The Pennsylvania State University
The Graduate School

SHOCKS AND WAVE INTERACTIONS FOR COMPRESSIBLE FLOW

A Dissertation in
Mathematics
by
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Submitted in Partial Fulfillment
of the Requirements
for the Degree of

Doctor of Philosophy

August 2008

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Abstract

We study the compressible Euler system in one (1-D) or several (multi-D) space dimensions. For the 1-D system we focus on Riemann problems and interactions while for multi-D we consider boundary value problems with spherical or cylindrical symmetry (which transforms the system into 1-D). In 1-D we consider the *full* system, which includes the energy equation, and for multi-D we also consider isentropic flow.

In Chapter 1 we consider 1-D flow of ideal polytropic gases. We resolve all but two possible interactions of elementary waves (shocks, contact discontinuities, and centered rarefactions). This has been done by several authors to varying degrees of completeness. In addition to solving for the outgoing waves we obtain various relations between strengths of incoming waves and outgoing waves.

As an application of resolving the interactions, we create a specific interaction pattern by carefully choosing the initial data which is done in Chapter 2. This particular pattern is motivated by examples by Young and Jenssen for certain non-physical systems for which the solution may blow up in sup-norm and/or total variation (for carefully chosen data). In the case of the 1-D Euler system, the chosen data generates two contact discontinuities and repeated reflections of shocks between them. We also impose absorbing boundaries outside the central region in order to make the analysis tractable. By using the relations of the incoming waves and outgoing waves strengths, we are able to show that there will be only a finite number of interactions in any finite time interval and that all conserved quantities are uniformly bounded. This gives that these special solutions of the full Euler system are defined for all time and are uniformly bounded. Further, we comment briefly on a particular scaling of the dependent variables in the Euler system. This provides a large data result for certain types of "scaled up" data.

Chapter 3 and Chapter 4 are devoted to the construction of stationary solutions of both the barotropic and full, compressible Euler system in several space dimensions with spherical or cylindrical symmetry, for a general equation of state. For given Dirichlet data on a sphere or a cylinder we first construct smooth and radially symmetric solutions to the Euler equations in the exterior domain. On the other hand, stationary smooth solutions in the interior domain necessarily become sonic and cannot be continued beyond a critical inner radius. We then use these solutions to construct entropy-satisfying shocks for the Euler equations in the region between two concentric spheres (or cylinders). In addition, for given data on the two concentric spheres (or cylinders) we investigate when a shock solution exists, and how the shock location is related to the boundary data. These shock solutions provide the first step in the construction of smooth solutions to the Navier-Stokes system which are shown to converge to the shock solutions of the Euler system in the small viscosity limit. Only the analysis of the shock solutions for the Euler system will be discussed here.

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e	Internal energy, p. 10
E	Specific total energy, p. 10
γ	Adiabatic exponent, p. 12
$\overset{>}{J}$	Down contact, p. 16
$\overset{<}{J}$	Up contact, p. 16
κ	Function of the adiabatic exponent, p. 17
ν	Function of the adiabatic exponent, p. 17
p	Pressure, p. 10
$\overset{\leftharpoonup}{R}$	Backward rarefaction, p. 16
$\overset{\rightharpoonup}{R}$	Forward rarefaction, p. 16
R_i	i-rarefaction curve, p. 6
ρ	Density, p. 10
$\overset{\leftharpoonup}{S}$	Backward shock, p. 15
$\overset{\rightharpoonup}{S}$	Forward shock, p. 15
S_i	i-shock curve, p. 8
σ_i	Shock speed of i-shock, p. 8

σ_- Shock speed of backward shock, p. 13

σ_+ Speed of forward shock, p. 13

τ Specific Volume, p. 10

u Velocity, p. 10

$\overleftharpoon{W}$ Backward wave curve, p. 17

$\overrightharpoon{W}$ Forward wave curve, p. 17

W_i i-wave curve, p. 9

ζ Function of the adiabatic exponent, p. 17

Acknowledgments

I would like to thank my advisor, Helge K. Jenssen , for his suggestions and guidance that he provided throughout my studies. I have learned a great deal from his insights and knowledge for which I am forever indebted. It has been a privilege to be his student. I would also like to thank my committee members Alberto Bressan, Alexi Novikov, Eric Mockensturm and Yuxi Zheng for their time. Lastly, I would like to thank Becky Halpenny for all her administrative help.

Chapter 1

Full 1-D Euler Equations

1.1 Introduction

1.1.1 Background

Hyperbolic partial differential equations are born out of the study of natural phenomena such as wave motion, transport phenomena, and fluid flow. In the study of gas dynamics there are various models with particular properties for various flow regimes. We will focus on both boundary value problems and initial value problems for the full, compressible Euler system. This is a system of conservation laws which models the conservation of mass, momentum and energy in gas flow. Historically and practically the Euler system is the most important system of conservation laws. Unlike elliptic and parabolic systems whose solutions remain smooth, hyperbolic equations tend to develop discontinuities. Due to this, the analysis of hyperbolic equations is typically involved and ad hoc.

For scalar conservation laws, in one or several space dimensions, a well developed theory for existence and uniqueness of solutions exists [35]. For systems the theory is much less compete. For 1-D systems it has been shown that if the data are close to equilibrium, as measured in total variation, then there is a global in time unique weak solution [6]. A natural question is whether there exists weak solutions for large data that are global in time, an issue of obvious importance in real world applications. It has been demonstrated by explicit counter-examples not to hold for hyperbolic systems in general, even in the strictly hyperbolic regime. More precisely, Joly-Metivier-Rauch [21], Young [38] and Jenssen [20] have constructed examples (not motivated by physics) in which blow-up in sup-norm and/or total variation can occur in finite time. The hope is that the structure of the 1-D Euler system, and other physical systems, will guarantee the existence of global in time weak solutions, even for large data. For systems with more than one space dimension much is known but there is currently no general theory for the existence of global in time solutions.

We will be interested in "large data" (i.e. not necessarily of small variation or sup-norm) for the 1-D Euler system. By studying specific systems one can hope to exploit the structure of the system and gain insight for solutions with large data. The works by Nishida [26] and Nishida & Smoller [27] on the isothermal and isentropic equations, respectively, provide global existence for large, but bounded, variation data: any bounded variation data in the case of isothermal flow, and data with $(\gamma - 1) \times$ (total variation of data) sufficiently small, in the case of isentropic flow (γ being the adiabatic exponent). These results were extended to the full Euler system by Liu [24] and Temple [36]. More recently Temple & Young [37] have established existence for the full Euler system up to an arbitrary time for data with large total variation and sufficiently small sup-norm.

One interaction pattern of interest is motivated by the aforementioned concrete examples by Young [38] and Jenssen [20] for non-physical systems. The pattern involves three regions where the middle region has repeated reflections of shocks. This seems to be the simplest pattern involving repeated reflections and if the initial data for these systems are chosen judiciously, then the solutions blow up in sup-norm and/or total variation in finite time. All known examples of this type involve wave patterns with infinitely many waves created in finite time.

We show that a similar pattern can occur for the Euler system, with carefully chosen data, but that there are only finitely many waves created in finite time, [15]. Furthermore, the existence a global in time solution in this case which does not blow up in sup-norm or total variation will be proven (see Proposition 2.2.2). This indicates that the Euler system is better behaved than general systems. This brings one to ask if the methods used can be generalized to other patterns to obtain global in time solutions.

A long term goal is to consider more general large data for the Euler system.

Motivated by the Glimm's use of a specific "Lyapunov" functional, it is natural to search for further decay properties. Towards this end we conduct a detailed analysis of wave interactions involving two waves. We consider interactions involving shocks, centered rarefactions and contact discontinuities. These interaction have been studied by others, [7], [11], [35] and [32], to varying degrees of completeness. Of these, Chang and Hsiao, [7], have the most complete analysis. However we would like more information than is provided in theses studies. In particular, we are interested in the following:

(i) to accurately resolve all possible interactions involving two waves,

(ii) to estimate strengths of the outgoing waves compared to that of the incoming waves,

(iii) to identify when vacuum formation occurs,

(iv) to identify possible decay properties.

The analysis involves studying highly nonlinear algebraic expressions and proves to be quite involved. Another difficulty comes from the wave curves being defined piecewise by the rarefaction and shock curves which have different algebraic expressions (see (1.49)-(1.51)). At several points we are therefore compelled to resort to a case-by-case analysis.

The analysis of resolving interactions involves the study of ten interactions in three groups (see section 1.3.1). Of these, the group involving one wave overtaking another wave is the most challenging. Specifically, in the cases where a shock and a rarefaction overtake each other a complete breakdown of all the possible cases does not seem to known. More precisely, there are multiple possibilities for the outgoing waves, as recorded in [7] (see table 1.5), and we have performed simulations to confirm these. On the other hand, the question of precisely characterizing the outgoing waves is still open. Apart from this we provided detailed answers to (i), (ii) and (iii) above. We also record various decay properties for specific interaction. However, it remains as an outstanding challenge to synthesize these into a Lyaponuv functional for the Euler system. This would require further analysis and also the consideration of other quantities (eg. entropy) which we have not considered here.

We will also be studying boundary value problems for the 2-D and 3-D compressible Euler system, with a general equation of state. We consider two concentric cylinders/spheres as the boundaries and constructed a concentric stationary shock located between the two boundaries. In Chapter 3 we will consider barotropic flow (i.e. pressure is a function of density alone) and in Chapter 4 non-barotropic flow. After constructing stationary shocks we find necessary and sufficient conditions for the existence of a stationary shock given the boundary conditions. The main results of Chapter 3 are found in Propositions 3.2.1, 3.2.2, 3.2.3 and 3.2.4, and Theorems 3.3.1, 3.3.2 and 3.4.1. Similarly, the main results of Chapter 4 can be found in Propositions 4.2.1 and 4.2.2, and Theorems 4.3.1, 4.3.2 and 4.4.1. This is part of a collaborative work with Jenssen and Williams in which the profile of viscous stationary solutions to the compressible Navier-Stokes equations are shown to converge to the profile of inviscid stationary shocks. The analysis applies to both the barotropic, [16], and non-barotropic flow, [17]. This is a natural extension of Gilbarg's [18] classical 1-D construction of viscous planar shock fronts and their convergence to step-shocks.

The construction and analysis of these 2/3-D stationary symmetric shocks is a first step in analyzing more general (dynamic) solutions that are radially symmetric. For example, it is an open problem to determine the exact solution in the case of a focussing shock for the 2/3-D Euler system. Thus, it is natural to first perform a detailed study of the stationary shocks.

1.1.2 Preliminaries

We briefly review some basic theory for by 1-D conservation laws and definitions which will be used throughout.

Hyperbolicity, weak solutions Consider a 1-D non-linear system of conservation laws,

$$U_t + F(U)_x = 0, \tag{1.1}$$

where x is the spatial variable, t is time, $(x,t) \in \mathbf{R} \times [0,\infty)$, $U(x,t) \in \mathbf{R}^n$ and $F : \Omega \subset \mathbf{R}^n \to \mathbf{R}^n$ is C^2. Here U is the vector of conserved quantities and F is the flux. The system is said to be hyperbolic if the Jacobian $DF(U)$ has a full set of eigenvectors with corresponding real eigenvalues for each $U \in \Omega$. Let the right eigenvectors be given by

$$r_1(U), r_2(U), \cdots, r_n(U). \tag{1.2}$$

Then the corresponding real eigenvalues for each $U \in \Omega$ may be ordered,

$$\lambda_1(U) \leq \lambda_2(U) \leq \cdots \leq \lambda_n(U). \tag{1.3}$$

If the eigenvalues are distinct at a state U then the system is said to be strictly hyperbolic at U. If $\nabla\lambda_i(U) \cdot r_i(U) = 0$ for all $U \in \Omega$ then we say the i^{th} family is linearly degenerate. Likewise, we say the i^{th} family is genuinely nonlinear if $\nabla\lambda_i(U) \cdot r_i(U) \neq 0$ for all $U \in \Omega$. The last two definitions we first given by Lax, [22].

Given initial data $U_0(x) = U(x, 0)$, solutions may develop discontinuities in finite time and thus making it necessary to consider *weak* solutions. A measurable function $U(x, t)$ is a weak solution of (1.1) if

$$\int_{\mathbf{R}^+} \int_{\mathbf{R}} \Big(\phi_t U + \phi_x F(U) \Big) dx dt + \int_{\mathbf{R}} \phi(x, 0) U(x, 0) dx = 0 \tag{1.4}$$

for all C^∞-smooth functions $\phi(x, t)$ with compact support in $\mathbf{R} \times [0, \infty)$.

Since arbitrarily smooth data can become discontinuous in finite time, we will consider discontinuous initial data. The simplest initial data with a discontinuity is one where the data is constant to the left and right of a location x_0. Riemann was the first to study problems of this type and thus equation (1.1) with initial data is called a Riemann problem, [30]. Consider (1.1), and assume the system is strictly hyperbolic, with Riemann initial data

$$U_0(x) = \begin{cases} U_l & \text{if } x < 0, \\ U_r & \text{if } x > 0. \end{cases} \tag{1.5}$$

As a first step we will search for continuous solutions. Notice that both the initial data and the differential equation in (1.1) are self-similar in the sense that they are invariant under the map $x \to kx$ and $x \to kt$, for any fixed $k > 0$. Thus we search for solutions of the form

$$U(x,t) = W\left(\frac{x}{t}\right) = W(\xi),$$

where $\xi = \frac{x}{t}$. Substituting into the equation in (1.1) one finds

$$DF(W)\frac{dW}{d\xi} = \xi\frac{dW}{d\xi}\,.$$

Hence, $\frac{dW}{d\xi}$ is an eigenvector of $DF(W)$ with eigenvalue ξ. Suppose the i^{th}-family is genuinely nonlinear and

$$\lambda_i\left(W(\xi)\right) = \xi\,, \quad \frac{dW}{d\xi}(\xi) = r_i\left(W(\xi)\right)\,.$$

We choose the direction of the right eigenvector such that

$$\nabla\lambda_i(U)\cdot r_i(U) > 0\,.$$

and the magnitude by $|r_i(U)| \equiv 1$. Thus λ_i increases along the i^{th} eigenfield in the direction chosen. If we further assume that $W\left(\lambda_i(U_l)\right) = U_l$ and $W\left(\lambda_i(U_r)\right) = U_r$, with $\lambda_i(U_l) < \lambda_i(U_r)$, then for fixed t the function $W\left(\frac{x}{t}\right)$ will continuously connect U_l to U_r as x increases. Also, one finds that $\lambda_i\left(W\left(\frac{x}{t}\right)\right)$ increases as x increases. This yields the solution

$$U(x,t) = \begin{cases} U_l & \text{if } x < t\lambda_i(U_l), \\ W\left(\frac{x}{t}\right) & \text{if } t\lambda_i(U_l) < x < t\lambda_i(U_r), \\ U_r & \text{if } x > t\lambda_i(U_r). \end{cases} \tag{1.6}$$

Such a solution is called an *i-rarefaction*. Fixing the left state U_l, there are right states which can be connected to the left state by moving along the integral curve of the eigenvector field $r_i(U)$. Notice, since $\nabla\lambda_i(U)\cdot r_i(U) > 0$ we have that the eigenvalue, λ_i, increases along the integral curve. This integral curve is the i-rarefaction curve. With $|r_i(U)| \equiv 1$ we have that the i-rarefaction curve moves with unit speed and can be parametrized by $\epsilon_i > 0$, for ϵ_i small. We denote it by $R_i(\epsilon_i)(U_l)$, where $R_i(0)(U_l) = U_l$. Notice, if the i^{th} family is linearly degenerate then by moving along the the integral curve corresponding to the eigenvector r_i, λ_i is constant along the curve. That is, λ_i is constant along $R_i(\epsilon_i)(U_l)$ for $\epsilon_i > 0$ and $\epsilon_i < 0$. Such solutions will be discussed below.

Another type of solution is piecewise constant and involves a discontinuity separating the left and right states. Fixing the left state, U_l, consider right states, U_r satisfying the Rankine-Hugoniot condition

$$\sigma(U_r - U_l) = F(U_r) - F(U_l),$$

for some real σ. From the weak form of the equation, (1.4), it follows that if U_l, U_r and σ satisfy the Rankine-Hugoniot condition then

$$U(x,t) = \begin{cases} U_r & \text{if } x > \sigma t, \\ U_l & \text{if } x < \sigma t. \end{cases} \tag{1.7}$$

is a weak solution to (1.1) with initial data (1.5), where σ is the shock speed.

Further, for a given left state the set of right states satisfying

$$\mathcal{H}(U_l) := \{U_r \mid \exists \sigma \in \mathbf{R} \text{ such that } \sigma(U_r - U_l) = F(U_r) - F(U_l)\}$$

is known as the Hugoniot locus. For a strictly hyperbolic system the set of right states in the Hugoniot locus form n curves passing through U_l, which are C^2 (see [6]).

Admissibility criteria, shocks Having extend to weak solutions, uniqueness has been lost. It is easy to give examples demonstrating the non-uniqueness of weak solutions. In particular, for a progressing discontinuity the Rankine-Hugoniot condition is not enough to ensure that (1.7) is a unique solution. Thus we need an admissibility criterion to regain uniqueness. There are several selection criteria that one could employ such as vanishing viscosity, and for a single discontinuity there is the Lax condition. The latter condition is particularly useful and easy to apply. Before stating the Lax condition notice, if $(sU_r + (1-s)U_l)$ is in the domain of F for $0 \leq s \leq 1$, then the Rankine-Hugoniot condition yields

$$\begin{aligned}\sigma(U_r - U_l) = F(U_r) - F(U_l) &= \int_0^1 \frac{d}{ds} F(sU_r + (1-s)U_l) ds \\ &= \int_0^1 DF(sU_r + (1-s)U_l) ds \cdot (U_r - U_l). \end{aligned} \tag{1.8}$$

Therefore σ is an eigenvalue value of the average matrix,

$$\int_0^1 DF(sU_r + (1-s)U_l) ds. \tag{1.9}$$

If the system is strictly hyperbolic the eigenvalues of the average matrix can be ordered,

$$\sigma_1 < \sigma_2 < \cdots < \sigma_n \, .$$

For a strictly hyperbolic system an i-shock is called admissible if its shock speed σ_i satisfies

$$\lambda_i(U_l) > \sigma_i > \lambda_i(U_r) \, . \tag{1.10}$$

With this we also have that

$$U(x,t) = \begin{cases} U_r & \text{if } x > \sigma_i t, \\ U_l & \text{if } x < \sigma_i t. \end{cases} \tag{1.11}$$

is an admissible i-shock solution to (1.1) if (1.10) is satisfied. Fixing the left state U_l, there are right states which can be connected to the left state by an i-shock. This yields a curve called the i-shock curve and can be parametrized by unit speed with the variable ϵ_i, for $\epsilon_i < 0$ and where ϵ_i is small in magnitude. We denote the i-shock curve by $S_i(\epsilon_i)(U_l)$, where $S_i(0)(U_l) = U_l$. If the i^{th} family is linearly degenerate then, with some abuse of notation, the curve S_i for $\epsilon_i \gtrless 0$ and the curve R_i for $\epsilon_i \lesseqgtr 0$ coincide. Such curves are called contact discontinuity curves, and the corresponding solution, obtained by connecting the left state to a right state along the wave curve, is called a contact discontinuity solution. With this we now define the i-wave curve to be given by

$$W_i(\epsilon_i)(U_l) := \begin{cases} R_i(\epsilon_i)(U_l) & \text{if } \epsilon_i > 0, \\ S_i(\epsilon_i)(U_l) & \text{if } \epsilon_i < 0. \end{cases} \tag{1.12}$$

It is a general fact that W_i is a C^2 curve, see [35].

Riemann problem We now define solutions to Riemann problems with the use of wave curves. We assume that (1.1) is strictly hyperbolic.Then given a left state U_l and a right state U_r there exists a weak solution consisting of at most $(n+1)$ constant states connected by at most n waves if there exists values of ϵ_1 through ϵ_n such that

$$U_r = W_n(\epsilon_n)\left(W_{n-1}(\epsilon_{n-1})(\cdots(W_1(\epsilon_1)(U_l))\cdots)\right) . \tag{1.13}$$

That is, the left state can be connected to the right state by(at most $(n-1)$) intermediate states along wave curves. Starting from the left state U_l we may find the intermediate state $W_1(\epsilon_1)(U_l)$ and continuing we find the next intermediate state $W_2(\epsilon_2)\left((W_1(\epsilon_1)(U_l)\right)$ and so forth until we reach U_r. The right hand side of (1.13) is function of $(\epsilon_1, \cdots, \epsilon_n)$ and if we define

$$\Lambda(\epsilon_1, \cdots, \epsilon_n) = W_n(\epsilon_n)\left(W_{n-1}(\epsilon_{n-1})(\cdots(W_1(\epsilon_1)(U_l))\cdots)\right) ,$$

then assuming the system to be strictly hyperbolic one finds $D\Lambda(0, \cdots, 0)$ to have full rank and thus by the inverse function theorem to be locally invertible. This gives Lax's Theorem. That is, suppose a system is strictly hyperbolic, where each family is linearly degenerate or genuinely nonlinear, on a given open domain Ω. Then for each left state $U_l \in \Omega$ there exists a $\delta > 0$ such that for all U_r with $|U_r - U_l| < \delta$, there exists an admissible solution of the above form, with initial data

$$U_0(x) = \begin{cases} U_l & \text{if } x < 0, \\ U_r & \text{if } x > 0. \end{cases}$$

This is a local statement, but for some systems all Riemann problems are solvable. This is the case with the Euler system (allowing for vacuum formation in the solution).

1.2 The Euler Equations

Consider the one-dimesional system of Euler equations in Lagrangian coordinates:

$$\begin{aligned} \tau_t - u_x &= 0 \\ u_t + p_x &= 0 \\ E_t + (pu)_x &= 0, \end{aligned} \tag{1.14}$$

where u is velocity, p is pressure, $\tau = \frac{1}{\rho}$ is specific volume, ρ is density, $E = e+u^2/2$ is specific total energy and $e(\tau, p)$ is internal energy, where we assume $e_p > 0$ and $e_\tau + p > 0$. The relation $p = p(e, \tau)$ is the *equation of state* and depends on the particular gas being considered. Specifically, (1.14) is a family of systems which depends on the equation of state. We make the standard assumption that given any two thermodynamic variables the remaining may be found and that

$$e_p > 0\,, \qquad e_\tau + p > 0\,.$$

What follows are general properties while later we will specialize to ideal, polytropic gasses.

When considering smooth solutions one may make a change of dependent variables. In particular, we may use the conservation of mass and momentum and that the internal energy is a function of specific volume and pressure to replace the system with

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ e_\tau & e_p & u \end{pmatrix} \begin{pmatrix} \tau \\ p \\ u \end{pmatrix}_t + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & u & p \end{pmatrix} \begin{pmatrix} \tau \\ p \\ u \end{pmatrix}_x = 0.$$

Since we are making the standard assumption that $e_p > 0$ and $e_\tau + p > 0$ we may rewrite the above as,

$$\begin{pmatrix} \tau \\ p \\ u \end{pmatrix}_t + \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & \frac{e_\tau + p}{e_p} \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \tau \\ p \\ u \end{pmatrix}_x = 0. \tag{1.15}$$

In this coordinate system the characteristic equation is, $\lambda(\lambda^2 - \frac{e_\tau + p}{e_p}) = 0$ giving the real and distinct eigenvalues

$$\lambda_- := -\sqrt{\frac{e_\tau + p}{e_p}}\,, \tag{1.16}$$

$$\lambda_0 := 0\,, \tag{1.17}$$

$$\lambda_+ := +\sqrt{\frac{e_\tau + p}{e_p}}\,. \tag{1.18}$$

Therefore the system (1.14) is strictly hyperbolic. We may choose the corresponding right eigenvectors to be

$$r_- := (1, -\lambda_-^2, -\lambda_-)^T \tag{1.19}$$

$$r_0 := (1, 0, 0)^T \tag{1.20}$$

$$r_+ := (1, -\lambda_+^2, -\lambda_+)^T, \tag{1.21}$$

respectively.

1.2.1 Ideal polytropic gasses

Restricting ourselves to ideal polytropic gasses we have that the equation of state is given by

$$p = \frac{RT}{\tau} \qquad \text{and} \qquad e = c_\nu T,$$

where T is temperature and both R and c_ν are positive constants. Combining these two equations gives us,

$$e = \frac{\tau p}{\gamma - 1}, \tag{1.22}$$

where $\gamma := c_\nu R^{-1}$ is the adiabatic exponent and is greater than 1. Using relation (1.22), (1.16)-(1.18) can be written as,

$$\lambda_- = -\sqrt{\frac{\gamma p}{\tau}}\,, \tag{1.23}$$

$$\lambda_0 = 0\,, \tag{1.24}$$

$$\lambda_+ = \sqrt{\frac{\gamma p}{\tau}}\,. \tag{1.25}$$

Similarly (1.19)-(1.21) can be written as,

$$r_- = (1, -\frac{\gamma p}{\tau}, \sqrt{\frac{\gamma p}{\tau}})^T\,, \tag{1.26}$$

$$r_0 = (1, 0, 0)^T\,, \tag{1.27}$$

$$r_+ = (1, -\frac{\gamma p}{\tau}, -\sqrt{\frac{\gamma p}{\tau}})^T. \tag{1.28}$$

From this we obtain,

$$\nabla\lambda_- \cdot r_- = \frac{(\gamma+1)}{2}\sqrt{\frac{\gamma p}{\tau^3}} > 0\,,$$

$$\nabla\lambda_+ \cdot r_+ = -\frac{(\gamma+1)}{2}\sqrt{\frac{\gamma p}{\tau^3}} < 0\,,$$

showing genuine nonlinearity in the "+" and "$-$" families. The "0" family is linearly degenerate.

1.2.2 Jump Discontinuities

The Rankine-Hugoniot conditions take the form

$$\sigma[u] = [p]\,, \tag{1.29}$$

$$\sigma[\tau] = -[u]\,, \tag{1.30}$$

$$\sigma[E] = [pu]\,, \tag{1.31}$$

where $[\cdot]$ = right state $-$ left-state.

Notice, (1.29)-(1.31) are satisfied if $\sigma = [u] = [p] = 0$. Since $E = \frac{\tau p}{\gamma - 1} + \frac{u^2}{2}$ there is a jump if and only if $[\tau] \neq 0$. This gives a contact discontinuity in the "0" family. We will denote a contact where specific volume increases across the jump, (from left to right) by $\overset{<}{J}$ and a contact where specific volume decreases across the jump across the contact and by $\overset{>}{J}$. That is,

$$u = u_0, \qquad p = p_0\,, \qquad \tau_0 \gtrless \tau \qquad \begin{cases} > : \text{down-contact } \overset{>}{J}\,, \\ < : \text{up-contact } \overset{<}{J}\,. \end{cases} \tag{1.32}$$

Considering the case where $\sigma \neq 0$, we have $[\tau] \neq 0, [p] \neq 0$ and $[u] \neq 0$. Combine (1.29)-(1.31) gives,

$$\sigma^2 = -\frac{[p]}{[\tau]}. \tag{1.33}$$

Taking the square-root of both sides define,

$$\sigma_+ \;:=\; \sqrt{-\frac{[p]}{[\tau]}} \qquad \text{(forward shock speed)}, \tag{1.34}$$

$$\sigma_- \;:=\; -\sqrt{-\frac{[p]}{[\tau]}} \qquad \text{(backward shock speed)}. \tag{1.35}$$

1.2.3 Shock Curves

We begin by finding the equation of the shock curves in the (p, τ)-plane. From (1.31) we obtain, by using $E = e + \frac{u^2}{2}$, that

$$\sigma[e] + \frac{\sigma}{2}[u^2] = [pu]. \tag{1.36}$$

Let the left state be denoted by (p_0, u_0, e_0) and the right state by (p, u, e). With this (1.31) becomes,

$$\sigma(e - e_0) + \frac{\sigma}{2}(u - u_0)(u + u_0) = pu - p_0 u_0.$$

From (1.29) we have $\sigma(u-u_0) = (p-p_0)$. Substituting one finds,

$$2\sigma[e] = pu - p_0u_0 + p_0u - pu_0.$$

Relation (1.30) yields $u = u_0 - \sigma[\tau]$. Substituting we obtain,

$$2\sigma[e] = -\sigma(p+p_0)[\tau].$$

Since $\sigma \neq 0$ one finds,

$$[e] + \frac{1}{2}(p+p_0)[\tau] = 0. \qquad \text{(Hugoniot relation)}$$

Using relation (1.22) yields

$$\tau(p+\mu^2 p_0) = \tau_0(p_0+\mu^2 p), \tag{1.37}$$

where

$$\mu^2 := \frac{\gamma-1}{\gamma+1} \in (0,1). \tag{1.38}$$

Equation (1.37) gives the shock curves in the (p,τ)-plane corresponding to both σ_+ and σ_-.

To find the shock curves in the (p,u)-plane we use equations (1.30) and (1.31) to obtain,

$$[u] = \sigma_\pm[p] = \pm[p]\sqrt{-\frac{[\tau]}{[p]}}.$$

If we use relations (1.34)-(1.35) and write the jumps with left state (τ_0, p_0, u_0) and right state (τ, p, u) we find,

$$u = u_0 \pm (p-p_0)\sqrt{\frac{\tau_0-\tau}{p-p_0}}.$$

From (1.37) we have that along shock curves $\tau = \frac{\tau_0(p_0+\mu^2 p)}{(p+\mu^2 p_0)}$. Substituting yields,

$$u = u_0 \pm (p-p_0)\sqrt{\frac{(1-\mu^2)\tau_0}{p+\mu^2 p_0}}\,, \tag{1.39}$$

where the + and − correspond to forward and backward shocks, respectively. Forward and backward shocks will be denoted by $\overrightarrow{S}$ and $\overleftarrow{S}$, respectively. Combining (1.37) and (1.39) we define,

$$\tau = \frac{\tau_0(p_0 + \mu^2 p)}{p + \mu^2 p_0} \tag{1.40}$$

$$u = u_0 \pm (p - p_0)\sqrt{\frac{(1-\mu^2)\tau_0}{p + \mu^2 p_0}} \qquad \begin{cases} + : \overrightarrow{S}, \\ - : \overleftarrow{S}\,. \end{cases} \tag{1.41}$$

It is well known that for shocks to be admissible they must be compressive. See [35]. That is, a particle overtaken by the shock experiences an increase in pressure. Using the fact that shocks are compressive along with (1.40) and (1.41) gives the admissible shock curves

$$\tau = \frac{\tau_0(p_0 + \mu^2 p)}{p + \mu^2 p_0} \tag{1.42}$$

$$u = u_0 \pm (p - p_0)\sqrt{\frac{(1-\mu^2)\tau_0}{p + \mu^2 p_0}} \qquad \begin{cases} + : \overrightarrow{S} \quad p_0 > p\,, \\ - : \overleftarrow{S} \quad p_0 < p\,. \end{cases} \tag{1.43}$$

1.2.4 Rarefaction curves

To find the rarefaction curves centered at (τ_0, p_0, u_0) we must solve for the integral curves corresponding to the eigenvectors r_- and r_+, from (1.26) and (1.28) respectively. More specifically if we parametrize the integral curves by τ then we need to solve

$$\frac{dp}{d\tau} = \frac{\gamma p}{\tau}\,,$$

$$\frac{du}{d\tau} = \pm\sqrt{\frac{\gamma p}{\tau}}\,,$$

where + and − correspond to r_- and r_+, respectively, with given initial data $p(\tau_0) = p_0$ and $u(\tau_0) = u_0$. Solving the differential equation for p one finds that $p(\tau)$ is a one to one function and thus may be inverted. With this we thus parametrize by by p yielding.

$$\tau = \frac{\tau_0^\gamma p_0}{\tau^\gamma} \tag{1.44}$$

$$u = u_0 \pm \frac{\sqrt{1-\mu^4}}{\mu^2} \tau_0^{\frac{1}{2}} p_0^{\frac{1}{2\gamma}} (p^{\frac{\gamma-1}{2\gamma}} - p_0^{\frac{\gamma-1}{2\gamma}}) \quad \begin{cases} + : \overrightarrow{R} & p_0 < p\,, \\ - : \overleftarrow{R} & p_0 > p\,. \end{cases} \tag{1.45}$$

where $\overrightarrow{R}$ is the forward rarefaction and $\overleftarrow{R}$ is the backward rarefaction.

