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FORMALITY AND KONTSEVICH–DUFLO THEOREM

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Abstract

Kontsevich's formality theorem states that there exists an L_∞ quasi-isomorphism from the dgla $T^\bullet_{\text{poly}}(M)$ of polyvector fields on a smooth manifold M to the dgla $D^\bullet_{\text{poly}}(M)$ of polydifferential operators on M, which extends the classical Hochschild–Kostant–Rosenberg map. The construction of Kontsevich formality morphism involves Fedosov type resolutions of the dglas of polyvector fields and polydifferential opartors on smooth manifolds. We introduce, for every $\mathbb{Z}$-graded manifold, a formal exponential map defined in a purely algebraic way and study its properties. As an application, we give a simple new construction of a Fedosov type resolution of the algebra of smooth functions of $\mathbb{Z}$-graded manifolds and we extend the Emmrich–Weinstein theorem to the context of $\mathbb{Z}$-graded manifolds.

We also extend Kontsevich's formality theorem to *Lie pairs*, a framework which includes a range of diverse geometric contexts such as complex manifolds, foliations, and $\mathfrak{g}$-manifolds (that is, manifolds endowed with an action of a Lie algebra $\mathfrak{g}$). The spaces $\operatorname{tot}\left(\Gamma(\Lambda^\bullet A^\vee)\otimes_R \mathcal{T}^\bullet_{\text{poly}}\right)$ and $\operatorname{tot}\left(\Gamma(\Lambda^\bullet A^\vee)\otimes_R \mathcal{D}^\bullet_{\text{poly}}\right)$ associated with a Lie pair (L,A) each carry an L_∞-algebra structure canonical up to L_∞ quasi-isomorphism. These two spaces serve as replacements for the spaces of polyvector fields and polydifferential operators, respectively. Their corresponding cohomology groups $\mathbb{H}^\bullet_{\text{CE}}(A,\mathcal{T}^\bullet_{\text{poly}})$ and $\mathbb{H}^\bullet_{\text{CE}}(A,\mathcal{D}^\bullet_{\text{poly}})$ are Gerstenhaber algebras. We establish the following formality theorem for Lie pairs: there exists an L_∞ quasi-isomorphism from $\operatorname{tot}\left(\Gamma(\Lambda^\bullet A^\vee)\otimes_R \mathcal{T}^\bullet_{\text{poly}}\right)$ to $\operatorname{tot}\left(\Gamma(\Lambda^\bullet A^\vee)\otimes_R \mathcal{D}^\bullet_{\text{poly}}\right)$ whose first Taylor coefficient is equal to $\operatorname{hkr}\circ(\operatorname{td}^\nabla_{L/A})^{\frac{1}{2}}$. Here the Todd cocycle $(\operatorname{td}^\nabla_{L/A})^{\frac{1}{2}}$ acts on $\operatorname{tot}\left(\Gamma(\Lambda^\bullet A^\vee)\otimes_R \mathcal{T}^\bullet_{\text{poly}}\right)$ by contraction. As a consequence, we prove a Kontsevich–Duflo type theorem for Lie pairs: the Hochschild–Kostant–Rosenberg map twisted by the square root of the Todd class of the Lie pair (L,A) is an isomorphism of Gerstenhaber algebras from $\mathbb{H}^\bullet_{\text{CE}}(A,\mathcal{T}^\bullet_{\text{poly}})$ to $\mathbb{H}^\bullet_{\text{CE}}(A,\mathcal{D}^\bullet_{\text{poly}})$. As applications, we establish formality theorems and Kontsevich–Duflo type theorems for complex manifolds, foliations, and $\mathfrak{g}$-manifolds. In the case of complex manifolds, we recover the Kontsevich–Duflo theorem of complex geometry.

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Chapter 1 Introduction

In this dissertation research, we investigate geometric and algebraic structures closely related to Kontsevich formality theorem which was invented to solve deformation quantization problem for Poisson manifolds. Deformation quantization is a problem originating from physics which was proposed by Bayen, Flato, Fronsdal, and Lichnerowicz and Sternheimer in [5, 6] as a transition from classical mechanics to quantum mechanics. More precisely, for a given Poisson manifold $(M, \{,\})$, deformation quantization concerns *star products*, i.e., $\hbar$-linear associative products on the space $C^\infty(M)[\![\hbar]\!]$ whose restrictions to $C^\infty(M)$ are of the form

$$f * g = fg + \hbar\{f, g\} + \hbar^2 B_2(f, g) + \cdots ,$$

where the coefficients B_i are bidifferential operators on M. In a purely algebraic context, the problem of deforming commutative associative algebras was considered by Gerstenhaber in [25] who establishes the link with Hochschild cohomology.

In the late 1990's, Kontsevich revolutionized the field of deformation quantization with his formality theorem: there exists an L_∞ quasi-isomorphism from the dgla $T^\bullet_{\text{poly}}(M)$ of polyvector fields on a smooth manifold M to the dgla $D^\bullet_{\text{poly}}(M)$ of polydifferential operators on M whose first 'Taylor coefficient' is the classical Hochschild–Kostant–Rosenberg map. More precisely, Kontsevich gave an explicit formula for the formality quasi-isomorphism in the case $M = \mathbb{R}^d$ and then outlined how the result can be generalized to arbitrary smooth manifolds [30]. On the other hand, in [61], Tamarkin gave a different proof of Kontsevich's formality theorem for $\mathbb{R}^d$ via operadic methods. Later, Dolgushev gave a detailed proof of the globalization to arbitrary smooth manifolds of Kontsevich's formality quasi-isomorphism for $\mathbb{R}^d$ based on Fedosov's patching technique [18]. As an important application, Kontsevich's formality theorem implies the existence of deformation quantizations for any smooth Poisson manifolds [14, 18, 30, 61].

The formality theorem also led to several new developments in geometry and algebra. In particular, Kontsevich proposed a new proof of Duflo's theorem [20] in Lie theory based on his formality theorem. For every Lie algebra $\mathfrak{g}$, the symmetrization map $\mathrm{pbw} : S(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$ is an isomorphism of $\mathfrak{g}$-modules called Poincaré–Birkhoff–Witt isomorphism. The induced isomorphism $\mathrm{pbw} : S(\mathfrak{g})^{\mathfrak{g}} \to \mathcal{U}(\mathfrak{g})^{\mathfrak{g}}$ between subspaces of $\mathfrak{g}$-invariants does not intertwine the obvious multiplications on $S(\mathfrak{g})^{\mathfrak{g}}$ and $\mathcal{U}(\mathfrak{g})^{\mathfrak{g}}$. However, it can be modified so as to become an isomorphism of associative algebras. The Duflo element $J \in \hat{S}(\mathfrak{g}^\vee)$ of a Lie algebra $\mathfrak{g}$ is the formal polynomial on $\mathfrak{g}$ defined by $J(x) = \det\left(\frac{1-e^{-\mathrm{ad}_x}}{\mathrm{ad}_x}\right)$, for all $x \in \mathfrak{g}$. Considered as a translation-invariant formal differential operator on $\mathfrak{g}^\vee$, the square root of the Duflo element defines a transformation $J^{\frac{1}{2}} : S(\mathfrak{g}) \to S(\mathfrak{g})$. A remarkable theorem due to Duflo [20] asserts that the composition $\mathrm{pbw} \circ J^{\frac{1}{2}} : S(\mathfrak{g})^{\mathfrak{g}} \to \mathcal{U}(\mathfrak{g})^{\mathfrak{g}}$ is an isomorphism of associative algebras. Duflo's theorem generalizes a fundamental result of Harish-Chandra regarding the center of the universal enveloping algebra of a semi-simple Lie algebra. Duflo's original proof was based on deep and sophisticated techniques of representation theory including Kirillov's orbit method. As an application of his formality construction, Kontsevich proposed a new proof of Duflo's theorem by means of the induced associative algebra structure on tangent cohomology at a Maurer–Cartan element. Indeed, Kontsevich's approach [30] has led to an extension of Duflo's theorem: *For every finite dimensional Lie algebra* $\mathfrak{g}$*, the map* $\mathrm{pbw} \circ J^{\frac{1}{2}} : H^\bullet_{\mathrm{CE}}(\mathfrak{g}, S(\mathfrak{g})) \to H^\bullet_{\mathrm{CE}}(\mathfrak{g}, \mathcal{U}(\mathfrak{g}))$ *is an isomorphism of graded associative algebras.* The classical Duflo theorem is the isomorphism of the cohomologies in degree 0. A detailed proof of the above extended Duflo theorem was given by Pevzner–Torossian [54] (see also [42, 43]).

Kontsevich discovered a similar phenomenon in complex geometry. Recall that the Hochschild cohomology groups $HH^\bullet(X)$ of a complex manifold X are defined as the groups $\mathrm{Ext}^\bullet_{\mathcal{O}_{X\times X}}(\mathcal{O}_\Delta, \mathcal{O}_\Delta)$ [12]. Gerstenhaber–Shack [26] derived an isomorphism of cohomology groups $\mathrm{hkr} : \mathbb{H}^\bullet(X, \Lambda^\bullet T_X) \to HH^\bullet(X)$ from the classical Hochschild–Kostant–Rosenberg map. This isomorphism fails to intertwine the multiplications in both cohomologies but can be tweaked so as to produce an isomorphism of associative algebras. More precisely, Kontsevich [30] obtained the following theorem: *The composition* $\operatorname{hkr} \circ (\operatorname{Td}_X)^{\frac{1}{2}} : \mathbb{H}^\bullet(X, \Lambda^\bullet T_X) \to HH^\bullet(X)$, *were the symbol* Td_X *denotes the Todd class of the complex manifold* X*, is an isomorphism of associative algebras.* The multiplications on $\mathbb{H}^\bullet(X, \Lambda^\bullet T_X)$ and $HH^\bullet(X)$ are respectively the wedge product and the Yoneda product. Calaque–Van den Bergh [11] wrote a detailed proof of Kontsevich's theorem, and showed that the map $\operatorname{hkr} \circ (\operatorname{Td}_X)^{\frac{1}{2}}$ actually preserves the Gerstenhaber algebra structures on both cohomologies.

Applying Kontsevich's formality theorem, we prove new formality theorems and Kontsevich–Duflo type theorems in several geometric contexts including foliations, smooth manifolds with Lie algebra actions, complex manifolds, and Lie pairs [36–38]. In the process, we obtain an enhanced version of Dolgushev–Fedosov type resolutions. We develop a different approach to and an extended version of Dolgushev's globalization construction [18] via a formal exponential map for graded manifolds, which we introduce as a replacement for the ∞-jet of the classical geodesic exponential map of smooth manifolds [35].

In Chapter 2, we summarize the necessary theories, including the theories about homological perturbation, Lie pairs (i.e. Lie algebroid with Lie subalgebroid) and Fedosov dg Lie algebroids. In Chapter 3, we define and investigate formal exponential map for graded manifolds. The relation between formal exponential map and Dolgushev's globalization construction in deformation quantization is explored in this chapter. In Chapter 4, we prove a formality theorem for Lie pairs. As an application, we obtain Kontsevich–Duflo type theorem for Lie pairs. Below is a brief summary of this thesis.

Formal exponential map for graded manifolds

On a smooth manifold, the choice of an affine connection determines an exponential map by way of the geodesic equation. However, the geodesic approach does not transpose in a straightforward way to the graded manifold context. It is thus natural to wonder what could replace efficiently the exponential map for graded manifolds.

We solve this problem in Chapter 3 (also see [35]) by forgetting about geodesics and resorting to the fiberwise infinite-order jet of the exponential evaluated along the zero section of T_M, which admits a purely algebraic description that does carry over to the graded context. Let $\mathcal{M}$ be a $\mathbb{Z}$-graded manifold, and let $\mathcal{D}(\mathcal{M})$ denote its algebra of differential operators. The *formal exponential map* associated to a connection ∇ on $T_{\mathcal{M}}$ is the morphism of left $C^\infty(\mathcal{M})$-modules

$$\mathrm{pbw}^\nabla : \Gamma(ST_{\mathcal{M}}) \to \mathcal{D}(\mathcal{M})$$

inductively defined by the relations

$$\begin{aligned}\mathrm{pbw}^\nabla(f) &= f, \quad \forall f \in C^\infty(\mathcal{M}) \\ \mathrm{pbw}^\nabla(X) &= X, \quad \forall X \in \Gamma(T_{\mathcal{M}}) \\ \mathrm{pbw}^\nabla(X_0 \odot \cdots \odot X_n) &= \frac{1}{n+1}\sum_{k=0}^{n} \epsilon_k \left\{ X_k \cdot \mathrm{pbw}^\nabla(X^{\{k\}}) - \mathrm{pbw}^\nabla\left(\nabla_{X_k}(X^{\{k\}})\right)\right\},\end{aligned}$$

where $X_0, \ldots, X_n$ are homogeneous vector fields, $\epsilon_k = (-1)^{|X_k|(|X_0|+\cdots+|X_{k-1}|)}$ and $X^{\{k\}} = X_0 \odot \cdots \odot X_{k-1} \odot X_{k+1} \odot \cdots \odot X_n$. The map pbw^∇ is an isomorphism of *co*algebras which coincides with the ∞-jet of the geodesic exponential map along the zero section of tangent bundle in the case of classical manifolds [35].

Theorem 1. *The formal exponential map* $\mathrm{pbw}^\nabla : \Gamma(S(T_{\mathcal{M}})) \to \mathcal{D}(\mathcal{M})$ *is an isomorphism of filtered coalgebras over* $C^\infty(\mathcal{M})$.

Applying the iterative technique conceived by Fedosov [23] for the deformation quantization [5,6] of symplectic manifolds, Emmrich & Weinstein [22] constructed, for every smooth manifold M, a flat connection on the completed symmetric tensor algebra $\widehat{S}(T_M^\vee)$ of the cotangent bundle of M. Furthermore, they proved, once again by Fedosov's iterative method, that for any smooth function f on M, there exists a unique flat section of $\widehat{S}(T_M^\vee)$ whose term of degree 0 (for the natural graduation of $\widehat{S}(T_M^\vee)$ determined by the symmetric tensor power) is equal to f. Doing so, they obtained an augmentation map $\tau^\nabla : C^\infty(M) \to \Gamma(\widehat{S}(T_M^\vee))$. Moreover, Emmrich & Weinstein [22] proved that the image $\tau^\nabla(f)$ of a function $f \in C^\infty(M)$ does coincide with the fiberwise infinite-order jet along the zero section of T_M of $f \circ \exp$, where exp denotes the classical geodesic exponential map associated to some affine connection on the manifold M. Their proof resorted to a complicated argument involving Ehresmann connections on analytic manifolds. One application of our formal exponential map is a direct and much more transparent proof of the Emmrich–Weinstein theorem. Indeed, we give a simple proof of an extension of the Emmrich–Weinstein theorem to graded manifolds, of which the classical Emmrich–Weinstein theorem is a special case.

The construction of flat connections by Fedosov's iterative method due to Dolgushev for classical smooth manifolds [18] can be extended to the realm of graded manifolds. Actually, we show that, for a graded manifold $\mathcal{M}$, the Fedosov flat connection D^{∇} on $\hat{S}(T_{\mathcal{M}}^{\vee})$ constructed from a torsionfree connection ∇ on $T_{\mathcal{M}}$ by Dolgushev–Fedosov's iteration method can be recovered in a straightforward manner by making use of our formal exponential map $\operatorname{pbw}^{\nabla}:\Gamma\Big(S(T_{\mathcal{M}})\Big)\to\mathcal{D}(\mathcal{M})$ associated with the chosen torsionfree connection ∇. Our construction goes as follows. The Lie algebra $\Gamma(T_{\mathcal{M}})$ of vector fields on $\mathcal{M}$ acts infinitesimally from the left on $\mathcal{D}(\mathcal{M})$ by composition of differential operators. Identifying $\mathcal{D}(\mathcal{M})$ with the symmetric algebra $\Gamma\Big(S(T_{\mathcal{M}})\Big)$ via the formal exponential map $\operatorname{pbw}^{\nabla}$ and transferring this infinitesimal action through $\operatorname{pbw}^{\nabla}$, we obtain a flat connection $\nabla^{\natural}$ on the vector bundle $S(T_{\mathcal{M}})$:

$$\nabla^{\natural}_{X}S:=(\operatorname{pbw}^{\nabla})^{-1}\Big(X\cdot\operatorname{pbw}^{\nabla}(S)\Big),\qquad\forall\,X\in\Gamma(T_{\mathcal{M}}),\quad\forall\,S\in\Gamma(ST_{\mathcal{M}}).$$

We prove that the covariant differential of the flat connection induced by $\nabla^{\natural}$ on the dual bundle $\hat{S}(T_{\mathcal{M}}^{\vee})$ coincides with the coboundary operator D^{∇} constructed by the iteration method [35].

Theorem 2. *Let $\mathcal{M}$ be a finite-dimensional graded manifold, let ∇ be a torsion-free connection on $T_{\mathcal{M}}$, and let $\nabla^{\natural}$ be the corresponding flat connection on $\hat{S}(T_{\mathcal{M}}^{\vee})$ defined as above. The covariant differential of the flat connection $\nabla^{\natural}$ coincides with the Dolgushev-Fedosov's coboundary operator D^{∇}, i.e. $d^{\nabla^{\natural}}=D^{\nabla}$.*

As a consequence, we prove an extension of the Emmrich–Weinstein theorem to graded manifolds: the augmentation map $\tau^{\nabla}:C^{\infty}(\mathcal{M})\to\Gamma(\hat{S}(T_{\mathcal{M}}^{\vee}))$ which identifies smooth functions on $\mathcal{M}$ with $\nabla^{\natural}$-flat sections of the bundle $\hat{S}(T_{\mathcal{M}}^{\vee})$ is a Taylor expansion twisted by the formal exponential map. When $\mathcal{M}$ is an ordinary smooth manifold, we recover the classical Emmrich–Weinstein theorem [22].

Since $\Omega^{\bullet}(\mathcal{M},\hat{S}(T_{\mathcal{M}}^{\vee}))$ may be regarded as the algebra of functions on the graded manifold $T_{\mathcal{M}}[1]\oplus T_{\mathcal{M}}$, it follows from the homological perturbation lemma that the dg manifold $\mathcal{M}$ with support M and trivial homological vector field is weakly equivalent to the dg manifold $T_{\mathcal{M}}[1]\oplus T_{\mathcal{M}}$ with support M and the operator D^{∇}

as homological vector field [35]. Inspired by this observation, we construct Fedosov dg Lie algebroids for Lie pairs and dg manifolds [36, 37].

Formality theorem for Lie pairs

In Chapter 4, we extend Kontsevich formality theorem to *Lie pairs*, a framework that encompasses a wide range of geometric contexts including complex manifolds, foliations, and manifolds equipped with Lie algebra $\mathfrak{g}$-actions.

By a *Lie pair* (L, A), we mean an inclusion $A \hookrightarrow L$ of Lie algebroids. Recall that a *Lie* $\Bbbk$-*algebroid* is a $\Bbbk$-vector bundle $L \to M$ together with a bundle map $\rho : L \to TM \otimes_{\mathbb{R}} \Bbbk$ called *anchor* and a Lie bracket $[-,-]$ on the sections of L such that $\rho : \Gamma(L) \to \mathfrak{X}(M) \otimes \Bbbk$ is a morphism of Lie algebras and $[X, fY] = f[X,Y] + \big(\rho(X)f\big)Y$ for all $X, Y \in \Gamma(L)$ and $f \in C^\infty(M, \Bbbk)$. A $\Bbbk$-vector bundle $L \to M$ is a Lie $\Bbbk$-algebroid if and only if $\Gamma(L)$ is a *Lie–Rinehart algebra* [55] over the commutative ring $C^\infty(M, \Bbbk)$.

Lie pairs arise naturally in a number of classical areas of mathematics such as complex geometry, foliation theory, and Lie theory. A complex manifold X determines a Lie pair over $\mathbb{C}$: $L = T_X \otimes \mathbb{C}$ and $A = T_X^{0,1}$. A foliation on a smooth manifold M determines a Lie pair over $\mathbb{R}$: $L = T_M$ and A is the integrable distribution on M tangent to the foliation. A manifold equipped with a Lie algebra $\mathfrak{g}$-action naturally gives rise to a Lie pair as well [4, 38].

Given a Lie pair (L, A), the quotient L/A is naturally an A-module. When $L = T_M$ is the tangent bundle to a manifold M and $A = T_{\mathcal{F}}$ is an integrable distribution on M, the A-action on $L/A = T_M/T_{\mathcal{F}} = N_{\mathcal{F}}$ is given by the Bott connection [7]. Associated with a Lie pair (L, A), there exist two L_∞ algebras $\operatorname{tot}\Big(\Gamma(\Lambda^\bullet A^\vee) \otimes_R \mathcal{T}_{\mathrm{poly}}^\bullet\Big)$ and $\operatorname{tot}\Big(\Gamma(\Lambda^\bullet A^\vee) \otimes_R \mathcal{D}_{\mathrm{poly}}^\bullet\Big)$, which are canonical up to L_∞ quasi-isomorphisms and serve as replacements for the spaces of polyvector fields and polydifferential operators respectively. Their corresponding cohomology groups $\mathbb{H}_{\mathrm{CE}}^\bullet(A, \mathcal{T}_{\mathrm{poly}}^\bullet)$ and $\mathbb{H}_{\mathrm{CE}}^\bullet(A, \mathcal{D}_{\mathrm{poly}}^\bullet)$ are canonically Gerstenhaber algebras [60].

More precisely, denoting the algebra of smooth functions on the manifold M by R, we set $\mathcal{T}_{\mathrm{poly}}^k = \Gamma(\Lambda^{k+1}(L/A))$ for $k \geqslant 0$, $\mathcal{T}_{\mathrm{poly}}^{-1} = R$, and $\mathcal{T}_{\mathrm{poly}}^\bullet = \bigoplus_{k=-1}^\infty \mathcal{T}_{\mathrm{poly}}^k$. The Bott A-connection on L/A makes every $\mathcal{T}_{\mathrm{poly}}^k$ an A-module. Thus we have a complex of A-modules with trivial differential

$$0 \longrightarrow \mathcal{T}^{-1}_{\mathrm{poly}} \xrightarrow{0} \mathcal{T}^{0}_{\mathrm{poly}} \xrightarrow{0} \mathcal{T}^{1}_{\mathrm{poly}} \xrightarrow{0} \mathcal{T}^{2}_{\mathrm{poly}} \xrightarrow{0} \cdots,$$

whose Chevalley–Eilenberg hypercohomology cochain complex is denoted $\mathrm{tot}\Big(\Gamma(\Lambda^\bullet A^\vee)\otimes_R \mathcal{T}^\bullet_{\mathrm{poly}}\Big)$. Similarly, we set $\mathcal{D}^\bullet_{\mathrm{poly}} = \bigoplus_{k=-1}^{\infty} \mathcal{D}^k_{\mathrm{poly}}$ where $\mathcal{D}^{-1}_{\mathrm{poly}} = R$, $\mathcal{D}^0_{\mathrm{poly}} = \frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}$, and $\mathcal{D}^k_{\mathrm{poly}}$ with $k \geqslant 1$ is the tensor product $\mathcal{D}^0_{\mathrm{poly}} \otimes_R \cdots \otimes_R \mathcal{D}^0_{\mathrm{poly}}$ of $(k+1)$ copies of the left R-module $\mathcal{D}^0_{\mathrm{poly}}$. Multiplication in $\mathcal{U}(L)$ from the left by elements of $\Gamma(A)$ induces an A-module structure on the quotient $\frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}$. This action of A on $\mathcal{D}^0_{\mathrm{poly}}$ extends naturally to an action of A on $\mathcal{D}^k_{\mathrm{poly}}$ for each k. In fact, $\mathcal{D}^0_{\mathrm{poly}}$ is a cocommutative coassociative coalgebra over R whose comultiplication $\Delta : \mathcal{D}^0_{\mathrm{poly}} \to \mathcal{D}^0_{\mathrm{poly}} \otimes_R \mathcal{D}^0_{\mathrm{poly}}$ is a morphism of A-modules. One can form the Hochschild complex in terms of the comultiplication $\Delta : \mathcal{D}^0_{\mathrm{poly}} \to \mathcal{D}^0_{\mathrm{poly}} \otimes_R \mathcal{D}^0_{\mathrm{poly}}$:

$$0 \longrightarrow \mathcal{D}^{-1}_{\mathrm{poly}} \xrightarrow{d_{\mathscr{H}}} \mathcal{D}^{0}_{\mathrm{poly}} \xrightarrow{d_{\mathscr{H}}} \mathcal{D}^{1}_{\mathrm{poly}} \xrightarrow{d_{\mathscr{H}}} \mathcal{D}^{2}_{\mathrm{poly}} \xrightarrow{d_{\mathscr{H}}} \cdots,$$

which is a complex of A-modules. Its Chevalley–Eilenberg hypercohomology cochain complex is denoted $\mathrm{tot}\Big(\Gamma(\Lambda^\bullet A^\vee) \otimes_R \mathcal{D}^\bullet_{\mathrm{poly}}\Big)$. For instance, for the Lie pair $L = T_X \otimes \mathbb{C}$ and $A = T^{0,1}_X$ stemming from a complex manifold X, $\mathrm{tot}\Big(\Gamma(\Lambda^\bullet A^\vee) \otimes_R \mathcal{T}^\bullet_{\mathrm{poly}}\Big)$ and $\mathrm{tot}\Big(\Gamma(\Lambda^\bullet A^\vee) \otimes_R \mathcal{D}^\bullet_{\mathrm{poly}}\Big)$ become the standard dglas $\Big(\Omega^{0,\bullet}(\mathcal{T}^\bullet_{\mathrm{poly}}(X)), \bar\partial\Big)$ and $\Big(\Omega^{0,\bullet}(\mathcal{D}^\bullet_{\mathrm{poly}}(X)), \bar\partial + d_{\mathscr{H}}\Big)$, respectively. And their corresponding Chevalley–Eilenberg hypercohomology groups $\mathbb{H}^\bullet_{\mathrm{CE}}(A, \mathcal{T}^\bullet_{\mathrm{poly}})$ and $\mathbb{H}^\bullet_{\mathrm{CE}}(A, \mathcal{D}^\bullet_{\mathrm{poly}})$ are isomorphic to the sheaf cohomology group $H^\bullet(X, \Lambda^\bullet T_X)$, and the Hochschild cohomology group $HH^\bullet(X)$, respectively.

The skew-symmetric extension of the natural inclusion $\Gamma(L/A) \hookrightarrow \mathcal{D}^0_{\mathrm{poly}}$ yields a morphism of Chevalley–Eilenberg hypercohomology cochain complexes $\mathrm{hkr} : \mathrm{tot}\Big(\Gamma(\Lambda^\bullet A^\vee) \otimes_R \mathcal{T}^\bullet_{\mathrm{poly}}\Big) \to \mathrm{tot}\Big(\Gamma(\Lambda^\bullet A^\vee) \otimes_R \mathcal{D}^\bullet_{\mathrm{poly}}\Big)$, called *Hochschild–Kostant–Rosenberg* map. For a Lie pair (L, A), the induced morphism on the level of cohomologies, $\mathrm{hkr} : \mathbb{H}^\bullet_{\mathrm{CE}}(A, \mathcal{T}^\bullet_{\mathrm{poly}}) \to \mathbb{H}^\bullet_{\mathrm{CE}}(A, \mathcal{D}^\bullet_{\mathrm{poly}})$, is an isomorphism of vector spaces but it preserves neither the Lie algebra structures nor the associative algebra structures. In order to obtain a morphism of Gerstenhaber algebras, the Hochschild–Kostant–Rosenberg map hkr must be modified: it must be tweaked by the square root of the Todd cocycle of the Lie pair in the formality theorem.

The Atiyah class of a Lie pair (L, A) [16] simultaneously extends both the classical Atiyah class of holomorphic vector bundles [1] and the Molino class of foliations [52]. Let ∇ be an L-connection on L/A extending the Bott A-representation. The curvature of ∇ induces a section $R^{\nabla}_{1,1} \in \Gamma(A^\vee \otimes A^\perp \otimes \operatorname{End}(L/A))$, whose cohomology class $\alpha_{L/A} \in H^1_{\mathrm{CE}}(A, A^\perp \otimes \operatorname{End}(L/A))$ does not depend on the choice of L-connections ∇, and is called the Atiyah class of the Lie pair (L, A).

Given a Lie pair (L, A), a Todd cocycle may be defined in terms of the Atiyah cocycle in the exact same way it is defined in the classical case of complex geometry. The Todd cocycle of a Lie pair (L, A) is the Chevalley-Eilenberg cocycle

$$\operatorname{td}^{\nabla}_{L/A} = \det\left(\frac{R^{\nabla}_{1,1}}{1 - e^{-R^{\nabla}_{1,1}}}\right) \in \bigoplus_{k=0} \Gamma(\Lambda^k A^\vee \otimes \Lambda^k (L/A)^\vee). \tag{1.1}$$

Its cohomology class $\operatorname{Td}_{L/A} \in \bigoplus_{k=0} H^k_{\mathrm{CE}}(A, \Lambda^k (L/A)^\vee)$ is the Todd class of the Lie pair (L, A). See [36] for details.

The main theorem of our paper [36] is the following formality theorem for Lie pairs:

Theorem 3 (Formality theorem for Lie pairs)**.** *Let (L, A) be a Lie pair and let ∇ be a torsion-free L-connection on $B = L/A$. There exists an L_∞ quasi-isomorphism*

$$\mathcal{I} : \operatorname{tot}\left(\Gamma(\Lambda^\bullet A^\vee) \otimes_R \mathcal{T}^\bullet_{\mathrm{poly}}\right) \to \operatorname{tot}\left(\Gamma(\Lambda^\bullet A^\vee) \otimes_R \mathcal{D}^\bullet_{\mathrm{poly}}\right) \tag{1.2}$$

with first Taylor coefficient $\mathcal{I}_1 : \operatorname{tot}\left(\Gamma(\Lambda^\bullet A^\vee) \otimes_R \mathcal{T}^\bullet_{\mathrm{poly}}\right) \to \operatorname{tot}\left(\Gamma(\Lambda^\bullet A^\vee) \otimes_R \mathcal{D}^\bullet_{\mathrm{poly}}\right)$ satisfying the following properties:

- *$\mathcal{I}_1$ preserves the associative algebra structures (wedge and cup product, respectively) up to homotopy;*
- *$\mathcal{I}_1 = \operatorname{hkr} \circ (\operatorname{td}^{\nabla}_{L/A})^{\frac{1}{2}}$, where $(\operatorname{td}^{\nabla}_{L/A})^{\frac{1}{2}}$ acts on $\operatorname{tot}\left(\Gamma(\Lambda^\bullet A^\vee) \otimes_R \mathcal{T}^\bullet_{\mathrm{poly}}\right)$ by contraction.*

As an immediate consequence, we proved the following theorem.

Theorem 4 (Kontsevich-Duflo type theorem for Lie pairs)**.** *For any Lie pair (L, A), the map*

$$\operatorname{hkr} \circ \operatorname{Td}^{\frac{1}{2}}_{L/A} : \mathbb{H}^\bullet_{\mathrm{CE}}(A, \mathcal{T}^\bullet_{\mathrm{poly}}) \to \mathbb{H}^\bullet_{\mathrm{CE}}(A, \mathcal{D}^\bullet_{\mathrm{poly}})$$

(where $\operatorname{Td}^{\frac{1}{2}}_{L/A} \in \bigoplus_{k=0} H^k_{\mathrm{CE}}(A, \Lambda^k B^\vee)$, acts on $\mathbb{H}^\bullet_{\mathrm{CE}}(A, \mathcal{T}^\bullet_{\mathrm{poly}})$ by contraction) is an isomorphism of Gerstenhaber algebras.

In particular, we obtain a formality theorem and a Kontsevich–Duflo type theorem for pairs of Lie algebras — neither of which was previously known. Moreover, Theorems 3 and 4 can be applied to various geometric situations including complex manifolds, $\mathfrak{g}$-manifolds and foliations.

Formality theorem for complex manifolds

Specializing Theorem 4 to the case of complex manifolds, we recover the classical Kontsevich–Duflo theorem [30]. More precisely, consider the Lie pair $L = T_X \otimes \mathbb{C}$ and $A = T_X^{0,1}$ associated with a complex manifold X.

The L_∞ algebras of polyvector fields and polydifferential operators reduce to the dglas $\Omega^{0,\bullet}(X, \mathcal{T}^\bullet_{\text{poly}}(X))$ and $\Omega^{0,\bullet}(X, \mathcal{D}^\bullet_{\text{poly}}(X))$, respectively [36].

Thus, Theorem 3 becomes

Theorem 5 (Formality theorem for complex manifolds)**.** *Given a complex manifold X and a choice of free $T_X^{1,0}$-connection $\nabla^{1,0}$ on $T_X^{1,0}$, there exists an L_∞ quasi-isomorphism*

$$\mathcal{I} : \Omega^{0,\bullet}\Big(X, \mathcal{T}^\bullet_{\text{poly}}(X)\Big) \to \Omega^{0,\bullet}\Big(X, \mathcal{D}^\bullet_{\text{poly}}(X)\Big)$$

with first Taylor coefficient $\mathcal{I}_1$ satisfying the following two properties:

- *$\mathcal{I}_1$ preserves the associative algebra structures up to homotopy;*
- *$\mathcal{I}_1 = \operatorname{hkr} \circ \Big(\operatorname{td}_X^{\bar\partial + \nabla^{1,0}}\Big)^{\frac{1}{2}}$, where the square root of the Todd cocycle $\operatorname{td}_X^{\bar\partial + \nabla^{1,0}} \in \bigoplus_{k=0} \Omega^{k,k}(X)$ acts on $\Omega^{0,\bullet}\Big(X, \mathcal{T}^\bullet_{\text{poly}}(X)\Big)$ by contraction.*

Theorem 4 becomes

Theorem 6 (Kontsevich-Duflo theorem for complex manifolds [11, 30])**.** *For every complex manifold X, the composition*

$$\operatorname{hkr} \circ (\operatorname{Td}_X)^{\frac{1}{2}} : \mathbb{H}^\bullet_{\text{sheaf}}(X, \Lambda^\bullet T_X) \to HH^\bullet(X)$$

is an isomorphism of Gerstenhaber algebras. It is understood that the square root of the Todd class

$$(\operatorname{Td}_X) \in \bigoplus_{k=0} H^{k,k}(X) \cong H^k_{\text{sheaf}}(X, \Omega^k_X)$$

acts on $\mathbb{H}^\bullet_{\text{sheaf}}(X, \Lambda^\bullet T_X)$ by contraction.

Formality theorem for $\mathfrak{g}$-manifolds

Let M be a $\mathfrak{g}$-manifold, i.e. a smooth manifold endowed with an infinitesimal action $\mathfrak{g} \ni a \mapsto \widehat{a} \in \mathfrak{X}(M)$. Every $\mathfrak{g}$-manifold M determines in a canonical way a matched pair of Lie algebroids $(\mathfrak{g} \ltimes M) \bowtie T_M$ (see e.g. [51, Example 5.5] or [40]). In this case, the L_∞ algebras of polyvector fields and polydifferential operators reduce to the dglas $\operatorname{tot}\Big(\Lambda^\bullet \mathfrak{g}^\vee \otimes_{\Bbbk} \mathcal{T}^\bullet_{\text{poly}}(M)\Big)$ and $\operatorname{tot}\Big(\Lambda^\bullet \mathfrak{g}^\vee \otimes_{\Bbbk} \mathcal{D}^\bullet_{\text{poly}}(M)\Big)$, respectively [36,38].

Given an affine connection ∇ on M, the Atiyah 1-cocycle associated with ∇ is the cocycle $\operatorname{At}^\nabla_{M/\mathfrak{g}} \in \mathfrak{g}^\vee \otimes \Gamma(T^\vee_M \otimes \operatorname{End} T_M)$ defined by

$$\operatorname{At}^\nabla_{M/\mathfrak{g}}(a, X) = \mathcal{L}_{\widehat{a}} \circ \nabla_X - \nabla_X \circ \mathcal{L}_{\widehat{a}} - \nabla_{\mathcal{L}_{\widehat{a}} X},$$

for all $a \in \mathfrak{g}$ and $X \in \mathfrak{X}(M)$. The associated Todd cocycle is defined by the same formula as in (1.1): $\operatorname{td}^\nabla_{M/\mathfrak{g}} \in \bigoplus_{k=0} \Lambda^k \mathfrak{g}^\vee \otimes \Omega^k(M)$. Its corresponding Chevalley–Eilenberg cohomology class is the *Todd class* $\operatorname{Td}_{M/\mathfrak{g}} \in \bigoplus_{k=0} H^k_{\text{CE}}(\mathfrak{g}, \Omega^k(M))$.

Theorems 3 and 4 become

Theorem 7 (Formality theorem for $\mathfrak{g}$-manifolds)**.** *Given a $\mathfrak{g}$-manifold M and an affine torsion-free connection ∇ on M, there exists an L_∞ quasi-isomorphism $\mathcal{I}$ from the dgla* $\operatorname{tot}\Big(\Lambda^\bullet \mathfrak{g}^\vee \otimes_{\Bbbk} \mathcal{T}^\bullet_{\text{poly}}(M)\Big)$ *to the dgla* $\operatorname{tot}\Big(\Lambda^\bullet \mathfrak{g}^\vee \otimes_{\Bbbk} \mathcal{D}^\bullet_{\text{poly}}(M)\Big)$ *with first 'Taylor coefficient' $\mathcal{I}_1$ satisfying the following two properties:*

1. *$\mathcal{I}_1$ is, up to homotopy, an isomorphism of associative algebras (and hence induces an isomorphism of associative algebras of the cohomologies);*
2. *$\mathcal{I}_1 = \operatorname{hkr} \circ (\operatorname{td}^\nabla_{M/\mathfrak{g}})^{\frac{1}{2}}$, where $(\operatorname{td}^\nabla_{M/\mathfrak{g}})^{\frac{1}{2}} \in \bigoplus_{k=0} \Lambda^k \mathfrak{g}^\vee \otimes \Omega^k(M)$ acts on* $\operatorname{tot}\Big(\Lambda^\bullet \mathfrak{g}^\vee \otimes_{\Bbbk} \mathcal{T}^\bullet_{\text{poly}}(M)\Big)$ *by contraction.*

Theorem 8 (Kontsevich–Duflo type theorem for $\mathfrak{g}$-manifolds)**.** *Given a $\mathfrak{g}$-manifold M, the map*

$$\operatorname{hkr} \circ \operatorname{Td}^{\frac{1}{2}}_{M/\mathfrak{g}} : \mathbb{H}^\bullet_{\text{CE}}\Big(\mathfrak{g}, \mathcal{T}^\bullet_{\text{poly}}(M)\Big) \xrightarrow{\cong} \mathbb{H}^\bullet_{\text{CE}}\Big(\mathfrak{g}, \mathcal{D}^\bullet_{\text{poly}}(M)\Big)$$

is an isomorphism of Gerstenhaber algebras. It is understood that the square root $\operatorname{Td}^{\frac{1}{2}}_{M/\mathfrak{g}}$ of the Todd class $\operatorname{Td}_{M/\mathfrak{g}} \in \bigoplus_{k=0} H^k_{\text{CE}}\Big(\mathfrak{g}, \Omega^k(M)\Big)$ acts on $\mathbb{H}^\bullet_{\text{CE}}\Big(\mathfrak{g}, \mathcal{T}^\bullet_{\text{poly}}(M)\Big)$ by contraction.

Formality theorem for regular foliations

Let $\mathcal{F}$ be a regular foliation of a smooth manifold M. The tangent bundle of $\mathcal{F}$ is a subbundle of T_M, denoted $T_\mathcal{F}$, whose sections are closed under the Lie bracket of vector fields. Therefore, $(T_M, T_\mathcal{F})$ is a Lie pair. Its quotient $N_\mathcal{F} = T_M/T_\mathcal{F}$ is the normal bundle of the foliation $\mathcal{F}$. The Chevalley–Eilenberg Lie algebroid cohomology $H^\bullet_{\mathrm{CE}}(T_\mathcal{F}, \mathfrak{M})$ with coefficients in a $T_\mathcal{F}$-module $\mathfrak{M}$ is the leafwise de Rham cohomology $H^\bullet_{\mathrm{dR}}(\mathcal{F}, \mathfrak{M})$ of the foliation $\mathcal{F}$ with coefficients in the module $\mathfrak{M}$.

Let ∇ be a T_M-connection on $N_\mathcal{F}$ extending the Bott $T_\mathcal{F}$-connection. The connection ∇ determines the Atiyah cocycle $R^\nabla_{1,1} \in \Gamma(T^\vee_\mathcal{F} \otimes T^\perp_\mathcal{F} \otimes \mathrm{End}(N_\mathcal{F}))$ and Todd cocycle $\mathrm{td}^\nabla_\mathcal{F} \in \bigoplus_{k=0}^\infty \Gamma(\Lambda^k T^\vee_\mathcal{F} \otimes \Lambda^k T^\perp_\mathcal{F})$. The Atiyah class $\alpha_\mathcal{F} = [R^\nabla_{1,1}] \in H^1_{\mathrm{dR}}(\mathcal{F}, T^\perp_\mathcal{F} \otimes \mathrm{End}(N_\mathcal{F}))$, is precisely the class introduced by Molino in [52].

Theorems 3 and 4 become

Theorem 9 (Formality theorem for foliations)**.** *Let $\mathcal{F}$ be a regular foliation on a smooth manifold M. Given a splitting of the short exact sequence $0 \to T_\mathcal{F} \to T_M \to N_\mathcal{F} \to 0$ and a torsion-free T_M-connection ∇ on $N_\mathcal{F}$, there exists an L_∞ quasi-isomorphism*

$$\mathcal{I} : \mathrm{tot}\Big(\Gamma(\Lambda^\bullet T^\vee_\mathcal{F}) \otimes_R \mathcal{T}^\bullet_{\mathrm{poly}}(N_\mathcal{F})\Big) \to \mathrm{tot}\Big(\Gamma(\Lambda^\bullet T^\vee_\mathcal{F}) \otimes_R \mathcal{D}^\bullet_{\mathrm{poly}}(N_\mathcal{F})\Big)$$

with first 'Taylor coefficient' $\mathcal{I}_1$ satisfying the following two properties:

1. *$\mathcal{I}_1$ preserves the associative algebra structures (wedge and cup product, respectively) up to homotopy;*
2. *$\mathcal{I}_1 = \mathrm{hkr} \circ (\mathrm{td}^\nabla_\mathcal{F})^{\frac{1}{2}}$, where $(\mathrm{td}^\nabla_\mathcal{F})^{\frac{1}{2}} \in \bigoplus_{k=0}^\infty \Gamma(\Lambda^k T^\vee_\mathcal{F} \otimes \Lambda^k T^\perp_\mathcal{F})$ acts on $\mathrm{tot}\Big(\Gamma(\Lambda^\bullet T^\vee_\mathcal{F}) \otimes_R \mathcal{T}^\bullet_{\mathrm{poly}}(N_\mathcal{F})\Big)$ by contraction.*

Theorem 10 (Kontsevich-Duflo type theorem for foliations)**.** *Given a regular foliation $\mathcal{F}$ on a smooth manifold M, the map*

$$\mathrm{hkr} \circ \mathrm{Td}^{\frac{1}{2}}_\mathcal{F} : \mathbb{H}^\bullet_{\mathrm{dR}}(\mathcal{F}, \mathcal{T}^\bullet_{\mathrm{poly}}(N_\mathcal{F})) \xrightarrow{\cong} \mathbb{H}^\bullet_{\mathrm{dR}}(\mathcal{F}, \mathcal{D}^\bullet_{\mathrm{poly}}(N_\mathcal{F}))$$

is an isomorphism of Gerstenhaber algebras. It is understood that the square root $\mathrm{Td}^{\frac{1}{2}}_\mathcal{F}$ of the Todd class $\mathrm{Td}_\mathcal{F} \in \bigoplus_{k=0} H^k_{\mathrm{dR}}(\mathcal{F}, \Lambda^k T^\perp_\mathcal{F})$ acts on $\mathbb{H}^\bullet_{\mathrm{dR}}(\mathcal{F}, \mathcal{T}^\bullet_{\mathrm{poly}}(N_\mathcal{F}))$ by contraction.

Chapter 2 Preliminaries

In this chapter, we provide a short summary of the classical theories of homological perturbation, Lie pairs and Fedosov Lie algebroids.

2.1 Homological perturbation lemma

Roughly speaking, homological perturbation is an algebraic tool which allows us to perturb a deformation retract to another deformation retract. To be precise, we need some technical definitions from homological algebra. We say that a cochain complex (N, δ) *contracts* onto a cochain complex (M, d) if there exists two chain maps $\sigma : N \to M$ and $\tau : M \to N$ and an endomorphism $h : N \to N[-1]$ of the graded module N satisfying

$$\sigma\tau = \mathrm{id}_N,$$
$$\tau\sigma - \mathrm{id}_M = h\delta + \delta h$$

and

$$\sigma h = 0, \qquad h\tau = 0, \qquad hh = 0.$$

If, furthermore, the cochain complexes N and M are filtered and the maps σ, τ, and h preserve the filtration, the contraction is said to be filtered [21, Section 12].

A filtration $\cdots \subset F_{p-1}N \subset F_pN \subset F_{p+1}N \subset \cdots$ on a cochain complex N is said to be exhaustive if $N = \bigcup_p F_pN$ and complete if $N = \varprojlim \frac{N}{F_pN}$.

A *perturbation* of the differential δ of a filtered cochain complex

$$\cdots \longrightarrow N^{n-1} \xrightarrow{\delta} N^n \xrightarrow{\delta} N^{n+1} \longrightarrow \cdots$$

is an operator $\partial : F_pN \to F_{p-1}N$ lowering the filtration and satisfying

$$(\delta + \partial)(\delta + \partial) = 0$$

so that $\delta + \partial$ is a new differential on N.

We refer the reader to [28, Section 1] for a brief history of the following proposition.

Proposition 2.1.1 (Homological Perturbation [8])**.** *Let*

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & N^{n-1} & \xrightarrow{\delta} & N^n & \xrightarrow{\delta} & N^{n+1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\sigma} & & \\
\cdots & \longrightarrow & M^{n-1} & \xrightarrow{d} & M^n & \xrightarrow{d} & M^{n+1} & \longrightarrow & \cdots \\
 & & {\scriptstyle\tau}\downarrow & \swarrow{\scriptstyle h} & {\scriptstyle\tau}\downarrow & \swarrow{\scriptstyle h} & {\scriptstyle\tau}\downarrow & & \\
\cdots & \longrightarrow & N^{n-1} & \xrightarrow{\delta} & N^n & \xrightarrow{\delta} & N^{n+1} & \longrightarrow & \cdots
\end{array}$$

be a filtered contraction. Given a perturbation ∂ of the differential δ on N, if the filtrations on M and N are bounded, then the series

$$\vartheta := \sum_{k=0}^{\infty} \sigma\partial(h\partial)^k\tau = \sum_{k=0}^{\infty} \sigma(\partial h)^k\partial\tau$$

$$\breve{\sigma} := \sum_{k=0}^{\infty} \sigma(\partial h)^k$$

$$\breve{\tau} := \sum_{k=0}^{\infty} (h\partial)^k\tau$$

$$\breve{h} := \sum_{k=0}^{\infty} (h\partial)^k h = \sum_{k=0}^{\infty} h(\partial h)^k$$

converge, ϑ is a perturbation of the differential d on M, and

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & N^{n-1} & \xrightarrow{\delta+\partial} & N^{n} & \xrightarrow{\delta+\partial} & N^{n+1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle\breve{\sigma}} & & \downarrow{\scriptstyle\breve{\sigma}} & & \downarrow{\scriptstyle\breve{\sigma}} & & \\
\cdots & \longrightarrow & M^{n-1} & \xrightarrow{d+\vartheta} & M^{n} & \xrightarrow{d+\vartheta} & M^{n+1} & \longrightarrow & \cdots \\
 & & {\scriptstyle\breve{\tau}}\downarrow & \swarrow{\scriptstyle\breve{h}} & {\scriptstyle\breve{\tau}}\downarrow & \swarrow{\scriptstyle\breve{h}} & {\scriptstyle\breve{\tau}}\downarrow & & \\
\cdots & \longrightarrow & N^{n-1} & \xrightarrow{\delta+\partial} & N^{n} & \xrightarrow{\delta+\partial} & N^{n+1} & \longrightarrow & \cdots
\end{array}$$

constitutes a new filtered contraction.

2.2 Lie pairs

2.2.1 Connections and representations for Lie algebroids

Let M be a smooth manifold, let $L \to M$ be a Lie $\Bbbk$-algebroid with anchor map $\rho : L \to T_M \otimes_{\mathbb{R}} \Bbbk$, and let $E \to M$ be a vector bundle over $\Bbbk$. The algebra of smooth functions on M with values in $\Bbbk$ will be denoted R.

The traditional description of a (linear) L-connection on E is in terms of a *covariant derivative*

$$\Gamma(L) \times \Gamma(E) \to \Gamma(E) : (l, e) \mapsto \nabla_l e$$

characterized by the following two properties:

$$\nabla_{f\cdot l} e = f \cdot \nabla_l e, \tag{2.1}$$

$$\nabla_l (f \cdot e) = \rho(l) f \cdot e + f \cdot \nabla_l e, \tag{2.2}$$

for all $l \in \Gamma(L)$, $e \in \Gamma(E)$, and $f \in R$.

Remark 2.2.1. *A covariant derivative $\nabla : \Gamma(L) \times \Gamma(E) \to \Gamma(E)$ induces a covariant derivative $\nabla : \Gamma(L) \times \Gamma(S(E)) \to \Gamma(S(E))$ through the relation*

$$\nabla_l (e_1 \odot \cdots \odot e_n) = \sum_{k=1}^{n} e_1 \odot \cdots \odot \nabla_l e_k \odot \cdots \odot e_n,$$

for all $l \in \Gamma(L)$ and $e_1, \ldots, e_n \in \Gamma(E)$.

Remark 2.2.2. *A covariant derivative* $\nabla : \Gamma(L) \times \Gamma(S(E)) \to \Gamma(S(E))$ *induces a covariant derivative* $\nabla : \Gamma(L) \times \Gamma(\hat{S}(E^\vee)) \to \Gamma(\hat{S}(E^\vee))$ *through the relation*

$$\rho(l) \langle \sigma | s \rangle = \langle \nabla_l \sigma | s \rangle + \langle \sigma | \nabla_l s \rangle$$

for all $l \in \Gamma(L)$*,* $s \in \Gamma(S(E))$*, and* $\sigma \in \Gamma(\hat{S}(E^\vee))$*.*

A *representation of a Lie algebroid* L on a vector bundle $E \to M$ is a flat L-connection ∇ on E, i.e. a covariant derivative $\nabla : \Gamma(L) \times \Gamma(E) \to \Gamma(E)$ satisfying

$$\nabla_{l_1} \nabla_{l_2} e - \nabla_{l_2} \nabla_{l_1} e = \nabla_{[l_1, l_2]} e, \tag{2.3}$$

for all $l_1, l_2 \in \Gamma(L)$ and $e \in \Gamma(E)$. A vector bundle endowed with a representation of the Lie algebroid L is called an *L-module*. More generally, given a left R-module $\mathcal{M}$, by an *infinitesimal action* of L on $\mathcal{M}$, we mean a $\Bbbk$-bilinear map $\nabla : \Gamma(L) \times \mathcal{M} \to \mathcal{M}$, $(l, e) \mapsto \nabla_l e$ satisfying Equations (2.1), (2.2), and (2.3). In other words, ∇ is a representation of the Lie–Rinehart algebra $(\Gamma(L), R)$ [55].

Example 2.2.3. *Let* (L, A) *be a Lie pair, i.e. an inclusion* $A \hookrightarrow L$ *of Lie algebroids. The* Bott representation *of* A *on the quotient* L/A *is the flat connection defined by*

$$\nabla_a^{\mathrm{Bott}} q(l) = q\Big([a, l]\Big), \quad \forall a \in \Gamma(A), l \in \Gamma(L),$$

where q *denotes the canonical projection* $L \twoheadrightarrow L/A$*. Thus the quotient* L/A *of a Lie pair* (L, A) *is an* A*-module.*

Let L be a Lie algebroid over a smooth manifold M, and R be the algebra of smooth functions on M valued in $\Bbbk$. The Chevalley–Eilenberg differential

$$d_L : \Gamma(\Lambda^k L^\vee) \to \Gamma(\Lambda^{k+1} L^\vee)$$

defined by

$$\begin{aligned}\Big(d_L \omega\Big)(l_0, l_1, \cdots, l_k) = &\sum_{i=0}^{n} (-1)^i \rho(l_i) \Big(\omega(l_0, \cdots, \widehat{l_i}, \cdots, l_k)\Big) \\ &+ \sum_{i<j} (-1)^{i+j} \omega([l_i, l_j], l_0, \cdots, \widehat{l_i}, \cdots, \widehat{l_j}, \cdots, l_k)\end{aligned}$$

and the exterior product make $\bigoplus_{k\geqslant 0}\Gamma(\Lambda^k L^\vee)$ into a differential graded commutative R-algebra.

Given a Lie algebroid L of rank n, and an L-connection ∇ on a vector bundle $E \to M$, the covariant differential is the operator

$$d_L^\nabla : \Gamma(\Lambda^k L^\vee \otimes E) \to \Gamma(\Lambda^{k+1} L^\vee \otimes E)$$

that takes a section $\omega \otimes e$ of $\Lambda^k L^\vee \otimes E$ to

$$d_L^\nabla(\omega \otimes e) = (d_L \omega) \otimes e + \sum_{j=1}^{n} (\nu_j \wedge \omega) \otimes \nabla_{v_j} e,$$

where $v_1, v_2, \dots, v_n$ and $\nu_1, \nu_2, \dots, \nu_n$ are any pair of dual local frames for the vector bundles L and $L^\vee$. If the connection ∇ is flat, then d_L^∇ is a coboundary operator: $d_L^\nabla \circ d_L^\nabla = 0$.

Let L be a Lie algebroid. The symbol $\mathfrak{L}$ denotes the abelian category of left modules over $\mathcal{U}(L)$. Its bounded below derived category is denoted by $D^+(\mathfrak{L})$. The Chevalley–Eilenberg cohomology in degree k of a complex of $\mathcal{U}(L)$-modules $\mathcal{E}^\bullet$ is

$$\mathbb{H}^k_{\mathrm{CE}}(L, \mathcal{E}^\bullet) := \operatorname{Hom}_{D^+(\mathfrak{L})}(R, \mathcal{E}^\bullet[k]).$$

It is computed as the total cohomology in degree k of the double complex

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \uparrow{\scriptstyle \mathrm{id}\otimes(-1)^p d_{\mathcal{E}}} & & \uparrow{\scriptstyle \mathrm{id}\otimes(-1)^{p+1} d_{\mathcal{E}}} & & \\
\cdots & \longrightarrow & \Gamma(\Lambda^p L^\vee) \otimes_R \mathcal{E}^{q+1} & \xrightarrow{d_L^{\mathcal{E}}} & \Gamma(\Lambda^{p+1} L^\vee) \otimes_R \mathcal{E}^{q+1} & \xrightarrow{d_L^{\mathcal{E}}} & \cdots \\
 & & \uparrow{\scriptstyle \mathrm{id}\otimes(-1)^p d_{\mathcal{E}}} & & \uparrow{\scriptstyle \mathrm{id}\otimes(-1)^{p+1} d_{\mathcal{E}}} & & \\
\cdots & \longrightarrow & \Gamma(\Lambda^p L^\vee) \otimes_R \mathcal{E}^{q} & \xrightarrow{d_L^{\mathcal{E}}} & \Gamma(\Lambda^{p+1} L^\vee) \otimes_R \mathcal{E}^{q} & \xrightarrow{d_L^{\mathcal{E}}} & \cdots \\
 & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & &
\end{array}$$

Here we follow Koszul's sign convention: $\mathrm{id} \otimes d_{\mathcal{E}}(\omega \otimes e) = (-1)^p \omega \otimes d_{\mathcal{E}} e$, for all $\omega \in \Gamma(\Lambda^p L^\vee)$ and $e \in \mathcal{E}^\bullet$.

