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The primary goal of this research was to develop a better understanding of the large nonlinear viscoelastic deformation of electroactive elastomers subject to high electric fields. This was achieved through experimental characterizations of dielectric material as well as through the development of analytical and finite element hyperelastic and viscoelastic models. Polyacrylate and silicone (polydimethylsiloxane) dielectric materials were considered in this research. The constitutive equations for the large nonlinear hyperelastic deformation of a dielectric elastomer subjected to an electric field was developed using the Mooney-Rivlin and Ogden material models. The analytical model was validated for an annular actuator configuration. It is known that the polyacrylate VHB 4910 dielectric elastomer has significant viscoelastic properties. Hence, to model large nonlinear viscoelastic deformations, Christensen’s viscoelastic theory was used. The analytical viscoelastic material model was validated with experimental results for specimens undergoing uniaxial deformation. Furthermore, non-axisymmetric actuator configurations with cavities in the shapes of an ellipse and rectangle were investigated using a finite element model.

An analytical model for dielectric elastomers was developed using the theory of large elastic deformations known as hyperelasticity. From these results a better understanding of how a dielectric elastomer annulus deforms was obtained by understanding the effects of varying Maxwell pressures, pre-stretches, and inner radial pressures, and internal stresses. Through the use of the analytical model, it was found that the radial and circumferential stresses transitions from tensile to compressive stresses at a "critical" Maxwell pressure. The significance of this is the determination of the Maxwell pressure that will cause the elastomer to buckle or wrinkle. Also, the model not only showed that prestretching yields the benefit of greater strain, but that there exists an "optimum" pre-stretch range that would yield a larger percent change in radius for a given incremental increase in the effective Maxwell pressure. In addition, a fixed-free annulus with an inner radial pressure, which may simulate a portion of a simple fluid pump, was modeled. It was found that there exists an operating range of the inner radial
pressure and effective Maxwell pressures for the device to be physically as well as mathematically viable.

A nonlinear finite deformation viscoelastic model for dielectric elastomer membranes was developed using Christensen’s viscoelastic model in the stretch regime $1.5<\lambda<3$. This model is applicable for small and large nonlinear deformations. In this research, the Mooney-Rivlin elastic material model was utilized. Uniaxial creep tests were conducted to determine the material constants of an exponentially decaying relaxation modulus. The analytical model validated with experimental results from a constant load uniaxial tensile test. The degree of agreement was a function of the relaxation modulus, $g(t)$. A relaxation modulus was found and correlated very well with experimental data within one time constant of the relaxation modulus. The same relaxation modulus could not predict viscoelastic deformations beyond this time constant.

A single relaxation function that characterizes the material over a large time $[0,t]$ domain can be obtained by choosing a more complicated mathematical form of the relaxation modulus.

Furthermore, the general-purpose finite element (FE) software ABAQUS was used to develop an FE model of non-axisymmetric dielectric elastomer actuators undergoing large nonlinear viscoelastic deformations. Rectangular framed actuators with rectangular and elliptical cavities at the center were investigated. Hyperelastic and viscoelastic material properties were determined from uniaxial constant load tensile and creep test data, respectively. The FE model was validated using experimental data from actuators with uniaxial, in-plane axisymmetric, and in-plane non-axisymmetric deformations. The FE hyperelastic (time-independent) results for uniaxial, in-plane axisymmetric, and in-plane non-axisymmetric deformations correlated well with experimental data. The time-dependent FE results of activated (non-zero electric field) uniaxial and in-plane axisymmetric (annular configuration) deformations correlated well with experimental data. The FE and experimental correlations degraded for time-dependent deformations of non-axisymmetric geometries. The cause for this may be due to the limitations of FE models that use uniaxial test data to define the material.
properties. Utilization of test results from in-plane tensile tests and in-plane shear tests may improve the FE model to better simulate in-plane deformations.

Finally, the FE model was used to model activated strains of cylindrical silicone dielectric elastomer actuators as these materials exhibit negligible viscoelastic behaviors. The material properties were once again defined using uniaxial tensile test data. The activated strains of the cylindrical tubes were subjected to uniaxial tensile tests. As expected, the FE model correlated well uniaxial test data.
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ACKNOWLEDGEMENTS

I would like to thank my parents Won Il Yang and Chung Ok Yang who have taught me the importance of education and perseverance. Both of you have been my role models. I love and respect you both.

Thanks Allen for keeping the porch light on for me on the many evenings I returned home late.

To my Nico, Nick, Nicholas, Dae-Ho, Buddy, and son: “Hey Buddy, Mom did it! I don’t have to look at that ‘egg’ any more. Only you and I know what this means! I made it because of you. Thanks Buddy. I luv ya!”

And of course, I’d like to thank my best friend Eva who gave me emotional support when I was feeling down and celebrated with me on the many occasions when there was a reason to do so. You rock! To all my friends and acquaintances here in State College who have made this part of my life’s journey truly memorable.

My studies at Penn State have been an extremely rewarding experience that would not have been possible without the support of Drs. Frecker, Mockensturm, Sommer, and Snyder. Thank you!
Chapter 1

Introduction

1.1 Background

Electroactive polymers (EAPs) are materials that have strong electro-mechanical coupling. The earliest known experimental research of EAPs, in early 1880s, is attributed to Roentgen [1] and Julius [2] who measured the deformation of rubber upon application of an electric field. In 1925, Eguchi [3] devised a method to solidify a polymer melt solution in the presence of a high electric field. This resulted in a solid polymer whose surfaces were oppositely charged and remained poled even when it was removed from the electric field. Significant progress in this field was made in the late 1950s and 1960s by Fukada and his co-workers [4, 5]. They discovered that when the polymer polyvinylidene-fluoride (PVDF) was mechanically strained, a voltage differential across the thickness of the material was induced (this is the piezoelectric effect). During this time, there was a large interest in developing ultrasonic transducers, and the newly discovered PVDF received much interest due to its piezoelectric response even at microwave frequencies.

More recently, electroactive polymers discovered in the past ten years are capable of undergoing large strains as high as 300% when subjected to high electric fields (>100 MV/m). The drawbacks to these EAPs are their low specific elastic energy density which is in the range of 3.4 J/m$^3$ [6]. These new EAP materials are classified into two categories: ionic and electronic. Ionic EAPs require submersion in an electrolytic solution to achieve deformation through ion diffusion. Electronic EAPs need not be submerged in a solution and are commonly operated exposed to air. There are several deformation mechanisms for electronic EAPs. They include electrostrictive, electrostatic, piezoelectric, and ferroelectric effects [7].
Most progress on electronic EAP materials has occurred in the last ten years where commercial elastomeric materials have been discovered to exhibit high dielectric strain capabilities. Commercially available EAPs that have emerged include CF19-2186 silicone (NuSil, Carpinteria, California), HS3 silicone (Dow Corning, Midland, Michigan), VHB polyacrylate elastomer (3M Company, St. Paul, Minnesota), and polyurethane. A popular material used to develop dielectric elastomer actuators is the VHB polyacrylate elastomer that is known to have minimal electrostrictive effects [8]. The VHB material is available in sheet form and comes in various thicknesses. This is the material primarily considered in this research.

1.2 Deformation Mechanism

Dielectric elastomer actuators realize deformation through electrostatic stresses induced by an electric field applied across the thickness of the material. The operation of a dielectric elastomer actuator can be depicted as a compliant parallel-plate capacitor as shown in Figure 1. An isotropic and homogenous dielectric elastomer membrane with uniform thickness is placed between compliant electrodes (major surfaces). The difference between the outer and inner radii is much greater than the thickness of the elastomer so that the fringing fields at the inner and outer edges can be neglected. A voltage differential, \( V_s \), across the thickness of the material creates electrostatic forces that are perpendicular to the major surfaces and induce compressive stresses. It is assumed that the electrostatic force remain perpendicular to the major surfaces called Maxwell stresses. Maxwell compressive stress is defined as

\[
T^M = \frac{1}{2} \varepsilon_r \varepsilon_o E^2
\]

where \( \varepsilon_r \) is the relative dielectric constant of the DEA, \( \varepsilon_o \) is the permittivity of vacuum (8.850E-12 F/m), and \( E \) is the applied electric field.
It has been determined that prestretching the dielectric elastomer increases the elastomer’s ability to withstand a higher voltage before failure occurs (electrical breakdown voltage) [6, 9]. It should be noted that electrical failure is caused by the physical burning of the elastomer due to an electrical arc that is created across the positive and negative electrode surfaces. The reason for the improvement in the electrical breakdown voltage due to prestretch is unknown. However Kofod offers a qualitative explanation as follows: An unstretched elastomer consists of a network of cross-linked polymer chains that is highly unorganized. Upon prestretch, the polymer chains stretches and the network of chains align into a highly organized grid. The density of the grid increases with increasing prestretch resulting in a grid that decreases the tendency for the positive and negative charges to cause an arc across the thickness of the material. Hence, it is common for dielectric elastomers to be utilized in a prestretched configuration [9].

1.3 Actuator Configurations

The strains due to Maxwell stress can be recruited for actuation purposes using various actuator configurations as shown in Figure 2. They are grouped into either planar or three-dimensional configurations. Researchers at SRI International (Menlo Park, CA) have developed several notable actuators. Consider their six-legged robot, MERbot, which has evolved from using a planar actuator in the shape of a bow tie [10] to utilizing...
an actuator in the shape of a cylinder. The MERbot was developed by Pei et al. [11] and mimics a walking bug (reference Figure 3). The MERbot’s source of actuation is obtained from a dielectric film that is prestretched and patterned with compliant electrodes. The uniaxially prestretched film is then wrapped around a compressed spring to yield a cylindrical actuator with four active regions as shown in Figure 4. Axial prestretch is obtained when the spring is released from compression. Each spring roll actuator is 2.3 cm in diameter and 9 cm in length and weighs 29 g. The maximum lateral force achieved by each spring roll is 1.68 N at 7.7 kV. Articulation of each spring roll actuator is obtained through selected activation of a region, i.e. the actuator can be made to curl by having one active and three passive sections.

Planar actuators can also be actuated to deform out-of-plane. This is evidenced by the diaphragm pump developed by Pope et al. [12]. The diaphragm pump consists of a circular prestretched dielectric elastomer affixed to the frame of a cylindrical chamber. With a bias chamber pressure, cyclic activation of the prestretched membrane results in cyclic out of plane diaphragm response. SRI International has also investigated diaphragms for speaker applications [13]. In addition, a micro-diaphragm pump was developed by Xu et al. [14] using an unstretched electrostrictive material poly(vinylidene fluoride-trifluoroethylene).

Another novel actuator is the inchworm developed by Jung et al. [15]. The inchworm actuator provides linear motion and two rotational degrees-of-freedom to obtain a worm-like crawling motion. The inchworm actuator (reference Figure 5a) consists of eight individual modules (reference Figure 5b) stacked and assembled into a single inchworm unit. Each module contains 12 planar electroactive diaphragm units (reference Figure 5c, six on each side), and upon actuation deforms into a convex shape and provides the strain that will be recruited to achieve motion. The inchworm actuator is 20 mm in diameter by 45 mm in length and weighs 4.7 g. It provides a force in the 13-16 mN range when 2500 V is applied at a frequency range of 0.4-100 Hz. The interesting aspect of this design is the utilization of out-of-plane deformation along with the modular/stacked assembly to fabricate the mobile actuator.
Figure 2. Dielectric elastomer actuator configuration [16].

Figure 3. 2-DOF MERbot actuator [11].
Figure 4. Fabrication process of the 2-DOF spring-roll actuator [11].
Figure 5. Inchworm actuator assembly [15]: a) assembled inchworm (b) actuator module (c) actuator unit.
1.4 Performance Comparison Between EAPs and Natural Muscles

A qualitative comparison of the specific strain-actuation pressure (normalized per mass density) between a variety of smart materials and natural muscles was given by Pelrine et al. [17] (reference Figure 6). In this figure, electrostrictive and magnetic actuators exhibit high strains (1-50%) but low specific actuation pressure (less than 0.1 kPa-m³/kg). Piezoelectric and magnetostrictive actuators on the other hand have low strain response (less than 1%) but high specific actuation pressure (8-30 kPa-m³/kg). The EAPs exhibit both large strains and specific strain-actuation pressures near those of the natural muscles.

Full and Meijer [18] compared the VHB 4910 EAP material to natural muscles using the work-loop technique which is a method commonly used to assess the power output of natural muscles. A schematic of the work-loop technique is shown in Figure 7. The work-loop technique consists of measuring the force response of a muscle undergoing cyclic strain and muscle stimulation at distinct phases \((t_1-t_2-t_3-t_1)\) of the strain cycle (reference Figure 7A). The net work is calculated by taking the difference between the work done during the shortening (curve \(t_1-t_2-t_3\)) and lengthening (curve \(t_3-t_1\)) phases of strain (reference Figure 7B). The shape of the work-loop (reference Figure 7C) generated by the muscle is a function of the muscle actuation frequency. High actuation frequency results in a rectangular loop whereas low actuation frequency results in an elliptically shaped loop. This is because the power and work output of natural muscles are dependent upon the frequency at which the muscles contract and lengthen. Net work and net power output are presented as a function of frequency for natural muscle and the VHB 4910 elastomer in Figure 8 and Figure 9. The work capacity and power output of the VHB 4910 is approximately 3 J/kg and 35 W/kg, respectively. These values are comparable to the values obtained from the muscles of crabs, bees, rats and lizards.

The modern EAP materials have been called artificial muscle [19]. They are an attractive materials for the fabrication of small biomimetic actuators. They could also extend to larger devices such as artificial limbs and human organs but much research is
needed to increase the actuation force and energy density of the materials for these types of applications.

Many challenges in the EAP research community exist to ultimately develop actuators to perform work equivalent to that of human skeletal muscles. In this research, part of the challenge is approached through experimental and analytical venues to gain further understanding of the behavior of EAPs. The scope of this research is presented in the next section.

1.5 Research Summary

The primary goal of this research is to develop a better understanding of the large nonlinear deformation of electroactive elastomers subject to high electric fields. This is achieved through experimental characterization of the material as well as through the development of analytical hyperelastic and viscoelastic models. The hyperelastic and viscoelastic models are presented in Chapters 2 and 3, respectively. Chapter 2 develops the constitutive equations for the large nonlinear hyperelastic deformation of a dielectric elastomer subjected to an electric field. The Mooney-Rivlin and Ogden material models are utilized in the model. The analytical model is validated for an annular actuator configuration. It is known that the VHB 4910 dielectric elastomer has significant viscoelastic properties. Hence, Chapter 4 develops a model for the large nonlinear viscoelastic deformation using Christensen’s viscoelastic theory. The analytical model is validated with experimental results for specimens undergoing uniaxial deformation. Non-axisymmetric actuator configurations with cavities in the shapes of an ellipse and rectangle are investigated using a finite element model are presented in Chapter 4. The deformation of the major and minor chords of the cavities are modeled using the general-purpose finite element software ABAQUS and validated with experimental results. The knowledge gained from the above work is implemented to develop a finite element model for a tubular actuator that is presented in Chapter 5.
Figure 6. Strain versus actuation pressure/density for various materials [17].
Figure 7. Schematic representations of the work-loop technique. (A) Muscle stress and strain patterns for three cycles. (B) From left to right, the work done by the muscle on its environment, the work done on the muscle and the resulting net work output. (C) From left to right: different shapes of work loops that are exhibited at high, med, and low frequencies. [20].
Figure 8. Mass-specific muscle work per cycle as a function of frequency for vertebrate and invertebrate muscles (open circles). Data were obtained using the work-loop method [20]. Preliminary results show that EAPs (filled square) fall within the range of values for natural muscle [21].
Figure 9. Mass-specific power output comparison between natural muscles and EAPs [22].
Chapter 2

Analytical Hyperelastic Model

2.1 Phenomenological Models

Mechanical response of an ideally elastic material subject to a given loading can be described by a set of constitutive equations based on a stored internal energy potential function (or strain-energy density function) as first presented by Green [23, 24]. Such material behavior is known as Green-elastic or more recently has become to be known as hyperelastic as called by Truesdell and Noll [25]. Hyperelastic theories are based on phenomenological methods rather than on statistical mechanics methods. In the phenomenological methods, the macroscopic deformation is analyzed without regards to the chemical composition of the material. In the statistical method, interactions between individual atoms are analyzed to explain macroscopic material deformations.

For a hyperelastic deformation of a material that is isotropic in its undeformed state, the stored internal energy potential function (strain-energy per unit undeformed volume), $W$, is assumed to be solely a function of elastic stretch and neglects thermal effects

$$W = W(\lambda_r, \lambda_\theta, \lambda_z)$$  \hspace{1cm} Eq. 2

where $\lambda_r$, $\lambda_\theta$ and $\lambda_z$ are the principal stretch ratios in the cylindrical coordinate system [26, 27]. The strain-energy density function can also be expressed as functions of the stretch invariants, $I_1$, $I_2$, and $I_3$

$$I_1 = \lambda_r^2 + \lambda_\theta^2 + \lambda_z^2$$
$$I_2 = \lambda_r^2 \lambda_\theta^2 + \lambda_\theta^2 \lambda_z^2 + \lambda_z^2 \lambda_r^2$$
$$I_3 = \lambda_r^2 \lambda_\theta^2 \lambda_z^2$$  \hspace{1cm} Eq. 3
since they remain constant for orthogonal transformations. For incompressible materials, \( I_3=1 \) and the strain energy density function simplifies to

\[
W = \tilde{W}(I_1, I_2). \tag{4}
\]

An equivalent representation of the above strain-energy density function is

\[
W = \tilde{W}(I_1 - 3, I_2 - 3) \tag{5}
\]

since the value of the function is zero in the undeformed configuration where \( I_1 = I_2 = 3 \).

The general form of the strain energy density function of Eq. 5 can be expanded using an infinite Taylor series

\[
W(I_1 - 3, I_2 - 3) = \sum_{(r,s)=0}^{\infty} C_{rs} (I_1 - 3)^r (I_2 - 3)^s, \tag{6}
\]

where \( C_{rs} \) are material properties. The first order approximation of Eq. 6 results in the well known Mooney material model [28]

\[
W = C_{10}(I_1 - 3) + C_{01}(I_2 - 3) \tag{7}
\]

developed under the assumptions that Hooke’s law is applicable for simple shear and that the material is incompressible and isotropic in the undeformed configuration.

Different material models are born depending upon the mathematical form of the stored internal energy potential function. Rivlin’s [26] material model (or commonly referred to as the Mooney-Rivlin material model) develops on the model presented by Mooney. Rivlin assumes that the strain invariants consist of even powered stretch ratio terms to ensure that \( W \) always remains positive

\[
W = C_{10}(\lambda_r^2 + \lambda_o^2 + \lambda_z^2 - 3) + C_{01}(\lambda_r^2 + \lambda_o^2 + \lambda_z^2 - 3). \tag{8}
\]

Rivlin published a series of papers developing the general theory of large deformations and hyperelasticity using the strain energy density function. In the first [29] of his series of eight papers, Rivlin presents the stress-strain relationships for large deformation using the strain energy density function. The stress-strain relationships and the boundary
conditions are used to obtain the equations of motion in terms of the strain energy density function. His second paper [30] focused on comparing experimental results to theoretical predictions for materials in shear, uniaxial extension, compressions, and torsion. In his third and fourth papers [26, 27] large deformation of axisymmetric geometries (solid and hollow cylinders) were investigated. The axisymmetric geometry simplifies the analysis since the analytical solution of the deformation of material points along a radial line determines the solution for the entire surface. Rivlin’s eighth paper is the most applicable to this research [31]. In this paper the stress-stretch equations are presented in cylindrical coordinates for a thin axisymmetric annular material in terms of the strain energy density function.

Another material model was proposed by Ogden [32]. Ogden’s model placed less mathematical restriction on the stretch ratios by assuming that the stretch ratios were always positive and that this requirement need not be enforced mathematically. He therefore allowed the powers of the stretch ratio to take on any values

\[ W = \sum_{n} \mu_n \left( \lambda_n^{\alpha_n} + \lambda_n^{\beta_n} + \lambda_n^{\gamma_n} - 3 \right) \quad n = 1, 2, 3, ... \]  

\text{Eq. 9}

where \( \mu_n \) and \( \alpha_n \) are material constants determined experimentally (reference Section 2.7).

For small deformation, the strain-energy density represents the area under a stress-stretch curve

\[ W = \int \sigma(\lambda) d\lambda. \]  

\text{Eq. 10}

The constitutive equation (stress-stretch relationship) can be obtained by differentiating Eq. 10 with respect to the principal stretch ratios and solving the resulting equation for the stress \( \sigma \). Analogously, the constitutive equation for large deformation is obtained by using the principle of virtual work. The constitutive equation for the Mooney-Rivlin model is

\[ T_{ii} = 2 \left( \lambda_i^2 \frac{\partial W}{\partial I_1} - \frac{1}{\lambda_i^2} \frac{\partial W}{\partial I_2} \right) + p, \quad i = r, \theta, z \]  

\text{Eq. 11}
where \( T_{ii} \) is the principal Cauchy stress (true stress), \( \partial W / \partial I_1 \) and \( \partial W / \partial I_2 \) are the material properties that represent the changes in the stored internal energy potentials with respect to the two stretch invariants, and \( p \) is the unknown hydrostatic pressure. The two ratios, \( \partial W / \partial I_1 \) and \( \partial W / \partial I_2 \), are considered to be constants as per the Mooney material [4]. Similarly, the constitutive equation for the Ogden model is obtained by differentiating Eq. 9 with respect to the principal stretch ratios. The two term (\( n=2 \) in Eq. 9) constitutive equation for the Ogden model is

\[
T_{ii} = \frac{\partial W}{\partial \lambda_r} \lambda_1^{\alpha_1} + \frac{\partial W}{\partial \lambda_\theta} \lambda_2^{\alpha_2} + p, \quad i = r, \theta, z
\]

Eq. 12

where \( \frac{\partial W}{\partial \lambda_r} \) and \( \frac{\partial W}{\partial \lambda_\theta} \) are the material properties that represent the change in the stored internal energy potential with respect to the two principal stretch ratios.

The material properties can be determined by using a numerical method to fit the experimental stress-stretch data to the theoretical stress-stretch relations of Eq. 11 and Eq. 12 (discussed in detail in Section 2.7). Extensive tests to determine the constants for the Mooney-Rivlin model from different types of strain (uniaxial, biaxial, shear and torsion) were conducted by Rivlin and Saunders [33]. Their results showed that \( \partial W / \partial I_1 \) remained constant as a function of strain, but \( \partial W / \partial I_2 \) exhibited dependence with strain. This, however, contradicted the assumption that Hooke’s law was applicable for shear deformations in the hyperelastic theory as assumed by Mooney. Obata, Kawabata, and Kawai [34] investigated this discrepancy. Their results showed that for small principal stretch difference \( \lambda_\theta - \lambda_r \) (i.e., small strain), the degree of dependence of \( \partial W / \partial I_2 \) on strain was large. However the degree of dependency became smaller as the strain difference increased (i.e. large strain). The significance of these experimental results is that material constants determined from uniaxial test cannot properly predict deformations for different forms of strain (i.e. biaxial, shear, and torsion) unless the strains in these non-uniaxial deformations are large. In such cases where deformations are small, material constants need to be determined from biaxial test results. In this research, the annular dielectric membranes are prestretched to values greater than 120%. This is
sufficiently large and the material constants obtained from uniaxial tensile tests can be used to properly predict the equi-biaxial deformation response of the annular membrane (results are presented in Section 2.76).

2.2 Hyperelastic Material Model for Dielectric Elastomers

In this chapter, a hyperelastic constitutive model is used to describe the large nonlinear deformation of a dielectric elastomer subjected to an electric field across the thickness of the elastomer. The geometry investigated is of an annular ring that is prestretched radially (reference Figure 10) and then constrained at the outer peripheral radius.

![Figure 10. Schematic of a thin annulus with unconstrained boundaries in radial extension.](image)

A novel application using this configuration would be a stacked assembly of annular rings to achieve peristaltic pumping action or mobility through the sequential activation of each ring. Recently, Ingram and Hong [35] designed a mobile device with an annular cross section. Their mobile device is an elongated toroid structure called a Whole Skin Locomotion (WSL) and is shown Figure 11. The WSL gains motion by sequential radial compression at one end of the structure causing the inner surface at the opposite end to creep out and transform into the outer surface. The reverse action occurs at the contracting end where the outer surface rolls inward and becomes the inner surface.
Their design is currently in the proof-of-concept phase. The authors believe the EAP material has properties that is suitable for the WSL application. The WSL can take advantage of the elasticity of the electroactive polymers (EAPs) to maneuver through small openings and use the actuated strains to propel the WSL into motion. Potential applications include robotic endoscopes and mobile devices to traverse rough terrains.

Figure 11. Schematic of the Whole Skin Locomotion (a) bio-inspired from the motility mechanism of a monopodial amoeba [35].

The formulation of the hyperelastic constitutive model in this chapter is arranged as follows: The governing equations are separated into strain-displacement (kinematic) relations, equilibrium equations, and boundary conditions are presented in Sections 2.3-2.5. These relations are used to formulate the model (Section 2.6) and are solved numerically using Matlab’s boundary value problem solver BVP4C (Section 2.8). The elastic material properties are determined from constant load creep tests (Section 2.7) and validations of theoretical results to experimental data are discussed in Section 2.9.
2.3 Kinematic Equations

Consider a planar, homogeneous, and incompressible membrane in the reference configuration with an inner radius of $R_A$, outer radius of $R_B$, and thickness $H$, where $H \ll R_B$ (reference Figure 12).

Let a particle $Q$ on the annular membrane at time $t=0$ be defined by its reference coordinates $Q(R, \Theta, Z)$. At time $t>0$, the membrane is radially prestretched and fixed at the perimeter of the outer radius. Particle $Q$ has now moved to a new location on the current configuration and is defined as $q$. Particle $q$ is now defined in terms of its spatial coordinates $q(r, \theta, z)$. Then, the Lagrangian deformation gradient, $F$, is given by

$$F = q \nabla \hat{V}$$  \hspace{1cm} \text{Eq. 13}

where $q$ is the spatial coordinate of the point $q$

$$q = q(r, \theta, z)$$  \hspace{1cm} \text{Eq. 14}

$\nabla \hat{V}$ is the gradient operator, $r = r \hat{e}_r$, $q = \theta \hat{e}_\theta$, $z = z \hat{e}_z$ and $\hat{e}_r$, $\hat{e}_\theta$ and $\hat{e}_z$ are the basis unit vectors. Axisymmetric deformation implies the material is free of shear. Therefore the deformation gradient has no off-diagonal elements and is given by
For convenience, let

$$\lambda_r(R) = \frac{dr}{dR}, \quad \lambda_\theta(R) = \frac{r}{R}.$$  

Eq. 16

where $\lambda_r(R)$ and $\lambda_\theta(R)$ are the radial and circumferential principal stretch ratios, respectively. For isochoric motions, the Jacobian $(\det F)$ of the deformation gradient, $F$, is equal to one. Hence, the transverse stretch ratio, $\lambda_z(R)$, can be written in terms of the radial and circumferential stretch ratios

$$\lambda_z(R) = \frac{1}{\lambda_\theta(R) \lambda_r(R)}.$$  

Eq. 17

2.4 Equations of Motion

The equations of motion for a body are given by Cauchy’s first law of motion

$$\nabla \bullet (F^{-1})^T \tilde{T} F^{-1} + \rho b = \rho \frac{dv}{dt},$$  

Eq. 18

where $\tilde{T}$ is the 2nd Piola-Kirchoff stress tensor, $b$ is the body force, $v$ is the velocity, and $\rho$ is the density of the material. In the absence of body forces, the equations of motion for a thin membrane consist, in the static case, only of the non-trivial radial component

$$\frac{d\tilde{T}_{rr}}{dR} + \frac{1}{R} (\tilde{T}_{rr} - \tilde{T}_{\theta\theta}) = 0,$$  

Eq. 19

where $\tilde{T}_{rr}$ and $\tilde{T}_{\theta\theta}$ are the resultant radial and circumferential stress components (force per unit deformed length), respectively of the 2nd Piola-Kirchoff stress tensor. They are defined as
where $T_{\theta\theta}$ is the principal radial and circumferential Cauchy stresses, respectively, and $H$ is the undeformed thickness of the membrane.

2.5 Boundary Conditions

An annular membrane with an outer radius of $R_B$ in the reference configuration is prestretched and constrained at the outer perimeter in the current configuration to a radius $R_b$. Under this prescribed displacement, the kinematic boundary condition at the outer radius, $R_B$, for all time, $t$, is

$$\frac{\lambda_{\theta}}{\lambda_r} = \frac{r_b}{R_B} \quad \text{Eq. 21}$$

where $r_b$ is the deformed radius. Since the inner radius edge is free of traction, the boundary condition at $R_A$ is

$$\left[T_{rr}\right]_{R_A} = 0 \quad \text{Eq. 22}$$

The traction boundary conditions on both surfaces $\lambda_z$ of the annular membrane is due to the non-electrostrictive Maxwell stress effects

$$T^M = \frac{1}{2} \varepsilon_r \varepsilon_0 E \otimes E \quad \text{Eq. 23}$$

where $\varepsilon_r$ is the relative dielectric constant of the material, $\varepsilon_0$ is the permittivity of vacuum (8.850E-12 F/m), and $E$ is the electric field with units of V/m.
2.6 Model Formulation

For convenience, the strain energy density functions of the Mooney-Rivlin and two term Ogden models (Eq. 8 and Eq. 9, respectively) are

\[ W = C_1 (I_1 - 3) + C_2 (I_2 - 3) \]  Mooney-Rivlin \hspace{3cm} Eq. 24
\[ W = \frac{\mu_1}{\alpha_1} (I_1 - 3) + \frac{\mu_2}{\alpha_2} (I_2 - 3) \]  Ogden \hspace{3cm} Eq. 25

and the constitutive equations (Eq. 11 and Eq. 12) for both models are expressed in Table 1.

Table 1. Principal Cauchy stress-stretch component.

<table>
<thead>
<tr>
<th>Material Model</th>
<th>Mooney-Rivlin equation</th>
<th>Ogden equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T_{rr} ) = ( 2 (\lambda_r \frac{\partial W}{\partial I_1} - \frac{1}{\lambda_r} \frac{\partial W}{\partial I_2}) + p )</td>
<td>( \frac{\partial W}{\partial \lambda_r} \lambda_r^3 + \frac{\partial W}{\partial \lambda_\theta} \lambda_\theta^3 + p )</td>
<td>Eq. 26</td>
</tr>
<tr>
<td>( T_{\theta\theta} ) = ( 2 (\lambda_\theta \frac{\partial W}{\partial I_1} - \frac{1}{\lambda_\theta} \frac{\partial W}{\partial I_2}) + p )</td>
<td>( \frac{\partial W}{\partial \lambda_r} \lambda_r^3 + \frac{\partial W}{\partial \lambda_\theta} \lambda_\theta^3 + p )</td>
<td>Eq. 27</td>
</tr>
<tr>
<td>( T_{zz} ) = ( 2 (\lambda_z \frac{\partial W}{\partial I_1} - \frac{1}{\lambda_z} \frac{\partial W}{\partial I_2}) + p )</td>
<td>( \frac{\partial W}{\partial \lambda_r} \lambda_r^3 + \frac{\partial W}{\partial \lambda_\theta} \lambda_\theta^3 + p )</td>
<td>Eq. 28</td>
</tr>
</tbody>
</table>

\( T_{rr}, T_{\theta\theta}, T_{zz} \) are principal Cauchy stresses in the radial, circumferential, and transverse directions to the annulus, respectively, and \( p \) denotes the unknown hydrostatic pressure needed to enforce the incompressibility constraint \( \lambda_r \lambda_\theta \lambda_z = 1 \).

