oscillator system is formed, the dynamical phase-locking behavior can be probed - by adjusting the orbit size and rate of the synthesizer, phase-slips can be observed as integer multiples of $2\pi$. These experiments are described in a later chapter.

1.3.2.2 Non-Linearity in the Pole-Wound Oscillator

Ultimately, we are interested in the behavior of a non-linear atom-optical system. Since the atomic system is naturally quite complicated, it is important to first observe the behavior of a simple non-linear system with easily characterized nonlinearities. In the Adler and Kuramoto models described above, we saw that these non-linearities can lead to rational fractional phase-slipping, and in this step, we would like to observe that this indeed occurs in a simple driven non-linear electronic oscillator.
In the simplest analysis, we can consider oscillation as a result of the divergence of the closed-loop response, when a $\pi$ phase shift and a unity crossing happen simultaneously, as described in equation (1.49). This formula is derived under the small signal limit with assumption of linear open-loop gain. If the open-loop gain of the circuit is $g(\omega)$ and this function depends on the signal linearly, each time a small signal goes around the loop, it is amplified by a factor of $g(\omega)$. Here, $g(\omega)$ is a complex value since it carries a phase factor. Therefore, the closed-loop response can be calculated as:

$$G(\omega) = \sum_{n=0}^{\infty} (-g(\omega))^n = \lim_{n \to \infty} \frac{1 - (-g(\omega))^n}{1 + g(\omega)}$$ \hspace{1cm} (1.49)$$

The minus sign is taken since this oscillator is built in a feedback loop which initially carries a $\pi$ phase shift. One can also use a summing stage other than a difference in the feedback loop and derive a similar equation with only a difference in the sign in the denominator. When $|g(\omega)| < 1$, the closed-loop response $G(\omega)$ takes the form in equation (1.49). It is easy to see a divergence happens when $g(\omega) = -1$, which means an overall phase shift of $\pi$ when crossing unity gain.

In reality, this closed-loop gain cannot become infinitely large. The closed-loop signal inevitably will be clipped at some stage inside the loop, for example, at the supply voltage of an operational amplifier set. Such clipping also adds a distortion to the waveform, generating a series of harmonics in the spectrum. Naturally, both effects represent inevitable forms of non-linearity which clamp the signal swing of any oscillator once unstable.

Other non-linearities may set in even when the signal swing is not above a 'threshold value.' It is of our interests to have control over the non-linearity of the transfer function. To achieve that control, a non-linearity control module is inserted into the oscillating loop. This is a second FPGA programmed to, using analog/digital converters, take an input signal and calculate a non-linear polynomial based on it to produce output. It’s gate construction is simply a combination of several elemental digital adders and multipliers which output a polynomial up to the fifth power of the input signal as

$$V_{out} = a_1V_{in} + a_2V_{in}^2 + a_3V_{in}^3 + a_4V_{in}^4 + a_5V_{in}^5,$$ \hspace{1cm} (1.50)$$

where all the coefficient $a_i$ are adjustable simply through re-configuration of the
FPGA board. The coefficients can take positive and negative values to either enhance or suppress a certain power otherwise present in the loop at large signal swings. A detailed discussion of programming the FPGA for a non-linear polynomial calculator can be found in Appendix B. By controlling the non-linear coefficients in this portion of the loop, the equivalent parameters of the Adler model can in principle be stepped through different rational-fraction phase-lockings. The results of these experiments will be described in a later chapter.

From the standpoint of hunting for fractional-Hall type phases, it is important to be able to control the non-linearity of the transfer function of the electronic oscillator and understand how the phase-locking phenomena occur. In most interacting quantum many-body systems, non-linearity plays an essential role in the physics of how different phases form. Most such systems can be understood classically through some effective wave-like equation of motion. A good example of this can be found in the Gross-Pitaevskii equation (GPE):

$$\left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial \vec{r}^2} + V(\vec{r}) + \frac{4\pi\hbar^2 a_s}{m} |\psi(\vec{r})|^2 \psi(\vec{r})\right)\psi(\vec{r}) = \mu \psi(\vec{r}), \quad (1.51)$$

where the nonlinearity comes from the interaction between boson wave-functions introduced through cold collisions. In the case of GPE, the atomic order parameter $\psi$ interacts with itself. In the electronic Hall system, the non-linearity is a cross-coupling of the electromagnetic field equations for the magnetic field dynamics and the electronic wave-functions, known as the Chern-Simons system. Unfortunately, few of the many different versions of the dynamical equation of motion of this sort are known for the different fractional quantum Hall phases.

By observing the fractionalized phase-lockings introduced in the Laughlin-oscillator experiment, the non-linear mixing terms above can be inferred. In this way, we have an experimental platform for interrogating those relevant non-linearities. For this reason, we will try to understand injection-locking phenomena under varying conditions in the next section.

### 1.3.2.3 Injection-Locking and Phase Relations in the Pole-Wound Oscillator

As the gain of a feedback loop passes a certain threshold value corresponding to unit small-signal gain, a small amount of noise within a certain band can build up,
causing the system to oscillate in a self-sustained way. The injection of another signal into the loop can result in a variety of new behaviors, pulling the oscillation frequency closer to its own frequency, or even locking the oscillation frequency to itself and pulling its phase to a well-defined value under certain circumstances named 'synchronization' or 'injection-locking.'

In the classical regime, the study of coupled-oscillators can be traced back all the way to the famous Dutch scientist Christiaan Huygens. There are many mathematical and physical results derived from coupled oscillator models, and one can find some pedagogical discussion in [137]. Previously, we mentioned the most famous of these, referred to as Adler’s equation, which we will restate in its simplest limit with some more physically meaningful parameters,

\[
\frac{d\Delta \phi(t)}{dt} = \Delta f_0/(2\pi) - \frac{V_i f_0}{V 2Q} \sin(\Delta \phi(t)).
\] (1.52)

In this equation, $\Delta \phi(t)$ is the time dependent phase difference, and $\Delta f_0$ is the frequency difference between the injected signal and the free running oscillating frequency $f_0$. $V_i$ is the amplitude of the injected signal and $V$ is the amplitude of the oscillator when it runs at $f_0$ in its natural oscillation. $Q$ is a quantity determined by the transfer function of the oscillator - the Q-factor in the case of an LC resonator. A locking range can be obtained by setting the left-hand side to zero to represent a locked relative phase, and solving the equation to find

\[
|\frac{\Delta f_0}{f_0}| = \left| \frac{1}{2Q} \frac{V_i}{V} \sin(\Delta \phi(t)) \right| \leq \frac{1}{2Q} \frac{V_i}{V} = f_L,
\] (1.53)

which shows the dependence of the locking range on the relative amplitude of the injected signal and the natural oscillation amplitude at where the injection is summed into the oscillator. $f_L$ has been defined as the half-width of this locking range [132]. This formula of locking range apply not only in the cases of classic regime of coupled oscillator, but also in other cases such as injection locked lasers. For a fixed oscillator, if one plot the injection amplitude against its frequency, the locking range becomes a region which is called Arnold tongue. If the amplitude of the injection is fixed, i.e. the ratio of $V_i/V$ remain unchanged, the synchronization region shows a plateau structure seems very much like a quantization behavior.

In the 'moderate forcing limit', where the injection amplitude is comparable the natural oscillation amplitude, some higher order locking behavior can be observed.
Instead of a single Arnold tongue exhibited around $f_0$, multiple Arnold tongues show up around rational multiples of $f_0$,

$$\Omega = \frac{m}{n} f_0,$$

(1.54)

where $m$ and $n$ take non-zero integer values and $\Omega$ represents the center frequency of a new locking region. The plot of $\Omega/\omega$ versus $\omega$ consists of horizontal plateaus at all possible rational numbers, which might naturally remind one of the cosmetic appearance of the transverse Hall conductivity’s relation to filling factor in the fractional quantum Hall effect.

Close, but outside, the boundary of Arnold tongue, lies the injection-pulling region, in which the phase of the oscillator doesn’t follow the injection exactly, but is pulled away from its natural oscillation frequency. If one looks more closely at the oscillator dynamics in this range, one sees that this oscillator pulling is dominated by short events. There is a sporadic occurrence of fast phase slips between the oscillator and the injected signal - most of the time, the oscillator is quasi-locked to the injection, but during some short intervals, the differential phase rapidly slip by $2\pi$. The characteristic frequency in which this phase slip happens is:

$$f_b = \sqrt{(f_0 - f)^2 - f_L^2},$$

(1.55)

where $f$ is the frequency of the injected signal, and all other quantities are the same as defined above [136].

For the adiabatic manipulation of the two frequencies described in the pole-winding manipulation above, this phase slip is what one sees as the frequency contour takes the oscillator to the edge of the Adler tongue. This is what we will actual record as a measurement when running the experiments.

As demonstrated in previous sections, the two-frequency synthesizer outputs two complex waveforms along an ellipse in the complex plane with a fixed $\pi$ phase-difference. One of these two signals controls the pass-band of the active filter, and the other plays the role of an injection signal. As the amplitude of the center control signal is kept sufficiently high to naturally induce oscillation in the feedback loop, one can consider a pole in the closed-loop response is also moving in the complex plane, at a position nearly equal to that of the filter’s control signal.

In the 'frame' of this natural pole, the injection signal is orbiting around it.
As both the amplitude and frequency are being modulated, the injection signal is moving across the area of the Arnold tongue. If it stays entirely within the tongue, the system responds with a locked oscillator phase relationship to the driving signal, and their relative phase will stay zero. If the orbit is brought to the edge of the tongue, the relative phase will advance in short slips, potentially of sub-integral size. It is these sub-integral steps that we are interested. It is also worth addressing whether, in various non-linear configurations, differing geometric phases can be observed in the fully adiabatic limit that one stays entirely within the Arnold tongue. The result of experiments driving pole-winding in a non-linear electronic oscillator of this sort are described in later sections.

1.3.3 Pole Winding Interrogation of Oscillators - A Driven Opto-electronic Sagnac-Loop Oscillator and the Laughlin-Oscillator

It is worth restating that the strategy employed in this experiment is designed to take elements step by step. The ultimate goal of course is to construct a direct transport measurement in an atom-light system, where we embed the atomic-vapor system into the feedback loop to create oscillation. The expectation is to engineer coherent many-body state by excite the system into oscillation and such coherent state is controlled by certain filters that define the nonlinearity/topology of the whole system, and hence confine the topology of the atom-light many-body wave-functions. The whole process is following pump-and-probe scheme. By introducing the injection signal, we perturb the oscillation of the whole system and help the detection of the pole structure by observation of phase slip event.

Given the complexity of both the optical and cold atom optical parts of the apparatus, a second intermediary step can be accomplished by building the optical portion of the apparatus into the non-linear feedback loop. Since an important portion of the technical challenge is presented by the detection of extremely low light levels of laser-cooling pumping light, this confronts the already challenging prospect of putting shot-noise limited detection into the feedback loop.

In this second phase, the output optical Sagnac interferometer is placed into the previously purely electronic design. Instead of reading out the field generated by the resonator coil above using a pickup loop, instead an optical fiber is inserted
into the output Sagnac loop, threading the fiber through the coil’s interior. The very large magnetic field induces a Faraday rotation of the Sagnac light, causing a shift in its fringe pattern. The Sagnac interferometer phase is converted to an electrical signal in a heterodyne optical detector, and is fed back to the active filter to drive the coil. The intention is to create oscillation in this electro-optical system, with an optical fiber as a "gain medium."

In the third and ultimate phases, an atomic cloud will be introduced and we will attempt to stir the atoms by perturbing the magnetic field created from a pair of Helmholtz coils, detecting how the movements of the atoms affect the output Sagnac interferometer phase. This set-up also adopts the oscillator idea by feeding the resulting Sagnac phase signal into active filter and back to the drive of the coil. In the ultimate phase which has been discussed previously, the magnetic field is replaced by modulation of the input Sagnac interferometer to form a complete light-atom-light system, as a platform for direct transport measurement in the Laughlin-pump geometry. Details will be discussed in the next subsection.

1.3.3.1 Sagnac Interferometer Detail and Optical Phase Signal Conversion

In this experiment, an optical path is added to the pure electronic oscillator to form a slightly more complicated realization of flux winding experiment, which contains one Sagnac interferometer, in which the optical Faraday effect is induced directly in an optical fiber inside the interferometer (instead of using atoms).

Shown in figure 1.9 is this second phase of test design in this experiment, specifically the opto-electronic system. It is designed to establish the idea of constructing a quantum pump and probing with a driven oscillator. It is upgraded from the pure electronic driven oscillator by simply replacing the pick-up coil and non-linearity control module with an optical Sagnac interferometer, with a sensitive opto-electronic heterodyne detector added to convert the Sagnac phase into an electronic signal.

The Sagnac interferometer is formed by first splitting a 780nm laser beam on a polarizing beam splitter. The two beams then propagate down the identical interferometer path in opposite directions, reaching the polarizer again. A quarter wave plate is inserted on each end of the interferometer, just beside the polarizing beam splitter’s exit port. Light becomes circularly polarized between these two
quarter wave plates (and is linearly polarized between the cube and those quarter waveplates). Both arms then exit the interferometer either from the entrance port or the fourth port of the polarizing beam splitter cube and interfere with each other. In order to observe the Sagnac phase conveniently, we deliberately misalign the interferometer by a small amount to create linear interference fringes, which move horizontally as the Sagnac phase is changed.

For this test phase of the experiment, inside the Sagnac interferometer, there is a section which consists of a single mode optical fiber, which runs through a solenoid made from about roughly a thousand windings of copper wire. The solenoid generates a roughly uniform magnetic field inside the tube when current flows through the copper wire. The optical Faraday effect can be induced inside the fiber, using the natural (small) Verdet constant of optical glass. The purpose of this interaction is to mimic the interaction between an atomic cloud and the optical field in the vacuum chamber in the ultimate phase of experiment, where optical pumping effects induce a similar Faraday phase shift. The optical Faraday effect is non-reciprocal, therefore two arms of the interferometer experience different phase retardation for circularly polarized light. This results in the shift of the interference fringes in the output of the Sagnac interferometry.

The interference fringes formed by beams exiting the interferometer form an optical intensity profile translating the Sagnac phase into a translation of a linear fringe pattern. We wish to use robust photo-detection techniques to convert this Sagnac phase to an electric signal - this was done by the Sagnac phase detector block shown in figure 1.9. In the actual set up, this detector consists of several different devices which are carefully engineered to improve the sensitivity and to suppress noise. In fact, this 'phase detector' is actually an opto-electronic set-up to lock a photodetector to one minimum of the intensity profile, i.e. a dark fringe, measuring the angular displacement necessary to do so, and outputting this. An acousto-optical modulator is used for this, to diffract the beam with a controlled angle. The fringe is detected by collecting light in a single-mode photodiode, heterodyning it with a much stronger beam of slightly different (O(100MHz)) frequency to enhance sensitivity, and detecting this in a radio-frequency heterodyne receiver. The deflecting AOM frequency is dithered to move the fringe across the fiber input, and its dc-value is locked to a zero-crossing. The 'Sagnac phase detector' is essentially the control signal necessary to find this fringe position. The
technical detail of this Sagnac phase detection will be further discussed in the next chapter.

One may notice that the Verdet constant of an optical fiber is much smaller than an alkali atom vapor, therefore the optical Faraday effect can be more significant in atom-light system than in optical fiber. However, the optical fiber in a magnetic field still forms a simple "sample" that is easy enough to model and to setup, in order to mimic the atom-light interaction, and to test the driven oscillator scheme when optical path is included.

1.3.3.2 The Atom-Optical Laughlin Pump Oscillator

To incorporate atoms into the experiment, I developed a phase 3 of this experiment, which is a short-cut opto-electronic, plus magneto-atomic version of the analogous Laughlin pump oscillator. This phase of experiment is closer to the final design. As the magnetic field is driven adiabatically instead of the input Sagnac interferometer from the far-off resonant YAG light, we could also expect quanta from magnetic field being pumped to optical vortices through atom-light interaction, and by feeding the Sagnac phase into the magnetic coil, an oscillating loop is closed to apply the quEASE method.

Shown in figure 1.10 is the phase 3 version of the experiment design, in which atoms are presented and the oscillation loop is closed by using the optical pumping Sagnac phase signal to feed to a pair of Helmholtz coils which generate magnetic field perpendicular to the direction of optical pumping light propagation. The coils used here are one pair out of three that are designed to compensate the earth magnetic field and to optimize magnetic field in different stages of laser cooling and trapping. During Raman sideband cooling process, those coils are driven with a current that optimize the cooling process with the presence of optical pumping beams. Atoms are then released into a dipole potential formed by YAG beams. The Sagnac phase detection is done with atoms trapped in the dipole potential and also interacting with the optical pumping light field. By adding disturbance to one pair of coils that affect the cooling efficiency the most, we can perturb atoms and hence impact the optical pumping beam and the Sagnac phase through optical Faraday effect. In this set-up, the optical fiber path used in phase 2 is bypassed so the only gain in the system is through atom-light interaction.

The idea of phase 3 of the design is to verify that a driven oscillator scheme
can be realized when atom-light interaction is part of the feedback loop.
Figure 1.7. Diagram of a realization of a quantum pump through atom-light interactions. Two Sagnac interferometers for light can take the place of wires coupling currents into and making voltage measurements of an electronic Hall bar. Both the input and output (or optical pumping OP) Sagnac form wrapped loops, which intersect at the location of an atomic vapor (black dot). By driving the solenoid around a fiber in the input interferometer (YAG) loop, the Sagnac phase is modulated through the Faraday effect in the fiber’s glass. The input interferometer light, which forms a far-detuned dipole potential for trapped atoms, causes a modulation of the atomic vapor’s ground state polarization. This, in turn, modulates the polarization of the output interferometer light through an atomic Faraday effect. Effectively, one can see that the atomic vapor mediates an exchange of angular momentum between the two Sagnac interferometers. By feeding a filtered version of the signal induced in the output interferometer back to the solenoid modulating the input interferometer, an oscillator is formed. The oscillator can be probed by 2-frequency synthesizer, controlling both the pass-band of the filter, and a probing injection signal. Details of all these steps will be discussed in detail in the next chapters.
Figure 1.8. The pole-winding method of interrogating an oscillator. Here we show a simplified fully electronic oscillator, probed by controlling the relative frequency and amplitude of the oscillator’s gain loop and a probing injection signal. In this simplified technique, which will later be applied to the Laughlin Pump Oscillator in the previous figure, a loop is closed by feeding the induced EMF on a pick-up coil around a solenoid through a filter controlled by the 2-frequency synthesizer, which then drives the solenoid. The active reference filter determines a narrow pass-band at one of the synthesizer’s frequencies, which restricts the region where oscillation can happen in the closed loop. A non-linearity control module is inserted in the loop in order to adjust the large-signal closed loop transfer function. The 2-frequency synthesizer produces two sinusoidal signals, one which defines the closed loop natural pole and the other which serves as an injection signal, summed into the loop, and pulling or locking the system oscillation by driving the oscillator. Those two signals are frequency- and amplitude-modulated, to produce two complex-valued frequencies moving around each other on the complex plane. In the adiabatic limit, this orbiting movements results in a geometric phase, and dynamical frequency pulling which can be observed as quantized phase slips between the injection and closed-loop system response.
Figure 1.9. Diagram of a fully opto-electronic version of the Laughlin pump. This is a more advanced version of the pump oscillator, where angular momentum is pumped from an electronic oscillator inducing optical Faraday effects into an optical Sagnac interferometer. The Sagnac interferometer has a robust phase which can only be changed when there’s an object in the loop that can break time-reversal symmetry, e.g. an optical Faraday effect. In this design, one optic fiber inside the solenoid experiences a changing magnetic field, and modulates the optical Sagnac phase through the Faraday effect. This phase is measured at a Sagnac phase detector (described in detail in the next chapter) and is converted into an electronic signal. This signal is then fed to the active reference filter module, where it is filtered and then summed with an injection signal. The output of this active reference filter is amplified then applied to the solenoid, which closes the oscillator loop.
Figure 1.10. Diagram of a short-cut opto-electronic, plus magneto-atomic version of a quantum pump used in phase 3 experiments. This is a further refinement of the quantum pump scheme, and one step away from the ultimate version described above. In this design, atoms replace the optic fiber in the previous set-up, and the Faraday effect is induced directly in the cold atomic vapor, which yields a much stronger Faraday effect. Three pairs of Helmholtz coils produce a magnetic field at the location of the vapor, which is almost along the direction of the optical pumping beam, optimizing the Raman side cooling mechanism. Its modulation induces an optical Sagnac phase on the output interferometer made from optical pumping light. The signal is converted, filtered, amplified, and then fed to one pair of coils which generates the magnetic field perpendicular to the optical pumping beam. This oscillator loop is closed only if atoms are present, and oscillation of this type represents one milestone on the way to using quEASE methods to detect fractional Hall physics in cold-atom systems. The details of this "phase 3" experiment, the technical hardware, and results are contained in the rest of this thesis.
1.3.3.3 Progress in Building up the Analog Laughlin Pump Oscillator

The previous sections have demonstrated the goal of this experiment and the progressive routine of constructing apparatus to realize the analogous Laughlin pump in a light-atom system, which is embedded in an oscillator and probed by a pole-wound scheme of driven oscillation. The completion of this design would be the first direct diffusive transport measurement in a cold atom system and the application of quEASE method could open up some new pathways for quantum simulation.

All phases have been constructed along the way and the phase 1 has been fully investigated. The functionality of the driven oscillator scheme has been verified. Once light and atoms are introduced to the apparatus, some major hurdles appear, which prevent the system from showing clear signal of a quantum pump. The challenge of this experiment mainly originates from small signal detection as the light beam for the output Sagnac loop is a 1-100pW level. A heterodyne amplification is adopted in order to enhance the light detection, but the multiplicative noise of the system remains an obstacle in the completion of the analog atom-optical Laughlin pump oscillator. Some tricks have been attempted to overcome those noise, however, although the signal has shown some positive sign of promising feature, there is still a long way to go until a well resolved signal can be detected. The current progress of this experiment and some data taken at different phases of the apparatus will be discussed in chapter 3.
Chapter 2

The Experimental Apparatus for Laser Cooling and Trapping of Neutral Atoms

In the previous chapter, we introduced the ideas behind the experiment - building an oscillator to probe the existence of coherent many-body states in a cold atomic vapor by closing the loop on an analog of Laughlin’s charge pump built with light.

In this chapter, I will discuss the details of the apparatus for operating this experiment, focusing first on the portion used for producing cold vapors of neutral Rubidium-87 atoms, and then proceeding to the optical Laughlin pump and oscillator hardware.

The apparatus includes many nearly independent pieces - an ultrahigh vacuum chamber, a 780nm diode-laser system for cooling and trapping, a high-power fiber laser system for dipole trapping, an imaging system, electronic timing and control system, and optical phase detection system.

The vacuum system and laser systems are thoroughly discussed in my colleague Jianshi Zhao’s thesis [209], and therefore in this thesis, I will focus more on the subsystems that are directly related to the optical Laughlin pump experiments, describing the portions of the cold-atom system which either are necessary to understand in detail for the Laughlin pump, or portions which were altered or upgraded during the timeframe that the Laughlin pump was constructed.

In the following section, I will briefly discuss the atom source, including the vacuum system and laser table. In section 2.2 we will cover the setup of Sagnac interferometers. In section 2.3 we will discuss the optics and electronics directly related to signal detection and data recording of the transport measurement.
2.1 Atom Sources and Laser Trapping and Cooling Methods

The production of cold and quantum-degenerate atomic vapors has a rich history, and has developed rapidly over the last several decades since the demonstration of Doppler and sub-Doppler laser cooling. I will not review that extensive literature here, but will point the interested reader to the excellent descriptions of the methodology in [209] and the references found therein. In this section, I will describe one of the many ways such vapors can be produced, and in a context in which the Laughlin-pump experiment can be carried out, ultimately placing an ultra-cold vapor in an optical setup with sufficient numerical aperture to produce a fine-grained disordered potential with light, and around which the intersecting Sagnac-geometry described in the first chapter can be placed. We will also be concerned with the final temperature of the atoms, and that the maximum interrogation times over which the Laughlin-oscillator can be run are sufficient to accrue a reasonable amount of data in a given repetition of the experiment.

2.1.1 Precision Lasers for Cooling and Trapping

It is somewhat difficult to cover all of the technical details in the apparatus used in this experiment in microscopic detail, as many are comprised of assemblies of a surprisingly large number of interacting subcomponents. In this section, I will describe a smaller corner of the apparatus, specifically the construction of highly-stable and narrow-linewidth lasers used for trapping and cooling. Partly, this is to show some sense of the level of detail at which experiments like these must work, but partly I include this section to illustrate a portion of the apparatus where my contribution was more independent, and the information would be lost if it were not included here.

In the introduction, I gave several examples of quantum simulation based on atoms that interact through inter-atomic collisions. Though the Laughlin-pump itself is not of this type, many other techniques for generating Hall effects are, and since the collisional behavior of atoms is very species-dependent, we have designed the experimental apparatus ultimately to perform with both Rubidium-87 and Cesium-133 atoms, and to be able to perform these collision-mediated experiments.
as well. Since the optical spectra of atoms are also very different from one another, multiple laser systems are needed to work with more than one.

Although our previous experiments are all merely using Rubidium-87 atoms, we have planned to upgrade the apparatus to include Cesium atoms so that in the future we are able to either transition to experiments with Cesium atoms or study the mixture of different types of atoms. As part of this, constructing lasers that work at the wavelengths coupled to atomic transition of Cs-133 was necessary. In this section, I will present external cavity lasers that I designed and constructed from raw materials and assembled components. These lasers are based on an innovative design by Cook, et. al. [210], which has high stability and great performance, but modified substantially for ease of production, greater stability, and hands-free operation.

As described in detail below, our entire laser table of 780nm lasers for the Rubidium experiments are constructed based on several external-cavity diode lasers (ECDLs). Those ECDLs are use a so-called 'Littrow configuration', in which a laser beam from diode-laser-chip is collimated and sent to a diffraction grating. By carefully tuning the orientation of the grating, we can send the first order diffraction back to the laser diode, creating an optical feedback. Therefore, an "extended" or "external" cavity is formed between the grating and the facets of the laser diode, the rear of which has a high-reflection coating. The laser diode intra-cavity, external cavity, and the diffraction grating together defines the gain profile of this setup.

In our previous design, the laser diode cavity is machined from four pieces of beryllium copper, one of which creates a 'C' shaped clamp in which the collimating lens is placed, and another in which the laser is clamped. The diffraction grating is glued to a rectangular piezoelectric actuator (piezo), and the piezo is glued to separately machine copper block forming a specialized kinematic mirror mount. The C-clamps for the lens and diode are then mounted on one thin flat plate made of copper by metal screws. The plate is slotted over its length (requiring the assistance of an expert machinist) to form a vertically-deflecting kinematic mount. Finally, a thermo-electric cooler (TEC) sits underneath the copper plate and above a large aluminum base, which serves as a heat sink. This TEC is part of a feedback loop for stabilizing the temperature of this device, since laser diode gain profile and the external cavity length can be affected by thermal drift. The temperature feedback loop is able to lock the temperature within 10 mK. The piezo belongs
to another feedback loop, in which the phase error signal of the ECDL is fed to
the injection current of laser diode and control voltage of the piezo, for further
stabilizing the laser frequency. Our ECDLs typically have a single mode tuning
range of 10GHz.

Although this older design of ECDLs have reliably served several generations of
experiments, there are still some issues, such as alignment drift caused by difference
in thermal expansion between separated components made from different materials,
and metal relaxing due to the softness of aluminum and copper. The need to tune
these elements subtracts from the running time of the apparatus and leads to lost
time.