2.2.2 Atiyah class and Todd class of a Lie pair

Let (L, A) be a pair of Lie algebroids over $\Bbbk$. Consider the short exact sequence of vector bundles

$$0 \longrightarrow A \xrightarrow{\ i\ } L \xrightarrow{\ q\ } L/A \longrightarrow 0 \ .$$

Given L-connection ∇ on L/A, we define a bundle map $T^{\nabla} : \Lambda^2 L \to L/A$ by

$$T^{\nabla}(x, y) = \nabla_x q(y) - \nabla_y q(x) - q\Big([x, y]\Big), \quad \forall x, y \in \Gamma(L).$$

An L-connection ∇ on L/A is said to extend the Bott A-connection on L/A (see Example 2.2.3) if

$$\nabla_{i(a)} q(l) = \nabla^{\mathrm{Bott}}_a q(l) = q\Big([i(a), l]\Big), \quad \forall a \in \Gamma(A), l \in \Gamma(L).$$

Lemma 2.2.4. *The following assertions are equivalent:*

1. *The L-connection ∇ on L/A extends the Bott A-connection on L/A.*
2. *For all $a \in \Gamma(A)$ and $l \in \Gamma(L)$, we have $T^{\nabla}\Big(i(a), l\Big) = 0$.*
3. *There exists a unique bundle map $\beta^{\nabla} : \Lambda^2(L/A) \to L/A$ such that the diagram*

$$\begin{array}{ccc} \Lambda^2 L & \xrightarrow{\ T^{\nabla}\ } & L/A \\ {\scriptstyle q}\downarrow & \nearrow_{\beta^{\nabla}} & \\ \Lambda^2(L/A) & & \end{array}$$

   commutes.

Hence a torsion-free L-connections on L/A is necessarily an extension of the Bott A-connection.

Proof. Since $q \circ i = 0$, we have

$$0 = T^{\nabla}\Big(i(a), l\Big) = \nabla_{i(a)} q(l) - q\Big([i(a), l]\Big),$$

for all $a \in \Gamma(A)$ and $l \in \Gamma(L)$. $\square$

An L-connection ∇ on B is said to be torsion-free if $T^\nabla = 0$ (and hence $\beta^\nabla = 0$).

Lemma 2.2.5. *Given a Lie pair (L, A), there exist torsion-free L-connections on L/A.*

Sketch of proof. First, construct an L-connection ∇ on B using the usual partition of unity argument. Then, tweak ∇ so as to obtain an extension $\nabla' : \Gamma(L) \times \Gamma(L/A) \to \Gamma(L/A)$ of the Bott A-connection: choose a splitting $i \circ p + j \circ q = \mathrm{id}_L$ of the short exact sequence

$$0 \longrightarrow A \underset{p}{\overset{i}{\rightleftarrows}} L \underset{j}{\overset{q}{\rightleftarrows}} B \longrightarrow 0 \tag{2.4}$$

and set

$$\nabla'_l b = q\Big([i \circ p(l), j(b)]\Big) + \nabla_{j \circ q(l)} b.$$

Finally, obtain a torsion-free connection $\nabla'' : \Gamma(L) \times \Gamma(L/A) \to \Gamma(L/A)$ from ∇' by setting

$$\nabla''_l b = \nabla'_l b - \frac{1}{2}\beta^{\nabla'}\Big(q(l), b\Big).$$

□

Given a Lie pair (L, A) with quotient $B = L/A$, let ∇ be an L-connection on B extending the Bott A-representation. The curvature of ∇ is the bundle map $R^\nabla : \Lambda^2 L \to \mathrm{End}(L/A)$ defined by

$$R^\nabla(x, y) = \nabla_x \nabla_y - \nabla_y \nabla_x - \nabla_{[x,y]}, \quad \forall x, y \in \Gamma(L).$$

Since L/A is an A-module, its restriction to $\Lambda^2 A$ vanishes. Hence the curvature induces a section $R^\nabla_{1,1} \in \Gamma(A^\vee \otimes A^\perp \otimes \mathrm{End}(L/A))$ or, equivalently, a bundle map $R^\nabla_{1,1} : A \otimes (L/A) \to \mathrm{End}(L/A)$ given by

$$R^\nabla_{1,1}\Big(a; q(l)\Big) = R^\nabla(a, l) = \nabla_a \nabla_l - \nabla_l \nabla_a - \nabla_{[a,l]}, \quad \forall a \in \Gamma(A), l \in \Gamma(L).$$

Proposition 2.2.6 ([16]).

1. *The section $R^\nabla_{1,1} \in \Gamma(A^\vee \otimes A^\perp \otimes \mathrm{End}(L/A))$ is a 1-cocycle for the Lie algebroid A with values in the A-module $A^\perp \otimes \mathrm{End}(L/A)$.*
2. *The cohomology class $\alpha_{L/A} \in H^1_{\mathrm{CE}}(A, A^\perp \otimes \mathrm{End}(L/A))$ of the 1-cocycle $R^\nabla_{1,1}$ does not depend on the choice of L-connections extending the Bott A-action.*

We call $R_{1,1}^{\nabla}$ the *Atiyah cocycle* associated with the L-connection ∇. Its cohomology class

$$\alpha_{L/A} \in H^1_{\mathrm{CE}}\Big(A, A^{\perp} \otimes \mathrm{End}(L/A)\Big) = H^1_{\mathrm{CE}}\Big(A, B^{\vee} \otimes \mathrm{End}(B)\Big)$$

is called the *Atiyah class* of the Lie pair.

Choosing a splitting $i \circ p + j \circ q = \mathrm{id}_L$ of the short exact sequence (2.4), we can identify $\Lambda^2 L^{\vee}$ with the Whitney sum $\Lambda^2 A^{\vee} \oplus (A^{\vee} \otimes B^{\vee}) \oplus \Lambda^2 B^{\vee}$.

The following lemma will be needed later on.

Lemma 2.2.7. *Under the identification above, the curvature of ∇ decomposes as*

$$R^{\nabla} = \widetilde{R_{1,1}^{\nabla}} + R_{0,2}^{\nabla}$$

where $\widetilde{R_{1,1}^{\nabla}} \in \Gamma(\Lambda^2 L^{\vee} \otimes \mathrm{End}(B))$ denotes the skew-symmetrization of $R_{1,1}^{\nabla} \in \Gamma(A^{\vee} \otimes B^{\vee} \otimes \mathrm{End}(B))$, and $R_{0,2}^{\nabla} : \Lambda^2 L \to \mathrm{End}(B)$ is the bundle map defined by

$$R_{0,2}^{\nabla}(x, y) = R^{\nabla}\Big(j \circ q(x), j \circ q(y)\Big), \quad \forall x, y \in \Gamma(L).$$

The Todd cocycle of a Lie pair (L, A) is the Chevalley-Eilenberg cocycle

$$\mathrm{td}_{L/A}^{\nabla} = \det\left(\frac{R_{1,1}^{\nabla}}{1 - e^{-R_{1,1}^{\nabla}}}\right) \in \bigoplus_{k=0} \Gamma(\Lambda^k A^{\vee} \otimes \Lambda^k A^{\perp}),$$

$$\widetilde{\mathrm{td}}_{L/A}^{\nabla} = \det\left(\frac{R_{1,1}^{\nabla}}{e^{\frac{1}{2}R_{1,1}^{\nabla}} - e^{-\frac{1}{2}R_{1,1}^{\nabla}}}\right) \in \bigoplus_{k=0} \Gamma(\Lambda^k A^{\vee} \otimes \Lambda^k A^{\perp}).$$

The Todd class of a Lie pair (L, A) is the cohomology class

$$\mathrm{Td}_{L/A} = \det\left(\frac{\alpha_{L/A}}{1 - e^{-\alpha_{L/A}}}\right) \in \bigoplus_{k=0} H^k_{\mathrm{CE}}(A, \Lambda^k A^{\perp}),$$

$$\widetilde{\mathrm{Td}}_{L/A} = \det\left(\frac{\alpha_{L/A}}{e^{\frac{1}{2}\alpha_{L/A}} - e^{-\frac{1}{2}\alpha_{L/A}}}\right) \in \bigoplus_{k=0} H^k_{\mathrm{CE}}(A, \Lambda^k A^{\perp}).$$

In the particular case of the Lie pair comprised of the Lie $\mathbb{C}$-algebroids $L = T_X \otimes \mathbb{C}$ and $A = T_X^{0,1}$ associated with a complex manifold X, the quotient of the pair is $T_X^{1,0}$ and the Atiyah class and the Todd class of the pair are the classical Atiyah class of T_X and the classical Todd class of the complex manifold X.

2.2.3 Polydifferential operators

The universal enveloping algebra $\mathcal{U}(L)$ of the Lie algebroid L is a coalgebra over R [67]. Its comultiplication

$$\Delta : \mathcal{U}(L) \to \mathcal{U}(L) \otimes_R \mathcal{U}(L)$$

is characterized by the identities

$$\begin{aligned} \Delta(1) &= 1 \otimes 1; \\ \Delta(x) &= 1 \otimes x + x \otimes 1, \quad \forall x \in \Gamma(L); \\ \Delta(u \cdot v) &= \Delta(u) \cdot \Delta(v), \quad \forall u, v \in \mathcal{U}(L), \end{aligned}$$

where $1 \in R$ denotes the constant function on M with value 1 while the symbol $\cdot$ denotes the multiplication in $\mathcal{U}(L)$. We refer the reader to [67] for the precise meaning of the last equation above. Explicitly, we have

$$\begin{aligned} \Delta(l_1 \cdot l_2 \cdot \cdots \cdot l_n) &= 1 \otimes (l_1 \cdot l_2 \cdot \cdots \cdot l_n) \\ &\quad + \sum_{\substack{p+q=n \\ p,q \in \mathbb{N}}} \sum_{\sigma \in \mathfrak{S}_p^q} (l_{\sigma(1)} \cdot \cdots \cdot l_{\sigma(p)}) \otimes (l_{\sigma(p+1)} \cdot \cdots \cdot l_{\sigma(n)}) + (l_1 \cdot l_2 \cdot \cdots \cdot l_n) \otimes 1, \end{aligned}$$

for all $l_1, \ldots, l_n \in \Gamma(L)$.

Let (L, A) be a pair of Lie algebroids over $\Bbbk$. Writing $\mathcal{U}(L)\Gamma(A)$ for the left ideal of $\mathcal{U}(L)$ generated by $\Gamma(A)$, the quotient $\frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}$ is automatically an R-coalgebra since

$$\Delta\Big(\mathcal{U}(L)\Gamma(A)\Big) \subseteq \mathcal{U}(L) \otimes_R \Big(\mathcal{U}(L)\Gamma(A)\Big) + \Big(\mathcal{U}(L)\Gamma(A)\Big) \otimes_R \mathcal{U}(L).$$

Let $\mathcal{D}_{\text{poly}}^{-1}$ denote the algebra R of smooth functions on the manifold M, let $\mathcal{D}_{\text{poly}}^{0}$ denote the left $\mathcal{U}(A)$-module $\frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}$, let $\mathcal{D}_{\text{poly}}^{k}$ denote the tensor product $\mathcal{D}_{\text{poly}}^{0} \otimes_R \cdots \otimes_R \mathcal{D}_{\text{poly}}^{0}$ of $(k+1)$ copies of the left R-module $\mathcal{D}_{\text{poly}}^{0}$, and set $\mathcal{D}_{\text{poly}}^{\bullet} = \bigoplus_{k=-1}^{\infty} \mathcal{D}_{\text{poly}}^{k}$. Since $\mathcal{U}(A)$ is a Hopf algebroid, it follows that for each $k \geqslant -1$, $\mathcal{D}_{\text{poly}}^{k}$ is also naturally a $\mathcal{U}(A)$-module [67].

Lemma 2.2.8. *The $\mathcal{U}(A)$-module $\mathcal{D}_{\text{poly}}^{0}$ is a cocommutative coassociative coalgebra over R whose comultiplication is a morphism of $\mathcal{U}(A)$-modules.*

Since the comultiplication Δ is coassociative, the Hochschild operator $d_{\mathscr{H}}$: $\mathcal{D}_{\text{poly}}^{k-1} \to \mathcal{D}_{\text{poly}}^{k}$ defined by

$$\begin{aligned} d_{\mathscr{H}}(u_1 \otimes \cdots \otimes u_k) &= 1 \otimes u_1 \otimes \cdots \otimes u_k \\ &+ \sum_{i=1}^{k} (-1)^i u_1 \otimes \cdots \otimes \Delta(u_i) \otimes \cdots \otimes u_k + (-1)^{k+1} u_1 \otimes \cdots \otimes u_k \otimes 1, \end{aligned}$$

for all $u_1, u_2, \ldots, u_k \in \mathcal{D}_{\text{poly}}^{0}$, is a coboundary operator, i.e. $d_{\mathscr{H}}^2 = 0$.

Moreover, $d_{\mathscr{H}} : \mathcal{D}_{\text{poly}}^{k-1} \to \mathcal{D}_{\text{poly}}^{k}$ is a morphism of $\mathcal{U}(A)$-modules, since the comultiplication $\Delta : \mathcal{D}_{\text{poly}}^{0} \to \mathcal{D}_{\text{poly}}^{0} \otimes_R \mathcal{D}_{\text{poly}}^{0}$ is a morphism of $\mathcal{U}(A)$-modules. Therefore, the Hochschild complex

$$0 \longrightarrow \mathcal{D}_{\text{poly}}^{-1} \xrightarrow{d_{\mathscr{H}}} \mathcal{D}_{\text{poly}}^{0} \xrightarrow{d_{\mathscr{H}}} \mathcal{D}_{\text{poly}}^{1} \xrightarrow{d_{\mathscr{H}}} \mathcal{D}_{\text{poly}}^{2} \xrightarrow{d_{\mathscr{H}}} \cdots$$

is a complex of $\mathcal{U}(A)$-modules.

The Chevalley–Eilenberg cohomology in degree k of the Hochschild complex of the pair (L, A), which is defined as

$$\mathbb{H}_{\text{CE}}^{k}(A, \mathcal{D}_{\text{poly}}^{\bullet}) := \operatorname{Hom}_{D^+(\mathfrak{A})}(R, \mathcal{D}_{\text{poly}}^{\bullet}[k]),$$

can be computed as the degree k hypercohomology of the double complex

$$\begin{array}{ccccccc} & & \vdots & & \vdots & & \\ & & {\scriptstyle -\operatorname{id}\otimes d_{\mathscr{H}}}\uparrow & & {\scriptstyle \operatorname{id}\otimes d_{\mathscr{H}}}\uparrow & & \\ \cdots \longrightarrow & & \Gamma(\Lambda^p A^\vee) \otimes_R \mathcal{D}_{\text{poly}}^{q+1} & \xrightarrow{d_A^{\mathcal{U}}} & \Gamma(\Lambda^{p+1} A^\vee) \otimes_R \mathcal{D}_{\text{poly}}^{q+1} & \xrightarrow{d_A^{\mathcal{U}}} & \cdots \\ & & {\scriptstyle -\operatorname{id}\otimes d_{\mathscr{H}}}\uparrow & & {\scriptstyle \operatorname{id}\otimes d_{\mathscr{H}}}\uparrow & & \\ \cdots \longrightarrow & & \Gamma(\Lambda^p A^\vee) \otimes_R \mathcal{D}_{\text{poly}}^{q} & \xrightarrow{d_A^{\mathcal{U}}} & \Gamma(\Lambda^{p+1} A^\vee) \otimes_R \mathcal{D}_{\text{poly}}^{q} & \xrightarrow{d_A^{\mathcal{U}}} & \cdots \\ & & \uparrow & & \uparrow & & \\ & & \vdots & & \vdots & & \end{array} \tag{2.5}$$

The coboundary operator $d_A^{\mathcal{U}} : \Gamma(\Lambda^p A^\vee) \otimes \mathcal{D}_{\text{poly}}^{q} \to \Gamma(\Lambda^{p+1} A^\vee) \otimes \mathcal{D}_{\text{poly}}^{q}$ is defined by

$$\begin{aligned} d_A^{\mathcal{U}}(\omega \otimes u_0 \otimes \cdots \otimes u_q) &= (d_A \omega) \otimes u_0 \otimes \cdots \otimes u_q \\ &+ \sum_{j=1}^{\operatorname{rk}(A)} \sum_{k=0}^{q} (\alpha_j \wedge \omega) \otimes u_0 \otimes \cdots \otimes u_{k-1} \otimes a_j \cdot u_k \otimes u_{k+1} \otimes \cdots \otimes u_q, \end{aligned}$$

for all $\omega \in \Gamma(\Lambda^p A^\vee)$ and $u_1, u_2, \dots, u_k \in \mathcal{D}^0_{\text{poly}}$. Here $(a_i)_{i\in\{1,\dots,r\}}$ designates any local frame of A and $(\alpha_j)_{j\in\{1,\dots,r\}}$ the corresponding dual local frame of $A^\vee$.

Lemma 2.2.9. *For any Lie pair (L, A), the Hochschild cohomology $\mathbb{H}^\bullet_{\text{CE}}(A, \mathcal{D}^\bullet_{\text{poly}})$ is an associative algebra, whose multiplication stems from the tensor product of left R-modules $\otimes_R$ in $\mathcal{D}^\bullet_{\text{poly}}$.*

Remark 2.2.10. *Since, unlike the universal enveloping algebra $\mathcal{U}(L)$ of a Lie algebroid L, the space $\mathcal{D}^0_{\text{poly}}$ is not a Hopf algebroid, the Gerstenhaber bracket* (2.12) *does not extend to $\mathcal{D}^\bullet_{\text{poly}}$ (and $\Gamma(\Lambda^\bullet A^\vee) \otimes_R \mathcal{D}^\bullet_{\text{poly}}$). Therefore, unlike $\mathbb{H}^\bullet_{\text{CE}}(A, \mathcal{U}(L)^{\otimes \bullet+1})$, the Hochschild cohomology $\mathbb{H}^\bullet_{\text{CE}}(A, \mathcal{D}^\bullet_{\text{poly}})$ does not, a priori, admit a Gerstenhaber algebra structure. However, it turns out that $\Gamma(\Lambda^\bullet A^\vee) \otimes_R \mathcal{D}^\bullet_{\text{poly}}$ does actually admit an L_∞ structure and that its cohomology admits a Gerstenhaber algebra structure. These arise from what we call a Fedosov dg Lie algebroid associated with the Lie pair (L, A) — see Corollary 2.3.19.*

2.2.4 Polyvector fields

Given a Lie pair (L, A), let $\mathcal{T}^{-1}_{\text{poly}}$ denote the algebra R of smooth functions on the manifold M, set $\mathcal{T}^k_{\text{poly}} := \Gamma(\Lambda^{k+1}(L/A))$ for $k \geqslant 0$, and consider $\mathcal{T}^\bullet_{\text{poly}} = \bigoplus_{k=-1} \mathcal{T}^k_{\text{poly}}$ as a complex of $\mathcal{U}(A)$-modules with trivial differential:

$$0 \longrightarrow \mathcal{T}^{-1}_{\text{poly}} \xrightarrow{0} \mathcal{T}^{0}_{\text{poly}} \xrightarrow{0} \mathcal{T}^{1}_{\text{poly}} \xrightarrow{0} \mathcal{T}^{2}_{\text{poly}} \xrightarrow{0} \cdots$$

Its Chevalley–Eilenberg cohomology in degree k

$$\mathbb{H}^k_{\text{CE}}(A, \mathcal{T}^\bullet_{\text{poly}}) := \operatorname{Hom}_{D^+(\mathfrak{A})}(R, \mathcal{T}^\bullet_{\text{poly}}[k])$$

is computed as the degree k hypercohomology of the double complex

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & 0\uparrow & & 0\uparrow & & \\
\cdots & \longrightarrow & \Gamma(\Lambda^p A^\vee) \otimes_R \mathcal{T}^{q+1}_{\text{poly}} & \xrightarrow{d_A^{\text{Bott}}} & \Gamma(\Lambda^{p+1} A^\vee) \otimes_R \mathcal{T}^{q+1}_{\text{poly}} & \xrightarrow{d_A^{\text{Bott}}} & \cdots \\
 & & 0\uparrow & & 0\uparrow & & \\
\cdots & \longrightarrow & \Gamma(\Lambda^p A^\vee) \otimes_R \mathcal{T}^{q}_{\text{poly}} & \xrightarrow{d_A^{\text{Bott}}} & \Gamma(\Lambda^{p+1} A^\vee) \otimes_R \mathcal{T}^{q}_{\text{poly}} & \xrightarrow{d_A^{\text{Bott}}} & \cdots \\
 & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & &
\end{array} \tag{2.6}$$

The coboundary operator $d_A^{\text{Bott}} : \Gamma(\Lambda^p A^\vee) \otimes \mathcal{T}_{\text{poly}}^q \to \Gamma(\Lambda^{p+1} A^\vee) \otimes \mathcal{T}_{\text{poly}}^q$ is defined by

$$\begin{aligned} d_A^{\text{Bott}}(\omega \otimes b_0 \wedge \cdots \wedge b_q) &= (d_A \omega) \otimes b_0 \wedge \cdots \wedge b_q \\ &\quad + \sum_{j=1}^{\text{rk}(A)} \sum_{k=0}^{q} (\alpha_j \wedge \omega) \otimes b_0 \wedge \cdots \wedge b_{k-1} \wedge \nabla_{a_j}^{\text{Bott}} b_k \wedge b_{k+1} \wedge \cdots \wedge b_q, \end{aligned}$$

for all $\omega \in \Gamma(\Lambda^p A^\vee)$ and $b_0, b_1, \ldots, b_q \in \Gamma(L/A)$. Here $(a_i)_{i \in \{1,\ldots,r\}}$ designates any local frame of A and $(\alpha_j)_{j \in \{1,\ldots,r\}}$ the corresponding dual local frame of $A^\vee$.

In fact, the coboundary operator d_A^{Bott} extends to an L_∞ structure on $\Gamma(\Lambda^\bullet A^\vee) \otimes \mathcal{T}_{\text{poly}}^\bullet$ — see Corollary 2.3.14.

Again, a priori, $\mathbb{H}_{\text{CE}}^\bullet(A, \mathcal{T}_{\text{poly}}^\bullet)$ is only a vector space. The following lemma, however, is obvious.

Lemma 2.2.11. *For any Lie pair* (L, A), $\mathbb{H}_{\text{CE}}^\bullet(A, \mathcal{T}_{\text{poly}}^\bullet)$ *is an associative algebra, whose multiplication stems from the wedge product on* $\mathcal{T}_{\text{poly}}^\bullet$.

2.2.5 Hochschild–Kostant–Rosenberg isomorphism

The natural inclusion $\Gamma(L/A) \hookrightarrow \mathcal{D}_{\text{poly}}^0$ extends to a morphism of complexes of $\mathcal{U}(A)$-modules

$$\text{hkr} : \mathcal{T}_{\text{poly}}^\bullet \to \mathcal{D}_{\text{poly}}^\bullet$$

by skew-symmetrization:

$$\text{hkr}(b_1 \wedge \cdots \wedge b_n) = \frac{1}{n!} \sum_{\sigma \in S_n} \text{sgn}(\sigma) b_{\sigma(1)} \otimes b_{\sigma(2)} \otimes \cdots \otimes b_{\sigma(n)}, \quad \forall b_1, \cdots, b_n \in \Gamma(L/A).$$

Furthermore, hkr induces a morphism of double complexes:

Lemma 2.2.12. *The map*

$$\text{id} \otimes \text{hkr} : \left(\Gamma(\Lambda^\bullet A^\vee) \otimes_R \mathcal{T}_{\text{poly}}^\bullet, d_A^{\text{Bott}}, 0\right) \to \left(\Gamma(\Lambda^\bullet A^\vee) \otimes_R \mathcal{D}_{\text{poly}}^\bullet, d_A^{\mathcal{U}}, \pm \text{id} \otimes d_{\mathscr{H}}\right)$$

is a morphism of double complexes from (2.6) *to* (2.5).

The induced map between total cohomologies is called *Hochschild–Kostant–Rosenberg* map. Abusing notations, we will denote it hkr instead of id ⊗ hkr.

Proposition 2.2.13. *For any Lie pair (L, A), the Hochschild–Kostant–Rosenberg map*

$$\mathrm{hkr} : \mathbb{H}^{\bullet}_{\mathrm{CE}}(A, \mathcal{T}^{\bullet}_{\mathrm{poly}}) \to \mathbb{H}^{\bullet}_{\mathrm{CE}}(A, \mathcal{D}^{\bullet}_{\mathrm{poly}})$$

is an isomorphism.

Proposition 2.2.13 can be proved by a spectral sequence argument. Repeating the argument in [30, Theorem 4.10], one can prove the following lemma:

Lemma 2.2.14. *For each p, the map*

$$\mathrm{id} \otimes \mathrm{hkr} : \Big(\Gamma(\Lambda^p A^\vee) \otimes_R \mathcal{T}^{\bullet}_{\mathrm{poly}}, 0\Big) \to \Big(\Gamma(\Lambda^p A^\vee) \otimes_R \mathcal{D}^{\bullet}_{\mathrm{poly}}, \pm \,\mathrm{id} \otimes d_{\mathscr{H}}\Big)$$

is a quasi-isomorphism.

Proof of Proposition 2.2.13. Consider the spectral sequences associated with the filtrations

$$F^k\Big(\bigoplus_{\substack{p\geqslant 0\\ q\geqslant -1}} \Big(\Gamma(\Lambda^p A^\vee) \otimes_R \mathcal{T}^{q}_{\mathrm{poly}}\Big)\Big) = \bigoplus_{\substack{p\geqslant k\\ q\geqslant -1}} \Big(\Gamma(\Lambda^p A^\vee) \otimes_R \mathcal{T}^{q}_{\mathrm{poly}}\Big)$$

$$F^k\Big(\bigoplus_{\substack{p\geqslant 0\\ q\geqslant -1}} \Big(\Gamma(\Lambda^p A^\vee) \otimes_R \mathcal{D}^{q}_{\mathrm{poly}}\Big)\Big) = \bigoplus_{\substack{p\geqslant k\\ q\geqslant -1}} \Big(\Gamma(\Lambda^p A^\vee) \otimes_R \mathcal{D}^{q}_{\mathrm{poly}}\Big)$$

on the double complexes (2.6) and (2.5). The map induced by $\mathrm{id} \otimes \mathrm{hkr}$ between the E_0-terms of these two spectral sequences is precisely the quasi-isomorphism of Lemma 2.2.14. Therefore $\mathrm{id} \otimes \mathrm{hkr}$ induces isomorphism between the E_k-terms of the two spectral sequences for each k larger than or equal to 1. Since both filtrations are complete and exhaustive, it follows from the Eilenberg–Moore comparison theorem that $\mathrm{hkr} : \mathbb{H}^{\bullet}_{\mathrm{CE}}(A, \mathcal{T}^{\bullet}_{\mathrm{poly}}) \to \mathbb{H}^{\bullet}_{\mathrm{CE}}(A, \mathcal{D}^{\bullet}_{\mathrm{poly}})$ is an isomorphism. □

2.2.6 Atiyah and Todd cocycles/classes of a dg Lie algebroid

A $\mathbb{Z}$-graded manifold $\mathcal{M}$ with base manifold M is a sheaf $\mathcal{R}$ of $\mathbb{Z}$-graded, graded-commutative algebras $\{\mathcal{R}_U \mid U \subset M \text{ open}\}$ over M isomorphic to $C^\infty(U) \otimes \hat{S}(V^\vee)$ for sufficiently small open subsets U of M. Here $\hat{S}(V^\vee)$ denotes the graded algebra of formal polynomials on some $\mathbb{Z}$-graded vector space V. The underlying local $\mathbb{Z}$-graded manifold is normally denoted $U \times V_{\mathrm{formal}}$. By $C^\infty(\mathcal{M})$, we denote the

$\mathbb{Z}$-graded, graded-commutative algebra of global sections of $\mathcal{R}$. A dg manifold is a $\mathbb{Z}$-graded manifold endowed with a homological vector field, i.e. a vector field Q of degree $+1$ satisfying $[Q,Q]=0$. For instance, if A is a Lie $\Bbbk$-algebroid, then $A[1]$ is a dg manifold with the Chevalley–Eilenberg differential d_{CE} as homological vector field. According to Vaĭntrob [63], there is a bijection between Lie algebroid structures on a vector bundle $A\to M$ and homological vector fields on the $\mathbb{Z}$-graded manifold $A[1]$.

A dg vector bundle [31, 47] is a vector bundle in the category of dg manifolds. Given a vector bundle $\mathcal{E}\xrightarrow{\pi}\mathcal{M}$ of graded manifolds, its space of sections, denoted $\Gamma(\mathcal{E})$, is defined to be $\bigoplus_{j\in\mathbb{Z}}\Gamma(\mathcal{E})_j$, where $\Gamma(\mathcal{E})_j$ consists of sections of degree j, i.e. maps $s\in\operatorname{Hom}(\mathcal{M},\mathcal{E}[-j])$ such that $(\pi[-j])\circ s=\mathrm{id}_{\mathcal{M}}$. Here $\pi[-j]:\mathcal{E}[-j]\to\mathcal{M}$ is the natural map induced by π; see [47] for more details. When $\mathcal{E}\to\mathcal{M}$ is a dg vector bundle, the homological vector fields $Q^{\mathcal{E}}$ and $Q^{\mathcal{M}}$ on $\mathcal{E}$ and $\mathcal{M}$ naturally determine an operator $\mathcal{Q}$ of degree $+1$ on $\Gamma(\mathcal{E})$, making $\Gamma(\mathcal{E})$ a dg module over the dg algebra $C^\infty(\mathcal{M})$. Indeed, $\mathcal{E}\xrightarrow{\pi}\mathcal{M}$ is a graded vector bundle if and only if "the vector field $Q^{\mathcal{E}}$ projects onto $Q^{\mathcal{M}}$" i.e.

$$Q^{\mathcal{E}}\Big(\pi^*(f)\Big)=\pi^*\Big(Q^{\mathcal{M}}(f)\Big),\quad \forall f\in C^\infty(\mathcal{M})$$

and "the flow of $Q^{\mathcal{E}}$ preserves the linear structure of the fibers of π" i.e. the submodule $\Gamma(\mathcal{E}^\vee)$ of $C^\infty(\mathcal{E})$ comprised of all smooth functions on $\mathcal{E}$ "linear along the fibers of π" is stable under the derivation $Q^{\mathcal{E}}$. The restriction of $Q^{\mathcal{E}}$ to $\Gamma(\mathcal{E}^\vee)$ determines an operator $\mathcal{Q}$ on $\Gamma(\mathcal{E})$ through the relation

$$Q^{\mathcal{M}}\Big(\langle\zeta|e\rangle\Big)=\Big\langle Q^{\mathcal{E}}(\zeta)\Big|e\Big\rangle+(-1)^{|\zeta|}\langle\zeta|\mathcal{Q}(e)\rangle\,,\quad \forall\zeta\in\Gamma(\mathcal{E}^\vee),e\in\Gamma(\mathcal{E}).$$

Since $\pi^*\Big(C^\infty(\mathcal{M})\Big)$ and $\Gamma(\mathcal{E}^\vee)$ together generate the algebra $C^\infty(\mathcal{E})$ multiplicatively, knowledge of the vector field $Q^{\mathcal{M}}$ and the operator $\mathcal{Q}$ suffices to recover the homological vector field $Q^{\mathcal{E}}$.

In this case, the degree $+1$ operator $\mathcal{Q}$ on $\Gamma(\mathcal{E})$ gives rise to a cochain complex

$$\cdots\to\Gamma(\mathcal{E})_i\xrightarrow{\mathcal{Q}}\Gamma(\mathcal{E})_{i+1}\to\cdots,$$

whose cohomology group will be denoted by $H^\bullet(\Gamma(\mathcal{E}),\mathcal{Q})$.

A dg Lie algebroid is a Lie algebroid object in the category of dg manifolds.

For more details, we refer the reader to [45, 47], where dg Lie algebroids are called Q-algebroids. It is simple to see that if $\mathcal{M}$ is a dg manifold, then $T_{\mathcal{M}}$ is naturally a dg Lie algebroid.

The notion of Atiyah class of dg Lie algebroids was introduced and studied by Mehta–StiÃľnon–Xu [48]. It extends the notion of Atiyah class of a dg manifold, which was first investigated by Shoikhet [56] in relation with Kontsevich's formality theorem and Duflo's formula.

Let $\mathcal{A}$ be a dg Lie algebroid with anchor $\rho : \mathcal{A} \to T_{\mathcal{M}}$. An $\mathcal{A}$-connection on $\mathcal{A}$ is a map

$$\nabla : \Gamma(\mathcal{A}) \otimes \Gamma(\mathcal{A}) \to \Gamma(\mathcal{A})$$

of degree 0 satisfying

$$\nabla_{fX} Y = f \nabla_X Y$$
$$\nabla_X(fY) = \big(\rho(X)f\big)Y + (-1)^{|X||f|} f \nabla_X Y$$

for all $f \in C^\infty(\mathcal{M})$ and all (homogeneous) $X, Y \in \Gamma(\mathcal{A})$. The notation $|-|$ is used to denote the degree of the argument. Here the degree of ∇ is its degree as a map from $\Gamma(\mathcal{A}) \otimes \Gamma(\mathcal{A})$ to $\Gamma(\mathcal{A})$. More precisely, by saying that ∇ is a map of degree 0 we mean that $|\nabla_X Y| = |X| + |Y|$ for every pair of homogeneous elements X and Y. Connections always exist since the standard partition of unity argument holds in the context of graded manifolds.

Given a dg Lie algebroid $\mathcal{A}$, the associated operator $\mathcal{Q}$ of degree $+1$ on $\Gamma(\mathcal{A})$, and an $\mathcal{A}$-connection ∇ on $\mathcal{A}$, one defines a bundle map $\operatorname{At}^\nabla_{\mathcal{A}} : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ of degree $+1$ by

$$\operatorname{At}^\nabla_{\mathcal{A}}(X, Y) := \mathcal{Q}(\nabla_X Y) - \nabla_{\mathcal{Q}(X)} Y - (-1)^{|X|} \nabla_X\Big(\mathcal{Q}(Y)\Big), \quad \forall X, Y \in \Gamma(\mathcal{A}).$$

Alternatively, we may think of $\operatorname{At}^\nabla_{\mathcal{A}}$ as a section of degree $+1$ in $\Gamma(\mathcal{A}^\vee \otimes \operatorname{End}\mathcal{A})$. It is immediate that $\mathcal{Q}(\operatorname{At}^\nabla_{\mathcal{A}}) = 0$. Since $\mathcal{Q}^2 = 0$, we may thus regard $\operatorname{At}^\nabla_{\mathcal{A}}$ as a 1-cocycle in the cochain complex $\Big(\Gamma(\mathcal{A}^\vee \otimes \operatorname{End}\mathcal{A})_\bullet, \mathcal{Q}\Big)$.

Definition 2.2.15. *The 1-cocycle* $\operatorname{At}^\nabla_{\mathcal{A}} \in Z^1(\Gamma(\mathcal{A}^\vee \otimes \operatorname{End}\mathcal{A}), \mathcal{Q})$ *is called the Atiyah 1-cocycle of the dg Lie algebroid* $\mathcal{A}$ *with respect to the* $\mathcal{A}$*-connection* ∇ *on* $\mathcal{A}$.

It is simple to check that its cohomology class $\alpha_{\mathcal{A}} := [\operatorname{At}^\nabla_{\mathcal{A}}] \in H^1\Big(\Gamma(\mathcal{A}^\vee \otimes \operatorname{End}\mathcal{A}), \mathcal{Q}\Big)$ is independent of the choice of the connection ∇. The class $\alpha_{\mathcal{A}}$ is called the *Atiyah class* of the dg Lie algebroid $\mathcal{A}$. It is the obstruction class to the existence of a dg compatible $\mathcal{A}$-connection on $\mathcal{A}$. (See [48] for more details.)

The *Todd cocycle* $\operatorname{td}^\nabla_{\mathcal{A}}$ (or $\widetilde{\operatorname{td}}^\nabla_{\mathcal{A}}$) and *Todd class* $\operatorname{Td}_{\mathcal{A}}$ (or $\widetilde{\operatorname{Td}}_{\mathcal{A}}$) of a dg Lie algebroid $\mathcal{A}$ are defined as follows:

$$\operatorname{td}^\nabla_{\mathcal{A}} := \operatorname{Ber}\left(\frac{\operatorname{At}^\nabla_{\mathcal{A}}}{1-e^{-\operatorname{At}^\nabla_{\mathcal{A}}}}\right) \in \prod_{k\geqslant 0}\Gamma(\Lambda^k\mathcal{A}^\vee)_k,$$

$$\widetilde{\operatorname{td}}^\nabla_{\mathcal{A}} := \operatorname{Ber}\left(\frac{\operatorname{At}^\nabla_{\mathcal{A}}}{e^{\frac{1}{2}\operatorname{At}^\nabla_{\mathcal{A}}}-e^{-\frac{1}{2}\operatorname{At}^\nabla_{\mathcal{A}}}}\right) \in \prod_{k\geqslant 0}\Gamma(\Lambda^k\mathcal{A}^\vee)_k$$

$$\operatorname{Td}_{\mathcal{A}} := \operatorname{Ber}\left(\frac{\alpha_{\mathcal{A}}}{1-e^{-\alpha_{\mathcal{A}}}}\right) \in \prod_{k\geqslant 0} H^k\Big(\Gamma(\Lambda^k\mathcal{A}^\vee),\mathcal{Q}\Big),$$

$$\widetilde{\operatorname{Td}}_{\mathcal{A}} := \operatorname{Ber}\left(\frac{\alpha_{\mathcal{A}}}{e^{\frac{1}{2}\alpha_{\mathcal{A}}}-e^{-\frac{1}{2}\alpha_{\mathcal{A}}}}\right) \in \prod_{k\geqslant 0} H^k\Big(\Gamma(\Lambda^k\mathcal{A}^\vee),\mathcal{Q}\Big),$$

where $\Lambda^k\mathcal{A}$ denotes the dg vector bundle $S^k(\mathcal{A}^\vee[-1])[k]\to\mathcal{M}$. The definition of the Berezinian Ber can be found in [17]. It is known that $\operatorname{Td}_{\mathcal{A}}$ and $\widetilde{\operatorname{Td}}_{\mathcal{A}}$ can be expressed in terms of the scalar Atiyah classes $c_k := \frac{1}{k!}(\frac{i}{2\pi})^k \operatorname{str}\alpha^k_{\mathcal{A}} \in H^k\Big(\Gamma(\Lambda^k\mathcal{A}^\vee),\mathcal{Q}\Big)$. Here $\operatorname{str}:\operatorname{End}\mathcal{A}\to C^\infty(\mathcal{M})$ denotes the supertrace and $\operatorname{str}\alpha^k_{\mathcal{A}}\in\Gamma(\Lambda^k\mathcal{A}^\vee)$ since $\alpha^k_{\mathcal{A}}\in\Gamma(\Lambda^k\mathcal{A}^\vee)\otimes_{C^\infty(\mathcal{M})}\operatorname{End}\mathcal{A}$.

Example 2.2.16. *[48, Example 3.4] Consider the tangent dg Lie algebroid $T_{\mathcal{M}}$ of a dg manifold $\mathcal{M}=(\mathcal{V},Q)$, where $\mathcal{V}=\mathbb{R}^m\times V_{\mathrm{formal}}$ for some finite dimensional $\mathbb{Z}$-graded vector space V over $\Bbbk$. The algebra of functions on the graded manifold $\mathcal{V}$ is $C^\infty(\mathcal{V})=C^\infty(\mathbb{R}^m)\otimes\hat{S}(V^\vee)$. Let $(z_1,\cdots,z_N)$ be a choice of coordinate functions on $\mathcal{V}$. Writing the homological vector field Q as $Q=\sum_k Q_k\frac{\partial}{\partial z_k}$, the Atiyah 1-cocycle associated with the trivial connection $\nabla^{\mathrm{trivial}}_{\frac{\partial}{\partial z_i}}\frac{\partial}{\partial z_j}=0$ admits the simple expression*

$$\operatorname{At}^{\mathrm{trivial}}_{T_{\mathcal{V}}}\left(\frac{\partial}{\partial z_i},\frac{\partial}{\partial z_j}\right)=(-1)^{|z_i|+|z_j|}\sum_k\frac{\partial^2 Q_k}{\partial z_i\partial z_j}\frac{\partial}{\partial z_k}.$$

Hence the Atiyah 1-cocycle $\operatorname{At}^{\mathrm{trivial}}_{T_{\mathcal{V}}}$ captures the second- and higher-order information contained in the homological vector field.

In this paper, we are particularly interested in an important class of dg Lie algebroids, namely the Fedosov dg Lie algebroids associated with a Lie pair (L,A). See the Appendix or [59] for more details.

2.2.7 Atiyah and Todd cocycles/classes of the Fedosov dg Lie algebroid

Let (L, A) be a Lie pair over a smooth manifold M. Each choice of (1) a splitting of the short exact sequence of vector bundles $0 \to A \to L \to B \to 0$ and (2) a torsion-free L-connection on B determines a homological vector field Q on the graded manifold $\mathcal{M} = L[1] \oplus B$ — see Theorem 2.3.7 in the Appendix. Any such dg manifold $(\mathcal{M}, Q)$ is called a Fedosov dg manifold associated with the Lie pair (L, A). The pullback $\mathcal{F} \to \mathcal{M}$ of the quotient bundle $B \to M$ through the canonical projection $\mathcal{M} \to M$ is a dg Lie subalgebroid of the tangent dg Lie algebroid $T_{\mathcal{M}} \to \mathcal{M}$ — see Proposition 2.3.9 in the Appendix. Any such dg Lie algebroid $\mathcal{F} \to \mathcal{M}$ is called a Fedosov dg Lie algebroid associated with the Lie pair (L, A).

Since $\mathcal{F} \to \mathcal{M}$ is both the pullback of the vector bundle $B \to M$ through the canonical map $\mathcal{M} \twoheadrightarrow M$ and a vector subbundle of $T_{\mathcal{M}} \to \mathcal{M}$, we have the inclusions

$$\begin{array}{ccccccc} \Gamma(B) & \hookrightarrow & C^\infty(\mathcal{M}) \otimes_{C^\infty(M)} \Gamma(B) & \overset{\cong}{\longleftrightarrow} & \Gamma(\mathcal{F} \to \mathcal{M}) & \hookrightarrow & \mathfrak{X}(\mathcal{M}) \\ b & \longmapsto & 1 \otimes b & & \longmapsto & & \hat{b} \end{array}$$

Indeed, $\Gamma(\mathcal{F} \to \mathcal{M})$ is the $C^\infty(\mathcal{M})$-submodule of $\mathfrak{X}(\mathcal{M})$ generated by $\Gamma(B)$. In particular, if $\partial_1, \dots, \partial_r$ is a local frame for B and $\chi_1, \dots, \chi_r$ is the dual local frame for $B^\vee$, then $\hat{\partial}_k$ is the vector field $\frac{\partial}{\partial \chi_k}$ on $\mathcal{M}$, i.e. the derivation $\lambda \otimes \chi^M \mapsto \lambda \otimes M_k \chi^{M - e_k}$ of $C^\infty(\mathcal{M}) \cong \Gamma(\Lambda L^\vee \otimes \hat{S}(B^\vee))$.

There exists a *canonical* $\mathcal{F}$-connection on $\mathcal{F}$ characterized by the relation

$$\nabla^{\text{can}}_{\hat{b}} \hat{c} = 0, \quad \forall b, c \in \Gamma(B).$$

Definition 2.2.17. *The Atiyah 1-cocycle* $\operatorname{At}^{\text{can}}_{\mathcal{F}} \in Z^1(\Gamma(\mathcal{M}; \mathcal{F}^\vee \otimes \operatorname{End} \mathcal{F}), \mathcal{Q})$ *corresponding to the canonical connection* ∇^{can} *is called the canonical Atiyah 1-cocycle.*

Since $\Gamma(\mathcal{M}; \mathcal{F}^\vee \otimes \operatorname{End} \mathcal{F})$ can be identified with $\Gamma(\Lambda^\bullet L^\vee \otimes \hat{S} B^\vee \otimes B^\vee \otimes \operatorname{End} B)$, the canonical Atiyah cocycle is essentially the tensor product of an endomorphism of B and an element of $\Gamma(\Lambda^1 L^\vee \otimes \hat{S} B^\vee \otimes \Lambda^1 B^\vee)$. Therefore, the Berezinian appearing in the expression for the Todd cocycle of ∇^{can} is simply the classical determinant and the *canonical Todd cocycle* is

$$\operatorname{td}_{\mathcal{F}}^{\mathrm{can}} := \det\left(\frac{\operatorname{At}_{\mathcal{F}}^{\mathrm{can}}}{1-e^{-\operatorname{At}_{\mathcal{F}}^{\mathrm{can}}}}\right) \in \prod_{k\geqslant 0} \Gamma(\Lambda^k L^\vee \otimes \hat{S}B^\vee \otimes \Lambda^k B^\vee). \tag{2.7}$$

Similarly,

$$\widetilde{\operatorname{td}}_{\mathcal{F}}^{\mathrm{can}} := \det\left(\frac{\operatorname{At}_{\mathcal{F}}^{\mathrm{can}}}{e^{\frac{1}{2}\operatorname{At}_{\mathcal{F}}^{\mathrm{can}}}-e^{-\frac{1}{2}\operatorname{At}_{\mathcal{F}}^{\mathrm{can}}}}\right) \in \prod_{k\geqslant 0} \Gamma(\Lambda^k L^\vee \otimes \hat{S}B^\vee \otimes \Lambda^k B^\vee). \tag{2.8}$$

Lemma 2.2.18. *Given any local frame $\partial_1, \cdots, \partial_r$ for B, the canonical Atiyah 1-cocycle $\operatorname{At}_{\mathcal{F}}^{\mathrm{can}} : \mathcal{F}\otimes\mathcal{F}\to\mathcal{F}$ of the Fedosov dg Lie algebroid $\mathcal{F}\to\mathcal{M}$ admits the local expression*

$$\operatorname{At}_{\mathcal{F}}^{\mathrm{can}}(\hat{\partial}_i, \hat{\partial}_j) = \sum_{k=1}^{r} \hat{\partial}_i(\hat{\partial}_j f_k)\cdot\hat{\partial}_k, \tag{2.9}$$

where the functions $f_k \in C^\infty(\mathcal{M})$ are the components of the vector field $X^\nabla = \sum_{k=1}^r f_k \cdot \hat{\partial}_k$ of Theorem 2.3.7 relative to the chosen frame.

Proof. Recall that the coboundary operator $\mathcal{Q}$ on the sections of the Fedosov dg Lie algebroid is the restriction of the Lie derivative $\mathcal{Q} = [Q, \cdot]$ in $\mathfrak{X}(\mathcal{M})$ with the homological vector field $Q \in \mathfrak{X}(\mathcal{M})$ appearing in Theorem 2.3.7. We have

$$\begin{aligned}\operatorname{At}_{\mathcal{F}}^{\mathrm{can}}(\hat{\partial}_i, \hat{\partial}_j) &= \mathcal{Q}(\nabla^{\mathrm{can}}_{\hat{\partial}_i}\hat{\partial}_j) - \nabla^{\mathrm{can}}_{\mathcal{Q}(\hat{\partial}_i)}\hat{\partial}_j - (-1)^0\nabla^{\mathrm{can}}_{\hat{\partial}_i}\Big(\mathcal{Q}(\hat{\partial}_j)\Big)\\ &= -\nabla^{\mathrm{can}}_{\hat{\partial}_i}\Big(\mathcal{Q}(\hat{\partial}_j)\Big).\end{aligned}$$

Now $\mathcal{Q}(\hat{\partial}_j) = [Q, \hat{\partial}_j] = -[\delta, \hat{\partial}_j] + [d_L^\nabla, \hat{\partial}_j] + [X^\nabla, \hat{\partial}_j]$ and it is easy to show that $[\delta, \hat{\partial}_j] = 0$. Therefore,

$$\operatorname{At}_{\mathcal{F}}^{\mathrm{can}}(\hat{\partial}_i, \hat{\partial}_j) = -\nabla^{\mathrm{can}}_{\hat{\partial}_i}([d_L^\nabla, \hat{\partial}_j]) - \nabla^{\mathrm{can}}_{\hat{\partial}_i}([X^\nabla, \hat{\partial}_j]).$$

Given local coordinates $(x_1, \ldots, x_m)$ on the base manifold M; a local frame $\eta_1, \ldots, \eta_l$ for L; the dual local frame $\lambda_1, \ldots, \lambda_l$ for $L^\vee$; the local frame $\partial_1, \ldots, \partial_r$ for B; and the dual local frame $\chi_1, \ldots, \chi_r$ for $B^\vee$, the vector field d_L^∇ on $\mathcal{M}$ decomposes as the sum

$$d_L^\nabla = \sum_j a_i^j \lambda_i \frac{\partial}{\partial x_j} - \frac{1}{2}\sum_{i,j,k} c_{ij}^k \lambda_i\lambda_j \frac{\partial}{\partial \lambda_k} - \sum_{i,j,k} \Gamma_{ij}^k \lambda_i \chi_j \frac{\partial}{\partial \chi_k},$$

where $a_i^j = \langle dx_j | \rho(\eta_i)\rangle$, $c_{ij}^k = \langle \lambda_k | [\eta_i, \eta_j]\rangle$, and $\Gamma_{ij}^k = \langle \chi_k | \nabla_{\eta_i}\partial_j\rangle$ are functions in

$C^\infty(M)$ encoding the anchor and the Lie bracket of the Lie algebroid L and the L-connection on B. Therefore, since $\hat{\partial}_j = \frac{\partial}{\partial \chi_j}$, we have $[d_L^\nabla, \hat{\partial}_j] = \sum_{ik} \Gamma_{ij}^k \lambda_i \cdot \hat{\partial}_k$ and thus $\nabla^{\text{can}}_{\hat{\partial}_i}([d_L^\nabla, \hat{\partial}_j]) = 0$. Likewise, we have $[X^\nabla, \hat{\partial}_j] = -\sum_k \hat{\partial}_j(f_k) \cdot \hat{\partial}_k$ and thus $\nabla^{\text{can}}_{\hat{\partial}_i}([X^\nabla, \hat{\partial}_j]) = -\sum_k \hat{\partial}_i(\hat{\partial}_j f_k) \cdot \hat{\partial}_k$. $\square$

Proposition 2.2.19. *The quasi-isomorphism*

$$\sigma_\natural : \Big(\Gamma(\Lambda^\bullet L^\vee) \otimes_R \mathfrak{T}^{r,s}_{\text{poly}}(B), \mathcal{L}_Q\Big) \to \Big(\Gamma(\Lambda^\bullet A^\vee) \otimes_R \mathcal{T}^{r,s}_{\text{poly}}, d_A^{\nabla^{\text{Bott}}}\Big)$$

of Proposition 2.3.11 takes $\text{At}^{\text{can}}_{\mathcal{F}}$, $\text{td}^{\text{can}}_{\mathcal{F}}$, *and* $\widetilde{\text{td}}^{\text{can}}_{\mathcal{F}}$ *to* $R^\nabla_{1,1}$, $\text{td}^\nabla_{L/A}$, *and* $\widetilde{\text{td}}^\nabla_{L/A}$, *respectively:*

$$\sigma_\natural(\text{At}^{\text{can}}_{\mathcal{F}}) = R^\nabla_{1,1}, \qquad \sigma_\natural(\text{td}^{\text{can}}_{\mathcal{F}}) = \text{td}^\nabla_{L/A}, \qquad \sigma_\natural(\widetilde{\text{td}}^{\text{can}}_{\mathcal{F}}) = \widetilde{\text{td}}^\nabla_{L/A}\,.$$

Proof. It suffices to prove the first statement; the second and third statements are immediate consequences of the first. Lemma 2.2.18, together with the natural identification of $\Gamma(\mathcal{M}; \mathcal{F}^\vee \otimes \operatorname{End}\mathcal{F})_1$ with $C^\infty(\mathcal{M}) \otimes_{C^\infty(M)} \Gamma(B^\vee \otimes B^\vee \otimes B)$, implies that

$$\text{At}^{\text{can}}_{\mathcal{F}} = \sum_{i,j,k=1}^{r} \hat{\partial}_i(\hat{\partial}_j f_k) \otimes (\chi_i \otimes \chi_j \otimes \partial_k),$$

where $\hat{\partial}_i(\hat{\partial}_j f_k) \in C^\infty(\mathcal{M}) = \Gamma(L^\vee \otimes \hat{S}(B^\vee))$ and $\chi_i \otimes \chi_j \otimes \partial_k \in \Gamma(B^\vee \otimes B^\vee \otimes B)$.

According to Theorem 2.3.7, we have $X^\nabla = \sum_{k=1}^r (\sum_{t=2}^\infty f_k^{(t)}) \otimes \partial_k$ with $f_k^{(t)} \in \Gamma(L^\vee \otimes S^t(B^\vee))$. Moreover, $X_2 = \sum_k f_k^{(2)} \otimes \partial_k$ decomposes as the sum of $h_\natural(\widetilde{R^\nabla_{1,1}}) = \sum_k f_k^{(1,1)} \otimes \partial_k$ and $h_\natural(R^\nabla_{0,2}) = \sum_k f_k^{(0,2)} \otimes \partial_k$, where $f_k^{(1,1)} \in \Gamma(p^\top(A^\vee) \otimes S^2 B^\vee)$ and $f_k^{(0,2)} \in \Gamma(q^\top(B^\vee) \otimes S^2 B^\vee)$.

Since $\hat{\partial}_i(\hat{\partial}_j f_k^{(t)}) \in \Gamma(L^\vee \otimes S^{t-2} B^\vee)$, we have $\sigma\Big(\hat{\partial}_i(\hat{\partial}_j f_k^{(t)})\Big) = 0$ for $t \geqslant 3$. Since $\hat{\partial}_i(\hat{\partial}_j f_k^{(0,2)}) \in \Gamma(q^\top(B^\vee))$, we also have $\sigma\Big(\hat{\partial}_i(\hat{\partial}_j f_k^{(0,2)})\Big) = 0$.