For a membrane subjected to an electric field, the stress in the transverse direction due to Maxwell stress is

\[ T_{zz} = -T_{kk}^M \]  Eq. 29

where the negative sign indicates compression and \( T_{kk}^M \) is as defined in Eq. 28.

The constitutive equations and the incompressibility condition are used to formulate two non-linear ordinary differential equations (ODEs) that are to be solved numerically for the two principal stretch ratios, \( \lambda_r \) and \( \lambda_\theta \). The two ODEs are obtained
by a series of algebraic manipulations. Substitution of Eq. 29 into Eq. 28 yields an expression that can be solved for the unknown hydrostatic pressure \( p \) by considering the case of zero surface traction (\( T_{zz} = 0 \) in Eq. 28). The expression for \( p \) is then substituted into Eq. 26 and Eq. 27 to yield the radial and circumferential stresses for the Mooney-Rivlin model

\[
T_{rr} = 2 \left( \lambda_1^2 C_1 - \frac{1}{\lambda_1^2} C_2 \right) - T_{zz}^M
\]

\[
T_{\theta\theta} = 2 \left( \lambda_2^2 C_1 - \frac{1}{\lambda_2^2} C_2 \right) - T_{zz}^M
\]

where \( C_1 = \partial W / \partial I_1 \) and \( C_2 = \partial W / \partial I_2 \) and the Ogden model

\[
T_{rr} = \mu_1 (\lambda_1^2 - \lambda_1^2) + \mu_2 (\lambda_1^2 - \lambda_1^2)
\]

\[
T_{\theta\theta} = \mu_1 (\lambda_2^2 - \lambda_2^2) + \mu_2 (\lambda_2^2 - \lambda_2^2)
\]

where \( \mu_1 = \partial W / \partial \lambda_r \) and \( \mu_2 = \partial W / \partial \lambda_\theta \). The resultant radial and circumferential stresses (force per unit deformed length), \( \tilde{T}_{rr} \) and \( \tilde{T}_{\theta\theta} \), respectively, in the deformed annulus are

\[
\tilde{T}_{rr} = h \lambda_z T_{rr}, \quad \text{Eq. 32}
\]

\[
\tilde{T}_{\theta\theta} = h \lambda_z T_{\theta\theta}, \quad \text{Eq. 33}
\]

and \( T_{rr} \) and \( T_{\theta\theta} \) are as defined in Eq. 30 and Eq. 31, respectively, and the deformed thickness is defined by \( h = H \lambda_z \). The first ODE is obtained by Eq. 19 for \( d\lambda_z / dR \). The resulting ODEs are

\[
\frac{d\lambda_z}{dR} = \frac{\lambda_z}{\lambda_z(3\lambda_z^2 + \lambda_r^2)} \left( \frac{d\lambda_z}{dR} - \frac{1}{\lambda_z^2} \left[ (3\lambda_z^2 - \lambda_r^2) + \frac{C_2}{C_1} \lambda_\theta^2 (\lambda_r^2 + \lambda_z^2) \right] \right)
\]

\[
\text{Eq. 34}
\]

for the Mooney-Rivlin model where \( T_{zz}^M = T_{zz}^M / C_1 \) represents the normalized Maxwell stress. The ODE for the Ogden model
where $T_{zz}^M = T_{zz} / \mu_1$.

The second ODE for both models is obtained by differentiating the principal circumferential stretch ratio $\lambda_\theta$ with respect to the undeformed radius, $R$ to yield

$$
\frac{d\lambda_\theta}{dR} = \frac{\lambda_r}{R} - \frac{\lambda_\theta}{R}.
$$

Eq. 36

The two ODEs using the prescribed boundary conditions are solved numerically using Matlab’s boundary value solver, BVP4C, for the unknown principal stretch ratios $\lambda_r$ and $\lambda_\theta$. The numerical algorithm is covered in the next section.

### 2.7 Determination of Material Constants

Equilibrium data from constant load uniaxial tensile tests of 11 specimens after 189 hours were used to determine the stress-stretch relationship (reference Figure 16). The VHB 4910 material specimens (1 mm thickness) were cut as specified in Figure 13. One end of the specimen was fixed to a fixture and the opposite end was loaded with weights as shown in Figure 14. The VHB 4910 material is commonly used as a very high bond adhesive and the tacky surface was used to adhere the test specimen to the fixture. The loads ranged from 0.6 to 1.2 N and the equilibrium stretch ratios were in the regime of $1.5 < \lambda < 3.0$. One test specimen, which is not shown in Figure 16, with a 1.75 N load failed near the midsection of the specimen approximately 20 minutes after the load was applied. The stretch at the time of failure is unknown since the stretch was not continuously measured. This failure may indicate non-elastic behavior. Since the model being developed assumes elastic behavior the stretch regime investigated in this research was limited to a stretch range of $1.5 < \lambda < 3.0$. The equilibrium stretch is defined as the
deformed length of B in Figure 13 divided by the original length after 189 hours. Digital images were used to measure (by counting pixels) the deformed lengths of the specimens. All tests were conducted at 70±3° F and 68±5% relative humidity.

Figure 13. Schematic of the uniaxial tensile specimen used for the constant load creep tests. A= 25.4 mm., B= 76.2 mm, C=25.4 mm, D=19.1 mm, E= Radius of 2.4 mm.
A non-linear curve fit algorithm utilizing Matlab’s optimization routine FMINSEARCH was written to fit experimental data to the uniaxial stress-stretch constitutive equations of the Mooney-Rivlin and Ogden material models. The algorithm solves for the materials constants $C_1$ and $C_2$ of the Mooney-Rivlin material model and the constants $\mu_1$, $\mu_2$, $\alpha_1$, and $\alpha_2$ of the Ogden material model. A flow chart of the algorithm is shown in Figure 15 and the corresponding Matlab code is available in Appendix A.

The uniaxial constitutive equation for both models are obtained by substituting into the appropriate constitutive equations (Eq. 26 and Eq. 28). Using a change of variable from $\lambda_r$ to $\lambda$ to indicate the principal uniaxial stretch ratio yields

$$\lambda^2_\theta = \lambda^2_z = 1 / \lambda_r$$  \hspace{1cm} \text{Eq. 37}$$

into the appropriate constitutive equations (Eq. 26 and Eq. 28). Using a change of variable from $\lambda_r$ to $\lambda$ to indicate the principal uniaxial stretch ratio yields

$$T = 2(\lambda^2 - \frac{1}{\lambda})(C_1 + \frac{C_2}{\lambda})$$  \hspace{1cm} \text{Eq. 38}$$

for the Mooney-Rivlin model and

$$T = \mu_1(\lambda^\alpha_1 - (1 / \lambda)^{\alpha_2/2}) + \mu_2(\lambda^\alpha_1 - (1 / \lambda)^{\alpha_2/2})$$  \hspace{1cm} \text{Eq. 39}$$
for the Ogden model.

The Mooney-Rivlin model is used to illustrate the nonlinear curve fit algorithm. The material constants of the uniaxial Mooney-Rivlin hyperelastic constitutive equation (Eq. 31) is obtained by minimizing the objective function

$$\text{Min } C_1, C_2 \left( \frac{\sqrt{\sum_{i=1}^{N} \left( T_i^{\text{theoretical}} - T_i^{\text{experimental}} \right)^2}}{N} \right)$$  \hspace{1cm} \text{Eq. 40}

where the true experimental stress $T_i^{\text{experimental}}$ is obtained by multiplying the engineering stress (applied load divided by the undeformed area) by the stretch obtained after 189 hours

$$T_i^{\text{experimental}} = \text{Applied Load} \ast \lambda_i^{\text{experimental}}.$$  \hspace{1cm} \text{Eq. 41}

and $\text{RMSE}$ means the root mean square error. The notation $T_i^{\text{experimental}}$ in Eq. 33 denotes discrete values of true uniaxial stress. The total number of theoretical and experimental true stress comparisons is represented by $N$ and is equal to 11. Matlab’s optimization function FMINSEARCH is an unconstrained nonlinear optimization routine. The algorithm was used supplied with different initial guess values to obtain the best $R^2$ value was obtained. It solves for the values of $C_1$ and $C_2$ that will minimize the objective function so that an $R^2$ value of 0.86. A similar algorithm is used to determine the Ogden material constants and material constants for both models are presented in Table 2.

Table 2. Mooney-Rivlin and Ogden elastic material constants.

<table>
<thead>
<tr>
<th>Mooney-Rivlin</th>
<th>Ogden</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_1=8843$ Pa</td>
<td>$\alpha_1=0.5$</td>
</tr>
<tr>
<td>$C_2=4584$ Pa</td>
<td>$\alpha_2=1.64$</td>
</tr>
<tr>
<td>$\mu_1=54.9$ kPa</td>
<td>$\mu_2=40.4$ kPa</td>
</tr>
</tbody>
</table>

For numerical computation convenience, Maxwell stress and the inner radial pressure are normalized by the material constant $C_1$. 


The electric field $E$ (Eq. 23) is normalized by substituting Eq. 42 into Eq. 23 to obtain

$$E = \sqrt{2C_1 \frac{T^{M}_{zz}}{\varepsilon_0 \varepsilon_r}} = 2.77 \times 10^7 \sqrt{\frac{T^{M}_{zz}}{\varepsilon_0 \varepsilon_r}}. \quad \text{Eq. 44}$$

It should be noted that the derived model is only applicable for thin (membrane) annuli in tension.

---

Step 1: Provide the initial guess solutions for $C_1$ and $C_2$

Step 2: Invoke the optimization function FMINSEARCH

Step 3: Use the solution values for $C_1$ and $C_2$

Step 4: Calculate RMSE

If the RSME is the minimum, then the material constants $C_1$ and $C_2$ have been determined and terminate the algorithm.

If not, FMINSEARCH generate new guess values and repeat Step 4.

Figure 15: Flow chart showing the numerical algorithm used to solve for the hyperelastic Mooney-Rivlin material constants $C_1$ and $C_2$.

Uniaxial tensile tests can be conducted in several different methods: constant load test, constant strain rate with stress relaxation, and constant strain with stress relaxation. It is worthy to note that material constants determined from these various methods may not result in the same values. A problem associated with the constant strain rate and constant strain with stress relaxation tests is the false representation of the elastic behavior. In these types of tests, a falsely high degree of nonlinearity that is proportional...
to the strain rate is observed [36]. In Figure 16, results from several constant strain rate tests at various rates are compared to the results of eleven constant load tests recorded at equilibrium (no tests were conducted using the constant strain test). The false nonlinearity in the stress-stretch relationship obtained from the constant strain rate test can be observed in the trend of the stress-stretch curves converging toward the curve from the constant load tests. In this research, the constant load test is used to determine of the elastic material constants because of its simplicity with regards to test equipment requirements as well as the accuracy in representing the elastic response of the material.

Figure 16: Cauchy stress vs. uniaxial stretch ratio, $\lambda_{11}$. Experimental data (solid triangle) are results from the constant load uniaxial tensile test. Comparison between constant stretch rate and constant load creep test results. Constant stretch rate data provided by 3M Industrial Adhesives and Tapes Division.
2.8 Numerical Algorithm

The two ODEs presented in Section 2.6 are a boundary value problem (BVP) that can be solved numerically in Matlab as either an initial value problem (IVP) using the shooting method or a BVP using the collocation method. The IVP and BVP solvers in Matlab require that an initial guess for $\lambda_r(r)$ and $\lambda_d(r)$ be provided. If a poor guess is provided, the IVP solver in Matlab will arrive at a wrong solution since multiple solutions exist. Matlab’s BVP solver, BVP4C, was chosen over the IVP solver (i.e., ODE45) because it is reported to be robust and to cope well with poor initial guess of the solution [37]. A numerical algorithm using BVP4C was written to solve the two ODEs for the unknown variables $\lambda_r(r)$ and $\lambda_d(r)$. The flow chart of the algorithm is shown in Figure 17 and the Matlab code is provided in Appendix B. A thorough explanation of the BVP4C solver can be found by in a paper by Shampine [37] available on the website cited in the bibliography.

The theoretical models can now be used to better understand and visualize the dielectric material’s behavior at various prestretches, Maxwell stresses, and inner radial pressures, $\Pi$. 
2.9 Discussion and Results

2.9.1 Annulus Inner Radial Pressure Equal to Zero

One interesting prediction of the model is that the radial and circumferential stresses do not remain positive throughout the membrane for all dielectric pressures as shown in Figure 18 and Figure 19. Rather both principal stresses transition from tensile to compressive throughout the membrane at a certain Maxwell pressure. This normalized Maxwell pressure is a function of the prestretch in the membrane and is referred to as the critical Maxwell pressure. This transition occurs because the outer edge of the annulus is
restrained from expanding as it would for a free outer boundary with a dielectric pressure applied. With no prestretch in the membrane, the critical Maxwell pressure is zero (reference Figure 20) and any amount of dielectric pressure will create negative (compressive) traction at the fixed outer boundary causing the membrane to wrinkle. Increasing the prestretch increases the critical Maxwell pressure. The effect of Maxwell pressure, in effect, works towards relieving the stress induced by the prestretch. For a given prestretch, applying Maxwell pressure that is less than the critical pressure will work towards relieving the stress induced by the prestretch. However, at some dielectric pressure level, the tensile stress caused by the prestretch will be equal to the Maxwell pressure. Further increase will causing compressive tractions at the outer boundary resulting in wrinkling. Tensioning the outer edge but allowing it to expand as it wishes would remove the difficulties presented by compressive stress; this may be difficult to implement practically.

Prestretching the elastomer has the benefit of yielding a larger deformed inner radius for a given Maxwell pressure (reference Figure 21 and Figure 22). However, an increment in Maxwell pressure provides continually smaller increments in percent change in inner radius as the Maxwell pressure increases. This is due to the stiffening of the elastomer upon prestretch. If the Maxwell pressure is limited by the critical value at which principal stresses become negative, Figure 23 shows that the optimum prestretch range is between approximately 1.1 and 2.5. Beyond 2.5 prestretch, greater relative change in inner radius can be obtained but at the expense of rapidly increasing electric field.

2.9.2 Annulus Inner Radial Pressure Equal to Zero

A unique application for an actuator with a stacked system of annular components would be a peristaltic pump as previously discussed. Hence it is interesting to see the effects of varying the inner radial pressure. Applying a uniform inner radial pressure at the center of the annulus will result in a compressive radial stress applied on the inner walls. As expected, for a given dimensionless Maxwell pressure, say 15 (line (a) of
Figure 24), there exists a maximum normalized inner radial pressure ($\Pi = 7$ in Figure 24) such that the Maxwell stress is no longer adequate to sustain the compressive effects of the inner radial pressure. This causes the inner radius to deform beyond the fixed outer radius of 2, which is not physically possible. Likewise, for a Maxwell pressure, say of 115 (line (e) in Figure 24), inner radial pressures less than approximately 0.25 results in a collapse of the inner radius.

A non-zero inner radial pressure, $\Pi$, works against the Maxwell pressure’s ability to decrease the dimension of the inner radius. Therefore, a higher Maxwell pressure is needed to obtain the same change in the inner radius (reference Figure 25).

### 2.10 Conclusion

A mathematical model for dielectric elastomers was developed using the theory of large elastic deformations. From these results a better understanding of how a dielectric elastomer annulus deforms was obtained by understanding the effects of varying Maxwell pressures, pre-stretches, and inner radial pressures at the inner radius and internal stresses.

Using the derived mathematical models for a fixed-free annulus with zero inner radial pressure configuration, the radial and circumferential stresses actually transitions from tensile to compressive at a "critical" Maxwell pressure. Also, the model not only showed that prestretching yields the benefit of greater strain, but that there exists an "optimum" pre-stretch range that would yield a larger percent change in radius for a given incremental increase in the Maxwell pressure. In addition, a fixed-free annulus with an inner radial pressure, which may simulate a portion of a simple fluid pump, was modeled. It was found that there exists an operating range of the inner radial pressure and Maxwell pressures for the device to be physically as well as mathematically viable.

The modeling approach presented does not account for viscoelastic behavior present in the polyacrylate dielectric elastomer. In the next chapter, an analytical model that accounts for viscoelastic behavior is presented.
Figure 18. Normalized radial Cauchy stress $T_{rr}$ for various normalized Maxwell pressures with an elastomer prestretch of 2. $T_{zz}^M$: (a) 0 (b) 5 (62 MV/m) (c) 10 (87.7 MV/m) (d) 15 (107 MV/m) (e) 20 (124 MV/m) (f) 25 (139 MV/m) (g) 30 (152 MV/m).
Figure 19. Normalized circumferential stress, $T_{\theta\theta}$, for various normalized Maxwell pressures with an elastomer prestretch of 2. $T_{\theta\theta}^M$ (a) 0 (b) 5 (62 MV/m) (c) 10 (87.7 MV/m) (d) 15 (107 MV/m) (e) 20 (124 MV/m) (f) 25 (139 MV/m) (g) 30 (152 MV/m).
Figure 20. Normalized radial stress, $T_r$, at the outer radius versus prestretch (at zero Maxwell pressure).
Figure 21. Deformed radius $r$ for various normalized Maxwell pressure, $T_{zz}^M$, for elastomer with a prestretch of 2. $T_{zz}^M$: (a) 0 (b) 5 (62 MV/m) (c) 10 (87.7 MV/m) (d) 15 (107 MV/m) (e) 20 (124 MV/m) (f) 25 (139 MV/m) (g) 30 (152 MV/m).
Figure 22. Deformed radius for various Maxwell pressures, $T_{zz}^M$, for elastomer with a prestretch of 3. $T_{zz}^M$: (a) 0 (b) 5 (62 MV/m) (c) 10 (87.7 MV/m) (d) 15 (107 MV/m) (e) 20 (124 MV/m) (f) 25 (139 MV/m) (g) 30 (152 MV/m).
Figure 23. Cross plot of percent change in inner radius versus prestretch and normalized critical Maxwell pressures, $T^M$. 
Figure 24. Deformed radius $r$ versus normalized inner radial pressure, $\Pi$, for an annular membrane with a prestretch of 2. This graph indicates the acceptable operating range of the Maxwell pressure, $T_{zz}^M$. $T_{zz}^M$: (a) 15 (107 MV/m); (b) 20 (124 MV/m); (c) 25 (139 MV/m); (d) 30 (152 MV/m); (e) 115 (297 MV/m).
Figure 25. Percent change in inner radius as a function of the normalized inner radial pressure, $\Pi$, in the center of the annulus and normalized Maxwell pressures, $T^M_{zz}$, at a prestretch of 2. $T^M_{zz}$: (a) 15 (107 MV/m); (b) 20 (124 MV/m); (c) 25 (139 MV/m); (d) 30 (152 MV/m).
Chapter 3

Analytical Nonlinear Viscoelastic Model

This chapter focuses on the development of a viscoelastic constitutive model for dielectric elastomers and is organized as follows. An introduction to viscoelasticity is briefly provided in Section 3.1 using mechanical models comprised of springs and dashpots. This will provide a general understanding of viscoelastic behaviors as well as introduce the creep phenomena. Two methods of formulating constitutive models are by means of using internal variable and functionals and are discussed in detail in Sections 3.2 and 3.3, respectively. In Section 3.4 a phenomenological constitutive model based upon the use of functionals is formulated for dielectric elastomer membranes using Christensen’s nonlinear viscoelastic theory. The resulting integro-differential equation consisting of functionals is approximated by discretizing the integrals with respect to time using Feng’s time-efficient recursion formula. The recursive forms of the principal Cauchy stresses for a membrane in uniaxial extension are derived in detail in Section 3.5. Also included in this section are the recursive forms of the radial and circumferential Cauchy stresses for an annular membrane subjected to Maxwell stress. The viscoelastic material property (relaxation modulus) is determined using results from a constant load creep test and is presented in Section 3.6. The theoretical predictions are compared to experimental data and presented in Section 3.7 through Section 3.9.

3.1 Introduction to Viscoelasticity

Mechanical response of solid materials to a load can be classified into four ideal categories: elastic, viscoelastic, plastic, and viscoplastic. Schematics of typical strain versus time graphs are shown in Figure 26. Strains in elastic materials are independent of time (rate-independent) and are fully recovered the instant the constant load is removed. Viscoelastic materials are similar in response to elastic materials except that strains are
rate-dependent and a creep phenomenon is observed. Viscoelastic materials may or may not exhibit instantaneous strain when a load is applied. A plastic material undergoes permanent deformation when it is stressed (typically beyond an elastic limit) and exhibits equilibrium hysteresis after load removal. Plastic materials have a load-path–dependent response but the response does not depend on the rate this path is traveled. Finally, viscoplastic materials also exhibit permanent deformations; however, they are both path- and rate-dependent.

![Viscoelastic and Viscoplastic Materials](image)

Figure 26: Material response due to a constant tensile load can be categorized into four categories: elastic, viscoelastic, plastic, and viscoplastic.

![Viscoelastic Strain Response](image)

Figure 27. A viscoelastic strain response is characterized by increasing strain as a function of time due to an applied constant load. Once the load is removed, there is an elastic and viscoelastic response characterized by the instantaneous change in strain followed by the time-dependent deformation, respectively.

In this chapter, viscoelastic responses of dielectric elastomers are of interest. Viscoelastic behavior is seen in many materials such as wood, metals, concrete, biological muscles, and polymers. The primary characteristic of viscoelastic materials is
the stress' dependence on time. Materials typically follow Coleman and Noll's [38] principle of fading memory such that the current stress state is influenced more by the recent deformation history than the deformations that have occurred in the far past. A material with this characteristic is referred to as a simple material as termed by Truesdell and Noll [25].

Macroscopic viscoelastic response can be understood through the use of one-dimensional mechanical models consisting of various combinations of the basic spring and dashpot elements as shown in Table 3. The springs allow for instantaneous strain response due to a constant applied load and instantaneous recovery of strain upon removal of the load. The dashpot element provides the time dependent or viscous response. Various combinations of these elements can be configured to exhibit viscoelastic behaviors such as instantaneous strain, delayed elasticity, and creep. The stress-strain constitutive equation qualitatively describes the time-dependent material’s response. The relaxation modulus, \( g(t) \), describes the stress relaxation in the material and is determined from experimental data.

A Maxwell fluid model consists of a spring and dashpot connected in series that can respond instantaneously to an applied step load of \( \sigma_0 \). The stress relaxation of the material is described by a decaying exponential function with the time constant of \( \eta/k \). The larger the time constant, the longer it takes for the material’s stress to reach an equilibrium state. The Kelvin model consists of a spring and dashpot in parallel and does not allow for a step increase in strain. It is possible that a material’s viscoelastic response cannot be described by a single time constant and may require several time constants. It is known that for a polymeric material, stress relaxation is very rapid in the very early part of the loading process then drops dramatically thereafter [39]. Therefore, a material’s response can be better described by a combination of elements. For example, consider the \( 2N+1 \) parameter model consisting of the \( N \) Maxwell elements in parallel with a spring. The \( N \) elements mean that the relaxation modulus will have \( N \) relaxation times \( \tau_i \)

\[
\tau_i = \frac{k_i}{\eta_i}, \quad i = 1, \ldots, N
\]

Eq. 45
and \( N \) relaxation coefficients \( k_i \) that make up the relaxation spectrum. However, a continuous relaxation spectrum \( (N = \infty) \) may be required for the theoretical model to correlate well with experiment. It is here where the weakness of using mechanical analogs to describe viscoelastic material behaviors becomes apparent.

Table 3. Viscoelastic mechanical models consisting of spring and dashpot elements where \( \sigma \) is the applied load, \( k \) is the spring constant, \( \eta \) is the damping coefficient, \( \varepsilon \) is the displacement of the spring, and \( \delta(t) \) is the Dirac delta function. Adapted from Flugge [40] and Haupt [39].

<table>
<thead>
<tr>
<th>Schematic</th>
<th>Model</th>
<th>Constitutive equation and relaxation modulus</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maxwell fluid</td>
<td>( \sigma ) + ( \frac{\eta}{k} \dot{\sigma} = \eta \dot{\varepsilon} )</td>
<td>Relaxation Modulus: ( g(t) = ke^{-t/(\eta/k)} )</td>
</tr>
<tr>
<td>Kelvin solid</td>
<td>( \sigma = k\varepsilon + \eta \dot{\varepsilon} )</td>
<td>Relaxation Modulus: ( g(t) = k + \eta \delta(t) )</td>
</tr>
<tr>
<td>2( N+1 ) parameter</td>
<td>Constitutive: ( \sigma = k_o \varepsilon + \sum_{i=1}^{N} k_i (\varepsilon - \varepsilon_i) )</td>
<td>Relaxation Modulus: ( g(t) = k_o + \sum_{i=1}^{N} k_i e^{-t/(\eta_i/k_i)} )</td>
</tr>
</tbody>
</table>

Viscoelastic theories can be phenomenological-based by analyzing macroscopic deformations or they can be based upon statistical mechanics in which the microscopic deformations are used to explain macroscopic response. Phenomenological viscoelastic theories can be subcategorized into theories using rate-dependent functionals or internal
state variables. This research uses a phenomenological theory, however, both are presented in this chapter for completeness. The concepts of rate-dependent functionals and internal variables to formulate constitutive equations are discussed in the next section.

### 3.2 Viscoelastic Theory With Internal State Variables

The concept of internal state variables used to develop constitutive equations is general and is not limited to viscoelastic theories. It is suitable in the development of constitutive equations that include temperature, strain rate, plasticity, degradation, and aging. Internal state variables describe a system’s state and do not have the ability to possess inertia. To illustrate, consider a system consisting of a Maxwell fluid model (spring and dashpot in series) connected in parallel to a spring with a spring constant of $k_1$ (reference Figure 28).

![Figure 28. A 3-parameter model consisting of a Maxwell fluid model connected in parallel with a spring.](image)

The spring constant and damping coefficient of the Maxwell elements are $k_2$ and $\eta$, respectively. The strain in the spring parallel with the Maxwell fluid model is $\varepsilon_1$ and the strains in the Maxwell spring and dashpot elements are $\varepsilon_2 = \varepsilon_1 - q$ and $\varepsilon_3 = q$, respectively. Substitution of the strain expressions into the force equilibrium equation yields

$$\sigma = k_1\varepsilon_1 + k_2\varepsilon_2$$  \hspace{1cm} \text{Eq. 46}
and the compatibility equation
\[ \eta \dot{\varepsilon}_3 = k_2 \varepsilon_2 \] 
Eq. 47

which provides the two governing equations for the system
\[ \sigma(t) = k_1 \varepsilon_1 + k_2 (\varepsilon_1 - q(t)) \] 
Eq. 48

and
\[ \dot{q}(t) = \frac{k_2}{\eta} (\varepsilon_1 - q(t)) \] 
Eq. 49

The variable \( q \) is the internal state variable of the system and Eq. 49 is known as the evolution equation of the time dependent behavior of \( q(t) \). The simple one-dimensional system is expanded for the general three-dimensional case next.

The second Piola-Kirchoff stress tensor can be expressed as a function using internal state variables is
\[ \tilde{T} = f(E, q_1, \ldots, q_N) \] 
Eq. 50

where and \( E \) is the Lagrangian strain tensor \( E = \frac{1}{2} F^T F - I \) and \( q_i \) are the internal variables. The internal variables can be solved by developing a set of \( N \) evolution equations
\[ \dot{q}_i(t) = f_i(E(t), q_1(t), \ldots, q_N(t)), \ i = 1, \ldots, N. \] 
Eq. 51

where \( q(t) \) are the time-dependent behaviors of the internal variable \( q \). The time rate of change of the internal variables, \( \dot{q}_i(t) \), are also a function of the Lagrangian strain and the same internal variables previously mentioned.

The evolution equations describe the stress relaxation of the material and must possess features that reflect this process. One requirement is that the equilibrium solutions for each internal variable \( \bar{q}_i \) exists for the system of equations for each state of strain \( E \)
\[ f_i(E, \bar{q}_1, \ldots, \bar{q}_N) = 0, \ i = 1, \ldots, N. \] 
Eq. 52
The second requirement is that the solutions to the evolution equations must approach the equilibrium solutions when the system is subjected to a constant strain tensor $E$ and initial conditions of each internal variable $q_i(t_0)$. The substitution of Eq. 51 into Eq. 52 yields the equilibrium relations. Hence, the system of equations consisting of the kinematic relations, boundary conditions, and the equilibrium equations is sufficient to model the deformation of a material.

### 3.3 Viscoelastic Theory with Rate-dependent Functionals

Consider a uniaxial deformation of a thin isotropic viscoelastic material. The nonlinear stress-strain relationship of the material can be represented using Volterra’s integral of the second kind [41]

$$u(t) = f(t) + \int_0^t K(t - s)u(s)\,ds, \text{ for all } s < t$$  \hspace{1cm} \text{Eq. 53}

where $s$ is a time in the past history, $t$ is the current time, $K(t - s)$ is the kernel or integral operator that operates on $u(s)$, $f(t)$ is the current elastic state of stress, and all functions are continuous and differentiable on $[0,t]$. Then, the current stress, $u(t)$, is the sum of the current elastic stress, $f(t)$, and the cumulative effects of stress that have occurred before time $t$ are given by the integral $\int_0^t K(t - s)u(s)\,ds$.

Mathematically, the dependence of the current state of a system on past history is termed “hereditary” and the integral is often termed as a hereditary integral (also known as a Stieltjes integral). In continuum mechanics, a material with such dependency is called a ‘simple material’ by Truesdell and Noll [25]. The behavior is referred to as the principle of fading memory by Coleman and Noll [38].

For a three-dimensional treatment of viscoelasticity based upon functionals, consider a simple, isotropic material whose current stress is dependent upon past strain history. The general constitutive relation [42, 43] for an isotropic solid describing such stress dependency is
\[
T = f(B) + \sum_{s=0}^{\infty} (G(t-s); B(t)), \quad s < t
\]

where \( T \) is the Cauchy stress tensor, \( B \) is the left Cauchy-Green tensor \( B = FF^T \), \( f(B) \) is the isotropic elastic stress response at equilibrium, the functional \( \mathcal{I} \) represents the memory part of the total stress \( T \), and \( G(t-s) \) is the relative strain difference history. The rate-dependent memory functional \( \mathcal{I} \) approaches zero as \( t \) approaches infinity and therefore has no contribution to the final stress at equilibrium.

The purpose herein is to mathematically define the terms in the general constitutive equation above. The relative strain difference history, \( G(t-s) \), is defined as

\[
G(t-s) = 2F^{-T}(t)(E(t-s) - E(t))F^{-1}(t)
\]

where \( F \) is the deformation gradient and \( E = \frac{1}{2}(F^T F - I) \) is the Lagrangian strain tensor (also called Green’s strain tensor). The functional \( \mathcal{I} \) which is now linear in \( E \) must be approximated since it cannot be evaluated explicitly. This is accomplished by using approximation procedures presented by Chacon and Rivlin [44]. Through the use Stone-Weierstrass and Riesz representation theorems that are valid for linear functionals the general constitutive equations are obtained. The Stone-Weierstrass theorem states that a real continuous functional \( \mathcal{I} \) can be approximated by a polynomial in a set of real continuous linear scalar valued functions

\[
f(i) = \sum_{s=0}^{\infty} \mathcal{I}(E(t-s), E(t)), \quad i = 1, 2, \ldots, n.
\]

The Riesz representation theorem can then be used to represent the linear functionals as Steiltjes (hereditary) integrals with bounded limits

\[
f(i) = \int_{0}^{\infty} E(t-s)dg_{(i)}(s)
\]

where \( dg_{(i)}(s) = 0, s < 0 \) and represent the material properties. An alternative form of Eq. 57 can be obtained by using integration by parts

\[
f(i) = \int_{-\infty}^{t} g_{(i)}(t-s) \partial E(t-s) \partial s, \quad s > 0.
\]
Next, it is necessary to mathematically define $f(B)$, the isotropic elastic stress response at equilibrium. Any elastic model could be used. For illustrative purpose, the simple neo-Hookean elastic stress model is used and is expressed in indicial notation as

$$x_{i,K}x_{j,L}g_{e}\delta_{KL}$$  \hspace{1cm} Eq. 59

where $g_{e}$ is the neo-Hookean elastic material constant and $\delta_{KL}$ is the Kronecker delta.