To improve the performance of ECDL from those aspects, we decided to reduce
the number of individual components for constructing the Littrow configuration
mount, while maintaining as many degrees of freedom for the purpose of optical
alignment as required. The moving components of the ECDLs are cut and machined
from a single 304 stainless steel sheet using rapid fabrication techniques based
on CNC plasma cutting and milling. The choice of 304 stainless steel is based
on its better elasticity and smaller coefficient of thermal expansion, compared to
aluminum and copper, with a small compromise on the easiness of machining and
lower thermal conductivity. With the help of a plasma cutter and a CNC milling
machine, machining stainless steel becomes substantially less time-consuming, and
many lasers can be constructed in a comparatively short time.

Three 5.5 inch × 5 inch × 0.5 inch stainless steel bulk pieces are cut into the
same polygon shape, forming three layers of the ECDL structure. The top layer is
essentially just an enclosing lid, with cutouts for accommodating a D-sub connector
which have wires connected to the laser diode, thermistors, and the piezo. The
middle layer is where we create the Littrow configuration, with kinetic flexure arms
- threaded holes are machined in this piece to accommodate actuator screws for the
flexure (differential screws). The pivot points for the flexure arms are machined to
0.125 inch thin to allow a reasonable flexibility. A diffraction grating is glued to
the front side of the thin flexure arm, facing a rectangular block which is attached
to the bottom layer. This flexure arm can be pushed by two fine thread adjustment
screws, which provide two degrees of freedom on the horizontal plane.

The Littrow configuration is completed by the laser diode and the lens tube
installed inside this rectangular mounting block. The mounting block is fixed onto
a flexible stand in the bottom layer, where a C-shaped hollow region is carved out around the stand and the stand itself is thinned to 0.125 inch. Another fine thread set screw sits inside the top layer and pushes against the mounting block, providing adjustment of vertical angle to the collimated laser beam. The Littrow angle can be easily calculated as the first order diffraction overlaps with incident beam \( \theta = \sin^{-1}(\lambda/2d) \), where \( \lambda \) is the optical wavelength and \( d \) is the grating period. As the gratings used in these new ECDLs are 1800 grooves/mm, one can find out \( \theta = 50.1^\circ \). This angle is not easy to achieve if metal is machined manually, however, such difficulty is vastly reduced by using a plasma cutter to remove the bulk of material with arbitrary shaping - a carbide end mill on a CNC machine can quickly finish the raw material to the correct structure and dimensions. The surface of the flexure arms was finished by HSS end mill to ensure a sufficient flatness for gluing the diffraction grating, and good reproducibility between lasers.

The top and bottom sides of all three layers, and all sides of the mounting block are also precisely machined by end mill and fly cutter, which ensures good thermal contact when all these components are assembled together. Four 1/4 inch – 20 tapped holes in the middle layer and four matching clear holes in both top and bottom layers piece are made so that three pieces can be clamped together by stainless steel screws. Two aluminum plates are attached to this three-layer ECDL from top and bottom, serving as both lids and heat sinks. A rubber sheet is used as a gasket in between aluminum lids and stainless steel ECDL components, preventing direct thermal conduction. This whole assembly forms an enclosed space, aside from one hole on the side of the middle layer for laser beam to exit the device, therefore, air currents that can affect the laser performance are largely reduced. Figure 2.1 shows the CAD drawing for the ECDLs. More details and mechanical drawings of the ECDL design can be found in Appendix C.

Much of the apparatus could also be described at this level of detail, but since many of these details are similar to other apparatus designs that can be found in many other students’ thesis and academic literature, I will try for the remainder of this thesis to work at a higher level of description, except for the more unique components, specific to the Laughlin-pump idea.
Figure 2.1. External Cavity Diode Laser schematics. Flexure arm structure is machined from a stainless steel bulk material with the help of CNC plasma cutter and CNC milling machine. Those arms provide three degrees of freedom, allowing one coarse and one fine horizontal adjustment, and one vertical adjustment, all of which are achieved by fine thread (1/4 inch − 80, part number F25SS100) set screws. This design is an application of Littrow configuration and is inspired by Dan Steck’s high stability laser design [210]. An infrared laser beam is emitted from an edge emitting diode, and is collimated by a strong lens. A mount is designed for accommodating the laser diode and collimating lens, which allows wires connected to the diode from the back of the mount, as well as some room for sliding the lens in the mount to optimize collimation. Collimated beam hits a diffraction grating, which is attached to the front side of the thin flexure arm by optic glue. By adjusting knobs that control the vertical and horizontal degrees of freedom, one can couple the first order diffraction back to the laser diode, forming an external cavity between the diffraction grating and the high reflection rear end of the diode chip. Such resonator design narrows the laser line-width to below 1MHz and allows the laser mode to be tuned through the diffraction grating, which can be done by adjusting those two fine thread set screws.
2.1.2 Atom Sources and the Optical System for Cooling and Trapping

Figure 2.2 shows the schematic of the overall experimental apparatus, without the Laughlin-pump architecture. Two ultra-high-vacuum compatible Pyrex glass cells are located at the top and bottom of the vacuum chamber, connected by largely stainless steel vacuum parts. These two cells, each made from bonded optical flats, form a source cell, where atoms are initially loaded from a room-temperature thermal vapor, and a UHV cell, where experiments are conducted after the vapor is transported over the separation of 80cm between them. Optics and magnetic coils are built around each cell to construct magneto-optical traps (MOTs), as well as perform other laser trapping and cooling techniques. Rubidium-87 atoms are initially generated in source cell from a getter source, collected in a source MOT and further cooled to sub-microkelvin temperatures, transferred into the UHV cell using gravity, captured by a few-mm UHV MOT in the center of an optical microscope, and then further cooled and loaded into a high-power optical speckle potential where the transport measurements are done. I will go through each part of this system, following the order of operating the experiment.

2.1.2.1 Cooling and Trapping Optical System

The optical systems around the source cell provide all 780nm laser beams for laser trapping, cooling and detection. The lasers are from a separate laser table, which includes custom built external cavity diode lasers (ECDL), a laser lock system, a tapered amplifier system and an optical switch-yard. Details of the laser systems can be found in Zhao’s thesis and the master’s thesis of Rene Jacome [209]. Briefly, one ECDL, called a ‘Reference laser’ is frequency locked to the \( F = 3 \rightarrow F' = 3/4 \) crossover resonance of Rubidium-85, by using a modified form of modulation-transfer saturated absorption spectroscopy [211]. The other two ECDLs, respectively named the ‘MOT laser’ and ‘repumping laser’ are offset-locked to this reference laser through beat-note locking, measuring and stabilizing their frequency difference in the microwave range [212]. The MOT laser frequency is locked near the atomic resonance frequency corresponding to the \( F = 2 \rightarrow F' = 3 \) in Rubidium-87, and the repumper laser is locked to the \( F = 1 \rightarrow F' = 2 \) of the Rubidium-87 D2 line. Both MOT and repumper are stabilized down to less than a 100kHz line-width as
Figure 2.2. Schematics of source of ultra-cold Rubidium-87 atoms. Rb atoms evaporates from a getter source, and are cooled and trapped in a Magneto-optical trap in the source cell. About $6 \times 10^{10}$ atoms are captured in the MOT. A Raman sideband cooling sequence is then applied and then atoms are dropped into the ultra-high vacuum (UHV) cell which is 80cm below, with the help of a vertical guiding beam to confine the transverse expansion. In the UHV cell, another MOT is applied, which is able to captures $12 \times 10^5$ atoms. Then atoms are further cooled by a Raman sideband cooling sequence where Optical Pumping beam is applied, while being cooled, atoms are also released into a 3-D disordered potential formed by 2 pairs of counter-propagating YAG beam. The details of YAG lasers are illustrated in the next subsection.
measured over a 100ms time.

The MOT and an intensity-controlled repumper beam are combined at a polarizing beam splitting cube with a power ratio of 10 : 1, and then the combined beam is delivered to a tapered amplifier (TA) set up by a polarization maintaining (PM) optic fiber, such that all beams derived from this amplifier can produce light near the cycling transition of Rubidium-87, as well as a smaller repumping fraction. The TA outputs about 400mW and then the output beam is coupled into a PM fiber which sends light into a "switch-yard" with 55% efficiency. The optical switch-yard splits the beam from TA output into multiple beam paths to control both their frequencies and intensities through acousto-optic modulators (AOMs) and mechanical shutters, in order to run various operations throughout the experiment. Most AOMs on the switch-yard are in a double pass configuration [213] in order to improve the stability. Finally, each beam is coupled into a single mode PM fiber and delivered to the experiment table optical setup. Here, we label all the beams based on their functions in the experiments. The diagram of the switch-yard table is shown in figure 2.3.

The switch-yard provides modulated optical power for several functions in the experiment:

- **Source MOT**: this path contains both cycling transition and repumper laser frequencies, and is responsible for loading MOT and polarization gradient cooling (PGC) in the source cell. This path is in a double pass AOM configuration using the +1 order with the MOT AOM frequency set at 100MHz. It takes roughly 100mW from the TA output at full intensity.

- **Source Cell Raman Sideband Cooling (RSC)**: RSC in the source cell uses four beams in a tetrahedral geometry. One AOM controls the vertical lattice beam (Vertical Lattice AOM) and another AOM (Source Cell Lattice AOM) controls the other three beams. These two AOMs share the same RF source to ensure a common frequency. They are in single pass configurations at 80MHz using the +1 order. All beams are derived from the zeroth order of the MOT AOM path.

- **Detection**: this path is used for diagnostics in the source cell and UHV cell, including time of flight (TOF) and absorption imaging. The AOM (Detection
AOM) frequency is the same as the MOT AOM. The power in this path is around 10mW.

- **UHV MOT**: this path is used for MOT loading and PGC in the UHV cell. The UHV MOT AOM frequency is 100MHz.

- **RSC Optical Pumping**: this path is for the optical pumping light ($F = 1 \rightarrow F' = 0$) during RSC. It is taken from the repumper laser before combining with MOT laser. The power is around 1 to 2mW. The RSC OP AOM operates at 80MHz in double pass using the +1 order.

- **MOT Repumper**: this path is responsible for controlling the repumper laser independently before combining with the MOT laser. A portion of the repumper laser (1mW) goes through a double-pass AOM (Repumper AOM) setup at 95MHz (+1 order). Then it combines with the MOT laser through a PBS, following with another PBS to project them onto the same polarization before coupling to a single mode PM fiber for TA injection.
Figure 2.3. Schematics of source of the laser table: A Diagram of the optical switch-yard. Laser beams are input from both ECDLs and TA Output. It consists of many optical paths with each path responsible for one or several steps in cooling and trapping atoms. The power of each beam path is dynamically controlled by an AOM and a mechanical shutter. Specific purpose of each path is discussed in the main text. [209]
2.1.2.2 Atom Source, Source Magneto Optical Trap, and Pre-Cooling of Atoms

A 'room-temperature' rubidium atomic vapor is produced from alkali metal getter dispensers [214], which dispense metal atoms while absorbing other active chemicals generated during the evaporation. Therefore, a relatively clean rubidium atom vapor is created in this way. We have both Rubidium getters and Cesium-133 getters installed in the source region of the apparatus, where the pressure is around $O(10^{-9})$ torr. The atoms dispensed directly from the getter source are created at the getter by Joule heating, but atoms participating in the vapor in the cell are loaded both directly from the source, or after interacting with cell walls, so the atomic vapor is at or above room temperature $\sim 300$ K.

The experiment starts with a MOT loading step in the source cell. Three pairs of circularly polarized counter-propagating beams, which are 24MHz red-detuned from the $F = 2 \rightarrow F' = 3$ transition, along with a pair of quadrupole coils, create a standard three-dimensional MOT [215–220]. The red-detuned light damps the motion of atoms moving towards it through an effect easy to understand by Doppler effect [19, 21, 221], and therefore referred to as "Doppler-cooling." The quadrupole coils generate a magnetic field gradient which causes a position-dependent Zeeman shift of the ground state magnetic sublevels, which together with optical pumping effects, form a restoring force [22, 222]. Together, the MOT they form resembles a strongly damped harmonic oscillator. Since a small portion of atoms may go through an off-resonance transition $F = 2 \rightarrow F' = 2$ and ends up at $F = 1$ ground state which doesn’t scatter MOT light any more, a repumper beam coupling $F = 1 \rightarrow F' = 2$ is applied to pump atoms back into the $F = 2$ manifold. About $8 \times 10^8$ atoms are loaded into the MOT, with temperature cooled to $145\mu K$, confined in a region of several hundred $\mu m$ across.

Following this, polarization gradient cooling (PGC) is applied to further lower the temperature through optical molasses [223–227] effects to approximately $\sim 4\mu K$. The same beam configuration for the MOT is used here, but the magnetic field gradient is switched-off. The light shift of the ground state magnetic sublevels depends on both magnetic quantum numbers and the polarization of light field, which results in a spatial potential. Atoms climb the potential hill and lose kinetic energy. At local maximum, atoms are pumped to opposite magnetic states and potential energy are removed by scattering photons with a higher frequency. The
result of PGC is an average effect after atoms repeatedly lose kinetic energy after climbing potential hills. Since this process is sensitive to the background magnetic field, which can produce an offset to these magnetic sublevels, in the actual experiment, MOT quadrupole coils are turned off, and six 'compensation coils' are turned-on to eliminate this external magnetic field. Over 2ms, the detuning of the MOT beams is ramped (using lock electronics) from 96MHz to 183MHz to decrease the light shift, and the intensity of the beam is ramped down to 1/10 of its initial value. As the PGC beams are held at these parameters, the atoms reach a final temperature of $\sim 4\mu K$ at the end of the optical molasses sequence.

Ultimately, these atoms need to be placed in a context in which the Laughlin-pump experiment can be performed. This requires both an interrogation time sufficiently long to build up and probe oscillator dynamics, and an optical geometry which facilitates the high numerical aperture necessary to produce disordered potentials. Neither of these constraints can be met in the geometry and vapor conditions of source chamber, since the necessity for sufficiently fast loading requires large MOT beams, restricting optical access, and the presence of thermal background atoms limits the interrogation time to below a hundred milliseconds. For this reason, atoms must be transferred into a new cell geometry to perform the experiment. This is done by allowing atoms to drop into a much higher vacuum cell at the lower end of the experiment, where they are captured in another MOT.

The transverse size of the cloud during this drop expands to a size which scales as the square root of temperature due to thermal motion. To ensure the cloud doesn’t expand by too much after being transferred from the source cell to the UHV cell through its 80cm free fall, a further cooling process is necessary. A three-dimensional Raman sideband cooling step is applied, which, at least in principle, can reduce temperature below the limit determined by a single photon recoil [228–230], since it is a dark state cooling scheme. Atoms are prepared in the $F = 1$ hyperfine ground state manifold by holding them in low intensity MOT beams for 1 ms without the presence of a repumper beam. A 3D optical lattice potential is generated by four moderately-detuned beams in a tetrahedral geometry. A small (of order one hundred milligauss) and uniform magnetic field is applied, which is controlled at a value so that the vibrational quanta in the lattice potential is equal to the Zeeman splitting between magnetic sublevels.
\[ \Delta E_z = g_F \mu_B B = \hbar \omega \quad (2.1) \]

Raman sideband cooling is a cooling scheme usually considered in the Lamb-Dicke regime, in which the vibrational splitting of tightly bound atoms is much larger than photon recoil energy. This cooling scheme relies on spontaneous anti-Stokes Raman scattering to remove energy from each atom. The existence of a dark state ensures an irreversible process that atoms accumulate in this state and are no longer resonant with to the optical pumping field.

Briefly, the pumping and cooling cycle of the degenerate Raman sideband cooling in optical lattice in this experiment can be described as following: atoms are confined in discrete deep traps on each site of the optical lattice formed by the tetrahedral lattice beams, which accommodates a series of well-resolved vibrational levels. An optical pumping beam, which consists of mostly \( \sigma^+ \) polarization and a little \( \pi \) polarization, drives \( F = 1 \rightarrow F' = 0 \) transition, while making \( |F = 1, m = 1, \nu = 0\rangle \) a dark state. Atoms are pumped from \( |F = 1, m = 1, \nu\rangle \) and \( |F = 1, m = 0, \nu\rangle \) to \( |F = 0, m = 0, \nu\rangle \) by \( \sigma^+ \) component and \( \pi \) component respectively. Atoms then spontaneously decay to \( |F = 1, m, \nu\rangle \).

The optical lattice not only provides a scaler potential, but also a vector light shift, which resembles an effective magnetic field that drives the two photon Raman transition, coupling the atomic state \( |F = 1, m, \nu - 1\rangle \) and \( |F = 1, m - 1, \nu\rangle \), which are made to be degenerate by the condition in equation 2.1. Therefore, atoms then cycle through those transitions, until decaying into the dark state through pumping transitions, which is off-resonant to both the optical lattice light and the optical pumping light.

A cooling cycle of degenerate Raman sideband cooling can on average carries away energy equal to one vibrational splitting by this pumping process. In the current setting of RSC in this experiment, the lattice beam is 20MHz blue-detuned from the \( F = 2 \rightarrow F' = 2 \) transition, and the lattice depth is \( \sim 50\text{kHz} \). The optical pumping beam is 12MHz blue-detuned from the stated transition. An external magnetic field is optimized at 80 mG and its orientation is nearly aligned to the optical pumping beam with a small angle to add a small component of \( \pi \) polarization.
2.1.2.3 Guided Transfer into the UHV MOT, Cooling and Imaging at High Numerical Aperture

The atoms are then released from the source cell and transferred into the UHV region under near free fall. The 0.8 m distance between source to UHV region corresponds to an 0.4s traveling time. It is easy to calculate that at an average thermal velocity of 6mm/s (consistent with the cooling results above), which corresponds to a 2.4mm thermal expansion on the cloud size when it reaches the UHV cell. A vertical guiding dipole potential is created by a several watt far-detuned (1064nm) laser beam, which provides a transverse confinement for the atoms along the path connecting chambers. This guiding light enhances the transfer efficiency by a factor of 2 per drop.

A second MOT recaptures atoms in the UHV cell, whose pressure is at most $10^{-12}$ torr. Three counter-propagating beams, roughly 3mm in diameter, and a pair of quadrupole coils together form this UHV MOT, and it is directly connected to the source MOT through the guiding beam, which allows a direct loading into the UHV MOT. Each release from source MOT results in about $10^7$ atoms loaded into the small volume roughly 3mm across. An optical molasses is applied to further cool atoms, followed by a generalized version of the Raman sideband cooling (RSC) scheme above, where the "lattice potential" is replaced by the three-dimensional disordered potential formed by roughly 20W of laser speckle (described in more detail below). This generalized Raman sideband cooling itself is an interesting topic, in which the group has observed new phenomena closely related to emergent dynamical phase transitions, gauge-field physics and holographic-scaling principles. Details of this may be found in Jianshi Zhao’s thesis.

The time sequences described above require a considerably complex control over many independent devices in the laboratory. This is accomplished using a timing system, which controls all electronics and devices which communicates with all AOMs, mechanical shutters, laser current controllers, data pipelines, etc. and eventually creates a timed sequence of events, which forms a typical experiment. More information about the 780nm optical table, laser-cooling setups, and timing system can be found in Jianshi Zhao’s thesis.

Though not used in the Laughlin pump experiment itself, it is worth mentioning the imaging system built around the UHV cell. Four identical microscope objectives (USMC WM-020NIR) are placed around the UHV cell, facing each of the four
sides of the cell, and allowing for the imaging of atoms from four vantage points with individual numerical apertures of approximately 0.4. All of the objectives are anti-reflection-coated with an overall 50% transmission for both 780nm and 1064nm light. In addition, by splitting paths and relaying images through a pair of rear lenses, we are able to simultaneously image atoms at a larger field of view (FOV) with low magnification of 2.5, and a smaller FOV with higher magnification of 24. Both fluorescence imaging and absorption imaging can be used in the current setup.

2.1.3 Characterizing Disordered Optical Potentials and Their Non-Reciprocal Modulation

Since disorder is important for stabilizing quantized transport in the Hall effects, I will discuss this portion of the apparatus in considerably more detail than the more standard laser-cooling techniques described in the last section. As was described in the previous subsection, the final destination for the laser-cooling processes with atoms is the three-dimensional disordered potential created by a high-power (40W) 1064nm Yb fiber-laser coupled to the vacuum chamber through multimode optical fibers. One multimode fiber contains $O(10^5)$ waveguide modes, and by changing the coupling conditions, we can easily adjust the amplitudes and phases of the modes carried in the fiber. The interference of these many modes creates a complicated three-dimensional intensity and polarization profile inside the vacuum apparatus, which can be modified by deliberately mechanically perturbing the fibers. In this section, I will describe the incorporation of this disordered light effect into the interior of a Sagnac interferometer forming the input interferometer of the optical Laughlin pump geometry.

2.1.3.1 Optical Table Layout for the Operation and Calibration of the Laughlin Pump Input Interferometer

Figure 2.4 shows the diagram of the high-power 1064nm laser table. A highly-stable single mode Nd:YAG laser (Lightwave Electronics NPRO 126) outputs 80mW optical power at 1064nm, seeding a 40W Yb-fiber amplifier (Nufern NuAmp SUB-1174-22). This output beam is split in two paths by a single-pass AOM - the first order of the AOM diffraction seeds the speckle-potential path, which is later coupled
into multimode fibers and creates the three-dimensional disordered potential in the vacuum chamber.

Disorder in optical systems itself has a rich history, and a portion of it is closely tied to the method of excitation we will use in the Laughlin-pump measurement scheme. The multimode optical fibers used to deliver light to the atomic vapor in this experiment, through propagation in a medium with multiple randomly positioned scatters form a type of disordered medium. Since it is precisely this medium in which we will later introduce a Faraday effect using a strong magnetic field, it is important to have a sense for the coherence properties of the light under this effect. An important tool to diagnose this effect involves the topic of coherent back scattering (CBS) [231], which recognizes the fact that, no matter how complicated a scattering pathway, a time-reversed copy of it always exists, and by optical reciprocity shares the same path-length. Due to this fact, the interference of modes always exhibits one point in the interference pattern which exhibits constructive interference, provided optical reciprocity is not broken. By monitoring intensity precisely at this spot, the size of a non-reciprocal Faraday effect can be inferred.

A portion of the laser table is designed for detecting the Faraday effect induced by modulating a solenoid around the multimode fiber in this path. Along the marker beam path, a laser beam is coupled into a single mode fiber. The fiber coupler at the output side is pointing at a mirror as a retro-reflector, which bounces the beam back to the fiber coupler, creating a reciprocal path. A small portion of the marker beam will be coupled into the CBS path, creating a marker for the alignment of CBS detection. Neither the CBS detection path or marker beam path is intended to be used in the actual experiment of quantum simulation of Hall-like transport measurements. The purpose of the CBS detection path is for calibrating relatively small Faraday effects by monitoring the coherent backscattered beam, which will be discussed later.

The speckle laser beam (after propagating past a triangular-shaped Sagnac interferometer used for another experiment) is split into two paths, which are ultimately coupled into two counter-propagating beam pairs at the optical cell, and named (QP/NQP) by their orientation around the UHV cell relative to magnetic trapping coils used in the MOT stages (the quadrupole 1064nm beam propagates along the direction of the magnetic field gradient present in MOT sequence, created
by the pair of quadrupole coils, and the non-quadrupole beam propagates along a perpendicular direction). Each path is also split into its two propagation directions at a non-polarizing beam splitter, and then these two branches are both sent to multimode fibers, producing disorder potential in the UHV cell. The multimode fibers (Thorlabs MHP910L02, NA 0.22) are large core step-index design, and contain $O(\sim 10^5)$ different modes.

After the multimode fibers, the 1064nm beams are combined with MOT beams on the QP axis, using dichromatic mirrors, which are reflective for 780nm and transmissive at 1064nm wavelengths. On both axes, the 1064nm beams are projected into the UHV cell through imaging system optics, including the microscope objectives described above. As is clear from the figure, both the quadrupole path and non-quadrupole path form Sagnac loops, since each path contains a loop and a pair of counter-propagating beams are in the loop. These loops/Sagnac interferometers are the input half of the interferometer pair used in the quEASE method of direct transport measurement for fractional quantum Hall physics. Details regarding the setup of input Sagnac interferometers will be discussed in next section.
Figure 2.4. High-power YAG laser table diagram. Laser beam is produced in a seed laser, amplified in a fiber amplifier, then form several paths for different functions. See detailed discussion in the main text.
2.1.3.2 Characterizing Static Disorder and Thermal Energy Scales

The non-reciprocal modulation properties of disordered potential can be characterized by CBS, but the static properties of its disorder are also important. There are many ways to characterize weakly disordered systems in ultracold atoms [232] and condensed matter systems [233,234], but the characterization of disorder which is not perturbative to a disordered system is somewhat more challenging. Based on recent successes, I will describe an alternative viewpoint based on the information carried in a static potential.

Previous experiments done by Jianshi Zhao have revealed that a generalized Raman sideband cooling mechanism can be created when atoms are loaded into this three-dimensional disordered potential and also dissipatively coupled to a weak optical pumping beam. The presence of this weak dissipative process can induce a phase transition between sub- and super-recoil cooling by generalized RSC. When the magnetic field is tuned to a critical orientation during the generalized RSC process, atoms after cooling show the decay of the number/density of atoms decay following power law instead of an exponential function, which normally happens in the number loss behavior. Such critical phenomenon is closely related to holographic principles and dynamic instability [209].

For our purposes, we need only review the method used here to characterize disorder. To do so, we can take a closer look at the multimode fiber used in the input interferometer, and the three-dimensional disordered potential formed by superposition of the quadrupole-axis and non-quadrupole-axis beams. Figure 2.5(a) gives a nice illustration of the 1064nm beam coupled into multimode fiber, formation of speckle beam profiles, and a representational cross section of it. The multimode fibers have a core of size 0.91mm diameter, which allows good coupling efficiency over a large region in the parameter space, i.e. a large area and a large solid angle for delivering the beam into the fiber. For a very short fiber, insufficient scattering occurs, and the launched mode amplitudes are preserved. When a single mode laser beam is coupled into a sufficiently long multimode fiber, a very large amount of scattering occurs, and the output of the fiber is a power-equilibrated mixture of all possible modes. If the fiber is intermediate in length, the actual distribution depends on how the laser beam is launched into the fiber, and we deliberately choose a fiber in this intermediate range.

By varying the fiber launch condition, we can change the mode distribution...
in the fiber. Since the information content depends on the mode structure, this would effectively change the scaling behavior, which results in a different power law decaying constant. Figure 2.5 (b) present different mode structures, with one equilibrium mode distribution, and two types of non-equilibrium distributions. The equilibrium distribution is achieved with optimized fiber coupling condition, as strong lens focusing the beam to the fiber core while the numeric aperture of the beam matching that of the fiber. If a collimated beam is injected into the fiber along fiber axis, the optical mode distribution favors modes with low radial wave vectors, showing as blank space around a circle at the aperture plane; if a focused beam is injected into the fiber with an inclination to the fiber axis, the optical mode distribution favors high radial wave vector modes, resulting in a hollow center at the aperture plane.
Figure 2.5. YAG speckle potential and mode content analysis. (a) A single mode beam propagates through a multimode fiber and an equilibrium distribution of all possible fiber modes are generated. Through four microscope objectives in the imaging setup, the disordered wavefronts are projected into the UHV cell, forming a far-off-resonant disordered potential for atoms to be cooled by a generalized Raman sideband cooling process, through coupling to optical pumping light. (b) The disordered wavefronts contains information, which can be varied through launching the multimode fiber differently. This different information content can be observed by imaging the light intensity profile in the aperture plane. Subfigure (a, c, e) are images of intensity distribution in the aperture planes and (b, d, f) are those in the imaging plane. Here red, blue, green correspond to distribution of equilibrium, low-K only, and high-K only optical modes, respectively. The plot g shows how the exponential of Shannon information $F_s$ in a region scales as the radius $l/2$ of such region. These exponents are believed to be associated with a phase transition observed in generalized Raman sideband cooling.
As a result of measuring these properties of the disorder potential, temperatures in the dissipative cooling process, and the scaling laws for power-law transport, one can surmise that the coherent extent of atoms’ deBroglie wavepacket extends over a volume which contains roughly 15-25 continuous degrees-of-freedom in the description of the potential. This is an "information-entropic" way of characterizing the combined thermal and disorder characteristics of the atomic system. It is somewhat more generic than the more specific models often employed in Hubbard- or Anderson-like models [235,236], and can be used for a more general type of comparison between systems.