1.2.5 Contact Discontinuities

Since the "0" family is linear degenerate the rarefaction curve and shock curve correspond. The contact discontinuity centered at at (τ_0, p_0, u_0) is given by

$$\tau = \tau\,, \tag{1.46}$$

$$p = p_0\,, \tag{1.47}$$

$$u = u_0\,. \tag{1.48}$$

1.2.6 Wave curves in (τ, u, p)-space

Given the admissible shock and rarefaction curves for the "$-$" and "$+$" families, and since the "0" family is linearly degenerate, we can now define the wave curves. For the purpose of analysis we will reparametrize the contact discontinuities, shock curves and rarefaction curves using pressure ratios as the parameter for backward and forward waves, while we use specific volume ratios as parameter for the contacts. Again, we will write $\overrightarrow{S}$, $\overrightarrow{R}$, and $\overleftarrow{S}$, $\overleftarrow{R}$ for forward and backward shocks and rarefactions, respectively, and contact discontinuities where τ increases and τ decreases by $\overset{<}{J}$ (up contact) and $\overset{>}{J}$ (down contact) respectively.

For forward waves we denote the pressure ratio by 'f', for backward waves we denote the pressure ratio by 'b' and for contacts we denote the specific volume ratio by 'c'.

Using (1.42) -(1.43), (1.44) -(1.45) and (1.46)-(1.48) a calculation shows that the new parametrization for the wave curves centered at (τ_0, u_0, p_0) (cf. [35], [7]):

$$\overleftharpoon{W}(b;\tau_0,u_0,p_0) = \begin{pmatrix} \overleftharpoon{\phi}(b)\tau_0 \\ u_0 - \overleftharpoon{\psi}(b)\sqrt{\tau_0 p_0} \\ b p_0 \end{pmatrix}, \qquad \begin{cases} b < 1 \ \leftrightarrow \ \overleftharpoon{R} \\ b > 1 \ \leftrightarrow \ \overleftharpoon{S}, \end{cases} \tag{1.49}$$

$$\overset{\lesseqgtr}{J}(c;\tau_0,u_0,p_0) = \begin{pmatrix} c\tau_0 \\ u_0 \\ p_0 \end{pmatrix}, \qquad 1 \lesseqgtr c, \tag{1.50}$$

$$\overrightharpoon{W}(f;\tau_0,u_0,p_0) = \begin{pmatrix} \overrightharpoon{\phi}(f)\tau_0 \\ u_0 + \overrightharpoon{\psi}(f)\sqrt{\tau_0 p_0} \\ f p_0 \end{pmatrix}, \qquad \begin{cases} f < 1 \ \leftrightarrow \ \overrightharpoon{S} \\ f > 1 \ \leftrightarrow \ \overrightharpoon{R}, \end{cases} \tag{1.51}$$

Here

$$\overleftharpoon{\phi}(b) = \begin{cases} b^{-1/\gamma} & b < 1, \\ \frac{1+\mu^2 b}{b+\mu^2} & b > 1, \end{cases} \qquad \overleftharpoon{\psi}(b) = \begin{cases} \nu\big(b^{\zeta} - 1\big) & b < 1, \\ \frac{\kappa(b-1)}{\sqrt{b+\mu^2}} & b > 1. \end{cases}$$

$$\overrightharpoon{\phi}(f) = \begin{cases} f^{-1/\gamma} & f > 1, \\ \frac{1+\mu^2 f}{f+\mu^2} & f < 1, \end{cases} \qquad \overrightharpoon{\psi}(f) = \begin{cases} \nu\big(f^{\zeta} - 1\big) & f > 1, \\ \frac{\kappa(f-1)}{\sqrt{f+\mu^2}} & f < 1. \end{cases}$$

The constants are given by

$$\zeta = \frac{\gamma-1}{2\gamma}, \qquad \kappa = \sqrt{1-\mu^2} > 0, \qquad \nu = \frac{\sqrt{1-\mu^4}}{\mu^2}. \tag{1.52}$$

Recall, $\mu^2 = \frac{\gamma-1}{\gamma+1}$ by (1.38).

Note that the functions $\overleftharpoon{\phi}(q)$ and $\overrightharpoon{\phi}(q)$ have the same shape (decreasing and convex up, Figure 1.1), as do the functions $\overleftharpoon{\psi}(q)$ and $\overrightharpoon{\psi}(q)$ (increasing and convex down, Figure 1.2). Each of these pairs of functions have second order contact at $q = 1$.

Notice the relations

$$\overleftarrow{\phi}\left(\frac{1}{q}\right) = \frac{1}{\overrightarrow{\phi}(q)}\,, \tag{1.53}$$

$$\sqrt{q\,\overleftarrow{\phi}(q)}\,\overrightarrow{\psi}\left(\frac{1}{q}\right) = -\overleftarrow{\psi}(q)\,, \tag{1.54}$$

and

$$\sqrt{q\,\overrightarrow{\phi}(q)}\,\overleftarrow{\psi}\left(\frac{1}{q}\right) = -\overrightarrow{\psi}(q)\,. \tag{1.55}$$

We also state two lemmas which will be used later.

Lemma 1.2.1. *Let* $\phi(x) = \frac{1+\mu^2 x}{x+\mu^2}$. *Given positive numbers* α_1, β_1, α_2, β_2 *where* $\alpha_1\beta_1 = \alpha_2\beta_2$ *and* $\alpha_1 + \beta_1 > \alpha_2 + \beta_2$, *then* $\phi(\alpha_1)\phi(\beta_1) > \phi(\alpha_1)\phi(\beta_1)$ *if and only if* $\alpha_1\beta_1 > 1$.

Proof. To easy the notation let $k = \alpha_1\beta_1 = \alpha_2\beta_2$, $x = \alpha_1 + \beta_1$ and $y = \alpha_2 + \beta_2$. With this, one finds that $\phi(\alpha_1)\phi(\beta_1) > \phi(\alpha_1)\phi(\beta_1)$ is equivalent to

$$\frac{1 + \mu^2 x + \mu^4 k}{k + \mu^2 x + \mu^4} > \frac{1 + \mu^2 y + \mu^4 k}{k + \mu^2 y + \mu^4}\,.$$

With some algebraic manipulation we have this to be equivalent to

$$k(x - y) > (x - y)\,.$$

By assumption $x = (\alpha_1 + \beta_1) > y = (\alpha_2 + \beta_2)$, and thus the inequality holds if and only if $k = \alpha_1\beta_1 > 1$. □

Lemma 1.2.2. *Let* $G(x) = x^{\frac{1}{\alpha}}\left(\frac{1+\mu^2 x}{x+\mu^2}\right)$ *for* $x > 0$. *Then* $G(x)$ *is an increasing function for all* $\alpha \geq 1$, *and* $G(1) = 1$.

1.3 Riemann Problems

Consider the Riemann problem with left state (τ_0, u_0, p_0) and right state (τ, u, p). For outgoing forward and backward waves we denote the pressure ratio by 'F' and

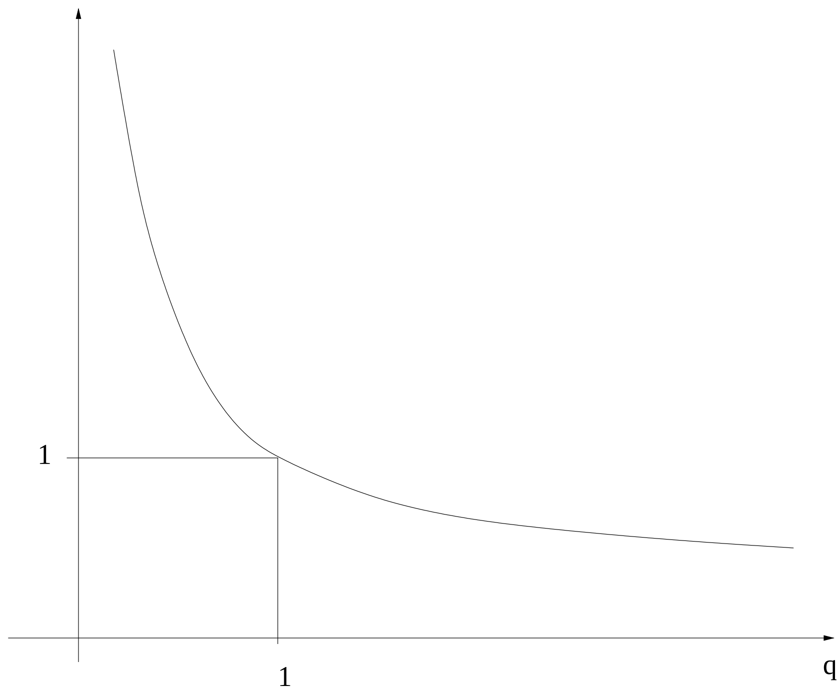

Figure 1.1. Graph of the backward and forward function ϕ (schematic)

'B' respectively, and for the outgoing contact we denote the specific volume ratio by 'C'. With these parameters we have the equations,

$$\begin{aligned} \tau &= C\overleftarrow{\phi}(B)\overrightarrow{\phi}(F)\tau_0 && (1.56)\\ \frac{u-u_0}{\sqrt{\tau_0 p_0}} &= \overrightarrow{\psi}(F)\sqrt{C\,B\,\overleftarrow{\phi}(B)} - \overleftarrow{\psi}(B) && (1.57)\\ p &= BFp_0\,. && (1.58) \end{aligned}$$

Using the first and last equations together with (1.54) gives the following equation for B,

$$\overleftarrow{\psi}(B) + \sqrt{\frac{\tau p}{\tau_0 p_0}}\overleftarrow{\psi}\left(\frac{Bp_0}{p}\right) = \frac{u_0 - u}{\sqrt{\tau_0 p_0}}\,. \tag{1.59}$$

We define the left hand side of (1.59) as the function $\mathcal{R}(B)$,

$$\mathcal{R}(B) := \overleftarrow{\psi}(B) + \sqrt{\frac{\tau p}{\tau_0 p_0}}\overleftarrow{\psi}\left(\frac{Bp_0}{p}\right)\,, \qquad B > 0\,.$$

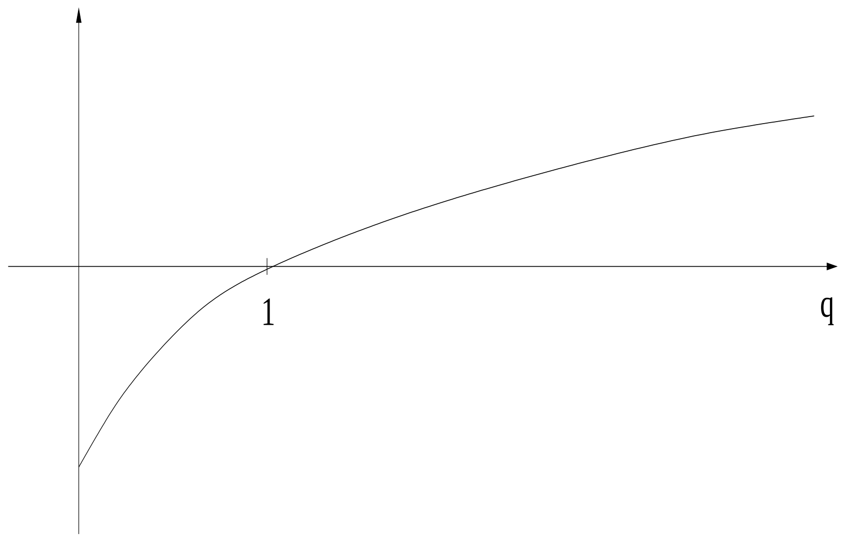

Figure 1.2. Graph of the forward and backward functions ψ (schematic)

The function $\overleftarrow{\psi}$ is increasing, tends to $+\infty$ at $+\infty$, and is convex down. It follows that $\mathcal{R}$ has the same properties.

It should be noted that all Riemann problems for the Euler system are solvable, if we allow for vacuum. The solution will not contain vacuum if and only if the solution B of the equation

$$\mathcal{R}(B) = \frac{\bar{u} - u}{\sqrt{\bar{\tau}\bar{p}}}$$

satisfies $B > 0$. In the case of vacuum there will be a rarefaction fan connecting the left state to the right state through a vacuum, $\rho = 0$ ($\tau = \infty$). If the left state is given by U_l and the right state is given by U_r, then left state will be connected to the vacuum by a backward rarefaction starting with speed $\lambda(U_l)$ and ending with speed zero, and the right state will be connected to the vacuum (from right to left) by a forward rarefaction starting with speed $\lambda(U_r)$ and ending with speed zero (see Figure 1.3).

We have made a specific choice of parametrization for the wave curves which seems to be desirable in the analysis of interactions, relative strengths and the possibility of vacuum formation. The investigation will be analytic as opposed to graphical. We further make the choice to solve for the backward outgoing wave strength first. The result of this analysis will the be used to study the outgoing

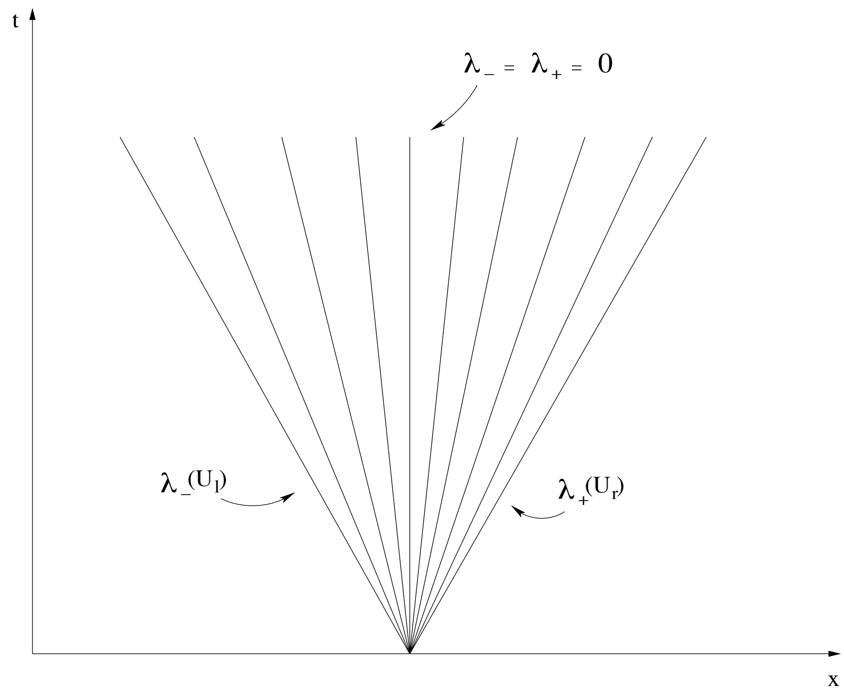

Figure 1.3. Left and right states connected via rarefactions with vacuum.

forward wave strength. With these two strengths determined we then analyze the outgoing contact. In finding the outgoing strengths and relations some analysis will be easy while others will be complicated due to the fact that rarefaction and shock portions of the wave curves take different forms. To see this consider (1.49) and (1.51), where the relative parameter is greater than or less than one.

1.3.1 Interactions

We want to consider all possible interactions of shocks, centered rarefactions and contacts for the Euler system. By symmetry there are ten essentially different interactions.

1a: $\overrightarrow{S}\overleftarrow{S}$

1b: $\overrightarrow{S}\overleftarrow{R}$

1c: $\overrightarrow{R}\overleftarrow{R}$

2a: $\overleftarrow{S}\overleftarrow{S}$

2b: $\overleftarrow{S}\overleftarrow{R}$

2c: $\overleftarrow{R}\overleftarrow{S}$

3a: $\overrightarrow{R}\overset{>}{J}$

3b: $\overrightarrow{R}\overset{<}{J}$

3c: $\overrightarrow{S}\overset{>}{J}$

3d: $\overrightarrow{S}\overset{<}{J}$

Group 1 deals with head on interactions, group 2 involves interactions where a wave overtakes a wave, and group 3 is concerned with interactions involving contacts. The conclusions drawn the analysis of interaction 3.c will be used Chapter 3.The grouping of these interactions into the three groups reflects the following fact: when solving the equations for the outgoing parameters the equations for these will be the same within each group, as long as the equations are written in terms of the functions ϕ and ψ.

Group 2 is the most challenging and interaction 2.a is the only interaction in this group treated in detail. Interactions 2.b and 2.c have been analyzed via computer simulation and show that any combination of outgoing backward and forward waves is possible. Also, group 2 does not contain the case of a rarefaction over taking a rarefaction, $\overleftarrow{R}\overleftarrow{R}$. This is due to the fact that these rarefactions will not interact. Indeed, if the left state U_l is connected to an intermediate state U_m via a rarefaction, which in turn is connected to the right state U_r by a rarefaction, then the rarefaction on the left has ending speed $\lambda_{-}(U_m)$ which is equal to the starting speed of the rarefaction on the right. See Figure 1.4.

Interactions involving rarefactions are resolved as in the Glimm scheme. That is, we resolve the Riemann problem where the left state is given by the data to the left of the left incoming wave, and the right state is given by the data to the right of the right incoming wave. For interactions involving only shocks and contacts the interactions are resolved exactly.

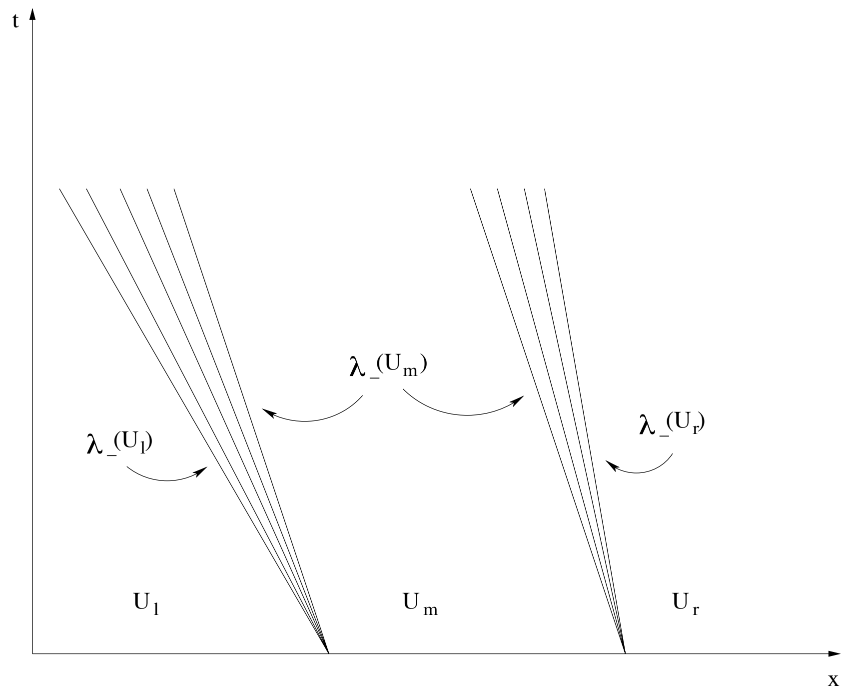

Figure 1.4. Neighboring rarefactions of the same family do not interact.

Notation: We use lower case letters for incoming parameter values: 'f' for forward waves, 'c' for contacts, and 'b' for backward waves. For group 2 where backward wave overtakes a backward wave we will use incoming parameter values b_1 and b_2. Similarly, we use upper case for outgoing parameter values: 'F' for forward waves, 'C' for contacts, and 'B' for backward waves.

We study each group individually by first defining the relation between incoming and outgoing wave strengths. We next analyze when an interaction may create a vacuum within the group. We then determine what the outgoing waves are in terms shocks, rarefactions and contacts. Finally, we determine some relations between the incoming and outgoing wave strengths.

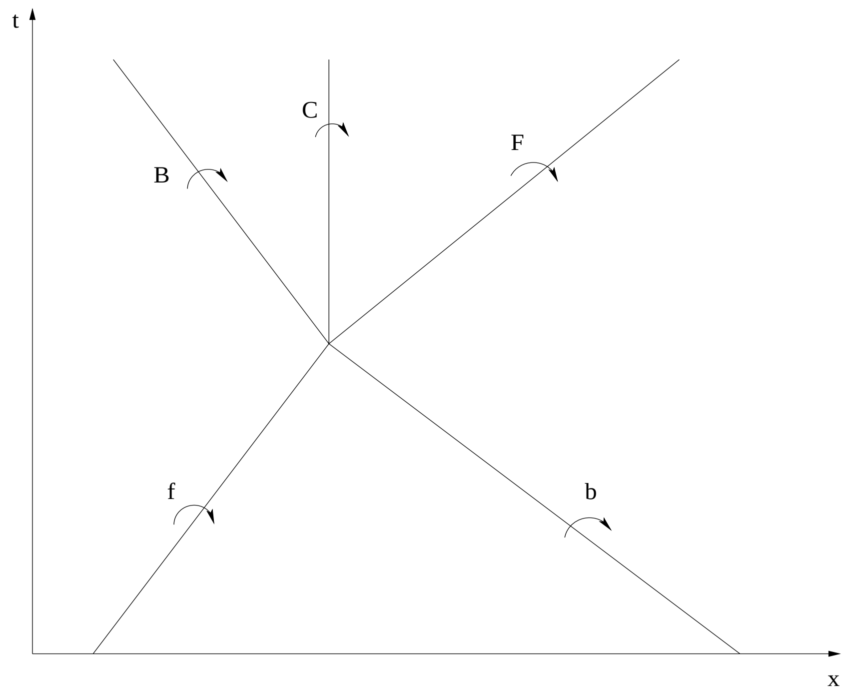

Figure 1.5. Incoming and outgoing parameters for head on interactions (group 1).

1.3.2 Interactions in group 1. Head on interactions.

Let the incoming parameter values be f and b, and let B, C, F denote the parameter values for the outgoing waves. See Figure 1.5. Computing the right state before the interaction in terms of f, b yields

$$\tau = \overleftarrow{\phi}(b)\overrightarrow{\phi}(f)\tau_0 \tag{1.60}$$

$$\frac{u-u_0}{\sqrt{\tau_0 p_0}} = \overrightarrow{\psi}(f) - \sqrt{f\overrightarrow{\phi}(f)}\,\overleftarrow{\psi}(b) \tag{1.61}$$

$$p = bfp_0\,. \tag{1.62}$$

Combining these relations with (1.56)-(1.58) gives

$$BF = bf\,, \tag{1.63}$$

$$\overleftarrow{\phi}(b)\overrightarrow{\phi}(f) = C\overleftarrow{\phi}(B)\overrightarrow{\phi}(F)\,, \tag{1.64}$$

$$\overleftarrow{\psi}(B) + \sqrt{bf\,\overleftarrow{\phi}(b)\overrightarrow{\phi}(f)}\;\overleftarrow{\psi}\left(\frac{B}{bf}\right) \;=\; \overleftarrow{\psi}(b)\sqrt{f\,\overrightarrow{\phi}(f)} - \overrightarrow{\psi}(f)\,. \qquad (1.65)$$

Notice, we have eliminated the unknowns F and C in (1.65) in order to find an equation for B in terms of f and b. These relations will be used to analyze the outgoing waves. Notice, in defining

$$\mathcal{F}_1(\alpha; b, f) \equiv \mathcal{F}_1(\alpha) \;=\; \overleftarrow{\psi}(\alpha) + \frac{\overleftarrow{\psi}(b)\overrightarrow{\psi}(f)}{\overrightarrow{\psi}\left(\frac{1}{b}\right)\overleftarrow{\psi}\left(\frac{1}{f}\right)}\;\overleftarrow{\psi}\left(\frac{\alpha}{bf}\right), \qquad (1.66)$$

we have that $\mathcal{F}_1$ is increasing, tending to $+\infty$ as α approaches $+\infty$, and is convex down. By using (1.54) and (1.55) we can rewrite equation (1.65) for B as

$$\mathcal{F}_1(B) \;=\; -\overrightarrow{\psi}(f)\left[\frac{\overleftarrow{\psi}(b)}{\overleftarrow{\psi}\left(\frac{1}{f}\right)} + 1\right]. \qquad (1.67)$$

Possible vacuum formation.

To analyze when a vacuum may form we use that $\mathcal{F}_1$ is an increasing function. Therefore one concludes that no vacuum will be produced by the interaction if and only if $\mathcal{F}_1(B) > \mathcal{F}_1(0)$. Evaluating (1.66) for $\alpha = 0$ and using relation (1.67) one finds this to be equivalent to

$$\overrightarrow{\psi}(f)\left[\frac{\overleftarrow{\psi}(b)}{\overleftarrow{\psi}\left(\frac{1}{f}\right)} + 1\right] < \nu\left(1 + \frac{\overleftarrow{\psi}(b)\overrightarrow{\psi}(f)}{\overrightarrow{\psi}\left(\frac{1}{b}\right)\overleftarrow{\psi}\left(\frac{1}{f}\right)}\right). \qquad (1.68)$$

Interaction 1.a: $\overrightarrow{\mathbf{S}}\overleftarrow{\mathbf{S}}$

For these type of interactions we have $b > 1$ and $f < 1$. With this we have the left hand side of (1.68) is negative while the right hand side is positive. Thus there can be no vacuum formation for an interaction of type 1.a.

Interaction 1.b: $\overrightarrow{\mathbf{S}}\overleftarrow{\mathbf{R}}$

For interactions of type 1.b we have $b, f < 1$. With this and evaluating one finds that (1.68) to be equivalent to

$$-\nu b^{\zeta}\sqrt{\frac{f(1+\mu^2 f)}{f+\mu^2}} - \frac{\kappa(1-f)}{\sqrt{f+\mu^2}} < b^{\zeta}\sqrt{\frac{f(1+\mu^2 f)}{f+\mu^2}} + \nu\left[1-\sqrt{\frac{f(1+\mu^2 f)}{f+\mu^2}}\right]. \tag{1.69}$$

Notice we can apply lemma 1.2.2 to the function $\frac{f(1+\mu^2 f)}{f+\mu^2}$ yielding that

$$1-\sqrt{\frac{f(1+\mu^2 f)}{f+\mu^2}} > 0\,,$$

since $f < 1$. With this and that $b, f < 1$ we conclude that the left hand side of (1.69) is negative and the right hand side is positive. Thus, vacuum formation for interactions of type 1.b is not possible.

Interaction 1.c: $\overrightarrow{\mathbf{R}}\overleftarrow{\mathbf{R}}$

For these interactions we have $b < 1$ and $f > 1$. Evaluating we find that (1.68) is equivalent to

$$b^{\zeta} + \frac{1}{f^{\zeta}} > 1. \tag{1.70}$$

Certainly one can choose $b < 1$ and $f > 1$ such that $b^{\zeta} + \frac{1}{f^{\zeta}} \leq 1$ and thus vacuum formation is possible for interactions of type 1.c. However we can find apriori bounds on b and f to ensure that no vacuum is formed. By (1.52) we have $\zeta \in (0, \frac{1}{2})$ and since $b < 1$ and $f > 1$ yields

$$b^{\zeta} + \frac{1}{f^{\zeta}} > b^{\frac{1}{2}} + \frac{1}{f^{\frac{1}{2}}}\,, \tag{1.71}$$

Therefore if $b > \frac{1}{4}$ and $f < 4$ then $b^{\zeta} + \frac{1}{f^{\zeta}} > 1$, which ensures that no vacuum can form.

1.3.2.1 Outgoing backward waves

We now study the outgoing waves and show that the backward outgoing wave is a shock (rarefaction) if the backward incoming wave is a shock (rarefaction). Likewise, we find the forward outgoing wave will be a shock (rarefaction) if the forward incoming wave is a shock (rarefaction). For outgoing contacts we find that for interaction 1.a $C > 1$ if and only if $bf > 1$ and for interaction 1.b $C > 1$ while in 1.c there will be no outgoing contact.

Interaction 1.a: $\overrightarrow{\mathbf{S}}\overleftarrow{\mathbf{S}}$

For these interactions we have $b > 1$ and $f < 1$. The outgoing backward wave will be a shock if and only if $B > 1$. Since $\mathcal{F}_1$ is strictly increasing this is equivalent to showing $\mathcal{F}_1(B) > \mathcal{F}_1(1)$. Using (1.66) and (1.67) we have that $\mathcal{F}_1(B) > \mathcal{F}_1(1)$ if and only if

$$-\overrightarrow{\psi}(f)\Big[\frac{\overleftarrow{\psi}(b)}{\overleftarrow{\psi}(\frac{1}{f})}+1\Big] > \frac{\overleftarrow{\psi}(b)\overrightarrow{\psi}(f)}{\overrightarrow{\psi}(\frac{1}{b})\overleftarrow{\psi}(\frac{1}{f})}\overleftarrow{\psi}(\frac{1}{bf}). \tag{1.72}$$

Notice, we do not know if $bf \geq 1$ and therefore we have two cases to consider.

Case (1). Assume $bf \geq 1$. Evaluating (1.72) we find that the left hand side is strictly greater than zero while the right hand side is less than or equal to zero. Thus (1.72) holds telling us that $B > 1$.

Case (2). Assume $bf < 1$. Observe that (1.72) will hold if

$$-\overrightarrow{\psi}(f) > \frac{\overleftarrow{\psi}(b)\overrightarrow{\psi}(f)}{\overrightarrow{\psi}(\frac{1}{b})\overleftarrow{\psi}(\frac{1}{f})}\overleftarrow{\psi}(\frac{1}{bf}).$$

Evaluating, one finds the expression to be equivalent to

$$(1-f) > (1-bf)\sqrt{\frac{1+\mu^2 b}{b+\mu^2}}\sqrt{\frac{1+\mu^2 f}{1+\mu^2 bf}}.$$

Since $b > 1$ we have

$$\sqrt{\frac{1+\mu^2 b}{b+\mu^2}} < 1, \qquad \sqrt{\frac{1+\mu^2 f}{1+\mu^2 bf}} < 1, \qquad (1-f) > (1-bf),$$

and thus the inequality holds. Therefore (1.72) holds, giving $B > 1$.

In both cases, $bf \geq 1$ and $bf < 1$, we have that the outgoing backward wave is a shock.