Finally, Eq. 51 and Eq. 52 are substituted into Eq. 54 to yield the expanded form of the general constitutive equation in indicial notation

$$T_{ij} = -p\delta_{ij} + x_{i,K}x_{j,L}(g_{e}\delta_{KL} + \int_{0}^{t} g_{1}(t-\tau)\frac{\partial E_{KL}(\tau)}{\partial \tau} d\tau + \int_{0}^{t} \int_{0}^{t} g_{2}(t-\tau)\frac{\partial E_{KL}(\tau)}{\partial \tau} \frac{\partial E_{MN}(\tau)}{\partial \eta} d\tau d\eta + ...), \tau < t$$  \hspace{1cm} Eq. 60

where $T_{ij}$ are the Cartesian components of the Cauchy stress tensor $T$, $g_{i}(t-\tau)$, $i=1, 2, ..., n$ are time dependent material functions, and $E_{KL}$ are the Cartesian components of the Green’s strain tensor. Explicit evaluation of this general constitutive equation is not possible. Various viscoelastic theories exist to evaluate this expression. In this research, Christensen’s viscoelastic theory, considered to be the simplest viscoelastic theory available [36], is utilized.

### 3.3.1 Christensen’s Viscoelastic Theory

Christensen’s viscoelastic theory simplifies the constitutive equation (Eq. 60) by using the physical concepts of the kinetic theory of elasticity so that higher ordered terms can be truncated. The theory is valid for an isotropic material undergoing a sufficiently slow, quasi-static deformation in isothermal conditions. Christensen’s imposes two conditions in the development of his viscoelastic theory:

i) For a quasi-static deformation, the theory reduces to the kinetic theory (Gaussian statistics) of rubber elasticity. This means that viscoelastic
effects have no contribution to the final state of stress when the system reaches equilibrium.

ii) The theory is applicable for systems subjected to a constant load. It does not apply to systems subjected to a constant strain.

For a one-dimensional system subjected to a constant load the Cauchy stress $T$ is a function of $\alpha$ and strain $E$

$$T = f(\alpha, E)$$

Eq. 61

where $\alpha$ is a time scale that can be accelerated or retarded. The accelerated strain history can then be defined as

$$E(\alpha t), \; \alpha > 1$$

Eq. 62

relative to $E(t)$ and a retarded strain history can be defined as

$$E(\alpha t), \; \alpha < 1.$$  

Eq. 63

In this system the change in Cauchy stress (Eq. 61) with respect to an accelerated time $\alpha$ must equal to zero, $\frac{\partial T}{\partial \alpha} = 0$, as required by condition ii above

$$\frac{\partial |f(\alpha, E)|}{\partial \alpha} + \frac{\partial |f(\alpha, E)|}{\partial E} \frac{dE}{d\alpha} = 0.$$  

Eq. 64

Simply, it just means that at accelerated times, or at long times, stress becomes time-independent. Solving Eq. 64 for $dE/d\alpha$

$$\frac{dE}{d\alpha} = -\frac{\partial |f(\alpha, E)|}{\partial \alpha} \sqrt{|f(\alpha, E)|}.$$  

Eq. 65

The negative change in strain with respect to the accelerated time, $dE/d\alpha$, indicates that the magnitude of strain is continually decreasing. This also implies that the viscoelastic effects at equilibrium ($t=\infty$) do not contribute to the final stress of the system, which satisfies Christensen’s first requirement. Thus, the higher ordered terms of Eq. 60 can be neglected and the resulting constitutive equation is expressed as

$$T_y = -p\delta_y + x_{i,k}x_{j,t} (g_{e} \delta_{KL} + \int_0^t g_e (t - \tau) \frac{\partial E_{KL}(\tau)}{\partial \tau} d\tau), \; \tau < t$$  

Eq. 66
with the change of variables of \( g_1 = g \) to clearly indicate that it represents the viscoelastic relaxation modulus.

In the next section, a phenomenological constitutive model for dielectric elastomers is developed using the results of Christensen’s viscoelastic theory for an annular geometric configuration. In Section 3.5, the constitutive equations are used to obtain the stress-stretch relationship for a two-dimensional annular membrane in radial tension. The resulting stress-stretch relationship is curve fit to experimental constant load creep test data to determine the relaxation modulus (Section 3.6).

### 3.4 Development of a Phenomenological Constitutive Model For Dielectric Elastomers

In developing the viscoelastic constitutive model for an annular dielectric elastomer membrane the material is assumed to be an isotropic, homogenous, and incompressible membrane undergoing quasi-static deformation in isothermal conditions. The annular membrane is constrained at the outer radius, free of traction at the inner radius, and is subjected to an electric field across the thickness of the material. Under these conditions, the kinematic relations, boundary conditions, and equilibrium equations are the same as presented in Chapter 2 except now they are also functions of time \( t \). The equations are repeated here without derivation for the viscoelastic case.

**Kinematic relations:**

\[
\lambda_r(R,t) = \frac{dr(t)}{dR} \quad \text{principal radial stretch ratio} \quad \text{Eq. 67}
\]

\[
\lambda_\theta(R,t) = \frac{r(t)}{R} \quad \text{principal circumferential stretch ratio} \quad \text{Eq. 68}
\]

\[
\lambda_z(R,t) = \frac{1}{\lambda_r(R,t)\lambda_\theta(R,t)} \quad \text{principal transverse stretch ratio} \quad \text{Eq. 69}
\]

**Radial (quasi-static) equilibrium equation:**

\[
\frac{dT_{rr}(R,t)}{dR} + \frac{1}{R}(T_{rr}(R,t) - T_{\theta\theta}(R,t)) = 0 \quad \text{Eq. 70}
\]
Boundary conditions:

\[
\left[ \lambda_r \right]_{\theta=0} = \frac{r_h}{R_y} \quad \text{prestretch at the outer radius} \quad \text{Eq. 71}
\]

\[
\left[ T_{tr} \right]_{\theta=0} = 0 \quad \text{principal radial Cauchy stress at the inner radius} \quad \text{Eq. 72}
\]

\[
T_{zz} = -T_{zz}^M \quad \text{transverse Maxwell stress on the major surfaces of the membrane} \quad \text{Eq. 73}
\]

The equations for \( d\lambda_r / dR \) and \( d\lambda_\theta / dR \) are obtained in a similar manner as presented in Chapter 2 for the hyperelastic model and is described next.

The Cauchy stress-stretch relations (Eq. 66) of Christensen’s viscoelastic model with the neo-Hookean hyperelastic model is repeated here for convenience

\[
T_{ij}(r,t) = -p\delta_{ij} + r_{i,k}r_{j,l}g_e\delta_{KL} + \int_0^t g_e(t-\tau) \frac{\partial E_{KL}(\tau)}{\partial \tau} d\tau \quad \text{Eq. 74}
\]

where the term \( r_{i,k}r_{j,l}g_e\delta_{KL} \) represents the neo-Hookean elastic stress, \( r_{i,k} = \lambda_i(r,t) \) is the deformation gradient in the principal radial and circumferential directions for \( i=r \) and \( \theta \), respectively. The stretch ratio \( \lambda_i \) and Cauchy stress \( T_{ij} \) are functions of the undeformed radius, \( R \), and time, \( t \). For brevity the terms \( (R,t) \) are excluded from all expressions herein.

The neo-Hookean model of Eq. 74 is a first order model and cannot describe highly nonlinear stress-stretch relationships [45]. Since the choice of an elastic model is arbitrary, the results of the theoretical model can be improved by using the Mooney-Rivlin material model. For the principal stresses in an axisymmetric stress state, the Mooney-Rivlin constitutive relations simplify to

\[
T_{ii}^{\text{elastic}} = 2\left(\lambda_i^2 \frac{\partial W}{\partial I_1} - \frac{1}{\lambda_i^2} \frac{\partial W}{\partial I_2}\right) + p, \quad i = r, \theta, z \quad \text{Eq. 75}
\]

where \( \partial W / \partial I_1 \) and \( \partial W / \partial I_2 \) are constants. Substitution of this equation into Eq. 74 yields
Comparison of experimental and theoretical results indicates that a higher ordered elastic model such as the Ogden model was not needed; this is discussed in Section 3.7.

The remainder of this section is devoted to expressing the radial equilibrium equation (Eq. 70) in terms of $\lambda_r$, $\lambda_\theta$, and $d\lambda_\theta / dR$. This is accomplished by determining the expressions for the principal radial Cauchy stress $T_r$, its derivative with respect to $R$, and the principal Cauchy stress $T_{\theta\theta}$ in terms of the unknown variables $\lambda_r$, $\lambda_\theta$, and $d\lambda_\theta / dR$. The resulting equilibrium equation is an integro-differential equation that will be cast into a form suitable for numerical integration and is presented in the next section.

For a membrane subjected to a Maxwell stress on its major surfaces, the transverse principal Cauchy stress is

$$T_{zz} = -T_{zz}^M.$$  \hspace{1cm} \text{Eq. 77}$$

The transverse principal Cauchy stress expression (Eq. 76, $i = z$) is used to obtain an expression for the hydrostatic pressure $p$

$$p = T_{zz}^M + 2 (\lambda_z^2 \frac{\partial W}{\partial I_1} - \frac{1}{\lambda_z} \frac{\partial W}{\partial I_2}) + \frac{\lambda_z^2}{2} \int_0^t g_r(t - \tau) \frac{\partial \lambda_z^2(\tau)}{\partial \tau} d\tau.$$  \hspace{1cm} \text{Eq. 78}$$

where $\lambda_z^2 = 1/(\lambda_r^2 \lambda_\theta^2)$. This result is then substituted into Eq. 76 for $i = r$ to yield the principal radial Cauchy stress

$$T_{rr} = -T_{zz}^M + 2 (\lambda_r^2 - 1/(\lambda_r^2 \lambda_\theta^2))(\frac{\partial W}{\partial I_1} + \lambda_\theta^2 \frac{\partial W}{\partial I_2}) +$$

$$\frac{\lambda_r^2}{2} \int_0^t g_r(t - \tau) \frac{\partial \lambda_r^2}{\partial \tau} d\tau -$$

$$\frac{1}{2 \lambda_r \lambda_\theta^2} \int_0^t g_r(t - \tau) \frac{\partial (1/(\lambda_r^2 \lambda_\theta^2))}{\partial \tau} d\tau.$$  \hspace{1cm} \text{Eq. 79}$$
and its first derivative with respect to the undeformed radius $R$

$$\frac{dT_{zz}}{dR} = 4(\lambda_r^2 + \lambda_r \lambda_\theta^2 + \lambda_\theta^2) \left( \frac{\partial W}{\partial I_1} + \lambda_\theta^2 \frac{\partial W}{\partial I_2} \right) + 4(\lambda_r^2 - \frac{1}{\lambda_r^2}) (\lambda_r \lambda_\theta \frac{\partial W}{\partial I_z})$$

$$\lambda_r \lambda_\theta \int_0^t g_r(t - \tau) \frac{\partial \lambda_\theta^2}{\partial \tau} d\tau +$$

$$\frac{1}{2 \lambda_r^2} \lambda_\theta^2 \int_0^t g_r(t - \tau) \frac{\partial (1/(\lambda_r^2 \lambda_\theta^2))}{\partial \tau} d\tau +$$

$$\frac{\lambda_r^2}{\lambda_r^2 \lambda_\theta^2} \int_0^t g_r(t - \tau) \frac{\partial (1/(\lambda_r^2 \lambda_\theta^2))}{\partial \tau} d\tau.$$

The principal circumferential Cauchy stress is determined in the same manner

$$T_{\theta \theta} = -T_{zz} + 2(\lambda_r^2 - \lambda_\theta^2) \left( \frac{\partial W}{\partial I_1} + \lambda_\theta^2 \frac{\partial W}{\partial I_2} \right) + \frac{\lambda_\theta^2}{2} \int_0^t g_r(t - \tau) \frac{\partial \lambda_\theta^2}{\partial \tau} d\tau$$

$$- \frac{1}{2 \lambda_r^2 \lambda_\theta^2} \int_0^t g_r(t - \tau) \frac{\partial (1/(\lambda_r^2 \lambda_\theta^2))}{\partial \tau} d\tau.$$  \text{Eq. 81}

Now, Eq. 79 can be expressed in terms of its unknown variables $\lambda_r$, $\lambda_\theta$ and $d\lambda_\theta / dR$ by employing Eq. 79 through Eq. 81. The resulting integro-differential equation will be cast into a form suitable for numerical integration.

A second equation needed to solve for the two unknown principal stretch ratios is obtained by differentiating the principal circumferential stretch ratio $\lambda_\theta$ with respect to the undeformed radius, $R$

$$\frac{d\lambda_\theta}{dR} = \frac{\lambda_r}{R} \frac{\lambda_\theta}{R}.$$  \text{Eq. 82}

Unlike the solution method used in the hyperelastic case, the integro-differential equation and Eq. 89 must be solved through numerical integration for the unknown principal stretch ratios. The numerical integration procedure can be presented clearly using the uniaxial case and is the topic of the next section.
3.5 Numerical Integration

Integro-differential equations cannot be solved explicitly and require numerical integration. Use of quadrature rules is a valid solution approach. However, this involves solving a large number of equations simultaneously. This method is high in computation cost, as the value of each integral at each time step needs to be stored for the entire limits of integration, 0 to t. A recursion formula presented by Feng [46] circumvents this problem by approximating the current value of the integral at time \( t_{n+\Delta t} \) as a function of the integral at the previous time \( t_n \). The recursive form for the uniaxial Cauchy stress equation is presented in detail next. This is followed by the derivation of the recursive forms of the constitutive equations (Eq. 79 and Eq. 81) for the two dimensional case of an annular membrane (Section 3.5.2).

3.5.1 Uniaxial Case

An expression for the uniaxial Cauchy stress is given

\[
T_{11} = -T_{33}^M + 2 \left( \lambda_1 \frac{1}{\lambda_1} \frac{\partial W}{\partial I_1} + \frac{1}{\lambda_1} \frac{\partial W}{\partial I_2} \right) + \frac{\lambda_1^2}{2} \int_0^t g_v(t-\tau) \frac{\partial \lambda_1^2}{\partial \tau} d\tau + \frac{1}{2\lambda_1} \int_0^t g_v(t-\tau) \frac{\partial (1/\lambda_1)}{\partial \tau} d\tau
\]

Eq. 83

where \( \lambda_1 \) is the uniaxial stretch ratio.

The time difference kernel or the relaxation modulus \( g_v(t-\tau) \) of the hereditary integral of \( \dot{\lambda}_1 \) is determined through experiments that are phenomenological in nature. The relaxation modulus is mathematically modeled using an exponential function that decays to zero at \( t=\infty \)

\[
g_v(t-\tau) = g_1 + g_2 e^{-(t-\tau)/\tau_1}, \quad \tau < t
\]

Eq. 84

where \( g_1, g_2, \) and \( \tau_1 \) are material constants, \( \tau \) is a time in the past history, and \( t \) is the current time. The material constants are determined from experimental data and are
discussed in Section 3.6. The rationale of selecting the relaxation modulus in the form presented in Eq. 84 will be explained in detail in Section 3.6. For now, it is used to illustrate the mathematical procedure needed to express the uniaxial Cauchy stress equation in recursive form.

The relaxation modulus can be rewritten in a form suitable for numerical integration first by evaluating it at the current time \( t_n + \Delta t \)

\[
g_1(t_n + \Delta t - \tau) = g_1 + g_2 e^{-(t_n+\Delta t-\tau)/\tau_1}, \tau < t. \tag{Eq. 85}
\]

This result is employed into Eq. 83 to yield the Cauchy stress for the uniaxial case

\[
T_{11}(t_n + \Delta t) = -T_{33}^M + 2 \left( \lambda_1^2 \frac{1}{\lambda_1} \right) \left( \frac{\partial W}{\partial I_1} + \frac{1}{\lambda_1} \frac{\partial W}{\partial I_2} \right) + \frac{\lambda_1^2}{2} \int_{t_n}^{t_n + \Delta t} (g_1 + g_2 e^{-(t_n+\Delta t-\tau)/\tau_1}) \frac{\partial \lambda_1^2}{\partial \tau} d\tau - \frac{1}{2\lambda_1} \int_{t_n}^{t_n + \Delta t} (g_1 + g_2 e^{-(t_n+\Delta t-\tau)/\tau_1}) \frac{\partial (1/\lambda_1)}{\partial \tau} d\tau, \tau < t. \tag{Eq. 86}
\]

The integrals are divided into time intervals 0 to \( t_n \) and \( t_n \) to \( t_n + \Delta t \)

\[
T_{11}(t_n + \Delta t) = -T_{33}^M + 2 \left( \lambda_1^2 \frac{1}{\lambda_1} \right) \left( \frac{\partial W}{\partial I_1} + \frac{1}{\lambda_1} \frac{\partial W}{\partial I_2} \right) + \frac{\lambda_1^2}{2} \int_{0}^{t_n} g_1 \frac{\partial \lambda_1^2}{\partial \tau} d\tau + \frac{\lambda_1^2}{2} \int_{t_n}^{t_n + \Delta t} g_2 e^{-(t_n+\Delta t-\tau)/\tau_1} \frac{\partial \lambda_1^2}{\partial \tau} d\tau - \frac{1}{2\lambda_1} \int_{0}^{t_n} g_1 \frac{\partial (1/\lambda_1)}{\partial \tau} d\tau - \frac{1}{2\lambda_1} \int_{t_n}^{t_n + \Delta t} g_2 e^{-(t_n+\Delta t-\tau)/\tau_1} \frac{\partial (1/\lambda_1)}{\partial \tau} d\tau, \tau < t. \tag{Eq. 87}
\]

Integrals that do not have the exponential term are explicitly evaluated.
The two hereditary integrals with limits of integration of 0 to $t_n$ in Eq. 88 can be rewritten as

$$T_{11}(t_n + \Delta t) = -T_{33}^M + 2 \left( \frac{\lambda_1^2}{\lambda_1} - \frac{1}{\lambda_1^2} \right) \left( \frac{\partial W}{\partial I_1} + \frac{1}{\lambda_1} \frac{\partial W}{\partial I_2} \right) + \frac{\lambda_1^2}{2} g_1(\lambda_1^2(t_n) - \lambda_1^2(0)) + \frac{\lambda_1^2}{2} \int_0^{t_n} g_2 e^{-(t_n + \Delta t - \tau)/\tau_1} \frac{\partial \lambda_1^2}{\partial \tau} d\tau + \frac{\lambda_1^2}{2} g_1(\lambda_1^2(t_n + \Delta t) - \lambda_1^2(t_n)) + \frac{\lambda_1^2}{2} \int_{t_n}^{t_n + \Delta t} g_2 e^{-(t_n + \Delta t - \tau)/\tau_1} \frac{\partial \lambda_1^2}{\partial \tau} d\tau - \frac{1}{2 \lambda_1} g_1(\frac{1}{\lambda_1(t_n)} - \frac{1}{\lambda_1(0)}) - \frac{1}{2 \lambda_1} \int_0^{t_n} g_2 e^{-(t_n + \Delta t - \tau)/\tau_1} \frac{\partial (1/\lambda_1)}{\partial \tau} d\tau - \frac{1}{2 \lambda_1} g_1(\frac{1}{\lambda_1(t_n + \Delta t)} - \frac{1}{\lambda_1(t_n)}) - \frac{1}{2 \lambda_1} \int_{t_n}^{t_n + \Delta t} g_2 e^{-(t_n + \Delta t - \tau)/\tau_1} \frac{\partial (1/\lambda_1)}{\partial \tau} d\tau, \tau < t.$$  

Eq. 88

$$e^{-\Delta t/\tau_1} \int_0^{t_n} g_2 e^{-(t_n - \tau)/\tau_1} \frac{\partial \lambda_1^2}{\partial \tau} d\tau$$  

Eq. 89

and

$$e^{-\Delta t/\tau_1} \int_0^{t_n} g_2 e^{-(t_n - \tau)/\tau_1} \frac{\partial (1/\lambda_1^2)}{\partial \tau} d\tau$$  

Eq. 90

where the values of the integrals are known since they represent the values of the hereditary integrals $\int_0^{t_n + \Delta t} g_2 e^{-(t_n + \Delta t - \tau)/\tau_1} \frac{\partial \lambda_1^2}{\partial \tau} d\tau$ and $\int_0^{t_n + \Delta t} g_2 e^{-(t_n + \Delta t - \tau)/\tau_1} \frac{\partial (1/\lambda_1^2)}{\partial \tau} d\tau$ at the previous time $t_n$.

The task of expressing the hereditary integrals of Eq. 88 that are evaluated on the interval $t_n + \Delta t$ into recursion forms is accomplished by using the second mean value theorem for integration:

If $f: [a, b] \rightarrow \mathbb{R}$ is a positive and monotonically decreasing function and $\varphi: [a, b] \rightarrow \mathbb{R}$ is an integrable function, then there exists a number $x$ in $(a, b]$ such that

$$\int_a^b f(t) \varphi(t) \, dt = \left( \lim_{t \to a^+} \int_a^b f(t) \, dt \right) \int_a^b \varphi(t) \, dt.$$  

Eq. 91
By applying the mean value theorem, the integrals evaluated on the interval $t_n + \Delta t$ can be approximated as

\[
\int_{t_n}^{t_n + \Delta t} g_2 e^{-(t_n - \tau)/\tau_1} \frac{\partial \lambda_1^2}{\partial \tau} d\tau = g_2 e^{-\Delta t/2} \int_{t_n}^{t_n + \Delta t} \frac{\partial \lambda_1^2}{\partial \tau} d\tau
\]

Eq. 92

and

\[
\int_{t_n}^{t_n + \Delta t} g_2 e^{-(t_n - \tau)/\tau_1} \frac{\partial (1/\lambda)}{\partial \tau} d\tau = g_2 e^{-\Delta t/2} \int_{t_n}^{t_n + \Delta t} \frac{\partial (1/\lambda)}{\partial \tau} d\tau.
\]

Eq. 93

Furthermore, Equations 3.50 and 3.51 can be explicitly evaluated to yield

\[
\int_{t_n}^{t_n + \Delta t} g_2 e^{-(t_n - \tau)/\tau_1} \frac{\partial \lambda_1^2}{\partial \tau} d\tau = g_2 e^{-\Delta t/2} (\lambda_1^2(t_n + \Delta t) - \lambda_1^2(t_n))
\]

Eq. 94

and

\[
\int_{t_n}^{t_n + \Delta t} g_2 e^{-(t_n - \tau)/\tau_1} \frac{\partial (1/\lambda)}{\partial \tau} d\tau = g_2 e^{-\Delta t/2} (\lambda_1^2(t_n + \Delta t) - \lambda_1^2(t_n)),
\]

Eq. 95

respectively. All equations are now in recursive forms suitable for numerical integration.

To summarize, the uniaxial Cauchy stress integral equation (Eq. 83) was cast into a recursion form first by dividing the limits of integration into time intervals (Eq. 87). This resulted in integrals that could either be explicitly evaluated or were approximated (Eq. 94 and Eq. 95) using the second mean value theorem for integration. The final uniaxial Cauchy stress equation in recursive form is

\[
T_{11}(t_n + \Delta t) = -T_{33}^M + 2 (\lambda_1^2(t_n + \Delta t) - \lambda_1^2(0)) - \frac{1}{\lambda_1(t_n + \Delta t)} \frac{\partial W}{\partial I_1} + \frac{1}{\lambda_1(t_n + \Delta t)} \frac{\partial W}{\partial I_2} +
\]

\[
\frac{\lambda_1^2}{2} g_1(\lambda_1^2(t_n + \Delta t) - \lambda_1^2(0)) - \frac{1}{2\lambda_1} g_1(\frac{1}{\lambda_1(t_n + \Delta t)} - \frac{1}{\lambda_1(0)}) +
\]

\[
\frac{\lambda_1^2}{2} \left( g_2 \frac{\partial \lambda_1^2}{\partial \tau} \right)_{t_n} + \frac{\lambda_1^2}{2} g_2 e^{-\Delta t/2} (\lambda_1^2(t_n + \Delta t) - \lambda_1^2(t_n)) -
\]

\[
\frac{1}{2\lambda_1} \left( g_2 \frac{\partial (1/\lambda_1)}{\partial \tau} \right)_{t_n} -
\]

\[
\frac{1}{2\lambda_1} g_2 e^{-\Delta t/2} (1/\lambda_1(t_n + \Delta t) - 1/\lambda_1(t_n + \Delta t)), \quad \tau < t.
\]

Eq. 96
To solve this equation numerically, initial conditions of the hereditary integrals must be specified. Since the system is in equilibrium and is independent of time at time $t_0$, the initial conditions of both hereditary integrals are simply zero.

The above equation is the constitutive equation for the uniaxial case. It can be used to predict the stretch of a dielectric elastomer membrane that is subjected to a constant load and Maxwell stress across the thickness of the membrane. A flow chart illustrating the numerical integration algorithm needed to solve the equation for the unknown stretch $\lambda_1$ is shown in Figure 29. The corresponding algorithm written using Mathematica is provided in Appendix C.
**Enter material constants and properties**

Elastic material constants, relaxation modulus, dielectric constant of the material, applied constant tensile load, prestretch of uniaxial specimen at start of the test, and undeformed thickness of the membrane.

**Set initial conditions**

All values of the hereditary integrals at time $t_0=0$.

**Begin loop for time $t=t_i$, $i=1,\ldots,N$**

1. Solve the following equation for $\lambda_1$ using the Mathematica command SOLVE.

$$
0 = T_{11}(t_n + \Delta t) - T_{33}^M + \\
2 \left( \lambda_1^2(t_n + \Delta t) - \frac{l}{\lambda_1(t_n + \Delta t)} \right) \left( \frac{1}{\lambda_1(t_n + \Delta t)} + \frac{1}{\lambda_1(t_n + \Delta t)} \right) + \\
\frac{\lambda_1^2}{2} g_1(2 \lambda_1^2(t_n + \Delta t) - \lambda_1^2(0)) - \frac{1}{2 \lambda_1} g_1(\lambda_1(t_n + \Delta t) - \lambda_1(0)) + \\
\frac{\lambda_1^2}{2} g_2 \left( \frac{\partial \lambda_1^2}{\partial \tau} \right)_{t_n} + \frac{\lambda_1^2}{2} g_2 e^{-\lambda_1^2/2} \left( \lambda_1^2(t_n + \Delta t) - \lambda_1^2(t_n) \right) - \\
\frac{1}{2 \lambda_1} g_1 \left( 1 / \lambda_1 \right)_{t_n} - \\
\frac{1}{2 \lambda_1} g_2 e^{-\lambda_1^2/2} (1 / \lambda_1(t_n + \Delta t) - 1 / \lambda_1(t_n + \Delta t)), \quad \tau < t
$$

2. After the solution has been found, assign current values of $\lambda_1$ as the previous values for use in the next time step.

3. Calculate the new Maxwell stress due to thinning of the membrane

End Loop

Figure 29. Flow chart of the numerical integration algorithm to solve for the principal stretch ratio $\lambda_1$. 
3.5.2 Annular Case

The Cauchy stress equations (Eq. 79 and Eq. 81) and the time rate of change of the radial Cauchy stress, $T_r^\prime$, (Eq. 80) can be rewritten in recursion form following the same mathematical procedures as for the uniaxial case. The resulting equations are shown without derivation.

Principal radial Cauchy stress equation in recursion form:

$$
T_r(t_n + \Delta t) = -T_{zz}^\prime + 2(\lambda_1^2 - \lambda_\theta^2)(\frac{\partial W}{\partial l_1} + \lambda_\theta^2 \frac{\partial W}{\partial l_2}) + \\
\frac{\lambda_1^2}{2} g_1(\lambda_1^2(t_n + \Delta t) - \lambda_1^2(0)) - \\
\frac{1}{2\lambda_1^2\lambda_\theta^2} g_1\left(\frac{\partial^2}{\partial \tau^2}(t_n + \Delta t) - \frac{1}{\lambda_1^2(0)}\lambda_1^2(0)\right) + \\
\frac{\lambda_1^2}{2} \int_0^{t_n} g_2 e^{-t_n - \tau} \frac{\partial^2}{\partial \tau^2} + \frac{\lambda_1^2}{2} g_2 e^{-\Delta t/2} (\lambda_1^2(t_n + \Delta t) - \lambda_1^2(t_n)) - \\
\frac{1}{2\lambda_1^2\lambda_\theta^2} \int_0^{t_n} g_2 e^{-(t_n - \tau)} \frac{\partial(1/\lambda_1^2\lambda_\theta^2)}{\partial \tau} d\tau - \\
\frac{1}{2\lambda_1^2\lambda_\theta^2} g_1 e^{-\Delta t/2} (\lambda_1^2(t_n + \Delta t) - \lambda_1^2(0)) - \frac{1}{\lambda_1^2(0)}(t_n + \Delta t), \quad \tau < t.
$$

Eq. 97

First derivative of the principal radial Cauchy stress equation with respect to $R$:

$$
T_{rr}^\prime = 4(\lambda_1^\prime \lambda_\theta^\prime - \lambda_\theta^\prime \lambda_1^\prime) l_n + \Delta t \left(\frac{\partial W}{\partial l_1} + 2\lambda_\theta^2 \frac{\partial W}{\partial l_2}\right) r_{n+1} + \\
+ (\lambda_1^\prime \lambda_\theta^\prime) r_{n+1} \left(g_1 \lambda_1^\prime \lambda_\theta^\prime\right) r_{n+1} + Ce^{-\Delta t/2} \left(\lambda_1^\prime \lambda_\theta^\prime - \lambda_\theta^\prime \lambda_1^\prime\right) \\
+ \lambda_1^2(t_n + \Delta t) e^{-\Delta t/2} \left(g_1 \lambda_1^\prime \lambda_\theta^\prime\right) r_{n+1} + Ce^{-\Delta t/2} \left(\lambda_1^\prime \lambda_\theta^\prime - \lambda_\theta^\prime \lambda_1^\prime\right) \\
+ \frac{1}{\lambda_1^2(\lambda_\theta^2)} l_n r_{n+1} \left(e^{-\Delta t/2} \left(g_1 \lambda_1^\prime \lambda_\theta^\prime\right) r_{n+1} + Ce^{-\Delta t/2} \left(\lambda_1^\prime \lambda_\theta^\prime - \lambda_\theta^\prime \lambda_1^\prime\right) r_{n+1}\right) \\
+ \frac{1}{\lambda_1^2(\lambda_\theta^2)} l_n r_{n+1} \left(e^{-\Delta t/2} \left(g_1 \lambda_1^\prime \lambda_\theta^\prime\right) r_{n+1} + Ce^{-\Delta t/2} \left(\lambda_1^\prime \lambda_\theta^\prime - \lambda_\theta^\prime \lambda_1^\prime\right) r_{n+1}\right) \\
+ \lambda_1^2 t_n r_{n+1} \left(g_1 \lambda_1^\prime \lambda_\theta^\prime\right) r_{n+1} + Ce^{-\Delta t/2} \left(\lambda_1^\prime \lambda_\theta^\prime - \lambda_\theta^\prime \lambda_1^\prime\right) r_{n+1}\right) \\
+ \left(\frac{\lambda_1^\prime}{\lambda_1^2(\lambda_\theta^2)}\right) r_{n+1} \left(g_1 \lambda_1^\prime \lambda_\theta^\prime\right) r_{n+1} + \left(\frac{\lambda_1^\prime}{\lambda_1^2(\lambda_\theta^2)}\right) r_{n+1} \left(g_1 \lambda_1^\prime \lambda_\theta^\prime\right) r_{n+1}.
$$

Eq. 98
Principal circumferential Cauchy stress equation in recursion form:

\[ T_{\theta\theta}(t_n + \Delta t) = -T_z(t_n)^M + 2 \left( \lambda_\theta^2 - \lambda_z^2 \right) \left( \frac{\partial W}{\partial I_1} + \lambda_z^2 \frac{\partial W}{\partial I_2} \right) + \]

\[ \frac{\lambda_\theta^2}{2} g_1 (\lambda_\theta^2 (t_n + \Delta t) - \lambda_\theta^2 (0)) - \]

\[ \frac{1}{2 \lambda_\theta^2 \lambda_\theta} g_1 \left( \frac{\lambda_\theta^2 (t_n + \Delta t)}{\lambda_\theta^2 (0)} - \lambda_\theta^2 (0) \right) + \]

\[ \frac{\lambda_\theta^2}{2} \int_0^{t_n} g_2 e^{-\left(\frac{t_n - \tau}{\tau_1}\right)} \frac{\partial^2 \lambda_\theta}{\partial \tau^2} d\tau + \frac{\lambda_\theta^2}{2} g_2 e^{-\Delta t/2} (\lambda_\theta^2 (t_n + \Delta t) - \lambda_\theta^2 (t_n)) - \]

\[ \frac{1}{2 \lambda_\theta^2 \lambda_\theta} \int_0^{t_n} g_2 e^{-\left(\frac{t_n - \tau}{\tau_1}\right)} \frac{\partial (1/ \lambda_\theta^2)}{\partial \tau} d\tau - \]

\[ \frac{1}{2 \lambda_\theta^2 \lambda_\theta} g_2 e^{-\Delta t/2} \left( \frac{1}{\lambda_\theta^2 (t_n + \Delta t)} \lambda_\theta^2 (t_n + \Delta t) - \frac{1}{\lambda_\theta^2 (0)} \lambda_\theta^2 (0) \right), \quad \tau < t. \]

Recall in the last part of Section 3.4 the expression for \( d\lambda/\partial R \) was expressed as an integro-differential equation. Here, the integro-differential equation has been cast into a recursion form suitable for numerical integration. An algorithm was written in Matlab to solve for the unknown principal stretch ratios. The flow chart of the numerical algorithm is shown in Figure 30 and the Matlab program is provided in Appendix D.
Figure 30. Flow chart of the numerical integration algorithm to solve for the principal stretch ratios \( \lambda_r \) and \( \lambda_\theta \).