2.2 Operation of the Input and Output Sagnac Interferometers

Both the far-detuned disorder-generating beams and the optical pumping beam for generalized RSC are set up as Sagnac interferometers. These two loops intersect at the location of the atomic vapor, creating an atom-light version of the Hall experiment. In this section, we will discuss the details of both Sagnac interferometers’ construction as well as some detail on operation of the actual experiment, exciting and interrogating modes in the light-atom system corresponding to a direct mimic of the transport measurements in electron systems.

2.2.1 Input Sagnac and Detection of the Optical Faraday Effect Using Coherent Back-Scattering

The far-detuned input interferometer path itself actually contains two Sagnac interferometers as both quadrupole and non-quadruple path form closed loops with laser beams propagating in both directions. This is necessary to create an isotropic disordered potential, with no preferred axis along which the disorder is characteristically of a different scale. As illustrated in figure 2.6, each Sagnac loop contains a portion of multimode fiber inside a solenoid. All four beams from both arms of both far-detuned speckle paths are combined into the imaging system, and projected to the location of UHV MOT through the microscopy objectives.

The solenoid was made by winding about 1000 turns of magnet wires along a 55cm long, 2.5cm diameter PVC pipe. It was driven by a high-power audio
amplifier (Pyle Audio PT8000CH, 1kW maximum output). Driving the solenoid resonantly at 820Hz using low-loss capacitors, and at the maximum output of the audio amplifier, we measured magnetic field of amplitude $0.03\, \text{T}$ is induced in the solenoid. The Verdet constant of the optical fiber glass is

$$V_{\text{eff}} = (0.142 \pm 0.004) \times 10^{-28} \nu^2 \text{rad/T} \cdot \text{m},$$

where $\nu$ is the optical frequency [237]. For a 1064nm beam, the value of the Verdet constant is 1.12. Since the input interferometer beams have linear polarization before being coupled into the multimode fibers, we can easily calculate the rotation angle of the polarization caused by the optical Faraday effect $\beta = 0.02\, \text{rad}$. For a single mode laser entering the Sagnac through a polarizing beam splitter, the Faraday effect causes an amplitude modulation of on polarization component as $\sin \beta \approx \beta = 0.02$.

It is difficult to directly observe the effect of this relatively small modulation in a complicated disorder (speckle) pattern. In order to verify Faraday effect, we used the coherent backscattering (CBS) light as the probe. As described briefly above, CBS is one example of a fundamental phenomenon which arises from universal principle of reciprocity. A constructive interference is established as quantum or classical waves propagates along time-reserved paths inside disordered media. Such coherent effects can be found in weak localization and metal-insulator transitions. [238]. Due to the nature of reciprocity, CBS is robust to any realization of disorder and can be suppressed when reciprocity is broken. [239,240]. Without the magnetic field induced in the solenoid causing optical Faraday effect on the fiber, there’s nothing break the reciprocity hence the CBS light maintains robust. Since both CBS light and the Sagnac interferometer are only sensitive to the break of reciprocity, the detection of CBS light is naturally a good probe of optical Faraday effect inside the YAG Sagnac.

The CBS detection path is shown on the top left side of figure 2.4. In principle the CBS should go all the way back to the OI. To make a practical measurement of the CBS, we took it for granted that no PBS has an absolute extinction, causing by small misalignment between half wave-plate and the property of the PBS crystal. In reality, there’s a small amount of light leak through a PBS which we can use as the signal for CBS detection. For the convenience of aligning optics, we added a marker beam path, which is an intentionally made time-reversed path, by using a
mirror to reflect a collimated beam from a fiber coupler back into this fiber. Due to optical reciprocity, the marker beam is naturally a coherent backscattering light.

The coherence of CBS indicates that only one single mode propagates along the time-reversed path as the incident beam. The major challenge in the CBS detection is the low intensity of light in the CBS mode, but once this light is found and separated from the rest, it experiences a macroscopic fractional modulation of its intensity. When setting up the optics, I first delivered most laser power into the marker beam path by turning off the speckle AOM. The CBS from the marker beam is bright enough to set up mirrors and CBS detection fiber couplers in the CBS path according to its alignment. Since the multimode fiber takes a single mode laser and distributes into $10^5$ modes, and this process at least happens twice in light propagation, this corresponds to many orders of magnitude attenuation in the power contained in one mode. Loss on all the optic components and extinction on the PBS adds approximately another two orders of suppression.

Even though the incident beam is at about 12W when injected into speckle AOM, the CBS from this disordered media which can be received by the CBS detection is less than 1pW. To improve the signal to noise ratio at this low light detection, we turned the $\lambda/2$ waveplate before the PBS slightly away from the orientation optimized for S-polarization to allow 100mW P-polarization to be transmitted. By adding a $\lambda/4$ waveplate and a mirror after the PBS in the transmission path, we can reflect this 100mW light back and turn it to S-polarization. Therefore, this beam can overlap with the CBS detection path, which effectively create a heterodyne measurement, providing an inherent gain of roughly a few by $10^5$.

The modulation of the magnetic field inside the solenoid induces optical Faraday effect on the multimode fiber, which affects the CBS beam as amplitude modulation (AM). A low frequency spectrum analyzer (Stanford Research, model: SR 785) is used to identify this signal of AM, as the solenoid is driven around 1kHz band. An immediate problem occurred was that the spectrum analyzer could pick up noise at exactly this frequency, coupled through air and ground connections. We added a chopper wheel in the far-detuned interferometer path chopping at 2.4kHz, which contributed another AM and generated sideband at the sum/difference of these two frequencies. The observation of these sidebands is a clear evidence that Faraday effect is induced on the YAG speckle beams, and is roughly in agreement with the 20 milliradian level calculated from the field and Verdet constant.
Figure 2.6. Diagram of YAG Sagnac Setup and Plot of Mode Content Measurement. The detailed YAG Sagnac diagram - By seeding a fiber amplifier with a 80mW YAG beam, we produced a 12W high-power 1064nm single mode laser beam. An Acousto-Optic Modulator upshifts the laser frequency by 85.6MHz. Then this beam is split into four beams by two non-polarizing beamsplitter. Each is then coupled into a multi-mode fiber, which contains up to 106 different modes. YAG laser beams are then launched into UHV cell, each of which is focused to the same position by a microscopic objective. Such setup creates a speckly dipole potential in three dimensions. Each fiber coupler around the UHV cell is also receiving the light from its opposite fiber launch, coupling it all the way back to the beam splitters, which forms two YAG Sagnac Interferometers. Both Sagnac Interferometers have one arm going through a solenoid, which is for creating optical Faraday effect.
2.2.2 Sensitive Photo-Detection in the Output Sagnac Interferometer

The design of the Laughlin pump experiment requires two Sagnac loops which intersect through atomic cloud. We discussed the input interferometer (which is actually formed by two far-detuned Sagnac loops) above. The output loop is formed by the beam responsible for optical pumping in generalized Raman sideband cooling of the atomic vapor. I will explain the details of this optical pumping Sagnac interferometer setup for the actual quantum Hall experiment, and then discuss a modified path including an extra optical fiber, used to simulating the Faraday effect when light propagates through atomic gas as a calibration step.

2.2.2.1 The Output Sagnac Path

Figure 2.7 shows the diagram of the optical pumping Sagnac interferometer. The optical pumping beam is delivered to the experiment table through a fiber from the switchyard, discussed in previous sections. A λ/2 waveplate and a polarizing beam splitting cube split the light power evenly in reflection and transmission from the cube. The λ/4 waveplate changes the linearly polarized beam into circularly polarized, for the pumping process in RSC. The slanted beams are showing that the beam path is not in a same plane, and both vertical and horizontal angles are included, due to the constraint in the space near the UHV cell. These two λ/4 waveplates are tuned so that the polarization of the beam is rotated by 90° after going around the entire loop, which ensure both arms of the Sagnac interferometer leave the loop from the fourth port of the PBS cube. There is a weak focusing lens L1 before the PBS cube, which puts a focal point at the middle of the Sagnac loop, compensating for the divergence of this Gaussian beam and, while ensuring that the beam comes around the loop has the same diameter as the incident beam, for the purpose of easier alignment of the Sagnac.

The two arms have identical optical path length when atoms are absent, owing to optical reciprocity. When atoms are present, beams propagating in both directions in the atomic vapor experience optical Faraday effect, which change the relative phase between these two arms. Atomic vapors are well known to have large Verdet constants [241], compared to other transparent materials, and it has been used, for example, in the innovation of new optical isolators for atomic micro-electro-
mechanical-systems (MEMS), which coincidentally adopted Rubidium-87 with light coupled to its D2 transition \[242\]. For the circularly polarized beam, the Faraday effect causes a small difference in the propagation speeds of left and right circularly polarized light. These two beams then experience a differential phase shift, which results in a difference of their optical path length. To translate the Sagnac phase into a detectable signal, we deliberately misaligned the OP Sagnac interferometer by a small amount, to create difference in the exiting angle for the two arms. The two outputs of the optical pumping interferometer leave the PBS with orthogonal polarization, and a polarizer is installed before a detection fiber, with 45° rotation of orientation from both beams. Components of each beam transmit through the polarizer, and then form an interference pattern. Because of the slight angular misalignment, this takes the form of a linear fringe pattern, and the Sagnac phase appears as a displacement of it.

By coupling the wavefront into an optical fiber which then delivers light to a photo-detector, we can monitor the intensity profile and use that as indication of the OP Sagnac interferometer phase. This is included in the Sagnac phase-detection setup, which, in the actual experiment, contains a more complex design of optics and electronics. It will be discussed further in next section.
Figure 2.7. Detailed optical pumping Sagnac 3D diagram (ultimate path). A 780nm laser is produced by ECDL sources. Laser beam goes through a switchyard for optical control and is then launched into an optical fiber as the source of the Optical Pumping Sagnac source beam. A half wave plate is right after the fiber launch for balancing the beam power into two arms of the Sagnac interferometer. A weak plano-convex lens focuses both arms at the middle of this Sagnac, compensating the divergence of Gaussian beams, in order to match the beam sizes everywhere in the loop. Two quarter wave plates are inserted in both arms after the polarizing beam splitter, to convert linearly polarized beam to circularly polarized beam. Then if the atoms are inducing Faraday effect, the result is a difference in the relative phase between two arms of the Sagnac Interferometer. The fourth port of the Sagnac Interferometer is connected to the detection path which is shown in next section.
2.2.2.2 Calibration using Faraday Phase Modulation in the Output Sagnac

Though the atomic vapor has large Verdet constant, and therefore can permit easy observation of optical Faraday effect [241], it is not a simple means to test components of the Laughlin pump, as the pulsed operation of cold atomic vapor source works at relatively low duty cycles. For this reason, a temporary step was created in which an optical fiber was inserted and used to induce Faraday effects directly in the output interferometer.

This allowed the output interferometer phase detection block to be tested straightforwardly. For testing purposes, I added two non-polarizing beam splitters in the Sagnac loop to reflect beams to create alternative paths, shown in figure 2.8. A beam block was inserted between two NPBS to prevent beam from propagating in the original Sagnac path. A fiber receives beams from both directions, forming a closed loop and completing the Sagnac structure. The idea of this fiber is to induce Faraday effects in the fiber glass using a magnetic field generated by a solenoid, to mimic a similar effect between atoms and the OP beams. This fiber runs through the solenoid once, is wrapped around, and runs through the solenoid again. This is repeated nine times so that Faraday effect on the fiber can be increased by a factor of ten.

The solenoid used in this setup is made in the same fashion as the one used in the input interferometer / speckle path. In order to match the output impedance for the audio amplifier, a low loss capacitor was added in series with the solenoid. The solenoid, which can be treated as an inductor, and the capacitor together form an LC-resonator. We chose the capacitance so that the resonance frequency is around 820Hz, which is far enough from 60Hz and its harmonics to avoid pick-up noise. When driving the solenoid at that frequency with the maximum output power of the audio amplifier, a magnetic field with amplitude of 5mT is generated inside the solenoid, measured by a three-winding pick-up coil and an oscilloscope. For a linear polarized beam, this magnetic field amplitude causes a Faraday rotation of 0.1rad.
Figure 2.8. Detailed optical pumping Sagnac (fiber path). Here is an alternative way of setting up the optical pumping Sagnac Interferometer, in which the Faraday effect is induced in an optical fiber instead of atomic cloud. Both arms of the interferometer are deflected by NPBS, and then are launched in to fiber couplers. The path for the ultimate design is blocked at middle point. This optical fiber contains beams propagating in both directions. Optical fiber runs through a solenoid with thousands of windings of coil, which can generate a magnetic field up to 0.005T, corresponding to a phase angle about 0.1 rad, calculated based on a typical Verdet constant of optic fiber material.
2.3 Output Sagnac Phase Detection

As the ultimate signal of the direct transport measurement in the Laughlin pump, the interferometer phase of the OP Sagnac needs to be detected. The output interferometer is operated with a total optical flux (power) which is limited to one-to one-hundred-picowatt levels by the requirements of the cooling process in the atomic vapor, and thus is somewhat demanding to detect. The detection setup I devised includes a heterodyne amplifying scheme, a sensitive radiofrequency (RF) photodetector, an RF vector-analyzer, a deflecting AOM scanner, and a phase lock circuit. All of these are combined together to form a measurement of the output Sagnac phase, and are ultimately built into the driven Laughlin oscillator loop, using the active filter and audio amplifier drive of the input interferometer’s Faraday-effect coil. Each part of the setup will be discussed in the following subsections. A diagram of the complete setup is shown in figure 2.9.

2.3.1 Heterodyne Detection and Vector Analyzer

The Sagnac phase measurement is made by monitoring the fringe pattern through a detection fiber, and the optical power received indicates the interferometer phase. Since the optical pumping beam is at sub-nano watt level during the experiment, we constructed a heterodyne scheme to enhance the signal. Heterodyne detection simply mixes an input signal (here, a laser beam) with a well-defined, and stronger, 'local oscillator' signal at a different frequency, and the output signal is at either the sum or the difference of these two frequencies [153]. Since this signal occurs with an amplitude at the geometric mean of the two signals, the power contained in it is also larger, defining the 'heterodyne gain.'

The left side of the diagram shows optics on the laser table switchyard that are relevant to this measurement. One incident beam is split into the optical pumping path and MOT repumper path. The zero-th order diffraction of the MOT repumper path is taken as the local oscillator for the heterodyne, which is also referred to as the 'reference beam.' The optical pumping beam is produced in a double pass AOM [213] at 79.9MHz and is delivered from the switchyard to the output Sagnac through an optical fiber. A copy of the optical pumping AOM RF-signal is sent to a frequency doubler, and then to the RF port of a mixer (Mini circuit ZMFM-3). Compared to the reference beam, the optical pumping beam is
upshifted in frequency by 159.8MHz.

The output beam from the optical pumping Sagnac first goes through a single pass AOM, where its optical frequency is downshifted by 81.3MHz. A copy of the Sagnac AOM RF signal is amplified and then send to the LO-port of the mixer. The reference beam sent to the detection fiber contains $\sim 100\,\mu W$ and the Sagnac output beam is at sub-nanowatt levels. The mixture of these two laser beams creates a RF-frequency beatnote at $2 \times 79.9 - 81.3 = 78.5$MHz, and an RF-signal at this same frequency is also created through mixing two RF electronic signals split from AOMs’ driving electronics. The optical signal is received on a high-speed, small-area silicon photo-detector (Thorlabs 671BFC-0114). We use a series of RF amplifiers including a low-noise-figure Trontech W110F (55dB, NF $\sim 0.7$dB) and two Mini-circuits ZFL-LN(20dB) to amplify this signal up into an operational range for the next stages. In addition, some bandpass filters are used to clean up spurious radiatively-coupled noise signals so that the signal-to-spurious suppression is at least 30dB. The 78.5MHz signal from optical path, when mixed with the 78.5MHz signal created through mixing signals from AOM drivers, results in a DC signal, corresponding to the optical power received by the detection fiber.

The heterodyne method of detection is phase-sensitive, but we only need or want a detection of optical power. This introduces a small complication, since the phase-sensitivity also introduces sensitivity to acoustic noise in the reference arm path. Since there is no well-defined relative phase between the optical beat-note and the RF path, we introduced a vector analyzer with an IQ-modulator to eliminate sensitivity to the phase noise. For this an IQ-modulator (mini-circuit ZFMIQ-91M) is used to extract both quadratures of the phase-sensitive optical beatnote (an IQ modulator is similar to a RF mixer, but instead of creating one signal carries the sum and difference from signals inputted to LO and RF ports, it creates two quadratures.) We then use a vector analyzer stage, which squares both I and Q signals then sum these two arms to give the total magnitude, which is no longer sensitive to the optical path-lengths.

It is worth modeling the steps above a little more formally. Since the detection fiber receives a mixture of reference and OP Sagnac output beams, and the photo diode converts this optical power into electrical current, we can reason that the portion of the signal produced by the photo-detector generating the heterodyne frequency is
\[ V_{PD} \propto I_{PD} \propto \eta \vec{E}_{\text{ref}} \cos(\omega_{\text{ref}} t) \cdot \vec{E}_{\text{Sagnac}} \cos(\omega_{\text{OP}_{\text{Sagnac}}} t + \phi), \quad (2.3) \]

where \( \eta \) is the quantum efficiency of the photodiode. \( \vec{E}_{\text{ref}} \) and \( \vec{E}_{\text{Sagnac}} \) are the electric fields of the reference beam and the optical pumping beam incident onto the detection fiber. \( \omega_{\text{ref}} \) and \( \omega_{\text{OP}_{\text{Sagnac}}} \) are the optical frequencies of the laser beams and \( \phi \) is the relative phase between them. The photodiode responses to RF frequency so that the signal received can be expressed as

\[ V_{\text{RF}} = A_{\text{RF}} \cos(\omega_{\text{RF}} t + \phi), \quad (2.4) \]

where \( \omega_{\text{RF}} = \omega_{\text{OP}_{\text{Sagnac}}} - \omega_{\text{ref}} \). Here the subscript 'RF' denotes that this signal is delivered into the RF port of the IQ-modulator. According to equation 2.3, we have the relation:

\[ A_{\text{RF}} \propto |\vec{E}_{\text{Sagnac}}| = \sqrt{I_{\text{Sagnac}}}, \quad (2.5) \]

where \( \sqrt{I_{\text{Sagnac}}} \) is the intensity of the OP Sagnac output beam.

On the upper left corner of the diagram, a copy of RF signal from the optical pumping AOM driver is doubled and then mixed with a RF signal copied from the Sagnac AOM driver, creating a beatnote:

\[ V_{\text{beat}} = A_{\text{OP}} \cos(2 \Omega_{\text{OP}} t + \Phi_{\text{OP}}) \times A_{\text{Sagnac}} \cos(\Omega_{\text{Sagnac}} t + \Phi_{\text{Sagnac}}), \quad (2.6) \]

where the \( A_{\text{OP}} \) and \( A_{\text{Sagnac}} \) are amplitudes of those RF electronic signals, \( \Omega_{\text{OP}} \) and \( \Omega_{\text{Sagnac}} \) are radio frequencies, and \( \Phi_{\text{OP}} \) and \( \Phi_{\text{Sagnac}} \) are phases. \( V_{\text{beat}} \) is then low-pass filtered, and then delivered to the LO port of the IQ-modulator, in the following form:

\[ V_{\text{LO}} = A_{\text{LO}} \cos[(2 \Omega_{\text{OP}} - \Omega_{\text{Sagnac}}) t + \Phi_{\text{OP}} - \Phi_{\text{Sagnac}}] \quad (2.7) \]

Since the optical pumping Sagnac output beam is \( 2 \Omega_{\text{OP}} \) upshifted in frequency from the reference beam, and then \( \Omega_{\text{Sagnac}} \) downshifted in frequency, we can easily obtain \( \omega_{\text{RF}} = \omega_{\text{OP}_{\text{Sagnac}}} - \omega_{\text{ref}} = 2 \Omega_{\text{OP}} - \Omega_{\text{Sagnac}} = 78.5\text{MHz} \). Therefore, the output signals from the I-port and Q-port are at DC, described as

\[ V_{\text{I}} = A_{\text{I}} \cos(\Phi_{\text{OP}} - \Phi_{\text{Sagnac}} - \phi) \quad (2.8) \]
and

\[ V_Q = A_Q \sin(\Phi_{OP} - \Phi_{Sagnac} - \phi) \]  \quad (2.9)

Here \( A_I \) and \( A_Q \) are proportional to \( A_{RF} \) hence is proportional to \( \sqrt{I_{Sagnac}} \). The RF phase and optical phase have no well-defined relation, therefore \( \Phi_{OP} - \Phi_{Sagnac} - \phi \) can drift over time, which causes phase noise to the optical power measurement. It is clear that if only the heterodyne amplified signal is recorded, i.e. a signal from either I-port or Q-port, the optical power measurement is susceptible to this phase noise.

This noise is suppressed by combining \( V_I \) and \( V_Q \) in a vector analyzer, which simply squares each signal by using an analog multiplier (AD633), then adds the squared signals together. Then in principle, the phase noise is canceled through the trigonometry identity. \( A_I \) and \( A_Q \) should be balanced and are both proportional to \( \sqrt{I_{Sagnac}} \), now we substitute them with \( A_{Sagnac} \) for simplicity. Then the output of the vector analyzer

\[ V_{out} = 2A_{Sagnac}^2 \propto I_{Sagnac} \]  \quad (2.10)

Therefore, the vector analyzer output represents the Sagnac optical power reliably. The multiplier stages only cause less than 1% error in the band we operate the apparatus at.
Figure 2.9. Optical Sagnac phase detection diagram - optical pumping beam is created from a reference beam through a double pass AOM, labeled as OP AOM, creating a 160MHz frequency shift. The Sagnac Interferometer output beam emitted from fourth port of the PBS is then up-shifted 81.3MHz by another AOM, labeled as Sagnac AOM. This nW beam is then mixed with the reference beam which contains 100µW optical power, result in a 78.7MHz beat signal, whose power is proportional to the Sagnac output beam power. Two RF signals from doubling of Optical pumping AOM drive and a copy of Sagnac AOM drive are mixed together to recreate this 78.5MHz beat. A Heterodyne scheme and a vector analyzer are constructed to combine these two paths to compress the phase noise from two optical paths. To increase the sensitivity of measuring the Sagnac phase, this Sagnac Interferometer is slightly misaligned and interference fringes are launched into the detection fiber launch, then the output of the vector analyzer is fed to the Sagnac AOM driver to lock the fiber launch to a dark fringe. The control signal of this phase lock is fed to the Active Filter to close this driven oscillator loop.
2.3.2 Sagnac Fringe and Phase Lock

Since we are ultimately after the Sagnac phase, not just the optical power, the apparatus consists of a few more steps and parts. The Sagnac AOM mentioned in previous section is a part of a feedback loop which by locks a dark fringe to the detection fiber by varying the AOM frequency. The reason that this locking scheme is needed is that those amplification stages create multiplicative noise that scales up with the optical power of the Sagnac fringes, and by locking the phase detector to a dark fringe, this set up is most sensitive to phase variation caused by break of reciprocity inside the OP Sagnac Interferometer. In this subsection we will discuss this fringe lock.

We created fringe patterns by slightly misaligned the Sagnac interferometer, and an image of the fringe pattern is shown in figure 2.10(a). The structure of the fringe pattern can be changed by adjusting how the interferometer is misaligned. The best contrast ratio between the bright fringe and the dark fringe is 12 : 1. In order to lock to a dark fringe, a dither signal at 101.3kHz is sent to the Sagnac AOM driver FM port from a function generator. We tune the amplitude of the dither so that the AOM frequency is modulated by 100kHz at this dither frequency. The purpose of the dither signal is to extract the slope, by mixing the dithered optical signal (vector analyzer output) with a copy of the dither signal.

Assume $I(\omega_{Sagnac})$ is the function of the detected optical power in terms of the Sagnac AOM frequency and $\omega_{Sagnac}$ is around 80MHz. A small dither adds a modulation to this optical power and now it is $I(\omega_{Sagnac} + \delta\omega\cos(2\pi f_D t))$, where $\delta\omega$ is the dither amplitude and $f_D$ is the dither frequency. For small $\delta\omega$, a Taylor expansion to the first order is sufficient:

$$I'(\omega_{Sagnac} + \delta\omega\cos(2\pi f_D t)) \approx I'(\omega_{Sagnac}) + \frac{dI'}{d\omega_{Sagnac}}\delta\omega\cos(2\pi f_D t))$$  \hspace{1cm} (2.11)

This dither-induced signal on the heterodyne detector output is then amplified, and mixed with an amplified copy of the dither at frequency $f_D$ in an analog mixer circuit, based on the Mini-circuit SRA-6 frequency mixer. The mixer output is low-pass filtered with a knee at 500Hz, which keeps only the DC signal proportional to the first order derivative $dI'/d\omega_{Sagnac}$.

Figure 2.10(b) shows an oscilloscope trace of the vector analyzer output (channel
1. An 82Hz sawtooth sweep of the deflecting AOM frequency is summed with the dither signal at the FM port of the Sagnac AOM driver. This sweep (channel 2) modulates the AOM frequency over a range ±5MHz, which results in a larger variation of the optical power as the Sagnac fringe pattern is scanned across the detection fiber aperture. The vector analyzer clearly shows a 'bright-dark-bright' structure of the intensity profile (the inversion of the signal is caused by an inverting operational amplifier stage in the vector analyzer circuit.) Most important is the region close to the dark fringe, where the analog mixer output correctly shows the slope of the intensity profile curve passing through zero. In operation, this 'error signal' is fed back to the AOM deflector, locking the signal to the dark fringe position. The correction signal is then taken as an output, reflecting the angle of deflection necessary to return the dark fringe to a fixed position.

The Sagnac phase lock module which accomplishes this in figure 2.9 is simply an integrator which gains up the vector signal and feeds it back to the FM port of the Sagnac AOM. When the phase lock circuit is engaged, the sweep is simultaneously turned off. Multiplicative intensity noise on the Sagnac beam, heterodyne reference beam or introduced in the photo-detector can be rejected by the lock circuit, since the AOM frequency necessary to lock to the dark fringe center is unchanged. This feedback loop has a 22kHz unity gain point, and can adequately follow modulation induced by the atomic vapor at the chosen center operation frequency for the Laughlin oscillator.
Figure 2.10. CCD images of optical pumping Sagnac fringes and oscilloscope trace of vector analyzer output as fringes are periodically scanned. Error signal is produced by dithering the AOM driver at 101.3kHz. Dither signal is from a function generator. A synchronized copy of the dither is mixed with the optical Sagnac signal to create a first order derivative.
2.3.3 The Pole-Wound Driven Oscillator and Active Filter

The output Sagnac interferometer phase signal must be fed back to the setup in order to close a loop for oscillation to occur in the Laughlin pump. The principle of building up an oscillation for probing and detecting a coherent many-body state have been discussed in the first chapter. In all different testing phases of this experiment, a two-frequency synthesizer (generating the pole-winding signals) and an active filter are the essential part for the study with the driven oscillator picture.

The basic concept of the two-frequency synthesizer is demonstrated early in this thesis and the technical detail is discussed in Appendix A. This section will start with the design of the active filter and then describe the actual measurement in this driven oscillator picture.