Interaction 1.b: $\overrightarrow{\mathbf{S}}\overleftarrow{\mathbf{R}}$

Since the incoming forward wave is a shock and the backward wave is a rarefaction we have $b, f < 1$. We show $F_1(B) < F_1(1)$, which yields $B < 1$ since $\mathcal{F}_1$ is strictly increasing. Using (1.66) and(1.67) we have that $\mathcal{F}_1(B) < \mathcal{F}_1(1)$ if and only if

$$-\overrightarrow{\psi}(f)\Big[\frac{\overleftarrow{\psi}(b)}{\overleftarrow{\psi}(\frac{1}{f})} + 1\Big] < \frac{\overleftarrow{\psi}(b)\overrightarrow{\psi}(f)}{\overrightarrow{\psi}(\frac{1}{b})\overleftarrow{\psi}(\frac{1}{f})}\overleftarrow{\psi}(\frac{1}{bf}).$$

Dividing by $\frac{-\overleftarrow{\psi}(b)\overrightarrow{\psi}(f)}{\overrightarrow{\psi}(\frac{1}{b})\overleftarrow{\psi}(\frac{1}{f})}$ we have that the above is equivalent to

$$\overrightarrow{\psi}(\frac{1}{b}) + \frac{\overrightarrow{\psi}(\frac{1}{b})\overleftarrow{\psi}(\frac{1}{f})}{\overleftarrow{\psi}(b)} + \overleftarrow{\psi}(\frac{1}{bf}) > 0. \tag{1.73}$$

Using the fact that $bf, b, f < 1$ and evaluating we find that (1.73) is equivalent to

$$\nu(b^{-\zeta} - 1) + \frac{\kappa(\frac{1}{bf} - 1)}{\sqrt{\frac{1}{bf} + \mu^2}} - \frac{\kappa b^{-\zeta}(\frac{1}{f} - 1)}{\sqrt{\frac{1}{f} + \mu^2}} > 0. \tag{1.74}$$

Recall $\zeta = \frac{\gamma-1}{2\gamma}$ and thus $\zeta \in (0, \frac{1}{2})$. Also note that $\frac{1}{f}, \frac{1}{b} > 1$ giving us

$$1 < b^{-\zeta},$$

and

$$\frac{b^{-\zeta}(\frac{1}{f}-1)}{\sqrt{\frac{1}{f}+\mu^2}} < \frac{b^{-\frac{1}{2}}(\frac{1}{f}-1)}{\sqrt{\frac{1}{f}+\mu^2}}.$$

Therefore (1.74) will hold if

$$\frac{(\frac{1}{f}-1)}{\sqrt{\frac{1}{f}+\mu^2}} < \frac{(\frac{1}{bf}-1)}{\sqrt{\frac{1}{b^2 f}+\frac{\mu^2}{b}}}. \tag{1.75}$$

To show this to be true consider $f \in (0,1)$ fixed, and define

$$G(x) = \frac{(\frac{x}{f}-1)}{\sqrt{\frac{x^2}{f}+\mu^2 x}}.$$

Taking the derivative we find

$$G'(x) = \frac{2x + x\mu^2 + f\mu^2}{2f(\frac{x^2}{f}+\mu^2 x)^{\frac{3}{2}}} > 0, \qquad \forall x \geq 1.$$

Therefore $G(1) < G(\frac{1}{b})$ telling us that (1.75) holds and therefore so does (1.74), proving $B < 1$.

Interaction 1.c: $\overrightarrow{\mathbf{R}}\overleftarrow{\mathbf{R}}$

We now show $B < 1$. We do this by using the fact that $\mathcal{F}_1$ is increasing. Combining (1.66) and(1.67) we have that $\mathcal{F}_1(B) < \mathcal{F}_1(1)$ if and only if

$$-\overrightarrow{\psi}(f)\left[\frac{\overleftarrow{\psi}(b)}{\overleftarrow{\psi}(\frac{1}{f})}+1\right] < \frac{\overleftarrow{\psi}(b)\overrightarrow{\psi}(f)}{\overrightarrow{\psi}(\frac{1}{b})\overleftarrow{\psi}(\frac{1}{f})}\overleftarrow{\psi}(\frac{1}{bf}). \tag{1.76}$$

Notice we cannot evaluate $\overleftarrow{\psi}(\frac{1}{bf})$ since we do not know whether or not $bf \leq 1$. Thus we have two cases to consider.

Case(1). Assume $bf \leq 1$. Then by evaluating we find that the left hand side of (1.76) is strictly less than zero while the right hand side is greater or equal to zero and thus (1.76) holds.

Case(2). Assume $bf > 1$. Evaluating (1.76) one equivalently finds

$$2f^{\varsigma}(b^{\varsigma} - 1) < 0,$$

which holds since $b < 1$.

Since (1.76) is true for both cases we have $B < 1$, and thus the outgoing backward wave is a rarefaction.

1.3.2.2 Outgoing forward waves

Interaction 1.a: $\overrightarrow{\mathbf{S}}\overleftarrow{\mathbf{S}}$

From (1.63) we know $F = \frac{bf}{B}$, since $B > 1$. Therefore $F < 1$ if and only if $bf < B$. Since $\mathcal{F}_1$ is an increasing function this is equivalent to

$$\mathcal{F}_1(B) > \mathcal{F}_1(bf), \quad \Leftrightarrow \quad -\overrightarrow{\psi}(f)\left[\frac{\overleftarrow{\psi}(b)}{\overleftarrow{\psi}(\frac{1}{f})} + 1\right] > \overleftarrow{\psi}(bf)\,. \tag{1.77}$$

As above, we have two cases to consider since we do not know if $bf \leq 1$ or $bf > 1$.

Case (1). Assume $bf \leq 1$. Evaluating (1.77) we find that the right hand side is less than or equal to zero and the left hand side is strictly greater than zero. Therefore (1.77) holds and $F < 1$.

Case (2). Assume $bf > 1$. Evaluating, (1.77) becomes

$$\frac{\kappa(b-1)}{\sqrt{b+\mu^2}}\sqrt{\frac{f(1+\mu^2 f)}{f+\mu^2}} + \frac{\kappa(1-f)}{\sqrt{f+\mu^2}} > \frac{\kappa(bf-1)}{\sqrt{bf+\mu^2}},$$

$$\Leftrightarrow \quad (bf-f)\sqrt{\frac{1+\mu^2 f}{bf+\mu^2 f}} + (1-f) > (bf-1)\sqrt{\frac{f+\mu^2}{bf+\mu^2}}. \tag{1.78}$$

Since $f < 1$ (1.78) will hold if

$$(bf - f)\sqrt{\frac{1+\mu^2 f}{bf+\mu^2 f}} > (bf-1)\sqrt{\frac{f+\mu^2}{bf+\mu^2}}. \tag{1.79}$$

Notice,

$$(bf - f) > (bf - 1),$$

and thus (1.79) will be true if

$$\sqrt{\frac{1+\mu^2 f}{bf+\mu^2 f}} > \sqrt{\frac{f+\mu^2}{bf+\mu^2}}, \qquad \Leftrightarrow \qquad bf(1-\mu^2) + \mu^2(1+f) > 0\,. \tag{1.80}$$

Since $\mu^2 < 1$ (1.80) holds and thus $F < 1$.
In both cases, $bf \geq 1$ and $bf < 1$, we have that the outgoing forward wave is a shock.

Interaction 1.b: $\overrightarrow{\mathbf{S}}\overleftarrow{\mathbf{R}}$
Again we use (1.63) to obtain $F = \frac{bf}{B}$, since $B > 0$. Therefore $F < 1$ if and only if $bf < B$. Since $\mathcal{F}_1$ is an increasing function this is equivalent to $\mathcal{F}_1(bf) < \mathcal{F}_1(B$. Evaluating one finds

$$\nu\big((bf)^\zeta - 1\big) < \nu(b^\zeta - 1)\sqrt{\frac{f+\mu^2 f^2}{f+\mu^2}} + \kappa\frac{1-f}{\sqrt{f+\mu^2}}.$$

Using the fact $b, f < 1$ the above will hold if

$$\big(1 - (bf)^\zeta\big) > (1-b^\zeta)\sqrt{\frac{f+\mu^2 f^2}{f+\mu^2}}.$$

This inequality does hold since

$$\big(1 - (bf)^\zeta)\big) > (1 - b^\zeta),$$

and

$$\sqrt{\frac{f+\mu^2 f^2}{f+\mu^2}} < 1.$$

Thus $F < 1$, telling us that the out going forward wave is a shock. Also notice that since $BF = bf$ we have that $F > 0$.

Interaction 1.c: $\overrightarrow{\mathbf{R}}\overleftarrow{\mathbf{R}}$

Assuming $B > 0$, we have $F = \frac{bf}{B} > 1$ is equivalent to $B < bf$. Since $\mathcal{F}_1$ is an increasing function this is equivalent to $\mathcal{F}_1(B) < \mathcal{F}_1(bf)$. Using (1.66) and (1.67) have that $\mathcal{F}_1(B) < \mathcal{F}_1(bf)$ holds if and only if

$$-\overrightarrow{\psi}(f)\left[\frac{\overleftarrow{\psi}(b)}{\overleftarrow{\psi}(\frac{1}{f})} + 1\right] < \overleftarrow{\psi}(bf). \tag{1.81}$$

We cannot just evaluate $\overleftarrow{\psi}(bf)$ since we do not know whether or not $bf \geq 1$.

Case (1). Assume $bf \geq 1$. Then the left hand side of (1.81) is less than zero while the right hand side is greater than or equal to zero and thus (1.81) holds.

Case (2). Assume $bf < 1$. Then evaluating (1.81) becomes,

$$2(1 - f^{\zeta}) < 0,$$

which is true since $f > 1$.

Therefore (1.81) is true for both cases yielding $F > 1$, and thus the outgoing forward wave is a rarefaction.

Notice in the limiting case of vacuum formation we have $B = 0$ and therefore by (1.63), $F = \infty$.

Before continuing we make one further observation. Assuming no vacuum and using that $B, b < 1$, $F, f > 1$ and $BF = bf$ we may evaluate (1.65) finding

$$B^\zeta = b^\zeta f^\zeta - f^\zeta + 1 < b^\zeta\,,$$

since $f > 1$ and $b < 1$. Thus $B < b$, and again using $BF = bf$ we conclude $F > f$.

1.3.2.3 Outgoing contacts

Interaction 1.a: $\overrightarrow{\mathbf{S}}\overleftarrow{\mathbf{S}}$

To determine when $C > 1$ we will use a relation between B and b. We begin by fixing $f \in (0, 1)$. Using equations (1.66), (1.67) and relations (1.54), (1.55) we define the function $G(b)$ by

$$\begin{aligned} G(b) &:= \mathcal{F}_1(b) - \mathcal{F}_1(B) \\ &= \overrightarrow{\psi}(f) - \overrightarrow{\psi}(f)\sqrt{b\overleftarrow{\phi}(b)} + \overleftarrow{\psi}(b) - \overleftarrow{\psi}(b)\sqrt{f\overrightarrow{\phi}(f)}, \end{aligned} \tag{1.82}$$

for $b > 1$. Direct computations show that

$$G'(b) > 0, \qquad \text{and} \qquad \lim_{b\to 1} G(b) = 0.$$

Therefore we have

$$G(b) = \mathcal{F}_1(b) - \mathcal{F}_1(B) > 0 \quad \Leftrightarrow \quad b > B\,, \tag{1.83}$$

since $\mathcal{F}_1$ is an increasing function.

Using (1.83) and the fact that $BF = bf$ we find

$$F > f. \tag{1.84}$$

From this analysis we have

$$f < F < 1 < B < b\,.$$

With this and the fact that $bf = BF$ one finds

$$\begin{aligned} & (B-F)^2 < (b-f)^2 \\ \Leftrightarrow\quad & (B+F)^2 < (b+f)^2, \qquad (1.85) \\ \Leftrightarrow\quad & B+F < b+f\,. \qquad (1.86) \end{aligned}$$

Applying lemma 1.2.1 we have

$$\overrightarrow{\phi}(f)\overleftarrow{\phi}(b) > \overleftarrow{\phi}(B)\overrightarrow{\phi}(F), \tag{1.87}$$

if and only if $bf > 1$. Combining this with (1.64) we conclude that $C \gtrless 1$ if and only if $bf \gtrless 1$. Notice, if $bf = 1$ there is no outgoing contact.

Interaction 1.b: $\overrightarrow{\mathbf{S}}\overleftarrow{\mathbf{R}}$

From (1.64) we have $C > 1$ if and only if

$$\overrightarrow{\phi}(F)\overleftarrow{\phi}(B) < \overleftarrow{\phi}(b)\overrightarrow{\phi}(f).$$

Evaluating and using $BF = bf$ one equivalently finds

$$F^{\frac{1}{\gamma}}\left(\frac{1+\mu^2 F}{F+\mu^2}\right) < f^{\frac{1}{\gamma}}\left(\frac{1+\mu^2 f}{f+\mu^2}\right). \tag{1.88}$$

By lemma 1.2.2 one finds (1.88) holds if and only if $F < f$, which is equivalent to $b < B$. To show $b < B$ we again use the function $\mathcal{F}_1$ and show $\mathcal{F}_1(b) < \mathcal{F}_1(B)$. From (1.66) and (1.67) this is equivalent to

$$\overleftarrow{\psi}(b) + \frac{\overleftarrow{\psi}(b)\overrightarrow{\psi}(f)}{\overrightarrow{\psi}(\frac{1}{b})} < -\overrightarrow{\psi}(f)\left[\frac{\overleftarrow{\psi}(b)}{\overleftarrow{\psi}(\frac{1}{f})} + 1\right]. \tag{1.89}$$

Evaluating the expression and with some basic algebraic manipulation we obtain that (1.89) is equivalent to

$$\frac{(1-b^{\varsigma})}{\sqrt{f+\mu^2}}\left(\kappa(1-f) + \nu\Big(\sqrt{f+\mu^2} - \sqrt{f+\mu^2 f^2}\Big)\right) > 0, \tag{1.90}$$

which holds since $b, f < 1$. Therefore (1.88) holds which gives $C > 1$. Notice that we have also shown $F < f$ and $b < B$.

Interaction 1.c: $\overrightarrow{\mathbf{R}}\overleftarrow{\mathbf{R}}$

From (1.64) we have $C = 1$ if and only if

$$\overrightarrow{\phi}(F)\overleftarrow{\phi}(B) = \overleftarrow{\phi}(b)\overrightarrow{\phi}(f).$$

At this point we know $f, F > 1$ and $b, B < 1$ and thus we may evaluate yielding

$$\frac{1}{b^{\frac{1}{\gamma}} f^{\frac{1}{\gamma}}} = \frac{C}{B^{\frac{1}{\gamma}} F^{\frac{1}{\gamma}}} .$$

Since $bf = BF$ we conclude $C = 1$ and thus there is no outgoing contact.

1.3.2.4 Summary of group 1

We give three tables which summarize our findings. The first table gives when vacuum formation is possible.

Interaccction	Incoming Waves	Vacuum
1.a	$\overrightarrow{S}\overleftarrow{S}$	No vac.
1.b	$\overrightarrow{S}\overleftarrow{R}$	No vac.
1.c	$\overrightarrow{R}\overleftarrow{R}$	No vac. $\Leftrightarrow b^{\zeta} + \frac{1}{f^{\zeta}} > 1$

Table 1.1. When vacuum formation is possible (group 1).

We now summarize in a table the the incoming and outgoing waves. The table also includes any conditions for the outgoing waves.

Interaction	Incoming Waves	Outgoing Waves	Condition
1.a	$\overrightarrow{S}\overleftarrow{S}$	$\overleftarrow{S}\overset{\gtrless}{J}\overrightarrow{S}$	$\overset{\gtrless}{J} \Leftrightarrow bf \lesseqgtr 1$
1.b	$\overrightarrow{S}\overleftarrow{R}$	$\overleftarrow{R}\overset{<}{J}\overrightarrow{S}$	NA
1.c	$\overrightarrow{R}\overleftarrow{R}$	$\overleftarrow{R}\overrightarrow{R}$	NA

Table 1.2. Incoming and outgoing waves (group 1).

The last table gives the relations that were found between the incoming and outgoing waves.

Interaction	Incoming Waves	Relations of strengths
1.a	$\overrightarrow{S}\overleftarrow{S}$	$f < F < 1 < B < b$ and $(b+f) > (B+F)$
1.b	$\overrightarrow{S}\overleftarrow{R}$	$F < f$ and $b < B$
1.c	$\overrightarrow{R}\overleftarrow{R}$	$B < b < 1 < f < F$ and $(B+F) > (b+f)$

Table 1.3. Relative strengths of incoming and outgoing waves (group 1).

1.3.3 Interactions in group 2. Waves overtaking waves.

Let the incoming parameter values be b_1 and b_2 from left to right respectively. Let B, C, F denote the parameter values for the outgoing waves. See Figure 1.6.

Computing the right state before the interaction in terms of b_1, b_2 yields

$$\tau = \overleftarrow{\phi}(b_1)\overleftarrow{\phi}(b_2)\tau_0 \tag{1.91}$$

$$\frac{u_0 - u}{\sqrt{\tau_0 p_0}} = \overleftarrow{\psi}(b_1) + \sqrt{b_1 \overleftarrow{\phi}(b_1)}\,\overleftarrow{\psi}(b_2) \tag{1.92}$$

$$p = b_1 b_2 p_0 \,. \tag{1.93}$$

Combining these relations with (1.56)-(1.58) gives

$$BF = b_1 b_2 \,, \tag{1.94}$$

$$\overleftarrow{\phi}(b_1)\overleftarrow{\phi}(b_2) = C\overleftarrow{\phi}(B)\overrightarrow{\phi}(F) \,, \tag{1.95}$$

$$\overleftarrow{\psi}(B) + \sqrt{b_1 b_2\, \overleftarrow{\phi}(b_1)\overleftarrow{\phi}(b_2)}\; \overleftarrow{\psi}\left(\frac{B}{b_1 b_2}\right) = \overleftarrow{\psi}(b_2)\sqrt{b_1\, \overleftarrow{\phi}(b_1)} + \overleftarrow{\psi}(b_1) \,. \tag{1.96}$$

As in group 1 these relations will be used in the analysis of the outgoing waves. If we define $\mathcal{F}_2(\alpha) \equiv \mathcal{F}_2(\alpha; b_1, b_2)$ by

$$\mathcal{F}_2(\alpha) = \overleftarrow{\psi}(\alpha) + \sqrt{b_1 b_2\, \overleftarrow{\phi}(b_1)\overleftarrow{\phi}(b_2)}\; \overleftarrow{\psi}\left(\frac{\alpha}{b_1 b_2}\right) \,, \tag{1.97}$$

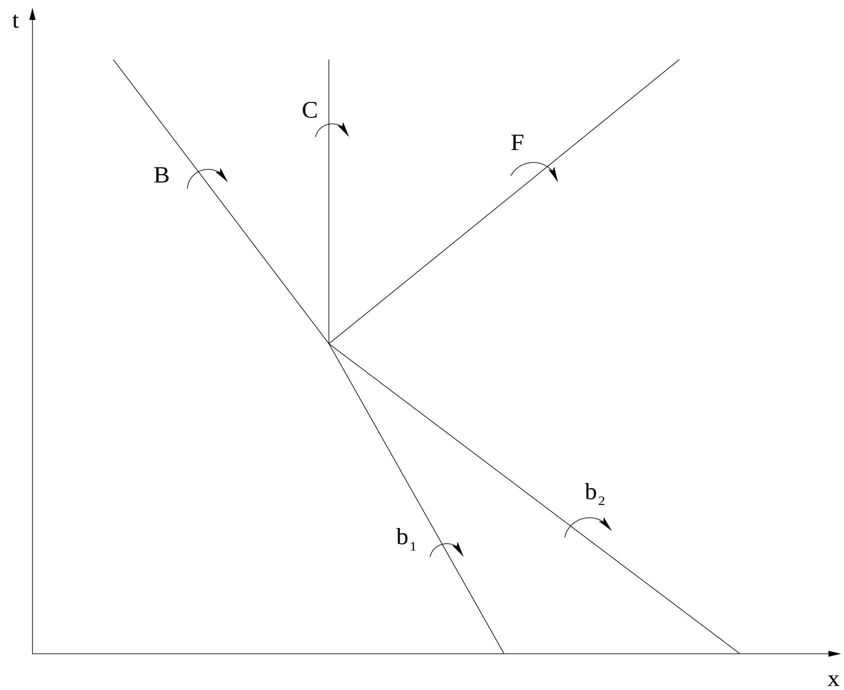

Figure 1.6. Incoming and outgoing parameters for waves overtaking waves (group 2).

then from the properties of $\overleftarrow{\psi}$ we have that $\mathcal{F}_2$ is increasing, tends to $+\infty$ at $+\infty$, and is convex down. Using (1.54) we obtain

$$\mathcal{F}_2(\alpha) = \overleftarrow{\psi}(\alpha) + \frac{\overleftarrow{\psi}(b_1)\overleftarrow{\psi}(b_2)}{\overrightarrow{\psi}\left(\frac{1}{b_1}\right)\overrightarrow{\psi}\left(\frac{1}{b_2}\right)}\ \overleftarrow{\psi}\left(\frac{\alpha}{b_1 b_2}\right), \tag{1.98}$$

and therefore we may rewrite (1.96), the equation for B, as

$$\mathcal{F}_2(B) = \overleftarrow{\psi}(b_1)\left[1 - \frac{\overleftarrow{\psi}(b_2)}{\overrightarrow{\psi}\left(\frac{1}{b_1}\right)}\right]. \tag{1.99}$$

Possible vacuum formation.

We will use that $\mathcal{F}_2$ is an increasing function. With this one finds that no vacuum will be formed by an interaction if and only if $\mathcal{F}_2(B) > \mathcal{F}_2(0)$. Evaluating (1.98) for $\alpha = 0$ and using relation (1.99) one finds that this is equivalent to

$$-\nu - \nu\sqrt{b_1 b_2 \overleftarrow{\phi}(b_1)\overleftarrow{\phi}(b_2)} \;<\; \overleftarrow{\psi}(b_2)\sqrt{b_1\,\overleftarrow{\phi}(b_1)} + \overleftarrow{\psi}(b_1)\,. \tag{1.100}$$

Interaction 2.a: $\overleftarrow{\mathbf{S}}\overleftarrow{\mathbf{S}}$

Interactions of type 2.a have $b_1, b_2 > 1$, giving that the left hand side of (1.100) is negative while the right hand side is positive and thus no vacuum formation is possible for interactions of type 2.a.

Interaction 2.b: $\overleftarrow{\mathbf{S}}\overleftarrow{\mathbf{R}}$

For these interactions one has $b_1 > 1$ and $b_2 < 1$. Evaluating we find that (1.100) is equivalent to

$$\frac{-\kappa(b_1 - 1)}{\sqrt{b_1 + \mu^2 b_1^2}} - \frac{\nu\sqrt{b_1 + \mu^2}}{\sqrt{b_1 + \mu^2 b_1^2}} < \nu(2b_2^{\zeta} - 1)\,. \tag{1.101}$$

Clearly the right hand side is an increasing function b_2, and as b_2 approaches zero the right hand side approaches $-\nu$, giving a lower bound. Similarly, as b_1 approaches infinity the left hand side approaches $\frac{-\kappa}{\mu}$. Using the definitions of κ and ν in (1.52) one finds that $\frac{\kappa}{\mu} < \nu$, thus sufficiently small values of b_2 and sufficiently large values of b_1, inequality (1.101) will be violated giving vacuum formation.

Next we show that we choose b_1 and b_2 such that the interaction will not form a vacuum. Assume $\frac{-\kappa}{\mu}$ is the upper bound of the left hand side of (1.101) (this will be shown to be true below). Then, if we are able to choose b_2 such that

$$\nu(2b_2^{\zeta} - 1) > \frac{-\kappa}{\mu} \tag{1.102}$$

no vacuum can form by the interaction. Using the definitions of of κ, μ and γ one finds that (1.102) will hold if

$$b_2^{\zeta} > \frac{1}{2}\,.$$

Again using (1.52) we find this will hold if $b_2 > \frac{1}{4}$.

We now show that $\frac{-\kappa}{\mu}$ is in fact the upper bound of the left hand side of (1.101). Notice that $\frac{-\kappa}{\mu}$ being the upper bound is equivalent to

$$\frac{\mu\kappa(b_1 - 1)}{\sqrt{b_1 + \mu^2 b_1^2}} + \frac{\mu\nu\sqrt{b_1 + \mu^2}}{\sqrt{b_1 + \mu^2 b_1^2}} > \kappa\,.$$

Since $\mu < 1$ this inequality will hold if

$$\frac{\mu^2\kappa(b_1 - 1)}{\sqrt{b_1 + \mu^2 b_1^2}} + \frac{\mu\nu\sqrt{b_1 + \mu^2}}{\sqrt{b_1 + \mu^2 b_1^2}} > \kappa\,.$$

Once again using the definitions of κ and ν we find this to be equivalent to

$$\mu\sqrt{1-\mu^4}\sqrt{b_1+\mu^2} + \mu^2\sqrt{1-\mu^2}(b_1 - 1) > \mu\sqrt{1-\mu^2}\sqrt{b_1 + \mu^2 b_1^2}\,. \qquad (1.103)$$

Since $\mu^2 < 1$ we have that (1.103) will hold if

$$\mu\sqrt{1-\mu^2}\sqrt{b_1+\mu^2} + \mu^2\sqrt{1-\mu^2}(b_1 - 1) > \mu\sqrt{1-\mu^2}\sqrt{b_1 + \mu^2 b_1^2}\,.$$

Since $b_1 > 1$ the left hand side and right hand side are positive and thus by squaring both sides one finds this to be equivalent to

$$(b_1 - 1)\left[\sqrt{b_1 + \mu^2} - \mu\right] > 0\,.$$

Since $b_1 > 0$ and $\mu < 1$ the inequality holds, thus proving the upper bound of the left hand side of (1.101) to be $\frac{-\kappa}{\mu}$.

Interaction 2.c: $\overleftharpoon{\mathbf{R}}\overleftharpoon{\mathbf{S}}$

Type 2.c interactions have $b_2 > 1$ and $b_1 < 1$. Using that $\overleftharpoon{\psi}$ is an increasing function we have that $\overleftharpoon{\psi}(b_1) > \overleftharpoon{\psi}(0) = -\nu$. With this and rewriting (1.100) as

$$-\nu\sqrt{b_1 b_2\,\overleftharpoon{\phi}(b_1)\overleftharpoon{\phi}(b_2)} \;<\; \overleftharpoon{\psi}(b_2)\sqrt{b_1\,\overleftharpoon{\phi}(b_1)} + \left(\overleftharpoon{\psi}(b_1) + \nu\right)\,,$$

we have that the left hand side is negative while the right hand side is positive. Therefore no vacuum formation is possible by interactions of type 2.c.

Due to the inherent difficulties involved in the analysis of this group we will only analyze the outgoing forward and backward waves of interactions of type 2.a in full detail. For this interaction the outgoing contact will be analyzed in the case where the outgoing forward wave is a shock. An analysis for the outgoing contact where the outgoing forward wave is a rarefaction my be found in [7].

The main complication in the analysis of this group stems from the fact that the outgoing forward and backward waves not only depend on the strengths of the incoming waves, but also on the adiabatic exponent, γ.

Before analyzing interaction 2.a we will find a general relation between b_1 and B dependent upon whether the $b_2 \gtrless 1$. Using the monotonicity of $\mathcal{F}_2$ we have that $B \gtrless b_1$ if and only if $\mathcal{F}_2(B) \gtrless \mathcal{F}_2(b_1)$. Using (1.98) and (1.99) we have that $\mathcal{F}_2(B) \gtrless \mathcal{F}_2(b_1)$ is equivalent to

$$\begin{aligned} & \overleftharpoon{\psi}(b_1)\left[1 - \frac{\overleftharpoon{\psi}(b_2)}{\overrightharpoon{\psi}(\frac{1}{b_1})}\right] \gtrless \overleftharpoon{\psi}(b_1) + \frac{\overleftharpoon{\psi}(b_1)\overleftharpoon{\psi}(b_2)}{\overrightharpoon{\psi}(\frac{1}{b_1})\overrightharpoon{\psi}(\frac{1}{b_2})}\overleftharpoon{\psi}(\frac{1}{b_2}), \\ \Leftrightarrow \quad & \frac{-\overleftharpoon{\psi}(b_1)\overleftharpoon{\psi}(b_2)}{\overrightharpoon{\psi}(\frac{1}{b_1})} \gtrless \frac{\overleftharpoon{\psi}(b_1)\overleftharpoon{\psi}(b_2)}{\overrightharpoon{\psi}(\frac{1}{b_1})\overrightharpoon{\psi}(\frac{1}{b_2})}\overleftharpoon{\psi}(\frac{1}{b_2}). \end{aligned} \tag{1.104}$$

Notice, if $b_2 > 1$ ($b_2 < 1$) then the left hand side is positive (negative) while the right hand side is negative (positvie). Therefore,

$$B \gtrless b_1 \quad \Leftrightarrow \quad b_2 \gtrless 1\,. \tag{1.105}$$

1.3.4 Interaction 2a: Backward shock overtaking backward shock

For this interaction we have $b_1, b_2 > 1$. The analysis the outgoing backward, forward and contact waves will be done individually. We will show that the outgoing backward wave is shock while the outgoing forward wave can be a shock or rarefaction depending on the relative strengths of the incoming shocks and the value of γ. We then show that the outgoing contact is a down-contact.

Backward outgoing wave is a shock.
Since $b_2 > 1$ we have $B > b_1$, from (1.105). For this interaction we have $b_1 > 1$ and thus $B > 1$. Thus the outgoing backward wave is a shock.

A direct computation using (1.98) and (1.99) also gives the relation $\mathcal{F}_2(B) > \mathcal{F}_2(b_2)$ which is equivalent to $B > b_2$. Thus $B > b_1, b_2$ and since $BF = b_1 b_2$ one finds $F < b_1, b_2$.

Forward outgoing wave can be a shock or rarefaction.
We know $F = \frac{b_1 b_2}{B}$ and thus $F > 1$ if and only if $b_1 b_2 > B$. Since $\mathcal{F}_2$ is an increasing function this is equivalent to $\mathcal{F}_2(b_1 b_2) > \mathcal{F}_2(B)$. Recall $\overleftarrow{\psi}(1) = 0$, and thus from (1.98) and (1.99) we have

$$\mathcal{F}_2(b_1 b_2) > \mathcal{F}_2(B) \quad \Leftrightarrow \quad \overleftarrow{\psi}(b_1 b_2) > \overleftarrow{\psi}(b_1)\left[1 - \frac{\overleftarrow{\psi}(b_2)}{\overrightarrow{\psi}(\frac{1}{b_1})}\right]. \tag{1.106}$$

Evaluating, we have (1.106) is equivalent to

$$\frac{b_1 b_2 - 1}{\sqrt{b_1 b_2 + \mu^2}} > \frac{b_1 - 1}{\sqrt{b_1 + \mu^2}} + \frac{(b_2 - 1)}{\sqrt{b_2 + \mu^2}}\sqrt{\frac{b_1 + b_1^2 \mu^2}{b_1 + \mu^2}}.$$

Multiplying both sides of the equation by

$$\sqrt{b_1 b_2 + \mu^2}\sqrt{b_1 + \mu^2}\sqrt{b_2 + \mu^2}$$

we obtain

$$
\begin{aligned}
(b_1b_2-1)\sqrt{b_1+\mu^2}\sqrt{b_2+\mu^2} &> (b_1-1)\sqrt{b_1b_2+\mu^2}\sqrt{b_2+\mu^2} \\
&+ (b_2-1)\sqrt{b_1+b_1^2\mu^2}\sqrt{b_1b_2+\mu^2}. \quad (1.107)
\end{aligned}
$$

Notice that the left hand side and right hand side are greater than zero and therefore by squaring both sides and with some algebraic manipulation we have (1.107) is equivalent to

$$
\begin{aligned}
&(b_1-1)(b_2-1)\Big(b_1^2b_2^2 - b_1^2b_2^2\mu^2 + 2b_1^2b_2\mu^2 + b_1b_2 + 3b_1b_2\mu^2 + 2b_1\mu^4\Big) \\
&\quad > (b_1-1)(b_2-1)\Big(2(b_1b_2+\mu^2)\sqrt{b_1+b_2^2\mu^2}\sqrt{b_2+\mu^2}\Big). \quad (1.108)
\end{aligned}
$$

Dividing both sides by $(b_1-1)(b_2-1)$, which is positive, we have that (1.108) will hold if and only if

$$
b_1^2b_2^2 - b_1^2b_2^2\mu^2 + 2b_1^2b_2\mu^2 + b_1b_2 + 3b_1b_2\mu^2 + 2b_1\mu^4 > 2(b_1b_2+\mu^2)\sqrt{b_1+\mu^2}\sqrt{b_2+\mu^2}.
$$

Again, the left hand side and the right hand side are positive and thus by squaring both sides we obtain an equivalent expression given by

$$
b_1(b_1b_2-1)^2\Big(\big[b_1(1-\mu^2)^2\big]b_2^2 - 4\mu^4(b_1+1)b_2 - 4\mu^6\Big) > 0,
$$

$$
\Leftrightarrow \quad \big[b_1(1-\mu^2)^2\big]b_2^2 - 4\mu^4(b_1+1)b_2 - 4\mu^6 > 0. \quad (1.109)
$$

Notice that if we consider b_1 and μ^2 fixed and let b_2 vary, then (1.109) is where the graph of a parabola lies above the b_2-axis. This parabola is concave up and at $b_2 = 0$ this point on the parabola lies below the b_2-axis. Therefore, since we are only interested in values of $b_2 > 1$ we only need to consider the positive root of the parabola. That is, (1.109) holds if and only if

$$b_2 > \frac{2\mu^3}{b_1(1-\mu^2)^2}\Big[\mu(b_1+1) + \sqrt{\mu^2(b_1^2+1) + b_1(1+\mu^4)}\Big]. \tag{1.110}$$

At this point we have found that the reflected wave is a rarefaction, $F > 1$, if and only if (1.110) holds.