**Enter constants**

Elastic and viscoelastic material constants, dielectric constant of the material, undeformed inner and outer radii, prestretch, and applied voltage.

**Set initial conditions**

All values of the hereditary integrals are zero at time \( t=0 \).

**Begin loop for time \( t=t_i, i=1, \ldots, N \)**

1. Use the BVP4C solver to solve \( d\lambda_r/dR \) and \( d\lambda_\theta/dR \) for the unknown variables \( \lambda_r \) and \( \lambda_\theta \).
2. After the solution has been found, assign the current values of \( \lambda_r, \dot{\lambda}_r, \lambda_\theta, \dot{\lambda}_\theta \), and all hereditary integrals as previous values (for use in the next time step).
3. Calculate the new Maxwell stress due to thinning of the membrane.

**End Loop**
3.6 Determination of Material Relaxation Modulus

3.6.1 Experimental Setup of a Uniaxial Constant Load Creep Test

A constant load uniaxial creep test was conducted on a sheet of VHB 4910 (1 mm thickness) elastomer in the shape as shown in Figure 31. One end of the specimen was attached to a fixture and the opposite end was subjected to a constant load of 0.57 N (reference Figure 32). The VHB 4910 material is commonly used as a very high bond adhesive and this property was used to adhere the test specimen to the fixture. The displacement of a constant load of 0.57 N was monitored. This was accomplished by using an ILD-1800 laser displacement sensor (Micro Epsilon, Raleigh, North Carolina) connected to a Siglab Model 20-42 data acquisition system (Spectral Dynamics, San Jose, California). Data was sampled for 12 hours at 2 kHz. A manual reading of the displacement was recorded after 168 hours. All experiments were conducted at $70 \pm 2^\circ F$ and $18 \pm 2\%$ relative humidity.

Figure 31. Schematic of the uniaxial tensile specimen used for the constant load creep tests. $A= 25.4$ mm., $B= 76.2$ mm., $C=25.4$ mm., $D=25.4$ mm., $E= \text{Radius of } 2.4$ mm.
3.6.2 Uniaxial Constant Load Creep Test Results

The results of the constant load creep tests are shown in Figure 33. The primary stage of creep is observed by a rapid and linear decrease in stretch rate (reference Figure 34) immediately after the start of the test. After approximately 20 seconds, 98% of the stress relaxation has occurred. At this point, the stretch rate has decreased dramatically to $3.3 \times 10^{-4}$ sec$^{-1}$ and continues to decrease nonlinearly as a function of time. Deformation continued even after 168 hours (not shown) at which time the uniaxial stretch ratio of the specimen was calculated to be 1.65. The nonlinear decreasing stretch rate seems contradictory to what is shown in Figure 34 but is clarified graphically in Figure 35 the time scale is magnified. A secondary stage of creep [47, 48] characterized by a constant stretch rate is absent as is the case in many polymers [47, Pg. 13]. At 0.57 N, there was
no existence of a third creep stage that marks the failure of the elastomer, even after 1344 hours (eight weeks).

Figure 33. Uniaxial tensile test result. Viscoelastic creep of a tensile specimen subjected to a constant load of 0.57 N and zero electric field.
Figure 34. Experimental result. Viscoelastic stretch rate of tensile specimen subjected to a constant load of 0.57 N and zero electric field. Stretch rate below the solid line is less than $3.3 \times 10^{-4}$ sec.

Figure 35. Experimental result. Nonlinear secondary stage of creep.
3.6.3 Determination of the Relaxation Modulus

Two well known single time relaxation models, Kohlrausch [49] and Debye [50], are considered. Kohlrausch’s low frequency (long time) stretched exponential is most often used to describe various relaxation phenomena [51, 52] as well as stress relaxation in polymers [53], [54]. Through trial and error, it was found that modified forms of these models through the addition of a constant provided a better fit to experimental data.

Debye standard:

\[ g_v(t) = g_1 e^{-t/\tau_1} \]  

Eq. 100

Debye modified:

\[ g_v(t) = g_0 + g_1 e^{-t/\tau_1} \]  

Eq. 101

Kohlrausch standard:

\[ g_v(t) = g_1 e^{-t/(\alpha \tau_1)} \]  

Eq. 102

\[ 0 < \alpha < 1 \]

Kohlrausch modified:

\[ g_v(t) = g_0 + g_1 e^{-t/(\alpha \tau_1)} \]  

Eq. 103

\[ 0 < \alpha < 1 \]

For all forms, the time constant, \( \tau_1 \), indicates the time required for the normalized stress with respect to \( g_1 \) to relax to \( 1/e \). There is no interpretation of the \( \alpha \) term in the Kohlrausch model.

An algorithm (Appendix D) utilizing Matlab’s optimization routine FMINSEARCH is used to determine the best values of the variables \( g_0, g_1, \) and \( \tau_1 \) by minimizing the root mean square error (RMSE) objective function

\[
\text{RMSE} = \sqrt{\frac{\sum_{i=1}^{N} (T^\text{theoretical}_i - T^\text{experimental}_i)^2}{N}}
\]

Eq. 104

where \( T^\text{theoretical}_i \) and \( T^\text{experimental}_i \) are discrete theoretical (Eq. 96) and experimental uniaxial Cauchy stresses, respectively, and \( N \) is the total number of theoretical and experimental
comparisons evaluated on the time interval $[0, t]$. Cauchy stress $T_{\text{experimental}}$ is obtained by using the following relation

$$T_{\text{experimental}}(t_n + \Delta t) = \sigma_{\text{eng}}^{\text{experimental}}(t_n + \Delta t)$$  \hspace{1cm} \text{Eq. 105}$$

where $\sigma_{\text{eng}}$ is the engineering stress due to constant load.

The algorithm proved to be sensitive to the number of evaluations, $N$, and the time interval $t=[0,t]$ chosen. The relaxation modulus using the modified Debye and Kohlrausch forms were calculated using a time step of 0.01 seconds with 60000 evaluations ($N$), on a time interval of $t = [0, 60]$

$$g_v(t) = 18,567 + 63,968e^{-t/5.8}$$  \hspace{1cm} \text{Eq. 106}$$

and

$$g_s(t) = 18,583 + 64,423e^{-t/3.353}$$  \hspace{1cm} \text{Eq. 107}$$

respectively. Correlation coefficients of $R^2=0.76$ were obtained for both models indicating that either form of the relaxation modulus was suitable. Since the relaxation modulus based upon the modified Debye was simpler in form, it was used in the development of the constitutive model.

Correlation between theoretical and experimental results was sensitive to the number of evaluations, $N$, and the time interval $t=[0,t]$ for which the objective function was minimized. To demonstrate this dependence, two relaxation moduli were calculated. Relaxation modulus $g_a$ was obtained by minimizing the objective function over the time interval $t=[0,60]$ with $N=6x10^4$ to yield

$$g_a(t) = 38 + 129.8e^{-t/5.6}$$  \hspace{1cm} \text{Eq. 108}$$

The second relaxation modulus, $g_b$, was obtained by minimizing the objective function over a large time interval of $t=[0,1000]$ with $N=1x10^6$ to yield

$$g_b(t) = 6.1 + 32e^{-t/103.2}.$$  \hspace{1cm} \text{Eq. 109}$$
Both relaxation moduli were obtained using a time step of 0.01 seconds. These relaxation moduli are used in the uniaxial viscoelastic model for dielectric elastomer membranes. The theoretical results are compared to experimental uniaxial constant load tests at 0.57 N and 1.2 N and are presented next.

### 3.7 Results and Discussion – Uniaxial Case

Theoretical results from the uniaxial model are compared to the results of the constant load tests at 0.57 N and 1.2 N. These loads represent the lower and upper end of the load spectrum investigated in this research.

Figure 36 shows the experimental and theoretical uniaxial viscoelastic response for a constant load of 0.57 N. For the $g_a(t)$ relaxation modulus, good agreement exists near the time constant of $\tau_a=5.6$ seconds. For times greater than approximately eight seconds, correlation between theoretical and experimental results begins to deteriorate. The single time relaxation may not be able to capturing both the rapid decrease in the stretch rate at the onset of the test followed by a 98% decrease in the stretch relaxation within 10 seconds of test start. Using the relaxation modulus $g_b(t)$ gave better correlation between theoretical and experimental for times greater than $\tau_a$. However, this was at the expense of poor correlation at times near the onset of the applied load.

A similar comparison was made to a test conducted with a uniaxial constant load of 1.2 N (reference Figure 37). This comparison demonstrates the ability of the relaxation modulus obtained that was obtained from an experiment with a constant load of 0.5 N to predict the viscoelastic deformations at a higher load. Similar trends were observed in comparison to the 0.57 N constant load test. However, the quality of agreement decreased as evidenced by the lack of agreement even near the time constants $\tau_a$ and $\tau_b$ ($\tau_b=103$ seconds) for both relaxation moduli $g_a(t)$ and $g_b(t)$, respectively.

In conclusion, using the relaxation modulus $g_a(t)$ to predict viscoelastic deformation correlates well with experimental data within near the time constant of $\tau_a=5.6$ seconds. It is this time domain $0< t< \tau_a$ that is of interest in terms of applications to...
EAP actuator cycling times. To develop a model to predict viscoelastic deformations for a large time span of \([0,t]\) it may be necessary to include additional exponential terms in the mathematical form of the relaxation modulus. Increasing the number of exponential terms, however, increases the number of relaxation constants, \(g_i\) and the corresponding time constants, \(\tau_i\). This poses a challenge in obtaining a suitable relaxation function due to the sensitivity of the FMINSEARCH algorithm to initial guess values required for these constants.

The uniaxial analytical model can now be used to investigate to determine the viscoelastic response of a uniaxial specimen subjected to a simultaneous electric field and constant load (reference Figure 38) near the time constant of the relaxation modulus. For a specimen that is subjected to a constant load of 35 kPa and an initial electric field of 40 MV/m, theoretical results using the relaxation function \(g_\alpha(t)\) (Eq. 108) show an 18% increase in stretch within eight seconds of the applied step voltage. This increase is not only attributed to the viscoelastic nature of the elastomer but also due to the time varying electric field resulting from the thinning of the membrane which has been accounted for in the uniaxial viscoelastic model.
Figure 36. Comparison of uniaxial constant load (0.57 N) creep test with theoretical predictions. (a) Theoretical prediction using relaxation modulus $g_a(t)$ (b) Theoretical prediction using relaxation modulus $g_b(t)$.

Figure 37. Comparison of uniaxial constant load (1.2 N) creep test with theoretical predictions. (a) Theoretical prediction using relaxation modulus $g_a(t)$ (b) Theoretical prediction using relaxation modulus $g_b(t)$. 
3.8 Experimental Setup of Annular Actuators

The annular actuators were constructed from 3M VHB™ 4910. A 50.8 mm diameter circle was drawn on each of three squares (15.2 cm²) of the dielectric material with the center located by two diametrical lines at 0° and 90° (reference Figure 39a). A 7.9 mm diameter circle, concentric to the 50.8 mm diameter circle, was cut using a steel punch. The material was radially stretched over a PVC pipe (12.9 cm outer diameter, wall thickness of 5.6 mm (reference Figure 39b). The points located every 45° on the dielectric material were systematically stretched and affixed to the corresponding scribed locations on the surface of the PVC fixture to achieve a uniform radial stretch.
To electrically activate the entire annular region of the actuator, 846-80G conductive carbon grease (MG Chemicals, Surrey, B.C., Canada) was evenly applied onto the top and bottom surfaces. In order to avoid electrical shorts, the conductive grease was applied 3 mm short of the inner radius on both surfaces. Electrical leads were made from strips (3 x 50 mm$^2$) of double-sided carbon tape (SPI Supplies Incorporated, West Chester, PA) and 1181 EMI copper foil shielding tape (3M Company, St. Paul, Minnesota). They were placed 180° apart on the top and bottom surface of the actuator as shown in Figure 40a and Figure 40b, respectively. A Trek 610D high voltage power supply (Trek Inc., Medina, New York) was connected to the leads and used to electrically load the actuator.

The time-dependent displacement of a point on the inner radius was monitored by directing the ILD-1800 laser displacement sensor to a target placed on the edge of the inner radius (reference Figure 41). The laser displacement connected to the Siglab Model 20-42 data acquisition system at displacement data was captured at a frequency of 5 kHz. Each actuator was tested at a step voltage of 4 kV, 5 kV, and 6 kV for 60 seconds. The dwell times between peak voltages were approximately five minutes. Based upon preliminary test data, five minutes was adequate to allow the actuators to reach mechanical equilibrium before the next increase in voltage was applied.
3.9 Results and Discussion - Annular Actuators

Constant voltage creep tests of prestretched annular actuators have been conducted and the results are shown in Figure 42 through Figure 45. Less than 4% strain
in the inner radius was observed for an actuator with an inner radius prestretch of 2.46 (reference Figure 42, 6 kV). Hence, larger values of the inner radius prestretch (up to 4.65) were investigated and are shown in Figure 43 through Figure 45. Dielectric breakdown failures across the thickness of the membrane were observed on two actuators (reference Figure 44 and Figure 45) The failures were not immediate and occurred as small holes on the surface of the membrane. The failures were not unexpected as the results from the uniaxial constant load tests showed material failure for uniaxial stretch greater than 3.0 (Section 2.6). Hence for actuators which utilize prestretch of values greater than 3, the actuator’s reliability and life may be adversely affected.

Figure 42. Constant voltage creep test results for an annular actuator with an inner radius prestretch of 2.46.
Figure 43. Constant voltage creep test for an annular actuator with an inner radius prestretch of 3.49. (Note: Actuator tested at 4 kV was inadvertently tested for 90 seconds).
Figure 44. Constant voltage creep test for an annular actuator with an inner radius prestretch of 4.65. Solid heavy line indicates the actuator experienced dielectric breakdown failure at approximately 23 seconds.
Theoretical viscoelastic results of an annular actuator are obtained using the algorithm in Appendix D. The annular actuator has a prestretch of one and is subjected to a constant voltage of 4 kV, 5 kV, and 6 kV are shown in Figure 41 through Figure 45, respectively. The algorithm solved for the principal radial and circumferential stretch ratios every 0.1 seconds however only results at every 0.5 seconds are shown in these figures. The Mooney-Rivlin material constants used in the numerical algorithm were $C_1 = 8.84$ kPa; $C_2 = 4.58$ kPa (reference Chapter 2, Section 2.6) and the relaxation modulus was

$$g_r(t) = 38 + 129.8e^{-t/5.6} \text{ kPa.}$$

Eq. 110

The undeformed thickness of the membrane was 0.5 mm.
In all three figures, as time progressed the system moved towards a state of elastic equilibrium as observed by the converging profiles of the radial and circumferential stretch ratios. The radial stretch rates at the inner radius are calculated for three voltage levels (4 kV, 5 kV, and 6 kV) and are shown in Figure 49. The slope of the stretch rate versus time curve increases negatively as higher voltages are applied. It is possible to lessen the viscoelastic effects by increasing the prestretch of the material before a voltage is applied.

Figure 46. Theoretical viscoelastic response of an annular membrane subjected to a simultaneous outer radius prestretch of 1.45 and constant voltage of 4 kV. The radial and circumferential stretch ratios are shown at every 0.5 second increments. As time increases the circumferential
Figure 47. Theoretical viscoelastic response of an annular membrane subjected to a simultaneous outer radius prestretch of 1.45 and constant voltage of 5 kV. The radial and circumferential stretch ratios are shown at every 0.5 second increments.
Figure 48. Theoretical viscoelastic response of an annular membrane subjected to a simultaneous outer radius prestretch of 1.45 and constant voltage of 6 kV. The radial and circumferential stretch ratios are shown at every 0.5 second increments.
A nonlinear finite deformation viscoelastic model for dielectric elastomer membranes was developed using Christensen’s viscoelastic model in the stretch regime $1.5<\lambda<3$. This model is applicable for small and large nonlinear deformations. Also, the model reduces to a hyperelastic model of choice as $t$ approaches infinity. In this research, the Mooney-Rivlin elastic material model was utilized. Uniaxial creep tests were conducted and compared to theoretical predictions. The degree of agreement was a function of the relaxation modulus, $g(t)$. A single relaxation function that characterizes the material over a large time $[0,t]$ domain could provide better correlation between analytical and

\[ \frac{\dot{\lambda}}{\lambda} \]

Figure 49. Theoretical radial stretch rates $\frac{\dot{\lambda}}{\lambda}$ at the inner radius.

### 3.10 Conclusions

A nonlinear finite deformation viscoelastic model for dielectric elastomer membranes was developed using Christensen’s viscoelastic model in the stretch regime $1.5<\lambda<3$. This model is applicable for small and large nonlinear deformations. Also, the model reduces to a hyperelastic model of choice as $t$ approaches infinity. In this research, the Mooney-Rivlin elastic material model was utilized. Uniaxial creep tests were conducted and compared to theoretical predictions. The degree of agreement was a function of the relaxation modulus, $g(t)$. A single relaxation function that characterizes the material over a large time $[0,t]$ domain could provide better correlation between analytical and
experimental data. This involves assuming a relaxation function that involves multiple terms.
Chapter 4

Finite Element and Experimental Analyses of Non-axisymmetric Dielectric Elastomer Actuators

4.1 Brief Introduction to Viscoelasticity

In the previous chapter, constitutive relations for the viscoelastic responses of axisymmetric dielectric elastomer actuators were developed. In this chapter, deformations of non-axisymmetric actuators are addressed. Interesting DEAs with non-axisymmetric geometries were recently proposed by Kofod [55] in the general shapes of squares and rectangles utilizing a flexible frame. These actuators deform out-of-plane when an electric field is applied across the thickness of the DEA. Another actuator in a shape of a saddle has been developed by Artificial Muscle Incorporated, Menlo Park, CA. The saddle actuator is incorporated at the center of a flapping wing assembly which causes the wings to flutter upon cyclic voltage application [56].

These unique actuator configurations amongst others demonstrate a need for developing models for non-axisymmetric DEAs. Currently, modeling the strain response of DEAs has been limited to axisymmetric or uniaxial DEA geometries and loading. Furthermore, these models either neglect the highly nonlinear deformation or viscoelastic response of the dielectric elastomer or assume small strain response. Pelrine et al. [6] and Kofod [9] modeled the uniaxial strain response of prestretched dielectric elastomers. Goulbourne et al. [57] used a higher order hyperelastic Ogden material model to describe the large nonlinear deformation of axisymmetric dielectric membranes undergoing inflation. Wissler et al. [58] addressed the planar deformation of a circular membrane using a FE model. Carpi [59] models the dielectric response of silicone dielectric elastomer tubes using small strain theory. Modeling the deformation of a rectangular strip of silicone dielectric elastomer (Dow Corning, HS III RTV) using the finite element software ANSYS was addressed by Yang et al. [60].
Simple analytical solutions for general non-axisymmetric geometries are not possible. Hence, this research is dedicated to developing an FE model for DEAs of non-axisymmetric geometries using the general purpose FE software ABAQUS. In Chapter 3 an analytical model for large nonlinear deformations of DEAs fabricated with the VHB 4910 dielectric elastomer was developed. These experimental results (reference Figure 16 in Chapter 2 and Figure 33 in Chapter 3) are utilized in the FE model to determine the hyperelastic and viscoelastic material constants. In this chapter, non-axisymmetric DEAs are fabricated using the VHB 4905 dielectric elastomer. The square dielectric elastomer sheets with circular and square cavities at the center are prestretched and affixed onto rigid frames to form elliptical and rectangular cavities. The DEAs are then subjected to electric fields up to 6.5kV (16 MV/m) and results after approximately 90 seconds are used to validate the FE model.

4.2 Finite Element Model

4.2.1 Hyperelastic Material Models and Constants

Experimental results from uniaxial time-independent (reference Figure 16 in Chapter 2) tensile tests are used within ABAQUS to evaluate various hyperelastic material models that best fit the experimental data. Nominal stress and nominal strain data pairs are required in ABAQUS for the evaluation. The Cauchy stress and the corresponding principal stretch data from the uniaxial tensile tests presented in Chapter 2 (reference Figure 16) were converted to nominal.

The hyperelastic material models considered were the polynomial, Mooney-Rivlin, Ogden, and reduced polynomial. These material models utilize a strain energy function to formulate the stress-strain relationships. The dielectric material is assumed to be isotropic and incompressible and operating under isothermal conditions.

The polynomial strain energy function has the following form
where $U$ is the strain energy potential per reference volume, $I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$ and $I_2 = \lambda_1^2 \lambda_2^2 \lambda_3^2$ are the first and second strain invariants which are functions of the principal stretch ratios ($\lambda_1$, $\lambda_2$, and $\lambda_3$), respectively, $J_{el}$ is the elastic volume strain which is zero for incompressible materials, $D_i$ determines the compressibility of the material which is also zero for the model considered here due again to the incompressibility assumption, and the $C_{ij}$ terms are material constants. If $N=1$, the polynomial strain energy potential reduces to that of the Mooney-Rivlin form

$$U = C_{10} (I_1 - 3) + C_{01} (I_2 - 3).$$

Eq. 112

The reduced polynomial strain energy function is obtained by setting all $C_{ij}$ with $j \neq 0$ to zero to yield

$$U = \sum_{i=1}^{N} C_{i0} (I_1 - 3)^i.$$

Eq. 113

The Ogden strain energy function is given by

$$U = \sum_{i=1}^{N} \frac{2\mu_i}{\alpha_i^2} \left( \lambda_1^{\alpha_i} + \lambda_2^{\alpha_i} + \lambda_3^{\alpha_i} - 3 \right).$$

Eq. 114

where $\alpha_i$ and $\mu_i$ are material constants. It is linear with the material constants $\mu_i$ but highly nonlinear with the powers $\alpha_i$ of the principal stretch ratios. Hence it is capable of capturing highly nonlinear stress-strain characteristics.

The strain energy function for the Arruda-Boyce model is based upon the stress-stretch relationship of an elastic network that uses an 8-chain polymer model to capture the three-dimensional response of polymers. This is accomplished by using Langevin statistics to describe the rapid increase in stress as the distorted polymer chain stretch approaches a locking or limiting stretch $\lambda_m$. The Arruda-Boyce strain energy function is given by

$$U = \mu \sum_{i=1}^{8} \frac{C_i}{\lambda_m^{2i-2}} (I_1^i - 3^i).$$

Eq. 115
where the terms $C_i (C_1=1/20, C_2=1/20, C_3=11/1050, C_4=19/7000, \text{and } C_5=519/673750)$ results from a series expansion of the inverse Langevin function, and $\mu$ is related to the initial shear modulus $\mu_0$ by the following relation

$$\mu_0 = \mu (1 + \frac{3}{5\lambda_m^2} + \frac{99}{175\lambda_m^4} + \frac{513}{875\lambda_m^6} + \frac{42039}{67375\lambda_m^8}).$$  

Eq. 116

The FE results of the hyperelastic material evaluations consist of utilizing curve-fitting algorithms to determine the material constants that would best fit the experimental data. The results of these evaluations are shown graphically in Figure 50 and Figure 51 with material constants for each model specified in Table 4. Figure 50 shows that Ogden material model of $N=1$ through 4 exhibits approximately the same stress-strain relationships as profiles are close to one another. Figure 51 shows that the material models for the polynomial $N=1$ (Mooney-Rivlin) and the reduced polynomial ($N=3$) yield similar stress-strain responses. However, the Arruda-Boyce model yields a stress-strain response that is not as stiff in comparison to all the models.

<table>
<thead>
<tr>
<th>Material model</th>
<th>Material constants</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Ogden $N=1$</strong></td>
<td>$\mu_1 = 25792.7$ Pa; $\alpha_1 = 1.61$</td>
</tr>
<tr>
<td><strong>Ogden $N=2$</strong></td>
<td>$\mu_1 = 12011.3$ Pa; $\mu_2 = 18714.1$ Pa; $\alpha_1 = 2.24$; $\alpha_2 = -2.02$</td>
</tr>
<tr>
<td><strong>Ogden $N=3$</strong></td>
<td>$\mu_1 = 13631.0$ Pa; $\mu_2 = 283.5$ Pa; $\mu_3 = 16330.0$ Pa; $\alpha_1 = 2.00$; $\alpha_2 = 4.00$; $\alpha_3 = -2.00$</td>
</tr>
<tr>
<td><strong>Ogden $N=4$</strong></td>
<td>$\mu_1 = 10329.3$ Pa; $\mu_2 = -16.5$ Pa; $\mu_3 = 8285.8$ Pa; $\mu_4 = 14189.3$ Pa; $\alpha_1 = 2.00$; $\alpha_2 = 4.00$; $\alpha_3 = -2.00$; $\alpha_4 = -4.00$</td>
</tr>
<tr>
<td><strong>Arruda-Boyce</strong></td>
<td>$\mu = 22057.5$ Pa; $\mu_0 = 22057.5$; $\lambda_m = 6047.8$;</td>
</tr>
<tr>
<td><strong>Mooney-Rivlin</strong></td>
<td>$C_{10} = 7920.6$; $C_{01} = 6690.9$</td>
</tr>
<tr>
<td><strong>Reduced polynomial $N=3$</strong></td>
<td>$C_{10} = 12664.1$ Pa; $C_{20} = -411.4$ Pa; $C_{30} = 23.2$ Pa;</td>
</tr>
</tbody>
</table>
Figure 50. Material evaluation results for the Ogden material model for $N=1$ through 4.
4.2.2 Viscoelastic Material Model and Constants

The viscoelastic responses are captured using multiple terms of the Prony series that represents the time-dependent stress relaxation of the material. The relaxation modulus can be described as a sum of a series of terms known as the Prony series

$$G(t) = G_0 \left(1 + \sum_{i=1}^{N} G_i e^{-t/\tau_i}\right).$$

Eq. 117
where $G_i$ are material constants that are determined from experimental results, $\tau_i$ is the relaxation time constant, and $G_0 = G_{\infty} + \sum_{i=1}^{n} G_i$ is the instantaneous shear modulus. The normalized shear modulus is defined as

$$ g(t) = \frac{G(t)}{G(t_0)} \quad \text{Eq. 118} $$

ABAQUS requires normalized shear compliance, the inverse of the normalized shear modulus, and time data pairs be provided to define the viscoelastic behavior. Normalized shear compliance is defined as

$$ j(t) = \frac{J(t)}{J(t_0)} \quad \text{Eq. 119} $$

where $J(t) = \gamma(t) / \tau_0$, $J(t_0) = \gamma(t_0) / \tau_0$, $\gamma(t)$ is the shear strain, $t_0 = 0$ that begins immediately after the instantaneous response, and $\tau_0$ is the nominal stress (or engineering stress).

The normalized shear compliance and time data pairs are obtained from the uniaxial tensile creep test (reference Figure 33). It is important to exclude the instantaneous deformation response of the elastomer due to the constant load when defining the material within ABAQUS. The region of time comprising the instantaneous response can be determined by using the creep test data. The stretch rate was calculated and then plotting against time (Figure 52). The VHB dielectric elastomer showed a dramatic decrease in stretch rate within the first several seconds. This indicates that instantaneous deformations are dominant in this time region as opposed to time-dependent deformations. The time at which the instantaneous region ended was defined to occur when the stretch rate no longer exhibited an approximately linear profile. It can be seen from Figure 52 that this occurs at approximately 2.5 seconds.

Long-term characteristics can be defined for a material by specifying a long-term normalized shear compliance value (SHRINF in ABAQUS)

$$ j(t = \infty) = \frac{J(t = \infty)}{J(t_0)} \quad \text{Eq. 120} $$

Specification of the long-term normalized shear compliance is optional in the case when there is sufficiently long creep test data available.
Figure 52. Stretch rate versus time results from a uniaxial tensile test showing the instantaneous deformation region. The instantaneous time-independent deformation region was defined to occur when the stretch rate no longer exhibited a linear profile. The solid line in this figure has been added to aid this visual.

Determination of the long-term normalized shear compliance is problematic for the VHB dielectric elastomer, as the material continues to deform even after 189 hours (reference Figure 3, Chapter 3). This indicates that the material has not reached mechanical equilibrium. It could also indicate that the viscoelastic behavior may be coupled with other mechanical phenomena. Due to these possibilities the long-term normalized compliance value was not used. Instead, the uniaxial creep test data of 727 seconds in duration was utilized. Discussion on the validity of this approach is discussed next.

The FE viscoelastic material evaluation result is shown in Figure 53. There exists very good correlation between the FE results and experimental data as their profiles overlap one another. The time-dependent deformation is captured using three terms of the
Prony series for which the constants are defined in Table 5. The FE simulations investigated in this research are within 60 seconds, which is significantly less than the available duration of the creep test data. Hence, using the creep tests data and foregoing the use of the long-term normalized shear compliance is justified.

Table 5. Prony series material constants obtained from the viscoelastic material evaluation option EVAL available in ABAQUS.