2.3.3.1 The Active Reference Filter

Figure 2.11 shows the schematic of the active reference filter. It is a new design created specifically for interrogation of the Laughlin pump, to create a precisely tuned and variable center bandpass filter. The basic idea of this active filter is to mix down an input signal with a center control signal, filter out high frequencies and then mix with the center control frequency again to reproduce the original input signal, filtered by a passband centered at the control frequency. The center control signal and its quadrature copy are split into two arms, and re-weighted in order to suppress the introduction of feed-through noise from the multipliers.

The FPGA inside the two-frequency synthesizer is programmed to produce two complex waveforms with complex frequencies move around each other in the complex plane. One signal is sent into the active filter and defines its center of the passband, the other signal is summed-in in a later stage in the oscillator loop, for the pole-winding driven oscillator scheme described in the previous chapter.

We can take a closer look at its operation. First, we assume the center control signal is split into its quadratures as

\[ V_{c1} = V_0 \cos(\omega_c t), \quad V_{c2} = V_0 \sin(\omega_c t) \]  \hspace{1cm} (2.12)

and the input signal is \( V_s \cos(\omega_s t + \phi_s) \). At the first pair of multipliers, two signals are generated,
\[ V_1 \propto \frac{1}{2} V_0 V_s \{ \cos[(\omega_c + \omega_s)t + \phi_s] + \cos[(\omega_c - \omega_s)t - \phi_s] \} \]  \hspace{1cm} (2.13) \\
and 
\[ V_2 \propto \frac{1}{2} V_0 V_s \{ \sin[(\omega_c + \omega_s)t + \phi_s] + \sin[(\omega_c - \omega_s)t - \phi_s] \}. \]  \hspace{1cm} (2.14)

Setting the low pass filter knee somewhere between 10Hz and 100Hz, the component with the sum frequency can be attenuated, for example, by a factor of roughly 82 to 8.2 when center frequency is at 820Hz. For simplicity, we only carry the low frequency into next step.

These two arms are crossed and mixed with the other quadratures, which yields

\[ V'_1 \propto G(|\omega_c - \omega_s|)^{1/2} V_0 V_s \cos[(\omega_c - \omega_s)t - \phi_s] \times V_0 \sin(\omega_c t) \]  \hspace{1cm} (2.15)\]

and

\[ V'_2 \propto G(|\omega_c - \omega_s|)^{1/2} V_0 V_s \sin[(\omega_c - \omega_s)t - \phi_s] \times V_0 \cos(\omega_c t) \]  \hspace{1cm} (2.16)

Here, \( G(|\omega_c - \omega_s|) \) is the transfer function of the low pass filter. As these two signals are input into the inverting and non-inverting inputs of an opamp (LM741) configured as a differential amplifier, their difference is formed at its output as

\[ V_{out} = V'_1 - V'_2 \propto \frac{1}{2} G(|\omega_c - \omega_s|) V_0^2 V_s \sin(-\omega_s t - \phi_s) \]  \hspace{1cm} (2.17)

Therefore, the output signal experiences a 90° phase shift and the overall transfer function is symmetric about \( \omega_c \) and it creates a single pole roll-off on both sides of \( \omega_c \). The advantage of this active filter is that the pass-band is controlled by the center control signal which defines the center frequency. The bandwidth can be easily adjusted by swapping components in the RC filter between the mixers.

It is also obvious that the amplitude of the output signal depends on the square of the control signal amplitude, which means the control signal can be simultaneously used to control the open loop gain of the active filter. In the actual experiment, \( \omega_c \) is the real part of a complex frequency generated in the two-frequency synthesizer.
and the pass-bandwidth is set at $\sim 100$Hz.

Figure 2.11. Active filter schematic. A driven active filter is designed to create a bandpass filter with a resonance frequency controlled by an external signal and bandwidth fixed by two RC filters. A control signal defines the center of the passband and the transfer function of this filter is determined by the amplitude of the external control signal. Both control signal and detected signal are split into two quadratures. One signal is selected from each path and then they are mixed in one out of four analog multipliers. One can show that the sum of these two arms is not affected by the relative phase between the control signal and detected signal. The schematic also shows an injection path, which is used for the driven oscillator scheme described in chapter 1. In addition, a digitizer is included for recording the detected signal.
In principle, instigating oscillation at a measured threshold gain is a viable way of interrogating the response function that we are interested in - since the response function of the many-body system forms the cross-interferometer portion of the open loop gain, measuring the point at which the full open-loop gain becomes unity with known gain in the rest of the loop measures this response function. However, due to the multiplicative noise contained in the apparatus, particularly that introduced by heterodyne amplification noise, we encountered difficulties in persistently exciting the system to oscillation, starting from phase two type experiments and beyond.

Inspired by the picture of Berry’s geometric phase, in which a system picks up a phase when one parameter undergoes an adiabatic variation around a closed loop, we invented the "pole winding" driven oscillator picture described in the previous chapter, with an injection signal adiabatically moving around the natural pole of the closed-loop system as a probe. This relative motion is actually done through moving both the center control for the active filter and the injection signal along the same orbit, with a constant $\pi$ phase delay. Relatively, as the natural pole approximately follows the motion of the center control signal, the injection signal orbits around the natural pole along a contour with radius double compared to the original orbit that is programmed.

The phase slips between the system’s closed-loop response and the injection signal is an indicator of both the structure of the natural pole contained inside this loop, and the properties of adiabaticity governing its response. The latter effect governs behavior at the edge of the Arnold tongue, and leads to some number of phase-slips over a single period of the pole-winding orbit. An integer phase slip is anticipated in a dominantly linear system, similar to the picture of an integer Hall effect, and fractionalized phase slips are expected when the system contains higher-order non-linear behavior. In the actual experiment, the orbit frequency under which the injection signal and center control signal move around each other is scanned, and eventually a plot of an accumulated relative phase between system response and the injection is plotted against the orbit frequency.

2.3.3.2 Phased Development of Laughlin’s Pump Oscillator

As discussed at length in the previous chapter, the goal is to infer the non-linearities present in the cross-interferometer response function. By building this cross-interferometer gain stage into an oscillator, we expect oscillation builds up when
a coherent many-body state is established in the light-atom system. For this to occur, the output of the active reference filter needs to be fed back into the system to close the loop of the oscillator. Previously, I described a staged-approach to this problem, introducing a series of experiments intended to check various aspects of this scheme one at a time.

In phase 1, an apparatus is produced with pure electronics, a signal mimicking the Sagnac phase is taken from a small pickup coil around a solenoid. This signal bypasses all the optical path discussed above and is directly inputted into the active filter. Through an impedance buffer and an audio amplifier, this signal is fed back as the drive of the solenoid, changing magnetic field in the pick-up coil.

In phase 2, an apparatus is produced with only the optical pumping Sagnac and with an optical fiber simulating the Faraday response of the atoms, the Sagnac phase lock tight optical phase signal to a dark fringe and the control signal of this lock circuit is duplicated and delivered to the audio amplifier, which in this case drives the solenoid in optical Sagnac fiber path, varying the Faraday effect in this fiber.

In phase 3, an apparatus is produced with the optical pumping Sagnac and with atoms in the UHV cell, the phase lock control signal, through the active filter, impedance buffer and audio amplifier, and is delivered to a pair of Helmholtz coils next to UHV cell, which directly modulates the external magnetic field, and hence modulating the optical pumping Sagnac phase through an optical Faraday effect in the atomic vapor.

In the ultimate phase apparatus, two interferometers, both input and output, intersect at where the atomic cloud is, and the phase lock (for optical pumping Sagnac phase) control signal is eventually used to drive a solenoid that varies the optical Faraday effect in the multimode fibers that changes the input Sagnac interferometer phase. The modulation in the input Sagnac phase affects the effective magnetic field generated from vector light shift from far-off resonance input Sagnac beam coupled to atomic spin states, which results in the modulation of optical Faraday effect as optical pumping beam propagates through atoms and experiences a phase shift.
### 2.3.3.3 Phase Slip Measurement and Data Recording

In each phase of the experiment, the data we wish to record is an accumulated relative phase between the injection signal and the system response as the injection signal orbits around the pole. The last portion of the experiment hardware performs this measurement and stores the results.

To accomplish this, first we digitize both signals - the injection signal is generated in the FPGA-based two-frequency synthesizer, which naturally generates a digital waveform for the injection, whose top bit forms a binary representation of the waveform. The system response is digitized by first passing it through a fast comparator (Analog Devices LT1016), as a "digitizer" in figure 2.11. Afterward, these two digital signals are processed in the FPGA - during every cycle of the injection signal, the frequency difference is measured using time-difference counter comparing the delay between their transition edges. This differential frequency is accumulated until one orbit cycle is finished.

There are four relevant frequencies used in the FPGA-based measurement:

- a 'fast clock' frequency at 22.7MHz
- the natural oscillation frequency of the Laughlin-Pump is chosen around 820Hz, and the center control and injection are modulated around this center frequency
- the orbit frequency for pole-winding is set near 10Hz and varied
- the orbit frequency is scanned at a modulation rate near 0.01Hz

These three time-scales sufficiently different in scale as to decouple from one another, leading to relatively small fractional errors stemming from commensurability of frequencies. We represent the two waveforms under comparison as sinusoids $V_1 = \sin(2\pi f_1 t)$ and $V_2 = \sin(2\pi f_2 t - \phi)$, where $f_1$ and $f_2$ are frequencies of these signals and $\phi$ is the instantaneous relative phase. $f_1$ and $f_2$ can be considered as approximately constant within a few pulses, and we have

$$\frac{|f_1 - f_2|}{|f_c|} \ll 1,$$

where $f_c$ is a frequency that signals $V_1$ and $V_2$ are modulated around.
The accumulated relative phase in one pole-winding orbit cycle is an integration of the difference frequency. To calculate this, we will use an (integrating) counter synchronized to the transitions in the two signals. Regarding $V_1$ as the injection signal which we use as reference to trigger one counter for differential frequency, on the rising edge of $V_1$ the counter resets to 0 and starts counting up until it sees an edge on $V_2$. The counter is paused until the falling edge of $V_1$ triggers it again and it starts to count down. When it sees another edge from $V_2$, it stops counting. The value of this counter is then added to an accumulator which resets when the complex frequency has gone around one full cycle along the orbit. The counter value at each point in this process can be summarized as

$$
t = 0 : \text{counter} = 0
$$

$$
t_1 = \frac{\phi}{f_2} : \text{counter} = f_t \times t_1 = f_t \cdot \frac{\phi}{2\pi f_2}
$$

$$
t_2 = \frac{1}{2f_1} : \text{counter} = f_t \cdot \frac{\phi}{2\pi f_2}
$$

$$
t_3 = \frac{\pi + \phi}{2\pi f_2} : \text{counter} = f_t \left[ \frac{\phi}{2\pi f_2} - (t_3 - t_2) \right] = f_t \left( \frac{1}{2f_1} - \frac{1}{2f_2} \right)
$$

Here, $f_t$ is the fast clock frequency. Under the approximation in equation 2.18, the counter value latched into the accumulator is $\frac{f_t}{f_c}(f_2 - f_1)$, which is proportional to the differential frequency within that cycle of injection signal. It is worth noticing that this counter is able to count across zero which means it can handle either sign of the difference frequency. This counting scheme works properly with the only exceptions when (1) edges are closely lined up between the two signals, and (2) only one edge or more than two edges of $V_2$ appear during one cycle of $V_1$. Both cases can happen at different portions of the waveform. The fix is rather simple - to use $V_1$ and its quadrature as two references and set up two counters identical to the description above. Most of time, these two counters provide the same result, except when cases (1) or (2) happen. In those cases, one counter will show a value beyond the half-width of the period of $V_1$, which can be rejected, and the other counter is guaranteed to represent the frequency difference correctly.
Figure 2.12. Differential frequency measurement. Two frequencies are presented in the chart and a fast counter which resets at edges of $V_1$ is used to obtain the instantaneous differential frequency between $V_1$ and $V_2$ and the detail of the counting scheme is explained in the main text.
Chapter 3

Pole-wound Oscillator Dynamics and the Optical Laughlin Pump - Some First Experiments and Results

In this chapter I will report laboratory progress on the Laughlin Oscillator using the concepts developed in the first chapters of this thesis and the apparatus design and build described in the preceding chapter. Data will be presented following the chronological order, corresponding to the development of the experiment from simple electronic driven oscillator (phase 1 in the language of previous chapters) to optical Laughlin pump with cross interferometers (phase 3). The results from phase 1a verify the oscillator dynamics measurements expected through the phase slip model described in previous chapters, showing integer phase slipping in a dominantly linear oscillator. In phase 1b experiments, controlled electronic non-linearity was introduced, and there is weak positive indication for the development of non-integral weight to the phase-slip distribution, though truly resolved fractional behavior is not definitively resolved. Phase 2 measurements are also reported on the behavior of an optical version of the oscillator dynamics, and give some positive indication for integer slip dynamics, which is expected for the dominantly linear system like the phase 2 model would predict. For phases 3 and 4, both introducing atomic vapor into the closed-loop oscillator dynamics, I will present preliminary data taken with both the magnetic-field induced loop-gain (which shortcuts the cross-interferometer gain) and with cross-loop dynamics instigated by driving the Faraday-effect through the cross-interferometer gain mechanism realizing the first attempted cold-atom Laughlin Pump Oscillator experiment. As yet, this data is inconclusive due to the comparatively large noise introduced by the low-light level optical detection, combined with the very low integration time that could be accumulated at the time of this thesis submission. To conclude, I will outline
prospects for future measurements of this type, and suggest directions for refinement beyond simply extending integration time.

3.1 Phase 1 Experiments - Simplest Electronic Driven Oscillators

The first phase of this experiment interrogated a pure electronic driven oscillator, and was separated into two sub-phases - 1a, in which a largely linear (aside from saturation effects) oscillator was built and probed using the pole-winding interrogation technique. In phase 1b, controlled non-linearity is added and varied to identify fractionalized phase-slipping phenomena.

The data taken for the pure classical electronic driven oscillator is already reasonably interesting. We will first see that the picture of two poles orbiting around each other, and leading to integral phase slips at the boundary of an Arnold tongue is well established in this driven oscillator. I will present phase relation between the injection signal and the system response, which shows quantized plateau structures, and give some experimental overview of how and when phase-slipping occurs in the driven system.

3.1.1 Spectral Analysis of Pole-Winding Synthesizer Signals, and Complex Oscillator Orbits

Prior to running phase-slipping sequences in oscillators, an important first check is to verify that the pole-winding synthesizer generates the appropriate signals. This can be done by accumulating samples of its output, digitizing, and finding its spectral properties in a sliding window.

The active reference filter has its center frequency defined by a signal being both amplitude and frequency modulated, and can be expressed as

\[ s(t) = \text{Re}(V_0 e^{i\tilde{\omega}t}), \quad \tilde{\omega} = \omega_0 + \omega_r \sin(\Omega t + \Phi) + i \omega_i \cos(\Omega t + \Phi), \quad (3.1) \]

where \( V_0 \) is a real amplitude of voltage, \( t \) is time, and \( \tilde{\omega} \) is the complex frequency, moving along an ellipse on the complex plane centered at \( (\omega_0, 0) \) and having major and minor axes \( \omega_r \) and \( \omega_i \) oriented along the real and imaginary axes. The complex
frequency moves along this ellipse periodically with an "orbit" frequency $\Omega$. Here, $\Phi$ is the initial phase of the complex frequency, and the filter control signal and injection have a $\pi$ relative phase. In the experiment, $\omega_0$ is carefully chosen to be $2\pi \times 820$ Hz, the major axis of the complex ellipse $\omega_r$ is $2\pi \times 3$ Hz and the minor axis $\omega_i$ is $2\pi \times 0.5$ Hz, and the orbit frequency $\Omega$ is scanned from $0.5$ Hz to $4.5$ Hz.

This waveform can also be expressed as

$$s(t) = V_0 e^{-\omega_i \sin(\Omega t + \Phi) \times t} \text{Re}(e^{i(\omega_0 + \omega_r \sin(\Omega t + \Phi) \times t)})$$

(3.2)

to show the amplitude modulation determined by $\omega_i$ and frequency modulation determined by $\omega_r$ more clearly.

To ensure that a natural oscillation can build up and to see how it is correlated with the center control signal of the active filter, we turned off the injection and turned up the gain at active filter until the loop oscillates, and such oscillation persists during the whole modulation period $2\pi/\Omega$. Both waveforms of the center control signal and the system response are recorded with an audio card with a sampling rate of 192 kHz, which corresponds to more than 200 data points per cycle. This density of data points is sufficient for running a spectral analysis to extract the amplitude and frequency.

Figure 3.1 shows the complex frequencies $\tilde{\omega}$ of the center control and the system response as a natural oscillation in the complex plane. The red data points represent the center control signal, which faithfully reconstruct the elliptical orbit determined by the parameters in the FPGA-based waveform synthesis (described in Appendix A). The data points are spread over roughly three complete orbit periods. The blue data points show the behavior of the phase 1a pure electronic (dominantly-linear) oscillator responding to this change in filter center. The shape of this contour deviates from a perfect ellipse, which can be explained by the change in the small-signal transfer function when the amplitude of the center control signal is modulated.
Figure 3.1. Spectral Analysis of Driven Oscillator Injection Signal (Red) and System Response (Blue). To study the behavior of the simple pure electronic driven oscillator in Phase 1a and Phase 1b, we used an audio card with sampling rate of 192kHz to record the waveforms of both injection and the system response signal. Using a sliding window, we fit these data points with an exponentially-enveloped (complex frequency) sinusoidal function, to extract both real and imaginary parts of the complex frequency. (a) shows an accumulated data as several orbits are completed. (b) shows the same data with the connection between injection and system response annotated by a joining line within a full orbit cycle. (c) show how, in the complex plane, the filter drive and system injection are driven in the pole-winding scheme. A phase slip happens when the injection signal is at the right end of the ellipse, where it has its minimum amplitude. Since the amplitude of the oscillator’s natural response is saturated above the oscillation threshold, and the injection signal is small, the width of the Arnold tongue is small, and a phase-slip occurs.
3.1.2 Phase 1a Relative Phase Measurements

Figure 3.2 shows data for the accumulated relative phase between the injection signal and system natural oscillation in a dominantly linear oscillator. We use the word 'dominantly' here to distinguish between the natural saturation behavior that any nominally linear system must eventually show - in practice, no oscillator can be purely linear, and must eventually show strong odd-order non-linearities in its open-loop transfer function. In this case (phase 1a), we apply an electronic system which has no additional modification to these natural non-linearities, and deliberately clamp the system with a comparator at a fixed gain. Since in Phase 1b we will introduce into the same loop a deliberate non-linearity with controllable even and odd order coefficients, using a second FPGA-based calculator (or 'non-linearity module'), Phase 1a will use the same module with all non-linear coefficients set to zero.

In figure 3.2 then, the "nonlinearity module" is actually a linear stage with a constant proportional gain, as all the coefficients \(a_i\) in equation 1.50 are set to zero except \(a_1 = \text{const}\). We turned the system gain high enough so that during the full orbit cycle, this feedback loop maintained its oscillation.

These plots are displayed, from top to bottom, in phase-order during the orbital period, with measurements of accumulated phase taken at different places in the orbit, which can be viewed as the injection signal is at certain polar angle \(\theta\) in the complex frequency plane.

In each plot, the horizontal axis is the orbit frequencies and the vertical axis is the accumulated phase (in unit of \(2\pi\)). At the end of each cycle of the injection signal, one measurement of the accumulated phase between the system response and the injection was recorded, following the measurement scheme described in previous chapter. The accumulated phase is reset to zero when the orbit phase angle is a multiple of \(2\pi\). The data recording is based on a DSP through FPGA. For the vertical axis, an 8-bit integer is being used, which sets the range of the accumulated phase to be \(\pm 9.3\pi\). For the horizontal axis, the orbit frequency/period in this DSP is represented as a 32-bit floating number following IEEE 754 standards, and the scanning of this frequency is done by modulating the 8 most significant bits in the 24-bit of the mantissa, at a much slower timescale compared to the orbital movement.
The measurement begins when $\theta$ reaches a multiple of $2\pi$ and the accumulated phase starts to deviate from zero. From the plots, when $\theta$ is close to zero (first plot), all the data points at all different orbit frequencies distribute along the horizontal zero line. As $\theta$ is increasing, the left side of the plot, i.e. when orbit frequency is small, the accumulated relative phase between injection and system oscillation clearly deviates from the starting line, while for the higher orbit frequency, the growth of this phase is much smaller.

This could be understood in the language of a driven oscillator, which can be described by the Adler model. An injection locked oscillator can lock to injection signal with a fixed relative phase. The change to the system gain or the injection signal can result in a different relative phase that the system is locked to, if such change is not large enough to knock the system out from the locking range. If the changes are large enough, the oscillator will be in the quasi-locking region during a portion of the orbit period. Therefore, for smaller orbit frequency the orbital period is longer and there is a longer time for the amplitude and frequency of injection and system response to change from their initial values, which in result change the natural pole location and locking range by a relatively large amount. On the other hand, for larger orbit frequency, the time-dependent change to the locking range/Arnold tongue structure is not strong enough to knock the system out of the locking range, then one would expect the accumulated phase undergoes some deviation from zero but eventually return to zero at the end of the orbital cycle.

In actual experiment, since both frequency and amplitude of injection and filter drive are being modulated, the modulation to the Arnold tongue is not associated with the orbit frequency in a simple way. Therefore, in these plots we can observe that there are multiple regions that the driven oscillator synchronizes to the injection at the end of the orbit, as accumulated phase at $\theta \to 1$ is around 0. Such behavior can be observed at $0.68 < \Omega < 0.72$ and $0.76 < \Omega < 0.86$.

When the modulation to Arnold tongue is large enough to push the system away from the locking range, namely the injection-pulled regime, the system shows a quasi-lock behavior, which means for most of its orbit, it follows the injection signal. During a rather short amount of time, a $2\pi$ phase slip happens rapidly [136]. If this under-driven oscillator carries un-modulated signal, this phase slip happens at a frequency $f_b$ defined as
\[ f_b = \sqrt{(f_0 - f_{inj})^2 - f_L^2}, \]  

(3.3)  

where \( f_0 \) is the natural oscillation frequency, \( f_{inj} \) is the injection frequency and \( f_L \) is the half width of the locking range. For a modulated driven oscillator system, if the modulation periodically takes the system in and out of the locking range, the phase slip should synchronize with this modulation. Such behavior can be observed in the data when \( 0.86 < \Omega < 0.94 \) and \( 1.05 < \Omega < 1.1 \).

However, this argument lacks the ability to explain those sudden changes in the behavior of the accumulated phase, near \( x = 0.68, 0.72, 0.78, 0.86 \). At those points, clearly the first order derivative show discontinuity and the accumulated phase present some plateau structures. In the small orbit frequency region where \( x < 0.8 \), the average values of those three steps show exact separation of \( 2\pi \).
Figure 3.2. Accumulated Phase Versus Orbit Frequency in Phase 1a Oscillator. The accumulated phase between the system response and the injection signal is plotted against the pole-winding orbit frequency. From top to bottom, each plot shows data taken at different values of the orbit angle $\theta/2\pi$. In this measurement, no artificial non-linearity is added using the non-linearity module (see discussion in the main text.)
3.1.3 Phase 1b Measurements - the Effect of Non-Linearity on Phase-Slapping

As introduced before, one way to view this crossed interferometer design of direct transport measurement with the cold atomic gas is to consider it as a nonlinear optical effect, as atoms behave as a media for one laser beam to couple to the other. It is interesting in the pure electronic driven oscillator system to investigate how non-linearity affects the phase response.

Non-linearity should emerge as long as an oscillation is built up. Although the discussion of oscillation in the feedback loop always tend to relate an oscillation with a pole in transfer function and unstable gain, a real system cannot reach infinity and therefore the system response is always clipped at some stage in the loop. This clipping naturally generates a nonlinear response function. One can obtain an evidence in a simple electronic oscillator by turning up the gain beyond oscillation threshold, and a distortion of the waveform of oscillation can be observed.

It is handy to control the nonlinearity contained in the transfer function of this feedback loop, simply by reprogramming the FPGA which plays the role of the nonlinearity module. We have taken experiments with an extensive parameter set to interrogate how nonlinearity affect the accumulated phase after complex signal has gone around the elliptical orbit.

In Figure 3.3, first order and third order coefficients $a_1$ and $a_3$ are taking nonzero values, hence the transfer function for the nonlinear module is

$$V_{out} = a_1 V_{in} + a_3 V_{in}^3$$

(3.4)

The values of $a_1$ and $a_3$ are selected based on the behavior of the system. The linear coefficient $a_1$ is fixed at a number, which is determined so that at a maximum input of the FPGA nonlinear module, with only the linear gain, the output will be 80% of its maximum output, which provides sufficiently good resolution, meanwhile allow space for varying higher order coefficients in the transfer function.
Figure 3.3. Accumulated phase of the Phase 1b pole-wound oscillator while varying the cubic non-linearity using the non-linearity module. All data points are accumulated phase as the complex frequencies for filter reference signal and injection have gone around the pole-winding orbit once (see discussion in the main text.)
From the plots shown, it is clear that the strength of nonlinearity affects the driven oscillation system. It may suggest that the nonlinearity has to be sufficiently small for the plateau structure corresponding to an integer of $2\pi$ phase slip to appear clearly. When the transfer function deviates from small nonlinearity limit, the plateaus tend to smooth out. This 'smooth out' may truly be the vanishing of the exactly quantized plateaus, or it actually consists of a few more sub-plateaus which corresponds to fractional phase slips. The later one is hardly distinguishable from the previous one due to the current data recording system doesn’t have high enough resolution.

In order to see how different order of nonlinearity affects the accumulative phase, we take the similar data set with the first and fifth power nonzero. Now the transfer function is:

$$V_{out} = a_1 V_{in} + a_5 V_{in}^5$$

The results contain some similarity to the data with coefficient of the cubic term varied. In the small non-linearity limit, quantized plateau structures can be easily identified, corresponding to integers of $2\pi$ phase slips. As the strength of nonlinearity becomes larger, integer plateaus tend to merge. In this set of data, this 'merging' seems more like fractionalized sub-plateaus rather than becoming smooth. It also requires better resolution in phase to seek out those features.

In figure 3.6 and figure 3.7, plots are showing data taken in the similar fashion as one nonlinear term is varied, with an extra term in the polynomial processor in order to eliminate the unintended nonlinearity from saturation in the electronic loop. This extra term seems to help the resolution of the plateau structures, however, selecting value for this extra term requires more carefulness to produce more meaningful results.
$V_{\text{out}} = V_{\text{in}} + 10^{-10} \cdot a_5 V_{\text{in}}^5$

**Figure 3.4.** Accumulated phase of the Phase 1b pole-wound oscillator while varying the fifth-order non-linearity using the non-linearity module. All data points are accumulated phase as the complex frequencies for filter reference signal and injection have gone around the pole-winding orbit once (see discussion in the main text.) Data is continued in the next plot.
Figure 3.5. Accumulated phase of the Phase 1b pole-wound oscillator while varying the fifth-order non-linearity using the non-linearity module. All data points are accumulated phase as the complex frequencies for filter reference signal and injection have gone around the pole-winding orbit once (see discussion in the main text.) Data is continued from the previous plot.
Figure 3.6. Accumulated phase of the Phase 1b pole-wound oscillator while attempting to cancel the natural third power and varying the fifth-order non-linearity using the non-linearity module. All data points are accumulated phase as the complex frequencies for filter reference signal and injection have gone around the pole-winding orbit once (see discussion in the main text.)

\[ V_{out} = V_{in} + 15 \cdot 10^{-6} V_{in}^3 + 10^{-10} \cdot a_5 V_{in}^5 \]
Figure 3.7. Accumulated phase of the Phase 1b pole-wound oscillator while canceling the fifth power and varying the third-order non-linearity using the non-linearity module. All data points are accumulated phase as the complex frequencies for filter reference signal and injection have gone around the pole-winding orbit once (see discussion in the main text.) Data is continued in the next plot.
3.2 Phase 2-4 Measurements: The Opto-Electronic and Atom-Optical Laughlin Pump Systems

In the previous section, I presented data taken in the pure electronic system, where some clear quantized phase slips were observed. As soon as we move to the optical apparatus, the multiplicative noise due to amplifying low light levels immediately becomes the largest obstacle to obtaining meaningful data. For the merit of running the experiment, we achieved our goal of building all different stages of the apparatus and finished taking data with each setup. Here, in Figure 3.8 we present the data taken from a phase 2 apparatus, in which a Faraday effect is induced in an optical fiber, which is for simulating the same effect from atom onto optical pumping Sagnac interferometer. The same pole cycling scheme is tested and similar data of accumulative phase versus orbit frequency is recorded. In this figure, we successfully identified one plateau structure and this transition happens when the orbit frequency is around 3.3Hz. This plateau structure corresponds to a phase slip of roughly 6π, and it is reasonable to believe that there are richer structures contained in this transition.