Therefore, we obtain

$$\begin{aligned}
\sigma_\natural(\text{At}^{\text{can}}_{\mathcal{F}}) &= \sigma_\natural\Big(\sum_{i,j,k=1}^{r} \hat{\partial}_i(\hat{\partial}_j f_k) \otimes (\chi_i \otimes \chi_j \otimes \partial_k)\Big) \\
&= \sum_{i,j,k=1}^{r} \sigma\Big(\sum_{t=2}^{\infty} \hat{\partial}_i(\hat{\partial}_j f_k^{(t)})\Big) \otimes (\chi_i \otimes \chi_j \otimes \partial_k) \\
&= \sum_{i,j,k=1}^{r} \hat{\partial}_i(\hat{\partial}_j f_k^{(1,1)}) \otimes (\chi_i \otimes \chi_j \otimes \partial_k)
\end{aligned}$$

$$
\begin{aligned}
&= 2\sum_{k=1}^{r} f_k^{(1,1)} \otimes \partial_k \\
&= 2h_\natural(\widetilde{R_{1,1}^\nabla}) \\
&= R_{1,1}^\nabla.
\end{aligned}
$$

□

Corollary 2.2.20. *1. The isomorphism* $\sigma_\natural : H^1\Big(\Gamma(\mathcal{M}; \mathcal{F}^\vee \otimes \operatorname{End}\mathcal{F}), \mathcal{Q}\Big) \to H^1_{\mathrm{CE}}(A, B^\vee \otimes \operatorname{End} B)$ *induced by the quasi-isomorphism of Proposition 2.3.11 (for* $r = 2$ *and* $s = 1$*) takes the Atiyah class* $\alpha_\mathcal{F}$ *of the Fedosov dg Lie algebroid* $\mathcal{F}$ *to the Atiyah class* $\alpha_{L/A}$ *of the Lie pair* (L, A)*:*

$$\sigma_\natural(\alpha_\mathcal{F}) = \alpha_{L/A}.$$

2. The isomorphism $\sigma_\natural : H^k\Big(\Gamma(\mathcal{M}; \Lambda^k\mathcal{F}^\vee), \mathcal{Q}\Big) \to H^k_{\mathrm{CE}}(A, \Lambda^k B^\vee)$ *induced by the quasi-isomorphism of Theorem 2.3.14 takes the Todd class of the Fedosov dg Lie algebroid* $\mathcal{F}$ *to the Todd class of the Lie pair* (L, A)*:*

$$\sigma_\natural(\mathrm{Td}_\mathcal{F}) = \mathrm{Td}_{L/A}, \qquad \sigma_\natural(\widetilde{\mathrm{Td}}_\mathcal{F}) = \widetilde{\mathrm{Td}}_{L/A}.$$

2.3 Fedosov dg Lie algebroids

In this section, we recall basic ingredients needed to establish our main result (Theorem 4.1.1) in Section 4.1. For details, we refer the interested reader to [59].

2.3.1 DGLAs associated to dg Lie algebroids

One can make sense of polyvector fields and polydifferential operators for a dg Lie algebroid just as one does for ordinary Lie algebroids. Both give rise to dglas and their cohomology groups are in fact Gerstenhaber algebras. More precisely, a k-vector field on a dg Lie algebroid $\mathcal{L} \to \mathcal{M}$ is a section of the vector bundle $\Lambda^k\mathcal{L} \to \mathcal{M}$ while a k-differential operator is an element of $\mathcal{U}(\mathcal{L})^{\otimes k}$, the tensor product (as left $C^\infty(\mathcal{M})$-modules) of k copies of the universal enveloping algebra $\mathcal{U}(\mathcal{L})$.

It is clear that the differential $\mathcal{Q} : \Gamma(\mathcal{L}) \to \Gamma(\mathcal{L})$ and the homological vector field $\mathcal{Q} : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ extend naturally to a degree +1 differential $\mathcal{Q} : \Gamma(\Lambda^{k+1}\mathcal{L}) \to \Gamma(\Lambda^{k+1}\mathcal{L})$ and the the Lie algebroid strucure on $\mathcal{L}$ yields a Schouten bracket

$$[-,-]:\Gamma(\Lambda^{u+1}\mathcal{L})\otimes\Gamma(\Lambda^{u+1}\mathcal{L})\to\Gamma(\Lambda^{u+v+1}\mathcal{L}).$$

Proposition 2.3.1. *Let $\mathcal{L}$ be a dg Lie algebroid over $\mathcal{M}$.*

1. *When endowed with the differential $\mathcal{Q}$, the wedge product, and the Schouten bracket, the space of 'polyvector fields' $\Gamma(\Lambda^{\bullet+1}\mathcal{L})$ is a differential Gerstenhaber algebra — whence a dgla.*

2. *When endowed with the wedge product and the Schouten bracket, the cohomology $H^\bullet\Big(\Gamma(\Lambda^{\bullet+1}\mathcal{L}),\mathcal{Q}\Big)$ is a Gerstenhaber algebra.*

Adapting the definition given for Lie algebroids, one can define the universal enveloping algebra of a dg Lie algebroid. The universal enveloping algebra of a dg Lie algebroid $\mathcal{L}\to\mathcal{M}$ is a dg Hopf algebroid $\mathcal{U}(\mathcal{L})$ over the dga $C^\infty(\mathcal{M})$. For each $k\geqslant 0$, the dg structure on the dg Lie algebroid $\mathcal{L}\to\mathcal{M}$ determines a differential $\mathcal{Q}:\mathcal{U}(\mathcal{L})^{\otimes k+1}\to\mathcal{U}(\mathcal{L})^{\otimes k+1}$ of degree +1. A Hochschild coboundary differential

$$d_{\mathscr{H}}:\mathcal{U}(\mathcal{L})^{\otimes k}\to\mathcal{U}(\mathcal{L})^{\otimes k+1}$$

and Gerstenhaber bracket

$$[\![-,-]\!]:\mathcal{U}(\mathcal{L})^{\otimes u+1}\otimes\mathcal{U}(\mathcal{L})^{\otimes v+1}\to\mathcal{U}(\mathcal{L})^{\otimes u+v+1}\tag{2.10}$$

can be defined by the following explicit algebraic expressions:

$$\begin{aligned}d_{\mathscr{H}}(u_1\otimes\cdots\otimes u_k)=1\otimes u_1\otimes\cdots\otimes u_k\\ +\sum_{i=1}^{k}(-1)^i u_1\otimes\cdots\otimes\Delta(u_i)\otimes\cdots\otimes u_k\\ +(-1)^{k+1}u_1\otimes\cdots\otimes u_k\otimes 1,\end{aligned}\tag{2.11}$$

and

$$[\![\phi,\psi]\!]=\phi\star\psi-(-1)^{uv}\psi\star\phi\in\mathcal{U}(\mathcal{L})^{\otimes u+v+1},\tag{2.12}$$

where $\phi \star \psi \in \mathcal{U}(\mathcal{L})^{\otimes u+v+1}$ is defined by

$$\phi \star \psi = \sum_{k=0}^{u} (-1)^{kv} d_0 \otimes_R \cdots \otimes_R d_{k-1} \otimes_R (\Delta^v d_k) \cdot \psi \otimes_R d_{k+1} \otimes_R \cdots \otimes_R d_u$$

if $\phi = d_0 \cdot d_1 \cdots d_u$ for some $d_0, d_1, \ldots, d_u \in \mathcal{U}(\mathcal{L})$.

We refer the reader to [67] for the precise meaning of the product $(\Delta^v d_k) \cdot \psi$ in $\mathcal{U}(\mathcal{L})^{\otimes v+1}$ appearing in the last equation above.

Proposition 2.3.2. *Let $\mathcal{L}$ be a dg Lie algebroid over $\mathcal{M}$.*

1. *When endowed with the differential $\mathcal{Q} + d_{\mathscr{H}}$ and the Gerstenhaber bracket* (2.10)*, $\mathcal{U}(\mathcal{L})^{\otimes \bullet+1}$ is a dgla.*

2. *When endowed with the cup product (i.e. the tensor product $\otimes_{C^\infty(\mathcal{M})}$) and the Gerstenhaber bracket, the Hochschild cohomology $H^\bullet\Big(\mathcal{U}(\mathcal{L})^{\otimes \bullet+1}, \mathcal{Q} + d_{\mathscr{H}}\Big)$, is a Gerstenhaber algebra.*

Recall that, for a Lie pair (L, A), if a splitting $j : B(= L/A) \to L$ of the short exact sequence $0 \to A \to L \to B \to 0$ is given, whose image $j(B)$ happens to be a Lie subalgebroid of L, then A and B are said to form a *matched pair of Lie algebroids* — see [51] for more details. In such a situation, we write $L = A \bowtie B$ to highlight that A and B play symmetric roles as a pair of complementary Lie subalgebroids of the Lie algebroid L.

Lemma 2.3.3. *If $A \bowtie B$ is a matched pair of Lie algebroids, then $(A[1] \oplus B, d_A^{\mathrm{Bott}})$ is a dg Lie algebroid over $(A[1], d_A)$.*

Proof. A classical result of Mackenzie [41] asserts that, if $A \bowtie B$ is a matched pair of Lie algebroids over a smooth manifold M, then

$$\begin{array}{ccc} A \bowtie B & \longrightarrow & B \\ \downarrow & & \downarrow \\ A & \longrightarrow & M \end{array}$$

is a double Lie algebroid. Moreover, Gracia-Saz and Mehta [27] proved that, given a double Lie algebroid

$$\begin{array}{ccc} D & \longrightarrow & B \\ \downarrow & & \downarrow \\ A & \longrightarrow & M \end{array},$$

the graded vector bundle $D[1] \to A[1]$ is automatically a differential graded Lie algebroid. □

Here the dg manifold structures on $(A[1]\oplus B, d_A^{\mathrm{Bott}})$ and $(A[1], d_A)$ result from the Lie algebroid structures on $A\oplus B\to B$ and $A\to M$, respectively. In what follows, denote by $\mathcal{B}$ the dg manifold $A[1]\oplus B$. The space of sections of $\mathcal{B}\to A[1]$ can be naturally identified with $\Gamma(\Lambda^\bullet A^\vee\otimes B)$. The bracket on $\Gamma(\Lambda^\bullet A^\vee\otimes B)$ is defined in terms of the Bott representation of B on $\Lambda A^\vee$ by

$$[\xi_1\otimes b_1, \xi_2\otimes b_2] = \xi_1\wedge\xi_2\otimes[b_1,b_2] + \xi_1\wedge\nabla^{\mathrm{Bott}}_{b_1}\xi_2\otimes b_2 - \nabla^{\mathrm{Bott}}_{b_2}\xi_1\wedge\xi_2\otimes b_1 \quad (2.13)$$

for all $\xi_1,\xi_2\in\Gamma(\Lambda^\bullet A^\vee)$ and $b_1,b_2\in\Gamma(B)$, while the anchor map $\Gamma(\Lambda^\bullet A^\vee\otimes B)\xrightarrow{\bar\rho}\mathrm{Der}(\Lambda^\bullet A^\vee)$ is defined by

$$\bar\rho(\xi\otimes b)(\eta) = \xi\wedge\nabla^{\mathrm{Bott}}_b\eta, \quad (2.14)$$

for all $\xi,\eta\in\Gamma(\Lambda^\bullet A^\vee)$ and $b\in\Gamma(B)$. Finally, the differential on the space of sections of $\mathcal{B}\to A[1]$ is simply the Chevalley–Eilenberg differential $d_A^{\mathrm{Bott}}:\Gamma(\Lambda^\bullet A^\vee\otimes B)\to\Gamma(\Lambda^{\bullet+1}A^\vee\otimes B)$. corresponding to the Bott representation of A on B.

According to Proposition 2.3.1, the dg Lie algebroid $\mathcal{B}$ induces a differential Gerstenhaber algebra structure on $\Gamma(\Lambda^{\bullet+1}\mathcal{B})\cong\Gamma(\Lambda^\bullet A^\vee\otimes\Lambda^{\bullet+1}B)$. Its differential is the Chevalley–Eilenberg differential

$$d_A^{\mathrm{Bott}}:\Gamma(\Lambda^\bullet A^\vee\otimes\Lambda^{\bullet+1}B)\to\Gamma(\Lambda^{\bullet+1}A^\vee\otimes\Lambda^{\bullet+1}B), \quad (2.15)$$

corresponding to the Bott representation of A on ΛB and its Lie bracket is the Schouten bracket of the dg Lie algebroid $\mathcal{B}\to A[1]$ — essentially the extension of Equations (2.13) and (2.14) by the graded Leibniz rule.

Next, consider the universal enveloping algebra $\mathcal{U}(\mathcal{B})$ of the dg Lie algebroid $\mathcal{B}$, which is a dg Hopf algebroid over $(\Gamma(\Lambda^\bullet A^\vee), d_A)$. It is clear that $\mathcal{U}(\mathcal{B})\cong\Gamma(\Lambda^\bullet A^\vee)\otimes_R\mathcal{U}(B)$, and $\mathcal{U}(\mathcal{B})^{\otimes k+1}\cong\Gamma(\Lambda^\bullet A^\vee)\otimes_R\mathcal{U}(B)^{\otimes k+1}$. Under this identification, the differential $\mathcal{Q}:\mathcal{U}(\mathcal{B})^{\otimes k+1}\to\mathcal{U}(\mathcal{B})^{\otimes k+1}$ becomes the Chevalley–Eilenberg differential

$$d_A^{\mathcal{U}}:\Gamma(\wedge^\bullet A^\vee)\otimes_R\mathcal{U}(B)^{\otimes k+1}\to\Gamma(\wedge^{\bullet+1}A^\vee)\otimes_R\mathcal{U}(B)^{\otimes k+1}. \quad (2.16)$$

Here the A-module structure on $\mathcal{U}(B)$ follows from the canonical identification of $\mathcal{U}(B)$ with $\frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}$ — the Lie algebroid A acts on the latter by mutiplication from the left — and extends to an A-module structure on $\mathcal{U}(B)^{\otimes k+1}$ in the natural way. As a consequence, the total differential $\mathcal{Q} + d_{\mathscr{H}}$ on $\mathcal{U}(\mathcal{B})^{\otimes \bullet+1}$ coincides with $d_A^{\mathcal{U}} + d_{\mathscr{H}}$ on $\operatorname{tot}\Big(\Gamma(\Lambda^\bullet A^\vee) \otimes_R \mathcal{U}(B)^{\bullet+1}\Big)$.

The following proposition summarizes the discussion above:

Proposition 2.3.4. *Suppose (A, B) is a matched pair of Lie algebroids.*

1. *When endowed with the differential d_A^{Bott} as in* (2.15) *and the Schouten bracket defined by Equations* (2.13)-(2.14)*, $\Gamma(\wedge^\bullet A^\vee \otimes \wedge^{\bullet+1} B)$ is a differential Gerstenhaber algebra, whence a dgla.*

2. *When endowed with the wedge product and the Schouten bracket, the cohomology $\mathbb{H}^\bullet_{\mathrm{CE}}(A, \Lambda^{\bullet+1} B)$ is a Gerstenhaber algebra.*

3. *When endowed with the differential $d_A^{\mathcal{U}} + d_{\mathscr{H}}$ (see* (2.11) *and* (2.16)*) and the Gerstenhaber bracket, $\Big(\operatorname{tot}\Big(\Gamma(\Lambda^\bullet A^\vee) \otimes_R \mathcal{U}(B)^{\otimes \bullet+1}\Big)$ is a dgla.*

4. *When endowed with the cup product and the Gerstenhaber bracket, the Hochschild cohomology $\mathbb{H}^\bullet_{\mathrm{CE}}\Big(A, \mathcal{U}(B)^{\otimes \bullet+1}\Big)$, i.e. the cohomology of the complex $\Big(\operatorname{tot}(\Lambda^\bullet A^\vee \otimes_R \mathcal{U}(B)^{\otimes \bullet+1}, d_A^{\mathcal{U}} + d_{\mathscr{H}}\Big)$, is a Gerstenhaber algebra.*

Remark 2.3.5. *Note that the Gerstenhaber bracket on* $\operatorname{tot}\Big(\Gamma(\Lambda^\bullet A^\vee) \otimes_R \mathcal{U}(B)^{\otimes \bullet+1}\Big)$ *is* not *the obvious extension of the Gerstenhaber bracket on $\mathcal{U}(B)^{\otimes \bullet+1}$ obtained by tensoring with the commutative associative algebra $\Gamma(\Lambda^\bullet A^\vee)$. In fact, to write down an explicit formula — which is quite involved — one needs to use the Bott representation of B on $\Gamma(\Lambda^\bullet A^\vee)$.*

2.3.2 Fedosov dg Lie algebroids

Let (L, A) be a Lie pair. We use the symbols B to denote the quotient vector bundle L/A and r to denote its rank.

Consider the endomorphism δ of the vector bundle $\Lambda^\bullet L^\vee \otimes \hat{S} B^\vee$ defined by

$$\delta(\omega \otimes \chi^J) = \sum_{m=1}^{r} \Big(q^\top(\chi_m) \wedge \omega\Big) \otimes J_m \, \chi^{J-e_m},$$

for all $\omega \in \Lambda L^\vee$ and $J \in \mathbb{N}^r$. Here $\{\chi_k\}_{k=1}^r$ denotes an arbitrary local frame for the vector bundle $B^\vee$, the symbol e_m denotes the multi-index $(0, \cdots, 0, 1, 0, \cdots, 0)$

having its single nonzero entry in m-th position, and

$$\chi^J = \underbrace{\chi_1 \odot \cdots \odot \chi_1}_{J_1 \text{ factors}} \odot \underbrace{\chi_2 \odot \cdots \odot \chi_2}_{J_2 \text{ factors}} \odot \cdots \otimes \underbrace{\chi_r \odot \cdots \odot \chi_r}_{J_r \text{ factors}}$$

if $J = (J_1, J_2, \cdots, J_r)$.

The operator δ is a derivation of degree $+1$ of the graded commutative algebra $\Gamma(\Lambda^\bullet L^\vee \otimes \hat{S}B^\vee)$ and satisfies $\delta^2 = 0$. The resulting cochain complex

$$\cdots \longrightarrow \Lambda^{n-1}L^\vee \otimes \hat{S}B^\vee \xrightarrow{\delta} \Lambda^n L^\vee \otimes \hat{S}B^\vee \xrightarrow{\delta} \Lambda^{n+1}L^\vee \otimes \hat{S}B^\vee \longrightarrow \cdots$$

deformation retracts onto the trivial complex

$$\cdots \longrightarrow \Lambda^{n-1}A^\vee \xrightarrow{0} \Lambda^n A^\vee \xrightarrow{0} \Lambda^{n+1}A^\vee \longrightarrow \cdots$$

Indeed, for every choice of splitting $i \circ p + j \circ q = \mathrm{id}_L$ of the short exact sequence

$$0 \longrightarrow A \underset{p}{\overset{i}{\rightleftarrows}} L \underset{j}{\overset{q}{\rightleftarrows}} B \longrightarrow 0 \tag{2.17}$$

and its dual

$$0 \longrightarrow B^\vee \underset{j^\top}{\overset{q^\top}{\rightleftarrows}} L^\vee \underset{p^\top}{\overset{i^\top}{\rightleftarrows}} A^\vee \longrightarrow 0 \ ,$$

the chain maps

$$\sigma : \Lambda^\bullet L^\vee \otimes \hat{S}B^\vee \to \Lambda^\bullet A^\vee$$

and

$$\tau : \Lambda^\bullet A^\vee \to \Lambda^\bullet L^\vee \otimes \hat{S}B^\vee$$

respectively defined by

$$\sigma(\omega \otimes \chi^J) = \begin{cases} \omega \otimes \chi^J & \text{if } v = 0 \text{ and } |J| = 0 \\ 0 & \text{otherwise,} \end{cases}$$

for all $\omega \in p^\top(\Lambda^u A^\vee) \otimes q^\top(\Lambda^v B^\vee)$, and

$$\tau(\alpha) = p^\top(\alpha) \otimes 1,$$

for all $\alpha \in \Lambda^\bullet(A^\vee)$, satisfy

$$\sigma\tau = \mathrm{id} \qquad \text{and} \qquad \mathrm{id} - \tau\sigma = h\delta + \delta h,$$

where the homotopy operator

$$h : \Lambda^\bullet L^\vee \otimes \hat{S}B^\vee \to \Lambda^{\bullet-1} L^\vee \otimes \hat{S}B^\vee$$

is defined by

$$h(\omega \otimes \chi^J) = \begin{cases} \frac{1}{v+|J|} \sum_{k=1}^r (\iota_{j(\partial_k)}\omega) \otimes \chi^{J+e_k} & \text{if } v \geqslant 1 \\ 0 & \text{if } v = 0 \end{cases}$$

for all $\omega \in p^\top(\Lambda^u A^\vee) \otimes q^\top(\Lambda^v B^\vee)$. Here $\{\partial_k\}_{k=1}^r$ denotes the local frame for B dual to $\{\chi_k\}_{k=1}^r$. Note that the operator h is *not* a derivation of the algebra $\Gamma(\Lambda^\bullet L^\vee \otimes \hat{S}B^\vee)$. Also, we note that $h\tau = 0$, $\sigma h = 0$, and $h^2 = 0$.

Lemma 2.3.6. *Let (L, A) be a Lie pair and let ∇ be an L-connection on B extending the Bott A-connection. The torsion T^∇ of ∇ vanishes (see Proposition 2.2.4) if and only if $\delta d_L^\nabla + d_L^\nabla \delta = 0$.*

Consider the four maps $\delta_\natural$, $\sigma_\natural$, $h_\natural$, and $\tau_\natural$

$$\Gamma(\Lambda^\bullet A^\vee \otimes B) \underset{\tau_\natural}{\overset{\sigma_\natural}{\leftrightarrows}} \Gamma(\Lambda^\bullet L^\vee \otimes \hat{S}B^\vee \otimes B) \underset{h_\natural}{\overset{\delta_\natural}{\rightleftarrows}} \Gamma(\Lambda^{\bullet+1} L^\vee \otimes \hat{S}B^\vee \otimes B)$$

defined by

$$\delta_\natural(\omega \otimes \sigma \otimes b) = \delta(\omega \otimes \sigma) \otimes b, \qquad \sigma_\natural(\omega \otimes \sigma \otimes b) = \sigma(\omega \otimes \sigma) \otimes b,$$
$$h_\natural(\omega \otimes \sigma \otimes b) = h(\omega \otimes \sigma) \otimes b, \qquad \tau_\natural(\alpha \otimes b) = \tau(\alpha) \otimes b,$$

for all $\alpha \in \Gamma(\Lambda A^\vee)$, $\omega \in \Gamma(\Lambda L^\vee)$, $\sigma \in \Gamma(\hat{S}B^\vee)$, and $b \in \Gamma(B)$.

Theorem 2.3.7 ([59])**.** *Let (L, A) be a Lie pair with quotient $B = L/A$. We interpret the sections of the bundle $L^\vee \otimes \hat{S}B^\vee \otimes B$ as derivations of the algebra $\Gamma(\Lambda^\bullet L^\vee \otimes \hat{S}B^\vee)$ in the natural way. Given a splitting of the short exact sequence*

(2.17) *and a torsion-free L-connection ∇ on B, there exists a* unique *derivation*

$$X^{\nabla} \in \Gamma(L^{\vee} \otimes \hat{S}^{\geqslant 2} B^{\vee} \otimes B),$$

satisfying $h_{\natural}(X^{\nabla}) = 0$ and such that the derivation $Q : \Gamma(\Lambda^{\bullet} L^{\vee} \otimes \hat{S} B^{\vee}) \to \Gamma(\Lambda^{\bullet+1} L^{\vee} \otimes \hat{S} B^{\vee})$ defined by

$$Q = -\delta + d_L^{\nabla} + X^{\nabla}$$

satisfies $Q^2 = 0$. Moreover, writing X_k for the component of X^{∇} in $L^{\vee} \otimes S^k B^{\vee} \otimes B$, we have $X^{\nabla} = \sum_{k=2}^{\infty} X_k$ with

$$X_2 = h_{\natural}(R^{\nabla}) = h_{\natural}(\widetilde{R_{1,1}^{\nabla}}) + h_{\natural}(R_{0,2}^{\nabla}).$$

As a consequence, $(\mathcal{M} = L[1] \oplus B, Q = -\delta + d_L^{\nabla} + X^{\nabla})$ is a dg manifold, which we call a Fedosov dg manifold *associated with the Lie pair (L, A).*

Sketch of proof. Suppose there exists such an X^{∇} and consider its decomposition $X^{\nabla} = \sum_{k=2}^{\infty} X_k$, where $X_k \in \Gamma(L^{\vee} \otimes \hat{S}^k B^{\vee} \otimes B)$. Then $D = -\delta + d_L^{\nabla} + X_2 + X_{\geqslant 3}$ with $X_{\geqslant 3} = \sum_{k=3}^{\infty} X_k$ and

$$\begin{aligned} D^2 &= \delta^2 - \left(\delta d_L^{\nabla} + d_L^{\nabla} \delta\right) + \left\{d_L^{\nabla} d_L^{\nabla} - \delta X_2 - X_2 \delta\right\} \\ &\quad + \left\{d_L^{\nabla} X^{\nabla} + X^{\nabla} d_L^{\nabla} + X^{\nabla 2} - \delta X_{\geqslant 3} - X_{\geqslant 3} \delta\right\} \\ &= \delta^2 - [\delta, d_L^{\nabla}] + \left\{R^{\nabla} - [\delta, X_2]\right\} + \left\{[d_L^{\nabla} + \tfrac{1}{2} X^{\nabla}, X^{\nabla}] - [\delta, X_{\geqslant 3}]\right\}. \end{aligned}$$

For degree reasons, the requirement $D^2 = 0$ is equivalent to the pair of equations

$$[\delta, X_2] = R^{\nabla} \qquad \text{and} \qquad [\delta, X_{\geqslant 3}] = [d_L^{\nabla} + \tfrac{1}{2} X^{\nabla}, X^{\nabla}].$$

Note that $\sigma_{\natural}(X_2) = 0$ and $\sigma_{\natural}(X_{\geqslant 3}) = 0$, since $X_2, X_{\geqslant 3} \in \Gamma(L^{\vee} \otimes \hat{S}^{\geqslant 2} B^{\vee} \otimes B)$, and also that $h_{\natural}(X_2) = 0$ and $h_{\natural}(X_{\geqslant 3}) = 0$, as $h_{\natural}(X^{\nabla}) = 0$. Since $\delta_{\natural} h_{\natural} + h_{\natural} \delta_{\natural} = \mathrm{id} - \tau_{\natural} \sigma_{\natural}$, we obtain $h_{\natural} \delta_{\natural}(X_2) = X_2$ and $h_{\natural} \delta_{\natural}(X_{\geqslant 3}) = X_{\geqslant 3}$.

It follows that

$$X_2 = h_{\natural} \delta_{\natural}(X_2) = h_{\natural}([\delta, X_2]) = h_{\natural}(R^{\nabla})$$

while

$$X_{\geqslant 3} = h_\natural \delta_\natural (X_{\geqslant 3}) = h_\natural([\delta, X_{\geqslant 3}]) = h_\natural [d_L^\nabla + \tfrac{1}{2} X^\nabla, X^\nabla].$$

Projecting the latter equation onto $\Gamma(L^\vee \otimes \hat{S}^{k+1} B^\vee \otimes B)$, we obtain

$$X_{k+1} = h_\natural \Big(d_L^\nabla \circ X_k + X_k \circ d_L^\nabla + \sum_{\substack{p+q=k+1 \\ 2 \leqslant p,q \leqslant k-1}} X_p \circ X_q \Big), \qquad \text{for } k \geqslant 2,$$

which shows that the higher terms of $X^\nabla = \sum_{k=2}^\infty X_k$ can be computed iteratively starting from $X_2 = h_\natural(R^\nabla)$. The derivation X^∇ is thus uniquely determined by the torsion-free connection ∇. □

The Fedosov dg manifold $(\mathcal{M}, Q)$ of Theorem 2.3.7 was also obtained independently by Batakidis–Voglaire [4] in the case of matched pairs.

Remark 2.3.8. *When L is the tangent bundle to a smooth manifold and A is its trivial subbundle of rank 0, Theorem 2.3.7 reduces to a classical theorem of Emmrich–Weinstein [22] (see also [18]). In the particular case of the Lie pair comprised of the complex Lie algebroids $L = T_X \otimes \mathbb{C}$ and $A = T^{0,1}X$ associated with a complex manifold X, Theorem 2.3.7 reduces to Theorem 5.9 in [9].*

The identification of $C^\infty(M)$ with the subalgebra $\Gamma(\wedge^0 L^\vee \otimes S^0(B^\vee))$ of $C^\infty(\mathcal{M}) = \Gamma(\wedge L^\vee \otimes \hat{S}(B^\vee))$ determines a surjective submersion $\mathcal{M} \twoheadrightarrow M$. Let $\mathcal{F} \to \mathcal{M}$ denote the pullback of the vector bundle $B \to M$ through $\mathcal{M} \twoheadrightarrow M$. It is a graded vector bundle whose total space $\mathcal{F}$ is the graded manifold with support M associated with the graded vector bundle $L[1] \oplus B \oplus B \to M$. Its space of sections $\Gamma(\mathcal{F} \to \mathcal{M})$ is canonically indentified with $C^\infty(\mathcal{M}) \otimes_{C^\infty(M)} \Gamma(B) = \Gamma(\Lambda^\bullet L^\vee \otimes \hat{S}(B^\vee) \otimes B)$. It is naturally a vector subbundle of $T_\mathcal{M} \to \mathcal{M}$; the inclusion $\Gamma(\mathcal{F} \to \mathcal{M}) \hookrightarrow \mathfrak{X}(\mathcal{M})$ takes the section $(\lambda \otimes \chi^J) \otimes \partial_k \in C^\infty(\mathcal{M}) \otimes_{C^\infty(M)} \Gamma(B)$ of the vector bundle $\mathcal{F} \to \mathcal{M}$ to the derivation $\mu \otimes \chi^M \mapsto \lambda \wedge \mu \otimes M_k \chi^{J+M-e_k}$ of $C^\infty(\mathcal{M})$.

Proposition 2.3.9. *The pullback $\mathcal{F} \to \mathcal{M}$ of the vector bundle $B \to M$ to the Fedosov dg manifold $(\mathcal{M}, Q)$ is a dg Lie subalgebroid of the tangent dg Lie algebroid $T_\mathcal{M} \to \mathcal{M}$.*

In other words, $\mathcal{F}$ is a dg foliation of the dg manifold $(\mathcal{M}, Q)$. Each such dg

Lie algebroid $\mathcal{F} \to \mathcal{M}$ is called a *Fedosov dg Lie algebroid* associated with the Lie pair (L, A).

2.3.3 Dolgushev–Fedosov type quasi-isomorphisms on $\mathcal{T}_{\mathrm{poly}}^{\bullet,\bullet}$ and $\mathcal{T}_{\mathrm{poly}}^{\bullet}$

Below we describe an extension of Dolgushev–Fedosov type quasi-isomorphisms [18] to the context of Lie pairs. Actually, a stronger result holds: the quasi-isomorphisms are contractions.

Set $\mathcal{T}_{\mathrm{poly}}^{r,s} := \Gamma((B^\vee)^{\otimes r} \otimes B^{\otimes s})$ and let $\mathfrak{T}_{\mathrm{poly}}^{r,s}(B)$ denote the space of formal vertical tensors of type (r, s) on the vector bundle $B \to M$, i.e.

$$\mathfrak{T}_{\mathrm{poly}}^{r,s}(B) = \Gamma(\hat{S}(B^\vee)) \otimes_R \mathcal{T}_{\mathrm{poly}}^{r,s}.$$

It is simple to see that

$$\Gamma(\mathcal{M}; (\mathcal{F}^\vee)^{\otimes r} \otimes \mathcal{F}^{\otimes s}) \cong \Gamma(\Lambda^\bullet L^\vee) \otimes_R \mathfrak{T}_{\mathrm{poly}}^{r,s}(B) \cong \Gamma(\Lambda^\bullet L^\vee \otimes \hat{S}B^\vee) \otimes_R \mathcal{T}_{\mathrm{poly}}^{r,s}.$$

Since Q is a homological vector field on the graded manifold $\mathcal{M} = L[1] \oplus B$, the Lie derivative $\mathcal{L}_Q$ is a coboundary operator on the space $T_{\mathcal{M}}^{r,s}$ of tensors of type (r, s) on $\mathcal{M}$. The Lie derivative $\mathcal{L}_Q$ stabilizes the subspaces of tensors of type (r, s) "tangent to the dg Lie subalgebroid $\mathcal{F}$ of $T_{\mathcal{M}}$."

Lemma 2.3.10. *The subspace $\Gamma(\Lambda^\bullet L^\vee) \otimes_R \mathfrak{T}_{\mathrm{poly}}^{r,s}(B)$ of $T_{\mathcal{M}}^{r,s}$ is stable under $\mathcal{L}_Q$.*

By $\sigma_\natural$, we denote the map $\sigma \otimes \mathrm{id}$:

$$\Gamma(\Lambda^\bullet L^\vee) \otimes_R \mathfrak{T}_{\mathrm{poly}}^{r,s}(B) \cong \Gamma(\Lambda^\bullet L^\vee \otimes \hat{S}B^\vee) \otimes_R \mathcal{T}_{\mathrm{poly}}^{r,s} \xrightarrow{\sigma\otimes\mathrm{id}} \Gamma(\Lambda^\bullet A^\vee) \otimes_R \mathcal{T}_{\mathrm{poly}}^{r,s}.$$

We have the following Dolgushev–Fedosov type quasi-isomorphism [18].

Proposition 2.3.11 ([59])**.** *For each type (r, s), the chain map*

$$\left(\Gamma(\Lambda^\bullet L^\vee) \otimes_R \mathfrak{T}_{\mathrm{poly}}^{r,s}(B), \mathcal{L}_Q\right) \xrightarrow{\sigma_\natural} \left(\Gamma(\Lambda^\bullet A^\vee) \otimes_R \mathcal{T}_{\mathrm{poly}}^{r,s}, d_A^{\mathrm{Bott}}\right)$$

is a quasi-isomorphism.

Indeed, a stronger result was proved in [59]. We only need this result for polyvector fields.

Set $\mathcal{T}^k_{\mathrm{poly}} := \Gamma(\Lambda^{k+1}B)$ and let $\mathfrak{T}^k_{\mathrm{poly}}(B)$ denote the space of formal vertical $(k+1)$-vector fields on B, i.e.

$$\mathfrak{T}^k_{\mathrm{poly}}(B) = \Gamma(\hat{S}(B^\vee)) \otimes_R \mathcal{T}^k_{\mathrm{poly}}.$$

Note that $\mathcal{T}^k_{\mathrm{poly}} \subset \mathcal{T}^{0,k+1}_{\mathrm{poly}}$ and $\mathfrak{T}^k_{\mathrm{poly}}(B) \subset \mathfrak{T}^{0,k+1}_{\mathrm{poly}}$. Then

$$\Gamma(\Lambda^\bullet L^\vee) \otimes_R \mathfrak{T}^k_{\mathrm{poly}}(B) \cong \Gamma(\Lambda^\bullet L^\vee \otimes \hat{S}B^\vee) \otimes_R \mathcal{T}^k_{\mathrm{poly}}.$$

Denote by $\sigma_\natural$ the map

$$\Gamma(\Lambda^\bullet L^\vee) \otimes_R \mathfrak{T}^k_{\mathrm{poly}}(B) \cong \Gamma(\Lambda^\bullet L^\vee \otimes \hat{S}B^\vee) \otimes_R \mathcal{T}^k_{\mathrm{poly}} \xrightarrow{\sigma\otimes\mathrm{id}} \Gamma(\Lambda^\bullet A^\vee) \otimes_R \mathcal{T}^k_{\mathrm{poly}}$$

Theorem 2.3.12 ([59]). *There exists a contraction*

$$\left(\mathrm{tot}\left(\Gamma(\Lambda^\bullet A^\vee) \otimes_R \mathcal{T}^\bullet_{\mathrm{poly}}\right), d_A^{\mathrm{Bott}}\right) \underset{\sigma_\natural}{\overset{\check{\tau}_\natural}{\rightleftarrows}} \left(\mathrm{tot}\left(\Gamma(\Lambda^\bullet L^\vee) \otimes_R \mathfrak{T}^\bullet_{\mathrm{poly}}(B)\right), \mathcal{L}_Q\right) \circlearrowright \check{h}_\natural$$

Consider the Fedosov dg Lie algebroid $\mathcal{F} \to \mathcal{M}$ of Section 2.3.2. It is clear that

$$\Gamma(\mathcal{M}; \Lambda^k \mathcal{F}) \cong \Gamma(\Lambda^\bullet L^\vee) \otimes_R \mathfrak{T}^k_{\mathrm{poly}}(B).$$

Applying Proposition 2.3.1 to the dg Lie subalgebroid $\mathcal{F}$ of $T_{\mathcal{M}}$, we obtain

Proposition 2.3.13. *1. Since the subspace $\Gamma(\Lambda^\bullet L^\vee) \otimes_R \mathfrak{T}^k_{\mathrm{poly}}(B)$ of the space $T^k_{\mathrm{poly}}(\mathcal{M})$ of $(k+1)$-vector fields on $\mathcal{M} = L[1] \oplus B$ is stable under $\mathcal{L}_Q$, we obtain a cochain complex*

$$\cdots \longrightarrow \Gamma(\Lambda^u L^\vee) \otimes_R \mathfrak{T}^k_{\mathrm{poly}} \xrightarrow{\mathcal{L}_Q} \Gamma(\Lambda^{u+1} L^\vee) \otimes_R \mathfrak{T}^k_{\mathrm{poly}} \longrightarrow \cdots$$

for each $k \geqslant -1$.

2. The total complex $\left(\mathrm{tot}\left(\Gamma(\Lambda^\bullet L^\vee) \otimes_R \mathfrak{T}^\bullet_{\mathrm{poly}}(B)\right), \mathcal{L}_Q\right)$ is a differential Gerstenhaber algebra, whence a dgla.

It follows from the homotopy transfer theorem for L_∞ algebras (see Lemma 4.1.25) applied to the contraction $\sigma_\natural$ that the dgla structure carried by tot $\big(\Gamma(\Lambda^\bullet L^\vee) \otimes_R$

$\mathfrak{T}^{\bullet}_{\text{poly}}(B)\Big)$ determines an L_∞ algebra structure on tot $\Big(\Gamma(\Lambda^\bullet A^\vee)\otimes_R \mathcal{T}^{\bullet}_{\text{poly}}\Big)$. Moreover, since the retraction $\sigma_\natural$ intertwines the associative algebra structures, we immediately obtain the following corollary of Proposition 2.3.13.

Corollary 2.3.14 ([59]). *Given a Lie pair (L, A), each choice of a splitting $j : B \to L$ of the short exact sequence of vector bundles $0 \to A \to L \to B \to 0$ and of a torsion-free L-connection ∇ on B determines*

1. *an L_∞ algebra structure on* tot $\Big(\Gamma(\Lambda^\bullet A^\vee)\otimes_R \mathcal{T}^{\bullet}_{\text{poly}}\Big)$ *with the operator d_A^{Bott} as unary bracket*
2. *and a Gerstenhaber algebra structure on $\mathbb{H}^\bullet_{\text{CE}}(A, \mathcal{T}^\bullet_{\text{poly}})$, the cohomology of the complex*

$$\Big(\operatorname{tot}\Big(\Gamma(\Lambda^\bullet A^\vee)\otimes_R \mathcal{T}^{\bullet}_{\text{poly}}\Big), d_A^{\text{Bott}}\Big).$$

A priori, the L_∞ algebra structure on tot $\Big(\Gamma(\Lambda^\bullet A^\vee)\otimes_R \mathcal{T}^{\bullet}_{\text{poly}}\Big)$ in Corollary 2.3.14 is not canonical; it depends on a choice of quasi-isomorphic 'Dolgushev–Fedosov type' replacement for the complex $\Big(\operatorname{tot}\Big(\Gamma(\Lambda^\bullet A^\vee)\otimes_R \mathcal{T}^{\bullet}_{\text{poly}}\Big), d_A^{\text{Bott}}\Big)$ via a Fedosov dg Lie algebroid $\mathcal{F} \to \mathcal{M}$. The construction of the Fedosov differential involves the choice of a torsion-free connection $\nabla : \Gamma(L) \times \Gamma(B) \to \Gamma(B)$ and a splitting $j : B \to L$ of the short exact sequence of vector bundles $0 \to A \to L \to B \to 0$. However, one can prove that different choices yield isomorphic L_∞ algebra structures on tot $\Big(\Gamma(\Lambda^\bullet A^\vee)\otimes_R \mathcal{T}^{\bullet}_{\text{poly}}\Big)$ [3]. Therefore, the induced Gerstenhaber algebra structure on $\mathbb{H}^\bullet_{\text{CE}}(A, \mathcal{T}^\bullet_{\text{poly}})$ is indeed canonical. Moreover, when the Lie pair happens to be a matched pair, the transferred L_∞ algebra structure on tot $\Big(\Gamma(\Lambda^\bullet A^\vee)\otimes_R \mathcal{T}^{\bullet}_{\text{poly}}\Big)$ is precisely the dgla structure described in Proposition 2.3.4.

Proposition 2.3.15. *Under the hypotheses of Corollary 2.3.14 and the additional assumption that $j(B)$ is a Lie subalgebroid of L — i.e. $L = A \bowtie B$ is a matched pair — the L_∞ algebra* tot $\Big(\Gamma(\Lambda^\bullet A^\vee)\otimes_R \mathcal{T}^{\bullet}_{\text{poly}}\Big)$ *and the Gerstenhaber algebra $\mathbb{H}^\bullet_{\text{CE}}(A, \mathcal{T}^\bullet_{\text{poly}})$ of Corollary 2.3.14 coincide respectively with the dgla $\Gamma(\Lambda^\bullet A^\vee \otimes \Lambda^{\bullet+1} B)$ and the Gerstenhaber algebra $\mathbb{H}^\bullet_{\text{CE}}(A, \Lambda^{\bullet+1} B)$ of Proposition 2.3.4.*

2.3.4 Dolgushev–Fedosov type quasi-isomorphism on $\mathcal{D}^\bullet_{\text{poly}}$

Let $C^n := \operatorname{Hom}_{\Bbbk}(\mathscr{C}^{\otimes n+1}, \mathscr{C})$ denote the space of Hochschild n-cochains of the algebra $\mathscr{C} := C^\infty(L[1] \oplus B)$. The Gerstenhaber bracket of two cochains $\phi \in C^u$

and $\psi \in C^v$ is the cochain

$$[\![\phi, \psi]\!] = \phi \star \psi - (-1)^{uv} \psi \star \phi \in C^{u+v}$$

where $\phi \star \psi \in C^{u+v}$ is defined by

$$\begin{aligned}(\phi \star \psi)&(a_0 \otimes a_1 \otimes \cdots \otimes a_{u+v}) \\ &= \sum_{k=0}^{u} (-1)^{kv} \phi\Big(a_0 \otimes \cdots \otimes a_{k-1} \otimes \psi(a_k \otimes \cdots \otimes a_{k+v}) \otimes a_{k+1+v} \otimes \cdots \otimes a_{u+v}\Big),\end{aligned}$$

for all $a_0, a_1, \ldots, a_{u+v} \in \mathscr{C}$. The Gerstenhaber bracket satisfies the graded Jacobi identity. Since the multiplication m in $C^\infty(L[1] \oplus B)$ is associative, we have $[\![m, m]\!] = 0$ and the standard Hochschild coboundary operator $[\![m, -]\!]$ turns $C^\bullet$ into a cochain complex.

The space $D^\bullet_{\text{poly}}(L[1] \oplus B)$ of polydifferential operators on $L[1] \oplus B$ is a subspace of $C^\bullet$ closed under the Gerstenhaber bracket. Note that $Q \in \mathfrak{X}(L[1] \oplus B) \subset D^0_{\text{poly}}(L[1] \oplus B)$ and $m \in D^1_{\text{poly}}(L[1] \oplus B)$.

Lemma 2.3.16. *We have* $[\![Q + m, -]\!]^2 = 0$.

Proof. We have $[\![m, m]\!] = 0$ since the multiplication m is associative, $[\![Q, m]\!] = 0$ since Q is a derivation of m, and $[\![Q, Q]\!] = 0$ since Q is a homological vector field. The conclusion follows from the Jacobi identity. □

Let $\mathscr{D}^k_{\text{poly}}$ denote the space of formal vertical $(k+1)$-polydifferential operators on the vector bundle B, and let $\mathscr{D}^\bullet_{\text{poly}} = \bigoplus_{k=-1}^{\infty} \mathscr{D}^k_{\text{poly}}$. Set $\mathscr{S} = \Gamma(\hat{S}(B^\vee))$. There exists a canonical isomorphism

$$\Gamma(\hat{S}(B^\vee) \otimes \underbrace{S(B) \otimes \cdots \otimes S(B)}_{k+1 \text{ factors}}) \xrightarrow[\cong]{\varphi} \mathscr{D}^k_{\text{poly}} \ .$$

To $\chi^I \otimes \partial^{J_0} \otimes \cdots \otimes \partial^{J_k} \in \Gamma(\hat{S}(B^\vee) \otimes \underbrace{S(B) \otimes \cdots \otimes S(B)}_{k+1 \text{ factors}})$, the isomorphism φ associates the polydifferential operator

$$\mathscr{S}^{\otimes k+1} \ni \chi^{M_0} \otimes \cdots \otimes \chi^{M_k} \longmapsto \chi^I \cdot \partial^{J_0}(\chi^{M_0}) \cdots \partial^{J_k}(\chi^{M_k}) \in \mathscr{S}.$$

The algebra of functions $C^\infty(L[1] \oplus B)$ is a module over its subalgebra $\Gamma(\Lambda^\bullet L) \equiv$

$\Gamma(\Lambda^\bullet L^\vee \otimes S^0(B^\vee))$. The subspace of $D^\bullet_{\text{poly}}(L[1]\oplus B)$ comprised of all $\Gamma(\Lambda^\bullet L^\vee)$-multilinear polydifferential operators is easily identified to $\Gamma(\Lambda^\bullet L^\vee)\otimes_R \mathscr{D}^\bullet_{\text{poly}}$. It is simple to see that the universal enveloping algebra $\mathcal{U}(\mathcal{F})$ of the Fedosov dg Lie algebroid $\mathcal{F}\to\mathcal{M}$ is naturally identified with $\Gamma(\Lambda^\bullet L^\vee)\otimes_R \mathscr{D}^0_{\text{poly}}$, which is a dg Hopf algebroid over $C^\infty(\mathcal{M})\cong\Gamma(\Lambda^\bullet L^\vee\otimes \hat{S}B^\vee)$. Moreover, $\mathcal{U}(\mathcal{F})$ is a dg Hopf subalgebroid of $D^0_{\text{poly}}(L[1]\oplus B)$. Note that

$$\mathcal{U}(\mathcal{F})^{\otimes k+1}\cong\Gamma(\Lambda^\bullet L^\vee)\otimes_R \mathscr{D}^k_{\text{poly}}.$$

Thus, as a consequence of Proposition 2.3.2, we have the following

Proposition 2.3.17. *1. The subspace $\Gamma(\Lambda^\bullet L^\vee)\otimes_R \mathscr{D}^\bullet_{\text{poly}}$ of $D^\bullet_{\text{poly}}(L[1]\oplus B)$ is stable under the Hochschild coboundary operator $[\![Q+m,-]\!]$.*

2. *The triple $\Big(\operatorname{tot}\Big(\Gamma(\Lambda^\bullet L^\vee)\otimes_R \mathscr{D}^\bullet_{\text{poly}}\Big),[\![Q+m,-]\!],[\![,]\!]\Big)$ is a dgla.*

3. *The cohomology group $H^\bullet\Big(\operatorname{tot}\Big(\Gamma(\Lambda^\bullet L^\vee)\otimes_R \mathscr{D}^\bullet_{\text{poly}}\Big),[\![Q+m,-]\!]\Big)$ is a Gerstenhaber algebra.*

Now consider the map

$$\sigma_\natural:\Gamma(\Lambda^u L^\vee)\otimes_R \mathscr{D}^v_{\text{poly}}\to\Gamma(\Lambda^u A^\vee)\otimes_R \mathcal{D}^v_{\text{poly}}$$

defined by the commutative diagram

$$\begin{array}{ccc} \Gamma(\Lambda^u L^\vee)\otimes_R \mathscr{D}^v_{\text{poly}} & \xrightarrow{\sigma_\natural} & \\ {\scriptstyle \mathrm{id}\otimes\varphi}\uparrow\equiv & & \Gamma(\Lambda^u A^\vee)\otimes_R \mathcal{D}^v_{\text{poly}} . \\ \Gamma(\Lambda^u L^\vee\otimes\hat{S}B^\vee\otimes(SB)^{\otimes v+1}) & \xrightarrow{\sigma\otimes\mathrm{pbw}^{\otimes v+1}} & \end{array}$$

The map $\sigma_\natural$ is a quasi-isomorphism of Dolgushev–Fedosov type similar to the classical Fedosov resolution of the polydifferential operators on a smooth manifold obtained by Dolgushev [18].

Theorem 2.3.18 ([59])**.** *We have a contraction*

$$\Big(\operatorname{tot}\left(\Gamma(\Lambda^\bullet A^\vee)\otimes_R \mathcal{D}^\bullet_{\text{poly}}\right),d^{\mathcal{U}}_A+d_{\mathscr{H}}\Big)\underset{\sigma_\natural}{\overset{\check{\tau}_\natural}{\rightleftarrows}}\Big(\operatorname{tot}\left(\Gamma(\Lambda^\bullet L^\vee)\otimes_R \mathscr{D}^\bullet_{\text{poly}}\right),[\![Q+m,-]\!]\Big)\circlearrowleft\check{h}_\natural$$

Corollary 2.3.19 ([59]). *Given a Lie pair (L, A), each choice of a splitting $j : B \to L$ of the short exact sequence of vector bundles $0 \to A \to L \to B \to 0$ and of a torsion-free L-connection ∇ on B determines*

1. *an L_∞ algebra structure on* $\operatorname{tot}\left(\Gamma(\Lambda^\bullet A^\vee) \otimes_R \mathcal{D}^\bullet_{\text{poly}}\right)$ *with the operator $d^{\mathcal{U}}_A + d_{\mathscr{H}}$ as unary bracket*

2. *and a Gerstenhaber algebra structure on $\mathbb{H}^\bullet_{\text{CE}}(A, \mathcal{D}^\bullet_{\text{poly}})$, the cohomology of the complex*
$$\left(\operatorname{tot}\left(\Gamma(\Lambda^\bullet A^\vee) \otimes_R \mathcal{D}^\bullet_{\text{poly}}\right), d^{\mathcal{U}}_A + d_{\mathscr{H}}\right).$$

A priori, the L_∞ algebra structure on $\operatorname{tot}\left(\Gamma(\Lambda^\bullet A^\vee) \otimes_R \mathcal{D}^\bullet_{\text{poly}}\right)$ in Corollary 2.3.19 is not canonical; it depends on a choice of quasi-isomorphic 'Dolgushev–Fedosov type' replacement for the complex $\left(\operatorname{tot}\left(\Gamma(\Lambda^\bullet A^\vee) \otimes_R \mathcal{D}^\bullet_{\text{poly}}\right), d^{\mathcal{U}}_A + d_{\mathscr{H}}\right)$ via a Fedosov dg Lie algebroid $\mathcal{F} \to \mathcal{M}$. The construction of the Fedosov differential involves the choice of a torsion-free connection $\nabla : \Gamma(L) \times \Gamma(B) \to \Gamma(B)$ and a splitting $j : B \to L$ of the short exact sequence of vector bundles $0 \to A \to L \to B \to 0$. However, one can prove that different choices yield isomorphic L_∞ algebra structures on $\operatorname{tot}\left(\Gamma(\Lambda^\bullet A^\vee) \otimes_R \mathcal{D}^\bullet_{\text{poly}}\right)$ [3]. Therefore, the induced Gerstenhaber algebra structure on $\mathbb{H}^\bullet_{\text{CE}}(A, \mathcal{D}^\bullet_{\text{poly}})$ is indeed canonical. Moreover, when the Lie pair happens to be a matched pair, the transferred L_∞ algebra structure on $\operatorname{tot}\left(\Gamma(\Lambda^\bullet A^\vee) \otimes_R \mathcal{D}^\bullet_{\text{poly}}\right)$ is precisely the dgla structure described in Proposition 2.3.4.

Proposition 2.3.20. *Under the hypotheses of Corollary 2.3.19 and the additional assumption that $j(B)$ is a Lie subalgebroid of L — i.e. $L = A \bowtie B$ is a matched pair — the L_∞ algebra* $\operatorname{tot}\left(\Gamma(\Lambda^\bullet A^\vee) \otimes_R \mathcal{D}^\bullet_{\text{poly}}\right)$ *and the Gerstenhaber algebra $\mathbb{H}^\bullet_{\text{CE}}(A, \mathcal{D}^\bullet_{\text{poly}})$ of Corollary 2.3.19 coincide respectively with the dgla $\Gamma(\wedge^\bullet A^\vee) \otimes_R \mathcal{U}(B)^{\bullet+1}$ and the Gerstenhaber algebra $\mathbb{H}^\bullet_{\text{CE}}(A, \mathcal{U}(B)^{\bullet+1})$ of Proposition 2.3.4.*

Chapter 3
Formal exponential map for graded manifolds

Introduction

We introduce, for every $\mathbb{Z}$-graded manifold, a formal exponential map defined in a purely algebraic way and advocate for its use in applications.

Although the geodesic exponential map $\exp : T_M \to M \times M$ associated to an affine connection ∇ does not transpose in a straightforward way to the graded manifold context, its fiberwise infinite-order jet evaluated along the zero section of T_M admits a purely algebraic description which does carry over to the $\mathbb{Z}$-graded context. We prove that the resulting map is an algebraically well-defined isomorphism of *co*algebras $\operatorname{pbw} : \Gamma\Big(S(T_{\mathcal{M}})\Big) \to \mathcal{D}(\mathcal{M})$, where $\mathcal{D}(\mathcal{M})$ denotes the associative algebra of differential operators on $\mathcal{M}$, called "*formal exponential map*." This map may be considered as a replacement for the geodesic exponential map of classical differential geometry. For instance, the formal exponential map was exploited recently in relation with the Atiyah class of dg-manifolds, which is the obstruction class of the existence of dg-compatible connections [48]. Moreover, it plays an important role in the construction of associated L_∞ algebras [34, 48]. We refer the reader to Definition 3.2.1 for the precise definition of the formal exponential map.

Applying the iterative technique conceived by Fedosov [24] for the deformation quantization [5, 6] of symplectic manifolds, Emmrich & Weinstein [22] constructed, for every smooth manifold M, a flat connection on the completed symmetric tensor algebra $\hat{S}(T_M^\vee)$ of the cotangent bundle of M. Furthermore, they proved, once again by the Fedosov iterative method, that for any smooth function f on M, there exists a unique flat section of $\hat{S}(T_M^\vee)$ whose term of degree 0 (for the natural graduation of $\hat{S}(T_M^\vee)$ determined by the symmetric tensor power) is equal to f. Doing so, they obtained an augmentation map $\tau : C^\infty(M) \to \Omega^0(M, \hat{S}(T_M^\vee))$. Moreover, Emmrich & Weinstein [22] proved that this map τ constructed by the Fedosov iterative method does coincide with the fiberwise infinite-order jet along the zero section of T_M of the classical geodesic exponential map associated to an affine connection on the manifold M. Their proof resorted to a complicated argument involving Ehresmann connections on analytic manifolds. One application of our formal exponential map is a direct and much more transparent proof of the Emmrich–Weinstein theorem. Indeed, we give a simple proof of an extension of the Emmrich–Weinstein theorem to $\mathbb{Z}$-graded manifolds, of which the classical Emmrich–Weinstein theorem is a special case.