<table>
<thead>
<tr>
<th>Prony series</th>
<th>$g_1 = 0.286; g_2 = 0.199; g_3 = 0.152; $</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau_1 = 3.79; \tau_2 = 29.73; \tau_3 = 275.25;$</td>
<td></td>
</tr>
</tbody>
</table>
4.2.3 Geometry

The VHB 4905 (0.5 mm thickness) dielectric elastomer is modeled as a homogenous, isotropic, incompressible material with large nonlinear viscoelastic (time-dependent) response; the current stress is dependent upon the past strain history [25]. Plane stress is assumed as the thickness of the membrane is much less than its major dimensions. The first-order, axisymmetric, hybrid solid elements (CAX4H) are used in the model.

4.2.4 Kinematic Boundary Conditions

The dielectric elastomers are initially prestretched in the 1-direction while preventing the top and bottom edges of the actuators from deforming in the 2- direction (see Figure 54). After the prescribed prestretch, the outer boundaries of the model are constrained in the 1- and 2- directions. The inner edges of the cavities remain traction free while Maxwell stress acts on the major surfaces of the actuator in the 3- direction.

In Figure 55, a schematic of the electrically active (solid black) and inactive (light grey) regions of the membrane are shown for both actuator configurations. The inactive regions remain so during the entire FE simulation. These inactive regions that bound the inner free edges simulate actual test conditions where inactive regions were necessary to avoid electrical breakdown of the air surrounding the actuator.
Figure 54. FE models of the non-axisymmetric DEAs before prestretch. Boundary conditions imposed during the prestretching process of the dielectric elastomer are shown. a) Model with a rectangular cavity at the center. b) Model with a circular cavity at the center.

Figure 55. Schematic of the FE model showing depicting the electrically active (solid black) and inactive regions (light grey). The width of the inactive region is 2.5 mm before prestretch for both actuator configurations. (a) Square actuator configuration before prestretch. (b) Circle actuator configuration before prestretch

4.2.5 Finite Element Loading

Dielectric elastomer actuators realize deformation through electrostatic and electrostrictive stresses [61-63] induced by an electric field applied across the thickness, $t$, of the material. The VHB 4905 dielectric elastomer used in this research is known to have minimal electrostrictive effects [8] and thus electrostriction is considered to be
negligible. The thickness of the membrane is much less than its major dimensions. Hence, plane stress is assumed and fringe effects at the fixed outer and the free inner perimeters of the actuators can be neglected. Application of a voltage differential, $V_s$, across the thickness of the material creates electrostatic stress called Maxwell stress. The Maxwell compressive stress tensor, $T^M$, is given by

$$ T^M = \varepsilon_r \varepsilon_0 \mathbf{E} \otimes \mathbf{E} - \frac{\varepsilon_r \varepsilon_0}{2} \mathbf{E} \cdot \mathbf{E} $$

Eq. 121

where $\varepsilon_r$ is the relative dielectric constant of the DEA, $\varepsilon_0$ is the permittivity of vacuum (8.854E-12 F/m), and $\mathbf{E}$ is the electric field vector (volts/meter). The electric field $\mathbf{E}$ is assumed to be directed only in the 3-direction since the thickness is much less than the major dimension. In a Cartesian coordinate system, this results in the 1- and 2-components of the Maxwell stress being accounted as being negligible and the stress in the 3-direction given by

$$ T^M_3 = \frac{1}{2} \varepsilon_r \varepsilon_0 E^2. $$

Eq. 122

The expression for the relative dielectric constant of the DEA can be determined using the definition for the capacitance of a parallel-plate capacitor

$$ C = \varepsilon_0 \varepsilon_r A_{area} / t $$

Eq. 123

where $C$ is the capacitance of the DEA that is experimentally obtained, $A_{area}$ is the electrically active area, and $t$ is the thickness.

The variation of the Maxwell stress due to the thinning of the membrane is accounted for using the ABAQUS command DLOAD. The DLOAD command is used to prescribe a distributed loading on the plane by taking into account the local thickness of the dielectric elastomer. Hence, the value of the Maxwell stress at each integration point of the finite element by using Eq. 122.

To model the initial prestretch followed by the instantaneous application of the electric field, three load steps are utilized in ABAQUS. The first load step consists of the hyperelastic prestretch. It simulates the actual test conditions where the prestretched DEA was allowed to reach mechanical equilibrium. To model the viscoelastic response two additional load steps are required. The first is a viscoelastic response that has a very short duration of time to account for the instantaneous response. The next step consists of a 10
second time period during which the instantaneous Maxwell stress is held locally constant.

4.3 Experimental Setup

To validate the FE model, DEAs were fabricated and tested. The actuators were made using the VHB 4905 (0.5 mm thickness) polyacrylate dielectric elastomer. Prestretched actuators with elliptical and rectangular cavities were investigated (see Figure 56a and Figure 56b). These geometries were obtained by prestretching the elastomer sheets with either a circular or square cavity at the center onto a rigid frame. To achieve a uniform prestretch a grid consisting of perpendicular lines was first drawn onto one side of the elastomer as shown in Figure 56c and Figure 56d. Taking advantage of the sticky/tacky surface features of the VHB material, stretching blocks were temporarily adhered near and parallel to lines A-C and B-D on the elastomer sheet (not shown). Then line A-C on the elastomer was affixed to edge A-C on the frame. This was followed by a uniaxial stretch so that line B-D of the elastomer could be affixed onto edge B-D on the frame. Finally, lines A-B and C-D on the elastomer were manually stretched and affixed to edges A-B and C-D, respectively, on the frame.
Figure 56. (a) A typical dielectric elastomer with a rectangular cavity. (b) A typical dielectric elastomer with an elliptical cavity. (c) The dielectric elastomer with grids drawn on one side to aid the prestretching process. In this figure, a square at the center has been cut out. (d) A typical rigid test frame fixture with a center cavity. Dark lines have been drawn to accentuate the cavity.

To electrically activate the dielectric elastomer actuator, 846-80G conductive carbon grease was applied onto the top and bottom surfaces. A 2.5 mm perimeter along the inner free edge was not electroded to avoid electrical breakdown of the air surrounding the actuator. Flexible double-sided carbon tape (SPI Supplies Incorporated, West Chester, PA) and 1181 EMI copper foil shielding tape (3M Company, St. Paul, MN) were used as electrical leads on both major surfaces. A Trek 610D high voltage power supply (Trek Inc., Medina, New York) was connected to the leads to electrically load the actuator. Table 6 shows the geometries of four actuator configurations (Rectangle 1, Rectangle 2, Ellipse 1, and Ellipse 2) before the elastomer was prestretched. A schematic of a typical experimental setup is shown in Figure 57 that is similar to that of the procedure outlined in Section 3.6 of Chapter 3. The displacements as a function of time at locations A and B (reference Figure 58) were collected. Three actuators for each configuration were fabricated. Preliminary experimental tests showed that viscoelastic response of the elastomer from the prestretch was absent after 30
minutes. Hence, all actuators were brought to mechanical equilibrium before electric fields were applied. Each actuator was subjected to three square-wave voltage signals with increasing peak voltages of 6.0 and 6.5 kV (13.5 and 16 MV/m, respectively) for approximately 30 seconds. The dwell times between each voltages were approximately five minutes. The purpose of the dwell was to allow the actuator to return to a state of mechanical equilibrium. The dwell duration is less in comparison to the 30 minutes that was required for the actuator to reach mechanical equilibrium after the prestretch. This is because the planar strains due to Maxwell stress are smaller in comparison to the planar strains resulting from the prestretch process.

Figure 57. A typical experimental setup to obtain the displacement of a point at the inner cavity of dielectric elastomer. Here a rectangular dielectric elastomer is configuration is shown.
Table 6. Geometry and prestretch imposed on the actuators.

<table>
<thead>
<tr>
<th>Actuator configuration</th>
<th>Undeformed outer dimensions of the actuator ( L \times W ) (mm²)</th>
<th>Undeformed cavity dimensions</th>
<th>Hyperelastic prestretch ratios of the outer dimensions of the actuator</th>
<th>Capacitance after prestretch (nF)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rectangle 1</td>
<td>50.8 x 50.8</td>
<td>Square cavity: 25.4 x 25.4 mm²</td>
<td>[\lambda_1 = 1.60, \lambda_2 = 1.0]</td>
<td>0.48</td>
</tr>
<tr>
<td>Rectangle 2</td>
<td>50.8 x 50.8</td>
<td>Square cavity: 25.4 x 25.4 mm²</td>
<td>[\lambda_1 = 1.80, \lambda_2 = 1.0]</td>
<td>0.50</td>
</tr>
<tr>
<td>Ellipse 1</td>
<td>50.8 x 50.8</td>
<td>Circular cavity: Radius=9.5 mm</td>
<td>[\lambda_1 = 1.40, \lambda_2 = 1.0]</td>
<td>0.48</td>
</tr>
<tr>
<td>Ellipse 2</td>
<td>50.8 x 50.8</td>
<td>Circular cavity: Radius=12.7 mm</td>
<td>[\lambda_1 = 1.40, \lambda_2 = 1.0]</td>
<td>0.48</td>
</tr>
</tbody>
</table>

Figure 58. Prestretched actuators indicating points AA and BB of the major and minor chords, respectively, of the cavities.

4.5 Results and Discussion

The material evaluation within ABAQUS yielded several potential material models. They included the Ogden, Arruda-Boyce, reduced polynomial, and the Mooney-Rivlin material models. The results of these models are compared to the uniaxial creep tensile test data (reference Figure 8 in Chapter 3) and are shown in Figure 60 and Figure 61. All models except for the Arruda-Boyce case show good correlation with the experimental data. The best correlations are obtained from the reduced polynomial of order \( N=3 \) and the Ogden of order \( N=1 \) material models.
Figure 60. FE results for the Arruda-Boyce, reduced polynomial ($N=3$), and the Mooney-Rivlin material models. The FE models are compared to experimental data for a uniaxial creep tensile creep test using normalized shear compliance and time data pairs up to 727 seconds. The FE model consisted of 969 elements.
Figure 61. FE results for the Ogden material models \((N=1\) to \(4\)). The FE models are compared to experimental data for a uniaxial creep tensile creep test using normalized shear compliance and time data pairs up to 727 seconds. The FE model consisted of 969 elements.

The Ogden \((N=1)\) and the reduced polynomial \((N=3)\) material models seemed to be closer in agreement to the experimental data. However, the Mooney-Rivlin material model seemed to capture the stretch-time profile very similar to the experimental data. Hence this material model was used to conduct a FE mesh refinement analysis. The results of the mesh analysis are shown in Figure 62. It can be seen that a model with 7752 elements is sufficient as the change in the deformed length is negligible for FE models with larger number of elements. The FE results using 7752 elements were then again used with the Mooney-Rivlin, reduced polynomial \((N=3)\), and Ogden \((N=1, 3)\) material models and compared to experimental data. The results of this comparison are shown in
Figure 63 with negligible change in correlation. This indicates that the use of 7752 elements is sufficient for the model.

Figure 62. Mesh analysis for the total deformed length of a dielectric elastomer strip (reference Figure 31, Chapter 3) due to a constant load (long-term).
Figure 63. FE results using the Ogden ($N=1, 2$), Mooney-Rivlin, reduced polynomial ($N=3$) material models with a finer mesh consisting of 7752 elements.

The FE model that has been developed using uniaxial tensile test data was also validated with experimental data from a radially stretched DEA with an annular configuration as discussed in Section 2.2 of Chapter 2. The FE model consisted of an annular sheet with a hole at the center. The inner and outer radii were 25 mm and 70 mm, respectively. The DEA was subjected to a hyperelastic outer radial prestretch of 1.27. The thickness of the dielectric elastomer material was 0.005 mm. The FE and experimental results for the deformation at the inner radius is presented in Figure 64. The reduced polynomial ($N=3$) material model shows better agreement than the Ogden ($N=3$) material model. Both models, however, do not exhibit nonlinearity in the decrease of the inner radius dimension as in the experimental data. This may be indicative of the limitations of using uniaxial test data to predict biaxial deformation.
Figure 64. FE and experimental results of an annular dielectric elastomer subjected to 3000 V. Open circle: Ogden ($N=3$); X: Reduced polynomial ($N=3$).

The FE model was then validated with experimental data of the Rectangle 2 non-axisymmetric configurations. In order to accomplish this a mesh refinement study using the Mooney-Rivlin material model was conducted. The change in the dimension of the major chord AA (reference Figure 58) was analyzed for this purpose. The dielectric elastomer actuator of a Rectangle 2 configuration was prestretched and subjected to an instantaneous voltage of 3000 V for 10 seconds. The mesh analysis showed only a negligible difference of 0.008 mm for the deformed length after 10 seconds for models with 225 and 2945 elements. Table 7 shows the results of four different meshes and their corresponding dimensions of the major chord AA. Due to numerical computation time, a mesh resulting in 893 elements was selected, as the gain in accuracy was negligible for meshes with larger number of elements. A mesh of 986 elements was used for the Ellipse 1 and 2 configurations.
Time-independent experimental results of the major and minor chord dimensions, AA and BB, respectively, of all samples and the corresponding FE results are shown in Table 7. The average and the standard deviation based upon a population of three samples are calculated and presented in Table 8. For all configurations, the FE model predictions of the major chord dimensions, AA, correlate well with experimental data showing only a maximum error of 5.2%. A maximum error of 8.8% is observed for the minor chord dimension predicted by the FE model at location BB. Except for the major chord dimension of the Ellipse 1 configuration, the FE predicts that the material is stiffer than it actually is. This is evidenced by the generally lower predicted values in the minor and major chord dimensions compared to the experimental values.

Time-dependent experimental results of the displacement at the major and minor chord points A and B of the Rectangle 2 configuration are shown in Figure 65 through Figure 69. The FE results are validated using the average values of these test data and the results are shown in through Figure 69 through Figure 72. For completeness, only data up to 10 seconds are shown due to actuators failing before of 30 seconds. The correlation between the FE and experimental data at locations A is much better than the correlation at location B. A possible explanation is that the uses of the uniaxial test data to predict biaxial deformations are limited. Obata et al. [33] showed that elastic material properties obtained from uniaxial tests can be used in hyperelastic material models to yield good correlation between experimental and theoretical predictions. However, for small principal stretch difference (i.e. small strain), material properties need to be determined from biaxial tests to properly predict small deformations. The results of this research are in agreement with Obata et al. Time-independent deformations involving large hyperelastic deformations between 140 and 150% show better correlation than the time-dependent deformations that had a maximum strain of approximately 3% (Rectangle 2 actuator configuration, location B). Hence, it is possible that using biaxial test results to determine the Prony series constants may improve the correlation of the viscoelastic FE model with experimental data.

The reduced polynomial ($N=3$) material model seemed to generally model deformations for uniaxial, axisymmetric, and non-axisymmetric actuator (at location A
only) geometries well. It does not seem to capture the nonlinear displacement profile for
the axisymmetric condition. However this may be due to the model being developed
using only tensile test data. Its correlation to experimental data is better than the Ogden
(N=3) material model and is therefore selected as the model of choice for the finite
element model.
Table 7. Results of mesh refinement analysis for a rectangular dielectric elastomer actuator configuration.

<table>
<thead>
<tr>
<th>Number of elements</th>
<th>Dimension of chord AA (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>225</td>
<td>50.84088</td>
</tr>
<tr>
<td>893</td>
<td>50.84092</td>
</tr>
<tr>
<td>1875</td>
<td>50.84092</td>
</tr>
<tr>
<td>2945</td>
<td>50.84903</td>
</tr>
</tbody>
</table>

Table 8. Experimental and FE hyperelastic results of the deformed major and minor chord dimensions, AA and BB, respectively.

<table>
<thead>
<tr>
<th></th>
<th>Major chord AA (mm)</th>
<th>Minor chord BB (mm)</th>
<th>Major chord AA (mm)</th>
<th>Minor chord BB (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Ellipse 1</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sample 1</td>
<td>35.0</td>
<td>25.4</td>
<td>Sample 1</td>
<td>59.6</td>
</tr>
<tr>
<td>Sample 2</td>
<td>37.1</td>
<td>24.0</td>
<td>Sample 2</td>
<td>57.5</td>
</tr>
<tr>
<td>Sample 3</td>
<td>36.6</td>
<td>24.7</td>
<td>Sample 3</td>
<td>58.2</td>
</tr>
<tr>
<td>Mean</td>
<td>36.2 ± 0.9</td>
<td>24.7 ± 0.6</td>
<td>Mean</td>
<td>58.4 ± 0.9</td>
</tr>
<tr>
<td>FE</td>
<td>38.1</td>
<td>24.6</td>
<td>FE</td>
<td>56.8</td>
</tr>
<tr>
<td>FE error</td>
<td>5.2%</td>
<td>-0.4%</td>
<td>FE error</td>
<td>-2.7%</td>
</tr>
<tr>
<td><strong>Ellipse 2</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sample 1</td>
<td>46.8</td>
<td>31.8</td>
<td>Sample 1</td>
<td>68.4</td>
</tr>
<tr>
<td>Sample 2</td>
<td>46.4</td>
<td>30.4</td>
<td>Sample 2</td>
<td>69.9</td>
</tr>
<tr>
<td>Sample 3</td>
<td>46.8</td>
<td>30.5</td>
<td>Sample 3</td>
<td>66.2</td>
</tr>
<tr>
<td>Mean</td>
<td>46.7 ± 0.2</td>
<td>30.9 ± 0.6</td>
<td>Mean</td>
<td>68.2 ± 1.5</td>
</tr>
<tr>
<td>FE</td>
<td>45.9</td>
<td>30.2</td>
<td>FE</td>
<td>67.1</td>
</tr>
<tr>
<td>FE error</td>
<td>-1.7%</td>
<td>-2.3%</td>
<td>FE error</td>
<td>-1.2%</td>
</tr>
<tr>
<td><strong>Rectangle 1</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sample 1</td>
<td></td>
<td></td>
<td>Sample 1</td>
<td>34.9</td>
</tr>
<tr>
<td>Sample 2</td>
<td></td>
<td></td>
<td>Sample 2</td>
<td>33.0</td>
</tr>
<tr>
<td>Sample 3</td>
<td></td>
<td></td>
<td>Sample 3</td>
<td>33.8</td>
</tr>
<tr>
<td>Mean</td>
<td></td>
<td></td>
<td>Mean</td>
<td>33.9 ± 0.8</td>
</tr>
<tr>
<td>FE</td>
<td></td>
<td></td>
<td>FE</td>
<td>31.0</td>
</tr>
<tr>
<td>FE error</td>
<td></td>
<td></td>
<td>FE error</td>
<td>-8.6%</td>
</tr>
<tr>
<td><strong>Rectangle 2</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sample 1</td>
<td></td>
<td></td>
<td>Sample 1</td>
<td>35.3</td>
</tr>
<tr>
<td>Sample 2</td>
<td></td>
<td></td>
<td>Sample 2</td>
<td>34.4</td>
</tr>
<tr>
<td>Sample 3</td>
<td></td>
<td></td>
<td>Sample 3</td>
<td>36.3</td>
</tr>
<tr>
<td>Mean</td>
<td></td>
<td></td>
<td>Mean</td>
<td>35.3 ± 0.8</td>
</tr>
<tr>
<td>FE</td>
<td></td>
<td></td>
<td>FE</td>
<td>32.2</td>
</tr>
<tr>
<td>FE error</td>
<td></td>
<td></td>
<td>FE error</td>
<td>-8.8%</td>
</tr>
</tbody>
</table>
Figure 65. Rectangle 2 configuration. Displacements at location A for five samples that have each been tested three times at 6000 V.
Figure 66. Rectangle 2 configuration. Displacements at location B for four samples that have each been tested three times at 6000 V.
Figure 67. Rectangle 2 configuration. Displacements at location A for five samples that have each been tested three times at 6500 V.
Figure 68. Rectangle 2 configuration. Displacements at location B for five samples that have each been tested three times at 6500 V.
Figure 69. Rectangle 2 Rectangle 2 configuration. Experimental and FE results for the displacement at location A due to a constant voltage of 6000 V. Experimental data are shown with +/- 1 standard deviation bars.
Figure 70. Rectangle 2 configuration. Experimental and FE results for the displacement at location B due to a constant voltage of 6000 V. Experimental data are shown with +/- 1 standard deviation bars.
Figure 71. Rectangle 2 configuration. Experimental and FE results for the displacement at location A due to a constant voltage of 6500 V. Experimental data are shown with +/- 1 standard deviation bars.
Figure 72. Rectangle 2 configuration. Experimental and FE results for the displacement at location B due to a constant voltage of 6500 V. Experimental data are shown with +/- 1 standard deviation bars.

4.6 Conclusions

The FE model using the general-purpose software ABAQUS was developed for actuators undergoing large nonlinear viscoelastic deformations uniaxially, axisymmetrically, and non-axisymmetrically. The hyperelastic material was defined using nominal stress and nominal strain data pairs from a uniaxial constant load tensile test results after 189 hours. The viscoelastic material was defined using results from a constant load uniaxial creep test 727 seconds in duration. The FE model was validated against experimental results for uniaxial and in-plane deformations. The FE model
correlated fairly well with viscoelastic deformations for uniaxial and axisymmetric deformations. The FE correlation to experimental data at the major chord locations AA was good. However, the agreement of the dimension of the minor chord BB was poor. This may be indicative of the limitations on utilizing uniaxial test data to predict in-plane deformations.
Chapter 5

Silicone Dielectric Elastomer Actuators

5.1 Introduction

In the previous three chapters hyperelastic and viscoelastic models for the VHB polyacrylate dielectric elastomer subjected to Maxwell stress were developed. The hyperelastic analytical model using the Mooney-Rivlin material model developed (Chapter 2) showed good agreement with experimental data. In Chapter 3 an analytical model using Christensen’s material model was developed to model the viscoelastic response of the VHB dielectric elastomer. In Chapter 4, a finite element (FE) model was developed using the general-purpose finite element software ABAQUS to model the viscoelastic response for the same material. Results from Chapters 3 and 4 showed that FE hyperelastic predictions for actuators with axisymmetric geometries correlated well with experimental data. Analytical uniaxial viscoelastic response correlated well with experimental data within one time constant of the relaxation modulus. The FE viscoelastic predictions did not correlate as well with experimental data.

In this chapter, a silicone dielectric elastomer with minimal viscoelastic effects is considered. Viscoelastic behavior translates to viscous losses due to stress relaxation that is an undesirable feature for actuator applications. In the past two years, researchers have aggressively sought alternative materials with minimal viscoelastic behaviors as well as improved electromechanical properties. Furthermore, researchers have been utilizing various commercially available silicone (polydimethylsiloxane) dielectric elastomers as they exhibit minimal viscoelastic behaviors [64]. They also have very stable mechanical properties in a large temperature range [65]. Section 5.2 highlights some of the results of these efforts. The elegance of both the hyperelastic as well as the viscoelastic models for dielectric elastomers that have been developed is their applicability to any rubber like, isotropic, homogeneous material. Hence, this chapter focuses on developing hyperelastic
a finite element model for a cylindrical dielectric elastomer actuator using silicone tubing.

**5.2 An Alternative Dielectric Elastomer: Silicone (Polydimethylsiloxane)**

Silicone (polydimethylsiloxane) is a promising material for dielectric elastomer actuator application due to its minimal viscous losses and stable performance in a large operating temperature range. Due to these properties, silicone dielectric elastomers are favorable material candidate for dielectric elastomer actuator applications. Improving the mechanical properties of silicone dielectric elastomers such as its Young’s modulus and relative dielectric constant for actuator applications have been investigated by various researchers in the EAP research community [60, 66, 67]. It is beneficial to have an elastomer with a low Young’s modulus. A decrease in Young’s modulus can be obtained by altering the degree of cross-linking of the polymer chains by varying the amount and type of hardener used during silicone processing [64]. With a lower Young’s modulus, a larger transverse actuated strain can be achieved for the same Maxwell stress since the transverse strain is inversely proportional to the stiffness of the dielectric elastomer. Also an increase in the dielectric constant can be obtained by using a filler material during silicone processing [64].

Of notable mention is research conducted by Zhang et al. [64]. Zhang compares the mechanical properties of silicone dielectric material DC3481 (Suter-Kunststoffe, Jegenstorf, Switzerland) and compares them to the VHB polyacrylate dielectric elastomer. Six different silicones were made by mixing the silicone fluid DC3481 with various percentages of a hardener. Their result showed that with 5% (weight for weight) of hardener 81-F yielded the most favorable mechanical properties. Using this blend, a 140% biaxially prestrained circular actuator with fixed outer boundaries was subjected to high voltages. At 15, 30, and 40 mV, strains of 2, 7.5, and 17.5%, respectively, were realized. Also, the viscoelasticity of the DC3481 silicone was significantly lower than the polyacrylate VHB dielectric elastomer. The time dependent strain of the DC3481 material reached equilibrium within three seconds whereas the VHB material did not
reach equilibrium even after 200 seconds. Also, the DC3481 silicone showed fairly consistent Young’s modulus values of 0.15 to 0.30 MPa in the temperature range of -25° C to 150° C. The VHB material showed significant variation in Young’s modulus (0.4 to 500 MPa) in the same temperature range.

Research utilizing silicone (polydimethylsiloxane) was completed by Carpi [68, 69]. Carpi attempted to lower the required actuation field by attempting to increase the relative dielectric constant. Since Maxwell stress is directly proportional to the relative dielectric constant of the material, an increase in the relative dielectric constant can in effect lower the required actuation field to yield the same Maxwell stress. Carpi achieves this by mixing a ceramic titanium dioxide powder (99%-pure rutile-type titanium dioxide) with a three-part silicone (Cine-Skin A/B/C 50% silicone, Burman Industries, California). The resulting polymer/ceramic dielectric composite material showed a significant increase in the relative dielectric constant of the material. At all levels of strain, the Young’s modulus of the polymer/ceramic dielectric composite material was much lower than the Young’s modulus of silicone without any titanium dioxide ceramic. At 200% stretch, the Young’s moduli were 16 kPa and 40 kPa for the composite and noncomposite material, respectively. These values are much lower than the VHB material whose values have been reported to be in the 0.1-20 MPa range [70].

Carpi also utilized commercially available silicone tubes (Detakta GmbH & Co. Norderstedt, Germany) to model small actuated strains of the tubes with 5% uniaxial prestretch [59]. He also utilized silicone (TC-5005 A/B-C, BJB Enterprises Inc., U.S.A.) to fabricate a contractile linear actuator [68]. The actuator consists of two helixes that have been coupled together and encased using the same silicone material. Each helix was machined from a solid silicone tube and only one of the helix had its major surfaces electroded. A maximum compressive strain up to 3% at 15 μV was realized.

Schlaak et al. [71] spin coated multiple layers (25 μm) of silicone (P7670, Wacker Elastosil). Equipment capable of applying the silicone as well as selectively applying the electrode onto the spin coated silicone surface is utilized. A circular actuator comprising of up to 57 layers was fabricated to realize out-of-plane deformation upon electrical activation. Strains as high as 20% for prestrained actuators were realized. Possible
applications for such an actuator consist of tactile displays for tele-manipulation or Braille displays.

In this research, silicone tubes from Detakta are characterized and hyperelastic deformations are modeled in ABAQUS. Although there are many silicone dielectric elastomers available, Detakta silicone tubes were selected due to the commercial availability of a wide range of tube diameters and wall thicknesses. In the foregoing sections, the material properties of the silicone tubes are determined from constant load uniaxial tensile test results. These results are then evaluated to determine the best material model using the EVALUATE command in ABAQUS. Finally, experimental and FE results of hyperelastic and activated deformations are compared.

5.3 Experimental Setup

Two experimental arrangements were conducted using the Detakta silicone tubes (Detakta, Norderstedt, Germany). In the first experimental setup, constant load uniaxial tensile tests using Detakta silicone tube (type 3003) were used to obtain the stress-strain relationships that would be needed to determine the materials constants for various material models. In the second arrangement, Detakta silicone tubes (type 2502) were fabricated so that positive and negative electrodes comprising the inner and outer surfaces of the tube, respectively, were formed. These tubes were subjected to various voltages and the deformation in length was monitored. The purpose for the second experimental arrangement was used to validate the FE model for non-zero Maxwell stresses.

For the first experimental arrangement, the material constants of the silicone (polymdimethylsiloxane) were determined by using the stress-strain relationships obtained from 67 constant load uniaxial tensile tests. Sixty-seven Detakta silicone tubes of type 3003 were each cut to 203.2 mm lengths with an additional 25.4 mm sections on both ends reserved for clamping purposes. Deformation of a 58.93 mm length region at the mid-section of each tube was measured by counting pixels from digital images. Each tube was clamped on one end to a test fixture and mass was hung on the opposite end (reference Figure 73). The mechanical equilibrium uniaxial stretch ratios were in the
regime of $1.0 < \lambda < 1.5$. All tests were conducted at $70\pm3^\circ$ F and $68\pm5\%$ relative humidity.

In the second experimental arrangement, four Detakta silicone tubes (type 2502) were cut to 304.8 mm lengths. Using a syringe, 846-80G conductive carbon grease (MG Chemicals, Surrey, B.C., Canada) was squeezed into each silicone tube so as to serve as a positive electrode surface. A flexible conductive tape (2 x 50 mm$^2$) was inserted half way into each tube. The remaining 25 mm portion of each tape served as the electrical lead to the interior electrode surface. The top 25.4 mm of each silicone tube was clamped onto a test fixture (reference Figure 73). A strip of conductive tape (2 x 50 mm$^2$) was wrapped twice around each tube with the remaining unwrapped portion serving as the negative lead. Then the entire outer surface including the conductive tape was coated with the 846-80G conductive grease to create the negative electrode surface. Capacitance of each undeformed tube was measured using a multimeter.

Binder clips loaded with a constant mass was hung onto the end of each tube. The clamps and the binder clips were placed so that a free length of 254 mm was attained. Each tube was subjected to a sequence of constant step voltages from 4 kV to 10 kV using a Trek 610D high voltage power supply (Trek Inc., Medina, New York). A null voltage between each 1 kV increase was utilized to allow the tube to attain mechanical equilibrium. It is known that viscoelastic effects of silicone are negligible [64]. However, to avoid any time-dependent change in the length of the tube, voltages were applied 189 hours after the constant loads were applied. The changes in lengths of the tubes due to the voltage were monitored by directing the ILD-1800 laser (Micro Epsilon, Raleigh, North Carolina) displacement sensor to a target placed on the bottom of the hanging mass. The voltage change of the laser displacement sensor indicating a change in length of the tube was monitored using a hand held voltmeter.
Figure 73. Experimental setup for testing the silicone tubes at high voltages.
5.4 Results and Discussion - Uniaxial Tensile Tests

In order to obtain the stress-strain relationships from the constant load uniaxial tensile tests, it was necessary to determine the inner and outer diameters of the undeformed Detakta tubes. Two different methods were used to obtain these dimensions. One method was to use a digital caliper to measure the outer diameter and to use a steel pin of various diameters to measure the inner diameter. The second method was to use a 6-mega-pixel camera to capture the image of a tube along side a ruler and to note the outer diameter measurement using the ruler in the image. The average dimensions of the inner and outer diameters by both methods yielded different values (reference Table 9). The measurements of each tube (from digital images as well as experiment) are available in Appendix E. The experimental values correlated well with Detakta’s specification but the dimensions from the digital images did not. Hence, the experimental values were used in the calculation of the nominal stress values in obtaining the nominal stress-strain relationship shown in Figure 74.

Table 9. Dimensions of the inner and outer diameters of Detakta silicone tube type 3003.