We now analyze (1.110) more closely. Let the right hand side of (1.110) be denoted by $G(\mu^2, b_1)$. Simple calculations show that

$$\frac{\partial G}{\partial \mu^2} > 0, \qquad \text{and} \qquad \frac{\partial G}{\partial b_1} < 0, \tag{1.111}$$

when $\mu^2 \in (0,1)$ and $b_1 > 1$. Also notice that

$$\lim_{b_1 \downarrow 1} G(\tfrac{1}{4}, b_1) = 1. \tag{1.112}$$

Using (1.111), (1.112) and the fact that $b_2 > 1$ we have that

$$b_2 > G(\mu^2, b_1) \qquad \text{if } \mu^2 < \frac{1}{4}, \tag{1.113}$$

and thus $F > 1$ if $\mu^2 < \frac{1}{4}$.

Note: $\mu^2 < \frac{1}{4}$ if and only if $\gamma < \frac{5}{3}$.

One last thing to notice is that given any $b_1 > 1$

$$\lim_{\mu^2 \downarrow 0} G(\mu^2, b_1) = 0, \qquad \text{and} \qquad \lim_{\mu^2 \uparrow 1} G(\mu^2, b_1) = \infty.$$

Combining this with the fact that $\frac{\partial G}{\partial \mu^2} > 0$ tells us that given any $b_1, b_2 > 1$ there exist values of $\mu^2 \in (0,1)$, such that $F > 1$, $F = 1$ and $F < 1$.

Outgoing contact.

For this interaction the outgoing contact is a down contact. To begin we will use a partial ordering of the outgoing wave strengths and the incoming wave strengths. From the analysis of the outgoing backward wave we know $B > b_1, b_2$ and $F < b_1, b_2$. Therefore we have the partial ordering, $F < b_1, b_2 < B$. This gives $(b_2 - b_1)^2 < (B - F)^2$, and thus by using $b_1 b_2 = BF$ one finds

$$b_2 + b_1 < B + F\,. \tag{1.114}$$

In the analysis of the outgoing contact there are two cases to consider, $F \leq 1$ and $F > 1$.

Case(1). For $F \leq 1$ if we evaluate (1.95) one finds

$$\left(\frac{1+\mu^2 b_1}{b_1+\mu^2}\right)\left(\frac{1+\mu^2 b_2}{b_2+\mu^2}\right) = C\left(\frac{1+\mu^2 B}{B+\mu^2}\right)\left(\frac{1+\mu^2 F}{F+\mu^2}\right).$$

Thus $C < 1$ if and only if

$$\left(\frac{1+\mu^2 b_1}{b_1+\mu^2}\right)\left(\frac{1+\mu^2 b_2}{b_2+\mu^2}\right) < \left(\frac{1+\mu^2 B}{B+\mu^2}\right)\left(\frac{1+\mu^2 F}{F+\mu^2}\right). \tag{1.115}$$

Using the fact that $BF = b_1 b_2 > 1$ and applying lemma 1.2.1 we conclude that (1.115) holds and thus $C < 1$ in this case.

Case(2). For $F < 1$ see [7], pages 119-121. Their analysis shows that $C < 1$ in this case as well.

1.3.4.1 Summary of group 2

We again give three tables which summarize when vacuum formation is possible, the incoming and outgoing waves, and the known relations between the incoming and outgoing wave strengths. For the most complete analysis of interactions of type 2.b and 2.c, see [7]. Again, computer simulations have been written that confirm the findings in [7].

The first table is concerned with when vacuum formation is possible.

In the next table we summarize the incoming and outgoing waves.

Interaction	Incoming waves	Vacuum
2.a	$\overleftharpoon{S}\overleftharpoon{S}$	No vac.
2.b	$\overleftharpoon{S}\overleftharpoon{R}$	No vac. $\Leftrightarrow$ (1.101)
2.c	$\overleftharpoon{R}\overleftharpoon{S}$	No vac.

Table 1.4. When vacuum formation is possible (group 2).

Interaction	Incoming Waves	Outgoing Waves	Condition
2.a	$\overleftharpoon{S}\overleftharpoon{S}$	$\overleftharpoon{S}\overset{>}{J}\overrightharpoon{S}$ or $\overleftharpoon{S}\overset{>}{J}\overrightharpoon{R}$	see (1.110)
2.b	$\overleftharpoon{S}\overleftharpoon{R}$	$\overleftharpoon{S}\overset{<}{J}\overrightharpoon{S}$ or $\overleftharpoon{S}\overset{<}{J}\overrightharpoon{R}$ or $\overleftharpoon{R}\overset{<}{J}\overrightharpoon{S}$ or $\overleftharpoon{R}\overset{<}{J}\overrightharpoon{R}$	
2.c	$\overleftharpoon{R}\overleftharpoon{S}$	$\overleftharpoon{S}\overset{<}{J}\overrightharpoon{S}$ or $\overleftharpoon{S}\overset{<}{J}\overrightharpoon{R}$ or $\overleftharpoon{R}\overset{<}{J}\overrightharpoon{S}$ or $\overleftharpoon{R}\overset{<}{J}\overrightharpoon{R}$	

Table 1.5. Incoming and outgoing waves (group 2),

Lastly, we give the relations of the incoming and outgoing wave strengths.

Interaction	Incoming Waves	Relations of strengths
2.a	$\overleftharpoon{S}\overleftharpoon{S}$	$F < b_1, b_2 < B < b$
2.b	$\overleftharpoon{S}\overleftharpoon{R}$	$b_1 > 1,\ b_2 < 1,\ B < b_1$
2.c	$\overleftharpoon{R}\overleftharpoon{S}$	$b_1 < 1,\ b_2 > 1,\ B > b_1$

Table 1.6. Relative strengths of incoming and outgoing waves (group 2).

1.3.5 Interactions in group 3. Interaction involving a contact.

Let the incoming parameter values be f and c, and let B, C, F denote the parameter values for the outgoing waves. See Figure 1.7.

Computing the right state before the interaction in terms of f, c yields

$$\tau = c\overrightharpoon{\phi}(f)\tau_0 \tag{1.116}$$

$$\frac{u_0 - u}{\sqrt{\tau_0 p_0}} = -\overrightharpoon{\psi}(f) \tag{1.117}$$

$$p = f p_0\,. \tag{1.118}$$

Combining these relations with (1.56)-(1.58) gives

$$BF = f\,, \tag{1.119}$$

$$c\overrightarrow{\phi}(f) = C\overleftarrow{\phi}(B)\overrightarrow{\phi}(F)\,, \tag{1.120}$$

$$\overleftarrow{\psi}(B) + \sqrt{cf\,\overrightarrow{\phi}(f)}\,\overleftarrow{\psi}\left(\frac{B}{f}\right) = -\overrightarrow{\psi}(f)\,. \tag{1.121}$$

Define $\mathcal{F}_3(\alpha) \equiv \mathcal{F}_3(\alpha; f, c)$ by

$$\mathcal{F}_3(\alpha) = \overleftarrow{\psi}(\alpha) + \sqrt{cf\overrightarrow{\phi}(f)}\,\overleftarrow{\psi}\left(\frac{\alpha}{f}\right). \tag{1.122}$$

From the properties of $\overleftarrow{\psi}$ we have that $\mathcal{F}_3$ is increasing, tends to $+\infty$ at $+\infty$, and is convex down. From (1.121) we can rewrite the equation for B as

$$\mathcal{F}_3(B) = -\overrightarrow{\psi}(f). \tag{1.123}$$

Possible vacuum formation.

For this interaction group there will be no vacuum formation if and only if $\mathcal{F}_3(B) > \mathcal{F}_3(0)$, since $\mathcal{F}_3$ is increasing. Evaluating (1.122) for $\alpha = 0$ and using relation (1.123) one finds this to be equivalent to

$$\overrightarrow{\psi}(f) < \nu + \nu\sqrt{cf\overrightarrow{\phi}(f)}\,. \tag{1.124}$$

Interactions 3a and 3b: $\overrightarrow{\mathbf{R}}\overset{>}{\mathbf{J}}$, $\overrightarrow{\mathbf{R}}\overset{<}{\mathbf{J}}$

For these types of interaction we have $f > 1$. Evaluating one finds that (1.124) will hold if and only if

$$2 + f^{\zeta}\left(\sqrt{c} - 1\right) > 0\,. \tag{1.125}$$

Thus, if $c > 1$ this inequality holds. Therefore, interactions of type 3.b no vacuum can form. On the other hand, interactions of type 3.a $c < 1$ and relation (1.125)

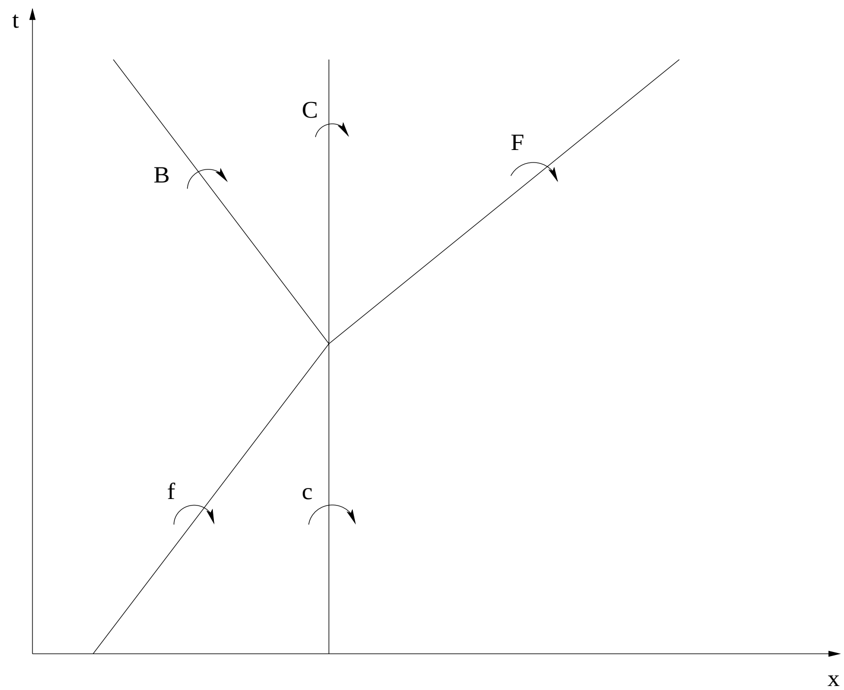

Figure 1.7. Incoming and outgoing parameters for interactions involving contacts (group 3).

is equivalent to

$$f^{\zeta} < \frac{2}{1-\sqrt{c}}\,. \tag{1.126}$$

Notice, for $c < 1$ fixed we may choose f large enough to violate this inequality and thus vacuum formation can occur for interactions of type 3.a. Albeit, we are able to find upper bounds on f which guarantees this does not occur. Using the fact that $c < 1$ we have

$$2 < \frac{2}{1-\sqrt{c}}\,.$$

Therefore if $f < 2^{\frac{1}{\zeta}}$ then (1.126) will hold. Using the definition of ζ in (1.52) we conclude that if $f < 4$ then $f < 2^{\frac{1}{\zeta}}$ and no vacuum can form.

Interactions 3.c and 3.d: $\overset{\rightharpoonup}{\mathbf{S}}\overset{>}{\mathbf{J}}$, $\overset{\rightharpoonup}{\mathbf{S}}\overset{<}{\mathbf{J}}$

For these interactions $f < 1$, and thus the left hand side of (1.124) is negative while the right hand side is positive. Therefore the can be no vacuums formed for interactions of types 3.c and 3.d.

In the analysis that follows we show that if the incoming forward wave is a shock (rarefaction) and interacts with a up contact then the reflected backward wave will be a rarefaction (shock). Similarly we find that if the incoming forward wave is a shock (rarefaction) and interacts with an down contact then the reflected backward wave is a shock (rarefaction). For the outgoing forward wave we show it will be a shock (rarefaction) if the incoming forward wave is a shock (rarefaction), independent of whether or not the incoming contact is an up contact or a down contact. In the analysis of the outgoing contacts we show that for interactions 3.a, 3.c and 3.b the outgoing contacts is the type as the incoming, while in interaction 3.b the contact can be either an up or down contact.

1.3.5.1 Outgoing backward waves.

Interactions 3a and 3b: $\overset{\rightharpoonup}{\mathbf{R}}\overset{>}{\mathbf{J}}$, $\overset{\rightharpoonup}{\mathbf{R}}\overset{<}{\mathbf{J}}$

Since $\mathcal{F}_3$ is increasing $B \lesseqgtr 1$ is equivalent to $\mathcal{F}_3(B) \lesseqgtr \mathcal{F}_3(1)$. Therefore, by (1.55), (1.122) and (1.123), $B \lesseqgtr 1$ if and only if

$$-\overset{\rightharpoonup}{\psi}(f) \lesseqgtr \sqrt{cf\overset{\rightharpoonup}{\phi}(f)}\,\overset{\leftharpoonup}{\psi}\left(\frac{1}{f}\right) = -\overset{\rightharpoonup}{\psi}(f)\sqrt{c}. \tag{1.127}$$

Since $f > 1$ we have $-\overset{\rightharpoonup}{\psi}(f) < 0$, and (1.127) holds if and only if

$$c \lesseqgtr 1. \tag{1.128}$$

Thus for interaction 3.a and 3.b, if the contact is a up contact the backward wave is a shock, and if the contact is a down contact then the backward wave is a rarefaction.

Interactions 3c and 3d: $\overset{\rightharpoonup}{\mathbf{S}}\overset{>}{\mathbf{J}}$, $\overset{\rightharpoonup}{\mathbf{S}}\overset{<}{\mathbf{J}}$

Since $f < 1$ we know that $-\overset{\rightharpoonup}{\psi}(f) > 0$, and from the monotonicity of $\mathcal{F}_3$ that $B \gtreqless 1$

if and only if

$$\mathcal{F}_3(B) \gtrless \mathcal{F}_3(1) \quad \Leftrightarrow \quad -\overset{\rightharpoonup}{\psi}(f) \gtrless \sqrt{cf\overset{\rightharpoonup}{\phi}(f)}\,\overset{\leftharpoonup}{\psi}(\frac{1}{f}).$$

From relation (1.55) this is equivalent to

$$-\overset{\rightharpoonup}{\psi}(f) \gtrless -\overset{\rightharpoonup}{\psi}(f)\sqrt{c} \quad \Leftrightarrow \quad c \lessgtr 1. \tag{1.129}$$

Thus, if the specific volume decreases across the contact then the backward outgoing wave is a shock, and if it increases across the contact then the backward outgoing wave is a rarefaction.

1.3.5.2 Outgoing forward waves.

Interactions 3a and 3b: $\overset{\rightharpoonup}{\mathbf{R}}\overset{>}{\mathbf{J}}$, $\overset{\rightharpoonup}{\mathbf{R}}\overset{<}{\mathbf{J}}$

Assuming no vacuum formation for interaction 3.a, we have $F = \frac{f}{B} > 1$ and that $F > 1$ if and only if $f > B$. From (1.122) and (1.123) and the fact that $\mathcal{F}_3$ is increasing we have that $f > B$ if and only if

$$\mathcal{F}_3(f) > \mathcal{F}_3(B) \quad \Leftrightarrow \quad \overset{\leftharpoonup}{\psi}(f) > -\overset{\rightharpoonup}{\psi}(f), \tag{1.130}$$

which holds since $\overset{\leftharpoonup}{\psi}(f) > 0$ and $-\overset{\rightharpoonup}{\psi}(f) < 0$. Thus $F > 1$ and the outgoing forward wave is a rarefaction. Also notice that since $BF = f$ we have for 3.a that $B < 1 < f < F$ and for 3.b $f > B, F$.

Interactions 3c and 3d: $\overset{\rightharpoonup}{\mathbf{S}}\overset{>}{\mathbf{J}}$, $\overset{\rightharpoonup}{\mathbf{S}}\overset{<}{\mathbf{J}}$

Again, we have $FB = f$ and therefore $F < 1$ if and only if $f < B$. Using the fact that $\mathcal{F}_3$ is strictly increasing we find that $f < B$ if and only if

$$\begin{aligned} & \mathcal{F}_3(f) < \mathcal{F}_3(B), \\ \Leftrightarrow \quad & \overset{\rightharpoonup}{\psi}(f) < -\overset{\rightharpoonup}{\psi}(f), \end{aligned}$$

which is true since $\overset{\rightharpoonup}{\psi}(f) < 0$. Thus the outgoing forward wave is a shock, $F < 1$.

Using $BF = f$ we also have that for 3.c $F < f < 1 < B$ and for 3.d $f < F, B$.

1.3.5.3 Outgoing contacts.

Interaction 3.a: $\overset{\rightharpoonup}{\mathbf{R}}\overset{>}{\mathbf{J}}$

If we assume no vacuum formation, then we may use $c, B < 1$ and $f, F > 1$ and evaluate (1.120) to find,

$$cf^{\frac{-1}{\gamma}} = C(BF)^{\frac{-1}{\gamma}} \quad \Leftrightarrow \quad c = C, \tag{1.131}$$

since $f = BF$ and therefore $C < 1$.

Interaction 3.b: $\overset{\rightharpoonup}{\mathbf{R}}\overset{<}{\mathbf{J}}$

To analyze when $C \gtrless 1$ we use $c, B > 1$ and that $f = BF > B > 1$. With this, then from (1.120) we find $C \gtrless 1$ if and only if

$$c\frac{(1+\mu^2 B)}{(B+\mu^2)B^{\frac{1}{\gamma}}} \gtrless 1\,. \tag{1.132}$$

Defining $H(x) = \frac{(1+\mu^2 x)}{(x+\mu^2)x^{\frac{1}{\gamma}}}$ then by lemma 1.2.2 we have that $H(x)$ is a decreasing function for $x > 0$. Also note that $H(1) = 1$ and as x approaches infinity $H(x)$ approaches zero. Therefore, for fixed $c^{-1} \in (0, 1)$ there exists a unique $\hat{B}$ such that

$$cH(\hat{B}) = 1\,.$$

With c fixed, if we choose $f < \hat{B}$ then $B < f < \hat{B}$ and therefore $cH(B) > 1$, or equivalently $C > 1$.

We now show that there exists $c, f > 1$ such that $cH(B) < 1$. In particular we will show is that we can choose $c, f > 1$ to make B so large that $cH(B) < 1$. To this end we define

$$G(x; f, c) := \mathcal{F}_3(x) + \overset{\rightharpoonup}{\psi}(f)\,,$$

where f and c are parameters. Notice that G is an increasing function in x. From

(1.123) we have $G(B; f, c) = 0$. With c fixed we consider values of f where $f > \hat{B}$. With this one finds

$$G(\hat{B}; f, c) = \overleftarrow{\psi}(\hat{B}) + \nu \left[\left(\sqrt{c}\hat{B}^{\zeta} - 1 \right) + f^{\zeta} \left(1 - \sqrt{c} \right) \right] .$$

Since $c > 1$ we have that as f increases $G(\hat{B}; f, c)$ decreases without bound. Thus with c fixed there exists an f such that the root of $G(x; f, c)$ is greater than $\hat{B}$. Since G is an increasing function in x and B is the root of $G(x; f, c)$ this proves there exists $c, f > 1$ such that $cH(B) < 1$.

Interaction 3.c and 3.d: $\overrightarrow{\mathbf{S}}\overset{>}{\mathbf{J}}$, $\overrightarrow{\mathbf{S}}\overset{<}{\mathbf{J}}$

The technique used to prove that the outgoing contact is the same type as the incoming is the same for both of these interactions. However, since we will be evaluating we will look at them individually.

For interaction 3.c we have $f, F, \frac{f}{B}, c < 1$ while $B > 1$. From (1.120) one concludes that $C < 1$ if and only if

$$\sqrt{c\overrightarrow{\phi}(f)} < \sqrt{\overleftarrow{\phi}(B)\overrightarrow{\phi}(F)} = \sqrt{\overleftarrow{\phi}(B)\overrightarrow{\phi}(\frac{f}{B})} . \tag{1.133}$$

From (1.121) one finds the left hand side to be

$$\sqrt{c\overrightarrow{\phi}(f)} = \frac{-\overrightarrow{\psi}(f) - \overleftarrow{\psi}(B)}{\kappa(B - f)} \sqrt{B + \mu^2 f} .$$

While evaluating the right hand side gives,

$$\sqrt{\overleftarrow{\phi}(B)\overrightarrow{\phi}(\frac{f}{B})} = \frac{\sqrt{B + \mu^2 f}\,\sqrt{1 + \mu^2 B}}{\sqrt{f + \mu^2 B}\,\sqrt{B + \mu^2}} .$$

Thus (1.133) will hold if and only if

$$-\overrightarrow{\psi}(f) - \overleftarrow{\psi}(B) < \frac{\kappa(B - f)\sqrt{1 + \mu^2 B}}{\sqrt{f + \mu^2 B}\,\sqrt{B + \mu^2}} .$$

Consider f fixed and let B vary taking values greater than or equal to one. Since

$\overleftarrow{\psi}$ is an increasing function one concludes that the left hand side is a decreasing function of B. A direct computation shows that the right hand side is an increasing function of B. Also note that that when $B = 1$ there is equality and thus for $B > 1$, (1.133) holds proving that the outgoing contact is a down-contact, $C < 1$.

Similiarly, for interaction 3.d we have $f, F, B, \frac{f}{B} < 1$ while $c > 1$, and that $C > 1$ if and only if

$$\sqrt{c\overrightarrow{\phi}(f)} < \sqrt{\overleftarrow{\phi}(B)\overrightarrow{\phi}(\frac{f}{B})}\,. \tag{1.134}$$

Using relation (1.121) and evaluating one finds that (1.134) is equivalent to

$$\kappa(1-f) + \nu(1-B^{\xi})\sqrt{f+\mu^2} > \frac{\kappa(B-f)\sqrt{f+\mu^2}}{B^{\frac{1}{2\gamma}}\sqrt{f+\mu^2 B}}\,.$$

Again, considering f fixed and letting B vary taking values greater than or equal to one we find by direct computation that the left hand side is decreasing and the right hand side is increasing as functions of B. One also finds that there is equality when $B = 1$. Thus, for $B < 1$ (1.134) holds and we conclude the the outgoing contact is an up-contact, $C > 1$.

1.3.5.4 Summary of group 3

We will summarize the findings of group 3 in tables. We begin with when vacuum formation is possible.

Interaction	Incoming waves	Vacuum
3.a	$\overrightarrow{R}\overset{>}{J}$	No vac. $\Leftrightarrow b^{\zeta}(1-\sqrt{c}) < 2$
3.b	$\overrightarrow{R}\overset{<}{J}$	No vac.
3.c	$\overrightarrow{S}\overset{>}{J}$	No vac.
3.d	$\overrightarrow{S}\overset{<}{J}$	No vac.

Table 1.7. When vacuum formation is possible (group 3).

The next table gives the incoming and outgoing waves. along with any applicable conditions for the outgoing waves.

Interaction	Incoming Waves	Outgoing Waves	Condition
3.a	$\overrightarrow{R}\overset{>}{J}$	$\overleftarrow{R}\overset{>}{J}\overrightarrow{R}$	NA
3.b	$\overrightarrow{R}\overset{<}{J}$	$\overleftarrow{S}\overset{\gtrless}{J}\overrightarrow{R}$	$\overset{\gtrless}{J} \Leftrightarrow B \gtrless \hat{B}$
3.c	$\overrightarrow{S}\overset{>}{J}$	$\overleftarrow{S}\overset{>}{J}\overrightarrow{S}$	NA
3.d	$\overrightarrow{S}\overset{<}{J}$	$\overleftarrow{R}\overset{<}{J}\overrightarrow{S}$	NA

Table 1.8. Incoming and outgoing waves (group 3).

Lastly, we write the relations we have found between the incoming and outgoing waves. We also include two results from Chapter 3 where we show that for interaction 3.c, $B < \frac{1}{f}$ and $c < C$.

Interaction	Incoming Waves	Relations of strengths
3.a	$\overrightarrow{R}\overset{>}{J}$	$B < 1 < f < F$
3.b	$\overrightarrow{R}\overset{<}{J}$	$F < f$
3.c	$\overrightarrow{S}\overset{>}{J}$	$F < f < 1 < B < \frac{1}{f}$, $c < C < 1$
3.d	$\overrightarrow{S}\overset{<}{J}$	$f < B$

Table 1.9. Relative strengths of incoming and outgoing waves (group 3).

1.3.6 No Vacuum Formation Criteria

We observe that in each of the three groups vacuum formation is possible for one of the interactions. Notice that in each case one of the incoming waves is a rarefaction. In each of these cases where a vacuum formation is possible we have found that if the incoming backward wave is a rarefaction and the strength of b is such that $b > \frac{1}{4}$ and/or if the incoming forward wave is a rarefaction and the strength of f is such that $f < 4$, then no vacuum can form. Thus we have a simple "universal" criterion (independent of data and system parameter γ) on the strengths of rarefactions which guarantees that no vacuum can form via interactions involving two waves.

Chapter 2

Case study for Large Data

2.1 Introduction

We will use the results of interaction 3.c, $\overset{\rightarrow>}{SJ}$, from the previous chapter to create a specific interaction pattern. There will be a slight change in notation of incoming and outgoing waves. This is due to the fact that we will be interested in infinitely many interactions and thus in infinitely many incoming and outgoing waves. The notation will be defined below. We will also find further relations on incoming and outgoing waves that where not found, but mentioned, for interactions of type 3.c.

Consider again the one-dimensional Euler system in Lagrangian coordinates:

$$\tau_t - u_x = 0 \tag{2.1}$$

$$u_t + p_x = 0 \tag{2.2}$$

$$E_t + (pu)_x = 0\,. \tag{2.3}$$

For initial data with sufficiently small total variation Glimm's theorem [19] applies and guarantees existence of a global-in-time weak, entropy admissible solution. The solution can be constructed by various methods: Glimm's original random choice method [19], Liu's deterministic version [23], the wave-front tracking schemes of DiPerna [13], Bressan [5], and Risebro [31], and the more recent wave tracing methods of Bianchini and Bressan [4], [2], [3].

The Euler system plays a distinguished role among hyperbolic systems and much effort has been invested in extending Glimm's existence result to larger classes of data. For "generic" data the solutions are exceedingly complicated with a myriad of interactions and complicated wave patterns.

2.1.1 Related results

For the 2×2-system of isothermal gas dynamics, i.e. (2.1) and (2.2) with $p = p(\tau) = a^2/\tau$ (disregarding (2.3)), Nishida [26] established global existence for any bounded BV_{loc} data. The key property of this system is that its wave curves are invariant under translations; see [34] for a generalization to 2×2-systems with this property. Nishida's result initiated a series of related results for more general 2×2-systems, [1], [12], [27]. For example, for isentropic gas dynamics, i.e. (2.1) and (2.2) with $p = p(\tau) = a^2/\tau^\gamma$, with $\gamma > 1$ (disregarding (2.3)), Nishida and Smoller [27] established existence for possibly large data that satisfy

$$(\gamma - 1) \cdot (\text{Total variation of data}) \qquad \text{is sufficiently small.} \tag{2.4}$$

The same authors extended this type of results to the mixed initial-boundary value problem for isentropic gas dynamics [28]. In particular they treated the case of a "double-piston" and provided conditions on the data ((2.4) together with strict separation of the pistons) that guarantee existence of global weak solutions.

Liu [24], [25] provided a significant extension of these results to the full Euler system (2.1)-(2.3). For the Cauchy problem for an ideal polytropic gas with equation of state

$$p = p(v, S) = a^2 v^{-\gamma} \exp\left[(\gamma - 1)S/R\right] ,$$

where S denotes entropy, and with $\gamma \in (1, 5/3]$, existence of a global solution is shown [24] under the condition that

$$(\gamma - 1) \cdot \left[\text{T.V.}u(x,0) + \text{T.V.}\tau(x,0) + \text{T.V.}S(x,0)\right] \qquad \text{is sufficiently small.} \tag{2.5}$$

In [25] Liu extended this result to the initial-boundary problem and he also showed that for the initial value problem, (2.5) can be replaced by the weaker condition

$$(\gamma - 1) \cdot \left[\text{T.V.}u(x,0) + \text{T.V.}p(x,0)\right] \qquad \text{is sufficiently small.} \tag{2.6}$$

This last result was extended to general $\gamma > 1$ by Peng [29]. Temple [36] considered more general 1-parameter families of gas dynamical equations of state $e_\varepsilon(\tau, S)$ with the property that

$$e_0(\tau, S) = -\ln \tau + S/R\,.$$

A generalization of Liu's result for the Cauchy problem was established through a careful analysis where also the role of ε is recorded.

Temple and Young [37] have established existence for the full Euler system (and more generally for 3×3-systems with a Riemann invariant) up to an arbitrary time for data with large total variation and sufficiently small sup-norm. In an interesting recent work Edens [14] has considered linearly degenerate models for gas dynamics (introduced earlier by Temple and Young) where the interaction coefficients have the same signs as for compressible Euler. Special solutions are constructed where waves interact at regularly spaced grid points. An analysis of the spectral properties of the map which updates the strengths of 1 and 3 waves at interactions demonstrates decay of total variation.

We finally mention the result of Rykov [33] who considered general BV data. Provided one has an a priori bound on the L^∞ norm of approximate solutions (generated by Glimm's scheme, say), and provided the approximations stay bounded away from vacuum, then the total variation of the approximations remains uniformly bounded.

Given the general form of Glimm's theorem it is natural to consider other systems of conservation laws and to investigate various large data regimes for these. However, recent examples show that no result similar to Glimm's theorem can hold at this level of generality. In particular [20], [21], [38] give examples of special (unphysical) systems of conservation laws whose solutions may suffer blowup of amplitude and/or total variation. It is not known if similar pathological behavior can occur for physical systems. Such systems are typically equipped with a convex entropy and one could hope that their additional structure would prevent this type of breakdown. The most interesting system in this respect is the Euler system.