The construction of flat connections by Fedosov's iterative method was subsequently extended to the realm of $\mathbb{Z}$-graded manifolds by Cattaneo & Felder for the purpose of quantizing coisotropic submanifolds [13]. We show that the Fedosov flat connection D on $\hat{S}(T_{\mathcal{M}}^\vee)$ constructed by Cattaneo & Felder for a graded manifold $\mathcal{M}$ starting from a torsionfree connection ∇ on $T_{\mathcal{M}}$ can be recovered in a straightforward manner by making use of the 'formal exponential map' $\operatorname{pbw} : \Gamma\Big(S(T_{\mathcal{M}})\Big) \to \mathcal{D}(\mathcal{M})$ associated to the chosen torsionfree connection ∇ on $T_{\mathcal{M}}$. Our construction goes as follows. The Lie algebra $\Gamma(T_{\mathcal{M}})$ of smooth vector fields on $\mathcal{M}$ acts infinitesimally from the left on $\mathcal{D}(\mathcal{M})$ by composition of differential operators. Identifying $\mathcal{D}(\mathcal{M})$ with the symmetric algebra $\Gamma\Big(S(T_{\mathcal{M}})\Big)$ via the formal exponential map pbw and transferring this infinitesimal action through pbw, we obtain a flat connection $\nabla^\natural$ on the vector bundle $S(T_{\mathcal{M}})$. We prove that the covariant differential of the flat connection induced by $\nabla^\natural$ on the dual bundle $\hat{S}(T_{\mathcal{M}}^\vee)$ coincides with the coboundary operator D constructed by Cattaneo & Felder — see Theorem 3.5.6. As a consequence, we prove an extension of the Emmrich–Weinstein theorem [22] to $\mathbb{Z}$-graded manifolds: the augmentation map τ which identifies smooth functions on $\mathcal{M}$ with $\nabla^\natural$-flat sections of the bundle $\hat{S}(T_{\mathcal{M}}^\vee)$ is a Taylor expansion twisted by the formal exponential map — see Corollary 3.5.7. When $\mathcal{M}$ is an ordinary smooth manifold, we recover the classical Emmrich–Weinstein theorem [22].

In 2005, while globalizing Kontsevich's formality theorem [30] from local charts to whole smooth manifolds, Dolgushev [18] proved that the Fedosov flat connection D on $\hat{S}(T_M^\vee)$ and the augmentation map $\tau : C^\infty(M) \to \Omega^0(M, \hat{S}(T_M^\vee))$, which both stem from a choice of torsionfree connection ∇ on the tangent bundle T_M (see Propositions 3.5.1 and 3.5.2), fit into an exact sequence

$$0 \longrightarrow C^\infty(M) \xrightarrow{\tau} \Omega^0(M, \hat{S}(T_M^\vee)) \xrightarrow{D} \Omega^1(M, \hat{S}(T_M^\vee)) \xrightarrow{D} \cdots \tag{3.1}$$

providing a resolution of the algebra $C^\infty(M)$ of smooth functions on M. Dolgushev's result was subsequently extended to the realm of graded manifolds by Cattaneo & Felder [13].

In this paper, we give a simple new proof of the exactness of the sequence (3.1), in the context of $\mathbb{Z}$-graded manifolds, based on homological perturbation, the principle of which is briefly recalled in the Appendix. Although this result is well-known to experts, we believe that this new proof could be of interest to the reader. The resulting Dolgushev–Fedosov resolution of $C^\infty(\mathcal{M})$ can be used to globalize Kontsevich's local formality theorem [30] in the context of $\mathbb{Z}$-graded mabifolds (see [13]).

Finally, we note that since $\Omega^\bullet(\mathcal{M}, \hat{S}(T_\mathcal{M}^\vee))$ may be regarded as the algebra of functions on the graded manifold $T_\mathcal{M}[1] \oplus T_\mathcal{M}$, the (graded manifold version of) exact sequence (3.1) means that the dg-manifold $\mathcal{M}$ with support M and trivial homological vector field is weakly equivalent to the dg-manifold $T_\mathcal{M}[1] \oplus T_\mathcal{M}$ with support M and the operator D as homological vector field. Throughout this paper, the structure sheaf $\mathcal{O}_{T_\mathcal{M}}$ of the $\mathbb{Z}$-graded manifold $T_\mathcal{M}$ is understood to be a sheaf over the smooth manifold M. Likewise, the graded manifold $T_\mathcal{M}[1] \oplus T_\mathcal{M}$ is a sheaf over M.

Notations

Some remarks concerning notations are necessary.

By default, in this paper, "graded" means $\mathbb{Z}$-graded.

We use the symbol $\Bbbk$ to denote the field of either real or complex numbers.

Given a module M over a ring, the symbol $\hat{S}(M)$ denotes the $\mathfrak{m}$-adic completion of the symmetric algebra $S(M)$, where $\mathfrak{m}$ is the ideal of $S(M)$ generated by M.

The *Koszul sign* $\epsilon(\sigma; X_1, X_2, \cdots, X_n)$ of a permutation σ of homogeneous elements $X_1, X_2, \ldots, X_n$ of a graded vector space V is determined by the equality

$$X_{\sigma(1)} \odot X_{\sigma(2)} \odot \cdots \odot X_{\sigma(n)} = \epsilon(\sigma; X_1, X_2, \cdots, X_n)\; X_1 \odot X_2 \odot \cdots \odot X_n$$

in the (graded) symmetric algebra $S(V)$.

Let $\mathcal{M}$ be a finite-dimensional graded manifold, let $(x_i)_{i\in\{1,\ldots,n\}}$ be a set of local coordinates on $\mathcal{M}$ and let $(y_j)_{j\in\{1,\ldots,n\}}$ be the induced local frame of $T^{\vee}_{\mathcal{M}}$ regarded as fiberwise linear functions on $T_{\mathcal{M}}$.

We use the symbol $\mathbb{N}$ to denote the set of positive integers and the symbol $\mathbb{N}_0$ for the set of nonnegative integers. Given a multi-index $I = (i_1, i_2, \cdots, i_n) \in \mathbb{N}_0^n$, we adopt the following multi-index notations:

$$\begin{aligned} I! &= i_1! i_2! \cdots i_n! & \tau_{\leqslant k} I &= (i_1, \cdots, i_k, 0, \cdots, 0) \\ |I| &= i_1 + i_2 + \cdots + i_n & \tau_{<k} I &= (i_1, \cdots, i_{k-1}, 0, \cdots, 0) \\ y^I &= (y_1)^{i_1} (y_2)^{i_2} \cdots (y_n)^{i_n} & \tau_{>k} I &= (0, \cdots, 0, i_{k+1}, \cdots, i_n) \end{aligned}$$

$$\partial_x^I = \underbrace{\partial_{x_1} \odot \cdots \odot \partial_{x_1}}_{i_1 \text{ factors}} \odot \underbrace{\partial_{x_2} \odot \cdots \odot \partial_{x_2}}_{i_2 \text{ factors}} \odot \cdots \odot \underbrace{\partial_{x_n} \odot \cdots \odot \partial_{x_n}}_{i_n \text{ factors}}$$

$$\partial_{\overleftarrow{x}}^I = \underbrace{\partial_{x_n} \odot \cdots \odot \partial_{x_n}}_{i_n \text{ factors}} \odot \underbrace{\partial_{x_{n-1}} \odot \cdots \odot \partial_{x_{n-1}}}_{i_{n-1} \text{ factors}} \odot \cdots \odot \underbrace{\partial_{x_1} \odot \cdots \odot \partial_{x_1}}_{i_1 \text{ factors}}$$

We use the symbol e_k to denote the multi-index all of whose components are equal to 0 except for the k-th which is equal to 1. Thus $\partial_x^{e_k} = \partial_{x_k}$.

The de Rham exterior differential d is an operator of degree $+1$ while the interior product i_X with a homogeneous vector field X of degree $|X|$ is an operator of degree $-1 - |X|$. The element

$$dx_{i_1} \wedge \cdots \wedge dx_{i_p} \otimes y^J \tfrac{\partial}{\partial y_q}$$

of $\Omega^p\Big(\mathcal{M}, S^{|J|}(T^{\vee}_{\mathcal{M}}) \otimes T_{\mathcal{M}}\Big)$ is of degree

$$\sum_{k=1}^{p} (1 + |x_{i_k}|) + \sum_{k=1}^{n} J_k\, |y_k| - |y_q|\,,$$

where $|x_k|$ (resp. $|y_q|$) denotes the degree of the coordinate function x_k (resp. y_q).

3.1 Connections on graded manifolds

A $\mathbb{Z}$-graded manifold $\mathcal{M}$ consists of a smooth manifold M (called the support of the graded manifold) and a sheaf of $\mathbb{Z}$-graded, graded-commutative algebras $\mathcal{O}_{\mathcal{M}}$ over M such that every point of M admits an open neighborhood U for which $\mathcal{O}_{\mathcal{M}}(U)$ is isomorphic to $C^\infty(U) \otimes \hat{S}(V^\vee)$, where V is a fixed $\mathbb{Z}$-graded vector space and $\hat{S}(V^\vee)$ denotes the formal power series on V. We say that the graded manifold $\mathcal{M}$ is finite-dimensional if $\dim M$ and $\dim V$ are finite. We use the notation $C^\infty(\mathcal{M})$ to denote the algebra $\mathcal{O}_{\mathcal{M}}(M)$. We refer the reader to [46, Chapter 2] for a short introduction to $\mathbb{Z}$-graded manifolds.

Definition 3.1.1. *Let $\mathcal{E} \to \mathcal{M}$ be a vector bundle in the category of graded manifolds. A connection on $\mathcal{E} \to \mathcal{M}$ is a $\Bbbk$-linear map*

$$\nabla : \Gamma(T_{\mathcal{M}}) \otimes \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E})$$

of degree 0 such that

$$\nabla_{fX} S = f \nabla_X S,$$
$$\nabla_X (fS) = X(f) S + (-1)^{|X||f|} f \nabla_X S,$$

for all $f \in C^\infty(\mathcal{M})$, $X \in \Gamma(T_{\mathcal{M}})$ and $S \in \Gamma(\mathcal{E})$.

The covariant differential associated to a connection ∇ is the map

$$d^\nabla : \Omega^\bullet(\mathcal{M}, \mathcal{E}) \to \Omega^{\bullet+1}(\mathcal{M}, \mathcal{E})$$

of degree +1 satisfying

$$\nabla_X S = \iota_X (d^\nabla S),$$
$$d^\nabla(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{|\alpha|} \alpha \wedge d^\nabla \beta,$$

for all $X \in \Gamma(T_{\mathcal{M}})$, $S \in \Gamma(\mathcal{E})$, $\alpha \in \Omega(\mathcal{M})$ and $\beta \in \Omega(\mathcal{M}, \mathcal{E})$.

The curvature of a connection ∇ is the 2-form $R^\nabla \in \Omega^2(\mathcal{M}, \mathrm{End}(\mathcal{E}))$ defined by

$$R^\nabla(X,Y) = (-1)^{|Y|-1}\Big\{\nabla_X\nabla_Y - (-1)^{|X||Y|}\nabla_Y\nabla_X - \nabla_{[X,Y]}\Big\},$$

for all homogeneous $X, Y \in \Gamma(T_\mathcal{M})$ so that, for all $\omega \in \Omega(\mathcal{M}, \mathcal{E})$, we have $(d^\nabla)^2\omega = R^\nabla \wedge \omega$.

If $\mathcal{E} = T_\mathcal{M}$, the torsion of ∇ is the 2-form $T^\nabla \in \Omega^2(\mathcal{M}, T_\mathcal{M})$ defined by

$$T^\nabla(X,Y) = \nabla_X Y - (-1)^{|X||Y|}\nabla_Y X - [X,Y],$$

for all homogeneous $X, Y \in \Gamma(T_\mathcal{M})$.

Let $\mathcal{M}$ be finite dimensional, let $(x_i)_{i\in\{1,\dots,n\}}$ be a set of local coordinates on $\mathcal{M}$ and let $(y_j)_{j\in\{1,\dots,n\}}$ be the induced local frame of $T^\vee_\mathcal{M}$ regarded as fiberwise linear functions on $T_\mathcal{M}$. As in [18], we define

$$\delta : \Omega^p(\mathcal{M}, S^q T^\vee_\mathcal{M}) \to \Omega^{p+1}(\mathcal{M}, S^{q-1}T^\vee_\mathcal{M})$$

and

$$\delta^{-1} : \Omega^p(\mathcal{M}, S^q T^\vee_\mathcal{M}) \to \Omega^{p-1}(\mathcal{M}, S^{q+1}T^\vee_\mathcal{M})$$

by

$$\delta = \sum_{i=1}^n dx_i \otimes \frac{\partial}{\partial y_i} \qquad \text{and} \qquad \delta^{-1} = \frac{1}{p+q}\sum_{i=1}^n i_{\frac{\partial}{\partial x_i}} \otimes y_i$$

or, more precisely,

$$\delta(\omega\otimes f) = \sum_{i=1}^n (-1)^{\left|\frac{\partial}{\partial y_i}\right||\omega|}\, dx_i \wedge \omega \otimes \frac{\partial}{\partial y_i}(f)$$

and

$$\delta^{-1}(\omega\otimes f) = \tfrac{1}{p+q}\sum_{i=1}^n (-1)^{\left|y^i\right||\omega|} i_{\frac{\partial}{\partial x_i}}\omega \otimes y_i \cdot f$$

for all homogeneous $\omega \in \Omega^p(\mathcal{M})$ and for all $f \in \Gamma(S^q T^\vee_\mathcal{M})$. It's not difficult to check that the operators δ and δ^{-1} are well defined, i.e. independent of the choice of local coordinates, and can be extended to $\Omega^\bullet\Big(\mathcal{M}, \mathrm{End}(\hat{S}(T^\vee_\mathcal{M}))\Big)$. The operator δ has degree $+1$ while the operator δ^{-1} has degree -1. Note that the operators δ and δ^{-1} are *not* inverse of each other.

3.2 Formal exponential map

The exponential maps defined in terms of geodesics of a connection for ordinary smooth manifolds does not generalize straightforwardly to graded manifolds as the latter only exist through their algebras of functions. However, it turns out that the the fiberwise ∞-order jet of the geodesic exponential map admits a purely algebraic description, which extends readily to the context of graded manifolds [34].

Definition 3.2.1. *Let $\mathcal{M}$ be a graded manifold and let $\mathcal{D}(\mathcal{M})$ denote its algebra of differential operators. The formal exponential map associated to a connection ∇ on $T_{\mathcal{M}}$ is the morphism of left $C^\infty(\mathcal{M})$-modules*

$$\mathrm{pbw} : \Gamma(ST_{\mathcal{M}}) \to \mathcal{D}(\mathcal{M}),$$

inductively defined by the relations

$$\mathrm{pbw}(f) = f, \quad \forall f \in C^\infty(\mathcal{M})$$
$$\mathrm{pbw}(X) = X, \quad \forall X \in \Gamma(T_{\mathcal{M}}),$$

and, for all $n \in \mathbb{N}$ and any homogeneous elements $X_0, \ldots, X_n$ of $\Gamma(T_{\mathcal{M}})$,

$$\mathrm{pbw}(X_0 \odot \cdots \odot X_n) = \tfrac{1}{n+1} \sum_{k=0}^{n} \epsilon_k \left\{ X_k \cdot \mathrm{pbw}(X^{\{k\}}) - \mathrm{pbw}\left(\nabla_{X_k}(X^{\{k\}})\right) \right\}, \quad (3.2)$$

where $\epsilon_k = (-1)^{|X_k|(|X_0|+\cdots+|X_{k-1}|)}$ and $X^{\{k\}} = X_0 \odot \cdots \odot X_{k-1} \odot X_{k+1} \odot \cdots \odot X_n$.

Lemma 3.2.2. *The formal exponential map is well-defined.*

Sketch of proof. Let ∇ be a connection on the tangent bundle $T_{\mathcal{M}}$ to a graded manifold $\mathcal{M}$.

Consider the sequence $(E_n)_{n \in \mathbb{N}_0}$ of maps

$$E_n : \underbrace{\Gamma(T_{\mathcal{M}}) \times \Gamma(T_{\mathcal{M}}) \times \cdots \times \Gamma(T_{\mathcal{M}})}_{n \text{ factors}} \to \mathcal{D}(\mathcal{M})$$

starting with the natural inclusions $E_0 : C^\infty(\mathcal{M}) \to \mathcal{D}(\mathcal{M})$ and $E_1 : \Gamma(T_{\mathcal{M}}) \to$

$\mathcal{D}(\mathcal{M})$ and defined recursively by the relation

$$E_n(X_1,\dots,X_n) = \tfrac{1}{n}\Bigg\{ \sum_{k=1}^{n} \epsilon\; X_k \cdot E_{n-1}(X_1,\cdots,\widehat{X_k},\cdots,X_n) \\ - \sum_{l<k} \epsilon\; E_{n-1}(X_1,\cdots,\nabla_{X_k}X_l,\cdots,\widehat{X_k},\cdots,X_n) \\ - \sum_{k<l} \epsilon\; E_{n-1}(X_1,\cdots,\widehat{X_k},\cdots,\nabla_{X_k}X_l,\cdots,X_n)\Bigg\},$$

for $n \geqslant 2$. The symbol ϵ appearing in each term of the sum above denotes the Koszul sign $\epsilon(\sigma; X_1, X_2, \cdots, X_n)$ of the permutation σ of the order in which the homogeneous elements $X_1, X_2, \dots, X_n$ of $\Gamma(T_{\mathcal{M}})$ appear in that term.

By induction on n, show that each E_n is multilinear over $C^\infty(\mathcal{M})$. The morphism of left $C^\infty(\mathcal{M})$-modules

$$E : \Gamma\Bigg(\bigoplus_{n\in\mathbb{N}_0} (T_{\mathcal{M}})^{\otimes n} \Bigg) \to \mathcal{D}(\mathcal{M})$$

determined by the sequence of maps $(E_n)_{n\in\mathbb{N}_0}$ vanishes on the ideal generated by all elements of the form

$$X_1 \otimes \cdots \otimes X_k \otimes X_{k+1} \otimes \cdots \otimes X_n - (-1)^{|X_k||X_{k+1}|} X_1 \otimes \cdots \otimes X_{k+1} \otimes X_k \otimes \cdots \otimes X_n$$

for all homogeneous elements $X_1, \dots, X_n$ of $\Gamma(T_{\mathcal{M}})$. The induced morphism of $C^\infty(\mathcal{M})$-modules from the symmetric algebra $\Gamma\big(S(T_{\mathcal{M}})\big)$ to the algebra $\mathcal{D}(\mathcal{M})$ of differential operators is the formal exponential map:

$$\begin{array}{ccc} \Gamma\big(\bigoplus_{n\in\mathbb{N}_0}(T_{\mathcal{M}})^{\otimes n}\big) & & \\ & \overset{E}{\searrow} & \\ \big\downarrow & & \mathcal{D}(\mathcal{M}) \\ & \underset{\text{pbw}}{\nearrow} & \\ \Gamma\big(S(T_{\mathcal{M}})\big) & & \end{array}$$

□

For a classical (i.e. nongraded) manifold M, this map pbw is the fiberwise ∞-order jet of the exponential map $\exp : T_M \to M \times M$ associated to the connection ∇ — whence the terminology 'formal exponential map.'

Proposition 3.2.3 ([34]). *If $\mathcal{M}$ is a classical smooth manifold M (i.e. $V = 0$),* *then*

$$\begin{aligned}\operatorname{pbw}(X_0\odot\cdots\odot X_k)(f)\\ &= \left.\frac{d}{dt_0}\right|_0 \left.\frac{d}{dt_1}\right|_0 \cdots \left.\frac{d}{dt_k}\right|_0 f\Big(\exp(t_0X_0+t_1X_1+\cdots+t_kX_k)\Big)\end{aligned}$$

for all $X_0, X_1, \cdots, X_k \in \Gamma(T_M)$ *and* $f \in C^\infty(M)$.

3.3 Properties of the formal exponential map

The algebra $\mathcal{D}(\mathcal{M})$ of differential operators on $\mathcal{M}$ can be thought of as the universal enveloping algebra $\mathcal{U}(T_{\mathcal{M}})$ of $T_{\mathcal{M}}$ (regarded as a Lie algebroid) [50, 53, 55] and admits a natural filtration by the order of the differential operators.

Using (3.2), it is straightforward to prove by induction on n that,

$$\operatorname{pbw}(X_1\odot\cdots\odot X_n)\in\mathcal{U}^{\leqslant n}(T_{\mathcal{M}}),$$

for all $n\in\mathbb{N}$ and $X_1,\dots,X_n\in\Gamma(T_{\mathcal{M}})$. In other words, the map pbw respects the filtrations on $\Gamma(ST_{\mathcal{M}})$ and $\mathcal{U}(T_{\mathcal{M}})$.

We introduce the functor Gr which takes a filtered vector space

$$\cdots\subset\mathscr{A}^{\leqslant k-1}\subset\mathscr{A}^{\leqslant k}\subset\mathscr{A}^{\leqslant k+1}\subset\cdots$$

to the associated graded vector space

$$\operatorname{Gr}\left(\mathscr{A}\right)=\bigoplus_k\frac{\mathscr{A}^{\leqslant k}}{\mathscr{A}^{\leqslant k-1}}.$$

Rinehart proved that, for every Lie algebroid L, the symmetrization map

$$\operatorname{sym}:\Gamma(S^\bullet(L))\to\operatorname{Gr}^\bullet\Big(\mathcal{U}(L)\Big),$$

defined by

$$X_1\odot\cdots\odot X_n\mapsto\frac{1}{n!}\sum_{\sigma\in S_n}\epsilon\;X_{\sigma(1)}\cdots X_{\sigma(n)},\quad\forall X_1,\dots,X_n\in\Gamma(L),$$

is an isomorphism of graded vector spaces [55].

The next lemma asserts that

$$\mathrm{Gr}(\mathrm{pbw}) = \mathrm{sym}\,.$$

Lemma 3.3.1. *For all $n \in \mathbb{N}$ and $X_1, \dots, X_n \in \Gamma(T_{\mathcal{M}})$,*

$$\mathrm{pbw}(X_1 \odot \cdots \odot X_n) - \frac{1}{n!} \sum_{\sigma \in S_n} \epsilon\ X_{\sigma(1)} \cdot X_{\sigma(2)} \cdot \cdots \cdot X_{\sigma(n)}$$

is an element of $\mathcal{U}(T_{\mathcal{M}})^{\leqslant n-1}$.

Sketch of proof. It follows from (3.2) that

$$\begin{aligned}\mathrm{pbw}(X_1 \odot \cdots \odot X_n) - \frac{1}{n} \sum_{k=1}^{n} \epsilon\ X_k \cdot \mathrm{pbw}(X_1 \odot \cdots \odot \widehat{X_k} \odot \cdots \odot X_n) \\ = -\frac{1}{n} \sum_{k=1}^{n} \epsilon\ \ \mathrm{pbw}\left(\nabla_{X_k}(X_1 \odot \cdots \odot \widehat{X_k} \odot \cdots \odot X_n)\right)\end{aligned}$$

belongs to $\mathcal{U}^{\leqslant n-1}(T_{\mathcal{M}})$ as

$$\nabla_{X_k}(X_1 \odot \cdots \odot \widehat{X_k} \odot \cdots \odot X_n) \in \Gamma(S^{n-1}(T_{\mathcal{M}}))$$

and pbw respects the filtrations. The result follows by induction on n. □

The functor Gr has the following remarkable property: given a homomorphism $\phi : V \to W$ of filtered vector spaces, if the associated morphism of graded vector spaces $\mathrm{Gr}(\phi)$ is an isomorphism and the filtrations on V and W are both exhaustive and complete,[1] then ϕ itself is an isomorphism. Therefore, we have proved the following proposition:

Proposition 3.3.2. *The formal exponential map*

$$\mathrm{pbw} : \Gamma(ST_{\mathcal{M}}) \to \mathcal{U}(T_{\mathcal{M}})$$

is an isomorphism of filtered left $C^\infty(\mathcal{M})$-modules.

[1] A filtration $\cdots \subset F^p(V) \subset F^{p+1}(V) \subset \cdots$ on a vector space V is said to be exhaustive if $\bigcup_p F^p(V) = V$ and complete if the natural map $V \to \varprojlim_p \frac{V}{F^p(V)}$ is an isomorphism.

Note that both $\Gamma(ST_{\mathcal{M}})$ and $\mathcal{U}(T_{\mathcal{M}})$ are coalgebras over $C^\infty(\mathcal{M})$. The comultiplication

$$\Delta : \mathcal{U}(T_{\mathcal{M}}) \to \mathcal{U}(T_{\mathcal{M}}) \otimes_{C^\infty(\mathcal{M})} \mathcal{U}(T_{\mathcal{M}})$$

is characterized by the identities

$$\begin{aligned} \Delta(1) &= 1 \otimes 1; \\ \Delta(X) &= 1 \otimes X + X \otimes 1, \quad \forall X \in \Gamma(T_{\mathcal{M}}); \\ \Delta(U \cdot V) &= \Delta(U) \cdot \Delta(V), \quad \forall U, V \in \mathcal{U}(T_{\mathcal{M}}), \end{aligned} \tag{3.3}$$

and is compatible with the natural filtration of $\mathcal{U}(T_{\mathcal{M}})$. Here the symbol $\otimes$ denotes the tensor product over $C^\infty(\mathcal{M})$, the symbol 1 denotes the constant function 1, and the symbol $\cdot$ denotes the multiplication in $\mathcal{U}(T_{\mathcal{M}})$. We refer the reader to [67] for the precise meaning of Equation (3.3).

More explicitly, for all homogeneous elements $X_1, \cdots, X_k \in \Gamma(T_{\mathcal{M}})$, we have

$$\begin{aligned} \Delta(X_1 \cdots X_k) = 1 \otimes (X_1 \cdots X_k) + (X_1 \cdots X_k) \otimes 1 \\ + \sum_{\substack{p+q=k \\ p,q \in \mathbb{N}}} \sum_{\sigma \in S_{p,q}} \epsilon(\sigma; X)(X_{\sigma(1)} \cdots X_{\sigma(p)}) \otimes (X_{\sigma(p+1)} \cdots X_{\sigma(k)}), \end{aligned}$$

where $\epsilon(\sigma; X)$ denotes the Koszul sign of the permutation σ of the homogeneous elements $X_1, \cdots, X_k \in \Gamma(T_{\mathcal{M}})$ and $S_{p,q}$ denotes the space of (p, q)-shuffles.

Similarly, the comultiplication

$$\Delta : \Gamma(ST_{\mathcal{M}}) \to \Gamma(ST_{\mathcal{M}}) \otimes \Gamma(ST_{\mathcal{M}})$$

is given by

$$\begin{aligned} \Delta(X_1 \odot \cdots \odot X_k) = 1 \otimes (X_1 \odot \cdots \odot X_k) + (X_1 \odot \cdots \odot X_k) \otimes 1 \\ + \sum_{\substack{p+q=k \\ p,q \in \mathbb{N}}} \sum_{\sigma \in S_{p,q}} \epsilon(\sigma; X)(X_{\sigma(1)} \odot \cdots \odot X_{\sigma(p)}) \otimes (X_{\sigma(p+1)} \odot \cdots \odot X_{\sigma(k)}). \end{aligned}$$

The symbol $\odot$ denotes the symmetric product in $\Gamma(ST_{\mathcal{M}})$.

Theorem 3.3.3. *The formal exponential map* $\operatorname{pbw} : \Gamma(S(T_{\mathcal{M}})) \to \mathcal{U}(T_{\mathcal{M}})$ *is an isomorphism of filtered coalgebras over* $C^\infty(\mathcal{M})$.

Proof. We need to prove that

$$\Delta \circ \mathrm{pbw} = (\mathrm{pbw} \otimes \mathrm{pbw}) \circ \Delta.$$

Given $n \in \mathbb{N}$ and homogeneous elements $X_0, X_1, \dots, X_n$ of $\Gamma(T_{\mathcal{M}})$, set

$$X^{\{k\}} = X_0 \odot \cdots \odot X_{k-1} \odot X_{k+1} \odot \cdots \odot X_n$$

and $\epsilon_k = (-1)^{|X_k|(|X_0|+\cdots+|X_{k-1}|)}$ for all $k \in \{0, \dots, n\}$.

We have

$$\begin{aligned}
&\Delta \circ \mathrm{pbw}(X_0 \odot \cdots \odot X_n) \\
&\quad = \frac{1}{n+1} \sum_{k=0}^{n} \epsilon_k \ \Delta\Big(X_k \cdot \mathrm{pbw}(X^{\{k\}}) - \mathrm{pbw}\left(\nabla_{X_k}(X^{\{k\}})\right)\Big) \\
&\quad = \frac{1}{n+1} \sum_{k=0}^{n} \epsilon_k \Big\{ \Delta(X_k) \cdot \Delta \circ \mathrm{pbw}(X^{\{k\}}) - \Delta \circ \mathrm{pbw}\left(\nabla_{X_k}(X^{\{k\}})\right)\Big\} \\
&\quad = \frac{1}{n+1} \sum_{k=0}^{n} \epsilon_k \Big\{ (1 \otimes X_k + X_k \otimes 1) \cdot (\mathrm{pbw} \otimes \mathrm{pbw}) \circ \Delta(X^{\{k\}}) \\
&\qquad\quad - (\mathrm{pbw} \otimes \mathrm{pbw}) \circ \Delta\Big(\nabla_{X_k}(X^{\{k\}})\Big)\Big\} \\
&\quad = \frac{1}{n+1} (\mathscr{A} + \mathscr{B} - \mathscr{C} - \mathscr{D})
\end{aligned}$$

where

$$\mathscr{A} = \sum_{p=-1}^{n-1} \sum_{\substack{\sigma \in S_{n+1} \\ \sigma(0) < \cdots < \sigma(p) \\ \sigma(p+2) < \cdots < \sigma(n)}} \epsilon \ \mathrm{pbw}(X_{\sigma(0)} \odot \cdots \odot X_{\sigma(p)}) \otimes X_{\sigma(p+1)} \cdot \mathrm{pbw}(X_{\sigma(p+2)} \odot \cdots \odot X_{\sigma(n)})$$

$$\mathscr{B} = \sum_{p=0}^{n} \sum_{\substack{\sigma \in S_{n+1} \\ \sigma(1) < \cdots < \sigma(p) \\ \sigma(p+1) < \cdots < \sigma(n)}} \epsilon \ X_{\sigma(0)} \cdot \mathrm{pbw}(X_{\sigma(1)} \odot \cdots \odot X_{\sigma(p)}) \otimes \mathrm{pbw}(X_{\sigma(p+1)} \odot \cdots \odot X_{\sigma(n)})$$

$$\begin{aligned}
\mathscr{C} = \sum_{p=1}^{n} \sum_{\substack{\sigma \in S_{n+1} \\ \sigma(2) < \cdots < \sigma(p) \\ \sigma(p+1) < \cdots < \sigma(n)}} \epsilon \ \mathrm{pbw}\left((\nabla_{X_{\sigma(0)}} X_{\sigma(1)}) \odot X_{\sigma(2)} \odot \cdots \odot X_{\sigma(p)} \right) \\
\otimes \mathrm{pbw}(X_{\sigma(p+1)} \odot \cdots \odot X_{\sigma(n)})
\end{aligned}$$

$$\mathscr{D} = \sum_{p=-1}^{n-2} \sum_{\substack{\sigma\in S_{n+1} \\ \sigma(0)<\cdots<\sigma(p) \\ \sigma(p+3)<\cdots<\sigma(n)}} \epsilon \;\; \mathrm{pbw}(X_{\sigma(0)} \odot \cdots \odot X_{\sigma(p)}) \\ \otimes \mathrm{pbw}\Big((\nabla_{X_{\sigma(p+1)}} X_{\sigma(p+2)}) \odot X_{\sigma(p+3)} \odot \cdots \odot X_{\sigma(n)}\Big).$$

Note that, in each term, the factor ϵ denotes the Koszul sign of the permutation σ applied to the homogeneous elements $X_0, X_1, \ldots, X_n$ of $\Gamma(T_{\mathcal{M}})$ in that term.

Setting

$$\mathfrak{S}_p = \sum_{\substack{\sigma\in S_{n+1} \\ \sigma(0)<\cdots<\sigma(p) \\ \sigma(p+1)<\cdots<\sigma(n)}} \epsilon \;\; \mathrm{pbw}(X_{\sigma(0)} \odot \cdots \odot X_{\sigma(p)}) \otimes \mathrm{pbw}(X_{\sigma(p+1)} \odot \cdots \odot X_{\sigma(n)}),$$

it follows from (3.2) that

$$\mathscr{B} - \mathscr{C} = \sum_{p=0}^{n} (p+1)\mathfrak{S}_p \qquad \text{and} \qquad \mathscr{A} - \mathscr{D} = \sum_{p=-1}^{n-1} (n-p)\mathfrak{S}_p.$$

Therefore, we obtain

$$\begin{aligned} \mathscr{B} - \mathscr{C} + \mathscr{A} - \mathscr{D} &= \sum_{p=0}^{n} (p+1)\mathfrak{S}_p + \sum_{p=-1}^{n-1} (n-p)\mathfrak{S}_p \\ &= \sum_{p=-1}^{n} (p+1)\mathfrak{S}_p + \sum_{p=-1}^{n} (n-p)\mathfrak{S}_p = (n+1) \sum_{p=-1}^{n} \mathfrak{S}_p \end{aligned}$$

and

$$\begin{aligned} \Delta \circ \mathrm{pbw}(X_0 \odot \cdots \odot X_n) &= \frac{1}{n+1}(\mathscr{B} - \mathscr{C} + \mathscr{A} - \mathscr{D}) \\ &= \sum_{p=-1}^{n} \mathfrak{S}_p = (\mathrm{pbw} \otimes \mathrm{pbw}) \circ \Delta\Big(X_0 \odot \cdots \odot X_n\Big) \end{aligned}$$

since

$$\Delta\Big(X_0 \odot \cdots \odot X_n\Big) = \sum_{p=-1}^{n} \sum_{\substack{\sigma\in S_{n+1} \\ \sigma(0)<\cdots<\sigma(p) \\ \sigma(p+1)<\cdots<\sigma(n)}} \epsilon \; (X_{\sigma(0)} \odot \cdots \odot X_{\sigma(p)}) \otimes (X_{\sigma(p+1)} \odot \cdots \odot X_{\sigma(n)}).$$

□

3.4 Highest order terms of the formal exponential map

The purpose of this section is to establish the following technical result.

Proposition 3.4.1. *Let $\mathcal{M}$ be a graded manifold. The formal exponential map associated to a* torsion-free *connection ∇ on $T_{\mathcal{M}}$ satisfies*

$$\operatorname{pbw}(X_0 \odot \cdots \odot X_n) \equiv X_0 \cdots X_n - \sum_{j<k} \epsilon\ X_0 \cdots \widehat{X_j} \cdots \widehat{X_k} \cdots X_n \cdot (\nabla_{X_j} X_k) \mod \mathcal{U}^{\leqslant n-1}(T_{\mathcal{M}}) \tag{3.4}$$

and

$$\operatorname{pbw}^{-1}(X_0 \cdots X_n) \equiv X_0 \odot \cdots \odot X_n + \sum_{j<k} \epsilon\ X_0 \odot \cdots \odot \widehat{X_j} \odot \cdots \odot \widehat{X_k} \odot \cdots \odot X_n \odot (\nabla_{X_j} X_k) \mod \Gamma(S^{\leqslant n-1} T_{\mathcal{M}}) \tag{3.5}$$

for all $n \in \mathbb{N}$ and all homogeneous elements $X_0, \dots, X_n$ in $\Gamma(T_{\mathcal{M}})$.

Throughout this section, we make use of the following simplified notations:

- The symbol ϵ appearing in the terms of a sum will always denote the Koszul sign $\epsilon(\sigma; X_0, X_1, \cdots, X_n)$ of the permutation σ of the order in which the homogeneous elements $X_0, X_1, \dots, X_n$ of $\Gamma(T_{\mathcal{M}})$ appear in that term.
- For every subset $\{i_1, i_2, \dots, i_k\}$ of $\{0, 1, \dots, n\}$, the symbol $X^{\{i_1, i_2, \dots, i_k\}}$ denotes what remains of the product $X_0 \cdots X_n$ after its factors $X_{i_1}, X_{i_2}, \dots, X_{i_k}$ have been erased.

Proposition 3.4.2. *For all homogeneous elements Y, Z of $\Gamma(T_{\mathcal{M}})$, we have*

$$\operatorname{pbw}(Y \odot Z) = Y \cdot Z - \nabla_Y Z + \tfrac{1}{2} T^{\nabla}(Y, Z) \tag{3.6}$$

and

$$\operatorname{pbw}^{-1}(Y \cdot Z) = Y \odot Z + \nabla_Y Z - \tfrac{1}{2} T^{\nabla}(Y, Z). \tag{3.7}$$

Proof. It follows directly from Equation (3.2) that

$$\begin{aligned}\operatorname{pbw}(Y \odot Z) =& \tfrac{1}{2}\Big\{Y \operatorname{pbw}(Z) + (-1)^{|Y||Z|} Z \operatorname{pbw}(Y) \\ & \qquad - \operatorname{pbw}(\nabla_Y Z) - (-1)^{|Y||Z|} \operatorname{pbw}(\nabla_Z Y)\Big\} \\ =& \tfrac{1}{2}\{Y \cdot Z + (-1)^{|Y||Z|} Z \cdot Y - \nabla_Y Z - (-1)^{|Y||Z|} \nabla_Z Y\} \\ =& \tfrac{1}{2}\Big\{2YZ - [Y,Z] - 2\nabla_Y Z + (\nabla_Y Z - (-1)^{|Y||Z|}\nabla_Z Y)\Big\} \\ =& Y \cdot Z - \nabla_Y Z + \tfrac{1}{2}\Big\{\nabla_Y Z - (-1)^{|Y||Z|}\nabla_Z Y - [Y,Z]\Big\} \\ =& Y \cdot Z - \nabla_Y Z + \tfrac{1}{2} T^{\nabla}(Y,Z).\end{aligned}$$

Therefore, we obtain

$$\begin{aligned} Y \cdot Z =& \operatorname{pbw}(Y \odot Z) + \underbrace{\nabla_Y Z + \tfrac{1}{2} T^{\nabla}(Y,Z)}_{\in \Gamma(T_{\mathcal{M}})} \\ =& \operatorname{pbw}(Y \odot Z) + \operatorname{pbw}\Big(\nabla_Y Z + \tfrac{1}{2} T^{\nabla}(Y,Z)\Big)\end{aligned}$$

and, applying pbw^{-1} to both sides,

$$\operatorname{pbw}^{-1}(Y \cdot Z) = Y \odot Z + \nabla_Y Z + \tfrac{1}{2} T^{\nabla}(Y,Z). \qquad \square$$

Lemma 3.4.3. *For all homogeneous elements $X_1, \ldots, X_n$ in $\Gamma(T_{\mathcal{M}})$, we have*

$$X_1 \cdots X_n \equiv \epsilon\ X_1 \cdots X_{k-1} X_{k+1} X_k X_{k+2} \cdots X_n \mod \mathcal{U}^{\leqslant n-1}(T_{\mathcal{M}}).$$

Proof. It follows from $Y \cdot Z = (-1)^{|Y||Z|} Z \cdot Y + [Y,Z]$ that

$$\begin{aligned} X_1 \cdots X_n =& \epsilon\ X_1 \cdots X_{k-1} X_{k+1} X_k X_{k+2} \cdots X_n \\ & + \underbrace{X_1 \cdots X_{k-1} [X_k, X_{k+1}] X_{k+2} \cdots X_n}_{\in\ \mathcal{U}^{\leqslant n-1}(T_{\mathcal{M}})}. \qquad \square \end{aligned}$$

Corollary 3.4.4. *For all homogeneous elements $X_1, \ldots, X_n$ in $\Gamma(T_{\mathcal{M}})$, we have*

$$X_1 \cdots X_n \equiv \epsilon\ X_1 \cdots \widehat{X_k} \cdots X_n \cdot X_k \mod \mathcal{U}^{\leqslant n-1}(T_{\mathcal{M}})$$

and, for all $\sigma \in S_n$,

$$X_1 \cdots X_n \equiv \epsilon\ X_{\sigma(1)} \cdots X_{\sigma(n)} \mod \mathcal{U}^{\leqslant n-1}(T_{\mathcal{M}}).$$

Corollary 3.4.5. *For all homogeneous elements $X_1, \ldots, X_n$ in $\Gamma(T_{\mathcal{M}})$, we have*

$$\mathrm{pbw}(X_1 \odot \cdots \odot X_n) \equiv X_1 \cdots X_n \mod \mathcal{U}^{\leqslant n-1}(T_{\mathcal{M}}) \tag{3.8}$$

and

$$X_1 \odot \cdots \odot X_n \equiv \mathrm{pbw}^{-1}(X_1 \cdots X_n) \mod \Gamma(S^{\leqslant n-1}T_{\mathcal{M}}). \tag{3.9}$$

Proof. It follows from Lemma 3.3.1 and Corollary 3.4.4 that

$$\mathrm{pbw}(X_1 \odot \cdots \odot X_n) \equiv \frac{1}{n!} \sum_{\sigma \in S_n} \epsilon\ X_{\sigma(1)} \cdots X_{\sigma(n)} \equiv X_1 \cdots X_n \mod \mathcal{U}^{\leqslant n-1}(T_{\mathcal{M}}). \quad \square$$

Lemma 3.4.6. *For all homogeneous elements $X_0, \ldots, X_n$ in $\Gamma(T_{\mathcal{M}})$, we have*

$$\sum_{k=0}^{n} \epsilon\ \mathrm{pbw}\left(\nabla_{X_k}(X_0 \odot \cdots \odot \widehat{X_k} \odot \cdots \odot X_n)\right) \equiv \sum_{j<k} \epsilon\ X_0 \cdots \widehat{X_j} \cdots \widehat{X_k} \cdots X_n \cdot \left(2\nabla_{X_j} X_k - [X_j, X_k] - T^{\nabla}(X_j, X_k)\right) \mod \mathcal{U}^{\leqslant n-1}(T_{\mathcal{M}}).$$

Proof. It follows from Equation (3.8) that

$$\sum_{k=1}^{n} \mathrm{pbw}\left(Z_1 \odot \cdots \odot Z_{k-1} \odot (\nabla_Y Z_k) \odot Z_{k+1} \odot \cdots \odot Z_n\right) \equiv \sum_{k=1}^{n} Z_1 \cdots Z_{k-1} \cdot (\nabla_Y Z_k) \cdot Z_{k+1} \cdots Z_n \mod \mathcal{U}^{\leqslant n-1}(T_{\mathcal{M}}).$$

Therefore, for every homogeneous elements $X_0, \dots, X_n$ in $\Gamma(T_{\mathcal{M}})$, we have

$$\begin{aligned}
&\epsilon\ \operatorname{pbw}\Big(\nabla_{X_k}(X_0 \odot \cdots \odot \widehat{X_k} \odot \cdots \odot X_n)\Big) \\
&\qquad \equiv \sum_{j=0}^{k-1} \epsilon\ X_0 \cdots X_{j-1}(\nabla_{X_k} X_j) X_{j+1} \cdots \widehat{X_k} \cdots X_n \\
&\qquad\quad + \sum_{j=k+1}^{n} \epsilon\ X_0 \cdots \widehat{X_k} \cdots X_{j-1}(\nabla_{X_k} X_j) X_{j+1} \cdots X_n \mod \mathcal{U}^{\leqslant n-1}(T_{\mathcal{M}}).
\end{aligned}$$

Summing over k, we obtain

$$\begin{aligned}
&\sum_{k=0}^{n} \epsilon\ \operatorname{pbw}\Big(\nabla_{X_k}(X_0 \odot \cdots \odot \widehat{X_k} \odot \cdots \odot X_n)\Big) \\
&\qquad \equiv \sum_{j<k} \epsilon\ X_0 \cdots X_{j-1}(\nabla_{X_k} X_j) X_{j+1} \cdots \widehat{X_k} \cdots X_n \\
&\qquad\quad + \sum_{k<j} \epsilon\ X_0 \cdots \widehat{X_k} \cdots X_{j-1}(\nabla_{X_k} X_j) X_{j+1} \cdots X_n \mod \mathcal{U}^{\leqslant n-1}(T_{\mathcal{M}}).
\end{aligned}$$

The desired result follows from Corollary 3.4.4 and the definition of the torsion T^{∇}. □

Lemma 3.4.7. *For all homogeneous elements $X_0, \dots, X_n$ in $\Gamma(T_{\mathcal{M}})$, we have*

$$\begin{aligned}
&\sum_{k=0}^{n} \epsilon\ X_k \cdot X_0 \cdots \widehat{X_k} \cdots X_n \\
&\qquad \equiv (n+1) X_0 \cdots X_n + \sum_{j<k} \epsilon\ X_0 \cdots \widehat{X_j} \cdots \widehat{X_k} \cdots X_n \cdot [X_k, X_j] \\
&\qquad\qquad \mod \mathcal{U}^{\leqslant n-1}(T_{\mathcal{M}}).
\end{aligned}$$

Proof. We have

$$\begin{aligned}
\sum_{k=0}^{n} \epsilon\ X_k \cdot X^{\{k\}} &= \sum_{k=0}^{n} \left\{ X_0 \cdots X_n + \sum_{j=0}^{k-1} \epsilon\ X_0 \cdots X_{j-1}[X_k, X_j] X_{j+1} \cdots \widehat{X_k} \cdots X_n \right\} \\
&= (n+1) X_0 \cdots X_n + \sum_{j<k} \epsilon\ X_0 \cdots X_{j-1}[X_k, X_j] X_{j+1} \cdots \widehat{X_k} \cdots X_n.
\end{aligned}$$

The desired result follows from Corollary 3.4.4. □

Proof of Proposition 3.4.1. We will proceed by induction on n. The result is true for $n = 1$ by Equation (3.6). Now, the induction hypothesis

$$\operatorname{pbw}(Z_1 \odot \cdots \odot Z_n) \equiv Z_1 \cdots Z_n - \sum_{j<k} \epsilon\ Z_1 \cdots \widehat{Z_j} \cdots \widehat{Z_k} \cdots Z_n \cdot (\nabla_{Z_j} Z_k) \mod \mathcal{U}^{\leqslant n-2}(T_{\mathcal{M}})$$

implies

$$\begin{aligned}\sum_{k=0}^{n} \epsilon\ X_k \cdot \operatorname{pbw}(X_0 \odot \cdot \odot \widehat{X_k} \odot \cdots \odot X_n) \\ \equiv \sum_{k=0}^{n} \epsilon\ X_k \cdot X_0 \cdots \widehat{X_k} \cdots X_n - \sum_{k=0}^{n} \sum_{\substack{i<j \\ k \notin \{i,j\}}} \epsilon\ X_k \cdot X^{\{i,j,k\}} \cdot (\nabla_{X_i} X_j) \\ \mod \mathcal{U}^{\leqslant n-1}(T_{\mathcal{M}}).\end{aligned}$$

Then, making use of Lemma 3.4.7 and Lemma 3.4.4, we obtain

$$\begin{aligned}&\sum_{k=0}^{n} \epsilon\ X_k \cdot \operatorname{pbw}(X_0 \odot \cdot \odot \widehat{X_k} \odot \cdots \odot X_n) \\ &\equiv (n+1) X_0 \cdots X_n + \sum_{j<k} \epsilon\ X^{\{j,k\}} [X_k, X_j] - \sum_{k=0}^{n} \sum_{\substack{i<j \\ k \notin \{i,j\}}} \epsilon\ X^{\{i,j\}} (\nabla_{X_i} X_j) \\ &\equiv (n+1) X_0 \cdots X_n - \sum_{j<k} \epsilon\ X^{\{j,k\}} [X_j, X_k] - (n-1) \sum_{i<j} \epsilon\ X^{\{i,j\}} (\nabla_{X_i} X_j) \\ &\equiv (n+1) X_0 \cdots X_n - \sum_{j<k} \epsilon\ X_0 \cdots \widehat{X_j} \cdots \widehat{X_k} \cdots X_n \Big\{ [X_j, X_k] + (n-1) \nabla_{X_j} X_k \Big\} \\ &\qquad \mod \mathcal{U}^{\leqslant n-1}(T_{\mathcal{M}}).\end{aligned}$$

Combining this last result with the Equation (3.2) and Lemma 3.4.6, we finally obtain

$$\begin{aligned}\operatorname{pbw}(X_0 \odot \cdots X_n) &= \tfrac{1}{n+1} \sum_{k=0}^{n} \Big\{ \epsilon\ X_k \cdot \operatorname{pbw}(X^{\{k\}}) - \epsilon\ \operatorname{pbw}\Big(\nabla_{X_k}(X^{\{k\}})\Big) \Big\} \\ &\equiv X_0 \cdots X_n - \tfrac{1}{n+1} \sum_{j<k} \epsilon X^{\{j,k\}} \Big\{ (n-1) \nabla_{X_j} X_k + 2 \nabla_{X_j} X_k - T^{\nabla}(X_j, X_k) \Big\} \\ &\equiv X_0 \cdots X_n - \sum_{j<k} \epsilon X_0 \cdots \widehat{X_j} \cdots \widehat{X_k} \cdots X_n \Big\{ \nabla_{X_j} X_k - \tfrac{1}{n+1} T^{\nabla}(X_j, X_k) \Big\} \\ &\qquad \mod \mathcal{U}^{\leqslant n-1}(T_{\mathcal{M}}).\end{aligned}$$

The proof of Equation (3.4) is complete since $T^\nabla = 0$. Finally, applying pbw^{-1} to both sides of Equation (3.4) and making use of Equation (3.9) yields Equation (3.5). □

3.5 Emmrich-Weinstein theorem for graded manifolds

Let $\mathcal{M}$ be finite dimensional, let $(x_i)_{i\in\{1,\dots,n\}}$ be a set of local coordinates on $\mathcal{M}$ and let $(y_j)_{j\in\{1,\dots,n\}}$ be the induced local frame of $T^\vee_{\mathcal{M}}$ regarded as fiberwise linear functions on $T_{\mathcal{M}}$. As in [18], we define

$$\delta : \Omega^p(\mathcal{M}, S^q T^\vee_{\mathcal{M}}) \to \Omega^{p+1}(\mathcal{M}, S^{q-1} T^\vee_{\mathcal{M}})$$

and

$$\delta^{-1} : \Omega^p(\mathcal{M}, S^q T^\vee_{\mathcal{M}}) \to \Omega^{p-1}(\mathcal{M}, S^{q+1} T^\vee_{\mathcal{M}})$$

by

$$\delta = \sum_{i=1}^n dx_i \otimes \frac{\partial}{\partial y_i} \qquad \text{and} \qquad \delta^{-1} = \frac{1}{p+q} \sum_{i=1}^n i_{\frac{\partial}{\partial x_i}} \otimes y_i$$

or, more precisely,

$$\delta(\omega \otimes f) = \sum_{i=1}^n (-1)^{\left|\frac{\partial}{\partial y_i}\right||\omega|}\, dx_i \wedge \omega \otimes \frac{\partial}{\partial y_i}(f)$$

and

$$\delta^{-1}(\omega \otimes f) = \tfrac{1}{p+q} \sum_{i=1}^n (-1)^{|y^i||\omega|} i_{\frac{\partial}{\partial x_i}} \omega \otimes y_i \cdot f$$

for all homogeneous $\omega \in \Omega^p(\mathcal{M})$ and for all $f \in \Gamma(S^q T^\vee_{\mathcal{M}})$. It's not difficult to check that the operators δ and δ^{-1} are well defined, i.e. independent of the choice of local coordinates, and can be extended to $\Omega^\bullet\Big(\mathcal{M}, \mathrm{End}(\hat{S}(T^\vee_{\mathcal{M}}))\Big)$. The operator δ has degree $+1$ while the operator δ^{-1} has degree -1. Note that the operators δ and δ^{-1} are *not* inverse of each other.

Following the construction of Dolgushev [18], we prove the the following proposition.

Proposition 3.5.1. *Given a torsion-free connection ∇ on the tangent bundle $T_{\mathcal{M}}$*

of a graded manifold $\mathcal{M}$, there exists a unique element

$$A^{\nabla} = \sum_{i=1}^{n} \sum_{\substack{J\in\mathbb{N}_0^n \\ |J|\geqslant 2}} \sum_{k=1}^{n} A^i_{J,k} dx_i \otimes y^J \frac{\partial}{\partial y_k}$$

of degree +1 in $\Omega^1(\mathcal{M}, \hat{S}^{\geqslant 2}(T^\vee_{\mathcal{M}}) \otimes T_{\mathcal{M}})$ such that $\delta^{-1}(A^\nabla) = 0$ and the operator

$$D : \Omega^\bullet(\mathcal{M}, \hat{S}(T^\vee_{\mathcal{M}})) \to \Omega^{\bullet+1}(\mathcal{M}, \hat{S}(T^\vee_{\mathcal{M}}))$$

of degree $+1$ defined by $D = -\delta + d^\nabla + A^\nabla$ satisfies $D \circ D = 0$. Here the symbol d^∇ denotes the covariant differential of the connection on $\hat{S}(T^\vee_{\mathcal{M}})$ induced by the connection ∇ on $S(T_{\mathcal{M}})$.

Thus we obtain the cochain complex

$$\Omega^0(\mathcal{M}, \hat{S}(T^\vee_{\mathcal{M}})) \xrightarrow{D} \Omega^1(\mathcal{M}, \hat{S}(T^\vee_{\mathcal{M}})) \xrightarrow{D} \Omega^2(\mathcal{M}, \hat{S}(T^\vee_{\mathcal{M}})) \xrightarrow{D} \cdots \quad (3.10)$$

Note that A^∇ can be thought of as a 1-form on $\mathcal{M}$ valued in fiberwise formal vector fields on $T_{\mathcal{M}}$ and hence acts on $\Omega^\bullet(\mathcal{M}, \hat{S}(T^\vee_{\mathcal{M}}))$, the forms on $\mathcal{M}$ valued in fiberwise formal functions on $T_{\mathcal{M}}$.

Consider the linear map $\sigma : \Omega^\bullet(\mathcal{M}, \hat{S}(T^\vee_{\mathcal{M}})) \to C^\infty(\mathcal{M})$ of degree 0 characterized by the relations

$$\begin{gathered} \sigma(f \otimes 1) = f, \quad \forall f \in C^\infty(\mathcal{M}); \\ \sigma(\omega \otimes y^J) = 0, \quad \forall \omega \in \Omega^{\geqslant 1}(\mathcal{M}), \forall J \in \mathbb{N}_0^n; \\ \sigma(f \otimes y^J) = 0, \quad \forall f \in \Omega^0(\mathcal{M}), \forall J \in \mathbb{N}_0^n \text{ s.t. } |J| \geqslant 1. \end{gathered} \quad (3.11)$$

Proposition 3.5.2. *For every $f \in C^\infty(\mathcal{M})$, there exists a unique $\xi \in \Gamma\Big(\hat{S}(T^\vee_{\mathcal{M}})\Big)$ such that $\sigma(\xi) = f$ and $D(\xi) = 0$.*

Hence, there exists a unique map

$$\tau : C^\infty(\mathcal{M}) \to \Omega^0(\mathcal{M}, \hat{S}(T^\vee_{\mathcal{M}}))$$

of degree 0 satisfying $\sigma \circ \tau = \mathrm{id}_{C^\infty(\mathcal{M})}$ and $D \circ \tau = 0$. Furthermore, due to Proposition 3.5.2, one can easily check that τ preserves the algebra structures.

Corollary 3.5.3. *The map* $\tau : C^\infty(\mathcal{M}) \to \Omega^0(\mathcal{M}, \hat{S}(T^\vee_{\mathcal{M}}))$ *is a morphism of algebras.*

The cochain complex (3.10) may be augmented to

$$0 \longrightarrow C^\infty(\mathcal{M}) \xrightarrow{\tau} \Omega^0(\mathcal{M}, \hat{S}(T^\vee_{\mathcal{M}})) \xrightarrow{D} \Omega^1(\mathcal{M}, \hat{S}(T^\vee_{\mathcal{M}})) \xrightarrow{D} \cdots$$

Inspired by the work of Emmrich & Weinstein [22], we proceed to prove that the coboundary operator D and the map τ may be obtained directly from the formal exponential map pbw.

We start by defining a connection $\nabla^{\natural}$ on $ST_{\mathcal{M}}$ by

$$\nabla^{\natural}_X S := \operatorname{pbw}^{-1}\Big(X \cdot \operatorname{pbw}(S)\Big)$$

for all $X \in \Gamma(T_{\mathcal{M}})$ and $S \in \Gamma(ST_{\mathcal{M}})$.