<table>
<thead>
<tr>
<th></th>
<th>Average inner diameter (mm)</th>
<th>Average outer diameter (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Digital image</td>
<td>-</td>
<td>3.39 +/- 0.06</td>
</tr>
<tr>
<td>Experimental</td>
<td>2.97</td>
<td>3.56 +/- 0.04</td>
</tr>
<tr>
<td>Detakta</td>
<td>3.00</td>
<td>3.60</td>
</tr>
</tbody>
</table>
The results from the first experimental arrangement consisting of 67 constant load uniaxial tensile tests measured after 189 hours are shown in Figure 74. Fairly large sample-to-sample test variations were observed. This could be attributed to experimental errors in demarcating the 203.2 mm free length section and the 58.93 mm section at the mid-length of the tube. It could also possibly be attributed to the method used to measure the deformed lengths of the tube. In all the uniaxial tensile test measurements, digital images of the deformed tube were used to calculate the total number of pixels and subsequently converted to units of length using a pixel per unit length ratio. As previously mentioned, there may be some error involved in this method.
The results of the second experimental arrangement consisting of voltage applications to the silicone tubes are presented in Figure 75 and Figure 76. The data is presented in terms of nominal stresses and nominal strains for convenience, as nominal values are required in the finite element software ABAQUS material definition section.

For the two tests that failed during the 8 kV voltage applications, audible noise indicating imminent dielectric breakdown started to occur at 6 kV and failed when a step voltage of 8 kV was applied. Before the onset of failures, audible shorting sounds were present near the top and bottom 50 mm sections of the tubes. In addition small electrical sparks could be seen near the bottom 50 mm sections during this time. Ultimate failure occurred for all tubes when a hole burned through the thickness of the silicone tube. The remaining two specimens were able to withstand voltages up to 10 kV.

The two tubes that did not fail were monitored for five minutes after the final 10 kV voltage supply was returned to zero. Both tubes did not return back to their original deformed lengths, i.e. the length before any voltages were applied. After five minutes, the two samples had nominal strains (%) of 0.34 and 0.38. This may indicate that there may possibly be permanent deformations that may have occurred due to the voltage application.

The activated strains (strains due to the application of voltages) were less than 1%, which is quite small. Higher strains could possibly attained with the use of tubes with thinner wall thicknesses. Detakta silicone tubes with thinner wall thicknesses (0.1 mm thicknesses) were experimentally evaluated. However, inserting the conductive grease to fabricate the positive electrode surface at the inner wall was problematic. The inner diameters of these tubes were a mere 0.5 mm. Even application of the conductive grease could not be obtained without avoiding permanent deformation. This was because the grease had to be manually squeezed down the length of the tube causing permanent change in length of the tube.

It is noted that the length of the tube during the constant step voltage application the tube length did not reach equilibrium; rather, it fluctuated. Figure 75 and Figure 76 represents the lowest and highest values, respectively, of the nominal strain (%) of each tube during the constant step voltage applications. It is also interesting to note the larger
difference in nominal strains between the two tests that did not fail for voltages greater than 8 kV than for voltages less than this value. Recall that the audible noise, indicative of possible failure, started during the 6 kV applications with final failure occurring at 8 kV for two of the tubes. It may be possible that the large difference may due to changes in the material cause by the high electrical loads.

Figure 75. Nominal strain (%) versus applied voltage. Experimental results of four Detakta silicone tubes (type 2502) with each tube subjected to voltages ranging from 4 kV to 10 kV. This figure represents the lowest values of the deformed lengths of each tube during voltage application.
Figure 76. Nominal strain (%) versus applied voltage. Experimental results of four Detakta silicone tubes (type 2502) with each tube subjected to voltages ranging from 4 kV to 10 kV. This figure represents the highest values of the deformed lengths of each tube during voltage application.

5.5 Finite Element Model

ABAQUS has the ability to determine the best hyperelastic material model that would fit experimental stress-strain data using various forms of the strain energy functions. This is accomplished by using the EVALUATE option. The hyperelastic material models available within ABAQUS that were considered were the polynomial, Ogden, and reduced polynomial. Based upon the results of the material evaluation, the Mooney-Rivlin (polynomial $N=1$) and the reduced polynomial of $N=3$ proved to provide the best fit to experimental data and were selected as the material model implemented in the FE model. These two models were used to generate FE results against and compared
to experimental data for Detakta tubes subjected to constant loads and various voltages as discussed in the previous sections. In the next section the different material models and the results of ABAQUS material evaluations are discussed. This is followed by the validation of the FE results with experimental data.

5.5.1 Hyperelastic Material Models and Material Constants

The hyperelastic material models considered were the polynomial, Mooney-Rivlin, Ogden, and reduced polynomial. These material models utilize a strain energy function to formulate the stress-strain relationships. The dielectric material is assumed to be isotropic and incompressible and operating under isothermal conditions.

The polynomial strain energy function has the following form

$$U = \sum_{i+j=1}^{N} C_{ij} (I_1 - 3)(I_2 - 3) + \sum_{i=1}^{2} \frac{1}{D_i} (J_{el} - 1)^{2i}$$

Eq. 124

where \( U \) is the strain energy potential per reference volume, \( I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \) and \( I_2 = \lambda_1^2 \lambda_2^2 \lambda_3^2 \) are the first and second strain invariants which are functions of the principal stretch ratios (\( \lambda_1, \lambda_2, \) and \( \lambda_3 \)), respectively, \( J_{el} \) is the elastic volume strain which is zero for incompressible materials, and \( D_i \) determines the compressibility of the material which is also zero for the model considered here due again to the incompressibility assumption. The \( C_{ij} \) terms are material constants that are determined experimentally. If \( N=1 \), the polynomial strain energy potential reduces to that of the Mooney-Rivlin form

$$U = C_{10} (I_1 - 3) + C_{01} (I_2 - 3).$$

Eq. 125

The reduced polynomial strain energy function is obtained by setting all \( C_{ij} \) with \( j \neq 0 \) to zero to yield

$$U = \sum_{i=1}^{N} C_{i0} (I_1 - 3)^i.$$

Eq. 126
The Ogden strain energy function is a higher ordered model and is able to capture highly nonlinear stress-strain characteristics since the powers of the principal stretch ratios can take on any value. The strain energy function takes on the form

$$U = \sum_{i=1}^{N} \frac{2\mu}{\alpha_i} \left( \lambda_1^{\alpha_i} + \lambda_2^{\alpha_i} + \lambda_3^{\alpha_i} - 3 \right)$$

Eq. 127

where $\alpha_i$ and $\mu_i$ are material constants.

For hyperelastic material definition, ABAQUS requires the nominal stress and nominal strain (change in length divided by the undeformed length) test data be provided. Hence, the nominal stress-strain relationship shown in Figure 74 was used in ABAQUS so that the material evaluation option command could be utilized.

5.5.2 Geometry

A Detakta silicone dielectric elastomer tube of type 2502 with an inner and outer radii of 1.321 and 1.620 mm, respectively, and length of 254 mm was modeled as a homogenous, isotropic, incompressible material operating under isothermal conditions. Viscoelastic effects were considered negligible. Due to the axial symmetry of the tube, only a plane spanning the length of the tube was used for the model pictured in Figure 77. Four-node solid hybrid elements (CAX4H) are used in the finite element model consisting of 48,656 elements.

5.5.3 Boundary and Loading Conditions

A schematic of the silicone tube that is modeled in ABAQUS is shown in Figure 77. Two step loads are utilized in the ABAQUS to model the prestretch and the application of the electric field. The first load step consists of a uniaxial hyperelastic prestretch imposed by a constant traction at the bottom end of the tube in the 2-direction. The clamped end of the silicone tubes are constrained from movements in the radial and axial or 1- and 2-directions, respectively, as shown in Figure 77a. The second load step
consists of a uniform surface traction on the inside surface of the tube applied radially (1-direction) outward to represent the Maxwell stress effects (reference Figure 77b). The top end remains constrained in the 1- and 2- direction during this loading. The bottom end remains constrained in the 2- direction and is free in the 1- direction.

The magnitude of the applied traction representing Maxwell stress is calculated using electromagnetic theory [72]. The expression for Maxwell stress in tensor form is

\[ \mathbf{T}^M = \varepsilon_r \varepsilon_0 \mathbf{E} \otimes \mathbf{E} - \varepsilon_r \varepsilon_0 \mathbf{E} \cdot \mathbf{E} \cdot \mathbf{I} \]  

Eq. 128

where \( \varepsilon_r \) is the relative dielectric constant of the silicone dielectric elastomer, \( \varepsilon_0 \) permittivity of vacuum (8.85 x 10^{-12} \text{ F/m}), \( \mathbf{E} \) is the electric field tensor, and \( \mathbf{I} \) is the identity tensor, respectively. The relative dielectric constant \( \varepsilon_r \) can be determined by using the capacitance per unit length relation for a cylindrical capacitor.
\[
\frac{C}{L} = \frac{2\pi \varepsilon_r \varepsilon_o}{\ln(R_b / R_a)} \tag{Eq. 129}
\]

where \(R_a\) and \(R_b\) are the undeformed inner and outer radii, and \(L\) is the undeformed length of the cylindrical capacitor, respectively, and the \(C\) is the capacitance.

The electric field \(\mathbf{E}\) for a cylindrical capacitor can be determined by using the relation

\[
\mathbf{E} = \frac{\lambda}{2\pi \varepsilon_r \varepsilon_o r} \hat{\mathbf{e}}_r \tag{Eq. 130}
\]

where \(\lambda\) is the surface charge density per unit deformed length of the cylinder and \(r\) is the deformed radius of the deformed cylinder. Since the cylindrical dielectric elastomer is subjected to various voltage potentials \(V\) across the radial thickness of the cylinder this provides motivation to convert the electric field expression to that of a voltage potential. This is achieved by integrating the electric field expression with limits of integration from the inner radius to the outer radius to yield

\[
V = \frac{\lambda}{2\pi \varepsilon_r \varepsilon_o} \ln \frac{b}{a}. \tag{Eq. 131}
\]

This expression can then be solved for the surface charge density \(\lambda\), substituted into the electric field relation to yield

\[
\mathbf{E} = \frac{V}{r \ln(r_b / r_a)} \hat{\mathbf{e}}_r. \tag{Eq. 132}
\]

The in-plane Maxwell stress components are considered to be negligible in comparison to the in-plane elastic stress. Hence, the radial component of the electric field into the Maxwell stress equation (Eq. 128) gives the radial traction expression

\[
T_r^M = \frac{\varepsilon_r \varepsilon_o V^2}{2} \left( \frac{1}{h^2} + \frac{1}{ha} + \frac{1}{12a^2} \right) \quad \text{for} \quad r_a < r < r_b. \tag{Eq. 133}
\]

If the thickness of the cylindrical tube is much smaller than the inner radius \(r_a\), by Taylor series expansion, the following expressions are obtained for the radial surface tractions at the inner and outer surfaces, respectively

\[
T_{r=a}^M = \frac{\varepsilon_r \varepsilon_o V^2}{2} \left( \frac{1}{h^2} + \frac{1}{ha} + \frac{13}{12a^2} - \frac{7h}{6a^3} \right),
\]

\[
T_{r=b}^M = \frac{\varepsilon_r \varepsilon_o V^2}{2} \left( \frac{1}{h^2} - \frac{1}{ha} + \frac{13}{12a^2} - \frac{7h}{6a^3} \right) \tag{Eq. 134}
\]
where $h$ is the deformed radial thickness. The higher ordered terms can be neglected and all but the first terms of both expressions remain. This then ultimately reduces to the Maxwell stress form for a parallel-plate capacitor.

Using the FE model, predictions for the deformed tube length and radii due to a constant load of 1.3 N (474.8 kPa nominal stress) are presented in the next section. In addition the FE model is validated with experimental data for non-zero Maxwell stresses.

### 5.6 Results and Discussion

Experimental data from 67 constant load uniaxial tensile tests conducted using Detakta silicone tubes (type 3003) were used to determine the most suitable hyperelastic material model. This was facilitated by using the EVALUATE option within ABAQUS. The results of the evaluation indicated several suitable material models and the material constants for these models are presented in Appendix F. The selection of the best material model was made by comparing the FE and experimental results. The deformed tube length and deformed outer radius due to a constant uniaxial load (1.3 N) was investigated for this purpose. The results are shown in Table 10. The experimental values of the deformed lengths and outer radii obtained from five silicone tubes type 2502 are available in Appendix G

From this comparison, the Mooney-Rivlin and reduced polynomial of $N=3$ yield the best correlations to experimental data. The material constants for both models are presented in Table 11. The FE predictions have less than 2% errors for both the deformed length and outer radius dimensions as summarized in Table 12.
Table 10. Experimental and FE results of uniaxial tensile tests of Detakta silicone tube (type 2502) subjected to a constant load of 1.3 N.

<table>
<thead>
<tr>
<th>Material Model</th>
<th>Deformed Length (mm)</th>
<th>Deformed OR (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Polynomial N=1 (Mooney-Rivlin)</td>
<td>294.06</td>
<td>1.506</td>
</tr>
<tr>
<td>Polynomial N=2</td>
<td>289.24</td>
<td>1.518</td>
</tr>
<tr>
<td>Ogden</td>
<td></td>
<td></td>
</tr>
<tr>
<td>N=2</td>
<td>289.97</td>
<td>1.516</td>
</tr>
<tr>
<td>N=3</td>
<td>290.26</td>
<td>1.515</td>
</tr>
<tr>
<td>N=4</td>
<td>290.27</td>
<td>1.515</td>
</tr>
<tr>
<td>N=5</td>
<td>290.33</td>
<td>1.515</td>
</tr>
<tr>
<td>Reduced Polynomial</td>
<td></td>
<td></td>
</tr>
<tr>
<td>N=3</td>
<td>294.29</td>
<td>1.505</td>
</tr>
<tr>
<td>N=4</td>
<td>292.05</td>
<td>1.511</td>
</tr>
<tr>
<td>N=5</td>
<td>290.09</td>
<td>1.514</td>
</tr>
<tr>
<td>N=6</td>
<td>290.01</td>
<td>1.516</td>
</tr>
<tr>
<td>Experimental Average</td>
<td>299.39</td>
<td>1.510</td>
</tr>
</tbody>
</table>

Table 11. Mooney-Rivlin and the reduced polynomial material constant outputs from ABAQUS material evaluation.

Mooney-Rivlin (Polynomial N=1)

\[
C_{10} = -174,473 \\
C_{01} = 868,213
\]

Reduced polynomial (N=3)

\[
C_{10} = 604,494 \\
C_{20} = -211,927 \\
C_{30} = 47,501
\]
Table 12. FE and experimental comparisons of the deformed dimensions of the length and outer diameter of the tubes.

<table>
<thead>
<tr>
<th></th>
<th>Deformed length (mm)</th>
<th>Deformed outer diameter (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average</td>
<td>299.38</td>
<td>3.02</td>
</tr>
<tr>
<td>Standard deviation</td>
<td>1.28</td>
<td>0.02</td>
</tr>
<tr>
<td>Mooney-Rivlin (Polynomial N=1)</td>
<td>294.06 (1.7% error)</td>
<td>3.01 (0.33 % error)</td>
</tr>
<tr>
<td>Reduced polynomial (N=3)</td>
<td>294.2 (1.7% error)</td>
<td>3.01 (0.33 % error)</td>
</tr>
</tbody>
</table>

Figure 78 shows ABAQUS material evaluation results of the two material models to the experimental test data. The Mooney-Rivlin and reduced polynomial (N=3) material model match reasonably well for nominal strains between 0 to 0.1 and 0.2 to 0.4 indicating that the FE model can be used with a certain level of confidence in these strain regions. For nominal strains larger than 0.4 the models can be improved with additional test data. Experimental data for strains between 0.1 and 0.2 deviate away from the material model profile. However, experimental and finite element result comparisons for the deformed lengths of silicone tubes at a strain level of 0.15 show good correlation (reference Table 13). The FE result correlates well with experimental data and is within 2% of experimental values. Therefore, the model developed should be applicable for the entire strain range between 0 to 0.4.

To validate the FE model for non-zero Maxwell stresses, Maxwell stresses in the range of 2.7 to 17.2 kPa (4 kV to 10 kV, respectively) were applied. These Maxwell stress values were calculated using Eq. 133 requiring the determination of the relative dielectric constant \( \varepsilon_r \). Substitution of the average experimental values of the deformed inner and outer radii (1.321 and 1.60 mm, respectively) of silicone tube type 2502 and the average experimental capacitance per unit length of 0.78 nF/m were used in Eq. 129 to yield a relative dielectric constant value of \( \varepsilon_r = 2.87 \).
Figure 78. Results of the Mooney-Rivlin (polynomial $N=1$) and reduced polynomial ($N=3$) FE models in comparison to experimental data. Open circle: Mooney-Rivlin (polynomial $N=1$); Open square: Reduced polynomial $N=3$; x: Experimental test data.
Figure 79 and Figure 80 show the comparison of experimental and FE results. Included in these figures are FE predictions utilizing a dielectric constant value of $\varepsilon_r = 3.0$ of the Detakta silicone tube as reported by Carpi [59]. The material model that correlates well with experimental data is the Mooney-Rivlin material model with a dielectric constant of $\varepsilon_r = 3.0$. Also, the FE model correlates well for strains induced by voltages less than or equal to 7 kV. However, it does not seem to correlate well with experimental data at voltages greater than 7 kV. This may possibly be due to the electrical effects on the silicone tubes as previously mentioned in Section 5.4.

Table 13. FE and experimental comparison. Undeformed Detakta silicone tubes (type 2502) of lengths 254 mm are subjected to constant loads of 1.3 N. Temperature was 70.4 F and 38.9 % relative humidity.

<table>
<thead>
<tr>
<th>Deformed length (mm)</th>
<th>Deformed outer diameter (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sample 1</td>
<td>300.74</td>
</tr>
<tr>
<td>Sample 2</td>
<td>297.66</td>
</tr>
<tr>
<td>Sample 3</td>
<td>300.04</td>
</tr>
<tr>
<td>Sample 4</td>
<td>300.04</td>
</tr>
<tr>
<td>Sample 5</td>
<td>298.45</td>
</tr>
<tr>
<td>Average</td>
<td>299.38</td>
</tr>
<tr>
<td>Standard deviation</td>
<td>1.28</td>
</tr>
</tbody>
</table>

FE
- (Reduced polynomial $N=3$) 294.2 (1.7% error) 3.01 (0.33 % error)
- (Mooney-Rivlin) 294.06 (1.7% error) 3.01 (0.33 % error)
Figure 79. Nominal strain (%) versus applied voltage. Finite element and experimental comparisons of four Detakta silicone tubes (type 2502) subjected to a constant engineering load of 1.3 N and various voltages ranging from 4 kV to 10 kV. This figure represents the *lowest* nominal strains (%) during the step voltage applications.
Figure 80. Nominal strain (%) versus applied voltage. Finite element and experimental comparisons of four Detakta silicone tubes (type 2502) subjected to a constant engineering load of 1.3 N and various voltages ranging from 4 kV to 10 kV. This figure represents the highest nominal strains (%) during the step voltage applications.

5.7 Conclusions

A finite element model of silicone tubes subjected to constant loads and Maxwell stress was developed. The material model used in the finite element model that yielded the best fit to uniaxial experimental data was the Mooney-Rivlin material model. The FE results for zero Maxwell stress correlated very well with experimental data. Additionally, FE results for non-zero Maxwell stresses that correspond to voltages less than 8 kV showed good correlation. The FE results for Maxwell stresses for voltages greater than or equal to 8 kV did not show good correlation. It is possible that the hyperelastic material
model may not be able to capture the phenomena resulting from the high Maxwell stresses.

It was noted that when constant step voltages were applied the deformed length did not reach mechanical equilibrium. Rather the length fluctuated more so at the higher levels of Maxwell stresses (8kV and higher) than a lower levels. Of the four tubes tested, two failed at the 8 kV applications. The remaining two tubes were able to withstand up to 10 kV. The final lengths of these tubes after the voltage was dropped to zero did not return to their original lengths. This also adds to the argument that at higher Maxwell stresses a hyperelastic material model that includes other deformation modes may be needed.
Chapter 6

Conclusions and Recommendations for Future Research

The primary purpose of this research was to develop analytical and finite element models to aid in modeling the large nonlinear viscoelastic deformations of dielectric elastomer actuators. The analytical and FE models were validated with experimental results from uniaxial and equi-biaxial tests (annular configuration). Furthermore, the FE model was validated with experimental results from general biaxial deformations of non-axisymmetric actuator configurations. Difficulties in obtaining an analytical viscoelastic model were encountered as more than one relaxation modulus was needed to capture the time-dependent deformation response. In addition, a more complex material model might be needed to capture the large nonlinear viscoelastic deformation response. Furthermore, development of a material model based upon uniaxial test data has limitations (as discussed in Chapter 4) and may have attributed to these difficulties.

Improvements to the existing models (analytical and FE) can be obtained primarily through comprehensive material testing protocols. It is known that the material definitions for the hyperelastic and viscoelastic properties are critical. Hence, uniaxial as well as biaxial and planar tension (pure shear) data should be utilized. Also, it is imperative that experimental data be obtained in the stress and stretch regime similar to that of the operating conditions of the dielectric elastomer actuator. It is also critical to obtain long-term deformations since complete mechanical equilibrium is desired to model the hyperelastic response. For viscoelastic dielectric elastomer materials this time frame may be quite long. As in the case of the VHB material, even after 189 hours, evidence of creep was present. Furthermore, all available material models within ABAQUS should be considered before selecting a material model of choice. As a final note, it should be noted that creating a viscoelastic FE model in ABAQUS requires two load steps. The first viscoelastic step accounts for the instantaneous response that is very short in duration. The second viscoelastic step accounts for the time-dependent (creep). With the
incorporation of these recommendations correlation between FE and experimental are expected to improve.

The research presented in the previous chapter involved silicone dielectric elastomer materials. The FE model developed correlated well with uniaxial tensile experimental data at zero as well as non-zero electric fields. Degradation in this correlation was seen for voltages greater than 7 kV. Incorporating other modes of deformations such as plastic deformation could make an improvement to the FE model.


Appendix A

Matlab Code to Determine Elastic Material Constants

Code to determine the Mooney-Rivlin material constants
function FindMRConstants()
clear all
global errorsquared r
tic
% Code written by: Eunice Yang June 2005
% This code determines the Mooney-Rivlin material constants
% Using long term uniaxial constant load test data
% Stretch range from 1.5<\lambda<3.0
% Reference tensiletests2.xls
% User input required: L1,L2,lhs,n (stress-stretch data from experimental tests)

format short g
[x,fval] = fminsearch(@(x) RateIndependentUniaxialCauchyStress(x),[-1,-1]) % mooney rivlin constants
plot(errorsquared)
toc
r

function rmse= RateIndependentUniaxialCauchyStress(x)
global errorsquared r
L1=1.14;    %Curve fit of constant load uniaxial test data (strip34)
L2=.0386;   %Data fit to a power law of the form stretch = L1*t^L2
n=5000;     %Number of samples in algorithm
            %Mooney-Rivlin constants will change as a function of n

for p=1:n
i=p*200000000;
stretch=L1*p^L2;
    rhs(p)=2*(stretch^2-1/stretch)*(x(1)+x(2)/stretch);
    lhs(p)=20700*stretch^1.9696;   % Calculated Cauchy Stress (from
                         % stretch=L1*p^L2)
    % true stress=20700*stretch^(1.9696)
    % ref. tensiletests2.xls after 189 hrs.
    errorsquared(p)=(rhs(p)-lhs(p))^2;
end
rmse=sqrt(sum(errorsquared)/n);
r = corr2(lhs,rhs);
**Code to determine the Ogden material constants**

```matlab
function FindOgdenConstants()
clear all
global errorsquared r
tic
% Code written by: Eunice Yang June 2005
% This code determines the Mooney-Rivlin material constants
% Using long term uniaxial constant load test data
% Stretch range from 1.5<\lambda<3.0
% Reference tensiletests2.xls
% User input required: L1,L2,lhs,n (stress-stretch data from experimental tests)

format short g
[x,fval] = fminsearch(@(x)
    RateIndependentUniaxialCauchyStress(x),[1,1,1,1,1]) % Ogden Constants
plot(errorsquared)
toc

r

function rmse= RateIndependentUniaxialCauchyStress(x)
global errorsquared r
L1=1.14;    %Curve fit of constant load uniaxial test data (strip34)
L2=.0386;   %Data fit to a power law of the form stretch = L1*t^L2
n=5000;     %Number of samples in algorithm
    %Mooney-Rivlin constants will change as a function of n

for p=1:n
    i=p*200000000;
    stretch=L1*p^L2;
    rhs(p)= x(1)*(stretch^x(2)-(1/stretch)^(x(3)/2))+ ... 
      x(4)*(stretch^x(2)-(1/stretch)^(x(3)/2))
    lhs(p)=20700*stretch^1.9696;   % Calculated Cauchy Stress (from
                      % stretch=L1*p^L2)
    errorsquared(p)=(rhs(p)-lhs(p))^2;
end
rmse=sqrt(sum(errorsquared)/n);
    % ref. tensiletests2.xls after 189 hrs.
    r = corr2(lhs,rhs);
```
Appendix B

Matlab Code to Determine the Principal Stretch Ratios Due to Varying Maxwell Stress and Prestretch

function []=ANNULARELASTIC()
% Hyperelastic Mooney-Rivlin material model.
% Geometry: An annular membrane fixed at the outer peripheral boundary and free at
% the inner radius boundary.
% This function calculates the deformed radius, rho, for various electric
% fields, and prestretches applied independently or simultaneously.
clear all
% Material properties for 3M(Tm) VHB 4910 Acrylic foam elastomer***
Eo=8.854e-12;   % free space (vacuum) permittivity [Farads/m]
Er=14.7;             % relative dielectric constant
C1=8842.5;        % Mooney-Rivlin material constants
C2=4583.8;
alpha=(C2/C1);
% End material props for VHB 4910 Acrylic ***********************
%**********  USER DEFINED PARAMETERS ******************
InnerRadius=.5*.0254;            % [m]
OuterRadius=1.125*.0254;     % [m]
Prestretch=1.2;                        % Non-dimensional
EFPressDimensionless=[1.5]; %(Maxwell stress/C1)
%********** END OF USER DEFINED PARAMETERS***********
options = bvpset('Stats','on','RelTol',1e-4,'Nmax',500);
N=300  % generates N mesh points between a and b in the function linspace(a,b,N)
[mEFPressDimensionless,nEFPressDimensionless]=size(EFPressDimensionless);
hold on
for j=1:nEFPressDimensionless
    [x,y,yprime,MatrixSize,EFPress]=AnnulusFixedFreehC1(Prestretch, InnerRadius, ...
        OuterRadius,EFPressDimensionless(j),C1,C2,alpha,Eo,Er,N);
    [a,b]=size(x);
    meshxIC=x.'/OuterRadius;  % Non-dimensionalize
    SOLyIC=y.';
    SOLypIC=yprime.';
    hold on
    subplot(2,1,1);plot(meshxIC,SOLyIC);
    title('Mooney Rivlin Material Model (0 Field)')
ylabel('\lambda_1,\lambda_2');
function [x,y,yprime,MatrixSize,EFPress]=AnnulusHyperElastic(Prestretch, …
InnerRadius, OuterRadius,EFPressDimensionless,C1,C2,alpha,Eo,Er,N)
  options = bvpset('Stats','on','RelTol',1e-4,'Nmax',500);
  EF=sqrt(4*C1*EFPressDimensionless/(Eo*Er));  % conversion to dimensional electric
  % field [V/m]
  EFPress=0.5*Eo*Er*EF^2;  % Maxwell stress
  % positive term here because negative sign
  % for compression taken into account in
  % constitutive equations
  solinit = bvpinit(linspace(InnerRadius,OuterRadius,N),@AnnulusInit);
  sol = bvp4c(@AnnulusODE,@AnnulusBC,solinit,options,EFPress, …
              C1,C2,alpha,Prestretch);
  x=sol.x;  % x represents points along the radius
  y=sol.y;  % y is a 2xN matrix.
  % 1st row represents lambda1 and 2nd row represents lambda2
  yprime=sol.yp;
  [m,n]=size(x);  % determines the mesh size for the derived solution;
  MatrixSize=n;
  meshxIC=x.;
  SOLyIC=y.;
  SOLypIC=yprime.;
  meshxIC
  % ------------------------------------------------------------------------
  function dydx=AnnulusODE(x,y,EFPress,C1,C2,alpha,Prestretch)
  % where x corresponds to the original variable r (radius)
  % where y corresponds to the original variables
  % y(1) = lambda1, y(2) = lambda2
  % this expression of Dlambda1 obtained using method from section 2 and
  % solving for dlambda1 by substitution of the T1Prime, T2Prime, etc.
  dlambda1=((1/2).*C1.^(-1).*EFPress.*y(1).^(-2).*x+2.*(1+alpha.*y(2).^2).*x+ ... 
             (-1).*y(1).^(-1).*y(2).^(-2)*y(1).^(-2)*y(2).^(-2)).*(1+alpha.*y(2).^2).*x ... 
             +2.*y(1).^(-4).*y(2).^(-2).*y(1).^(-2)*y(2).^(-2)*y(2).^(-2)).*(1+alpha.*y(2).^2).*x ...
             y(1).^(-1).*y(1).^(-1).*y(1).^(-2).*y(2).^(-2).*y(2).^(-2).*y(2).^(-2)).*(1+alpha.*y(2).^2).*x ...
             y(1).^(-1).*y(1).^(-1).*y(1).^(-2).*y(2).^(-2).*y(2).^(-2).*y(2).^(-2)).*(1+alpha.*y(2).^2).*x ...
             y(1).^(-1).*y(1).^(-1).*y(1).^(-2).*y(2).^(-2).*y(2).^(-2).*y(2).^(-2)).*(1+alpha.*y(2).^2).*x ...
             y(1).^(-1).*y(1).^(-1).*y(1).^(-2).*y(2).^(-2).*y(2).^(-2).*y(2).^(-2)).*(1+alpha.*y(2).^2).*x ...
             y(1).^(-1).*y(1).^(-1).*y(1).^(-2).*y(2).^(-2).*y(2).^(-2).*y(2).^(-2)).*(1+alpha.*y(2).^2).*x ...
             y(1).^(-1).*y(1).^(-1).*y(1).^(-2).*y(2).^(-2).*y(2).^(-2).*y(2).^(-2)).*(1+alpha.*y(2).^2).*x ...
             y(1).^(-1).*y(1).^(-1).*y(1).^(-2).*y(2).^(-2).*y(2).^(-2).*y(2).^(-2)).*(1+alpha.*y(2).^2).*x ...
             y(1).^(-1).*y(1).^(-1).*y(1).^(-2).*y(2).^(-2).*y(2).^(-2).*y(2).^(-2)).*(1+alpha.*y(2).^2).*x ...
             y(1).^(-1).*y(1).^(-1).*y(1).^(-2).*y(2).^(-2).*y(2).^(-2).*y(2).^(-2)).*(1+alpha.*y(2).^2).*x ...
             y(1).^(-1).*y(1).^(-1).*y(1).^(-2).*y(2).^(-2).*y(2).^(-2).*y(2).^(-2)).*(1+alpha.*y(2).^2).*x ...
             y(1).^(-1).*y(1).^(-1).*y(1).^(-2).*y(2).^(-2).*y(2).^(-2).*y(2).^(-2)).*(1+alpha.*y(2).^2).*x ...
             y(1).^(-1).*y(1).^(-1).*y(1).^(-2).*y(2).^(-2).*y(2).^(-2).*y(2).^(-2)).*(1+alpha.*y(2).^2).*x ...
             y(1).^(-1).*y(1).^(-1).*y(1).^(-2).*y(2).^(-2).*y(2).^(-2).*y(2).^(-2)).*(1+alpha.*y(2).^2).*x ...
             y(1).^(-1).*y(1).^(-1).*y(1).^(-2).*y(2).^(-2).*y(2).^(-2).*y(2).^(-2)).*(1+alpha.*y(2).^2).*x ...
             y(1).^(-1).*y(1).^(-1).*y(1).^(-2).*y(2).^(-2).*y(2).^(-2).*y(2).^(-2)).*(1+alpha.*y(2).^2).*x ...
             y(1).^(-1).*y(1).^(-1).*y(1)^...
\[1).*(y(1).*x.^(-1)+(-1).*y(2).*x.^(-1)).*x+(-2).*alpha.*y(1).^(-1).*(y(1).^2+ ... \\
(-1).*y(1).^(-2).*y(2).^(-2)).*y(2).*(y(1).*x.^(-1)+(-1).*y(2).*x.^(-1)).*x+ ... \\
(-2).*y(1).^(-3).*y(2).^(-3).*(1+alpha.*y(2).^2).*y(1).*(y(1).^2+(-1).*y(2).*x.^(-1))+ ... \\
(-1).*y(2).^(-3).*y(2).^(-3).*(1+alpha.*y(2).^2).*y(1).*(y(1).^2+(-1).*y(2).*x.^(-1)).*x+y(1).^(-1).*(y(1).^2+(-1).*y(2).*x.^(-1)).*x;
\]

\[dydx=[dlambda1 \quad %dlambda1/dr \]
\[y(1)/x-y(2)/x]; \quad %dlambda2/dr\]

%-------------------------------------------------------------------------

function res = AnnulusBC(ya,yb,EFPress,C1,C2,alpha,Prestretch)
%rivlin8bc Boundary conditions:
% RES = RIVLIN8BC(YA,YB) returns a column vector RES of the
% residual in the boundary conditions resulting from the
% approximations YA and YB to the solution at the ends of
% the interval [a b]. The BVP is solved when RES = 0.
% The components of y correspond to the original variables
% y(1) = lambda1, y(2) = lambda2
K1=EFPress/(2*C1);
Root=sqrt((1/2).*K1.*ya(2).^2.*(ya(2).^2+alpha.*ya(2).^4).^(-1)+ ... \\
(1/2).*ya(2).*ya(2).^2+alpha.*ya(2).^4).^(-1)+ ... \\
sqrt(4+8.*alpha.*ya(2).^2+K1.*ya(2).^2+ ... \\
4.*alpha.*ya(2).^4));
res=[ya(1)-Root \\
yb(2)-Prestretch];
%-------------------------------------------------------------------------

function v = AnnulusInit(x)
v=[3.2314*x^5-25.792*x^4+82.09*x^3-130.65*x^2+104.64*x-33.146 \\
-32.73*x^5+260.73*x^4-824.81*x^3+1295.9*x^2-1012.4*x+316.66];
%-------------------------------------------------------------------------
Appendix C

Matlab Code to Plot the Stretch Versus Time Relationship for a Uniaxial Constant Load Condition

This code will plot stretch vs. time for the constant load uniaxial condition. The results will be used to compare with experimental data (with and without electric field).

\[ g(t) = A0 + A1 \ e^{-t/b1}; \]

Viscoelastic Theory: Christensen with Mooney-Rivlin hyperelastic material model.