The blowup examples in [20], [38] were based on particular interaction patterns where repeated reflections create infinitely many waves in finite time. By judiciously choosing initial data one can generate solutions where wave strengths are amplified at each interaction, leading to finite time blowup.

In this chapter we construct special solutions of the full Euler system with a particular interaction pattern of the same type as in [20], [38]. We then investigate if such solutions can produce growth, or even finite time blowup, of amplitudes or variation. We consider the case of an ideal and polytropic gas where shocks undergo repeated reflections as they interact with two approaching contact waves. Our analysis shows that no blowup occurs for solutions of this type, at least when the transmitted shock waves are absorbed and thus prevented from interfering with the basic pattern of shocks and contacts, see Figure 2.1. Imposing absorbing boundary conditions outside the central region amounts to disregarding interactions of shocks in the same family and renders the analysis tractable.

We begin by setting up the interaction pattern and deriving some of its properties in sections 2.2 and 2.2.1. The main part of the analysis is concerned with deriving a bound for the pressure in section 2.2.3. Once this is done it is easy to show that the solution is globally defined and remains uniformly bounded. In section 2.2.4 we comment briefly on a particular scaling of the dependent variables in the Euler system which provides a (weak) large data result for certain types of "scaled up" data.

2.2 Shock-contact interaction pattern

We will considering ideal polytropic gasses where the equation of state is

$$e = \frac{\tau p}{\gamma - 1}\,, \qquad \gamma > 1\,. \tag{2.7}$$

In our study of interactions in chapter 1 we found that an interaction of a backward shock with an up-contact results in a transmitted backward shock, an up-contact, and a reflected forward shock:

$$\overset{<}{J}\overset{\leftarrow}{S} \quad \mapsto \quad \overset{\leftarrow}{S}\overset{<}{J}\vec{S}\,. \tag{2.8}$$

Similarly,

$$\vec{S}\overset{>}{J} \quad \mapsto \quad \overset{\leftarrow}{S}\overset{>}{J}\vec{S}\,. \tag{2.9}$$

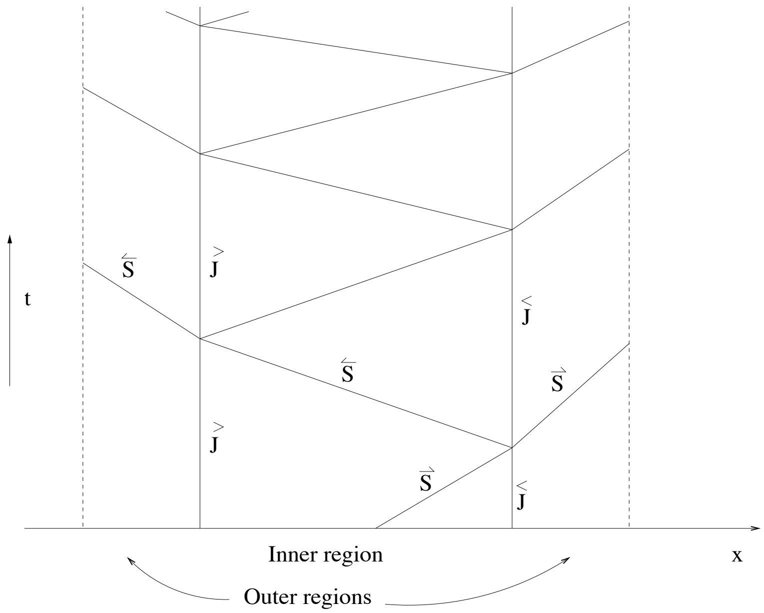

Figure 2.1. Interaction pattern with repeated reflections.

These interactions will be used to set up an interaction pattern where reflected shocks traverse an inner region between two contacts in which the density is lower than in the outside regions on the left and right. See Figure 2.1. As a consequence of Lax' entropy inequalities the transmitted shocks in the outer regions will approach each other. In an interaction between two shocks of the same family the type (shock/rarefaction) of the reflected wave depends on the value of γ, and may also depend on their relative strengths. In either case the wave reflected from these same-family interactions would interact either with the "next" transmitted shock and/or with a contact. The resulting pattern quickly becomes complicated and does not seem amenable to a direct analysis. In order to render the time evolution computationally tractable we avoid this issue altogether by inserting absorbing boundaries in the outer regions (along the dotted lines in Figure 2.1). While it is clear from (2.8) and (2.9) that the resulting pattern (now with boundary condition imposed) will only involve forward and backward shocks together with the two contacts, it is not immediately clear that it can be continued indefinitely in time. Specifically we need to show that the speeds of the shock waves in the solution do not increase without bound and that the flow variables remain bounded. As we shall see they are all uniformly bounded in time. A similar bound on the speeds

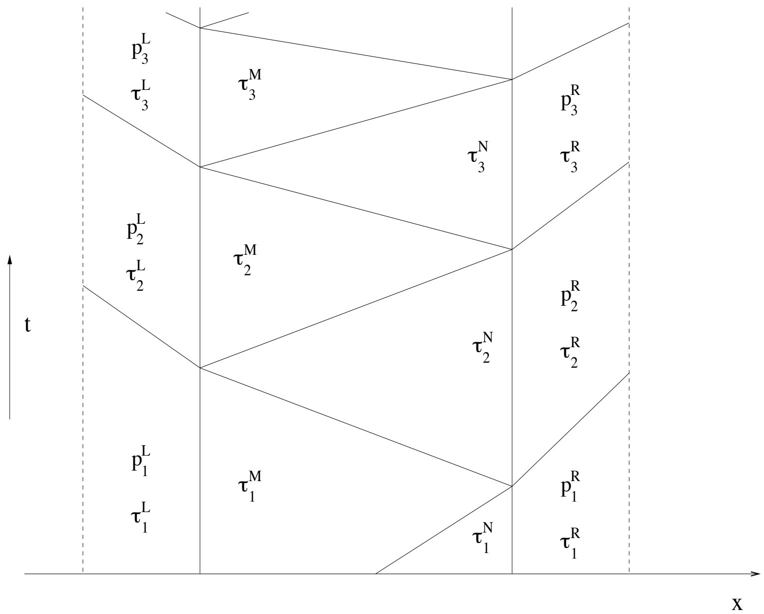

Figure 2.2. Interaction pattern with states resulting from interactions.

of the transmitted shocks will show that we can impose the boundary conditions at two fixed locations (in Lagrangian coordinates), as indicated in Figure 2.1.

In the next subsection we derive some general properties on the densities, pressures and velocities in the outer and inner regions. We then show that, provided the pressures in the three regions remain uniformly bounded, then shocks speeds, particle velocities, densities, as well as specific volumes, all remain uniformly bounded. In the final step we show that pressure is indeed uniformly bounded.

2.2.1 Properties of the interaction pattern

General porperties. We consider ideal polytropic g In what follows we use superscripts L and R to denote states in the outer left and right regions, respectively. For the inner region we use superscripts M and N; the various constant states resulting from interactions are indexed as in Figure 2.2.

Since shocks are compressive (i.e. a particle experiences an increase in pressure and density as it crosses a shock, [11]) we obtain the following series of inequalities:

$$\tau_1^L > \tau_2^L > \cdots , \tag{2.10}$$

$$\tau_1^R > \tau_2^R > \cdots , \tag{2.11}$$

$$\tau_1^N > \tau_1^M > \tau_2^N > \tau_2^M > \cdots \,. \tag{2.12}$$

Each sequence is decreasing and bounded below, and thus convergent. Furthermore, the density in each region is bounded away from vacuum. Compression gives that

$$p_1^R < p_1^L < p_2^R < p_2^L < \cdots \,. \tag{2.13}$$

As velocity and pressure do not change across contacts we have

$$u_i^L = u_i^M \,, \qquad p_i^L = p_i^M \,, \qquad u_i^R = u_i^N \,, \qquad p_i^R = p_i^N \,.$$

By (2.13) and (1.43) we have

$$u_{i+1}^L - u_i^L = -(p_{i+1}^L - p_i^L)\sqrt{\frac{(1-\mu^2)\tau_i^L}{p_{i+1}^L + \mu^2 p_i^L}} < 0 \,,$$

such that

$$u_1^L > u_2^L > \cdots \,.$$

Similarly one finds for the outer right region and for the inner region that

$$u_1^R < u_2^R < \cdots \,, \quad \text{and} \quad u_i^R < u_i^L \quad \forall i, \quad \text{and in fact} \quad u_i^R < u_j^L \qquad \forall i,j \,. \tag{2.14}$$

It follows that $\{u_i^L\}$ is a decreasing and bounded sequence, while $\{u_i^R\}$ is an increasing and bounded sequence. Thus they both converge.

Next we show that the temperature, $\theta = K^{-1}p\tau$, increases in each region. Consider a shock in any of the three regions and let p_i, τ_i, θ_i and p_{i+1}, τ_{i+1}, θ_{i+1} be the pressure, specific volume, and temperature of a fluid particle before and after passing through the shock, respectively. The relation between pressures and specific volumes is given by (1.42). The relationship is symmetric in p and p_0 and thus takes the same form for forward and for backward shocks. Since $p_{i+1} > p_i$, equation (1.42) yields

$$\theta_{i+1} = K^{-1}p_{i+1}\tau_{i+1} \quad = \quad K^{-1}p_{i+1}\frac{\tau_i(p_i + \mu^2 p_{i+1})}{(p_{i+1} + \mu^2 p_i)}$$

$$> \quad K^{-1} p_{i+1} \frac{\tau_i(p_i + \mu^2 p_i)}{(p_{i+1} + \mu^2 p_{i+1})} = K^{-1} p_i \tau_i = \theta_i \,. \quad (2.15)$$

2.2.2 Properties assuming pressure is bounded.

We will now *assume* that pressure is uniformly bounded above. By (2.13) this implies that the pressure in each region converges to a common value $\bar{p}$:

$$\bar{p} := \lim_{i\to\infty} p_i^L = \lim_{i\to\infty} p_i^R \,. \quad (2.16)$$

From above we know that the velocities in the right and left region are converging to finite values. Expression (1.43) and assuming pressure is bonded gives that the particle velocities in each region are converging to the same velocity,

$$\lim_{i\to\infty} (u_i^L - u_i^R) = \lim_{i\to\infty} (p_i^L - p_i^R) \sqrt{\frac{(1-\mu^2)\tau_i^N}{p_i^L + \mu^2 p_i^R}} = 0 \,. \quad (2.17)$$

We proceed to use the bound on pressure to bound the other flow variables in the three regions. For concreteness consider the outer left region. From (2.15), and using that τ_i^L is a convergent sequence by (2.10), we find that

$$0 \; < \; p_1^L \tau_1^L \; < \; \lim_{i\to\infty} \left(p_i^L \tau_i^L\right) \; = \; \bar{p} \cdot \left(\lim_{i\to\infty} \tau_i^L \right) , \quad (2.18)$$

which shows that τ_i^L converges to a finite positive value. A similar analysis applies in the other regions. Therefore there is a $\bar{\tau} > 0$ such that

$$\tau_i^L,\, \tau_i^R,\, \tau_i^N,\, \tau_i^M > \bar{\tau} \qquad \forall i \,. \quad (2.19)$$

Next consider the speed of shock waves in any of the three regions. Let p_i, τ_i and p_{i+1}, τ_{i+1} be the pressure and specific volume of a fluid particle before and after passing through a shock, respectively. By (1.34), (1.35) and (1.42) the shock speed σ_i is given by

$$|\sigma_i| = \sqrt{\frac{p_{i+1} - p_i}{\tau_i - \tau_{i+1}}} = \sqrt{\frac{p_{i+1} + \mu^2 p_i}{\tau_i (1-\mu^2)}} \,.$$

Since $p_i\,, p_{i+1} < \bar{p}$ and $\tau_i > \bar{\tau}$ we have

$$|\sigma_i| < \sqrt{\frac{\bar{p} + \mu^2 \bar{p}}{\bar{\tau}(1-\mu^2)}} = \sqrt{\frac{\bar{p}\gamma}{\bar{\tau}}} \qquad \forall i\,. \tag{2.20}$$

This shows that all shock speeds are uniformly bounded. Therefore, on any finite time interval there is only a finite number of shock-contact interactions. It follows that the interaction pattern exists for all times, and that p, τ and u converge to finite values in all regions. Notice that the specific volume is uniformly bounded away from 0 as well as away from $+\infty$ (i.e., no vacuum occurs). As shock speeds are bounded it also follows that the absorbing boundaries in the outer regions may be prescribed at fixed locations (in Lagrangian coordinates).

2.2.3 Boundedness of pressure

Parametrization of shock curves.

To investigate how the pressure increases we will use the parametrization of the wave curves in (1.49)-(1.51). Here the shocks are parametrized by pressure ratios p_+/p_-, where p_+, p_- denote pressure on the right and on the left, respectively. Similarly, contacts are parametrized by the ratio of specific volumes. For a forward shock we denote the pressure ratio by f, and for a backward shock we denote the pressure ratio by b. Since fuid particles experience an increase in pressure when crossing a shock we have $b > 1$ and $f < 1$. For contacts we denote the specific volume ratio by c. Since there are no rarefactions to consider the harpoons over the ϕ's and ψ's may be dropped. This provides the following parametrization of shock-curves and contact-curves in (τ, u, p)-space centered at (τ_0, u_0, p_0) (see [35], [7]):

$$\overleftharpoon{S}(b;\tau_0,u_0,p_0) = \begin{pmatrix} \phi(b)\tau_0 \\ u_0 - \psi(b)\sqrt{\tau_0 p_0} \\ bp_0 \end{pmatrix}, \qquad b > 1\,, \tag{2.21}$$

$$\overset{\lessgtr}{J}(c;\tau_0,u_0,p_0) = \begin{pmatrix} c\tau_0 \\ u_0 \\ p_0 \end{pmatrix}, \qquad 1 \lessgtr c\,, \tag{2.22}$$

$$\vec{S}(f;\tau_0,u_0,p_0) = \begin{pmatrix} \phi(f)\tau_0 \\ u_0 + \psi(f)\sqrt{\tau_0 p_0} \\ f p_0 \end{pmatrix}, \qquad f < 1\,, \tag{2.23}$$

where

$$\phi(\alpha) = \frac{1+\mu^2\alpha}{\alpha+\mu^2}\,, \qquad \psi(\alpha) = \frac{\kappa(\alpha-1)}{\sqrt{\alpha+\mu^2}}\,, \qquad \kappa = \sqrt{1-\mu^2}\,. \tag{2.24}$$

As above, note that the function $\psi(\alpha)$ is strictly increasing (see Figure 1.2) and satisfies the relation

$$\sqrt{\alpha\phi(\alpha)}\psi\left(\frac{1}{\alpha}\right) = -\psi(\alpha)\,. \tag{2.25}$$

Single interactions

Consider the Riemann problem that occurs when a forward (backward) shock interacts with a down-contact (up-contact). Let the outgoing forward and backward shocks have pressure ratios F and B, respectively, and let C denote the ratio of specific volumes across the outgoing contact. See figure 2.3. From (2.8) and (2.9) we thus have $F < 1$, $B > 1$, and $C \gtreqless 1$ if and only if $c \gtreqless 1$. Traversing the various waves before and after interaction, and using (2.21)-(2.23), yields the following relationships between incoming and outgoing ratios:

$$c\phi(f) = C\phi(B)\phi(F) \tag{2.26}$$

$$\psi(f) = \psi(F)\sqrt{CB\phi(B)} - \psi(B) \tag{2.27}$$

$$f = BF\,. \tag{2.28}$$

Eliminating F and C, and using (2.25), one obtains the following equation for the pressure ratio across the reflected (backward, outgoing) wave:

$$\psi(B) + \sqrt{cf\phi(f)}\psi\left(\frac{B}{f}\right) + \psi(f) = 0\,. \tag{2.29}$$

Define $\mathcal{F}(\alpha; f, c)$ to be the expression on the left-hand side of (2.29):

$$\mathcal{F}(\alpha; f, c) := \psi(\alpha) + \sqrt{cf\phi(f)}\,\psi\left(\frac{\alpha}{f}\right) + \psi(f)\,. \tag{2.30}$$

Since ψ is an increasing function it follows that $\alpha \mapsto \mathcal{F}(\alpha; f, c)$ is an increasing function for fixed values of f and c. As $F(0; f, c) < 0$ it follows that (2.29) has a unique root B. We proceed to estimate B in terms of the incoming waves. Consider

$$\mathcal{F}\left(\frac{1}{f}; f, c\right) = \psi\left(\frac{1}{f}\right) + \sqrt{cf\phi(f)}\,\psi\left(\frac{1}{f^2}\right) + \psi(f)\,. \tag{2.31}$$

Since $f < 1$ we have $\psi\left(\frac{1}{f^2}\right) > 0$, while (2.24) and (2.25) shows that

$$\psi\left(\frac{1}{f}\right) + \psi(f) = \psi\left(\frac{1}{f}\right)\left[1 - \sqrt{f\phi(f)}\right] > 0\,, \tag{2.32}$$

giving

$$\mathcal{F}(\frac{1}{f}; f, c) > 0\,. \tag{2.33}$$

Since $\mathcal{F}(\cdot; f, c)$ is an increasing function with a unique zero B we conclude that

$$B < \frac{1}{f}\,. \tag{2.34}$$

Next consider the ratio

$$\frac{c}{C} = \frac{\phi(B)\phi(F)}{\phi(f)}\,.$$

Evaluating the ϕ's and using that $B = \frac{f}{F}$ we find that $c < C$ if and only if

$$(F + \mu^2 f)(1 + \mu^2 F)(f + \mu^2) < (f + \mu^2 F)(1 + \mu^2 f)(F + \mu^2)\,.$$

Expanding the products and using the fact that $F < f < 1$ (since $B > 1$) one finds that this inequality holds, giving

$$c < C < 1\,.$$

Next consider a backward shock interacting with an up-contact. This is exactly the mirror image of the previous case. Denote the incoming backward shock strength by b', incoming contact by c', and the outgoing strengths by B', C', F'. Then, with $f = 1/b'$, $c = 1/c'$, $B = 1/F'$, $C = 1/C'$, and $F = 1/B'$, we have the same result

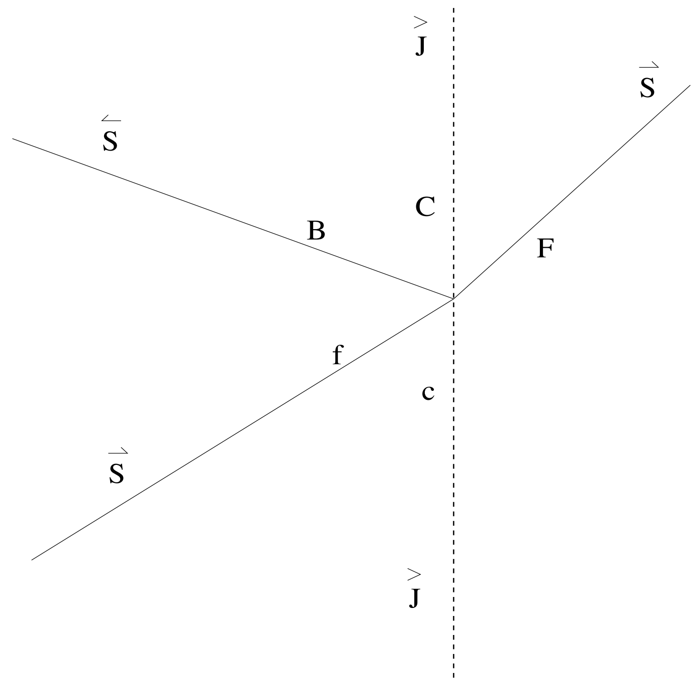

Figure 2.3. Interaction pattern with states resulting from interactions.

as above. This gives,

$$\frac{1}{F'} < b' \,, \qquad c' > C' > 1 \,.$$

2.2.3.1 Absorbing boundaries

A similar analysis of shocks interacting with contact discontinuities between gasses with different adiabatic exponents γ shows the following. Consider a forward shock interacting with a contact discontinuity with strengths f and c, respectively. On the left of the contact let the adiabatic exponent be denoted by γ_0. Then it can be shown that for there to be no reflected wave the adiabatic exponent on the right of the contact, γ_1, is given by

$$\gamma_1 = c\gamma_0 + (c-1)\frac{1-f}{1+f} \,. \tag{2.35}$$

Notice, since $\gamma_1 > 1$ we must have

$$1 < c\gamma_0 + (c-1)\frac{1-f}{1+f}\,,$$

which is equivalent to

$$c > \frac{2}{(\gamma_0+1)+f(\gamma_0-1)}\,,$$

where the right hand side is less than one. Therefore, for given values of γ_0 and f, in order for there to be no backward outgoing wave from the interaction we must choose $c \in \left(\frac{2}{(\gamma_0+1)+f(\gamma_0-1)}\,, 1\right) \bigcup (1, \infty)$. With c and f now fixed, (2.35) defines γ_1.

We also find that the outgoing strength of the contact, C, is given by

$$C = \frac{c\gamma_0(1+f)-(2-c)(1-f)}{\gamma_0(1+f)-(1-f)}\,.$$

Notice, $C \gtrless 1$ if and only if $c \gtrless 1$.

Since the pressure does not change across contact discontinuities and there is no reflected backward wave we have that the outgoing forward wave is a shock.

We now construct the absorbing right boundary as follows. Let γ_0 denote the adiabatic exponent in the region between the left and right boundaries and define the pressure ratios in the right region by

$$F_i := \frac{p_i^L}{p_{i+1}^L}\,.$$

The strength of the contact at the right boundary we will choose to always have strength $C > 1$.

Before the first forward shock interacts with the boundary we will choose the adiabatic exponent, γ_1, on the right of the boundary to be

$$\gamma_1 = C\gamma_0 + (C-1)\frac{1-F_1}{1+F_1}\,,$$

and the specific volume, τ_1, to be

$$\tau_1 = C\tau_1^R \,.$$

By construction this will absorb the incoming forward shock. Immediately after the interaction we will change the adiabatic exponent, γ_2, on the right of the boundary to be

$$\gamma_2 = C\gamma_0 + (C-1)\frac{1-F_2}{1+F_2}\,,$$

and the specific volume, τ_2, to be

$$\tau_2 = C\tau_2^R \,.$$

This again will absorb the incoming forward shock and we may continue this process. Notice that the γ_i's are bounded between γ_0 and $C(\gamma_0 - 1)$.

A similar analysis holds for the left boundary.

2.2.3.2 Interaction pattern

We proceed to use the estimates derived above to bound the pressure that results from multiple reflections. We define the pressure ratios for the inner region and specific volume ratios across contacts by

$$f_i := \frac{p_i^R}{p_i^L}\,, \qquad b_i := \frac{p_{i+1}^R}{p_i^L}\,, \qquad c_i^L := \frac{\tau_i^M}{\tau_i^L}\,, \qquad c_i^R := \frac{\tau_i^R}{\tau_i^N}\,,$$

see Figure 2.2. Using the above observations we have the monotone sequences

$$\frac{1}{f_1} > b_1 > \frac{1}{f_2} > b_2 \cdots > 1\,, \tag{2.36}$$

$$c_1^R < c_2^R < \cdots < 1\,, \tag{2.37}$$

$$c_1^L > c_2^L > \cdots > 1\,. \tag{2.38}$$

For reference we state the following elementary lemma which will be used to derive a uniform bound on the pressures.

Lemma 2.2.1. *Let* $\Omega = (0, \alpha_1) \times (0, \infty)$ *and assume* $G \in C^1(\bar{\Omega})$ *is such that* *(i)* $G(\alpha, 1) > 0$ *for all* $\alpha \in (0, \alpha_1)$,

(ii) $G(0, \beta) = 0$ *for all* $\beta > 0$,

(iii) $\partial_\alpha G(0, 1) > 0$.

Then there exists a $\delta > 0$ *such that* $G(\alpha, \beta) > 0$ *for all* $(\alpha, \beta) \in (0, \alpha_1) \times [1-\delta, 1+\delta]$.

Consider first the interactions of forward shocks with down contacts and set

$$x_i := \frac{1}{f_i} - 1\,.$$

From (2.36) we have

$$x_1 > x_2 > \cdots > 0. \tag{2.39}$$

Now define the function

$$G(x, y) := \mathcal{F}(1 + xy; \frac{1}{1+x}, c_1^R)\,, \qquad \text{for } x \in [0, x_1] \text{ and } y > 0, \tag{2.40}$$

which satisfies the assumptions of Lemma 2.2.1:

$$G(x, 1) = \mathcal{F}(1 + x; \frac{1}{1+x}, c_1^R) > 0 \qquad \forall x \in (0, x_1]\,, \tag{2.41}$$

$$G(0, y) = \mathcal{F}(1; 1, c_1^R) = 0 \qquad \forall y > 0\,, \tag{2.42}$$

$$\partial_x G(0, 1) = \frac{2\kappa c_1^R}{\sqrt{1+\mu^2}} > 0\,. \tag{2.43}$$

(Relation (2.41) follows from (2.33), while (2.42) and (2.43) follow by direct calculations.) It follows that there exists a $\delta_1 > 0$ such that

$$G(x, y) = \mathcal{F}(1 + xy; \frac{1}{1+x}, c_1^R) > 0\,,$$

for all $(x, y) \in (0, x_1] \times [1 - \delta_1, 1 + \delta_1]$. (Note that δ_1 depends on f_1 and c_1^R).

Now consider the ith interaction of a forward shock and down-contact in Figure 2.2. The parameters of the incoming shock and contact are $x_i = 1/f_i - 1$ and c_i, respectively. By (2.36) and (2.37) we have $x_i \in (0, x_1)$ and $c_1^R < c_i^R < 1$. We also observe from the definition of $\mathcal{F}$ in (2.30) that $c \mapsto \mathcal{F}(\alpha; f, c)$ is increasing in c for fixed α and f. It follows that

$$\mathcal{F}(1+x_i y; \frac{1}{1+x_i}, c_i^R) \ \geq \ \mathcal{F}(1+x_i y; \frac{1}{1+x_i}, c_1^R) \ = \ G(x_i, y) \ > \ 0\,, \tag{2.44}$$

for $y \in [1-\delta_1, 1+\delta_1]$. Since $\alpha \mapsto \mathcal{F}(\alpha; \frac{1}{1+x_i}, c_i^R)$ is increasing, and since $\mathcal{F}(b_i; \frac{1}{1+x_i}, c_i^R) = 0$ by construction, it follows from (2.44) that

$$b_i < 1 + x_i y \qquad \text{for } y \in [1-\delta_1, 1+\delta_1]. \tag{2.45}$$

Next consider interactions of backward shocks with up-contacts and set

$$\hat{x}_i := b_i - 1 > 0.$$

From (2.36) we have

$$\hat{x}_1 > \hat{x}_2 > \cdots > 0\,, \tag{2.46}$$

and we define

$$H(x,y) := \mathcal{F}(1+xy; \frac{1}{1+x}, \frac{1}{c_1^L})\,,$$

where $x \in [0, \hat{x}_1]$ and $y > 0$. Repeating the argument above shows that there is a $\delta_2 > 0$ such that

$$\frac{1}{f_{i+1}} < 1 + \hat{x}_i y, \tag{2.47}$$

for all $y \in [1-\delta_2, 1+\delta_2]$. Note that δ_2 depends on c_1^L and b_1, while b_1 depends on f_1 and c_1^R (see (2.29)). Thus δ_2 depends on c_1^L, c_1^R and f_1.

We can now give a uniform bound on the pressure in the constructed solution. For clarity we repeat the relevant definitions and properties from above:

$$f_i := \frac{p_i^R}{p_i^L}\,, \qquad b_i := \frac{p_{i+1}^R}{p_i^L}\,, \qquad x_i := \frac{1}{f_i} - 1\,, \qquad \hat{x}_i := b_i - 1\,,$$

$$b_i < 1 + x_i y \quad \text{for all } y \in [1-\delta_1, 1+\delta_1]\,, \tag{2.48}$$

$$\frac{1}{f_{i+1}} < 1 + \hat{x}_i y \quad \text{for all } y \in [1-\delta_2, 1+\delta_2]\,. \tag{2.49}$$

Set $\delta := \min(\delta_1, \delta_2)$ and let $\bar{y} := 1 - \delta$. Using the above definitions and inequalities one inductively finds that

$$\begin{aligned}
1+x_1 &= \frac{1}{f_1}\,, \\
1+\hat{x}_1 &= b_1 \overset{(2.48)}{<} 1+x_1\bar{y}\,, \qquad (2.50)\\
1+x_2 &= \frac{1}{f_2} \overset{(2.49)}{<} 1+\hat{x}_1\bar{y} \overset{(2.50)}{<} 1+x_1\bar{y}^2\,, \qquad (2.51)\\
1+\hat{x}_2 &= b_2 \overset{(2.48)}{<} 1+x_2\bar{y} \overset{(2.51)}{<} 1+x_1\bar{y}^3\,, \\
&\vdots \\
1+x_n &= \frac{1}{f_n} \overset{(2.49)}{<} 1+\hat{x}_{n-1}\bar{y} < 1+x_1\bar{y}^{2n-2}\,, \\
1+\hat{x}_n &= b_n \overset{(2.48)}{<} 1+x_n\bar{y} < 1+x_1\bar{y}^{2n-1}\,,
\end{aligned}$$

etc. We therefore have that

$$\begin{aligned}
p^R_{n+1} &= p^R_1 \frac{1}{f_1} b_1 \cdots \frac{1}{f_n} b_n \\
&= p^R_1 (1+x_1)(1+\hat{x}_1)\cdots(1+x_n)(1+\hat{x}_n) \\
&< p^R_1 (1+x_1)(1+x_1\bar{y})\cdots(1+x_1\bar{y}^{2n-2})(1+x_1\bar{y}^{2n-1}) \\
&= p^R_1 \prod_{m=0}^{2n-1}(1+x_1\bar{y}^m)\,.
\end{aligned}$$

Since $\bar{y} \in (0,1)$, $\prod_{m=0}^{\infty}(1+x_1\bar{y}^m)$ converges. This shows that the pressure in the solutions constructed above remains uniformly bounded in time.

We summarize our findings as follows:

Proposition 2.2.2. *Given the Euler system (2.1), (2.2), (2.3) describing one-dimensional, compressible flow of an ideal polytropic gas (2.7). Consider any Cauchy problem where the data produce two contact waves, together with an admissible shock between them, and where the density is lower in the region between the contacts than in the outer regions; see Figure 2.1. Then, by imposing absorbing boundary conditions at fixed locations (in Lagrangian coordinates and depending on the data) to the left and to the right of the two contacts, the resulting flow is defined for all times and consist of only shocks and contacts. Furthermore, the pressure, density, particle velocities and shock speeds are all uniformly bounded in* *time, the bounds depending on the initial data and the given parameters* γ, K.

2.2.4 Scaling invariance of the Euler system

In the analysis above we found it useful to parametrize wave curves by ratios of pressures and specific volumes. One reason to consider this choice for measuring wave strengths is the fact that the Euler system is formally invariant under the scaling

$$(\tau, u, p, E) \mapsto (\zeta\tau, \zeta u, \zeta p, \zeta^2 E) \qquad \zeta > 0\,,$$

which preserves such ratios. While the four variables above are not independent, this scaling is consistent for ideal polytropic gases. More generally, the Euler system is invariant under the scaling $(\tau, u, p) \mapsto (\zeta\tau, \zeta u, \zeta p)$ provided the internal energy function $e(\tau, p)$ satisfies

$$e(\zeta\tau, \zeta p) = \zeta^2 e(\tau, p)\,. \tag{2.52}$$

This clearly holds for ideal polytropic gases (2.7). A direct calculation shows that if (2.52) holds, then the characteristic speeds λ_0, $\lambda_\pm$, as well as shock speeds, are invariant as well.