Lemma 3.5.4. *The connection* $\nabla^{\natural}$ *is flat.*

Abusing notations, we will use the same symbol $\nabla^{\natural}$ to denote the induced flat connection on the dual bundle $\hat{S}(T^\vee_{\mathcal{M}})$.

Remark 3.5.5. *Equation* (3.7) *may be rewritten as*

$$\nabla^{\natural}_Y Z = Y \odot Z + \nabla_Y Z - \tfrac{1}{2} T^{\nabla}(Y, Z), \quad \forall Y, Z \in \Gamma(T_{\mathcal{M}}).$$

Theorem 3.5.6. *The operator* $D : \Omega^\bullet(\mathcal{M}, \hat{S}(T^\vee_{\mathcal{M}})) \to \Omega^{\bullet+1}(\mathcal{M}, \hat{S}(T^\vee_{\mathcal{M}}))$ *constructed in Proposition 3.5.1 is the covariant differential associated to the flat connection* $\nabla^{\natural}$*, i.e.* $D = d^{\nabla^{\natural}}$.

The proof of Theorem 3.5.6 is deferred to Section 3.6.

The operator D on $\Omega^\bullet(\mathcal{M}, \hat{S}(T^\vee_{\mathcal{M}}))$ induces similar operators on $\Omega^\bullet(\mathcal{M}, \mathcal{T}_{\text{poly}})$ and $\Omega^\bullet(\mathcal{M}, \mathcal{D}_{\text{poly}})$. See for instance [13] and [18]. Here $\mathcal{T}_{\text{poly}}$ and $\mathcal{D}_{\text{poly}}$ denote the bundles of fiberwise polyvector fields and fiberwise polydifferential operators on $T_{\mathcal{M}}$, respectively. These induced operators may be used to 'globalize' Kontsevich's (local) formality theorem. Like D, both operators can be constructed directly from the map pbw.

As a corollary of Theorem 3.5.6, we obtain an extension of a result of Emmrich & Weinstein to graded manifolds (see [22]).

Corollary 3.5.7. *Let $\mathcal{M}$ be finite dimensional, let $(x_i)_{i\in\{1,\dots,n\}}$ be a set of local coordinates on $\mathcal{M}$ and let $(y_j)_{j\in\{1,\dots,n\}}$ be the induced local frame of $T^\vee_{\mathcal{M}}$ regarded as fiberwise linear functions on $T_{\mathcal{M}}$. For all $f \in C^\infty(\mathcal{M})$, we have*

$$\tau(f) = \sum_{I\in\mathbb{N}_0^n} \tfrac{1}{I!} y^I \otimes \operatorname{pbw}\left(\underleftarrow{\partial_x^I}\right)(f),$$

where

$$\underleftarrow{\partial_x^I} = \underbrace{\partial_{x_n} \odot \cdots \odot \partial_{x_n}}_{i_n \text{ factors}} \odot \underbrace{\partial_{x_{n-1}} \odot \cdots \odot \partial_{x_{n-1}}}_{i_{n-1} \text{ factors}} \odot \cdots \odot \underbrace{\partial_{x_1} \odot \cdots \odot \partial_{x_1}}_{i_1 \text{ factors}}$$

for $I = (i_1, i_2, \dots, i_n) \in \mathbb{N}_0^n$.

Proof. Straightforward computations yield

$$\sigma\left(\sum_{I\in\mathbb{N}_0^n} \tfrac{1}{I!} y^I \otimes \operatorname{pbw}\left(\underleftarrow{\partial_x^I}\right)(f)\right) = f$$

and

$$d^{\nabla^\natural}\left(\sum_{I\in\mathbb{N}_0^n} \tfrac{1}{I!} y^I \otimes \operatorname{pbw}\left(\underleftarrow{\partial_x^I}\right)(f)\right) = 0$$

for all $f \in C^\infty(\mathcal{M})$. Since $d^{\nabla^\natural} = D$ and τ is the only map from $C^\infty(\mathcal{M})$ to $\Omega^0(\mathcal{M}, \hat{S}(T^\vee_{\mathcal{M}}))$ satisfying $\sigma \circ \tau = \mathrm{id}_{C^\infty(\mathcal{M})}$ and $D \circ \tau = 0$, the desired result follows. □

Specializing to classical (i.e. nongraded) manifolds, we recover the result of Emmrich & Weinstein:

Corollary 3.5.8 (Emmrich–Weinstein)**.** *For a smooth (nongraded) manifold M, the map*

$$\tau : C^\infty(M) \to \Omega^0(M, \hat{S}(T^\vee_{\mathcal{M}}))$$

satisfies

$$\tau(f) = \sum_{I\in\mathbb{N}_0^n} \tfrac{1}{I!} \operatorname{pbw}\left(\partial_x^I\right)(f) \otimes y^I$$

for all $f \in C^\infty(M)$.

3.6 Proof of Theorem 3.5.6

Lemma 3.6.1. *Let $\mathcal{M}$ be a graded manifold and let ∇ be a torsionfree connection on $T_{\mathcal{M}}$. For all $n \in \mathbb{N}$ and $X_0, \ldots, X_n \in \Gamma(T_{\mathcal{M}})$, we have*

$$\nabla^{\natural}_{X_0}(X_1 \odot \cdots \odot X_n) \equiv X_0 \odot X_1 \odot \cdots \odot X_n + \nabla_{X_0}(X_1 \odot \cdots \odot X_n) \mod \Gamma(S^{\leqslant n-1}T_{\mathcal{M}}).$$

Proof. According to Proposition 3.4.1, we have

$$\operatorname{pbw}(X_1 \odot \cdots \odot X_n) \equiv X_1 \cdots X_n - \sum_{j<k} \epsilon\ X^{\{j,k\}} \cdot (\nabla_{X_j} X_k) \mod \mathcal{U}^{\leqslant n-2}(T_{\mathcal{M}}).$$

Multiplying from the left by X_0, we obtain

$$\begin{aligned} &X_0 \cdot \operatorname{pbw}(X_1 \odot \cdots \odot X_n) \\ &\quad \equiv X_0 \cdot X_1 \cdots X_n - \sum_{0<j<k} \epsilon\ X^{\{j,k\}} \cdot (\nabla_{X_j} X_k) \mod \mathcal{U}^{\leqslant n-1}(T_{\mathcal{M}}) \\ &\quad \equiv \left\{ X_0 \cdots X_n - \sum_{j<k} \epsilon\ X^{\{j,k\}} \cdot (\nabla_{X_j} X_k) \right\} + \sum_{k=1}^{n} \epsilon\ X^{\{0,k\}} \cdot (\nabla_{X_0} X_k) \\ &\qquad\qquad \mod \mathcal{U}^{\leqslant n-1}(T_{\mathcal{M}}). \end{aligned}$$

Making use of Equations (3.4) and (3.8), the last congruence above becomes

$$\begin{aligned} &X_0 \cdot \operatorname{pbw}(X_1 \odot \cdots \odot X_n) \\ &\quad \equiv \operatorname{pbw}(X_0 \odot \cdots \odot X_n) + \sum_{k=1}^{n} \epsilon\ \operatorname{pbw}\left(X_1 \odot \cdots \odot \widehat{X_k} \odot \cdots \odot X_n \odot (\nabla_{X_0} X_k) \right) \\ &\qquad\qquad \mod \mathcal{U}^{\leqslant n-1}(T_{\mathcal{M}}) \\ &\quad \equiv \operatorname{pbw}\left(X_0 \odot \cdots \odot X_n + \sum_{k=1}^{n} \epsilon\ X_1 \odot \cdots \odot (\nabla_{X_0} X_k) \odot \cdots \odot X_n \right) \\ &\qquad\qquad \mod \mathcal{U}^{\leqslant n-1}(T_{\mathcal{M}}) \\ &\quad \equiv \operatorname{pbw}\left(X_0 \odot \cdots \odot X_n + \nabla_{X_0}(X_1 \odot \cdots \odot X_n) \right) \qquad \mod \mathcal{U}^{\leqslant n-1}(T_{\mathcal{M}}). \end{aligned}$$

Therefore, we have

$$\nabla^{\lightning}_{X_0}(X_1 \odot \cdots \odot X_n) = \operatorname{pbw}^{-1}\Big(X_0 \cdot \operatorname{pbw}(X_1 \odot \cdots \odot X_n)\Big)$$
$$\equiv X_0 \odot X_1 \odot \cdots \odot X_n + \nabla_{X_0}(X_1 \odot \cdots \odot X_n) \mod \Gamma(S^{\leqslant n-1}T_{\mathcal{M}}). \quad \square$$

Lemma 3.6.1 above leads us to consider the map

$$\Theta^\nabla : \Gamma\Big(S(T_{\mathcal{M}})\Big) \to \Omega^1\Big(\mathcal{M}, S(T_{\mathcal{M}})\Big)$$

of degree $+1$ defined by

$$i_X \Theta^\nabla(S) = \nabla^{\lightning}_X S - X \odot S - \nabla_X S, \tag{3.12}$$

for all $X \in \Gamma(T_{\mathcal{M}})$ and $S \in \Gamma(ST_{\mathcal{M}})$.

Lemma 3.6.1 asserts that, if ∇ is torsionfree, $i_X\Theta^\nabla$ maps $\Gamma\Big(S^k T_{\mathcal{M}}\Big)$ to $\Gamma(S^{\leqslant k-1}T_{\mathcal{M}})$, for all $X \in \Gamma(T_{\mathcal{M}})$ and $k \in \mathbb{N}_0$.

Proposition 3.6.2. *For every $X \in \Gamma(T_{\mathcal{M}})$, the operator $i_X\Theta^\nabla$ is a coderivation of the coalgebra $\Gamma(ST_{\mathcal{M}})$.*

Proof. The operators $\nabla^{\lightning}_X$, ∇_X, and $S \mapsto X \odot S$ are coderivations of $\Gamma(ST_{\mathcal{M}})$. $\square$

Proposition 3.6.3. *If $T^\nabla = 0$, then $\Theta^\nabla(S) = 0$ for all $S \in \Gamma(S^{\leqslant 1}T_{\mathcal{M}})$.*

Proof. It follows from the definitions and Remark 3.5.5 that $\Theta^\nabla(1) = 0$ and

$$i_Y \Theta^\nabla(Z) = \nabla^{\lightning}_Y Z - Y \odot Z - \nabla_Y Z = -\tfrac{1}{2} T^\nabla(Y, Z),$$

for all $Y, Z \in \Gamma(T_{\mathcal{M}})$. $\square$

Proposition 3.6.4. *For all $n \in \mathbb{N}$ and all homogeneous $X_0, \ldots, X_n \in \Gamma(T_{\mathcal{M}})$, we have*

$$\sum_{k=0}^{n} \epsilon \; i_{X_k}\Theta^\nabla(X_0 \odot \cdots \odot \widehat{X_k} \odot \cdots \odot X_n) = 0.$$

Proof. Applying pbw^{-1} to both sides of Equation (3.2), we obtain

$$(n+1)\; X_0 \odot \cdots \odot X_n = \sum_{k=0}^{n} \epsilon \; \Big\{ \operatorname{pbw}^{-1}\Big(X_k \cdot \operatorname{pbw}(X^{\{k\}})\Big) - \nabla_{X_k}(X^{\{k\}}) \Big\},$$

which we may rewrite as

$$\sum_{k=0}^{n} \epsilon \; X_k \odot X^{\{k\}} = \sum_{k=0}^{n} \epsilon \; \Big\{ \nabla^{\lightning}_{X_k}(X^{\{k\}}) - \nabla_{X_k}(X^{\{k\}}) \Big\}.$$

The desired result follows from Equation (3.12). □

Now consider the map

$$\Xi^\nabla : \Gamma(S^\bullet(T_\mathcal{M}^\vee)) \to \Omega^1\Big(\mathcal{M}, \hat{S}^{\geqslant\bullet+1}(T_\mathcal{M}^\vee)\Big)$$

of degree +1 defined by

$$\Big\langle S\Big|i_X\Xi^\nabla(\sigma)\Big\rangle = (-1)^{|S||X|}\Big\langle i_X\Theta^\nabla(S)\Big|\sigma\Big\rangle, \tag{3.13}$$

for all homogeneous $X \in \Gamma(T_\mathcal{M})$, $S \in \Gamma(ST_\mathcal{M})$, and $\sigma \in \Gamma(\hat{S}(T_\mathcal{M}^\vee))$. Here

$$\Gamma(ST_\mathcal{M}) \otimes_{C^\infty(\mathcal{M})} \Gamma(\hat{S}(T_\mathcal{M}^\vee)) \xrightarrow{\langle -|-\rangle} C^\infty(\mathcal{M})$$

is the duality pairing defined by

$$\langle X_1 \odot \cdots \odot X_p | \alpha_1 \odot \cdots \odot \alpha_q\rangle = \begin{cases} \sum_{\sigma\in S_p} \epsilon\ i_{X_1}\alpha_{\sigma(1)} \cdot i_{X_2}\alpha_{\sigma(2)} \cdots i_{X_p}\alpha_{\sigma(p)} & \text{if } p = q \\ 0 & \text{if } p \neq q \end{cases}$$

for all homogeneous $X_1, \dots, X_p \in \Gamma(T_\mathcal{M})$ and $\alpha_1, \dots, \alpha_q \in \Gamma(T_\mathcal{M})$. The factor ϵ in the equation above denotes the Koszul sign of the permutation of homogeneous elements

$$X_1, X_2, \dots, X_p, \alpha_1, \alpha_2, \dots, \alpha_p \quad \longmapsto \quad X_1, \alpha_{\sigma(1)}, X_2, \alpha_{\sigma(2)}, \dots, X_p, \alpha_{\sigma(p)}.$$

A straightforward computation yields the following lemma.

Lemma 3.6.5. *Let $\mathcal{M}$ be finite dimensional, let $(x_i)_{i\in\{1,\dots,n\}}$ be a set of local coordinates on $\mathcal{M}$ and let $(y_j)_{j\in\{1,\dots,n\}}$ be the induced local frame of $T_\mathcal{M}^\vee$ regarded as fiberwise linear functions on $T_\mathcal{M}$. For all $I, J \in \mathbb{N}_0^n$ such that $|I| = |J|$, we have*

$$\Big\langle \overleftarrow{\partial_x^I}\Big| y^J\Big\rangle = I!\ \delta_{I,J}.$$

Lemma 3.6.6. *For all homogeneous $X \in \Gamma(T_\mathcal{M})$, $S \in \Gamma(ST_\mathcal{M})$, and $\sigma \in \Gamma\Big(\hat{S}(T_\mathcal{M}^\vee)\Big)$, we have*

$$\langle S|i_X\delta(\sigma)\rangle = (-1)^{|S||X|}\langle X \odot S|\sigma\rangle.$$

Proof. It suffices to prove the relation for $S = \underset{\leftarrow}{\partial_x^I}$, $\sigma = y^J$, and $X = \frac{\partial}{\partial x_l}$. We have

$$\begin{aligned}
\left\langle \underset{\leftarrow}{\partial_x^I} \middle| i_{\frac{\partial}{\partial x_l}} \delta(y^J) \right\rangle &= \left\langle \underset{\leftarrow}{\partial_x^I} \middle| \sum_{k=1}^n i_{\frac{\partial}{\partial x_l}} dx_k \ \frac{\partial}{\partial y_k}(y^J) \right\rangle \\
&= \left\langle \underset{\leftarrow}{\partial_x^I} \middle| \frac{\partial}{\partial y_l}(y^J) \right\rangle \\
&= \left\langle \underset{\leftarrow}{\partial_x^I} \middle| (-1)^{|y_l||y^{\tau_{<l}J}|} \ j_l \ y^{J-e^l} \right\rangle \\
&= (-1)^{|y_l||y^{\tau_{<l}J}|} \ j_l \ I! \ \delta_{I,J-e^l} \\
&= (-1)^{|y_l||y^{\tau_{<l}J}|} \ J! \ \delta_{I+e_l,J} \\
&= (-1)^{\left|\frac{\partial}{\partial x_l}\right|\left|\partial_x^{\tau_{<l}I}\right|} \left\langle \underset{\longleftarrow}{\partial_x^{I+e_l}} \middle| y^J \right\rangle \\
&= \left\langle \underset{\leftarrow}{\partial_x^I} \odot \frac{\partial}{\partial x_l} \middle| y^J \right\rangle \\
&= (-1)^{\left|\underset{\leftarrow}{\partial_x^I}\right|\left|\frac{\partial}{\partial x_l}\right|} \left\langle \frac{\partial}{\partial x_l} \odot \underset{\leftarrow}{\partial_x^I} \middle| y^J \right\rangle .
\end{aligned}$$

□

Proposition 3.6.7. *The operator $i_X \Xi^\nabla$ is a derivation of the algebra $\Gamma\Big(\hat{S}(T_{\mathcal{M}}^\vee)\Big)$ for every $X \in \Gamma(T_{\mathcal{M}})$.*

Proof. The result follows immediately from Proposition 3.6.2 since the algebra $\Gamma\Big(\hat{S}(T_{\mathcal{M}}^\vee)\Big)$ is dual to the coalgebra $\Gamma(ST_{\mathcal{M}})$ and $i_X \Xi^\nabla$ is the transpose of $i_X \Theta^\nabla$ according to Equation (3.13). □

Hence Ξ^∇ may be regarded as an element of $\Omega^1\Big(\mathcal{M}, \hat{S}(T_{\mathcal{M}}^\vee) \otimes T_{\mathcal{M}}\Big)$.

Proposition 3.6.8. *If $T^\nabla = 0$, then $\Xi^\nabla \in \Omega^1\Big(\mathcal{M}, \hat{S}^{\geqslant 2}(T_{\mathcal{M}}^\vee) \otimes T_{\mathcal{M}}\Big)$.*

Proof. Let $(x_i)_{i\in\{1,\dots,n\}}$ be a set of local coordinates on $\mathcal{M}$ and let $(y_j)_{j\in\{1,\dots,n\}}$ be the induced local frame of $T_{\mathcal{M}}^\vee$ regarded as fiberwise linear functions on $T_{\mathcal{M}}$. Since $i_X \Xi^\nabla$ is a derivation of the algebra $\Gamma\Big(\hat{S}(T_{\mathcal{M}}^\vee)\Big)$, which is generated by $y_1, \dots, y_n$, we have

$$i_X \Xi^\nabla = \sum_{k=1}^n i_X \Xi^\nabla(y_k) \frac{\partial}{\partial y_k},$$

with

$$i_X\Xi^\nabla(y_k) = \sum_{I\in\mathbb{N}_0^n} \frac{1}{I!}(1\otimes y^I)\cdot(\left\langle \overleftarrow{\partial_x^I}\middle| i_X\Xi^\nabla(y_k)\right\rangle \otimes 1) \qquad \text{by Lemma 3.6.5,}$$

$$= \sum_{I\in\mathbb{N}_0^n} \frac{(-1)^{\left|\overleftarrow{\partial_x^I}\right||X|}}{I!}(1\otimes y^I)\cdot(\left\langle i_X\Theta^\nabla\left(\overleftarrow{\partial_x^I}\right)\middle| y_k\right\rangle \otimes 1) \quad \text{by Equation (3.13).}$$

Since $T^\nabla = 0$, it follows from Proposition 3.6.3 that $i_X\Xi^\nabla \in \Gamma\left(\hat{S}^{\geqslant 2}(T_{\mathcal{M}}^\vee)\otimes T_{\mathcal{M}}\right)$ as $\Theta^\nabla\left(\overleftarrow{\partial_x^I}\right) = 0$ for $|I| \leqslant 1$. □

Proposition 3.6.9. $\delta^{-1}(\Xi^\nabla) = 0$

Proof. Let $(x_i)_{i\in\{1,\dots,n\}}$ be a set of local coordinates on $\mathcal{M}$ and let $(y_j)_{j\in\{1,\dots,n\}}$ be the induced local frame of $T_{\mathcal{M}}^\vee$ regarded as fiberwise linear functions on $T_{\mathcal{M}}$.

From

$$\Xi^\nabla = \sum_{k=1}^n \sum_{J\in\mathbb{N}_0^n} \frac{1}{J!}\,(1\otimes y^J)\cdot(\left\langle \overleftarrow{\partial_x^J}\middle|\Xi^\nabla(y_k)\right\rangle \otimes \frac{\partial}{\partial y_k}),$$

we obtain

$$\begin{aligned}
\delta^{-1}(\Xi^\nabla) &= \sum_{k=1}^n \sum_{J\in\mathbb{N}_0^n} \sum_{l=1}^n \frac{1}{J!}\; y_l y^J \left\langle \overleftarrow{\partial_x^J}\middle| i_{\frac{\partial}{\partial x_l}}\Xi^\nabla(y_k)\right\rangle \frac{\partial}{\partial y_k} \\
&= \sum_{k=1}^n \sum_{J\in\mathbb{N}_0^n} \sum_{l=1}^n \frac{1}{J!}(-1)^{|y_l|\left|y^{\tau_{\leqslant l}J}\right|} y^{J+e_l}(-1)^{\left|\frac{\partial}{\partial x_l}\right|\left|\partial_x^J\right|} \left\langle i_{\frac{\partial}{\partial x_l}}\Theta^\nabla\left(\overleftarrow{\partial_x^J}\right)\middle| y_k\right\rangle \frac{\partial}{\partial y_k} \\
&= \sum_{k=1}^n \sum_{J\in\mathbb{N}_0^n} \sum_{l=1}^n \frac{1}{J!}\; y^{J+e_l}(-1)^{\left|\frac{\partial}{\partial x_l}\right|\left|\partial_x^{\tau_{>l}J}\right|} \left\langle i_{\frac{\partial}{\partial x_l}}\Theta^\nabla\left(\overleftarrow{\partial_x^J}\right)\middle| y_k\right\rangle \frac{\partial}{\partial y_k} \\
&= \sum_{k=1}^n \sum_{M\in\mathbb{N}_0^n} \frac{1}{M!}\; y^M \left\langle \sum_{l=1}^n m_l(-1)^{\left|\frac{\partial}{\partial x_l}\right|\left|\partial_x^{\tau_{>l}M}\right|} i_{\frac{\partial}{\partial x_l}}\Theta^\nabla\left(\overleftarrow{\partial_x^{M-e_l}}\right)\middle| y_k\right\rangle \frac{\partial}{\partial y_k}.
\end{aligned}$$

It follows directly from Proposition 3.6.4 that

$$\sum_{l=1}^n m_l\;(-1)^{\left|\frac{\partial}{\partial x_l}\right|\left|\partial_x^{\tau_{>l}M}\right|}\; i_{\frac{\partial}{\partial x_l}}\Theta^\nabla\left(\overleftarrow{\partial_x^{M-e_l}}\right) = 0$$

for every $M = (m_1,\dots,m_n)\in\mathbb{N}_0^n$. □

Proof of Theorem 3.5.6. The connections ∇ and $\nabla^\natural$ defined on $S(T_{\mathcal{M}})$ induce connections on the dual bundle $\hat{S}(T^\vee_{\mathcal{M}})$:

$$\left\langle \nabla^\natural_X S \middle| \sigma \right\rangle + (-1)^{|X||S|} \left\langle S \middle| \nabla^\natural_X \sigma \right\rangle = X\Big(\langle S|\sigma\rangle \Big) = \langle \nabla_X S|\sigma\rangle + (-1)^{|X||S|} \langle S|\nabla_X \sigma\rangle .$$

Therefore, we obtain

$$\begin{aligned} \left\langle \nabla^\natural_X S - \nabla_X S \middle| \sigma \right\rangle &= (-1)^{|X||S|} \left\langle S \middle| \nabla_X \sigma - \nabla^\natural_X \sigma \right\rangle \\ \left\langle X \odot S + i_X \Theta^\nabla(S) \middle| \sigma \right\rangle &= (-1)^{|X||S|} \left\langle S \middle| i_X\Big(d^\nabla \sigma - d^{\nabla^\natural} \sigma \Big) \right\rangle \end{aligned}$$

and, making use of Lemma 3.6.6 and Equation (3.13),

$$\left\langle S \middle| i_X \Big(\delta\sigma + \Xi^\nabla \sigma \Big) \right\rangle = \left\langle S \middle| i_X \Big(d^\nabla \sigma - d^{\nabla^\natural} \sigma \Big) \right\rangle$$

or, equivalently,

$$d^{\nabla^\natural} = -\delta + d^\nabla - \Xi^\nabla .$$

Since $\delta^{-1}(\Xi^\nabla) = 0$ (Proposition 3.6.9) and $d^{\nabla^\natural} \circ d^{\nabla^\natural} = 0$ (Proposition 3.5.4), Theorem 3.5.1 asserts that $A^\nabla = -\Xi^\nabla$ and $D = d^{\nabla^\natural}$. □

Corollary 3.6.10. $A^\nabla = -\Xi^\nabla$

3.7 Dolgushev–Fedosov resolution via homological perturbation

In this section, we give a new proof of the following theorem of Cattaneo & Felder [13] and Dolgushev [18] via homological perturbation (see Appendix).

Theorem 3.7.1. *Suppose $\mathcal{M}$ is a graded manifold of finite dimension and ∇ is a torsion-free connection on $T_{\mathcal{M}}$ satisfying $\delta R^\nabla = 0$. Then the associated cochain complex*

$$\Omega^0(\mathcal{M}, \hat{S}(T^\vee_{\mathcal{M}})) \xrightarrow{D} \Omega^1(\mathcal{M}, \hat{S}(T^\vee_{\mathcal{M}})) \xrightarrow{D} \Omega^2(\mathcal{M}, \hat{S}(T^\vee_{\mathcal{M}})) \xrightarrow{D} \cdots$$

together with the augmentation map $\tau : C^\infty(\mathcal{M}) \to \Omega^0(\mathcal{M}, \hat{S}(T^\vee_{\mathcal{M}}))$ is a resolution *of* $C^\infty(\mathcal{M})$*. Moreover, we have*

$$\tau = \sum_{n=0}^{\infty} \Big(\delta^{-1} \circ (d^\nabla + A^\nabla)\Big)^n \circ i,$$

where i *denotes the canonical inclusion of* $C^\infty(\mathcal{M})$ *into* $\Omega^0(\mathcal{M}, \hat{S}(T^\vee_{\mathcal{M}}))$*.*

Proof. It is not difficult to check that the cochain complex $\Big(\Omega^\bullet(\mathcal{M}, \hat{S}(T^\vee_{\mathcal{M}})), -\delta\Big)$ deformation retracts onto $C^\infty(\mathcal{M})$: the canonical inclusion $i : C^\infty(\mathcal{M}) \to \Omega^0(\mathcal{M}, \hat{S}(T^\vee_{\mathcal{M}}))$ and the linear map $\sigma : \Omega^\bullet(\mathcal{M}, \hat{S}(T^\vee_{\mathcal{M}})) \to C^\infty(\mathcal{M})$ characterized by Equations (3.11) satisfy

$$\sigma i = \mathrm{id}_{C^\infty(\mathcal{M})} \qquad \text{and} \qquad \delta\delta^{-1} + \delta^{-1}\delta = \mathrm{id}_{\Omega^\bullet(\mathcal{M}, \hat{S}(T^\vee_{\mathcal{M}}))} - i\sigma.$$

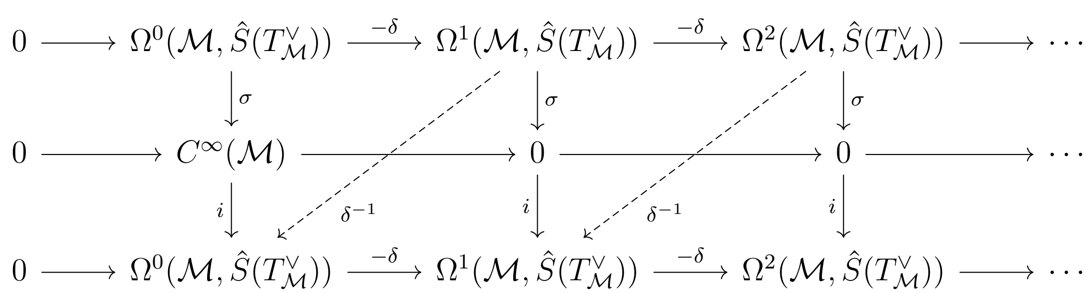
Furthermore, the maps σ, i, and δ^{-1} respect the exhaustive, complete, descending filtrations on $C^\infty(\mathcal{M})$ and the complex $\Big(\Omega^\bullet(\mathcal{M}, \hat{S}(T^\vee_{\mathcal{M}})), -\delta\Big)$ respectively defined by

$$\mathcal{F}^m = \begin{cases} C^\infty(\mathcal{M}) & \text{if } m \leqslant 0 \\ 0 & \text{if } m > 0 \end{cases}$$

and

$$\mathscr{F}^m = \prod_{p+q \geqslant m} \Omega^p\Big(\mathcal{M}, S^q(T^\vee_{\mathcal{M}})\Big).$$

More precisely, the diagram



is a filtered contraction.

The operator $\partial := d^\nabla + A^\nabla$ is a perturbation of the differential $-\delta$ on $\Omega^\bullet(\mathcal{M}, \hat{S}(T^\vee_{\mathcal{M}}))$, for $D = -\delta + \partial$ satisfies $D \circ D = 0$ and $\partial(\mathscr{F}^m) \subset \mathscr{F}^{m+1}$ for all $m \in \mathbb{N}_0$.

Hence homological perturbation (see Appendix) yields the contraction

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega^0(\mathcal{M},\hat{S}(T^\vee_{\mathcal{M}})) & \xrightarrow{D} & \Omega^1(\mathcal{M},\hat{S}(T^\vee_{\mathcal{M}})) & \xrightarrow{D} & \Omega^2(\mathcal{M},\hat{S}(T^\vee_{\mathcal{M}})) & \longrightarrow & \cdots \\
 & & \big\downarrow \breve{\sigma} & & \big\downarrow \breve{\sigma} & & \big\downarrow \breve{\sigma} & & \\
0 & \longrightarrow & C^\infty(\mathcal{M}) & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \breve{\imath} \big\downarrow & \swarrow h & \breve{\imath} \big\downarrow & \swarrow h & \breve{\imath} \big\downarrow & & \\
0 & \longrightarrow & \Omega^0(\mathcal{M},\hat{S}(T^\vee_{\mathcal{M}})) & \xrightarrow{D} & \Omega^1(\mathcal{M},\hat{S}(T^\vee_{\mathcal{M}})) & \xrightarrow{D} & \Omega^2(\mathcal{M},\hat{S}(T^\vee_{\mathcal{M}})) & \longrightarrow & \cdots
\end{array}
$$

where

$$\breve{\sigma} = \sum_{k=0}^{\infty} \sigma(\partial\delta^{-1})^k \qquad\qquad \breve{\imath} = \sum_{k=0}^{\infty} (\delta^{-1}\partial)^k i$$

and

$$h = \sum_{k=0}^{\infty} (\delta^{-1}\partial)^k \delta^{-1}.$$

In particular, we have $D\breve{\imath} = 0$ and, since $\sigma\delta^{-1} = 0$,

$$\sigma\breve{\imath} = \sigma \sum_{n=0}^{\infty} (\delta^{-1}\partial)^n i = \sum_{n=0}^{\infty} \sigma(\delta^{-1}\partial)^n i = \sigma i = \mathrm{id}_{C^\infty(\mathcal{M})} \,.$$

Therefore,

$$\tau = \breve{\imath} = \sum_{n=0}^{\infty} (\delta^{-1}\partial)^n i = \sum_{n=0}^{\infty} (\delta^{-1}(d^\nabla + A))^n i.$$

We note that $\breve{\sigma} = \sigma$ since $\sigma\partial\delta^{-1} = 0$. □

Recall that a dg-manifold is a graded manifold $\mathcal{M}$ endowed with a vector field Q of degree $+1$ such that $[Q,Q] = 0$. Hence the algebra of functions $C^\infty(\mathcal{M})$ on a dg-manifold $(\mathcal{M}, Q)$ is a cochain complex with the vector field Q as coboundary operator. Two dg-manifolds $(\mathcal{M}_1, Q_1)$ and $(\mathcal{M}_2, Q_2)$ are said to be weakly equivalent if their associated cochain complexes $(C^\infty(\mathcal{M}_1), Q_1)$ and $(C^\infty(\mathcal{M}_2), Q_2)$ are quasi-isomorphic. In this terminology, Theorem 3.7.1 can be rephrased as follows: the chain map τ is a weak equivalence of dg-manifolds from $(\mathcal{M}, 0)$ to $(T_{\mathcal{M}}[1] \oplus T_{\mathcal{M}}, D)$.

Chapter 4
Formality and Kontsevich–Duflo theorem for Lie pairs

Introduction

In the late 1990's, Kontsevich revolutionized the field of deformation quantization with his formality theorem: there exists an L_∞ quasi-isomorphism from the dgla $T^\bullet_{\mathrm{poly}}(M)$ of polyvector fields on a smooth manifold M to the dgla $D^\bullet_{\mathrm{poly}}(M)$ of polydifferential operators on M extending the classical Hochschild–Kostant–Rosenberg map. Indeed, the formality theorem implies the existence of deformation quantizations for every smooth Poisson manifold [18, 30, 61]. In his paper [30], Kontsevich gave an explicit formula for the formality quasi-isomorphism in the case $M = \mathbb{R}^d$ and then outlined how the result can be generalized to arbitrary smooth manifolds. Later, Dolgushev gave a detailed proof of the globalization to arbitrary smooth manifolds of Kontsevich's formality quasi-isomorphism for $\mathbb{R}^d$ based on Fedosov's patching technique [18, 24].

In this paper, we extend Kontsevich's formality theorem to *Lie pairs*, a framework which includes a wide range of diverse geometric contexts including complex manifolds, foliations, and $\mathfrak{g}$-manifolds. By a *Lie pair* (L, A), we mean an inclusion $A \hookrightarrow L$ of Lie $\Bbbk$-algebroids over a smooth manifold M. (Throughout the paper, we use the symbol $\Bbbk$ to denote either of the fields $\mathbb{R}$ and $\mathbb{C}$.) Recall that a *Lie $\Bbbk$-algebroid* is a $\Bbbk$-vector bundle $L \to M$, whose space of sections is endowed with a Lie bracket $[-,-]$, together with a bundle map $\rho : L \to TM \otimes_{\mathbb{R}} \Bbbk$ called *anchor* such that $\rho : \Gamma(L) \to \mathfrak{X}(M) \otimes \Bbbk$ is a morphism of Lie algebras and

$[X, fY] = f[X,Y] + \Big(\rho(X)f\Big)Y$ for all $X, Y \in \Gamma(L)$ and $f \in C^\infty(M, \Bbbk)$. A $\Bbbk$-vector bundle $L \to M$ is a Lie algebroid if and only if $\Gamma(L)$ is a *Lie–Rinehart algebra* [55] over the commutative ring $C^\infty(M, \Bbbk)$. Lie pairs arise naturally in a number of subdisciplines of mathematics such as complex geometry, foliation theory, and Lie theory. A complex manifold X determines a Lie pair (over $\mathbb{C}$): $L = T_X \otimes \mathbb{C}$ and $A = T_X^{0,1}$. A foliation on a smooth manifold M determines a Lie pair (over $\mathbb{R}$): $L = TM$ and A is the integrable distribution on M tangent to the foliation. A manifold equipped with an action of a Lie algebra $\mathfrak{g}$ gives rise to a Lie pair in a natural way (see [51, Example 5.5] and [39, 40]).

Given a Lie pair (L, A), the quotient L/A is naturally an A-module. When L is the tangent bundle to a manifold M and A is an integrable distribution on M, the A-action on L/A is given by the Bott connection [7]. The spaces $\operatorname{tot}\Big(\Gamma(\Lambda^\bullet A^\vee) \otimes_R \mathcal{T}^\bullet_{\text{poly}}\Big)$ and $\operatorname{tot}\Big(\Gamma(\Lambda^\bullet A^\vee) \otimes_R \mathcal{D}^\bullet_{\text{poly}}\Big)$ associated with a Lie pair (L, A) serve as replacements for the spaces of polyvector fields and polydifferential operators respectively. Each one of them carries an L_∞ algebra structure canonical up to L_∞ quasi-isomorphism. Their corresponding cohomology groups $\mathbb{H}^\bullet_{\text{CE}}(A, \mathcal{T}^\bullet_{\text{poly}})$ and $\mathbb{H}^\bullet_{\text{CE}}(A, \mathcal{D}^\bullet_{\text{poly}})$ are Gerstenhaber algebras [59].

Denoting the algebra of smooth functions on the manifold M by R, we set $\mathcal{T}^k_{\text{poly}} = \Gamma(\Lambda^{k+1}(L/A))$ for $k \geqslant 0$, $\mathcal{T}^{-1}_{\text{poly}} = R$, and $\mathcal{T}^\bullet_{\text{poly}} = \bigoplus_{k=-1}^\infty \mathcal{T}^k_{\text{poly}}$. The Bott A-connection on L/A makes every $\mathcal{T}^k_{\text{poly}}$ an A-module. We can thus consider the complex of A-modules with trivial differential

$$0 \longrightarrow \mathcal{T}^{-1}_{\text{poly}} \xrightarrow{0} \mathcal{T}^{0}_{\text{poly}} \xrightarrow{0} \mathcal{T}^{1}_{\text{poly}} \xrightarrow{0} \mathcal{T}^{2}_{\text{poly}} \xrightarrow{0} \cdots .$$

Its Chevalley–Eilenberg hypercohomology cochain complex is denoted $\operatorname{tot}\Big(\Gamma(\Lambda^\bullet A^\vee) \otimes_R \mathcal{T}^\bullet_{\text{poly}}\Big)$. Similarly, we set $\mathcal{D}^\bullet_{\text{poly}} = \bigoplus_{k=-1}^\infty \mathcal{D}^k_{\text{poly}}$ where $\mathcal{D}^{-1}_{\text{poly}} = R$, $\mathcal{D}^0_{\text{poly}} = \frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}$, and $\mathcal{D}^k_{\text{poly}}$ with $k \geqslant 1$ is the tensor product $\mathcal{D}^0_{\text{poly}} \otimes_R \cdots \otimes_R \mathcal{D}^0_{\text{poly}}$ of $(k+1)$ copies of the left R-module $\mathcal{D}^0_{\text{poly}}$. Multiplication in $\mathcal{U}(L)$ from the left by elements of $\Gamma(A)$ induces an A-module structure on the quotient $\frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}$. This action of A on $\mathcal{D}^0_{\text{poly}}$ extends naturally to an action of A on $\mathcal{D}^k_{\text{poly}}$ for each k. In fact, $\mathcal{D}^0_{\text{poly}}$ is a cocommutative coassociative coalgebra over R whose comultiplication $\Delta : \mathcal{D}^0_{\text{poly}} \to \mathcal{D}^0_{\text{poly}} \otimes_R \mathcal{D}^0_{\text{poly}}$ is a morphism of A-modules. Therefore the Hochschild complex

$$0 \longrightarrow \mathcal{D}^{-1}_{\mathrm{poly}} \xrightarrow{d_{\mathscr{H}}} \mathcal{D}^{0}_{\mathrm{poly}} \xrightarrow{d_{\mathscr{H}}} \mathcal{D}^{1}_{\mathrm{poly}} \xrightarrow{d_{\mathscr{H}}} \mathcal{D}^{2}_{\mathrm{poly}} \xrightarrow{d_{\mathscr{H}}} \cdots$$

determined by the comultiplication $\Delta : \mathcal{D}^0_{\mathrm{poly}} \to \mathcal{D}^0_{\mathrm{poly}} \otimes_R \mathcal{D}^0_{\mathrm{poly}}$ is a complex of A-modules. Its Chevalley–Eilenberg hypercohomology cochain complex is denoted $\mathrm{tot}\left(\Gamma(\Lambda^\bullet A^\vee) \otimes_R \mathcal{D}^\bullet_{\mathrm{poly}}\right)$.

For instance, for the Lie pair $L = T_X \otimes \mathbb{C}$ and $A = T^{0,1}_X$ stemming from a complex manifold X, the pair od spaces $\mathrm{tot}\left(\Gamma(\Lambda^\bullet A^\vee) \otimes_R \mathcal{T}^\bullet_{\mathrm{poly}}\right)$ and $\mathrm{tot}\left(\Gamma(\Lambda^\bullet A^\vee) \otimes_R \mathcal{D}^\bullet_{\mathrm{poly}}\right)$ are precisely the standard dglas $\left(\Omega^{0,\bullet}(\mathcal{T}^\bullet_{\mathrm{poly}}(X)), \bar{\partial}\right)$ and $\left(\Omega^{0,\bullet}(\mathcal{D}^\bullet_{\mathrm{poly}}(X)), \bar{\partial} + d_{\mathscr{H}}\right)$. The corresponding Chevalley–Eilenberg hypercohomology groups $\mathbb{H}^\bullet_{\mathrm{CE}}(A, \mathcal{T}^\bullet_{\mathrm{poly}})$ and $\mathbb{H}^\bullet_{\mathrm{CE}}(A, \mathcal{D}^\bullet_{\mathrm{poly}})$ are isomorphic to the sheaf cohomology group $\mathbb{H}^\bullet(X, \Lambda^\bullet T_X)$ and the Hochschild cohomology group $HH^\bullet(X)$, respectively.

The skew-symmetric extension of the natural inclusion $\Gamma(L/A) \hookrightarrow \mathcal{D}^0_{\mathrm{poly}}$ to the complex of A-modules $\mathcal{T}^\bullet_{\mathrm{poly}}$ yields a morphism of A-modules $\mathrm{hkr} : \mathcal{T}^\bullet_{\mathrm{poly}} \to \mathcal{D}^\bullet_{\mathrm{poly}}$. The induced morphism of Chevalley–Eilenberg hypercohomology cochain complexes $\mathrm{hkr} : \mathrm{tot}\left(\Gamma(\Lambda^\bullet A^\vee) \otimes_R \mathcal{T}^\bullet_{\mathrm{poly}}\right) \to \mathrm{tot}\left(\Gamma(\Lambda^\bullet A^\vee) \otimes_R \mathcal{D}^\bullet_{\mathrm{poly}}\right)$, which is also called *Hochschild–Kostant–Rosenberg* map, is actually a quasi-isomorphism. It is thus natural to ask whether hkr can be extended to an L_∞ quasi-isomorphism analogous to Kontsevich's formality quasi-isomorphism for smooth manifolds. The answer is negative in general and the reason is quite simple. For a smooth manifold M, the Hochschild–Kostant–Rosenberg map induces an isomorphism of Lie algebras (in fact an isomorphism of Gerstenhaber algebras) from the polyvectors fields $\mathcal{T}^\bullet_{\mathrm{poly}}(M)$ on M equipped with the Schouten bracket to the Hochschild cohomology $H^\bullet(\mathcal{D}^\bullet_{\mathrm{poly}}(M), d_{\mathscr{H}})$ equipped with the Gerstenhaber bracket. However, for a Lie pair (L, A), the morphism in cohomology $\mathrm{hkr} : \mathbb{H}^\bullet_{\mathrm{CE}}(A, \mathcal{T}^\bullet_{\mathrm{poly}}) \to \mathbb{H}^\bullet_{\mathrm{CE}}(A, \mathcal{D}^\bullet_{\mathrm{poly}})$ induce by the Hochschild–Kostant–Rosenberg map no longer preserves neither the Lie algebra nor the associative algebra structures. The Hochschild–Kostant–Rosenberg map hkr must indeed be modified; it must be tweaked by the square root of the Todd cocycle of the Lie pair.

The Atiyah class of a Lie pair (L, A) was introduced and studied by Chen–StiÃľnon–Xu in [16]. It captures the obstruction to the existence of 'compatible' L-connections on L/A extending the Bott A-representation. The Atiyah class of Lie pairs is a simultaneous extension of both the classical Atiyah class of holomorphic vector bundles [1] and the Molino class of foliations [52]. As was first observed for holomorphic vector bundles by Kapranov [29], the Atiyah class of Lie pairs is the source of homotopy Lie algebras [16, 33, 34]. Let us briefly recall its definition. Given a Lie pair (L, A) with quotient $B = L/A$, choose an L-connection ∇ on B extending the Bott A-representation. The curvature of ∇ induces a section $R^{\nabla}_{1,1} \in \Gamma(A^\vee \otimes A^\perp \otimes \operatorname{End}(L/A))$, which is a Chevalley–Eilenberg 1-cocycle for the Lie algebroid A with values in the A-module $A^\perp \otimes \operatorname{End}(L/A)$. Its cohomology class $\alpha_{L/A} \in H^1_{\mathrm{CE}}(A, A^\perp \otimes \operatorname{End}(L/A))$ does not depend on the choice of L-connection ∇ and is called Atiyah class of the Lie pair (L, A).

We can assign a Todd cocycle — defined in terms of the Atiyah cocycle — with each Lie pair (L, A) in the exact same way the Todd cocycle of a complex manifold is derived from its Atiyah cocycle. The Todd cocycle of a Lie pair (L, A) is the Chevalley-Eilenberg cocycle

$$\operatorname{td}^{\nabla}_{L/A} = \det\left(\frac{R^{\nabla}_{1,1}}{1 - e^{-R^{\nabla}_{1,1}}}\right) \in \bigoplus_{k=0} \Gamma(\Lambda^k A^\vee \otimes \Lambda^k A^\perp). \tag{4.1}$$

Its cohomology class $\operatorname{Td}_{L/A} \in \bigoplus_{k=0} H^k_{\mathrm{CE}}(A, \Lambda^k A^\perp)$ is the Todd class of the Lie pair (L, A). See Section 2.2.2 for details.

The main goal of this paper is to establish the following formality theorem for Lie pairs: *There exists an L_∞ quasi isomorphism from* $\operatorname{tot}\left(\Gamma(\Lambda^\bullet A^\vee) \otimes_R \mathcal{T}^\bullet_{\mathrm{poly}}\right)$ *to* $\operatorname{tot}\left(\Gamma(\Lambda^\bullet A^\vee) \otimes_R \mathcal{D}^\bullet_{\mathrm{poly}}\right)$ *whose first Taylor coefficient is equal to* $\operatorname{hkr} \circ (\operatorname{td}^{\nabla}_{L/A})^{\frac{1}{2}}$*, with* $(\operatorname{td}^{\nabla}_{L/A})^{\frac{1}{2}} \in \bigoplus_{k=0} \Gamma(\Lambda^k A^\vee \otimes \Lambda^k A^\perp)$ *acting on* $\operatorname{tot}\left(\Gamma(\Lambda^\bullet A^\vee) \otimes_R \mathcal{T}^\bullet_{\mathrm{poly}}\right)$ *by contractions.* See Theorem 4.1.1.

As an immediate consequence, we obtain the following Kontsevich–Duflo type theorem for Lie pairs: *Given a Lie pair (L, A), the map* $\operatorname{hkr} \circ \operatorname{Td}^{\frac{1}{2}}_{L/A}$: $\mathbb{H}^\bullet_{\mathrm{CE}}(A, \mathcal{T}^\bullet_{\mathrm{poly}}) \to \mathbb{H}^\bullet_{\mathrm{CE}}(A, \mathcal{D}^\bullet_{\mathrm{poly}})$*, is an isomorphism of Gerstenhaber algebras.* See Theorem 4.1.2.

Our result is very much inspired by Kontsevich's seminal work [30], in which it is highlighted that the classical Duflo theorem is one of many consequences of the formality construction. For every Lie algebra $\mathfrak{g}$, the symmetrization map $\operatorname{pbw} : S(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$ is an isomorphism of $\mathfrak{g}$-modules called PoincarÃľ–Birkhoff–Witt isomorphism. The induced isomorphism $\operatorname{pbw} : S(\mathfrak{g})^{\mathfrak{g}} \to \mathcal{U}(\mathfrak{g})^{\mathfrak{g}}$ between subspaces of $\mathfrak{g}$-invariants does not intertwine the obvious multiplications on $S(\mathfrak{g})^{\mathfrak{g}}$ and $\mathcal{U}(\mathfrak{g})^{\mathfrak{g}}$. However, it can be modified so as to become an isomorphism of associative algebras.

The Duflo element $J \in \hat{S}(\mathfrak{g}^\vee)$ of a Lie algebra $\mathfrak{g}$ is the formal polynomial on $\mathfrak{g}$ defined by $J(x) = \det\left(\frac{1-e^{-\mathrm{ad}_x}}{\mathrm{ad}_x}\right)$, for all $x \in \mathfrak{g}$. Considered as a translation-invariant formal differential operator on $\mathfrak{g}^\vee$, the square root of the Duflo element defines a transformation $J^{\frac{1}{2}} : S(\mathfrak{g}) \to S(\mathfrak{g})$. A remarkable theorem due to Duflo [20] asserts that the composition $\mathrm{pbw} \circ J^{\frac{1}{2}} : S(\mathfrak{g})^{\mathfrak{g}} \to \mathcal{U}(\mathfrak{g})^{\mathfrak{g}}$ is an isomorphism of associative algebras. Duflo's theorem generalizes a fundamental result of Harish-Chandra regarding the center of the universal enveloping algebra of a semi-simple Lie algebra. Duflo's original proof was based on deep and sophisticated techniques of representation theory including Kirillov's orbit method. As an application of his formality construction, Kontsevich proposed a new proof of Duflo's theorem by means of the induced associative algebra structure on tangent cohomology at a Maurer–Cartan element. Indeed, Kontsevich's approach [30] has led to an extension of Duflo's theorem: *For every finite dimensional Lie algebra* $\mathfrak{g}$*, the map* $\mathrm{pbw} \circ J^{\frac{1}{2}} : H^\bullet_{\mathrm{CE}}(\mathfrak{g}, S(\mathfrak{g})) \to H^\bullet_{\mathrm{CE}}(\mathfrak{g}, \mathcal{U}(\mathfrak{g}))$ *is an isomorphism of graded associative algebras.* The classical Duflo theorem is the isomorphism of the cohomologies in degree 0. A detailed proof of the above extended Duflo theorem was given by Pevzner–Torossian [54] (see also [42, 43]).

Kontsevich discovered a similar phenomenon in complex geometry. Recall that the Hochschild cohomology groups $HH^\bullet(X)$ of a complex manifold X are defined as the groups $\mathrm{Ext}^\bullet_{\mathcal{O}_{X\times X}}(\mathcal{O}_\Delta, \mathcal{O}_\Delta)$ [12]. Gerstenhaber–Shack [26] derived an isomorphism of cohomology groups $\mathrm{hkr} : \mathbb{H}^\bullet(X, \Lambda^\bullet T_X) \to HH^\bullet(X)$ from the classical Hochschild–Kostant–Rosenberg map. This isomorphism fails to intertwine the multiplications in both cohomologies but can be tweaked so as to produce an isomorphism of associative algebras. More precisely, Kontsevich [30] obtained the following theorem: *The composition* $\mathrm{hkr} \circ (\mathrm{Td}_X)^{\frac{1}{2}} : \mathbb{H}^\bullet(X, \Lambda^\bullet T_X) \to HH^\bullet(X)$, *were the symbol* Td_X *denotes the Todd class of the complex manifold* X*, is an isomorphism of associative algebras.* The multiplications on $\mathbb{H}^\bullet(X, \Lambda^\bullet T_X)$ and $HH^\bullet(X)$ are respectively the wedge product and the Yoneda product. Calaque–Van den Bergh [11] wrote a detailed proof of Kontsevich's theorem, and showed that the map $\mathrm{hkr} \circ (\mathrm{Td}_X)^{\frac{1}{2}}$ actually preserves the Gerstenhaber algebra structures on both cohomologies. We refer the reader to [19, 62] for a related result.

Hence Kontsevich's formality revealed a hidden connection between two areas of mathematics: complex geometry and Lie theory. The mysterious and surprising similarity between the Todd class of a complex manifold and the Duflo element of a Lie algebra — two seemingly unrelated objects — led to further exciting developments. In 1998, Shoikhet [56] announced the so called Kontsevich–Shoikhet theorem (Theorem 4.1.17), which explains the deep ties between Lie theory and complex geometry and provides a unified framework for their study. The theorem states that a formula of Duflo type holds for every dg manifold $(\mathbb{R}^{m|n}, Q)$. See [10] for a detailed proof.

Our approach is inspired by Dolgushev's proof of Kontsevich's global formality theorem for smooth manifolds [18] and relies heavily on the Fedosov dg Lie algebroid constructed by two of the authors in [59] (and independently by Batakidis–Voglaire in the special case of matched pairs [4]). Roughly speaking, a Fedosov dg Lie algebroid associated with a Lie pair (L, A) is a dg Lie algebroid whose associated spaces of polyvector fields and polydifferential operators are homotopy equivalent to $\operatorname{tot}\left(\Gamma(\Lambda^\bullet A^\vee) \otimes_R \mathcal{T}^\bullet_{\mathrm{poly}}\right)$ and $\operatorname{tot}\left(\Gamma(\Lambda^\bullet A^\vee) \otimes_R \mathcal{D}^\bullet_{\mathrm{poly}}\right)$, respectively (in a style reminiscent of Dolgushev's Fedosov resolutions [18]). More precisely, having chosen some additional geometric data, one can endow the graded manifold $\mathcal{M} = L[1] \oplus L/A$ with a structure of dg manifold $(\mathcal{M}, Q)$ quasi-isomorphic to $(A[1], d_A)$. We call any such a dg manifold $(\mathcal{M}, Q)$ a 'Fedosov dg manifold associated with the Lie pair (L, A).' The Fedosov dg Lie algebroid $\mathcal{F}$ is a certain dg Lie subalgebroid of the tangent dg Lie algebroid $T_{\mathcal{M}}$ of the Fedosov dg manifold $(\mathcal{M}, Q)$. In other words, $\mathcal{F}$ is the dg Lie algebroid encoding a certain dg foliation of $(\mathcal{M}, Q)$. Since a Lie algebroid can be thought of as an extension of the tangent bundle of a manifold, the notions of polyvector fields and polydifferential operators admit extensions to the context of a Lie algebroid and these each carry a natural dgla structure [66, 67]. Likewise, the notions of polyvector fields and polydifferential operators can be extended in an appropriate sense to the context of a dg Lie algebroid, yielding two dglas $\mathcal{T}^\bullet_{\mathrm{poly}}$ and $\mathcal{D}^\bullet_{\mathrm{poly}}$ whose corresponding cohomology groups are naturally Gerstenhaber algebras. The "polyvector fields" and "polydifferential operators" associated to the Fedosov dg Lie algebroid $\mathcal{F}$ can be viewed geometrically as polyvector fields and polydifferential operators tangent to the dg foliation on the Fedosov dg manifold $(\mathcal{M}, Q)$. In fact, one can identify the "polyvector fields" and "polydifferential operators" on $\mathcal{F}$ to $(\Gamma(\Lambda^\bullet L^\vee) \otimes_R \mathfrak{T}^\bullet_{\mathrm{poly}},\ [Q, \cdot],\ [-,-])$ and $(\Gamma(\Lambda^\bullet L^\vee) \otimes_R \mathscr{D}^\bullet_{\mathrm{poly}},\ [\![Q, \cdot]\!] + d_{\mathscr{H}},\ [\![-,-]\!])$, respectively, where $\mathfrak{T}^\bullet_{\mathrm{poly}}$ denotes the formal polyvector fields and $\mathscr{D}^\bullet_{\mathrm{poly}}$ the formal polydifferential operators tangent to the fibers of the vector bundle $L/A \to M$.