3.2.1 Phase 2 Measurements - The Opto-Electronic Laughlin Pump System

In phase 2 of the experiment, the optical output interferometer, including the Sagnac phase detection and heterodyne optical detector described in the previous chapter were incorporated into the oscillator loop.

Figure 3.8 shows the results of phase 2 experiments. For reference, accumulated phase increments of 2π are marked with dotted red lines. The data in all cases shows a strong increase of accumulated relative phase as the orbit frequency is increased, though this behavior is measured with the sweep unidirectional in time. After the accrual of this data, it was recognized that during the experiment run, the output interferometer light exhibited a stronger than expected thermal drift during the interrogation sequence, likely leading to a varying open-loop gain and poorer correlation between the active reference filter pole and the free-running oscillation. Though not clearly shown by the data, there is still perhaps some weak clustering of data indicating integer plateauing at sufficiently low injection amplitudes, beginning
at a level of 140mV. Further data would be necessary to confirm this behavior and resolve sub-steps in the phase-slipping behavior. Presumably, this system is largely linear, leading to integral phase slips.

Better noise suppression, and longer integration times are likely necessary for making progress in the study of these systems. Some improvements are relatively straightforward to accomplish - for example, either locking the intensity of the output interferometer light before injection into the Sagnac setup, and/or better suppressing the sensitivity to intensity in the phase measurement would improve the oscillator gain stability, likely reducing the phase drift seen in the preliminary measurements.
Figure 3.8. The Phase 2 oscillator response - the output Sagnac interferometer is driven directly by inducing a Faraday effect in a fiber. The accumulative phase shows preliminary evidence to contain at least one transition at 3.3Hz, and the value of this sudden phase change corresponds to somewhere in the range of $3 - 6\pi$ of phase slip, depending of the injection amplitude. Further improvement on the experiment requires better techniques of stabilizing interferometer input power, suppressing multiplicative noise and higher resolution in the phase measurement.
3.2.2 Phase 3 Measurements - Direct Output Interferometer Modulation

Phases 3 and 4 require the use of the cold atomic vapor, using first a direct induction of the Faraday effect in the vapor during the generalized RSC processes through a modulated magnetic field, and finally in phase 4 the complete Laughlin pump oscillator geometry with cross-interferometer gain. Both stages require considerably more experiment time than phases 1 and 2 due to the lower available duty cycle when working with the atomic vapor. In addition, the interrogation time with the vapor is limited to shorter bursts, as the vapor exhibits a finite lifetime during the cooling process instigated with the Laughlin pump geometry.

Nevertheless, it was possible to acquire some preliminary data for these configurations. Figure 3.9 shows the result from phase 3 measurements, now comparing function of the oscillator with and without the presence of atoms. To generate this data, a weak magnetic field modulation was produced during the dissipative cooling process using Helmholtz coils to modulate not the vector light shift caused by input interferometer light, but a uniform, true magnetic field that adds to it. This produces a similar effect in the gas as does modulation of spatially-varying vector light shift, without the additional complexity of driving a Faraday effect in the input interferometer fibers. The atoms can be held during the cooling process for several tens of seconds, but to acquire sufficient data including a sweep over the pole-winding orbit frequency, the orbit frequency scan was adjusted to have a center frequency of 3.5Hz.
Figure 3.9. Phase 3 experimental results - the input interferometer is short-cut by inducing a Faraday effect directly in the output interferometer using the atomic vapor and a uniform, time-modulated magnetic field to modify the dissipative cooling process. Due to larger noise-floor and weaker gains present in atomic-vapor experiments, we have not yet achieved the same level of stability as in pure electronic oscillators or the opto-electronic oscillator of Phase 2, in which data are separated into sub-parts, based on orbit angle. Two plots show the comparison between cases with atoms (above) and without atoms (below) for a single orbit sample phase, and it is clear that the system behavior is different in these two situations. Without atoms, the phase wanders over large values randomly, consistent with low loop gain, and with no coherent oscillation building up. With atoms, sufficient gain exists to create an oscillation, a necessary condition for quEASE methods of simulation to move forward.
3.2.3 Phase 4 Measurements - The Full Laughlin Pump Geometry

In the full Laughlin pump geometry, modulation of the atomic vapor is induced from the input interferometer, and the effect measured at the output interferometer, closing the loop through the active reference filter, and back to modulation of the input interferometer using Faraday-induced modulation in its optical fibers. This phase represents the ultimate conclusion of the phased-development of the experiment, a fully closed Optical Laughlin Pump Oscillator.

Figure 3.10 shows the results from the full Laughlin pump. Data was accumulated for each of these experiments in a single sweep of the orbit frequency over a one-second duration experiment probing a single production of the cold atomic vapor. Again, data was taken with and without a vapor present, and shows a strong indication of appreciable open-loop gain and coherent oscillation only with the atomic vapor present. This is important, as development of stable oscillation in the atomic system is one milestone along the path to implementation of quEASE methods, incorporating an atomic vapor into a sensitive feedback loop.

Unfortunately, like the phase 2 and 3 experiments, the background noise-floor is sufficiently high and the open-loop small-signal gain sufficiently low that oscillation is not a clear enough tone to extract a stable relative phase in the pole-winding scheme, allowing us to extract the integral or non-integral phase-slipping. However, for a first demonstration of the optical Laughlin Pump oscillator, this is not a bad starting point.

Reducing background noise sources and increasing the small-signal gain, particularly the cross-interferometer portion of it, can likely be accomplished in a number of simple ways. For example, the number of atoms contained in the vapor can likely be increased by one order-of-magnitude by improving the transport, capture and cooling schemes described in an earlier chapter of the thesis. Another likely simple improvement is to run the generalized version of Raman sideband cooling at higher optical pumping powers. The optical power used currently is limited by optimal pump power in a balanced counter-propagating geometry, and with the Raman coupling level achievable with the far-off-resonant beams derived from the 1064nm laser. Raman sideband cooling introduces a very weak but relevant radiation pressure, and it is likely higher pumping levels can be achieved with better balance.
Figure 3.10. The Phase 4 oscillator data, incorporating the full Optical Laughlin Pump Oscillator. Plots on the left show the result without an atomic vapor present, and on the right with the vapor. Similar to Phase 3 measurements, the much higher noise floor and higher multiplicative noise prevent us from resolving the dynamic of phase slips, but evidence does exist for coherent oscillation with gain mediated through the atomic vapor. Two sets of data are presented here, which differ by the active filter control amplitude (controlling overall gain) and injection amplitude (controlling width of the Arnold tongue), hence these two situations have different locking ranges.

of beam powers in the Sagnac. Moving to closer detuned Raman-coupling beams can improve the size of the vector light shift by likely another order-of-magnitude, leading to substantially stronger input interferometer modulation of the vapor properties, and simultaneously higher optical pumping rates.
Appendix A

Two Frequency Synthesizer

The two-frequency synthesizer is a FPGA based circuit that can produce two signals with complex frequencies described as

\[ s(t) = \text{Re}(V_0e^{i\tilde{\omega}t}), \quad \tilde{\omega} = \omega_0 + \omega_r \sin(\Omega t + \Phi) + i\omega_i \cos(\Omega t + \Phi), \]  

(A.1)

The FPGA we used is a Saxo-Q Acquisition board produced by KNJN, which is based on an Altera’s Cyclone II FPGA chip with part number EP2C5T144. We used Quartus II as the EDA for programming the board in Verilog.

Computing an exponential function could take a huge amount of resource and it could be very time consuming. We developed a method which turns the FPGA into a soft processor only contains a 32-bit floating point multiplier and a 32-bit floating point adder.

The Taylor series for an exponential function is

\[ e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \]  

(A.2)

This expansion is valid for complex value of \( x \) and it converges for all \( x \).

For exponential function with pure imaginary argument, it is easy to see that the function wraps around when the argument exceeds \( 2\pi \). As \( 2\pi \approx 6.28 \) and we proved that it only requires 37 terms so that the series sum and the actual value differs by less than \( 10^{-6} \).

In the case where \( x \) has real component bigger than 1, this exponential growth brought a small difficulty. However, in the experiment, \( \omega_i \) is much smaller than \( \omega_0 \) and \( \omega_r \), therefore the deviation only appears to be problematic after a long time of
operation.

The actual process of this calculation is long and tedious, which requires a flow chart to demonstrate the operation in each step. We take it for granted that the timescale of the carrier cycles and timescale of the orbiting dynamics differs by more than 2 orders of magnitude which makes it safe to separate the calculation of these two dynamics under two different clocks. The 820Hz carrier is updated under a 22kHz clock which ensures about 24 data points to form a complete cycle of a sinusoidal waveform. The 3Hz orbiting is updated under a 1kHz clock. There is a faster clock runs at 22MHz, which is for calculating the value of terms in the series which add up to an exponential function.

At a certain time the phase $\tilde{\omega}t$ is calculated, which combined with prefactor $i$ can be represent as the general form of a complex number $a + ib$, since $\tilde{\omega}$ is a complex frequency. Then the question of these whole calculation becomes calculating $e^{a+ib}$ at several different clock-domains. Then in each fast clock cycle, a term in the series is computed. The $(n+1)_{th}$ term in the series can be calculated with the $n_{th}$ term and the common factor $a + ib$. We keep a number which keeps accumulating those products and eventually it approaches the exponential function. Following that strategy, we turned a computation of an exponential function into serially computing a product of two complex numbers and then a sum of two complex numbers, which can be easily turned to product and sum of real numbers and different address to distinguish between real part and imaginary part.

Here I present the Verilog code for producing these complex frequencies. It is worth to notice that this code also contains a serial communication module, which is used for recording data in my experiment.

```verilog
module FX2_DDS_Q(
    FX2_CLK, FX2_FD, FX2_SLRD, FX2_SLWR, FX2_flags,
    FX2_PA_2, FX2_PA_3, FX2_PA_4, FX2_PA_5, FX2_PA_6, FX2_PA_7,
    ADC_dataA, ADC_DACCTRL, clk_ADC_out, ADC_dataB,
    DAC_clk_in, DAC_clk_out, DACs_data_out, DACc_data_out, ref_clk_in, IO1, IO2
);

input FX2_CLK;
inout [7:0] FX2_FD;
input [2:0] FX2_flags;
output FX2_SLRD, FX2_SLWR;
```

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output FX2_PA_2;
output FX2_PA_3;
output FX2_PA_4;
output FX2_PA_5;
output FX2_PA_6;
input FX2_PA_7;

input DAC_clk_in;
input ref_clk_in;
output DAC_clk_out;
output [9:0] DACs_data_out;
output [9:0] DACc_data_out;

input [7:0] ADC_dataA; // first ADC channel
input [7:0] ADC_dataB; // second ADC channel
output ADC_DACCTRL;
wire clk_acq = DAC_clk_in;
assign clk_ADC_out = DAC_clk_in;
output clk_ADC_out;

output IO2;
input IO1;

wire FIFO_CLK = FX2_CLK;

wire FIFO2_empty = ~FX2_flags[0];
wire FIFO2_data_available = ~FIFO2_empty;
wire FIFO3_empty = ~FX2_flags[1];
wire FIFO3_data_available = ~FIFO3_empty;
wire FIFO4_full = ~FX2_flags[2];
wire FIFO4_ready_to_accept_data = ~FIFO4_full;
wire FIFO5_full = ~FX2_PA_7;
wire FIFO5_ready_to_accept_data = ~FIFO5_full;
assign FX2_PA_3 = 1'b1;

wire FIFO_RD, FIFO_WR, FIFO_PKTEND, FIFO_DATAIN_OE, FIFO_DATAOUT_OE;
wire FX2_SLRD = ~FIFO_RD;
wire FX2_SLWR = ~FIFO_WR;
assign FX2_PA_2 = ~FIFO_DATAIN_OE;
assign FX2_PA_6 = ~FIFO_PKTEND;
wire [1:0] FIFO_FIFOADR;
assign {FX2_PA_5, FX2_PA_4} = FIFO_FIFOADR;

wire [7:0] FIFO_DATAIN = FX2_FD;
wire [7:0] FIFO_DATAOUT;
assign FX2_FD = FIFO_DATAOUT_OE ? FIFO_DATAOUT : 8'hZZ;

assign FIFO_FIFOADR = 2'b00; // select FIFO2
assign FIFO_RD = 1'b1; // always read
assign FIFO_WR = 1'b0; // never write
assign FIFO_DATAOUT = 8'h00; // never write, so this value is not used
assign FIFO_DATAIN_OE = 1'b1; // always read data
assign FIFO_DATAOUT_OE = 1'b0; // never output data
assign FIFO_PKTEND = 1'b0;

reg [3:0] timeout;
always @(posedge FIFO_CLK) if(FIFO2_data_available) timeout <= 4'hF; else if(!timeout) timeout <= timeout - 4'h1;
wire timeout_complete = (timeout == 4'h1) && !FIFO2_data_available;

reg [31:0] FIFO_value;
always @(posedge FIFO_CLK) if(FIFO2_data_available) FIFO_value[31:24] <= FIFO_DATAIN;
always @(posedge FIFO_CLK) if(FIFO2_data_available) FIFO_value[23:16] <= FIFO_value[31:24];
always @(posedge FIFO_CLK) if(FIFO2_data_available) FIFO_value[15: 8] <= FIFO_value[23:16];
always @(posedge FIFO_CLK) if(FIFO2_data_available) FIFO_value[ 7: 0] <= FIFO_value[15: 8];

//generate a master clock signal, internal VCO is at 75.757 MHZ and the allowed maximum clk for running multipliers is 227.271 MHZ, which is three times the VCO frequency
//master_clk period is 4.40003344ns
//cmu_clk period is 44.0003344
wire master_clk_reset;
wire master_clk_locked;
wire master_clk;
PLLL_master PLL_master_inst ( .areset ( master_clk_reset ),
    .inclk0 ( DAC_clk_in ),
    .c0 ( master_clk ),
    .locked ( master_clk_locked )
);
// create a cmu_cmd_clk and casu_cmd_clk, and have casu_clk delayed by one master_clk
reg[3:0] cmu_clk_counter = 0;
reg cmu_clk;
reg casu_clk;
parameter cmu_period = 4'b1001; // actual period is (9+1)*master_period = 44ns
always@(posedge master_clk) begin
    if (master_clk_locked == 1) begin
        if (cmu_clk_counter == cmu_period - 4' b0010) begin
            cmu_clk <= 0;
            cmu_clk_counter <= cmu_clk_counter + 4' b0001;
        end
        else if (cmu_clk_counter == cmu_period) begin
            cmu_clk_counter <= 0;
            cmu_clk <= 1'b1;
        end
        else if (cmu_clk_counter == 0) begin
            cmu_clk_counter <= cmu_clk_counter + 4' b0001;
            casu_clk <= 1;
        end
        else if (cmu_clk_counter == cmu_period - 1) begin
            cmu_clk_counter <= cmu_clk_counter + 4' b0001;
            casu_clk <= 0;
        end
        else cmu_clk_counter <= cmu_clk_counter + 4' b0001;
    end
end

wire[7:0] DAC_regdata[3:0];
assign DAC_regdata[0] = 8'b10000000; // minimum values (corresponds to maximum vertical pos + max
assign DAC_regdata[1] = 8'b10000000;
assign DAC_regdata[2] = 8'b10000000;
assign DAC_regdata[3] = 8'b10000000;
parameter psdac = 8;
reg[psdac+9:0] DAC_cnt; always @(posedge cmu_clk) DAC_cnt <= DAC_cnt + 18'd1;
wire[15:0] DAC_data = {5'b11111, DAC_cnt[psdac+9:psdac+8], 1'b1, DAC_regdata[DAC_cnt[psdac+9:psdac+8]]};
reg ADC_DACCTRL; always @(posedge cmu_clk) ADC_DACCTRL <= &DAC_cnt[psdac+7:psdac+5] & (~DAC_cnt[p]...
reg[7:0] ADC_latch; // use this to read from ADC_data and this only needs to be updated slowly

// waveform generation starts from here

// following parameters and initialization is based on master_clk 227.271 MHZ, which yields a period of 4.4 ns and since cmu_clk is 10 times slow, 44 ns is the cycle period
// now add in the other waveform, which has a non-zero PHI to begin with, start with rewriting the instructions
// Use the slow time T_clk for updating the frequency by updating slow T-- pole moving around on the complex plane
// waveform is Amp_dp*Exp(j*(w_c + amp_comp_freq*Exp(j*OMEGA*T))*t), amp_comp_freq is small compare to w_c so complex frequency is small compared to real part when t is small

reg[6:0] T_count = 7’d0;
parameter T_up = 7’d37;
reg[7:0] dp_count = 8’d0;
parameter dp_c_up = 7’d37;
parameter dp_delta_up = 8’d119;
reg[3:0] step_count = 4’d0;
wire[31:0] t_inc; // increment of lab time
assign t_inc = 32’b001101111101101100010010001; // t_inc = cmu_clk_period*(6+4*(dp_c_up+1)+6*(dp_delta_up-1)-2)+9 = 44*647=28468 ns = 28468E-9
wire[31:0] phc_i_inc;
assign phc_i_inc = 32’b000111000010110011001101010; // phc_i_inc = w_c*T inc = 2*pi*f_c*T_inc = 0.146673169, f_c = 820
wire[31:0] PHY_i_inc;
assign PHY_i_inc = {1’b0,8’b01111001,ADC_latch,15’b001000101011010}; // 2.3-4.7 HZ

wire[31:0] twopi_fp;
assign twopi_fp = 32’b0100000011001001000011111011011; // 6.2831853

reg[7:0] inv_addr;
wire[31:0] inv_lut;
wire[31:0] amp_comp_freq; // this is the amplitude of the small complex part of the frequency

reg[31:0] wdelaptaph_l = 32’d0;
reg[31:0] wdelaptaph_r = 32’b00111111000000101000111101011100; // 0.5 same as cmu_addr == 28 inside
reg[31:0] PHY_l =32’b0011111111001001000011111011011; // pi/2
wire[31:0] amp_dp;
assign amp_dp = 32'b0100000111000000001111001100110011; // this is the amplitude in the final waveform

reg[31:0] phc_i =32'd0;
reg[31:0] phdelta_r = 32'd0;
reg[31:0] phdelta_i = 32'd0;
reg[31:0] Ms_r;
reg[31:0] Ms_i; // s means slow dynamics
reg[31:0] Sigma_r;
reg[31:0] Sigma_i;
reg[31:0] Q_rr; //Qs are shared by all three calculations, also sr_inc, dp1_rr
reg[31:0] Q_ii; // also si_inc, dp1_ii
reg[31:0] Q_r1; // also dp2_rr
reg[31:0] Q_r; // also dp2_ii
reg[31:0] sc_r;
reg[31:0] sc_i;
reg[31:0] mc_r; // mc_r will read terms for calculation carrier data mc_r/i and envelop data mdelta_r/i
reg[31:0] mc_i;
reg[31:0] s1_r;
reg[31:0] s1_i;
reg[31:0] s2_r;
reg[31:0] s2_i;

reg[31:0] dp1_out; // = 32'b010000011111010000000000000000000; // dp_out_1(t=0) = amp_dp
reg[31:0] dp2_out; // = 32'b010000100000000000000000000000000; // dp_out_1(t=0) = amp_dp
reg sign_indicator; // for fliping sign of power series term of one waveform

reg[5:0] addr_que[0:2];
reg[5:0] casu_addr_que[0:2];
reg addsub_selector;

rom      rom_inst (
    .address ( inv_addr ),
    .clock ( master_clk ),
    .q ( inv_lut )
);
reg [31:0] phase_i;
wire phase_reset;
wire [31:0] piovertwo_fp;
assign piovertwo_fp = 32' b001111111100100001111111011011; //1.57...
reg [31:0] phase_compare = 32' b010000001100100100001111111011011; // 2pi, this number toggles between 2pi and 0.5 pi, 0.5 pi is only used for orbit phase
reg piovertwo_cross = 0; // now it’s actually set to reset phase measurement at pi/2

fp_compare32 fp_compare32_phase_i(
    .clock ( master_clk ),
    .dataa ( phase_i ),
    .datab ( phase_compare ),
    .ageb ( phase_reset )
);

/////////////////////////////////////////////////////

wire [31:0] CMU_IO [0:11];
wire [31:0] CASU_IO [0:13];

CMU cmu_inst (.fast_clk(master_clk), .cmu_cmd_clk(cmu_clk), .addr_0(addr_que[0]), .addr_1(addr_que[1]), .Inp_0(wdeltaph_r), .Inp_1(wdeltaph_i), .Inp_2(t_inc), .Inp_3(PHY_i), .Inp_4(Ms_r), .Inp_5(Ms_i), .Inp_6(mc_r), .Inp_7(mc_i), .Inp_8(Q_ri), .Inp_9(Q_ii), .Inp_10(Q_rr), .Inp_11(Sigma_r), .Inp_12(Sigma_i), .Inp_13(amp_dp), .IO_0(CMU_IO[0]), .IO_1(CMU_IO[1]), .IO_2(CMU_IO[2]), .IO_3(CMU_IO[3]), .IO_4(CMU_IO[4]), .IO_5(CMU_IO[5]), .IO_6(CMU_IO[6]), .IO_7(CMU_IO[7]), .IO_8(CMU_IO[8]), .IO_9(CMU_IO[9]), .IO_10(CMU_IO[10]), .IO_11(CMU_IO[11]));//, .IO_12(test));

CASU casu_inst (.casu_fast_clk(master_clk), .casu_cmd_clk(casu_clk), .as_selector(addsub_selector), .casu_Inp_0(PHY_i), .casu_Inp_1(twopi_fp), .casu_Inp_2(phc_i), .casu_Inp_3(phc_i_inc), .casu_Inp_4(Ms_r), .casu_Inp_5(Ms_i), .casu_Inp_6(mc_r), .casu_Inp_7(mc_i), .casu_Inp_8(Q_ri), .casu_Inp_9(Q_ii), .casu_Inp_10(Q_rr), .casu_Inp_11(Sigma_r), .casu_Inp_12(Sigma_i), .casu_Inp_13(amp_dp), .casu_Inp_20(PHY_i_inc), .casu_Inp_21(wdeltaph_r), .casu_Inp_22(wdeltaph_i), .casu_Inp_23(phdelta_r), .casu_Inp_24(phdelta_i), .casu_IO_0(CASU_IO[0]), .casu_IO_1(CASU_IO[1]), .casu_IO_2(CASU_IO[2]), .casu_IO_3(CASU_IO[3]), .casu_IO_4(CASU_IO[4]), .casu_IO_5(CASU_IO[5]), .casu_IO_6(CASU_IO[6]), .casu_IO_7(CASU_IO[7]), .casu_IO_8(CASU_IO[8]), .casu_IO_9(CASU_IO[9]), .casu_IO_10(CASU_IO[10]), .casu_IO_11(CASU_IO[11]), .casu_IO_12(CASU_IO[12]), .casu_IO_13(CASU_IO[13]);

reg [9:0] SI_1;
reg [9:0] SI_2;
reg fptosi_addr;

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wire[9:0] si_read;

CFPTOSI cfptosi_inst(.conv_clk(master_clk), .conv_addr(fptosi_addr), .conv_inp0(dpi_out), .conv_input(dpi_input));

always@(posedge cmu_clk) begin
  if(T_count == 0&&dp_count == 0 ) begin
    if( step_count == 0) begin
      step_count = step_count + 4'd1;
      Ms_r = 32'd0;
      Sigma_r = 32'b00111111110000000000000000000000; // 1 in fp
      Sigma_i = 32'd0;
      phase_i = PHY_i;
      //wdeltaph_r = CMU_IO[0]; // need to find a way to flip sign
      addr_que[0] = 1; // phdelta_i_inc = wdelta_i * t_inc
      addr_que[1] = 2;
      addr_que[2] = 1;
      casu_addr_que[0] = 0;// PHY_i = PHY_i - twopi_fp, Do this step anyway so there's
      casu_addr_que[1] = 1;
      casu_addr_que[2] = 0;
      addsub_selector = 0;
    end
    else if( step_count == 1) begin
      step_count = step_count + 4'd1;
      wdelta_i = CMU_IO[1]; // phdelta_i_inc ( wdelta_i )
      if(phase_reset == 1) begin
        PHY_i = CASU_IO[0]; // Only take this value if phase needs to reset
        Ms_i = CASU_IO[0];
        // piovertwo_cross = 0;
        //PHY_i_twopi_trigger = 1;
      end
      else Ms_i = PHY_i;
    end
  end
  addr_que[0] = 0; // -phdelta_r_inc = -wdelta_r * t_inc ; remember wdelta_r = -wdelta
  addr_que[1] = 2;
  addr_que[2] = 0;
casu_addr_que[0] = 2; // phc_i = phc_i + phc_i_inc
casu_addr_que[1] = 3;
casu_addr_que[2] = 1;
addsub_selector = 1;
end
else if(step_count == 2) begin
    step_count = step_count + 4'd1;
    wdeltaph_r = CMU_IO[0]; // phdelat_r_inc(wdeltaph_r)
    phc_i = CASU_IO[1];

    addr_que[0] = 3; // Qs_i = PHY_i * Ms_i
    addr_que[1] = 5;
    addr_que[2] = 4;

    casu_addr_que[0] = 4; // Sigma_r = Sigma_r + Ms_r
    casu_addr_que[1] = 6;
    casu_addr_que[2] = 2;
    addsub_selector = 1;
end
else if (step_count == 3) begin
    step_count = step_count + 4'd1;
    Q_ii = CMU_IO[4];
    Sigma_r = CASU_IO[2];
    inv_addr = T_count + 7'd2;
    // PHY_i_two1pi_trigger = 0;

    addr_que[0] = 3; // Qs_ir = PHY_i * Ms_r
    addr_que[1] = 4;
    addr_que[2] = 5;

    casu_addr_que[0] = 5; // Sigma_i = Sigma_i + Ms_i
    casu_addr_que[1] = 7;
    casu_addr_que[2] = 3;
    addsub_selector = 1;
end
else if (step_count == 4) begin
    step_count = step_count + 4'd1;
Q_ir = CMU_IO[5];
Sigma_i = CASU_IO[3];
phase_i = phc_i;
sc_r = 32'b00111111110000000000000000000000;//fp 1
sc_i = 32'd0;
addr_que[0] = 6; // Ms_r = -Qii * 1/N
addr_que[1] = 8;
addr_que[2] = 6;

casu_addr_que[0] = 2;//phc_i- 2pi
casu_addr_que[1] = 1;
casu_addr_que[2] = 0;
addsub_selector = 0;

end

else if (step_count ==5) begin
step_count = 4'd0;
dp_count = dp_count + 8'd1;
//Ms_r = {-CMU_IO[6][31],CMU_IO[6][30:0]};
Ms_r = CMU_IO[6];
if (phase_reset ==1) begin
phc_i = CASU_IO[0];
mc_i = CASU_IO[0];
end
else mc_i = phc_i;

addr_que[0] = 7; // Ms_i = Qir * 1/N
addr_que[1] = 8;
addr_que[2] = 7;

casu_addr_que[0] = 29;//0- Ms_r
casu_addr_que[1] = 6;
casu_addr_que[2] = 12;
addsub_selector = 0;

end

end

else if((T_count>0 && T_count <= T_up) && dp_count == 0) begin
if(step_count ==0) begin
  step_count = step_count + 4'd1;
  casu_addr_que[0] = 2;//phc_i = phc_i+ phc_i_inc
  casu_addr_que[1] = 3;
  casu_addr_que[2] = 1;
  addsub_selector = 1;
end
else if(step_count ==1) begin
  step_count = step_count + 4'd1;
  phase_i = CASU_IO[1];
  phc_i = CASU_IO[1];
  casu_addr_que[0] = 2;//phc_i = phc_i- 2pi
  casu_addr_que[1] = 1;
  casu_addr_que[2] = 0;
  addsub_selector = 0;
end
else if(step_count ==2) begin
  step_count = step_count + 4'd1;
  if(phase_reset ==1) begin
    phc_i = CASU_IO[0];
    mc_i = CASU_IO[0];
  end
  else mc_i = phc_i;
  inv_addr = T_count+ 8'd2;
  addr_que[0] = 3; // Qs_ii = PHY_i * Ms_i
  addr_que[1] = 5;
  addr_que[2] = 4;
  casu_addr_que[0] = 4;//Sigma_r = Sigma_r + Ms_r
  casu_addr_que[1] = 6;
  casu_addr_que[2] = 2;
  addsub_selector = 1;
end
else if(step_count ==3) begin
  step_count = step_count + 4'd1;
Q_{ii} = \text{CMU\_IO}[4];
\text{Sigma}_r = \text{CASU\_IO}[2];