It follows that if $U(x,t) = \big(\tau(x,t),\, u(x,t),\, p(x,t)\big)$ is any (weak or classical) solution of the Euler system (2.1), (2.2), (2.3) with initial data $U_0(x) = \big(\tau_0(x),\, u_0(x),\, p_0(x)\big)$, then $V(x,t) := \zeta U(x,t)$ $(\zeta > 0)$ is again a solution with data $V_0(x) = \zeta U_0(x)$. Furthermore, the wave patterns and interactions are identical in the two solutions, and it is clear that $V(x,t)$ is an entropy admissible weak solution if and only if the same holds for $U(x,t)$. As a consequence we see that once we have a solution to the Euler system (with an equation of state satisfying (2.52)), then we can obtain other solutions with arbitrarily large (or small) amplitudes, and variations, by applying the scaling above.

As a final observation we note that it is natural to try and argue the other way as well. That is: given large BV data for the Euler system, we can scale these down (ζ small) to obtain BV data with small variation; then apply Glimm's theorem to obtain a global-in-time weak solution; and finally scale up again (by $1/\zeta$). However, this argument is flawed as the amount of "allowable variation" of the data in Glimm's theorem depend on where the data are located in (τ, u, p)-space. In particular, as the set where the data lie shrinks towards the origin in (τ, u, p)-space under the down-scaling, so does the amount of variation allowed by Glimm's theorem. Thus there is no guarantee that Glimm's theorem applies to the scaled-down data.

Chapter 3

Symmetric shocks in stationary barotropic flow on a bounded domain

3.1 Introduction

We will now be considering the multi-D Euler system with a general equations of state. However, we are only interested in solutions with cylindrical or spherical symmetry which will reduce the problem to a system of ODEs.

Consider the domain between two concentric spheres $r = a$ and $r = b$, where $a < b$, and imagine that a compressible fluid is injected with a prescribed constant density ρ_a and constant radial velocity u_a at the inner boundary $r = a$. Depending on how fast the fluid is allowed to exit at the outer boundary, fluid may or may not pile up in the interior and a shock may or may not form. Similarly, one can consider the case where fluid is injected radially at the outer boundary, or the cases where spheres are replaced by cylinders.

As a first step in constructing stationary shock solutions of this type to the Euler equations, we first construct *inner solutions*, that is, smooth solutions defined everywhere in the exterior $r \geq a$ of a sphere $r = a$ with data (ρ_a, u_a) prescribed at the inner boundary, and *outer solutions*, which are smooth and defined inside $r = b$ when data (ρ_b, u_b) is prescribed at the outer boundary. Note that the stationary equations describing the flow reduce to ODEs under the symmetry assumption, see (3.3)-(3.6). We find that inner solutions remain subsonic (resp., supersonic) everywhere if they are subsonic (resp., supersonic) at $r = a$. A similar result holds for outer solutions, with the interesting difference that there is a critical inner radius at which the flow becomes sonic and beyond which the stationary solution cannot be extended. In the cylindrically symmetric (CS) case we allow swirling flows with nonzero angular (v) and axial (w) components, but we find that it is only the *radial* Mach number that is relevant for classifying solutions (and for determining the critical radius in the case of outer solutions). The main results on inner and outer solutions are summarized in Propositions 3.2.1, 3.2.2, 3.2.3, and 3.2.4.

In section 3.3 we show how to build symmetric, entropy-satisfying shock solutions to the Euler equations by using either inner or outer solutions. In each case, since we consider only stationary solutions and due to the compressibility of admissible shocks, there is only a single shock in these solutions. The main results are summarized in Theorems 3.3.1 and 3.3.2.

Section 3.4 addresses the following question: Taking a, b, and data at $r = a$ as fixed, can one formulate necessary and sufficient conditions on the flow variables at $r = b$ which guarantee the existence of a stationary, weak solution of the barotropic Euler equations with these boundary values, and which contains a single shock at *some* location $\bar{r} \in (a, b)$? The answer is provided, for cylindrically symmetric flow with or without swirl, in Theorem 3.4.1.

Remark 3.1.1. *The inviscid solutions we build have been studied, for isentropic flow, also by Chen and Glimm [8] - [9] in their analysis of the initial value problem on exterior domains. In these works the shock solutions serve as building blocks in a Godunov type scheme and their analysis requires detailed local* L^∞ *estimates. The work applies to more general barotropic flows and we are interested in properties in the large.*

3.1.1 Equations

The barotropic Euler equations express the conservation of mass and the balance of momentum. In Eulerian coordinates the equations in $\mathbb{R}^3$ take the form

$$\rho_t + \mathrm{div}(\rho \mathbf{U}) = 0 \tag{3.1}$$

$$(\rho \mathbf{U}^i)_t + \mathrm{div}(\rho \mathbf{U}^i \mathbf{U}) + P(\rho)_{x_i} = 0, \quad i = 1,2,3\,. \tag{3.2}$$

Here $x = (x_1, x_2, x_3)$ and $\rho, \mathbf{U} = (\mathbf{U}^1, \mathbf{U}^2, \mathbf{U}^3)$, and $P(\rho)$ are the density, velocity, and pressure, respectively.

In the case of spherical or cylindrical symmetry the density and velocities at a point depend only on time and the radial distance to either the origin or to the x_3-axis. We refer to these as the spherically symmetric (SS) and the cylindrically symmetric (CS) cases, respectively. We let (u, v, w) be the velocity components in either spherical or cylindrical coordinates. We set $r = |x|$ in the SS case, while $r = \sqrt{x_1^2 + x_2^2}$ in the CS case. In either case, with a slight abuse of notation we write $\rho(x,t) = \rho(r,t)$, *etc.* Thus

$$\mathbf{U}(x,t) = u(r,t)\frac{x}{r}\,, \qquad v = w \equiv 0$$

in the SS case, while

$$\mathbf{U}(x,t) = u(r,t)\frac{(x_1, x_2, 0)}{r} + v(r,t)\frac{(-x_2, x_1, 0)}{r} + w(r,t)(0,0,1)\,,$$

in the CS case. The equations (3.1)-(3.2) then take the form

$$\rho_t + (\rho u)_\xi = 0, \tag{3.3}$$

$$(\rho u)_t + (\rho u^2)_\xi - \frac{\rho v^2}{r} + P(\rho)_r = 0, \tag{3.4}$$

$$(\rho v)_t + (\rho u v)_\xi + \frac{\rho u v}{r} = 0, \tag{3.5}$$

$$(\rho w)_t + (\rho u w)_\xi = 0, \tag{3.6}$$

where $\partial_\xi \equiv \partial_r + m/r$, and $m = 1$ (CS case) or $m = 2$ (SS case). The compressible

Euler equations are derived, for example, in [32].

3.1.2 Setup and assumptions

Our first task is to construct stationary profiles for the barotropic Euler equations. We treat both the SS and CS cases in domains which are bounded by concentric and fixed spheres or cylinders with radii $b > a > 0$. The solutions are constructed to take on given values at the inner or outer boundaries $\{r = a\}$ and $\{r = b\}$. To analyze stationary solutions we will make the following assumptions about the pressure $P(\rho)$:

(A1) The function $\rho \mapsto P(\rho)$ is a twice differentiable on $(0, +\infty)$ with

$$P'(\rho) > 0 \quad \text{for all } \rho > 0\,. \tag{3.7}$$

(A2)

$$P''(\rho) \geq 0 \quad \text{for all } \rho > 0\,. \tag{3.8}$$

(A3)

$$\lim_{\rho \downarrow 0} P'(\rho) = 0\,. \tag{3.9}$$

Remark 3.1.2. *1. We note that the pressure function of a polytropic ideal gas for isentropic flow, i.e.* $P(\rho) = K\rho^\gamma$ *with* $\gamma > 1$*, satisfies all of (A1)-(A3).*

2. Observe that (A1) and (A2) imply

$$\lim_{\rho \to +\infty} \int_1^\rho \frac{P'(\sigma)}{\sigma}\, d\sigma = +\infty, \tag{3.10}$$

a fact we will use often in what follows.

We denote by c the local sound speed,

$$c = c(\rho) := \sqrt{P'(\rho)}\,.$$

3.2 Stationary solutions of the barotropic Euler equations

ODE system for spherically symmetric flow In the SS case the barotropic Euler equations reduce to the ODE system

$$\frac{d(\rho u r^2)}{dr} = 0, \tag{3.11}$$

$$u\frac{du}{dr} + \frac{1}{\rho}\frac{d(P(\rho))}{dr} = 0\,, \tag{3.12}$$

and the Rankine-Hugoniot conditions reduce to

$$[\rho u] = 0\,, \qquad \left[P(\rho) + \rho u^2\right] = 0\,.$$

ODE system for cylindrically symmetric flow In the CS case the barotropic Euler equations reduce to the ODE system

$$\frac{d(\rho u r)}{dr} = 0, \tag{3.13}$$

$$\frac{d(\rho u^2 r)}{dr} - \rho v^2 + r\frac{d(P(\rho))}{dr} = 0, \tag{3.14}$$

$$\frac{d(\rho u r v)}{dr} + \rho u v = 0, \tag{3.15}$$

$$\frac{d(\rho u r w)}{dr} = 0\,, \tag{3.16}$$

and the Rankine-Hugoniot conditions reduce to

$$[\rho u] = 0 \qquad \left[P(\rho) + \rho u^2\right] = 0\,, \tag{3.17}$$

together with

$$[v] = 0\,, \qquad [w] = 0\,. \tag{3.18}$$

Inner and outer solutions We are seeking stationary solutions defined in regions between fixed, concentric spheres or cylinders with radii $b > a > 0$. Different situations occur according to whether the Dirichlet data for ρ and u, v, w are prescribed at $r = a$ or at $r = b$. Solution with data prescribed at the inner (outer) boundary will be referred to as inner (outer) solutions.

Remark 3.2.1. *For barotropic flow the problem of solving the stationary equations will be reduced a single algebraic equation. One could also analyze the ODEs more directly. Indeed, this approach leads to a separable ODE for the velocity in the case of the full Euler system for an ideal gas (see [17]). While this is not true for the barotropic case, we note that the ODEs for the radial velocity are*

$$u_r = \frac{2uc^2}{r(u^2 - c^2)}\,, \qquad\qquad u_r = \frac{u(v^2 + c^2)}{r(u^2 - c^2)} \tag{3.19}$$

in the SS and CS cases, respectively. Thus, sonic points are singular points for the ODEs.

3.2.1 Inner solutions in the spherically symmetric case

We now consider the case with given Dirichlet data ρ_a, u_a at the inner boundary $r = a$, and we seek a smooth, stationary solution to the barotropic Euler equations (3.11)-(3.12) in the region $r \geq a$. Note that we are not (yet) saying anything about the sign or size of u_a. The ODE (3.11) yields

$$u(r) = \frac{C_a}{\rho(r)r^2} \qquad \text{for } r \geq a, \text{ where} \qquad C_a := \rho_a u_a a^2\,. \tag{3.20}$$

Notice that this implies that if $u_a \gtrless 0$, then $u(r) \gtrless 0$ for all $r \geq a$. We will only consider the case where $\rho_a > 0$ and $u_a \neq 0$. To analyze the ODE (3.12) it is convenient to introduce the function $\Pi : \mathbb{R}^2_+ \to \mathbb{R}$ defined by

$$\Pi(\rho_2, \rho_1) := \int_{\rho_1}^{\rho_2} \frac{2P'(\sigma)}{\sigma}\, d\sigma\,. \tag{3.21}$$

We note that (3.10) amounts to $\Pi(\rho, \rho_1) \to +\infty$ as $\rho \to \infty$. From (3.12) we get

$$\frac{d}{dr}\big[u(r)^2 + \Pi(\rho(r), \rho_a)\big] = 0\,.$$

Since $\Pi(\rho_a, \rho_a) = 0$, it follows that

$$u(r)^2 + \Pi(\rho(r), \rho_a) \equiv u_a^2, \qquad \text{for } r \geq a\,. \tag{3.22}$$

Using (3.20) to eliminate $u(r)$ we thus get that $\rho = \rho(r)$ satisfies

$$\frac{C_a^2}{\rho(r)^2 r^4} + \Pi(\rho(r), \rho_a) \equiv u_a^2 \qquad \text{for } r \geq a\,. \tag{3.23}$$

We need to show that equation (3.23) can be solved for $\rho(r)$ in terms of r when $r \geq a$. Let us define the function

$$\phi(\rho, \rho_a, u_a) := \rho^2\big[u_a^2 - \Pi(\rho, \rho_a)\big]\,, \tag{3.24}$$

such that (3.23) takes the form

$$\phi(\rho, \rho_a, u_a) = \frac{C_a^2}{r^4}\,. \tag{3.25}$$

We now consider ρ_a and u_a as fixed and let $' = \frac{d}{d\rho}$. With $\phi(\rho) \equiv \phi(\rho, \rho_a, u_a)$ and $\Pi(\rho) \equiv \Pi(\rho, \rho_a)$ we thus have

$$\phi'(\rho) = 2\rho\big[u_a^2 - \big(\Pi(\rho) + P'(\rho)\big)\big]\,. \tag{3.26}$$

From (A1) and (A2) it follows that the map $\rho \mapsto \Pi(\rho) + P'(\rho)$ is strictly increasing on $(0, \infty)$, and from (3.10) it follows that it tends to $+\infty$ as $\rho \uparrow \infty$. Note that (A3) together with the definition of Π, and the fact that $\rho_a > 0$, implies $u_a^2 > 0 > (\Pi(\rho) + P'(\rho))|_{\rho=0}$. Thus, by (3.26), $\phi'(\rho) > 0$ for $\rho > 0$ sufficiently small, while $\phi'(\rho) \downarrow -\infty$ as $\rho \uparrow \infty$. It follows that there are unique ρ-values $0 < \rho_* < \rho_0$ such that

$$\phi'(\rho_*) = 0\,, \qquad \phi(\rho_0) = 0\,.$$

Using (A4) it also follows that both $\phi(\rho)$ and $\phi'(\rho)$ vanish as $\rho \downarrow 0$. The graph of ϕ thus looks like in Figure 3.1. As $\phi(\rho_a) = \rho_a^2 u_a^2 > 0$ it follows that $\rho_a < \rho_0$. Note that, by construction, we have $\phi(\rho_a) < \phi(\rho_*)$. We observe that ρ_* and ρ_0 depend on a, ρ_a, u_a, and that ρ_* is implicitly given by

$$\partial_\rho \phi(\rho, \rho_a, u_a)\Big|_{\rho=\rho^*} = 0\,.$$

For given a, ρ_a, and u_a there are thus two possibilities: $\rho_* < \rho_a$ or $\rho_* > \rho_a$. ¿From the figure it's clear that $\rho_a \gtrless \rho_*$ if and only if $\phi'(\rho_a) \lessgtr 0$. By (3.26) and the

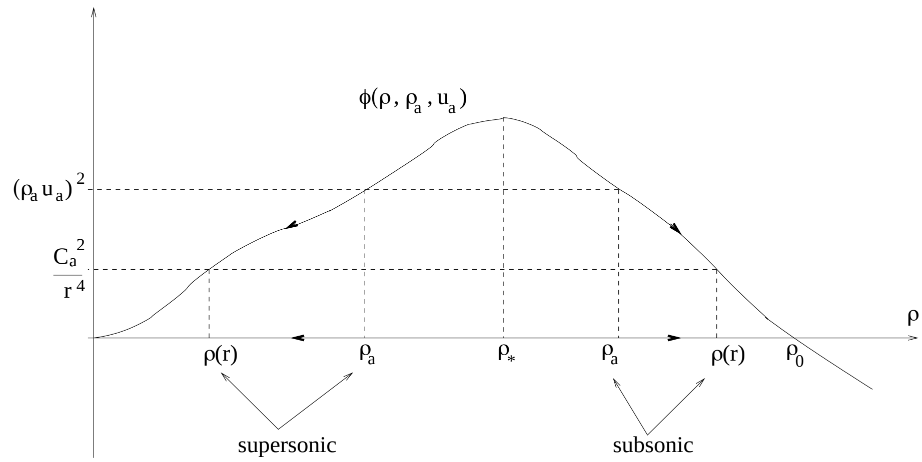

Figure 3.1. Inner solutions. The function $\phi(\rho, \rho_a, u_a)$. Arrows indicate direction as r increases from $r = a$. The function $\psi(\rho, \rho_a, u_a, v_a)$ in Section 3.2.2 has the same form.

fact that $\Pi(\rho_a) = 0$, we see that $\phi'(\rho_a) > 0$ ($\phi'(\rho_a) < 0$) if and only if the flow is supersonic (subsonic) at $r = a$.

Returning to equation (3.25) we see that its right-hand side is a strictly decreasing function of r. The two cases are thus given as follows ($c_a^2 := c(\rho_a)^2 = P'(\rho_a)$):

- Subsonic case: $|u_a| < c_a$. In this case we find a unique, smooth solution $\rho(r)$ of (3.25) for all $r > a$, with $\frac{d\rho(r)}{dr} > 0$;

- Supersonic case: $|u_a| > c_a$. In this case we find a unique, smooth solution $\rho(r)$ of (3.25) for all $r > a$, with $\frac{d\rho(r)}{dr} < 0$.

It is clear from Figure 3.1 that these smooth solutions are defined for all $r \geq a$.

Consider the subsonic case and recall the convexity assumption (A2). This condition implies that the sound speed along the profile, $c(\rho(r))$, is an increasing function of r in this case. On the other hand, since $\Pi(\rho, \rho_a)$ is increasing with respect to ρ, it follows from (3.22) that $|u(r)|$ is a strictly decreasing function of r in this case. Thus, the Mach number $M(r) := |u(r)|/c(\rho(r))$ decreases as r increases: if the flow is subsonic at $r = a$, then the same is true for all $r \geq a$. A similar argument applies in the supersonic case: if the flow is supersonic at $r = a$, then the same is true for all $r \geq a$. Note that these conclusions are independent of the direction of flow, i.e. we get a solution for both inflow ($u_a > 0$) and outflow ($u_a < 0$) boundary data. Summing up we have:

Proposition 3.2.1. (Existence of spherically symmetric stationary inner solutions) *Consider the stationary barotropic Euler equations with spherical symmetry (3.11)-(3.12) in the exterior of a sphere with radius* $a > 0$*, and with prescribed Dirichlet data* $\rho_a > 0$, $u_a \neq 0$ *at* $r = a$*. Assume that the pressure* P *satisfies the assumptions (A1)-(A3) and that the data are non-sonic (i.e.* $u_a^2 \neq P'(\rho_a)$*).*

Then (3.11)-(3.12) have a unique smooth solution defined for all $r \geq a$*. The resulting flow is strictly subsonic/supersonic for all* $r \geq a$ *if and only if it is strictly subsonic/supersonic at the inner boundary* $r = a$*.*

Remark 3.2.2. *If the data at* $r = a$ *are sonic, then there are two solutions to (3.11)-(3.12) defined for* $r \geq a$ *- one supersonic and one subsonic.*

3.2.2 Inner solutions in the cylindrically symmetric case

Next we construct inner solutions for CS flow: given Dirichlet data $\rho_a > 0$, $u_a \neq 0$, v_a, w_a at the inner boundary $r = a$, we seek a smooth, stationary solution to the barotropic Euler equations (3.13)-(3.16) in the region $r \geq a$. From (3.13) we see that (3.16) is satisfied with $w \equiv w_a$. By using (3.13) in (3.14) and (3.15) we reduce the remaining equations to

$$\begin{aligned} \rho u r &\equiv C_a \,, && (3.27)\\ r v &\equiv D_a \,, && (3.28)\\ u\frac{du}{dr} - \frac{D_a^2}{r^3} + \frac{1}{\rho}\frac{d(P(\rho))}{dr} &= 0 \,, && (3.29) \end{aligned}$$

where

$$C_a = a\rho_a u_a \,, \qquad D_a = a v_a \,. \qquad (3.30)$$

Defining $\Pi(\rho_2, \rho_1)$ as in (3.21) and integrating (3.29) once, we get that the density $\rho(r)$ satisfies the algebraic equation

$$\frac{1}{r^2} = \frac{\rho(r)^2}{C_a^2 + D_a^2\,\rho(r)^2}\left[V_a^2 - \Pi(\rho(r), \rho_a)\right], \qquad \text{where } V_a^2 := u_a^2 + v_a^2. \qquad (3.31)$$

To analyze this equation we define the function

$$\psi(\rho) \equiv \psi(\rho, \rho_a, u_a, v_a) := \frac{\rho^2}{C_a^2 + D_a^2\,\rho^2}\left[V_a^2 - \Pi(\rho, \rho_a)\right],$$

where the dependence on u_a and v_a are given by (3.30). When $D_a = 0$ this reduces to the function ϕ defined in (3.24). We get that

$$\psi'(\rho) = \frac{2\rho}{\left(C_a^2 + D_a^2\,\rho^2\right)^2}\left\{C_a^2 V_a^2 - \left[C_a^2\big(\Pi(\rho, \rho_a) + P'(\rho)\big) + D_a^2\,\rho^2 P'(\rho)\right]\right\}. \quad (3.32)$$

As in the SS case (see the argument above for $\phi(\rho)$) we have that the map

$$\rho \mapsto C_a^2\big(\Pi(\rho, \rho_a) + P'(\rho)\big) + D_a^2\,\rho^2 P'(\rho) \quad (3.33)$$

is strictly increasing, tends to $+\infty$ as $\rho \to +\infty$, and tends to a strictly negative value as $\rho \downarrow 0$. (Recall that we assume $\rho_a > 0$). Also, from (3.10), it follows that $\psi(\rho) < 0$ for ρ sufficiently large. Hence, just as for $\phi(\rho)$, we have that $\psi(\rho)$ is positive for small positive ρ, tends to 0 as $\rho \downarrow 0$, and that there are unique values $0 < \rho_* < \rho_0$ for which

$$\psi'(\rho_*) = 0\,, \qquad \psi(\rho_0) = 0\,.$$

As $\psi(\rho_a) > 0$ it follows that $\rho_0 > \rho_a$. The situation is thus the same as for the case without a tangential velocity component, and $\psi(\rho) \equiv \psi(\rho, \rho_a, u_a, v_a)$ has the same shape as $\phi(\rho, \rho_a, u_a)$ in Figure 3.1.

Returning to the algebraic equation (3.31) for $\rho(r)$ we observe that equality holds at $r = a$ by definition. As r increases from $r = a$ we see that the properties of ψ guarantees a solution $\rho(r)$, defined for all $r \geq a$. Just as in the case with no tangential velocity there are two cases: $\rho_a \gtrless \rho_*$, which is the case if and only if $\psi'(\rho_a) \lesseqgtr 0$, which holds if and only if $u_a^2 \lesseqgtr P'(\rho_a) \equiv c_a^2$. Note that the tangential velocity is irrelevant at this point. We refer to these cases as *radially* super/sub-sonic. To analyze the sonicity we define the

$$\textit{Radial Mach number} = M_{rad}(r) := \frac{|u(r)|}{c(r)}\,,$$

as well as the

$$\textit{(proper) Mach number} = M(r) := \frac{V(r)}{c(r)},$$

where $V(r) := \sqrt{u(r)^2 + v(r)^2}$. We thus have two cases:

- Radially subsonic case: $|u_a| < c(a)$. In this case we find a unique, smooth solution $\rho(r)$ of (3.31) for all $r > a$, with $\frac{d\rho(r)}{dr} > 0$;
- Radially supersonic case: $|u_a| > c(a)$. In this case we find a unique, smooth solution $\rho(r)$ of (3.31) for all $r > a$, with $\frac{d\rho(r)}{dr} < 0$.

Consider the radially subsonic case where $\frac{d\rho(r)}{dr} > 0$. From (3.27), (3.28), and (3.31) we have

$$V(r)^2 + \Pi(\rho(r), \rho_a) \equiv V_a^2 . \tag{3.34}$$

Since $\Pi(\rho, \rho_a)$ is increasing with respect to ρ, it follows that the speed $V(r)$ is a strictly decreasing function of r in this case. As $\rho u r \equiv C_a$ it follows that $|u(r)|$ is decreasing in this case, while the sound speed $c(r)$ increases with r. Thus, in the radially subsonic case we have that *both* the radial and the proper Mach numbers are decreasing as r increases from $r = a$.

Next, consider the radially supersonic case where $\frac{d\rho(r)}{dr} < 0$. Again, (3.34) holds and we conclude that the speed $V(r)$ is strictly increasing in this case. Also, since $\rho(r)$ decreases, so does the sound speed $c(r)$. The flow, which is supersonic at $r = a$ (as it is radially supersonic there) therefore becomes more supersonic as r increases from $r = a$. We proceed to show that the flow remains also *radially* supersonic as r increases. Multiplying (3.31) by $C_a^2/\rho(r)^2$, and using (3.34) we get that

$$u(r)^2 = \frac{C_a^2 V(r)^2}{C_a^2 + D_a^2 \rho(r)^2} .$$

As $V(r)$ increases with r, in the the present case, while $\rho(r)$ decreases, it follows that $|u(r)|$, and thus $M_{rad}(r)$, increases as r increases.

Thus, stationary, cylindrically symmetric flow (possibly with swirl) which is *radially* super- or sub-sonic at the inner boundary $r = a$, becomes increasingly so as r increases. Summing up we have:

Proposition 3.2.2. (Existence of cylindrically symmetric stationary inner solutions) *Consider the stationary barotropic Euler equations with cylindrical symmetry*

(3.13)-(3.16) in the exterior of a cylinder with radius $a > 0$, and with prescribed Dirichlet data $\rho_a > 0, u_a \neq 0, v_a, w_a$ at $r = a$. Assume that the pressure satisfies assumptions (A1)-(A3) and that the data are radially non-sonic ($u_a^2 \neq P'(\rho_a)$).

Then (3.13)-(3.16) have a unique, smooth solution defined for all $r \geq a$. The flow is subsonic/supersonic throughout $r > a$ if and only if it is subsonic/supersonic at $r = a$.

Remark 3.2.3. *As in the SS case, if the data are sonic at $r = a$ then there are two smooth, stationary solutions defined in $r > a$ - one supersonic and one subsonic.*

3.2.3 Outer solutions in the spherically symmetric case

We now give Dirichlet data $\rho_b > 0, u_b \neq 0$ at an *outer* boundary $r = b$, and we seek a smooth, stationary solution to the Euler equations. Differently from inner solutions which were defined everywhere outside the inner boundary, outer solutions are not defined for all $r < b$: there is a critical radius $r^* = r^*(b, \rho_b, u_b) \in (0, b]$ where the flow becomes sonic and beyond which a stationary solution cannot be extended. In order to construct a solution in a nontrivial interval we will assume that the data are strictly super- or subsonic at $r = b$.

We define the functions Π and ϕ as in Section 3.2.1. An entirely similar analysis shows that the density profile $\rho(r)$ in the present case is given as the solution to the algebraic equation

$$\frac{C_b^2}{r^4} = \phi(\rho(r), \rho_b, u_b) \qquad \text{where} \quad C_b := \rho_b u_b b^2 \,. \tag{3.35}$$

Again as in Section 3.2.1: setting $\phi(\rho) \equiv \phi(\rho, \rho_b, u_b)$ we get that there are unique ρ-values $0 < \rho^* < \rho^0$ (each depending on b, ρ_b, and u_b) such that $\phi'(\rho^*) = 0\,, \phi(\rho^0) = 0$. Note that, by construction, $\phi(\rho_b) < \phi(\rho^*)$. As $\phi(\rho_b) = \rho_b^2 u_b^2 > 0$ it follows that $\rho^0 > \rho_b$. The situation thus looks like in Figure 3.2, and we get the following two possibilities:

- Subsonic case: $|u_b| < c_b$. In this case we find a unique, smooth solution $\rho(r)$ to (3.35) with $\frac{d\rho(r)}{dr} > 0$;
- Supersonic case: $|u_b| > c_b$. In this case we find a unique, smooth solution $\rho(r)$ to (3.35) with $\frac{d\rho(r)}{dr} < 0$.

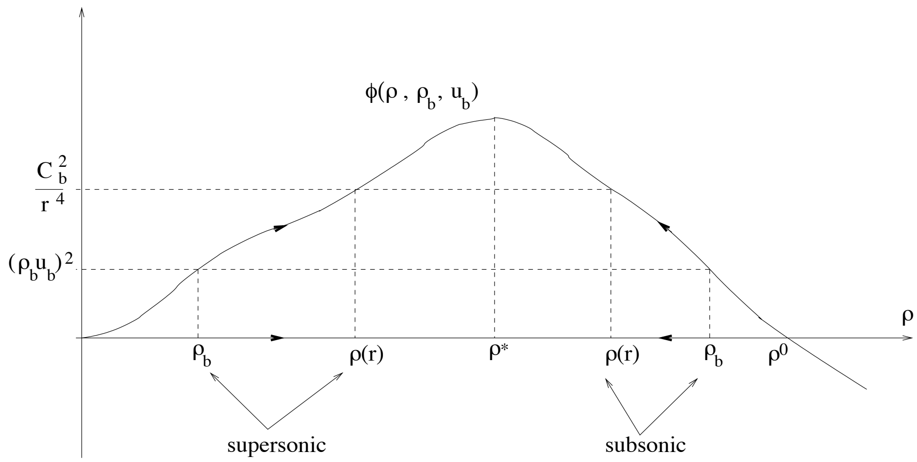

Figure 3.2. Outer solutions. The function $\phi(\rho, \rho_b, u_b)$. Arrows indicate direction as r decreases from $r = b$. The function $\psi(\rho, \rho_b, u_b, v_b)$ in Section 3.2.4 has the same form.

So far the analysis is similar to the analysis of inner solutions. However, as r decreases from $r = b$, there is a limiting value of the radius $r = r^* = r^*(b, \rho_b, u_b)$ below which (3.35) does not have a solution $\rho(r)$. Observe that r^* is given by $\rho(r^*) = \rho^*$, and that ρ^* is implicitly given by $\phi'(\rho^*) = 0$. Using (3.26) (with u_b instead of u_a and with ϕ and Π as in this section) we have $u_b^2 - \Pi(\rho^*, \rho_b) = P'(\rho^*)$, and it follows that $\phi(\rho^*) = \rho^{*2}(u_b^2 - \Pi(\rho^*, b)) = \rho^{*2} P'(\rho^*)$. By (3.35) the limiting value $r = r^*$ is thus given implicitly by

$$\rho^{*2} P'(\rho^*) = \frac{C_b^2}{r^{*4}} = \rho^{*2} u^{*2} .$$

That is, as we let r decrease from $r = b$ we reach the limiting radius at the sonic point.

To analyze the sonicity of flow in outer SS solutions we can argue in a similar manner as for inner solutions to reach the following conclusions: if the flow is subsonic (supersonic) at $r = b$, then the flow becomes less subsonic (supersonic) as r decreases. We summarize our findings for outer SS solutions:

Proposition 3.2.3. (Existence of spherically symmetric stationary outer solutions) *Consider the stationary barotropic Euler equations with spherical symmetry (3.11)-(3.12) inside a sphere with radius $b > 0$, and with prescribed Dirichlet data $\rho_b > 0$, $u_b \neq 0$ at $r = b$. Assume that the pressure satisfies the assumptions (A1)-(A3) and that the data are non-sonic ($u_b^2 \neq P'(\rho_b)$).*

Then there is a critical inner radius $r^* = r^*(b, \rho_b, u_b) > 0$ where the flow becomes sonic, and below which there is no solution of the equations. For each fixed $\bar{r} > r^*$ the equations have a unique smooth solution defined for $\bar{r} \le r \le b$. The resulting flow is strictly subsonic (supersonic) throughout $[\bar{r}, b]$ if and only if it is strictly subsonic (supersonic) at the outer boundary $r = b$.