By applying Kontsevich formality theorem fiberwisely to $\mathcal{F} \to \mathcal{M}$, we prove that there exists an L_∞ quasi-isomorphism

$$\Phi : (\Gamma(\Lambda^\bullet L^\vee) \otimes_R \mathfrak{T}^\bullet_{\mathrm{poly}},\ [Q, \cdot],\ [-,-]) \\ \to (\Gamma(\Lambda^\bullet L^\vee) \otimes_R \mathscr{D}^\bullet_{\mathrm{poly}},\ [\![Q, \cdot]\!] + d_{\mathscr{H}},\ [\![-,-]\!]).$$

This L_∞ quasi-isomorphism Φ is in fact a sequence of maps $(\Phi_n)_{n=1}^\infty$ — its 'Taylor coefficients' — the first amongst which is a quasi-isomorphism of cochain complexes

$$\Phi_1 : \Gamma(\Lambda^\bullet L^\vee) \otimes_R \mathfrak{T}^\bullet_{\mathrm{poly}} \to \Gamma(\Lambda^\bullet L^\vee) \otimes_R \mathscr{D}^\bullet_{\mathrm{poly}}.$$

The latter induces an isomorphism of Lie algebras on the level of cohomologies. A standard argument of Kontsevich, Manchon–Torossian, and Mochizuki [30,42,43,49] suffices to prove that Φ_1 intertwines the associative multiplications carried by the cohomologies as well. Hence, in cohomology, Φ_1 really is an isomorphism of Gerstenhaber algebras.

Next, we apply the Kontsevich–Shoikhet theorem (Theorem 4.1.17) in order to prove that Φ_1 is essentially the fiberwise HKR map enhanced by the Todd class of the Fedosov dg Lie algebroid. More precisely, we prove that $\Phi_1 : \Gamma(\Lambda^\bullet L^\vee) \otimes_R \mathfrak{T}^\bullet_{\mathrm{poly}} \to \Gamma(\Lambda^\bullet L^\vee) \otimes_R \mathscr{D}^\bullet_{\mathrm{poly}}$ is the composition

$$\Phi_1 = \mathrm{hkr} \circ (\widetilde{\mathrm{td}}^{\mathrm{can}}_{\mathcal{F}})^{\frac{1}{2}}$$

of the natural extension hkr of the fiberwise Hochschild–Kostant–Rosenberg map $\mathfrak{T}^\bullet_{\mathrm{poly}} \to \mathscr{D}^\bullet_{\mathrm{poly}}$ and the action on $\Gamma(\Lambda^\bullet L^\vee) \otimes_R \mathfrak{T}^\bullet_{\mathrm{poly}}$ (by contraction) of the Todd cocycle $\widetilde{\mathrm{td}}^{\mathrm{can}}_{\mathcal{F}}$ of the Fedosov dg Lie algebroid $\mathcal{F}$ associated with the canonical connection defined by Equation (2.8).

Then our main theorem essentially follows from a careful combination of the above results together with various L_∞ quasi-isomorphisms. Our approach was largely influenced and indeed relies on several standard techniques pioneered by Kontsevich in his seminal paper [30] and expounded at greater length in subsequent literature [10,11,56]. However, we emphasize the role of Fedosov dg Lie algebroids as it sheds new light on and indeed provides transparent understanding of Kontsevich's global formality theorem and, in particular, the Kontsevich–Duflo phenomenon.

In Section 4.2, we apply our results to a number of interesting classes of examples of Lie pairs, namely those arising from complex manifolds, from regular foliations, and from $\mathfrak{g}$-manifolds. In each case, we obtain a formality theorem and a Kontsevich–Duflo type theorem. In the case of lie pairs stemming from complex manifolds, we recover the Kontsevich–Duflo theorem of complex geometry [11, 30]. As far as we know, the formality and Kontsevich–Duflo type theorems obtained for geometric situations such as foliations and $\mathfrak{g}$-manifolds are new. In the future, we plan to investigate the implications of the formality theorem in deformation quantization, in particular for the special instances of Lie pairs listed above.

Terminology and notations

Natural numbers

We use the symbol $\mathbb{N}$ to denote the set of positive integers and the symbol $\mathbb{N}_0$ for the set of nonnegative integers.

Field $\Bbbk$ and ring R

We use the symbol $\Bbbk$ to denote the field of either real or complex numbers. The symbol R always denotes the algebra of smooth functions on M with values in $\Bbbk$.

Tensor products

For any two R-modules P and Q, we write $P \otimes_R Q$ to denote the tensor product of P and Q as R-modules and $P \otimes Q$ to denote the tensor product of P and Q regarded as $\Bbbk$-modules.

Completed symmetric algebra

Given a module $\mathcal{M}$ over a ring, the symbol $\hat{S}(\mathcal{M})$ denotes the $\mathfrak{m}$-adic completion of the symmetric algebra $S(\mathcal{M})$, where $\mathfrak{m}$ is the ideal of $S(\mathcal{M})$ generated by $\mathcal{M}$.

Duality pairing

For every vector bundle $E \to M$, we define a duality pairing

$$\Gamma(\hat{S}(E^\vee)) \times \Gamma(S(E)) \to R$$

by

$$\langle \nu_1 \otimes \cdots \otimes \nu_p | v_1 \otimes \cdots \otimes v_q \rangle = \begin{cases} \sum_{\sigma \in S_p} \prod_{k=1}^{p} \left\langle \nu_k \middle| v_{\sigma(k)} \right\rangle & \text{if } p = q, \\ 0 & \text{otherwise.} \end{cases}$$

Multi-indices

Let $E \to M$ be a smooth vector bundle of finite rank r, let $(\partial_i)_{i \in \{1,\dots,r\}}$ be a local frame of E and let $(\chi_j)_{j \in \{1,\dots,r\}}$ be the dual local frame of $E^\vee$. Thus, we have $\langle \chi_i | \partial_j \rangle = \delta_{i,j}$. Given a multi-index $I = (I_1, I_2, \cdots, I_r) \in \mathbb{N}_0^r$, we adopt the following multi-index notations:

$$I! = I_1! \cdot I_2! \cdots I_r!$$

$$|I| = I_1 + I_2 + \cdots + I_r$$

$$\partial^I = \underbrace{\partial_1 \odot \cdots \odot \partial_1}_{I_1 \text{ factors}} \odot \underbrace{\partial_2 \odot \cdots \odot \partial_2}_{I_2 \text{ factors}} \odot \cdots \odot \underbrace{\partial_r \odot \cdots \odot \partial_r}_{I_r \text{ factors}}$$

$$\chi^I = \underbrace{\chi_1 \odot \cdots \odot \chi_1}_{I_1 \text{ factors}} \odot \underbrace{\chi_2 \odot \cdots \odot \chi_2}_{I_2 \text{ factors}} \odot \cdots \odot \underbrace{\chi_r \odot \cdots \odot \chi_r}_{I_r \text{ factors}}$$

We use the symbol e_k to denote the multi-index all of whose components are equal to 0 except for the k-th which is equal to 1. Thus $\chi^{e_k} = \chi_k$.

Shuffles

A (p,q)-shuffle is a permutation σ of the set $\{1, 2, \cdots, p+q\}$ such that $\sigma(1) < \sigma(2) < \cdots < \sigma(p)$ and $\sigma(p+1) < \sigma(p+2) < \cdots < \sigma(p+q)$. The symbol $\mathfrak{S}_p^q$ denotes the set of (p,q)-shuffles.

Graduation shift

Given a graded vector space $V = \bigoplus_{k \in \mathbb{Z}} V^{(k)}$, $V[i]$ denotes the graded vector space obtained by shifting the grading on V according to the rule $(V[i])^{(k)} = V^{(i+k)}$. Accordingly, if $E = \bigoplus_{k \in \mathbb{Z}} E^{(k)}$ is a graded vector bundle over M, $E[i]$ denotes the graded vector bundle obtained by shifting the degree in the fibers of E according to the above rule.

Koszul sign

The Koszul sign $\mathrm{sgn}(\sigma; v_1, \cdots, v_n)$ of a permutation σ of homogeneous vectors $v_1, v_2, \ldots, v_n$ of a $\mathbb{Z}$-graded vector space $V = \bigoplus_{n\in\mathbb{Z}} V_n$ is determined by the equality

$$v_{\sigma(1)} \odot v_{\sigma(2)} \odot \cdots \odot v_{\sigma(n)} = \mathrm{sgn}(\sigma; v_1, \cdots, v_n) \cdot v_1 \odot v_2 \odot \cdots \odot v_n$$

in the graded commutative algebra $S(V)$.

L_∞ algebra

An $L_\infty[1]$ algebra [15,32,44,58] is a $\mathbb{Z}$-graded vector space $V = \bigoplus_{n\in\mathbb{Z}} V_n$ endowed with a sequence $(Q_k)_{k=1}^{\infty}$ of linear maps $Q_k : S^k(V) \to V[1]$ satisfying the generalized Jacobi identities

$$\sum_{p+q=n} \sum_{\sigma\in\mathfrak{S}_p^q} \mathrm{sgn}(\sigma; v_1, \cdots, v_n)\ Q_{1+q}\Big(Q_p(v_{\sigma(1)}, \cdots, v_{\sigma(p)}), v_{\sigma(p+1)}, \cdots, v_n\Big) = 0$$

for each $n \in \mathbb{N}$ and for all homogeneous vectors $v_1, v_2, \ldots, v_n \in V$. In particular, the first map Q_1 is a coboundary operator on V.
Alternatively, an $L_\infty[1]$ algebra is a $\mathbb{Z}$-graded vector space $V = \bigoplus_{n\in\mathbb{Z}} V_n$ together with a ($\mathbb{k}$-linear) coderivation Q of degree $+1$ of the symmetric coalgebra $S(V)$ satisfying the properties $Q \circ Q = 0$ and $Q(1) = 0$.

This coderivation Q determines the sequence of multibrackets $(Q_k)_{k=1}^{\infty}$ through commutative diagrams

$$\begin{array}{ccc} S(V) & \xrightarrow{Q} & S(V)[1] \\ \uparrow & & \downarrow \\ S^k(V) & \xrightarrow[Q_k]{} & S^1(V)[1]. \end{array}$$

Dualizing the coalgebra $S(V)$, we obtain the algebra $\hat{S}(V^\vee)$, which can be thought of as the algebra of functions on the graded manifold $V[-1]$. The derivation of the algebra $\hat{S}(V^\vee)$ dual to the coderivation Q of $S(V)$ can be regard as a homological vector field on $V[-1]$.
A $\mathbb{Z}$-graded vector space V is an L_∞ algebra if and only if $V[1]$ is an $L_\infty[1]$ algebra.

L_∞ morphism

Let V and W be two $L_\infty[1]$ algebras. An L_∞ morphism from V to W is a morphism $\Phi : S(V) \to S(W)$ of coalgebras over $\Bbbk$ intertwining the coderivations of $S(V)$ and $S(W)$ and satisfying $\Phi(1) = 0$. Every L_∞ morphism Φ is entirely determined by its sequence of 'Taylor coefficients' $(\Phi_k)_{k=1}$, the k-th of which is the composition

$$S^k(V) \hookrightarrow S(V) \xrightarrow{\Phi} S(W) \twoheadrightarrow S^1(W), \qquad \Phi_k : S^k(V) \to S^1(W)$$

and its first Taylor coefficient Φ_1 is a chain map from the cochain complex (V, Q_1) to the chain complex (W, Q'_1). An L_∞ quasi-isomorphism is an L_∞ morphism whose first Taylor coefficient happens to be a quasi-isomorphism of cochain complexes.

The definitions of L_∞ algebras and L_∞ morphisms can also be formulated in terms of exterior tensor algebras, which are isomorphic to the symmetric tensor algebras via the *décalage* map. Later in the paper, we adopt the exterior algebra point of view.

Lie algebroid

In this paper 'Lie algebroid' always means 'Lie $\Bbbk$-algebroid' unless specified otherwise.

Contraction

Let $(C, \eth)$ and (K, d) be two cochain complexes. A contraction of (K, d) onto $(C, \eth)$ consists of a pair of chain maps $\tau : C \to K$ and $\sigma : K \to C$ together with a chain homotopy operator $h : K \to K[-1]$ satisfying

$$\sigma\tau = \mathrm{id}_C; \qquad \mathrm{id}_K - \tau\sigma = dh + hd;$$
$$\sigma h = 0; \qquad h^2 = 0; \qquad \text{and} \qquad h\tau = 0.$$

We symbolize such a contraction by a diagram

$$(C, \eth) \underset{\sigma}{\overset{\tau}{\rightleftarrows}} (K, d) \circlearrowleft h\,.$$

4.1 Formality theorem for Lie pairs

4.1.1 Statements of main theorems

We are ready to state the main theorems of the paper. Let (L, A) be a Lie pair. According to Corollary 2.3.14 and Corollary 2.3.19, $\operatorname{tot}\left(\Gamma(\Lambda^\bullet A^\vee) \otimes_R \mathcal{T}^\bullet_{\mathrm{poly}}\right)$ and $\operatorname{tot}\left(\Gamma(\Lambda^\bullet A^\vee) \otimes_R \mathcal{D}^\bullet_{\mathrm{poly}}\right)$ are L_∞ algebras with d_A^{Bott} and $d_A^{\mathcal{U}} + d_{\mathscr{H}}$ as their respective unary brackets. Moreover, their cohomologies $\mathbb{H}^\bullet_{\mathrm{CE}}(A, \mathcal{T}^\bullet_{\mathrm{poly}})$ and $\mathbb{H}^\bullet_{\mathrm{CE}}(A, \mathcal{D}^\bullet_{\mathrm{poly}})$ carry canonical Gerstenhaber algebra structures. The main result of the paper is the following

Theorem 4.1.1 (Formality theorem for Lie pairs)**.** *Let (L, A) be a Lie pair. Assume the associated graded vector spaces* $\operatorname{tot}\left(\Gamma(\Lambda^\bullet A^\vee) \otimes_R \mathcal{T}^\bullet_{\mathrm{poly}}\right)$ *and* $\operatorname{tot}\left(\Gamma(\Lambda^\bullet A^\vee) \otimes_R \mathcal{D}^\bullet_{\mathrm{poly}}\right)$ *are endowed with their inherited L_∞ algebras — see Corollaries 2.3.14 and 2.3.19. Then, there exists an L_∞ quasi-isomorphism*

$$\mathcal{I} : \operatorname{tot}\left(\Gamma(\Lambda^\bullet A^\vee) \otimes_R \mathcal{T}^\bullet_{\mathrm{poly}}\right) \to \operatorname{tot}\left(\Gamma(\Lambda^\bullet A^\vee) \otimes_R \mathcal{D}^\bullet_{\mathrm{poly}}\right)$$

with first Taylor coefficient $\mathcal{I}_1 : \operatorname{tot}\left(\Gamma(\Lambda^\bullet A^\vee) \otimes_R \mathcal{T}^\bullet_{\mathrm{poly}}\right) \to \operatorname{tot}\left(\Gamma(\Lambda^\bullet A^\vee) \otimes_R \mathcal{D}^\bullet_{\mathrm{poly}}\right)$ *satisfying the following two properties:*

1. *$\mathcal{I}_1$ preserves the associative algebra structures (wedge and cup product, respectively) up to homotopy;*
2. *$\mathcal{I}_1 = \mathrm{hkr} \circ (\mathrm{td}^\nabla_{L/A})^{\frac{1}{2}}$, where $(\mathrm{td}^\nabla_{L/A})^{\frac{1}{2}} \in \bigoplus_{k=0} \Gamma(\Lambda^k A^\vee \otimes \Lambda^k A^\perp)$ acts on* $\operatorname{tot}\left(\Gamma(\Lambda^\bullet A^\vee) \otimes_R \mathcal{T}^\bullet_{\mathrm{poly}}\right)$ *by contraction.*

As an immediate consequence, we have the following

Theorem 4.1.2 (Kontsevich-Duflo type theorem for Lie pairs)**.** *Given a Lie pair (L, A), the map*

$$\mathrm{hkr} \circ \mathrm{Td}^{\frac{1}{2}}_{L/A} : \mathbb{H}^\bullet_{\mathrm{CE}}(A, \mathcal{T}^\bullet_{\mathrm{poly}}) \to \mathbb{H}^\bullet_{\mathrm{CE}}(A, \mathcal{D}^\bullet_{\mathrm{poly}})$$

is an isomorphism of Gerstenhaber algebras — the square root of the Todd class

$$\mathrm{Td}^{\frac{1}{2}}_{L/A} \in \bigoplus_{k=0} H^k_{\mathrm{CE}}(A, \Lambda^k A^\perp)$$

acts on $\mathbb{H}^\bullet_{\mathrm{CE}}(A, \mathcal{T}^\bullet_{\mathrm{poly}})$ *by contraction.*

To prove Theorem 4.1.1, our method is to use Fedosov dg Lie algebroids $\mathcal{F} \to \mathcal{M}$ [59] associated to Lie pairs (see the Appendix for details). This is a dg Lie algebroid over the dg manifold $(\mathcal{M}, Q)$, where $\mathcal{M} = L[1] \oplus L/A$ and Q is a homological vector field on $\mathcal{M}$, called the Fedosov differential. More precisely, a Fedosov dg Lie algebroid $\mathcal{F}$ is a dg Lie subalgebroid of the tangent dg Lie algebroid $T_{\mathcal{M}}$ of the Fedosov dg manifold $(\mathcal{M}, Q)$. In other words, $\mathcal{F}$ is the dg Lie algebroid encoding a dg foliation of $(\mathcal{M}, Q)$.

Since a Lie algebroid can be thought of as an extension of the tangent bundle of a manifold, the notions of polyvector fields and polydifferential operators admit extensions to the context of a Lie algebroid and these each carry a natural dgla structure [66, 67]. Likewise, the notions of polyvector fields and polydifferential operators can be extended in a appropriate sense to the context of a dg Lie algebroid. The "polyvector fields" and "polydifferential operators" associated to a Fedosov dg Lie algebroid $\mathcal{F}$ can be viewed geometrically as polyvector fields and polydifferential operators tangent to the dg foliation on the Fedosov dg manifold $(\mathcal{M}, Q)$. In fact, one can identify the dglas of "polyvector fields" and "polydifferential operators" on $\mathcal{F}$ to $(\Gamma(\Lambda^\bullet L^\vee) \otimes_R \mathfrak{T}^\bullet_{\mathrm{poly}}, [Q, \cdot])$ and $(\Gamma(\Lambda^\bullet L^\vee) \otimes_R \mathscr{D}^\bullet_{\mathrm{poly}}, [\![Q, \cdot]\!])$, respectively, where $\mathfrak{T}^\bullet_{\mathrm{poly}}$ denotes the formal polyvector fields and $\mathscr{D}^\bullet_{\mathrm{poly}}$ the formal polydifferential operators tangent to the fibers of the vector bundle $L/A \to M$.

In fact, according to Corollary 2.3.14 and Corollary 2.3.19, the L_∞ structures on $\mathrm{tot}\left(\Gamma(\Lambda^\bullet A^\vee) \otimes_R \mathcal{T}^\bullet_{\mathrm{poly}}\right)$ and $\mathrm{tot}\left(\Gamma(\Lambda^\bullet A^\vee) \otimes_R \mathcal{D}^\bullet_{\mathrm{poly}}\right)$ are indeed obtained by the homotopy transfer from the dgla structures on $(\Gamma(\Lambda^\bullet L^\vee) \otimes_R \mathfrak{T}^\bullet_{\mathrm{poly}}, [Q, \cdot])$ and $(\Gamma(\Lambda^\bullet L^\vee) \otimes_R \mathscr{D}^\bullet_{\mathrm{poly}}, [\![Q, \cdot]\!])$, respectively (see [59]). Therefore, as a key step, we apply Kontsevich formality theorem to the Fedosov dg Lie algebroid $\mathcal{F}$ and establish the following

Theorem 4.1.3. *There exists an* L_∞ *quasi-isomorphism*

$$\Psi : (\Gamma(\Lambda^\bullet L^\vee) \otimes_R \mathfrak{T}^\bullet_{\mathrm{poly}},\ [Q, \cdot],\ [-,-]) \\ \to (\Gamma(\Lambda^\bullet L^\vee) \otimes_R \mathscr{D}^\bullet_{\mathrm{poly}},\ [\![Q, \cdot]\!] + d_{\mathscr{H}},\ [\![-,-]\!])$$

from the dgla of "polyvector fields" on $\mathcal{F}$ *to the dgla of "polydifferential operators" on* $\mathcal{F}$ *with first Taylor coefficient* $\Psi_1 : \Gamma(\Lambda^\bullet L^\vee) \otimes_R \mathfrak{T}^\bullet_{\mathrm{poly}} \to \Gamma(\Lambda^\bullet L^\vee) \otimes_R \mathscr{D}^\bullet_{\mathrm{poly}}$ *satisfying the following two properties:*

1. Ψ_1 *preserves the associative algebra structures (wedge and cup product, respectively) up to homotopy;*

2. $\Psi_1 = \mathrm{hkr} \circ (\mathrm{td}_{\mathcal{F}}^{\mathrm{can}})^{\frac{1}{2}}$*, where* hkr *denotes the natural extension of the fiberwise Hochschild–Kostant–Rosenberg map* $\mathfrak{T}_{\mathrm{poly}}^{\bullet} \to \mathscr{D}_{\mathrm{poly}}^{\bullet}$ *and* $(\mathrm{td}_{\mathcal{F}}^{\mathrm{can}})^{\frac{1}{2}}$ *the action of the square root of the canonical Todd cocycle* $\mathrm{td}_{\mathcal{F}}^{\mathrm{can}}$ *of the Fedosov dg Lie algebroid* $\mathcal{F}$ *on* $\Gamma(\Lambda^{\bullet} L^{\vee}) \otimes_R \mathfrak{T}_{\mathrm{poly}}^{\bullet}$ *by contraction.*

4.1.2 Kontsevich formality morphism for Lie pairs

4.1.2.1 Tangent L_∞ algebras

Let $\mathfrak{g}$ and $\mathfrak{g}'$ be two L_∞ algebras and let Q and Q' denote the corresponding homological vector fields on the associated dg manifolds $\mathfrak{g}[1]$ and $\mathfrak{g}'[1]$. An L_∞ morphism $\mathscr{U} : \mathfrak{g} \to \mathfrak{g}'$ is, by definition, a morphism of dg manifolds $\mathfrak{g}[1] \to \mathfrak{g}'[1]$, which means that the homomorphism of algebras $\mathscr{U}^* : C^\infty(\mathfrak{g}'[1]) \to C^\infty(\mathfrak{g}[1])$ intertwines the derivations: $\mathscr{U}^* \circ Q' = Q \circ \mathscr{U}^*$. Such an L_∞ morphism $\mathscr{U}$ is entirely determined by its so-called 'Taylor coefficients,' which are a sequence $(\mathscr{U}_n)_{n=1,2,\cdots}$ of morphisms of graded vector spaces

$$\mathscr{U}_n : \Lambda^n \mathfrak{g} \to \mathfrak{g}'[1-n].$$

A Maurer–Cartan (MC) element of an L_∞ algebra $(\mathfrak{g}, Q)$ is an element $\omega \in \mathfrak{g}_1$ (of degree 1) satisfying

$$\sum_{j=1}^{\infty} \frac{1}{j!} Q_j(\omega^j) = 0,$$

where $\omega^j = \omega \wedge \cdots \wedge \omega \in \Lambda^j \mathfrak{g}$. In particular, the MC elements ω of a dgla satisfy the classical Maurer–Cartan equation

$$d\omega + \frac{1}{2}[\omega, \omega] = 0.$$

Given a MC element ω of an L_∞ structure Q on a graded vector space $\mathfrak{g}$, there is a new L_∞ algebra structure Q_ω on $\mathfrak{g}$ called tangent L_∞ algebra [30]: the Taylor coefficients of Q_ω satisfy

$$(Q_\omega)_n(\gamma) = \sum_{j=0}^{\infty} \frac{1}{j!} Q_{n+j}(\omega^j \wedge \gamma), \quad \forall \gamma \in \Lambda^n \mathfrak{g}.$$

In general, the convergence of the summations above is an issue that has to be addressed. However, if $\mathfrak{g}$ is a dgla with differential d and Lie bracket $[-,-]$, the sums are finite and the tangent L_∞ algebra is again a dgla with the same bracket but with the modified differential $d_\omega = d + [\omega, -]$.

We use the symbol $T_\omega\mathfrak{g}$ to distinguish the tangent L_∞ algebra at ω from the original L_∞ algebra $\mathfrak{g}$.

Given an L_∞ morphism of dglas $\mathscr{U} : \mathfrak{g} \to \mathfrak{g}'$ and a MC element ω of $\mathfrak{g}$, consider the element $\mathscr{U}(\omega)$ of $\mathfrak{g}'$ defined by

$$\mathscr{U}(\omega) = \sum_{j=1}^{\infty} \frac{1}{j!} \mathscr{U}_j(\omega^j) \tag{4.2}$$

assuming the summation converges. Then $\mathscr{U}(\omega)$ is a Maurer–Cartan element of $\mathfrak{g}'$ and therefore both $T_\omega\mathfrak{g}$ and $T_{\mathscr{U}(\omega)}\mathfrak{g}'$ are dglas. There is a tangent L_∞ morphism

$$\mathscr{U}_\omega : T_\omega\mathfrak{g} \to T_{\mathscr{U}(\omega)}\mathfrak{g}'$$

defined through the relations

$$(\mathscr{U}_\omega)_n(\gamma) = \sum_{j=0}^{\infty} \frac{1}{j!} \mathscr{U}_{n+j}(\omega^j \wedge \gamma), \quad \forall \gamma \in \Lambda^n \mathfrak{g}. \tag{4.3}$$

Provided the summations in the r.h.s. of Equation (4.3) converge, $\mathscr{U}_\omega$ is indeed a well defined L_∞ morphism. One may read [30, 68] for more details.

4.1.2.2 Kontsevich formality morphism for $\mathbb{k}^d$

In this section, we briefly recall the definition of Kontsevich's formality morphism for $\mathbb{k}^d$ (where $\mathbb{k}$ is either $\mathbb{R}$ or $\mathbb{C}$), which we need later on. For more details, the reader may want to refer to Kontsevich's original paper [30].

Kontsevich's formality morphism is an L_∞ quasi-isomorphism

$$\mathscr{U} : \mathcal{T}^\bullet_{\text{poly}}(\mathbb{k}^d) \to \mathcal{D}^\bullet_{\text{poly}}(\mathbb{k}^d)$$

between the two dglas $\mathcal{T}^\bullet_{\text{poly}}(\mathbb{k}^d)$ and $\mathcal{D}^\bullet_{\text{poly}}(\mathbb{k}^d)$. Its 'Taylor coefficients' are of the form

$$\mathscr{U}_n = \sum_{m \geqslant 0} \sum_{\Gamma \in \mathscr{G}_{n,m}} W_\Gamma \mathscr{U}_\Gamma, \tag{4.4}$$

where $\mathscr{G}_{n,m}$ denotes the set of admissible graphs of type (n, m), W_Γ is a number called Kontsevich weight of the graph Γ, and $\mathscr{U}_\Gamma$ is a map which assembles n polyvector fields into a single polydifferential operator in a way determined by the graph Γ.

We will now describe $\mathscr{G}_{n,m}$, $\mathscr{U}_\Gamma$, and W_Γ successively.

Admissible graphs

A directed graph Γ is a pair of (finite) sets V_Γ and E_Γ together with two maps $s,\ t : E_\Gamma \to V_\Gamma$. The elements of V_Γ are called vertices. The elements of E_Γ are called edges. Each edge $e \in E_\Gamma$ starts at its source $s(e) \in V_\Gamma$ and ends at its target $t(e) \in V_\Gamma$. Given a vertex $v \in V_\Gamma$, we use the symbol $\operatorname{Out}(v)$ to denote the set $s^{-1}(v)$ of all edges starting at v and we use the symbol $\operatorname{In}(v)$ to denote the set $t^{-1}(v)$ of all edges ending at v.

An admissible graph of type (n, m) is a directed graph $\Gamma = (V_\Gamma, E_\Gamma)$ with labels on its vertices and edges satisfying the following requirements.

1. The set of vertices is partitioned into two subsets: $V_\Gamma = V_\Gamma^1 \sqcup V_\Gamma^2$. The elements of V_Γ^1 are called vertices of the first type or aerial vertices. The elements of V_Γ^2 are called vertices of the second type or terrestrial vertices.

2. For all $e \in E_\Gamma$, $s(e) \in V_\Gamma^1$.

3. For all $e \in E_\Gamma$, $s(e) \neq t(e)$.

4. No two edges have the same source and the same target.

5. The aerial vertices are labelled by the symbols $1, 2, 3, \ldots, n$ while the terrestrial vertices are labelled by the symbols $\bar{1}, \bar{2}, \bar{3}, \ldots, \bar{m}$.

6. For every vertex $k \in V_\Gamma^1$ of the first type, the elements of $\operatorname{Out}(k)$ are labelled by the symbols $e_k^1, e_k^2, e_k^3, \ldots$.

Assembling a polydifferential operator from polyvector fields according to an admissible graph.

Fix an admissible graph $\Gamma \in \mathscr{G}_{n,m}$. Each choice of a vertex $v \in V_\Gamma$ and a map $I : E_\Gamma \to \{1, \cdots, d\}$ determines a constant differential operator

$$D_I^v := \prod_{e \in \mathrm{In}(v)} \frac{\partial}{\partial x_{I(e)}}$$

on $\mathbb{k}^d$. Furthermore, each choice of an aerial vertex $k \in V_\Gamma^1$ and a map $I : E_\Gamma \to \{1, \cdots, d\}$ determines a map

$$\mathcal{T}_{\mathrm{poly}}^\bullet(\mathbb{k}^d) \ni \gamma \mapsto \gamma^{I(\mathrm{Out}(k))} \in C^\infty(\mathbb{k}^d)$$

through the relation

$$\gamma^{I(\mathrm{Out}(k))} = \langle dx_{I(e_k^1)} \otimes \cdots \otimes dx_{I(e_k^{|\mathrm{Out}(k)|})} | \ \mathrm{alt}(\gamma) \rangle.$$

The bundle map $\mathrm{alt} : \Lambda^\bullet T_{\mathbb{k}^d} \to \bigotimes^\bullet T_{\mathbb{k}^d}$ is the antisymmetrization

$$\xi_1 \wedge \cdots \wedge \xi_r \overset{\mathrm{alt}}{\longmapsto} \sum_{\sigma \in S_r} \mathrm{sgn}(\sigma) \ \xi_{\sigma(1)} \otimes \cdots \otimes \xi_{\sigma(r)}.$$

The admissible graph Γ with n aerial and m terrestrial vertices is a recipe for assembling n polyvector fields $\gamma_1, \gamma_2, \ldots, \gamma_n$ on $\mathbb{k}^d$ into an m-differential operator $\mathscr{U}_\Gamma(\gamma_1, \cdots, \gamma_n)$ on $\mathbb{k}^d$, which is defined by

$$(f_1, \cdots, f_m) \xmapsto{\mathscr{U}_\Gamma(\gamma_1, \cdots, \gamma_n)} \sum_{I: E_\Gamma \to \{1, \cdots, d\}} \left(\prod_{k=1}^n D_I^k \left(\gamma_k^{I(\mathrm{Out}(k))} \right) \right) \left(\prod_{l=1}^m D_I^{\bar{l}} \big(f_l \big) \right), \quad (4.5)$$

for all $f_1, \ldots, f_m \in C^\infty(\mathbb{k}^d)$. We note that $\gamma^{I(\mathrm{Out}(k))} = 0$ if $\gamma \in \mathcal{T}_{\mathrm{poly}}^r(\mathbb{k}^d)$ with $r + 1 \neq |\mathrm{Out}(k)|$. Therefore, $\mathscr{U}_\Gamma(\gamma_1, \cdots, \gamma_n) = 0$ if $\gamma_1, \cdots, \gamma_n$ are homogeneous elements of $\mathcal{T}_{\mathrm{poly}}^\bullet(\mathbb{k}^d)$ and $|\gamma_1| + \cdots + |\gamma_n| + n \neq |E_\Gamma|$.

Configuration spaces and their compactifications

The Kontsevich weights are obtained from integrals over compactified configuration spaces.

Let $\mathbb{H}^+ = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ denote the hyperbolic plane and let $\overline{\mathbb{H}^+}$ denote its closure in $\mathbb{C}$. The group $G_2 := \mathbb{R}^+ \ltimes \mathbb{R} = \{z \mapsto az + b \mid a, b \in \mathbb{R},\ a > 0\}$ acts on the configuration space

$$\mathrm{Conf}^+_{n,m} := \{(z_1, \cdots, z_n, q_1, \cdots, q_m) \in (\mathbb{H}^+)^n \times \mathbb{R}^m \mid z_i \neq z_j \text{ if } i \neq j\ ;\ q_1 < \cdots < q_m\}.$$

The quotient $C^+_{n,m} = \mathrm{Conf}^+_{n,m} / G_2$ is a manifold of dimension $2n + m - 2$. Fixing z_1 at $i = \sqrt{-1}$, we may identify $C^+_{n,m}$ with an open subset of $\mathbb{C}^{n-1} \times \mathbb{R}^m$ and transfer the standard orientation of the affine space to $C^+_{n,m}$.

We now proceed with the compactification of $C^+_{n,m}$.

Let Conf_n be the space of configurations of n distinct points $z_1, z_2, \ldots, z_n$ in $\mathbb{C}$. The group $G_3 := \mathbb{R}^+ \ltimes \mathbb{C}$ acts on Conf_n by dilations and translations. The quotient $C_n := \mathrm{Conf}_n / G_3$ is a manifold which we embed into $(S^1)^{n(n-1)} \times (\mathbb{RP}^2)^{n(n-1)(n-2)}$ by recording all possible angles $\arg(z_i - z_j)$ and homogeneous coordinate triples $[|z_i - z_j| : |z_j - z_k| : |z_k - z_i|]$.

Since $C^+_{n,m}$ is itself embedded into C_{2n+m} by the map

$$(z_1, \cdots, z_n, q_1, \cdots, q_m) \mapsto (z_1, \cdots, z_n, \overline{z_1}, \cdots, \overline{z_n}, q_1, \cdots, q_m),$$

we obtain an embedding

$$C^+_{n,m} \hookrightarrow C_{2n+m} \hookrightarrow (S^1)^{N_1} \times (\mathbb{RP}^2)^{N_2}$$

with $N_1 = (2n + m)(2n + m - 1)$ and $N_2 = (2n + m)(2n + m - 1)(2n + m - 2)$. The desired compactification of $C^+_{n,m}$ is the closure $\overline{C^+_{n,m}}$ of the image of the above embedding.

Kontsevich weight of an admissible graph.

Consider the hyperbolic angle function $\varphi : \overline{\mathbb{H}^+} \times \overline{\mathbb{H}^+} \to S^1$ defined by $\varphi(z, w) = \frac{1}{2\pi} \arg\left(\frac{z-w}{\bar{z}-w}\right)$.

Given an admissible graph $\Gamma \in \mathscr{G}_{n,m}$, define a function $\varphi_e : C^+_{n,m} \to S^1$ for each edge $e \in E_\Gamma$ by

$$\varphi_e(z_1, \cdots, z_n, z_{\bar{1}}, \cdots, z_{\bar{m}}) = \varphi(z_{s(e)}, z_{t(e)})$$

and a differential form κ_Γ of degree $|E_\Gamma|$ on $C^+_{n,m}$ by

$$\kappa_\Gamma = \bigwedge_{e\in E_\Gamma} d\varphi_e,$$

where $d\varphi_e$ denotes the pullback of the standard volume on S^1 through φ_e. In the exterior product, the 1-forms are multiplied according to the lexicographic order $e_1^1, e_1^2, \ldots, e_2^1, e_2^2, \ldots$ of the edges of the graph. Note that κ_Γ extends smoothly to $\overline{C^+_{n,m}}$. Integrating κ_Γ over the (oriented) compactified configuration space, we obtain the Kontsevich weight of the graph Γ:

$$W_\Gamma = \prod_{k=1}^{n} \frac{1}{|\operatorname{Out}(k)|!} \int_{\overline{C^+_{n,m}}} \kappa_\Gamma.$$

Obviously, the Kontsevich weight of a graph $\Gamma \in \mathscr{G}_{n,m}$ is zero if $|E_\Gamma| \neq \dim(\overline{C^+_{n,m}}) = 2n+m-2$.

Kontsevich formality theorem for $\Bbbk^d$

For all graphs $\Gamma \in \mathscr{G}_{n,m}$ and all homogeneous polyvector fields $\gamma_1, \ldots, \gamma_n$ on $\Bbbk^d$, we know that $\mathscr{U}_\Gamma(\gamma_1, \cdots, \gamma_n)$ is a homogeneous element of degree $m-1$ in $\mathcal{D}^\bullet_{\text{poly}}(\Bbbk^d)$ and that $W_\Gamma \mathscr{U}_\Gamma(\gamma_1, \cdots, \gamma_n) \neq 0$ only when $|\gamma_1| + \cdots + |\gamma_n| + n = |E_\Gamma| = 2n+m-2$. It follows that $\mathscr{U}_n : \bigotimes^n \mathcal{T}^\bullet_{\text{poly}}(\Bbbk^d) \to \mathcal{D}^\bullet_{\text{poly}}(\Bbbk^d)$ is a map of degree $1-n$.

Consider the case $n = 1$. There are $m!$ distinct graphs $\Gamma \in \mathscr{G}_{1,m}$ satisfying the property $|E_\Gamma| = 2 \cdot 1 + m - 2 = m$. In each such graph, each one of the m terrestrial vertices is the target of a single edge starting from the unique aerial vertex. Any two such graphs only differ by the labelling of the m edges. Moreover, all such graphs have the same weight $W_\Gamma = (m!)^{-2}$. It follows that the 'first Taylor coefficient' $\mathscr{U}_1$ of the formality map is precisely the Hochschild–Kostant–Rosenberg map:

$$\mathcal{T}^\bullet_{\text{poly}}(\Bbbk^d) \xrightarrow{\mathscr{U}_1 = \text{hkr}} \mathcal{D}^\bullet_{\text{poly}}(\Bbbk^d).$$

Theorem 4.1.4 (Kontsevich formality theorem [30])**.** *The maps $(\mathscr{U}_n)_{n=1}^\infty$ defined above are the 'Taylor coefficients' of an L_∞ quasi-isomorphism*

$$\mathscr{U} : \mathcal{T}^\bullet_{\text{poly}}(\Bbbk^d) \to \mathcal{D}^\bullet_{\text{poly}}(\Bbbk^d)$$

satisfying the following additional properties.

1. *The first Taylor coefficient of $\mathscr{U}$ is the Hochschild–Kostant–Rosenberg map* hkr.

2. *The formality morphism $\mathscr{U}$ is $GL(\Bbbk^d)$-equivariant.*

3. *For all $n \geqslant 2$ and $\xi_1, \cdots, \xi_n \in \mathcal{T}^0_{\text{poly}}(\Bbbk^d)$, we have*

$$\mathscr{U}_n(\xi_1, \cdots, \xi_n) = 0.$$

4. *Provided ξ is a linear vector field on $\Bbbk^d$ and $n \geqslant 2$, we have*

$$\mathscr{U}_n(\xi, \eta_2, \cdots, \eta_n) = 0$$

for all $\eta_2, \cdots, \eta_n \in \mathcal{T}^\bullet_{\text{poly}}(\Bbbk^d)$.

Furthermore, the formality morphism $\mathscr{U}$ can be defined for $\Bbbk^d_{\text{formal}}$ as well.

Remark 4.1.5. *With suitable sign corrections, Kontsevich's formality theorem was generalized to $\mathbb{Z}$-graded manifolds by Cattaneo–Felder [13]. Later, these sign corrections were given a simple operadic explanation.*

4.1.2.3 Fiberwise formality map

Let (L, A) be a Lie pair over a smooth manifold M. As before, set $R = C^\infty(M)$. The quotient $B = L/A$ is a vector bundle over M whose fibers are all (noncanonically) isomorphic to $\Bbbk^d$. Next we apply Kontsevich's formality theorem (essentially) fiberwisely to a Fedosov dg Lie algebroid. See Section 2.3.2 for the construction of Fedosov dg Lie algebroids.

Since the formality morphism $\mathscr{U} : \mathcal{T}^\bullet_{\text{poly}}(\Bbbk^d_{\text{formal}}) \to \mathcal{D}^\bullet_{\text{poly}}(\Bbbk^d_{\text{formal}})$ is $GL(\Bbbk^d)$-equivariant (see Theorem 4.1.4 (2)), there exist R-linear maps

$$\mathscr{U}^f_n : \Lambda^n \mathfrak{T}^\bullet_{\text{poly}} \to \mathscr{D}^\bullet_{\text{poly}}[1-n]$$

whose restrictions to each fiber of $B \to M$ coincide with the Taylor coefficients

$$\mathscr{U}_n : \Lambda^n \mathcal{T}^\bullet_{\text{poly}}(\Bbbk^d_{\text{formal}}) \to \mathcal{D}^\bullet_{\text{poly}}(\Bbbk^d_{\text{formal}})[1-n]$$

of $\mathscr{U}$. Extending them $\Gamma(\Lambda^\bullet L^\vee)$-linearly, we obtain R-linear maps

$$\mathscr{U}_n^f : \Lambda^n \operatorname{tot}\Big(\Gamma(\Lambda^\bullet L^\vee) \otimes_R \mathfrak{T}^\bullet_{\text{poly}}\Big) \to \operatorname{tot}\Big(\Gamma(\Lambda^\bullet L^\vee) \otimes_R \mathscr{D}^\bullet_{\text{poly}}\Big)[1-n].$$

Consider the difference

$$\omega = Q - d_L^\nabla \in \Gamma(L^\vee) \otimes_R \mathfrak{T}^0_{\text{poly}} \subset \mathcal{T}^0_{\text{poly}}(L[1] \oplus B)$$

of the derivations $Q = d_L^{\nabla^\sharp} = -\delta + d_L^\nabla + X^\nabla$ and d_L^∇ of the algebra

$$\Gamma(\Lambda L^\vee \otimes \hat{S}(B^\vee)) = C^\infty(L[1] \oplus B)$$

appearing in Theorem 2.3.7. We do *not* claim that ω is a MC element for any dgla structure on $\operatorname{tot}(\Gamma(\Lambda^\bullet L^\vee) \otimes_R \mathfrak{T}^\bullet_{\text{poly}})$. Nevertheless, we define a sequence $(\Phi_n)_{n=1,2,\cdots}$ of R-linear maps

$$\Phi_n : \Lambda^n\Big(\operatorname{tot}(\Gamma(\Lambda^\bullet L^\vee) \otimes_R \mathfrak{T}^\bullet_{\text{poly}})\Big) \to \operatorname{tot}(\Gamma(\Lambda^\bullet L^\vee) \otimes_R \mathscr{D}^\bullet_{\text{poly}})[1-n]$$

by

$$\Phi_n(\gamma) = \sum_{j=1}^{\infty} \mathscr{U}^f_{n+j}(\omega^j \wedge \gamma), \quad \forall \gamma \in \Lambda^n\Big(\operatorname{tot}(\Gamma(\Lambda^\bullet L^\vee) \otimes_R \mathfrak{T}^\bullet_{\text{poly}})\Big).$$

Lemma 4.1.6. *The maps $(\Phi_n)_{n=1}^\infty$ are well defined.*

Proof. Suppose $\gamma_k \in \Gamma(\Lambda^\bullet L^\vee) \otimes_R \mathfrak{T}^{r_k}_{\text{poly}}$ for $k \in \{1, 2, \cdots, n\}$. Since

$$\mathscr{U}_{n+j} : \Lambda^{n+j}\mathcal{T}^\bullet_{\text{poly}}(\mathbb{k}^d_{\text{formal}}) \to \mathcal{D}^\bullet_{\text{poly}}(\mathbb{k}^d_{\text{formal}})$$

is a map of degree $1-(n+j)$ and $\omega \in \Gamma(L^\vee) \otimes \mathfrak{T}^0_{\text{poly}}$, we have

$$\mathscr{U}^f_{n+j}(\underbrace{\omega \wedge \cdots \wedge \omega}_{j \text{ factors}} \wedge \gamma_1 \wedge \cdots \wedge \gamma_n) \in \Gamma(\Lambda^\bullet L^\vee) \otimes_R \mathscr{D}^{r_1+\cdots+r_n+1-(n+j)}_{\text{poly}}.$$

As j increases, $r_1 + \cdots + r_n + 1 - (n+j)$ eventually becomes smaller than -1 forcing $\mathscr{U}^f_{n+j}(\omega \wedge \cdots \wedge \omega \wedge \gamma_1 \wedge \cdots \wedge \gamma_n)$ to vanish. Therefore, only finitely many of the terms of $\Phi_n(\gamma_1 \wedge \cdots \wedge \gamma_n)$ are not zero. □

Although ω is not a MC element, the maps $(\Phi_n)_{n=1}^\infty$ still define an L_∞ morphism.

Proposition 4.1.7. *The maps $(\Phi_n)_{n=1}^{\infty}$ are the Taylor coefficients of an L_∞ morphism of dglas*

$$\Phi : \Big(\operatorname{tot}(\Gamma(\Lambda^\bullet L^\vee) \otimes_R \mathfrak{T}^\bullet_{\mathrm{poly}}), [Q, \cdot], [-,-] \Big) \\ \to \Big(\operatorname{tot}(\Gamma(\Lambda^\bullet L^\vee) \otimes_R \mathscr{D}^\bullet_{\mathrm{poly}}), [\![Q, \cdot]\!] + d_{\mathscr{H}}, [\![-,-]\!] \Big).$$

We will need the following well known lemma.

Lemma 4.1.8. *Let $(C, \bar{d})$ be a cdga, and $\mathscr{U} : (\mathfrak{g}, d, [-,-]) \to (\mathfrak{g}', d', [-,-]')$ be an L_∞ morphism of dglas*

1. *Then $(C \otimes \mathfrak{g}, \bar{d} \otimes \mathrm{id} + \mathrm{id} \otimes d, [-,-])$ and $(C \otimes \mathfrak{g}', \bar{d} \otimes \mathrm{id} + \mathrm{id} \otimes d', [-,-]')$ are dglas*

2. *and the C-linear extension of $\mathscr{U}$*

$$\widehat{\mathscr{U}} : (C \otimes \mathfrak{g}, \bar{d} \otimes \mathrm{id} + \mathrm{id} \otimes d, [-,-]) \to (C \otimes \mathfrak{g}', \bar{d} \otimes \mathrm{id} + \mathrm{id} \otimes d', [-,-]')$$

   is an L_∞ morphism of dglas.

Proof of Proposition 4.1.7. Choosing a local trivialization $B|_U \cong U \times \mathbb{k}^d$ of the vector bundle B over an open subset U of M yields identifications

$$\Big(\Gamma(\Lambda^\bullet L^\vee) \otimes_R \mathfrak{T}^\bullet_{\mathrm{poly}} \Big)|_U \cong \Gamma(U; \Lambda^\bullet L^\vee) \otimes_{\mathbb{k}} \mathcal{T}^\bullet_{\mathrm{poly}}(\mathbb{k}^d_{\mathrm{formal}})$$
$$\Big(\Gamma(\Lambda^\bullet L^\vee) \otimes_R \mathscr{D}^\bullet_{\mathrm{poly}} \Big)|_U \cong \Gamma(U; \Lambda^\bullet L^\vee) \otimes_{\mathbb{k}} \mathcal{D}^\bullet_{\mathrm{poly}}(\mathbb{k}^d_{\mathrm{formal}}).$$

According to Lemma 4.1.8, the restrictions to U of the maps $(\mathscr{U}^f_n)_{n=1,2,\cdots}$ constructed earlier are the Taylor coefficients of an L_∞ morphism of dglas

$$\begin{array}{c} \Big(\operatorname{tot}(\Gamma(U; \Lambda^\bullet L^\vee) \otimes_{\mathbb{k}} \mathcal{T}^\bullet_{\mathrm{poly}}(\mathbb{k}^d_{\mathrm{formal}})), d_L \otimes \mathrm{id}_{\mathcal{T}^\bullet_{\mathrm{poly}}(\mathbb{k}^d_{\mathrm{formal}})}, [-,-] \Big) \\ \Big\downarrow \mathscr{U}^f_U \\ \Big(\operatorname{tot}(\Gamma(U; \Lambda^\bullet L^\vee) \otimes_{\mathbb{k}} \mathcal{D}^\bullet_{\mathrm{poly}}(\mathbb{k}^d_{\mathrm{formal}})), d_L \otimes \mathrm{id}_{\mathcal{D}^\bullet_{\mathrm{poly}}(\mathbb{k}^d_{\mathrm{formal}})} + \mathrm{id} \otimes d_{\mathscr{H}}, [\![-,-]\!] \Big). \end{array}$$

In the chosen local trivialization $B|_U \cong U \times \mathbb{k}^d$ of the vector bundle B over the open subset U of M, we may compare Q with the derivation $d_L \otimes \mathrm{id}_{\mathbb{k}[[\chi_1, \cdots, \chi_d]]}$ of the algebra

$$\Gamma(U;\Lambda^\bullet L^\vee\otimes\hat{S}B^\vee)\cong\Gamma(U;\Lambda^\bullet L^\vee)\otimes_{\Bbbk}\Bbbk[[\chi_1,\cdots,\chi_d]].$$

Since $Q^2=0$, their difference

$$\varpi=Q-d_L\otimes\mathrm{id}\in\Gamma(U;L^\vee)\otimes_{\Bbbk}\mathcal{T}^0_{\mathrm{poly}}(\Bbbk^d_{\mathrm{formal}})$$

is a MC element of the dgla

$$\mathrm{tot}\left(\Gamma(\Lambda^\bullet L^\vee)\otimes_R\mathfrak{T}^\bullet_{\mathrm{poly}}\right)|_U\cong\mathrm{tot}\left(\Gamma(U;\Lambda^\bullet L^\vee)\otimes_{\Bbbk}\mathcal{T}^\bullet_{\mathrm{poly}}(\Bbbk^d_{\mathrm{formal}})\right)$$

endowed with the differential $d_L\otimes\mathrm{id}_{\mathcal{T}^\bullet_{\mathrm{poly}}(\Bbbk^d_{\mathrm{formal}})}$ and the $\Gamma(U;\Lambda L^\vee)$-multilinear extension of the Schouten bracket on $\mathcal{T}^\bullet_{\mathrm{poly}}(\Bbbk^d_{\mathrm{formal}})$.

Since

$$\begin{aligned}\mathscr{U}^f_U(\varpi)&=\sum_{j=1}^{\infty}\frac{1}{j!}\mathscr{U}^f_j(\varpi^j) && \text{by Equation (4.2)}\\&=\mathscr{U}^f_1(\varpi) && \text{by Theorem 4.1.4 (3)}\\&=\mathrm{hkr}(\varpi) && \text{by Theorem 4.1.4 (1)}\\&=\varpi,\end{aligned}$$

we obtain the tangent L_∞ morphism of $\mathscr{U}^f_U$ at ϖ:

$$\begin{aligned}\mathscr{U}^f_{U,\varpi}:\Big(\mathrm{tot}(\Gamma(\Lambda^\bullet L^\vee)\otimes_R\mathfrak{T}^\bullet_{\mathrm{poly}})|_U,Q,[-,-]\Big)&\\ \to\Big(\mathrm{tot}(\Gamma(\Lambda^\bullet L^\vee)\otimes_R\mathscr{D}^\bullet_{\mathrm{poly}})|_U,Q+d_{\mathscr{H}},[\![-,-]\!]\Big)&.\end{aligned}$$

Adapting the argument used for Φ earlier, one can show that the tangent L_∞ morphism $\mathscr{U}^f_{U,\varpi}$ is well defined.

Since the map Φ_n depends only locally on its arguments, we may consider its restriction to the open subset U of M. We claim that the n-th Taylor coefficient of the L_∞ morphism $\mathscr{U}^f_{U,\varpi}$ is the restriction of Φ_n to U. Indeed, one easily checks that $\omega-\varpi$ is (the tensor product of a section of $L^\vee$ over U with) a *linear* vertical vector field on $\Bbbk^d_{\mathrm{formal}}$ and it then follows from Theorem 4.1.4 (4) that, for all

$\gamma \in \Lambda^n\Big(\operatorname{tot}(\Gamma(\Lambda^\bullet L^\vee) \otimes_R \mathfrak{T}^\bullet_{\text{poly}})|_U\Big)$,

$$\Phi_n(\gamma) = \sum_{j=1}^{\infty} \mathscr{U}^f_{n+j}(\omega^j \wedge \gamma) = \sum_{j=1}^{\infty} \mathscr{U}^f_{n+j}(\varpi^j \wedge \gamma) = \Big(\mathscr{U}^f_{U,\varpi}\Big)_n(\gamma).$$

This shows that $(\Phi_n)_{n=1,2,\cdots}$ is the sequence of Taylor coefficients of an L_∞ morphism

$$\Phi : \Big(\operatorname{tot}(\Gamma(\Lambda^\bullet L^\vee) \otimes_R \mathfrak{T}^\bullet_{\text{poly}}), [Q, \cdot], [-,-]\Big) \\ \to \Big(\operatorname{tot}(\Gamma(\Lambda^\bullet L^\vee) \otimes_R \mathscr{D}^\bullet_{\text{poly}}), [\![Q, \cdot]\!] + d_{\mathscr{H}}, [\![-,-]\!]\Big)$$

defined globally on M. □

Our construction of the quasi-isomorphism Φ is essentially the same as the one given by Dolgushev in [18] except that we define its Taylor coefficients Φ_n globally from the get-go rather than by glueing local data.

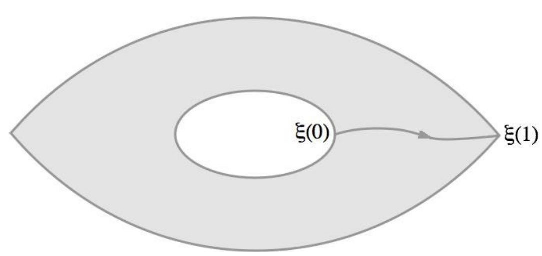
4.1.3 Algebraic homomorphism property

In this section, we sketch a proof why Φ_1 is a morphism of associative algebras up to homotopy. For more details, the reader may want to consult [11, 30, 42, 43, 49].

4.1.3.1 Kontsevich's eye

The compactified configuration space $\overline{C^+_{2,0}}$, which is customarily called 'Kontsevich's eye,' is represented in Figure 4.1. Its boundary admits the following

Figure 4.1: 'Kontsevich's eye'



decomposition in strata:

$$\partial(\overline{C^+_{2,0}}) = C_{1,0} \sqcup C_{1,1} \sqcup C_{1,1} \sqcup C_{0,2}.$$

The stratum $C_{1,0}$ — the pupil of the eye — is reached when the two aerial vertices z_1 and z_2 merge. The first copy of $C_{1,1}$ — the upper eyelid — is reached when the aerial vertex z_1 approaches the real line. The second copy of $C_{1,1}$ — the lower eyelid — is reached when the aerial vertex z_2 approaches the real line. The stratum $C_{0,2}$ is made of two points — the corners of the eye. The left corner is reached when the vertices z_1 and z_2 each approach a distinct point of the real line simultaneously and z_1 is the leftmost of the two points. The right corner is reached when the vertices z_1 and z_2 each approach a distinct point of the real line simultaneously and z_1 is the rightmost of the two points.

4.1.3.2 Vanishing lemma

Given a configuration space $C^+_{n,m}$ with $n \geqslant 2$, consider the projection $\pi : C^+_{n,m} \to C^+_{2,0}$, which forgets all but the first two of the n aerial points in $\mathbb{H}^+$ and all m points on the real line. More precisely, consider its continuous extension $\overline{\pi} : \overline{C^+_{n,m}} \to \overline{C^+_{2,0}}$ to the compactified configuration spaces.