Modified Debye relaxation

\[ Q(t) = q1 - q2 \ e^{-t/(\tau_2)}; \]

User entries required: Material constants: mr1, mr2, q1, q2, \tau_2, er, DELTAT, StressEng

Data output: Stretch ratio in the direction of applied load. Varying thickness is taken into account.

```matlab
mr1 = 9003.5; (* Mooney-Rivlin constants *)
mr2 = 4641.3;
A0 = 18582.7 (* Modified debye constants *)
A1 = 64423;
b1 = 5.8;
divFactor = A1;
a0 = A0 / divFactor;
a = A1 / divFactor; (*Nondimensionalize*)
MR1 = mr1 / divFactor;
MR2 = mr2 / divFactor;
E0 = 8.854 \times 10^{-12}; (* Free space (vacuum) permittivity *)
er = 4.7; (* relative dielectric constant *)
DELTAT = .1; (* time step *)
StressEng = 29816 / divFactor; (*Applied constant load*)
prestretch = 1;
(* -----Initial conditions -----*)
AGprevious12 = 0;
AGprevious22 = 0;
lambdadetZero = 1;
lambdaxprevious = 1;
(*--------------------------------*)
(* convolution integral \( q1 * \int_{-\infty}^{t} \frac{d}{dx} \) *)
BF = 200 \times 10^4; (* V/m*) (* Applied electric field V/m*)
BF = 0;
```
Md = - 0.5 * Eo * Er * (EF)^2 / divFactor; (* initial Maxwell stress value *)

(* convolution integral \( \left( g_1 e^{-\frac{t}{\Delta t_1}} \right) \) *)

\[
\text{AGK12}[y] = e^{-\left(\frac{\Delta t_1}{2}\right)} \text{AGKprevious12} + e^{-\left(-\frac{\Delta t_1}{2}\right)} \left( \frac{1}{y} - \frac{1}{\text{lambaDprevious}} \right);
\]

\[
\text{AGK1}[y] = a_0 \left( \frac{1}{y} - \frac{1}{\text{lambaDzero}} \right) * \text{AGK12}[y];
\]

(* convolution integral \( \left( g_1 e^{-\frac{t^2}{\Delta t_2}} \right) \) *)

\[
\text{AGK22}[y] = e^{-\left(\frac{\Delta t_2}{2}\right)} \text{AGKprevious22} + e^{-\left(-\frac{\Delta t_2}{2}\right)} \left( y^2 - \text{lambaDprevious}^2 \right);
\]

\[
\text{AGK2}[y] = a_0 \left( y^2 - \text{lambaDzero}^2 \right) * \text{AGK22}[y];
\]

(* hyperelastic stress-stretch eq.*)

\[
\text{sigma11}[y] = 2 * \left( y^2 - \frac{1}{y} \right) * \left( \text{NR1} + \frac{\text{NR2}}{y} \right); (* \text{NR1 and NR2 are the mooney-rivlin matl constants} *)
\]

(* uniaxial viscoelastic stress-stretch eq.*)

\[
\text{func}[y] := \text{soly} = y /. \text{Solve}\left[ 0 = \text{-StressEng} * y - \text{Md} + \text{sigma11}[y] = \frac{\text{AGK1}[y]}{2 * y} + \frac{y^2 \text{AGK2}[y]}{2}, y \right];
\]

\[
\text{lambaD} = \text{soly}[[6]];\]

For[t = 1, t < 200, {Table[func[y], {y, 0, 1, 0.01}], t++,

\[
\text{AGKprevious12} = e^{-\left(\frac{\Delta t_1}{2}\right)} \text{AGKprevious12} + e^{-\left(-\frac{\Delta t_1}{2}\right)} \left( \frac{1}{\text{func}[y]} - \frac{1}{\text{lambaDprevious}} \right);
\]

\[
\text{AGKprevious22} = e^{-\left(\frac{\Delta t_2}{2}\right)} \text{AGKprevious22} + e^{-\left(-\frac{\Delta t_2}{2}\right)} \left( \text{func}[y]^2 - \text{lambaDprevious}^2 \right);
\]

\[
\text{AGK12}[y] = e^{-\left(\frac{\Delta t_1}{2}\right)} \text{AGKprevious12} + e^{-\left(-\frac{\Delta t_1}{2}\right)} \left( \frac{1}{\text{func}[y]} - \frac{1}{\text{lambaD}} \right);
\]

\[
\text{AGK1}[y] = a_0 \left( \frac{1}{y} - \frac{1}{\text{lambaDzero}} \right) * \text{AGK12}[y];
\]

\[
\text{AGK22}[y] = e^{-\left(\frac{\Delta t_2}{2}\right)} \text{AGKprevious22} + e^{-\left(-\frac{\Delta t_2}{2}\right)} \left( y^2 - \text{lambaD}^2 \right);
\]

\[
\text{AGK2}[y] = a_0 \left( y^2 - \text{lambaDzero}^2 \right) * \text{AGK22}[y];
\]

Md = 0; (* approximate dimensionless maxwell stress based upon previous thickness *)

SessionTime[]

Print[soly]
Appendix D

Matlab Code to Simulate the Viscoelastic Response of An Annular Membrane Due to Maxwell Stress

function AnnularMembraneV15_11()
clear all

% copied from AnnularMembraneV15_9.m
global Eo H Voltage r2 Md CURRENT_TIME DELTAT Prestretch UndeformedInnerRadius
global yPrime1previous yPrime2previous mxPREVIOUS xPREVIOUS AnnularAreaPrestretched
global AKKprevious1 AKKprevious2 AKKprevious3 AKKprevious4 AKKprevious5 AKKprevious6
global AKKprevious12 AKKprevious22
global AKKprevious32 AKKprevious42
global AKKprevious52 AKKprevious62
global y1previous y2previous
global y1atZero y2atZero yPrime1atZero yPrime2atZero
global R1 R2 r3 MR1 MR2
global y1previousTempIC y2previousTempIC yPrime1previousTempIC yPrime2previousTempIC
global AKKprevious1TempIC AKKprevious2TempIC AKKprevious3TempIC AKKprevious4TempIC
global AKKprevious5TempIC AKKprevious6TempIC
global AKKprevious12TempIC AKKprevious22TempIC
global AKKprevious32TempIC AKKprevious42TempIC
global AKKprevious52TempIC AKKprevious62TempIC
global AKKprevious1AtPrevxi AKKprevious2AtPrevxi AKKprevious3AtPrevxi
global y1previousAtPrevxi
global y2previousAtPrevxi yPrime2previousAtPrevxi
global SOLyIC SOLypIC meshxIC Prestretch case1 case2 case3

% Relaxation function= R1+R2*Exp[-t*R3]
tic
%-----------------------SECTION 0 ------------------------------------------%
% Purpose of this section is to initialize variables and provide user
% required values.

%load('c:\Act46ElasticSol.mat'); % Use Mooney_RivlinV1_4.m to generate the elastic initial conditions.
% Hyperelastic solution for an annulus
% prestretched so that lambda2 at
% IR=2.4554 and OR=1.5556
load('c:\Prestretchof1.mat') % Use file initialconditions1.m to generate a prestretch=1 data set.
% Use timestep=0.1; CASE3
%load('c:\PrestretchOfl_2.mat')(mlCs,nlCs)=size(SOLyIC);
[a,b]=size(meshxIC);
%---------------------- Annulus actuator specs (*USER INPUT*)---------------------
Prestretch=SOLyIC(a,2);
OriginalInnerRadius=.15625;
OriginalOuterRadius=1.25;
PrestretchedInnerRadius=OriginalInnerRadius*SOLyIC(1,2)*.0254; % I.R. after prestretch equilibrium reached
PrestretchedOuterRadius=OriginalOuterRadius*SOLyIC(a,2)*.0254; % O.R. after prestretch equilibrium reached
OriginalThickness=0.0005; % [m] Membrane's original thickness in stress and strain free state (before prestretch)
AnnularAreaPrestretched=pi*(PrestretchedOuterRadius^2-PrestretchedInnerRadius^2);
Voltage=5000; % [Volts]
UndeformedInnerRadius=meshxIC(1,1);
UndeformedOuterRadius=1;
%------- MATERIAL CONSTANTS FROM A CREEP TEST-------------------

%---------------------------------------------------------------

mr1=8842.5;
mr2=abs(4583.8);
r1=38000;
r2=129800;
r3=abs(1/5.6);
fval =2071.8 R2 =.762
R1=r1/(r2);
R2=r2/(r2);
MR1=mr1/(r2);
MR2=mr2/(r2);
%--------------------------------------------------------------------
Eo=8.850e-12; % Free space (vacuum) permittivity
Capacitance=.5e-9;

h=OriginalThickness/(SOLyIC(1,1)*SOLyIC(1,2)); % [meters] reference configuration thickness
(config
% after prestretch) (* USER INPUT *)
Er=Capacitance*h/(Eo*AnnularAreaPrestretched);
EF=Voltage/h
Md=abs(0.5*Eo*Er*EF^2/r2);

%........ USER SELECT ONE OF THE FOLLOWING TIME STEPS, [sec]
%........ USER SELECT CASE1,CASE2,CASE3 1=TRUE 0=FALSE
% Operation of code is highly sensitive to the initial guess value in
% function odeINIT. Choose one of the following cases:
% The following time steps should be tried.
case1=0; % odeINIT guess assumes unity values for dlambda1/dr and dlambda2/dr
% This is the case if Prestretch=1 and Voltage=0
% Use Root(4)
case2=0;  % odeINIT guess assumes non unity values for dlambda1/dr and dlambda2/dr
% This is the case if Prestretch>1 && Voltage>=0
% Use Root(2)
case3=1;  % odeINIT guess assumes non unity values for dlambda1/dr and dlambda2/dr
% This is the case if Prestretch=1 && Voltage>0
% Use Root(1)

[mICs,nICs]=size(SOLyIC);   % from loaded file specified in SEC 0
xPREVIOUS=meshxIC;
[mxPREVIOUS,nxPREVIOUS]=size(SOLyIC);

%--------- INITIAL CONDITIONS ------------------------------

AKKprevious12=zeros(mICs,1);
AKKprevious22=zeros(mICs,1);
AKKprevious32=zeros(mICs,1);
AKKprevious42=zeros(mICs,1);
AKKprevious52=zeros(mICs,1);
AKKprevious62=zeros(mICs,1);
y1previous=SOLyIC(:,1);
y2previous=SOLyIC(:,2);
yPrime1previous=SOLypIC(:,1);
yPrime2previous=SOLypIC(:,2);

%---------------------------------------------------------------------

options=bvpset('RelTol',1e-3,'AbsTol',1e-3,'Stats','on','NMax',10000);
N=500;   %<-- increasing this made the sol.yp smooth at the outer radius.
solinit=bvpinit(linspace(meshxIC(1,1),meshxIC(a,1),N),@odeINIT);
totalTimeIteration=80;
hold on
CURRENT_TIME=0;
for i=1:totalTimeIteration
    CURRENT_TIME=DELTAT+CURRENT_TIME;
sol=bvp4c(@odeFUN,@odeBC,solinit,options);

%-------------- RECALCULATE EF DUE TO THINNING OF MEMBRANE ------------

h=OriginalThickness/(SOLy(1,1)*SOLy(1,2));   % [meters] reference configuration thickness (config
% after prestretch) (* USER INPUT *)
CurrentInnerRadius=OriginalInnerRadius*SOLy(1,2)*.0254;
CurrentOuterRadius=OriginalOuterRadius*SOLy(a,2)*.0254;
AnnularAreaPrestretched=pi*(CurrentOuterRadius^2-CurrentInnerRadius^2);
Er=Capacitance*h/(Eo*AnnularAreaPrestretched);
EF=Voltage/h
Md=abs(0.5*Eo*Er*EF^2/r2);

%---------------- END OF RECALCULATING EF DUE TO THINNING OF MEMBRANE ---

hold on
lambda2history(i,1)=CURRENT_TIME;
lambda2history(i,2)=SOLy(1,2);
lambda2history
plot((sol.x.')/UndeformedOuterRadius,SOLy);
xlabel('r/ro');
ylabel('lambda1,lambda2');
lambda2atID(i,1)=SOLy(1,2);
grid on
legend('lambda1','lambda2');

%****************************************************************
%--------------SECTION 1 ---------------------------------------%
% The purpose of this section is to generate the values of the
% all variables used in the two ODEs at t=2*DELTAT (i.e. next time step)
% which are evaluated at t=1*DELTAT

if i==1
  [mxNOW,ncol]=size(sol.x');
  [mxPREVIOUS,nPREVIOUS]=size(meshxIC);
  xNOW=sol.x';
  xPREVIOUS=meshxIC;
  clear j
  % Note: Variables with the name "AKKprevious" are the values
  % of the convolution integral at the previous time step.

  % The "Temp" variables below are values at the previous time step.
  % The variables ending in "Temp" need to be interpolated to the mesh
  % for the solution that has just been calculated. Also, the
  % "AKKprevious" variables at this point in time are all
  % zeros due to initial conditions. So,
  % the only variables that need to be interpolated are the SOLyIC
  % SOLypIC y1atzero y2atzero yPrime1atZero and yPrime2atZero

  y1previousTempIC(1,1)=SOLyIC(1,1);
y2previousTempIC(1,1)=SOLyIC(1,2);
yPrime1previousTempIC(1,1)=SOLypIC(1,1);
yPrime2previousTempIC(1,1)=SOLypIC(1,2);
y1atZeroTempIC(1,1)=SOLyIC(1,1);
y2atZeroTempIC(1,1)=SOLyIC(1,2);
yPrime1atZeroTempIC(1,1)=SOLypIC(1,1);
yPrime2atZeroTempIC(1,1)=SOLypIC(1,2);
AKKprevious12TempIC(1,1)=0;
AKKprevious22TempIC(1,1)=0;
AKKprevious32TempIC(1,1)=0;
AKKprevious42TempIC(1,1)=0;
AKKprevious52TempIC(1,1)=0;
AKKprevious62TempIC(1,1)=0;

  % INTERPOLATE:
  for r=2:mxNOW
    AKKprevious12TempIC(r,1)=0;
    AKKprevious22TempIC(r,1)=0;
    AKKprevious32TempIC(r,1)=0;
    AKKprevious42TempIC(r,1)=0;
    AKKprevious52TempIC(r,1)=0;
    AKKprevious62TempIC(r,1)=0;
  end

end
% Criterion for finding x (the current mesh location)
% in between two consecutive mesh from the from the
% previous soln.
s=find(xPREVIOUS>=xNOW(r),1);

% Once criterion is met, linearly interpolate the following variables.

% Forward difference
y1atZeroTempIC(r,1)=((SOLyIC(s-1,1)-SOLyIC(s,1))/(xPREVIOUS(s-1,1)-xPREVIOUS(s,1)))*abs(xNOW(r,1)-xPREVIOUS(s-1,1))+SOLyIC(s-1,1);
y2atZeroTempIC(r,1)=((SOLyIC(s-1,2)-SOLyIC(s,2))/(xPREVIOUS(s-1,1)-xPREVIOUS(s,1)))*abs(xNOW(r,1)-xPREVIOUS(s-1,1))+SOLyIC(s-1,2);
yPrime1atZeroTempIC(r,1)=((SOLypIC(s-1,1)-SOLypIC(s,1))/(xPREVIOUS(s-1,1)-xPREVIOUS(s,1)))*abs(xNOW(r,1)-xPREVIOUS(s-1,1))+SOLypIC(s-1,1);
yPrime2atZeroTempIC(r,1)=((SOLypIC(s-1,2)-SOLypIC(s,2))/(xPREVIOUS(s-1,1)-xPREVIOUS(s,1)))*abs(xNOW(r,1)-xPREVIOUS(s-1,1))+SOLypIC(s-1,2);
y1previousTempIC(r,1)=((SOLyIC(s-1,1)-SOLyIC(s,1))/(xPREVIOUS(s-1,1)-xPREVIOUS(s,1)))*abs(xNOW(r,1)-xPREVIOUS(s-1,1))+SOLyIC(s-1,1);
y2previousTempIC(r,1)=((SOLyIC(s-1,2)-SOLyIC(s,2))/(xPREVIOUS(s-1,1)-xPREVIOUS(s,1)))*abs(xNOW(r,1)-xPREVIOUS(s-1,1))+SOLyIC(s-1,2);
yPrime1previousTempIC(r,1)=((SOLypIC(s-1,1)-SOLypIC(s,1))/(xPREVIOUS(s-1,1)-xPREVIOUS(s,1)))*abs(xNOW(r,1)-xPREVIOUS(s-1,1))+SOLypIC(s-1,1);
yPrime2previousTempIC(r,1)=((SOLypIC(s-1,2)-SOLypIC(s,2))/(xPREVIOUS(s-1,1)-xPREVIOUS(s,1)))*abs(xNOW(r,1)-xPREVIOUS(s-1,1))+SOLypIC(s-1,2);
end
for j=1:mxNOW
    % Formulate previous convolution integral values by
    % assigning current values of variables as previous values so
    % that it can be used in the next time step iteration.
    % Convolution integrals are assigned as follows:
    % AKKcurrent1=(g1* d(L1^2)/dT), AKKcurrent2=(g1* d(L2^2)/dT),
    % AKKcurrent3=(g1* d(1/(L1^2 L2^2)/dT), AKKcurrent4=(g1*d(L2'/(L1^2 L2^3)/dT)
    % AKKprevious5=(g1* d(L1L1')/dT)
    AKKprevious12(j,1)=AKKprevious12TempIC(j,1).*exp((-1).*DELTAT.*r3)+exp((-1/2).*DELTAT.*r3).* ...
    R2.*(SOLy(j,1).^2+(-1).*y1previousTempIC(j,1).^2);  
    AKKprevious22(j,1)=AKKprevious22TempIC(j,1).*exp((-1).*DELTAT.*r3)+exp((-1/2).*DELTAT.*r3).* ...
    R2.*(SOLy(j,2).^2+(-1).*y2previousTempIC(j,1).^2);  
    AKKprevious32(j,1)=AKKprevious32TempIC(j,1).*exp((-1).*DELTAT.*r3)+exp((-1/2).*DELTAT.*r3).* ...
    R2.*(SOLy(j,1).^(-2).*SOLy(j,2).^(-2)+(-1).*y1previousTempIC(j,1).^(-2).*y2previousTempIC(j,1).^(-2));
    AKKprevious42(j,1)=AKKprevious42TempIC(j,1).*exp((-1).*DELTAT.*r3)+exp((-1/2).*DELTAT.*r3).* ...
    R2.*(SOLy(j,2).*SOLy(j,1).^(-2).*SOLy(j,2).^(-3)+(-1).*y1previousTempIC(j,1).^(-2).*y2previousTempIC(j,1).^(-3).*yPrime2previousTempIC(j,1));
    AKKprevious52(j,1)=AKKprevious52TempIC(j,1).*exp((-1).*DELTAT.*r3)+exp((-1/2).*DELTAT.*r3).* ...
    R2.*(SOLy(j,1).*SOLyp(j,1)+(-1).*y1previousTempIC(j,1).*yPrime1previousTempIC(j,1));
    AKKprevious62(j,1)=AKKprevious62TempIC(j,1).*exp((-1).*DELTAT.*r3)+exp((-1/2).*DELTAT.*r3).* ...
    R2.*(SOLy(j,1).^(-3).*SOLyp(j,1).*SOLy(j,2).^(-2)+(-1).*y1previousTempIC(j,1).^(-3).*yPrime1previousTempIC(j,1));
end
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\[ \text{AKKprevious1}(j,1) = R1 \cdot (\text{SOLy}(j,1)^{\cdot 2} + (-1) \cdot y1atZeroTempIC(j,1)^{\cdot 2}) + \text{AKKprevious12}(j,1); \]
\[ \text{AKKprevious2}(j,1) = R1 \cdot (\text{SOLy}(j,2)^{\cdot 2} + (-1) \cdot y2atZeroTempIC(j,1)^{\cdot 2}) + \text{AKKprevious22}(j,1); \]
\[ \text{AKKprevious3}(j,1) = R1 \cdot (\text{SOLy}(j,1)^{-2} \cdot \text{SOLy}(j,2)^{-2} + (-1) \cdot y1atZeroTempIC(j,1)^{-2} \cdot y2atZeroTempIC(j,1)^{-2}) + \text{AKKprevious32}(j,1); \]
\[ \text{AKKprevious4}(j,1) = R1 \cdot (\text{SOLyp}(j,2) \cdot \text{SOLy}(j,1)^{-2} \cdot \text{SOLy}(j,2)^{-3} - yPrime2atZeroTempIC(j,1) \cdot y1atZeroTempIC(j,1)^{-2} \cdot y2atZeroTempIC(j,1)^{-3}) + \text{AKKprevious42}(j,1); \]
\[ \text{AKKprevious5}(j,1) = R1 \cdot (\text{SOLy}(j,1) \cdot \text{SOLyp}(j,1) - y1atZeroTempIC(j,1) \cdot yPrime1atZeroTempIC(j,1)) + \text{AKKprevious52}(j,1); \]
\[ \text{AKKprevious6}(j,1) = R1 \cdot (\text{SOLyp}(j,1) / (\text{SOLy}(j,1)^{\cdot 3} \cdot \text{SOLy}(j,2)^{\cdot 2}) - (yPrime1atZeroTempIC(j,1) / (y1atZeroTempIC(j,1)^{\cdot 3} \cdot y2atZeroTempIC(j,1)^{\cdot 2})) + \text{AKKprevious62}(j,1); \]

\[ y1previous(j,1) = \text{SOLy}(j,1); \]
\[ y2previous(j,1) = \text{SOLy}(j,2); \]
\[ \text{SOLyp} = \text{real}((\text{sol.yp}')); \]
\[ yPrime1previous(j,1) = \text{SOLyp}(j,1); \]
\[ yPrime2previous(j,1) = \text{SOLyp}(j,2); \]
\[ y1atZeroTemp(j,1) = \text{SOLyIC}(j,1); \]
\[ y2atZeroTemp(j,1) = \text{SOLyIC}(j,2); \]
\[ yPrime1atZeroTemp(j,1) = \text{SOLypIC}(j,1); \]

\%--------------SECTION 2 ---------------------------------------\%
\% The purpose of this section is to generate the values of the
\% all variables used in the two ODEs for \( t = 3 \cdot \text{DELTAT} \) and larger (i.e. next time step)
\% which are evaluated at the previous time step.
\% Note: Variables with the name "AKKprevious" are the values
\% of the convolution integral at the previous time step.
\% The "Temp" variables below are values at the previous time step.
\% The variables ending in "Temp" need to be interpolated to the mesh
\% for the solution that has just been calculated.

if i > 1
    clear mrow ncol
    [mxNOW,ncol]=size(sol.x');
    xNOW=sol.x';
    
    AKKprevious12Temp(1,1)=AKKprevious12(1,1);
    AKKprevious22Temp(1,1)=AKKprevious22(1,1);
    AKKprevious32Temp(1,1)=AKKprevious32(1,1);
    AKKprevious42Temp(1,1)=AKKprevious42(1,1);
    AKKprevious52Temp(1,1)=AKKprevious52(1,1);
    AKKprevious62Temp(1,1)=AKKprevious62(1,1);
    y1previousTemp(1,1)=SOLy(1,1);
    y2previousTemp(1,1)=SOLy(1,2);
    yPrime1previousTemp(1,1)=SOLyp(1,1);
    yPrime2previousTemp(1,1)=SOLyp(1,2);
    y1atZeroTempIC(1,1)=SOLyIC(1,1);
    y2atZeroTempIC(1,1)=SOLyIC(1,2);
    yPrime1atZeroTempIC(1,1)=SOLypIC(1,1);
yPrime2atZeroTemp(1,1)=SOLypIC(1,2);

for r=2:mxNOW % The previous values need to be calculated for
% the current solution's mesh location

% Criterion for finding x (the current mesh location)
% in between two consecutive mesh from the from the
% previous soln.
s=find(xPREVIOUS>=xNOW(r,1));
ICradius=find(meshxIC>=xNOW(r,1));

% Once criterion is met, linearly interpolate the following variables.