3.2.4 Outer solutions in the cylindrically symmetric case

We proceed to analyze solutions of the system (3.13)-(3.15) which takes on given Dirichlet data $\rho_b > 0$, u_b, v_b at the outer boundary $r = b$. (We have now set $w \equiv w_b$). The analysis follows the same steps as in the earlier sections.

We define Π as in Section 3.2.1 and observe that the same analysis as in Section 3.2.2 shows that the density profile $\rho(r)$ in the present case satisfies the algebraic equation

$$\frac{1}{r^2} = \psi(\rho) \equiv \psi(\rho, \rho_b, u_b, v_b) \tag{3.36}$$

where

$$\psi(\rho) \equiv \psi(\rho, \rho_b, u_b, v_b) := \frac{\rho^2}{C_b^2 + D_b^2\,\rho^2}\left[V_b^2 - \Pi(\rho, \rho_b)\right],$$

and with $V_b^2 := u_b^2 + v_b^2$, $C_b = b\rho_b u_b$, and $D_b = bv_b$. An argument entirely similar to the one in Section 3.2.2 gives unique values $0 < \rho^* < \rho^0$ (now depending on ρ_b, u_b, v_b) such that

$$\psi'(\rho^*) = 0\,, \qquad \psi(\rho^0) = 0\,.$$

As $\psi(\rho_b) > 0$ it follows that $\rho^0 > \rho_b$. The situation is thus the same as for the case without a tangential velocity component, and $\psi(\rho) \equiv \psi(\rho, \rho_b, u_b, v_b)$ has the same shape as $\phi(\rho, \rho_b, u_b)$ in Figure 3.2. We have the two cases: $\rho_b \gtrless \rho^*$, which holds if and only if $M_{rad}(r) \lessgtr 1$, where

$$M_{rad}(r) \;=\; \textit{Radial Mach number} := \frac{|u(r)|}{c(r)}\,.$$

We thus have the two cases for any $\bar{r} > r^*$:

- Radially subsonic case: $|u_b| < c(b)$. In this case we find a unique, smooth solution $\rho(r)$ of (3.36) on $[\bar{r}, b]$, with $\frac{d\rho(r)}{dr} > 0$;

- Radially supersonic case: $|u_b| > c(b)$. In this case we find a unique, smooth solution $\rho(r)$ of (3.36) on $[\overline{r}, b]$, with $\frac{d\rho(r)}{dr} < 0$.

Consider the radially subsonic case where $\frac{d\rho(r)}{dr} > 0$. Combining the ODEs with data given at $r = b$ we get that

$$V(r)^2 + \Pi(\rho(r), \rho_b) \;\equiv\; V_b^2\,, \tag{3.37}$$

where $V(r) := \sqrt{u(r)^2 + v(r)^2}$. Using the convexity of the pressure we get that the sound speed $c(r)$ increases with r. Thus, in the radially subsonic case we have that *both* the radial and the proper Mach numbers increases as r decreases from $r = b$. In particular, the flow becomes *less* radially subsonic as the particles flow towards the origin. A similar argument shows that supersonic flow becomes less radially supersonic as r decreases from $r = b$.

Thus, stationary, cylindrically symmetric flow (possibly with swirl) which is *radially* super- or sub-sonic at the outer boundary $r = b$, becomes less so as r decreases from $r = b$.

As in the case without a tangential velocity component we see from Figure 3.2 that, as r decreases from $r = b$, there is a limiting value of the radius $r = r^* = r^*(b, \rho_b, u_b)$ below which (3.36) does not have a solution $\rho(r)$. We proceed to show that this occurs exactly where the flow becomes *radially* sonic (i.e. $M_{rad}(r^*) = 1$). First observe that r^* is given by $\rho(r^*) = \rho^*$, and that ρ^* is implicitly given by $\psi'(\rho^*) = 0$. Using (3.32) (with C_b, D_b instead of C_a, D_a and with $\Pi = \Pi(\cdot, \rho_b)$) we get that

$$C_b^2 V_b^2 = C_b^2 \left[\Pi(\rho^*, \rho_b) + P'(\rho^*)\right] + D_b^2 \rho^{*2} P'(\rho^*)\,.$$

Using the relations $\Pi(\rho^*, \rho_b) = V_b^2 - V^{*2}$, $C_b = u^* \rho^* r^*$, $D_b = v^* r^*$, it follows that

$$u^{*2} = P'(\rho^*) =: c^{*2}\,.$$

Summarizing we have:

Proposition 3.2.4. (Existence of spherically symmetric stationary outer solutions) *Consider the stationary barotropic Euler equations with cylindrical symmetry (3.13)-(3.16) inside a sphere with radius $b > 0$, and with prescribed Dirichlet data*

$\rho_b > 0$, $u_b \neq 0$, v_b, w_b at $r = b$. Assume that the pressure satisfies the assumptions (A1)-(A3) and that the data are radially non-sonic ($u_b^2 \neq P'(\rho_b)$).

Then there is a critical inner radius $r^* = r^*(b, \rho_b, u_b, v_b) > 0$ where the flow becomes radially sonic, and below which there is no solution of the equations. For each fixed $\bar{r} > r^*$ the equations have a unique smooth solution defined for $\bar{r} \leq r \leq b$. The resulting flow is radially subsonic (supersonic) throughout $[\bar{r}, b]$ if and only if it is radially subsonic (supersonic) at the outer boundary $r = b$.

Remark 3.2.4. *A calculation shows that r^* is an increasing function of $|v_b|$ for fixed values of b, ρ_b, u_b. That is, faster rotation of the fluid decreases the interval of existence of a stationary solution.*

3.3 Stationary solutions with shocks

We next use the inner and outer solutions to construct *weak* solutions with spherical or cylindrical symmetry, and with a single stationary, entropy admissible shock. As with the inner and outer solutions in the previous section the construction will depend on whether we work outward with data given $r = a$, or inward with data given at $r = b$. The two cases are treated separately in the two next subsections. In the last subsection we consider the problem of deciding existence and possible location of a shock when data are provided at both $r = a$ and at $r = b$.

We note that the Rankine-Hugoniot relations for density and momentum are identical for SS and CS flow. It follows from (3.18) that only the radial part of the velocity changes across a discontinuity in the CS case. We thus treat the two cases as one case, bearing in mind that $v = w \equiv 0$ in the SS case

3.3.1 Shock solution built from inner solutions

We assume we are given Dirichlet data $\rho_a > 0, u_a, v_a, w_a$ at the inner boundary $r = a$ (a being fixed from now on) and we assume that the flow is either strictly *radially* supersonic or strictly *radially* subsonic at $r = a$. That is, we assume $u_a^2 \gtrless c(a)^2$, where $c(a)^2 = P'(\rho_a)$.

Next we fix any radius $b > a$ together with any intermediate radius $\bar{r} \in (a, b)$. According to the proposeitions above we can solve (3.11)-(3.12) in the SS case, or (3.13)-(3.16) in the CS case, for $r \in (a, \bar{r})$ with the given values at $r = a$ as initial data. This provides the values $\rho(\bar{r}-)$, $u(\bar{r}-)$, $v(\bar{r}-)$, $w(\bar{r}-)$. The Rankine-Hugoniot conditions then give $\rho(\bar{r}+)$, $u(\bar{r}+)$, $v(\bar{r}+)$, $w(\bar{r}+)$ (see below), which we use as initial data for (3.11)-(3.12) (or (3.13)-(3.16)) in the outer region $r \in (\bar{r}, b)$. Appealing once more to the earlier discussion we obtain a stationary solution defined for all $r \in [a, b]$ and with a single discontinuity at any intermediate location.

It remains to verify that $\rho(\bar{r}+)$, $u(\bar{r}+)$, $v(\bar{r}+)$, $w(\bar{r}+)$ are uniquely determined by the Rankine-Hugoniot relations, and to analyze the admissibility of the resulting solutions. As selection criteria we impose that the flow should be *compressive*: a fluid particle suffers an increase in density as it crosses a shock. Let $\bar{\rho} = \rho(\bar{r}_-)$, $\bar{u} = u(\bar{r}_-)$, and let $\hat{\rho} = \rho(\bar{r}_+)$, $\hat{u} = u(\bar{r}_+)$. Defining $F(\rho) := P(\rho) - P(\bar{\rho})$ and $G(\rho) := \bar{\rho}\bar{u}^2(1 - \bar{\rho}/\rho)$, the Rankine-Hugoniot conditions are

$$F(\hat{\rho}) = G(\hat{\rho}) \,, \qquad \hat{u} = \frac{\bar{\rho}\bar{u}}{\hat{\rho}} \,. \tag{3.38}$$

We have

$$F'(\rho) = P'(\rho) \,, \qquad \text{and} \qquad G'(\rho) = \left(\frac{\bar{\rho}\bar{u}}{\rho}\right)^2 \,.$$

It follows that the flow at $\bar{r}_-$ is radially supersonic if and only if $G'(\bar{\rho}) > F'(\bar{\rho})$, and in this case the Rankine-Hugoniot conditions have a unique nontrivial solution $\hat{\rho} > \bar{\rho}$. On the other hand, the flow at $\bar{r}_-$ is radially subsonic if and only if $G'(\bar{\rho}) < F'(\bar{\rho})$, and in this case the Rankine-Hugonoiot conditions have a unique nontrivial solution $\hat{\rho} < \bar{\rho}$. (These conclusions are consequences of the convexity assumption (A2)).

Recall that $\bar{u} \gtrless 0$ if and only if $u_a \gtrless 0$ and that the radial sonicity is conserved as we move away from the origin (for an inner solution, which is what we consider here). It follows that if the flow is supersonic at the inner boundary $r = a$, then the flow there must be into the domain. Similarly, if the flow is subsonic at the inner boundary $r = a$, then the flow there must be out of the domain.

Having determined the flow in $(a, \bar{r})$, as well as the values of the flow variables at $r = r_+$, we can now find a unique stationary and smooth solution in the outer region $\bar{r} < r < b$.

Proposition 3.3.1. (Stationary symmetric shocks built from inner solutions) *Consider the barotropic Euler equations with spherical (or cylindrical) symmetry in the domain between two concentric spheres (cylinders) with radii $a < b$, and with prescribed density $\rho_a > 0$ and velocities $u_a \neq 0$, v_a, w_a at $r = a$. Assume that flow at $r = a$ is (radially) non-sonic and that the pressure satisfies the assumptions (A1)-(A3). Given any radius $\bar{r} \in (a, b)$.*

Then there is a unique weak admissible solution with a single shock located at $\bar{r}$ if and only if, either, the flow is radially supersonic at $r = a$ and directed into the domain (i.e. $u_a > 0$), or the flow is radially subsonic at $r = a$ and directed out of the domain (i.e. $u_a < 0$). In the former case the flow is (radially) supersonic in $(a, \bar{r})$ and (radially) subsonic in $(\bar{r}, b)$, while the opposite holds in the latter case.

3.3.2 Shock solution built from outer solution

The procedure for constructing stationary, symmetric solutions with an admissible shock from data given at the outer boundary, is similar to, but slightly more involved than, the procedure in the previous subsection. To formulate the result we need to identify the two critical radii involved. For concreteness let's consider the SS case. First, from the data at $r = b$ we calculate $\rho_1^* = \rho_1^*(\rho_b, u_b)$ from the equation

$$\partial_\rho \phi(\rho, \rho_b, u_b)\Big|_{\rho=\rho_1^*} = 0\,, \tag{3.39}$$

where ϕ is defined in (3.24). We then calculate $r_1^* = r_1^*(\rho_b, u_b)$ from

$$\frac{C_b^2}{r_1^{*4}} = \phi(\rho_1^*, \rho_b, u_b) \qquad \text{where} \quad C_b = \rho_b u_b b^2\,. \tag{3.40}$$

Now, given any intermediate radius $\bar{r} \in (r_1^*, b)$ we know that we can solve the flow equations on $(\bar{r}, b)$ to find $(\hat{\rho},\, \hat{u}) := (\rho(\bar{r}_+),\, u(\bar{r}_+))$. The earlier analysis shows that the flow will be strictly super- or subsonic at $\bar{r}_+$ if and only if it is so at $r = b$. As in the case of inner solutions the Rankine-Hugoniot relations now determine a unique state $(\bar{\rho},\, \bar{u})$ at $\bar{r}_-$ which connects to $(\hat{\rho},\, \hat{u})$. The flow at $\bar{r}_-$ is supersonic (subsonic) if and only if the flow at $\bar{r}_+$ is subsonic (supersonic). Repeating the above analysis we find that the flow can be extended inwards until a second critical radius $r_2^* < r_1^*$ which is determined as above (with with data $\bar{r}$, $\bar{\rho}$, $\bar{u}$ instead of b, ρ_b, u_b). (For the

CS case there are similarly two critical radii r_1^* and r_2^* determined in the same way from the data at $r = b$.) Finally, the admissibility condition dictates the direction of the flow in the same way as in Section 3.3.1. Summarizing we have:

Proposition 3.3.2. (Stationary symmetric shocks built from outer solutions) *Consider the barotropic Euler equations with spherical (or cylindrical) symmetry in the domain between two concentric spheres (cylinders) with radii $a < b$, and with prescribed density $\rho_b > 0$ and velocities $u_b \neq 0$, v_b, w_b at $r = b$. Assume that flow at r = b is (radially) non-sonic and that the pressure satisfies the assumptions (A1)-(A3). Given any radius $\bar{r} \in (r_1^*, b)$ and assume that $a \in (r_2^*, r_1^*)$, where r_1^*, r_2^* are determined as above.*

Then there is a unique weak admissible solution, defined on (a, b) and with a single shock located at $\bar{r}$ if and only if, either, the flow is radially supersonic at $r = b$ and directed into the domain (i.e. $u_b < 0$), or the flow is radially subsonic at $r = b$ and directed out of the domain (i.e. $u_b < 0$). In the former case the flow is (radially) supersonic in $(\bar{r}, b)$ and (radially) subsonic in $(a, \bar{r})$, while the opposite holds in the latter case.

3.4 When can a shock solution be found?

We next consider the possibility of finding shock solutions for given boundary data. For concreteness we treat supersonic CS flow and ask: Given data $(\rho_a,\, u_a,\, v_a,\, w_a)$ at $r = a$ together with data
$(\rho_b,\, u_b,\, v_b,\, w_b)$ at $r = b$; does there exist a solution of the stationary barotropic Euler equations with these boundary data and with an admissible shock located at some intermediate radius $\bar{r} \in (a, b)$? To analyze this question we fix a, b, and $(\rho_a,\, u_a,\, v_a,\, w_a)$, and then formulate necessary and sufficient conditions on ρ_b, u_b, v_b, w_b which guarantee the existence of a solution with a shock in (a, b).

From the earlier analysis we know that $\rho(r)u(r)r \equiv C_a := \rho_a u_a a$, $rv(r) \equiv D_a := av_a$, and $w(r) \equiv w_a$ along any stationary solution (smooth or not). Thus, a necessary condition for the existence of a shock solution with "final" data ρ_b, u_b, v_b, w_b at $r = b$ is that

$$\rho_b u_b = \frac{C_a}{b}\,, \qquad v_b = \frac{D_a}{b}\,, \qquad w_b = w_a\,. \tag{3.41}$$

We choose to work with the density as the primary unknown so that the issue becomes: what final densities ρ_b can be attained for a solution with a shock at $\bar{r} \in (a, b)$. For concreteness we consider the case with (radially) supersonic inflow at $r = a$, that is, $u_a > 0$ and $u_a^2 > c_a^2 = P'(\rho_a)$.

To see how the final density ρ_b depends on the shock location $\bar{r}$, we first observe that the ODE for the density takes the same form in the two intervals $(a, \bar{r})$ and $(\bar{r}, b)$, and it is independent of $\bar{r}$. Indeed, from (3.29) it follows that

$$\frac{d\rho}{dr} = \frac{\rho(v^2 + u^2)}{r(c^2 - u^2)} . \tag{3.42}$$

From the earlier analysis we know that the flow remains (radially) subsonic for all $r > \bar{r}$, whence (3.42) is a well-behaved ODE with unique solutions. Thus, if $\rho_1(r)$, $\rho_2(r)$ are two smooth solutions with $\rho_1(s) > \rho_2(s)$ for some s, then necessarily $\rho_1(r) > \rho_2(r)$ for all $r > s$.

We can use this to infer how ρ_b varies with the shock location $\bar{r}$. Specifically we will show that an increase in $\bar{r}$ implies a lower ending value for the density at $r = b$, see Figure 3.3. Let $\rho_1(r)$ denote the solution to (3.42) for $r > \bar{r}$, and let $\hat{\rho}(r)$ denote the density immediately on the outside of the shock. It follows from the uniqueness of solutions to (3.42) that an increase in $\bar{r}$ implies a lower ending value for the density at $r = b$ if and only if

$$\hat{\rho}'(\bar{r}) \, < \, \rho_1'(\bar{r}+) , \tag{3.43}$$

which we will show to hold. From (3.42) we have

$$(\hat{c}^2 - \hat{u}^2)\rho_1'(\bar{r}+) = \frac{\hat{\rho}(\bar{v}^2 + \hat{u}^2)}{\bar{r}} , \tag{3.44}$$

where a bars (hats) denote evaluation immediately on the inside (outside) of the shock. To express $\hat{\rho}'(\bar{r})$ we use the Rankine-Hugoniot relations. As $\rho u = C_a/r$ throughout we get from $(3.17)_2$ that

$$P(\hat{\rho}) + \frac{C_a^2}{\hat{\rho}\,\bar{r}^2} = P(\bar{\rho}) + \frac{C_a^2}{\bar{\rho}\,\bar{r}^2} .$$

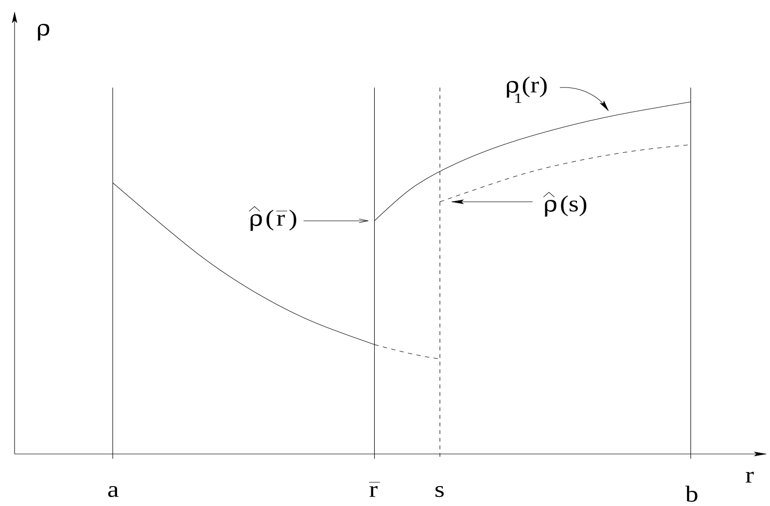

Figure 3.3. Configuration in the case of supersonic inflow at $r = a$.

Taking the derivative with respect to $\bar{r}$ and rearranging gives

$$(\hat{c}^2 - \hat{u}^2)\hat{\rho}' = \frac{2C_a^2}{\bar{r}^3}\Big(\frac{1}{\hat{\rho}} - \frac{1}{\bar{\rho}}\Big) + (\bar{c}^2 - \bar{u}^2)\bar{\rho}' = \frac{2C_a^2}{\bar{r}^3}\Big(\frac{1}{\hat{\rho}} - \frac{1}{\bar{\rho}}\Big) + \frac{\bar{\rho}(\bar{v}^2 + \bar{u}^2)}{\bar{r}}, \quad (3.45)$$

where we have used that (3.42) holds throughout (a, b). From (3.44) and (3.45), and recalling that $\hat{c}^2 - \hat{u}^2 > 0$, we get that (3.43) holds if and only if $C_a^2(\bar{\rho} - \hat{\rho}) < D_a^2\bar{\rho}\hat{\rho}(\hat{\rho} - \bar{\rho})$, which holds since the shock is compressive ($\hat{\rho} > \bar{\rho}$).

It follows that the minimal value α for ρ_b is attained by placing the shock at $\bar{r} = b-$, while the maximal value for ρ_b is attained by placing the shock at $\bar{r} = a+$. We summarize our findings in:

Theorem 3.4.1. (Possible shocks in cylindrically symmetric flow) *Consider the stationary, cylindrically symmetric, barotropic Euler equations. Given radii $a < b$ and data ρ_a, u_a, v_a, w_a which corresponds to supersonic inflow at $r = a$ (i.e. $u_a^2 > c_a^2$, $u_a > 0$).*

Then there is a finite interval (α, β) of ρ_b-values that can be reached from the data at $r = a$ through a stationary, compressive shock located at some location $\bar{r} \in (a, b)$. α, β depend only on a, ρ_a, u_a, v_a, and b, and there is a one-to-one correspondence between ρ_b values in (α, β) and shock locations in (a, b).

Remark 3.4.1. *A similar analysis applies to the case of CS flow with (radially) subsonic data given at $r = a$, as well as to the case of SS flows.*

Chapter 4

Symmetric shocks in stationary non-barotropic flow on a bounded domain

4.1 Introduction

This chapter extends to non-barotropic flows the constructions in the previous chapter of stationary solutions to the full Euler system with spherical or cylindrical symmetry. The physical setup is the same as in Chapter 3. We consider the domain between two concentric spheres or cylinders $r = a$ and $r = b$, where $a < b$. into which a compressible fluid is injected with a prescribed constant density ρ_a, velocity $\mathbf{U}_a$ and temperature θ_a at the inner boundary $r = a$, or at the outer boundary with constant data ρ_b, $\mathbf{U}_b$ and θ_b.

As in the previous chapter, inner solutions and outer solutions will be used to build stationary shock solutions. The conclusions found will mirror those of Chapter 3 where the main results on inner and outer solutions are summarized in Propositions 4.2.1 and 4.2.2.

In section 4.3 we again show how to build symmetric, entropy-satisfying shock solutions to the Euler equations by using either inner or outer solutions. In each case, since we consider only stationary solutions and due to the compressibility of admissible shocks, there is only a single shock in these solutions. These results are summarized in Theorems 4.3.1 and 4.3.2.

Section 4.4 addresses the following question: Taking a, b, and data at $r = a$ as fixed, can one formulate necessary and sufficient conditions on the flow variables at $r = b$ which guarantee the existence of a stationary, weak solution of the barotropic Euler equations with these boundary values, and which contains a single shock at *some* location $\bar{r} \in (a, b)$. The answer is provided, for cylindrically symmetric flow with or without swirl, in Theorem 4.4.1.

Remark 4.1.1. *Since we consider stationary solutions we find that the entropy is constant throughout smooth regions. In this sense the situation reduces to the barotropic case already considered in in Chapter 3, and one could use part of that analysis to establish existence of stationary profiles also in the case of the "full" system. However, we would then need to consider the energy balance separately. Also, the Rankine-Hugoniot relations are genuinely different in the full case we consider here. We therefore work with the full system throughout. This amounts to solving ODEs instead of algebraic equations as in the previous chapter.*

Remark 4.1.2. *In the case of smooth, inviscid flow without swirl of an ideal polytropic gas in a cone was analyzed in [11]. Below we extend the analysis to more general equations of state, and in the CS case we also consider flows with swirl.*

4.1.1 Equations

The full (non-barotropic) compressible Euler equations express the conservation of mass and the balance of momentum and of energy. In Eulerian coordinates the equations in $\mathbb{R}^3$ take the form

$$\rho_t + \operatorname{div}(\rho \mathbf{U}) = 0 \tag{4.1}$$

$$(\rho \mathbf{U}^i)_t + \operatorname{div}(\rho \mathbf{U}^i \mathbf{U}) + p_{x_i} = 0 \tag{4.2}$$

$$(\rho E)_t + \operatorname{div}\big((\rho E + p(\rho, S))\mathbf{U}\big) = 0\,. \tag{4.3}$$

Here $x \in \mathbb{R}^3$ is the spatial coordinate, $t > 0$ is time, and $\rho, \mathbf{U} = (\mathbf{U}^1, \mathbf{U}^2, \mathbf{U}^3)$, θ, S are the density, velocity, temperature, entropy, respectively. The specific total energy is then $E = \frac{1}{2}|\mathbf{U}|^2 + e$, where the specific internal energy $e = e(\rho, S)$ satisfies the fundamental relations $e_S = \theta > 0$, $e_\rho = p/\rho^2 > 0$.

For spherical (cylindrical) symmetric flow the density, velocities, and temperature depend only on time and the radial distance to the origin (x_3-axis). We refer to these as the spherically symmetric (SS) and the cylindrically symmetric (CS) cases, respectively. We let (u, v, w) be the velocity components in either spherical or cylindrical coordinates. We set $r = |x|$ in the SS case, while $r = \sqrt{x_1^2 + x_2^2}$ in the CS case. In either case, with a slight abuse of notation we write $\rho(x, t) = \rho(r, t)$, *etc.* Thus,

$$\begin{aligned}
\mathbf{U}(x,t) &= \frac{u(r,t)}{r}x\,, \qquad v = w \equiv 0 \qquad \text{(SS case)} \\
\mathbf{U}(x,t) &= \frac{u(r,t)}{r}(x_1, x_2, 0) + \frac{v(r,t)}{r}(-x_2, x_1, 0) + w(r,t)(0,0,1) \qquad \text{(CS case).}
\end{aligned}$$

The equations (4.1)-(4.3) reduce to (see [32])

$$\rho_t + (\rho u)_\xi = 0 \tag{4.4}$$

$$(\rho u)_t + (\rho u^2)_\xi - \frac{\rho v^2}{r} + p_r = 0 \tag{4.5}$$

$$(\rho v)_t + (\rho u v)_\xi + \frac{\rho u v}{r} = 0 \tag{4.6}$$

$$(\rho w)_t + (\rho u w)_\xi = 0 \tag{4.7}$$

$$(\rho e)_t + (\rho u e)_\xi + p u_\xi = 0, \tag{4.8}$$

where $\partial_\xi = \partial_r + m/r$, $m = 1$ in the CS case and $m = 2$ in the SS case.

4.1.2 Setup and assumptions

We treat simultaneously the SS and CS cases in domains which are bounded by concentric and fixed spheres or cylinders with radii $b > a > 0$. The solutions take on prescribed data at the inner or outer boundaries $\{r = a\}$ and $\{r = b\}$. We make the following standard assumptions, see [11]. Pressure is considered as a function of entropy and density,

$$p := f(\rho, S)\,, \tag{4.9}$$

where we assume that f is a C^2 map such that

$$f_\rho > 0\,, \qquad f_{\rho\rho} \geq 0\,, \qquad \text{and} \qquad f_S > 0\,. \tag{4.10}$$

Furthermore we assume

$$e \to 0\,, \quad p/\rho \to 0\,, \quad c \to 0 \qquad\qquad \text{as } \rho \to 0, \tag{4.11}$$

and that

$$c \to \infty \qquad\qquad \text{as } \rho \to \infty, \tag{4.12}$$

where $c = \sqrt{f_\rho}$ is the local sound speed.

4.2 Stationary solutions of the full Euler equations

ODE system for spherically/cylindrically symmetric flow Assuming that the constant mass flux $\rho u r^m$ is nonzero, the stationary Euler equations for SS or CS flow reduce to the ODE system

$$\frac{d(\rho u r^m)}{dr} = 0 \tag{4.13}$$

$$\rho u \frac{du}{dr} - \rho \frac{v^2}{r} + \frac{dp}{dr} = 0 \tag{4.14}$$

$$r\,\frac{dv}{dr} + v = 0 \tag{4.15}$$

$$\frac{dw}{dr} = 0 \tag{4.16}$$

$$\frac{d}{dr}\left[(\rho E + p) u r^m\right] = 0\,, \tag{4.17}$$

while the Rankine-Hugoniot conditions across a stationary discontinuity reduce to

$$[\rho u] = 0\,, \qquad \left[p + \rho u^2\right] = 0\,, \qquad [\rho u v] = 0\,, \qquad [\rho u w] = 0\,, \qquad [(\rho E + p) u] = 0\,. \tag{4.18}$$

For stationary flow we have that

$$\frac{d}{dr}(\rho u r^m S) = 0\,.$$

As the mass flux is nonzero the entropy takes on the same constant value throughout any region where the flow is smooth. The pressure is thus a function of ρ alone for smooth flows: $p = f(\rho)$.

4.2.1 Inner solutions for SS/CS flow

We prescribe Dirichlet data $\rho_a > 0$, $\theta_a > 0$, $u_a \neq 0$, and v_a, w_a on the inner boundary $r = a$, and seek a smooth, stationary solution to the Euler equations in the region $r \geq a$. We refer to this as an *inner solution*. The data at the inner boundary determine the value $S = S_a$ of the specific entropy at $r = a$. Above we found that the entropy is constant where the flow is smooth which gives

$$p = f(\rho) \equiv f(\rho, S_a)\,.$$

This effectively reduces the problem of existence of an inner solution to the corresponding question for *barotropic* flow. This was treated in detail in the previous chapter under the hypotheses that pressure $p = P(\rho)$ is an increasing, convex function of density with $\lim_{\rho\downarrow 0} P'(\rho) = 0$. In the present context these assumptions are satisfied due to $(4.10)_1$, $(4.10)_2$, and $(4.11)_3$, respectively.

For convenience we briefly outline the arguments from Chapter 3 for the existence of inner solutions. Equation (4.15) yields $r\,v \equiv D_a := v_a a$. Substituting into (4.14) and integrating once, we obtain

$$u^2 + v^2 + \Pi(\rho) \equiv u_a^2 + v_a^2 =: V_a^2\,, \tag{4.19}$$

where we have defined the function

$$\Pi(\rho) = \Pi(\rho, \rho_a, S_a) := \int_{\rho_a}^{\rho} \frac{2 f_\rho(\rho, S_a)}{\sigma}\, d\sigma\,. \tag{4.20}$$

From (4.13) we get $\rho u r^m \equiv C_a := \rho_a u_a a^m$. Together with $rv \equiv D_a$ and (4.19) this shows that the denisty $\rho = \rho(r)$ along the profile is given implicitly as the solution

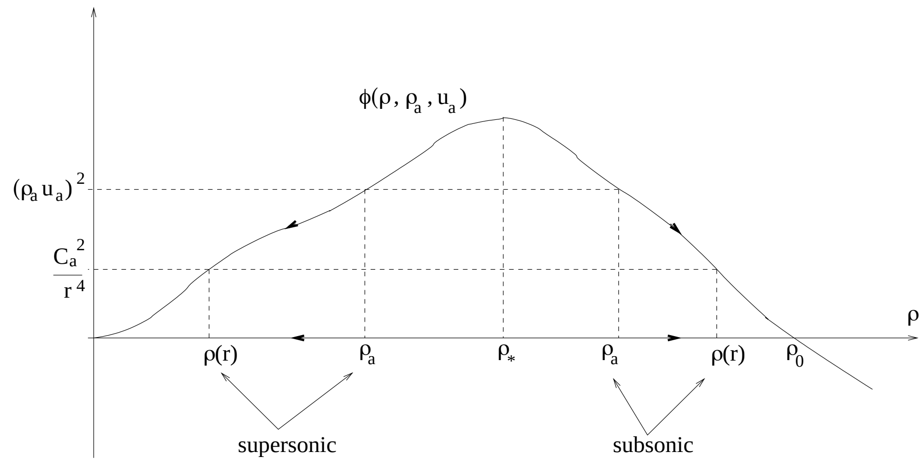

Figure 4.1. Inner solutions. The function $\phi(\rho, \rho_a, u_a)$. Arrows indicate direction as r increases from $r = a$.

of the algebraic equation

$$\frac{1}{r^m} = \frac{\rho^2}{C_a^2 + \rho^2 D_a^2 \delta_{m,1}} \left[V_a^2 - \Pi(\rho)\right] =: \Phi(\rho, \rho_a, u_a, v_a, S_a) \,. \tag{4.21}$$

(The Kronecker delta $\delta_{m,1}$ is used in order to treat both SS and CS flow at the same time.) In Chapter 3 it was shown that, as a consequence of our assumptions on the pressure function $p = f(\rho, S)$, the function $\Phi(\rho) = \Phi(\rho, \rho_a, u_a, v_a, S_a)$ has the form as in Figure 4.1, and that (4.21) defines an inner solution $\rho(r)$ for all $r \geq a$. The radial velocity along the profile is then given by $u(r) = \frac{C_a}{r^m \rho(r)}$.