Now choose a smooth path $\xi : [0,1] \to \overline{C^+_{2,0}}$ starting from a point on the inner boundary of Kontsevich's eye and ending at the right corner. The inverse image of $\xi([0,1])$ under π in $\overline{C^+_{n,m}}$ is a compact subspace denoted $Z_{n,m}$. Kontsevich assigns a weight

$$\widetilde{W}_\Gamma = \prod_{k=1}^{n} \frac{1}{|\operatorname{Out}(k)|!} \int_{Z_{n,m}} j^*(\kappa_\Gamma)$$

to each admissible graph $\Gamma \in \mathscr{G}_{n,m}$. The symbol j denotes the embedding of $Z_{n,m}$ into $\overline{C^+_{n,m}}$. Since $\dim Z_{n,m} = 2n + m - 3$ and κ_Γ is an $|E_\Gamma|$-form, the weight $\widetilde{W}_\Gamma$ is zero unless $|E_\Gamma| = 2n + m - 3$.

The following vanishing lemma is analogous to Theorem 4.1.4 (4).

Lemma 4.1.9. *If $\xi \in \mathcal{T}^0_{\text{poly}}(\mathbb{k}^d)$ is a vector field* linear on $\mathbb{k}^d$ *and $\Gamma \in \mathscr{G}_{n,m}$ is an admissible graph with $n \geqslant 3$, then*

$$\widetilde{W}_\Gamma \mathscr{U}_\Gamma(\eta_1, \cdots, \eta_{n-1}, \xi) = 0$$

for all $\eta_1, \cdots, \eta_{n-1} \in \mathcal{T}^\bullet_{\text{poly}}(\mathbb{k}^d)$.

All ingredients of the proof can be found in Kontsevich's original paper [30].

Recall the hyperbolic angle function $\varphi : \overline{\mathbb{H}^+} \times \overline{\mathbb{H}^+} \to S^1$ defined by $\varphi(z,w) = \frac{1}{2\pi} \arg\left(\frac{z-w}{\bar{z}-w}\right)$. Given a point z_0 of $\overline{\mathbb{H}^+}$, let $d\varphi(z,z_0)$ denote the pullback to $\mathbb{H}^+$ of the standard volume form on S^1 through the function $\mathbb{H}^+ \ni z \mapsto \varphi(z, z_0) \in S^1$. Likewise let $d\varphi(z_0, z)$ denote the pullback to $\mathbb{H}^+$ of the standard volume form on S^1 through the function $\mathbb{H}^+ \ni z \mapsto \varphi(z_0, z) \in S^1$.

Lemma 4.1.10 ([30, Lemmas 7.3, 7.4, and 7.5]). *1. For every pair of distinct points z_1 and z_2 in $\mathbb{H}^+$, we have*

$$\int_{z\in\mathbb{H}^+\smallsetminus\{z_1,z_2\}} d\varphi(z_1, z) \wedge d\varphi(z, z_2) = 0.$$

2. For every pair of points $z_1 \in \mathbb{H}^+$ and $z_2 \in \mathbb{R} = \partial(\mathbb{H}^+)$, we have

$$\int_{z\in\mathbb{H}^+\smallsetminus\{z_1\}} d\varphi(z_1, z) \wedge d\varphi(z, z_2) = 0.$$

3. For every point z_0 in $\mathbb{H}^+$, we have

$$\int_{z\in\mathbb{H}^+\smallsetminus\{z_0\}} d\varphi(z_0, z) \wedge d\varphi(z, z_0) = 0.$$

Proof of Lemma 4.1.9. Since $\xi \in \mathcal{T}^0_{\text{poly}}(\Bbbk^d)$, we have $\xi^{I(\text{Out}(n))} = 0$ for all maps $I : E_\Gamma \to \{1, \cdots, d\}$ unless $|\text{Out}(n)| = 1$. Moreover, if $|\text{In}(n)| > 1$, the order of the differential operator D^n_I is at least two, no matter which map $I : E_\Gamma \to \{1, \cdots, d\}$ is considered, and the function $D^n_I \xi^{I(\text{Out}(n))}$ vanishes as ξ is linear. Hence $\mathscr{U}_\Gamma(\eta_1, \cdots, \eta_{n-1}, \xi) = 0$ unless $|\text{Out}(n)| = 1$ and $|\text{In}(n)| \leqslant 1$. We may thus assume without loss of generality that $\text{Out}(n) = \{e^1_n\}$ and $|\text{In}(n)| \in \{0, 1\}$.

Let's assume for now that $\text{Out}(n) = \{e^1_n\}$ and $\text{In}(n) = \{e'\}$ — we will treat the other case later. Consider the graph $\Delta \in \mathscr{G}_{n-1,m}$ obtained from $\Gamma \in \mathscr{G}_{n,m}$ by removing the n-th aerial vertex n and all the edges starting or ending at it. We have

$$\kappa_\Gamma = \pm d\varphi_{e'} \wedge d\varphi_{e^1_n} \wedge F^*_n(\kappa_\Delta),$$

where $F_n : C^+_{n,m} \twoheadrightarrow C^+_{n-1,m}$ is the projection which forgets the n-th aerial point z_n of a configuration $(z_1, \cdots, z_n; q_1, \cdots, q_m)$. Making use of Fubini's theorem, we obtain

$$\widetilde{W}_\Gamma = \int_{Z_{n,m}} j^*(\kappa_\Gamma) = \pm \int_{Z_{n,m}} j^*\Big(d\varphi_{e'} \wedge d\varphi_{e^1_n} \wedge F^*_n(\kappa_\Delta)\Big) = \pm \int_{F_n(Z_{n,m})} f \cdot j^*(\kappa_\Delta),$$

where f denotes the function on $C^+_{n-1,m}$ obtained by integration of the 2-form

$d\varphi_{e'} \wedge d\varphi_{e_n^1}$ along the fibers of $F_n : C^+_{n,m} \twoheadrightarrow C^+_{n-1,m}$. Since

$$f(z_1, \cdots, z_{n-1}; z_{\bar{1}}, \cdots, z_{\bar{m}}) = \int_{z_n \in \mathbb{H}^+ \smallsetminus \{z_1, \cdots, z_{n-1}\}} d\varphi(z_{s(e')}, z_n) \wedge d\varphi(z_n, z_{t(e_n^1)}),$$

it follows from Lemma 4.1.10 that $\widetilde{W}_\Gamma = 0$.

Finally, we turn our attention to the situation where $\mathrm{Out}(n) = \{e_n^1\}$ and $\mathrm{In}(n) = \varnothing$. We start with making two observations.

1. For all $e \in E_\Gamma$ with $e \neq e_n^1$, the aerial vertex n is neither the source nor the target of the edge e and, consequently, the function $\varphi_e : C^+_{n,m} \to S^1$ is constant along the fibers of the projection $F_n : C^+_{n,m} \twoheadrightarrow C^+_{n-1,m}$ which forgets the n-th aerial point z_n of a configuration $(z_1, \cdots, z_n; q_1, \cdots, q_m)$.

2. Each fiber of the projection $F_n : C^+_{n,m} \twoheadrightarrow C^+_{n-1,m}$ is diffeomorphic to $\mathbb{H}^+$ punctured at $n-1$ points and is foliated by its intersections with the level sets of the function $\varphi_{e_n^1} : C^+_{n,m} \to S^1$. In other words, $C^+_{n,m}$ is foliated by the fibers of F_n, which are themselves foliated by curves along which the function $\varphi_{e_n^1}$ is constant. Obviously, the subspace $Z_{n,m}$ of $C^+_{n,m}$ is a union of such curves.

It follows from these observations that $Z_{n,m}$ is foliated by curves along which all functions φ_e for all edges $e \in E_\Gamma$ are constant. Therefore, the component of the form $j^*(\bigwedge_{e \in E_\Gamma} d\varphi_e)$ of degree $\dim(Z_{n,m}) = 2n + m - 3$ vanishes. Hence $\widetilde{W}_\Gamma = \int_{Z_{n,m}} j^*(\kappa_\Gamma) = 0$. □

4.1.3.3 Homotopy operator

For every admissible graph $\Gamma \in \mathscr{G}_{n,m}$, the operator

$$\mathscr{U}_\Gamma : \mathcal{T}^\bullet_{\mathrm{poly}}(\mathbb{k}^d_{\mathrm{formal}})^{\otimes n} \to \mathcal{D}^{m-1}_{\mathrm{poly}}(\mathbb{k}^d_{\mathrm{formal}})$$

defined by Equation (4.5) is $GL_d(\mathbb{k})$-equivariant. Therefore there exists an R-linear map

$$\mathscr{U}^f_\Gamma : \left(\mathfrak{T}^\bullet_{\mathrm{poly}}\right)^{\otimes n} \to \mathscr{D}^{m-1}_{\mathrm{poly}}$$

whose restrictions to each fiber of $B \to M$ coincide with $\mathscr{U}_\Gamma$. Extending the latter $\Gamma(\Lambda^\bullet L^\vee)$-multilinearly, we obtain an R-linear operator

$$\mathscr{U}^f_\Gamma : (\Gamma(\Lambda^\bullet L^\vee) \otimes_R \mathfrak{T}^\bullet_{\mathrm{poly}})^{\otimes n} \to \Gamma(\Lambda^\bullet L^\vee) \otimes_R \mathscr{D}^{m-1}_{\mathrm{poly}}.$$

Using the maps $\mathscr{U}_\Gamma^f$, the weights $\widetilde{W}_\Gamma$, and the difference

$$\omega = Q - d_L^\nabla \in \Gamma(L^\vee) \otimes_R \mathfrak{T}^0_{\text{poly}}$$

of the derivations Q and d_L^∇ appearing in Theorem 2.3.7, we define an operator

$$H : \Big(\Gamma(\Lambda^\bullet L^\vee) \otimes_R \mathfrak{T}^\bullet_{\text{poly}}\Big) \times \Big(\Gamma(\Lambda^\bullet L^\vee) \otimes_R \mathfrak{T}^\bullet_{\text{poly}}\Big) \to \Gamma(\Lambda^\bullet L^\vee) \otimes_R \mathscr{D}^\bullet_{\text{poly}}$$

by

$$H(\alpha, \beta) = \sum_{\substack{n\geqslant 0\\ m\geqslant 0}} \frac{1}{n!} \sum_{\Gamma\in\mathscr{G}_{2+n,m}} (-1)^{m+1} \widetilde{W}_\Gamma \mathscr{U}_\Gamma^f(\alpha, \beta, \underbrace{\omega, \cdots, \omega}_{n \text{ arguments}}). \tag{4.6}$$

Lemma 4.1.11. *The operator H is well defined.*

Proof. It suffices to prove that the infinite sum in the r.h.s. of Equation (4.6) converges for pairs of homogeneous elements $\alpha \in \Gamma(\Lambda^\bullet L^\vee) \otimes_R \mathfrak{T}^a_{\text{poly}}$ and $\beta \in \Gamma(\Lambda^\bullet L^\vee) \otimes_R \mathfrak{T}^b_{\text{poly}}$.

We know that, for all $\Gamma \in \mathscr{G}_{2+n,m}$,

- $\mathscr{U}_\Gamma^f(\alpha, \beta, \underbrace{\omega, \cdots, \omega}_{n \text{ arguments}}) = 0$ unless $|E_\Gamma| = (a+1) + (b+1) + n(0+1)$
- and $\widetilde{W}_\Gamma = 0$ unless $|E_\Gamma| = \dim(Z_{2+n,m})$.

Therefore, for all $\Gamma \in \mathscr{G}_{2+n,m}$, we have $\widetilde{W}_\Gamma \mathscr{U}_\Gamma^f(\alpha, \beta, \underbrace{\omega, \cdots, \omega}_{n \text{ arguments}}) = 0$ unless

$$a + b + n + 2 = |E_\Gamma| = 2n + m + 1.$$

We may thus rewrite the r.h.s. of Equation (4.6) as the *finite* sum

$$(-1)^{a+b} \sum_{n=0}^{a+b+1} \frac{(-1)^n}{n!} \sum_{\Gamma\in\mathscr{G}_{2+n,a+b+1-n}} \widetilde{W}_\Gamma \mathscr{U}_\Gamma^f(\alpha, \beta, \underbrace{\omega, \cdots, \omega}_{n \text{ arguments}}).$$ □

Proposition 4.1.12. *For all $\alpha \in (\Gamma(\Lambda^\bullet L) \otimes \mathfrak{T}^a_{\text{poly}})$ and $\beta \in (\Gamma(\Lambda^\bullet L) \otimes \mathfrak{T}^b_{\text{poly}})$, we have*

$$\begin{aligned}\Phi_1(\alpha \cup \beta) - \Phi_1(\alpha) \cup \Phi_1(\beta) \\ = (Q + d_{\mathscr{H}})\Big(H(\alpha, \beta)\Big) - H\Big(Q(\alpha), \beta\Big) + (-1)^a H\Big(\alpha, Q(\beta)\Big).\end{aligned}$$

Sketch of proof. Recall from the proof of Theorem 4.1.7 that, though ω is not a Maurer–Cartan element, its restriction to any open subset of the manifold M over which the vector bundle B is trivial is equal to the sum of a Maurer–Cartan element and a linear vector field. It follows from Lemma 4.1.9, the definitions and locality of Φ_1 and H that, for the purpose of this proof, ω may be treated as if it were a Maurer–Cartan element. The rest of the proof is then virtually identical to a difficult computation due to Manchon and Torossian [42, 43, ThÃĺorÃĺme 4.6] — see also Mochizuki's work [49, Equation 56]. There is only one significant difference with [42, 43, ThÃĺorÃĺme 4.6]: their Poisson bivector $\hbar\gamma$ must be replaced by our vector field ω. This is responsible for the discrepancy in the number of edges of the admissible graphs appearing here and in [42, 43, ThÃĺorÃĺme 4.6]. □

Remark 4.1.13. *The operator H defined above, which is implicit in [42, 43, ThÃĺorÃĺme 4.6] and [49, Equation 56], was made explicit in [11, Proposition 9.1].*

4.1.4 Explicit formula for Φ_1

Consider the first 'Taylor coefficient'

$$\operatorname{tot}\Big(\Gamma(\Lambda^\bullet L^\vee)\otimes_R \mathfrak{T}^\bullet_{\mathrm{poly}}(B)\Big)\xrightarrow{\Phi_1}\operatorname{tot}\Big(\Gamma(\Lambda^\bullet L^\vee)\otimes_R \mathscr{D}^\bullet_{\mathrm{poly}}(B)\Big)$$

of the L_∞ morphism Φ constructed in Section 4.1.2.3.

In this section, we prove the following

Proposition 4.1.14. *The map $\Phi_1:\Gamma(\Lambda^\bullet L^\vee)\otimes_R \mathfrak{T}^\bullet_{\mathrm{poly}}\to\Gamma(\Lambda^\bullet L^\vee)\otimes_R \mathscr{D}^\bullet_{\mathrm{poly}}$ is the modification of the Hochschild–Kostant–Rosenberg map by (the square root of) the canonical Todd cocycle:*

$$\Phi_1=\operatorname{hkr}\circ(\widetilde{\operatorname{td}}^{\mathrm{can}}_{\mathcal{F}})^{\frac{1}{2}}.$$

Suppose that an open subset U of M diffeomorphic to $\mathbb{R}^m$ is the domain of a coordinate chart of M over which the vector bundles L and B are trivial. The algebra of functions of the graded manifold $\mathcal{V}=\mathbb{R}^m\otimes\Bbbk^{r|l}_{\mathrm{formal}}$ obtained by restriction of the Fedosov dg manifold $\mathcal{M}$ to the support U is

$$C^\infty(\mathcal{V})=C^\infty(\mathbb{R}^m)\otimes\hat{S}\Big((\Bbbk^r\oplus\Bbbk^l[1])^\vee\Big).$$

Here m is the dimension of the manifold M while r is the rank of the vector bundle B and l is the rank of the vector bundle L.

There are natural injections

$$\begin{array}{ccc} & & C^\infty(\mathbb{R}^m \times \Bbbk^{0|l}_{\text{formal}}) \otimes \mathcal{T}^0_{\text{poly}} \Bbbk^{r|0}_{\text{formal}} \\ & & \big\downarrow \mathscr{I} \\ \mathfrak{X}(\mathbb{R}^m \times \Bbbk^{0|l}_{\text{formal}}) & \hookrightarrow & \mathfrak{X}(\mathcal{V}). \end{array}$$

The restriction of the Fedosov homological vector field $Q \in \mathfrak{X}(\mathcal{M})$ to U is the sum

$$Q = d_L + \varpi$$

of $d_L \in \mathfrak{X}(\mathbb{R}^m \times \Bbbk^{0|l}_{\text{formal}})$ and $\varpi \in C^\infty(\mathbb{R}^m \times \Bbbk^{0|l}_{\text{formal}}) \otimes \mathcal{T}^0_{\text{poly}} \Bbbk^{r|0}_{\text{formal}}$ as observed in Section 4.1.2.3.

Lemma 4.1.15. *The sum $Q = Q_1 + Q_2$ of two vector fields $Q_1 \in \mathfrak{X}(\mathbb{R}^m \times \Bbbk^{0|l}_{\text{formal}}) \subset \mathfrak{X}(\mathcal{V})$ and $Q_2 \in C^\infty(\mathbb{R}^m \times \Bbbk^{0|l}_{\text{formal}}) \otimes \mathcal{T}^0_{\text{poly}} \Bbbk^{r|0}_{\text{formal}} \subset \mathfrak{X}(\mathcal{V})$ is a homological vector field on $\mathcal{V} = \mathbb{R}^m \otimes \Bbbk^{r|l}_{\text{formal}}$ if and only if **(1)** Q_1 is a homological vector field on $\mathbb{R}^m \times \Bbbk^{0|l}_{\text{formal}}$ (i.e. Q_1 is of degree $+1$ and $Q_1^2 = 0$) and **(2)** Q_2 satisfies the Maurer–Cartan equation $\mathcal{L}_{Q_1} Q_2 + \frac{1}{2}[Q_2, Q_2] = 0$.*
Moreover, in this case, $(C^\infty(\mathbb{R}^m \times \Bbbk^{0|l}_{\text{formal}}), Q_1)$ is a cdga and Q_2 is a MC element in $C^\infty(\mathbb{R}^m \times \Bbbk^{0|l}_{\text{formal}}) \otimes \mathcal{T}^\bullet_{\text{poly}}(\Bbbk^r_{\text{formal}})$ endowed with its L_∞ algebra structure determined by Q_1.

According to Vaĭntrob [63], Condition (1) in Lemma 4.1.15 above means that the trivial vector bundle $\mathbb{R}^m \times \Bbbk^l \to \mathbb{R}^m$ carries a Lie algebroid structure.

It follows from Lemma 4.1.15 that $\left(C^\infty(\mathbb{R}^m \times \Bbbk^{0|l}_{\text{formal}}), d_L\right)$ is a cdga and ϖ is a Maurer–Cartan element of the dgla $C^\infty(\mathbb{R}^m \times \Bbbk^{0|l}_{\text{formal}}) \otimes \mathcal{T}^0_{\text{poly}} \Bbbk^{r|0}_{\text{formal}}$ determined by the differential $d_L \otimes \mathrm{id}_{\mathcal{T}^0_{\text{poly}} \Bbbk^{r|0}_{\text{formal}}} = \mathcal{L}_{d_L}$ and the restriction of the Schouten bracket in $\mathcal{T}^0_{\text{poly}} \mathcal{V}$.

In Section 4.1.2.3, we proved that Φ_1 depends only locally on its arguments and that its restriction to U is

$$C^\infty(\mathbb{R}^m \times \Bbbk^{0|l}_{\text{formal}}) \otimes \mathcal{T}^\bullet_{\text{poly}}(\Bbbk^r_{\text{formal}}) \xrightarrow{\left(\mathscr{U}^f_{U,\varpi}\right)_1} C^\infty(\mathbb{R}^m \times \Bbbk^{0|l}_{\text{formal}}) \otimes \mathcal{D}^\bullet_{\text{poly}}(\Bbbk^r_{\text{formal}}),$$

the first Taylor coefficient of the tangent L_∞ morphism to $\mathscr{U}_U^f$ at the Maurer–Cartan element ϖ.

Therefore, to establish Proposition 4.1.14, it suffices to prove that

$$\left(\mathscr{U}^f_{U,\varpi}\right)_1 = \operatorname{hkr}\circ\left(\widetilde{\operatorname{td}}^{\mathrm{can}}_{\mathcal{F}|_U}\right)^{\frac{1}{2}}$$

in every coordinate chart U of M over which the vector bundles B and L are trivial.

Note that the dglas $C^\infty(\mathbb{R}^m\times\Bbbk^{0|l}_{\mathrm{formal}})\otimes\mathcal{T}^\bullet_{\mathrm{poly}}(\Bbbk^r_{\mathrm{formal}})$ and $C^\infty(\mathbb{R}^m\times\Bbbk^{0|l}_{\mathrm{formal}})\otimes\mathcal{D}^\bullet_{\mathrm{poly}}(\Bbbk^r_{\mathrm{formal}})$, the restrictions of $\Gamma(\Lambda^\bullet L^\vee)\otimes_R\mathfrak{T}^\bullet_{\mathrm{poly}}$ and $\Gamma(\Lambda^\bullet L^\vee)\otimes_R\mathscr{D}^\bullet_{\mathrm{poly}}$ over U, are dg Lie subalgebras of $\left(\mathcal{T}^\bullet_{\mathrm{poly}}\mathcal{V},\mathcal{L}_Q,[,]\right)$ and $\left(\mathcal{D}^\bullet_{\mathrm{poly}}\mathcal{V},d_{\mathscr{H}}+[\![Q,]\!],[\![,]\!]\right)$, respectively.

Now, consider the Kontsevich formality L_∞ quasi-isomorphism

$$(\mathcal{T}^\bullet_{\mathrm{poly}}\mathcal{V},0,[-,-])\xrightarrow{\mathscr{U}^{\mathcal{V}}}(\mathcal{D}^\bullet_{\mathrm{poly}}\mathcal{V},d_{\mathscr{H}},[\![-,-]\!])$$

devised for $\mathcal{V}=\mathbb{R}^m\times\Bbbk^{r|l}_{\mathrm{formal}}$ in [13, Appendix].

Since $[Q,Q]=0$, the vector field $Q\in\mathcal{T}^0_{\mathrm{poly}}\mathcal{V}$ is a Maurer–Cartan element of the dgla $\mathcal{T}^\bullet_{\mathrm{poly}}\mathcal{V}$ and we can consider the tangent L_∞ morphism $\mathscr{U}^{\mathcal{V}}_Q$ defined per Equation (4.3). Since Q is a vector field, it follows from Theorem 4.1.4 (4) that $\mathscr{U}^{\mathcal{V}}(Q)=Q\in\mathcal{D}^0_{\mathrm{poly}}\mathcal{V}$, where $\mathscr{U}^{\mathcal{V}}(Q)$ is given by the graded version of Equation (4.2) as in [13]. Hence we obtain the L_∞ quasi-isomorphism

$$(\mathcal{T}^\bullet_{\mathrm{poly}}\mathcal{V},[Q,-],[-,-])\xrightarrow{\mathscr{U}^{\mathcal{V}}_Q}(\mathcal{D}^\bullet_{\mathrm{poly}}\mathcal{V},d_{\mathscr{H}}+[\![Q,-]\!],[\![-,-]\!]).$$

Lemma 4.1.16. *In the category of cochain complexes of $\Bbbk$-modules, the diagram*

$$\begin{array}{ccc}\left(\mathcal{T}^\bullet_{\mathrm{poly}}(\mathcal{V}),[Q,-]\right) & \xrightarrow{\left(\mathscr{U}^{\mathcal{V}}_Q\right)_1} & \left(\mathcal{D}^\bullet_{\mathrm{poly}}(\mathcal{V}),d_{\mathscr{H}}+[\![Q,-]\!]\right)\\ \mathscr{I}\big\uparrow & & \mathscr{I}\big\uparrow\\ C^\infty(\mathbb{R}^m\times\Bbbk^{0|l}_{\mathrm{formal}})\otimes\mathcal{T}^\bullet_{\mathrm{poly}}\Bbbk^r_{\mathrm{formal}} & \xrightarrow[\left(\mathscr{U}^f_{U,\varpi}\right)_1]{} & C^\infty(\mathbb{R}^m\times\Bbbk^{0|l}_{\mathrm{formal}})\otimes\mathcal{D}^\bullet_{\mathrm{poly}}\Bbbk^r_{\mathrm{formal}}\end{array}$$

is commutative.

Proof. Let $\gamma\in C^\infty(\mathbb{R}^m\times\Bbbk^{0|l}_{\mathrm{formal}})\otimes\mathcal{T}^k_{\mathrm{poly}}\Bbbk^r_{\mathrm{formal}}$ be a $(k+1)$-vector field. It follows from Equations (4.3), (4.4), and (4.5) that

$$\begin{aligned}\left(\mathscr{U}_Q^{\mathcal{V}}\right)_1(\gamma) &= \sum_{j=0}^{\infty} \frac{1}{j!} \mathscr{U}_{1+j}^{\mathcal{V}}(Q \wedge \cdots \wedge Q \wedge \gamma) \\ &= \sum_{j=0}^{\infty} \frac{1}{j!} \sum_{m \geqslant 0} \sum_{\Gamma \in \mathscr{G}_{1+j,m}} W_\Gamma \mathscr{U}_\Gamma^{\mathcal{V}}(Q \wedge \cdots \wedge Q \wedge \gamma)\end{aligned}$$

and

$$\begin{aligned}\left(\mathscr{U}_{U,\varpi}^{f}\right)_1(\gamma) &= \sum_{j=0}^{\infty} \frac{1}{j!} \left(\mathscr{U}_U^{f}\right)_{1+j}(\varpi \wedge \cdots \wedge \varpi \wedge \gamma) \\ &= \sum_{j=0}^{\infty} \frac{1}{j!} \mathscr{U}_{1+j}^{\mathcal{V}}(\varpi \wedge \cdots \wedge \varpi \wedge \gamma) \\ &= \sum_{j=0}^{\infty} \frac{1}{j!} \sum_{m \geqslant 0} \sum_{\Gamma \in \mathscr{G}_{1+j,m}} W_\Gamma \mathscr{U}_\Gamma^{\mathcal{V}}(\varpi \wedge \cdots \wedge \varpi \wedge \gamma).\end{aligned}$$

Therefore, since $Q = d_L + \varpi$, it suffices to prove that, provided $X_1, X_2, \dots, X_j \in \{d_L, \varpi\}$ and $X_p = d_L$ for at least one $p \in \{1, 2, \dots, j\}$, the expression $W_\Gamma \mathscr{U}_\Gamma^{\mathcal{V}}(X_1 \wedge X_2 \wedge \cdots \wedge X_j \wedge \gamma)$ vanishes for all $\Gamma \in \mathscr{G}_{j+1,m}$.

Given a graph $\Gamma \in \mathscr{G}_{1+j,m}$, we know that

- $W_\Gamma = 0$ if $|E_\Gamma| \neq 2j + m$;
- $\mathscr{U}_\Gamma(X_1, \cdots, X_j, \gamma) = 0$ if $|E_\Gamma| \neq j + k + 1$;
- and $\mathscr{U}_\Gamma(X_1, \cdots, X_j, \gamma) = 0$ if the number of edges starting from the $(j+1)$-th aerial vertex is different from $k + 1$ (since γ is a $(k+1)$-vector field).

Therefore, if $\Gamma \in \mathscr{G}_{1+j,m}$, we have $W_\Gamma \mathscr{U}_\Gamma(X_1, \cdots, X_j, \gamma) = 0$ unless $|\mathrm{Out}(v_{j+1})| = k + 1 = j + m$. In other words, since $(j+1) + m$ is the total number of vertices of the graph Γ, we have $W_\Gamma \mathscr{U}_\Gamma(X_1, \cdots, X_j, \gamma) = 0$ unless a single edge runs from the $(j+1)$-th aerial vertex of Γ to each one of the other $j + m$ vertices of Γ.

However, if an edge e' of Γ starts at the $(j+1)$-th aerial vertex v_{j+1} (the aerial vertex corresponding to γ) and ends at v_p (the aerial vertex corresponding to $X_p = d_L$), the factor $D_I^{v_p}(X_p^{I(\mathrm{Out}(p))})$ appearing in each term of the expansion (4.5) of $\mathscr{U}_\Gamma(X_1, \cdots, X_j, \gamma)$ must vanish. Indeed, $D_I^{v_p}$ is a composition of one or more partial derivatives w.r.t. the coordinates on $\Bbbk^r_{\text{formal}}$ containing $\frac{\partial}{\partial x_{I(e')}}$ at the very least, while $X_p^{I(\mathrm{Out}(p))} = (d_L)^{I(\mathrm{Out}(p))}$ is a function in the subalgebra $C^\infty(\mathbb{R}^m \times \Bbbk^{0|l}_{\mathrm{formal}})$ of $C^\infty(\mathcal{V})$.

The proof is complete. □

The following theorem was first announced by Shoikhet in [56]. We refer the interested reader to [10, 57] for more details.

Theorem 4.1.17 (Kontsevich–Shoikhet [56]). *The first 'Taylor coefficient' $\left(\mathscr{U}_Q^{\mathcal{V}}\right)_1 : \mathcal{T}^\bullet_{\mathrm{poly}}\mathcal{V} \to \mathcal{D}^\bullet_{\mathrm{poly}}\mathcal{V}$ of the tangent L_∞ quasi-isomorphism $\mathscr{U}_Q^{\mathcal{V}}$ is the modification*

$$\left(\mathscr{U}_Q^{\mathcal{V}}\right)_1 = \mathrm{hkr} \circ \left(\widetilde{\mathrm{td}}_{T_{\mathcal{V}}}^{\mathrm{trivial}}\right)^{\frac{1}{2}}$$

of the Hochschild–Kostant–Rosenberg map by (the square root of) the Todd cocycle $\widetilde{\mathrm{td}}_{T_{\mathcal{V}}}^{\mathrm{trivial}} \in \prod_{k=0}^\infty \Omega^k(\mathcal{V})$ of the dg manifold $(\mathcal{V}, Q)$ associated with the trivial connection as in Example 2.2.16. The Todd cocycle acts on $\mathcal{T}^\bullet_{\mathrm{poly}}\mathcal{V}$ by contraction.

Lemma 4.1.18. *For all $s \in \mathbb{N}$, we have*

$$\mathrm{tr}\left((\mathrm{At}^{\mathrm{can}}_{\mathcal{F}|_U})^s\right) = \mathscr{I}^\top \mathrm{str}\left((\mathrm{At}^{\mathrm{trivial}}_{T_{\mathcal{V}}})^s\right).$$

Proof. Let $U \xrightarrow{(x_1,\dots,x_m)} \mathbb{R}^m$ be a local chart of M.

Let $\partial_1, \dots, \partial_r$ be a local frame for $B \to M$ over U and let $\chi_1, \dots, \chi_r$ be the dual local frame for $B^\vee \to M$. Likewise, let $\eta_1, \dots, \eta_l$ be a local frame for $L \to M$ over U and let $\lambda_1, \dots, \lambda_l$ be the dual local frame for $L^\vee \to M$ with the degree shift: $|\lambda_j| = 1$.

The restrictions to U of the anchor map $\rho : L \to T_M$, the Lie bracket on $\Gamma(L)$, the bundle map $q : L \to B$, and the L-connection ∇ on B admit local expressions

$$\rho(\eta_j) = \sum_{i=1}^m \rho_{ij} \frac{\partial}{\partial x_i} \qquad [\eta_i, \eta_j] = \sum_{k=1}^l c_{ij}^k \eta_k$$
$$q(\eta_j) = \sum_{i=1}^r q_{ij} \partial_i \qquad \nabla_{\eta_i}\partial_j = \sum_{k=1}^r \Gamma_{ij}^k \partial_k$$

where ρ_{ij}, c_{ij}^k, q_{ij}, and Γ_{ij}^k are functions of the coordinates $x_1, \dots, x_m$.

Then $(z_1, \dots, z_{m+l+r}) = (x_1, \dots, x_m, \lambda_1, \dots, \lambda_l, \chi_1, \dots, \chi_r)$ are coordinates on

$\mathcal{V} = \mathbb{R}^m \times \Bbbk^{r|l}_{\mathrm{formal}}$ whose degrees are

$$|x_i| = 0, \qquad |\lambda_j| = 1, \qquad |\chi_k| = 0,$$

for $i \in \{1, \cdots, m\}$, $j \in \{1, \cdots, l\}$, and $k \in \{1, \cdots, r\}$. The homological vector field on $\mathcal{V}$ is the sum $Q = -\delta + d_L^\nabla + X^\nabla$ of

$$\delta = \sum_{k=1}^{r} \sum_{j=1}^{l} q_{kj} \lambda_j \frac{\partial}{\partial \chi_k},$$
$$d_L^\nabla = \sum_{j=1}^{m} \sum_{i=1}^{l} \rho_{ji} \lambda_i \frac{\partial}{\partial x_j} - \frac{1}{2} \sum_{i,j,k=1}^{l} c_{ij}^k \lambda_i \lambda_j \frac{\partial}{\partial \lambda_k} - \sum_{i=1}^{l} \sum_{j,k=1}^{r} \Gamma_{ij}^k \lambda_i \chi_j \frac{\partial}{\partial \chi_k},$$

and

$$X^\nabla = \sum_{k=1}^{r} f_k \frac{\partial}{\partial \chi_k},$$

where the functions $f_k \in C^\infty(\mathcal{V})$ are linear in the λ-coordinates and have trivial jet of order 1 w.r.t. the χ-coordinates as per Theorem 2.3.7.

Let $\hat{z}_1, \cdots, \hat{z}_{m+l+r}$ be the local frame of $T_{\mathcal{V}}^\vee$ dual to $\frac{\partial}{\partial z_1}, \cdots, \frac{\partial}{\partial z_{m+l+r}}$. This local frame is essentially $dz_1, \cdots, dz_{m+l+r}$, but they have different degrees: $|\hat{z}_i| = |dz_i| - 1 = |z_i|$. It follows from Lemma 2.2.18 that

$$\mathrm{At}^{\mathrm{can}}_{\mathcal{F}|_U} = \sum_{i,j,k=1}^{r} \frac{\partial^2 f_k}{\partial \chi_i \partial \chi_j} \hat{\chi}_i \otimes \left(\hat{\chi}_j \otimes \frac{\partial}{\partial \chi_k} \right) \tag{4.7}$$

and from Example 2.2.16 that the Atiyah 1-cocycle of the dg Lie algebroid $T_{\mathcal{V}}$ associated with the trivial connection $\nabla^{\mathrm{trivial}}_{\frac{\partial}{\partial z_i}} \frac{\partial}{\partial z_j} = 0$ is

$$\mathrm{At}^{\mathrm{trivial}}_{T_{\mathcal{V}}} = \sum_{i,j,k=1}^{m+l+r} (-1)^{|z_i|+|z_j|} \frac{\partial^2 \big(Q(z_k)\big)}{\partial z_i \partial z_j} \hat{z}_i \otimes \left(\hat{z}_j \otimes \frac{\partial}{\partial z_k} \right).$$

Then, we have

$$\big(\mathscr{I}^\top \otimes \mathrm{id}\big)(\mathrm{At}^{\mathrm{trivial}}_{T_{\mathcal{V}}}) = \sum_{i=1}^{r} \sum_{j,k=1}^{m+l+r} (-1)^{|z_j|} \frac{\partial^2 \big(Q(z_k)\big)}{\partial \chi_i \partial z_j} \hat{\chi}_i \otimes \left(\hat{z}_j \otimes \frac{\partial}{\partial z_k} \right).$$

Since $Q(x_k) = \sum_{i=1}^{l} \rho_{ki}\lambda_i$ and the functions ρ_{ki} depend on the x-coordinates only, we have

$$\frac{\partial^2 \Big(Q(x_k)\Big)}{\partial \chi_i \partial z_j} = 0.$$

Since $Q(\lambda_k) = -\frac{1}{2}\sum_{i,j=1}^{l} c_{ij}^k \lambda_i\lambda_j$ and the functions c_{ij}^k depend on the x-coordinates only, we have

$$\frac{\partial^2 \Big(Q(\lambda_k)\Big)}{\partial \chi_i \partial z_j} = 0.$$

Since $Q(\chi_k) = -\sum_{j=1}^{l} q_{kj}\lambda_j - \sum_{i,j} \Gamma_{ij}^k \lambda_i \chi_j + f_k$, the functions q_{kj} and Γ_{ik}^j depend on the x-coordinates only, we have

$$\frac{\partial^2 \Big(Q(\chi_k)\Big)}{\partial \chi_i \partial \chi_j} = \frac{\partial^2 f_k}{\partial \chi_i \partial \chi_j}.$$

Therefore, the matrix representation of $\Big(\mathscr{I}^\top \otimes \mathrm{id}\Big)(\mathrm{At}_{T_\mathcal{V}}^{\mathrm{trivial}})$ with respect to the frame $(\frac{\partial}{\partial z_1}, \cdots, \frac{\partial}{\partial z_{m+l+r}})$ is

$$\Big(\mathscr{I}^\top \otimes \mathrm{id}\Big)(\mathrm{At}_{T_\mathcal{V}}^{\mathrm{trivial}}) = \begin{bmatrix} 0 & 0 & * \\ 0 & 0 & * \\ 0 & 0 & \frac{\partial^2 f_k}{\partial \chi_i \partial \chi_j}\hat{\chi}_i \end{bmatrix}. \tag{4.8}$$

It follows immediately from Equations (4.7) and (4.8) that the image of $\mathrm{At}_{\mathcal{F}}^{\mathrm{can}}$ under the composition

$$\mathcal{F}^\vee \otimes \mathrm{End}(\mathcal{F}) \xrightarrow{(\dots)^s} \Lambda^s \mathcal{F}^\vee \otimes \mathrm{End}(\mathcal{F}) \xrightarrow{\mathrm{id}\otimes\mathrm{tr}} \Lambda^s \mathcal{F}^\vee$$

and the image of $(\mathscr{I}^\top \otimes \mathrm{id})(\mathrm{At}_{T_\mathcal{V}}^{\mathrm{trivial}})$ under the composition

$$\mathcal{F}|_U^\vee \otimes \mathrm{End}(T_\mathcal{V}) \xrightarrow{(\dots)^s} \Lambda^s \mathcal{F}|_U^\vee \otimes \mathrm{End}(T_\mathcal{V}) \xrightarrow{\mathrm{id}\otimes\mathrm{str}} \Lambda^s \mathcal{F}|_U^\vee$$

are equal:

$$(\mathrm{id}\otimes\mathrm{tr})\Big((\mathrm{At}_{\mathcal{F}}^{\mathrm{can}})^s\Big) = (\mathrm{id}\otimes\mathrm{str})\Big(((\mathscr{I}^\top \otimes \mathrm{id})(\mathrm{At}_{T_\mathcal{V}}^{\mathrm{trivial}}))^s\Big).$$

Since the diagram

$$\begin{array}{ccccc}
T_{\mathcal{V}}^{\vee} \otimes \operatorname{End}(T_{\mathcal{V}}) & \xrightarrow{(\dots)^s} & \Lambda^s T_{\mathcal{V}}^{\vee} \otimes \operatorname{End}(T_{\mathcal{V}}) & \xrightarrow{\mathrm{id}\otimes \mathrm{str}} & \Lambda^s T_{\mathcal{V}}^{\vee} \\
\downarrow{\scriptstyle \mathscr{I}^{\top}\otimes \mathrm{id}} & & \downarrow{\scriptstyle \mathscr{I}^{\top}\otimes \mathrm{id}} & & \downarrow{\scriptstyle \mathscr{I}^{\top}} \\
\mathcal{F}|_U^{\vee} \otimes \operatorname{End}(T_{\mathcal{V}}) & \xrightarrow{(\dots)^s} & \Lambda^s \mathcal{F}|_U^{\vee} \otimes \operatorname{End}(T_{\mathcal{V}}) & \xrightarrow{\mathrm{id}\otimes \mathrm{str}} & \Lambda^s \mathcal{F}|_U^{\vee}
\end{array}$$

commutes, we obtain the desired conclusion:

$$(\mathrm{id}\otimes \mathrm{tr})\Big((\mathrm{At}_{\mathcal{F}}^{\mathrm{can}})^s\Big) = \mathscr{I}^{\top}\Bigg((\mathrm{id}\otimes \mathrm{str})\Big((\mathrm{At}_{T_{\mathcal{V}}}^{\mathrm{trivial}})^s\Big)\Bigg). \qquad \square$$

Since the Todd cocycle can be expressed in terms of scalar Atiyah cocycles $\frac{1}{s!}(\frac{i}{2\pi})^s \operatorname{str}(\mathrm{At}^s)$, we have the following immediate corollary.

Corollary 4.1.19. *The diagram*

$$\begin{array}{ccc}
\mathcal{T}_{\mathrm{poly}}^{\bullet}(\mathcal{V}) & \xrightarrow{\widetilde{\mathrm{td}}_{T_{\mathcal{V}}}^{\mathrm{trivial}}} & \mathcal{T}_{\mathrm{poly}}^{\bullet}(\mathcal{V}) \\
\uparrow{\scriptstyle \mathscr{I}} & & \uparrow{\scriptstyle \mathscr{I}} \\
C^{\infty}(\mathbb{R}^m \times \Bbbk_{\mathrm{formal}}^{0|l}) \otimes \mathcal{T}_{\mathrm{poly}}^{\bullet}\Bbbk_{\mathrm{formal}}^{r} & \xrightarrow[\widetilde{\mathrm{td}}_{\mathcal{F}|_U}^{\mathrm{can}}]{} & C^{\infty}(\mathbb{R}^m \times \Bbbk_{\mathrm{formal}}^{0|l}) \otimes \mathcal{T}_{\mathrm{poly}}^{\bullet}\Bbbk_{\mathrm{formal}}^{r}
\end{array}$$

commutes.
Here $\widetilde{\mathrm{td}}_{\mathcal{F}|_U}^{\mathrm{can}} \in \prod_{k=0}^{\infty} C^{\infty}(\mathbb{R}^m \times \Bbbk_{\mathrm{formal}}^{0|l}) \otimes \Omega^k(\Bbbk_{\mathrm{formal}}^{r|0})$ is the Todd cocycle of the restriction to U of the Fedosov Lie algebroid $\mathcal{F}$ associated with the canonical connection while $\widetilde{\mathrm{td}}_{T_{\mathcal{V}}}^{\mathrm{trivial}} \in \prod_{k=0}^{\infty} \Omega^k(\mathcal{V})$ is the Todd cocycle of the dg manifold $(\mathcal{V}, Q)$ associated with the trivial connection as in Example 2.2.16.
The Todd cocycles act by contraction on the spaces of polyvector fields.

Finally we have

$$\begin{aligned}
\mathscr{I} \circ \Big(\mathscr{U}_{U,\varpi}^{f}\Big)_1 &= \Big(\mathscr{U}_{Q}^{\mathcal{V}}\Big)_1 \circ \mathscr{I} && \text{(by Lemma 4.1.16)} \\
&= \mathrm{hkr} \circ \Big(\widetilde{\mathrm{td}}_{T_{\mathcal{V}}}^{\mathrm{trivial}}\Big)^{\frac{1}{2}} \circ \mathscr{I} && \text{(by Theorem 4.1.17)} \\
&= \mathrm{hkr} \circ \mathscr{I} \circ \Big(\widetilde{\mathrm{td}}_{\mathcal{F}|_U}^{\mathrm{can}}\Big)^{\frac{1}{2}} && \text{(by Corollary 4.1.19)} \\
&= \mathscr{I} \circ \mathrm{hkr} \circ \Big(\widetilde{\mathrm{td}}_{\mathcal{F}|_U}^{\mathrm{can}}\Big)^{\frac{1}{2}}
\end{aligned}$$

Therefore, in every coordinate chart U of M over which the vector bundles B and

L are trivial, we have

$$\left(\mathscr{U}^{f}_{U,\varpi}\right)_1 = \operatorname{hkr}\circ\left(\widetilde{\operatorname{td}}^{\operatorname{can}}_{\mathcal{F}|_U}\right)^{\frac{1}{2}}.$$

The proof of Proposition 4.1.14 is complete.

4.1.5 Proof of Theorem 4.1.3

The difference between Equations (2.8) and (2.7) is the factor $e^{\frac{1}{2}\operatorname{tr}\operatorname{At}^{\operatorname{can}}_{\mathcal{F}}}$. We start with considering $\operatorname{tr}\operatorname{At}^{\operatorname{can}}_{\mathcal{F}} \in \Gamma(\mathcal{F}^\vee)_1$.

Lemma 4.1.20. *We have*

$$\operatorname{tr}\operatorname{At}^{\operatorname{can}}_{\mathcal{F}} = d_{\mathcal{F}}(\operatorname{div} X^\nabla),$$

where $\operatorname{div} X^\nabla \in C^\infty(\mathcal{M}) = \Gamma(L^\vee \otimes \hat{S}B^\vee)$ *is the divergence of the formal vertical vector field* X^∇. *(Since the vector field* X^∇ *is tangent to the fibers and the fibers are vector spaces, we do not need to specify a volume form in order to make sense of the divergence of* X^∇. *Indeed, on a vector space, the divergence is canonically defined.) More explicitly,* $\operatorname{div} X^\nabla = \sum_k \hat{\partial}_k f_k$.

Proof. By Equation (2.9),

$$\operatorname{tr}\operatorname{At}^{\operatorname{can}}_{\mathcal{F}} = \sum_{i,k=1}^{r} \hat{\partial}_i(\hat{\partial}_k f_k)\chi_i = d_{\mathcal{F}}(\operatorname{div} X^\nabla). \qquad \square$$

Lemma 4.1.21. *Let* $\mathcal{A} \to \mathcal{M}$ *be a dg Lie algebroid. Let* $\mathcal{Q}$ *denote the endomorphism of* $\Gamma(\mathcal{A})$ *encoding the dg structure and let* $d_{\mathcal{A}}$ *denote the Chevalley–Eilenberg differential. If* $\xi \in \Gamma(\mathcal{A}^\vee)$ *satisfies* $d_{\mathcal{A}} = 0$ *and* $\mathcal{Q} = 0$, *then the contraction with* ξ *is a derivation of the differential Gerstenhaber algebra* $\left(\Gamma(\Lambda^\bullet\mathcal{A}), [-,-], \mathcal{Q}\right)$.

Applying Lemma 4.1.21 to the Fedosov dg Lie algebroid $\mathcal{F}$ and the section $\operatorname{tr}\left(\operatorname{At}^{\operatorname{can}}_{\mathcal{F}}\right)$ of $\mathcal{F}^\vee$ and noting that $\Gamma(\Lambda^\bullet\mathcal{F}) \cong \operatorname{tot}\left(\Gamma(\Lambda^\bullet L^\vee)\otimes_R \mathfrak{T}^\bullet_{\operatorname{poly}}\right)$ and $\mathcal{Q} = [Q,-]$, we obtain

Corollary 4.1.22. *1. The contraction by* $\operatorname{tr}\operatorname{At}^{\operatorname{can}}_{\mathcal{F}}$ *is a derivation of the differential Gerstenhaber algebra*

$$\left(\operatorname{tot}\left(\Gamma(\Lambda^\bullet L^\vee)\otimes_R \mathfrak{T}^\bullet_{\operatorname{poly}}\right), [Q,-]\right).$$

2. *The contraction by* $e^{\frac{1}{2}\operatorname{tr}\operatorname{At}^{\mathrm{can}}_{\mathcal{F}}}$ *is an automorphism of the differential Gerstenhaber algebra*

$$\Big(\operatorname{tot}\Big(\Gamma(\Lambda^\bullet L^\vee)\otimes_R \mathfrak{T}^\bullet_{\mathrm{poly}}\Big), [Q,-]\Big).$$

The composition $\Psi=\Phi\circ e^{\frac{1}{2}\operatorname{tr}\operatorname{At}^{\mathrm{can}}_{\mathcal{F}}}$ of the L_∞ quasi-isomorphism

$$\Phi:\operatorname{tot}(\Gamma(\Lambda^\bullet L^\vee)\otimes_R \mathfrak{T}^\bullet_{\mathrm{poly}})\to\operatorname{tot}(\Gamma(\Lambda^\bullet L^\vee)\otimes_R \mathscr{D}^\bullet_{\mathrm{poly}})$$

constructed in Proposition 4.1.7 of Φ with the contraction operator $e^{\frac{1}{2}\operatorname{tr}\operatorname{At}^{\mathrm{can}}_{\mathcal{F}}}$, which is an automorphism of the dgla $\operatorname{tot}(\Gamma(\Lambda^\bullet L^\vee)\otimes_R \mathscr{D}^\bullet_{\mathrm{poly}})$ according to Corollary 4.1.22, is an L_∞ quasi-isomorphism Ψ from $\operatorname{tot}(\Gamma(\Lambda^\bullet L^\vee)\otimes_R \mathfrak{T}^\bullet_{\mathrm{poly}})$ to $\operatorname{tot}(\Gamma(\Lambda^\bullet L^\vee)\otimes_R \mathscr{D}^\bullet_{\mathrm{poly}})$. Its first Taylor coefficient is

$$\Psi_1=\Phi_1\circ e^{\frac{1}{2}\operatorname{tr}\operatorname{At}^{\mathrm{can}}_{\mathcal{F}}}=\operatorname{hkr}\circ(\widetilde{\operatorname{td}}{}^{\mathrm{can}}_{\mathcal{F}})^{\frac{1}{2}}\circ e^{\frac{1}{2}\operatorname{tr}\operatorname{At}^{\mathrm{can}}_{\mathcal{F}}}=\operatorname{hkr}\circ(\operatorname{td}^{\mathrm{can}}_{\mathcal{F}})^{\frac{1}{2}}$$

— we used Proposition 4.1.14 in the second equality. The proof of Theorem 4.1.3 is thus complete.

4.1.6 Proof of Theorem 4.1.1

A straightforward computation yields the following lemma.

Lemma 4.1.23. *The following diagram commutes.*

$$\begin{array}{ccc}
\operatorname{tot}\Big(\Gamma(\Lambda^\bullet L^\vee)\otimes_R \mathfrak{T}^\bullet_{\mathrm{poly}}\Big) & \xrightarrow{\operatorname{hkr}} & \operatorname{tot}\Big(\Gamma(\Lambda^\bullet L^\vee)\otimes_R \mathscr{D}^\bullet_{\mathrm{poly}}\Big)\\
{\scriptstyle\sigma_\natural}\big\downarrow & & \big\downarrow{\scriptstyle\sigma_\natural}\\
\operatorname{tot}\Big(\Gamma(\Lambda^\bullet A^\vee)\otimes_R \mathcal{T}^\bullet_{\mathrm{poly}}\Big) & \xrightarrow[\operatorname{hkr}]{} & \operatorname{tot}\Big(\Gamma(\Lambda^\bullet A^\vee)\otimes_R \mathcal{D}^\bullet_{\mathrm{poly}}\Big)
\end{array}$$

The following result is an immediate consequence of Proposition 2.2.19.

Corollary 4.1.24. *The following diagram commutes.*

$$\begin{array}{ccc}
\operatorname{tot}\Big(\Gamma(\Lambda^\bullet L^\vee)\otimes_R \mathfrak{T}^\bullet_{\mathrm{poly}}\Big) & \xrightarrow{(\operatorname{td}^{\mathrm{can}}_{\mathcal{F}})^{\frac{1}{2}}} & \operatorname{tot}\Big(\Gamma(\Lambda^\bullet L^\vee)\otimes_R \mathfrak{T}^\bullet_{\mathrm{poly}}\Big)\\
{\scriptstyle\sigma_\natural}\big\downarrow & & \big\downarrow{\scriptstyle\sigma_\natural}\\
\operatorname{tot}\Big(\Gamma(\Lambda^\bullet A^\vee)\otimes_R \mathcal{T}^\bullet_{\mathrm{poly}}\Big) & \xrightarrow[(\operatorname{td}^{\nabla}_{L/A})^{\frac{1}{2}}]{} & \operatorname{tot}\Big(\Gamma(\Lambda^\bullet A^\vee)\otimes_R \mathcal{T}^\bullet_{\mathrm{poly}}\Big)
\end{array}$$

According to Theorem 2.3.12, we have a contraction

$$\Big(\operatorname{tot}\Big(\Gamma(\Lambda^\bullet A^\vee)\otimes_R \mathcal{T}^\bullet_{\mathrm{poly}}\Big), d_A^{\mathrm{Bott}}\Big) \underset{\sigma_\natural}{\overset{\breve{\tau}_\natural}{\rightleftarrows}} \Big(\operatorname{tot}\Big(\Gamma(\Lambda^\bullet L^\vee)\otimes_R \mathfrak{T}^\bullet_{\mathrm{poly}}(B)\Big), \mathcal{L}_Q\Big) \circlearrowright \breve{h}_\natural\,.$$

The r.h.s. $\operatorname{tot}\Big(\Gamma(\Lambda^\bullet L^\vee)\otimes_R \mathfrak{T}^\bullet_{\mathrm{poly}}(B)\Big)$ is a dgla while the l.h.s. $\operatorname{tot}\Big(\Gamma(\Lambda^\bullet A^\vee)\otimes_R \mathcal{T}^\bullet_{\mathrm{poly}}\Big)$ inherits an L_∞ structure from the dgla structure of the r.h.s. by homotopy transfer.

Lemma 4.1.25 (Homotopy transfer of L_∞ structures [2, Theorem 1.9])**.** *Let $(C, \eth)$ and (K, d) be two cochain complexes and let*

$$(C,\eth) \underset{\sigma}{\overset{\tau}{\rightleftarrows}} (K,d) \circlearrowright h$$

be a contraction of (K, d) onto $(C, \eth)$. Given an L_∞ algebra structure on K (with d as unary bracket), there exists a 'transferred' L_∞ algebra structure on C and a pair of L_∞ quasi-isomorphisms $T : C \to K$ and $\Sigma : K \to C$ having the chain maps τ and σ as respective first Taylor coefficients.

Lemma 4.1.25 asserts the existence of an L_∞ quasi-isomorphism $\mathcal{T}$ having $\breve{\tau}_\natural$ as first Taylor coefficient.

According to Theorem 2.3.18, there is also a contraction

$$\Big(\operatorname{tot}\left(\Gamma(\Lambda^\bullet A^\vee)\otimes_R \mathcal{D}^\bullet_{\mathrm{poly}}\right), d_A^{\mathcal{U}} + d_{\mathscr{H}}\Big) \underset{\sigma_\natural}{\overset{\breve{\tau}_\natural}{\rightleftarrows}} \Big(\operatorname{tot}\left(\Gamma(\Lambda^\bullet L^\vee)\otimes_R \mathscr{D}^\bullet_{\mathrm{poly}}\right), [\![Q+m, -]\!]\Big) \circlearrowright \breve{h}_\natural\,.$$

Again, the l.h.s. inherits an L_∞ structure from the dgla structure of the r.h.s. by homotopy transfer. Lemma 4.1.25 asserts the existence of an L_∞ quasi-isomorphism $\Sigma_\natural$ having $\sigma_\natural$ as first Taylor coefficient.