\text{addr\_que}[0] = 3; // Qs_{ir} = \text{PHY}_i \ast \text{Ms}_r
\text{addr\_que}[1] = 4;
\text{addr\_que}[2] = 5;
\text{casu\_addr\_que}[0] = 5; // \Sigma_i = \Sigma_i \ast \text{Ms}_i
\text{casu\_addr\_que}[1] = 7;
\text{casu\_addr\_que}[2] = 3;
\text{addsub\_selector} = 1;
end
else if (step\_count == 4) begin
\text{step\_count} = \text{step\_count} + 4'd1;
Q_{ir} = \text{CMU\_IO}[5];
\text{Sigma}_i = \text{CASU\_IO}[3];
\text{sc}_r = 32'b00111111100000000000000000000000; // fp 1
\text{sc}_i = 32'd0;
\text{addr\_que}[0] = 6; // Ms_r = -Q_{ii} \ast 1/N
\text{addr\_que}[1] = 8;
\text{addr\_que}[2] = 6;
end
else if (step\_count == 5) begin
\text{step\_count} = 4'd0;
\text{dp\_count} = \text{dp\_count} + 8'd1;
\text{Ms}_r = \text{CMU\_IO}[6];
// Ms_r = \{-\text{CMU\_IO}[6][31], \text{CMU\_IO}[6][30:0]\};
\text{addr\_que}[0] = 7; // Ms_i = Q_s_{ir} \ast 1/N
\text{addr\_que}[1] = 8;
\text{addr\_que}[2] = 7;
\text{casu\_addr\_que}[0] = 29; // 0 - Ms_r
\text{casu\_addr\_que}[1] = 6;
\text{casu\_addr\_que}[2] = 12;
\text{addsub\_selector} = 0;
else if( (0 <= T_count && T_count <= T_up) && dp_count == 1 ) begin
  if(step_count == 0) begin
    step_count = step_count + 4'd1;
    Ms_i = CMU_IO[7];
    Ms_r = CASU_IO[12];
    mc_r = 32'd0; // mc_r = phc_r = 0;
    inv_addr = dp_count + 8'd1;
    fptosi_addr = 0;

    addr_que[0] = 9; // qc_ii = phc_i * mc_i
    addr_que[1] = 11;
    addr_que[2] = 4; // use the same Q_ii port

    casu_addr_que[0] = 18; // sc_r = sc_r + mc_r
    casu_addr_que[1] = 8;
    casu_addr_que[2] = 4;
    addsub_selector = 1;
  end
  else if(step_count == 1) begin
    step_count = step_count + 4'd1;
    Q_ii = CMU_IO[4];
    sc_r = CASU_IO[4];
    SI_1 = si_read;

    addr_que[0] = 9; // qc_ir = phc_i * mc_r
    addr_que[1] = 10;
    addr_que[2] = 5;

    casu_addr_que[0] = 19; // sc_i = sc_i + mc_i
    casu_addr_que[1] = 9;
    casu_addr_que[2] = 5;
    addsub_selector = 1;
  end
  else if(step_count == 2) begin
    step_count = step_count + 4'd1;
    end
  end
end
Q_ir = CMU_IO[5];
sc_i = CASU_IO[5];
fptos1_addr = 1;

addr_que[0] = 6; // mc_r = -qc_ii * 1/n
addr_que[1] = 8;
addr_que[2] = 8;

end
else if (step_count ==3) begin
  step_count = 4'd0;
  dp_count = dp_count + 8'd1;
  mc_r = CMU_IO[8];
  //mc_r = {-CMU_IO[8][31],CMU_IO[8][30:0]};
  SI_2 = si_read;

  addr_que[0] = 7; // mc_i = qc_ir+ 1/n
  addr_que[1] = 8;
  addr_que[2] = 9;

  casu_addr_que[0] = 29; // mc_r = 0 -mc_r
  casu_addr_que[1] = 8;
  casu_addr_que[2] = 12;
  addsub_selector = 0;
end
end

else if((0<= T_count && T_count<= T_up) && (1< dp_count && dp_count< dp_c_up)) begin
  if(step_count ==0) begin
    step_count = step_count + 4'd1;
    mc_i = CMU_IO[9];
    mc_r = CASU_IO[12];
    inv_addr = dp_count + 8'd1;

    addr_que[0] = 9; // qc_ii = phc_i * mc_i
    addr_que[1] = 11;
    addr_que[2] = 4; // use the same Q_ii port
casu_addr_que[0] = 18;  // sc_r = sc_r + mc_r
casu_addr_que[1] = 8;
casu_addr_que[2] = 4;
addsub_selector = 1;
end
else if( step_count ==1) begin
  step_count = step_count + 4'd1;
  Q_ii = CMU_IO[4];
  sc_r = CASU_IO[4];

  addr_que[0] = 9;  // qc_ir = phc_i * mc_r
  addr_que[1] = 10;
  addr_que[2] = 5;

  casu_addr_que[0] = 19;  // sc_i = sc_i + mc_i
  casu_addr_que[1] = 9;
casu_addr_que[2] = 5;
  addsub_selector = 1;
end
else if( step_count ==2) begin
  step_count = step_count + 4'd1;
  Q_ir = CMU_IO[5];
  sc_i = CASU_IO[5];

  addr_que[0] = 6;  // mc_r = -qc_ii * 1/n
  addr_que[1] = 8;
  addr_que[2] = 8;
end
else if ( step_count ==3) begin
  step_count = 4'd0;
  dp_count = dp_count + 8'd1;
  mc_r = CMU_IO[8];
  // mc_r = {~CMU_IO[8][31],CMU_IO[8][30:0]};

  addr_que[0] = 7;  // mc_i = qc_ir* 1/n
  addr_que[1] = 8;
addr_que[2] = 9;

casu_addr_que[0] = 29; // mc_r = 0
casu_addr_que[1] = 8;
casu_addr_que[2] = 12;
addsub_selector = 0;
end

else if ((0 <= T_count && T_count <= T_up) && (dp_count == dp_c_up)) begin
  if (step_count == 0) begin
    step_count = step_count + 4'd1;
    mc_i = CMU_IO[9];
    mc_r = CASU_IO[12];
    inv_addr = dp_count + 8'd1;

    addr_que[0] = 9; // qc_ii = phc_i * mc_i
    addr_que[1] = 11;
    addr_que[2] = 4; // use the same Q_ii port

    casu_addr_que[0] = 18; // sc_r = sc_r + mc_r
    casu_addr_que[1] = 8;
casu_addr_que[2] = 4;
addsub_selector = 1;
  end
  else if (step_count == 1) begin
    step_count = step_count + 4'd1;
    Q_ii = CMU_IO[4];
    sc_r = CASU_IO[4];

    addr_que[0] = 9; // qc_ir = phc_i * mc_r
    addr_que[1] = 10;
    addr_que[2] = 5;

    casu_addr_que[0] = 19; // sc_i = sc_i + mc_i
    casu_addr_que[1] = 9;
casu_addr_que[2] = 5;
addsub_selector = 1;
  end
end
else if(step_count == 2) begin
    step_count = step_count + 4'd1;
    Q_i_r = CMU_IO[5];
    sc_i = CASU_IO[5];

    addr_que[0] = 6; // mc_r = -qc_i * 1/n
    addr_que[1] = 8;
    addr_que[2] = 8;

    casu_addr_que[0] = 24; // phdelta_i = phdelta_i + phdelta_i_inc
    casu_addr_que[1] = 22;
    casu_addr_que[2] = 7;
    addsub_selector = 1;
end
else if (step_count == 3) begin
    step_count = 4'd0;
    dp_count = dp_count + 8'd1;
    mc_r = CMU_IO[8];
    // mc_r = {~ CMU_IO[8][31], CMU_IO[8][30:0]};
    phdelta_i = CASU_IO[7];

    addr_que[0] = 7; // mc_i = qc_i * 1/n
    addr_que[1] = 8;
    addr_que[2] = 9;

    casu_addr_que[0] = 29; // mc_r = 0 -mc_r
    casu_addr_que[1] = 8;
    casu_addr_que[2] = 12;
    addsub_selector = 0;
end
else if((0<= T_count && T_count<= T_up) && (dp_count == dp_c_up + 8'd1)) begin
    if(step_count == 0) begin
        step_count = step_count + 4'd1;
        mc_i = CMU_IO[9];
        mc_r = CASU_IO[12];
        // phdelta_r = CASU_IO[6];
casu_addr_que[0] = 18; // sc_r = sc_r + mc_r
casu_addr_que[1] = 8;
casu_addr_que[2] = 4;
addsub_selector = 1;
end
else if (step_count == 1) begin
  step_count = step_count + 4'd1;
  sc_r = CASU_IO[4];

casu_addr_que[0] = 19; // sc_i = sc_i + mc_i
casu_addr_que[1] = 9;
casu_addr_que[2] = 5;
addsub_selector = 1;
end
else if (step_count == 2) begin
  step_count = step_count + 4'd1;

  // mc_r = phdelta_r;
  sc_i = CASU_IO[6];
s1_r = 32'b00111111111000000000000000000000; // fp 1/n
  s1_i = 32'd0;
s2_r = 32'b00111111111000000000000000000000;
s2_i = 32'd0;

casu_addr_que[0] = 23; // phdelta_r = phdelta_r -(- phdelta_r_inc)
casu_addr_que[1] = 21;
casu_addr_que[2] = 6;
addsub_selector = 0; // pole moving clockwise
// addsub_selector = 1; // pole moving counter-clockwise
end
else if (step_count == 3) begin
  step_count = 4'd0;
  dp_count = dp_count + 8'd1;
  phdelta_r = CASU_IO[6];
  mc_r = CASU_IO[6]; // get mc_r ready for calculating the envelope, now mc_r ready
  // phase_i = phdelta_i;
  phase_i = (phdelta_i[31] == 1'b0)? phdelta_i[30:0]: phdelta_i[31:0];
casu_addr_que[0] = 24;// phdelta_i = phdelta_i - 2pi
casu_addr_que[1] = 1;
casu_addr_que[2] = 0;
//addsub_selector = 0;
addsub_selector = (phdelta_i[31] == 1'b0)?1'b0:1'b1;// if phdelta_i possessive, then subtract 2pi, if phdelta_i negative, then add 2pi
end
end

else if((0 <= T_count && T_count <= T_up)&&( dp_count == dp_c_up + 8'd2 )) begin
    if(step_count ==0) begin
        step_count = step_count + 4'd1;
        if(phase_reset ==1 ) begin
            phdelta_i = CASU_IO[0];
            mc_i = CASU_IO[0];
        end
        else mc_i = phdelta_i;
        addr_que[0] = 12;// Q_rr = phdelta_r * mc_r
        addr_que[1] = 10;
        addr_que[2] = 2;
    end
    else if(step_count ==1) begin
        step_count = step_count + 4'd1;
        Q_rr = CMU_IO[2];
        s1_r = CASU_IO[8];
        addr_que[0] = 14;// s1_r = s1_r + mc_r
        casu_addr_que[0] = 14;
        casu_addr_que[1] = 8;
        casu_addr_que[2] = 8;
        addsub_selector = 1;
    end
end

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else if (step_count == 2) begin
    step_count = step_count + 4'd1;
    Q_ii = CMU_IO[4];
    s1_i = CASU_IO[9];
    sign_indicator = 1'b1;
    inv_addr = dp_count - dp_c_up;

    addr_que[0] = 12; // Q_ri = phdelta_r * mc_i
    addr_que[1] = 11;
    addr_que[2] = 3;
    casu_addr_que[0] = 10; // Q_ii = sr_inc = Q_rr - Q_ii
    casu_addr_que[1] = 11;
    casu_addr_que[2] = 13;
    addsub_selector = 0;
end
else if (step_count == 3) begin
    step_count = step_count + 4'd1;
    Q_ri = CMU_IO[3];
    Q_ii = CASU_IO[13]; // sr_inc

    addr_que[0] = 13; // Q_ir = phdelta_i * mc_r
    addr_que[1] = 10;
    addr_que[2] = 5;
    casu_addr_que[0] = 16; // s2_r = s2_r + (-1)^((dp_count - dp_c_up - 8'd1)*mc_r, minus
    casu_addr_que[1] = 8;
    casu_addr_que[2] = 10;
    addsub_selector = (sign_indicator == 1'b1)? 1'b0 : 1'b1;
end
else if (step_count == 4) begin
    step_count = step_count + 4'd1;
    Q_ir = CMU_IO[5];
    s2_r = CASU_IO[10];

    addr_que[0] = 6; // mc_r = Q_ii(sr_inc) * 1/n;
    addr_que[1] = 8;
    addr_que[2] = 8;
    casu_addr_que[0] = 12; // Q_ir = si_inc = Q_ir + Q_ri
    casu_addr_que[1] = 13;
casu_addr_que[2] = 12;
addsub_selector = 1;
else if (step_count == 5) begin
    step_count = 4'd0;
    dp_count = dp_count + 8'd1;
    mc_r = CMU_IO[9];
    Q_ir = CASU_IO[12];
    addr_que[0] = 7;//mc_i = Q_ir(s_i_inc) * 1/n;
    addr_que[1] = 8;
    addr_que[2] = 9;
    casu_addr_que[0] = 17;//s2_i = s2_i + (-1)^(dp_count - dp_c_up - 8'd1)*mc_i
    casu_addr_que[1] = 9;
    casu_addr_que[2] = 11;
    addsub_selector = (sign_indicator == 1'b1)? 1'b0 : 1'b1;
end
else if((0 <= T_count && T_count <= T_up) && (dp_c_up + 8'd2 < dp_count && dp_count < dp_delta_up)) begin
    if(step_count == 0) begin
        step_count = step_count + 4'd1;
        mc_i = CMU_IO[9];
        s2_i = CASU_IO[11];
        addr_que[0] = 12;//Q_rr = phdelta_r * mc_r
        addr_que[1] = 10;
        addr_que[2] = 2;
        casu_addr_que[0] = 14;//s1_r = s1_r + mc_r
        casu_addr_que[1] = 8;
        casu_addr_que[2] = 8;
        addsub_selector = 1;
    end
    else if(step_count == 1) begin
        step_count = step_count + 4'd1;
        Q_rr = CMU_IO[2];
        s1_r = CASU_IO[8];
end
addr_que[0] = 13;// Q_ii = phdelta_i * mc_i
addr_que[1] = 11;
addr_que[2] = 4;
casu_addr_que[0] = 15;// s1_i = s1_i + mc_i
casu_addr_que[1] = 9;
casu_addr_que[2] = 9;
addsub_selector = 1;
end
else if(step_count ==2) begin
  step_count = step_count + 4'd1;
  Q_ii = CMU_IO[4];
  s1_i = CASU_IO[9];
  sign_indicator = ~sign_indicator;
  inv_addr = dp_count - dp_c_up;

  addr_que[0] = 12;// Q_ri = phdelta_r * mc_i
  addr_que[1] = 11;
  addr_que[2] = 3;
casu_addr_que[0] = 10;// Q_ii = sr_inc = Q_rr - Q_ii
casu_addr_que[1] = 11;
casu_addr_que[2] = 13;
  addsub_selector = 0;
end
else if(step_count ==3) begin
  step_count = step_count + 4'd1;
  Q_ri = CMU_IO[3];
  Q_ii = CASU_IO[13];// sr_inc

  addr_que[0] = 13;// Q_ir = phdelta_i * mc_r
  addr_que[1] = 10;
  addr_que[2] = 5;
casu_addr_que[0] = 16;// s2_r = s2_r + (-1)^(dp_count-dp_c_up - 8'd1)*mc_r, minus
  casu_addr_que[1] = 8;
casu_addr_que[2] = 10;
  addsub_selector = (sign_indicator == 1'b1)?1'b0:1'b1;
end
else if(step_count ==4) begin

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step_count = step_count + 4'd1;
Q_ir = CMU_IO[5];
s2_r = CASU_IO[10];

addr_que[0] = 6;//mc_r = Q_ii(sr_inc) * 1/n;
addr_que[1] = 8;
addr_que[2] = 8;
casu_addr_que[0] = 12;//Q_ir = si_inc = Q_ir + Q_ri
casu_addr_que[1] = 13;
casu_addr_que[2] = 12;
addsub_selector = 1;
end

else if (step_count == 5) begin
step_count = 4'd0;
dp_count = dp_count + 8'd1;
mc_r = CMU_IO[8];
Q_ir = CASU_IO[12];

addr_que[0] = 7;//mc_i = Q_ir(si_inc) * 1/n;
addr_que[1] = 8;
addr_que[2] = 9;
casu_addr_que[0] = 17;//s2_i = s2_i + (-1)^(dp_count - dp_c_up - 8'd1)*mc_i
casu_addr_que[1] = 9;
casu_addr_que[2] = 11;
addsub_selector = (sign_indicator == 1'b1)? 1'b0 : 1'b1;
end

else if (0 <= T_count && T_count < T_up) && (dp_count == dp_delta_up) begin
if (step_count == 0) begin
step_count = step_count + 4'd1;
mc_i = CMU_IO[9];
s2_i = CASU_IO[11];
//ref_clk = -ref_clk;

casu_addr_que[0] = 14;//si_r = s1_r + mc_r
casu_addr_que[1] = 8;
casu_addr_que[2] = 8;
end
addsub_selector = 1;
else if(step_count ==1) begin
    step_count = step_count + 4'd1;
s1_r = CASU_IO[8];
sign_indicator = ~sign_indicator;

    addr_que[0] = 18;//dp1_rr(Q_rr) = sc_r * s1_r
    addr_que[1] = 14;
    addr_que[2] = 2;
    casu_addr_que[0] = 15;//s1_i = s1_i + mc_i
    casu_addr_que[1] = 9;
    casu_addr_que[2] = 9;
    addsub_selector = 1;
end
else if(step_count ==2) begin
    step_count = step_count + 4'd1;
    Q_rr = CMU_IO[2];
s1_i = CASU_IO[9];

    addr_que[0] = 19;//dp1_ii(Q_ii) = sc_i * s1_i
    addr_que[1] = 16;
    addr_que[2] = 4;
    casu_addr_que[0] = 16;//s2_r = s2_r + (-1)^((dp_count - dp_c_up - 8'd1)*mc_r,
    casu_addr_que[1] = 8;
    casu_addr_que[2] = 10;
    addsub_selector = (sign_indicator == 1'b1)?1'b0 :1'b1;
end
else if(step_count ==3) begin
    step_count = step_count + 4'd1;
    Q_ii = CMU_IO[4];
s2_r = CASU_IO[10];//s2_r

    addr_que[0] = 18;//dp2_rr(Q_ri) = sc_r * s2_r
    addr_que[1] = 16;
    addr_que[2] = 3;
    casu_addr_que[0] = 17;//s2_i = s2_i + (-1)^((dp_count - dp_c_up - 8'd1)*mc_i
    casu_addr_que[1] = 9;

casu_addr_que[2] = 11;
addsub_selector = (sign_indicator == 1'b1)? 1'b0:1'b1;
end
else if(step_count == 4) begin
step_count = step_count + 4'd1;
Q_ri = CMU_IO[3];
s2_i = CASU_IO[11];

addr_que[0] = 19;// dp2_ii(Q_ri) = sc_i * s2_i
addr_que[1] = 17;
addr_que[2] = 5;
casu_addr_que[0] = 10;//dp1(s1_r) = dp1_rr(Q_rr) - dp1_ii(Q_i1)
casu_addr_que[1] = 11;
casu_addr_que[2] = 8;
addsub_selector = 0;
end
else if(step_count == 5) begin
step_count = step_count + 4'd1;
Q_ir = CMU_IO[5];
s1_r = CASU_IO[8];//dp_1

addr_que[0] = 23;// dp1_out = amp * dp1(s1_r)
addr_que[1] = 14;
addr_que[2] = 10;
casu_addr_que[0] = 12;//dp2(s2_r) = dp2_rr(Q_ri) - dp2_ii(Q_ir)
casu_addr_que[1] = 13;
casu_addr_que[2] = 10;
addsub_selector = 0;
end
else if(step_count == 6) begin
step_count = step_count + 4'd1;

dp1_out = CMU_IO[10];
//xor_out = (~CMU_IO[10][31]) ^ ID1;// for fp sign bit 1 is negative
s2_r = CASU_IO[10];//dp_2

addr_que[0] = 23;// dp2_out = amp * dp2
addr_que[1] = 16;
addr_que[2] = 11;
```markdown
end
else if (step_count == 7) begin
    step_count = step_count + 4'd1;
    dp2_out = CMU_IO[11];
end
else if (step_count == 8) begin
    step_count = 4'd0;
    dp_count = 8'd0;
    T_count = T_count + 7'd1;
end
else if ((T_count == T_up) && (dp_count == dp_delta_up)) begin
    if (step_count == 0) begin
        step_count = step_count + 4'd1;
        mc_i = CMU_IO[9];
        s2_i = CASU_IO[11];
        // ref_clk = ~ ref_clk;
        ///////////////
        phase_compare = piovertwo_fp;
        phase_i = PHY_i;
        addr_que[0] = 20; // wdeltaph_i = comp_amp_freq * Sigma_r
        addr_que[1] = 21;
        addr_que[2] = 1;
        casu_addr_que[0] = 14; // s1_r = s1_r + mc_r
        casu_addr_que[1] = 8;
        casu_addr_que[2] = 8;
        addsub_selector = 1;
    end
    else if (step_count == 1) begin
        step_count = step_count + 4'd1;
        wdeltaph_i = CMU_IO[1];
        s1_r = CASU_IO[8];
        sign_indicator = ~ sign_indicator;
        ///////////////
        piovertwo_cross = phase_reset;
        addr_que[0] = 18; // dp1_rr(Q_rr) = sc_r * s1_r
    end
else if (step_count == 2) begin
    step_count = step_count + 4'd1;
    dp1_out = CMU_IO[11];
    dp_count = 8'd0;
    T_count = T_count + 7'd1;
end
else if (step_count == 3) begin
    step_count = 4'd0;
    dp_count = 8'd0;
    T_count = T_count + 7'd1;
end
else if (step_count == 4) begin
    step_count = step_count + 4'd1;
    dp1_out = CMU_IO[11];
    dp_count = 8'd0;
    T_count = T_count + 7'd1;
end
else if (step_count == 5) begin
    step_count = 4'd0;
    dp_count = 8'd0;
    T_count = T_count + 7'd1;
end
else if (step_count == 6) begin
    step_count = step_count + 4'd1;
    dp1_out = CMU_IO[11];
    dp_count = 8'd0;
    T_count = T_count + 7'd1;
end
```

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addr_que[1] = 14;
addr_que[2] = 2;
casu_addr_que[0] = 15; // s1_i = s1_i + mc_i
casu_addr_que[1] = 9;
casu_addr_que[2] = 9;
addsub_selector = 1;
end
else if(step_count == 2) begin
  step_count = step_count + 4'd1;
  Q_rr = CMU_IO[2];
  s1_i = CASU_IO[9];
  phase_compare = twopi_fp;

daddr_HERE [0] = 19; // dp1_ii (Q_ii) = sc_i * s1_i
addr_que[1] = 15;
addr_que[2] = 4;
casu_addr_que[0] = 16; // s2_r = s2_r + (-1)^(dp_count - dp_c_up - 8'd1)*mc_r,
casu_addr_que[1] = 8;
casu_addr_que[2] = 10;
addsub_selector = (sign_indicator == 1'b1)? 1'b0 : 1'b1;
end
else if(step_count == 3) begin
  step_count = step_count + 4'd1;
  Q_ii = CMU_IO[4];
  s2_r = CASU_IO[10]; // s2_r

  addr_que[0] = 18; // dp2_rr(Q_ri) = sc_r * s2_r
  addr_que[1] = 16;
  addr_que[2] = 3;
  casu_addr_que[0] = 17; // s2_i = s2_i + (-1)^(dp_count - dp_c_up - 8'd1)*mc_i
  casu_addr_que[1] = 9;
casu_addr_que[2] = 11;
addsub_selector = (sign_indicator == 1'b1)? 1'b0 : 1'b1;
end
else if(step_count == 4) begin
  step_count = step_count + 4'd1;
  Q_ri = CMU_IO[3];

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s2_i = CASU_IO[11];

addr_que[0] = 19; // dp2_ii(Q_ir) = sc_i * s2_i
addr_que[1] = 17;
addr_que[2] = 5;
casu_addr_que[0] = 10; // dp1(s1_r) = dp1_rr(Q_rr) - dp1_ii(Q_i1)
casu_addr_que[1] = 11;
casu_addr_que[2] = 8;
addsub_selector = 0;
end
else if(step_count == 5) begin
    step_count = step_count + 4'd1;
    Q_ir = CMU_IO[5];
    s1_r = CASU_IO[8];

    addr_que[0] = 23; // dp1_out = amp * dp1(s1_r)
    addr_que[1] = 14;
    addr_que[2] = 10;
    casu_addr_que[0] = 12; // dp2(s2_r) = dp2_rr(Q_ri) - dp2_ii(Q_ir)
    casu_addr_que[1] = 13;
    casu_addr_que[2] = 10;
    addsub_selector = 0;
end
else if(step_count == 6) begin
    step_count = step_count + 4'd1;
    dp1_out = CMU_IO[10];
    // xor_out = (~CMU_IO[10][31]) ^ IO1;
    s2_r = CASU_IO[10];

    addr_que[0] = 23; // dp2_out = amp * dp_2
    addr_que[1] = 16;
    addr_que[2] = 11;
    casu_addr_que[0] = 0; // PHY_i = PHY_i + PHY_i_inc
    casu_addr_que[1] = 20;
    casu_addr_que[2] = 0;
    addsub_selector = 1;
end
else if(step_count == 7) begin
step_count = step_count + 4'd1;
dp2_out = CMU_IO[11];
PHY_1 = CASU_IO[0];