AKKprevious12Temp(r,1)=((AKKprevious12(s-1,1)-AKKprevious12(s,1))/ ...     
  (xPREVIOUS(s-1,1)-xPREVIOUS(s,1)))* ...     
  abs(xNOW(r,1)-xPREVIOUS(s-1,1))+AKKprevious12(s-1,1);

AKKprevious22Temp(r,1)=((AKKprevious22(s-1,1)-AKKprevious22(s,1))/ ...     
  (xPREVIOUS(s-1,1)-xPREVIOUS(s,1)))* ...     
  abs(xNOW(r,1)-xPREVIOUS(s-1,1))+AKKprevious22(s-1,1);

AKKprevious32Temp(r,1)=((AKKprevious32(s-1,1)-AKKprevious32(s,1))/ ...     
  (xPREVIOUS(s-1,1)-xPREVIOUS(s,1)))* ...     
  abs(xNOW(r,1)-xPREVIOUS(s-1,1))+AKKprevious32(s-1,1);

AKKprevious42Temp(r,1)=((AKKprevious42(s-1,1)-AKKprevious42(s,1))/ ...     
  (xPREVIOUS(s-1,1)-xPREVIOUS(s,1)))* ...     
  abs(xNOW(r,1)-xPREVIOUS(s-1,1))+AKKprevious42(s-1,1);

AKKprevious52Temp(r,1)=((AKKprevious52(s-1,1)-AKKprevious52(s,1))/ ...     
  (xPREVIOUS(s-1,1)-xPREVIOUS(s,1)))* ...     
  abs(xNOW(r,1)-xPREVIOUS(s-1,1))+AKKprevious52(s-1,1);

AKKprevious62Temp(r,1)=((AKKprevious62(s-1,1)-AKKprevious62(s,1))/ ...     
  (xPREVIOUS(s-1,1)-xPREVIOUS(s,1)))* ...     
  abs(xNOW(r,1)-xPREVIOUS(s-1,1))+AKKprevious62(s-1,1);

y1atZeroTemp(r,1)=((SOLyIC(ICradius-1,1)-SOLyIC(ICradius,1))/ ...     
  (meshxIC(ICradius-1,1)-meshxIC(ICradius,1)))* ...     
  abs(xNOW(r,1)-meshxIC(ICradius-1,1))+SOLyIC(ICradius-1,1);

y2atZeroTemp(r,1)=((SOLyIC(ICradius-1,2)-SOLyIC(ICradius,2))/ ...     
  (meshxIC(ICradius-1,1)-meshxIC(ICradius,1)))* ...     
  abs(xNOW(r,1)-meshxIC(ICradius-1,1))+SOLyIC(ICradius-1,2);

yPrime1atZeroTemp(r,1)=((SOLypIC(ICradius-1,1)-SOLypIC(ICradius,1))/ ...     
  (meshxIC(ICradius-1,1)-meshxIC(ICradius,1)))* ...     
  abs(xNOW(r,1)-meshxIC(ICradius-1,1))+SOLypIC(ICradius-1,1);

yPrime2atZeroTemp(r,1)=((SOLypIC(ICradius-1,2)-SOLypIC(ICradius,2))/ ...     
  (meshxIC(ICradius-1,1)-meshxIC(ICradius,1)))* ...     
  abs(xNOW(r,1)-meshxIC(ICradius-1,1))+SOLypIC(ICradius-1,2);

y1previousTemp(r,1)=((y1previous(s-1,1)-y1previous(s,1))/ ...     
  (xPREVIOUS(s-1,1)-xPREVIOUS(s,1)))* ...     
  abs(xNOW(r,1)-xPREVIOUS(s-1,1))+y1previous(s-1,1);
(xPREVIOUS(s-1,1)-xPREVIOUS(s-1,1)) * ...
abs(xNOW(r,1)-xPREVIOUS(s-1,1))+y1previous(s-1,1);

y2previousTemp(r,1)=((y2previous(s-1,1)-y2previous(s,1))/ ...
(xPREVIOUS(s-1,1)-xPREVIOUS(s,1)))* ...
abs(xNOW(r,1)-xPREVIOUS(s-1,1))+y2previous(s-1,1);

yPrime1previousTemp(r,1)=((yPrime1previous(s-1,1)-yPrime1previous(s,1))/ ...
(xPREVIOUS(s-1,1)-xPREVIOUS(s,1)))* ...
abs(xNOW(r,1)-xPREVIOUS(s-1,1))+yPrime1previous(s-1,1);

yPrime2previousTemp(r,1)=((yPrime2previous(s-1,1)-yPrime2previous(s,1))/ ...
(xPREVIOUS(s-1,1)-xPREVIOUS(s,1)))* ...
abs(xNOW(r,1)-xPREVIOUS(s-1,1))+yPrime2previous(s-1,1);

end

for p=1:mxNOW
  % Formulate previous convolution integral values by
  % assigning current values of variables as previous values so
  % that it can be used in the next time step iteration.
  % Convolution integrals are assigned as follows:
  % AKKcurrent1=(g1* d(L1^2)/dT), AKKcurrent2=(g1* d(L2^2)/dT),
  % AKKcurrent3=(g1* d(1/(L1^2 L2^2)/dT), AKKcurrent4=(g1*d(L2'/(L1^2 L2^3)/dT)
  % AKKprevious5=(g1* d(L1L1')/dT)
  AKKprevious12(p,1)=AKKprevious12Temp(p,1).*exp((-1).*DELTAT.*r3)+exp((-1/2).*DELTAT.*r3).* ...
    R2.*(SOLy(p,1).^2+(-1).*y1previousTemp(p,1).^2);
  AKKprevious22(p,1)=AKKprevious22Temp(p,1).*exp((-1).*DELTAT.*r3)+exp((-1/2).*DELTAT.*r3).* ...
    R2.*(SOLy(p,2).^2+(-1).*y2previousTemp(p,1).^2);
  AKKprevious32(p,1)=AKKprevious32Temp(p,1).*exp((-1).*DELTAT.*r3)+exp((-1/2).*DELTAT.*r3).* ...
    R2.*(SOLy(p,1).^(-2).*SOLy(p,2).^(-2)+(-1).*y1previousTemp(p,1).^(-2).*y2previousTemp(p,1).^(-2));
  AKKprevious42(p,1)=AKKprevious42Temp(p,1).*exp((-1).*DELTAT.*r3)+exp((-1/2).*DELTAT.*r3).* ...
    R2.*(SOLyp(p,2).*SOLy(p,1).^(-2).*SOLy(p,2).^(-3)+(-1).*y1previousTemp(p,1).^(-2).* ...
      y2previousTemp(p,1).^(-2).*yPrime2previousTemp(p,1));
  AKKprevious52(p,1)=AKKprevious52Temp(p,1).*exp((-1).*DELTAT.*r3)+exp((-1/2).*DELTAT.*r3).* ...
    R2.*(SOLy(p,1).*SOLyp(p,1)+(-1).*y1previousTemp(p,1).*yPrime1previousTemp(p,1));
  AKKprevious62(p,1)=AKKprevious62Temp(p,1).*exp((-1).*DELTAT.*r3)+exp((-1/2).*DELTAT.*r3).* ...
    R2.*(SOLy(p,1).^(-3).*SOLyp(p,1).*SOLy(p,2).^(-2)+(-1).*y1previousTemp(p,1).^(-3).* ...
      y2previousTemp(p,1).^(-2).*yPrime1previousTemp(p,1));

  AKKprevious11(p,1)=AKKprevious12(p,1)+R1.*(SOLy(p,1).^2+(-1).*y1atZeroTemp(p,1).^2);
\[ \begin{align*}
\text{AKKprevious3}(p,1) &= \text{AKKprevious32}(p,1) + R1 \cdot (\text{SOLy}(p,1)^{-2} \cdot \text{SOLy}(p,2)^{-2} + (-2) \cdot y1atZeroTemp(p,1)^{-2} \cdot \text{SOLy}(p,2)^{-2}); \\
\text{AKKprevious4}(p,1) &= \text{AKKprevious42}(p,1) + R1 \cdot (\text{SOLyp}(p,2) \cdot \text{SOLy}(p,1)^{-2} \cdot \text{SOLy}(p,2)^{-3} - yPrime2atZeroTemp(r,1)^2 \cdot y1atZeroTemp(p,1)^{-2} \cdot y2atZeroTemp(p,1)^{-3}); \\
\text{AKKprevious5}(p,1) &= \text{AKKprevious52}(p,1) + R1 \cdot (\text{SOLy}(p,1) \cdot \text{SOLyp}(p,1) - y1atZeroTemp(p,1) \cdot yPrime1atZeroTemp(p,1)); \\
\text{AKKprevious6}(p,1) &= \text{AKKprevious62}(p,1) + R1 \cdot (\text{SOLyp}(p,1) / (\text{SOLy}(p,1)^3 \cdot \text{SOLy}(p,2)^2) - (yPrime1atZeroTemp(p,1) / (y1atZeroTemp(p,1)^3 \cdot y2atZeroTemp(p,1)^2)); \\
y1previous(p,1) &= \text{SOLy}(p,1); \\
y2previous(p,1) &= \text{SOLy}(p,2); \\
\text{SOLyp} &= \text{real(sol.yp}'); \\
yPrime1previous(p,1) &= \text{SOLyp}(p,1); \\
yPrime2previous(p,1) &= \text{SOLyp}(p,2); \\
\end{align*} \]
\[ AKK_{\text{previous6}2\text{AtPrevxi}} = AKK_{\text{previous6}2(1,1)}; \]
\[ y_{1\text{atZeroAtPrevxi}} = \text{SOLyIC}(1,1); \]
\[ y_{2\text{atZeroAtPrevxi}} = \text{SOLyIC}(1,2); \]
\[ y_{\text{Prime1atZeroAtPrevxi}} = \text{SOLyIC}(1,1); \]
\[ y_{\text{Prime2atZeroAtPrevxi}} = \text{SOLyIC}(1,2); \]
\[ y_{1\text{previousAtPrevxi}} = y_{1\text{previous}(1,1)}; \]
\[ y_{2\text{previousAtPrevxi}} = y_{2\text{previous}(1,1)}; \]
\[ y_{\text{Prime1previousAtPrevxi}} = y_{\text{Prime1previous}(1,1)}; \]
\[ y_{\text{Prime2previousAtPrevxi}} = y_{\text{Prime2previous}(1,1)}; \]

\[ \text{end} \]

\[ \text{if } x > \text{UndeformedInnerRadius} \]
\[ r = \text{find}(x_{\text{PREVIOUS}}>x,1); \]
\[ IC\text{radius} = \text{find}(\text{meshxIC}>=x,1); \]

% Once criterion is met, linearly interpolate the following variables.
\[ AKK_{\text{previous12AtPrevxi}} = ((AKK_{\text{previous12}(r-1,1)} - AKK_{\text{previous12}(r,1)})/ ... (x_{\text{PREVIOUS}(r-1,1)} - x_{\text{PREVIOUS}(r)}))*\text{abs}(x - x_{\text{PREVIOUS}(r-1,1)}) + AKK_{\text{previous12}(r-1,1)}; \]
\[ AKK_{\text{previous22AtPrevxi}} = ((AKK_{\text{previous22}(r-1,1)} - AKK_{\text{previous22}(r,1)})/ ... (x_{\text{PREVIOUS}(r-1,1)} - x_{\text{PREVIOUS}(r)}))*\text{abs}(x - x_{\text{PREVIOUS}(r-1,1)}) + AKK_{\text{previous22}(r-1,1)}; \]
\[ AKK_{\text{previous32AtPrevxi}} = ((AKK_{\text{previous32}(r-1,1)} - AKK_{\text{previous32}(r,1)})/ ... (x_{\text{PREVIOUS}(r-1,1)} - x_{\text{PREVIOUS}(r)}))*\text{abs}(x - x_{\text{PREVIOUS}(r-1,1)}) + AKK_{\text{previous32}(r-1,1)}; \]
\[ AKK_{\text{previous42AtPrevxi}} = ((AKK_{\text{previous42}(r-1,1)} - AKK_{\text{previous42}(r,1)})/ ... (x_{\text{PREVIOUS}(r-1,1)} - x_{\text{PREVIOUS}(r)}))*\text{abs}(x - x_{\text{PREVIOUS}(r-1,1)}) + AKK_{\text{previous42}(r-1,1)}; \]
\[ AKK_{\text{previous52AtPrevxi}} = ((AKK_{\text{previous52}(r-1,1)} - AKK_{\text{previous52}(r,1)})/ ... (x_{\text{PREVIOUS}(r-1,1)} - x_{\text{PREVIOUS}(r)}))*\text{abs}(x - x_{\text{PREVIOUS}(r-1,1)}) + AKK_{\text{previous52}(r-1,1)}; \]
\[ AKK_{\text{previous62AtPrevxi}} = ((AKK_{\text{previous62}(r-1,1)} - AKK_{\text{previous62}(r,1)})/ ... (x_{\text{PREVIOUS}(r-1,1)} - x_{\text{PREVIOUS}(r)}))*\text{abs}(x - x_{\text{PREVIOUS}(r-1,1)}) + AKK_{\text{previous62}(r-1,1)}; \]

\[ y_{1\text{atZeroAtPrevxi}} = ((\text{SOLyIC}(IC\text{radius}-1,1) - \text{SOLyIC}(IC\text{radius},1))/ ... (\text{meshxIC}(IC\text{radius}-1,1) - \text{meshxIC}(IC\text{radius},1)))*\text{abs}(x - \text{meshxIC}(IC\text{radius}-1,1)) + \text{SOLyIC}(IC\text{radius}-1,1); \]
\[ y_{2\text{atZeroAtPrevxi}} = ((\text{SOLyIC}(IC\text{radius}-1,2) - \text{SOLyIC}(IC\text{radius},2))/ ... (\text{meshxIC}(IC\text{radius}-1,1) - \text{meshxIC}(IC\text{radius},1)))*\text{abs}(x - \text{meshxIC}(IC\text{radius}-1,1)) + \text{SOLyIC}(IC\text{radius},1); \]
\[ y_{\text{Prime1atZeroAtPrevxi}} = ((\text{SOLypIC}(IC\text{radius},1,1) - \text{SOLypIC}(IC\text{radius},1))/ ... (\text{meshxIC}(IC\text{radius}-1,1) - \text{meshxIC}(IC\text{radius},1)))*\text{abs}(x - \text{meshxIC}(IC\text{radius}-1,1)) + \text{SOLypIC}(IC\text{radius},1); \]
\[ y_{\text{Prime2atZeroAtPrevxi}} = ((\text{SOLypIC}(IC\text{radius},1,2) - \text{SOLypIC}(IC\text{radius},2))/ ... (\text{meshxIC}(IC\text{radius}-1,1) - \text{meshxIC}(IC\text{radius},1)))*\text{abs}(x - \text{meshxIC}(IC\text{radius}-1,1)) + \text{SOLypIC}(IC\text{radius},2); \]
\[ y_{\text{Prime1previousAtPrevxi}} = ((y_{\text{Prime1previous}(r-1,1)} - y_{\text{Prime1previous}(r,1)})/ ... (x_{\text{PREVIOUS}(r-1,1)} - x_{\text{PREVIOUS}(r)}))*\text{abs}(x - x_{\text{PREVIOUS}(r-1,1)}) + y_{\text{Prime1previous}(r-1,1)}; \]
\[ y_{\text{Prime2previousAtPrevxi}} = ((y_{\text{Prime2previous}(r-1,1)} - y_{\text{Prime2previous}(r,1)})/ ... (x_{\text{PREVIOUS}(r-1,1)} - x_{\text{PREVIOUS}(r)}))*\text{abs}(x - x_{\text{PREVIOUS}(r-1,1)}) + y_{\text{Prime2previous}(r-1,1)}; \]
\[ y_{1\text{previousAtPrevxi}} = ((y_{1\text{previous}(r-1,1)} - y_{1\text{previous}(r,1)})/ ... (x_{\text{PREVIOUS}(r-1,1)} - x_{\text{PREVIOUS}(r)}))*\text{abs}(x - x_{\text{PREVIOUS}(r-1,1)}) + y_{1\text{previous}(r-1,1)}; \]
\[ y_{2\text{previousAtPrevxi}} = ((y_{2\text{previous}(r-1,1)} - y_{2\text{previous}(r,1)})/ ... (x_{\text{PREVIOUS}(r-1,1)} - x_{\text{PREVIOUS}(r)}))*\text{abs}(x - x_{\text{PREVIOUS}(r-1,1)}) + y_{2\text{previous}(r-1,1)}; \]

\[ \text{end} \]

% Convolution integrals in recurrence form containing only previous --------
% time step terms. Convolution integral is denoted in my derivation as:
% (g1*phi), where g1 operates on phi.
% When m=1, then the variables evaluated at the 'previous' time (or at ti-1)
$$\lambda_2' = \frac{y(1)}{x} - \frac{y(2)}{x};$$

$$\text{AKKcurrent1} = \text{AKKprevious12AtPrevxi} \times \exp((-1) \cdot \Delta T \cdot \tau_3) + \exp((-1/2) \cdot \Delta T \cdot \tau_3) \cdot R_2 \cdot (y(1)^2 + y(2)^2);$$

$$\text{R1.} \cdot (y(1)^2 + y(2)^2);$$

$$\text{AKKcurrent2} = \text{AKKprevious22AtPrevxi} \times \exp((-1) \cdot \Delta T \cdot \tau_3) + \exp((-1/2) \cdot \Delta T \cdot \tau_3) \cdot R_2 \cdot (y(2)^2 + y(2)^2);$$

$$\text{R1.} \cdot (y(2)^2 + y(2)^2);$$

$$\text{AKKcurrent3} = \text{AKKprevious32AtPrevxi} \times \exp((-1) \cdot \Delta T \cdot \tau_3) + \exp((-1/2) \cdot \Delta T \cdot \tau_3) \cdot R_2 \cdot (y(1)^2 + y(2)^2);$$

$$\text{R1.} \cdot (y(1)^2 + y(2)^2);$$

$$\text{AKKcurrent4part1} = R_1 \cdot \frac{1}{(y(1)^2 \cdot y(2)^3)} + R_2 \cdot \frac{1}{(y(1)^2 \cdot y(2)^3)};$$

$$\text{AKKcurrent4part2} = -R_1 \cdot y(1) - \frac{1}{(y(1)^2 \cdot y(2)^2)};$$

$$\text{R1.} \cdot (y(1)^2);$$

$$\text{AKKcurrent5part1} = R_1 \cdot y(1) \cdot \exp((-1/2) \cdot \Delta T \cdot \tau_3) + \frac{1}{(y(1)^2 \cdot y(2)^2)};$$

$$\text{AKKcurrent5part2} = -R_1 \cdot y(1) \cdot \exp((-1/2) \cdot \Delta T \cdot \tau_3);$$

$$\text{R1.} \cdot (y(1)^2);$$

$$\text{y3} = \frac{1}{(y(1)^2)};$$

$$\text{sigma1elastic} = 2 \cdot y(1)^2 \cdot \text{MR1} - \text{MR2} / y(1)^2;$$

$$\text{sigma2elastic} = 2 \cdot y(2)^2 \cdot \text{MR1} - \text{MR2} / y(2)^2;$$

$$\text{sigma3elastic} = 2 \cdot y(3)^2 \cdot \text{MR1} - \text{MR2} / y(3)^2;$$

$$\text{sigma1V} = -M_d + (\text{sigma1elastic} - \text{sigma3elastic}) + R_1 \cdot 0.5 \cdot y(1)^2 \cdot y(1)^2;$$

$$\text{sigma2V} = -M_d + (\text{sigma2elastic} - \text{sigma3elastic}) + R_1 \cdot 0.5 \cdot y(2)^2 \cdot y(2)^2;$$

$$\text{sigma3V} = -M_d + (\text{sigma3elastic} - \text{sigma3elastic}) + R_1 \cdot 0.5 \cdot y(3)^2 \cdot y(3)^2;$$

$$\text{R1.} \cdot 0.5 \cdot y(1)^2 \cdot y(1)^2;$$

$$\text{A} = 4 \cdot y(1)^3 \cdot \text{MR1} + 4 \cdot y(1)^2 \cdot y(2)^2;$$

$$\text{B} = 4 \cdot y(1)^2 \cdot y(2);$$

$$\text{C} = y(1)^2 \cdot \text{AKKcurrent1} + y(1)^2 \cdot \text{AKKcurrent5part1} + \text{AKKcurrent3} \cdot (y(1)^3 \cdot y(2)^2) + \text{AKKcurrent6part1} \cdot (y(1)^3 \cdot y(2)^2);$$

$$\text{D} = -\text{sigma1V} \cdot y(1)^2;$$

$$\text{E} = y(1)^2 \cdot \text{AKKcurrent5part2} + \text{AKKcurrent6part2} \cdot (y(1)^2 \cdot y(2)^2) + \text{AKKcurrent4part2} \cdot (y(1)^2 \cdot y(2)^2);$$

$$\text{F} = R_1 \cdot y(1)^2 \cdot y(1)^2;$$
\[ G = R_1 \times \frac{1}{(y_1(1)^2 \times y_2(2)^3)} \times \frac{1}{(y_{1atZeroAtPrevxi}^2 \times y_{2atZeroAtPrevxi}^2) - 1.0}; \]
\[ H = R_1 \times y_1(1)^2 \times \left( \frac{y_{1atZeroAtPrevxi} \times y_{Prime1atZeroAtPrevxi}}{y_{1atZeroAtPrevxi}^3 \times y_{2atZeroAtPrevxi}^2} \right) + \frac{R_1 \times 1}{(y_1(1)^2 \times y_2(2)^2)} \times \left( \frac{y_{Prime1atZeroAtPrevxi}}{y_{1atZeroAtPrevxi}^3 \times y_{2atZeroAtPrevxi}^2} \right) + \frac{R_1 \times 1}{(y_1(1)^2 \times y_2(2)^2)} \times \left( \frac{y_{Prime2atZeroAtPrevxi}}{y_{1atZeroAtPrevxi}^2 \times y_{2atZeroAtPrevxi}^3} \right); \]
\[ \lambda_{1Prime} = \frac{-\lambda_{2Prime} \times (D + B + G) - E - H - y_1(1) \times (\sigma_1V - \sigma_2V)/(x \times y_2)}{(C + A + F)}; \]
\[ \frac{dy}{dx} = \begin{bmatrix} \lambda_{1Prime} \\ \lambda_{2Prime} \end{bmatrix}; \]

%--------- FUNCTION -------------------------
%********************************************

function v = odeINIT(x)

global case1 case2 case3

%%%%%%%%%%%%% USER SELECT CASE1 OR CASE2.  1=TRUE  0=FALSE
% case1:  % odeINIT guess assumes unity values for dlambda1/dr and dlambda2/dr
%          % This is the case if Prestretch=1 and Voltage=0
% case2:  % odeINIT guess assumes non unity values for dlambda1/dr and dlambda2/dr
%          % This is the case if Prestretch>1 && Voltage>=0
% case3:  % odeINIT guess assumes non unity values for dlambda1/dr and dlambda2/dr
%          % This is the case if Prestretch=1 && Voltage>0

if case1 == 1
    v = [1
         1];
end

if case2 == 1
    %Lambda1
    p1 = 2.9973;
    p2 = -7.9746;
    p3 = 7.2224;
    p4 = -1.1429;
    %Lambda2
    z1 = -1.5516;
    z2 = 4.227;
    z3 = -3.9693;
    z4 = 2.4928;

    v = [p1 \times x^3 + p2 \times x^2 + p3 \times x + p4
         z1 \times x^3 + z2 \times x^2 + z3 \times x + z4];
end

if case3 == 1
    %Lambda1
    p1 = 2.9973;
    p2 = -7.9746;
    p3 = 7.2224;
    p4 = -1.1429;
    %Lambda2
    z1 = -1.5516;
end
\[ z_2 = 4.227; \]
\[ z_3 = -3.9693; \]
\[ z_4 = 2.4928; \]

\[
v = [z_1 x^3 + z_2 x^2 + z_3 x^1 + z_4 + p_1 x^3 + p_2 x^2 + p_3 x^1 + p_4];
\]

end

%------------------------------------------------------------------------
%---------------------------FUNCTION----------------------------------------
%************************************************************************

function res=odeBC(ya,yb,PassToFunctions)

global Md CURRENT_TIME DELTAT Prestretch
global R1 R2 r3 MR1 MR2
global y2previousAtPrevxi y1previousAtPrevxi
global AKKprevious12 AKKprevious32
global y1previous y2previous
global yPrime1atZero yPrime2atZero
global SOLyIC SOLyIC

AKKPREVIOUS12ATPREVXI(1,1)=AKKprevious12(1,1);
AKKPREVIOUS32ATPREVXI(1,1)=AKKprevious32(1,1);
Y1PREVIOUSATPREVXI(1,1)=y1previous(1,1);
Y2PREVIOUSATPREVXI(1,1)=y2previous(1,1);
Y1ATZEROATPREVXI(1,1)=SOLyIC(1,1);
Y2ATZEROATPREVXI(1,1)=SOLyIC(1,2);

% From q16RadialConstitutiveEqnV4.nb
y1_0thpower=(-1/2).*R1.*ya(2).^(-4)+(-1/2).*exp((-1/2).*DELTAT.*r3).*R2.*ya(2).^(-4);
y1_2ndpower=(-1/2).*AKKPREVIOUS32ATPREVXI(1,1).*exp((-1).*DELTAT.*r3).*ya(2).^(-2)+( ... 
-0.5E0).*R1.*((0.1E0)+Y1ATZEROATPREVXI(1,1).^(-2).*Y2ATZEROATPREVXI(1,1).^(-2)).*ya(2).^(-2)+(1/2).*R1.*Y1ATZEROATPREVXI(1,1).^(-2).*Y2ATZEROATPREVXI(1,1).^(-2).*ya(2).^(-2)+( ... 
2.*ya(2).^(-2).*(MR1+MR2.*ya(2).^2);
y1_4thpower=-Md;
y1_8thpower=(1/2).*R1+(1/2).*exp((-1/2).*DELTAT.*r3).*R2;
y1_6thpower=(1/2).*AKKPREVIOUS12ATPREVXI(1,1).*exp((-1).*DELTAT.*r3)+(-1/2).*R1.* ... 
Y1ATZEROATPREVXI(1,1).^2+0.5E0.*R1.*((0.1E1)+Y1ATZEROATPREVXI(1,1).^2)+( ... 
-1/2).*exp((-1/2).*DELTAT.*r3).*R2.*Y1PREVIOUSATPREVXI(1,1).^2+2.*(MR1+ ... 
MR2.*ya(2).^2);
Coefficientof8thPower=1;
Coefficientof6thPower=y1_6thpower/y1_8thpower;
Coefficientof4thPower=y1_4thpower/y1_8thpower;
Coefficientof2thPower=y1_2ndpower/y1_8thpower;
Coefficientof0thPower=y1_0thpower/y1_8thpower;

EQ=[Coefficientof8thPower 0 Coefficientof6thPower 0 Coefficientof4thPower 0 Coefficientof2thPower 0 Coefficientof0thPower];
Root=roots(EQ);
y=;
res=[(ya(1)-Root(1))
     (yb(2)-Prestretch)];
### Appendix E

**Dimensions of the Outer Diameters of Silicone Detakta Tubes (Type 3003)**

Table 14. Dimensions of the undeformed outer diameters using digital images. Measurements are for Detakta silicone tubes (type 3003).

<table>
<thead>
<tr>
<th>Specimen ID</th>
<th>Undeformed inner diameter (mm)</th>
<th>Undeformed outer diameter (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sample 5</td>
<td>2.64</td>
<td>3.45</td>
</tr>
<tr>
<td>Sample 6</td>
<td>2.64</td>
<td>3.33</td>
</tr>
<tr>
<td>Sample 7</td>
<td>2.64</td>
<td>3.48</td>
</tr>
<tr>
<td>Sample 8</td>
<td>2.64</td>
<td>3.35</td>
</tr>
<tr>
<td>Sample 9</td>
<td>2.64</td>
<td>3.33</td>
</tr>
<tr>
<td>Sample 10</td>
<td>2.64</td>
<td>3.45</td>
</tr>
<tr>
<td>Sample 11</td>
<td>2.64</td>
<td>3.40</td>
</tr>
<tr>
<td>Sample 12</td>
<td>2.64</td>
<td>3.33</td>
</tr>
<tr>
<td>Sample 13</td>
<td>2.64</td>
<td>3.38</td>
</tr>
<tr>
<td>Sample 14</td>
<td>2.64</td>
<td>3.43</td>
</tr>
<tr>
<td>Average</td>
<td>2.64</td>
<td>3.39</td>
</tr>
</tbody>
</table>

Standard deviation: 0.06
Table 15. Dimensions of the undeformed outer diameters obtained experimentally. Measurements are for Detakta silicone tubes (type 3003). Stainless steel pins of varying diameters were used to determine the inner diameter. A digital caliper was used to measure the outer diameter.

<table>
<thead>
<tr>
<th>Specimen ID</th>
<th>Undeformed inner diameter (mm)</th>
<th>Undeformed outer diameter (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sample 15</td>
<td>2.97</td>
<td>3.50</td>
</tr>
<tr>
<td>Sample 16</td>
<td>2.97</td>
<td>3.59</td>
</tr>
<tr>
<td>Sample 17</td>
<td>2.97</td>
<td>3.56</td>
</tr>
<tr>
<td>Sample 18</td>
<td>2.97</td>
<td>3.57</td>
</tr>
<tr>
<td>Average</td>
<td>2.97</td>
<td>3.56</td>
</tr>
<tr>
<td>Standard deviation</td>
<td>0</td>
<td>0.04</td>
</tr>
</tbody>
</table>
Appendix F

Material Constants for Various Material Models for the Detakta Silicone Material

Table 16. ABAQUS material evaluation results for the Detakta silicone material.

<table>
<thead>
<tr>
<th>Material model</th>
<th>Material constants</th>
</tr>
</thead>
<tbody>
<tr>
<td>Polynomial $N=1$ (Mooney-Rivlin)</td>
<td>$C_{10} = -174472.7; C_{01} = 868213.3$</td>
</tr>
<tr>
<td>Polynomial $N=3$</td>
<td>Table 17. $C_{10} = -535017.8 \text{ Pa}; C_{20} = -466658.7 \text{ Pa}; C_{01} = 1358209.7 \text{ Pa}; C_{11} = 1732796.43 \text{ Pa}; C_{02} = -1506654.9 \text{ Pa}$</td>
</tr>
<tr>
<td>Ogden $N=2$</td>
<td>$\mu_1 = -3114268.3 \text{ Pa}; \mu_2 = 4835438.1 \text{ Pa}; \alpha_1 = 0.046; \alpha_2 = -2.65$</td>
</tr>
<tr>
<td>Ogden $N=3$</td>
<td>$\mu_1 = -2441529.2 \text{ Pa}; \mu_2 = 444771.9 \text{ Pa}; \mu_3 = 3654925.6 \text{ Pa}; \alpha_1 = 1.42; \alpha_2 = 3.47; \alpha_1 = -2.34$</td>
</tr>
<tr>
<td>Ogden $N=4$</td>
<td>$\mu_1 = -1772831.0 \text{ Pa}; \mu_2 = 325286.5 \text{ Pa}; \mu_3 = 1865994.8 \text{ Pa}; \mu_4 = 1247464.1 \text{ Pa}; \alpha_1 = 2.00; \alpha_2 = 4.00; \alpha_3 = -2.00; \alpha_4 = -4.00$</td>
</tr>
<tr>
<td>Ogden $N=5$</td>
<td>$\mu_1 = -5949222.5 \text{ Pa}; \mu_2 = 2698656.8 \text{ Pa}; \mu_3 = -312841.9 \text{ Pa}; \mu_4 = 8480563.7 \text{ Pa}; \mu_5 = -3332456.6 \text{ Pa}; \alpha_1 = 2.00; \alpha_2 = 4.00; \alpha_3 = 6.00; \alpha_4 = -2.00; \alpha_4 = -4.00$</td>
</tr>
<tr>
<td>Reduced Polynomial $N=3$</td>
<td>$C_{10} = 601708.9 \text{ Pa}; C_{20} = -216453.1 \text{ Pa}; C_{30} = 49208.5 \text{ Pa}$</td>
</tr>
<tr>
<td>Reduced Polynomial $N=4$</td>
<td>$C_{10} = 645465.9 \text{ Pa}; C_{20} = -368292.4 \text{ Pa}; C_{30} = 180734.6 \text{ Pa}; C_{40} = -33129.5 \text{ Pa}$</td>
</tr>
<tr>
<td>Reduced Polynomial $N=5$</td>
<td>$C_{10} = 675969.4 \text{ Pa}; C_{20} = -528753.6 \text{ Pa}; C_{30} = 422219.0 \text{ Pa}; C_{40} = -173812.7 \text{ Pa}; C_{50} = 27954.5 \text{ Pa}$</td>
</tr>
<tr>
<td>Reduced Polynomial $N=6$</td>
<td>$C_{10} = 704350.5 \text{ Pa}; C_{20} = -729179.7 \text{ Pa}; C_{30} = 882398.0 \text{ Pa}; C_{40} = -631614.4 \text{ Pa}; C_{50} = 233293.4 \text{ Pa}; C_{60} = -34064.1 \text{ Pa}$</td>
</tr>
</tbody>
</table>
Figure 81. ABAQUS material evaluation results for the Detakta silicone tube. The polynomial material model is compared with experimental data.
Figure 82. ABAQUS material evaluation results for the Detakta silicone tube. The Ogden material model is compared with experimental data.
Figure 83. ABAQUS material evaluation results for the Detakta silicone tube. The reduced polynomial material model is compared with experimental data.
### Appendix G

**Deformed Lengths and Outer Radii Obtained From Five Dettakta Silicone Tubes**  
*(Type 2502)*

Table 18. Deformed lengths and outer radii obtained from five Dettakta silicone tubes.

<table>
<thead>
<tr>
<th>Sample</th>
<th>Deformed length (mm)</th>
<th>Deformed outer diameter (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sample 1</td>
<td>300.74</td>
<td>3.00</td>
</tr>
<tr>
<td>Sample 2</td>
<td>297.66</td>
<td>3.04</td>
</tr>
<tr>
<td>Sample 3</td>
<td>300.04</td>
<td>3.03</td>
</tr>
<tr>
<td>Sample 4</td>
<td>300.04</td>
<td>3.02</td>
</tr>
<tr>
<td>Sample 5</td>
<td>298.45</td>
<td>3.03</td>
</tr>
<tr>
<td><strong>Average</strong></td>
<td><strong>299.38</strong></td>
<td><strong>3.02</strong></td>
</tr>
<tr>
<td><strong>Standard deviation</strong></td>
<td><strong>1.28</strong></td>
<td><strong>0.02</strong></td>
</tr>
</tbody>
</table>

**FE**  
(Reduced polynomial *N*=3)  

<table>
<thead>
<tr>
<th></th>
<th>Deformed length</th>
<th>Deformed outer diameter</th>
</tr>
</thead>
<tbody>
<tr>
<td>FE</td>
<td>294.2 (1.7% error)</td>
<td>3.01 (0.33 % error)</td>
</tr>
</tbody>
</table>

**FE**  
(Mooney-Rivlin)  

<table>
<thead>
<tr>
<th></th>
<th>Deformed length</th>
<th>Deformed outer diameter</th>
</tr>
</thead>
<tbody>
<tr>
<td>FE</td>
<td>294.06 (1.7% error)</td>
<td>3.01 (0.33 % error)</td>
</tr>
</tbody>
</table>
VITA

Eunice Eun-Young Yang

Eunice Eun-Young Yang earned a B.S. degree in Mechanical Engineering from the University of Hawaii in 1987. In 1991 she earned a M.S. degree in Mechanical Engineering from California State University Long Beach, in Long Beach, California. From 1987 to 1992 she was a Member of the Technical Staff at the Rockwell International Corporation in Canoga Park, California. She worked in the Project Office and the Turbomachinery Department for the Atlas and Delta II Expendable Rocket Engine Program. From 1992 to 1997 she co-owned and operated the United Paper Box Corporation in Santa Fe Springs, California. From 1996 to 2006 she co-owned and operated Boxware Company in State College, PA. United Paper Box Corporation and Boxware Company were businesses geared toward servicing the printing and folding carton industry.