Consider the L_∞ quasi-isomorphism

$$\mathcal{I} : \operatorname{tot}\Big(\Gamma(\Lambda^\bullet A^\vee)\otimes_R \mathcal{T}^\bullet_{\mathrm{poly}}\Big) \to \operatorname{tot}\Big(\Gamma(\Lambda^\bullet A^\vee)\otimes_R \mathcal{D}^\bullet_{\mathrm{poly}}\Big)$$

obtained as the composition

$$\mathcal{I} = \Sigma_\natural \circ \Psi \circ \mathcal{T}$$

of the L_∞ quasi-isomorphism $\Psi : \Gamma(\Lambda^\bullet L^\vee)\otimes_R \mathfrak{T}^\bullet_{\mathrm{poly}} \to \Gamma(\Lambda^\bullet L^\vee)\otimes_R \mathscr{D}^\bullet_{\mathrm{poly}}$ of Theorem 4.1.3 with the L_∞ quasi-isomorphisms $\mathcal{T}$ and $\Sigma_\natural$:

$$\begin{array}{ccc}
\operatorname{tot}\Big(\Gamma(\Lambda^\bullet L^\vee)\otimes_R \mathfrak{T}^\bullet_{\mathrm{poly}}(B)\Big) & \xrightarrow{\Psi} & \operatorname{tot}\Big(\Gamma(\Lambda^\bullet L^\vee)\otimes_R \mathscr{D}^\bullet_{\mathrm{poly}}(B)\Big) \\
\mathcal{T}\Big\uparrow & & \Big\downarrow \Sigma_\natural \\
\operatorname{tot}\Big(\Gamma(\Lambda^\bullet A^\vee)\otimes_R \mathcal{T}^\bullet_{\mathrm{poly}}\Big) & \underset{\mathcal{I}}{\dashrightarrow} & \operatorname{tot}\Big(\Gamma(\Lambda^\bullet A^\vee)\otimes_R \mathcal{D}^\bullet_{\mathrm{poly}}\Big).
\end{array}$$

Its first Taylor coefficient is

$$\begin{aligned}
\mathcal{I}_1 &= (\Sigma_\natural)_1 \circ \Psi_1 \circ (\mathcal{T})_1 \\
&= \sigma_\natural \circ (\operatorname{hkr} \circ (\operatorname{td}^{\mathrm{can}}_{\mathcal{F}})^{\frac{1}{2}}) \circ \breve{\tau}_\natural && \text{(by Theorem 4.1.3)} \\
&= \operatorname{hkr} \circ \sigma_\natural \circ (\operatorname{td}^{\mathrm{can}}_{\mathcal{F}})^{\frac{1}{2}} \circ \breve{\tau}_\natural && \text{(by Lemma 4.1.23)} \\
&= \operatorname{hkr} \circ (\operatorname{td}^{\nabla}_{L/A})^{\frac{1}{2}} \circ \sigma_\natural \circ \breve{\tau}_\natural && \text{(by Corollary 4.1.24)} \\
&= \operatorname{hkr} \circ (\operatorname{td}^{\nabla}_{L/A})^{\frac{1}{2}}
\end{aligned}$$

This concludes the proof of Theorem 4.1.1.

4.2 Applications

4.2.1 Complex manifolds

Let X be a complex manifold. Then $T_X \otimes \mathbb{C} \cong T_X^{0,1} \bowtie T_X^{1,0}$ is a matched pair of Lie algebroids (see Section 2.3.1 or [51] for the definition of matched pairs). Hence $(T_X \otimes \mathbb{C}, T_X^{0,1})$ is a Lie pair with quotient $T_X^{1,0}$. Its Bott representation is the flat $T_X^{0,1}$-connection on $T_X^{1,0}$ which encodes the holomorphic vector bundle structure of $T_X^{1,0}$; the sections of $T_X^{1,0}$ which are flat w.r.t. the $T_X^{0,1}$-connection are precisely the holomorphic sections of $T_X^{1,0}$. It is simple to see that the Bott A-connection on L/A, which is a flat $T_X^{0,1}$-connection on $T_X^{1,0}$, coincides exactly with the one making $T_X^{1,0}$ into a holomorphic vector bundle over X. In other words, the Chevalley–Eilenberg differential associated with the Bott representation of the Lie pair $(T_X \otimes \mathbb{C}, T_X^{0,1})$ is the Dolbeault operator

$$\bar{\partial} : \Omega^{0,\bullet}(T_X^{1,0}) \to \Omega^{0,\bullet+1}(T_X^{1,0}).$$

4.2.1.1 Atiyah and Todd classes of complex manifolds

A torsion-free $T_X \otimes \mathbb{C}$-connection ∇ on $T_X^{1,0}$ extending the Bott $T_X^{0,1}$-connection is necessarily the sum $\nabla = \bar{\partial} + \nabla^{1,0}$ — more precisely $d^{\nabla} = \bar{\partial} + d^{\nabla^{1,0}}$ — of the Dolbeault operator and a torsion-free $T_X^{1,0}$-connection $\nabla^{1,0}$ on $T_X^{1,0}$, i.e. a $\mathbb{C}$-bilinear map $\nabla^{1,0} : \Gamma(T_X^{1,0}) \times \Gamma(T_X^{1,0}) \to \Gamma(T_X^{1,0})$ satisfying the usual connection axioms and the condition

$$\nabla_X Y - \nabla_Y X = [X, Y], \quad \forall\, X, Y \in \Gamma(T_X^{1,0}).$$

The Atiyah cocycle associated with such a connection ∇ is the element $R_{1,1}^{\nabla} \in \Omega^{1,1}\Big(\operatorname{End}(T_X^{1,0})\Big)$ defined by

$$R_{1,1}^{\nabla}(a; b) = \nabla_a \nabla_b - \nabla_b \nabla_a - \nabla_{[a,b]}, \quad \forall\, a \in \Gamma(T_X^{0,1}),\ b \in \Gamma(T_X^{1,0}).$$

Its cohomology class $\alpha_X \in H^{1,1}\Big(X, \operatorname{End}(T_X^{1,0})\Big)$ is independent of the choice of the connection ∇ and is precisely the Atiyah class of the complex manifold X.

The Todd cocycle associated with the connection ∇ is

$$\operatorname{td}_X^{\nabla} = \det\left(\frac{R_{1,1}^{\nabla}}{1 - e^{-R_{1,1}^{\nabla}}}\right) \in \bigoplus_{k=0}^{\infty} \Omega^{k,0}\Big(\Lambda^k (T_X^{0,1})^{\vee}\Big) \cong \bigoplus_{k=0}^{\infty} \Omega^{k,k}(X).$$

Its cohomology class

$$\operatorname{Td}_X = \det\left(\frac{\alpha_X}{1 - e^{-\alpha_X}}\right) \in \bigoplus_{k=0}^{\infty} H^k\Big(X, \Lambda^k (T_X^{0,1})^{\vee}\Big).$$

is independent of the choice of the connection ∇ and is called the Todd class of the complex manifold X.

4.2.1.2 Polyvector fields and polydifferential operators on complex manifolds

Since $T_X^{0,1} \bowtie T_X^{1,0}$ is a matched pair, it follows from Corollary 2.3.14 that $\Omega^{0,\bullet}\Big(X, \mathcal{T}_{\text{poly}}^{\bullet}(X)\Big)$ is a differential Gerstenhaber algebra with the Dolbeault operator $\bar{\partial} : \Omega^{0,\bullet}\Big(X, \mathcal{T}_{\text{poly}}^{\bullet}(X)\Big) \to \Omega^{0,\bullet+1}\Big(X, \mathcal{T}_{\text{poly}}^{\bullet}(X)\Big)$ as differential, the wedge product as associative multiplication and the natural extension (see Equation (2.13))

$$[\xi_1 \otimes b_1, \xi_2 \otimes b_2] = \xi_1 \wedge \xi_2 \otimes [b_1, b_2] + \xi_1 \wedge \nabla^{\text{Bott}}_{b_1} \xi_2 \otimes b_2 - \nabla^{\text{Bott}}_{b_2} \xi_1 \wedge \xi_2 \otimes b_1,$$
$$\forall\, \xi_1, \xi_2 \in \Omega^{0,\bullet}(X),\ b_1, b_2 \in \Gamma(T_X^{1,0}),$$

of the Lie bracket on $\Gamma(T_X^{1,0})$ as graded Lie bracket.

Similarly, $\Omega^{0,\bullet}\Big(X, \mathcal{D}^\bullet_{\text{poly}}(X)\Big)$ is a dgla and its cohomology is a Gerstenhaber algebra (Corollary 2.3.19). Here the differential is $\bar\partial + \mathrm{id} \otimes d_{\mathscr{H}}$, where $\bar\partial : \Omega^{0,\bullet}(X, \mathcal{D}^\bullet_{\text{poly}}(X)) \to \Omega^{0,\bullet+1}(X, \mathcal{D}^\bullet_{\text{poly}}(X))$ is the Dolbeault operator, while the associative multiplication and the graded Lie bracket are given by Proposition 2.3.4.

Note that $\Big(\Omega^{0,\bullet}(X, \mathcal{T}^\bullet_{\text{poly}}(X)), \bar\partial\Big)$ is the Dolbeault resolution of the complex of sheaves

$$0 \to \mathcal{O}_X \xrightarrow{0} \Theta_X \xrightarrow{0} \Lambda^2\Theta_X \xrightarrow{0} \Lambda^3\Theta_X \to \cdots$$

of holomorphic polyvector fields over X, while $\Big(\Omega^{0,\bullet}(X, \mathcal{D}^\bullet_{\text{poly}}(X)), \bar\partial + \mathrm{id} \otimes d_{\mathscr{H}}\Big)$ is the Dolbeault resolution of the complex of sheaves

$$0 \to \mathcal{O}_X \to \mathscr{D}_X \xrightarrow{d_{\mathscr{H}}} \mathscr{D}_X^{\otimes 2} \xrightarrow{d_{\mathscr{H}}} \mathscr{D}_X^{\otimes 3} \to \cdots$$

of holomorphic polydifferential operators over X.

As a result, for the Lie pair $(L = T_X \otimes \mathbb{C}, A = T_X^{0,1})$, the cohomology

$$\mathbb{H}^\bullet_{\text{CE}}(A, \mathcal{T}^\bullet_{\text{poly}}) = \mathbb{H}(\Big(\Omega^{0,\bullet}(X, \mathcal{T}^\bullet_{\text{poly}}(X)), \bar\partial\Big)$$

is isomorphic to the sheaf cohomology $\mathbb{H}^\bullet_{\text{sheaf}}(X, \Lambda^\bullet\Theta_X)$ while the cohomology

$$\mathbb{H}^\bullet_{\text{CE}}(A, \mathcal{D}^\bullet_{\text{poly}}) = \mathbb{H}^\bullet\left(\Omega^{0,\bullet}(X, \mathcal{D}^\bullet_{\text{poly}}(X)), \bar\partial + \mathrm{id} \otimes d_{\mathscr{H}}\right)$$

is isomorphic to the Hochschild cohomology $HH^\bullet(X) \cong \mathrm{Ext}^\bullet_{\mathcal{O}_{X\times X}}(\mathcal{O}_\Delta, \mathcal{O}_\Delta)$ (see [12]) of the complex manifold X.

4.2.1.3 Formality theorem for complex manifolds

Theorems 4.1.1 and 4.1.2 imply the following two theorems.

Theorem 4.2.1 (Formality theorem for complex manifolds)**.** *Let X be a complex manifold. Choose a torsion-free $T_X^{1,0}$-connection $\nabla^{1,0}$ on $T_X^{1,0}$. There exists an L_∞*

quasi-isomorphism

$$\mathcal{I} : \Omega^{0,\bullet}\Big(X, \mathcal{T}^{\bullet}_{\mathrm{poly}}(X)\Big) \to \Omega^{0,\bullet}\Big(X, \mathcal{D}^{\bullet}_{\mathrm{poly}}(X)\Big)$$

with first taylor coefficient $\mathcal{I}_1$ satisfying the following two properties:

- *$\mathcal{I}_1$ preserves the associative algebra structures up to homotopy;*
- *$\mathcal{I}_1 = \mathrm{hkr} \circ \Big(\mathrm{td}_X^{\bar{\partial}+\nabla^{1,0}}\Big)^{\frac{1}{2}}$, where the square root of the Todd cocycle $\mathrm{td}_X^{\bar{\partial}+\nabla^{1,0}} \in \bigoplus_{k=0} \Omega^{k,k}(X)$ acts on $\Omega^{0,\bullet}\Big(X, \mathcal{T}^{\bullet}_{\mathrm{poly}}(X)\Big)$ by contraction.*

Theorem 4.2.2 (Kontsevich-Duflo theorem for complex manifolds)**.** *For every complex manifold X, the composition*

$$\mathrm{hkr} \circ (\mathrm{Td}_X)^{\frac{1}{2}} : \mathbb{H}^{\bullet}_{\mathrm{sheaf}}(X, \Lambda^{\bullet} T_X) \to HH^{\bullet}(X)$$

is an isomorphism of Gerstenhaber algebras. It is understood that the square root of the Todd class

$$(\mathrm{Td}_X) \in \bigoplus_{k=0} H^{k,k}(X) \cong H^k_{\mathrm{sheaf}}(X, \Omega^k_X)$$

acts on $\mathbb{H}^{\bullet}_{\mathrm{sheaf}}(X, \Lambda^{\bullet} T_X)$ by contraction.

The Kontsevich-Duflo theorem for complex manifolds is due to Kontsevich [30] (for associative algebra structures). See [11] for a detailed proof including Gerstenhaber algebra structures.

4.2.2 Lie algebra pairs

A Lie algebra pair is a Lie pair $(\mathfrak{g}, \mathfrak{h})$, where $\mathfrak{g}$ is a finite-dimensional Lie algebra and $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$.

4.2.2.1 Atiyah and Todd classes of Lie algebra pairs

A $\mathfrak{g}$-connection on $\mathfrak{g}/\mathfrak{h}$ is simply a bilinear map $\nabla : \mathfrak{g} \times \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}/\mathfrak{h}$. Its torsion is the linear map $T^{\nabla} : \Lambda^2 \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ defined by

$$T^{\nabla}(X, Y) = \nabla_X q(Y) - \nabla_Y q(X) - q([X, Y]), \quad \forall\, X, Y \in \mathfrak{g}.$$

The map $q : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ is the canonical projection.

Let ∇ be a $\mathfrak{g}$-connection on $\mathfrak{g}/\mathfrak{h}$ which extends the Bott $\mathfrak{h}$-connection: $\nabla_a^{\mathrm{Bott}} q(l) = q([a,l])$, for all $a \in \mathfrak{h}$ and $l \in \mathfrak{g}$. The Atiyah cocycle associated with ∇ is the bilinear map

$$R_{1,1}^{\nabla} : \mathfrak{h} \otimes \mathfrak{g}/\mathfrak{h} \to \mathrm{End}(\mathfrak{g}/\mathfrak{h})$$

defined by

$$R_{1,1}^{\nabla}(a; q(l)) = \nabla_a \nabla_l - \nabla_l \nabla_a - \nabla_{[a,l]}, \qquad \forall\, a \in \mathfrak{h},\ l \in \mathfrak{g}.$$

According to Proposition 2.2.6, the element $R_{1,1}^{\nabla} \in \mathfrak{h}^{\vee} \otimes \mathfrak{h}^{\perp} \otimes \mathrm{End}(\mathfrak{g}/\mathfrak{h})$ is a Chevalley–Eilenberg 1-cocycle for the Lie algebra $\mathfrak{h}$ with values in the $\mathfrak{h}$-module $\mathfrak{h}^{\perp} \otimes \mathrm{End}(\mathfrak{g}/\mathfrak{h})$. Its cohomology class $\alpha_{\mathfrak{g}/\mathfrak{h}} \in H^1_{\mathrm{CE}}\big(\mathfrak{h}, \mathfrak{h}^{\perp} \otimes \mathrm{End}(\mathfrak{g}/\mathfrak{h})\big)$ is independent of the choice of $\mathfrak{g}$-connection ∇ and is called the Atiyah class of the Lie algebra pair $(\mathfrak{g}, \mathfrak{h})$.

The Todd cocycle of the Lie algebra pair $(\mathfrak{g}, \mathfrak{h})$ associated with the connection ∇ is the Chevalley-Eilenberg cocycle

$$\mathrm{td}_{\mathfrak{g}/\mathfrak{h}}^{\nabla} = \det\left(\frac{R_{1,1}^{\nabla}}{1 - e^{-R_{1,1}^{\nabla}}}\right) \in \bigoplus_{k=0} \Lambda^k \mathfrak{h}^{\vee} \otimes \Lambda^k \mathfrak{h}^{\perp}.$$

The Todd class of the Lie algebra pair $(\mathfrak{g}, \mathfrak{h})$ is the corresponding Chevalley-Eilenberg cohomology class

$$\mathrm{Td}_{\mathfrak{g}/\mathfrak{h}} = \det\left(\frac{\alpha_{\mathfrak{g}/\mathfrak{h}}}{1 - e^{-\alpha_{\mathfrak{g}/\mathfrak{h}}}}\right) \in \bigoplus_{k=0} H^k_{\mathrm{CE}}(\mathfrak{h}, \Lambda^k \mathfrak{h}^{\perp}).$$

4.2.2.2 Polyvector fields and polydifferential operators on Lie algebra pairs

For a Lie algebra pair $(\mathfrak{g}, \mathfrak{h})$, it follows from Corollary 2.3.14 and Corollary 2.3.19 that both $\mathrm{tot}\big(\Lambda^{\bullet}\mathfrak{h}^{\vee} \otimes \Lambda^{\bullet+1}(\mathfrak{g}/\mathfrak{h})\big)$ and $\mathrm{tot}\left(\Lambda^{\bullet}\mathfrak{h}^{\vee} \otimes \left(\frac{\mathcal{U}(\mathfrak{g})}{\mathcal{U}(\mathfrak{g})\cdot\mathfrak{h}}\right)^{\otimes \bullet+1}\right)$ carry L_∞ algebra structures unique up to L_∞ quasi-isomorphisms. (Whenever $\mathfrak{g} = \mathfrak{h} \bowtie \mathfrak{m}$ is a matched pair, $\frac{\mathcal{U}(\mathfrak{g})}{\mathcal{U}(\mathfrak{g})\cdot\mathfrak{h}}$ is isomorphic to $\mathcal{U}(\mathfrak{m})$ and the two L_∞ algebras above are actually differential graded Lie algebras.)

The quotient $\mathfrak{g}/\mathfrak{h}$ of a Lie algebra pair $(\mathfrak{g}, \mathfrak{h})$ is an $\mathfrak{h}$-module with the action

$$a \cdot q(l) = \nabla_a^{\mathrm{Bott}} q(l) = q([a,l]), \quad \forall\, a \in \mathfrak{h},\ l \in \mathfrak{g}.$$

Again, $q : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ is the canonical projection. This action extends by the Leibniz rule to an $\mathfrak{h}$-action on $\mathcal{T}_{\text{poly}}^{\bullet} = \Lambda^{\bullet+1}(\mathfrak{g}/\mathfrak{h})$. Let $d_{\mathfrak{h}}^{\text{Bott}} : \Lambda^p \mathfrak{h}^{\vee} \otimes \Lambda^{q+1}(\mathfrak{g}/\mathfrak{h}) \to \Lambda^{p+1}\mathfrak{h}^{\vee} \otimes \Lambda^{q+1}(\mathfrak{g}/\mathfrak{h})$ be the corresponding Chevalley–Eilenberg differential. According to Corollary 2.3.14, the space $\operatorname{tot}\left(\Lambda^{\bullet}\mathfrak{h}^{\vee} \otimes \Lambda^{\bullet+1}(\mathfrak{g}/\mathfrak{h})\right)$ carries an L_∞ algebra structure, unique up to L_∞ quasi-isomorphism, with $d_{\mathfrak{h}}^{\text{Bott}}$ as first bracket. Furthermore, when endowed with the wedge product, the hypercohomology $\mathbb{H}_{\text{CE}}^{\bullet}\left(\mathfrak{h}, \Lambda^{\bullet+1}(\mathfrak{g}/\mathfrak{h})\right)$ becomes a Gerstenhaber algebra.

Similarly, the Lie algebra $\mathfrak{h}$ acts on $\mathcal{D}_{\text{poly}}^{0} = \frac{\mathcal{U}(\mathfrak{g})}{\mathcal{U}(\mathfrak{g})\cdot\mathfrak{h}}$ by left multiplication and henceforth on $\mathcal{D}_{\text{poly}}^{\bullet} = \left(\frac{\mathcal{U}(\mathfrak{g})}{\mathcal{U}(\mathfrak{g})\cdot\mathfrak{h}}\right)^{\otimes\bullet+1}$ as well. The Chevalley–Eilenberg differential associated with this action is denoted

$$d_A^{\mathcal{U}} : \Lambda^p\mathfrak{h}^{\vee} \otimes \mathcal{D}_{\text{poly}}^{q} \to \Lambda^{p+1}\mathfrak{h}^{\vee} \otimes \mathcal{D}_{\text{poly}}^{q}.$$

Meanwhile, the Hochschild differential $d_{\mathscr{H}} : \mathcal{D}_{\text{poly}}^{q} \to \mathcal{D}_{\text{poly}}^{q+1}$ extends to

$$\operatorname{id} \otimes d_{\mathscr{H}} : \Lambda^p\mathfrak{h}^{\vee} \otimes \mathcal{D}_{\text{poly}}^{q} \to \Lambda^{p}\mathfrak{h}^{\vee} \otimes \mathcal{D}_{\text{poly}}^{q+1}.$$

By Corollary 2.3.19, the graded vector space $\operatorname{tot}\left(\Lambda^{\bullet}\mathfrak{h}^{\vee} \otimes \left(\frac{\mathcal{U}(\mathfrak{g})}{\mathcal{U}(\mathfrak{g})\cdot\mathfrak{h}}\right)^{\otimes\bullet+1}\right)$ carries an L_∞ algebra structure, unique up to quasi-isomorphism, with $d_A^{\mathcal{U}} + \operatorname{id}\otimes d_{\mathscr{H}}$ as first bracket. When endowed with the cup product, the corresponding hypercohomology $\mathbb{H}_{\text{CE}}^{\bullet}\left(\mathfrak{h}, \left(\frac{\mathcal{U}(\mathfrak{g})}{\mathcal{U}(\mathfrak{g})\cdot\mathfrak{h}}\right)^{\otimes\bullet+1}\right)$ becomes a Gerstenhaber algebra.

The above L_∞ algebra structures depend on the choice of a splitting of the short exact sequence $0 \to \mathfrak{h} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{h} \to 0$ and a torsion-free $\mathfrak{g}$-connection on $\mathfrak{g}/\mathfrak{h}$. However, different choices induce quasi-isomorphic L_∞ algebras. Moreover, the first 'Taylor coefficient' of the L_∞ quasi-isomorphism is the identity map. Therefore, the Gerstenhaber algebra structures inherited by the cohomologies are in fact canonical [3, 59].

The natural map induced by skew-symmetrization (see Section 2.2.5)

$$\text{hkr} : \operatorname{tot}\left(\Lambda^{\bullet}\mathfrak{h}^{\vee} \otimes \Lambda^{\bullet+1}(\mathfrak{g}/\mathfrak{h})\right) \to \operatorname{tot}\left(\Lambda^{\bullet}\mathfrak{h}^{\vee} \otimes \left(\frac{\mathcal{U}(\mathfrak{g})}{\mathcal{U}(\mathfrak{g})\cdot\mathfrak{h}}\right)^{\otimes\bullet+1}\right)$$

is a quasi-isomorphism of cochain complexes.

4.2.2.3 Formality theorem for Lie algebra pairs

Theorems 4.1.1 and 4.1.2 imply the following corollaries.

Theorem 4.2.3 (Formality theorem for Lie algebra pairs)**.** *Let* $(\mathfrak{g}, \mathfrak{h})$ *be a Lie algebra pair. Given a splitting of the short exact sequence* $0 \to \mathfrak{h} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{h} \to 0$ *and a torsion-free* $\mathfrak{g}$*-connection* ∇ *on* $\mathfrak{g}/\mathfrak{h}$*, there exists an* L_∞ *quasi-isomorphism*

$$\mathcal{I} : \operatorname{tot}\left(\Lambda^\bullet \mathfrak{h}^\vee \otimes \Lambda^{\bullet+1}(\mathfrak{g}/\mathfrak{h})\right) \to \operatorname{tot}\left(\Lambda^\bullet \mathfrak{h}^\vee \otimes \left(\frac{\mathcal{U}(\mathfrak{g})}{\mathcal{U}(\mathfrak{g})\cdot\mathfrak{h}}\right)^{\otimes \bullet+1}\right)$$

with first 'Taylor coefficient' $\mathcal{I}_1$ *satisfying the following two properties:*

1. $\mathcal{I}_1$ *preserves the associative algebra structures (wedge and cup product, respectively) up to homotopy;*

2. $\mathcal{I}_1 = \operatorname{hkr} \circ (\operatorname{td}^\nabla_{\mathfrak{g}/\mathfrak{h}})^{\frac{1}{2}}$*, where*

$$(\operatorname{td}^\nabla_{\mathfrak{g}/\mathfrak{h}})^{\frac{1}{2}} \in \bigoplus_{k=0}^{\infty} \Lambda^k \mathfrak{h}^\vee \otimes \Lambda^k \mathfrak{h}^\perp = \bigoplus_{k=0} \Lambda^k \mathfrak{h}^\vee \otimes \Lambda^k (\mathfrak{g}/\mathfrak{h})^\vee$$

acts on $\operatorname{tot}\left(\Lambda^\bullet \mathfrak{h}^\vee \otimes \Lambda^{\bullet+1}(\mathfrak{g}/\mathfrak{h})\right)$ *by contraction.*

Theorem 4.2.4 (Kontsevich-Duflo type theorem for Lie algebra pairs)**.** *Given a Lie algebra pair* $(\mathfrak{g}, \mathfrak{h})$*, the map*

$$\operatorname{hkr} \circ \operatorname{Td}^{\frac{1}{2}}_{\mathfrak{g}/\mathfrak{h}} : \mathbb{H}^\bullet_{\mathrm{CE}}(\mathfrak{h}, \Lambda^{\bullet+1}(\mathfrak{g}/\mathfrak{h})) \xrightarrow{\cong} \mathbb{H}^\bullet_{\mathrm{CE}}\left(\mathfrak{h}, \left(\frac{\mathcal{U}(\mathfrak{g})}{\mathcal{U}(\mathfrak{g})\cdot\mathfrak{h}}\right)^{\otimes \bullet+1}\right)$$

is an isomorphism of Gerstenhaber algebras. It is understood that the square root $\operatorname{Td}^{\frac{1}{2}}_{\mathfrak{g}/\mathfrak{h}}$ *of the Todd class* $\operatorname{Td}_{\mathfrak{g}/\mathfrak{h}} \in \bigoplus_{k=0} H^k_{\mathrm{CE}}(\mathfrak{h}, \Lambda^k \mathfrak{h}^\perp)$ *acts on* $\mathbb{H}^\bullet_{\mathrm{CE}}(\mathfrak{h}, \Lambda^{\bullet+1}(\mathfrak{g}/\mathfrak{h}))$ *by contraction.*

4.2.3 $\mathfrak{g}$-manifolds

In this section, we consider the formality theorem for a $\mathfrak{g}$-manifold, i.e. a smooth manifold with a Lie algebra action (see [38] for more details). The case of trivial Lie algebra action was considered in [65]. Let M be a $\mathfrak{g}$-manifold with infinitesimal action $\mathfrak{g} \ni a \mapsto \hat{a} \in \mathfrak{X}(M)$. Every $\mathfrak{g}$-manifold M determines in a canonical way a matched pair of Lie algebroids $(\mathfrak{g} \ltimes M) \bowtie T_M$ (see e.g. [51, Example 5.5] or [40]).

The notation $\mathfrak{g} \ltimes M$ refers to the transformation Lie algebroid arising from the infinitesimal $\mathfrak{g}$-action on M. Therefore, we can form a Lie pair (L, A), where $L = (\mathfrak{g} \ltimes M) \bowtie T_M$ and $A = \mathfrak{g} \ltimes M$. In this case, the quotient L/A is isomorphic to T_M and the Bott A-connection on L/A is the map

$$\nabla^{\text{Bott}} : C^\infty(M, \mathfrak{g}) \otimes \mathfrak{X}(M) \to \mathfrak{X}(M)$$

defined by

$$\nabla^{\text{Bott}}_{f \cdot a} X = f \cdot [\hat{a}, X],$$

for all $a \in \mathfrak{g}$, $f \in C^\infty(M)$, and $X \in \mathfrak{X}(M)$.

4.2.3.1 Atiyah and Todd classes of $\mathfrak{g}$-manifolds

It is not difficult to that, for the Lie pair constituted of the Lie algebroid $L = (\mathfrak{g} \ltimes M) \bowtie T_M$ and its Lie subalgebroid $A = \mathfrak{g} \ltimes M$, a choice of L-connection on L/A extending the Bott A-connection is essentially a choice of affine connection on M. Moreover, the torsion of the L-connection on L/A reduces to the torsion of the corresponding affine connection on M.

Given an affine connection ∇ on M, the Atiyah 1-cocycle associated with ∇ is the map

$$R^\nabla_{1,1} : \mathfrak{g} \times \mathfrak{X}(M) \to \operatorname{End}_{\mathcal{R}} \mathfrak{X}(M)$$

defined by

$$R^\nabla_{1,1}(a, X) = \mathcal{L}_{\hat{a}} \circ \nabla_X - \nabla_X \circ \mathcal{L}_{\hat{a}} - \nabla_{\mathcal{L}_{\hat{a}} X},$$

for all $a \in \mathfrak{g}$ and $X \in \mathfrak{X}(M)$.

Following Proposition 2.2.6, we prove the following

Proposition 4.2.5. *1. The Atiyah cocycle $R^\nabla_{1,1} \in \mathfrak{g}^\vee \otimes \Gamma(T^\vee_M \otimes \operatorname{End} T_M)$ is a Chevalley–Eilenberg 1-cocycle of the $\mathfrak{g}$-module $\Gamma(T^\vee_M \otimes \operatorname{End} T_M)$.*

2. The cohomology class $\alpha_{M/\mathfrak{g}} \in H^1_{\text{CE}}(\mathfrak{g}, \Gamma(T^\vee_M \otimes \operatorname{End} T_M))$ of the 1-cocycle $R^\nabla_{1,1}$ does not depend on the choice of connection ∇.

The cohomology class $\alpha_{M/\mathfrak{g}}$ is called the Atiyah class of the $\mathfrak{g}$-manifold M. It is the obstruction class to the existence of a $\mathfrak{g}$-invariant connection on M, i.e. an affine connection ∇ on M satisfying

$$[\hat{a}, \nabla_X Y] = \nabla_{[\hat{a},X]} Y + \nabla_X [\hat{a}, Y]$$

for all $a \in \mathfrak{g}$ and $X, Y \in \mathfrak{X}(M)$. Note that if $\mathfrak{g}$ is a compact Lie algebra, $\alpha_{M/\mathfrak{g}}$ vanishes since $\mathfrak{g}$-invariant connections always exist.

In the context of $\mathfrak{g}$-manifolds, the Todd cocycle of a $\mathfrak{g}$-manifold M is the Chevalley–Eilenberg cocycle

$$\operatorname{td}^{\nabla}_{M/\mathfrak{g}} = \det\left(\frac{R^{\nabla}_{1,1}}{1 - e^{-R^{\nabla}_{1,1}}}\right) \in \bigoplus_{k=0} \Lambda^k \mathfrak{g}^\vee \otimes \Omega^k(M).$$

Its corresponding Chevalley–Eilenberg cohomology class is the *Todd class*

$$\operatorname{Td}_{M/\mathfrak{g}} = \det\left(\frac{\alpha_{M/\mathfrak{g}}}{1 - e^{-\alpha_{M/\mathfrak{g}}}}\right) \in \bigoplus_{k=0} H^k_{\mathrm{CE}}(\mathfrak{g}, \Omega^k(M)).$$

The spaces $\Omega^k(M)$, with $k \geq 0$, are endowed with their natural $\mathfrak{g}$-module structures. Since the Lie algebra $\mathfrak{g}$ is finite dimensional, the above expression for the Todd class $\operatorname{Td}_{M/\mathfrak{g}}$ reduces to a finite sum.

4.2.3.2 Polyvector fields and polydifferential operators on $\mathfrak{g}$-manifolds

The space of polyvector fields and the space of polydifferential operators on the Lie pair $\big((\mathfrak{g} \ltimes M) \bowtie T_M,\ \mathfrak{g} \ltimes M\big)$ are naturally isomorphic to $\operatorname{tot}\big(\Lambda^\bullet \mathfrak{g}^\vee \otimes_{\Bbbk} \mathcal{T}^\bullet_{\mathrm{poly}}\big(M\big)\big)$ and $\operatorname{tot}\big(\Lambda^\bullet \mathfrak{g}^\vee \otimes_{\Bbbk} \mathcal{D}^\bullet_{\mathrm{poly}}\big(M\big)\big)$, respectively. Here $\mathcal{T}^\bullet_{\mathrm{poly}}\big(M\big)$ denotes the space of ordinary polyvector fields on M, while $\mathcal{D}^\bullet_{\mathrm{poly}}\big(M\big)$ denotes the space of ordinary polydifferential operators on M. Since $(\mathfrak{g} \ltimes M) \bowtie T_M$ is a matched pair, it follows from Proposition 2.3.4 that both $\operatorname{tot}\big(\Lambda^\bullet \mathfrak{g}^\vee \otimes_{\Bbbk} \mathcal{T}^\bullet_{\mathrm{poly}}\big(M\big)\big)$ and $\operatorname{tot}\big(\Lambda^\bullet \mathfrak{g}^\vee \otimes_{\Bbbk} \mathcal{D}^\bullet_{\mathrm{poly}}\big(M\big)\big)$ are dglas.

We proceed to describe these dgla structures. The $\mathfrak{g}$-action on M and the Schouten bracket together determine a $\mathfrak{g}$-module structure on $\mathcal{T}^k_{\mathrm{poly}}$ for each $k \geq -1$:

$$a \cdot \gamma = [\hat{a}, \gamma] \qquad \forall\, a \in \mathfrak{g},\ \gamma \in \mathcal{T}^k_{\mathrm{poly}}(M).$$

Therefore, the complex with trivial differential

$$\cdots \to \mathcal{T}^{k}_{\mathrm{poly}}(M) \xrightarrow{0} \mathcal{T}^{k+1}_{\mathrm{poly}}(M) \to \cdots$$

is a complex of $\mathfrak{g}$-modules and we obtain the differential Gerstenhaber algebra

$$\Big(\operatorname{tot}(\Lambda^{\bullet}\mathfrak{g}^{\vee} \otimes_{\Bbbk} \mathcal{T}^{\bullet}_{\mathrm{poly}}\big(M)), d_{\mathrm{CE}} + 0, [-,-], \wedge\Big),$$

whose graded Lie bracket and product are defined by

$$[\alpha \otimes \mathcal{X}, \beta \otimes \mathcal{Y}] = (-1)^{q_1 p_2} \alpha \wedge \beta \otimes [\mathcal{X}, \mathcal{Y}]$$

and

$$(\alpha \otimes \mathcal{X}) \wedge (\beta \otimes \mathcal{Y}) = (-1)^{q_1 p_2} (\alpha \wedge \beta) \otimes (\mathcal{X} \wedge \mathcal{Y}),$$

for all $\alpha \otimes \mathcal{X} \in \Lambda^{p_1}\mathfrak{g}^{\vee} \otimes_{\Bbbk} \mathcal{T}^{q_1}_{\mathrm{poly}}(M)$ and $\beta \otimes \mathcal{Y} \in \Lambda^{p_2}\mathfrak{g}^{\vee} \otimes_{\Bbbk} \mathcal{T}^{q_2}_{\mathrm{poly}}(M)$.

Likewise, the $\mathfrak{g}$-action on M and the Gerstenhaber bracket together determine a $\mathfrak{g}$-module structure on $\mathcal{D}^{\bullet}_{\mathrm{poly}}$:

$$a \cdot \mu = [\![\hat{a}, \mu]\!] \qquad \forall\, a \in \mathfrak{g},\ \mu \in \mathcal{D}^{\bullet}_{\mathrm{poly}}(M).$$

Since the Gerstenhaber bracket satisfies the graded Jacobi identity, the infinitesimal $\mathfrak{g}$-action on $\mathcal{D}^{\bullet}_{\mathrm{poly}}(M)$ is compatible with the Hochschild differential. Consequently, the Hochschild complex

$$\cdots \to \mathcal{D}^{k}_{\mathrm{poly}}(M) \xrightarrow{d_{\mathscr{H}}} \mathcal{D}^{k+1}_{\mathrm{poly}}(M) \to \cdots$$

is a complex of $\mathfrak{g}$-modules. Next, we endow $\Lambda^{\bullet}\mathfrak{g}^{\vee} \otimes_{\Bbbk} \mathcal{D}^{\bullet}_{\mathrm{poly}}(M)$ with the differential $d_{\mathrm{CE}} + d_{\mathscr{H}}$, the cup product $\smile$, and the Gerstenhaber bracket $[\![-,-]\!]$ defined by

$$(\alpha \otimes \xi) \smile (\beta \otimes \eta) = (-1)^{q_1 p_2} (\alpha \wedge \beta) \otimes (\xi \smile \eta)$$
$$[\![\alpha \otimes \xi, \beta \otimes \eta]\!] = (-1)^{q_1 p_2} \alpha \wedge \beta \otimes [\![\xi, \eta]\!]$$

for all $\alpha \otimes \xi \in \Lambda^{p_1}\mathfrak{g}^{\vee} \otimes_{\Bbbk} \mathcal{D}^{q_1}_{\mathrm{poly}}(M)$ and $\beta \otimes \eta \in \Lambda^{p_2}\mathfrak{g}^{\vee} \otimes_{\Bbbk} \mathcal{D}^{q_2}_{\mathrm{poly}}(M)$. It follows from

Proposition 2.3.4 that

$$\Big(\operatorname{tot}(\Lambda^\bullet \mathfrak{g}^\vee \otimes_\Bbbk \mathcal{D}^\bullet_{\mathrm{poly}}(M)), d_{\mathrm{CE}} + \mathrm{id} \otimes d_{\mathscr{H}}, [\![-,-]\!]\Big)$$

is a dgla whose cohomology $\mathbb{H}^\bullet_{\mathrm{CE}}(\mathfrak{g}, \mathcal{D}^\bullet_{\mathrm{poly}}(M))$, endowed with the cup product and the Gerstenhaber bracket, is a Gerstenhaber algebra.

The $\Lambda^\bullet\mathfrak{g}$-linear extension $\mathrm{hkr} : \Lambda^\bullet \mathfrak{g}^\vee \otimes_\Bbbk \mathcal{T}^\bullet_{\mathrm{poly}}(M) \to \Lambda^\bullet \mathfrak{g}^\vee \otimes_\Bbbk \mathcal{D}^\bullet_{\mathrm{poly}}(M)$ of the classical HKR map of the smooth manifold M is a quasi-isomorphism of cochain complexes but does not preserve the Lie structures on cohomologies.

4.2.3.3 Formality theorem for $\mathfrak{g}$-manifolds

Theorem 4.1.1 and Theorem 4.1.2 imply the following:

Theorem 4.2.6 (Formality theorem for $\mathfrak{g}$-manifolds)**.** *Given a $\mathfrak{g}$-manifold M and an affine torsion-free connection ∇ on M, there exists an L_∞ quasi-isomorphism $\mathcal{I}$ from the dgla* $\operatorname{tot}\Big(\Lambda^\bullet \mathfrak{g}^\vee \otimes_\Bbbk \mathcal{T}^\bullet_{\mathrm{poly}}(M)\Big)$ *to the dgla* $\operatorname{tot}\Big(\Lambda^\bullet \mathfrak{g}^\vee \otimes_\Bbbk \mathcal{D}^\bullet_{\mathrm{poly}}(M)\Big)$ *with first 'Taylor coefficient' $\mathcal{I}_1$ satisfying the following two properties:*

1. *$\mathcal{I}_1$ is, up to homotopy, an isomorphism of associative algebras (and hence induces an isomorphism of associative algebras of the cohomologies);*

2. *$\mathcal{I}_1 = \mathrm{hkr} \circ (\mathrm{td}^\nabla_{M/\mathfrak{g}})^{\frac{1}{2}}$, where $(\mathrm{td}^\nabla_{M/\mathfrak{g}})^{\frac{1}{2}} \in \bigoplus_{k=0} \Lambda^k \mathfrak{g}^\vee \otimes \Omega^k(M)$ acts on* $\operatorname{tot}\Big(\Lambda^\bullet \mathfrak{g}^\vee \otimes_\Bbbk \mathcal{T}^\bullet_{\mathrm{poly}}(M)\Big)$ *by contraction.*

Theorem 4.2.7 (Kontsevich–Duflo type theorem for $\mathfrak{g}$-manifolds)**.** *Given a $\mathfrak{g}$-manifold M, the map*

$$\mathrm{hkr} \circ \mathrm{Td}^{\frac{1}{2}}_{M/\mathfrak{g}} : \mathbb{H}^\bullet_{\mathrm{CE}}\Big(\mathfrak{g}, \mathcal{T}^\bullet_{\mathrm{poly}}(M)\Big) \xrightarrow{\cong} \mathbb{H}^\bullet_{\mathrm{CE}}\Big(\mathfrak{g}, \mathcal{D}^\bullet_{\mathrm{poly}}(M)\Big)$$

is an isomorphism of Gerstenhaber algebras. It is understood that the square root $\mathrm{Td}^{\frac{1}{2}}_{M/\mathfrak{g}}$ of the Todd class $\mathrm{Td}_{M/\mathfrak{g}} \in \bigoplus_{k=0} H^k_{\mathrm{CE}}\Big(\mathfrak{g}, \Omega^k(M)\Big)$ acts on $\mathbb{H}^\bullet_{\mathrm{CE}}\Big(\mathfrak{g}, \mathcal{T}^\bullet_{\mathrm{poly}}(M)\Big)$ by contraction.

4.2.4 Foliations

Let $\mathcal{F}$ be a regular foliation of a smooth manifold M. The tangent bundle of $\mathcal{F}$ is a subbundle of T_M, denoted $T_{\mathcal{F}}$, whose sections are closed under the Lie bracket of vector fields. Therefore, $(T_M, T_\mathcal{F})$ is a Lie pair. Its quotient $N_\mathcal{F} = T_M/T_\mathcal{F}$ is the normal bundle of the foliation $\mathcal{F}$. We have the short exact sequence of vector bundles

$$0 \to T_\mathcal{F} \to T_M \xrightarrow{q} N_\mathcal{F} \to 0.$$

The Bott $T_\mathcal{F}$-connection on $N_\mathcal{F}$ is defined by

$$\nabla_a^{\mathrm{Bott}} q(l) = q([a, l]), \quad \forall\, a \in \Gamma(T_\mathcal{F}),\ l \in \mathfrak{X}(M).$$

The Chevalley–Eilenberg Lie algebroid cohomology $H^\bullet_{\mathrm{CE}}(T_\mathcal{F}, \mathfrak{M})$ with coefficients in a $T_\mathcal{F}$-module $\mathfrak{M}$ coincides exactly with the leafwise de Rham cohomology $H^\bullet_{\mathrm{dR}}(\mathcal{F}, \mathfrak{M})$ of the foliation $\mathcal{F}$ with coefficients in the module $\mathfrak{M}$.

4.2.4.1 Atiyah and Todd classes of foliations

Let ∇ be a T_M-connection on $N_\mathcal{F}$ extending the Bott $T_\mathcal{F}$-connection. The torsion of ∇ is the bundle map $T^\nabla : \Lambda^2 T_M \to N_\mathcal{F}$ defined by

$$T^\nabla(X, Y) = \nabla_X q(Y) - \nabla_Y q(X) - q([X, Y]),$$

for all vector fields X and Y on M. The Atiyah cocycle associated with ∇ is the bundle map $R^\nabla_{1,1} : T_\mathcal{F} \otimes N_\mathcal{F} \to \mathrm{End}(N_\mathcal{F})$ — or the corresponding section of $T^\vee_\mathcal{F} \otimes T^\perp_\mathcal{F} \otimes \mathrm{End}(N_\mathcal{F})$ — defined by

$$R^\nabla_{1,1}\big(a; q(l)\big) = \nabla_a \nabla_l - \nabla_l \nabla_a - \nabla_{[a,l]}, \qquad \forall\, a \in \Gamma(T_\mathcal{F}),\ l \in \Gamma(T_M).$$

According to Proposition 2.2.6, $R^\nabla_{1,1} \in \Gamma(T^\vee_\mathcal{F} \otimes T^\perp_\mathcal{F} \otimes \mathrm{End}(N_\mathcal{F}))$ is a leafwise de Rham closed 1-form with values in the $T_\mathcal{F}$-module $T^\perp_\mathcal{F} \otimes \mathrm{End}(N_\mathcal{F})$. Its cohomology class $\alpha_\mathcal{F} \in H^1_{\mathrm{dR}}(\mathcal{F}, T^\perp_\mathcal{F} \otimes \mathrm{End}(N_\mathcal{F}))$ is independent of the choice of T_M-connection ∇ extending the Bott $T_\mathcal{F}$-connection and is called the Atiyah class of the foliation $\mathcal{F}$. It is precisely the invariant of the foliation that was first introduced by Molino [52].

The Todd cocycle of the foliation $\mathcal{F}$ associated with the connection ∇ is the leafwise closed form

$$\mathrm{td}^\nabla_\mathcal{F} = \det\left(\frac{R^\nabla_{1,1}}{1 - e^{-R^\nabla_{1,1}}}\right) \in \bigoplus_{k=0} \Gamma(\Lambda^k T^\vee_\mathcal{F} \otimes \Lambda^k T^\perp_\mathcal{F}).$$

The Todd class of the foliation $\mathcal{F}$ is the corresponding cohomology class

$$\mathrm{Td}_{\mathcal{F}} = \det\left(\frac{\alpha_{\mathcal{F}}}{1-e^{-\alpha_{\mathcal{F}}}}\right) \in \bigoplus_{k=0} H^k_{\mathrm{dR}}(\mathcal{F}, \Lambda^k T^{\perp}_{\mathcal{F}}).$$

4.2.4.2 Transversal polyvector fields and transversal polydifferential operators on foliations

It follows from Corollary 2.3.14 and Corollary 2.3.19 applied to the Lie pair $(T_M, T_{\mathcal{F}})$ that both $\mathrm{tot}\left(\Gamma(\Lambda^\bullet T^\vee_{\mathcal{F}}) \otimes_R \mathcal{T}^\bullet_{\mathrm{poly}}(N_{\mathcal{F}})\right)$ and $\mathrm{tot}\left(\Gamma(\Lambda^\bullet T^\vee_{\mathcal{F}}) \otimes_R \mathcal{D}^\bullet_{\mathrm{poly}}(N_{\mathcal{F}})\right)$ can be endowed with L_∞ algebra structures, unique up to L_∞ quasi-isomorphisms. Here $\mathcal{T}^\bullet_{\mathrm{poly}}(N_{\mathcal{F}}) = \Gamma(\Lambda^{\bullet+1} N_{\mathcal{F}})$ can be considered as the space of polyvector fields transversal to $\mathcal{F}$. The first bracket on $\mathrm{tot}\left(\Gamma(\Lambda^\bullet T^\vee_{\mathcal{F}}) \otimes_R \mathcal{T}^\bullet_{\mathrm{poly}}(N_{\mathcal{F}})\right)$ is the leafwise de Rham differential d_{dR} with values in $\mathcal{T}^\bullet_{\mathrm{poly}}(N_{\mathcal{F}})$. Similarly, $\mathcal{D}^\bullet_{\mathrm{poly}}(N_{\mathcal{F}}) = \bigoplus_{k=-1}^{\infty} \mathcal{D}^k_{\mathrm{poly}}(N_{\mathcal{F}})$ can be considered as the space of polydifferential operators transversal to $\mathcal{F}$. Here $\mathcal{D}^{-1}_{\mathrm{poly}}(N_{\mathcal{F}})$ denotes the algebra R of smooth functions on the manifold M, $\mathcal{D}^0_{\mathrm{poly}}(N_{\mathcal{F}})$ denotes the left R-module $\frac{\mathcal{U}(T_M)}{\mathcal{U}(T_M)\cdot\Gamma(T_{\mathcal{F}})} \cong \frac{\mathcal{D}(M)}{\mathcal{D}(M)\cdot\Gamma(T_{\mathcal{F}})}$ of 'transverse differential operators,' and $\mathcal{D}^k_{\mathrm{poly}}(N_{\mathcal{F}})$ denotes the tensor product $\mathcal{D}^0_{\mathrm{poly}}(N_{\mathcal{F}}) \otimes_R \cdots \otimes_R \mathcal{D}^0_{\mathrm{poly}}(N_{\mathcal{F}})$ of $(k+1)$ copies of the left R-module $\mathcal{D}^0_{\mathrm{poly}}(N_{\mathcal{F}})$. (If there existed a foliation $\mathcal{F}'$ transverse to $\mathcal{F}$, the space $\mathcal{D}^0_{\mathrm{poly}}(N_{\mathcal{F}})$ would be isomorphic to $\mathcal{U}(T_{\mathcal{F}'})$, the space of differential operators in the direction of $\mathcal{F}'$.) A different construction of the L_∞ structure of Transversal polyvector fields can be found in [64].

For every $k \geq 0$, $\mathcal{D}^k_{\mathrm{poly}}(N_{\mathcal{F}})$ is naturally a left $\mathcal{U}(T_{\mathcal{F}})$-module and we can consider the associated leafwise de Rham differential

$$d_{\mathrm{dR}} : \Gamma(\Lambda^\bullet T^\vee_{\mathcal{F}}) \otimes_R \mathcal{D}^k_{\mathrm{poly}}(N_{\mathcal{F}}) \to \Gamma(\Lambda^{\bullet+1} T^\vee_{\mathcal{F}}) \otimes_R \mathcal{D}^k_{\mathrm{poly}}(N_{\mathcal{F}}).$$

Since $\frac{\mathcal{U}(T_M)}{\mathcal{U}(T_M)\cdot\Gamma(T_{\mathcal{F}})}$ is a coalgebra over R with an associative comultiplication

$$\Delta : \frac{\mathcal{U}(T_M)}{\mathcal{U}(T_M)\cdot\Gamma(T_{\mathcal{F}})} \to \frac{\mathcal{U}(T_M)}{\mathcal{U}(T_M)\cdot\Gamma(T_{\mathcal{F}})} \otimes_R \frac{\mathcal{U}(T_M)}{\mathcal{U}(T_M)\cdot\Gamma(T_{\mathcal{F}})},$$

there is a Hochschild differential

$$d_{\mathscr{H}} : \mathcal{D}^k_{\mathrm{poly}}(N_{\mathcal{F}}) \to \mathcal{D}^{k+1}_{\mathrm{poly}}(N_{\mathcal{F}}),$$

which extends to a $\Gamma(\Lambda^\bullet T^\vee_{\mathcal{F}})$-graded linear operator of degree $+1$ on $\mathrm{tot}\Big(\Gamma(\Lambda^\bullet T^\vee_{\mathcal{F}}) \otimes_R$

$\mathcal{D}^\bullet_{\mathrm{poly}}(N_\mathcal{F})\Big)$ still denoted $d_\mathscr{H}$ by abuse of notation. The first bracket of the L_∞ algebra structure on $\mathrm{tot}\Big(\Gamma(\Lambda^\bullet T_\mathcal{F}^\vee)\otimes_R \mathcal{D}^\bullet_{\mathrm{poly}}(N_\mathcal{F})\Big)$ is $d_{\mathrm{dR}}+d_\mathscr{H}$.

The L_∞ structures on $\mathrm{tot}\Big(\Gamma(\Lambda^\bullet T_\mathcal{F}^\vee)\otimes_R \mathcal{T}^\bullet_{\mathrm{poly}}(N_\mathcal{F})\Big)$ and $\mathrm{tot}\Big(\Gamma(\Lambda^\bullet T_\mathcal{F}^\vee)\otimes_R \mathcal{D}^\bullet_{\mathrm{poly}}(N_\mathcal{F})\Big)$ depend on the choice of a splitting of the short exact sequence $0\to T_\mathcal{F}\to T_M\to N_\mathcal{F}\to 0$ and a torsion-free T_M-connection on $N_\mathcal{F}$ extending the Bott $T_\mathcal{F}$-connection [59]. However, different choices induce quasi-isomorphic L_∞ algebra structures. Moreover, the first 'Taylor coefficient' of the L_∞ quasi-isomorphism is the identity map. Therefore, the resulting Gerstenhaber algebra structures on the cohomologies $\mathbb{H}^\bullet_{\mathrm{dR}}(\mathcal{F},\mathcal{T}^\bullet_{\mathrm{poly}}(N_\mathcal{F}))$ and $\mathbb{H}^\bullet_{\mathrm{dR}}(\mathcal{F},\mathcal{D}^\bullet_{\mathrm{poly}}(N_\mathcal{F}))$ are indeed canonical [3].

According to Section 2.2.5, skew-symmetrization induces a quasi-isomorphism of cochain complexes

$$\mathrm{hkr}:\mathrm{tot}\Big(\Gamma(\Lambda^\bullet T_\mathcal{F}^\vee)\otimes_R \mathcal{T}^\bullet_{\mathrm{poly}}(N_\mathcal{F})\Big)\to \mathrm{tot}\Big(\Gamma(\Lambda^\bullet T_\mathcal{F}^\vee)\otimes_R \mathcal{D}^\bullet_{\mathrm{poly}}(N_\mathcal{F})\Big).$$

4.2.4.3 Formality theorem for foliations

Theorem 4.1.1 and Theorem 4.1.2 imply the following

Theorem 4.2.8 (Formality theorem for foliations)**.** *Let $\mathcal{F}$ be a regular foliation on a smooth manifold M. Given a splitting of the short exact sequence $0\to T_\mathcal{F}\to T_M\to N_\mathcal{F}\to 0$ and a torsion-free T_M-connection ∇ on $N_\mathcal{F}$, there exists an L_∞ quasi-isomorphism*

$$\mathcal{I}:\mathrm{tot}\Big(\Gamma(\Lambda^\bullet T_\mathcal{F}^\vee)\otimes_R \mathcal{T}^\bullet_{\mathrm{poly}}(N_\mathcal{F})\Big)\to \mathrm{tot}\Big(\Gamma(\Lambda^\bullet T_\mathcal{F}^\vee)\otimes_R \mathcal{D}^\bullet_{\mathrm{poly}}(N_\mathcal{F})\Big)$$

with first 'Taylor coefficient' $\mathcal{I}_1$ satisfying the following two properties:

1. *$\mathcal{I}_1$ preserves the associative algebra structures (wedge and cup product, respectively) up to homotopy;*
2. *$\mathcal{I}_1=\mathrm{hkr}\circ(\mathrm{td}^\nabla_\mathcal{F})^{\frac{1}{2}}$, where $(\mathrm{td}^\nabla_\mathcal{F})^{\frac{1}{2}}\in\bigoplus_{k=0}^\infty\Gamma(\Lambda^k T_\mathcal{F}^\vee\otimes\Lambda^k T_\mathcal{F}^\perp)$ acts on $\mathrm{tot}\Big(\Gamma(\Lambda^\bullet T_\mathcal{F}^\vee)\otimes_R \mathcal{T}^\bullet_{\mathrm{poly}}(N_\mathcal{F})\Big)$ by contraction.*

Theorem 4.2.9 (Kontsevich-Duflo type theorem for foliations)**.** *Given a regular foliation $\mathcal{F}$ on a smooth manifold M, the map*

$$\mathrm{hkr}\circ\mathrm{Td}_\mathcal{F}^{\frac{1}{2}}:\mathbb{H}^\bullet_{\mathrm{dR}}(\mathcal{F},\mathcal{T}^\bullet_{\mathrm{poly}}(N_\mathcal{F}))\xrightarrow{\cong}\mathbb{H}^\bullet_{\mathrm{dR}}(\mathcal{F},\mathcal{D}^\bullet_{\mathrm{poly}}(N_\mathcal{F}))$$

is an isomorphism of Gerstenhaber algebras. It is understood that the square root $\mathrm{Td}_{\mathcal{F}}^{\frac{1}{2}}$ *of the Todd class* $\mathrm{Td}_{\mathcal{F}} \in \bigoplus_{k=0} H^k_{\mathrm{dR}}(\mathcal{F}, \Lambda^k T_{\mathcal{F}}^{\perp})$ *acts on* $\mathbb{H}^{\bullet}_{\mathrm{dR}}(\mathcal{F}, \mathcal{T}^{\bullet}_{\mathrm{poly}}(N_{\mathcal{F}}))$ *by contraction.*

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