//addr_que[0] = 20;// wdeltaph_r = - comp_amp_freq * Sigma_1, if looking at pure
addr_que[0] = 28;// pole moving on a ellipse in complex plane, now comp_amp_freq
addr_que[1] = 22;
addr_que[2] = 0;

end
else if (step_count == 8) begin
  step_count = 4'd0;
dp_count = 8'd0;
T_count = 7'd0;
wdeltaph_r = CMU_IO[0]; // wdeltaph_r = - wdelta;

end

end

10'b0000000001-(10'b1111111111 - SI_1 ) );// FP to int module provide signed integer
assign DACs_data_out = SI_1 + 10'b1000000000; // FP to int module provide signed integer

10'b0000000001-(10'b1111111111 - SI_2 ) );
assign DACc_data_out = SI_2 + 10'b1000000000;

// below is the measurement of relative frequency between response signal and the injection, injection
//response is input into IO1 and digitized there, but AC couple and using a pot to offset right at
//1. create square version of injection, assign it to IO2
//2. create xor of inj_square and response
//3. counter count up when xor high, count down next cycle, use a flipflop to do it
//4. PWM output such counter signal

// create a sampling clk using DAC_clk_in
reg sample_clk = 0;
reg [6:0] sample_counter = 0;
parameter sample_period = 7'd63;
always @(posedge DAC_clk_in) begin // sample_clk ~ 591855.4 Hz
    if(sample_counter == sample_period) begin
        sample_counter <= 7'd0;
        sample_clk <= ~sample_clk;
    end
    else sample_counter <= sample_counter + 7'd1;
end

errorMessage

*************************/
/// now try a different scheme, divide down both the digitized internal signal and the IO1 by 2, and create two quadratures
reg dp1_div_inp = 0;
reg dp1_sign_old;
reg dp1_sign;
always @(posedge sample_clk) begin
    dp1_sign<= dp1_out [31];
    dp1_sign_old<= dp1_sign;
    if({dp1_sign_old , dp1_sign} == 2'b10) dp1_div_inp<= ~dp1_div_inp;
    // else if({dp1_sign_old , dp1_sign} == 2'b01) dp1_div_outp <= ~ dp1_div_outp;
end
reg IO1_div = 0; // IO1 is noisy on its rising edge
reg IO1_div_outp =0;
reg IO1_old;
reg IO1_older;
reg IO1_oldest;
reg edge_lock = 0;

reg [6:0] edge_lock_counter;
parameter lock_window = 6'b111111;
always @(posedge sample_clk) begin // edge lock is to deal with ring downs
    IO1_old <= IO1;
    //IO1_older <= IO1_old;
    //IO1_oldest <= IO1_older;
end

always @(posedge sample_clk) begin // generate two divided signals from IO1 input, 90 phase shifted
    if(edge_lock==0) begin
if([{IO1_old, IO1} == 2'b10) begin
    IO1_div <= ~ IO1_div;
    edge_lock <= 1;
    edge_lock_counter <= 0;
end
else if([{IO1_old, IO1} == 2'b01) begin
    IO1_div_outp <= ~ IO1_div_outp;
    edge_lock <= 1;
    edge_lock_counter <= 0;
end
else if (edge_lock == 1) begin
    if (edge_lock_counter == lock_window) edge_lock <= 0;
    else edge_lock_counter <= edge_lock_counter + 7'd1;
end
end

/*
 * typical FPGA gate delay is 1 ns (routing may add more delay to it), which means at master_clk
 * //now try four phasor scheme, since the divide method may result in a sign flip (a sudden 180 phase
 * //counter is triggered on rising edge of dp1_divide, stop at any edge of the IO1_divide, pause using
 * reg dp1_div_old;
 * reg IO1_div_old; // edge detection
 * reg IO1_div_outp_old;
 * always @(posedge sample_clk) begin // use cmu_clk or sample_clk?
 *     dp1_div_old <= dp1_div_inp;
 *     IO1_div_old <= IO1_div;
 *     IO1_div_outp_old <= IO1_div_outp;
 * end
 */

reg signed [20:0] phase_accum;
wire signed [20:0] phase_output;
assign phase_output = phase_accum + 21'b0_0011_1111_1111_1111_1111_1111_1111_1111_1111_1111_1111_1111_1111_1111_1111_1111_1111;
reg PHY_i_twopi_trigger;
reg signed[15:0] counter_inp;
reg signed[15:0] counter_outp;
reg counter_lock1;
reg counter_lock2;
reg counter_flag1;
reg counter_flag2;
reg piovertwo_cross_old;

//reg errflag;
always@(posedge cmu_clk) piovertwo_cross_old <= piovertwo_cross;

//if(T_count == 0 && dp_count ==0 && step_count <2 && phase_reset ==1)
always@(posedge cmu_clk) begin
  if({dp1_div_old, dp1_div_inp} == 2'b01 && PHY_i_twopi_trigger == 1) ADC_latch <= ADC_dataA;
end

always@(posedge cmu_clk) begin
  if({dp1_div_old, dp1_div_inp} == 2'b01) begin // counter unlock. If orbit reaches 2pi during last cycle , reset accumulative phase and latch data
    counter_lock1 <= 0;
    counter_lock2 <= 0;
    counter_inp <= 16'd0; // try signed register to allow zero crossing
    counter_outp <= 16'd0;
    counter_flag1 <= 0;
    counter_flag2 <= 0;
    if(PHY_i_twopi_trigger == 1) begin // reset accumulated phase
      phase_accum <= 19'd0;//18'b01_1111_1111_1111_1111;
      //phase_output <= phase_accum + 19'b000_1111_1111_1111_1111;//21'd
      //phase_output <= phase_accum[15:8]+8'b0111_1111;//8'b0111_1111;//
      PHY_i_twopi_trigger <= 0;//
      //ADC_latch <= ADC_dataA;
      //slip_latch <= xor_counter[15:7]; // one full cycle is 27683 which
  end
else begin


if({piovertwo_cross_old,piovertwo_cross}==2'b01) begin
    PHY_i_twopi_trigger <= 1;
    //if(T_count == 0 && dp_count==0 && step_count<2 && phase_reset ==1) PHY_i_twopi_trigger <= 1;
    if(counter_flag1 == 0) phase_accum <= phase_accum + counter_inp;
    else if(counter_flag1 ==1 && counter_flag2 == 0) phase_accum <= phase_accum + counter_outp;
    //this has an assumption that two counter flags won't both raise, it's true theoretically
    //else errflag<=1;
end
else begin
    if(counter_flag1 == 0) phase_accum <= phase_accum + counter_inp;
    else if(counter_flag1 ==1 && counter_flag2 == 0) phase_accum <= phase_accum + counter_outp;
    //else errflag<=1;
end
else if({dp1_div_old,dp1_div_inp} == 2'b11) begin
    if({piovertwo_cross_old,piovertwo_cross}==2'b01) PHY_i_twopi_trigger <= 1;
    //if(T_count == 0 && dp_count==0 && step_count<2 && phase_reset ==1)
    PHY_i_twopi_trigger <= 1;
    if(counter_lock1 == 0) counter_inp <= counter_inp + 16'd1;
    if(counter_lock2 == 0) counter_outp <= counter_outp + 16'd1;
    if(IO1_div_old != IO1_div) counter_lock1 <= 1;
    if(IO1_div_outp_old != IO1_div_outp) counter_lock2 <= 1;
end
else if ({dp1_div_old,dp1_div_inp} == 2'b10) begin
    counter_lock1 <= 0;
    counter_lock2 <= 0;
end
else if({piovertwo_cross_old,piovertwo_cross}==2'b01) PHY_i_twopi_trigger <= 1;
//if(T_count == 0 && dp_count==0 && step_count<2 && phase_reset ==1)
PHY_i_twopi_trigger <= 1;
//for 820 Hz divided to 410 Hz, its full half cycle gives 27683 counts, raise flag if counter falls out of [1/8, 7/8]*27683 = [3461, 24222]
if(counter_inp < 16'b0000_1101_1000_0101 || counter_inp > 16'b0101_1110_1001_1110) counter_flag1 <=1;
if(counter_outp < 16'b0000_1101_1000_0101 || counter_outp > 16'b0101_1110_1001_1110) counter_flag2 <=1;
end

else if {{dp1_div_old, dp1_div_inp} == 2'b00} begin
    if {{piovertwo_cross_old, piovertwo_cross} == 2'b01} PHY_i_twopi_trigger <= 1;
    // if(T_count == 0 && dp_count == 0 && step_count < 2 && phase_reset == 1)
    PHY_i_twopi_trigger <= 1;
    if(counter_lock1 == 0) counter_inp <= counter_inp - 16'd1;
    if(counter_lock2 == 0) counter_outp <= counter_outp - 16'd1;
    if(IO1_div_old != IO1_div) counter_lock1 <= 1;
    if(IO1_div_outp_old != IO1_div_outp) counter_lock2 <= 1;
end

end

reg send_start;
// always@ master_clk_locked send_start = 1;
reg dataTrain_clk = 0;
reg [10:0] dataTrain_counter = 0; // 9 is too much for the speed is close to 11.5 KBps
reg [7:0] TX_out_test = 0;
// dataTrain_clk is the clock for updating data sending out to the TX(IO2) on FPGA, which runs at
// so the dataTrain_clk should be slower than the throughput. Sample_clk is at 600kHz,
// 600k/126 = 4K Hz
always@ (posedge cmu_clk) begin
    dataTrain_counter <= dataTrain_counter + 1;
    if(dataTrain_counter == 0) dataTrain_clk <= ~ dataTrain_clk;
end

///////////////////////////////////////////////////
always@ (posedge cmu_clk) begin
    if {{dp1_div_old, dp1_div_inp} == 2'b01} send_start <= 1;
    else if {{dp1_div_old, dp1_div_inp} == 2'b00 && data_state == 4'b0101} send_start <= 0;
end

reg [3:0] data_state = 4'b0000;
// engineer a train of data, three data together, separated with idle clock cycle of dataTrain_clk,
// make sure 0 is not available for all data
always@ (posedge dataTrain_clk) begin
    case (data_state)
4'b0000: begin // initialization identifier 0
    if(send_start) begin
        data_state <= 4'b0001;
        TX_out_test<= 8'b00000000;
    end
end
4'b0001: begin // first data
    data_state <= 4'b0010;
    TX_out_test<= (ADC_latch>2)?ADC_latch:3;
end
4'b0010: begin // first update identifier 1
    data_state <= 4'b0011;
    TX_out_test<= 8'b00000001;
end
4'b0011: begin // second data
    data_state <= 4'b0100;
    TX_out_test<= (phase_output[18:11]>2)?phase_output[18:11]:3;
end
4'b0100: begin // second update identifier 2
    data_state <= 4'b0101;
    TX_out_test<= 8'b00000010;
end
4'b0101:begin
    data_state <= 4'b0000;
end
endcase
end

wire send_busy;
async_transmitter TxD_data_out(
    .clk(cmu_clk),
    .TxD_start(send_start),
    .TxD_data(TX_out_test),
    .TxD(IO2),
    .TxD_busy(send_busy)
);
wire receive_ready;
wire[7:0] receive_data;
wire receive_idle;
wire receive_end;
async_receiver(
    .clk(cmu_clk),
    .RxD(ref_clk_in),
    .RxD_data_ready(receive_ready),
    .RxD_data(receive_data),
    .RxD_idle(receive_idle),
    .RxD_endofpacket(receive_end)
);

endmodule

/////////// create a central multiplier to run calculation in series
// need a way to trigger/reset the CMP so data are latched at the right time

module CMU(fast_clk, cmu_cmd_clk, addr_0, addr_1, addr_2,
            Inp_0, Inp_1, Inp_2, Inp_3, Inp_4, Inp_5, Inp_6, Inp_7, Inp_8, Inp_9,
            Inp_10, Inp_11, Inp_12, Inp_13, Inp_14, Inp_15, Inp_16, Inp_17, Inp_18, Inp_19,
            Inp_20, Inp_21, Inp_22, Inp_23,
            IO_0, IO_1, IO_2, IO_3, IO_4, IO_5, IO_6, IO_7, IO_8, IO_9,
            IO_10, IO_11);  //, IO_12);
input fast_clk;
input cmu_cmd_clk;
input[5:0] addr_0;
input[5:0] addr_1;
input[5:0] addr_2;
input[31:0] Inp_0;
input[31:0] Inp_1;
input[31:0] Inp_2;
input[31:0] Inp_3;
input[31:0] Inp_4;
input [31:0] Inp_5;
inout [31:0] IO_0;
input [31:0] Inp_6;
inout [31:0] IO_1;
input [31:0] Inp_7;
inout [31:0] IO_2;
input [31:0] Inp_8;
inout [31:0] IO_3;
input [31:0] Inp_9;
inout [31:0] IO_4;
input [31:0] Inp_10;
inout [31:0] IO_5;
input [31:0] Inp_11;
inout [31:0] IO_6;
input [31:0] Inp_12;
inout [31:0] IO_7;
input [31:0] Inp_13;
inout [31:0] IO_8;
input [31:0] Inp_14;
inout [31:0] IO_9;
input [31:0] Inp_15;
inout [31:0] IO_10;
input [31:0] Inp_16;
inout [31:0] IO_11;
input [31:0] Inp_17;
inout [31:0] IO_12;
input [31:0] Inp_18;
inout [31:0] IO_13;
input [31:0] Inp_19;
inout [31:0] IO_14;
input [31:0] Inp_20;
inout [31:0] IO_15;
input [31:0] Inp_21;
inout [31:0] IO_16;
input [31:0] Inp_22;
inout [31:0] IO_17;
input [31:0] Inp_23;
inout [31:0] IO_18;
wire [31:0] mult_1;
inout [31:0] IO_19;
wire [31:0] mult_2;
inout [31:0] IO_20;
wire [31:0] product;
inout [31:0] IO_21;
reg clk_en_sig;

// step 1 multiplier input selection
// mult_1 can be a constant when addr_0 = 29, can change that at the end
assign mult_1 = (addr_0 == 0)? Inp_0:(addr_0 == 1)? Inp_1:(addr_0 == 2)? Inp_2:(addr_0 == 3)? Inp_3:(addr_0 == ...:(addr_0 == 28)?32 'b00111111000000101000111101011100 :32 'bzzzzzzzzzzzzzzzzzzzzzzzzzzzzzzzz ;//28 refer to 0.51 fp
assign mult_2 = (addr_1 == 0)? Inp_0:(addr_1 == 1)? Inp_1:(addr_1 == 2)? Inp_2:(addr_1 == 3)? Inp_3:(addr_1 == ...:(addr_1 == 21)? Inp_21: (addr_1 == 22)? Inp_22: (addr_1 == 23)? Inp_23 :32 'bzzzzzzzzzzzzzzzzzzzzzzzzzzzzzzzz ;

FPMULT FPMULT_inst_CMU (
   . clk_en ( clk_en_sig ),
   . clock ( fast_clk ),
   . dataa ( mult_1 ),
   . datab ( mult_2 ),
   . result ( product )
);

reg cmu_clk_reg_delay ;
reg [5:0] addr_out ;
always@(negedge fast_clk) begin
    cmu_clk_reg_delay <= cmu_cmd_clk ;
    addr_out <= addr_2 ;
    if( cmu_cmd_clk == 1) clk_en_sig <= 1;
    else clk_en_sig <= 0 ;
end
assign IO_0 = ( addr_out == 0 && ({ cmu_cmd_clk , cmu_clk_reg_delay }== 2' b00 ||{ cmu_cmd_clk , cmu_clk_reg_delay }== 2' b10) )? product : 32 ' bzzzzzzzzzzzzzzzzzzzzzzzzzzzzzzzz ;
assign IO_1 = ( addr_out == 1 && ({ cmu_cmd_clk , cmu_clk_reg_delay }== 2' b00 ||{ cmu_cmd_clk , cmu_clk_reg_delay }== 2' b10) )? product : 32 ' bzzzzzzzzzzzzzzzzzzzzzzzzzzzzzzzz ;
assign IO_2 = ( addr_out == 2 && ({ cmu_cmd_clk , cmu_clk_reg_delay }== 2' b00 ||{ cmu_cmd_clk , cmu_clk_reg_delay }== 2' b10) )? product : 32 ' bzzzzzzzzzzzzzzzzzzzzzzzzzzzzzzzz ;
assign IO_3 = ( addr_out == 3 && ({ cmu_cmd_clk , cmu_clk_reg_delay }== 2' b00 ||{ cmu_cmd_clk , cmu_clk_reg_delay }== 2' b10) )? product : 32 ' bzzzzzzzzzzzzzzzzzzzzzzzzzzzzzzzz ;
assign IO_4 = ( addr_out == 4 && ({ cmu_cmd_clk , cmu_clk_reg_delay }== 2' b00 ||{ cmu_cmd_clk , cmu_clk_reg_delay }== 2' b10) )? product : 32 ' bzzzzzzzzzzzzzzzzzzzzzzzzzzzzzzzz ;
assign IO_5 = ( addr_out == 5 && ({ cmu_cmd_clk , cmu_clk_reg_delay }== 2' b00 ||{ cmu_cmd_clk , cmu_clk_reg_delay }== 2' b10) )? product : 32 ' bzzzzzzzzzzzzzzzzzzzzzzzzzzzzzzzz ;
assign IO_6 = ( addr_out == 6 && ({ cmu_cmd_clk , cmu_clk_reg_delay }== 2' b00 ||{ cmu_cmd_clk , cmu_clk_reg_delay }== 2' b10) )? product : 32 ' bzzzzzzzzzzzzzzzzzzzzzzzzzzzzzzzz ;
assign IO_7 = ( addr_out == 7 && ({ cmu_cmd_clk , cmu_clk_reg_delay }== 2' b00 ||{ cmu_cmd_clk , cmu_clk_reg_delay }== 2' b10) )? product : 32 ' bzzzzzzzzzzzzzzzzzzzzzzzzzzzzzzzz ;
assign IO_8 = ( addr_out == 8 && ({ cmu_cmd_clk , cmu_clk_reg_delay }== 2' b00 ||{ cmu_cmd_clk , cmu_clk_reg_delay }== 2' b10) )? product : 32 ' bzzzzzzzzzzzzzzzzzzzzzzzzzzzzzzzz ;
// assign IO_17 = (addr_2 == 17 && ({ cmu_cmd_clk, cmu_clk_reg_delay})== 2'b00 || {cmu_cmd_clk, cmu_clk_reg_delay}== 2'b10) ? product : 32'b0;

endmodule

// center add/sub processing unit
module CASU (casu_fast_clk, casu_cmd_clk, as_selector, casu_addr_0, casu_addr_1, casu_addr_2, casu_Inp_0, casu_Inp_1, casu_Inp_2, casu_Inp_3, casu_Inp_4, casu_Inp_5, casu_Inp_6, casu_Inp_7, casu_Inp_8, casu_Inp_9, casu_Inp_10, casu_Inp_11, casu_Inp_12, casu_Inp_13, casu_Inp_14, casu_Inp_15, casu_Inp_16, casu_Inp_17, casu_Inp_18, casu_Inp_19, casu_Inp_20, casu_Inp_21, casu_Inp_22, casu_Inp_23, casu_Inp_24, casu_IO_0, casu_IO_1, casu_IO_2, casu_IO_3, casu_IO_4, casu_IO_5, casu_IO_6, casu_IO_7, casu_IO_8, casu_IO_9, casu_IO_10, casu_IO_11, casu_IO_12, casu_IO_13, casu_IO_14, casu_IO_15, casu_IO_16, casu_IO_17, casu_IO_18, casu_IO_19, casu_IO_20, casu_IO_21, casu_IO_22, casu_IO_23, casu_IO_24, casu_IO_25, casu_IO_26, casu_IO_27, casu_IO_28, casu_IO_29, casu_IO_30, casu_IO_31);
input casu_fast_clk;
input casu_cmd_clk;
input as_selector; // choose between add or sub
input [5:0] casu_addr_0;
inpu[t[5:0] casu_addr_1;
input [5:0] casu_addr_2;
inpu[t[31:0] casu_Inp_0;
input [31:0] casu_Inp_1;
input [31:0] casu_Inp_2;
input [31:0] casu_Inp_3;
inpu[t[31:0] casu_Inp_4;
input [31:0] casu_Inp_5;
inpu[t[31:0] casu_Inp_6;
input [31:0] casu_Inp_7;
inpu[t[31:0] casu_Inp_8;
input [31:0] casu_Inp_9;
inpu[t[31:0] casu_Inp_10;
inpu[t[31:0] casu_Inp_11;
inpu[t[31:0] casu_Inp_12;
inpu[t[31:0] casu_Inp_13;
inpu[t[31:0] casu_Inp_14;
inpu[t[31:0] casu_Inp_15;
inpu[t[31:0] casu_Inp_16;
inpu[t[31:0] casu_Inp_17;
inpu[t[31:0] casu_Inp_18;
input[31:0] casu_Inp_19;
input[31:0] casu_Inp_20;
input[31:0] casu_Inp_21;
input[31:0] casu_Inp_22;
input[31:0] casu_Inp_23;
input[31:0] casu_Inp_24;
// input[31:0] casu_Inp_25;
inout[31:0] casu_IO_0;
inout[31:0] casu_IO_1;
inout[31:0] casu_IO_2;
inout[31:0] casu_IO_3;
inout[31:0] casu_IO_4;
inout[31:0] casu_IO_5;
inout[31:0] casu_IO_6;
inout[31:0] casu_IO_7;
inout[31:0] casu_IO_8;
inout[31:0] casu_IO_9;
inout[31:0] casu_IO_10;
inout[31:0] casu_IO_11;
inout[31:0] casu_IO_12;
inout[31:0] casu_IO_13;
wire[31:0] casu_data_1;
wire[31:0] casu_data_2;
wire[31:0] casu_add_sub;
reg casu_clk_en_sig;

//step 1 multiplier input selection
assign casu_data_1 = (casu_addr_0 == 0)?casu_Inp_0:(casu_addr_0 == 1)?casu_Inp_1:(casu_addr_0 == 2)?casu_Inp_2:(...:(casu_addr_0 == 29)?32 'd0 :32 'bzzzzzzzzzzzzzzzzzzzzzzzzzzzzzzzz;//(casu_addr_0 == 25)?casu_Inp_25 :32 'bZ;
assign casu_data_2 = (casu_addr_1 == 0)?casu_Inp_0:(casu_addr_1 == 1)?casu_Inp_1:(casu_addr_1 == 2)?casu_Inp_2:(...:(casu_addr_1 == 24)?casu_Inp_24 :32 'bzzzzzzzzzzzzzzzzzzzzzzzzzzzzzzzz;//(casu_addr_1 == 25)?casu_Inp_25 :32 'bZ;

fp_32_add_sub fp_32_add_sub_inst(
    .add_sub (as_selector ), // high is add, otherwise sub
    .clk_en (casu_clk_en_sig ),
    .clock (casu_fast_clk ),
    .dataa (casu_data_1 ),
    .datab (casu_data_2 ),
    .result (casu_add_sub )
)
reg casu_clk_reg_delay;
reg[5:0] casu_addr_out;
always@ (negedge casu_fast_clk) begin
    casu_clk_reg_delay <= casu_cmd_clk;
    casu_addr_out <= casu_addr_2;
    if(casu_cmd_clk == 1) casu_clk_en_sig <= 1;
    else casu_clk_en_sig <= 0;
end

assign casu_IO_0 = (casu_addr_out == 0 && {casu_cmd_clk, casu_clk_reg_delay} == 2' b00 || {casu_cmd_clk, casu_clk_reg_delay} == 2' b10) ? casu_add_sub : 32' bzzzzzzzzzzzzzzzzzzzzzzzzzzzzzzzz;
assign casu_IO_1 = (casu_addr_out == 1 && {casu_cmd_clk, casu_clk_reg_delay} == 2' b00 || {casu_cmd_clk, casu_clk_reg_delay} == 2' b10) ? casu_add_sub : 32' bzzzzzzzzzzzzzzzzzzzzzzzzzzzzzzzz;
assign casu_IO_2 = (casu_addr_out == 2 && {casu_cmd_clk, casu_clk_reg_delay} == 2' b00 || {casu_cmd_clk, casu_clk_reg_delay} == 2' b10) ? casu_add_sub : 32' bzzzzzzzzzzzzzzzzzzzzzzzzzzzzzzzz;
assign casu_IO_3 = (casu_addr_out == 3 && {casu_cmd_clk, casu_clk_reg_delay} == 2' b00 || {casu_cmd_clk, casu_clk_reg_delay} == 2' b10) ? casu_add_sub : 32' bzzzzzzzzzzzzzzzzzzzzzzzzzzzzzzzz;
assign casu_IO_4 = (casu_addr_out == 4 && {casu_cmd_clk, casu_clk_reg_delay} == 2' b00 || {casu_cmd_clk, casu_clk_reg_delay} == 2' b10) ? casu_add_sub : 32' bzzzzzzzzzzzzzzzzzzzzzzzzzzzzzzzz;
assign casu_IO_5 = (casu_addr_out == 5 && {casu_cmd_clk, casu_clk_reg_delay} == 2' b00 || {casu_cmd_clk, casu_clk_reg_delay} == 2' b10) ? casu_add_sub : 32' bzzzzzzzzzzzzzzzzzzzzzzzzzzzzzzzz;
assign casu_IO_6 = (casu_addr_out == 6 && {casu_cmd_clk, casu_clk_reg_delay} == 2' b00 || {casu_cmd_clk, casu_clk_reg_delay} == 2' b10) ? casu_add_sub : 32' bzzzzzzzzzzzzzzzzzzzzzzzzzzzzzzzz;
assign casu_IO_7 = (casu_addr_out == 7 && {casu_cmd_clk, casu_clk_reg_delay} == 2' b00 || {casu_cmd_clk, casu_clk_reg_delay} == 2' b10) ? casu_add_sub : 32' bzzzzzzzzzzzzzzzzzzzzzzzzzzzzzzzz;
assign casu_IO_8 = (casu_addr_out == 8 && {casu_cmd_clk, casu_clk_reg_delay} == 2' b00 || {casu_cmd_clk, casu_clk_reg_delay} == 2' b10) ? casu_add_sub : 32' bzzzzzzzzzzzzzzzzzzzzzzzzzzzzzzzz;
assign casu_IO_9 = (casu_addr_out == 9 && {casu_cmd_clk, casu_clk_reg_delay} == 2' b00 || {casu_cmd_clk, casu_clk_reg_delay} == 2' b10) ? casu_add_sub : 32' bzzzzzzzzzzzzzzzzzzzzzzzzzzzzzzzz;
assign casu_IO_10 = (casu_addr_out == 10 && {casu_cmd_clk, casu_clk_reg_delay} == 2' b00 || {casu_cmd_clk, casu_clk_reg_delay} == 2' b10) ? casu_add_sub : 32' bzzzzzzzzzzzzzzzzzzzzzzzzzzzzzzzz;
assign casu_IO_11 = (casu_addr_out == 11 && {casu_cmd_clk, casu_clk_reg_delay} == 2' b00 || {casu_cmd_clk, casu_clk_reg_delay} == 2' b10) ? casu_add_sub : 32' bzzzzzzzzzzzzzzzzzzzzzzzzzzzzzzzz;
assign casu_IO_12 = (casu_addr_out == 12 && {casu_cmd_clk, casu_clk_reg_delay} == 2' b00 || {casu_cmd_clk, casu_clk_reg_delay} == 2' b10) ? casu_add_sub : 32' bzzzzzzzzzzzzzzzzzzzzzzzzzzzzzzzz;
assign casu_IO_13 = (casu_addr_out == 13 && {casu_cmd_clk, casu_clk_reg_delay} == 2' b00 || {casu_cmd_clk, casu_clk_reg_delay} == 2' b10) ? casu_add_sub : 32' bzzzzzzzzzzzzzzzzzzzzzzzzzzzzzzzz;