Remark 4.2.1. *Note that the energy equation (4.17) has been applied in finding $\rho(r)$ and $u(r)$. In the analysis above we use that the specific entropy remains constant along the flow, a fact that depends on (4.17).*

Finally, the internal energy along the profile is obtained by substituting $\rho u r^m \equiv C_a$ into (4.17) and integrating. This yields Bernoulli's identity for steady flow:

$$\frac{u^2 + v^2}{2} + e + \frac{p}{\rho} \equiv B_a \,, \qquad \text{where } B_a := \frac{V_a^2}{2} + e_a + \frac{p_a}{\rho_a} \,. \tag{4.22}$$

We refer to the previous chapter for the analysis of the sonicity of the constructed solution. The conclusion is that if the flow is supersonic (subsonic) at $r = a$, then it becomes increasingly so as r increases from a: the radial Mach-number $M = \frac{|u|}{c}$ increases (decreases) as r increases.

We may also analyze the radial velocity by deriving an ODE for $u(r)$. From (4.14) we first calculate

$$\frac{dp}{dr} = f_\rho \frac{d\rho}{dr} = c^2 \frac{d\rho}{dr},$$

where we have used that the entropy is constant, $S \equiv S_a$. We then use $\rho(r)u(r)r^m \equiv C_a$ to compute $\frac{d\rho}{dr}$ and substitute into (4.14) to obtain

$$\frac{du}{dr} = \frac{u(v^2 + mc^2)}{r(u^2 - c^2)}, \tag{4.23}$$

where (suppressing the dependence of f on S) $c^2 = f_\rho\left(\frac{C_a}{ur^m}\right)$ and $v = \frac{D_a}{r}$. We observe from (4.22) and $e > 0$ that the velocity is uniformly bounded along the flow. The four possible velocity profiles of an inner solution, corresponding to whether $u_a \gtrless 0$ and $|u_a| \gtrless c_a$, are sketched in Figure 4.2 (for $r \geq a$). Finally, let us see that

$$\lim_{r\uparrow\infty} u^2 = \begin{cases} 2B_a & \text{for supersonic flow,} \\ 0 & \text{for subsonic flow.} \end{cases} \tag{4.24}$$

For supersonic flow we have (see Figure 4.1 that $\rho \downarrow 0$ as $r \uparrow \infty$. Assumptions $(4.11)_{1,2}$ then give $e + p/\rho \to 0$, and Bernoulli's identity yields $u^2 \to 2B_a$. For subsonic flow ρ is bounded away from zero, whence $\rho(r)u(r)r^m \equiv C_a$ shows that $|u| \downarrow 0$ as $r \uparrow \infty$. This establishes (4.24). We summarize the results in:

Proposition 4.2.1. (Existence of spherically/cylindrically symmetric inner solutions.) *Consider the full, stationary Euler equations with spherical/cylindrical symmetry (4.13) - (4.17) in the exterior of a sphere/cylinder with radius $a > 0$, and with prescribed Dirichlet data $\rho_a > 0$, $u_a \neq 0$, v_a, w_a, $\theta_a > 0$ at $r = a$. Assuming the data are radially non-sonic, then (4.13) -(4.17) have a unique solution defined for all $r \geq a$. The resulting flow is strictly supersonic (subsonic) with increasing (decreasing) Mach number as r increases, if and only if it is strictly supersonic (subsonic) at the inner boundary $r = a$. See Figure 4.1 and Figure 4.2.*

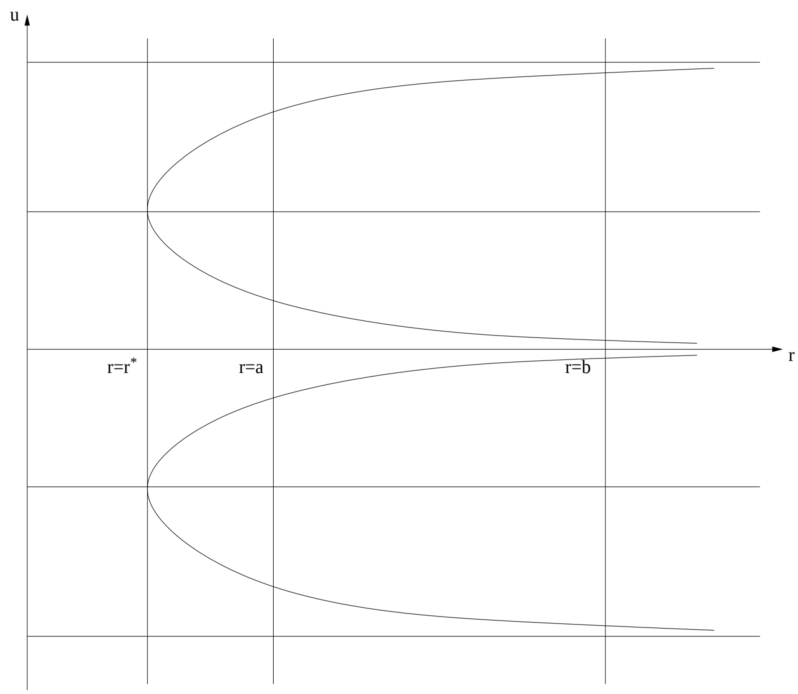

Figure 4.2. Radial velocity in stationary solution

4.2.2 Outer solutions for spherically/cylindrically symmetric flow

Next we consider Dirichlet data $\rho_b > 0$, $e_b > 0$, $u_b \neq 0$, v_b, w_b on the outer boundary $r = b$, and we seek a smooth stationary solution to the Euler equations in the region $r \leq b$. We refer to this as an outer solution. To obtain a solution on a non-trivial interval we assume that the data are radially non-sonic. As indicated in Figure 4.2 there is then a critical radius $r^* < b$ where the flow becomes sonic and beyond which the solution cannot be extended. The same analysis as above shows that

$$\rho u r^m \equiv C_b\,, \qquad rv \equiv D_b\,, \qquad w \equiv w_b\,, \qquad \frac{u^2+v^2}{2} + e + \frac{p}{\rho} \equiv B_b\,. \tag{4.25}$$

Arguing as above (see Chapter 3 for details) we get that the density profile again is given by an algebraic equation of the form (4.21) (with a replaced by b). Letting r decrease from $r = b$ it is clear from Figure 4.1 that there is a finite and strictly positive inner radius r^* beyond which the stationary solution can not be defined.

The situation for the velocity profile is given in Figure 4.2. The analysis in in the previous chapter shows that the flow becomes sonic at the critical radius. Summarizing we have:

Proposition 4.2.2. (Existence of spherically/cylidrically symmetric outer solution) *Consider the full, stationary Euler equations with spherical/cylindrical symmetry (4.13) - (4.17) in the interior of a sphere with radius* $b > 0$*, and with prescribed Dirichlet data* $\rho_b > 0$*,* $u_b \neq 0$*,* v_b*,* w_b*,* $\theta_b > 0$ *at* $r = b$*. Assuming the data are radially non-sonic, there is a critical radius* $r^* \in (0, b)$ *such that (4.13) - (4.17) have a unique solution defined for all* $r \in (r^*, b)$*. The resulting flow is strictly supersonic (subsonic) with decreasing (increasing) Mach number as* r *decreases on the interval* (r^*, b) *if and only if it is strictly supersonic (subsonic) at the outer boundary* $r = b$*. The flow becomes sonic at* $r = r^*$ *and cannot be extended as a stationary solution inside this radius. See Figure 4.2.*

4.3 Stationary solutions with shocks

Next we use the inner and outer solutions from above to construct symmetric *weak* solutions with a single stationary, admissible shock located at any intermediate location $\bar{r} \in (a, b)$.

4.3.1 Shocks built from inner solutions

Consider Dirichlet data as in Proposition 4.2.1 given on the inner boundary $r = a$. We fix an outer boundary at $r = b > a$ and choose an intermediate radius $\bar{r} \in (a, b)$ which will be the shock location. We next describe the assumptions on the equation of state that guarantee existence and uniqueness of an admissibility shock. As the Rankine-Hugoniot conditions are identical with those of *planar* shocks (oblique, in the case of swirl) we refer to [11] for the details of the arguments.

Let the specific volume be denoted $\tau = 1/\rho$ and set $\rho(\bar{r}\pm) = \rho_\pm$, etc. For (τ_-, p_-) fixed, the points of intersection (different from (τ_-, p_-)) between these two curves

$$-k^2 = \frac{p - p_-}{\tau - \tau_-}, \qquad k := \rho_- u_- = \rho_+ u_+ \qquad \text{(Rayleigh line)} \tag{4.26}$$

and

$$H(\tau,p) := e(\tau,p)-e(\tau_-,p_-)+\tfrac{1}{2}(\tau-\tau_-)(p+p_-) = 0\,, \qquad \text{(Hugoniot curve)} \quad (4.27)$$

provide the possible states at $\bar{r}+$ that satisfy the Rankine-Hugoniot conditions (4.18). (e is now considered as a function of τ and p.) We make the following standard assumptions on the Hugoniot curve $\{H(\tau,p)=0\}$:

A1. The pressure along the Hugoniot curve increases monotonically from 0 to $+\infty$ as the specific volume decreases from a maximal value $\tau_{max} \leq \infty$ to a minimal value $\tau_{min} \geq 0$. We denote the pressure along the Hugoniot curve by $p = G(\tau)$.

A2. Any straight line through (τ_-,p_-) which intersects the τ-axis at a point with $\tau \leq \tau_{max}$ intersects the Hugoniot curve at a unique point (different from (τ_-,p_-)).

As admissibility criteria we request that the entropy of a fluid particle should increase as it passes through a shock. It is demonstrated in [11] that A1 and A2, in conjunction with assumptions (4.10), imply that admissible shocks are such that fluid particles are compressed, and pass from supersonic to subsonic flow, as they traverse the shock surface.

Consider data at $r = a$ that are strictly supersonic (subsonic). By the earlier analysis there is a unique flow satisfying the stationary and symmetric Euler equations on $r \in [a,\bar{r})$, and it is supersonic (subsonic) also at $r = \bar{r}-$. It follows that the flow must be directed outward (inward), i.e. $u_a > 0$ ($u_a < 0$). Under our assumptions the data ρ_+, e_+, u_+, v_+ and w_+ at $\bar{r}+$ are uniquely determined by the Rankine-Hugoniot conditions, and corresponds to strictly subsonic (supersonic) flow. Again there is a unique subsonic (supersonic) flow satisfying the stationary and symmetric Euler equations in $r \in [\bar{r}, b]$. This yields a stationary solution on $[a,b]$ with a single, entropy admissible shock at $\bar{r}$.

Proposition 4.3.1. (Stationary symmetric shocks built from inner solutions) *Consider the full, stationary Euler equations with spherical (or cylindrical) symmetry in the domain between two concentric spheres (cylinders) with radii* $a < b$

with prescribed Dirichlet data $\rho_a > 0$, $e_a > 0$, u_a, v_a and w_a at $r = a$. Assume that the flow is radially non-sonic at $r = a$, and fix any $\bar{r} \in (a, b)$.

Then there is a unique weak admissible solution with a single shock located at $\bar{r}$ if and only if, either, the flow is radially supersonic at $r = a$ and directed into the domain (i.e. $u_a > 0$), or the flow is radially subsonic at $r = a$ and directed out of the domain (i.e. $u_a < 0$). In the former case the flow is (radially) supersonic in $(a, \bar{r})$ and (radially) subsonic in $(\bar{r}, b)$, while the opposite holds in the latter case.

4.3.2 Shocks built from outer solutions

Given data $\rho_b > 0$, u_b, v_b, w_b, and $e_b > 0$ at the outer boundary $r = b$. The construction of stationary, symmetric solutions for $r < b$ with an admissible shock at some location $\bar{r} < b$ is similar to above, the only restriction being that we need to place the shock at a location $\bar{r}$ where the flow is defined. That is, provided we choose $\bar{r} \in (r_1^*, b)$, where r_1^* is the critical radius at which the flow constructed by starting at $r = b$ and solving inward, becomes radially sonic. Assuming this, our assumptions on the equation of state and the Hugoniot curve guarantee unique values of the solution at $\bar{r}-$, and we can solve inward until we reach the critical radius $r_2^* > 0$ corresponding to these values.

Proposition 4.3.2. (Stationary symmetric shocks built from outer solutions) *Consider the full, stationary Euler equations with spherical (or cylindrical) symmetry in the domain between two concentric spheres (cylinders) with radii $a < b$ with prescribed Dirichlet data $\rho_b > 0$, u_b, v_b, w_b, and $e_b > 0$ at $r = b$. Assume that the flow is radially non-sonic at $r = b$ and given any $\bar{r} \in (r_1^*, b)$ and $a \in (r_2^*, \bar{r})$.*

Then there is a unique weak admissible solution, defined on $[a, b]$ and with a single shock located at $\bar{r}$ if and only if the flow is radially supersonic at $r = b$ and directed into the domain (i.e. $u_b < 0$), or the flow is radially subsonic at $r = b$ and directed out of the domain (i.e. $u_b > 0$). In the former case the flow is radially supersonic at $r = b$ and radially subsonic at $r = a$, while the opposite holds in the latter case.

4.4 When can a shock solution be found?

We next consider the possibility of finding shock solutions for given boundary data. We consider the following question: given ρ_a, u_a, v_a, w_a, e_a at the inner boundary, what are the possible states that can be reached at $r = b$ through an admissible, stationary shock located at some intermediate $\bar{r} \in (a, b)$?

From the earlier analysis we know that $\rho u r^m \equiv C_a$, $rv \equiv D_a$, and $w \equiv w_a$ along any stationary solution (smooth or not). Thus, a necessary condition for the existence of a shock solution with "final" data ρ_b, u_b, v_b, w_b, e_b at $r = b$ is that

$$\rho_b u_b = \frac{C_a}{b^m}\,, \qquad v_b = \frac{D_a}{b}\,, \qquad w_b = w_a\,. \tag{4.28}$$

We choose to work with the density as the primary unknown so that the issue becomes: what final densities ρ_b can be attained for a solution with a shock at some $\bar{r} \in (a, b)$. For concreteness we consider the case with (radially) supersonic inflow at $r = a$, that is, $u_a > 0$ and $u_a^2 > c_a^2$.

To see how the final density ρ_b depends on the shock location $\bar{r}$, we first observe that the ODE for the density takes the same form in the two intervals $(a, \bar{r})$ and $(\bar{r}, b)$, and it is independent of $\bar{r}$. Indeed, from (4.23) it follows that

$$\frac{d\rho}{dr} = \frac{\rho(v^2 + mu^2)}{r(c^2 - u^2)}\,. \tag{4.29}$$

From the earlier analysis we know that the flow remains (radially) subsonic for all $r > \bar{r}$, whence (4.29) is a well-behaved ODE with unique solutions. Thus, if $\rho_1(r)$, $\rho_2(r)$ are two smooth solutions with $\rho_1(s) > \rho_2(s)$ for some $s > \bar{r}$, then necessarily $\rho_1(r) > \rho_2(r)$ for all $r > s$.

We can use this to infer how ρ_b varies with the shock location $\bar{r}$. Specifically we will show that an increase in the shock location $\bar{r}$ implies a lower ending value for the density at $r = b$, see Figure 4.3. Let $\rho_1(r)$ denote the solution to (4.29) for $r > \bar{r}$ whose "data" at $\bar{r}+$ is $\hat{\rho}(\bar{r})$ = the density immediately on the outside of the shock. By uniqueness of solutions to (4.29), an increase in $\bar{r}$ implies a lower ending value for the density at $r = b$, if and only if

$$\hat{\rho}'(\bar{r}) \;<\; \rho_1'(\bar{r}+)\,. \tag{4.30}$$

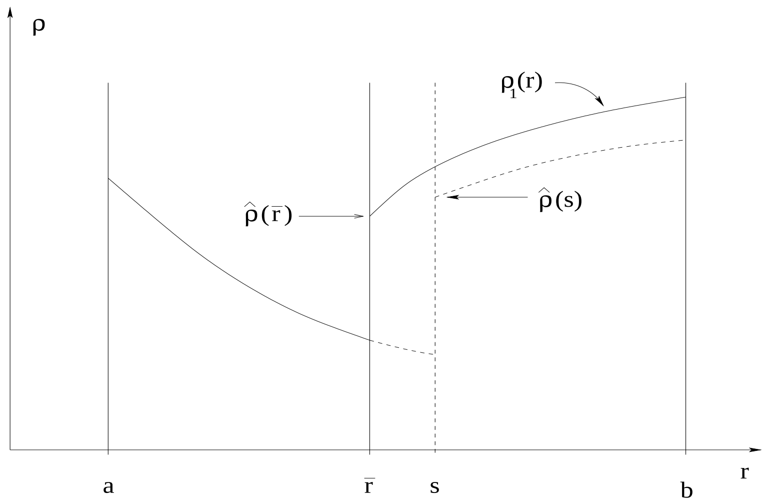

Figure 4.3. Configuration in the case of supersonic inflow at $r = a$.

We proceed to verify (4.30) by direct calculation. It is convenient now to regard the internal energy as a function of specific volume and pressure, $e = e(\tau, p)$. We start from the Rankine-Hugoniot relations

$$e(\hat\tau, \hat p) - e(\bar\tau, \bar p) + \tfrac{1}{2}(\hat\tau - \bar\tau)(\hat p + \bar p) = 0\,, \qquad \hat p - \bar p + k^2(\hat\tau - \bar\tau) = 0\,,$$

where $k = \hat\rho\hat u = \bar\rho\bar u = C_a/\bar r$, and bars (hats) denote evaluation immediately on the inside (outside) of the shock. Differentiating with respect to the shock location $\bar r$ ($' = \frac{d}{d\bar r}$) yields two linear equations for $\hat\tau' - \bar\tau'$ and $\hat p' - \bar p'$, and we get

$$\hat\tau' - \bar\tau' = \frac{1}{\Delta}\Big\{ \big[\bar e_\tau - \hat e_\tau\big]\bar\tau' + \big[(\bar e_p - \hat e_p) + (\bar\tau - \hat\tau)\big]\bar p' + \frac{2mk^2}{\bar r}(\bar\tau - \hat\tau)\big[\hat e_p + \tfrac{1}{2}(\hat\tau - \bar\tau)\big]\Big\}\,,$$

where

$$\Delta = \big[\hat e_\tau + \tfrac{1}{2}(\hat p - \bar p)\big] - k^2\big[\hat e_p + \tfrac{1}{2}(\hat\tau - \bar\tau)\big] = \frac{\hat c^2 - \hat u^2}{\hat\tau^2}\hat e_p\,.$$

Using that $\bar\tau' = -\bar\tau^2\bar\rho'$ and $\hat\tau' = -\hat\tau^2\hat\rho'$, we obtain

$$\hat\rho' = \frac{\bar\tau^2}{\hat\tau^2}\bar\rho' - \frac{1}{\hat\tau^2\Delta}\Big\{ \big[\bar e_\tau - \hat e_\tau\big]\bar\tau' + \big[(\bar e_p - \hat e_p) + (\bar\tau - \hat\tau)\big]\bar p' + \frac{2mk^2}{\bar r}(\bar\tau - \hat\tau)\big[\hat e_p + \tfrac{1}{2}(\hat\tau - \bar\tau)\big]\Big\}\,,$$

Next we use that $\bar p' = \bar c^2\bar\rho'$, collect terms multiplying $\bar\rho'$, and use that $e_\tau = -p +$

$c^2 e_p/\tau^2$. Rearranging gives

$$\begin{aligned}\hat{\rho}' &= \left\{\frac{\bar{\tau}^2}{\hat{\tau}^2} + \frac{\bar{\tau}^2}{\hat{c}^2 - \hat{u}^2}\left[\left(\frac{\bar{c}^2}{\bar{\tau}^2} - \frac{\hat{c}^2}{\hat{\tau}^2}\right)\hat{e}_p - \frac{\bar{c}^2 - \bar{u}^2}{\bar{\tau}^2}(\bar{\tau} - \hat{\tau})\right]\right\}\bar{\rho}' \\ &\quad - \frac{2mk^2(\bar{\tau} - \hat{\tau})}{\bar{r}\hat{\tau}^2\Delta}\left[\hat{e}_p + \tfrac{1}{2}(\hat{\tau} - \bar{\tau})\right].\end{aligned}$$

We want to show that this last expression is majorized by $\rho_1'(\bar{r}+)$, which is given by (4.29):

$$\rho_1'(\bar{r}+) = \frac{(\bar{v}^2 + m\hat{u}^2)}{\hat{\tau}\bar{r}(\hat{c}^2 - \hat{u}^2)}.$$

We substitute the two expressions into (4.30) and rearrange. In doing so we use the fact that the flow is subsonic at $\bar{r}+$ and also that $\hat{e}_p > 0$. This last relation follows from our assumption that pressure increases with increasing entropy for fixed density. Collecting terms that multiply $\hat{e}_p$ we obtain that (4.30) holds if and only if

$$\begin{aligned}&(\bar{v}^2 + m\bar{u}^2)(\hat{\tau} - \bar{\tau}) \ + mk^2\bar{\tau}(\bar{\tau} - \hat{\tau})^2 \\ < &\left[2mk^2\bar{\tau}(\bar{\tau} - \hat{\tau}) + (\bar{v}^2 + m\hat{u}^2) - \frac{(\bar{v}^2 + m\bar{u}^2)}{(\bar{c}^2 - \bar{u}^2)}\frac{(\bar{c}^2\hat{\tau}^2 - \hat{u}^2\bar{\tau}^2)}{\hat{\tau}^2}\right]\hat{e}_p\,.\end{aligned}$$

The left hand side equals $(\hat{\tau} - \bar{\tau})(\bar{v}^2 + mk^2\bar{\tau}\hat{\tau})$, which is negative since the shock is compressive. On the other hand the right hand side simplifies to $mk^2(\bar{\tau} - \hat{\tau})^2\hat{e}_p$, which is positive.

It follows that the minimal value α for the density at $r = b$ is attained by placing the shock at $\bar{r} = b-$, while the maximal value is attained by placing the shock at $\bar{r} = a+$. We summarize our findings in:

Theorem 4.4.1. (Possible shocks for outward symmetric flow) *Consider the stationary, symmetric, non-barotropic Euler equations (4.13) - (4.17). Consider radii* $a < b$ *and data* ρ_a, u_a, v_a, w_a, e_a *corresponding to supersonic inflow at* $r = a$ *(i.e.* $u_a^2 > c_a^2$, $u_a > 0$*).*

Then there is a finite interval (α, β) *of* ρ_b*-values that can be reached from the data at* $r = a$ *through a stationary, compressive shock located at some location* $\bar{r} \in (a, b)$. α, β *depend on* a, ρ_a, u_a, v_a, w_a, e_a *and* b, *and there is a one-to-one correspondence between* ρ_b *values in* (α, β) *and shock locations in* (a, b).

Bibliography

[1] N. S. Bahvalov, *The existence in the large of a regular solution of a quasilinear hyperbolic system*, Ž. Vyčisl. Mat. i Mat. Fiz. **10** (1970), 969–980. 0279443 (43 #5165) (Russian)

[2] Stefano Bianchini, *BV solutions of the semidiscrete upwind scheme*, Arch. Ration. Mech. Anal. **167** (2003), 1–81. 1967667 (2004k:35249)

[3] ______, *Hyperbolic limit of the Jin-Xin relaxation model*, Comm. Pure Appl. Math. **59** (2006), 688–753. 2172805 (2008b:35167)

[4] Stefano Bianchini and Alberto Bressan, *Vanishing viscosity solutions of nonlinear hyperbolic systems*, Ann. of Math. (2) **161** (2005), 223–342. 2150387 (2007i:35160)

[5] Alberto Bressan, *Global solutions of systems of conservation laws by wave-front tracking*, J. Math. Anal. Appl. **170** (1992), 414–432. 1188562 (93k:35166)

[6] ______, *Hyperbolic systems of conservation laws*, Oxford Lecture Series in Mathematics and its Applications, vol. 20, Oxford University Press, Oxford, 2000, ISBN 0-19-850700-3, The one-dimensional Cauchy problem. 1816648 (2002d:35002)

[7] Tung Chang and Ling Hsiao, *The Riemann problem and interaction of waves in gas dynamics*, Pitman Monographs and Surveys in Pure and Applied Mathematics, vol. 41, Longman Scientific & Technical, Harlow, 1989, ISBN 0-582-01378-X. 994414 (90m:35122)

[8] Gui-Qiang Chen and James Glimm, *Global solutions to the compressible Euler equations with geometrical structure*, Comm. Math. Phys. **180** (1996), 153–193. 1403862 (97j:35120)

[9] ______, *Global solutions to the cylindrically symmetric rotating motion of isentropic gases*, Z. Angew. Math. Phys. **47** (1996), 353–372. 1394912 (97h:76100)

[10] Earl A. Coddington and Norman Levinson, *Theory of ordinary differential equations*, McGraw-Hill Book Company, Inc., New York-Toronto-London, 1955. 0069338 (16,1022b)

[11] R. Courant and K. O. Friedrichs, *Supersonic Flow and Shock Waves*, Interscience Publishers, Inc., New York, N. Y., 1948. 0029615 (10,637c)

[12] Ronald J. DiPerna, *Existence in the large for quasilinear hyperbolic conservation laws*, Arch. Rational Mech. Anal. **52** (1973), 244–257. 0338576 (49 #3340)

[13] ______, *Global existence of solutions to nonlinear hyperbolic systems of conservation laws*, J. Differential Equations **20** (1976), 187–212. 0404872 (53 #8672)

[14] Thaddeus J. Edens, *Spectral analysis of a model of the compressible Euler equations*, Commun. Math. Sci. **5** (2007), 205–232. 2310640 (2008b:35172)

[15] E. Endres and H.K. Jenssen, *Compressible 1-D Euler Equations with Large Data: A Case Study*, submitted.

[16] E. Endres, H.K. Jenssen, and M. Williams, *Symmetric Euler and Navier-Stockes shocks in stationary barotropic flow on a bounded domain*, accepted, J. Differential Equations (2008).

[17] ———, *Symmetric Euler and Navier-Stockes shocks in stationary flow on a bounded domain: the non-barotropic case*, in preparation.

[18] D. Gilbarg, *The existence and limit behavior of the one-dimensional shock layer*, Amer. J. Math. **73** (1951), 256–274. 0044315 (13,401e)

[19] James Glimm, *Solutions in the large for nonlinear hyperbolic systems of equations*, Comm. Pure Appl. Math. **18** (1965), 697–715. 0194770 (33 #2976)

[20] Helge Kristian Jenssen, *Blowup for systems of conservation laws*, SIAM J. Math. Anal. **31** (2000), 894–908 (electronic). 1752421 (2001a:35114)

[21] J.-L. Joly, G. Métivier, and J. Rauch, *A nonlinear instability for* 3×3 *systems of conservation laws*, Comm. Math. Phys. **162** (1994), 47–59. 1272766 (95f:35145)

[22] P. Lax, *Hyperbolic systems of conservation laws, II.*, Comm. Pure Appl. Math. **10** (1957), 537–566.

[23] Tai Ping Liu, *The deterministic version of the Glimm scheme*, Comm. Math. Phys. **57** (1977), 135–148. 0470508 (57 #10259)

[24] ———, *Solutions in the large for the equations of nonisentropic gas dynamics*, Indiana Univ. Math. J. **26** (1977), 147–177. 0435618 (55 #8576)

[25] ———, *Initial-boundary value problems for gas dynamics*, Arch. Rational Mech. Anal. **64** (1977), 137–168. 0433017 (55 #5996)

[26] Takaaki Nishida, *Global solution for an initial boundary value problem of a quasilinear hyperbolic system*, Proc. Japan Acad. **44** (1968), 642–646. 0236526 (38 #4821)

[27] Takaaki Nishida and Joel A. Smoller, *Solutions in the large for some nonlinear hyperbolic conservation laws*, Comm. Pure Appl. Math. **26** (1973), 183–200. 0330789 (48 #9126)

[28] Takaaki Nishida and Joel Smoller, *Mixed problems for nonlinear conservation laws*, J. Differential Equations **23** (1977), 244–269. 0427852 (55 #882)

[29] Yue-Jun Peng, *Solutions faibles globales pour l'équation d'Euler d'un fluide compressible avec de grandes données initiales*, Comm. Partial Differential Equations **17** (1992), 161–187. 1151259 (93e:35083) (French, with English summary)

[30] B. Riemann, *Gesammelte Werke* (1896), pp. 149ff.

[31] Nils Henrik Risebro, *A front-tracking alternative to the random choice method*, Proc. Amer. Math. Soc. **117** (1993), 1125–1139. 1120511 (93e:35071)

[32] B. L. Roždestvenskiĭ and N. N. Janenko, *Systems of quasilinear equations and their applications to gas dynamics*, Translations of Mathematical Monographs, vol. 55, American Mathematical Society, Providence, RI, 1983, ISBN 0-8218-4509-8, Translated from the second Russian edition by J. R. Schulenberger. 694243 (85f:35127)

[33] Yu. G. Rykov, *On the Cauchy problem for systems of equations of one-dimensional gas dynamics*, Tr. Mosk. Mat. Obs. **59** (1998), 221–239. 1824509 (2001k:76080) (Russian, with Russian summary)

[34] Denis Serre, *Systems of conservation laws. 1*, Cambridge University Press, Cambridge, 1999, ISBN 0-521-58233-4, Hyperbolicity, entropies, shock waves; Translated from the 1996 French original by I. N. Sneddon. 1707279 (2000g:35142)

[35] Joel Smoller, *Shock waves and reaction-diffusion equations*, 2nd ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 258, Springer-Verlag, New York, 1994, ISBN 0-387-94259-9. 1301779 (95g:35002)

[36] J. Blake Temple, *Solutions in the large for the nonlinear hyperbolic conservation laws of gas dynamics*, J. Differential Equations **41** (1981), 96–161. 626623 (82i:35117)

[37] B. Temple and R. Young, *The large time stability of sound waves*, Comm. Math. Phys. **179** (1996), 417–466. 1400747 (97f:35132)

[38] Robin Young, *Exact solutions to degenerate conservation laws*, SIAM J. Math. Anal. **30** (1999), 537–558 (electronic). 1677943 (2000b:35165)

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