//input [7:0] casu_Inp_26;
//inout [31:0] casu_IO_12;
//assign casu_IO_12 = (casu_addr_2 == 11 && {casu_cmd_clk, casu_clk_reg_delay} == 2' b00 || {casu_cmd_clk, casu_clk_reg_delay} == 2' b10) ? casu_data_1 : casu_Inp_26;
//assign casu_IO_12 = casu_data_1;
endmodule

module CFPTDSI (conv_clk, conv_addr, conv_inp0, conv_inp1, sint_out);
input [31:0] conv_inp0;
inout [31:0] conv_inp1;
input conv_clk;
input conv_addr;
output [9:0] sint_out;

wire [31:0] conv_data;
assign conv_data = (conv_addr == 0)? conv_inp0:(conv_addr == 1)? conv_inp1:32'bz 

FPtoUI FPtoUI_waveform_real ( 
    .clock (conv_clk ),
    .dataa (conv_data ),
    .result ( sint_out )
);

endmodule

module async_transmitter ( 
    input clk,
    input TxD_start,
    input [7:0] TxD_data,
    output TxD,
    output TxD_busy
);

// Assert TxD_start for (at least) one clock cycle to start transmission of TxD_data 
// TxD_data is latched so that it doesn’t have to stay valid while it is being sent

//parameter ClkFrequency = 25000000; // 25MHz
parameter ClkFrequency = 22727250;
parameter Baud = 115200;

generate
    if(ClkFrequency<Baud*8 && (ClkFrequency % Baud!=0)) ASSERTION_ERROR PARAMETER_OUT_OF_RANGE
endgenerate

'ifdef SIMULATION

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wire BitTick = 1'b1; // output one bit per clock cycle

'else
wire BitTick;
BaudTickGen #(ClkFrequency, Baud) tickgen(.clk(clk), .enable(TxD_busy), .tick(BitTick));
'endif

reg [3:0] TxD_state = 0;
wire TxD_ready = (TxD_state ==0);
assign TxD_busy = ~TxD_ready;

reg [7:0] TxD_shift = 0;
always @(posedge clk)
begin
if(TxD_ready & TxD_start)
    TxD_shift <= TxD_data;
else
    if(TxD_state[3] & BitTick)
        TxD_shift <= (TxD_shift >> 1);
case(TxD_state)
4'b0000: if(TxD_start) TxD_state <= 4'b0100;
4'b0100: if(BitTick) TxD_state <= 4'b1000; // start bit
4'b1000: if(BitTick) TxD_state <= 4'b1001; // bit 0
4'b1010: if(BitTick) TxD_state <= 4'b1010; // bit 1
4'b1010: if(BitTick) TxD_state <= 4'b1011; // bit 2
4'b1100: if(BitTick) TxD_state <= 4'b1100; // bit 3
4'b1100: if(BitTick) TxD_state <= 4'b1101; // bit 4
4'b1110: if(BitTick) TxD_state <= 4'b1110; // bit 5
4'b1110: if(BitTick) TxD_state <= 4'b1111; // bit 6
4'b0010: if(BitTick) TxD_state <= 4'b0010; // bit 7
4'b0000: if(BitTick) TxD_state <= 4'b0011; // stop1
4'b0000: if(BitTick) TxD_state <= 4'b0000; // stop2
default: if(BitTick) TxD_state <= 4'b0000;
endcase
end

assign TxD = (TxD_state<4) | (TxD_state[3] & TxD_shift[0]); // put together the start, data and stop bits
endmodule
module async_receiver(
    input clk,
    input RxD,
    output reg RxD_data_ready = 0,
    output reg [7:0] RxD_data = 0, // data received, valid only (for one clock cycle) when RxD_data_ready is asserted
    output RxD_idle, // asserted when no data has been received for a while
    output reg RxD_endofpacket = 0 // asserted for one clock cycle when a packet has been detected
);

parameter ClkFrequency = 22727250;
parameter Baud = 115200;

parameter Oversampling = 8; // needs to be a power of 2
// we oversample the RxD line at a fixed rate to capture each RxD data bit at the "right" time
// 8 times oversampling by default, use 16 for higher quality reception

generate
    if( ClkFrequency < Baud * Oversampling ) ASSERTION_ERROR PARAMETER_OUT_OF_RANGE(" Frequency too low for current Baud rate and oversampling ");
    if( Oversampling < 8 || (( Oversampling & ( Oversampling -1))!=0)) ASSERTION_ERROR PARAMETER_OUT_OF_RANGE(" Invalid oversampling value ");
endgenerate

reg [3:0] RxD_state = 0;

ifndef SIMULATION
    wire RxD_bit = RxD;
    wire sampleNow = 1'b1; // receive one bit per clock cycle

else
    wire OversamplingTick;
    BaudTickGen #(ClkFrequency, Baud, Oversampling) tickgen(.clk(clk), .enable(1'b1), .tick(OversamplingTick))

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// synchronize RxD to our clk domain
reg [1:0] RxD_sync = 2'b11;
always @(posedge clk) if(OversamplingTick) RxD_sync <= {RxD_sync[0], RxD};

// and filter it
reg [1:0] Filter_cnt = 2'b11;
reg RxD_bit = 1'b1;
always @(posedge clk) if(OversamplingTick)
begin
  if(RxD_sync[1]==1'b1 && Filter_cnt != 2'b11) Filter_cnt <= Filter_cnt + 1'd1;
  else if(RxD_sync[1]==1'b0 && Filter_cnt != 2'b00) Filter_cnt <= Filter_cnt - 1'd1;
  if(Filter_cnt == 2'b11) RxD_bit <= 1'b1;
  else if(Filter_cnt == 2'b00) RxD_bit <= 1'b0;
end

// and decide when is the good time to sample the RxD line
function integer log2(input integer v); begin log2 = 0; while(v>>log2) log2 = log2 + 1; end endfunction
localparam l2o = log2(Oversampling);
reg [l2o-2:0] OversamplingCnt = 0;
always @(posedge clk) if(OversamplingTick) OversamplingCnt <= (RxD_state == 0) ? 1'd0 : OversamplingCnt + 1'd1;
wire sampleNow = OversamplingTick && (OversamplingCnt == Oversampling/2 - 1);

// now we can accumulate the RxD bits in a shift-register
always @(posedge clk)
case(RxD_state)
  4'b0000: if(~RxD_bit) RxD_state <= 'ifdef SIMULATION 4'b1000 'else 4'b0001 'endif;
  // start bit found?
  4'b0001: if(sampleNow) RxD_state <= 4'b1000; // sync start bit to sampleNow
  4'b1000: if(sampleNow) RxD_state <= 4'b1001; // bit 0
  4'b1001: if(sampleNow) RxD_state <= 4'b1010; // bit 1
  4'b1010: if(sampleNow) RxD_state <= 4'b1011; // bit 2
  // rest of 8 states for 2-bit sampled data
endcase

always @(posedge clk)
if(sampleNow && RxD_state[3]) RxD_data <= { RxD_bit , RxD_data[7:1]};

// reg RxD_data_error = 0;
always @(posedge clk)
begin
RxD_data_ready <= (sampleNow && RxD_state==4'b0010 && RxD_bit); // make sure a stop bit is received
// RxD_data_error <= (sampleNow && RxD_state==4'b0010 && ~ RxD_bit); // error if a stop bit is not received
end

ifdef SIMULATION
assign RxD_idle = 0;
else
reg[l2o+1:0] GapCnt = 0;
always @(posedge clk) if (RxD_state!=0) GapCnt<=0; else if(OversamplingTick & ~ GapCnt[ log2(Oversampling) +1]) GapCnt <= GapCnt + 1'h1;
assign RxD_idle = GapCnt[l2o+1];
always @(posedge clk) RxD_endofpacket <= OversamplingTick & ~GapCnt[l2o+1] & & GapCnt[l2o:0];
endif

endmodule
module ASSERTION_ERROR();
endmodule

//////////////////////////////////////////////////////////////
module BaudTickGen(
    input clk , enable ,
    output tick // generate a tick at the specified baud rate * oversampling
);
//parameter ClkFrequency = 25000000;
parameter ClkFrequency = 22727250;
parameter Baud = 115200;
parameter Oversampling = 1;

function integer log2(input integer v); begin log2 = 0; while (v >> log2) log2 = log2 + 1; end endfunction

localparam AccWidth = log2(ClkFrequency / Baud) + 8; // +/- 2% max timing error over a byte
reg [AccWidth-1:0] Acc = 0;
localparam ShiftLimiter = log2(Baud * Oversampling >> (31-AccWidth)); // this makes sure Inc calculation doesn't overflow
localparam Inc = ((Baud * Oversampling << (AccWidth - ShiftLimiter)) + (ClkFrequency >> (ShiftLimiter + 1)))
always @(posedge clk) if(enable) Acc <= Acc[AccWidth-1:0] + Inc[AccWidth:0]; else Acc <= Inc[AccWidth:0];
assign tick = Acc[AccWidth];
endmodule

///////////////////////////////////////////////////////////////////////////////////////////////
Appendix B

FPGA Non-Linear Polynomial Calculator

The non-linear module is a polynomial calculator based on a multiplier module in FPGA, whose goal is to calculate

\[ V_{out} = a_1 V_{in} + a_2 V_{in}^2 + a_3 V_{in}^3 + a_4 V_{in}^4 + a_5 V_{in}^5. \]  

(B.1)

The strategy of efficiently computing this polynomial is the same as calculating the series expansion of the exponential function in the two frequency synthesizer. The analog input of the board takes \( V_{in} \) and convert it to an 8-bit integer. This analogy input is a Flashy acquisition circuit based on an ADC8200 chip. The range and offset can be easily controlled by an external signal.

This 8-bit integer is then converted into a 32-bit floating point number and is then sent to a soft processor to serially compute the terms in the sum. The process of calculation is identical to that in the two-frequency synthesizer. All the coefficients are defined by registers in Verilog and can be easily changed. Then \( V_{out} \) is first computed into a 32-bit floating point and then converted to a 10-bit integer before being sent out through the analog output port. This board is capable to produce voltage range from 0 to 1V and it can be resolved to 1mV.

There is a custom built external circuit which offset and re-scale the AC signal \( V_{in} \) so that it can fit in the analog input of the FPGA acquisition board.

Here I present the Verilog code for this polynomial calculator.

// This is a processor which calculate a polynomial of the input, using serially running multipliers
// This code is created based on the center multiplier and center adder soft processor for the pol

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// Need to write a new instructional queue

module FX2_DDS_Q(
    FX2_CLK, FX2_FD, FX2_SLRD, FX2_SLWR, FX2_flags,
    FX2_PA_2, FX2_PA_3, FX2_PA_4, FX2_PA_5, FX2_PA_6, FX2_PA_7,
    ADC_dataA, ADC_DACCTRL, clk_ADC_out, ADC_dataB,
    DAC_clk_in, DAC_clk_out, DACs_data_out, DACc_data_out, ref_clk_in, IO1, IO2
);

input FX2_CLK;
input [7:0] FX2_FD;
inout [7:0] FX2_flags;
input FX2_SLRD, FX2_SLWR;

//output FX2_PA_0;
//output FX2_PA_1;
output FX2_PA_2;
output FX2_PA_3;
output FX2_PA_4;
output FX2_PA_5;
output FX2_PA_6;
input FX2_PA_7;

input DAC_clk_in;
input ref_clk_in;
output DAC_clk_out;
output [9:0] DACs_data_out;
output [9:0] DACc_data_out;

input [7:0] ADC_dataA; // first ADC channel
input [7:0] ADC_dataB; // second ADC channel
output ADC_DACCTRL;

wire clk_acq = DAC_clk_in; // used to be clk_ADC in flashy.v;
//wire clk_acq = FX2_CLK;
assign clk_ADC_out = DAC_clk_in; // used to be clk_ADC in flashy.v;
output clk_ADC_out;

assign DAC_clk_out = ~DAC_clk_in;
output IO2;
input IO1;

// Rename "FX2" ports into "FIFO" ports, to give them more meaningful names
// FX2 USB signals are active low, take care of them now
// Note: You probably don’t need to change anything in this section

// FX2 outputs
wire FIFO_CLK = FX2_CLK;
wire FIFO2_empty = ~ FX2_flags [0];
wire FIFO2_data_available = ~ FIFO2_empty;
wire FIFO3_empty = ~ FX2_flags [1];
wire FIFO3_data_available = ~ FIFO3_empty;
wire FIFO4_full = ~ FX2_flags [2];
wire FIFO4_ready_to_accept_data = ~ FIFO4_full;
wire FIFO5_full = ~ FX2_PA_7;
wire FIFO5_ready_to_accept_data = ~ FIFO5_full;
// assign FX2_PA_0 = 1'b1;
// assign FX2_PA_1 = 1'b1;
assign FX2_PA_3 = 1'b1;

// FX2 inputs
wire FIFO_RD, FIFO_WR, FIFO_PKTEND, FIFO_DATAIN_OE, FIFO_DATAOUT_OE;
wire FX2_SLRD = ~ FIFO_RD;
wire FX2_SLWR = ~ FIFO_WR;
assign FX2_PA_2 = ~ FIFO_DATAIN_OE;
assign FX2_PA_6 = ~ FIFO_PKTEND;
wire [1:0] FIFO_FIFOADR;
assign {FX2_PA_5, FX2_PA_4} = FIFO_FIFOADR;

// FX2 bidirectional data bus
wire [7:0] FIFO_DATAIN = FX2_FD;
wire [7:0] FIFO_DATAOUT;
assign FX2_FD = FIFO_DATAOUT_OE ? FIFO_DATAOUT : 8'hZZ;

// So now everything is in positive logic
// FIFO_RD, FIFO_WR, FIFO_DATAIN, FIFO_DATAOUT, FIFO_DATAIN_OE, FIFO_DATAOUT_OE, FIFO_PKTEND,
// FIFO2_empty, FIFO2_data_available
// FIFO3_empty, FIFO3_data_available
// FIFO4_full, FIFO4_ready_to_accept_data
// FIFO5_full, FIFO5_ready_to_accept_data

assign FIFO_FIFOADR = 2'b00;  // select FIFO2
assign FIFO_RD = 1'b1;  // always read
assign FIFO_WR = 1'b0;  // never write
assign FIFO_DATADOUT = 8'b00;  // never write, so this value is not used
assign FIFO_DATAIN_OE = 1'b1;  // always read data
assign FIFO_DATADOUT_OE = 1'b0;  // never output data
assign FIFO_PKTEND = 1'b0;

reg [3:0] timeout;
always @ (posedge FIFO_CLK) if(FIFO2_data_available) timeout <= 4'hF; else if (!timeout) timeout <= timeout - 4'h1;
wire timeout_complete = (timeout == 4'h1) && !FIFO2_data_available;

// create 32-bits value from FIFO
reg [31:0] FIFO_value;
always @ (posedge FIFO_CLK) if(FIFO2_data_available) FIFO_value[31:24] <= FIFO_DATAIN;
always @ (posedge FIFO_CLK) if(FIFO2_data_available) FIFO_value[23:16] <= FIFO_value[31:24];
always @ (posedge FIFO_CLK) if(FIFO2_data_available) FIFO_value[15:8] <= FIFO_value[23:16];
always @ (posedge FIFO_CLK) if(FIFO2_data_available) FIFO_value[7:0] <= FIFO_value[15:8];

// generate a master clock signal, internal VCO is at 75.757 MHZ and the allowed maximum clk for running multipliers is 227.271 MHZ, which is three times the VCO frequency
// master_clk period is 4.40003344 ns, remember the standard/conventional transient time for FPGA to transfer data is ~1 ns, depending on the structure and wires on board
wire master_clk_reset;
wire master_clk_locked;
wire master_clk;
PLL_master PLL_master_inst (
  .areset ( master_clk_reset ),
  .inclk0 ( DAC_clk_in ),
  .c0 ( master_clk ),
  .locked ( master_clk_locked )
);
//create a cmu_cmd_clk and casu_cmd_clk, and have casu_clk delayed by one master_clk,
//not very clear why it's necessary but when calculating exponential function as a serial expansion,
//probably no need for CMU and CASU, since the multiplier and adder module, and change input value
//minimum latency for multiplier is about 6 clock cycle and 7 for adder/subtractor

reg [3:0] cmu_clk_counter = 0;
reg cmu_clk;
reg casu_clk;
parameter cmu_period = 4'b1001; // actual period is (9+1)*master_period = 44ns
always @(posedge master_clk) begin
    if (master_clk_locked == 1) begin
        if (cmu_clk_counter == cmu_period - 4'b0010) begin
            cmu_clk <= 0;
            cmu_clk_counter <= cmu_clk_counter + 4'b0001;
        end
        else if (cmu_clk_counter == cmu_period) begin
            cmu_clk_counter <= 0;
            cmu_clk <= 1'b1;
        end
        else if (cmu_clk_counter == 0) begin
            cmu_clk_counter <= cmu_clk_counter + 4'b0001;
            casu_clk <= 1;
        end
        else if (cmu_clk_counter == cmu_period - 1) begin
            cmu_clk_counter <= cmu_clk_counter + 4'b0001;
            casu_clk <= 0;
        end
        else
            cmu_clk_counter <= cmu_clk_counter + 4'b0001;
    end
end

////////////////////////////////////////

// Minimal __ADC__ : DAC-control logic
// Required with some revisions of Flashy that don't work if the DAC is not set to a minimum value
// See the Flashy documentation for more details

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wire clk = FX2_CLK;
wire [7:0] DAC_regdata[3:0];
assign DAC_regdata[0] = 8'b11111110; // minimum values (corresponds to maximum vertical pos + max range)
assign DAC_regdata[1] = 8'b01111111;
assign DAC_regdata[2] = 8'b11111110;
assign DAC_regdata[3] = 8'b01111111;
parameter psdac = 8;
reg [psdac+9:0] DAC_cnt; always @(posedge cmu_clk) DAC_cnt <= DAC_cnt + 18'd1;
wire [15:0] DAC_data = {5'b11111, DAC_cnt[psdac+9:psdac+8], 1'b1, DAC_regdata[DAC_cnt[psdac+9:psdac+8]]};
reg ADC_DACCTRL; always @(posedge cmu_clk) ADC_DACCTRL <= &DAC_cnt[psdac+7:psdac+5] & (!DAC_cnt[psdac+4:psdac+1]);

//I change the DAC_regdata s above to (254,127) and now the ADC reads (0,1.5V) as (-0,250) that's probably good enough for this purpose

// in order to read ADC_dataA, the above minimum settings are necessary, V-range ~400 mV, V-pos --50 mV
// noticed some non-linearity when padding to a 8 bits out of 32 bits

// still not quite sure how ADC_data read from input voltage;

// the logical for this FPGA processor is simple, read input from ADC, convert to Floating point, 
//ax+bx^2+cx^3+dx^4, with preset parameters a,b,c,d as four floating poin

// following parameters and initialization is based on master_clk 227.271 MHZ, which yields a period of 4.4 ns
//
//reg[9:0] SI_1;
//reg[9:0] SI_2;
//reg fptosi_addr;
//wire[9:0] si_read;
//CFPTOSI cfptosi_inst(.conv_clk(master_clk), .conv_addr(fptosi_addr), .conv_inp0(dp1_out), .conv_
// convert integer read from ADC to fp to run calculation of polynomial
// probably want some scaling before and after to increase resolution

// polynomial processor from here

// first convert ADC_dataA into a signed integer and remove offset set by V-pos (I think it’s 8'b00000000; try it)
reg convert_clk_en;
reg signed[8:0] sig_int_offset; // this read ADC_data which is unsigned integer
reg signed[8:0] sig_int;
always@(posedge cmu_clk) begin
  sig_int_offset <= {1'b0,ADC_dataA};
  sig_int <= sig_int_offset - 9'b010000000; // remove offset
  convert_clk_en <= 1'b1;
end
wire[31:0] sig_fp;

INT2FP INT2FP_inst( // Conversion works on signed integer
  .clk_en (convert_clk_en&master_clk_locked),
  .clock (master_clk ),
  .dataa (sig_int ),
  .result (sig_fp)
);

// predefined parameters
wire[31:0] ConstA;
wire[31:0] ConstB;
wire[31:0] ConstC;
wire[31:0] ConstD;
wire[31:0] ConstE;
assign ConstA = 32'b01000000000000000000000000000000; // fp 2
assign ConstB = 32'd0; // 32'b00111111100000000000000000000000; //1SE-4
assign ConstC = 32'd0; // 32'b00011101110000000000000000000000; //1SE-4
assign ConstD = 32'd0; // 32'b00011101110000000000000000000000; //1SE-4
assign ConstE = 32'd0; // 32'b00011101110000000000000000000000; //1SE-4
assign ConstC = 32'b00110110110010010101001110011100 ; //6E-6 //32'b00110110100001100001111011111101

assign ConstD = 32'd0 ; //32'b001101000001011000011100101000100 ; //13E-8 //32'b001101000101101110111111111111111
assign ConstE = 32'd0 ; //32'b001101110101101111100110111111111111111111111111111

// for square: 1.5E-3: 32'b001111010110001001001101110100110 ; //2E-3: 32'b00111101100000110001001001101111

// for cube: //20E-6:b00110111011000110001001001101111
// 15E-6: b001101110001001010011100110111100 ; //12E-6: b001101110001001011001101111001100
// 10E-6: b001101110001001011001101111001100
// 8E-6: b001101110001001011001101111001100
// 6E-6: b001101110001001011001101111001100
// 4E-6: b001101110001001011001101111001100
// 2E-6: b001101110001001011001101111001100

// for fourth: 13E-8: b00110100000011011000100100110100
// 20E-8: b0011010000101010110111101010101

// for fifth://
// 20E-10: b001100001001011111111111111111111111111111111111111
// 16E-10: b001100001011111111111111111111111111111111111111111
// 12E-10: b001100001011111111111111111111111111111111111111111
// 8E-10: b001100001011111111111111111111111111111111111111111
// 6E-10: b001100001011111111111111111111111111111111111111111
// 4E-10: b001100001011111111111111111111111111111111111111111
// 2E-10: b001100001011111111111111111111111111111111111111111
// 1E-10: b001100001011111111111111111111111111111111111111111

////////////////////////////////////////////////////////
//setup multiplier
reg mult_clk_en ;
reg[31:0] mult_data_1;
reg[31:0] mult_data_2;
wire[31:0] product;
FPMULT FPMULT_inst_CMU (  
  .clk_en ( mult_clk_en ),  
  .clock ( master_clk ),  
  .dataa ( mult_data_1 ),  
  .datab ( mult_data_2 ),  
  .result ( product )  
);

// set up adder/subtractor  
reg casu_clk_en;  
reg [31:0] casu_data_1;  
reg [31:0] casu_data_2;  
reg as_selector; // always add  
wire [31:0] casu_add_sub;  
fp_32_add_sub fp_32_add_sub_inst (  
  .add_sub ( as_selector ), // high is add, otherwise sub  
  .clk_en ( casu_clk_en ),  
  .clock ( master_clk ),  
  .dataa ( casu_data_1 ),  
  .datab ( casu_data_2 ),  
  .result ( casu_add_sub )  
);

///////////
// Instructional queue for the polynomial
// Calculating the polynomial takes 6 steps (can't think about a faster way right now)
// Create a state machine, passing data to processor using blocking assignment
reg [3:0] que_state = 4'd0;

always @(posedge cmu_clk) begin  
  case (que_state)  
    4'b0000: begin  
      que_state <= 4'b0001;  
      // sig_int <= ADC_dataA;  
    end  
    4'b0001: que_state <= 4'b0010;  
    4'b0010: que_state <= 4'b0011;  
  endcase  
end
4'b0011: que_state <= 4'b0100;
4'b0100: que_state <= 4'b0101;
4'b0101: que_state <= 4'b0110;
4'b0110: que_state <= 4'b0111;
4'b0111: que_state <= 4'b1000;
4'b1000: que_state <= 4'b1001;
4'b1001: que_state <= 4'b0000;
default: que_state <= 4'b0000;
endcase
end

reg[31:0] data_out_fp;
always @(posedge cmu_clk) begin
  case(que_state)
    4'b0000: begin // x(E)
      mult_data_1 = sig_fp;
      mult_data_2 = ConstE;
      mult_clk_en = 1;
    end
    4'b0001: begin // D+x(E)
      casu_data_1 = product;
      casu_data_2 = ConstD;
      mult_clk_en = 0;
      casu_clk_en = 1;
      as_selector = 1;
    end
    4'b0010: begin // x(D+x(E))
      mult_data_1 = sig_fp;
      mult_data_2 = casu_add_sub;
      mult_clk_en = 1;
      casu_clk_en = 0;
    end
    4'b0011: begin // C+x(D+x(E))
      casu_data_1 = product;
      casu_data_2 = ConstC;
      mult_clk_en = 0;
      casu_clk_en = 1;
      as_selector = 1;
  endcase
end
4'b0100: begin // x(C+x(D+xE))
    mult_data_1 = sig_fp;
    mult_data_2 = casu_add_sub;
    mult_clk_en = 1;
    casu_clk_en = 0;
end

4'b0101: begin // B+x(C+x(D+xE))
    casu_data_1 = product;
    casu_data_2 = ConstB;
    mult_clk_en = 0;
    casu_clk_en = 1;
    as_selector = 1;
end

4'b0110: begin // x(B+x(C+x(D+xE)))
    mult_data_1 = sig_fp;
    mult_data_2 = casu_add_sub;
    mult_clk_en = 1;
    casu_clk_en = 0;
end

4'b0111: begin // A+x(B+x(C+x(D+xE)))
    casu_data_1 = product;
    casu_data_2 = ConstA;
    mult_clk_en = 0;
    casu_clk_en = 1;
    as_selector = 1;
end

4'b1000: begin // x(A+x(B+x(C+x(D+xE))))
    mult_data_1 = sig_fp;
    mult_data_2 = casu_add_sub;
    mult_clk_en = 1;
    casu_clk_en = 0;
end

4'b1001: begin
    data_out_fp = product;
end
endcase
end
/************************************/
//conver the fp into int for DAC output
//reg conver_clk_en =1;
wire[9:0] data_out_int;

FP2INT FP2INTinst(
    .clock (master_clk ),
    .dataa (data_out_fp ),
    .result (data_out_int),
    .nan ( nan_sig ),
    .overflow ( overflow_sig ),
    .underflow ( underflow_sig )
);

//// make sure to AC couple the output of this FPGA so no need to worry about offset problems

assign DACs_data_out = data_out_int + 10'l'b0000000000; // FP should have no offset, lift it for DAC
assign DACc_data_out = sig_int_offset;
assign IO2 = convert_clk_en;
//assign DACs_data_out = data_out_int + 10'b1000000000; // FP to int module provide signed integer
//assign DACc_data_out = SI_2 + 10'b1000000000;

endmodule
Appendix C
ECDL Design

In this Appendix section we present the design and mechanical details of the external cavity diode laser (ECDL). This compact ECDL device include three main layers. Figure C.1 shows the CAD drawing and mechanical drawing of the top layer. The unit of measurements in the mechanical drawing is inch. Five through holes with countersinks for attaching the top plate to the middle plate by using five 1/4 – 20 stainless steel screws. The rectangular hole and the other bigger one with an extended arm were first cut by plasma cutter, then were finished on a CNC milling machine. High precision machine tools were used to achieve 25µm precision. The complex hollow structures in the middle layer and bottom layer were also manufactured following that process. CNC milling machine was operated under G-code command lines, which were created by Mach3. Details of middle layer and bottom layer are shown in figure C.2 and figure C.3. In figure C.2, the laser diode mount, lens tube and diffraction grating are also presented, which form a Littrow configuration.
Figure C.1. ECDL top layer design

Figure C.2. ECDL middle layer design
Figure C.3. ECDL bottom layer design
Bibliography


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